CONFIDENCE INTERVAL METHODS IN ~~FOR~~ DISCRETE EVENT COMPUTER SIMULATION: THEORETICAL PROPERTIES AND PRACTICAL RECOMMENDATIONS

by

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ABSTRACT

Most of steady state simulation outputs are characterized by some degree of dependency between successive observations at different lags measured by the autocorrelation function. In such cases, classical statistical techniques based on independent, identical and normal random variables are not recommended in the construction of confidence intervals for steady state means. Such confidence intervals would cover the steady state mean with probability different from the nominal confidence level.

For the last two decades, alternative confidence interval methods have been proposed for stationary simulation output processes. These methods offer different ways to estimate the variance of the sample mean with final objective of achieving coverages equal to the nominal confidence level. Each sample mean variance estimator depends on a number of different parameters and the sample size.

In assessing the performance of the confidence interval methods, emphasis is necessarily placed on studying the actual properties of the methods in an empirical context rather than proving their mathematical properties. The testing process takes place in the context of an environment where certain statistical criteria, which measure the actual properties, are estimated through Monte Carlo methods on output processes from different types of simulation models.

Over the past years, however, different testing environments have been used. Different methods have been tested on different output processes under different sample sizes and

parameter values for the sample mean variance estimators. The diversity of the testing environments has made it difficult to select the most appropriate confidence interval method for certain types of output processes. Moreover, a catalogue of the properties of the confidence interval methods offers limited direct support to a simulation practitioner seeking to apply the methods to particular processes.

Five confidence interval methods are considered in this thesis. Two of them were proposed in the last decade. The other three appeared in the literature in 1983 and 1984 and constitute the recent research objects for the statistical experts in simulation output analysis. First, for the case of small samples, theoretical properties are investigated for the bias of the corresponding sample mean variance estimators on AR(1) and AR(2) time series models and the delay in queue in the M/M/1 queueing system. Then an asymptotic comparison for these five methods is carried out. The special characteristic of the above three processes is that the s^{th} lag autocorrelation coefficient is given by known difference equations.

Based on the asymptotic results and the properties of the sample mean variance estimators in small samples, several recommendations are given in making the following decisions:

I) The selection of the most appropriate confidence interval method for certain types of simulation outputs.

II) The determination of the best parameter values for the sample mean variance estimators so that the corresponding confidence interval methods achieve acceptable performances.

III) The orientation of the future research in confidence interval estimation for steady state autocorrelated simulation outputs.

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To My parents

Stavros and Popi

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CONFIDENCE INTERVAL METHODS FOR STEADY-STATE SIMULATIONS

1.1 INTRODUCTION

In studying real world systems, computer simulation has been accepted as a powerful technique to providing useful information to support decision making under conditions of uncertainty.

For the last three decades, different definitions have been offered by several authors of computer simulation. Some of them, which define simulation as the art for modelling discrete systems, are given by Balmer and Paul(1985). Others which give more emphasis to the stage of experimentation are the following:

> *" Simulation implies experimentation. However,instead of experimenting with the real world object, we experiment by means of the model of that object. "*
> *[Kleijnen(1974)]*

> *" In a simulation, we use a computer to evaluate a model numerically over a time period of interest and data are gathered to estimate the desired true characteristics of the model. "*
> *[Law & Kelton(1982b)]*

All the definitions describe computer simulation as an attempt to represent the operation of a system in a computer program. This can be achieved via a "valid" simulation model which depicts the relationships between the system entities. Then, by using the simulation program, alternative operating policies can be compared in a well organized experimentation. Thus the best policy to the management can be selected.

For several years, at the London School of Economics, the Computer Aided Simulation Modelling(CASM) project has been researching the appropriate software support to facilitate the processes of both developing simulation models and generating computer simulation programs. As examples, we mention the work of Doukidis and Paul(1985) on simulation problem formulation using expert systems, Chew(1986) on interactive simulation program generators and El Sheikh(1987) on simulation modelling using a relational data-base system called INGRESS.[For a more detailed description about CASM objectives see Balmer and Paul(1986)]. Research in the statistical aspects of discrete event simulation within the CASM project has been limited.

