London School of Economics and Political Science

Essays on the Social Welfare Effects of Fiscal Policy

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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For the avoidance of doubt, I certify that *all chapters* of this thesis consist *solely of my own work*. This thesis contains 42333 words.

Abstract

This thesis contains three chapters, each concerning the social welfare effects of fiscal policy. The first chapter examines the changes to government spending and tax rates that achieve a potentially substantial reduction in government debt at least cost to social welfare. The consequences of altering the speed of debt reduction are also examined. The second chapter considers how differences in monetary policy regime and stance may alter the optimal mix of spending and tax rate changes for government debt reduction. These two chapters make use of the representative agent theoretical framework. By contrast, the third chapter uses a framework of heterogeneous agents, in which there is a non-trivial distribution of wealth and income. The framework includes both aggregate and idiosyncratic uncertainty. The third chapter characterises a constrained efficient outcome in this framework. This outcome is treated as a welfare benchmark. The competitive equilibrium outcome is compared to this benchmark and shown to be constrained inefficient in certain circumstances. The sign and magnitude of constrained inefficiency in competitive equilibrium depends critically on the way in which aggregate shocks affect the distribution of idiosyncratic shocks. The extent of wealth inequality is also an important determining factor. Competitive equilibria under different fiscal policies are then considered as potential welfare improvements.

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Introduction

This thesis contains three chapters, each concerning the social welfare effects of fiscal policy.

The first chapter investigates what mix of government spending and tax rate changes finances a desired reduction in outstanding government debt at least cost to social welfare. The appropriate speed for debt reduction is also considered. These questions are investigated theoretically in a deterministic, neoclassical model with public and private capital. Government spending takes the form of investment in a productive public capital stock. The government solves a Ramsey (optimal fiscal policy) problem and chooses government spending and tax rates on private capital and wage income to engineer an exogenous downward path for the level of real government debt. The problem is solved for different exogenous downward paths for debt, each reflecting a different speed of debt reduction. Solutions are obtained using numerical methods, with parameters set in accordance with empirical studies. In a model with public capital but without private capital, a reduction in the level of government debt is optimally financed by lowering public spending. Labour tax rates are lowered as debt is retired. Increasing the speed of government debt reduction implies relatively lower spending and higher tax rates while debt is being paid down, leading to lower consumption. However, these fiscal policy measures can be reversed earlier because debt is paid down faster. Consumption also recovers faster. In a model with both private and public capital, the Ramsey planner prefers to finance debt reduction by sustained taxation of private capital income. The post tax rate of return on private capital falls. The real market interest rate on government bonds is also relatively low, because of a no arbitrage condition. This reduces the real burden of government interest payments on debt. Public spending is left relatively unaltered and labour income taxes are initially low. Faster debt reduction requires higher tax rates on private capital income, accompanied by lower real interest rates on government debt.

The second chapter considers how differences in monetary policy regime (and the stance of policy) can change the optimal combination of government spending changes and tax rate changes that achieve a given path for government debt reduction over a given timeframe. This question is addressed by solving a Ramsey problem in a theoretical model with monetary policy. The Ramsey planner chooses a path for productive government spending and distortionary labour income tax rates in order to achieve an exogenous downward path for the level of real government debt. I consider two monetary policy regimes. First, I allow the Ramsey planner to determine monetary policy optimally. Second, I constrain the Ramsey planner to have to choose a nominal interest rate consistent with an interest rate feedback rule, similar to those used to describe the behaviour of inflation targeting central banks. In both settings, I present the deterministic transition path of the economy as the government engineers a reduction in real government debt of between approximately five and ten percentage points of GDP. I find that the government chooses to finance some of the required debt reduction by lowering government spending (public investment). Distortionary tax rates are reduced as spending falls and debt is lowered. The real interest rate is generally higher when monetary policy is set in accordance with an interest rate feedback rule, compared with when monetary policy is chosen optimally by the government. Higher real interest rates increase the real value of debt repayments that the government must make. Reducing debt when monetary policy follows the rule requires higher tax rates and leads to lower levels of real consumption.

In the third chapter, the theoretical assumption that there is a representative consumer is relaxed. This chapter studies the constrained efficiency of saving / private capital accumulation in a model with incomplete markets for both aggregate and idiosyncratic risk. Understanding the efficiency (i.e. welfare properties) of competitive equilibria in a heterogeneous agent framework is important for designing fiscal policy that improves welfare. Constrained inefficiency arises because of the pecuniary externality of saving on factor prices, something that is not taken into account in individual consumer decisions. This chapter shows that the sign and magnitude of constrained inefficiency depends critically on the interaction between aggregate and idiosyncratic risk. That is, on how aggregate shocks change the distribution of the idiosyncratic shock, conditional on the aggregate shock realisation. For certain levels of wealth inequality, there can be constrained inefficiency due to under saving if bad (good) aggregate shocks make the realisation of bad (good) idiosyncratic employment shocks more likely, as may happen in recessions and booms. The aggregate shock realisation changes the skewness of the idiosyncratic shock's conditional distribution in this case. By contrast, there can be constrained inefficiency due to over saving if a bad (good) aggregate shock makes bad idiosyncratic employment shocks potentially more (less) severe, in the sense that the idiosyncratic shock has a larger (smaller) conditional variance. In both these examples, the unconditional distribution of the idiosyncratic shock can be approximately the same as in the case where aggregate and idiosyncratic shocks are fully independent. This shows the importance of studying efficiency in the presence of both aggregate and idiosyncratic risk, rather than only considering the latter. The sign and magnitude of constrained inefficiency also depends on the degree of wealth inequality. A tax / subsidy on the return to saving can induce different saving behaviour that improves efficiency, if accompanied by lump sum subsidies / taxes that balance the government budget but do not redistribute income. Re-distribution by fiscal transfers to consumers that are asset poor can be itself welfare improving. This can justify positive taxation on the rate of return to capital, even if saving / capital accumulation is inefficiently low (i.e. constrained inefficient) in a competitive equilibrium without fiscal policy.

Chapter 1 The Speed and Composition of Optimal Fiscal Policy for Government Debt Reduction

1.1 Introduction

A government seeking to engineer a reduction in its outstanding level of debt faces a number of key questions. First, governments must decide what mix of government spending and tax rate changes should be implemented in order to help finance government debt reduction. Presumably, governments wish to choose the policy mix that achieves the debt reduction objective with the minimum cost to the economy and society. Second, governments must choose a desired time profile for government debt reduction: that is, by how much should government debt fall in each period of time?

This chapter presents a theoretical investigation of these questions.

The approach of the chapter is first to take a particular downward path for the level of government debt as given, and determine what mix of government spending and tax rate changes should a benevolent government implement to achieve it? The theoretical framework used in the chapter is that of a deterministic, neoclassical model. There are publicly owned and privately owned capital stocks. Government spending takes the form of investment in the *productive* public capital stock. The government can raise revenue by levying distortionary taxes on income from private capital and from labour income, but does not have access to lump sum taxation. The government solves a Ramsey optimal policy problem, choosing a sequence of government spending and tax rates to maximise consumer welfare, subject to the requirement that the chosen sequence gives rise to a competitive equilibrium and provided the desired downward path for the level of government debt is achieved. It is assumed that the government can fully commit to an announced sequence of government spending and tax rates, that is referred to as a *fiscal policy*. It should be noted that there is no long run or steady state economic growth in the model.

The *speed* of debt reduction is considered by studying different downward paths for the level of government debt. Each downward path is specified exogenously and the Ramsey planner chooses a fiscal policy to achieve it. The paths are of the same length and involve reducing real government debt by the same amount. However, they differ as to the *timing* of debt reduction. Paths reflecting a relatively *fast* pace of debt reduction require more of the desired debt reduction to occur sooner.

A numerical simulation is then presented, using parameter values chosen to reflect the results of empirical studies. Specifically, what is shown is the deterministic transition path of the economy from a steady state in which the level of government debt is relatively high to one in which government debt is relatively low. Government debt falls as a percentage of gross domestic product (GDP) over the transition path. Along this transition path, government spending and tax rates are set at Ramsey optimal levels, subject to being consistent with a particular, exogenous downward path for the level of government debt from its initial condition to its target level in the new steady state. The deterministic transition path is solved for by simultaneously solving a system of equations describing the behaviour of the economy under optimally chosen fiscal policy, over a fixed time horizon. The values of government debt and other variables are initialised at their values in the steady state with a relatively high level of government debt. The values of these variables in the steady state with a relatively low level of government debt are imposed as terminal conditions in the final period of the transition path. The level of government debt follows a particular, exogenous downward path over the transition, reaching its target level at or before the end of the chosen time horizon. This numerical simulation is performed for different exogenous downward paths of government debt, reflecting different *speeds* of debt reduction. The results for optimal fiscal policy over the transition path can be compared for different speeds of debt reduction.

First, I consider a relatively slow pace of debt reduction, such that government debt falls from 60 per cent of GDP to 45 per cent of GDP in approximately ten years. In a model with *only* public capital (and without physical capital), the Ramsey planner finances debt reduction by setting public spending (i.e. public investment) sufficiently low such that the public capital stock is eroded over time. The tax rate on labour income is relatively high in the early periods of the transition path. This lowers the post tax wage and consumption declines. The labour tax rate is then lowered as the level of government debt approaches its new, lower target level. I compare these results with the choices of the Ramsey planner under a different exogenous path for government debt - one which involves a *faster* pace of debt reduction, in the sense that more of the desired debt reduction occurs sooner. In the early periods of the transition, government spending (i.e. public investment) is now lower and the labour tax rate is higher. Consumer income and the level of real consumption are both lower because of this. However, the position changes further along the transition path. Government debt nears its new, lower target level sooner. This allows labour tax rates to be reduced sooner and government spending to be increased. The after-tax wage and consumption recover faster. It seems that a faster pace of debt reduction requires more painful austerity in the short run (in terms of lower consumption), but that the costs of austerity are of shorter duration.

Next, I consider the full model with *both* private *and* public capital. This allows the government the ability to tax income from investment in private capital. The Ramsey planner chooses a positive tax rate on private capital income over much of the transition path. This reduces the post-tax rental rate of return on private capital investment. The real market interest rate on government bonds is also relatively low, reflecting a no arbitrage condition. This reduces the real value of interest payments on government debt. The planner principally uses revenue from private capital income taxation to finance debt reduction, something made easier by lower real interest rates on government bonds. The reduction in the post-tax rental rate of return on capital leads to lower private investment, eroding the private capital stock. Compared with the model with *only* public capital, government spending (i.e. public investment) is left relatively unaltered and the public capital stock relatively unaffected. Tax rates on labour income are also lower for much of the transition path.

A faster exogenous pace of debt reduction prompts the Ramsey planner to set both capital and labour income tax rates higher in the early periods of the transition. The higher capital income tax rate forces the rental rate on capital and the real interest rate on government bonds even lower. However, debt is paid down to be near its new target level more quickly, so the tax increases can be unwound sooner than if debt was reduced more slowly. This implies a faster recovery in the post-tax rental rate on capital, boosting private investment. The result is that the private capital stock does not fall for as long or by as much as when debt is reduced more slowly. Consumption is also higher than it otherwise would be, further along the transition path.

The remainder of this chapter is organised as follows: Section (1.2) describes the deterministic, neoclassical model that is the theoretical setting for the chapter. The Ramsey problem is formulated and the conditions characterising the economy's behaviour under Ramsey optimal fiscal policy are obtained. Section (1.3) presents the numerical simulation exercise, providing the deterministic transition path for the economy under Ramsey optimal fiscal policy, from an initial steady state with a relatively high level of government debt to one with a relatively low level of government debt. Section (1.4) concludes.

1.2 The Model

The optimal fiscal policy problem in this chapter is solved in the theoretical setting of a neoclassical model in discrete time indexed by $t = \{0, 1, 2, ...\}$. The model is deterministic, so that there is no uncertainty and agents have perfect foresight. I abstract from uncertainty because it is not the aim of the chapter to study the response of the economy to shocks in the neighbourhood of a particular steady state. Rather, the aim of the chapter is to study how optimal fiscal policy engineers the economy's transition from a steady state with a relatively high level of government debt to one with a relatively low level of government debt. Adding uncertainty to the model would not add anything essential to performing this exercise. There are three types of agent in the model: consumers, perfectly competitive firms and a benevolent government.

1.2.1 The Production Technology

The deterministic, neoclassical model has a single sector - that is, a single production technology. This technology is used to produce a homogeneous good that can be used for consumption, or investment in the private and public capital stocks. The number of units of the homogeneous good Y_t that can be produced each period is given by

$$Y_t = F(G_{t-1}, K_{t-1}, h_t) \tag{1.1}$$

where F is a function $F : \mathbb{R}^3_+ \longrightarrow \mathbb{R}_+$ in three inputs: the public capital stock G_{t-1} owned by the government that is predetermined each period, the privately owned capital stock K_{t-1} that is also predetermined and labour supply h_t. The function F is continuously differentiable in its arguments (i.e. each of the first order partial derivatives exist and is continuous) and also strictly increasing in its arguments, so that

$$\begin{aligned} F_G(G_{t-1}, K_{t-1}, h_t) &> 0 \qquad (1.2) \\ F_K(G_{t-1}, K_{t-1}, h_t) &> 0 \\ F_h(G_{t-1}, K_{t-1}, h_t) &> 0 \end{aligned}$$

It is further the case that

$$\begin{aligned} Y_t &= F(0, K_{t-1}, h_t) = 0 \qquad (1.3) \\ Y_t &= F(G_{t-1}, 0, h_t) = 0 \\ Y_t &= F(G_{t-1}, K_{t-1}, 0) = 0 \end{aligned}$$

The function F is strictly quasi-concave. The production technology each period also satisfies the following Inada conditions:

$$\begin{aligned}
\lim_{K\to\infty} F_K(G_{t-1}, K_{t-1}, h_t) &= 0 \\
\lim_{G\to\infty} F_G(G_{t-1}, K_{t-1}, h_t) &= 0 \\
\lim_{K\to 0} F_K(G_{t-1}, K_{t-1}, h_t) &= \infty \\
\lim_{G\to 0} F_G(G_{t-1}, K_{t-1}, h_t) &= \infty
\end{aligned} \tag{1.4}$$

It is assumed that the production function exhibits constant returns to scale in *private* inputs. Private capital and labour are private inputs because they are owned by private agents and not the government, as defined in section (1.2.2). I make this assumption for analytical convenience because it implies that perfectly competitive firms earn zero profits in competitive equilibrium. There are increasing returns to scale across all three inputs.[1] However, note that only the public and private capital stocks constitute *reproducible* inputs, since the endowment of available hours in the economy is fixed and constant: see section (1.2.2). There will be decreasing returns to scale in *reproducible* inputs, a necessary condition for a zero growth steady state.

1.2.2 Consumers

The endowment of available hours in the economy each period is normalised to one. Markets are complete, so it is possible to treat the economy as if there is a single representative consumer with an endowment of time or hours each period equal to one, with $0 \leq h_t \leq 1$. The consumer accumulates and owns a stock of private capital K^s_{t-1}, which is rented to firms for use in production. The representative consumer owns all shares in firms. Each period, the representative agent consumes an amount of the consumption good c_t, supplies an amount of labour hours h^s_t and saves in the form of both investment in its private capital stock i^K_t and demand for one

[1] The assumption of increasing returns to scale across all three inputs implies that the public capital stock is a stock of pure public goods, to which all firms have costless and unrestricted access. It is possible to relax the assumption that the public capital stock is non-rival and allow for "congestion" of the public capital stock. One way to achieve this would be to allow the ratio of the public to private capital stocks to enter the production function, rather than the level of the public capital stock. I leave this for future research.

period risk free government bonds b_t^d. In this model, the level of *outstanding government debt* at the beginning of period t equals the *level of government borrowing* the previous period. This is because of the assumption that government bonds reach maturity after one period. The terms *government debt* and *government borrowing* may be used interchangeably for this reason.

The representative consumer's problem is to choose in advance an infinite sequence $\{c_t, h_t^s, K_t^s, b_t^d\}_{t=0}^{\infty}$ of consumption, hours worked, private investment (implicitly by choosing next period private capital stock) and demand for government bonds to maximise the discounted present value of future utility

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(1-h_t)\right] \tag{1.5}$$

where β is the subjective discount factor $0 < \beta < 1$. The period utility functions $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $v : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are assumed bounded above, continuously differentiable in each argument and strictly increasing in each argument. The functions are assumed to be strictly quasi-concave. The following Inada conditions hold:

$$\begin{aligned} \lim_{c \to \infty} u'(c) &= 0 \\ \lim_{c \to 0} u'(c) &= \infty \\ \lim_{h \to 1} v'(1-h) &= \infty \end{aligned} \tag{1.6}$$

The representative consumer's maximisation problem is subject to the consumer's budget constraint each period

$$c_t + p_t^b b_t^d + K_t^s - (1-\delta^K)K_{t-1}^s \leq (1-\tau_t^h)w_t h_t^s + (1-\tau_{t-1}^K)r_t K_{t-1}^s + b_{t-1}^d \tag{1.7}$$

and also subject to the inequality constraint every period

$$0 \leq h_t^s \leq 1 \tag{1.8}$$

as well as the non-negativity constraints every period

$$c_t \geq 0 \tag{1.9}$$

$$b_t^d \geq 0 \tag{1.10}$$

$$K_t^s \geq 0 \tag{1.11}$$

plus the equation determining the accumulation of the private capital stock

$$K_t^s = i_t^K + (1 - \delta^K) K_{t-1}^s \tag{1.12}$$

with initial conditions b_{-1}^d and K_{-1}^s.

The consumer takes as given all prices and tax rates each period: w_t is the real wage, r_t is the rental rate paid to private capital leased by households to firms in period t, p_t^b is the time t price of a risk-free government bond paying one unit the next period. Note that the tax rate τ_{t-1}^K on private capital income received by consumers in period t is *predetermined*. I make this assumption because fixing the *initial* tax rate τ_{-1}^K is a convenient way of preventing the government from using the taxation of private capital income at time zero as a form of lump sum, non-distortionary taxation, confiscating all income from the pre-existing stock of private capital. Furthermore, the assumption of a pre-determined capital income tax rate prevents the government from confiscating all private capital income in the first period of the deterministic transition path from a steady state with a relatively high level of government debt to a steady state with a low level of government debt, as presented in section (1.3).

It turns out that the inequality constraint (1.8) and the non-negativity constraints (1.9), (1.10) and (1.11) will never bind because of the assumptions made about the production technology in subsection (1.2.1) and utility in subsection (1.2.2). The Inada conditions (1.6) show that consumption must always be positive and that hours worked can never equal one. The assumption in (1.2) that production is zero if any individual input is zero implies that labour supply and the private capital stock must always be positive. Otherwise, there would be no output and non-positive consumption, which has been ruled out. However, it is important to note that private investment i_t^K may be negative, allowing for disinvestment in the private capital stock. The inequality constraint on demand for government bonds (1.10) will never bind in competitive equilibrium because I will assume that government borrowing is exogenous, positive and on a downward path, in all competitive equilibria. Similarly, although in principle consumers are subject to an upper limit on government bond holdings to prevent the government running Ponzi schemes, this will never bind in competitive equilibrium. The consumer budget constraint (1.7) will hold with equality each period in equilibrium because it can never be optimal to leave resources unspent, given that the marginal utility of consumption $u'(c) > 0$ for all $c > 0$.

Solving the representative consumer's problem yields conditions that characterise optimising behaviour by consumers each period:

$$v'(1 - h_t^s) = u'(c_t)(1 - \tau_t^h)w_t \tag{1.13}$$

$$u'(c_t) = \beta \frac{u'(c_{t+1})}{p_t^b} \tag{1.14}$$

$$u'(c_t) = \beta u'(c_{t+1}) \left[(1 - \tau_t^k) r_{t+1} + (1 - \delta^K)\right] \tag{1.15}$$

as well as the limit on government bond holdings ruling out Ponzi schemes and the budget constraint (1.7) each period. Please see Appendix (A.1) for further information on the derivations. Equation (1.13) is an *intra*temporal optimality condition requiring that the marginal disutility of an additional hour worked equal the marginal utility of the additional after-tax real income that would be earned by this. Equation (1.14) is an Euler equation for government bonds, requiring that the marginal disutility of foregoing current period consumption for saving equals the present utility value of return on marginal additional saving next period. Finally, equation (1.15) is an Euler equation for private capital investment, requiring that the marginal disutility of foregoing current period consumption for private investment equals the present utility value of the next period marginal return on this investment, after tax. The combination of equations (1.14) and (1.15) implies a no-arbitrage condition: namely that the present utility value of next period marginal return on private capital investment and on government bond holdings (i.e. the two forms of savings) must be equal.

1.2.3 Firms

It is possible to view the economy as containing a single, representative firm. This is because there is only one type of good produced in the single sector economy and labour and private capital can be viewed as being supplied by a single representative consumer, with public capital provided by the government. The representative firm is assumed to behave in a perfectly competitive fashion, taking the price of its output Y_t as given each period. Further, the representative firm can be treated as solving a static, profit maximisation problem each period. This is because the firm does not make intertemporal decisions. Private capital services and labour services are rented from the representative consumer each period, with public capital provision being the responsibility of the government.

Normalising the price of output to one, the firm chooses labour demanded h_t^d, and private capital services demanded K_{t-1}^d each period to maximise period profit

$$Y_t - w_t h_t^d - r_t K_{t-1}^d \tag{1.16}$$

subject to

$$Y_t \leq F(G_{t-1}, K_{t-1}^d, h_t^d) \tag{1.17}$$

Because the firm is perfectly competitive, it is assumed that the firm takes the wage w_t and the rental rate r_t as given and does not take into account the effect of its choices of h_t^d and K_{t-1}^d on these factor prices when maximising profit. It can never be optimal for the firm not to use rented inputs, so that (1.17) holds with equality and can be substituted into (1.16). The profit maximisation problem can be expressed as

$$\max_{K_{t-1},h_t} F(G_{t-1}, K^d_{t-1}, h^d_t) - w_t h^d_t - r_t K^d_{t-1} \tag{1.18}$$

The first order conditions of this simple unconstrained maximisation problem are sufficient to characterise a unique maximum for two reasons. First, this is because the expression for period profit in (1.18) is strictly quasi-concave, because it is a linear combination of the production technology (1.1), which is assumed to be strictly quasi-concave. Second, the opportunity set for firms can be assumed to be convex. Available labour supply is necessarily bounded between zero and one, as shown in (1.8), while the public and private capital stocks are necessarily bounded below by zero. For convexity of the firm's opportunity set each period, it must only then be assumed that available public and private capital are bounded above by some arbitrary constants The conditions describing firm optimality behaviour are thus

$$w_t = F_h(G_{t-1}, K^d_{t-1}, h^d_t) \tag{1.19}$$

$$r_t = F_K(G_{t-1}, K^d_{t-1}, h^d_t) \tag{1.20}$$

These conditions (1.19) and (1.20) imply that labour and private capital are paid their marginal products in equilibrium. Profit each period in equilibrium can be expressed as

$$F(G_{t-1}, K^d_{t-1}, h^d_t) - F_h(G_{t-1}, K^d_{t-1}, h^d_t)h^d_t - F_K(G_{t-1}, K^d_{t-1}, h^d_t)K^d_{t-1} \tag{1.21}$$

In equilibrium, the profit of the representative, perfectly competitive firm is zero each period. This follows from the assumption in subsection (1.2.1) that the production technology exhibits constant returns to scale to the private inputs (labour h^d_t and private capital services K^d_{t-1}) and is thus homogeneous of degree one in the private inputs. By Euler's Theorem, the expression for period profit in (1.21) must equal zero.

1.2.4 The Government

The government spends by investing i^G_t each period in the public capital stock which becomes productive next period. In other words, the public capital stock is predetermined, so that the public capital stock available at the beginning of a period $t+1$ evolves according to

$$G_t = i_t^G + (1-\delta^G)G_{t-1} \tag{1.22}$$

where δ^G is the depreciation rate of public capital. The government levies a distortionary tax τ_t^h on labour income each period and a tax τ_{t-1}^K on private capital income earned by consumers in period t. This capital income tax is predetermined, so that the tax rate levied in period $t+1$ is chosen in time t. As explained in section (1.1), this feature prevents the government confiscating all capital income from consumers in the initial period of the deterministic transition of the economy under Ramsey optimal fiscal policy from a steady state with a high level of government debt to one with a low level of government debt. This is because the predetermined tax rate on capital income in this first period of the transition is the tax rate that prevailed in the initial steady state. The government is assumed not to have access to lump sum taxation. Otherwise, the Ramsey optimal fiscal policy would rely on non-distortionary lump sum taxation. The government borrows by issuing one period risk free bonds, b_t at price p_t^b, which require the government to pay one unit of the consumption good the next period $t+1$. Each period the government must satisfy its period budget constraint

$$G_t - (1-\delta^G)G_{t-1} + b_{t-1} \leq \tau_t^h w_t h_t + \tau_{t-1}^K r_t k_{t-1} + p_t^b b_t \tag{1.23}$$

1.2.5 Aggregate Resource Constraint

The economy must satisfy the aggregate resource constraint each period, which is given by

$$c_t + K_t - (1-\delta^K)K_{t-1} + G_t - (1-\delta^G)G_{t-1} \leq F(G_{t-1}, K_{t-1}, h_t) \tag{1.24}$$

This constraint will hold with equality in equilibrium because it can never be optimal to leave resources unspent, given that the marginal utility of consumption $u'(c) > 0$ for all $c > 0$. It has been shown in section (1.2.2) that the consumer's period budget constraint must also always hold with equality each period in equilibrium. Together, these two constraints imply that the government period budget constraint (1.23) must hold with equality each period.

1.2.6 Competitive Equilibrium

The statement of the competitive equilibrium will make use of the following definitions:

Definition 1 *Given initial conditions b_{-1}, K_{-1}, G_{-1} and τ^K_{-1}, a* ***Feasible Allocation*** *is a sequence $\{c_t, h_t, K_t\}_{t=0}^{\infty}$ together with a* ***Public Capital Stock Path*** *$\{G_t\}_{t=0}^{\infty}$ such that the aggregate resource constraint (1.24) is satisfied with equality each period. A* ***Tax Policy*** *is a sequence of tax rates $\left\{\tau^h_t, \tau^K_t\right\}_{t=0}^{\infty}$, while a* ***Government Borrowing Path*** *is a sequence $\{b_t\}_{t=0}^{\infty}$. A* ***Price Path*** *is a sequence $\left\{w_t, r_t, p^b_t\right\}_{t=0}^{\infty}$*

It is now possible to define a competitive equilibrium.

Definition 2 *A* ***Competitive Equilibrium*** *is a Feasible Allocation, a Tax Policy, a Government Borrowing Path and a Price Path such that each period the consumer optimality conditions (1.13),(1.14) and (1.15), the consumer budget constraint (1.7), the firm optimality conditions (1.19) and (1.20), the government budget constraint (1.23) and the aggregate resource constraint (1.24) all hold. Further, all markets clear each period and appropriate transversality conditions must hold.*

Note that market clearing implies that the government bond market and the markets for the supply of labour and private capital services each period all clear, so that

$$b^s_t = b^d_t = b_t \tag{1.25}$$

$$h^s_t = h^d_t = h_t \tag{1.26}$$

$$K^s_t = K^d_t = K_t \tag{1.27}$$

The transversality conditions necessary for a competitive equilibrium are

$$\lim_{j\to\infty} \beta^j u'(c_{t+j}) \left[(1-\tau^k_{t+j-1})r_{t+j} + (1-\delta^K)\right] K_{t+j} = 0 \tag{1.28}$$

$$\lim_{j\to\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} b_{t+j} = \lim_{j\to\infty} \left(\prod_{s=0}^{j-1} \left[p^b_{t+s}\right]\right) b_{t+j} = 0 \tag{1.29}$$

Note that the bond price p^b_t is the reciprocal of the gross interest rate on government bonds. It follows that (1.29) requires government debt to grow slower than the rate of interest in the limit, so that a No Ponzi condition is satisfied. This statement of competitive equilibrium follows the structure used by Farhi (2010).

All competitive equilibria considered in this chapter will consider an *exogenous* Government Borrowing Path of the form

$$b_t - \bar{b} = \rho \left(b_{t-1} - \bar{b} \right) \tag{1.30}$$

given an initial condition b_{-1}, where $\bar{b}$ is an exogenously specified target level of government borrowing. The parameter ρ determines the *speed* of debt reduction, where $0 < \rho < 1$, in the sense that a reduction in this parameter's value requires more of the debt reduction from b_{-1} to $\bar{b}$ to be achieved sooner. It is clear that the transversality condition on government bonds (1.29) is satisfied in all these competitive equilibria, provided the interest rate on government bonds is positive in the limit. Government debt is constant at level $\bar{b}$ in the limit, thus growing more slowly than the rate of interest and satisfying a No Ponzi condition. This *exogenous* downward path for government borrowing will be a constraint on the Ramsey planner in the following Ramsey problem. The Ramsey problem can be solved for different Government Borrowing Paths reflecting different *speeds* of debt reduction.

1.2.7 The Ramsey Problem

I define the optimal policy problem formally.

Definition 3 *Given initial conditions b_{-1}, K_{-1}, G_{-1} and τ^K_{-1}, the **Ramsey Problem** is to choose a Feasible Allocation, a Tax Policy, and a Price Path to maximise consumer welfare (as defined by equation (1.5)) subject to all the requirements in the definition of a competitive equilibrium being satisfied, with the Government Borrowing Path equal to the exogenous, downward path specified in equation (1.30).*

Effectively, the Ramsey planner (i.e. the benevolent government) maximises over the set of competitive equilibria, so that the chosen optimal *fiscal* policy - Public Capital Stock Path and Tax Policy - give rise to a competitive equilibrium. In order now to state the problem mathematically, some endogenous variables are eliminated from the competitive equilibrium conditions which constrain the Ramsey planner. First, the consumer optimality conditions (1.14) and (1.13) are used to isolate the period t price of government bonds p_t^b and the labour income tax rate τ_t^h

$$p_t^b = \beta \frac{u'(c_{t+1})}{u'(c_t)} \tag{1.31}$$

$$\tau_t^h = 1 - \frac{v'(1-h_t)}{u'(c_t)w_t} \tag{1.32}$$

Equations (1.31) and (1.32) are used to eliminate the bond price p_t^b and the labour income tax rate from the consumer period budget constraint (1.7), while the firm optimality conditions (1.19) and (1.20) are used to eliminate the wage w_t and the rental rate on private capital r_t. The consumer period budget constraint becomes

$$\begin{aligned} & c_t + \beta \frac{u'(c_{t+1})}{u'(c_t)} b_t + K_t - (1-\delta^K)K_{t-1} \qquad (1.33) \\ = \quad & \frac{v'(1-h_t)}{u'(c_t)} h_t + (1-\tau_{t-1}^K) F_K(G_{t-1}, K_{t-1}, h_t) K_{t-1} + b_{t-1} \end{aligned}$$

Equation (1.7) is now rearranged to isolate the forward looking term $u'(c_{t+1})$

$$\begin{aligned} & \frac{c_t u'(c_t)}{b_t} + \beta u'(c_{t+1}) + \frac{u'(c_t)}{b_t}(K_t - (1-\delta^K)K_{t-1}) \qquad (1.34) \\ = \quad & \frac{v'(1-h_t)}{b_t} h_t + \frac{u'(c_t)(1-\tau_{t-1}^K) F_K(G_{t-1}, K_{t-1}, h_t) K_{t-1}}{b_t} + \frac{u'(c_t) b_{t-1}}{b_t} \end{aligned}$$

I call this equation an *Implementability Condition* deriving from the consumer period budget constraint.

Further, the firm optimality condition (1.20) is used to eliminate the rental rate on private capital from the Euler equation for private capital (1.15)

$$u'(c_t) = \beta u'(c_{t+1}) \left[(1-\tau_t^k) F_K(G_t, K_t, h_{t+1}) + (1-\delta^K) \right] \tag{1.35}$$

This is also an Implementability Condition deriving from the Euler equation for private capital.

Mathematically, the Ramsey problem is then for the Ramsey planner (i.e. benevolent government) to choose an infinite sequence $\left\{c_t, h_t, K_t, G_t, \tau_t^K\right\}_{t=0}^{\infty}$ to maximise consumer welfare (1.5)

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(1-h_t) \right]$$

subject to the two Implementability Conditions (1.34) and (1.35)

$$\begin{aligned} & \frac{c_t u'(c_t)}{b_t} + \beta u'(c_{t+1}) + \frac{u'(c_t)}{b_t}(K_t - (1-\delta^K)K_{t-1}) \\ = \quad & \frac{v'(1-h_t)}{b_t}h_t + \frac{u'(c_t)(1-\tau^K_{t-1})F_K(G_{t-1},K_{t-1},h_t)K_{t-1}}{b_t} + \frac{u'(c_t)b_{t-1}}{b_t} \end{aligned}$$

$$u'(c_t) = \beta u'(c_{t+1})\left[(1-\tau^k_t)F_K(G_t,K_t,h_{t+1}) + (1-\delta^K)\right]$$

and subject to the aggregate resource constraint (1.24)

$$c_t + K_t - (1-\delta^K)K_{t-1} + G_t - (1-\delta^G)G_{t-1} = F(G_{t-1},K_{t-1},h_t)$$

plus the exogenous downward process for government borrowing (1.30)

$$b_t - \overline{b} = \rho\left(b_{t-1} - \overline{b}\right)$$

given initial conditions b_{-1}, K_{-1}, G_{-1} and τ^K_{-1}. The government budget constraint need not be stated as a constraint because it must hold with equality if both the consumer budget constraint (1.7) (which can be recovered from the Implementability Condition (1.34)) and the aggregate resource constraint (1.24) hold with equality every period. I call an infinite sequence $\left\{c_t, h_t, K_t, G_t, \tau^K_t\right\}_{t=0}^{\infty}$ that solves the Ramsey problem as stated above a **Ramsey Optimal Plan** or **Ramsey Economy.**

It is important to note that the opportunity set or feasible set of competitive equilibria defined by (1.34), (1.35), (1.24) and (1.30) over which the Ramsey planner maximises may not be convex. In such a case, the first order conditions of the Ramsey problem may not be sufficient to characterise a unique Ramsey economy. However, in the numerical simulations I present in section (1.3), the Ramsey plan presented is unique.

1.2.8 A Recursive Formulation to Solve the Ramsey Problem

The Ramsey problem as defined in section (1.2.7) is *not* recursive. Supposing that the Ramsey planner makes decisions sequentially, a recursive problem in a *deterministic* setting is one where the decision facing a Ramsey planner is the same each period. The decision problem is the same each period given the beginning of period value of *state* variables. These *state* variables encapsulate the effects of the initial conditions on the problem as well as the effects of all decisions made in periods prior to current time t. When the problem in a *deterministic* setting is recursive, the sequence of variables chosen *sequentially* each period by the Ramsey planner will be the same as the sequence chosen if all decisions (i.e. the entire infinite sequence) are made in the initial period. I have not yet defined state variables for the Ramsey problem in section (1.2.7). Recursive formulations are useful when seeking to apply numerical solution techniques to the model, although a recursive formulation is not strictly necessary to solve for the deterministic transition path that I present in section (1.3).

In the Ramsey problem defined in section (1.2.7), the Ramsey planner is constrained at each time t by conditions which contain forward looking terms (i.e. dated at time $t+1$). Specifically, the Implementability Condition derived from the consumer period budget constraint (1.34) contains the term $u'(c_{t+1})$, while the Implementability Condition derived from the Euler equation for private capital (1.35) contains the terms

$$\beta u'(c_{t+1})\left[(1-\tau_t^k)F_K(G_t, K_t, h_{t+1}) + (1-\delta^K)\right]$$

The presence of these forward looking terms among the time t constraints on the Ramsey planner implies that decisions made by the Ramsey planner in future periods limit the choices available to the Ramsey planner at time t. In other words, decisions taken today constrain the Ramsey planner's choices in previous periods. However, there is no mechanism in the Ramsey problem as it is defined in Section (1.2.7) that allows the Ramsey planner to take into account the effect of a *decision taken at time* t on the set of feasible choices available to the Ramsey planner in previous periods. Making decisions sequentially given the state variables will not be the same as if the Ramsey planner made all choices in the initial period, taking into account the effect that choices at time t have on the set of feasible choices in previous periods. Because of this, decisions made sequentially under the current formulation of the Ramsey problem may not be optimal. For more information, please see Kydland and Prescott (1977) and Kydland and Prescott (1980).

In order to obtain a recursive formulation of the Ramsey problem, I proceed by defining the following modified problem at time t. Given values for predetermined variables b_{t-1}, K_{t-1}, G_{t-1} and τ^K_{t-1}

$$\max_{\{c_{t+j},h_{t+j},K_{t+j},G_{t+j},\tau^K_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty}(\beta^t \left[u(c_{t+j}) + v(1-h_{t+j})\right]) \\ -\gamma_1 \left[u'(c_t)\right] - \gamma_2 \left[u'(c_t) \left[(1-\tau^k_{t-1})F_K(G_{t-1},K_{t-1},h_t) + (1-\delta^K)\right]\right] \tag{1.36}$$

subject to the constraints on the Ramsey problem for all periods from t onwards. These include the two Implementability Conditions (1.34) and (1.35)

$$\begin{aligned} & \frac{c_{t+j}u'(c_{t+j})}{b_{t+j}} + \beta u(c_{t+1+j}) + \frac{u'(c_{t+j})}{b_{t+j}}(K_{t+j} - (1-\delta^K)K_{t+j-1}) \\ = \; & \frac{v'(1-h_{t+j})}{b_{t+j}}h_{t+j} \\ & + \frac{u'(c_{t+j})(1-\tau^K_{t+j-1})F_K(G_{t+j-1},K_{t+j-1},h_{t+j})K_{t+j-1}}{b_{t+j}} + \frac{u'(c_{t+j})b_{t+j-1}}{b_{t+j}} \end{aligned}$$

$$u'(c_{t+j}) = \beta u'(c_{t+j+1}) \left[(1-\tau^k_{t+j})F_K(G_{t+j},K_{t+j},h_{t+j+1}) + (1-\delta^K)\right]$$

and subject to the aggregate resource constraint (1.24)

$$c_{t+j} + K_{t+j} - (1-\delta^K)K_{t+j-1} + G_{t+j} - (1-\delta^G)G_{t+j-1} = F(G_{t+j-1},K_{t+j-1},h_{t+j})$$

and the exogenous downward process for government borrowing (1.30)

$$b_{t+j} - \bar{b} = \rho\left(b_{t+j-1} - \bar{b}\right)$$

The difference between this problem and the one defined in section (1.2.7) is that I have added the forward looking components of the Implementability Conditions (1.34) and (1.35) from time $t-1$ to the objective function of the Ramsey planner, weighted by γ_1 and γ_2. This formulation of the problem can be solved every period t. In order to obtain a recursive formulation, it is necessary that the planner *commits* to update the weights γ_1 and γ_2 when solving the modified problem in period t with the values of the Lagrange multipliers attached to the time $t-1$ Implementability Conditions (1.34) and (1.35) from the modified problem solved the previous period, at time $t-1$. By committing to update the weights γ_1 and γ_2 in this way each period, the planner commits to take into account the effect of decisions at time t on the feasible choices available to the Ramsey planner in previous periods. This sequence of modified optimisation problems defines a recursive formulation for the Ramsey problem. The state variables each period t are the predetermined values b_{-1}, K_{-1}, G_{-1} and τ_{-1}^K *and* the values of the Lagrange multipliers attached to the time $t-1$ Implementability Conditions in the modified problem solved at time $t-1$. Given the values of these state variables, the modified problem is the *same* each period. The key feature allowing for this recursive formulation is the *augmenting* of the number of state variables with the Lagrange multipliers described, as pointed out by Kydland and Prescott (1980), Marcet and Marimon (2011) and Kumhof and Yakadina (2007). The solution to this sequence of modified problems can be obtained by solving a single problem using the method of Lagrange multipliers. The Lagrangian for this problem is:

$$\max_{\{c_t,h_t,K_t,G_t,\tau_t^K\}_{t=0}^{\infty}} L = \sum_{t=0}^{\infty} \beta^t \tag{1.37}$$

$$\left\{ \begin{array}{c} u(c_t) + v(1-h_t) - \mu_{t-1}^E u'(c_t) - \\ \mu_{t-1}^K u'(c_t)\left[(1-\tau_{t-1}^K)F_K(G_{t-1},K_{t-1},h_t) + (1-\delta^K)\right] + \\ \mu_t^K u'(c_t) - \mu_t^E \left[\frac{u'(c_t)c_t}{b_t} + \frac{u'(c_t)}{b_t}(K_t - (1-\delta^K)K_{t-1})\right] \\ +\mu_t^E \left[\frac{v'(1-h_t)h_t}{b_t} + \frac{(1-\tau_{t-1}^K)F_K(G_{t-1},K_{t-1},h_t)K_{t-1}u'(c_t)}{b_t} + \frac{u'(c_t)b_{t-1}}{b_t}\right] \\ +\mu_t^A \left[\begin{array}{c} F(G_{t-1},K_{t-1},h_t) - c_t - K_t \\ +(1-\delta^K)K_{t-1} - G_t + (1-\delta^G)G_{t-1} \end{array} \right] \end{array} \right\}$$

where μ_t^E and μ_t^K are the Lagrange multipliers at time t attached to the Implementability Conditions (1.34) and (1.35) derived from the consumer budget constraint and Euler equation for private capital respectively. The Lagrange multiplier μ_t^A is attached to the aggregate resource constraint at time t.

The necessary conditions for a unique solution to the Ramsey problem, or in other words, necessary conditions for a Ramsey Economy are given by the equations determining when the first order partial derivatives of the Lagrangian (1.37) vanish each period, together with the constraints on the Ramsey planner each period. These constraints include the exogenous downward path for government borrowing (1.30). The conditions characterising the Ramsey economy also include the transversality conditions for private capital (1.28) and government bond holdings (1.29). An additional transversality condition required to characterise the Ramsey economy is one for public capital, since it cannot be optimal to leave public capital positive as time tends to infinity:

$$\lim_{j \to \infty} \beta^j u'(c_{t+j}) G_{t+j} = 0 \tag{1.38}$$

From the sequence $\left\{c_t, h_t, K_t, G_t, \tau_t^K, \mu_t^E, \mu_t^K, \mu_t^A\right\}_{t=0}^{\infty}$ of endogenous variables and sequence of government borrowing $\{b_t\}_{t=0}^{\infty}$ satisfying these conditions it is then possible to recover a sequence $\left\{p_t^b, w_t, r_t, \tau_t^h\right\}_{t=0}^{\infty}$ for the price of government bonds, the rental rate on private capital, the wage and the labour income tax rate, using the competitive equilibrium conditions (1.31), (1.19), (1.20) and (1.32). The conditions characterising a Ramsey Economy and their full derivation are presented in **Appendix (**A.3**)**.

1.3 The Deterministic Transition Path Of The Ramsey Economy

In this section, I use numerical methods to compute the deterministic transition path of the Ramsey Economy from a steady state with a relatively high level of government debt to one with a relatively low level of government debt. Over the transition path, the level of debt falls as a percentage of GDP. This exercise captures the behaviour of the economy under Ramsey optimal fiscal policy, as the government engineers a potentially large reduction in the level of government debt, following an exogenous path for the level of real government debt. This exercise can be repeated for different exogenous paths for the level of government debt, each reflecting a different *speed* of debt reduction. By contrast, studying how the economy behaves in response to small shocks in the neighbourhood of a particular steady state seems unlikely to capture how large reductions in government debt are achieved.

This section of the chapter is divided as follows. First, I choose specific functional forms for period utility and for the production technology in section (1.3.1). Second, values are assigned to the parameters of the model in section (1.3.2). In section (1.3.3), I explain how the Ramsey Economy's deterministic transition path between steady states is computed using numerical methods. Fourth, I present the transition path between steady states of relatively high government debt and relatively low government debt in a model with public capital but *without* private capital: see section (1.3.4). Finally, I present the transition path in a model with both private and public capital. In both settings, I consider different speeds of government debt reduction. I also discuss the robustness of my results to different choices for parameter values.

It should be noted at this point that Lansing (1998) considers optimal fiscal policy in a model with public and private capital stocks. However, my work differs from Lansing (1998) in a number of key respects. First, the focus of Lansing's (1998) paper is not on government debt reduction and government borrowing is endogenous in that paper. Further, Lansing (1998) solves for policy functions giving the endogenous variables as functions of state variables, exogenous variables and shocks, by taking approximations of the model's equilibrium equations around a steady state. Lansing (1998) uses these policy functions to characterise the model's behaviour in the neighbourhood of the steady state in response to shocks. By contrast, I solve for the model's transition path between steady states, without recourse to local approximation methods. Guo and Lansing (1997) is a similar paper to Lansing (1998).

1.3.1 Specific Functional Forms

For the purposes of computing the deterministic transition path of the Ramsey Economy between steady states, I assume that period utility takes the form

$$u(c) + v(1-h) = \ln(c) + \varkappa \ln(1-h_t) \tag{1.39}$$

This implies that the intertemporal elasticity of substitution of consumption is one. The parameter $\varkappa$ affects the consumer labour supply choice and its value will influence the Frisch elasticity of labour supply with respect to the real wage (for a given marginal utility of consumption). I discuss the choice of the value for $\varkappa$ in section (1.3.2).

The production technology each period is assumed to take the multiplicative form

$$F(G_{-1}, K_{-1}, h) = G_{-1}^{\theta} K_{-1}^{\alpha} h^{1-\alpha} \tag{1.40}$$

where $0 < \alpha < 1$ and $\theta > 0$. The parameter θ governs the elasticity of output with respect to public capital. Consistent with the general functional form outlined in section (1.2.1), this specific functional form exhibits constant returns to scale in the private inputs: private capital K_{-1} and labour h. Again, I discuss the specific values chosen for these parameters in section (1.3.2).

1.3.2 Choice of Parameter Values

The *benchmark* choices for the model's parameters are displayed in the following table:

Table 1.3.2	
Benchmark Parameter Values	
θ	0.05
α	0.36
β	0.99
δ^G	0.02
δ^K	0.02
$\varkappa$	3
ρ	0.95

The most controversial choice is that for the parameter θ determining the elasticity of output with respect to public capital. There is a literature computing empirical estimates for this parameter. Unfortunately, the range of parameter estimates in this literature is quite wide. However, recent surveys of this literature such as Bom and Ligthart (2013) attempt to systematically reconcile the various estimates by understanding the differences in methodological approach taken in different papers. Bom and Ligthart (2013) find that values are most commonly in the narrower range of 0.05-0.1. Estimates nearer or over 0.1 are found when the measurement of public capital is restricted to infrastructure such as roads and ports. Lower estimates closer to

0.05 are found when the measurement of public capital is expanded to cover infrastructure such as schools and hospitals. Papers presenting real business cycle type models with public capital such as the recent paper of Leeper and Yang (2010) use the conservative value 0.05 for the parameter determining the elasticity of output to public capital, when solving models numerically. The well known paper of Baxter and King (1993) also uses the value 0.05. Following Leeper and Yang (2010) and Baxter and King (1993), I choose the conservative value 0.05 for θ as the benchmark. However, I will consider the robustness of my results to a higher value.

The parameter $\varkappa$ is set to three. This implies that the Frisch elasticity of labour supply with respect to the real wage (given constant marginal utility of consumption) is around 3.5 in the steady states that I consider.[2] Values of 3-4 for the Frisch elasticity of labour supply have been used before in the theoretical macroeconomics literature: see Schmitt-Grohe and Uribe (2004) and Kumhof and Yakadina (2007). It should be noted that there are micro-level estimates of the Frisch elasticity that are much lower: see Chetty and Weber (2011) and Peterman (2012). I test the robustness of my results by solving for the deterministic transition path of the Ramsey Economy using the lower value of *two* for the Frisch elasticity.

The value 0.95 is chosen as a benchmark for the parameter determining the rate at which the level of government borrowing must fall over the transition path. The autoregressive nature of the equation determining the downward path for government borrowing implies that the reduction in government borrowing is *frontloaded*: that is, the amount by which government borrowing is reduced each period falls over time. The level of government borrowing asymptotes to its new target value over the transition path. Nonetheless, the value 0.95 chosen for ρ produces a relatively gradual reduction in the level of government borrowing (debt), compared with values closer to

[2] The Frisch elasticity of labour supply with respect to the real wage (given constant marginal utility of consumption) is given by $\frac{\partial h}{\partial w}\frac{\overline{w}}{\overline{h}}\mid_{u'(\overline{c})}=\frac{1-\overline{h}}{\overline{h}}$ for the chosen specific functional form for utility. Setting $\varkappa = 3$ implies that hours worked in the steady states I consider $\overline{h}$ will be approximately 0.25, giving a Frisch elasticity of around 3.5. This approach of choosing Frisch elasticity of labour supply to match particular values of hours worked (implememted by choosing $\varkappa$) follows Prescott (2003).

zero. In my numerical simulation, government debt falls from 60 per cent to 45 per cent of GDP in approximately ten years. I also solve for the deterministic transition path of the Ramsey Economy setting ρ to be 0.75. This reflects a faster, frontloaded debt reduction path, with government debt falling from 60 to 45 per cent of GDP in less than five years. It should be noted that the exogenous downward paths for debt are the same *length*, under either value for ρ and the amount of debt reduction that occurs is the same. Lower values of ρ imply that more of the desired debt reduction occurs sooner.

I set the quarterly depreciation rate of public capital to be 2 per cent (corresponding to δ^G of 0.02). This is at the high end of the range of estimates of quarterly public capital depreciation rates in a recent International Monetary Fund study: see Serkan Arslanalp and Sze (2010). I set the quarterly depreciation rate of *private* capital to the same value. This is within the range of values for quarterly private capital depreciation rates used in the macroeconomics literature and is close to the value of 2.5 per cent (corresponding to δ^K of 0.025) that was used in the recent paper of Leeper and Yang (2010), to solve numerically a model that also contained public capital.

The chosen values for the remaining parameters are standard in the macroeconomic theory literature. The share of output paid to private capital α is set to 0.36, close to that in Leeper and Yang (2010), which also features a production technology with public capital and constant returns to *private* inputs. The subjective discount factor of consumers β is set to 0.99.

1.3.3 Solution Method

I present the deterministic transition path of the Ramsey Economy from a steady state with a relatively high level of government debt to one with a relatively low level of government debt. Over the transition path, government debt falls as a percentage of GDP. I undertake the following steps to do this:

(1) I derive again the equations characterising the Ramsey Economy described in section (1.2.8), this time using the specific functional forms chosen in section (1.3.1).

(2) The values of the model's parameters are set to those described in section (1.3.2).

(3) I solve for the steady state values of the endogenous variables

$$\left\{\overline{K}, \overline{G}, \overline{\tau}^K\right\}$$

in a steady state where the exogenous target level of government borrowing is b^{high}, which I set to a value such that level of government debt to GDP is around 60 per cent in this steady state.

(4) Initial conditions are required for b_0, K_0, G_0, $\overline{\tau}_0^K$, μ_0^E and μ_0^K. I set $b_0 = b^{high}$. The initial conditions for the private capital stock K_0, public capital stock G_0 and predetermined capital income tax rate $\overline{\tau}_0^K$ are set to their values in the steady state consistent with b^{high}, as solved for in the previous step. The remaining terms μ_0^E and μ_0^K are set to *zero.* These terms are the multipliers attaching to the constraints (1.34) and (1.35) on the Ramsey problem that contain forward looking terms. Marcet and Marimon (2011) explain that these multipliers μ_0^E and μ_0^K must be zero initially because there are no previous policy commitments with which the planner must be consistent.

(5) I simulate the transition path over $T = 200$ periods. In period one of the transition, the target level of government borrowing jumps exogenously from b^{high} to b^{low} ($b^{high} > b^{low}$). The target level remains at b^{low} thereafter. The level of government borrowing must fall in the first period of the transition path according to the equation (1.30) describing the exogenous downward path for government borrowing. Thereafter, government borrowing b_t asymptotes towards its new target level b^{low} over the transition path.

(6) There are eight equations for each of the T periods of the transition path, forming a large system of simultaneous non-linear equations. In the final period T, I set the values of the endogenous variables to their values in a steady state where the level of government borrowing is b^{low}. In this steady state, the level of government debt (as a percentage of GDP) is around 40 per cent. This is a terminal condition and the chosen length of the transition period $T = 200$ is sufficiently long such that the economy will be close to this new steady state by the time period T is reached. I then solve this system of equations for the sequence $\left\{c_t, h_t, K_t, G_t, \tau_t^K, \mu_t^E, \mu_t^K, \mu_t^A\right\}_{t=1}^{T-1}$. The solution can be obtained by using a solver of systems of simultaneous non-linear equations, such as that described in Juillard (1996) or Reiter (2005).[3]

(7) Different lengths T for the transition path can be experimented with, to ensure that the choice of T does not overly influence the results. In other words, the level of government borrowing should have time to fall from b^{high} to b^{low} according to the exogenous path given by equation (1.30) by the time T is reached, so that endogenous variables have had time to adjust. By the time T is reached, endogenous variables should be close to their values in the new steady state.

(8) This process can be repeated using different exogenous, downward paths for government debt reflecting different *speeds* of government debt reduction. Again, these different paths are all of the same length. A faster pace of debt reduction means that more of the desired debt reduction occurs sooner. The speed of government debt reduction is altered by changing the parameter ρ in (1.30).

3 The non-linear equation solver used to obtain the results presented in this chapter is that called by the "SIMUL" command in DYNARE Version 4.3. I have also solved the model using an alternative non-linear equation solver designed for systems with a very large number of equations. This alternative solver was developed by Michael Reiter and I am most grateful to him for providing me with the necessary computer code.

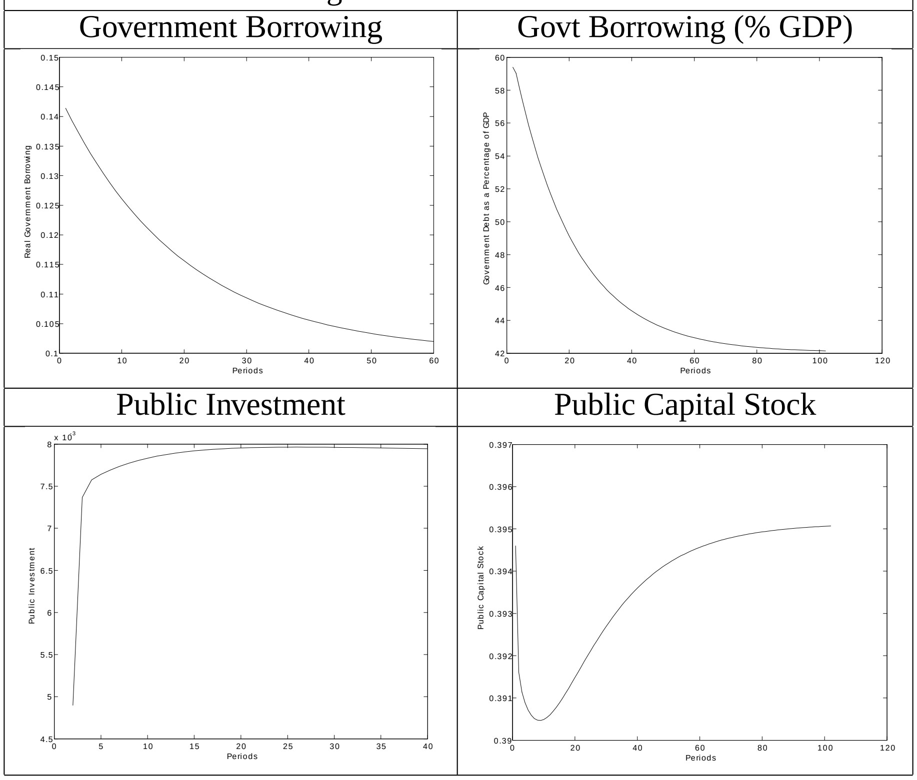
1.3.4 Government Debt Reduction In A Model Without Private Capital

I now consider the model with *only* public capital and abstract from private capital. First, I present the transition path of the Ramsey economy using the benchmark choices for parameter values, whereby government debt falls from 60 to 45 per cent of GDP in around ten years (NB: In the model, one period is one quarter). I then compare the results with Ramsey optimal policy under a faster, exogenous pace of debt reduction, so that more of the desired debt reduction is achieved earlier. Third, I consider the robustness of my results to alternative choices for parameter values.