This thesis describes research into confidence interval estimation for steady-state means of simulation output processes. This research can be considered as a continuation of the CASM project to the general area of the statistical analysis of simulation outputs.

Due to the problems of autocorrelation and initial transient state, the classical confidence interval estimator is not valid. For this reason, several confidence interval methods have been developed for estimating the variance of the sample mean in stationary autocorrelated processes. Five such methods are considered in this thesis. For both small and large sample sizes, the performance of the five methods is evaluated on different simulation output processes. This performance is measured by certain statistical criteria. For large sample sizes these criteria are computed analytically. For small samples, the criteria are estimated empirically using Monte Carlo methods.

In this introductory chapter, the theory of confidence interval estimation in steady-state simulation outputs is presented. In the next section, the problems of autocorrelation and initial transience, which do not allow the use of the classical estimator for constructing confidence intervals for true steady-state means, are discussed. Alternative ways to overcome these problems are briefly described in section three. A survey of fixed sample size confidence interval methods for steady-state means is provided in section four. Sections five and six give a more detailed description of the thesis objective and the structure of the remaining chapters respectively.

1.2 THE PROBLEMS OF AUTOCORRELATION AND INITIAL TRANSIENCE IN SIMULATION OUTPUT ANALYSIS

Let $\{X_t, t=1,2,3,...\}$ be a covariance stationary output process. Covariance stationary means that the mean and variance of the random variables X_t are stationary over time with common finite mean μ and common finite variance γ_0. Moreover, for a covariance stationary process, the covariances $Cov(X_t, X_{t+s})$ between X_t and X_{t+s} depends only on the lag s and not on the actual values at times t and t+s.

Consider a sample $X_1, X_2, ..., X_n$ of size n from $\{X_t\}$. At the stage of reporting the results of simulation experiments, the statistical measure often used is the sample mean

$$X_n = \frac{\sum_{t=1}^{n} X_t}{n}$$

as the best estimator of the steady state mean $\mu = E(X_t)$ for $t=1,2,3,\ldots$.

Consider the case where r number of replications of the process $\{X_t, 1 \leq t \leq n\}$ are generated by choosing the initial conditions to be identical to the steady-state conditions. Applying the above estimator to these replications, we produce r estimates $\bar{x}_{nj} (1 \leq j \leq r)$. These estimates will vary about the true steady-state mean μ. Consequently, any particular value may lie far away from μ, especially for sampling distributions characterized by large variances. Therefore, the report of any $\bar{x}_{nj}$ without any measure of its precision would provide misleading information at the stage of decision making.

The most familiar representation of precision is provided by the confidence interval for the steady-state mean μ. If X_1, $X_2, \ldots,$ X_n were independent, identical and normal random variables, the classical confidence interval estimator for μ would be

$$\bar{X}_n - t_{n-1,\alpha/2} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{X}_n + t_{n-1,\alpha/2} \frac{S}{\sqrt{n}}$$

where

$$S^2 = \frac{\sum_{t=1}^{n} \left[X_t - \bar{X}_n \right]^2}{n - 1} \tag{1.1}$$

is an unbiased estimator of the variance of the output process.

However, in most of simulation modelling, successive observations display forms of autocorrelation. Let $\rho_\tau = \mathrm{corr}(X_t, X_{t+\tau})$ be the τ^{th} lag theoretical autocorrelation coefficient for the process $\{X_t\}$. Fishman(1973b, 1978b) records the true variance of the sample mean as

$$V(\bar{X}_n) = \frac{\gamma_0}{n}\left[1 + 2\sum_{\tau=1}^{n-1}\left[1 - \frac{\tau}{n}\right]\rho_\tau\right] \tag{1.2}$$

Consider now the following estimator of the true variance of the sample mean

$$\widehat{V(\bar{X}_n)} = \frac{S^2}{n}\left[1 + 2\sum_{\tau=1}^{n-1}\left[1 - \frac{\tau}{n}\right]\rho_\tau\right] \tag{1.3}$$