1.3.4.1 Public Capital Only: The Benchmark Case

In the early part of the transition path, the government finances a reduction in borrowing by generating more revenue while keeping government spending relatively low. Spending in the form of public investment is set sufficiently low such that the public capital stock is eroded, because investment does not offset depreciation fully.

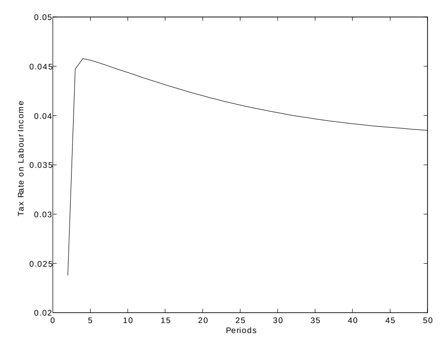
Figure 1.3.4.1 - Part One



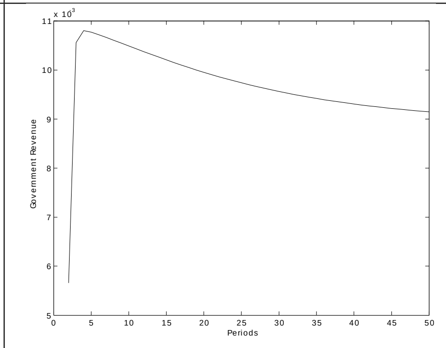
There is an increase in the rate of labour income taxation which causes government revenue to rise: Figure (1.3.4.1) (Part Two). The real interest rate on government bonds is initially low. In fact, it is negative for one period. This effectively constitutes taxation of consumers with holdings of government bonds, the only asset that can be used to smooth consumption over time in the absence of private capital. This low real interest rate reflects a high price for government bonds (since the interest rate is the reciprocal of the bond price). The amount of government bonds issued and available for consumers to use as savings instruments declines, hence the high bond price.

Figure 1.3.4.1 - Part Two

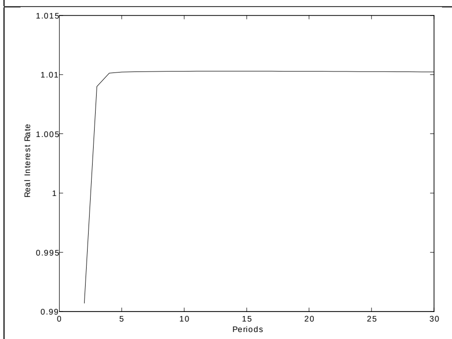
Labour Income Tax Rate



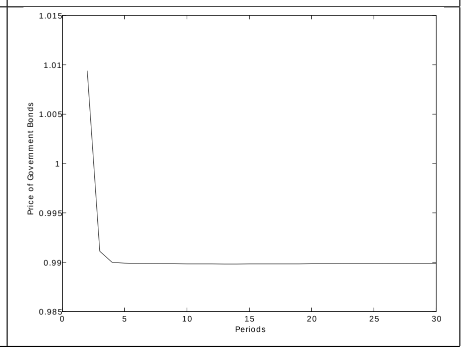
Revenue



Real Interest Rate



Bond Price



The post tax wage is lower because of the higher tax rate on labour income and the lower public capital stock:

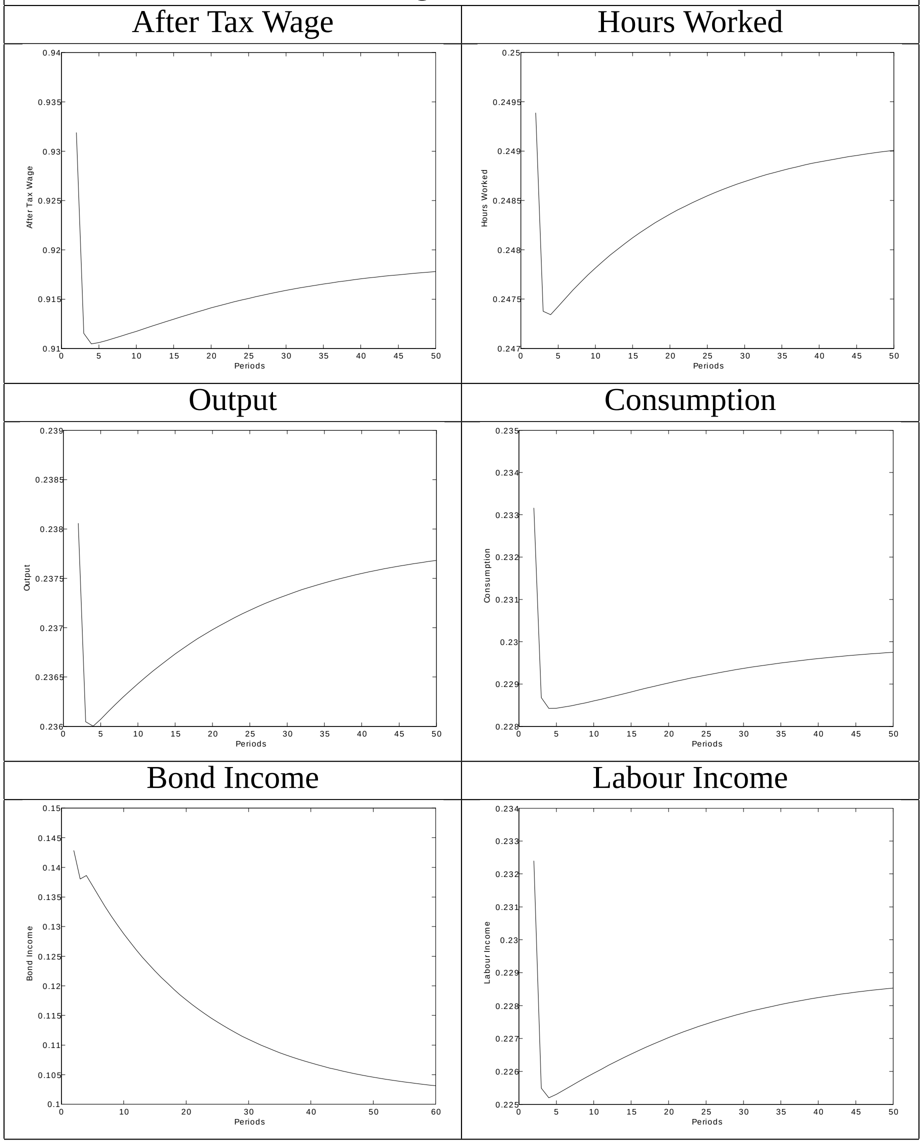
$$(1-\tau_t)w_t = (1-\tau_t)G_{t-1}^{\alpha} \tag{1.41}$$

Because of this, labour supply and output fall in the early part of the transition: Figure (1.41). Consumption also falls, reflecting lower income for consumers from labour and bond holdings. The fall in consumption is consistent with the initially low real interest rate, as shown by the Euler equation (1.14) (using the specific functional forms):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \frac{\beta}{p_t^b} = \beta R_t$$

where R_t is the real interest rate, the reciprocal of the bond price p_t^b.

Figure 1.41



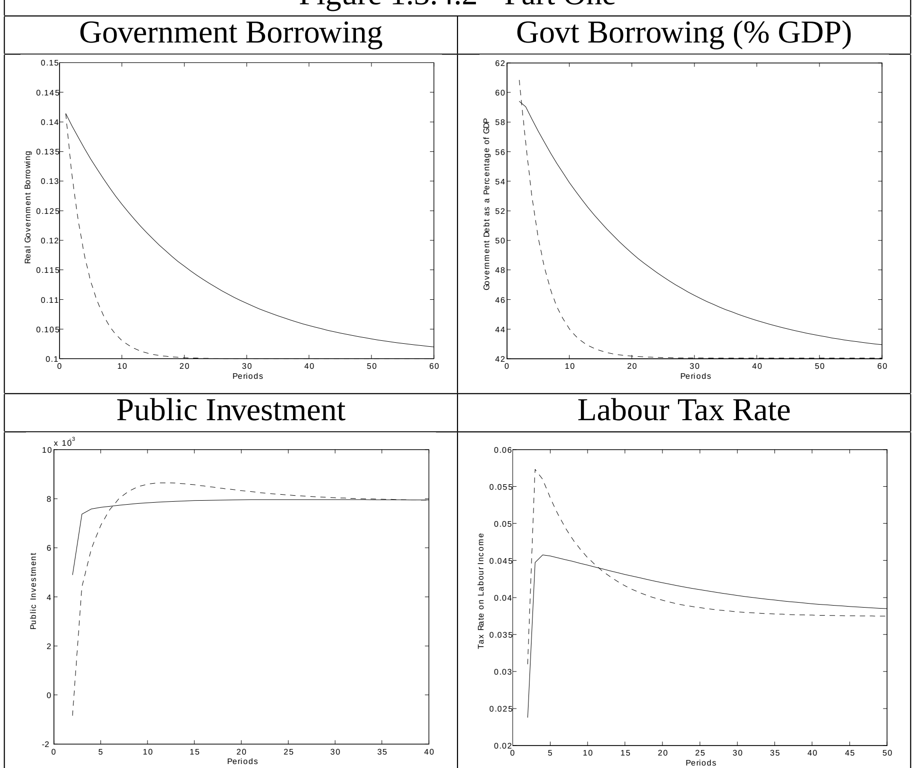
The position changes further along the transition path. As government debt nears its new, lower target value b^{low}, the amount by which government borrowing must fall each period is lower, according to the exogenous, frontloaded debt reduction path in (1.30). This allows the government to *lower* the labour income tax rate and *increase* government spending (i.e. public investment). The erosion of the public capital stock is somewhat reversed. The post tax wage rises, as do hours worked and output. This allows consumption to recover.

1.3.4.2 Public Capital Only: The Speed of Debt Reduction

I now compare the results in Subsection (1.3.4.1) with the deterministic transition path of the Ramsey economy when the Ramsey planner engineers a *faster* reduction in government borrowing from b^{high} to b^{low}. The deterministic transition path is the same length as above, but more of the desired debt reduction must now occur sooner. Specifically, the parameter ρ governing the speed of debt reduction in the exogenous process (1.30) is set to 0.75. Government debt now falls from 60 to 45 per cent of GDP in less than five years, rather than ten years in the model presented in Subsection (1.3.4.1) above.

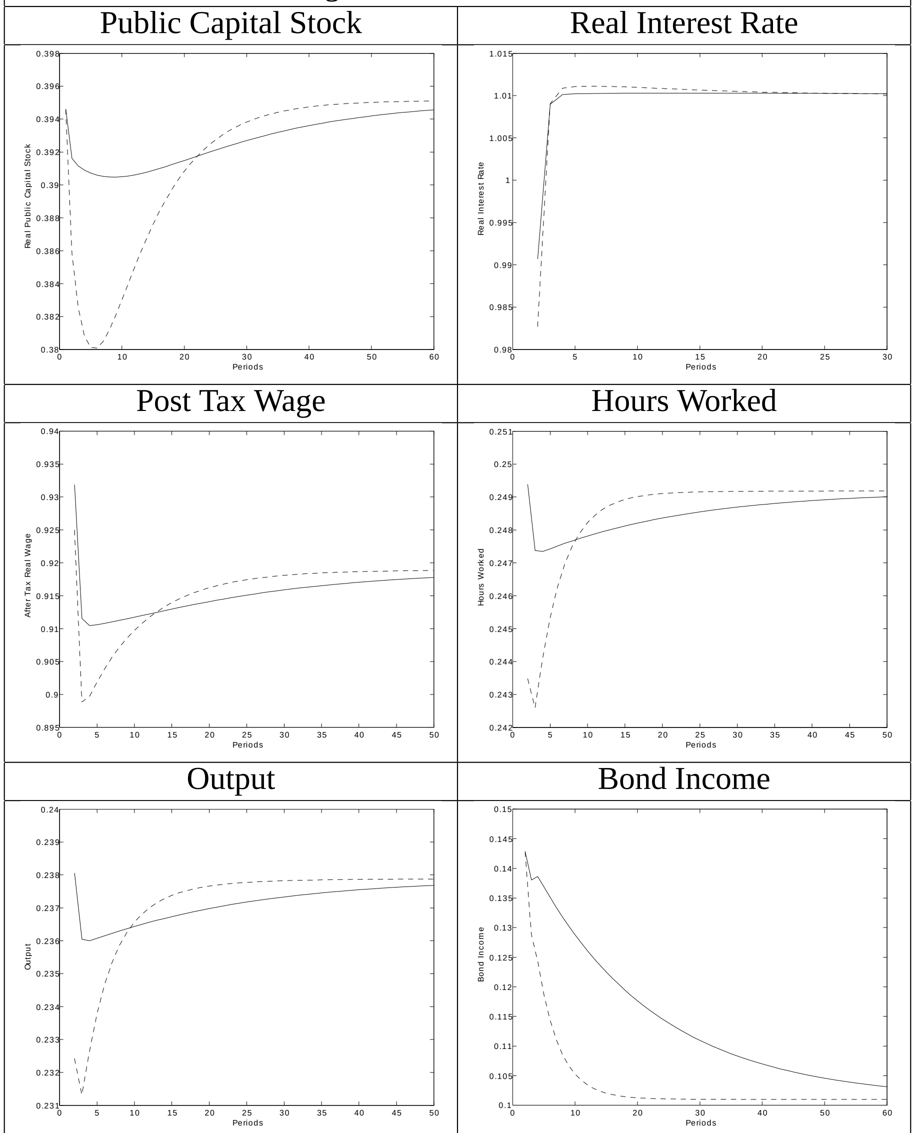
Forced to reduce debt faster, the Ramsey planner sets public investment lower initially. There is actually disinvestment in public capital and the public capital stock declines by more than when debt reduction occurred more gradually. There is a larger rise in the labour income tax rate and also more taxation of bond holdings, in the form of an even lower, negative real interest rate: Figure (1.3.4.2) (Part Two).

Figure 1.3.4.2 - Part One



Legend: — $\rho = 0.95$ (slow); $\cdots$ $\rho = 0.75$ (fast)

Figure 1.3.4.2 - Part Two

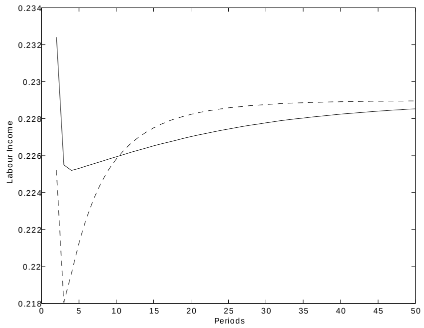


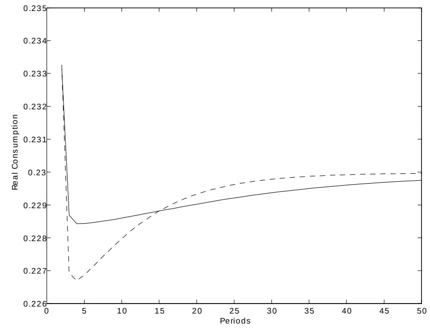
Legend: — $\boldsymbol{\rho = 0.95}$ (slow); $\cdots$ $\boldsymbol{\rho = 0.75}$ (fast)

The initial decline in the post tax wage is sharper and deeper, as is the fall in consumer income from bond holdings, which must be lower than they otherwise would be as the government reduces debt faster. The result is a sharper and deeper fall in consumption: Figure (1.3.4.2) (Part Three).

Figure 1.3.4.2 - Part Three

Labour Income	Consumption





Legend: — $\boldsymbol{\rho = 0.95}$ (slow); $\cdots$ $\boldsymbol{\rho = 0.75}$ (fast)

However, the faster pace of debt reduction means that debt nears its new, lower target level b^{low} sooner. This allows the labour income tax rate to be reduced faster than it would be if debt reduction occurred more gradually: Figure (1.3.4.2) (Part One). Similarly, public investment can rise more sharply and the public capital stock can be restored more quickly. This allows for a faster recovery in the post tax wage and thus a faster recovery of hours worked and output. Consumption also recovers faster.

Overall, it seems that a faster pace of debt reduction involves a greater short term cost (in terms of consumption). However, these austerity costs are of shorter duration and consumption recovers faster than if there was a slower pace of debt reduction.

1.3.4.3 Public Capital Only: Robustness to Alternative Parameter Values

I consider the robustness of the results presented in Subsection (1.3.4.1) along two dimensions.

(1) I consider a *more productive* public capital stock. I alter the benchmark choice of parameter values by setting θ, the elasticity of output to public capital, to 0.1, instead of 0.05. This is at the higher end of estimates for this elasticity in the empirical papers surveyed by Bom and Ligthart (2013). This should test whether a more productive public capital stock makes government spending cuts too costly for output and consumption, so that the Ramsey planner becomes reliant on tax rate increases to achieve debt reduction. The deterministic transition path of the Ramsey economy under this alternative choice of parameter values is presented in Appendix (A.4).

(2) I test robustness to a *lower* Frisch elasticity of labour supply. I do this by setting the parameter κ in the utility function to two, rather than three. This generates a steady state Frisch elasticity of *two*, rather than over three. This tests whether more *inelastic* labour supply allows the planner to rely on higher rates of income taxation to reduce debt, because the tax increases will be less distortionary to labour supply and output in these circumstances. The results are presented also in Appendix (A.4).

I find that the *qualitative* features of the results presented in Subsection (1.3.4.1) are largely unchanged under these alternative choices for parameter values. Government debt reduction is still financed by a combination of lower government spending (public investment) and higher labour income tax rates.

1.3.5 Government Debt Reduction: The Full Model with Private and Public Capital

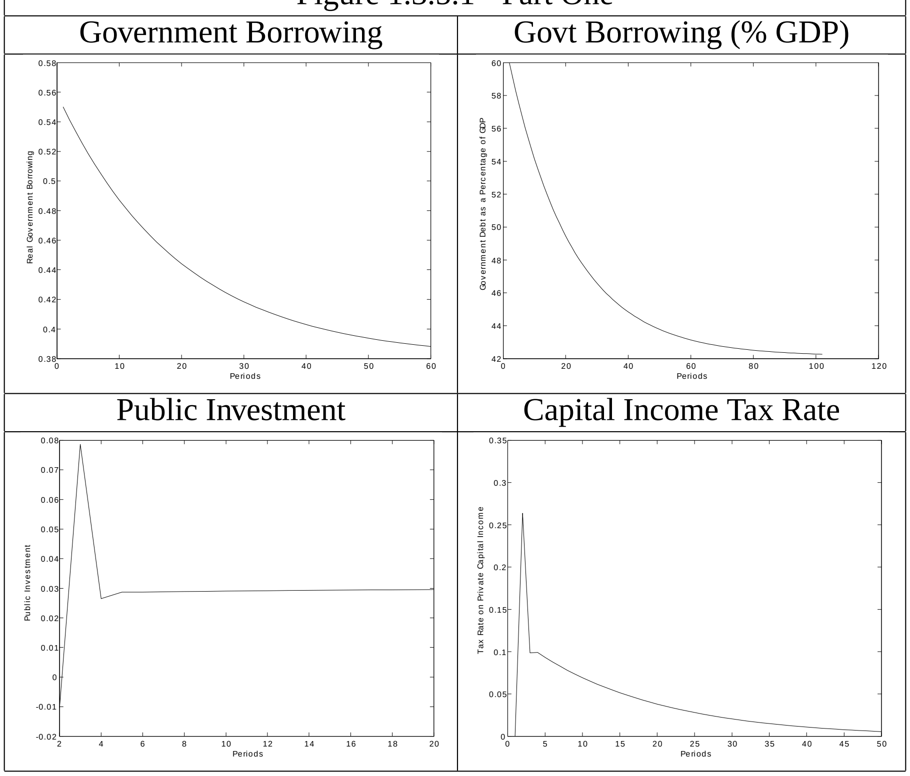
I now present the transition path in the model with both *private* capital and public capital. In this model, the government can tax income from investment in private capital, as well as labour income. Again, government debt falls from 60 to 45 per cent of GDP over the transition path.

1.3.5.1 Private and Public Capital: The Benchmark Case

In a model with *both* private and public capital, it is taxation of private capital income and its affect on the real interest rate on government bonds that are key to achieving the desired debt reduction.

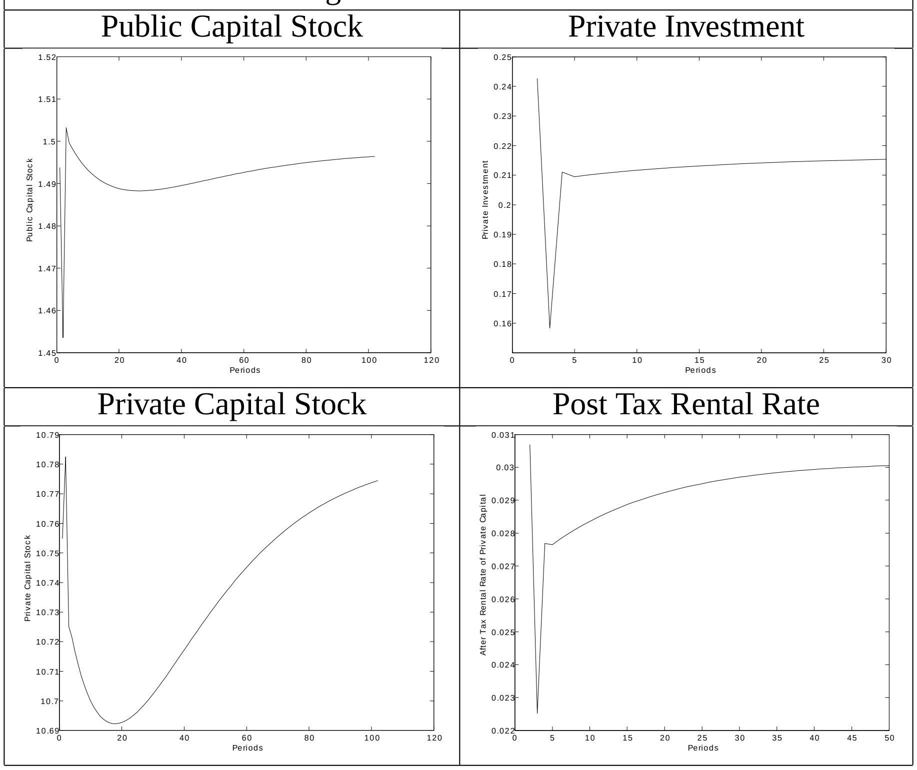
In the first period of the transition path, the capital income tax rate is fixed or predetermined τ_0^k at its value in the steady state corresponding to government debt being approximately 60 per cent of GDP. This prevents "surprise" taxation of pre-existing private capital in the first period. Because of this, the government finances the reduction in government borrowing required in the first period by setting government spending (i.e. public investment) at a relatively low level - there is disinvestment in public capital: Figure (1.3.5.1) (Part One).

Figure 1.3.5.1 - Part One



After the first period, the Ramsey planner is free to set the capital income tax rate. The planner chooses to set a positive rate of capital income taxation over much of the transition path. This causes government revenue to jump up in the second period. This is not "surprise" taxation of pre-existing capital, since the positive capital income tax rate persists over the transition path. It causes a fall in private capital investment and a protracted decline in the *private* capital stock: Figure (1.3.5.1) (Part Two).

Figure 1.3.5.1 - Part Two



The elevated tax rate on private capital income reduces the post tax rental rate of return on private capital. The real interest rate on government bonds will be low also, reflecting a no arbitrage condition obtained by combining the Euler equations (1.14) and (1.15) for government bonds and private capital income:

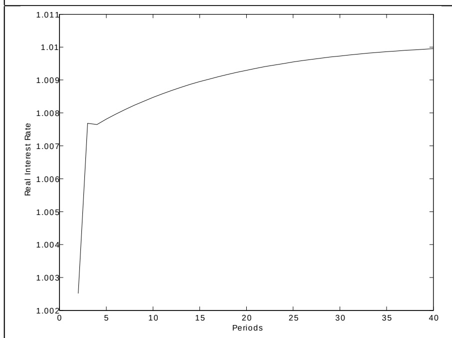
$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1}{p_t^b} = R_t = (1-\tau_t^k)r_{t+1} + (1-\delta^k) \tag{1.42}$$

using specific functional forms, where R_t is the real interest rate on government bonds and r_t is the rental rate of return on private capital.

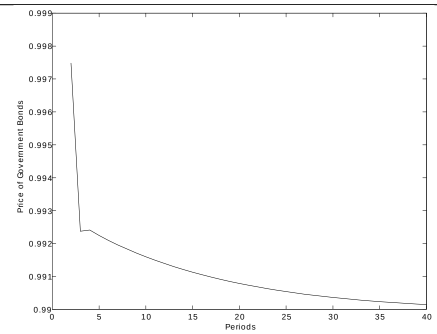
The real interest rate remains below its level in *any* steady (i.e. $R = \frac{1}{\beta}$) over much of the transition path: Figure (1.42) (Part Three). This reduces the real value of interest payments on government debt that the government must make.

Figure 1.42 - Part Three

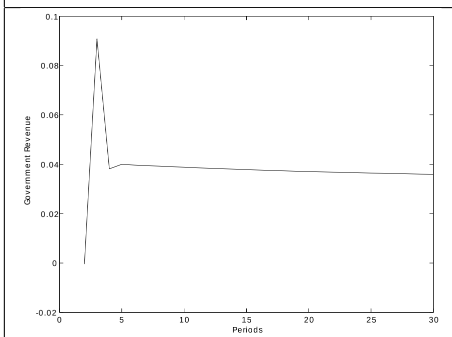
Real Interest Rate



Bond Price



Revenue



Labour Income Tax Rate

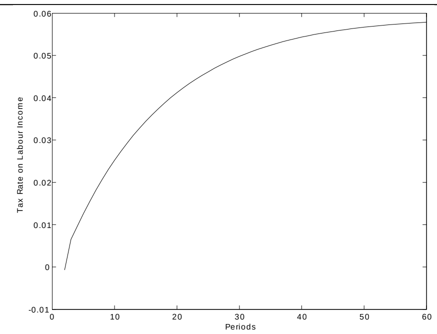
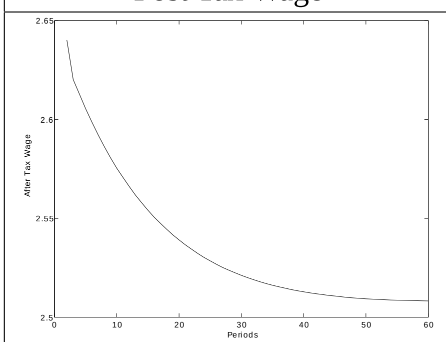
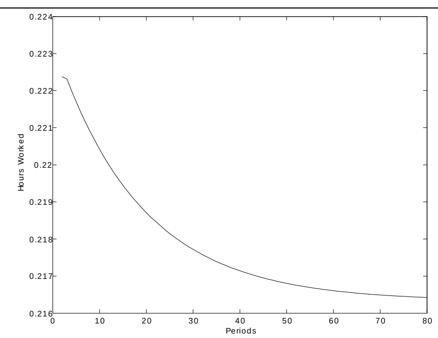


Figure 1.42 - Part Four

Post Tax Wage



Hours Worked



Output

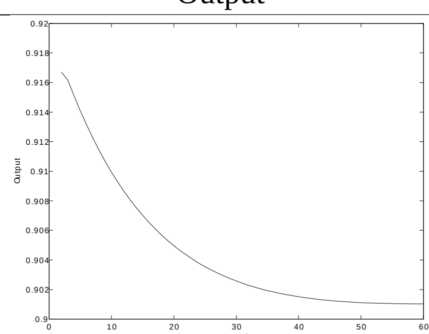
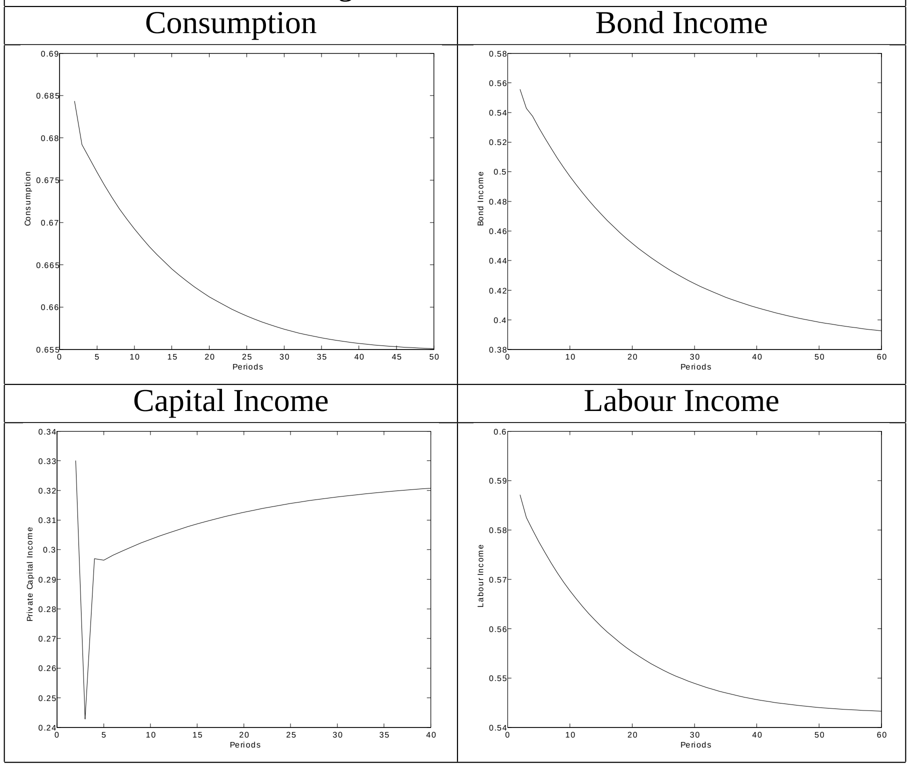


Figure 1.42 - Part Five



Access to revenue from private capital income taxation and the reduction in the real interest rate on government bonds are the key factors explaining how the government finances government debt reduction. Compared to the model with only public capital, there is little erosion of the *public* capital stock: Figure (1.3.5.1) (Part Two). Government spending (i.e. public investment) quickly settles at its ultimate steady state level of around 3-4 per cent of GDP. This is consistent with the average level of public investment computed for OECD countries by the recent IMF study of Serkan Arslanalp and Sze (2010). Labour income taxation plays a relatively small role in raising revenue for debt reduction, compared with the model containing only public capital. The labour income tax rate is close to zero in the early periods of the transition path: Figure (1.42) (Part Three). It then rises slowly as the tax rate on private capital income falls to zero in the ultimate steady state. The post tax wage falls eventually over the transition as the rate of labour income taxation rises and the level of private capital is eroded, as shown by

$$(1-\tau^h)w = (1-\tau^h)(1-\alpha)G_{-1}^{\theta}K_{-1}^{\alpha}h^{-\alpha} \tag{1.43}$$

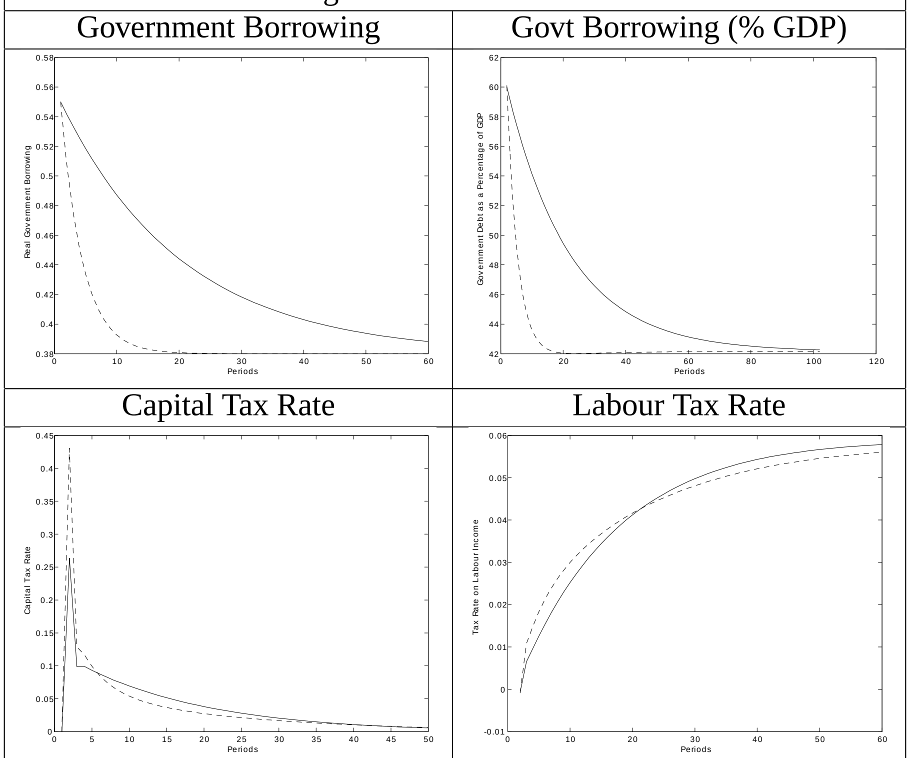
using specific functional forms. This is reflected in the fall in hours worked over the transition path. Lower labour supply and private capital stock lead to lower output: Figure (1.42) (Part Four). Consumers suffer lower labour income and also lower income from investment in government bonds and private capital. Real consumption falls over the transition, consistent with the low value of the real interest rate, as shown by the Euler equation (1.14):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \frac{\beta}{p_t^b} = \beta R_t$$

1.3.5.2 Private and Public Capital: The Speed of Debt Reduction

I compare the results in Subsection (1.3.5.1) with the Ramsey optimal fiscal policy for a faster pace of debt reduction: i.e. reducing government debt from 60 to 45 per cent of GDP in less than five years. Again, the length of the transition path remains the same as in Subsection (1.3.5.1). However, a larger share of the desired debt reduction now occurs in the *early* part of the transition path. The Ramsey planner achieves this by raising the capital income tax rate even higher.

Figure 1.3.5.2 - Part One



Legend: — $\boldsymbol{\rho = 0.95}$ (slow); $\cdots$ $\boldsymbol{\rho = 0.75}$ (fast)

The post tax rental rate of return on private capital and the real interest rate on government bonds are driven down lower, according to the no arbitrage condition (1.42). This further reduces real interest payments for the government: Figure (1.3.5.2) (Part Two). The tax rate on labour income is also higher in the early periods of the transition, compared with when debt is reduced more slowly. This makes the post tax wage lower: Figure (1.3.5.2) (Part Three). Consumer income from labour, government bond holdings and private capital investment are all lower in the early periods of the transition than they would be under slower debt reduction. Because of this, there is a sharper and deeper fall in consumption. Lower consumption is consistent with a lower real interest rate, according to the relationship described by the bond Euler equation (1.14).

Figure 1.3.5.2 - Part Two

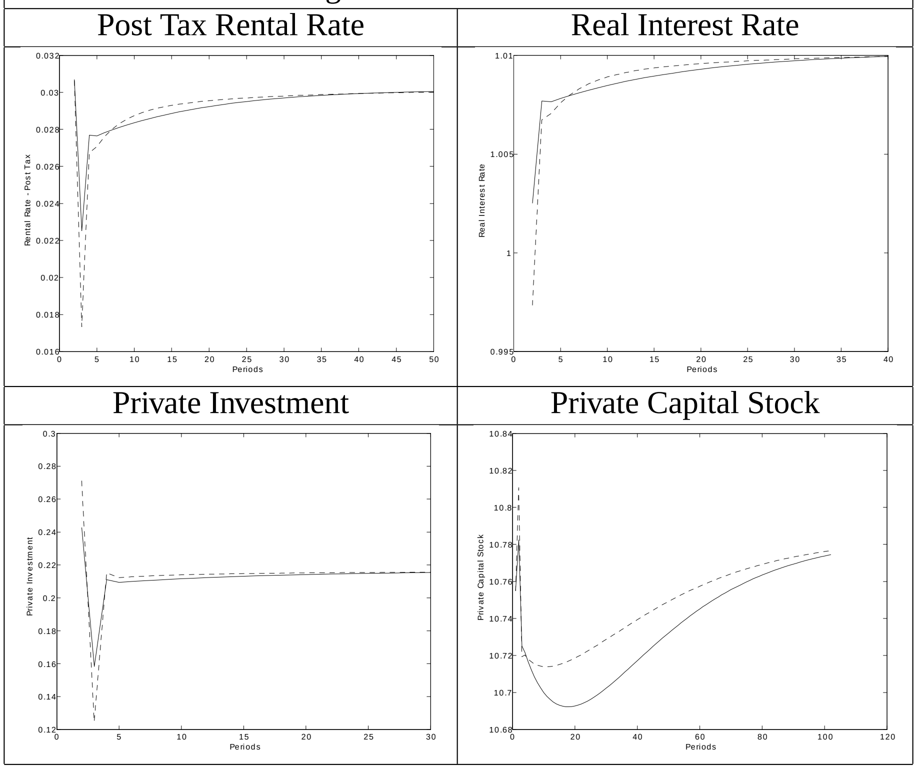
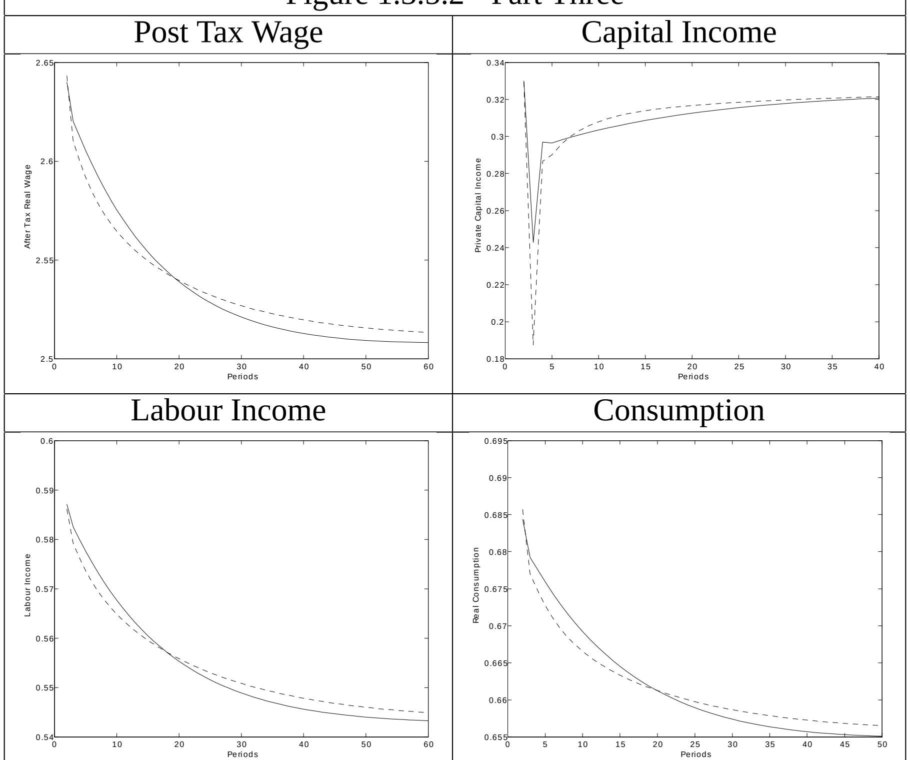


Figure 1.3.5.2 - Part Three

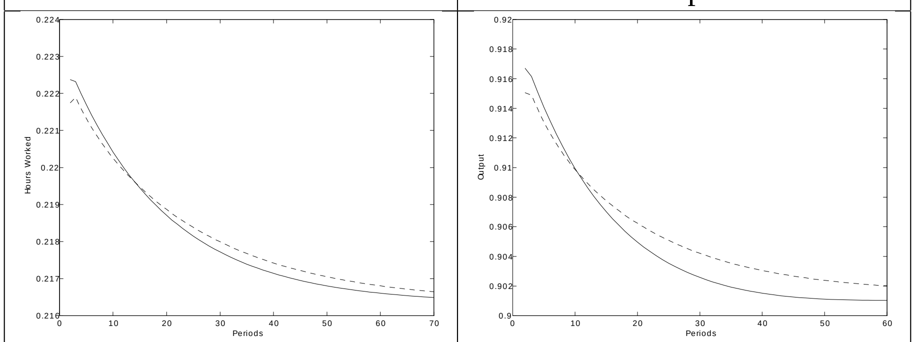


Legend: — $\rho = 0.95$ (slow); $\cdots$ $\rho = 0.75$ (fast)

However, the position changes considerably further along the transition path. Faster debt reduction ensures that debt is closer to its new, lower target level b^{low} sooner. This allows lower rates of capital and labour income taxation further along the transition path, than would otherwise be the case. This avoids the larger and more protracted fall in the private capital stock that occurs when debt reduction is slower, as in Subsection (1.3.5.1). It also boosts the post tax wage such that labour supply is higher than it would otherwise be, further along the transition path. It follows that output is also higher than it would otherwise be: Figure (1.3.5.2) (Part Four). Consumption is then higher further along the transition path than under a slower pace of debt reduction.

Figure 1.3.5.2 - Part Four

Hours Worked	Output



Legend: — $\boldsymbol{\rho = 0.95}$ (slow); $\cdots$ $\boldsymbol{\rho = 0.75}$ (fast)

It seems that a faster pace of debt reduction involves a larger consumption cost in the short term, as was the case in the model with only public capital. However, paying debt down faster allows fiscal policy to return to more normal settings faster. In turn, this allows the economy to recover from the effects of austerity faster than it otherwise would.

1.3.5.3 Private and Public Capital: Robustness to Alternative Parameter Values

The *qualitative* properties of Ramsey optimal policy over the transition path in the model with *both* private and public capital do *not* change when:

(1) The stock of public capital is more productive (i.e. when the elasticity of output to public capital θ is set to 0.1 rather than 0.05); and

(2) The Frisch elasticity of labour supply is lowered from three to two, by setting the parameter κ to 2 rather than 3.

Numerical solutions for the behaviour of the Ramsey economy over the transition path under these alternative parameter settings are presented in Appendix (A.5).

1.4 Conclusion

This chapter has presented a theoretical investigation of how a benevolent government chooses changes to government spending and tax rates to achieve an exogenously given, frontloaded path of government debt reduction. The optimal policy choices have been studied for different *speeds* of debt reduction. Specifically, different exogenous, downward paths for government debt have been considered, each of the same length. A *faster* pace of debt reduction requires more of the desired reduction in debt to occur sooner. The benevolent government solves a Ramsey problem in a deterministic, neoclassical model with public and private capital. Government spending takes the form of investment in the productive public capital stock. The chapter presents the deterministic transition path between a steady state with a relatively high level of government debt to one with a low level of government debt. Over the transition path, government debt falls as a percentage of GDP.

The optimal fiscal policy changes for achieving a reduction in government debt depend on the ability of the government to tax private capital income. In a model *without* private capital, a reduction in government debt is achieved by raising the labour income tax rate and setting government spending (i.e. public investment) to a sufficiently low level such that the public capital stock is eroded. Consumption falls as the post tax wage declines.

Imposing a faster pace of debt reduction requires a larger increase in the labour income tax rate and lower government spending in the early periods of the transition. There is a sharper and deeper fall in consumption. However, this more painful austerity in early periods allows more of the desired debt reduction to be achieved sooner. This allows the government to revert to more normal fiscal policy settings earlier in the transition, lowering the labour income tax rate and raising government spending (i.e. public investment). Post tax labour income recovers faster and so too does consumption.

Optimal fiscal policy for reducing government debt *changes* in a model with *both* private and public capital. Over much of the transition path, the government taxes income from private capital investment, but leaves government spending (i.e. public investment) relatively unaltered. Labour income tax rates are low in the early periods of the transition.

The reduction in the post tax rental rate of return on private capital reduces private investment. This erodes the private capital stock. The reduction in the post tax rental rate drives down the real interest rate on government bonds, reflecting a no arbitrage relationship. It is the reduction in the real value of government interest payments and the revenue from private capital income taxation that principally finance government debt reduction in this setting with both private and public capital. Consumption is driven down by reductions in consumer income from capital, labour and government bond holdings.

If a faster pace of debt reduction is imposed, capital income tax rates are set even higher in the early periods of the transition. This drives down even further the post tax rental rate on private capital and the real interest rate on government bonds. However, paying down more of the debt faster again allows the government to reduce both capital and labour income tax rates sooner. This boosts private investment and actually prevents the larger and more prolonged decline in the private capital stock that occurs under a slower pace of debt reduction. The post tax wage, labour supply, output and consumption all recover faster and are higher than they would otherwise be further along the transition path. It seems that incurring greater consumption costs of austerity in the short term allows for a faster recovery of consumption in later periods.

Chapter 2 Optimal Fiscal Policy For Reducing Government Debt Under Different Monetary Policy Regimes: How Hawkish Monetary Policy Leads To Higher Taxes and Lower Consumption

2.1 Introduction

Fiscal stimulus and assistance to the banking industry during the crisis of 2008-2009 are among the reasons why a number of governments in developed economies have accumulated large stocks of outstanding debt, as a percentage of gross domestic product (GDP). In fact, some of these governments entered the crisis of 2008-2009 with already high levels of debt. A number of these governments have announced an intention to reduce government budget deficits by particular dates (and potentially run surpluses), with the ultimate aim of lowering the ratio of outstanding government debt to GDP. Some of these governments have committed to changes to government spending and tax rates in order bring about these reductions in debt and deficits.

The Research Question And The Methodology

In essence, governments in this situation face two related questions. First, what combination of changes to government spending and tax rates will bring about the debt reduction within a desired timeframe at the least cost (or greatest benefit) to employment, output and consumer welfare? Second, what monetary policy allows for the level of government debt to be reduced, at least cost to employment, output and welfare? Also, if monetary policy is not set optimally in this sense, how does the chosen fiscal policy change and is this change harmful to the real economy and consumers? Finally, governments must consider what is the optimal speed of debt reduction, choosing by how much debt should be reduced each year into the future?

This chapter investigates the first three of these questions, but abstracts from the final question as to speed and timing of debt reduction. Specifically, this chapter presents a theoretical investigation of the optimal combination of changes to government spending and tax rates to achieve a given reduction in government debt according to a given timeline for this debt reduction. First, optimal fiscal policy is determined in the case when monetary policy can be chosen optimally by the government, jointly with fiscal policy. This is then compared to optimal fiscal policy in the case where monetary policy follows a feedback (or Taylor type) rule for the nominal interest rate, similar to those used to describe the behaviour of inflation targeting central banks.

The Type of Model

The theoretical setting for this chapter is a deterministic, cashless model with many of the features of a neoclassical model, with the addition of nominal rigidity in the setting of goods prices. The nominal rigidity arises by the assumption that there are real costs of price adjustment, which take the form outlined in Rotemberg (1982). The presence of a nominal rigidity allows for fiscal policy changes to impact upon aggregate demand. This setting also allows government borrowing to be modelled as occurring through the issuance of one period, risk free *nominal* bonds. Because of this assumption, the outstanding stock of government debt is equal to the level of government borrowing each period. The *real* return on government bonds (i.e. the real repayment burden) is determined by both the nominal interest rate and the inflation rate in the economy. Monetary policy can be modelled in this setting as the choice of the nominal interest rate on government borrowing. The government has no recourse to lump sum taxation but can levy a distortionary tax on labour income. Government spending takes the form of investment in a *productive public capital stock*, which is an input into the production of output along with labour. It should be noted that there is no long run or steady state economic growth in the model.

The Optimal Policy Problem Of The Government

The benevolent government solves a Ramsey problem, effectively choosing a sequence of government spending and tax rates to reduce the level of *real* government borrowing (i.e. the value of outstanding debt in terms of the economy's price level), which must follow an exogenous downward path. The Ramsey problem is solved in two cases. The first allows the government to jointly choose monetary policy, which I refer to as *optimal* monetary policy or *endogenous* monetary policy. In the second case, the Ramsey planner is constrained by the requirement that the nominal interest rate follow a standard, inflation targeting interest rate *feedback rule* (i.e. a Taylor type rule). This investigates whether monetary policy under the interest rate feedback rule is more or less contractionary than endogenous monetary policy (i.e. whether interest rates are higher or lower), as the government reduces real debt. The impact of these different monetary policies on optimal fiscal policy choices is then examined. Further, the chapter then presents a study of the different impacts on the real economy of these monetary policies and their corresponding optimal fiscal policies, as real government debt is reduced.

It is assumed throughout the chapter that the government is able to commit fully to an announced sequence of government spending, tax rates and nominal interest rates, constituting a fiscal policy and monetary policy. The Ramsey planner's problem is non-trivial. It is not optimal for the Ramsey planner to simply use inflation as a form of lump sum taxation, to reduce the real value of the government's nominal borrowing. This is because inflation has real costs in a model with nominal rigidities: see Schmitt-Grohe and Uribe (2004) for further discussion.

A *deterministic* setting is chosen for the model because the objective is to study a transition path along which there is potentially substantial government debt reduction, rather than to study the economy's response to shocks in the neighbourhood of a steady state. I solve numerically for this transition path by formulating a system of equations characterising the behaviour of the economy under Ramsey optimal fiscal policy over a *particular* horizon. I impose initial conditions which imply that government debt is relatively high. Terminal conditions correspond to the economy being in a steady state with a lower level of government debt. *Real* government debt (equivalent to real government borrowing with one period bonds) follows an exogenous downward path and reaches the level of real debt corresponding to the steady state *at or before* the end of the horizon. Government debt as a percentage of GDP falls over the transition path. The model is solved using parameter values chosen based on the results of empirical studies.

The Results: Government Spending and Tax Rates

It is found that the government lowers real government debt by reducing government spending (public investment). The stock of public capital ultimately falls to its final steady state value. The labour tax rate is gradually reduced from its initial level as public investment declines. The tax rate settles at a lower level in the ultimate steady state. This suggests that the reduction in public investment finances both debt reduction and a lower distortionary tax rate. This is the case both when monetary policy is chosen optimally and when chosen in accordance with an interest rate feedback rule.

In both cases, it should be noted that the Ramsey planner generates some surprise inflation in the initial period. This reduces the real value of debt repayment in the first period, by lowering the real interest rate. In other words, the government generates seigniorage revenue in the first period. This is a standard feature of the first period of a Ramsey plan, because the planner is not bound to any previous policy commitments in this period. The amount of surprise inflation is modest in this model, reflecting that inflation is assumed to have real economic costs.

Optimal Monetary Policy

The government chooses to lower the nominal interest rate in the first period of the transition path, when setting monetary policy optimally, unconstrained by a monetary policy rule. This helps lower the real interest rate and thus the real debt repayment burden of the government in the first period of the transition. It is a form of surprise taxation on bond holders (i.e. consumers). My results suggest that the Ramsey planner would be constrained by the zero lower bound on nominal interest rates, if the planner attempted a government debt reduction of more than around ten percentage points of GDP, using a frontloaded debt reduction path. Following the first period, the government then raises the nominal interest rate, to avoid a sharp fall in consumption.

Monetary Policy Under An Interest Rate Rule: Inflation Targeting

Monetary policy is *less* accommodating when it is chosen in accordance with the interest rate feedback rule than when chosen optimally by the government. By following the interest rate rule (with standard choices for its parameters), the government places a higher weight on price stability than on easing the real interest burden on government debt. There is no reduction of the nominal interest rate in the first period of the transition. The real interest rate is higher compared to the case when monetary policy is set optimally. The real value of government debt repayments is also higher. Because of this, labour tax rates must be initially set higher when monetary policy follows the rule, as opposed to when it is chosen optimally. Consumption is also lower over much of the transition path, reflecting a lower post-tax wage.

The remainder of this chapter is organised as follows. Section (2.2) presents the cashless model with nominal rigidity in a deterministic setting. Competitive equilibrium is defined and the Ramsey problem is formulated and solved. In section (2.3), I present the numerical solution for the model's deterministic transition under Ramsey optimal fiscal policy, from a state with a relatively high level of government debt to a steady state with a relatively low level of debt. Government debt as a percentage of GDP falls over the transition path. I do this twice: once when monetary policy is also determined optimally by the Ramsey planner and once when nominal interest rates must follow an interest rate feedback rule. Section (2.4) concludes.

2.2 The Model

Discrete time is indexed by $t = \{0, 1, 2, ...\}$ in the deterministic, cashless theoretical model similar in some respects to that in Schmitt-Grohe and Uribe (2004).[4] In this section, I define first

[4] It should be noted that optimal monetary policy may display some stabilisation bias in this model, because a role for money is not assumed.

the production technology and then describe the actions of the two types of agent in the model: consumers (who also act as producers) and a benevolent government. I then define competitive equilibrium and formulate the Ramsey problem of the government.

2.2.1 The Production Technology

I assume that there is a continuum of varieties of intermediate goods in this model, indexed by $i \in (0,1)$. It is further assumed that a single production technology is used to produce each of these varieties. This technology is used to produce Y_{it} units of variety i in period t according to

$$Y_{it} = F(G_{t-1}, h_{it}) \tag{2.1}$$

where F is a function $F : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ of two inputs: the public capital stock G_{t-1} owned by the government that is predetermined each period and the amount of labour h_{it} applied by a producer to the production of variety i in period t. The function F is continuously differentiable in its arguments (i.e. each of the first order partial derivatives exist and is continuous) and also strictly increasing in its arguments, so that

$$F_G(G_{t-1}, h_{it}) > 0, \quad F_h(G_{t-1}, h_{it}) > 0 \tag{2.2}$$

It is further the case that

$$Y_{it} = F(0, h_{it}) = 0, \quad Y_{it} = F(G_{t-1}, 0) = 0 \tag{2.3}$$

The function F is assumed to be strictly quasi-concave. The production technology each period also satisfies the following Inada conditions:

$$\lim_{G \to \infty} F_G(G_{t-1}, h_{it}) = 0, \quad \lim_{G \to 0} F_G(G_{t-1}, h_{it}) = \infty \tag{2.4}$$

It is assumed that the production function exhibits constant returns to scale in the *private, non-reproducible* input h_{it}, that is provided by a fixed quantity of consumers (see section (2.2.2)) and not the government. There are increasing returns to scale across both the public input (public capital) and the private input.[5] However, note that there will be decreasing returns to scale in the

[5] Public capital is a pure public good, to which any number of firms can have costless and unrestricted access.

sole reproducible input, public capital. This is a necessary condition for a zero growth steady state to exist in the model.

2.2.2 Consumers

It is assumed that each consumer is the monopolistically competitive producer of a single variety of intermediate good. This implies that there is a continuum of consumers indexed by $i \epsilon (0, 1)$. Each variety is produced according to the single production technology described in section (2.2.1). The consumer has free access to the predetermined public capital stock G_{t-1} and also *demands* and hires labour services h_{it} from a perfectly competitive labour market, where the real wage is w_t and is taken as given by the consumer. The consumer chooses price p_{it} to charge for the intermediate variety. The consumer also consumes an amount c_t of an aggregate consumption good, comprising the continuum of intermediate goods. This aggregate takes the Dixit and Stiglitz (1977) form:

$$c_t = \left(\int_0^1 c_{it}^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \tag{2.5}$$

where η is a parameter. The consumer chooses to *supply* h_t^s labour hours to the perfectly competitive labour market. Each consumer is assumed to have an endowment of time or hours each period that is normalised to one, so that $0 \leq h_t^s \leq 1$. The decisions about how much labour to *demand* for production and how much to *supply* to earn labour income are taken separately by the consumer and the consumer does not believe that its labour supply decision impacts upon the wage w_t or on the labour demand decision and vice versa. The consumer saves by demanding holdings of *nominal* government bonds B_t^d.

2.2.2.1 Consumers' Two Stage Maximisation Problem

The maximisation problem facing a consumer is to choose an infinite sequence of the consumption aggregate c_t, labour *supply* h_t^s, savings by *nominal* government bond holdings demanded B_t^d, labour *demanded* h_{it} for production and price of the produced intermediate good p_{it} to maximise the discounted present value of utility over an infinite horizon. This implies that the consumer chooses an infinite sequence of each variety c_{it} to consume. It is possible to solve this maximisation problem in two stages because the consumption aggregate c_t exhibits *homogenous separability*: see John-Green (1971) and Dixit and Stiglitz (1977) for further information.

Stage One of the Consumers' Problem The consumer determines how much of each variety c_{it} will be consumed to achieve *one* unit of consumption c_t of the aggregate good. The consumer does this by solving the following *intra*-temporal expenditure minimisation problem in a given period t

$$\min_{c_i} \int_0^1 p_i c_i di \tag{2.6}$$

subject to the constraint that

$$c_t = \left(\int_0^1 c_{it}^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} \geq 1 \tag{2.7}$$

which will hold with equality, since otherwise expenditure could be minimised further if all prices are positive. The first order conditions of this problem imply that the demand by the consumer for each variety of intermediate good is given by

$$c_i = (\frac{p_i}{P_t})^{-\eta} c_t \tag{2.8}$$

where P_t is the price of one unit of the consumption aggregate c_t

$$P_t = \left[\int_0^1 p_i^{1-\eta} di \right]^{\frac{1}{1-\eta}} \tag{2.9}$$

The own price elasticity of demand for variety i is given by $-\eta$ while the income elasticity of demand is one. The parameter η can also be shown to equal the elasticity of substitution between varieties.[6] Full derivations of the expenditure minimisation problem are available in **Appendix (**B.1**)**.

Stage Two of the Consumers' Problem Given the composition of the consumption aggregate c_t, the second stage of the consumers' maximisation problem is an *inter*-temporal problem. The

[6] The elasticity of substitution between varieties is given by $\frac{\partial \log(\frac{c_i}{c_j})}{\partial \log(\frac{p_i}{p_j})} = -\eta$.

consumer chooses an infinite sequence $\{c_t, h_t^s, b_t^d, h_{it}, p_{it}\}_{t=0}^{\infty}$ of consumption, labour (hours worked supplied), nominal government bond holdings demanded, labour services demanded for production and the price of the intermediate good produced to maximise the discounted present value of future utility

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(1-h_t)\right] \tag{2.10}$$

where β is the subjective discount factor $0 < \beta < 1$. The period utility functions $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $v : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are assumed bounded above, continuously differentiable in each argument and strictly increasing in each argument. The functions are assumed to be strictly quasi-concave. The following Inada conditions hold:

$$\lim_{c\to\infty} u'(c) = 0, \qquad \lim_{c\to 0} u'(c) = \infty, \qquad \lim_{h\to 1} v'(1-h) = \infty \tag{2.11}$$

The representative consumer's maximisation problem is subject to the consumer's *nominal* budget constraint each period

$$P_t c_t + B_t^d \leq P_t(1-\tau_t^h)w_t h_t^s + P_t\left[\frac{p_{it}}{P_t}q_{it} - w_t h_{it} - \frac{\theta}{2}\left(\frac{p_{it}}{p_{it-1}} - 1\right)^2\right] + R_{t-1}B_{t-1}^d \tag{2.12}$$

where R_{t-1} is the gross nominal interest rate on government bonds issued in period $t-1$ and τ_t^h is the distortionary tax rate on labour income. I introduce the variable q_{it} to denote the amount of the intermediate variety sold each period, which must equal the amount demanded by the economy at price p_{it}

$$q_{it} = (\frac{p_{it}}{P_t})^{-\eta} a_t \tag{2.13}$$

Aggregate demand a_t comprises purchases of an aggregate of varieties by all consumers and also by the government, something that will be explained further in section (2.2.3). The *real* profit of a consumer as an intermediate good producer at time t corresponds to the term

$$\frac{p_{it}}{P_t}q_{it} - w_t h_{it} - \frac{\theta}{2}\left(\frac{p_{it}}{p_{it-1}} - 1\right)^2 \tag{2.14}$$

where *real* costs to the producer of changing the price of the intermediate good in period t take a form following Rotemberg (1982):

$$\frac{\theta}{2}\left(\frac{p_{it}}{p_{it-1}}-1\right)^2 \tag{2.15}$$

with θ a parameter. Additional constraints on the consumers' problem include a requirement that production of the intermediate good meet demand at the posted price p_{it} each period:

$$F(G_{t-1},h_{it}) \geq (\frac{p_{it}}{P_t})^{-\eta}a_t \tag{2.16}$$

This will hold with equality always since it can never be profit maximising to produce more output than is sold. The consumer is also subject to the inequality constraint

$$0 \leq h_t^s \leq 1 \tag{2.17}$$

as well as the non-negativity constraints every period

$$c_t \geq 0 \tag{2.18}$$

$$B_t^d \geq 0 \tag{2.19}$$

This constraint (2.19) prevents the *consumer* running Ponzi schemes. The consumer is also assumed to be subject to some upper limit on bond holdings. The initial condition is $R_{-1}B_{-1}$.

It turns out that the inequality constraint (2.17) and the non-negativity constraints (2.18) and (2.19) will never bind in competitive equilibrium. The Inada conditions (2.11) show that consumption must always be positive and that hours worked can never equal one. In competitive equilibrium, hours worked can also never be zero. This is because it will be shown that all consumers supply the same amount of labour in competitive equilibrium, so that national output and income would be zero if labour supply was zero, as shown by (2.3). The inequality constraint on demand for government bonds (2.19) and the upper limit on bond holdings will never bind in competitive equilibrium because I will assume that *real* government borrowing is exogenous, positive and on a downward path, in all competitive equilibria. A consumer is assumed to not consider the effect of its pricing, production and labour supply decisions on the aggregate price level P_t and the real wage w_t. It should be noted that the parameter η must be greater than one η to ensure that the period utility functions in (2.10) are concave.

I solve the consumers' problem using the method of Lagrange multipliers.

The Lagrangian for the consumers' problem is

$$L = \sum_{t=0}^{\infty} \beta \left[\begin{array}{c} u(c_t) + v(1-h_t) \\ -\frac{\lambda_t}{P_t} \left\{ \begin{array}{c} P_t c_t + B_t^d - P_t(1-\tau_t^h) w_t h_t^s \\ -P_t \left[\frac{p_{it}}{P_t} q_{it} - w_t h_{it} - \frac{\theta}{2} \left(\frac{p_{it}}{p_{it-1}} - 1 \right)^2 \right] - R_{t-1} B_{t-1}^d \end{array} \right\} \\ -mc_t \lambda_t \left\{ (\frac{p_{it}}{P_t})^{-\eta} a_t - F(G_{t-1}, h_{it}) \right\} \end{array} \right] \tag{2.20}$$

where $\frac{\lambda_t}{P_t}$ is the multiplier on the consumer period budget constraint, implying that λ_t is the marginal utility of real wealth in period t. The multiplier on the constraint requiring that production meet demand is $mc_t \lambda_t$. It will be shown below that mc_t is the real marginal cost of producing one unit of an intermediate variety at time t. The first order necessary conditions for an interior maximum are given by the following:

$$\frac{\partial L}{\partial c_t} = u'(c_t) - \lambda_t = 0 \tag{2.21}$$

$$\frac{\partial L}{\partial h_t^s} = -v'(1-h_t^s) + \lambda_t(1-\tau_t^h) w_t = 0 \tag{2.22}$$

$$\frac{\partial L}{\partial B_t^d} = -\frac{\lambda_t}{P_t} + \beta \frac{\lambda_{t+1}}{P_{t+1}} R_t = 0 \tag{2.23}$$

$$\frac{\partial L}{\partial h_{it}} = -\lambda_t w_t + mc_t \lambda_t F_h(G_{t-1}, h_{it}) = 0 \tag{2.24}$$

$$\begin{aligned} \frac{\partial L}{\partial p_{it}} &= \lambda_t(1-\eta)p_{it}^{-\eta}(\frac{1}{P_t})^{1-\eta}a_t \\ &\quad -\lambda_t\theta(\frac{p_{it}}{p_{it-1}}-1)(\frac{1}{p_{it-1}}) \\ &\quad +\beta\lambda_{t+1}\theta(\frac{p_{it+1}}{p_{it}}-1)(\frac{p_{it+1}}{p_{it}^2}) \\ &\quad +\eta mc_t\lambda_t p_{it}^{-\eta-1}(\frac{1}{P_t})^{-\eta}a_t \\ &= 0 \end{aligned} \tag{2.25}$$

together with the consumer budget constraint (2.12) holding with equality each period, because it cannot be optimal to leave excess income unconsumed. There is also the initial condition $R_{-1}B_{-1}$. Further assumptions would need to be made to establish that these first order conditions are *sufficient* to characterise a unique, interior maximum for the infinite sequence $\{c_t, h_t^s, B_t^d, h_{it}, p_{it}\}_{t=0}^{\infty}$. However, the first order conditions do characterise a unique interior maximum in the numerical simulations I present in section (2.3).

These first order conditions embody optimising behaviour by consumers and can be interpreted as follows. Equation (2.21) implies that the marginal utility of consumption each period must equal the marginal utility of that period's wealth. The next condition (2.22) is an *intra*-temporal condition stating that consumers should supply labour up to the point where the marginal disutility of an additional unit of labour supplied equals the marginal utility value of the additional after tax income attained by doing this. Equation (2.23) is a standard Euler equation for government bond holdings, implying that consumers will demand bonds in period t up to the point where the marginal utility cost of purchasing the bond equals the *real* discounted marginal utility value of the bond's return in period $t+1$. The condition (2.24) can be rearranged to show that the *real* marginal cost mc_t of producing an additional unit of a consumer's differentiated intermediate good each period equals the real labour cost per unit of the good produced, given by

$$\frac{w_t}{F_h(G_{t-1}, h_{it})} \tag{2.26}$$

The first order condition with respect to the price for a consumer's intermediate good p_{it} implies that the price adjustment costs (related to the parameter θ) prevent the consumer setting the *real* price $\frac{p_{it}}{P_t}$ so as to equal a mark-up over *real* marginal cost mc_t. That is, if $\theta = 0$ and there were no price adjustment costs

$$\frac{p_{it}}{P_t} = \frac{\eta}{\eta - 1} mc_t \tag{2.27}$$

2.2.3 The Government

The government spends by investing I_t^G each period in the productive public capital stock G_{t-1}. The public capital stock is predetermined, so that the public capital stock available at the beginning of period $t+1$ equals

$$G_t = I_t^G + (1 - \delta^G) G_{t-1} \tag{2.28}$$

where δ^G is the depreciation rate of the public capital stock. One unit of investment in the public capital stock I_t^G is an aggregate of intermediate varieties, constructed in the same way as the consumption aggregate c_t so that

$$I_t^G = \left(\int_0^1 (I_{it}^G)^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} \tag{2.29}$$

The government is assumed to choose how much of each intermediate variety should be used to construct one unit of the investment good I_t^G by solving an expenditure minimisation problem, identical to that of the consumer: please see **Appendix (**B.1**)** for the derivations.

Recall that in section (2.2.2.1), I noted that a_t as aggregate demand would comprise consumption and government investment in productive public capital. It is now possible to define aggregate demand as

$$a_t \equiv c_t + I_t^G \tag{2.30}$$

The government raises revenue by levying a distortionary tax τ_t^h on consumer's labour income.

The government borrows by issuing one period, risk free *nominal* bonds. Bonds issued in period t pay the gross *nominal* interest rate R_t in period $t+1$. Each period the government must satisfy its *nominal* period budget constraint:

$$P_t\left[G_t-(1-\delta^G)G_{t-1}\right]+R_{t-1}B^S_{t-1}\leq P_t\tau^h_t w_t h_t+B^S_t \tag{2.31}$$

where $G_t-(1-\delta^G)G_{t-1}$ is real period t investment in public capital, as shown by (2.28). Gross nominal interest payments required on debt issued in period $t-1$ are given by $R_{t-1}B^S_{t-1}$. On the revenue side, tax takings from tax on labour income are $P_t\tau^h_t w_t h_t$ while the nominal amount raised by issuing bonds in period t is B^S_t. Defining the real value of government bond issuance in period t as

$$b^S_t\equiv\frac{B^S_t}{P_t} \tag{2.32}$$

the *real* period budget constraint of the government can be written as

$$\left[G_t-(1-\delta^G)G_{t-1}\right]+R_{t-1}\frac{b^S_{t-1}P_{t-1}}{P_t}\leq\tau^h_t w_t h_t+b^S_t \tag{2.33}$$

2.2.4 Aggregate Resource Constraint

The economy must satisfy its aggregate resource constraint each period, which in *real* terms is given by

$$c_t+G_t-(1-\delta^G)G_{t-1}+\frac{\theta}{2}(\pi_t-1)^2\leq F(G_{t-1},h_t) \tag{2.34}$$

implying that consumption c_t, investment in public capital by the government $G_t-(1-\delta^G)G_{t-1}$ and the real costs of price adjustment across the economy $\frac{\theta}{2}(\pi_t-1)^2$ must be less than or equal to real output each period, where

$$\pi_t\equiv\frac{P_t}{P_{t-1}} \tag{2.35}$$

is the *gross* inflation rate of the aggregate price level. This constraint (2.34) will hold with equality in equilibrium because it can never be optimal to leave resources unspent, given that the marginal utility of consumption $u'(c)>0$ for all $c>0$. It has been explained in section (2.2.2.1) that the consumer's period budget constraint must also always hold with equality each period in equilibrium. Together, these two constraints imply that the government's real period budget constraint (2.33) must hold with equality each period.

2.2.5 Competitive Equilibrium

It is possible to simplify the first order conditions of the consumers' problem stated in section (2.2.2.1). First, all consumers producing an intermediate good will charge the same price for that good. This is because all firms use the same production technology (2.1) and face the same real wage w_t, with labour the only private input. The price charged by each producer will equal the aggregate price level in the economy in equilibrium

$$p_{it} = P_t \tag{2.36}$$

All firms will hire the same amount of labour also for these reasons. There is a continuum of producers all hiring labour and a continuum of consumers supplying labour, so it must be that

$$h_{it} = h_t^s = h_t \tag{2.37}$$

The labour supply h_t^s of each consumer can be treated as the amount of labour hired by each producer every period. Because of these assumptions about price setting and hours worked, it is possible to treat the model as if there is a single representative consumer. These results can be imposed on the representative consumer's problem first order conditions, along with the constraint (2.16) that output equal the amount demanded. These conditions become

$$\frac{\partial L}{\partial c_t} : u'(c_t) = \lambda_t \tag{2.38}$$

$$\frac{\partial L}{\partial h_t^s} : v'(1 - h_t) = \lambda_t(1 - \tau_t^h)w_t \tag{2.39}$$

$$\frac{\partial L}{\partial B_t^d} : \lambda_t = \beta \frac{\lambda_{t+1}}{\pi_{t+1}} R_t \tag{2.40}$$

$$\frac{\partial L}{\partial h_{it}} : w_t = mc_t F_h(G_{t-1}, h_t) \tag{2.41}$$

$$\begin{aligned} \frac{\partial L}{\partial p_{it}} \quad : \quad & \lambda_t \theta \pi_t(\pi_t - 1) = \beta\theta\lambda_{t+1}\pi_{t+1}(\pi_{t+1} - 1) \\ & + \lambda_t \eta F(G_{t-1}, h_t) \left[mc_t - \left(\frac{\eta - 1}{\eta} \right) \right] \end{aligned} \tag{2.42}$$

The formal statement of the competitive equilibrium will also make use of the following definitions:

Definition 4 *Given initial conditions $R_{-1}b_{-1}, G_{-1}$, a* ***Feasible Allocation*** *is a sequence $\{c_t, h_t, \pi_t, mc_t\}_{t=0}^{\infty}$ together with a* ***Public Capital Stock Path*** *$\{G_t\}_{t=0}^{\infty}$ such that the aggregate resource constraint (2.34) is satisfied with equality each period. A* ***Tax Policy*** *is a sequence of tax rates $\{\tau_t^h\}_{t=0}^{\infty}$, while a* ***Real Government Borrowing Path*** *is a sequence $\{b_t\}_{t=0}^{\infty}$. A* ***Wage Path*** *is a sequence $\{w_t\}_{t=0}^{\infty}$ while a* ***Nominal Interest Rate Path*** *is a sequence of gross nominal interest rates $\{R_t\}_{t=0}^{\infty}$.*

It is now possible to define a competitive equilibrium formally.

Definition 5 *A* ***Competitive Equilibrium*** *is a Feasible Allocation (including a Public Capital Stock Path), a Tax Policy, a Government Borrowing Path, a Nominal Interest Rate Path and a Wage Path such that each period the first order conditions of the representative consumer's problem (2.38), (2.39) (2.40), (2.41) and (2.42), a real version of the consumer's budget constraint (2.12), the real government budget constraint (2.33) and the aggregate resource constraint (2.34) always hold. Further, all markets clear each period and an appropriate transversality condition must hold.*

Note that market clearing implies that the nominal government bond market and the market for labour each period all clear, so that

$$B_t^s = B_t^d = B_t \tag{2.43}$$

$$h_t^s = h_{it} = h_t \tag{2.44}$$

This last condition implies that the representative consumer's labour supply h_t^s equals the amount of labour demanded by a representative consumer producing an intermediate good.

The transversality condition on *nominal* government borrowing necessary for a competitive equilibrium is

$$\lim_{j\to\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} B_{t+j} = \lim_{j\to\infty} \prod_{s=0}^{j} \left[\frac{1}{\frac{R_{t+s-1}}{\pi_{t+s}}} \right] B_{t+j} = 0 \tag{2.45}$$

Note that the condition (2.45) requires the nominal value of government debt B_t to grow more slowly than the real rate of interest in the limit, satisfying a No Ponzi condition. This statement of competitive equilibrium follows the structure used by Farhi (2010).

All competitive equilibria considered in this chapter will consider an *exogenous* Real Government Borrowing Path of the form

$$b_t - \bar{b} = \rho\left(b_{t-1} - \bar{b}\right) \tag{2.46}$$

given an initial condition b_{-1}, where $\bar{b}$ is an exogenously specified target level of *real* government borrowing. The transversality condition on *nominal* government bond holdings (2.45) is satisfied in all these competitive equilibria, provided that the real rate of interest exceeds the rate of inflation in the limit. In the limit, the level of *nominal* government debt B grows at the rate of inflation, since the level of *real* government debt is constant. If the *real* rate of interest exceeds the inflation rate, then the level of *nominal* government bond holdings grows slower than the real rate of interest. This is the case in all numerical examples presented in section (2.3). The *exogenous* downward path for real government borrowing will be a constraint on the Ramsey planner in the following Ramsey problem.

Note that condition (2.42) from the representative consumer's first order condition with respect to price setting can be viewed as an *aggregate supply* relation or Phillips Curve, relating current inflation to future inflation and to current real marginal cost of production.

2.2.6 The Ramsey Problem

The statement of the Ramsey problem will make use of the following definition:

Definition 6 *A **Monetary Policy** is a infinite sequence of gross nominal interest rates* $\{R_t\}_{t=0}^{\infty}$.

It is now possible to state the Ramsey problem.

Definition 7 *Given initial conditions* $R_{-1}b_{-1}, G_{-1}$*, the **Ramsey Problem** is to choose a Feasible Allocation (including a Public Capital Stock Path), a Tax Policy and a Monetary Policy to maximise consumer welfare (as defined by equation (2.10)) subject to all the requirements in the definition of a competitive equilibrium being satisfied, with the Real Government Borrowing Path equal to the exogenous, downward path specified in equation (2.46).*

In other words, the Ramsey planner (i.e. the benevolent government) maximises over the set of competitive equilibria, so that the chosen Public Capital Stock Path (part of the Feasible

Allocation), Monetary Policy and Tax Policy give rise to a competitive equilibrium.

Before stating the problem mathematically, the competitive equilibrium conditions in Section (2.2.5) can be simplified by eliminating the real wage w_t and the tax rate on labour income τ_t^h. The representative consumer's first order condition (2.41) can be used to isolate w_t:

$$w_t = mc_t F_h(G_{t-1}, h_t)$$

while the intratemporal condition (2.39) can be manipulated to isolate τ_t^h:

$$\tau_t^h = 1 - \frac{v'(1-h_t)}{\lambda_t w_t} \tag{2.47}$$

These expressions can be used to eliminate w_t and τ_t^h from the real government period budget constraint (2.33), which becomes what I call an **Implementability Condition** derived from this constraint:

$$\left[G_t - (1-\delta^G)G_{t-1}\right] + R_{t-1}\frac{b_{t-1}}{\pi_t} - mc_t F_h(G_{t-1}, h_t)h_t + \frac{v'(1-h_t)h_t}{\lambda_t} = b_t \tag{2.48}$$

Mathematically, the **Ramsey Problem** is then for the Ramsey planner (i.e. benevolent government) to choose an infinite sequence $\{c_t, h_t, G_t, \pi_t, \lambda_t, mc_t, R_t\}_{t=0}^{\infty}$ to maximise consumer welfare (2.10)

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(1-h_t)\right]$$

subject *each period* to the Implementability Condition (2.48) derived from the real government budget constraint

$$\left[G_t - (1-\delta^G)G_{t-1}\right] + R_{t-1}\frac{b_{t-1}}{\pi_t} - mc_t F_h(G_{t-1}, h_t)h_t + \frac{v'(1-h_t)h_t}{\lambda_t} = b_t$$

and competitive equilibrium conditions (2.38), (2.40) and (2.42)

$$u'(c_t) = \lambda_t$$

$$\lambda_t = \beta\frac{\lambda_{t+1}}{\pi_{t+1}}R_t$$

$$\lambda_t\theta\pi_t(\pi_t - 1) = \beta\theta\lambda_{t+1}\pi_{t+1}(\pi_{t+1} - 1) + \lambda_t\eta F(G_{t-1}, h_t)\left[mc_t - \left(\frac{\eta-1}{\eta}\right)\right]$$

plus the real aggregate resource constraint (2.34)

$$c_t + G_t - (1-\delta^G)G_{t-1} + \frac{\theta}{2}(\pi_t - 1)^2 \leq F(G_{t-1}, h_t)$$

and the exogenous downward process for government borrowing (2.46)

$$b_t - \bar{b} = \rho\left(b_{t-1} - \bar{b}\right)$$

given initial conditions $R_{-1}b_{-1}$, G_{-1}. The real consumer period budget constraint need not be stated as a constraint because it must hold with equality if both the real government budget constraint (2.33) (which can be recovered from the Implementability Condition (2.48)) and the real aggregate resource constraint (2.34) hold with equality every period. I call an infinite sequence $\{c_t, h_t, G_t, \pi_t, \lambda_t, mc_t, R_t\}_{t=0}^{\infty}$ that solves the Ramsey problem as stated above a **Ramsey Optimal Plan** or **Ramsey Economy.**

It is important to note that the opportunity set or feasible set of competitive equilibria defined by (2.48), (2.38), (2.40), (2.42), (2.34) and (2.46) over which the Ramsey planner maximises may not be convex. In such a case, the first order conditions of the Ramsey problem may not be sufficient to characterise a unique Ramsey economy. However, in the numerical exercises I present in section (2.3), the Ramsey plans presented are unique.

In the Ramsey problem as defined above, Monetary Policy is set optimally. Alternatively, I will consider solving a Ramsey problem with an *additional constraint* on the Ramsey planner. Namely, this additional constraint requires that Monetary Policy be chosen such that the gross nominal interest rate is consistent with the following interest rate feedback rule or Taylor-type rule:

$$\ln(\frac{R_t}{R_{t-1}}) = \alpha_\pi \ln\left(\frac{\pi_t}{\pi_{t-1}}\right) + \alpha_{AD}\ln\left(\frac{a_t}{a_{t-1}}\right) \tag{2.49}$$

where a_t is aggregate demand. Comparing the solutions of the Ramsey problem when Monetary Policy is set optimally to when it is constrained to follow the rule (2.49) will help to illustrate whether Monetary Policy set in accordance with the feedback rule is optimal from the point of view of a Ramsey planner seeking to reduce real government debt. It should also reveal any different real economic effects of these two potentially different monetary policies.

2.2.7 A Recursive Formulation to Solve the Ramsey Problem

The Ramsey problem as defined in section (2.2.6) is *not* recursive.

In a *deterministic* setting, the problem of a Ramsey planner that makes decisions sequentially each period is a *recursive* problem if the decision facing the Ramsey planner is the *same* each period, given the beginning of period values of *state* variables. *State* variables encapsulate the effects of the initial conditions on the problem as well as the effects of all decisions made in periods prior to current time t. When the problem in a *deterministic* setting is recursive, the sequence of variables chosen *sequentially* each period by the Ramsey planner will be the same as the sequence chosen if all decisions (i.e. the entire infinite sequence) are made in the initial period. A recursive formulation is useful when applying numerical solution techniques to a deterministic model like that in this chapter, although it is not strictly necessary.

I follow the approach of Kydland and Prescott (1980), Marcet and Marimon (2011) and Kumhof and Yakadina (2007) and obtain a recursive formulation for the Ramsey problem in section (2.2.6) by *augmenting* the number of state variables with the Lagrange multipliers attached to the forward looking constraints on the Ramsey planner each period - that is, constraints containing terms indexed by time $t+1$. Specifically, these are the Euler equation for government bond holdings (2.40) and the Phillips Curve relation (2.42). The other state variables are the beginning of period values of the public capital stock G_{t-1} and the real government debt repayment obligations for that period $R_{t-1}b_{t-1}$. A detailed explanation of how I formulate this recursive Ramsey problem is given in **Appendix (**B.2**).** The Lagrangian that is used to solve the recursive Ramsey problem is:

$$\max_{\{c_t,h_t,G_t,\pi_t,\lambda_t,mc_t,R_t\}_{t=0}^{\infty}} L = \sum_{t=0}^{\infty} \beta^t \tag{2.50}$$

$$\left\{ \begin{array}{c} u(c_t) + v(1-h_t) - \mu^E_{t-1}\frac{\lambda_t}{\pi_t} + \\ \mu^P_{t-1}\lambda_t\theta\pi_t(\pi_t - 1) + \mu^E_t\frac{\lambda_t}{R_t} \\ +\mu^C_t\left[u'(c_t) - \lambda_t\right] \\ +\mu^P_t \left[\begin{array}{c} \lambda_t\eta F(G_{t-1}, h_t)\left[mc_t - \left(\frac{\eta-1}{\eta}\right)\right] \\ -\lambda_t\theta\pi_t(\pi_t - 1) \end{array} \right] \\ +\mu^G_t \left[\begin{array}{c} b_t - \left[G_t - (1-\delta^G)G_{t-1}\right] \\ -R_{t-1}\frac{b_{t-1}}{\pi_t} + mc_t F_h(G_{t-1}, h_t)h_t - \frac{v'(1-h_t)h_t}{\lambda_t} \end{array} \right] \\ +\mu^A_t \left[\begin{array}{c} F(G_{t-1}, h_t) - c_t \\ -G_t + (1-\delta^G)G_{t-1} - \frac{\theta}{2}(\pi_t - 1)^2 \end{array} \right] \end{array} \right\}$$

where μ^E_t and μ^P_t are the Lagrange multipliers at time t attached to the competitive equilibrium conditions (2.40) and (2.42) with forward looking terms. According to Marcet and Marimon (2011), Kumhof and Yakadina (2007) and Reiter (2005), these multipliers must be zero initially (i.e. $\mu^E_{-1} = 0 = \mu^P_{-1}$). This is because the planner in period *zero* need not behave consistently with the time $t-1$ versions of (2.40) and (2.42), containing forward looking terms. I impose this when solving the Ramsey problem numerically in Section (2.3). The Lagrange multiplier μ^G_t is attached to the Implementability Condition (2.48) derived from the government's real period budget constraint while μ^A_t is the Lagrange multiplier attached to the aggregate resource constraint (2.34) at time t. An additional Lagrange multiplier μ^C_t is attached to the competitive equilibrium condition (2.38).

The necessary conditions for a unique solution to the Ramsey problem, or in other words, necessary conditions for a Ramsey Economy are given by the equations determining when the first order partial derivatives of the Lagrangian vanish each period, together with the constraints (2.48), (2.38), (2.40), (2.42) and (2.34) on the Ramsey planner each period. Also necessary is the constraint determining the exogenous downward path for real government borrowing (2.46) and the initial conditions $R_{-1}b_{-1}$ and G_{-1}. The conditions characterising the Ramsey economy also include the transversality condition for government bond holdings (2.45). An additional transversality condition required to characterise the Ramsey economy is one for public capital, since it cannot be optimal to leave public capital positive as time tends to infinity:

$$\lim_{j\to\infty} \beta^j u'(c_{t+j})G_{t+j} = 0 \tag{2.51}$$

From the sequence $\left\{c_t, h_t, G_t, \pi_t, \lambda_t, mc_t, R_t, \mu_t^E, \mu_t^P, \mu_t^G, \mu_t^A, \mu_t^C\right\}_{t=0}^{\infty}$ of endogenous variables and sequence of real government borrowing $\{b_t\}_{t=0}^{\infty}$ satisfying these conditions it is then possible to recover a sequence $\left\{w_t, \tau_t^h\right\}_{t=0}^{\infty}$ for the real wage and the labour income tax rate, using the expressions (2.41) and (2.47). The conditions characterising a Ramsey Economy are presented in **Appendix (**B.3**)**. In that Appendix, I present the conditions characterising the Ramsey economy when the Ramsey planner is constrained to implement Monetary Policy consistent with an interest rate feedback rule in (2.49).

2.2.8 The Intuition of Optimal Fiscal Policy to Reduce Government Debt

The path of government spending (i.e. investment in public capital) and labour income tax rates chosen by the Ramsey planner (i.e. the government) must at all times satisfy the competitive equilibrium conditions (2.38), (2.39), (2.40), (2.41) and (2.42). This is also true for the path of nominal interest rates, whether they are chosen optimally by the government or set in accordance with the interest rate feedback rule (2.49). These competitive equilibrium conditions can be used to gain intuition about the effects of changes in government spending, tax rates and the nominal interest rate on the economy.

Government Spending

An increase in public spending has a direct one-for-one cost to the government budget, as can be seen from the government's real period budget constraint (2.33). All else equal, this has also the effect of raising labour productivity $F_h(G_{t-1}, h_t)$ and this lowers the real marginal cost of production, which using competitive equilibrium condition (2.41) can be expressed as

$$\frac{w_t}{F_h(G_{t-1}, h_t)} \tag{2.52}$$

This should put downward pressure on inflation which the Phillips Curve relation (2.42) shows is a function of real marginal cost and future inflation. However, higher labour productivity should raise the after tax real wage, assuming tax rates are constant. Labour supply should increase as a result. This should have a positive, second order effect on the government budget by raising government revenue from labour income taxation

$$\tau_t w_t h_t \tag{2.53}$$

This assumes that consumer income levels and the functional form for utility are such that the increase in labour supply because of the substitution effect of higher wages is not outweighed by any income and wealth effects.

Labour Tax Rates

An increase in the rate of labour income tax has a direct, positive effect on the budget, all else equal, as revealed by the government's real period budget constraint (2.33). However, the after tax real wage is lowered mechanically

$$(1-\tau_t)w_t \tag{2.54}$$

and this reduces labour supply, again assuming that substitution effects outweigh income and wealth effects. This has a negative, second order effect on the government period budget constraint by feeding back into lower revenue from income taxation, as implied by (2.53). This second order effect may change if there is further effect on the real wage caused by the increased scarcity of labour after the reduction in labour supply.

Nominal Interest Rates

An increase in the gross nominal interest rate R_{t-1} paid on beginning of period bond holdings raises the gross *real* interest rate

$$\frac{R_{t-1}}{\pi_t} \tag{2.55}$$

all else equal. A standard Euler equation for bond holdings can be written by combining competitive equilibrium conditions (2.38) and (2.40), giving

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \frac{R_t}{\pi_{t+1}} \tag{2.56}$$

The path of nominal interest rates must be always consistent with this Euler equation, irrespective of whether nominal interest rates are chosen optimally by the government or set in accordance with the interest rate feedback rule (2.49). The Euler equation indicates that consumption growth will be always proportional to the gross real interest rate, given that the utility function is assumed concave in section (2.2.2.1).

This intuition will be referred to when interpreting numerical results in section (2.3).

2.3 Government Debt Reduction: The Deterministic Transition Path Of The Ramsey Economy

In this section, I use numerical methods to compute the deterministic transition path of the Ramsey Economy from a state with a relatively high level of government debt to a steady state with a relatively low level of government debt. Government debt eventually falls as a percentage of GDP over the transition path. First, this computation will be done for the Ramsey Economy where monetary policy is chosen optimally. This exercise captures the behaviour of the economy under Ramsey optimal monetary and fiscal policy, where fiscal policy consists of a Public Capital Stock Path (implying a path for public investment spending) and a Tax Policy. Second, I perform this exercise for the Ramsey Economy where the Ramsey planner is constrained to implement monetary policy that satisfies the interest rate feedback rule in (2.49). By doing this, I investigate how optimally chosen monetary policy differs from monetary policy that satisfies the type of interest rate feedback rule that central banks in many developed economies are often thought to follow. This exercise also sheds light on how optimal fiscal policy to reduce real government debt differs between the two types of Ramsey Economy and shows the real economic effects of these differences.

This section of the chapter is divided as follows. First, I choose specific functional forms for period utility and for the production technology in section (2.3.1). Second, values are assigned to the parameters of the model in section (2.3.2). Third, I present the transition path from a state of relatively high government debt to a steady state with relatively low government debt under Ramsey optimal fiscal policy, with monetary policy also set optimally: see section (2.3.3). Fourth, I present the transition path in a model with optimal fiscal policy but where the Ramsey planner is constrained to implement a monetary policy that satisfies the interest rate feedback rule (2.49): see section (2.3.4). Finally, I analyse the robustness of my results to alternative choices for parameter values. A detailed explanation of the numerical method used to solve for the deterministic transition path is given in **Appendix (**B.4**)**.

2.3.1 Specific Functional Forms

For the purposes of computing the deterministic transition path of the Ramsey Economy, I assume that period utility takes the form

$$u(c) + v(1-h) = \ln(c) + \varkappa \ln(1-h_t) \tag{2.57}$$

This implies that the intertemporal elasticity of substitution of consumption is one. The parameter $\varkappa$ affects the consumer labour supply choice and its value will influence the Frisch elasticity of labour supply with respect to the real wage. I discuss the choice of the value for $\varkappa$ in section (2.3.2).

The production technology each period is assumed to take the multiplicative form

$$F(G_{-1}, h) = G_{-1}^{\alpha} h \tag{2.58}$$

where $\alpha > 0$. The parameter α governs the elasticity of output with respect to public capital. Consistent with the general functional form outlined in section (2.2.1), this specific functional form exhibits constant returns to scale in the private input: labour supply h. There are increasing returns to scale across both inputs.

2.3.2 Choice of Parameter Values: The Benchmark

The chosen values for the model's parameters are displayed in the following table:

Table 2.3.2

Parameter Values

α	0.05		θ	16.5		δ^G	0.02		α_π	1.5
β	0.99		η	6		$\varkappa$	3		α_{AD}	≈0
ρ	0.8									

Amongst the most controversial choices is that for the parameter α determining the elasticity of output with respect to public capital. There is a literature computing empirical estimates for this parameter. Unfortunately, the range of parameter estimates in this literature is quite wide. However, recent surveys of this literature such as Bom and Ligthart (2013) attempt to systematically reconcile the various estimates by understanding the differences in the approach taken in different papers. Bom and Ligthart (2013) find that values are most commonly in the narrower range of 0.05-0.1. Estimates nearer or over 0.1 are found when the measurement of public capital is restricted to infrastructure such as roads and ports. Lower estimates closer to 0.05 are found when the measurement of public capital is expanded to cover infrastructure such as schools and hospitals. Papers presenting real business cycle type models with public capital such as the recent paper of Leeper and Yang (2010) use the conservative value 0.05 for the parameter determining the elasticity of output to public capital, when solving models numerically. The well known paper of Baxter and King (1993) also uses the value 0.05. Following Leeper and Yang (2010) and Baxter and King (1993), I choose the conservative value 0.05 for α.

The parameter η determines the consumers' elasticity of substitution between varieties of the intermediate good and also the own price elasticity of demand for each intermediate good. I follow Schmitt-Grohe and Uribe (2004) and choose a value for η such that the mark up of the price of intermediate goods over marginal costs in a steady state of the Ramsey economy will be around 25 per cent, consistent with the estimates of these mark ups in Basu and Fernald (1997).

The parameter θ determines the extent to which price adjustment has real costs. Values chosen for this parameter can be over 50. This can be justified using Keen and Wang (2007), which discusses how the value of θ is related to the values of the elasticity of substitution between varieties η in New Keynesian models. Further, papers presenting New Keynesian models such as Ireland (2001) have used similarly high values for θ. I follow Schmitt-Grohe and Uribe (2004) and choose θ of 16.5, since the model in that paper has similar features to mine. This value for θ can be seen as reflecting a conservative estimate of the degree of price stickiness in a developed economy. The value is based on estimates of $\frac{\eta\theta}{(\eta-1)h}$ by Sbordone (2002), who analyses a New Keynesian Phillips Curve analogous to the one that could be derived from competitive equilibrium condition (2.42) in my model.

The parameter $\varkappa$ is set to three. This implies that the Frisch elasticity of labour supply with respect to the real wage is around 4 in the steady states that I consider.[7] Values of 3-4 for the Frisch elasticity of labour supply have been used before in the theoretical macroeconomics literature: see Schmitt-Grohe and Uribe (2004) and Kumhof and Yakadina (2007). It should be noted that there are micro-level estimates of the Frisch elasticity that are much lower: see Chetty and Weber (2011). Recent research by Peterman (2012) attempts to reconcile these competing estimates. Peterman (2012) finds that a Frisch elasticity of around 3 is obtained when an estimation methodology used in some microeconometric studies is modified to account for changes in *both* the total number of the employed (the extensive margin) as well as labour supply changes by those already employed (the intensive margin).

The value 0.8 is chosen for the parameter determining the rate at which the level of government borrowing must fall over the transition path. The autoregressive nature of the equation (2.46) determining the downward path for government borrowing implies that the reduction in government borrowing is *frontloaded*: that is, the amount by which government borrowing is reduced each period falls over time. The level of government borrowing asymptotes to its new target value over the transition path.

I set the quarterly depreciation rate of public capital to be 2 per cent (corresponding to δ^G of 0.02). This is at the high end of the range of estimates of quarterly public capital depreciation rates in a recent International Monetary Fund study: see Serkan Arslanalp and Sze (2010).

The parameters α_π and α_{AD} determine the response of the gross nominal interest rate to changes in inflation and aggregate demand respectively, under the interest rate feedback rule (2.49). I set α_π to be 1.5 and α_{AD} to be close to zero. This reflects the findings of Schmitt-Grohe

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7 The Frisch elasticity of labour supply with respect to the real wage (given constant marginal utility of wealth) is given by $\frac{\partial h}{\partial w}\frac{\overline{w}}{\overline{h}}\mid_{\overline{\lambda}}=\frac{1-\overline{h}}{\overline{h}}$ for the chosen specific functional form for utility. Setting $\varkappa = 3$ implies that hours worked in the steady states I consider $\overline{h}$ will be approximately 0.2, giving a Frisch elasticity of around 4. This approach of choosing Frisch elasticity of labour supply to match particular values of hours worked (implememted by choosing $\varkappa$) follows Prescott (2003).

and Uribe (2007) that interest rate feedback rules should react to inflation but *not* react strongly to changes in output, if the rules are to be optimal in terms of consumer welfare.

Finally, I set the quarterly subjective discount factor of consumers β to 0.99, as is standard in the literature.

2.3.3 Optimal Fiscal And Monetary Policy

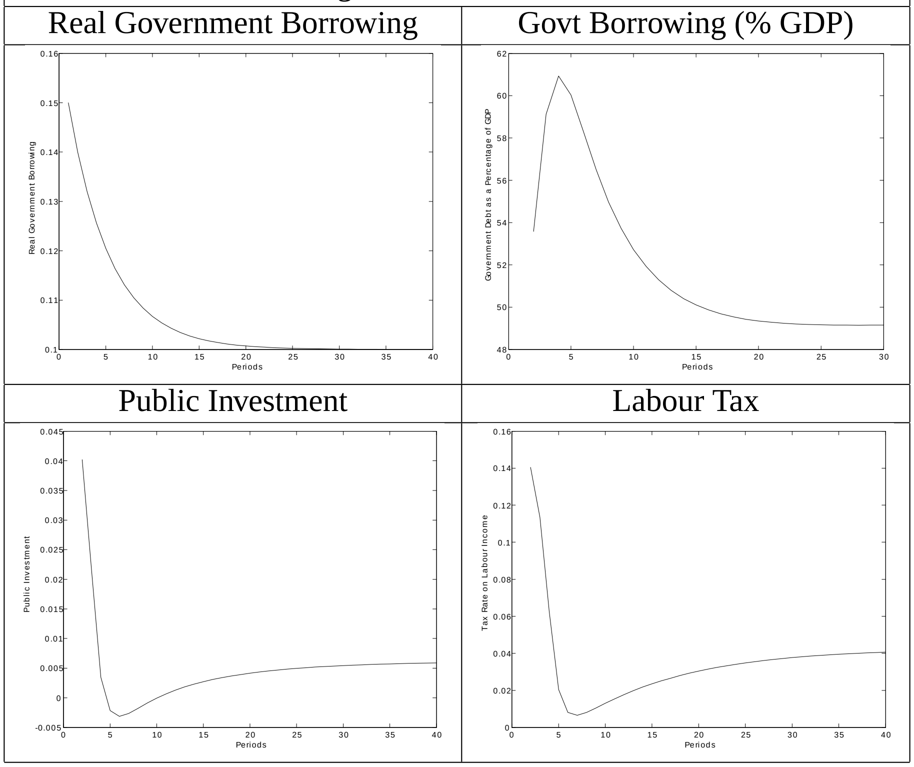
Figure (2.3.3) (Parts One and Two) presents the transition path for the Ramsey Economy when the government chooses monetary policy optimally . The economy moves from a state with relatively high government debt b^{high} until it reaches a steady state where the level of real government borrowing is b^{low}. Initial conditions must be set for real government debt b_{-1} (i.e. b^{high}), initial public capital G_{-1} and the gross nominal interest rate R_{-1}. I set b^{high} and G_{-1} such that government debt is around fifty-five percent of GDP in the first period of the transition path. The gross nominal interest rate R_{-1} is set to the value it will take in *any* steady state. Real government borrowing falls by more in period one of the transition than in any subsequent period. This is because of the assumed *frontloaded* path for real debt reduction: see equation (2.46).

The Ramsey planner reduces public spending (i.e. public investment) over the transition path, allowing the revenue saved to be used for debt reduction. In fact, public investment falls to zero (and is briefly negative) for some time. At these low level of public investment, the effects of public capital depreciation are not offset and the public capital stock is eroded (and of course, this happens if there is disinvestment). The labour tax rate is initially well above its ultimate steady state level, but is actually reduced over the horizon as public investment falls. The reduction in public investment spending funds not only debt reduction but also a reduction in the distortionary tax rate: see Figure (2.3.3) - Parts One and Two.

Erosion of the public capital stock reduces labour productivity as shown by equation (2.41) and thus reduces the real wage considerably. Despite the fall in the rate of tax over the transition, the post tax real wage falls also. All else equal, this should reduce labour supply, provided the substitution effect of the change in the post tax wage outweighs the income and wealth effects. The combined effect of a lower tax rate, lower real wages and reduced labour supply is to lower real government revenue from income taxation, all else equal. The reduction in real government debt occurs in an environment of falling government revenue.

It should be noted that there is some surprise inflation in the first period of the transition path. This lowers the real interest rate (or the real debt repayment burden) for the government in the first period, all else equal. An equivalent interpretation is that the planner generates seigniorage revenue in the first period of the transition path, all else equal. This seigniorage revenue may keep public investment higher than it would otherwise be in the first few periods of the transition path. In fact, public investment is still sufficiently large in the early periods for the stock of public capital actually to increase somewhat, before being eroded in the manner described above. Surprise inflation is a standard feature of Ramsey plans because the planner is not bound by any previous policy commitments in the first period of the transition. The amount of surprise inflation is modest in this model (only four or five percentage points), given that there are real economic costs of inflation, because of price adjustment costs (2.15). Inflation quickly falls to near zero and does not seem to play a dominant part in achieving the desired reduction in real government debt over the transition path.

Figure 2.3.3 - Part One



Unconstrained by an interest rate rule, the planner sets monetary policy by *reducing* the gross nominal interest rate in the first period of the transition path. This reduces the cost of debt repayment for the government. Note that the Ramsey planner would be constrained by the *zero lower bound*, if a larger debt reduction was attempted (as a percentage of GDP), following a similar frontloaded debt reduction path.

Figure 2.3.3 - Part Two

In subsequent periods of the transition path, the Ramsey planner raises the gross nominal interest rate, to ensure that the real interest rate is positive. This prevents the dramatic collapse of consumption growth that Euler equation (2.56) predicts would otherwise occur if the planner allowed the real interest rate to be negative beyond the first period of the transition. This reversal in the sign of the real interest rate explains the kink in the transition path for consumption.

Overall, the reduction in labour supply and the erosion of public capital cause total production / output to fall, as government debt is reduced over the transition to the steady state. Government debt rises initially as a percentage of GDP, before eventually falling to reach a lower level in the new steady state.

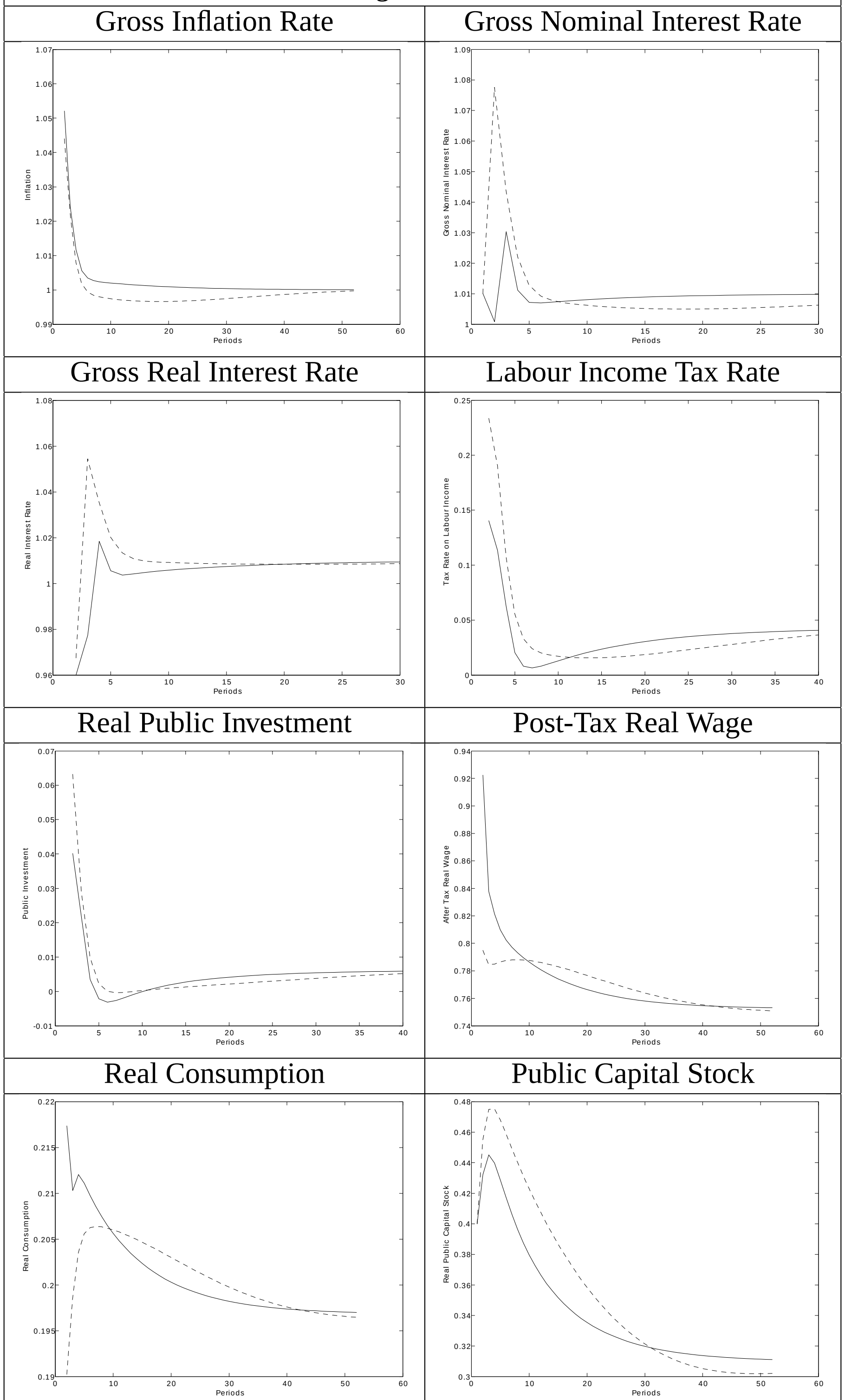
2.3.4 Optimal Fiscal Policy With Monetary Policy Rule

I now present the transition path for the Ramsey Economy when the government is constrained to set monetary policy in accordance with an interest rate feedback rule: see Figure (2.3.4). I use the same initial conditions as for the case of optimal monetary policy in Subsection (2.3.3). Government debt falls by around ten percentage points of GDP over the transition path, before reaching the new steady state level.

There is again a modest amount of surprise inflation in the first period of the transition path. However, the Ramsey planner must follow the interest rate rule, so that there is no reduction in the gross nominal interest rate in the first period. In fact, interest rates rise to counter the inflation. The real interest rate is higher in the early periods of the transition, compared with the case where the planner is unconstrained by the interest rate rule. In fact, there is deflation for some time while the nominal interest rate is elevated, driving up the real interest rate. The government's real interest repayment burden is consequentially higher. The tax rate is set higher in the early periods of the transition to help finance debt reduction under the higher real interest rate.

However, the higher tax rate makes the post-tax wage lower, compared with the case where the Ramsey planner is unconstrained by the interest rate rule. The combination of the lower post-tax wage and higher real interest rate mean that consumption is initially lower than when the planner is free to set the interest rate.

Figure 2.3.4



Legend: — optimal policy; · · · interest rate rule

Interestingly, public investment is initially higher under the interest rate rule than in the case where monetary policy is set optimally. This helps to avoid an even lower post-tax wage.

2.3.5 Robustness of Numerical Results

I consider the robustness of my numerical results to the chosen parameter values. I do this in the model where the Ramsey planner is constrained to follow an interest rate rule. I consider robustness along two dimensions:

(1) **Productivity of Public Capital:** I consider the case where the elasticity of output to public capital α is given by 0.1, rather than 0.05. This is among the higher values for this elasticity in the literature surveyed by Bom and Ligthart (2013). It is also one of the values used by Leeper and Yang (2010). Making the public capital stock more productive may make the Ramsey planner rely more heavily on taxation to reduce government debt, rather than on the reduction of public investment.

(2) **Frisch Elasticity of Labour Supply:** I experiment with lowering the Frisch elasticity of labour supply to *three* and also to *two*. I do this by setting the parameter κ to 2.5 and 3 respectively, with the price adjustment cost parameter raised to 21.875 and 26.25 in each case. This generates the desired Frisch elasticities in the steady state of a competitive equilibrium. A more *inelastic* labour supply may also make the Ramsey planner rely more on taxation to achieve debt reduction. This is because increases in the distortionary labour income tax rate τ should be less distortionary to labour supply.

Changing these parameters changes the ultimate steady state around which the Ramsey economy settles after real government debt has been reduced to the desired level. Initial conditions must be also altered to ensure that the Ramsey economy begins at a relatively high level of government debt (as a percentage of GDP), which then declines over the transition path.

However, **figures (**B.5**) to (**B.6**) in Appendices (**B.5**) and (**B.6**)** show that the qualitative features of my results do not change under either of the two parameter specifications discussed here. There is still a fall in public investment over the transition path (eroding the public capital stock), accompanied by a fall in the distortionary labour income tax rate.

2.4 Conclusions And Implications

This chapter has presented a theoretical investigation of what is the optimal combination of government spending and tax rate changes that should be implemented in order to achieve a desired reduction in the level of real government borrowing, at least cost to consumer welfare. This question has been investigated both with monetary policy chosen optimally and alternatively with monetary policy following an inflation targeting, interest rate feedback rule. The downward path of real government borrowing is treated as exogenous in this chapter, so no comment is made on the optimal speed of government debt reduction.

Specifically, I solved a Ramsey problem in a model with nominal rigidity, choosing an optimal path of public capital investment and tax rates to achieve the exogenous downward path for real government borrowing. Under the two alternative monetary policy regimes, I solved for the deterministic transition path of the Ramsey economy from a state with a relatively high level of real government debt (as a percentage of GDP) to a steady state with lower debt. Numerical results are obtained under choices for parameter values informed by empirical studies. However, the robustness of the qualitative results to different choices for certain parameter values has also been demonstrated.