Define also the ratio

$$c^2 = \frac{S^2/n}{\widehat{V(\bar{X}_n)}}$$

Providing that $X_1, X_2, \ldots, X_n$ are normal variables with $E(X_t)=\mu$ for $1 \leqslant t \leqslant n$, the actual confidence level achieved by the classical interval estimator is going to be

$$\Pr\left[-t_{n-1,\alpha/2} \leqslant \frac{\bar{X}_n - \mu}{S/\sqrt{n}} \leqslant t_{n-1,\alpha/2}\right] =$$

$$= \Pr\left[-ct_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{\widehat{V(\bar{X}_n)}}} < ct_{n-1,\alpha/2} \right] \tag{1.4}$$

When c is less(greater) than one, the probability defined in (1.4) is less(greater) than the nominal confidence level $(1-\alpha)$.

For instance, consider the first order autoregressive process{AR(1)} which has the form

$$X_t = \varphi X_{t-1} + \epsilon_t \ , \ t=1,2,3....$$

This process is stationary when $|\varphi|<1$. The ϵ_t's are independent and normal random variables with mean zero and common variance σ_p^2. For positive φ, the s_{th} lag theoretical autocorrelation coefficient is φ^s. Given that

$$\sum_{s=1}^{n-1} \varphi^s = \frac{\varphi(1-\varphi^{n-1})}{(1-\varphi)} \quad \text{and} \quad \sum_{s=1}^{n-1} s\varphi^s = \frac{\varphi\{1-n\varphi^{n-1}+(n-1)\varphi^n\}}{(1-\varphi)^2}$$

the constant c^2 will be

$$c^2 = \frac{1}{1 + 2\sum_{s=1}^{n-1}\left[1 - \frac{s}{n}\right]\rho_s} = \frac{1}{\frac{1+\varphi}{1-\varphi} - \frac{2\varphi(1-\varphi^n)}{n(1-\varphi)^2}} \tag{1.5}$$

Taking the square root of (1.5) and substituting it into (1.4), the actual confidence levels achieved by the classical confidence interval estimator can be computed analytically. From table (1.1) for positive φ, these levels are lower than the nominal confidence level 0.95. Moreover, they are decreasing as n is increasing.

TABLE 1.1
Actual confidence levels of the classical confidence interval estimator in the AR(1) when the nominal confidence level is 95%

φ	SAMPLE SIZES				
	10	20	50	100	∞
.10	0.9313	0.9277	0.9254	0.9246	0.9232
.50	0.8058	0.7738	0.7548	0.7485	0.7416
.80	0.6430	0.5606	0.5139	0.4998	0.4844
.90	0.5766	0.4628	0.3889	0.3667	0.3472
.999	0.5079	0.3559	0.2243	0.1594	0.0320

Let us now assume that the initial conditions for the process $\{X_t\}$ are not identical to the steady state conditions. In such cases, there is a transient period where the random variables $X_1, X_2, \ldots, X_n$ are distributed with mean $\mu_n = \mu + B_n$ where

$$\lim_{n\to\infty} B_n = 0$$

In the simulation literature, the factor B_n is called initialization bias. Providing that the random variables X_1, $X_2, \ldots, X_n$ are normally distributed

$$\Pr\left[-t_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu_n}{\sqrt{\widehat{V(\bar{X}_n)}}} < t_{n-1,\alpha/2} \right]$$

$$= \Pr\left[-t_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{\widehat{V(\bar{X}_n)}}} - \frac{\mu_n - \mu}{\sqrt{\widehat{V(\bar{X}_n)}}} < t_{n-1,\alpha/2} \right] \tag{1.6}$$

In (1.6), $(\bar{X}_n-\mu)/\{\widehat{V(\bar{X}_n)}\}^{\frac{1}{2}}$ is distributed as a noncentral t-distribution with noncentrality parameter $(\mu_n-\mu)/\{\widehat{V(\bar{X}_n)}\}^{\frac{1}{2}}$. From Owen(1965), it can be verified that the probabilities defined in (1.6) are lower than the nominal confidence level $(1-\alpha)$. Furthermore, the use of the classical estimator (1.1) instead of the estimator defined in (1.3) or the nonnormality of the random variables $X_1, X_2, \ldots, X_n$ makes the problem of initialization bias even more serious.