In general, real government debt reduction is achieved by a fall in public investment. Labour taxation plays a role. However, the fall in public investment allows for the reduction of both real government debt *and* the labour income tax rate over the deterministic transition path.

The chapter also highlights the trade-off between price stability and the cost to governments of real debt reduction. Setting monetary policy in accordance with an interest rate feedback rule (typical of an inflation targeting regime) can result in real interest rates on government debt being higher than a Ramsey planner (i.e. a benevolent fiscal authority) would otherwise want them to be. In fact, a Ramsey planner that is unconstrained by an interest rate feedback rule would seek a lower real interest rate (and lower real debt repayment burden for the government). This is despite the fact that there are real economic costs of inflation and that the Ramsey planner is trying to minimise costs to employment and consumer welfare.

Chapter 3 The Interaction Between Aggregate and Idiosyncratic Risk in an Incomplete Markets Model: Implications for Constrained Efficiency and Fiscal Policy

3.1 Introduction

Understanding the inefficiency of competitive equilibria in heterogeneous agent models is important for designing appropriate fiscal policies that improve welfare. An allocation is constrained efficient if it maximises a weighted average of expected consumer utilities (i.e. *utilitarian social welfare*), within the class of allocations that satisfy all budget constraints and where factor prices are determined in competitive markets. This is one potential benchmark against which fiscal policies that give rise to a competitive equilibrium can be assessed.

Generating a Pareto efficient outcome involves completing markets. Realistic tax and transfer policies usually fail to complete markets, so the constrained efficient outcome can be the more useful benchmark.

3.1.0.1 Constrained Inefficiency

First, in this chapter I characterise the constrained inefficiency of competitive equilibria in a neoclassical model, with incomplete markets for both aggregate risk and idiosyncratic risk to labour income, as well as consumer borrowing constraints. In doing this, I extend the analysis of Davila and Rios-Rull (2012) and Gottardi and Nakajima (2013).

I obtain intuition about the causes of constrained inefficiency by considering the marginal effect on utilitarian social welfare in competitive equilibrium of increasing aggregate saving / capital. If this marginal effect is positive, then it may be said that there is constrained inefficiency of competitive equilibrium due to *under saving* and vice versa. At the margin, I demonstrate that any efficiency improvement of higher aggregate saving / capital occurs through the pecuniary externality of saving on factor prices. Specifically, a marginal increase in capital lowers the interest rate and raises the wage. There are two competing forces at work. First, a higher wage and lower interest rate raises the proportion of every consumer's expected total income that is derived from labour income. Conditional on any realisation of the aggregate shock, this raises the *share* of income subject to uninsurable idiosyncratic risk. In isolation, this harms the utility of risk averse consumers. I call this the negative *idiosyncratic risk effect*. The second force relates to changes in expected total income. Conditional on any realisation of the aggregate shock, it is the case in the model that a lower interest rate and higher wage raise the expected total income of asset poor consumers: that is, those with lower than average saving. This raises their utility. For asset rich consumers, the reverse happens and their utility is lower. The expected income of the average consumer is unaffected: that is, the consumer with average saving. Intuitively, a higher wage and lower interest rate benefit those receiving the bulk of their income from labour, while it harms those receiving the bulk of their income from capital. Overall, I show that this has a positive effect on *utilitarian social welfare*, because the utilitarian social welfare function favours income redistribution of this type. I call this the positive *expected income effect and distribution effect*. These effects are not taken into account by individual consumers when making a saving choice. These external effects matter in an incomplete markets environment. The relative magnitude of the *idiosyncratic risk effect* and the *expected income effect* determines whether a marginal increase in capital raises utilitarian social welfare in competitive equilibrium (i.e. whether there is constrained inefficiency due to under saving).

I will show that the trade off between the *idiosyncratic risk effect* and the *expected income effects* depends on the level of wealth inequality and the way in which aggregate risk affects the distribution of idiosyncratic risk.

3.1.0.2 Interaction Between Aggregate and Idiosyncratic Risk

I study two cases in which aggregate shocks alter the *conditional* distribution of the idiosyncratic shock in different ways (i.e. conditional in the realisation of the aggregate shock). In both cases, the *unconditional* distribution of the idiosyncratic shock remains approximately the same as when aggregate and idiosyncratic shocks are fully independent. However, I demonstrate that the sign and magnitude of constrained inefficiency of competitive equilibria *differs* between the two cases. This illustrates the importance of studying efficiency in the presence of aggregate risk, allowing for different interactions with idiosyncratic risk.

In the first case, bad aggregate shocks make bad idiosyncratic shocks more likely than good idiosyncratic shocks. Under a good realisation of the aggregate shock, good idiosyncratic shocks are more likely. In this case, aggregate shocks change the *skewness* of the conditional distribution of the idiosyncratic shock. Empirically, it seems plausible that bad idiosyncratic employment outcomes like job loss become more likely in recessions. In this case, higher saving is welfare improving, for most levels of wealth inequality. The negative *idiosyncratic risk effect* of a higher wage and lower interest rate (if aggregate saving rises) is *lower* in absolute value. The correlation between aggregate and idiosyncratic risk reduces the extent of truly idiosyncratic risk, so the scaling up of labour income's share in total income is not as costly. There is constrained inefficiency due to *under* saving, because the positive *expected income and distribution effect* of higher saving dominates.

In the second case, the conditional distribution of the idiosyncratic shock has a relatively large variance under a bad aggregate shock and a relatively small variance under a good idiosyncratic shock. Again, the *unconditional* distribution of the idiosyncratic shock remains approximately the same as when the aggregate and idiosyncratic shocks are fully independent. Empirically, it seems plausible that bad idiosyncratic shocks can be larger in recessions, while lucky agents receiving good shocks do particularly well, widening the income distribution. The result is that there is *over* saving in competitive equilibrium for many levels of wealth inequality. The negative *idiosyncratic risk effect* of a higher wage and lower interest rate (if aggregate saving rises) is *higher* overall in this case and can dominate the positive *expected income and distribution effect* of higher saving.

3.1.0.3 Tax / Transfer Programs that Improve Efficiency

I consider specific types of tax / transfer schemes that may generate an efficiency improvement, compared with constrained inefficient competitive equilibria without fiscal policy. The fact that it is a pecuniary externality of saving that causes constrained inefficiency suggests that an efficiency improvement is possible if consumers are induced to save in a different way. I follow Gottardi and Nakajima (2013) in abstracting completely from government purchases. The only useful role that any tax / transfer scheme can play is the improvement of utilitarian social welfare. First, I derive the marginal effect on utilitarian social welfare of introducing these schemes. When there is constrained inefficiency due to under saving, a *subsidy* to the return on saving (financed by personalised lump sum taxes that do *not* redistribute income) can be welfare improving.

However, even if there is *under* saving in a constrained inefficient, competitive equilibrium without fiscal policy, the introduction of a *positive* tax rate on the return to saving improves efficiency if it is accompanied by sufficiently large re-distributive transfers from the asset rich to the asset poor. The efficiency benefit of the redistribution can outweigh the distortion to saving caused by the tax. A utilitarian social welfare function rewards re-distribution because it reduces wealth and income inequality, effectively providing some insurance against the bad shocks that lead to low income and low wealth.

The efficiency benefit of this redistribution will be larger in the presence of aggregate risk, compared with the effect in a model with only idiosyncratic risk. This is because bad aggregate shocks can lower the income ratio of the asset poor to the asset rich, provided that the bad shock lowers the wage by at least as much as the interest rate in percentage terms. In these circumstances, the re-distributive transfers improve welfare by more. Also, these re-distributive effects become more important when the distribution of initial wealth is more unequal.

I solve for the Ramsey optimal levels of tax / transfers within the classes or types of policies considered, using numerical methods. These are Ramsey problems in the sense that the optimal policies within each class cannot complete markets and attain the Pareto efficient outcome: the Ramsey planner chooses the second best. I compare the level of aggregate saving / capital accumulation under Ramsey optimal policy to the constrained efficient level in an economy without fiscal policy. The Ramsey optimal level of capital accumulation will be relatively close to the constrained efficient level if a *subsidy (tax)* on the return to capital is used to induce saving behaviour that improves utilitarian social welfare, financed by lump sum taxes (transfers) that do *not* redistribute income. By contrast, if transfers redistribute income, then this may have a welfare benefit that outweighs the distortionary effect of taxing saving. This can justify a positive rate of tax on the return to saving, driving the Ramsey optimal level of capital accumulation *below* the constrained efficient level.

3.1.0.4 Optimal Cyclicality of Transfers

I consider the role that re-distributive transfers can have as partial insurance against aggregate shocks. I find that it is never optimal for the size of these transfer payments to be counter-cyclical. A-cyclical or counter-cyclical transfers paid to asset poor consumers can reduce the incentive for these consumers to save, because they provide a form of partial insurance against bad aggregate shocks. Numerical results provide some evidence that Ramsey optimal transfers should in fact be slightly *pro-cyclical*.

I use a simple two period framework for the above analysis. This framework allows for useful theoretical analysis and avoids the formidable task of using numerical methods to solve infinite horizon, Ramsey optimal fiscal policy problems in the presence of both aggregate and idiosyncratic risk. To my knowledge, solving such a problem with an infinite horizon has not yet been attempted but would be a valuable topic of future research.

3.1.0.5 Related Literature on Optimal Fiscal Policy in Heterogeneous Agent Models

The literature on optimal fiscal policy has followed three broad approaches. In the macroeconomics literature, it is common to assume an exogenous path for government spending and to solve a Ramsey optimal policy problem, to find the levels of taxation that fund spending at minimum cost to utilitarian social welfare. The maximisation occurs only within a given class of policies (e.g. linear, proportional taxes on capital and labour income) and subject to satisfying the conditions characterising competitive equilibrium, including Euler equations. Ramsey problems search for the "second best," given that exogenous government spending must be financed using distortionary taxation instruments. Pareto efficiency is not attained. Classic papers solving Ramsey optimal fiscal policy problems in the representative agent setting include those of Chamley (1986) and Judd (1986). More recently, Erosa and Gervais (2002) and Garriga (2003) extended this analysis to the overlapping generations framework. In a deterministic model where agents differ by initial wealth, Saez (2013) analyses optimal non-linear capital taxation. In frameworks with incomplete markets for *idiosyncratic* labour income risk, Reiter (2004), Acikgoz (2014), Bakis and Poschke (2012) and Gottardi and Tomoyuki (2011) study various aspects of optimal fiscal policy by solving Ramsey problems. Gottardi and Nakajima (2013) and Panousi (2010) (building on the model of Angeletos (2007)) solve Ramsey problems in the presence of idiosyncratic risk to capital income. A classic paper that warrants particular mention is that of Aiyagari (1995), which concludes that the tax rate on income from capital should be positive in the presence of incomplete markets for idiosyncratic risk. Important for this result is the presence of government debt in the model, which allows the government to spread the burden of taxation over time.

Another approach in the macroeconomics literature is to use numerical methods to compute the utilitarian social welfare arising under various tax policies in competitive equilibrium, without solving a Ramsey problem. Examples of this approach include Benabou (2002), Domeij and Heathcote (2004), Conesa and Krueger (2006) and Conesa and Krueger (2009). These papers concern the optimal progressivity of income tax rates. This is the subject of ongoing research and recent papers include Krueger and Ludwig (2013), Krueger and Kindermann (2014), Fehr and Kindermann (2012) and Heathcote and Violante (2014).

The third approach is found in the public finance literature. Optimal income taxation has been studied in dynamic models where consumers are heterogeneous in their productivity, with individual productivity being private information. The objective of the planner is to redistribute income from high productivity to low productivity consumers with the minimum of distortion. Unlike in Ramsey problems, tax policies are not restricted to having a particular functional form (like a linear or progressive proportional tax rate) but can be specific to individuals and depend on an entire history of consumer income. There are a great many papers in this literature, including Werning (2007), Werning (2011), Farhi and Werning (2012), Farhi and Werning (2013), Findeisen and Sachs (2014), Golosov and Tsyvinski (2003) and Golosov and Tsyvinski (2013).

None of the papers mentioned so far allow a role for *aggregate risk*, together with idiosyncratic risk. There have been a number of papers in recent years that use numerical methods to solve for competitive equilibrium in models with both aggregate and idiosyncratic uncertainty. Among the first of these kind of papers were Krusell and Smith (1998) and Den Haan (1997). Fiscal policy has been considered in frameworks with both aggregate and idiosyncratic risk in papers by Heathcote (2005) and Costain and Reiter (2005), but a Ramsey problem does not appear to be solved.

I study an incomplete markets model with aggregate and idiosyncratic risk, computing the constrained efficient level of capital accumulation in the economy without fiscal policy. I solve for the Ramsey optimal levels of proportional tax on capital income under two classes of fiscal policy: one *without* redistributive transfers and one *with* redistributive transfers. I compare capital accumulation under Ramsey optimal policy with the constrained efficient level in the economy without fiscal policy.

3.1.0.6 Structure of the Chapter

Section (3.2) of the chapter presents the modelling framework. In Section (3.3), I consider the constrained inefficiency of competitive equilibria in the model. First, I do this by considering the marginal effect on utilitarian social welfare of an increase in aggregate saving. This effect is decomposed to demonstrate that the extent of inefficiency depends on the distribution of wealth across heterogeneous agents and also upon the interaction between aggregate and idiosyncratic risk. I also derive the conditions characterising the constrained efficient level of aggregate saving and solve for this level using numerical methods. Section (3.4) presents two main types of tax / transfers schemes and investigates whether they improve efficiency compared with the laissez faire equilibrium. The first scheme involves a tax on the return to saving and personalised lump sum transfers that rebate exactly the amount taxed (i.e. with no income redistribution). This scheme was considered by Gottardi and Nakajima (2013) and allows the pure substitution effect of the tax on savings decisions to be studied. The second type of tax / transfer schemes that I consider involves a tax on the return to saving accompanied by transfers that redistribute income. Transfers are subject to an asset based means test and are only paid to consumers that are relatively asset poor. For both of these types of tax / transfer schemes, I study their effect on efficiency by deriving their marginal effect on utilitarian social welfare in competitive equilibrium. I also derive the conditions that characterise the Ramsey optimal levels of tax / transfers within each type. For the tax / transfer scheme with re-distributive transfers, I solve the Ramsey problem allowing the transfers to be state contingent - i.e. to vary with the realisation of the aggregate shock. I compare the Ramsey optimal level of capital accumulation with the constrained efficient level in an economy without fiscal policy. I solve the Ramsey problems using numerical methods.

3.2 The Model

The modelling framework used in this chapter is a two period, one sector, neoclassical model. In this model, heterogeneous agents face borrowing constraints and both aggregate and idiosyncratic shocks against which they can only partially insure. In other words, there are incomplete insurance markets for both idiosyncratic and aggregate risk. There are only two types of agents in the model: consumers and firms.

3.2.1 Consumers

There are I types of consumers. There is a continuum of consumers of each type, with measure $(1/I)$. *Ex ante*, the types of consumer differ only by endowment of initial wealth $\Omega = \{\Omega^i\}_{i=1}^{I}$. Consumers of the same type are *ex ante* identical.

3.2.1.1 Period One

Consumers of type i take their endowment of initial wealth Ω^i as given and choose period one consumption c_1^i and saving a^i, such that

$$c_1^i + a^i \leq \Omega^i \tag{3.1}$$

It is assumed that consumers cannot borrow, so that consumers must choose saving $a^i \in [0, \Omega^i]$. Savings in the one sector model take the form of capital accumulation. The amount saved a becomes productive private capital in the second period. Consumers make no other choice in the first period and do not supply any labour in this period. There is no uncertainty of any kind in the first period.

3.2.1.2 Period Two

Consumers obtain income in the second period in two ways. First, they supply the capital accumulated in the previous period to a perfectly competitive capital market, where it earns rental rate of return r. At the end of the second period, the capital owned by consumers depreciates fully. Second, consumers supply inelastically an endowment of productive labour e to a perfectly competitive labour market, where it earns wage w. The endowment of productive labour is *random* and distributed independently but identically across *all* consumers. This is the sole source of *idiosyncratic* uncertainty in the model. Its distribution is described in detail in Subsection (3.2.4).

3.2.2 Firms

There is a continuum of perfectly competitive firms of measure one. Firms are inactive in the first period. In the second period, firms rent productive labour L and capital K in perfectly competitive factor markets, which the firms combine to produce output Y. Firms have access to a neoclassical production technology f :

$$Y = zf(K, L) \tag{3.2}$$

where z is total factor productivity. The function f is twice continuously differentiable in its arguments (i.e. each of the first and second order partial derivatives exist and is continuous) and also strictly increasing in its arguments, so that

$$\begin{aligned} f_K(K, L) &> 0 \qquad (3.3) \\ f_L(K, L) &> 0 \end{aligned}$$

Second order partial derivatives are assumed to be always

$$\begin{aligned} f_{KK}(K, L) &< 0 \qquad (3.4) \\ f_{LL}(K, L) &< 0 \end{aligned}$$

It is further the case that the cross-partial derivatives are assumed to be always positive

$$f_{KL}(K, L) = f_{LK}(K, L) > 0 \tag{3.5}$$

where the equality follows from Young's Theorem. In the extreme cases,

$$\begin{aligned} Y &= f(0, L) = 0 \qquad (3.6) \\ Y &= f(K, 0) = 0 \end{aligned}$$

The function f is assumed to be strictly quasi-concave. It is also assumed that the production function exhibits constant returns to scale in *private* inputs capital K and labour L. I make this assumption for analytical convenience because it implies that perfectly competitive firms earn zero profits in competitive equilibrium.

3.2.3 Aggregate Uncertainty

There is no aggregate uncertainty in the first period. In the second period, total factor productivity z is random and can take one of two values:

$$z = \begin{Bmatrix} z^G \text{ with probability } \rho \\ z^B \text{ with probability } 1-\rho \end{Bmatrix} \tag{3.7}$$

where $z^G > z^B > 0$ and $0 < \rho < 1$.

3.2.4 Idiosyncratic Uncertainty

There is no idiosyncratic uncertainty in the first period. In the second period, each agent's endowment of productive labour e is random. Productive labour e endowment has a discrete distribution over two values, e_H and e_L, with $e_H > e_L > 0$. The distribution is independent but identical across *all* consumers and is characterised by

$$e = \begin{Bmatrix} e_L \text{ with probability } \phi(z) \\ e_H \text{ with probability } 1-\phi(z) \end{Bmatrix} \tag{3.8}$$

The probability of e_L, conditional on the realisation of the aggregate shock is given by $\Pr(e_L \mid z) = \phi(z)$, where $0 < \phi(z) < 1$. The conditional distribution of the idiosyncratic endowment of productive labour can change with the realisation of the aggregate shock.

3.2.5 Consumers' Problem

The decision problem faced by the individual consumer of type i is to choose consumption c_1^i and saving a^i in the first period to maximise the discounted present value of expected utility, over the two period life of the model:

$$\max_{a^i \in [0, \Omega^i]} u(c_1^i) + \beta E_0[u(c_2^i)] \tag{3.9}$$

subject to a no borrowing constraint

$$a^i \geq 0 \tag{3.10}$$

and subject to the period budget constraint in period one

$$c_1^i + a^i \leq \Omega^i \tag{3.11}$$

plus the period two budget constraints for every possible combination of aggregate and idiosyncratic shock realisations

$$c_2^i \leq ra^i + we_n \tag{3.12}$$

where $n \in \{L, H\}$. Note that consumers receive only the net return ra^i on their savings or accumulated capital, because capital is assumed to depreciate fully at the end of the second period. Consumption in the second period is denoted by c_2. Consumers make no choices in period two, because it is assumed that labour and capital are supplied *inelastically* in this period, as described above in Subsection (3.2.1).

3.2.6 Concavity of the Objective Function

Consumer utility is assumed to be time separable, with period utility having the von Neumann Morgernstern form, so that expected utility is a probability weighted average of utilities for all realisations of aggregate and idiosyncratic shocks. The period utility function u is assumed bounded above, continuously differentiable and strictly increasing. It is also assumed to be strictly concave, so it is always the case that

$$\begin{aligned} u'(c) &> 0 \\ u''(c) &< 0 \end{aligned} \tag{3.13}$$

It follows that *expected* utility is strictly concave, since it is a linear combination of period utility for all realisations of the aggregate and idiosyncratic shock. The following Inada conditions hold:

$$\begin{aligned} \lim_{c \to \infty} u'(c) &= 0 \\ \lim_{c \to 0} u'(c) &= \infty \end{aligned} \tag{3.14}$$

3.2.6.1 Period One Inequality Constraints

Given an initial endowment of wealth, Ω^i, the no borrowing constraint (3.10) implies that the opportunity set for saving a is given by the closed interval $[0, \Omega]$. It is also clear that the period one budget constraint (3.11) must hold with equality, since it is optimal to consume any resources not saved, if period utility is strictly increasing. The consumer's decision thus reduces to a choice of saving $a^i \in [0, \Omega^i]$ that implies a level of period once consumption

$$c_1^i = \Omega^i - a^i$$

The Inada conditions on period utility u imply that it cannot be optimal for consumers to save all the initial wealth endowment Ω^i and consume nothing in the first period. By contrast, it cannot be optimal to consume all of the endowment Ω^i, *if* it is assumed that every type's endowment of initial wealth Ω^i is sufficiently large in comparison to the possible endowments of productive labour in the second period $\{e_L, e_H\}$. By sufficiently large, I mean large enough so that it must be optimal for the consumer to smooth consumption by saving at least some of initial endowment Ω^i, something the consumer will want to do because of the concavity of period utility. I impose this restriction for convenience, to ensure that the solution to the maximisation problem is in the interior of the consumer's opportunity set $[0, \Omega^i]$.

3.2.6.2 Period Two Inequality Constraints

The period two budget constraint (3.12) must hold with equality, since it is optimal to consume all income in this final period, if period utility is strictly increasing.

Given an endowment of initial wealth Ω^i, it is now clear that the individual consumer's problem is to maximise a strictly concave objective function over a convex opportunity set. There exists a global maximum to this optimisation problem, by the Weierstrass Theorem. Further, this global maximum must be unique by the Local-Global Theorem. First order conditions will be sufficient to characterise this maximum, which will be in the interior of the opportunity set $[0, \Omega^i]$.

3.2.6.3 Obtaining the First Order Conditions

For consumers of type i, the expectation term in the objective function (3.9) can be expanded using the joint distribution of the aggregate and idiosyncratic shocks:

$$\begin{aligned} & u(c_1^i) + \beta \Pr(z^G \cap e_L) u(c_2^i(z^G, e_L)) && (3.15) \\ & +\beta \Pr(z^G \cap e_H) u(c_2^i(z^G, e_H)) \\ & +\beta \Pr(z^B \cap e_L) u(c_2^i(z^B, e_L)) \\ & +\beta \Pr(z^B \cap e_H) u(c_2^i(z^B, e_H)) \end{aligned}$$

where $\Pr(z^j \cap e_n)$ is the joint probability of aggregate shock $j \in (G, B)$ and idiosyncratic shock $n \in (L, H)$. The term $c_2(z^j, e_n)$ denotes period two consumption when the aggregate shock is z^j and the idiosyncratic shock is e_n. The first period budget constraint (3.11) can be used to substitute out the c_1^i term. The period budget constraints (3.12) for all realisations of aggregate and idiosyncratic shocks can be used to substitute out the $c_2(z^j, e_i)$ terms.

$$\begin{aligned} & \max_{a \in [0,\Omega]} u(\Omega - a) && (3.16) \\ & +\beta \Pr(z^G \cap e_L) u(r(z^G)a + w(z^G)e_L) \\ & +\beta \Pr(z^G \cap e_H) u(r(z^G)a + w(z^G)e_H) \\ & +\beta \Pr(z^B \cap e_L) u(r(z^B)a + w(z^B)e_L) \\ & +\beta \Pr(z^B \cap e_H) u(r(z^B)a + w(z^B)e_H) \end{aligned}$$

Using summation notation, this can be written as

$$\max_{a^i \in [0,\Omega^i]} u(\Omega^i - a^i) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(z^j)a^i + w(z^j)e_n) \tag{3.17}$$

This is effectively a classical maximisation problem in the single variable of saving a^i. The optimal level of saving for the individual consumer is denoted a^* and this satisfies the first order condition that is sufficient to characterise the unique, interior maximum

$$\frac{d}{da} |_{a=a^*} = 0 = -u'(\Omega^i - a^{i*}) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j) u'(r(z^j)a^{i*} + w(z^j)e_n) \tag{3.18}$$

The amount of consumer saving is not a function of the aggregate or idiosyncratic shocks, because the consumer saving decision is made to maximise expected utility, before these shocks are realised. Because of this, the first order condition is the same for all consumers of the same type, where types differ by endowment of initial wealth Ω^i.

Proposition 1 *The level of saving a^i for an individual consumer is strictly increasing in the level of initial wealth Ω^i.*

Proof. The first order condition defines *implicitly* the optimal level of saving for the individual consumer $a(\Omega^i)$ as a function of initial wealth Ω^i in a neighbourhood that should encompass the I possible values of initial wealth $[\Omega^1, \Omega^2, ..., \Omega^I]$. Individual consumers take the wage and interest rate as given, with the decisions of any individual consumer having a negligible effect on aggregate quantities. The Implicit Function Theorem can be applied. Denoting the right hand side (RHS) of this first order condition (3.18) as a relation $F(\Omega^i, a^i)$ that is continuously differentiable in its arguments, it can be seen that

$$\frac{d}{da} |_{a=a^*} = F(\Omega^i, a^i) = 0 \tag{3.19}$$

Letting $a(\Omega^i)$ denote the *decision rule* describing optimal individual saving as a function of initial wealth, the first derivative can be expressed as

$$\begin{aligned} a'(\Omega^i) &= -\frac{\partial F(\Omega^i, a^i)/\partial \Omega}{\partial F(\Omega^i, a^i)/\partial a} \qquad (3.20) \\ &= -[\frac{-u''(\Omega^i - a)}{\left(\begin{array}{c} u''(\Omega - a) \\ +\beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j, e_n)^2 u''(r(z^j, e_n) a^{i*} + w(z^j, e_n) e_n) \end{array}\right)}] \\ &> 0 \end{aligned}$$

The denominator of the fraction in (3.20) is unambiguously *negative* because of the concavity of period utility. The numerator is clearly *positive*, also because of period utility's concavity. Overall, this makes clear that optimal consumer saving a^i is an everywhere *strictly increasing* function of initial wealth Ω^i: $a'(\Omega) > 0$. ■

Consumers with higher initial wealth have higher *levels* of saving. The intuition for this is that individuals are smoothing consumption across the two periods of the model, given their initial wealth.

3.2.7 Aggregation

3.2.7.1 Capital Accumulation Through Saving

The average per-consumer level of capital accumulated in the economy is denoted by k and is given by

$$k = (\frac{1}{I}) \sum_i a(\Omega^i) \tag{3.21}$$

which is a weighted average of consumer savings choices $a(\Omega^i)$, over the I different types of consumers. As noted in Subsection (3.2.5), the decision rule for saving is a function only of initial wealth, because consumers make the decision to maximise expected utility before aggregate and idiosyncratic shocks are realised. Consumers and firms with rational expectations know the distribution of initial wealth. Because of this, consumers and firms know average capital per person k with *certainty*.

3.2.7.2 Labour Supply

Consumers supply inelastically their idiosyncratic endowment of productive labour in the second period. The average per-consumer supply of productive labour in the second period is denoted $l(z)$ and is given by

$$l(z) = \phi(z)e_L + (1 - \phi(z))e_H \tag{3.22}$$

where $\phi(z) = \Pr(e_L \mid z)$, the probability of receiving idiosyncratic productive labour endowment e_L conditional on the realisation of the aggregate shock z.

Applying an appropriate law of large numbers, the fraction of the continuum of consumers with e_L (one of two possible productive labour endowments) is given by $\phi(z) = \Pr(e_L \mid z)$. This implies that average per-consumer productive labour supply $l(z)$ is known, conditional on the realisation of the aggregate shock z. Rational consumers and firms that know the distribution of the aggregate shock thus know the distribution of average per-consumer labour supply $l(z)$.

3.2.8 Firms' Problem

Perfectly competitive firms solve a static profit maximisation problem in the second period. Firms take as given the wage and rental rates that prevail in competitive factor markets, given realisations of the aggregate and idiosyncratic shocks z^j and e_n, $j \in \{G, B\}$, $n \in \{L, H\}$. The firm chooses amounts of capital K and productive labour L to rent in these factors markets, as well as a level of production Y, in order to maximise profit.

$$\max_{K,L} PY(z) - wL - rK \tag{3.23}$$

subject to production technology

$$Y \leq zf(K, L) \tag{3.24}$$

where P is the price of the homogeneous output. This price is normalised to one hereafter. The technology constraint will hold with equality because it can never be optimal to leave rented labour and capital not utilised. The constraint can then be used to substitute out Y from the objective function, so that the maximisation problem becomes

$$\max_{K,L} zf(K, L) - wL - rK \tag{3.25}$$

Taking the aggregate shock realisation z, the wage w and the interest rate r all as given, the objective function can be seen to be a linear combination of a strictly quasi concave neoclassical production function and thus the objective function is quasi concave. Given consumer saving choices, there is an upper limit on the amount of capital available. The distribution of the idiosyncratic productive labour shock described in Subsection (3.2.4) implies an upper limit on the amount of labour available. These limits constitute upper bounds on the firms' opportunity sets, with the lower bounds being given of course by zero. However, it can never be optimal for the firm to choose to rent either zero capital or zero labour. This would result in no output, as shown by conditions (3.6). I assume that the upper bounds on the opportunity set never bind the firm.

Since all firms are identical, it cannot be the case that only one firm rents all available resources.

In summary, the firm is maximising a strictly quasi concave objective function over a convex set. The first order conditions will be sufficient to characterise a unique profit maximum (in the interior of the firms' opportunity set), something that follows in part from the Weierstrass and Local-Global Theorems.

Given realisations z^j and e_n of the aggregate and idiosyncratic shocks in the second period, where $j \in (G, B)$ and $n \in (L, H)$, the first order conditions for the firms' problem are

$$\frac{\partial}{\partial K} \mid_{\substack{K=K^* \\ L=L^*}} = 0 = z^j f_K'(K, L) - r \quad (3.26)$$

$$\frac{\partial}{\partial L} \mid_{\substack{K=K^* \\ L=L^*}} = 0 = z^j f_L'(K, L) - w$$

The optimal firm choices of rented capital and productive labour are denoted by K^* and L^*.

3.2.9 Competitive Equilibrium

It is possible now to define a competitive equilibrium in the two period model.

Definition 8 *A **Competitive Equilibrium** is a **Decision Rule** for consumer saving $a(\Omega^i)$ (a function of initial wealth Ω^i that varies by consumer type) such that **(i)** the no borrowing constraints (3.10) are satisfied for all consumers; **(ii)** all consumer and firm budget constraints (3.11), (3.12) and (3.24) are satisfied with equality for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; **(iii)** the first order condition (3.18) is satisfied for all individual consumers; **(iv)** the first order conditions (3.26) for all firms are satisfied for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$.*

There is a continuum of firms of measure one and all are identical. All firms will choose to hire the same amount of capital and labour. In competitive equilibrium, this amount must be equivalent to the per-consumer amounts of labour $l(z)$ and capital k supplied, given that there is a continuum of consumers of measure one. These per-consumer average quantities were defined in Subsection (3.2.7). This implies that firms' first order conditions (3.26) must equal

$$r(z^j, k) = z^j f_k'(k, l(z^j)) \quad (3.27)$$

$$w(z^j, k) = z^j f_l'(k, l(z^j))$$

for all possible realisations of the aggregate shock z^j where $j \in (G, B)$. Recall from Subsection (3.2.7) that average per-consumer capital is known with certainty by rational consumers and firms. Recall also that average per-consumer productive labour supply depends only on the realisation of the aggregate shock z^j, so the distribution of $l(z)$ is known to rational consumers and firms. Given average per consumer saving k, this implies that the rental rate r and the wage rate w in competitive equilibrium differ depending on the realised value of the aggregate shock, so can be written as $r(z, k)$ and $w(z, k)$ respectively. Rational consumers and firms thus know the distribution of the rental rate and wage rate in competitive equilibrium, because the distribution of the aggregate shock is known. Factor prices can be written as functions of k and z in competitive equilibrium.

Given the realisation of the aggregate shock z^j, it is clear from (3.27) that capital and labour are paid their marginal products in competitive equilibrium. The expression for individual firm profit in competitive equilibrium, given a particular realisation of the aggregate shock z^j, $j \in (G, B)$, is

$$z^j f(k, l(z^j)) - r(z^j, k)k - w(z^j, k)l(z^j)$$

which is equivalent to

$$z^j f(k, l(z^j)) - z^j f'_k(k, l(z^j))k - z^j f'_l(k, l(z^j))l(z^j) \tag{3.28}$$

Recall that the firm has access to a neoclassical production technology (3.2) that exhibits constant returns to scale in private inputs. It follows from Euler's Theorem that firm profits are zero in competitive equilibrium, irrespective of the realisation of the aggregate shock.

It is possible to write the consumer's first order condition (3.18) and Euler equation in competitive equilibrium as

$$u'(\Omega^i - a^{i*}) = \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j, k) u'(r(z^j, k)a^{i*} + w(z^j, k)e_n) \tag{3.29}$$

for consumers of type $i \in \{1, 2, ..., I\}$.

3.3 Efficiency of Competitive Equilibrium

This section examines the normative properties or social welfare / efficiency of competitive equilibrium in the presence of both aggregate and idiosyncratic risk.

Definition 9 *Given a distribution of initial wealth (i.e. I types of initial wealth $\Omega^1, \Omega^2, ..., \Omega^I$), an **allocation** is a decision rule for savings $a(\Omega^i)$ and an associated level of average per-consumer accumulated capital k (as defined in Subsection (*3.2.7*)), which give rise to a competitive equilibrium, as defined in Subsection (*3.2.9*).*

The expected utility to an individual consumer with initial wealth Ω^i from following a decision rule $a(\Omega^i)$ is given by *indirect utility* U :

$$U(\Omega^i, a(\Omega^i), k) = u(\Omega^i - a(\Omega^i)) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(k, z^j) a(\Omega^i) + w(k, z^j) e_n) \quad (3.30)$$

The level of indirect utility produced by adopting $a(\Omega^i)$ will *differ* across types of consumers *only* because of differing initial levels of wealth Ω.

In order to assess *social* rather than individual welfare in the two period model (i.e. to assess the allocation that achieves social optimality), the indirect utility of all consumers must be taken into account, by considering a weighted average of indirect utilities over the *types* of consumers, differing by initial wealth:

$$SW = (\frac{1}{I}) \sum_i U(a(\Omega)) \quad (3.31)$$

Constrained efficiency is the notion of social optimality with which the competitive equilibrium allocation is compared.

Definition 10 *A competitive equilibrium **allocation** (with saving rule $a^*(\Omega^i)$) is **Constrained Efficient** if it produces the highest weighted average of consumer **indirect** utility U :*

$$(\frac{1}{I})\sum_i U(a^*(\Omega))$$

*amongst the **set** Υ of all possible allocations that: **(i)** satisfy all **consumer and firm budget constraints** (3.11), (3.12) and (3.24) for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; **(ii)** satisfy all **no borrowing constraints** (3.10); and **(iii)** take as given **factor prices** r and w that are determined in competitive markets.*

The set of allocations Υ referred to includes those where consumer Euler equations (i.e. those derived from consumer first order condition (3.18)) are *not* satisfied. (Of course, the competitive equilibrium allocation *does* satisfy the Euler equations) However, *excluded* from the set Υ are allocations that can only be attained by the social planner *completing* insurance markets for risk, by making transfers of wealth between agents after shocks are realised. Attaining a *Pareto efficient* allocation may require this. Allowing these allocations in the set would render the competitive equilibrium allocation in the incomplete markets model constrained *inefficient* by definition. For this reason, the notion of efficiency is a *constrained* one because any potentially efficient allocation must be achievable within a framework of incomplete markets for aggregate and idiosyncratic risk, without intervention by a social planner to complete the markets.

The competitive equilibrium allocation is *constrained inefficient* if a marginal change in average per-consumer capital k raises the weighted average of consumer indirect utilities (3.31), evaluated at competitive equilibrium values. In other words, a marginal change in aggregate saving (i.e. aggregate capital) that increases the weighted average of indirect utilities demonstrates constrained inefficiency of the competitive equilibrium allocation. To analyse this question, I proceed in two stages. In Subsection (3.3.1), I analyse the effect of a change in average per-consumer capital k on the indirect utility of an *individual* consumer in competitive equilibrium with a particular initial wealth Ω^i. The effect of a change in k on the weighted average of indirect utilities in competitive equilibrium - i.e. on utilitarian social welfare or efficiency - is analysed in Subsection (3.3.2). I show how this welfare effect depends upon the way in which bad aggregate shocks change the distribution of the idiosyncratic employment shock. In Subsection (3.3.3), I compute the constrained efficient level of average per consumer saving k by solving the appropriate social planner's problem.

3.3.1 Marginal Effect Of Aggregate Saving On An Individual Consumer

For a consumer with initial wealth Ω^i, the derivative of expected indirect utility with respect to average per consumer saving (i.e. capital) k, evaluated in competitive equilibrium is

$$\frac{\partial U(\Omega^i)}{\partial k}\mid_{Comp.Eq.}=\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial a}\frac{\partial a}{\partial k}+\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial k} \tag{3.32}$$

In competitive equilibrium, the term $\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial a}$ is zero, as shown by the first order condition of the individual consumers' problem (3.18). This means that the Envelope Theorem can be applied, so that (3.32) becomes

$$\frac{\partial U(\Omega^i)}{\partial k}\mid_{Comp.Eq.}=\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial k} \tag{3.33}$$

which implies

$$\frac{\partial U(\Omega^i)}{\partial k}\mid_{Comp.Eq.}=\beta\sum_j\sum_n \Pr(z^j\cap e_n)\left\{\begin{array}{c}\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial r}\frac{\partial r(z^j,k)}{\partial k}\\+\frac{\partial U(\Omega^i,a(\Omega^i),k)}{\partial w}\frac{\partial w(z^j,k)}{\partial k}\end{array}\right\} \tag{3.34}$$

using the fact that factor prices $r(z^j,k)$ and $w(z^j,k)$ are functions of k in competitive equilibrium, as discussed in Subsection (3.2.9) of Section (3.2). This expression demonstrates that the marginal effect on indirect expected utility of higher aggregate saving k operates through the pecuniary externality effect of a change in k on factor prices.

Davila and Rios-Rull (2012) and Gottardi and Nakajima (2013) make a similar point in an environment *without* aggregate risk. Individual consumers do not take these external effects into account when making savings decisions.

The external effects of saving on factor prices impact upon consumers as follows. Lowering the interest rate $r(z^j,k)$ *and* raising the wage $w(z^j,k)$ changes the *share* of second period income received from accumulated wealth (i.e. capital, $a(\Omega^i)$) and from labour income respectively. This is true for every consumer. First, this changes expected total income, conditional on a realisation of the aggregate shock. Whether or not this raises or lower's a consumer's expected income will depend upon whether they save more than average (i.e. the asset rich, with $a(\Omega^i)>k$), or less than average. I call this the *Expected Income Effect*. Second, changing the relative shares of total second period income received from accumulated wealth and labour income changes the share of income subject to idiosyncratic risk, conditional on a realisation of the aggregate shock. Labour income is subject to idiosyncratic risk; capital income is not. This affects expected utility of risk averse agents because risk is partially uninsurable. I call this the *Idiosyncratic Risk Effect*. Third, both labour income and capital income are subject to aggregate risk, since factor prices depend on the realisation of the aggregate shock. Changing the shares of total income received from capital and labour may change exposure to aggregate risk. I call this the *Aggregate Risk Insurance Effect*. The derivative (3.34) can be manipulated to demonstrate these effects.

First, the derivative in (3.34) can be re-written as

$$\frac{\partial U(\Omega^i)}{\partial k} \, |_{Comp.Eq.} = \beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left\{ \frac{\partial r(z^j,k)}{\partial k} a(\Omega^i) + \frac{\partial w(z^j,k)}{\partial k} e_n \right\}$$

Proposition 2 *The marginal effect of higher average saving k on indirect expected utility (i.e. the derivative in (3.34)) of a consumer with initial wealth Ω^i can be decomposed as:*

$$\beta E\left(u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)\left[k\left\{\frac{\partial r(z^j,k)}{\partial k} - E(\frac{\partial r(z^j,k)}{\partial k})\right\}\right]\right) + \tag{3.35}$$

$$\beta E\left(u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)\left[l(z^j)\frac{\partial w(z^j,k)}{\partial k} - E(l(z^j)\frac{\partial w(z^j,k)}{\partial k})\right]\right) +$$

$$\beta \sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j))\right] |_z \, \frac{\partial w(z^j,k)}{\partial k}\right) +$$

$$\beta \sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)\right] |_z \, (a^i - k)\frac{\partial r(z^j,k)}{\partial k}\right)$$

Proof. Please see Appendix (C.1).

■

3.3.1.1 Interpretation of the Decomposition

3.3.1.2 First Two Terms - Aggregate Risk Insurance Effect

$$\beta E\left(u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\left[k\left\{\frac{\partial r(z^j,k)}{\partial k}-E(\frac{\partial r(z^j,k)}{\partial k})\right\}\right]\right)+ \quad (3.36)$$
$$\beta E\left(u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\left[l(z^j)\frac{\partial w(z^j,k)}{\partial k}-E(l(z^j)\frac{\partial w(z^j,k)}{\partial k})\right]\right)$$

For all realisations of the aggregate shock z^j, $j\in(G,B)$, recall that an increase in average per consumer capital k *raises* the wage $w(z^j,k)$ and *lowers* the interest rate $r(z^j,k)$, as implied by (3.4) and (3.27). Because of this, a change in average capital per consumer k changes the *shares* of total income derived from capital and labour income, for an individual consumer. This changes the extent of exposure to risk arising from aggregate shocks to the wage and interest rate respectively. In order to obtain intuition for the expression in (3.36), note that the effect on expected utility of *replacing* a unit of second period income subject to *aggregate* risk with its expectation is:

$$\frac{dEU\left(\left(r(z^j,k)k+w(z^j,k)l(z^j)\right)(1-p)+pE\left(r(z^j,k)k+w(z^j,k)l(z^j)\right)\right)}{dp}\,|_{p=0} \quad (3.37)$$
$$= \quad -Eu'(\Omega^i,z^j,e_n)\left\{k\left[r(z^j,k)-E(r(z^j,k))\right]+\left[w(z^j,k)l(z^j)-E(w(z^j,k)l(z^j))\right]\right\}$$

The terms in (3.36) can be seen as the net utility cost of the change in exposure to *aggregate* risk - i.e. the change in exposure to aggregate risk brought about by changing the *share* of total income derived from wage income and capital income (depending on the interest rate) respectively.

Proposition 3 *The aggregate risk insurance effect* ***is zero*** *when the production function is h.d.1. in private inputs capital k and labour $l(z^j)$.*

Proof. Rearrange (3.36) to yield

$$Eu'(\Omega^i, z^j, e_n)\left\{k\frac{\partial r(z^j,k)}{\partial k} + l(z^j)\frac{\partial w(z^j,k)}{\partial k} - E\left[k\frac{\partial r(z^j,k)}{\partial k} + l(z^j)\frac{\partial w(z^j,k)}{\partial k}\right]\right\} \quad (3.38)$$

The production function $y = z^j f(k, l(z^j))$ is assumed homogenous of degree one (h.d.1) in private inputs, as discussed in Section (3.2). By Euler's Theorem, it must be the case that

$$y(z^j,k) = \frac{\partial z^j f(k,l(z^j))}{\partial k}k + \frac{\partial z^j f(k,l(z^j))}{\partial l}l(z^j) \quad (3.39)$$

It follows that in competitive equilibrium

$$y(z^j,k) = r(z^j,k)k + w(z^j,k)l(z^j) \quad (3.40)$$

for all $j \in (G,B)$, where $y(z^j)$ denotes the output of an individual firm and the interest rate and the wage are defined as in equations (3.27). Taking the derivative of individual firm output with respect to average per consumer saving k

$$\frac{\partial y(z^j,k)}{\partial k} = \frac{\partial r(z^j,k)}{\partial k}k + r(z^j,k) + \frac{\partial w(z^j,k)}{\partial k}l(z^j) \quad (3.41)$$

it follows that

$$0 = \frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j) \quad (3.42)$$

for all $j \in (G,B)$, because (3.27) shows that

$$\frac{\partial y(z^j,k)}{\partial k} = r(z^j,k)$$

This gives the result. ■

Intuitively, both capital income and labour income are exposed to aggregate risk. Changing the *shares* of total income obtained from capital and labour income thus leaves exposure to aggregate risk unchanged. The net utility effect is zero. However, aggregate risk can affect constrained efficiency of competitive equilibria in other ways, including by changing the distribution of idiosyncratic risk.

3.3.1.3 Third Term - Idiosyncratic Risk Effect

This term is

$$\beta \sum_j \Pr(z^j) \left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j)) \right] |_z \frac{\partial w(z^j,k)}{\partial k} \right) \tag{3.43}$$

Given any realisation of the aggregate shock z^j, for all $j \in (G,B)$, an increase in average per consumer capital *raises* the wage $w(z^j,k)$ and *lowers* the interest rate $r(z^j,k)$, as implied by (3.4) and (3.27). This raises the *share* of expected total income obtained through inelastic labour supply in the second period, conditional on the realisation of z^j, the aggregate shock. In turn, this raises the exposure of the consumer to risk from idiosyncratic shocks to the individual's productive labour endowment. Capital income is not subject to idiosyncratic risk. It can be shown that (3.43) is the *negative* of the expected utility effect of substituting a unit of risky labour income with its expectation, conditional on a particular value for the aggregate shock

$$\begin{aligned} & \frac{dE\left(U\left(\begin{array}{c} (r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)(1-p) \\ +pE\left(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n\right)|_z \end{array}\right)\right)|_z}{dp}|_{p=0} \qquad (3.44) \\ = & \frac{dE\left(U\left(\begin{array}{c} (r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)(1-p) \\ +p\left(r(z^j,k)a(\Omega^i)+w(z^j,k)E\left(e_n\right)|_z\right) \end{array}\right)\right)|_z}{dp}|_{p=0} \\ = & -E\left(u'(\Omega^i,z^j,e_n)(e_n-l(z_j))\right)|_z\, w(z^j,k) \end{aligned}$$

The *idiosyncratic risk effect* can be thought of as the utility cost of substituting a unit of risk free income for a unit subject to idiosyncratic risk, conditional on a realisation of the aggregate shock.

Proposition 4 *The* ***idiosyncratic risk effect*** *in (3.43) is* ***negative****.*

Proof. The effect of higher average capital k per consumer on the wage is positive $\frac{\partial w(z^j,k)}{\partial k} > 0$ for all z^j, as follows from (3.4) and (3.27). It is also the case that

$$E\left(u'(\Omega^i,z^j,e_n)(e_n-l(z_j))\right) \quad | \quad _z = \tag{3.45}$$

$$Cov\left(u'(\Omega^i,z^j,e_n)(e_n-l(z_j))\right) \quad | \quad _z + E\left(u'(\Omega^i,z^j,e_n)\right)|_z\, E((e_n-l(z_j)))\,|_z$$

The average deviation of a variable from its mean is zero, so $E((e_n - l(z_j)) \mid_z = 0$. For an individual consumer, the covariance of $u'(\Omega^i, z^j, e_n)(e_n - l(z_j))$ with respect to the idiosyncratic shock is negative, something that follows from the concavity of period utility

$$E\left(u'(\Omega^i, z^j, e_n)(e_n - l(z_j))\right) \mid_z = Cov\left(u'(\Omega^i, z^j, e_n)(e_n - l(z_j))\right) \mid_z < 0 \tag{3.46}$$

This gives the result. ■

The intuition is as above. A greater share of expected total income now comes from labour income, which is subject to idiosyncratic risk, unlike capital income which is risk free, conditional on a realisation of the aggregate shock. This has a negative effect on utility.

3.3.1.4 Final Term - Expected Income Effect

This term is

$$\beta \sum_j \Pr(z^j) \left(E\left[u'(r(z^j, k)a(\Omega^i) + w(z^j, k)e_n)\right] \mid_z (a^i - k) \frac{\partial r(z^j, k)}{\partial k} \right) \tag{3.47}$$

For any realisation of the aggregate shock, z^j, for all $j \in (G, B)$, an increase in average per consumer capital *raises* the wage $w(z^j, k)$ and *lowers* the interest rate $r(z^j, k)$, as implied by (3.4) and (3.27). Conditional on any realisation of the aggregate shock, this will raise the expected total income of the asset poor - i.e. consumers with below average saving. It will have the opposite effect on the asset rich. Intuitively, a higher wage and lower interest rate benefit those receiving the bulk of their income from labour, while it harms those receiving the bulk of their income from capital.

Proposition 5 *The **expected income effect** is **negative** for relatively asset rich consumers and **positive** for relatively asset poor consumers.*

Proof. For any realisation of the aggregate shock, z^j, for all $j \in (G, B)$, a marginal increase in average capital k per consumer *lowers* the interest rate $r(z^j, k)$ and *raises* the wage $w(z^j, k)$, as follows from (3.4) and (3.27). The term $E\left[u'(r(z^j, k)a(\Omega^i) + w(z^j, k)e_n)\right] \mid_z$ is always positive because of the concavity of period utility. It follows that the sign of the entire distribution effect term (3.47) depends on the sign of $a(\Omega^i) - k$. This will be *positive* for relatively asset rich consumers, with above average saving and *negative* for relatively asset poor consumers with below average savings. This gives the result. ■

In the model of Section (3.2), the overall effect on individual expected utility in competitive equilibrium (3.34) of a marginal change in k will depend on the net result of the *expected income effect* and the *idiosyncratic risk effect*. So far, I have considered the marginal effect on the expected utility of a *single* consumer in competitive equilibrium of an increase in average capital k per consumer. I now turn to the marginal effect on *utilitarian social welfare* in competitive equilibrium. If this effect is non-zero, then the level of saving in competitive equilibrium is *constrained inefficient*. If the marginal effect is positive, then this suggests that there is *under saving* in competitive equilibrium and vice versa.

3.3.2 Marginal Effect Of Aggregate Saving On Social Welfare

The specific form of utilitarian social welfare is given by the average of expected utility across the various types of consumer that differ ex ante only by initial wealth. The form is

$$SW = (\frac{1}{I})\sum_i U(\Omega^i) = (\frac{1}{I})\sum_i \left\{ u(c_1(\Omega^i)) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(c_2(\Omega^i, z^j, e_n)) \right\} \quad (3.48)$$

The objective is to understand the change in efficiency (compared with its level in a laissez faire competitive equilibrium) because of a change in the level of average saving k per consumer. This can be measured by the marginal effect on utilitarian social welfare of an increase in k, evaluated in laissez faire competitive equilibrium.

$$\frac{\partial SW}{\partial k} \mid_{Comp.Eq.} = (\frac{1}{I})\sum_i \frac{\partial U(\Omega^i)}{\partial k} \quad (3.49)$$

This is simply the average over consumer types of the marginal effects of a change in k on individual expected indirect utility, given by the decomposed expression (3.35), all evaluated in laissez faire competitive equilibrium. That is

$$\frac{\partial SW}{\partial k} \mid_{Comp.Eq.} = \quad (50)$$

$$(\frac{1}{I})\sum_i \beta E\left(u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[k \left\{ \frac{\partial r(z^j,k)}{\partial k} - E(\frac{\partial r(z^j,k)}{\partial k}) \right\} \right] \right) +$$

$$(\frac{1}{I})\sum_i \beta E\left(u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[l(z^j)\frac{\partial w(z^j,k)}{\partial k} - E(l(z^j)\frac{\partial w(z^j,k)}{\partial k}) \right] \right) +$$

$$(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j) \left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j)) \right] \mid_z \frac{\partial w(z^j,k)}{\partial k} \right)$$

$$(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j) \left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \right] \mid_z (a^i - k)\frac{\partial r(z^j,k)}{\partial k} \right)$$

The first two terms are weighted averages across consumer types of the aggregate risk insurance effect (3.36). In Proposition (3) of Subsection (3.3.1), it was demonstrated that this effect is *exactly* zero for an individual consumer. This must also be the case for the average across consumer types. This leaves the averages across consumer types of the *idiosyncratic risk effect* (3.43) and the *expected income effect* (3.47).

3.3.2.1 Expected Income Effect and Idiosyncratic Risk Effect

Proposition 6 *The **expected income effect** on social welfare is always **positive** and it is given by*

$$(\frac{1}{I})\sum_{i}\beta\sum_{j}\Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\right]\mid_z (a(\Omega^i)-k)\frac{\partial r(z^j,k)}{\partial k}\right) \tag{3.51}$$

Proof. It was shown in Proposition (5) of Subsection (3.3.1) that the sign of the expected income effect is *positive* for an individual consumer with below average saving (i.e. $a(\Omega^i) < k$, an asset poor consumer) and negative for an asset rich consumer. In the average over consumer types (3.51), the terms $(a(\Omega^i) - k)$ are weighted by the marginal utility of that consumer type's expected period two income,

$$E\left[u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\right]\mid_z$$

conditional on the realisation of the aggregate shock z^j, for all $j \in (G, B)$. This expected value differs across consumer types by $a(\Omega^i)$, which is increasing in initial wealth Ω^i, as shown by Proposition (1) in Section (3.2). It follows that expected period two income is lower for relatively asset poor consumers and higher for relatively asset rich consumers. Because of the concavity of period utility, the marginal utility weights are *larger* for relatively asset poor consumers than for relatively asset rich consumers. This gives greater weight in the weighted average to the terms associated with poor consumers, which are positive in sign. Therefore, the expected income effect on social welfare, (3.51), is positive in sign. This will be the case provided the distribution of initial wealth is not highly skewed in either direction. ■

Intuitively, expected total income (conditional on the realisation of the aggregate shock) is higher for asset poor consumers and lower for asset rich consumers, because of the marginal change in factor prices (lower interest rate, higher wage) when k is perturbed. The expected total income of the consumer with average saving k is *unchanged* because

$$0 = \frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j) \tag{3.52}$$

which implies

$$-\frac{\partial r(z^j,k)}{\partial k}k = \frac{\partial w(z^j,k)}{\partial k}E\left[e_n\right]|_z \tag{3.53}$$

noting that $E\left[e_n\right]|_z = l(z^j) = \phi(z)e_L + (1-\phi(z))e_H$, where $\phi(z) = \Pr(e_L \mid z)$. The relationship in (3.52) follows from Euler's Theorem (under a h.d.1. production technology), as discussed in Appendix (C.1). For the consumer with average saving k, the *reduction* in expected total income (conditional on a realisation of the aggregate shock) because of a lower interest rate is *exactly offset* by the *increase* in expected total income because of the higher wage. Overall, the change in factor prices (when k is perturbed) reduces inequality of expected income (conditional on any realisation of the aggregate shock), while leaving unchanged the mean of the distribution of expected income. This generates a net gain to *utilitarian social welfare*.