1.3 DIFFERENT WAYS TO OVERCOME THE PROBLEMS OF AUTOCORRELATION AND INITIAL TRANSIENT STATE

Fixed sample size confidence interval methods

For stationary simulation output processes, these methods produce different estimators for the variance of the sample mean, providing that the sample size n is fixed a-priori. We shall call these estimators "sample mean variance estimators". The derivation of these estimators is based on transforming the original output process into a new process which has desirable and known statistical properties. The final objective of these methods is to produce confidence intervals which will cover the steady state mean with probability equal to the nominal confidence level. However, the question which arises is why we do not use the estimator of the true variance of the sample mean defined in (1.3). The reason is that in most simulation output processes the theoretical autocorrelation coefficients are not known and as Law and Kelton(1982b) point out, the estimation of these coefficients is not recommended since

- for large n, the computing time to estimate $\rho_s (1 \leq s \leq n-1)$ is rather large and
- for s close to n, the estimation of ρ_s will be based only on few observations.

Moreover, for simulation output processes characterized by different autocorrelation structures, the sample size which guarantees the adequacy of a normal approximation is not known, although there are some indications that this is not a major problem[see for example Law(1977) and Kleijnen(1975,page 445)].

The technical details of these methods are described in the next section.

<u>Sequential confidence interval methods</u>

The objective of these methods is to determine the run length(sample size) of realizations of stationary simulation output processes which guarantees both an adequate correspondence between actual and nominal confidence levels and a prespecified absolute or relative precision. The last two terms are defined by Law(1983) as the half length of confidence intervals and the ratio of half length over the sample mean respectively.

Law and Kelton(1982a) distinguish these methods as regenerative and non-regenerative. The methods classified in the first type determine the run length by using the regenerative property, that is, they identify random points where the process probabilistically starts over again. Fishman's(1977) and Lavenberg and Sauer's(1977) methods belong to this category. The methods developed by Mechanic and Mckay(1966), Law and Carson(1978), Heidelberger and Welch(1981a) and Adam(1983) have been characterized as non-regenerative.

Law and Kelton(1982a) compared the performance of several sequential methods. For the output processes the authors used, the required run lengths for obtaining acceptable confidence intervals were quite large.

Truncation methods

Their objective is the elimination of initialization bias effects. These methods provide estimators for the time point $t^*(1 \leq t \leq n)$ for which

$$| E(X_t) - \mu | > e \quad \text{for} \quad t<t^*$$

and

$$| E(X_t) - \mu | \leq e \quad \text{for} \quad t \geq t^*$$

where e is a prespecified very small positive number.

Let $\{x_{ij}: 1 \leq i \leq n, 1 \leq j \leq r\}$ be r replications of the simulation output process $\{X_t\}$. For each replication, the initial conditions are exactly the same. Some of the truncation methods estimate t^* by applying the truncation rule to each replication. The methods of Fishman(1971,1973b), Schriber(1974) and Heidelberger and Welch(1983) can be classified into this category. Other methods estimate t^* from a pilot study which is carried out on a number of exploratory replications. Then the estimated value of t^* is used as the global truncation point in any other replication for which we use the same initial conditions. The methods of Conway(1963), Gordon(1969), Gafarian et al(1978) and Kelton and Law(1983) belong to this category.