Before considering the sign of the combined effect on social welfare (3.49) of increasing average capital k per consumer, it is necessary to consider the *idiosyncratic risk effect* on social welfare.

Proposition 7 *The **idiosyncratic risk effect** on social welfare is **negative** and is given by*

$$(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j))\right]|_z\; z^j \frac{\partial w(z^j,k)}{\partial k}\right) \tag{3.54}$$

Proof. It was shown in Proposition (4) of Subsection (3.3.1) that the idiosyncratic risk effect for an individual with initial wealth Ω^i is always negative. The effect on social welfare is a weighted average of these initial effects across consumer types. Therefore, the effect on social welfare must be also negative. ■

Conditional on a realisation of the aggregate shock, increasing k increases the wage $w(k,z^j)$ and reduces the interest rate $r(k,z^j)$, which increases the share of expected income derived from labour for all consumers. Labour income is subject to idiosyncratic risk, while capital income is not. This has a negative effect on utilitarian social welfare.

3.3.2.2 Overall Effect on Social Welfare

Proposition 8 *The marginal effect on social welfare (3.50) of higher average saving k per consumer will be **positive** if the positive expected income effect (3.51) outweighs the negative idiosyncratic risk effect (3.54).*

Proof. Proposition (6) establishes that the expected income effect is positive, while Proposition (7) shows that the idiosyncratic risk effect is negative. Proposition (3) of Subsection (3.3.1) showed that the aggregate risk insurance effects are zero for all consumers if the production function is h.d.1. This gives the result. ■

3.3.2.3 Effect of Initial Wealth Inequality

The marginal effect on social welfare (3.50) of higher average saving k per consumer should become increasingly positive as the variance of the initial wealth distribution rises. Intuitively, this is because the *expected income effect* on social welfare should be *magnified* as the variance of the distribution of initial wealth increases. Increasing the wage and decreasing the interest rate raises the expected income of the asset poor and lowers it for the asset rich, conditional on any realisation of the aggregate shock. This is a form of income redistribution (while preserving the mean expected income, as argued in the discussion of Proposition (6)). This income redistribution will be more beneficial to utilitarian social welfare, the more unequal is the distribution of initial wealth.

3.3.2.4 Interaction Between Aggregate and Idiosyncratic Risk

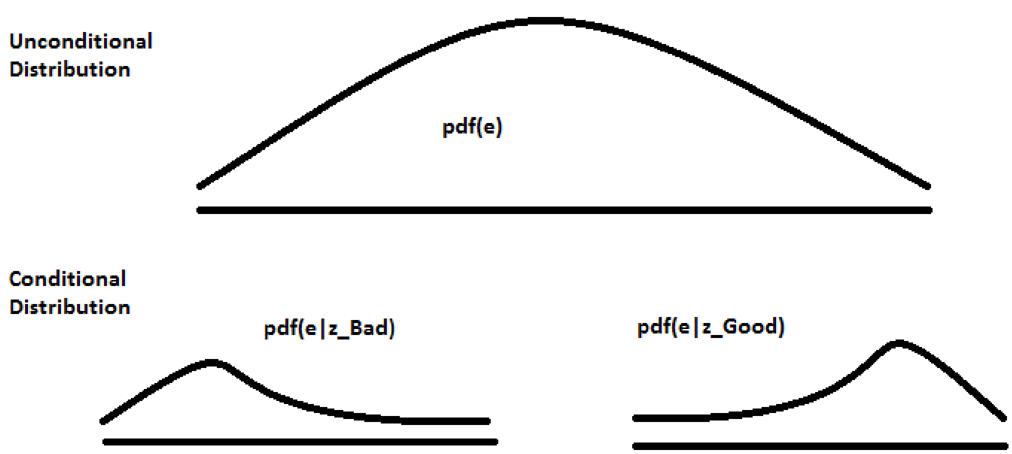
The interaction between aggregate and idiosyncratic risk also has important implications for the marginal effect (3.50) of higher k on social welfare. Principally, this is because different kinds of interaction can change significantly the *idiosyncratic risk effect* of higher k on social welfare (3.54). I consider two examples where the realisation of the aggregate shock changes the conditional distribution of the idiosyncratic shock (i.e. conditional on the realisation of the aggregate shock). In both cases, the unconditional distribution of the idiosyncratic shock is approximately the same as where aggregate and idiosyncratic risk are fully independent. This demonstrates the importance of studying efficiency in the presence of *both* aggregate and idiosyncratic risk, not just the latter.

3.3.2.5 Case One - Skewness

First, I consider the situation where the realisation of the aggregate shock influences the skewness of the conditional distribution of the idiosyncratic shock. Specifically, bad idiosyncratic shocks are more likely given a bad realisation of the aggregate shock and vice versa. Using discrete conditional distributions, this can be represented as:

$$\Pr(e_L \mid z^B) = \Pr(e_H \mid z^G) > \Pr(e_L \mid z^G) = \Pr(e_H \mid z^B) \tag{3.55}$$

I use continuous conditional distributions to give graphical intuition:



Intuitively, it seems plausible that a bad idiosyncratic employment shock becomes more likely if there has been a bad realisation of the aggregate shock, which in this model is a productivity shock that lowers economic output, as can happen in recession.

Proposition 9 *The negative idiosyncratic risk effect (of higher wage and lower interest rate when k rises)* ***approaches zero*** *in absolute value as the correlation between aggregate and idiosyncratic risk described in (3.55) $Corr(z^j, e_n)$ approaches one.*

Proof. $\Pr(e_H \mid z^G)$ and $\Pr(e_L \mid z^B)$ rise as $Corr(z^j, e_n)$ described in (3.55) rises (while $\Pr(e_H \mid z^B)$ and $\Pr(e_L \mid z^G)$ fall). Recall from equation (3.22) in Section Two that average labour supply per consumer is given by the weighted average $l(z) = \Pr(e_L \mid z)e_L + \Pr(e_H \mid z)e_H$. This implies that the absolute values $\left|e_L - l(z^B)\right|$ and $\left|e_H - l(z^G)\right|$ *fall* as $Corr(z^j, e_n)$ described in (3.55) rises. In the idiosyncratic risk effect on social welfare (3.54), the conditional expectation terms are

$$E\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j))\right] |_z \tag{3.56}$$

which can be written as

$$\sum_n \Pr(e_n \mid z^j)\left[u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)(e_n - l(z^j))\right] |_z$$

As $Corr(z^j, e_n)$ described in (3.55) rises, the dominant term is $\Pr(e_H \mid z^G)\left[\begin{array}{c} u'(r(z^G,k)a(\Omega^i) \\ +w(z^G,k)e_H)(e_H - l(z^G)) \end{array}\right]$ if $z^j = z^G$, or $\Pr(e_L \mid z^B)\left[\begin{array}{c} u'(r(z^B,k)a(\Omega^i) \\ +w(z^B,k)e_L)(e_L - l(z^B)) \end{array}\right]$ if $z^j = z^B$. These dominant terms approach zero as $Corr(z^j, e_n)$ described in (3.55) rises since $\left|e_L - l(z^B)\right|$ and $\left|e_H - l(z^G)\right|$ approach zero, provided the marginal utility terms are not explosive. ■

Intuitively, introducing correlation between idiosyncratic and aggregate shocks reduces the utility cost of increasing the share of expected income derived from labour and subject to idiosyncratic risk (which occurs when there is a marginal increase in k). The extent of purely idiosyncratic risk is lower. Hence the idiosyncratic risk effect should be lower.

It is now possible to discuss the marginal effect of a change in aggregate saving on utilitarian social welfare in competitive equilibrium.

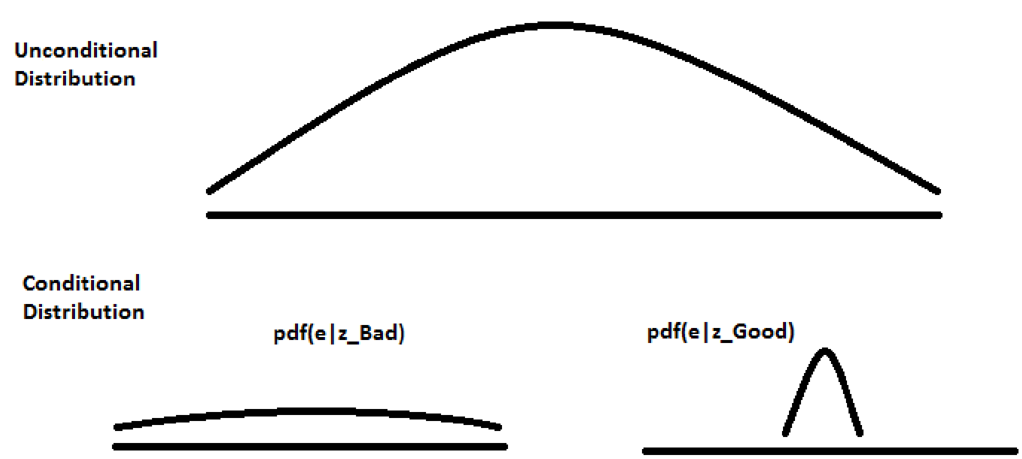
Proposition 10 *The expected income effect (3.51) dominates the idiosyncratic risk effect as the correlation between aggregate and idiosyncratic risk $Corr(z^j, e_n)$ of the type described in (3.55) approaches one.*

Proof. The extent of $Corr(z^j, e_n)$ has no significant impact on the expected income effect on social welfare, as is apparent from (3.51). Proposition (3) of Subsection (3.3.1) demonstrates that the aggregate risk insurance effects are *exactly* zero. Proposition (9) establishes that the idiosyncratic risk effect approaches zero as the correlation between aggregate and idiosyncratic risk $Corr(z^j, e_n)$ of the type described in (3.55) approaches one. This gives the result. ■

It was shown in Proposition (6) that the expected income effect is *positive*, while proposition (7) shows that the idiosyncratic risk effect is *negative*. Therefore, the marginal effect of higher k on social welfare in competitive equilibrium will be *positive* when the expected income effect dominates (there is *undersaving* in competitive equilibrium), something Proposition (10) shows will occur as the correlation between aggregate and idiosyncratic risk $Corr(z^j, e_n)$ of the type described in (3.55) rises.

3.3.2.6 Case Two - Variance

I consider the case where the variance of the idiosyncratic shock is relatively *large* when there is a bad realisation of the aggregate shock, but relatively *small* when there is a good realisation of the aggregate shock. Again, the unconditional distribution of the idiosyncratic shock is approximately the same as when aggregate and idiosyncratic risk are fully independent. I use continuous conditional distributions to give graphical intuition:



Empirically, it seems plausible that bad idiosyncratic shocks become potentially large in recessions, while lucky agents receiving good shocks do particularly well. I mimic this situation by modifying the discrete, conditional distribution of the idiosyncratic shock in the model such that under a *bad* aggregate shock:

$$e = \begin{Bmatrix} e_{HB} \text{ w. prob. } \frac{1}{2} \\ e_{LB} \text{ w. prob. } \frac{1}{2} \end{Bmatrix} \tag{3.57}$$

while under a *good* aggregate shock

$$e = \begin{Bmatrix} e_{HG} \text{ w. prob. } \frac{1}{2} \\ e_{LG} \text{ w. prob. } \frac{1}{2} \end{Bmatrix}$$

where

$$e_{HB} > e_{HG} > e_{LG} > e_{LB}$$

An informal argument about the effect of this assumption on the marginal welfare effect (3.50) of higher k is as follows. Recall that the negative *idiosyncratic risk effect* (3.54) of higher wage and lower interest rate (when k rises) is given by

$$(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)(e_n-l(z^j))\right]|_z\, z^j\frac{\partial w(z^j,k)}{\partial k}\right)$$

representing the utility cost of scaling up the share of expected income subject to idiosyncratic risk, conditional on a realisation of the aggregate shock. Under the assumptions made above in (3.57), the conditional variance of the idiosyncratic shock is relatively *large* under a bad realisation of the aggregate shock. The absolute values $\left|e_n - l(z^B)\right|$ should be relatively large in this case. The marginal utility weight $u'()$ is largest under a bad realisation of both the *idiosyncratic* and *aggregate* shocks, all else equal, by the concavity of marginal utility. This means that the negative term $e_{LB} - l(z^B)$ receives the greatest weight. The end result is a relatively large, negative *idiosyncratic risk effect* (3.54). All else equal, this makes it more likely that the negative *idiosyncratic risk effect* (3.54) of a marginal increase in k will dominate the positive *expected income effect* (3.51). If this is the case, the welfare marginal effect (3.50) of higher k is *negative* and there is constrained inefficiency due to *over saving* in competitive equilibrium.

Intuitively, increasing the variance of idiosyncratic risk in the bad state of the world (i.e. under a bad aggregate shock) increases the utility cost of scaling up labour income's share in expected total income, when k rises and increases the wage, but lowers the interest rate. This utility cost may be lower in a good state of the world (i.e. under a good aggregate shock), because of the much lower variance of the idiosyncratic shock in that situation. However, the utilitarian social welfare function gives greater weight to the situation under a bad aggregate shock, as shown by the marginal utility terms $u'()$ in the decomposition (3.50). The end result is that for certain levels of wealth inequality, there is constrained inefficiency due to *over* saving, the opposite of when aggregate and idiosyncratic risk are fully independent or correlated in a different way, such as in Case One.

3.3.3 The Constrained Efficient Level of Saving

I characterise the constrained efficient level of saving by formulating a social planner's problem. The social planner chooses a decision rule for period one consumer saving $a(\Omega^i)$ (that dictates a level of saving for every consumer) to maximise utilitarian social welfare

$$SW = (\frac{1}{I})\sum_i U(a(\Omega^i)) \tag{3.58}$$

subject to satisfying no borrowing constraints for all consumers

$$a(\Omega^i) \geq 0 \tag{3.59}$$

and subject to satisfying all period budget constraints for consumers for all realisations of aggregate and idiosyncratic shocks

$$c_1(\Omega^i) = \Omega^i - a(\Omega^i) \tag{3.60}$$

$$c_2(\Omega^i, z^j, e_n) = r(z^j, k)a(\Omega^i) + w(z^j, k)e_n \tag{3.61}$$

for all $j \in (G, B)$ and $n \in (L, H)$, while allowing factor prices to be set in competitive markets so that

$$r(z^j, k) = z^j f_k'(k, l(z^j)) \tag{3.62}$$

$$w(z^j, k) = z^j f_l'(k, l(z^j)) \tag{3.63}$$

and taking into account the effect of individual consumer saving $a(\Omega^i)$ on aggregate quantities

$$k = (\frac{1}{I})\sum_i a(\Omega^i) \tag{3.64}$$

$$l(z^j) = \Pr(e_L \mid z^j)e_L + \Pr(e_H \mid z^j)e_H \tag{3.65}$$

The notation $c_1(\Omega^i)$ and $c_2(\Omega^i)$ recognises that the expected consumption levels for each consumer will differ depending on initial wealth Ω^i, given any decision rule for saving $a(\Omega^i)$. The notation $c_2(\Omega^i, z^j, e_n)$ recognises that actual realisations of period two consumption depend not only on initial wealth but on specific realisations of aggregate and idiosyncratic shocks. The same can be said for the factor price notation $r\left(z^j,k\right)$ and $w(z^j,k)$.

The planner is *not* constrained by consumer Euler equations (derived from the consumers' problem first order condition (3.18)), which are necessary conditions for competitive equilibrium. However, the planner may *not* complete markets for aggregate and idiosyncratic risk by making transfers between consumers once shocks have been realised. The planner must respect all period consumer budget constraints and no borrowing constraints. Factor prices must be allowed to be determined in competitive markets given a decision rule for savings $a(\Omega^i)$. These restrictions on the planner follow from the notion of *constrained* efficiency discussed in Section Two.

I formulate the problem as a classical, unconstrained maximisation problem by **(i)** using the budget constraints (3.60) and (3.61) to substitute for $c_1(\Omega^i)$ and $c_2(\Omega^i)$; **(ii)** using (3.62) and (3.63) to substitute out factor prices $r(z^j,k)$ and $w(z^j,k)$; and **(iii)** using (3.64) to substitute out the average per-consumer level of saving (i.e. capital accumulation) k.

The first order conditions for the planner's problem are derived in Appendix **(**C.2**).** These first order conditions imply the following condition

$$\begin{aligned}
& u'(\Omega^i - a(\Omega^i)) \qquad (3.66)\\
= \; & \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j,k) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \\
& + \sum_{i=1}^{I} (\frac{1}{I}) \left\{ \beta \sum_j \sum_n \Pr(z^j \cap e_n) \left(\begin{array}{c} \frac{\partial r(z^j,k)}{\partial k} a(\Omega^i) \\ + \frac{\partial w(z^j,k)}{\partial k} e_n \end{array} \right) u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ + w(z^j,k)e_n \end{array} \right) \right\}
\end{aligned}$$

Recall that in a laissez-faire competitive equilibrium, the Euler equation for a consumer of type $i \in \{1,2,...,I\}$ is given by (3.29):

$$u'(\Omega^i - a^{i*}) = \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j,k) u'(r(z^j,k)a^{i*} + w(z^j,k)e_n)$$

Note that the right hand side (RHS) of the planner's condition (3.66) differs from the competitive equilibrium condition (3.29) by the term

$$\sum_{i=1}^{I}(\frac{1}{I})\left\{\beta\sum_{j}\sum_{n}\Pr(z^j\cap e_n)\left(\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^i)\\+\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right)u'\left(\begin{array}{c}r(z^j,k)a(\Omega^i)\\+w(z^j,k)e_n\end{array}\right)\right\} \tag{3.67}$$

If this term (3.67) is non-zero, the constrained efficient level of per consumer saving k must differ from the level arising in laissez-faire competitive equilibrium. Actually, this term (3.67) is identical to the marginal effect on social welfare in laissez-faire competitive equilibrium of a change in k. This is because it is an average over consumer types $i \in \{1, 2, ..., I\}$ of the marginal welfare effect on an individual consumer, which is given by (3.34). I showed in Subsection (3.3.2) that the marginal effect of higher k on social welfare depends on (**i**) the variance of the initial wealth distribution; and (**ii**) how the realisation of the aggregate shock changes the conditional distribution of the idiosyncratic shock.

3.3.4 Numerical Solutions - Competitive Equilibrium and Constrained Efficiency

In order to compute numerical solutions, I assume that period utility has the constant coefficient of relative risk aversion (CRRA) form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \tag{3.68}$$

where $\frac{1}{\sigma}$ is the constant inter-temporal elasticity of substitution of consumption. I assume that the production function takes the standard Cobb-Douglas form

$$z^j f(k, l(z^j)) = z^j k^{\alpha} l(z^j)^{1-\alpha} \tag{3.69}$$

for all $j \in (G, B)$, which is homogenous of degree one in private inputs k and $l(z^j)$, consistent with the assumption made in Section Two. The parameter α measures the share of output paid to the owners of capital in period two of the model. The share of output paid to labour is $(1-\alpha)$.

I assign the following values to parameters

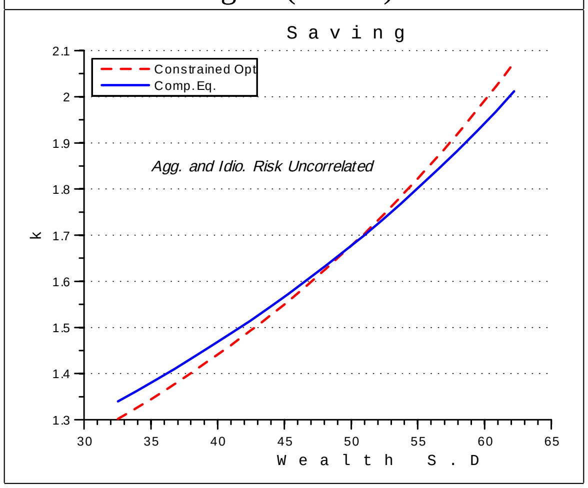
β	σ	α	I
0.99	3	0.36	2

The number of types of consumers I (which differ ex ante by initial wealth) is set to two. The two types can be thought of as *asset poor* and *asset rich* respectively. The discount factor β is set to 0.99 while the coefficient of relative risk aversion σ is set to three. These values are standard in the literature. A similarly standard choice is the value 0.36 for α, the share of output paid to capital. In this Subsection (3.3.4), I set the standard deviation of the idiosyncratic employment shock e_n to around sixty per cent of its mean. Unless otherwise specified, the standard deviation of the aggregate shock z^j is set to ten per cent of its mean.[8]

3.3.4.1 Saving - Constrained Efficient vs Competitive Equilibrium

Figure (3.3.4.1) compares the constrained efficient level of average per consumer saving k with its level in laissez-faire competitive equilibrium, as the variance of the distribution of initial wealth rises (along the horizontal axis). There is constrained inefficiency due to *over* saving if the difference between constrained efficient k and its competitive equilibrium level is negative (and vice versa).

Figure (3.3.4.1)



The sign and magnitude of constrained inefficiency *changes* as the variance of the initial wealth distribution $var(\Omega^i)$ rises (this is on the horizontal axis of the graphs). At low levels of initial wealth inequality, there is *over* saving, while there is *under* saving when wealth inequality is relatively high.

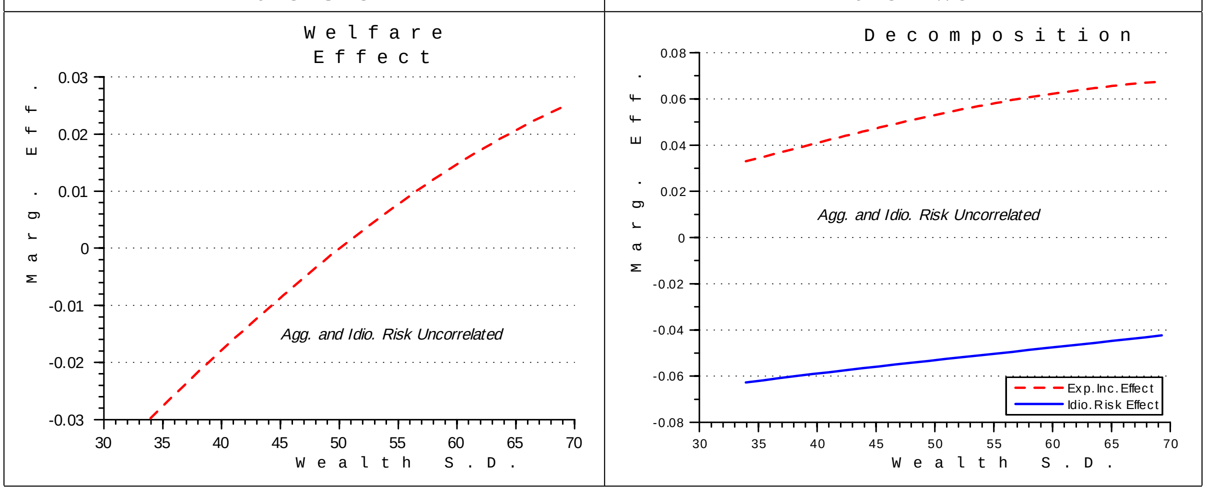
[8] All numerical solutions presented in this chapter were obtained using the non-linear equation solver developed by Michael Reiter. I am most grateful to him for providing me with the necessary computer code.

3.3.4.2 Explaining the Results using Local Analysis

These results are consistent with the local analysis presented in Subsection (3.3.2), where I discussed the marginal effect of higher k on social welfare in competitive equilibrium. The first panel of Figure (3.3.4.2) shows that this effect rises with the variance of initial wealth. It is negative at low levels of wealth inequality (suggesting *over* saving), but becomes positive when wealth inequality is relatively high (suggesting *under* saving). This is partly because the *Expected Income Effect* of lower interest rate and higher wage (because of higher k) becomes more important as $var(\Omega^i)$ rises: see the second panel of Figure (3.3.4.2). The expected income effect is positive because the lower interest rate and higher wage reduce inequality of expected income conditional on any realisation of the aggregate shock (while preserving the mean expected income). This becomes more important as $var(\Omega^i)$ rises.

Figure (3.3.4.2)

Panel One | Panel Two



3.3.4.3 Interaction between Aggregate and Idiosyncratic Risk

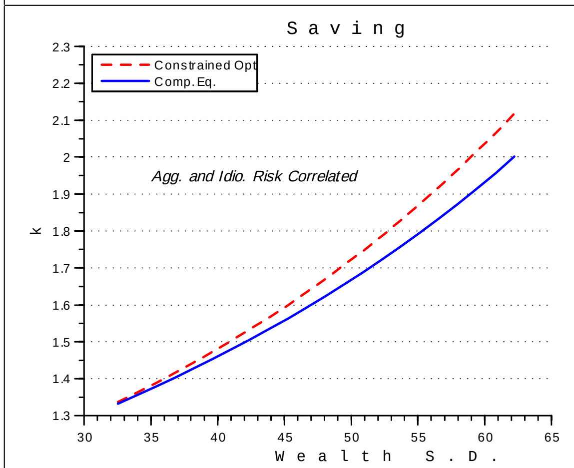
The following graphs compare again the constrained efficient level of average per consumer saving k with that in competitive equilibrium, as the variance of the initial wealth distribution increases (along the horizontal axis). The difference between the graphs is the way in which bad aggregate shocks affect the conditional distribution of the idiosyncratic employment shock.

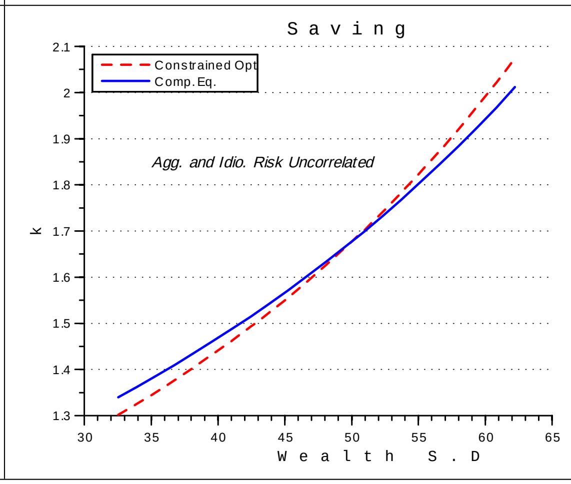
3.3.4.4 Case One - Skewness

The left panel of Figure (3.3.4.4) shows the situation where a bad aggregate shock makes the realisation of a bad idiosyncratic shock more likely. This is the case where the aggregate shock affects the *skewness* of the conditional distribution of the idiosyncratic shock, considered in Subsection (3.3.2) of Section (3.3). In the right panel of Figure (3.3.4.4), the distribution of the idiosyncratic shock is independent of the aggregate shock. If the bad aggregate shock makes a bad idiosyncratic shock more likely, the left panel of Figure (3.3.4.4) shows that there is constrained inefficiency due to *under* saving at all levels of wealth inequality shown in the graphs.

Figure (3.3.4.4)

Panel One	Panel Two



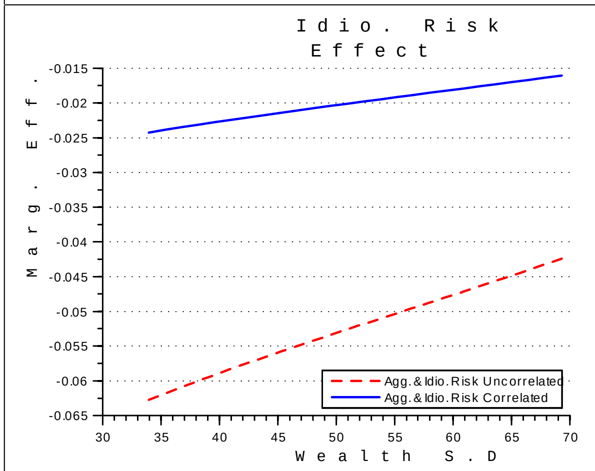


3.3.4.5 Explaining the Results using Local Analysis

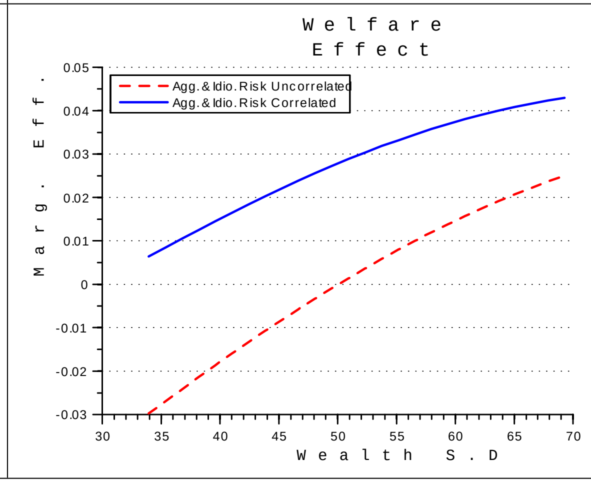
This is consistent with the local analysis in Subsection (3.3.2). The negative *Idiosyncratic Risk Effect* of a marginal increase in k is closer to zero when the realisation of a bad aggregate shock makes a bad idiosyncratic shock more likely: see panel one of Figure (3.3.4.5). Intuitively, the idiosyncratic risk effect captures the utility cost of scaling up the share of expected income obtained from labour, which is subject to idiosyncratic risk. This scaling up occurs because the wage rises (and the interest rate falls) when k increases. Capital income is not subject to idiosyncratic risk, conditional on a realisation of the aggregate shock. The utility cost of scaling up the share of income subject to idiosyncratic risk *falls* when there is correlation between the direction of the aggregate shock and the idiosyncratic shock. The extent of purely idiosyncratic risk is lower in this situation. This means that the marginal effect of higher k on welfare in competitive equilibrium is larger: see panel two of Figure (3.3.4.5).

Figure (3.3.4.5)

Panel One



Panel Two

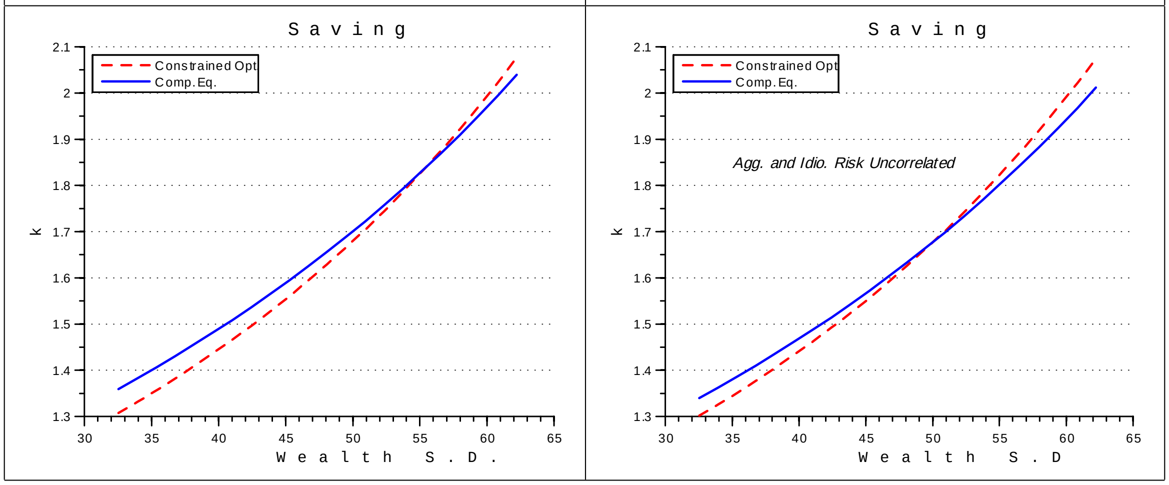


3.3.4.6 Case Two - Variance

In the left panel of Figure (3.3.4.6), a bad aggregate shock makes the *variance* of the idiosyncratic shock relatively large, while a good aggregate shock has the opposite effect. This is the case where the aggregate shock affects the *variance* of the conditional distribution of the idiosyncratic shock, considered in Subsection (3.3.2) of Section (3.3). In the right panel of Figure (3.3.4.6), the distribution of the idiosyncratic shock is again independent of the aggregate shock. The left panel of Figure (3.3.4.6) now shows that there is constrained inefficiency due to *over* saving over a somewhat *larger* range of wealth inequality levels (compared with the right panel), when the aggregate shock affects the conditional variance of the idiosyncratic shock.

Figure (3.3.4.6)

Panel One | Panel Two

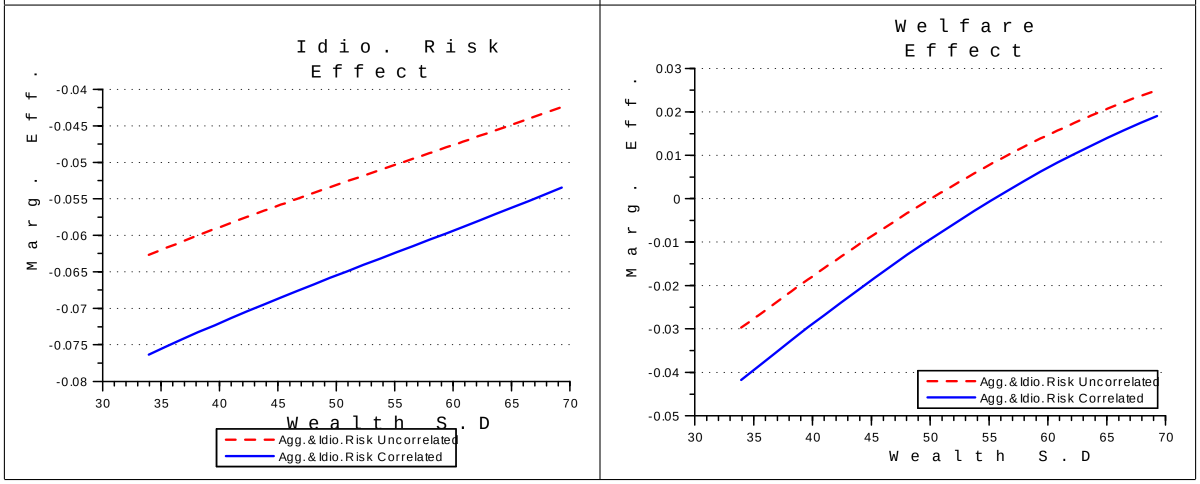


3.3.4.7 Explaining the Results using Local Analysis

This is consistent with the local analysis in Subsection (3.3.2). The negative *Idiosyncratic Risk Effect* of a marginal increase in k is *larger* in absolute value when the realisation of the aggregate shock changes the conditional variance of the idiosyncratic shock: see panel one of Figure (3.3.4.7). Intuitively, the utility cost of scaling up the share of expected income subject to idiosyncratic risk is more costly in this scenario. This scaling up occurs because the wage rises (and the interest rate falls) when k increases. This means that the marginal effect of higher k on welfare in competitive equilibrium is now *negative* over a relatively large range of initial wealth inequality levels: see panel two of Figure (3.3.4.7).

Figure (3.3.4.7)

Panel One	Panel Two



3.4 Achieving Constrained Efficiency Through Implementable Tax / Transfer Schemes

I now consider two types or classes of tax and fiscal transfer schemes, that are implementable in the sense that they can give rise to a competitive equilibrium. There are no government purchases that need to be financed. Therefore, the only potentially useful purpose that these schemes can have is to improve utilitarian social welfare.

First, I investigate whether a tax / subsidy on the return to saving is efficiency improving, accompanied by lump sum transfers / taxes that do *not* redistribute income. It is possible to generate an efficiency improvement in this way because constrained inefficiency can be caused by a pecuniary externality of saving. Inducing different saving behaviour should thus be a way to improve efficiency. The welfare / efficiency effect of introducing a tax / subsidy on the return to saving is affected by the variance of the distribution of initial wealth and the way in which aggregate shocks change the conditional distribution of the idiosyncratic shock.

The position changes when taxes are accompanied by transfers that *redistribute* income and wealth. Redistribution itself may improve utilitarian social welfare (effectively insuring consumers against low realisations of initial wealth), to such an extent that it offsets any negative efficiency effect of an otherwise distortionary tax. I show how this welfare improvement of redistribution depends on the variance of the aggregate shock.

Also, I solve for the optimal levels of tax rates and transfer payments within each type or class of scheme, by formulating Ramsey-type optimal fiscal policy problems. The level of average saving k per consumer generated by Ramsey optimal policy can be compared to the benchmark, constrained efficient level in a model without fiscal policy.

The presence of aggregate risk allows me to study whether it is optimal for the level of redistributive transfer payments to vary in a way that is pro-cyclical, a-cyclical or counter-cyclical. Cyclical variation in the size of transfer payments may effect the extent to which they constitute insurance for consumers against bad aggregate shocks (which lower wages and interest rates). In turn, this may affect the extent of precautionary saving by consumers.

Specifically, the two types of tax / transfer scheme studied are:

(1) A linear, proportional tax τ^k on the rate of return to saving $r(z^j, k)$ levied on consumers of all types (i.e. all initial wealth Ω^i). The *exact* amount of revenue raised from each consumer is *rebated* to the consumer using *personalised* lump sum transfers. These transfers eliminate any income effect or redistribution effect of taxation. This allows the *pure* substitution effect of the tax to be studied. Gottardi and Nakajima (2013) studies this type of taxation in an environment without aggregate risk.

(2) A non-linear proportional tax $\tau^k(\Omega^i)$ on the rate of return to saving $r(z^j, k)$. It is non-linear in the sense that the tax rate is positive *only* for those types of consumers with initial wealth exceeding a threshold level $\overline{\Omega}$, such that $\Omega^1 < ..\overline{\Omega}.. < \Omega^I$. Also, lump sum transfers are paid but are subject to an asset based means test. Transfers are only paid to those types of consumers with initial wealth *below* threshold $\overline{\Omega}$. Clearly, this transfer scheme entails significant redistribution. It is assumed that the government always runs a balanced budget, so that if the level of transfer payments is fixed exogenously, the tax rate must be endogenised to balance the budget, or vice versa.

It should be noted that Gottardi and Nakajima (2013) study some of the questions addressed in this Section in an environment *without* aggregate risk.

3.4.1 Tax / Transfer Scheme One: Pure Substitution Effect of Capital Tax

This type of scheme involves a linear, proportional tax rate τ^k on the rate of return to saving $r(z^j, k)$ that is constant and does not vary across types of consumers. The personalised lump sum transfers $T(\Omega^i, z^j, e_n)$ *do* vary across types of consumers and also with realisations of aggregate and idiosyncratic shocks (which affect the amount of labour and capital income). This ensures that the transfers exactly rebate the amount of tax collected from each consumer, whatever the realisation of the aggregate and idiosyncratic shock. Introducing this tax / transfer scheme leaves the expected total income of every consumer unchanged (for any given level of saving), but alters the rate of return on saving. The effect this has on saving is the pure substitution effect of the tax and may be used to induce saving behaviour that generates an efficiency improvement compared with competitive equilibrium.

A rational consumer of type i (with initial wealth Ω^i) takes the tax rate τ^k and factor prices $r(z^j, k)$ and $w(z^j, k)$ as given and chooses a level of saving $a(\Omega^i)$ to maximise expected utility:

$$\max_{a^i \in [0, \Omega^i]} u(c_1^i) + \beta E_0[u(c_2^i)] \tag{3.70}$$

subject to a no borrowing constraint

$$a^i \geq 0 \tag{3.71}$$

and subject to the period budget constraint in period one

$$c_1^i + a^i \leq \Omega^i \tag{3.72}$$

plus the period two budget constraints for every possible combination of aggregate and idiosyncratic shock realisations

$$c_2^i(z^j, e_n) \leq (1 - \tau^k) r(z^j, k) a^i + w(z^j, k) e_n + T(\Omega^i, z^j, e_n) \tag{3.73}$$

where $j \in \{G, B\}$ and $n \in \{L, H\}$. The period budget constraint will always bind under the assumptions discussed in Section Two, while the no borrowing constraint will never bind.

The first order condition is now given by

$$u'(\Omega^i - a^i) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)(1-\tau^k) r(z^j) u'((1-\tau^k) r(z^j) a^i + w(z^j) e_n + T(\Omega^i, z^j, e_n)) \tag{3.74}$$

The firms' problem is the same as described in Section Two. Full derivations are in Appendix (C.3).

Definition 11 *A* ***Competitive Equilibrium under Tax / Transfer Scheme One*** *is a tax rate τ^k, a* ***Decision Rule*** *for consumer saving $a(\Omega^i)$ (a function of initial wealth Ω^i that varies by consumer type) such that* ***(i)*** *the no borrowing constrains (3.71) are satisfied for all consumers;* ***(ii)*** *all consumer and firm budget constraints (3.72), (3.73) and (3.24) are satisfied with equality for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$;* ***(iii)*** *the first order condition (3.74) is satisfied for all individual consumers;* ***(iv)*** *the first order conditions (3.26) for all firms are satisfied for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; and* ***(v)*** *tax collected from every consumer is rebated to the consumer using a personalised transfer $\tau^k r(z^j, k) a(\Omega^i) = T(\Omega^i, z^j, e_n)$, implying that the government budget is always balanced.*

Utilitarian social welfare in this competitive equilibrium is again given by a weighted average across consumer types of expected utility, which takes the form

$$SW(\tau^k, k) = (\frac{1}{I}) \left\{ u(\Omega^i - a(\Omega^i)) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(k, z^j) a(\Omega^i) + w(k, z^j) e_n) \right\} \tag{3.75}$$

where $T(\Omega^i, z^j, e_n)$ has been substituted out using $\tau^k r(z^j, k) a(\Omega^i) = T(\Omega^i, z^j, e_n)$. After this substitution is made, it is apparent that this utilitarian social welfare function is identical in *competitive equilibrium* to its form (3.31) in *laissez faire* competitive equilibrium without fiscal policy.

In order to investigate whether introducing tax / transfer scheme one is efficiency improving, I compute the marginal effect on utilitarian social welfare (3.75) of introducing the tax / transfer scheme in competitive equilibrium.

3.4.1.1 Marginal Effect on Social Welfare of Introducing Tax / Transfer Scheme One

Introducing the tax on the return to saving in tax / transfer scheme one can have a *direct* effect on social welfare (say through redistribution) and an *indirect* effect through the pecuniary externality on factor prices of a change in saving brought about because of the distortionary tax.

Since utilitarian social welfare in competitive equilibrium under tax / transfer scheme one (3.75) has the same form as in *laissez faire* competitive equilibrium (with τ^k not appearing), it is clear that introducing the tax / transfer scheme has no *direct* marginal effect on utilitarian social welfare. This reflects the fact that the personalised lump sum transfers $T(\Omega^i, z^j, e_n)$ rebate all the tax paid by each consumer, so at the margin they eliminate any income or redistribution effect of the tax. At the margin, introducing tax / transfer scheme one leaves income *unchanged*, so welfare is unchanged, all else equal.

However, the linear, proportional tax τ^k on the return to saving should change the amount of average saving or capital accumulation in the economy at the margin. This changes factor prices and can generate a pecuniary externality that consumers do not take into account when making savings decisions. In order to evaluate this pecuniary externality, it is necessary to make an assumption about the impact of the tax on the level of average saving k at the margin.

Conjecture 11 *The linear proportional tax τ^k on the return to saving* ***reduces*** *average saving per consumer k:* $\frac{\partial k}{\partial \tau^k} < 0$

This will be the case in competitive equilibrium under Cobb-Douglas production technology and CRRA utility, as used in Subsection (3.4.1.7) of Section (3.3) to solve numerically for the competitive equilibrium.

The marginal effect on social welfare of introducing tax / transfer scheme one in competitive equilibrium will consist solely of the *indirect* effect, arising because of the pecuniary externality

$$\frac{\partial SW(\tau^k, k)}{\partial \tau^k} \mid_z = \underbrace{\frac{\partial SW}{\partial \tau^k}}_{=0} + \frac{\partial SW}{\partial k}\frac{\partial k}{\partial \tau^k} = \frac{\partial SW}{\partial k}\frac{\partial k}{\partial \tau^k} \tag{3.76}$$

This indirect effect is given by the *negative* of the marginal effect of additional saving k on utilitarian social welfare (3.50) discussed in Section Three. The change of sign occurs because $\frac{\partial k}{\partial \tau^k} < 0$. Note that the Envelope Theorem implies that changes in *individual* saving choices $a(\Omega^i)$ because of the tax have no utility effect at the margin, since consumers are already saving at a utility maximising point in competitive equilibrium.

The marginal effect of the tax / transfer scheme one on social welfare can be written as

$$\frac{\partial SW(\tau^k, k)}{\partial \tau^k} \mid_{Comp.Eq.} = \tag{77}$$

$$+(\frac{1}{I})\sum_i \beta E\left(u'(\Omega^i, z^j, e_n)\left[k\left\{\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k} - E(\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k})\right\}\right]\right)$$

$$+(\frac{1}{I})\sum_i \beta E\left(u'(\Omega^i, z^j, e_n)\left[l(z^j)\frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k} - E(l(z^j)\frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k})\right]\right)$$

$$+(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j)\left(E\left[u'(\Omega^i, z^j, e_n)(e_n - l(z^j))\right] \mid_z \frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k}\right)$$

$$+(\frac{1}{I})\sum_i \beta \sum_j \Pr(z^j)\left(E\left[u'(\Omega^i, z^j, e_n)\right] \mid_z (a^i - k)\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k}\right)$$

The first two terms are the aggregate risk insurance effects (3.36). Given that the production function is homogeneous of degree one, these terms continue to be exactly zero, as shown in Proposition (3) of Section Three. The third and fourth terms represent the *idiosyncratic risk effect* (3.54) and the *expected income effect* (3.51) respectively, as discussed in Subsection (3.3.2) of Section Three. However, their signs are now reversed. To see this, first consider the effect of the tax on factor prices.

Proposition 12 *It is the case that* $\frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k} < 0$ *and* $\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial \tau^k} > 0$ *for all* $j \in \{G, B\}$.

Proof. It was shown in (3.5) of Section Two that $\frac{\partial w(z^j,k)}{\partial k} > 0$ and $\frac{\partial r(z^j,k)}{\partial k} < 0$. The Conjecture (11) provides that $\frac{\partial k}{\partial \tau^k} < 0$. Combining these two outcomes gives the result. ■

The tax on the return to saving reduces average saving (i.e. accumulated capital) k per consumer at the margin, which lowers the wage. However, it increases the interest rate, because marginal returns to capital are higher at lower levels of k.

The sign of the idiosyncratic risk and expected income effects in response to introducing tax / transfer scheme one can now be established.

Proposition 13 *When tax / transfer scheme one is introduced, the* ***expected income effect*** *is* ***negative*** *and the* ***idiosyncratic risk effect*** *is* ***positive****.*

Proof. Proposition (6) in Section Three reveals that the expected income effect (3.51) of a marginal increase in saving k is positive. The expected income effect brought about by the tax is obtained by multiplying (3.51) by $\frac{\partial k}{\partial \tau^k} < 0$. Proposition (7) in Section Three demonstrates that the idiosyncratic risk effect (3.54) of a marginal increase in saving k is negative. The idiosyncratic risk effect brought about by the introduction of tax / transfer scheme one is obtained by multiplying (3.54) by $\frac{\partial k}{\partial \tau^k} < 0$. This gives the result. ■

The intuition is that the introduction of the tax reduces saving k, thus lowering the wage and increasing the interest rate given any realisation of the aggregate shock z^j, $j \in \{G, B\}$. For all consumers, this *reduces* the share of expected income obtained from labour and increases the share obtained from capital. This *reduces* the share of total income subject to idiosyncratic labour productivity shocks e_n, $n \in \{H, L\}$. Hence, the *idiosyncratic risk effect* on social welfare of introducing the tax is positive.

The higher interest rate and lower wage raise the expected total income of the asset rich (i.e. those with above average saving $a(\Omega^i) > k$) and lower that of the asset poor, conditional on any realisation of the aggregate shock. The expected total income of the average consumer (with saving k) is *unchanged* since

$$-\frac{\partial r(z^j, k)}{\partial k} k = \frac{\partial w(z^j, k)}{\partial k} E\left[e_n\right] |_z$$

as discussed earlier in Subsection (3.3.2). Overall, there is an increase in inequality of expected income, conditional on any realisation of the aggregate shock (while preserving the mean of the distribution of expected income). This is penalised by the utilitarian social welfare function used in this chapter. Hence, the *expected income effect* of introducing the tax is negative.

3.4.1.2 Effect of Initial Wealth Inequality

It should be the case that the negative expected income effect brought about by tax / transfer scheme one will dominate as the distribution of initial wealth becomes more unequal (i.e. as $var(\Omega^i)$ rises). Intuitively, as $var(\Omega^i)$ rises, the widening of inequality of expected income (conditional on any realisation of the aggregate shock) that occurs because of a higher interest rate and lower wage becomes relatively important (while the mean expected income is preserved). Because of this, the marginal effect of introducing the tax on utilitarian social welfare (3.77) should become increasingly negative as $var(\Omega^i)$ rises. Note that the effect of the initial wealth distribution on the expected income effect was discussed in more detail in Section (3.3).

3.4.1.3 Interaction Between Aggregate and Idiosyncratic Risk

The interaction between aggregate and idiosyncratic risk is also important for determining the impact of introducing tax / transfer scheme one. I consider the same two cases described in Subsection (3.3.2) of Section (3.3). In the first case, a bad realisation of the aggregate shock makes a bad idiosyncratic shock more likely (i.e. it affects the *skewness* of the conditional distribution of the idiosyncratic shock). In the second case, a bad realisation of the aggregate shock makes the variance of the conditional distribution of the idiosyncratic shock relatively large (vice versa for a good aggregate shock). In this case, the aggregate shock affects the conditional *variance* of the idiosyncratic shock.

3.4.1.4 Case One - Skewness

Proposition 14 *The marginal effect on social welfare (3.77) of introducing tax / transfer scheme one is unambiguously* ***negative*** *when the probability of a bad idiosyncratic shock (conditional on a bad realisation of the aggregate shock, as described in (3.55))* ***approaches one****.*

Proof. Proposition (9) of Section Three shows that the absolute value of the *idiosyncratic risk effect* (3.54) of higher saving k approaches zero as the probability of a bad (good) idiosyncratic shock (conditional on a bad (good) realisation of the aggregate shock) approaches one. This will also be true for the absolute value of the *idiosyncratic risk effect* of the tax, since this effect is the negative of (3.54), as shown by Proposition (13). As this occurs, the negative *expected income effect* of introducing the tax dominates. This gives the result. ∎

Intuitively, the positive welfare effect of reducing the share of expected income subject to idiosyncratic risk (caused because the tax lowers the wage and raises the interest rate) is less important when there is correlation between aggregate and idiosyncratic risk of the type described in (3.55). The extent of purely idiosyncratic risk is lower in these circumstances. Therefore, the marginal welfare effect (3.77) of introducing tax / transfer scheme one should become *negative* as the probability of a bad idiosyncratic shock (conditional on a bad aggregate shock) rises.

3.4.1.5 Case Two - Variance

The positive *idiosyncratic risk effect* of introducing tax / transfer scheme one should be *larger* when the aggregate shock affects the variance of the idiosyncratic shock. The *idiosyncratic risk effect* of introducing tax / transfer scheme one is the negative of the *idiosyncratic risk effect* of a marginal increase in k. In Subsection (3.3.2) of Section (3.3), I explained that the absolute value of this effect is larger when the aggregate shock changes the conditional variance of the idiosyncratic shock.

Intuitively, the utility benefit of scaling down the share of expected income subject to idiosyncratic risk (because the tax lowers the wage and raises the interest rate) is larger in the situation when a bad aggregate shock magnifies the variance of the idiosyncratic shock. This situation is given greatest weight by the utilitarian social welfare function.

All else equal, the positive *idiosyncratic risk effect* of introducing tax / transfer scheme one will be larger relative to the negative *expected income effect*. This will raise the marginal effect on social welfare (3.77) of introducing tax / transfer scheme one, making it more likely that a positive taxation rate on the return to saving is welfare improving. (i.e. reflecting the fact that there is more likely to be constrained inefficiency due to *over* saving in this case).

3.4.1.6 Ramsey Optimal Policy - Tax / Transfer Scheme One

I now proceed to solve for the Ramsey optimal level of taxation, within this *type* of tax / transfer scheme. This involves solving a social planner's problem. The social planner chooses a tax rate on the return to saving τ^k and a decision rule for saving (i.e. a level of saving $a(\Omega^i)$ for each consumer type) to maximise utilitarian social welfare

$$\max_{\tau^k, \{a(\Omega^i)\}_{i=1}^I} SW = (\frac{1}{I}) \sum_i \left\{ u(\Omega^i - a(\Omega^i)) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(k, z^j) a(\Omega^i) + w(k, z^j) e_n) \right\} \tag{3.78}$$

subject to satisfying no borrowing constraints for all consumers

$$a(\Omega^i) \geq 0 \tag{3.79}$$

and subject to satisfying the first order condition of the consumer's problem for each consumer type $i \in \{1, 2, ..., I\}$

$$u'(\Omega^i - a^i) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)(1 - \tau^k) r(z^j) u'((1 - \tau^k) r(z^j) a^i + w(z^j) e_n + T(\Omega^i, z^j, e_n)) \tag{3.80}$$

as well as all period budget constraints for consumers for all realisations of aggregate and idiosyncratic shocks

$$c_1(\Omega^i) = \Omega^i - a(\Omega^i) \tag{3.81}$$

$$c_2(\Omega^i, z^j, e_n) = (1 - \tau^k) r(z^j, k) a(\Omega^i) + w(z^j, k) e_n + T(\Omega^i, z^j, e_n)$$

for all $j \in (G, B)$ and $n \in (L, H)$, while allowing factor prices to be set in competitive markets so that

$$r(z^j, k) = z^j f_k'(k, l(z^j)) \tag{3.82}$$

$$w(z^j, k) = z^j f_l'(k, l(z^j))$$

and taking into account the effect of individual consumer saving $a(\Omega^i)$ on aggregate quantities

$$k = (\frac{1}{I}) \sum_i a(\Omega^i) \tag{3.83}$$

$$l(z^j) = \Pr(e_L \mid z^j) e_L + \Pr(e_H \mid z^j) e_H$$

In effect, the Ramsey planner chooses a tax rate τ^k on the return to saving that maximises utilitarian social welfare, subject to the requirement that the chosen policy gives rise to a competitive equilibrium. Full derivation of the first order conditions is in Appendix (C.3).

Definition 12 *A Ramsey Equilibrium under tax / transfer scheme one is a tax rate τ^k and a decision rule for saving (i.e. a level of saving $a(\Omega^i)$ for each consumer type) that maximises utilitarian social welfare (3.78) such that* ***(i)*** *the no borrowing constrains (3.79) are satisfied for all consumers;* ***(ii)*** *all consumer and firm budget constraints (3.81) and (3.24) are satisfied with equality for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$;* ***(iii)*** *the first order condition (3.80) is satisfied for all individual consumers;* ***(iv)*** *the first order conditions (3.26) for all firms are satisfied for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$.*

Constrained inefficiency of competitive equilibrium in an economy *without* fiscal policy suggests that welfare improvements are possible by changing consumer saving behaviour. Tax / transfer scheme one does this by manipulating the return on saving, but without redistributing any income across consumers. The Ramsey optimal *level* of tax / transfer scheme one gives rise to a competitive equilibrium. The level of average capital k generated by Ramsey policy in competitive equilibrium can be compared with the benchmark, constrained efficient level in the laissez faire economy without fiscal policy (recalling that constrained efficient k must satisfy budget constraints but not necessarily Euler equations). This will show whether a tax / transfer scheme that gives rise to a competitive equilibrium can manipulate saving decisions and generate an efficiency improvement.

I now use numerical methods to solve the Ramsey problem and I compare the level of average saving k under Ramsey optimal policy with the benchmark, *constrained* optimal level in the laissez-faire economy solved for in Section (3.3). I also compute the marginal effects on utilitarian social welfare (3.77) of introducing tax / transfer scheme one in competitive equilibrium.

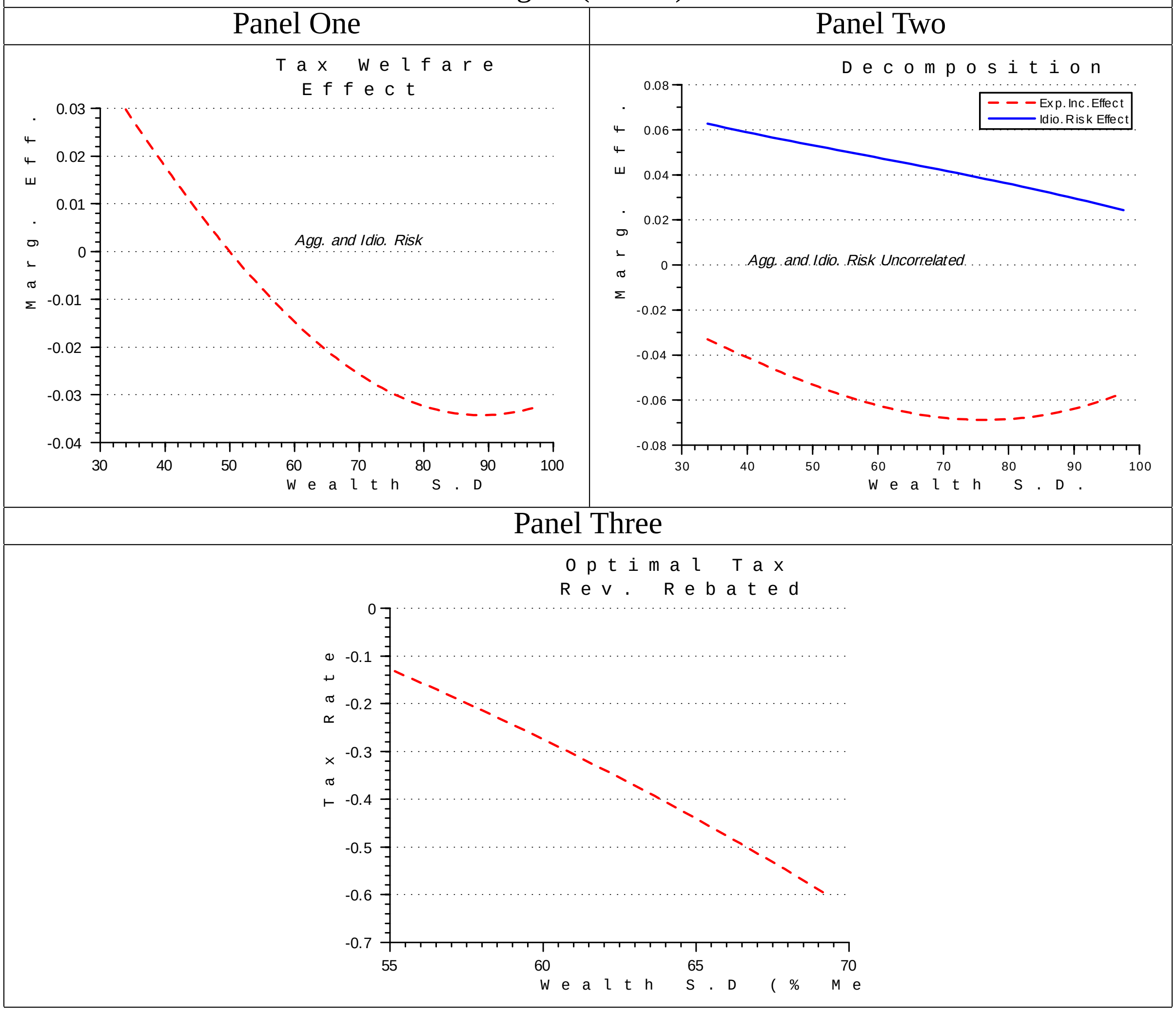
3.4.1.7 Numerical Solutions - Tax / Transfer Scheme One

The functional forms and parameter values used to obtain numerical solutions are the same as those used in Section (3.3). Unless otherwise specified, the standard deviation of the idiosyncratic shock is approximately sixty per cent of its mean. The standard deviation of the aggregate shock is around ten per cent of its mean. The tax reduces k at the margin, so raises the interest rate $r(z^j, k)$ and lowers the wage $w(z^j, k)$ at the margin, for all z^j. For any realisation of aggregate shock z^j, this lowers the share of expected total income derived from labour and subject to idiosyncratic risk (i.e. the positive idiosyncratic risk effect on welfare). Also, it raises expected total income for asset rich consumers (with above average saving $a(\Omega^i) > k$) and lowers it for asset poor consumers. Conditional on any realisation of the aggregate shock, this widens the distribution of expected income, while preserving the mean expected income: (i.e. the negative expected income effect on welfare).

Panel One of Figure (3.4.1.7) shows that the marginal effect on utilitarian social welfare of introducing the tax is positive when the variance of the distribution of initial wealth is relatively small (as shown on the horizontal axis). However, as the variance rises, the effect becomes negative. This is because the negative expected income effect of introducing the tax is larger when initial wealth is more unevenly distributed. The negative expected income effect dominates the positive idiosyncratic risk effect as $var(\Omega^i)$ rises. This is shown explicitly in Panel Two of Figure (3.4.1.7).

Panel Three of Figure (3.4.1.7) shows the Ramsey optimal level of the tax rate on the rate of return to saving (accompanied by lump sum transfers rebating exactly the amount of tax paid by each individual consumer). It is negative when the standard deviation of initial wealth is around fifty-five per cent of its mean or higher. This is consistent with the marginal welfare effect (3.77) of introducing the tax shown in Panel One. A negative rate of tax is a *subsidy* paid to the rate of return on saving. It will be financed by lump sum taxes, personalised to each consumer and exactly equal to the amount of subsidy received by the consumer

Figure (3.4.1.7)



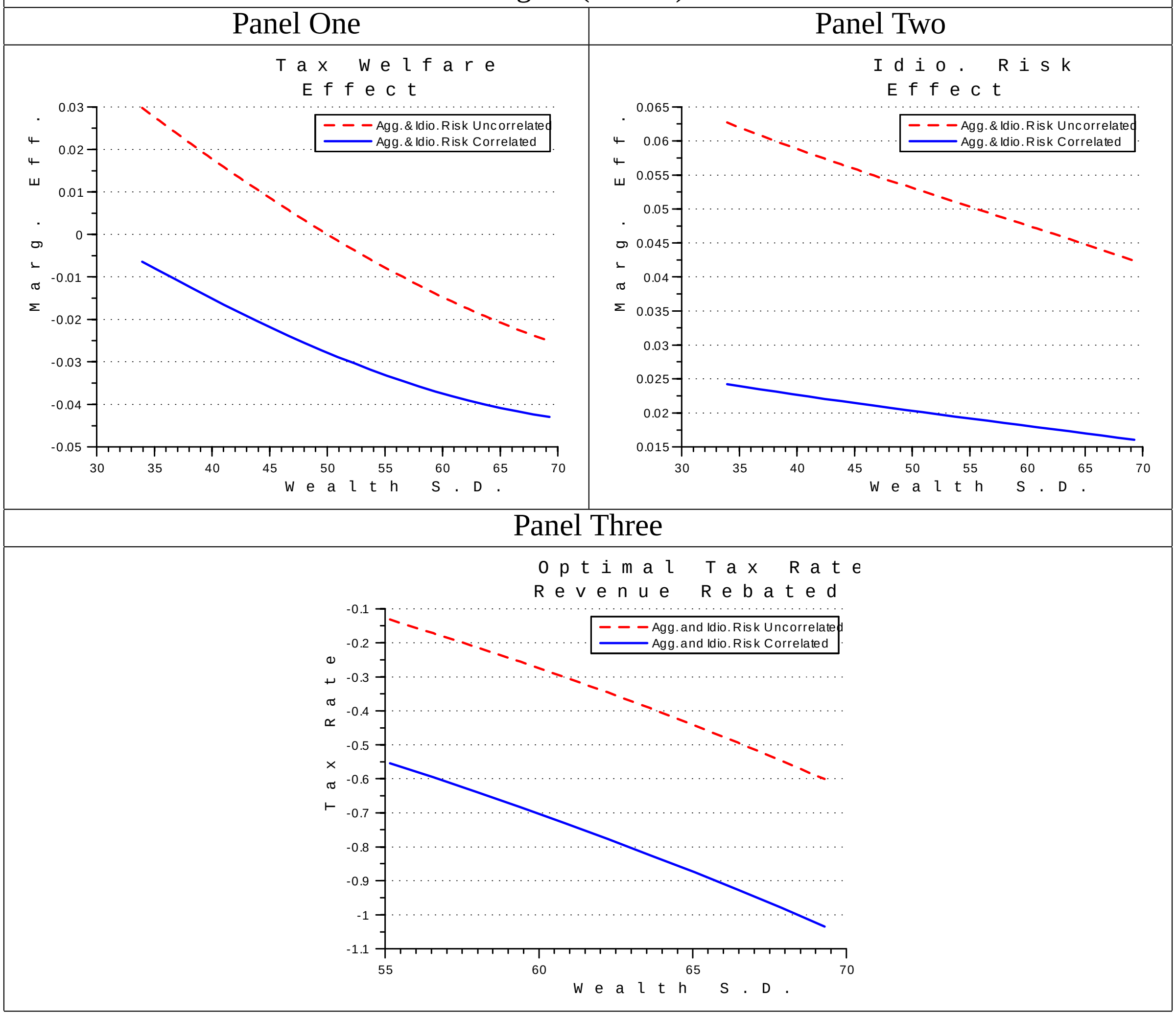
3.4.1.8 Interaction between Aggregate and Idiosyncratic Risk

3.4.1.9 Case One: Skewness

Panel One of Figure (3.4.1.9) shows that the marginal effect of introducing the tax on social welfare is *negative* over a *wider* range of initial wealth inequality levels, when correlation between idiosyncratic and aggregate risk $Corr(z^j, e_n)$ is such that the probability of a bad idiosyncratic shock rises when there is a bad realisation of the aggregate shock, as set out in (3.55). This is consistent with there being a smaller positive idiosyncratic risk effect (in absolute value) when

$Corr(z^j, e_n)$ increases, as is shown in Panel Two of Figure (3.4.1.9). Panel Three of Figure (3.4.1.9) shows the Ramsey optimal rate of taxation in this case. For all levels of initial wealth inequality shown, there should be a lower tax rate (or larger subsidy) on the rate of return to capital, compared with the case when aggregate and idiosyncratic risk are independent.

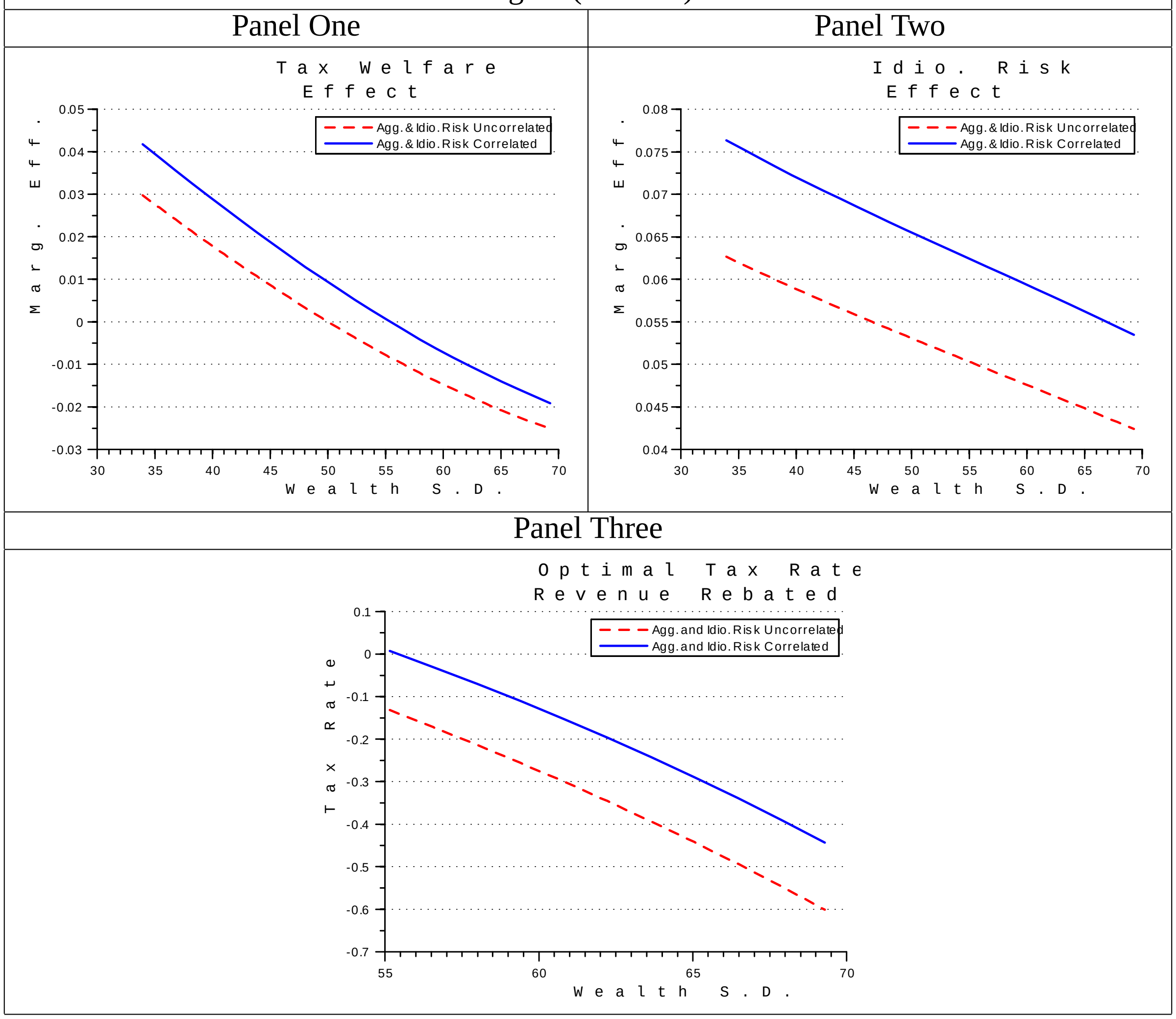
Figure (3.4.1.9)



3.4.1.10 Case Two - Variance

Panel One of Figure (3.4.1.10) shows the case where the realisation of the aggregate shock affects the conditional variance of the idiosyncratic shock. (i.e. when a bad aggregate shock makes the variance of the idiosyncratic shock relatively large; vice versa for a good aggregate shock). It shows that the marginal effect of introducing the tax on social welfare is *positive* over a *wider* range of initial wealth inequality levels, compared with where aggregate and idiosyncratic risk are fully independent. At higher levels of wealth inequality, there is a smaller negative effect. This is consistent with there being a *larger* positive idiosyncratic risk effect of introducing the tax, as is shown in Panel Two of Figure (3.4.1.10). Panel Three of Figure (3.4.1.10) shows the Ramsey optimal rate of taxation in this case. For all levels of initial wealth inequality shown, there should be a *higher* tax rate (or lower level of subsidy, in absolute value) on the rate of return to capital, compared with the case when aggregate and idiosyncratic risk are independent.

Figure (3.4.1.10)

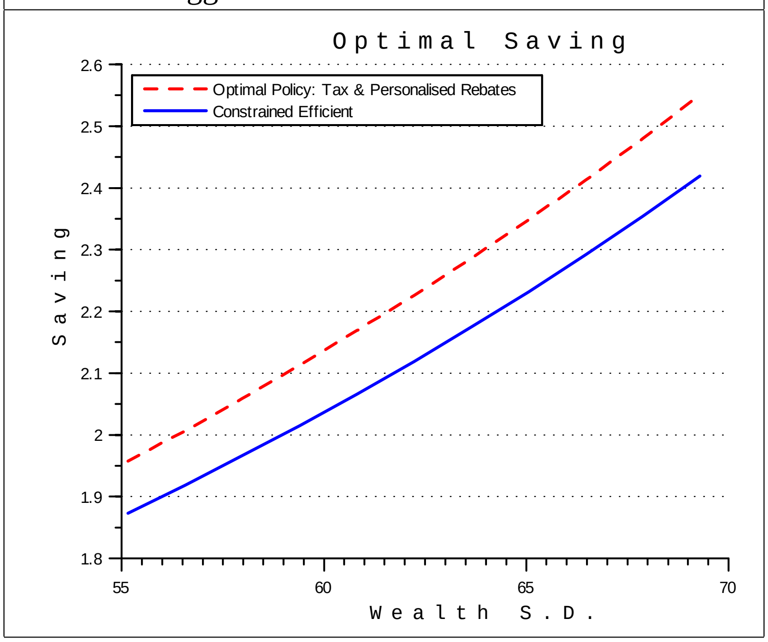


3.4.1.11 Saving - Ramsey Optimal vs Constrained Efficient

Figure (3.4.1.11) reveals that the level of average saving k per consumer in the Ramsey Equilibrium is relatively close to the constrained efficient level of saving in the economy without fiscal policy, solved for in Section Three, although somewhat below it. This suggests that changing the rate of return to saving in competitive equilibrium (using tax / transfer scheme one) comes reasonably close to the benchmark of constrained efficiency in a laissez-faire economy. It is perhaps unsurprising that the level of saving in the Ramsey economy does not coincide exactly with the constrained efficient level. The Ramsey economy must be a competitive equilibrium, whereas the constrained efficient benchmark need not satisfy consumer Euler equations.

Figure (3.4.1.11)

Agg. Risk Std. Dev. 10% of Mean



3.4.2 Tax / Transfer Scheme Two: Tax with Redistributive Transfers

This type of tax / transfer scheme combines a tax / subsidy that manipulates the rate of return on saving with redistributive transfers. Transfers that redistribute income from the asset rich to the asset poor can improve utilitarian social welfare, since they effectively insure consumers against low endowments of initial wealth. This may alter the rate of taxation / subsidy to the return on capital that is desirable. Tax / transfer scheme two again involves a distortionary tax on the return to saving. However, this tax $\tau^k(\Omega^i)$ is non-linear in the sense that it is only levied on consumers with initial wealth exceeding threshold $\overline{\Omega}$. This type of tax / transfer scheme also involves redistributive fiscal transfers that are subject to an asset based means test. Only consumers with initial wealth below threshold $\overline{\Omega}$ receive the transfers. In what follows, I make a simplifying assumption about the number of consumer types.

Definition 13 *The number of consumer types is assumed to be two $I = 2$, with $\Omega^R > \Omega^P$. Consumers of type R (with initial wealth Ω^R) are referred to as asset rich while consumers of type P (with initial wealth Ω^P) are referred to as asset poor.*

This implies that asset rich consumers pay the tax τ^k on the return to saving and asset poor consumers receive the redistributive transfers.

In the model with both aggregate and idiosyncratic risk, it is possible to consider transfers $T(z^j)$ that vary positively or inversely with the realisation of the aggregate shock z^j. Specifically, transfers can be pro-cyclical, a-cyclical or counter-cyclical. Transfers that are a-cyclical or counter-cyclical may provide partial insurance against aggregate shocks, so may affect the level of precautionary saving by consumers. I study this possibility.

In what follows, I proceed by fixing the level of the transfers for all realisations of the aggregate shock $T(z^j)$ and then endogenising the tax rate, to ensure that the government always runs a balanced budget.

3.4.2.1 Asset Rich Consumers

An asset rich consumer with initial wealth Ω^R takes the tax rate (contingent on the realisation of the aggregate shock z^j) $\tau^k(z^j)$ as given, along with factor prices $r(z^j,k)$, $w(z^j,k)$, to maximise :

$$\max_{a^R \in [0,\Omega^R]} u(c_1^R) + \beta E_0[u(c_2^R)] \tag{3.84}$$

subject to a no borrowing constraint

$$a^R \geq 0 \tag{3.85}$$

and subject to the period budget constraint in period one

$$c_1^R + a^R \leq \Omega^R \tag{3.86}$$

plus the period two budget constraints for every possible combination of aggregate and idiosyncratic shock realisations

$$c_2^R(z^j, e_n) \leq (1-\tau^k(z^j))r(z^j,k)a^R + w(z^j,k)e_n$$

where $j \in \{G,B\}$ and $n \in \{L,H\}$. Rational consumers take the tax rate $\tau^k(z^j)$ in each aggregate state z^j as given. The period budget constraint will always bind under the assumptions discussed in Section (3.2.5), while the no borrowing constraint will never bind.

The first order condition is given by

$$u'(\Omega^R - a^R) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)(1-\tau^k(z^j))r(z^j,k)u'((1-\tau^k(z^j))r(z^j,k)a^R + w(z^j,k)e_n) \tag{3.87}$$

Full derivations are in Appendix (C.4).

3.4.2.2 Asset Poor Consumers

The optimisation problem solved by an asset poor consumer with initial wealth Ω^P becomes:

$$\max_{a^P \in [0,\Omega^P]} u(c_1^P) + \beta E_0[u(c_2^P)] \tag{3.88}$$

subject to a no borrowing constraint

$$a^P \geq 0 \tag{3.89}$$

and subject to the period budget constraint in period one

$$c_1^P + a^P \leq \Omega^P \tag{3.90}$$

plus the period two budget constraints for every possible combination of aggregate and idiosyncratic shock realisations

$$c_2^P(z^j, e_n) \leq r(z^j, k)a^P + w(z^j, k)e_n + T(z^j)$$

where $j \in \{G, B\}$ and $n \in \{L, H\}$. Rational consumers take the level of transfers $T(z^j)$ in each aggregate state z^j as given. The period budget constraint will always bind under the assumptions discussed in Section Two, while the no borrowing constraint will never bind.

The first order condition is given by

$$u'(\Omega^P - a^P) = \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j, k) u'(r(z^j, k)a^P + w(z^j, k)e_n + T(z^j)) \tag{3.91}$$

Full derivations are in Appendix (C.4). The firms' problem is the same as described in Section (3.2.8).

Definition 14 *A **Competitive Equilibrium under Tax / Transfer Scheme Two** is a plan $T(z^j)$ for the payment of fiscal transfers contingent on the realisation of the aggregate shock z^j, for all $j \in (G, B)$, a **Decision Rule** for consumer saving $a(\Omega^i)$ (a function of initial wealth Ω^i that varies by consumer type) such that **(i)** the no borrowing constraints (3.85) and (3.89) are satisfied for all consumers; **(ii)** all consumer and firm budget constraints (3.86), (3.90) and (3.24) are satisfied with equality for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; **(iii)** the first order condition (3.87) is satisfied for all rich consumers, while first order condition (3.91) is satisfied for all asset poor consumers; **(iv)** the first order conditions (3.26) for all firms are satisfied for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; and **(v)** the tax rate $\tau^k(z^j)$ can vary with the realisation of the aggregate shock z^j, for all $j \in (G, B)$, so that the government budget is always balanced, i.e. $\tau^k(z^j)r(z^j, k)k = T(z^j)$ for all $j \in (G, B)$.*

3.4.2.3 Marginal Effect of Tax / Transfer Scheme Two on Efficiency

In competitive equilibrium, utilitarian social welfare is given by

$$SW(T,k) = (\frac{1}{2})\left\{ \begin{array}{c} u(\Omega^R - a^R) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u((1-\tau^k(z^j)) r(z^j,k) a^R + w(z^j,k) e_n) \\ + u(\Omega^P - a^P) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(z^j,k) a^P + w(z^j,k) e_n + T(z^j)) \end{array} \right\} \tag{3.92}$$

Substituting out $\tau^k(z^j)$ using $\tau^k(z^j) r(z^j,k) a(\Omega^R) = T(z^j)$ allows social welfare to be written as

$$SW(T,k) = (\frac{1}{2})\left\{ \begin{array}{c} u(\Omega^R - a^R) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u((1-\left\{\frac{T(z^j)}{r(z^j,k)a(\Omega^R)}\right\}) r(z^j,k) a^R + w(z^j,k) e_n) \\ + u(\Omega^P - a^P) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u(r(z^j,k) a^P + w(z^j,k) e_n + T(z^j)) \end{array} \right\} \tag{3.93}$$

Introducing tax / transfer scheme two in competitive equilibrium will have a *direct effect* (i.e. from redistribution) on social welfare at the margin. There will also be the *indirect (pecuniary externality) effect*, because introducing the tax / transfer scheme affects the level of average capital k in the economy at the margin. The marginal effect on utilitarian social welfare of introducing tax / transfer scheme two in a laissez faire competitive equilibrium is

$$\frac{\partial SW(T,k)}{\partial T} \mid_{\substack{comp.eq.\\ T=0}} = \frac{\partial SW}{\partial T} + \frac{\partial SW}{\partial k}\frac{\partial k}{\partial T} \tag{3.94}$$

Specifically, this is given by

$$\frac{\partial SW(T,k)}{\partial T} \mid_{\substack{comp.eq.\\ T=0}} = (\frac{1}{2})\left\{ \begin{array}{c} -\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k) a^R + w(z^j,k) e_n) \\ +\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k) a^P + w(z^j,k) e_n) \\ + \sum_{i=1}^2 \beta E\left(u'(\Omega^i, z^j, e_n)\left[k\left\{\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial T} - E(\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial T})\right\}\right]\right) \\ + \sum_{i=1}^2 \beta E\left(u'(\Omega^i, z^j, e_n)\left[l(z^j)\frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial T} - E(l(z^j)\frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial T})\right]\right) \\ + \sum_{i=1}^2 \beta \sum_j \Pr(z^j)\left(E\left[u'(\Omega^i, z^j, e_n)(e_n - l(z^j))\right] \mid_z \frac{\partial w(z^j,k)}{\partial k}\frac{\partial k}{\partial T}\right) \\ + \sum_{i=1}^2 \beta \sum_j \Pr(z^j)\left(E\left[u'(\Omega^i, z^j, e_n)\right] \mid_z (a^i - k)\frac{\partial r(z^j,k)}{\partial k}\frac{\partial k}{\partial T}\right) \end{array} \right\} \tag{3.95}$$

The first two terms in the expression represent the *direct effect* of tax / transfer scheme two on efficiency. These effects are non-zero because tax / transfer scheme two redistributes income, unlike the personalised lump sum transfers considered in Subsection (3.4.1) which rebated exactly the amount of tax paid by the consumer. Redistributing income from asset rich to asset poor consumers will itself improve utilitarian social welfare, since it effectively insures consumers against having low levels of initial wealth.

3.4.2.4 Redistribution Effect

Proposition 15 *The **Redistribution Effect** of tax / transfer scheme two is **positive** and is given by*

$$-\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j, k) a^R + w(z^j, k) e_n) \qquad (3.96)$$
$$+\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j, k) a^P + w(z^j, k) e_n)$$

Proof. Recall the assumption that $\Omega^R > \Omega^P$. It follows that $a(\Omega^R) > a(\Omega^P)$, since Proposition (1) established that saving $a(\Omega^i)$ is an increasing function of wealth, for a consumers' problem similar to the ones under tax / transfer scheme two. The two groups of terms in (3.96) differ only by the level of saving $a(\Omega^i)$. The second group of terms in (3.96) (which is positive) will be larger than the first (which is negative) because of the concavity of period utility. This reflects the greater "weight" given to the utility of asset poor consumers, by the utilitarian social welfare function. This gives the result. ■

If this Redistribution term dominates the others in (3.95), then the efficiency impact of introducing tax / transfer scheme two will be positive. It is important to identify factors that affect the magnitude of the Redistribution Effect. Intuitively, the greater is initial *wealth inequality*, the more redistributive transfers will be efficiency improving, as measured by a utilitarian social welfare function. Redistributive transfers can be seen as insurance against being allocated a *low* level of initial wealth, if this allocation is treated as random.

3.4.2.5 Pecuniary Externality Effects

The remaining terms in (3.95) represent the indirect effect of introducing tax / transfer scheme two on efficiency. They arise because the tax / transfer scheme involves a tax τ^k on the rate of return to saving for rich consumers, which affects the average level of saving k in the economy at the margin. Changing k has a pecuniary externality effect on factor prices that consumers do not take into account when making individual savings decisions.

Again, in order to study the indirect effect, it is necessary to make an assumption about the effect of the tax / transfer scheme on average saving k at the margin.

Conjecture 16 *The marginal impact on average saving k of introducing tax / transfer scheme two is negative $\frac{\partial k}{\partial T} < 0$ (noting that a positive transfer $T(z^j)$ implies a positive rate of taxation $\tau^k(z^j)$ under the assumption of a balanced government budget).*

The terms comprising the indirect effect are identical to those terms (3.77) arising from the introduction of tax / transfer scheme one. They include (**i**) the *expected income effect* (the sixth term in (3.95)) of a higher interest rate and lower wage, which raises the expected incomes of the asset rich with above average saving $a(\Omega^i) > k$, while lowering the expected incomes of the asset poor; and (**ii**) the *idiosyncratic risk effect* (the fifth term in (3.95)) of a higher interest rate and lower wage, *lowering* the share of expected total income subject to idiosyncratic risk for all consumers; and (**iii**) the aggregate risk insurance effects (the third and fourth terms in (3.95)) that describe changes in exposure to aggregate risk due to changes in the wage and interest rate, terms which continue to be exactly zero when the production function is homogeneous of degree one, as explained in Section (3.3.1). These effects were discussed in detail in Section (3.3.2).

Proposition 17 *The* ***expected income effect*** *because of introducing tax / transfer scheme two is* ***negative*** *while the* ***idiosyncratic risk effect*** *is* ***positive****.*

Proof. The negative expected income effect is given by

$$\sum_{i=1}^{2} \beta \sum_{j} \Pr(z^j) \left(E\left[u'(\Omega^i, z^j, e_n)\right] |_z \; (a^i - k) z^j \frac{\partial r(z^j, k)}{\partial k} \frac{\partial k}{\partial T} \right) \tag{3.97}$$

while the positive idiosyncratic risk effect is given by

$$\sum_{i=1}^{2} \beta \sum_{j} \Pr(z^j) \left(E\left[u'(\Omega^i, z^j, e_n)(e_n - l(z^j))\right] |_z \; z^j \frac{\partial w(z^j, k)}{\partial k} \frac{\partial k}{\partial T} \right) \tag{3.98}$$

The proof is exactly the same as that for Proposition (13), given in the discussion of tax / transfer scheme one. ■

The intuition about these effects is the same as for tax / transfer scheme one. The *expected income effect* is negative because a higher interest rate and lower wage raises the expected total income of the asset rich, conditional on any realisation of the aggregate shock, while lowering the expected total income of the asset poor. The expected total income of the average consumer (with saving k) is *unchanged* since

$$-\frac{\partial r(z^j, k)}{\partial k} k = \frac{\partial w(z^j, k)}{\partial k} E\left[e_n\right] |_z$$

as discussed earlier in Subsection (3.3.2). With the mean of the income distribution preserved, there is an increase in inequality of expected income (conditional on any realisation of the aggregate shock) that lowers utilitarian social welfare. The *idiosyncratic risk effect* is positive because the higher interest rate and lower wage reduce the share of expected income subject to idiosyncratic risk, for all consumers.

Proposition 18 *The marginal effect on social welfare (3.95) of tax / transfer scheme two will be* ***positive*** *if the positive redistribution effect (3.96) and idiosyncratic risk effect (3.98) outweigh the negative expected income effect (3.97).*

Proof. Propositions (15) and (17) show the redistribution effect and the idiosyncratic risk effect to be both positive, while the expected income effect is negative. This gives the result. ■

This suggests that if transfers redistribute income and wealth to a sufficient extent, then this can offset the distortionary effect of the tax (i.e. the negative expected income effect of a wider distribution of expected income, because of a higher interest rate and lower wage). This offsetting is possible because the utilitarian social welfare function rewards income and wealth equality, so redistribution is welfare enhancing.

3.4.2.6 The Impact of Aggregate Risk

The introduction of aggregate risk (i.e. the probability of bad aggregate shocks $z^j = z^B$) can change the relative magnitude of the *redistribution effect* (3.96), compared with the situation when $z^j = z^G$ always. Intuitively, a bad aggregate shock z^B can reduce both the interest rate and the wage. This can **lower** the income ratio of the asset poor to the asset rich when z^G falls to z^B, provided the percentage fall in the wage is at least as large as the fall in the interest rate. Since the asset poor receive the bulk of their income from labour, a larger percentage fall in the wage (compared with the interest rate) is more costly for the asset poor than for the asset rich. Redistribution through fiscal transfers is particularly beneficial for welfare in these circumstances.

An informal mathematical argument is as follows. First note that for given average per consumer saving k, the wage and interest rate should *fall* when there is a bad realisation of the aggregate shock z^B. This is the case when I solve the model numerically, later in this Subsection. The ratio of conditional expected marginal utilities between asset poor and asset rich consumers can **rise** when z^G falls to z^B (provided the percentage fall in the wage is at least as large as the fall in the interest rate)

$$\frac{E\left[u'(r(z^B,k)a(\Omega^P)\ +w(z^B,k)e_n)\right]|_z}{E\left[u'(r(z^B,k)a(\Omega^R)\ +w(z^B,k)e_n)\right]|_z} > \frac{E\left[u'(r(z^G,k)a(\Omega^P)\ +w(z^G,k)e_n)\right]|_z}{E\left[u'(r(z^G,k)a(\Omega^R)\ +w(z^G,k)e_n)\right]|_z}$$

noting that $a(\Omega^R) > a(\Omega^P)$ and holding the level of average capital k constant. Because of this, the terms of the redistribution effect (3.96) under a bad aggregate shock z^B

$$\begin{aligned}&-\beta\sum_n \Pr(e_n \mid z^B)u'(r(z^B,k)a^R + w(z^B,k)e_n)\\&+\beta\sum_n \Pr(e_n \mid z^B)u'(r(z^B,k)a^P + w(z^B,k)e_n)\end{aligned}$$

are larger in absolute value compared with those corresponding to the case of z^G, again holding k constant. This gives some illustration of how the possibility of bad aggregate shocks can magnify the redistribution effect on social welfare (3.96) of introducing tax / transfer scheme two.

This has implications for the marginal effect (3.95) of tax / transfer scheme two on social welfare. If the redistribution effect of tax / transfer scheme two (3.96) dominates the other effects in (3.95), then the introduction of aggregate risk (i.e. $\Pr(z^j = z^B) \neq 0$ compared with the situation when $z^j = z^G$ always) can magnify the positive marginal effect of tax / transfer scheme two on utilitarian social welfare, holding all else equal.

3.4.2.7 Cyclicality of Redistributive Transfers

In the presence of aggregate risk, it is possible to compare the effect on savings behaviour of transfers that vary pro-cyclically, a-cyclically and counter-cyclically. Transfers that are a-cyclical or counter-cyclical provide transfer recipients with some insurance against aggregate shocks. This may reduce the amount of precautionary savings undertaken by asset poor transfer recipients. One way to examine this issue is to solve for the optimal level of the redistributive transfers within the class considered in tax / transfer scheme two, by formulating a Ramsey problem. Specifically, I solve for the optimal level of the transfers $T(z^j)$ for all realisations of the aggregate shock, which will suggest how the transfers should vary with the aggregate shock. I also compute the Ramsey optimal level of average per consumer saving k and compare it to the constrained efficient level in the model without fiscal policy, solved for in Section (3.3).

The Ramsey planner chooses transfers $T(z^j)$ and saving rules for asset rich and asset poor consumers ($a(\Omega^R)$ and $a(\Omega^P)$) to maximise utilitarian social welfare

$$\max_{T(z^j),\{a(\Omega^i)\}_{i=1}^2} SW = (\frac{1}{2})\left\{\begin{array}{c} u(\Omega^R - a^R)+ \\ \beta Eu((1-\left\{\frac{T(z^j)}{r(z^j,k)a(\Omega^R)}\right\})r(z^j,k)a^R + w(z^j,k)e_n) \\ +u(\Omega^P - a^P) \\ +\beta Eu(r(z^j,k)a^P + w(z^j,k)e_n + T(z^j)) \end{array}\right\} \tag{3.99}$$

subject to satisfying no borrowing constraints for all consumers

$$a(\Omega^i) \geq 0 \tag{3.100}$$

and subject to satisfying the first order condition of the consumer's problem for both asset rich consumers

$$u'(\Omega^R - a^R) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)(1-\tau^k(z^j))r(z^j,k)u'((1-\tau^k(z^j))r(z^j,k)a^R + w(z^j,k)e_n) \tag{3.101}$$

and asset poor consumers

$$u'(\Omega^P - a^P) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)r(z^j,k)u'(r(z^j,k)a^P + w(z^j,k)e_n + T(z^j)) \tag{3.102}$$

as well as all period one budget constraints

$$c_1(\Omega^i) = \Omega^i - a(\Omega^i) \tag{3.103}$$

and period two constrains

$$c_2(\Omega^R, z^j, e_n) = (1 - \tau^k(z^j))r(z^j, k)a(\Omega^R) + w(z^j, k)e_n$$

for rich consumers and

$$c_2(\Omega^P, z^j, e_n) = r(z^j, k)a(\Omega^P) + w(z^j, k)e_n + T(z^j)$$

for poor consumers, for all $j \in (G, B)$ and $n \in (L, H)$, while allowing factor prices to be set in competitive markets so that

$$r(z^j, k) = z^j f_k^{'}(k, l(z^j)) \tag{3.104}$$

$$w(z^j, k) = z^j f_l^{'}(k, l(z^j))$$

and taking into account the effect of individual consumer saving $a(\Omega^i)$ on aggregate quantities

$$k = (\frac{1}{I}) \sum_i a(\Omega^i) \tag{3.105}$$

$$l(z^j) = \Pr(e_L \mid z^j)e_L + \Pr(e_H \mid z^j)e_H$$

with budget balance implying values for the tax rate

$$\tau^k(z^j)r(z^j, k)a(\Omega^R) = T(z^j) \tag{3.106}$$

Full derivations of the first order conditions are presented in the Appendix (C.4).

Definition 15 *A Ramsey Equilibrium under tax / transfer scheme two is a level for transfers $T(z^j)$ for each realisation of the aggregate shock z^j, $j \in (G, B)$ and decision rules for saving for each consumer type (i.e. $a(\Omega^R)$ and $a(\Omega^P)$) such that utilitarian social welfare (3.99) is maximised subject to **(i)** the no borrowing constrains (3.100) being satisfied for all consumers; **(ii)** all consumer and firm budget constraints (3.103) and (3.24) being satisfied with equality for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; **(iii)** the first order conditions (Euler equations) (3.101) and (3.102) being satisfied for all individual consumers; **(iv)** the first order conditions (3.26) for all firms being satisfied for all possible combinations of aggregate and idiosyncratic shocks z^j and e_n, where $j \in (G, B)$ and $n \in (L, H)$; and **(v)** the tax rate $\tau^k(z^j)$ for all realisations of the aggregate shock z^j, $j \in (G, B)$, being given by (3.106).*

I solve the Ramsey problem using numerical methods. I also solve for a competitive equilibrium under tax / transfer scheme two (as defined earlier) and compute the marginal effect on social welfare of introducing the tax / transfer scheme. Unlike tax / transfer scheme one, this tax / transfer scheme changes the rate of return on saving *and* also redistributes income. Redistributing income may generate a sufficiently large welfare improvement that it justifies a positive rate of taxation on saving return, even if this drives consumer saving choices further away from the constrained efficient choices in a laissez faire economy without fiscal policy.

3.4.2.8 Numerical Methods - Tax / Transfer Scheme Two

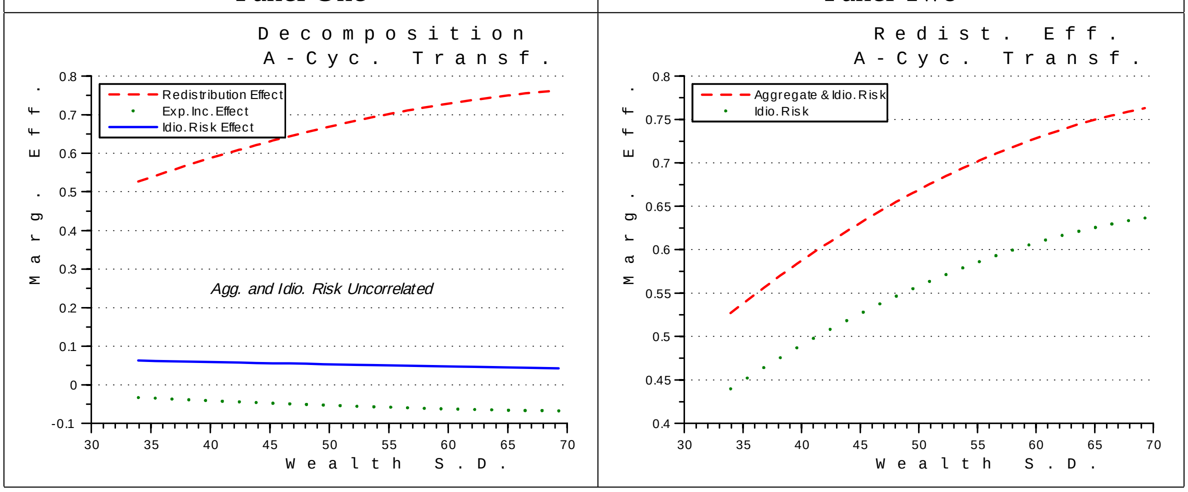
I use the same functional forms and parameter values as described in Section (3.3.4), in order to obtain numerical solutions. Again, the standard deviation of the idiosyncratic shock is approximately sixty per cent of its mean, while the standard deviation of the aggregate shock is approximately ten per cent of its mean.

3.4.2.9 Marginal Welfare Effect of Tax / Transfer Scheme Two

Panel One of Figure (3.4.2.9) demonstrates that the positive redistribution effect of introducing tax / transfer scheme two (together with the positive idiosyncratic insurance effect of a lower wage and higher interest rate) *dominate* the negative expected income effect (arising when a higher interest rate and lower wage increase the expected income of the asset rich and reduce that of the asset poor, but preserve the mean (conditional on any realisation of the aggregate shock)). The redistribution effect increases as the variance of the distribution of initial wealth $Var(\Omega^i)$ increases (shown on the horizontal axis), as does its dominance over the other effects.

Figure (3.4.2.9)

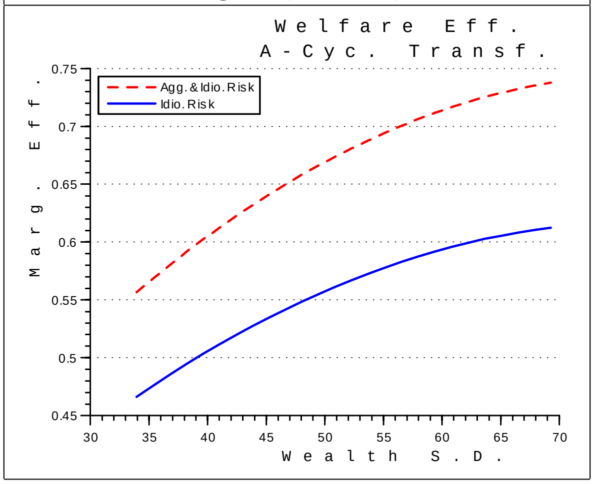
Panel One | Panel Two



3.4.2.10 Impact of Aggregate Risk

Consistent with my theoretical analysis, the redistribution effect is larger in the presence of aggregate risk, all else equal, compared with a model containing only idiosyncratic risk. Panel Two of Figure (3.4.2.9) above shows this. As expected, the marginal effect on efficiency of introducing tax / transfer scheme two is also greater in the presence of aggregate risk, because the introduction of aggregate risk magnifies the redistribution effect: see Figure (3.4.2.10).

Figure (3.4.2.10)

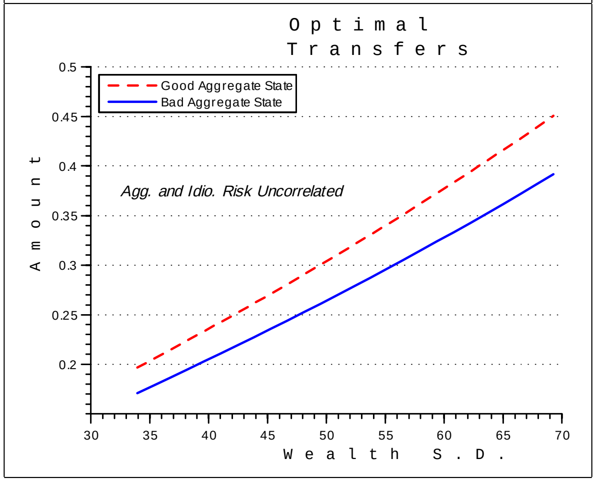


Introducing tax / transfer scheme two can be welfare improving, when the variance of the distribution of initial wealth $Var(\Omega^i)$ is relatively large. The efficiency benefit produced by the redistributive transfers outweighs the distortion of a positive tax on the return to saving in these circumstances, thus justifying a positive rate of capital income taxation. The presence of aggregate risk magnifies this effect.

3.4.2.11 Optimal Cyclicality of Transfers

Ramsey optimal transfers are found to be always positive when the variance of the distribution of initial wealth $Var(\Omega^i)$ is relatively large. Under the balanced budget assumption, this implies a positive rate of tax on savings return, consistent with the reasoning just outlined. Numerical solutions to the Ramsey problem indicate that the transfers should be mildly *pro-cyclical:* see Figure (3.4.2.11). Intuitively, a-cyclical or counter-cyclical transfers may reduce the amount of precautionary saving by asset poor consumers and this may not be beneficial for social welfare. Making the transfers pro-cyclical means that they will not act as insurance against bad aggregate shocks (to the wage and interest rate), so they should not reduce precautionary saving to the same extent.

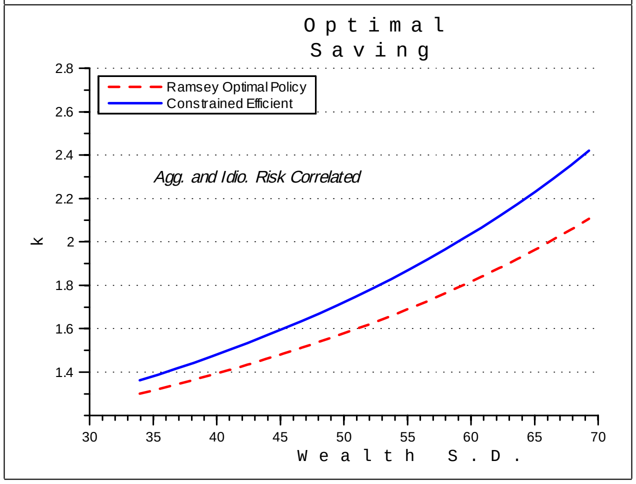
Figure (3.4.2.11)



3.4.2.12 Saving: Ramsey Optimal vs Constrained Efficient

Figure (3.4.2.12) shows that the Ramsey optimal policy of a positive tax rate (and transfers) can lead to a level of average saving that is *below* the constrained efficient level in a laissez faire economy without fiscal policy. This is particularly the case at relatively high levels of initial wealth inequality. This is an interesting result. The efficiency benefit of redistributive fiscal transfers seems to outweigh the distortion of a positive tax on the rate of return to saving. This is Ramsey optimal, despite the fact that it lowers the level of average saving k in the economy to below the constrained efficient level in an economy without fiscal policy.

Figure (3.4.2.12)



3.5 Conclusion

3.5.0.13 Constrained Inefficiency

In this chapter, the constrained inefficiency of competitive equilibria in the presence of incomplete markets for aggregate and idiosyncratic risk has been demonstrated. Constrained inefficiency arises because of the pecuniary externality of savings decisions on factor prices. This externality is not taken into account in individual decision making and this becomes problematic in a model with incomplete insurance markets for risk.

At the *margin*, an increase in aggregate saving raises the wage and lowers the interest rate, conditional on any realisation of the aggregate shock. This has two competing effects. First, this increases the *share* of expected total income derived from labour income and subject to idiosyncratic shocks, which has an efficiency cost: the *idiosyncratic risk effect*. Second, the rise in the wage and the fall in the interest rate *increases* expected income for those with below average saving $a(\Omega^i) < k$ (i.e. the asset poor), while income of the asset rich is *reduced*. The expected income of consumers with average saving k is preserved. This reduces inequality of expected income (conditional on any realisation of the aggregate shock) and is efficiency improving: the *expected income effect*.

When initial wealth inequality is high, the positive expected income effect dominates the negative idiosyncratic risk effect. A marginal increase in average saving raises utilitarian social welfare evaluated in competitive equilibrium:- hence there is constrained inefficiency of competitive equilibrium due to under saving. I solve the social planner's problem for the constrained efficient level of average saving and find that it exceeds the competitive equilibrium level when initial wealth inequality is large. The constrained efficient outcome can be used as a benchmark to judge fiscal policies aimed at improving welfare in the heterogeneous agent framework.

3.5.0.14 Interaction Between Aggregate and Idiosyncratic Risk

I considered two different ways in which aggregate shocks can change the conditional distribution of the idiosyncratic shock (that is, conditional on a realisation of the aggregate shock). In both cases, the *unconditional* distribution of the idiosyncratic shock is approximately the same as when aggregate and idiosyncratic risk are fully independent.

In the first case, a bad aggregate shock makes a bad idiosyncratic shock more likely, as may happen in recessions (and vice versa for a good aggregate shock). The aggregate shock affects the *skewness* of the idiosyncratic shock's conditional distribution. In this case, there is constrained inefficiency due to *under* saving over a wider range of wealth inequality levels, compared with when aggregate and idiosyncratic risk are independent. The correlation of the direction of aggregate and idiosyncratic shocks means that the extent of truly idiosyncratic risk is lower. This lowers the utility cost of scaling up the share of expected income subject to idiosyncratic risk (which occurs when an increase in k raises the wage and lowers the interest rate). The negative *idiosyncratic risk effect* of higher k is lower in absolute value.

In the second case, a bad aggregate shock makes the conditional variance of the idiosyncratic shock relatively *large*, while a good aggregate shock makes it relatively small. This may be also a feature of recessions and booms. In this case, the utility cost of scaling up the share of expected income subject to idiosyncratic risk is large (which occurs when higher k raises the wage and lowers the interest rate). This is because of the larger conditional variance of the idiosyncratic shock under a bad aggregate shock. This situation is given the highest weight by the utilitarian social welfare function. The result is that lower k improves welfare - i.e. there is constrained inefficiency due to *over* saving for a wider range of initial wealth inequality levels, compared with the case where aggregate and idiosyncratic risk are independent.

The key point is that in both cases, the *unconditional* distribution of the idiosyncratic shock is approximately the same as when aggregate and idiosyncratic risk are fully independent. This demonstrates the importance of studying constrained efficiency in an environment with *both* aggregate and idiosyncratic risk, rather than with only idiosyncratic risk.

3.5.0.15 The Substitution Effect of a Tax on the Return to Saving

Efficiency improvements should be possible if consumers can be induced to save differently. This follows because constrained inefficiency is caused by the pecuniary externality of saving on factor prices. Tax / transfer scheme one allows the pure substitution effect of a tax on the return to capital to be considered. This is because the amount of tax collected from each consumer is exactly rebated using personalised lump sum transfers, eliminating income or redistribution effects. At the margin, introducing the tax reduces capital accumulation. This harms efficiency when there is constrained inefficiency of competitive equilibrium due to *under* saving, as can be the case when bad aggregate shocks make bad idiosyncratic shocks more likely. The Ramsey optimal level of tax is negative in this situation (i.e. a subsidy should be paid to the return on capital). By contrast, a positive tax rate is welfare improving in the case of constrained inefficiency due to *over* saving. An example of when over saving occurs can be when the aggregate shock affects the conditional variance of the idiosyncratic shock. The Ramsey optimal level of taxation will be positive in this case.

The Ramsey optimal rate of taxation generates a level of average per consumer saving that is reasonably close to the constrained efficient level, although not equal to it. It is possible to generate an outcome close to the constrained efficient benchmark because this form of taxation changes the return on saving to reflect the pecuniary externalities discussed above, without redistributing income in any way.

3.5.0.16 Redistributive Transfers

The picture changes when a tax on the return to saving is accompanied by transfers that redistribute income from the asset rich to the asset poor (as in tax / transfer scheme two).

Redistributive transfers are efficiency improving as judged by a utilitarian social welfare function, effectively insuring consumers against low realisations of initial wealth. The positive efficiency effect of redistribution is magnified by the introduction of aggregate risk (i.e. the possibility of bad aggregate shocks). This is because bad aggregate shocks lower the interest rate and the wage (all else equal) and can lower the income ratio of the asset poor to the asset rich (provided the percentage fall in the wage is at least as large as that in the interest rate). The utility benefit of redistributive transfers is larger in the presence of this risk.

If the amount of redistribution is sufficiently large, the efficiency benefit of the transfers outweighs the efficiency cost of the tax on saving. In these circumstances, there can be an efficiency improvement obtained by taxing the return to saving (at least of asset rich consumers) in order to fund redistributive transfers. The consequence of the tax distortion is that the Ramsey optimal level of average per consumer saving is below the constrained efficient level in a laissez faire economy.

3.5.0.17 Pro-Cyclical Transfers

Within this class of tax / transfer schemes, Ramsey optimal policy suggests that the level of the transfer payments should not vary counter-cyclically (with the realisation of the aggregate shock). Counter-cyclical transfers and also a-cyclical transfers provide transfer recipients with partial insurance against bad realisations of the aggregate shock (which reduce income by lowering interest rates and wages). Partial insurance against bad aggregate shocks may reduce the incentive to engage in precautionary saving. Numerical solutions for Ramsey optimal policy suggest that transfers should be mildly *pro-cyclical*. This suggests that a-cyclical or counter-cyclical transfer payments would have an efficiency cost in these circumstances, perhaps because of their effects on precautionary saving.

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Appendix A for Chapter (1)

In this Appendix, I provide full derivations of the consumers problem, firms' problem and the conditions describing the Ramsey Economy.

A.1 Consumers' Problem

The first order conditions of the representative consumer's dynamic optimisation problem will be sufficient to characterise a unique interior maximum. There are two reasons for this. First, the consumer's objective function (1.5) is strictly quasi-concave. This is because it is time separable and period utility $u(c) + v(1-h)$ is assumed to be strictly quasi-concave. Second, the opportunity set of consumers is convex. The opportunity set is defined by consumer budget inequality constraint (1.7), the labour endowment inequality constraint (1.8) and non-negativity conditions (1.9), (1.10) and (1.11) *every* period. For the opportunity set to be convex, note that arbitrary upper bounds on public and private capital stocks must be assumed. The inequality constraints always bind for the reasons explained in section (1.2.2), while the nonnegativity constraints never bind.

The first order conditions of the consumer's problem are obtained by using the method of Lagrange multipliers. The consumer chooses a sequence $\{c_t, h_t^s, K_t^s, b_t^d\}_{t=0}^{\infty}$ to maximise the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} u(c_t) + v(1-h_t^s) \\ -\lambda_t \left[\begin{array}{c} c_t + p_t^b b_t^d + K_t^S - (1-\delta^K) K_{t-1}^S \\ -(1-\tau_t^h) w_t h_t^s - (1-\tau_{t-1}^K) r_t K_{t-1}^S - b_{t-1}^d \end{array} \right] \end{array} \right\} \tag{A.1}$$

where λ_t is the Lagrange multiplier attached to the representative consumer's period budget constraint at time t. The first order conditions of the consumer's problem are obtained when the first order partial derivatives of the Lagrangian vanish each period:

$$\frac{\partial L}{\partial c_t} = u'(c_t) - \lambda_t = 0 \tag{A.2}$$

$$\frac{\partial L}{\partial h_t^s} = -v'(1 - h_t^s) + \lambda_t(1 - \tau_t^h)w_t = 0 \tag{A.3}$$

$$\frac{\partial L}{\partial K_t^s} = -\lambda_t + \beta\lambda_{t+1}\left[(1 - \tau_t^K)r_{t+1} + (1 - \delta^K)\right] = 0 \tag{A.4}$$

$$\frac{\partial L}{\partial b_t^d} = -\lambda_t p_t^b + \beta\lambda_{t+1} = 0 \tag{A.5}$$

Eliminating the Lagrange multiplier λ_t gives the consumer optimality conditions (1.13), (1.14) and (1.15).

A.2 Firms' Problem

The representative, perfectly competitive firm produces a homogenous good using the economy's single production technology (1.1). The firm maximises profit. However, the profit maximising decisions of firms are not dynamic, because the firm rents private capital services and labour services from consumers each period. The representative firm can be thought of as solving a static profit maximisation problem each period

$$\max_{Y_t, K_{t-1}^d, h_t^d} P_t Y_t - w_t h_t^d - r_t K_{t-1}^d$$

subject to

$$Y_t \leq F(G_{t-1}, K_{t-1}^d, h_t^d)$$

where P_t is the price of output and Y_t is the quantity of the homogenous good the firm produces and sells each period. The output price is normalised to one $P_t \equiv 1$. The inequality constraint (1.17) must always bind, else factor inputs would not be fully utilised. Substituting for Y_t, the static profit maximisation problem reduces to a classic programming problem

$$\max_{K_{t-1},h_t} F(G_{t-1}, K^d_{t-1}, h^d_t) - w_t h^d_t - r_t K^d_{t-1}$$

The first order conditions are sufficient for a unique interior maximum. This is because the representative firm is maximising a strictly quasi-concave function over a convex set. The production technology (1.1) is strictly quasi-concave by the assumptions made in section (1.2.1). The firm's objective function (1.18) is strictly quasi-concave because it is a linear combination of the period production technology. To see that the opportunity set is convex, recall that labour supply is bounded between zero and one, as shown by (1.8). The public capital stock is bounded below by zero and the private capital stock available for rental by the representative firm is also bounded below by zero. For convexity of the opportunity set, it is necessary to assume that the public capital stock and private capital stocks are bounded above by some arbitrary constants. The conditions characterising the profit maximising quantities of labour h^d_t and private capital K^d_{t-1} demanded by the firm each period are given by

$$\frac{\partial}{\partial h^d_t} \mid_{h^d_t, K^d_{t-1}} = F_h(G_{t-1}, K^d_{t-1}, h^d_t) - w_t = 0 \tag{A.6}$$

$$\frac{\partial}{\partial K^d_{t-1}} \mid_{h^d_t, K^d_{t-1}} = F_K(G_{t-1}, K^d_{t-1}, h^d_t) - r_t = 0 \tag{A.7}$$

A.3 Ramsey Problem

The conditions determining when the first order partial derivatives of the Lagrangian (1.37) for the Ramsey problem *vanish* each period are as follows:

$$
\begin{aligned}
\frac{\partial L}{\partial c_t} &= u'(c_t) - \mu^E_{t-1}u''(c_t) \qquad \text{(A.8)}\\
&-\mu^K_{t-1}u''(c_t)\left[(1-\tau^K_{t-1})F_K(G_{t-1},K_{t-1},h_t)+(1-\delta^K)\right]\\
&+\mu^K_t u''(c_t)\\
&-\frac{\mu^E_t}{b_t}\left[u''(c_t)c_t+u'(c_t)+u''(c_t)\left(K_t-(1-\delta^K)K_{t-1}\right)\right]\\
&+\frac{\mu^E_t u''(c_t)}{b_t}\left[(1-\tau^K_{t-1})F_K(G_{t-1},K_{t-1},h_t)K_{t-1}+\frac{b_{t-1}}{b_t}\right]\\
-\mu^A_t &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial h_t} &= -v'(1-h_t) \qquad \text{(A.9)}\\
&-\mu^K_{t-1}u'(c_t)(1-\tau^K_{t-1})F_{Kh}(G_{t-1},K_{t-1},h_t)\\
&+\frac{\mu^E_t}{b_t}\left[\begin{array}{c}-v''(1-h_t)+v'(1-h_t)\\ +(1-\tau^K_{t-1})F_{Kh}(G_{t-1},K_{t-1},h_t)K_{t-1}u'(c_t)\end{array}\right]\\
&+\mu^A_t F_h(G_{t-1},K_{t-1},h_t)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial K_t} &= -\frac{\mu^E_t u'(c_t)}{b_t} - \mu^A_t \qquad \text{(A.10)}\\
&-\beta\mu^K_t u'(c_{t+1})(1-\tau^K_t)F_{KK}(G_t,K_t,h_{t+1})\\
&+\frac{\beta\mu^E_{t+1}u'(c_{t+1})(1-\delta^K)}{b_{t+1}}\\
&+\frac{\beta\mu^E_{t+1}u'(c_{t+1})K_tF_{KK}(G_t,K_t,h_{t+1})(1-\tau^K_t)}{b_{t+1}}\\
&+\beta\mu^A_{t+1}\left[F_K(G_t,K_t,h_{t+1})+(1-\delta^K)\right]\\
&= 0
\end{aligned}
$$

$$\frac{\partial L}{\partial G_t} = -\mu_t^A - \beta\mu_t^K u'(c_{t+1})(1-\tau_t^K)F_{KG}(G_t, K_t, h_{t+1}) + \frac{\beta\mu_{t+1}^E(1-\tau_t^K)u'(c_{t+1})K_t F_{KG}(G_t, K_t, h_{t+1})}{b_{t+1}} + \beta\mu_{t+1}^A\left[F_G(G_t, K_t, h_{t+1}) + (1-\delta^G)\right] = 0 \tag{A.11}$$

$$\frac{\partial L}{\partial \tau_t^K} = \beta\mu_t^K u'(c_{t+1})F_K(G_t, K_t, h_{t+1}) - \frac{\beta\mu_{t+1}^E K_t u'(c_{t+1})K_t F_K(G_t, K_t, h_{t+1})}{b_{t+1}} = 0 \tag{A.12}$$

The additional equations which must hold every period in the Ramsey Economy are as follows. First, the Implementability Condition (1.34) derived from the consumer budget constraint:

$$\frac{c_t u'(c_t)}{b_t} + \beta u'(c_{t+1}) + \frac{u'(c_t)}{b_t}(K_t - (1-\delta^K)K_{t-1}) = \frac{v'(1-h_t)}{b_t}h_t + \frac{u'(c_t)(1-\tau_{t-1}^K)F_K(G_{t-1}, K_{t-1}, h_t)K_{t-1}}{b_t} + \frac{u'(c_t)b_{t-1}}{b_t}$$

Second, the Implementability Condition (1.35) derived from the Euler equation for private capital:

$$u'(c_t) = \beta u'(c_{t+1})\left[(1-\tau_t^k)F_K(G_t, K_t, h_{t+1}) + (1-\delta^K)\right]$$

The Aggregate Resource Constraint (1.24) must hold with equality every period in the Ramsey Economy:

$$c_t + K_t - (1-\delta^K)K_{t-1} + G_t - (1-\delta^G)G_{t-1} = F(G_{t-1}, K_{t-1}, h_t)$$

The Ramsey Planner will satisfy the Transversality conditions (1.38), (1.28) and (1.29) in the Ramsey Economy:

$$\lim_{j\to\infty}\beta^j u'(c_{t+j})G_{t+j} = 0$$

$$\lim_{j\to\infty} \beta^j u'(c_{t+j}) \left[(1-\tau^k_{t+j-1})r_{t+j} + (1-\delta^K)\right] K_{t+j} = 0$$

$$\lim_{j\to\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} b_{t+j} = \lim_{j\to\infty} \left(\prod_{s=0}^{j-1} \left[p^b_{t+s}\right]\right) b_{t+j} = 0$$

Finally, given initial condition b_{-1}, government borrowing in the Ramsey Economy follows the exogenous path given by (1.30):

$$b_t - \bar{b} = \rho \left(b_{t-1} - \bar{b}\right)$$

A.4 Robustness: Model With Only Public Capital

This Appendix presents numerical results for the model with *only* public capital, under (i) a more productive public capital stock; and (ii) a lower Frisch elasticity of labour supply with respect to the real wage.

A.4.1 A More Productive Public Capital Stock

The following figures present the deterministic transition path in the model with *only* public capital, as the Ramsey planner reduces government debt at a relatively slow pace (i.e. debt takes around ten years to fall from 60 per cent of GDP to 45 per cent of GDP). The parameter ρ determining the speed of government debt reduction is set to 0.95. All parameters are the same as in the benchmark case set out in Subsection (1.3.2), *except* for the elasticity θ of output to public capital, which is set to 0.1, rather than 0.05.

Figure A.4.1

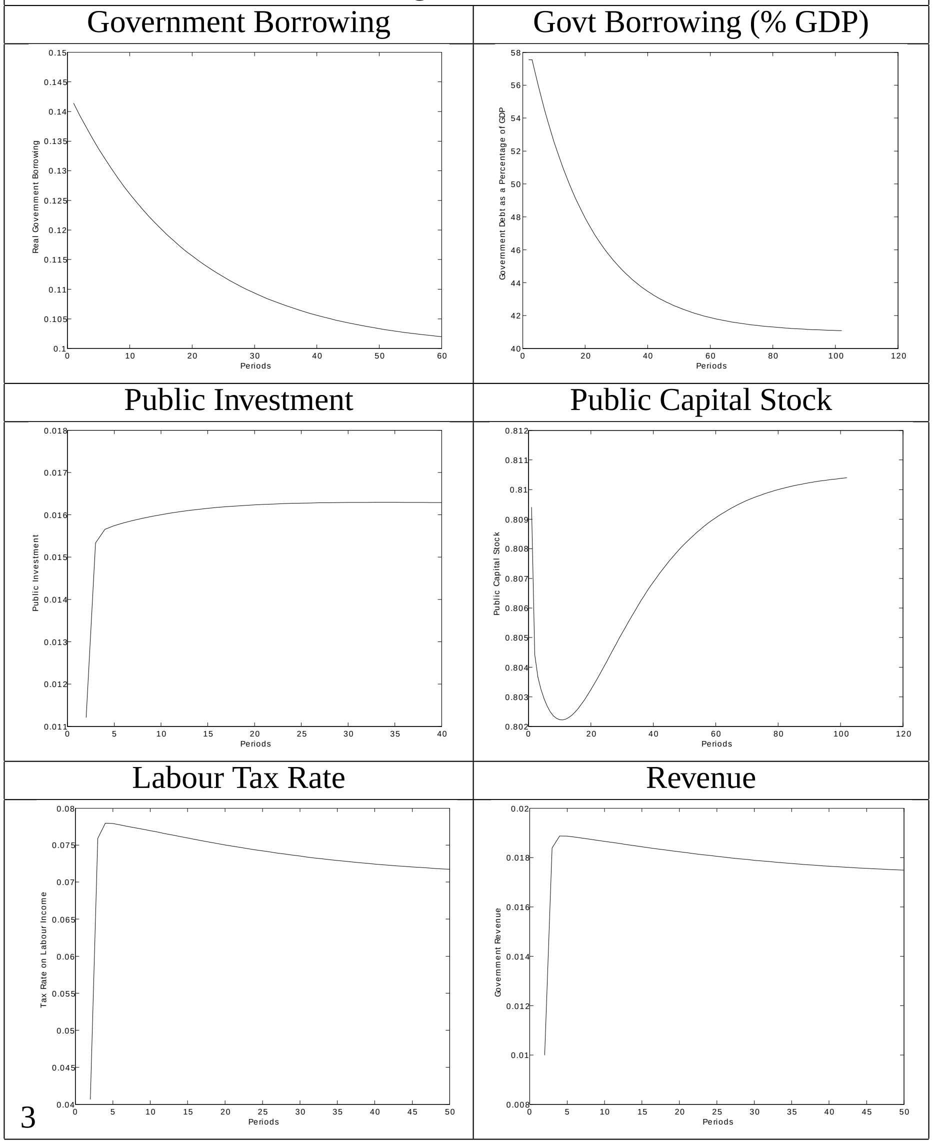
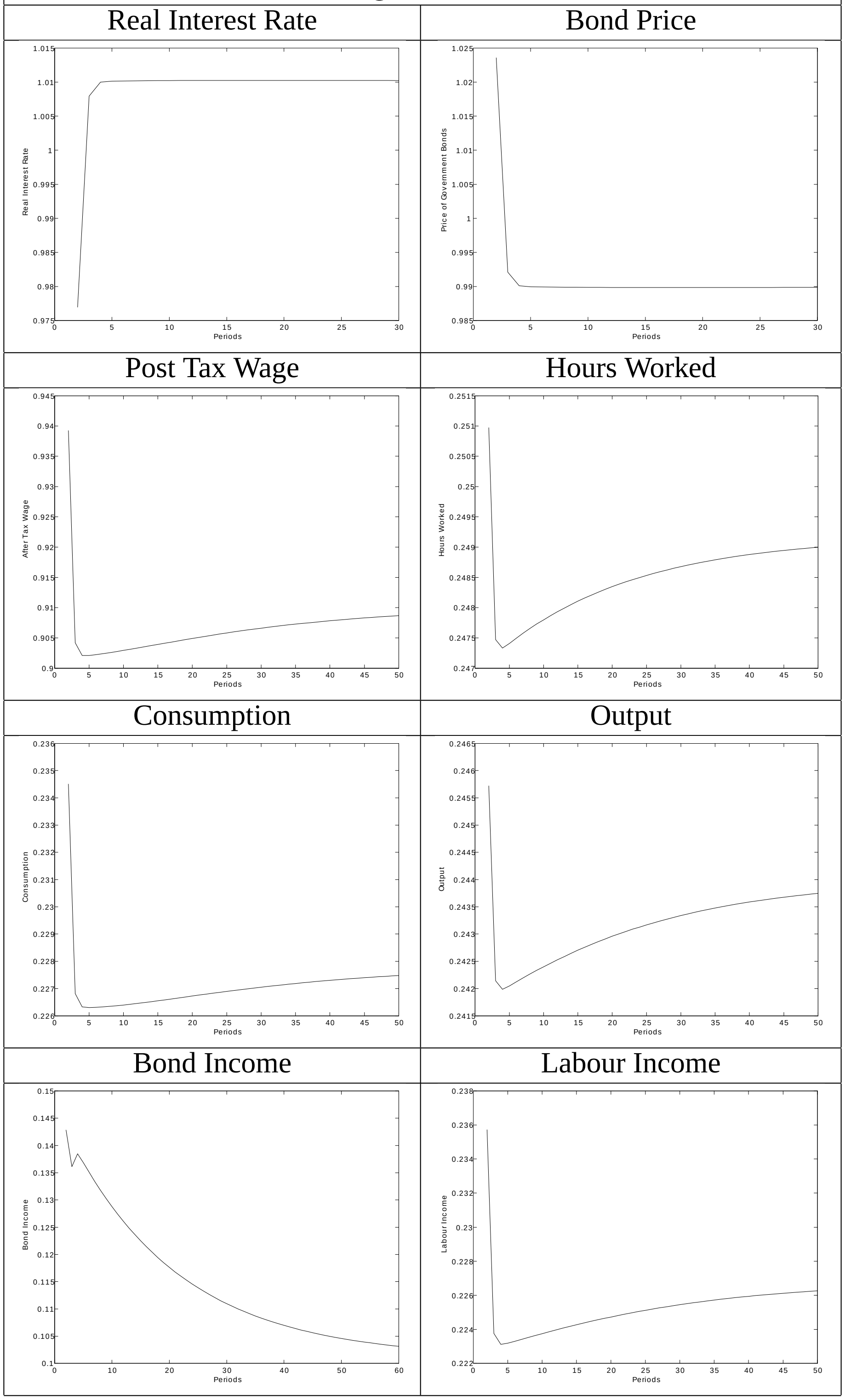

Government Borrowing
Govt Borrowing (% GDP)
Real Government Borrowing
Government Debt as a Percentage of GDP
Periods
Public Investment
Public Capital Stock
Public Investment
Public Capital Stock
Periods
Labour Tax Rate
Revenue
Tax Rate on Labour Income
Government Revenue
Periods
3

Figure A.4.1



A.4.2 Lower Frisch Elasticity of Labour Supply

The following figures present the deterministic transition path in the model with *only* public capital, as the Ramsey planner reduces government debt at a relatively slow pace (i.e. debt takes around ten years to fall from 45 per cent of GDP to 32 per cent of GDP). The parameter ρ determining the speed of government debt reduction is set to 0.95. All parameters are the same as in the benchmark case set out in Subsection (1.3.2), *except* for the parameter κ in the utility function which is set to two. This generates a Frisch elasticity of labour supply with respect to the real wage of *two* in the ultimate steady state reached by the Ramsey economy.

Figure A.4.2

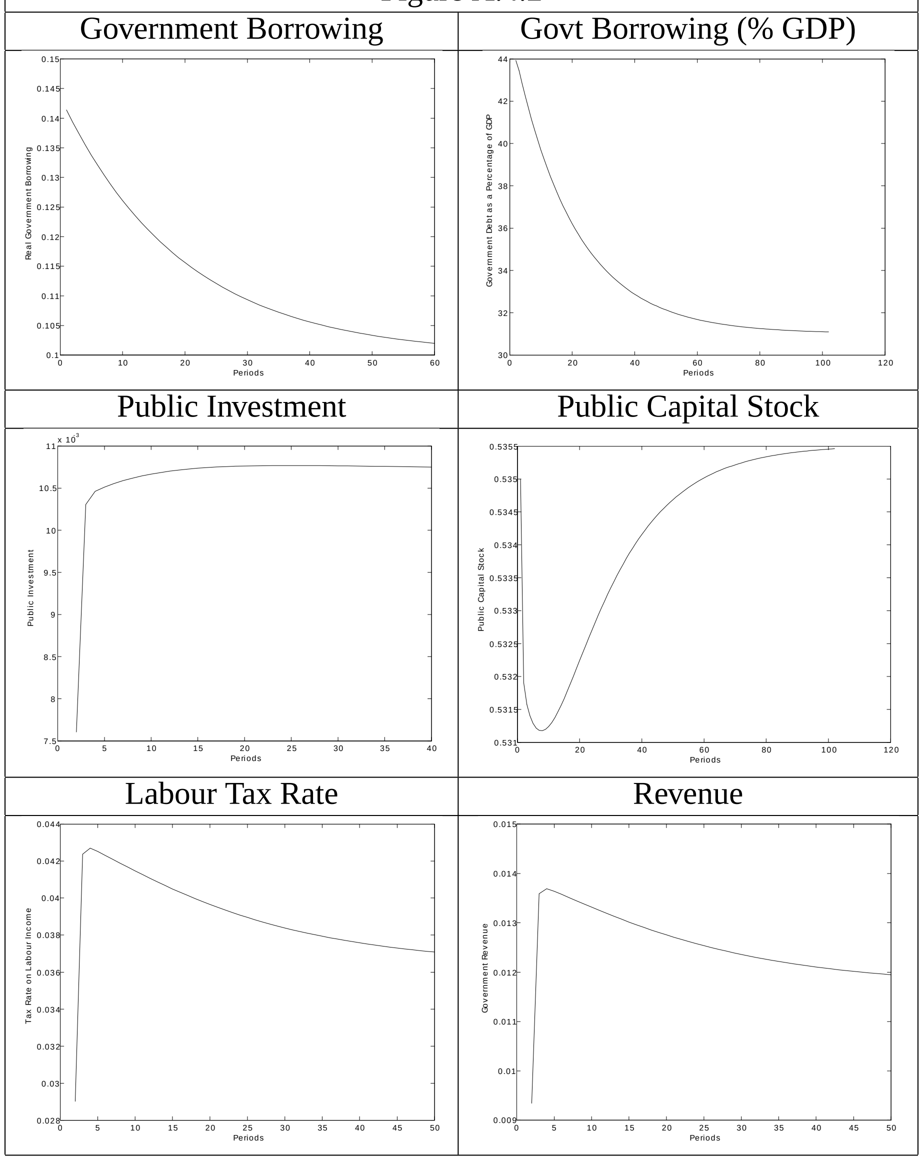
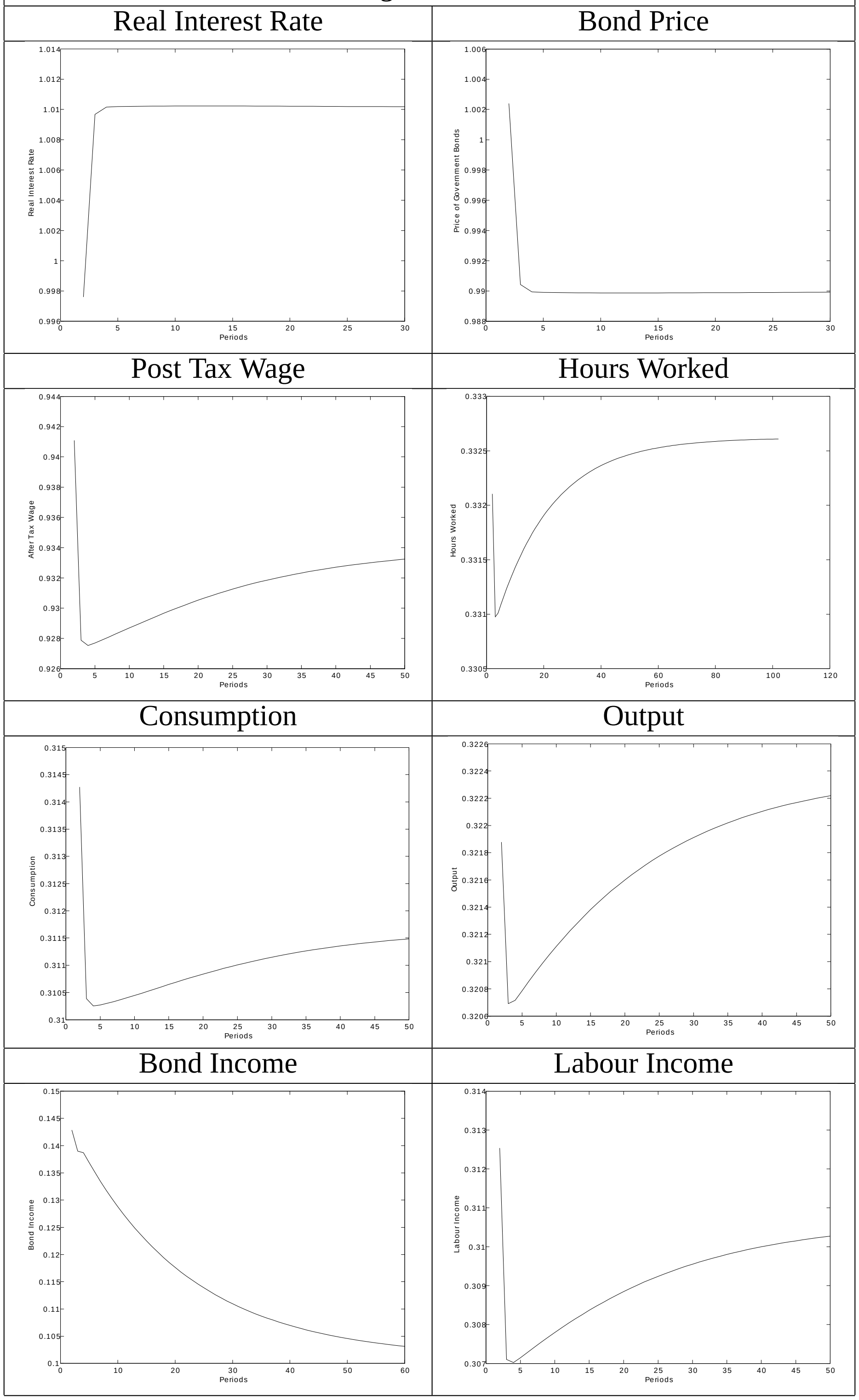


Figure A.4.2


Real Interest Rate
Real Interest Rate
Periods
Bond Price
Price of Government Bonds
Periods
Post Tax Wage
After Tax Wage
Periods
Hours Worked
Hours Worked
Periods
Consumption
Consumption
Periods
Output
Output
Periods
Bond Income
Bond Income
Periods
Labour Income
Labour Income
Periods

A.5 Robustness: Model With Private And Public Capital

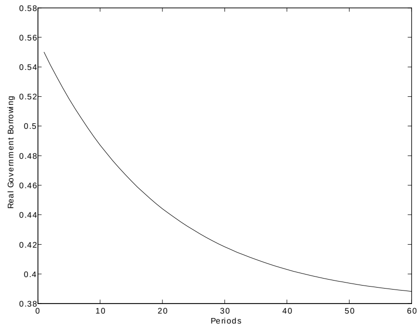
This Appendix presents numerical results for the model with *both* private *and* public capital, under (i) a more productive public capital stock; and (ii) a lower Frisch elasticity of labour supply with respect to the real wage.

A.5.1 A More Productive Public Capital Stock

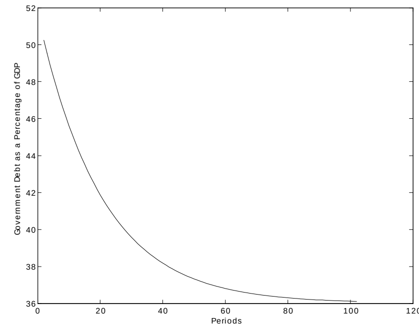
The following figures present the deterministic transition path in the model with *both* private *and* public capital, as the Ramsey planner reduces government debt at a relatively slow pace (i.e. debt takes around ten years to fall from 50 per cent of GDP to 38 per cent of GDP). The parameter ρ determining the speed of government debt reduction is set to 0.95. All parameters are the same as in the benchmark case set out in Subsection (1.3.2), *except* for the elasticity θ of output to public capital, which is set to 0.1, rather than 0.05.

Figure A.5.1

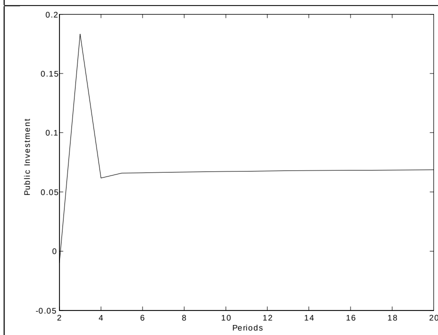
Government Borrowing



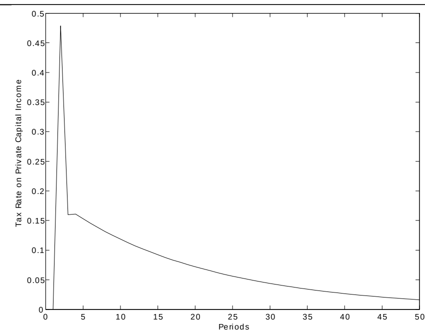
Govt Borrowing (% GDP)



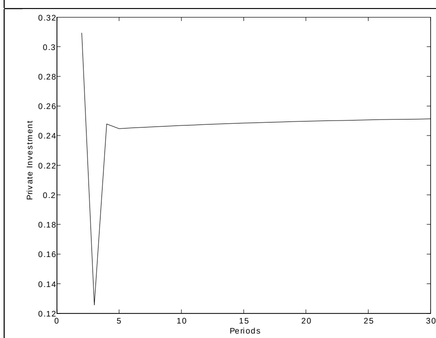
Public Investment



Capital Tax Rate



Private Investment



Private Capital Stock

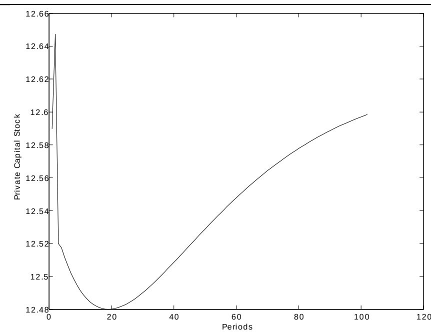
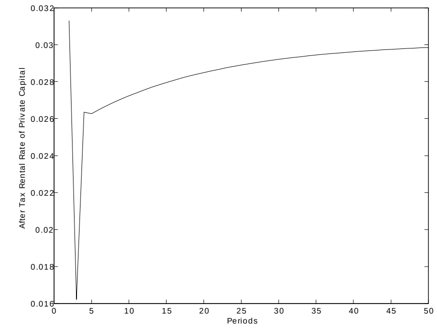
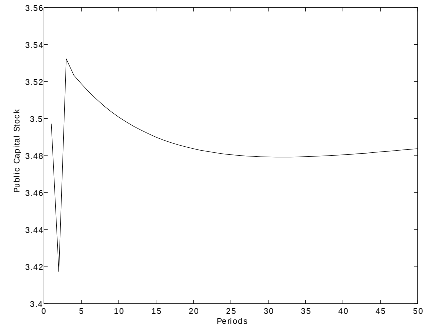


Figure A.5.1

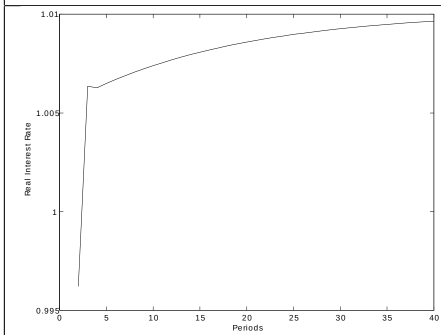
Post Tax Rental Rate



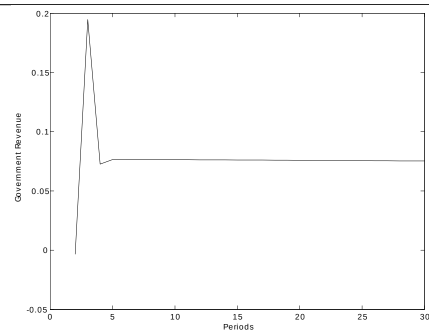
Public Capital Stock



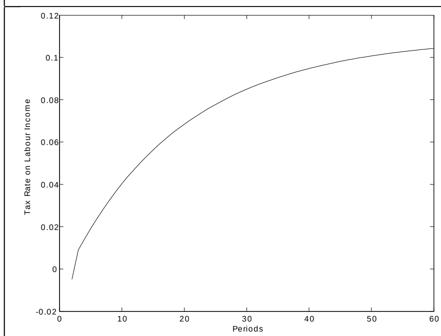
Real Interest Rate



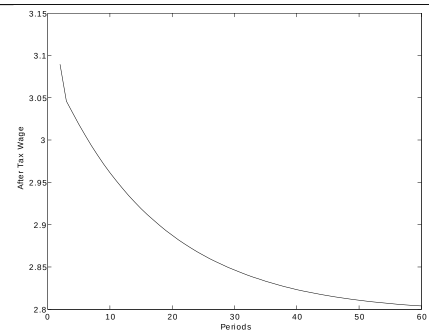
Revenue



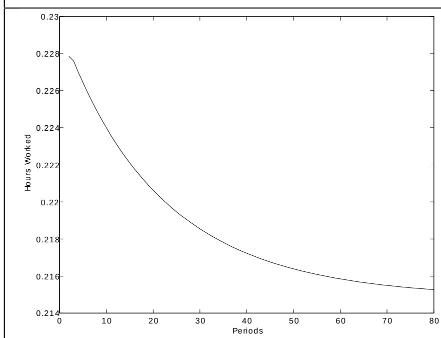
Labour Tax Rate



Post Tax Wage



Hours Worked



Output

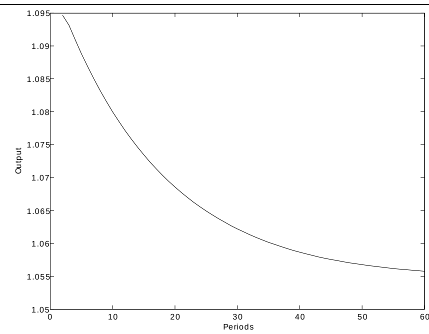
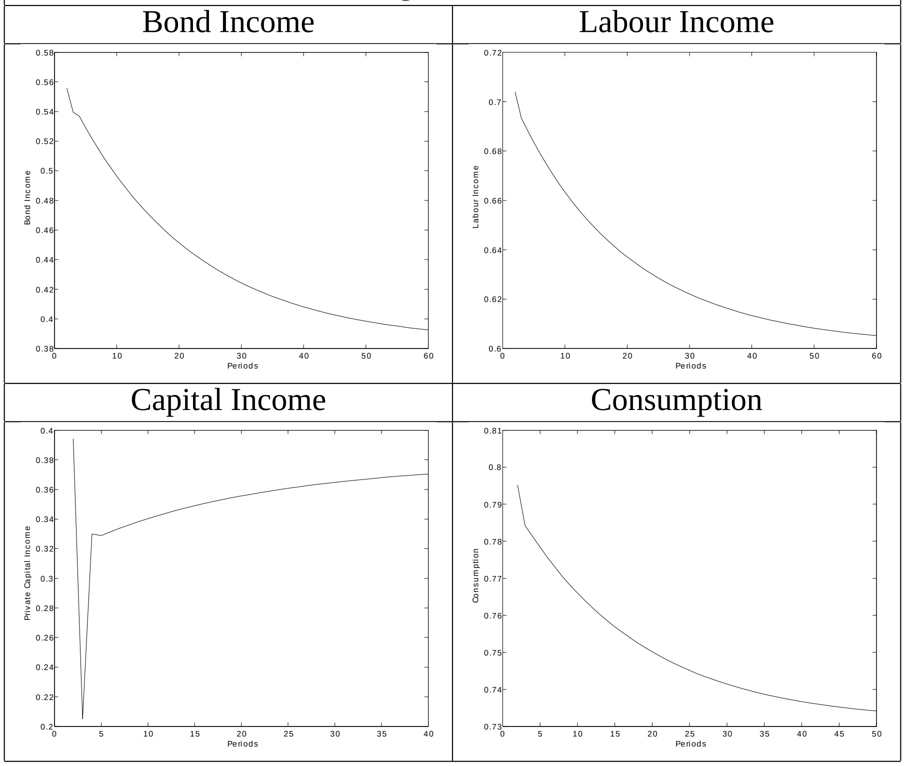


Figure A.5.1



A.5.2 Lower Frisch Elasticity of Labour Supply

The following figures present the deterministic transition path in the model with *both* private *and* public capital, as the Ramsey planner reduces government debt at a relatively slow pace (i.e. debt takes around ten years to fall from 43 per cent of GDP to 32 per cent of GDP). The parameter ρ determining the speed of government debt reduction is set to 0.95. All parameters are the same as in the benchmark case set out in Subsection (1.3.2), *except* for the parameter κ in the utility function which is set to two. This generates a Frisch elasticity of labour supply with respect to the real wage of *two* in the ultimate steady state reached by the Ramsey economy.

Figure A.5.2

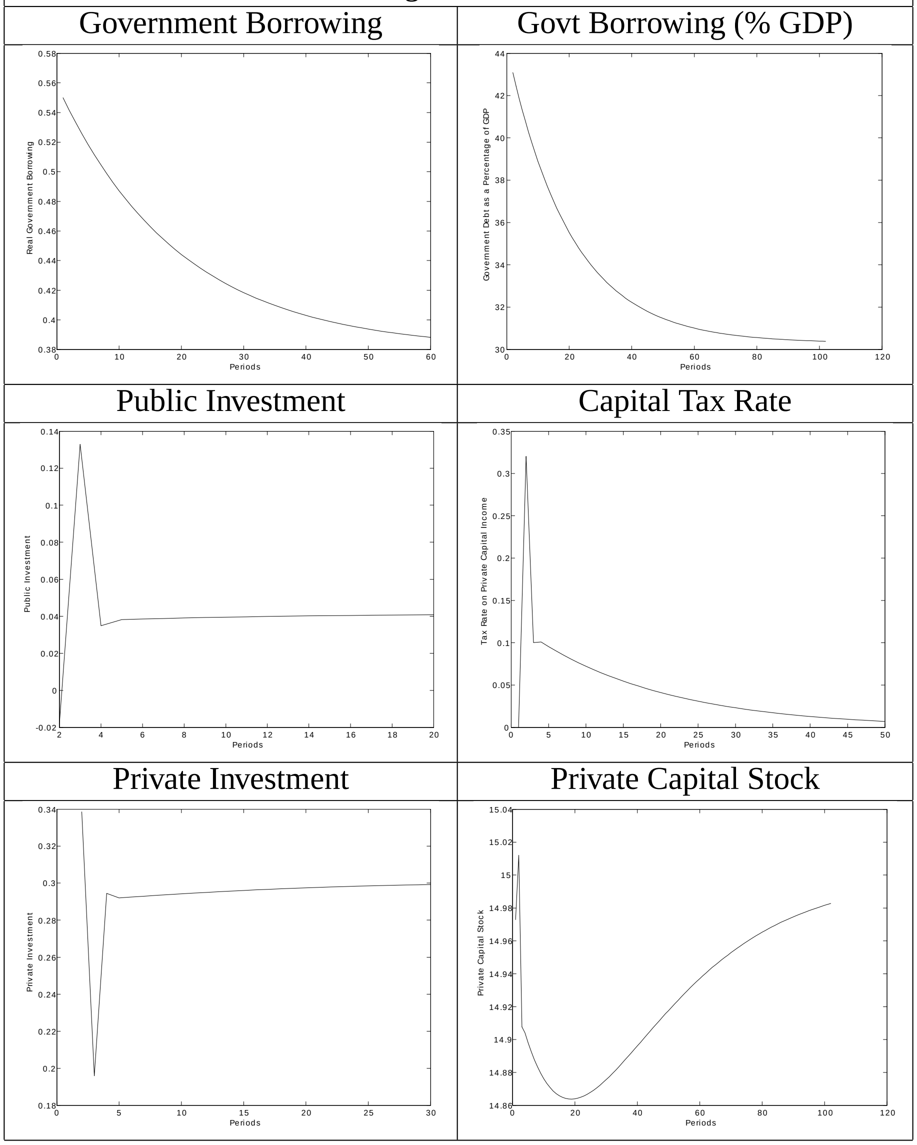

Government Borrowing
Govt Borrowing (% GDP)
Public Investment
Capital Tax Rate
Private Investment
Private Capital Stock

Figure A.5.2

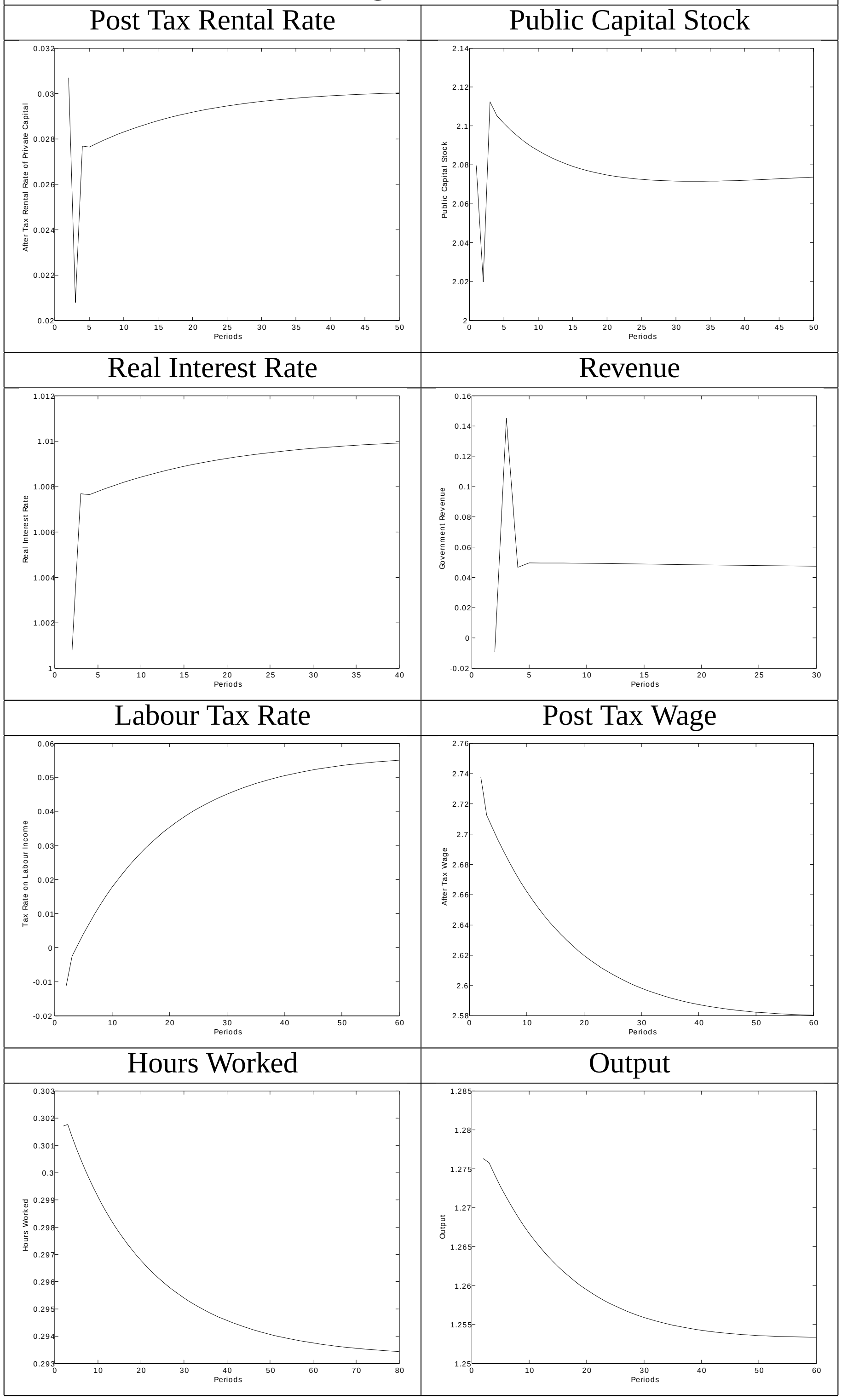
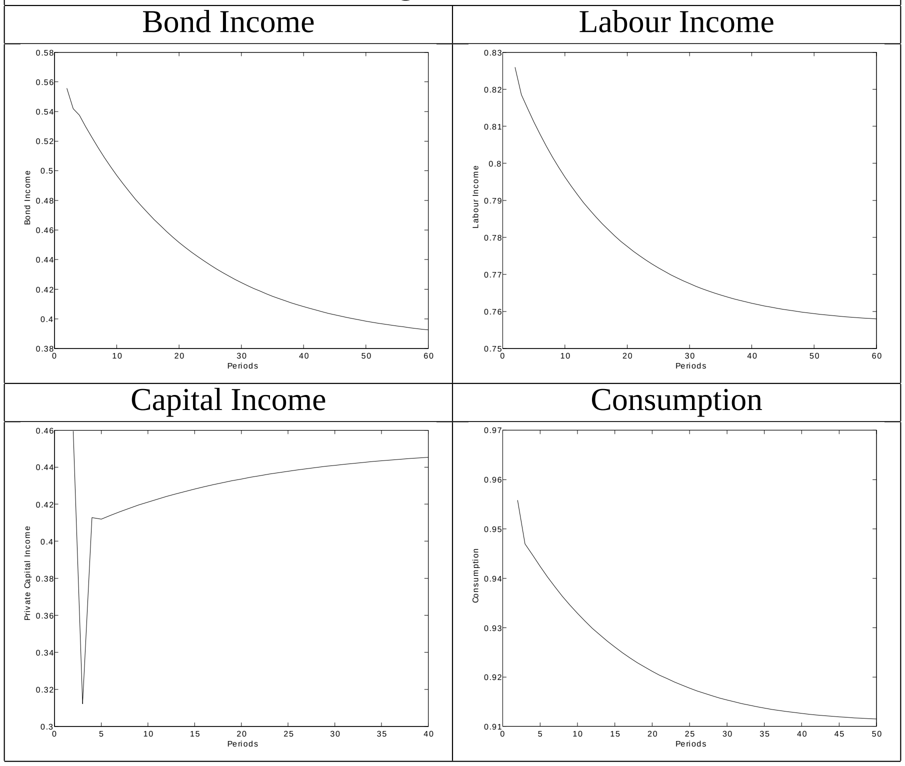

Post Tax Rental Rate
After Tax Rental Rate of Private Capital
Periods
Public Capital Stock
Public Capital Stock
Periods
Real Interest Rate
Real Interest Rate
Periods
Revenue
Government Revenue
Periods
Labour Tax Rate
Tax Rate on Labour Income
Periods
Post Tax Wage
After Tax Wage
Periods
Hours Worked
Hours Worked
Periods
Output
Output
Periods

Figure A.5.2


Bond Income
Labour Income
Capital Income
Consumption
Periods
Private Capital Income

Appendix B for Chapter (2)

B.1 Intratemporal Expenditure Minimisation By Consumers And The Government

Consumers and the government solve a static expenditure minimisation problem to determine how much of each intermediate good should be used to construct a unit of an aggregate consumption or public investment good. I solve the problem here using notation for the consumers' problem. The problem for the government is identical, although the notation should differ accordingly. The expenditure minimisation problem is

$$\min_{c_i} \int_0^1 p_i c_i di \tag{B.1}$$

subject to the constraint

$$\left(\int_0^1 c_i^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} = 1 \tag{B.2}$$

Expenditure is minimised subject to the constraint of forming one unit of the aggregate good. This is a classic programming problem that can be solved by the method of Lagrange multipliers. The Lagrangian is

$$L = \int_0^1 p_i c_i di - \lambda \left[\left(\int_0^1 c_i^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} - 1 \right] \tag{B.3}$$

The first order condition with respect to each variety i of intermediate good is

$$\frac{\partial L}{\partial c_i} = p_i - \lambda \frac{\eta}{\eta - 1} \left(\int_0^1 c_i^{\frac{\eta-1}{\eta}} di \right)^{\frac{1}{\eta-1}} \frac{\eta - 1}{\eta} c_i^{-\frac{1}{\eta}} = 0 \tag{B.4}$$

which simplifies to

$$p_i = \lambda \left(\int_0^1 c_i^{\frac{\eta-1}{\eta}} di \right)^{\frac{1}{\eta-1}} c_i^{-\frac{1}{\eta}}$$

I make use of the fact that the amount of consumption aggregate consumed in a given period is given by

$$c = \left(\int_0^1 c_i^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}$$

so that the first order condition can be further simplified to

$$p_i = \lambda c^{\frac{1}{\eta}} c_i^{-\frac{1}{\eta}}$$

Rearranging for c_i - the amount of intermediate good i used - produces

$$c_i = \left(\frac{p_i}{\lambda}\right)^{-\eta} c \tag{B.5}$$

In order to eliminate the Lagrange multiplier λ, I raise each side of (B.5) to the power $\frac{\eta-1}{\eta}$ and then integrate both sides of the equation over the continuum of intermediate goods $i \in (0, 1)$

$$\int_0^1 c_i^{\frac{\eta-1}{\eta}} di = \int_0^1 \left(\frac{p_i}{\lambda}\right)^{1-\eta} c^{\frac{\eta-1}{\eta}} di$$

It is possible to isolate the Lagrange multiplier λ, which equals

$$\lambda = \left[\int_0^1 (p_i)^{1-\eta} di\right]^{\frac{1}{1-\eta}} \tag{B.6}$$

The best way of interpreting the meaning of this Lagrange multiplier λ is to use (B.5) and (B.6) to write an expression for total minimised expenditure on one unit of the aggregate good c in a given period

$$\int_0^1 p_i c_i di = \int_0^1 p_i \left(\frac{p_i}{\lambda}\right)^{-\eta} c di$$

and simplify

$$\int_0^1 p_i c_i di = \frac{c}{\lambda^{-\eta}} \int_0^1 p_i^{1-\eta} di = \frac{c}{\lambda^{-\eta}} \lambda^{1-\eta}$$

to reveal that

$$\int_0^1 p_i c_i di = c\lambda$$

so that the Lagrange multiplier λ must be the price of one unit of the consumption aggregate c in any period - P_t - which is the price level in the economy

$$\lambda = P_t = \left[\int_0^1 (p_i)^{1-\eta} di\right]^{\frac{1}{1-\eta}} \tag{B.7}$$

It is now possible to eliminate the Lagrange multiplier λ from (B.5) to reveal an expression for the compensated demand by consumers for the intermediate good i

$$c_i = \left(\frac{p_i}{P_t}\right)^{-\eta} c \tag{B.8}$$

as a function of the intermediate good's price relative to the price level in the economy.

B.2 Obtaining A Recursive Formulation For The Ramsey Problem

The Ramsey problem as defined in section (2.2.6) is *not* recursive.

Supposing that the Ramsey planner makes decisions sequentially, a recursive problem in a *deterministic* setting is one where the decision facing a Ramsey planner is the same each period. The decision problem is the same each period given whatever the beginning of period value of *state* variables happens to be. *State* variables encapsulate the effects of the initial conditions on the problem as well as the effects of all decisions made in periods prior to current time t. When the problem in a *deterministic* setting is recursive, the sequence of variables chosen *sequentially* each period by the Ramsey planner will be the same as the sequence chosen if all decisions (i.e. the entire infinite sequence) are made in the initial period.

In the Ramsey problem defined in section (2.2.6), the Ramsey planner is constrained at each time t by conditions which contain forward looking terms (i.e. dated at time $t+1$). Specifically, the Euler equation for government bond holdings (2.40) contains the term $u'(c_{t+1})$ and π_{t+1}, while the aggregate supply relation (2.42) derived from representative consumer's first order condition with respect to price setting contains the terms π_{t+1} and λ_{t+1}. The presence of these forward looking terms among the time t constraints on the Ramsey planner implies that decisions made by the Ramsey planner in future periods limit the choices available to the Ramsey planner at time t. In other words, decisions taken today constrain the Ramsey planner's choices in previous periods. If the Ramsey planner makes decisions sequentially, there is no mechanism in the Ramsey problem as it is defined in Section (2.2.6) to take account of the effect of time t decisions on the set of feasible choices available to the Ramsey planner in previous periods. Making decisions sequentially given the state variables will not be the same as if the Ramsey planner made all choices in the initial period, taking into account the effect that choices at time t have on the set of feasible choices in previous periods. Because of this, decisions made sequentially under the current formulation of the Ramsey problem may not be optimal. For more information, please see Kydland and Prescott (1977) and Kydland and Prescott (1980).

In order to obtain a recursive formulation of the Ramsey problem, I proceed by defining the following modified problem at time t. Given values for predetermined variables $R_{-1}b_{-1}$ and G_{-1}

$$\max_{\{c_{t+j},h_{t+j},G_{t+j},mc_{t+j},\pi_{t+j},\lambda_{t+j},R_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty}(\beta^t \left[u(c_t)+v(1-h_t)\right])$$
$$-\gamma_1\left[\frac{\lambda_t}{\pi_t}\right]+\gamma_2\left[\lambda_t\theta\pi_t(\pi_t-1)\right]$$

subject to the constraints on the Ramsey problem for all periods from t onwards. These include the Implementability Conditions (2.48)

$$\begin{aligned}
& \left[G_{t+j}-(1-\delta^G)G_{t+j-1}\right]+R_{t+j-1}\frac{b_{t+j-1}}{\pi_{t+j}} \\
& -mc_{t+j}F_h(G_{t+j-1},h_{t+j})h_{t+j}+\frac{v'(1-h_{t+j})h_{t+j}}{\lambda_{t+j}} \\
= \quad & b_{t+j}
\end{aligned}$$

plus the competitive equilibrium conditions (2.38), (2.40) and (2.42)

$$u'(c_{t+j})=\lambda_{t+j}$$

$$\lambda_{t+j}=\beta\frac{\lambda_{t+j+1}}{\pi_{t+j+1}}R_{t+j}$$

$$\begin{aligned}
& \lambda_{t+j}\theta\pi_{t+j}(\pi_{t+j}-1) \\
= \quad & \beta\theta\lambda_{t+j+1}\pi_{t+j+1}(\pi_{t+j+1}-1) \\
& +\lambda_{t+j}\eta F(G_{t+j-1},h_{t+j})\left[mc_{t+j}-\left(\frac{\eta-1}{\eta}\right)\right]
\end{aligned}$$

and the aggregate resource constraint (2.34)

$$c_{t+j}+G_{t+j}-(1-\delta^G)G_{t+j-1}+\frac{\theta}{2}(\pi_{t+j}-1)^2=F(G_{t+j-1},h_{t+j})$$

plus the exogenous downward process for government borrowing (2.46)

$$b_{t+j} - \overline{b} = \rho(b_{t+j-1} - \overline{b})$$

The difference between this problem and the one defined in section (2.2.6) is that I have added the forward looking components of the competitive equilibrium conditions (2.40) and (2.42) from time $t-1$ to the objective function of the Ramsey planner, weighted by γ_1 and γ_2. This formulation of the problem can be solved every period t. In order to obtain a recursive formulation, it is necessary that the planner *commits* to update the weights γ_1 and γ_2 when solving the modified problem in period t with the values of the Lagrange multipliers attached to the time $t-1$ constraints (2.40) and (2.42), obtained when solving the problem at time $t-1$. By committing to update the weights γ_1 and γ_2 in this way each period, the planner commits to take into account the effect of decisions at time t on the feasible choices available to the Ramsey planner in previous periods. This sequence of modified optimisation problems defines a recursive formulation for the Ramsey problem. The state variables each period t are the predetermined values $R_{-1}b_{-1}$, G_{-1} *and* the value of the Lagrange multipliers attached to the time $t-1$ competitive equilibrium conditions (2.40) and (2.42) in the modified problem solved at time $t-1$. Given the values of these state variables, the modified problem is the *same* each period. The key feature allowing for this recursive formulation is the *augmenting* of the number of state variables with the Lagrange multipliers described, as pointed out by Kydland and Prescott (1980), Marcet and Marimon (2011) and Kumhof and Yakadina (2007). The solution to this sequence of modified problems can be obtained by solving a single problem using the method of Lagrange multipliers. The Lagrangian for this problem was given above by (2.50) and is:

$$\max_{\{c_t,h_t,G_t,\pi_t,\lambda_t,mc_t,R_t\}_{t=0}^{\infty}} L = \sum_{t=0}^{\infty} \beta^t$$

$$\left\{ \begin{array}{c} u(c_t) + v(1-h_t) - \mu^E_{t-1}\frac{\lambda_t}{\pi_t} + \\ \mu^P_{t-1}\lambda_t\theta\pi_t(\pi_t - 1) + \mu^E_t\frac{\lambda_t}{R_t} \\ +\mu^C_t\left[u'(c_t) - \lambda_t\right] \\ +\mu^P_t\left[\begin{array}{c} \lambda_t\eta F(G_{t-1},h_t)\left[mc_t - \left(\frac{\eta-1}{\eta}\right)\right] \\ -\lambda_t\theta\pi_t(\pi_t-1) \end{array}\right] \\ +\mu^G_t\left[\begin{array}{c} b_t - \left[G_t - (1-\delta^G)G_{t-1}\right] \\ -R_{t-1}\frac{b_{t-1}}{\pi_t} + mc_tF_h(G_{t-1},h_t)h_t - \frac{v'(1-h_t)h_t}{\lambda_t} \end{array}\right] \\ +\mu^A_t\left[\begin{array}{c} F(G_{t-1},h_t) - c_t \\ -G_t + (1-\delta^G)G_{t-1} - \frac{\theta}{2}(\pi_t - 1)^2 \end{array}\right] \end{array} \right\}$$

where μ_t^E and μ_t^P are the Lagrange multipliers at time t attached to the competitive equilibrium conditions (2.40) and (2.42) with forward looking terms. The Lagrange multiplier μ_t^G is attached to the Implementability Condition (2.48) derived from the government's real period budget constraint while μ_t^A is the Lagrange multiplier attached to the aggregate resource constraint (2.34) at time t. An additional Lagrange multiplier μ_t^C is attached to the competitive equilibrium condition (2.38).

B.3 Solving The Ramsey Problem

In this Appendix, I present the Ramsey problem when the Ramsey planner is required to set nominal interest rates in order to satisfy the interest rate feedback rule in (2.49). The Lagrangian for this problem is

$$\max_{\{c_t,h_t,G_t,\pi_t,\lambda_t,mc_t,R_t,a_t\}_{t=0}^{\infty}} L = \sum_{t=0}^{\infty} \beta^t \tag{B.9}$$

$$\left\{ \begin{array}{c} u(c_t) + v(1-h_t) - \mu_{t-1}^{E}\frac{\lambda_t}{\pi_t} + \\ \mu_{t-1}^{P}\lambda_t\theta\pi_t(\pi_t - 1) + \mu_t^{E}\frac{\lambda_t}{R_t} \\ +\mu_t^{C}\left[u'(c_t) - \lambda_t\right] \\ +\mu_t^{P}\left[\begin{array}{c} \lambda_t\eta F(G_{t-1},h_t)\left[mc_t - \left(\frac{\eta-1}{\eta}\right)\right] \\ -\lambda_t\theta\pi_t(\pi_t-1) \end{array} \right] \\ +\mu_t^{G}\left[\begin{array}{c} b_t - \left[G_t - (1-\delta^G)G_{t-1}\right] \\ -R_{t-1}\frac{b_{t-1}}{\pi_t} + mc_t F_h(G_{t-1},h_t)h_t - \frac{v'(1-h_t)h_t}{\lambda_t} \end{array} \right] \\ +\mu_t^{A}\left[\begin{array}{c} F(G_{t-1},h_t) - a_t \\ -\frac{\theta}{2}(\pi_t - 1)^2 \end{array} \right] \\ +\mu_t^{R}\left[\ln(\frac{R_t}{R_{t-1}}) - \alpha_\pi \ln\left(\frac{\pi_t}{\pi_{t-1}}\right) - \alpha_{AD}\ln\left(\frac{a_t}{a_{t-1}}\right)\right] \\ +\mu_t^{AD}\left[a_t - c_t - G_t + (1-\delta^G)G_{t-1}\right] \end{array} \right\}$$

where μ_t^R is an additional Lagrange multiplier at time t attached to the interest rate feedback rule (2.49) and μ_t^{AD} is a Lagrange multiplier at time t attached to the definition of aggregate demand (2.30).

The conditions determining when the first order partial derivatives of the Lagrangian vanish each period are as follows:

$$\frac{\partial L}{\partial c_t} = u'(c_t) + \mu_t^{C}u''(c_t) - \mu_t^{AD} = 0 \tag{B.10}$$

$$
\begin{aligned}
\frac{\partial L}{\partial h_t} &= -v'(1-h_t) + \mu_t^P \lambda_t \eta F_h(G_{t-1}, h_t) \left[mc_t - \left(\frac{\eta - 1}{\eta} \right) \right] \\
&\quad + \mu_t^G mc_t F_{hh}(G_{t-1}, h_t) h_t \\
&\quad + \mu_t^G mc_t F_h(G_{t-1}, h_t) \\
&\quad - \frac{\mu_t^G}{\lambda_t} \left[v'(1-h_t) - v''(1-h_t) h_t \right] \\
&\quad + \mu_t^A F_h(G_{t-1}, h_t) \\
&= 0
\end{aligned} \tag{B.11}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial G_t} &= -\mu_t^G - \mu_t^{AD} \\
&\quad + \beta \mu_{t+1}^P \lambda_{t+1} \eta F_G(G_t, h_{t+1}) \left[mc_{t+1} - \left(\frac{\eta - 1}{\eta} \right) \right] \\
&\quad + \beta \mu_{t+1}^G (1 - \delta^G) \\
&\quad + \beta \mu_{t+1}^G mc_{t+1} F_{hG}(G_t, h_{t+1}) h_{t+1} \\
&\quad + \beta \mu_{t+1}^A F_G(G_t, h_{t+1}) + \beta \mu_{t+1}^{AD} (1 - \delta^G) \\
&= 0
\end{aligned} \tag{B.12}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_t} &= -\frac{\mu_{t-1}^E}{\pi_t} + \mu_{t-1}^P \pi_t \theta \left(\pi_t - 1 \right) \\
&\quad + \frac{\mu_t^E}{R_t} + \mu_t^P \left[\eta F\left(G_{t-1}, h_t \right) \left[mc_t - \left(\frac{\eta - 1}{\eta} \right) \right] \right] \\
&\quad - \mu_t^P \theta \pi_t \left(\pi_t - 1 \right) \\
&\quad - \mu_t^C + \mu_t^G \frac{v'(1-h_t) h_t}{\lambda_t^2} \\
&= 0
\end{aligned} \tag{B.13}
$$

$$
\frac{\partial L}{\partial mc_t} = \mu_t^P \lambda_t \eta F\left(G_{t-1}, h_t \right) + \mu_t^G F_h\left(G_{t-1}, h_t \right) h_t = 0 \tag{B.14}
$$

$$\frac{\partial L}{\partial \pi_t} = \frac{\mu^E_{t-1}\lambda_t}{\pi_t^2} + \mu^P_{t-1}\lambda_t\theta\left(2\pi_t - 1\right) - \mu^P_t\lambda_t\theta\left(2\pi_t-1\right) + \frac{\mu^G_t R_{t-1}b_{t-1}}{\pi_t^2} - \mu^A_t\theta\left(\pi_t - 1\right) - \mu^R_t\alpha_\pi\left(\frac{\pi_{t-1}}{\pi_t}\right)\left(\frac{1}{\pi_{t-1}}\right) + \beta\alpha_\pi\mu^R_{t+1}\left(\frac{\pi_t}{\pi_{t+1}}\right)\left(\frac{\pi_{t+1}}{\pi_t^2}\right) = 0 \quad \text{(B.15)}$$

$$\frac{\partial L}{\partial R_t} = -\mu^E_t\frac{\lambda_t}{R_t^2} + \mu^R_t\left(\frac{R_{t-1}}{R_t}\right)\left(\frac{1}{R_{t-1}}\right) - \beta\mu^G_{t+1}\frac{b_t}{\pi_{t+1}} - \beta\mu^R_{t+1}\left(\frac{R_t}{R_{t+1}}\right)\frac{R_{t+1}}{R_t^2} = 0 \quad \text{(B.16)}$$

$$\frac{\partial L}{\partial a_t} = -\mu^A_t + \mu^{AD}_t - \mu^R_t\alpha_{AD}\frac{a_{t-1}}{a_t}\left(\frac{1}{a_{t-1}}\right) + \beta\mu^R_{t+1}\alpha_{AD}\left(\frac{a_t}{a_{t+1}}\right)\left(\frac{a_{t+1}}{a_t^2}\right) \quad \text{(B.17)}$$

The additional equations which must hold every period in the Ramsey Economy are as follows. First, the competitive equilibrium conditions (2.38), (2.40) and (2.42). Second the Implementability Condition (2.48) derived from the government budget constraint:

$$\left[G_t - (1-\delta^G)G_{t-1}\right] + R_{t-1}\frac{b_{t-1}}{\pi_t} - mc_tF_h(G_{t-1},h_t)h_t + \frac{v'(1-h_t)h_t}{\lambda_t} = b_t$$

Next, the aggregate resource constraint (2.34) must hold. Further, real government borrowing must follow the exogenous downward path in (2.46)

$$b_t - \bar{b} = \rho\left(b_{t-1} - \bar{b}\right)$$

The interest rate feedback rule (2.49) must be satisfied by the Ramsey planner

$$\ln(\frac{R_t}{R_{t-1}}) = \alpha_\pi \ln\left(\frac{\pi_t}{\pi_{t-1}}\right) + \alpha_{AD}\ln\left(\frac{a_t}{a_{t-1}}\right)$$

where $a_t \equiv c_t + G_t - (1-\delta^G)G_{t-1}$. Other conditions in the Ramsey economy are the transversality conditions (2.45) and (2.51), with the initial conditions $R_{-1}b_{-1}$, G_{-1} and μ^E_{-1}, μ^P_{-1} for the Lagrange multipliers attaching to constraints with forward looking terms.

The conditions describing the Ramsey Economy when monetary policy is set *optimally* (unconstrained by the interest rate rule (2.49)) are essentially the same as those presented in this Appendix, but with Lagrange multipliers μ_t^R and μ_t^{AD} set to zero in every period. Further, the interest rate rule itself (2.49) and the definition of aggregate demand (2.30) are omitted from the conditions characterising the Ramsey Economy, when monetary policy is set optimally by the Ramsey planner.

B.4 The Numerical Solution Method Used To Find The Deterministic Transition Path

I present the deterministic transition path of the Ramsey Economy from a state with a relatively high level of government debt to a steady state with relatively low level of government debt. Government debt as a percentage of GDP will be ultimately lower in the new steady state. I do this both in the Ramsey Economy where monetary policy is set optimally and again in the Ramsey Economy where the Ramsey planner is constrained to choose nominal interest rates in accordance with the feedback rule (2.49). Each time, I carry out the following steps to do this:

(1) I derive again the equations characterising the Ramsey Economy described in section (2.2.7), this time using the specific functional forms chosen in section (2.3.1).

(2) The values of the model's parameters are set to those described in section (2.3.2).

(3) I set initial conditions for the public capital stock G_0 and for the initial level of real government debt b^{high} such that the level of government debt to GDP is at the desired value in the early periods of the transition. The initial conditions for the lagrange multipliers μ_0^E and μ_0^Pon the constraints (2.40) and (2.42) with forward looking components are set to zero. According to Marcet and Marimon (2011), Kumhof and Yakadina (2007) and Reiter (2005), the choices of the Ramsey planner in the first period of the transition cannot constrain the planner's choices in previous periods, implying that these constraints are not binding and thus the multipliers should be set to zero.

(4) The level of government borrowing must fall in the first period of the transition path according to the equation (2.46) describing the exogenous downward path for government borrowing. Thereafter, real government borrowing b_t asymptotes towards its new target level b^{low} over the transition path.

(5) There are thirteen equations characterising the Ramsey Economy for each of the T periods of the transition path when monetary policy is set optimally, forming a large system of simultaneous non-linear equations. In the Ramsey Economy with the interest rate feedback rule, there are sixteen equations each period. In the final period T, I set the values of the endogenous variables to their values in a steady state where the level of government borrowing is b^{low}

$$\left\{\overline{c}, \overline{h}, , \overline{G}, \overline{\pi}, \overline{\lambda}, \overline{mc}, \overline{R}, \overline{\mu}^E, \overline{\mu}^P, \overline{\mu}^G, \overline{\mu}^A, \overline{\mu}^C\right\}$$

In this steady state, the level of government borrowing (as a percentage of GDP) is at a lower value than it was initially. The terminal condition for the level of real government borrowing is b^{low}. The chosen length of the transition period T must be sufficiently long such that the economy will be close to this new steady state by the time period T is reached.

(6) I set the duration of the transition to $T = 200$ and then solve this large system of equations for the sequence

$$\left\{c_t, h_t, G_t, \pi_t, mc_t, \lambda_t, R_t, \mu_t^E, \mu_t^P, \mu_t^G, \mu_t^A, \mu_t^C\right\}_{t=1}^{T-1}$$

in the case of optimal monetary policy. The solution can be obtained by using a solver of systems of simultaneous non-linear equations, such as that described in Reiter (2005) or Juillard (1996)[9].

(7) Different lengths T for the transition path can be experimented with, to ensure that the choice of T does not overly influence the results. In other words, the level of real government borrowing should have time to fall from b^{high} to b^{low} according to the exogenous path given by equation (2.46) by the time T is reached, so that endogenous variables have had time to adjust. By the time T is reached, endogenous variables should be close to their values in the new steady state.

[9] The non-linear equation solver used to obtain the results presented in this chapter is that called by the "SIMUL" command in DYNARE Version 4.3. I have also solved the model using an alternative non-linear equation solver designed for systems with a very large number of equations. This alternative solver was developed by Michael Reiter and I am most grateful to him for providing me with the necessary computer code.

B.5 Robustness: More Productive Public Capital

The following figures present the deterministic transition path, as the Ramsey planner reduces government debt subject to an interest rate feedback rule. All parameters are the same as in the benchmark case set out in Subsection (2.3.2), *except* for the elasticity α of output to public capital, which is set to 0.1, rather than 0.05.

Figure B.5

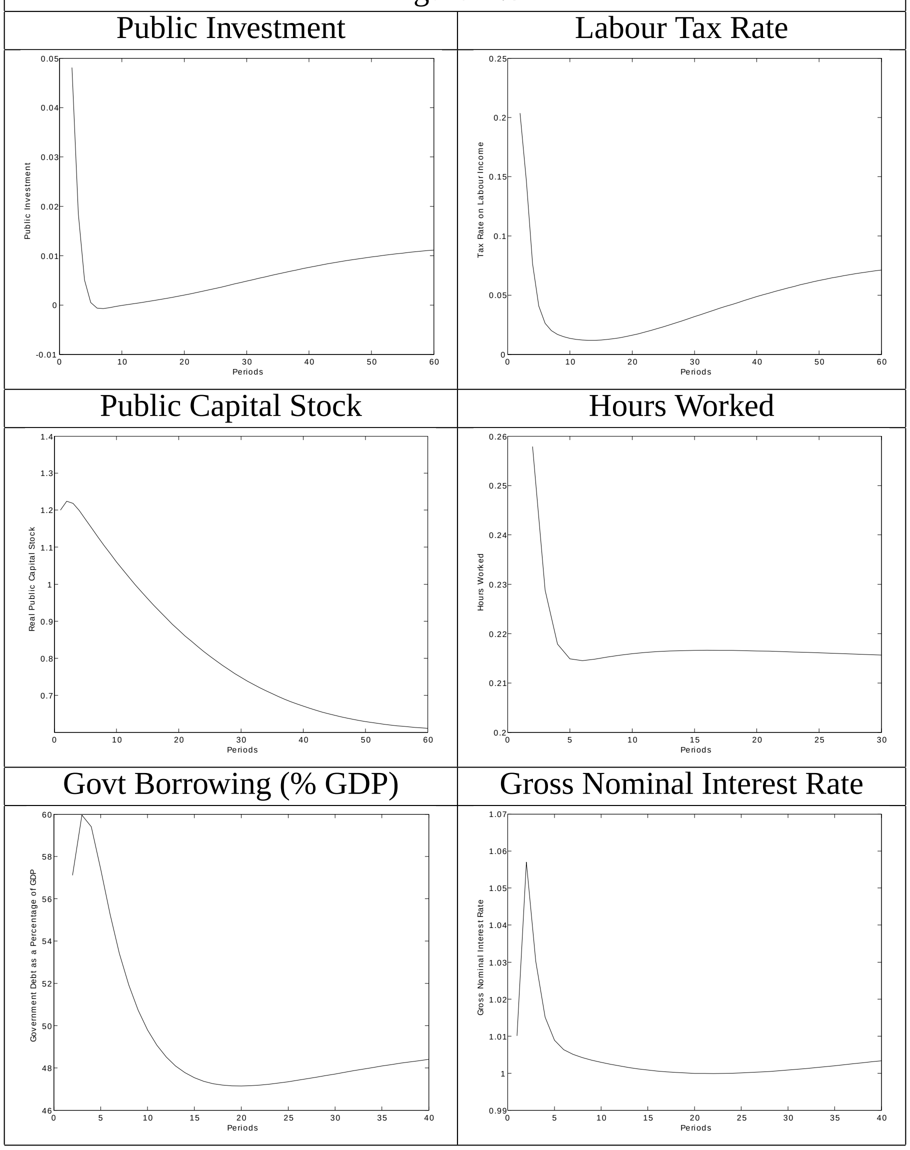
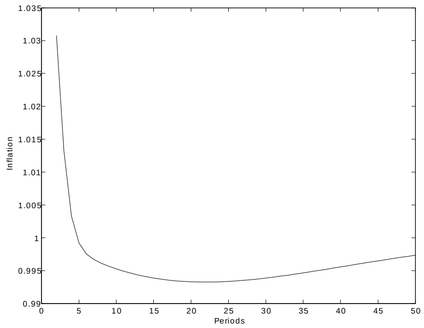
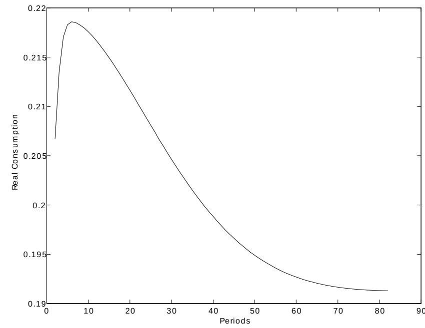


Figure B.5

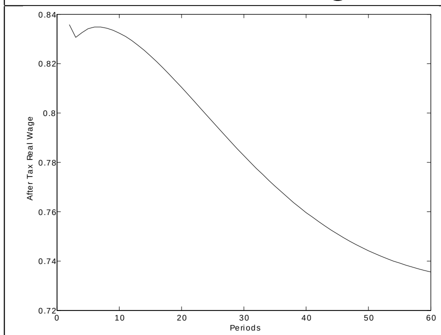
Gross Inflation Rate



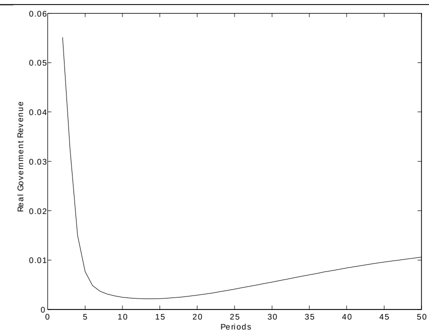
Consumption



Post-Tax Real Wage



Real Revenue



B.6 Robustness: Lower Frisch Elasticity of Labour Supply

The following figures present the deterministic transition path, as the Ramsey planner reduces government debt subject to an interest rate feedback rule. All parameters are the same as in the benchmark case set out in Subsection (2.3.2), *except* for the parameter κ in the utility function and the price adjustment cost parameter θ. Figure (B.6) (Parts One and Two) present the case when the Frisch elasticity of labour supply is *three* in the steady state of a competitive equilibrium, generated by setting κ to 2.5 and θ to 21.875.

Figure B.6 - Part One

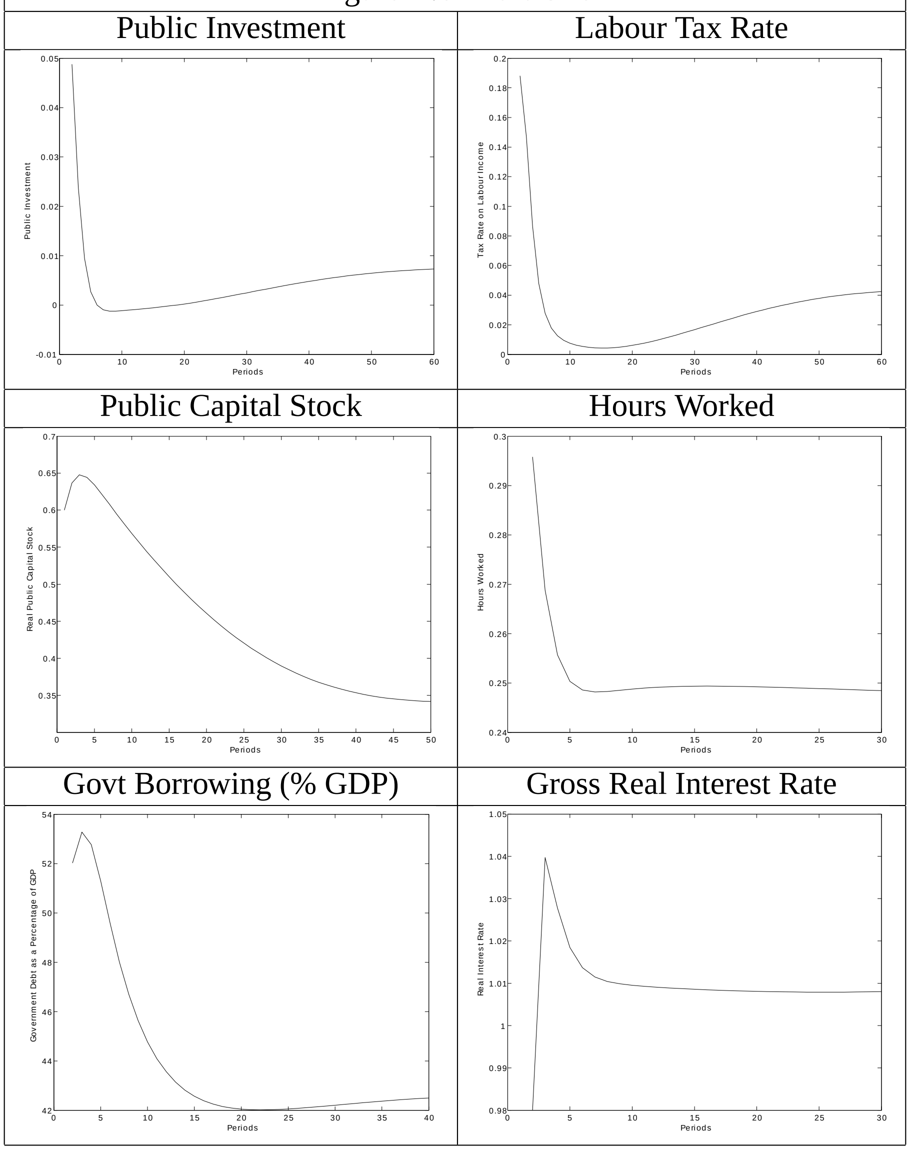


Figure B.6 - Part Two

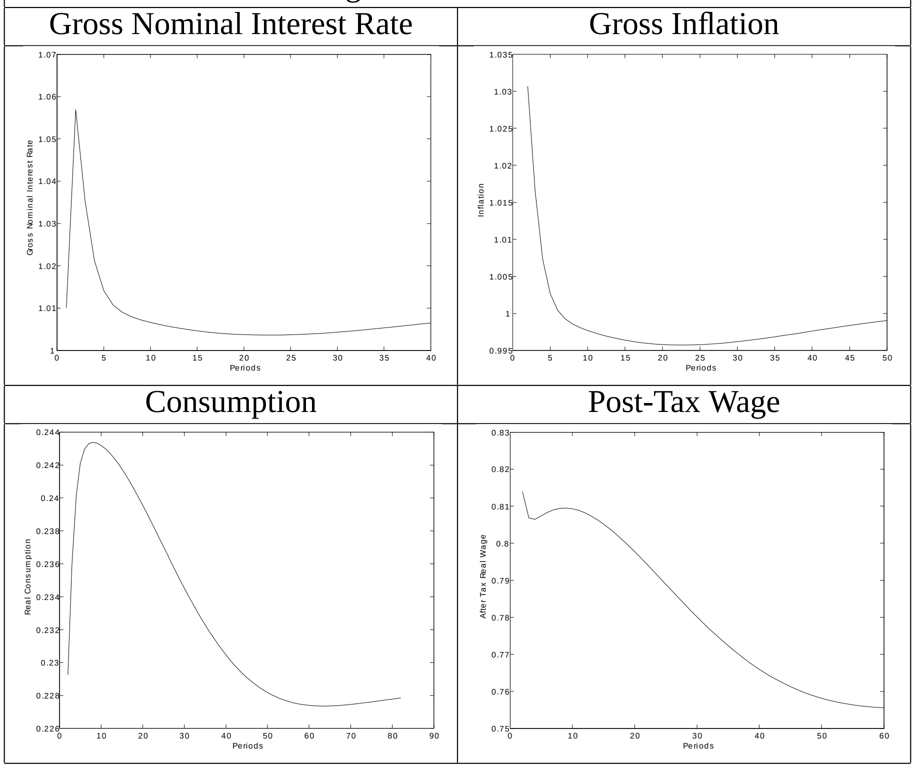


Figure (B.6) (Parts Three and Four) present the case when the Frisch elasticity of labour supply is *two* in the steady state of a competitive equilibrium, generated by setting κ to 2 and θ to 26.25.

Figure B.6 - Part Three

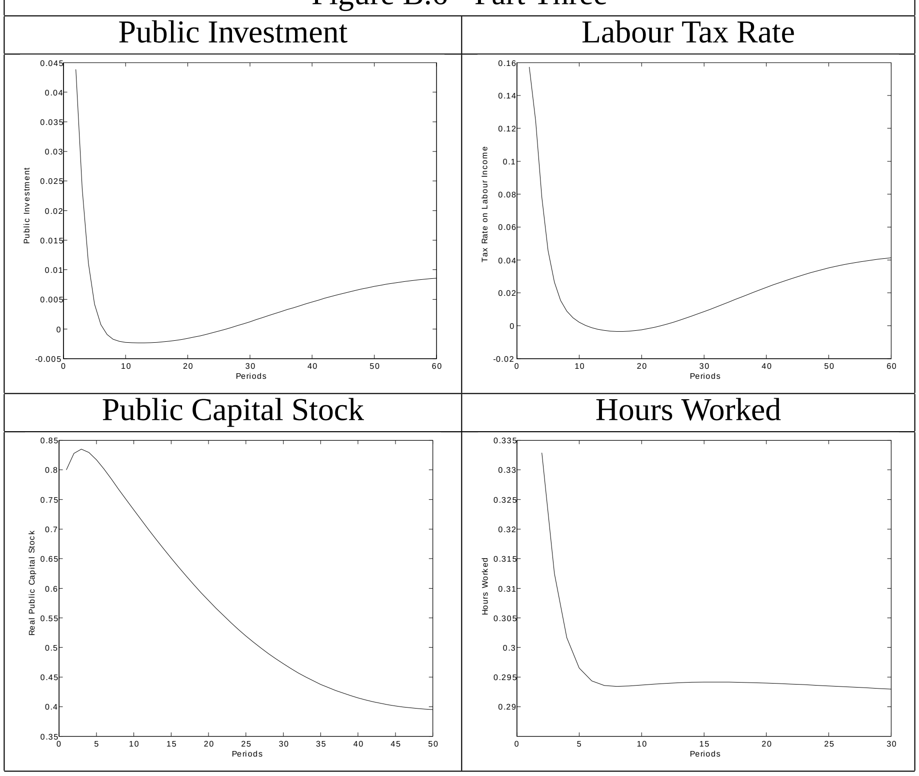
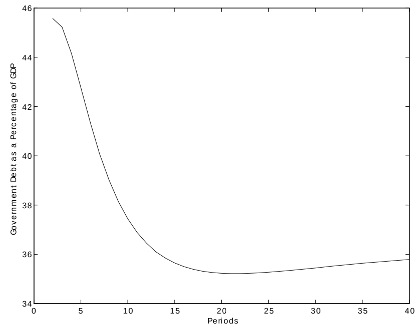
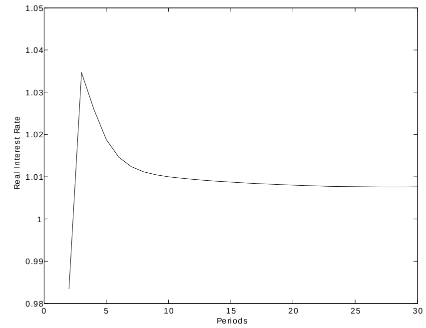
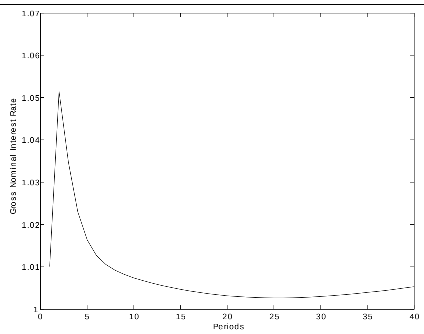
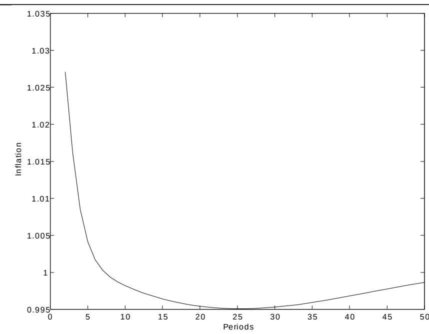
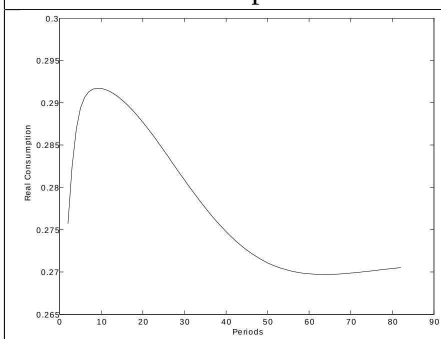
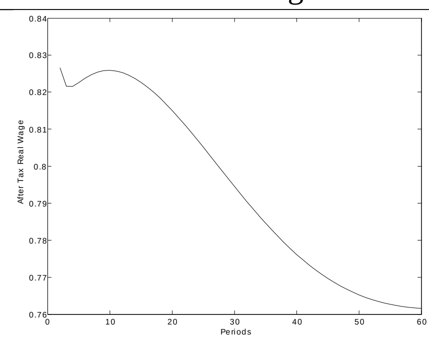


Figure B.6 - Part Four

Govt Borrowing (% GDP)	Gross Real Interest Rate
	

Gross Nominal Interest Rate	Gross Inflation Rate
	

Consumption	Post-Tax Wage
	

Appendix C for Chapter (3)

C.1 Decomposition

The marginal effect of higher average saving k on indirect expected utility of a consumer with initial wealth Ω^i can be decomposed as (3.35):

$$\beta E\left(u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\left[k\left\{\frac{\partial r(z^j,k)}{\partial k}-E(\frac{\partial r(z^j,k)}{\partial k})\right\}\right]\right)+$$
$$\beta E\left(u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\left[l(z^j)\frac{\partial w(z^j,k)}{\partial k}-E(l(z^j)\frac{\partial w(z^j,k)}{\partial k})\right]\right)+$$
$$\beta\sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)(e_n-l(z^j))\right]|_z\ \frac{\partial w(z^j,k)}{\partial k}\right)+$$
$$\beta\sum_j \Pr(z^j)\left(E\left[u'(r(z^j,k)a(\Omega^i)+w(z^j,k)e_n)\right]|_z\ (a^i-k)\frac{\partial r(z^j,k)}{\partial k}\right)$$

PROOF

First, some Algebra.

Proposition 19 *It is the case that*

$$0=\frac{\partial r(z^j,k)}{\partial k}k+\frac{\partial w(z^j,k)}{\partial k}l(z^j) \tag{C.1}$$

for all j$\in (G,B)$.

Proof. The production function $y=z^jf(k,l(z^j))$ is assumed homogenous of degree one (h.d.1) in private inputs, as discussed in Section (3.2). By Euler's Theorem, it must be the case that

$$y(z^j,k)=\frac{\partial z^jf(k,l(z^j))}{\partial k}k+\frac{\partial z^jf(k,l(z^j))}{\partial l}l(z^j) \tag{C.2}$$

It follows that in competitive equilibrium

$$y(z^j,k)=r(z^j,k)k+w(z^j,k)l(z^j) \tag{C.3}$$

for all $j\in(G,B)$, where $y(z^j)$ denotes the output of an individual firm and the interest rate and the wage are defined as in equations (3.27). Taking the derivative of individual firm output with respect to average per consumer saving k

$$\frac{\partial y(z^j,k)}{\partial k} = \frac{\partial r(z^j,k)}{\partial k}k + r(z^j,k) + \frac{\partial w(z^j,k)}{\partial k}l(z^j) \tag{C.4}$$

it follows that

$$0 = \frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j) \tag{C.5}$$

for all $j \in (G,B)$, because (3.27) shows that

$$\frac{\partial y(z^j,k)}{\partial k} = r(z^j,k)$$

This gives the result. ■

An important consequence is the following proposition.

Proposition 20 *It is the case that*

$$E\left[z^j\frac{\partial r(z^j,k)}{\partial k}k + z^j\frac{\partial w(z^j,k)}{\partial k}l(z^j)\right] = 0 \tag{C.6}$$

Proof. The unconditional expectation

$$\sum_j \Pr(z^j \cap e_n)\left(\frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j)\right) \tag{C.7}$$

must equal zero because (C.5) shows that

$$0 = \frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j)$$

for all $j \in (G,B)$. ■

Corollary 21 *The term* $\frac{\partial r(z^j,k)}{\partial k}k + \frac{\partial w(z^j,k)}{\partial k}l(z^j)$ *multiplied by* $\beta\Pr(z^j \cap e_n)u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)$ *is zero for all* $j \in (G,B)$ *and* $n \in (H,L)$. *The term* $E\left[z^j\frac{\partial r(z^j,k)}{\partial k}k + z^j\frac{\partial w(z^j,k)}{\partial k}l(z^j)\right]$ *multiplied by* $\beta\Pr(z^j \cap e_n)u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)$ *is zero for all* $j \in (G,B)$ *and* $n \in (H,L)$.

Take the marginal effect on social welfare (3.34) of higher k in competitive equilibrium

$$\frac{\partial U(\Omega^i)}{\partial k}\mid_{Comp.Eq.} = \beta\sum_j\sum_n \Pr(z^j \cap e_n)u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)\left\{\frac{\partial r(z^j,k)}{\partial k}a(\Omega^i) + \frac{\partial w(z^j,k)}{\partial k}e_n\right\}$$

add and subtract

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left\{ \frac{\partial r(z^j,k)}{\partial k} k + \frac{\partial w(z^j,k)}{\partial k} l(z^j) \right\}$$

and subtract

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left\{ E\left[\frac{\partial r(z^j,k)}{\partial k} k + \frac{\partial w(z^j,k)}{\partial k} l(z^j) \right] \right\}$$

so that the marginal effect of higher average capital per consumer k on utilitarian social welfare

becomes

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[k \left\{ \frac{\partial r(z^j,k)}{\partial k} - E(\frac{\partial r(z^j,k)}{\partial k}) \right\} \right]$$

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[l(z^j) \frac{\partial w(z^j,k)}{\partial k} - E(l(z^j) \frac{\partial w(z^j,k)}{\partial k}) \right]$$

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[(e_n - l(z^j)) \frac{\partial w(z^j,k)}{\partial k} \right]$$

$$\beta \sum_j \sum_n \Pr(z^j \cap e_n) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \left[(a^i - k) \frac{\partial r(z^j,k)}{\partial k} \right]$$

which gives the result.

C.2 Constrained Efficiency

The social planner chooses individual saving rules $\{a(\Omega^i)\}_{i=1}^{I}$(and a corresponding level of average saving k) to maximise a weighted average of expected utility over I consumer types:

$$\max_{\{a(\Omega^i)\}_{i=1}^{I},k} \sum_{i=1}^{I}(\frac{1}{I})\left\{u(\Omega^i - a(\Omega^i)) + \beta\sum_j\sum_n \Pr(z^j \cap e_n)u(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n)\right\} \tag{C.8}$$

subject to

$$\sum_{i=1}^{I}(\frac{1}{I})\left[a(\Omega^i)\right] = k \tag{C.9}$$

The first order conditions of the planner's problem can be obtained using Lagrange multipliers.

The Lagrangian is:

$$\begin{aligned} L \;=\; & \sum_{i=1}^{I}(\frac{1}{I})\left\{\begin{array}{c} u(\Omega^i - a(\Omega^i)) \\ +\beta\sum_j\sum_n \Pr(z^j \cap e_n)u(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \end{array}\right\} \\ & -\lambda\left\{\sum_{i=1}^{I}(\frac{1}{I})\left[a(\Omega^i)\right] - k\right\} \end{aligned} \tag{C.10}$$

where λ is the lagrange multiplier attaching to (C.9). The first order condition with respect to the savings $a(\Omega^i)$ of consumer with initial wealth Ω^i is

$$\frac{\partial L}{\partial a(\Omega^i)} = (\frac{1}{I})\left\{\begin{array}{c} -u'(\Omega^i - a(\Omega^i)) \\ +\beta\sum_j\sum_n \Pr(z^j \cap e_n)r(z^j,k)u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \end{array}\right\} - \frac{\lambda}{I} = 0 \tag{C.11}$$

while that with respect to k is

$$\frac{\partial L}{\partial k} = \lambda + \sum_{i=1}^{I}(\frac{1}{I})\left\{\beta\sum_j\sum_n \Pr(z^j \cap e_n)\left(\begin{array}{c} \frac{\partial r(z^j,k)}{\partial k}a(\Omega^i) \\ +\frac{\partial w(z^j,k)}{\partial k}e_n \end{array}\right)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n \end{array}\right)\right\} = 0 \tag{C.12}$$

Using (C.12) to substitute out the Lagrange multiplier λ from (C.11) gives the following condition necessary to characterise the constrained efficient k

$$
\begin{aligned}
& u'(\Omega^i - a(\Omega^i)) \qquad \text{(C.13)} \\
= \; & \beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j,k) u'(r(z^j,k)a(\Omega^i) + w(z^j,k)e_n) \\
& + \sum_{i=1}^{I} (\frac{1}{I}) \left\{ \beta \sum_j \sum_n \Pr(z^j \cap e_n) \left(\begin{array}{c} \frac{\partial r(z^j,k)}{\partial k} a(\Omega^i) \\ + \frac{\partial w(z^j,k)}{\partial k} e_n \end{array} \right) u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n \end{array} \right) \right\}
\end{aligned}
$$

C.3 Tax / Transfer Scheme One

C.3.1 Consumer's Problem

The consumers of type i choose a level of saving $a(\Omega^i)$ to maximise expected utility:

$$\max_{a(\Omega^i)} \left\{ u(\Omega^i - a(\Omega^i)) + \beta \sum_j \sum_n \Pr(z^j \cap e_n) u \begin{pmatrix} (1-\tau^k) r(z^j,k) a(\Omega^i) \\ +w(z^j,k)e_n + T(\Omega^i, z^j, e_n) \end{pmatrix} \right\} \quad \text{(C.14)}$$

where $c_1(\Omega^i)$ and $c_2(\Omega^i, z^j, e_n)$ have been substituted out using period budget constraints (3.72) and (3.73). This is an unconstrained optimisation problem in single variable $a(\Omega^i)$, with consumers taking the tax rate τ^k and factor prices $r(z^j,k)$, $w(z^j,k)$ as given. The first order condition is:

$$\frac{\partial}{\partial a(\Omega^i)} = \left\{ \begin{array}{c} -u'(\Omega^i - a(\Omega^i)) \\ +\beta \sum_j \sum_n \Pr(z^j \cap e_n)(1-\tau^k) r(z^j,k) u' \begin{pmatrix} (1-\tau^k) r(z^j,k) a(\Omega^i) \\ +w(z^j,k)e_n + T(\Omega^i, z^j, e_n) \end{pmatrix} \end{array} \right\} = 0 \quad \text{(C.15)}$$

which implies the Euler equation (3.74):

$$u'(\Omega^i - a^i) = \beta \sum_j \sum_n \Pr(z^j \cap e_n)(1-\tau^k) r(z^j,k) u' \begin{pmatrix} (1-\tau^k) r(z^j,k) a(\Omega^i) \\ +w(z^j,k)e_n + T(\Omega^i, z^j, e_n) \end{pmatrix}$$

C.3.2 Ramsey Problem

The first order conditions of the Ramsey problem are obtained using Lagrange multipliers. The Lagrangian is:

$$\begin{aligned} L &= (\frac{1}{I}) \sum_i \left\{ \begin{array}{c} u(\Omega^i - a(\Omega^i)) + \beta E u \begin{pmatrix} r(z^j,k) a(\Omega^i) \\ +w(z^j,k)e_n \end{pmatrix} \\ -\lambda^i \left[\begin{array}{c} u'(\Omega^i - a^i) \\ -\beta E (1-\tau^k) r(z^j,k) u' \begin{pmatrix} r(z^j,k) a(\Omega^i) \\ +w(z^j,k)e_n) \end{pmatrix} \end{array} \right] \end{array} \right\} \\ & \quad -\psi \left\{ \sum_{i=1}^{I} (\frac{1}{I}) \left[a(\Omega^i) \right] - k \right\} \end{aligned} \quad \text{(C.16)}$$

where λ^i is the Lagrange multiplier attaching to the Euler equation (3.74) of consumers of type $i \in \{1, 2, ..., I\}$ and ψ is the Lagrange multiplier attaching to (3.83) The government budget balance condition $T(\Omega^i, z^j, e_n) = \tau^k r(z^j,k) a(\Omega^i)$ has been used to substitute for $T(\Omega^i, z^j, e_n)$.

The first order condition with respect to saving $a(\Omega^i)$ of consumers of type i is given by:

$$\begin{aligned} \frac{\partial L}{\partial a(\Omega^i)} &= (\frac{1}{I})\sum_i \left\{ \begin{array}{c} -u'(\Omega^i - a(\Omega^i)) \\ +\beta E r(k,z^j)u'(r(k,z^j)a(\Omega^i) + w(k,z^j)e_n) \\ +\lambda^i \left[\begin{array}{c} u''(\Omega^i - a^i) \\ +\beta E(1-\tau^k)r(z^j,k)^2 u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n \end{array} \right) \end{array} \right] \end{array} \right\} \quad \text{(C.17)} \\ -\frac{\psi}{I} &= 0 \end{aligned}$$

while the condition with respect to average per consumer saving k is:

$$\frac{\partial L}{\partial k} = \quad \text{(18)}$$

$$(\frac{1}{I})\sum_i \left\{ \begin{array}{c} \left\{ \beta E \left(\begin{array}{c} \frac{\partial r(z^j,k)}{\partial k} a(\Omega^i) \\ +\frac{\partial w(z^j,k)}{\partial k} e_n \end{array} \right) u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n \end{array} \right) \right\} \\ +\lambda^i \left[\begin{array}{c} \beta E(1-\tau^k)r(z^j,k) \left(\begin{array}{c} \frac{\partial r(z^j,k)}{\partial k} a(\Omega^i) \\ +\frac{\partial w(z^j,k)}{\partial k} e_n \end{array} \right) u'' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n) \end{array} \right) \\ +\beta E(1-\tau^k)\frac{\partial r(z^j,k)}{\partial k} u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n) \end{array} \right) \end{array} \right] \end{array} \right\}$$

$$+\psi = 0$$

Finally, the first order condition with respect to the tax rat τ^k is:

$$\frac{\partial L}{\partial \tau^k} = -(\frac{1}{I})\sum_i \left\{ \lambda^i \left[\beta \sum_j \sum_n \Pr(z^j \cap e_n) r(z^j,k) u' \left(\begin{array}{c} r(z^j,k)a(\Omega^i) \\ +w(z^j,k)e_n) \end{array} \right) \right] \right\} = 0 \quad \text{(C.19)}$$

C.4 Tax / Transfer Scheme Two

C.4.1 Asset Rich Consumer's Problem

An asset rich consumer chooses a level of saving $a(\Omega^R)$ to maximise expected utility:

$$\max_{a(\Omega^R)}\left\{u(\Omega^R-a(\Omega^R))+\beta\sum_j\sum_n \Pr(z^j\cap e_n)u\left(\begin{array}{c}(1-\tau^k(z^j))r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n)\end{array}\right)\right\} \quad \text{(C.20)}$$

where $c_1(\Omega^R)$ and $c_2(\Omega^R,z^j,e_n)$ have been substituted out using period budget constraints (3.86). This is an unconstrained optimisation problem in single variable $a(\Omega^R)$, with consumers taking the tax rate $\tau^k(z^j)$ (for every realisation of the aggregate shock z^j) and factor prices $r(z^j,k)$, $w(z^j,k)$ as given. The first order condition is:

$$\frac{\partial}{\partial a(\Omega^R)}=\left\{\begin{array}{c}-u'(\Omega^R-a(\Omega^R))\\ +\beta\sum_j\sum_n \Pr(z^j\cap e_n)(1-\tau^k(z^j))r(z^j,k)u'\left(\begin{array}{c}(1-\tau^k(z^j))r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n\end{array}\right)\end{array}\right\}=0 \quad \text{(C.21)}$$

which implies the Euler equation (3.87):

$$u'(\Omega^R-a^R)=\beta\sum_j\sum_n \Pr(z^j\cap e_n)(1-\tau^k(z^j))r(z^j,k)u'\left(\begin{array}{c}(1-\tau^k(z^j))r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n\end{array}\right)$$

C.4.2 Asset Poor Consumer's Problem

An asset poor consumer chooses a level of saving $a(\Omega^P)$ to maximise expected utility:

$$\max_{a(\Omega^P)}\left\{u(\Omega^P-a(\Omega^P))+\beta\sum_j\sum_n \Pr(z^j\cap e_n)u\left(\begin{array}{c}r(z^j,k)a(\Omega^P)\\ +w(z^j,k)e_n+T(z^j))\end{array}\right)\right\} \quad \text{(C.22)}$$

where $c_1(\Omega^P)$ and $c_2(\Omega^P,z^j,e_n)$ have been substituted out using period budget constraints (3.90). This is an unconstrained optimisation problem in single variable $a(\Omega^P)$, with consumers taking the level of transfers $T(z^j)$ (for every realisation of the aggregate shock z^j) and factor prices $r(z^j,k)$, $w(z^j,k)$ as given. The first order condition is:

$$\frac{\partial}{\partial a(\Omega^P)}=\left\{\begin{array}{c}-u'(\Omega^P-a(\Omega^P))\\ +\beta\sum_j\sum_n \Pr(z^j\cap e_n)r(z^j,k)u'\left(\begin{array}{c}r(z^j,k)a(\Omega^P)\\ +w(z^j,k)e_n+T(z^j)\end{array}\right)\end{array}\right\}=0 \quad \text{(C.23)}$$

which implies the Euler equation (3.91):

$$u'(\Omega^P-a^P)=\beta\sum_j\sum_n \Pr(z^j\cap e_n)r(z^j,k)u'\left(\begin{array}{c}r(z^j,k)a(\Omega^P)\\ +w(z^j,k)e_n+T(z^j)\end{array}\right)$$

C.4.3 Ramsey Problem

The first order conditions of the Ramsey problem are obtained using Lagrange multipliers. The Lagrangian is:

$$
\begin{aligned}
L &= (\frac{1}{2})\left\{\begin{array}{c} u(\Omega^R - a^R) \\ +\beta E u((1-\left\{\frac{T(z^j)}{r(z^j,k)a(\Omega^R)}\right\})r(z^j,k)a^R + w(z^j,k)e_n) \\ +u(\Omega^P - a^P) + \beta E u(r(z^j,k)a^P + w(z^j,k)e_n + T(z^j)) \end{array}\right\} \qquad \text{(C.24)} \\
&\quad -\lambda^R \left\{\begin{array}{c} u'(\Omega^R - a^R) - \\ \beta E\left(\left(\begin{array}{c} 1- \\ \left\{\frac{T(z^j)}{r(z^j,k)a(\Omega^R)}\right\}\end{array}\right) r(z^j,k)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^R) \\ +w(z^j,k)e_n - T(z^j)\end{array}\right)\right)\end{array}\right\} \\
&\quad -\lambda^P\left\{u'(\Omega^P - a^P) - \beta E r(z^j,k)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^P) \\ +w(z^j,k)e_n + T(z^j)\end{array}\right)\right\} \\
&\quad -\psi\left\{\sum_{i=1}^{I}(\frac{1}{I})\left[a(\Omega^i)\right] - k\right\}
\end{aligned}
$$

where λ^R and λ^P are the Lagrange multipliers attaching to the Euler equations (3.87) and (3.91) of asset rich and asset poor consumers respectively. The symbol ψ is the Lagrange multiplier attaching to (3.105). The government budget balance condition $T(\Omega^i, z^j, e_n) = \tau^k(z^j)r(z^j,k)a(\Omega^R)$ has been used to substitute for $\tau^k(z^j)$.

The first order condition with respect to the saving of asset rich consumers $a(\Omega^R)$ is given by:

$$
\begin{aligned}
\frac{\partial L}{\partial a(\Omega^R)} &= (\frac{1}{2})\left\{\begin{array}{c} -u'(\Omega^R - a^R) \\ +\beta E r(z^j,k)u'(r(z^j,k)a^R + w(z^j,k)e_n - T(z^j))\end{array}\right\} \qquad \text{(C.25)} \\
&\quad +\lambda^R\left\{\begin{array}{c} u''(\Omega^R - a^R) + \beta E r(z^j,k)^2 u''\left(\begin{array}{c} r(z^j,k)a(\Omega^R) \\ +w(z^j,k)e_n - T(z^j)\end{array}\right) \\ +\beta E\left(\frac{T(z^j)}{a(\Omega^R)^2}\right)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^R) \\ +w(z^j,k)e_n - T(z^j)\end{array}\right) \\ -\beta E\left(\frac{T(z^j)}{a(\Omega^R)}\right) r(z^j,k)u''\left(\begin{array}{c} r(z^j,k)a(\Omega^R) \\ +w(z^j,k)e_n - T(z^j)\end{array}\right)\end{array}\right\} \\
-\frac{\psi}{2} &= 0
\end{aligned}
$$

while that for asset poor consumers is:

$$
\begin{aligned}
&(\frac{1}{2})\left\{-u'(\Omega^P - a^P) + \beta E r(z^j,k)u'(r(z^j,k)a^R + w(z^j,k)e_n + T(z^j))\right\} \qquad \text{(C.26)} \\
&+\lambda^P\left\{u''(\Omega^P - a^P) + \beta E r(z^j,k)^2 u''\left(\begin{array}{c} r(z^j,k)a(\Omega^P) \\ +w(z^j,k)e_n + T(z^j)\end{array}\right)\right\} \\
-\frac{\psi}{2} &= 0
\end{aligned}
$$

The first order condition with respect to average saving k per consumer is given by:

$$\begin{aligned}
\frac{\partial L}{\partial k} &= (\frac{1}{2})\left\{\begin{array}{c} \beta E\left[\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^R)\\ +\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right]u'\left(\begin{array}{c} r(z^j,k)a^R\\ +w(z^j,k)e_n-T(z^j)\end{array}\right)\\ +\beta E\left[\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^P)\\ +\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right]u'\left(\begin{array}{c} r(z^j,k)a^P\\ +w(z^j,k)e_n+T(z^j)\end{array}\right)\end{array}\right\} \\
&\quad +\lambda^R\left\{\begin{array}{c} \beta E\frac{\partial r(z^j,k)}{\partial k}u'\left(\begin{array}{c} r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n-T(z^j)\end{array}\right)\\ \beta E\left[\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^R)\\ +\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right]r(z^j,k)u''\left(\begin{array}{c} r(z^j,k)a^R\\ +w(z^j,k)e_n-T(z^j)\end{array}\right)\\ -\beta E\left[\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^R)\\ +\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right]\left(\frac{T(z^j)}{a(\Omega^R)}\right)u''\left(\begin{array}{c} r(z^j,k)a^R\\ +w(z^j,k)e_n-T(z^j)\end{array}\right)\end{array}\right\} \\
&\quad +\lambda^P\left\{\begin{array}{c} \beta E\frac{\partial r(z^j,k)}{\partial k}u'\left(\begin{array}{c} r(z^j,k)a(\Omega^P)\\ +w(z^j,k)e_n+T(z^j)\end{array}\right)\\ \beta E\left[\begin{array}{c}\frac{\partial r(z^j,k)}{\partial k}a(\Omega^P)\\ +\frac{\partial w(z^j,k)}{\partial k}e_n\end{array}\right]r(z^j,k)u''\left(\begin{array}{c} r(z^j,k)a^P\\ +w(z^j,k)e_n+T(z^j)\end{array}\right)\end{array}\right\} \\
+\Psi &= 0
\end{aligned} \tag{C.27}$$

The final first order conditions of the Ramsey problem are with respect to the level of transfers $T(z^j)$ (that can vary with the realisation of the aggregate shock) and are given by:

$$\begin{aligned}
\frac{\partial L}{\partial T(z^j)} &= (\frac{1}{2})\left\{\begin{array}{c} -\beta\Pr(z^j)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n-T(z^j))\end{array}\right)\\ +\beta\Pr(z^j)u'\left(\begin{array}{c} r(z^j,k)a(\Omega^P)\\ +w(z^j,k)e_n+T(z^j))\end{array}\right)\end{array}\right\} \\
&\quad +\lambda^R\left\{\begin{array}{c} -\beta\Pr(z^j)(\frac{1}{a(\Omega^R)})u'\left(\begin{array}{c} r(z^j,k)a(\Omega^R)\\ +w(z^j,k)e_n-T(z^j)\end{array}\right)+\\ \beta\Pr(z^j)(\frac{T(z^j)}{a(\Omega^R)})u''(r(z^j,k)a(\Omega^R)+w(z^j,k)e_n-T(z^j))\\ -\beta\Pr(z^j)r(z^j,k)u''(r(z^j,k)a(\Omega^R)+w(z^j,k)e_n-T(z^j))\end{array}\right\} \\
&\quad +\lambda^P\beta E r(z^j,k)u''(r(z^j,k)a(\Omega^P)+w(z^j,k)e_n+T(z^j)) \\
&= 0
\end{aligned} \tag{C.28}$$