1.4 FIXED SAMPLE SIZE CONFIDENCE INTERVAL METHODS

Let $\{X_t\}$ be a steady-state simulation output process. Suppose also that

$$E(X_t)=\mu < \infty \qquad t \geq 1$$

and

$$Var(X_t)=\gamma_0 < \infty \qquad t \geq 1$$

Fixed sample size confidence interval methods propose different estimators for the variance of the sample mean (sample mean variance estimators). Let $\hat{\sigma}_i^2$ be the sample mean variance estimator of the i^{th} method. Then, the confidence interval proposed by the i^{th} method will take the form

$$\bar{X}_n - t_{v_i,\alpha/2}\hat{\sigma}_i \leq \mu \leq \bar{X}_n + t_{v_i,\alpha/2}\hat{\sigma}_i$$

where $\bar{X}_n$ is the sample mean and v_i are the degrees of freedom according to the i^{th} method. Presented below are several confidence interval methods which have been developed for the last two decades.

Replication Method

Suppose we generate $k>1$ independent replications of the simulation output process $\{X_t\}$ by using independent streams of random numbers. The run length(sample size) of each replication is m. Define the sample mean of the j^{th} $(1<j<k)$ replication as

$$\bar{X}_{jm} = \frac{\sum_{t=1}^{m} X_{tj}}{m} \quad , \quad 1<j<k \tag{1.7}$$

where $\{X_{tj}\}$ is the t^{th} random variable on the j^{th} replication.

When m is large enough, the k sample means defined in (1.7) can be considered as independent, identical and normal random variables. Then, the sample mean variance estimator proposed by this method is given by

$$\hat{\sigma}^2_{IR} = \frac{1}{k(k-1)} \sum_{j=1}^{k} \left[\bar{X}_{jm} - \bar{X}_n \right]^2 \quad , \quad 1<j<k$$

The degrees of freedom are $v_{IR}=k-1$.

Nonoverlapping Batch Means Method

Let $\{X_t\}$ be a covariance stationary output process. The nonoverlapping batch means method is based on generating a single long replication of $\{X_t\}$. This replication is partitioned into $k>1$ contiguous and nonoverlapping batches. The size of each batch is m. The batch mean of each batch is defined as

$$\bar{X}_{j,m} = \frac{\sum_{t=1}^{m} X_{(j-1)m+t}}{m} \quad , \ 1 \leq j \leq k$$

Provided that m is large enough and $\sum_{s=-\infty}^{\infty} |\gamma_s| < \infty$, Law and Carson(1978) showed that the nonoverlapping batch means can be considered approximately uncorrelated. Furthermore, if we choose m large, the batch means can be considered approximately normal random variables. Then the sample mean variance estimator of this method is given by

$$\hat{\sigma}^2_{NB} = \frac{1}{k(k-1)} \sum_{j=1}^{k} \left[\bar{X}_{j,m} - \bar{X}_n \right]^2 \quad , \quad 1 \leq j \leq k$$

As in the replication method, the degrees of freedom are $v_{NB}=k-1$.

Overlapping Batch Means Method

Consider a single long replication of a covariance stationary output process $\{X_t\}$. Let n be the run length(sample size) of this replication. The j^{th} overlapping batch mean of size m is defined as

$$\bar{X}_j(m) = \frac{\sum_{t=0}^{m-1} X_{j+t}}{m} \quad , \quad 1 \leq j \leq n-m+1$$

For large m and n/m, Welch(1987) proposed the following estimator for the variance of the sample mean

$$\hat{\sigma}^2_{OB} = \frac{m}{n(n-m+1)} \sum_{j=1}^{n-m+1} \left[\bar{X}_j(m) - \bar{X}_n \right]^2$$

The degrees of freedom are 1.5((n/m)-1).

<u>Standardized Time Series Methods</u>

Let $\{X_t\}$ be a strictly stationary output process. Strictly stationary means that the joint distribution of $Xt_1, Xt_2, \ldots, Xt_n$ is the same as the joint distribution of $Xt_1+s, Xt_2+s, \ldots, Xt_n+s$ for every $t_1, t_2, \ldots, t_n$ and s. We also assume that this process is phi-mixing. Roughly speaking, any process is phi-mixing when $Corr(X_t, X_{t+s})$ is negligible for large s[see Law(1983)]. In fact, the phi-mixing property is satisfied by a wide class of processes including autoregressive, regenerative and m-dependent processes [see Schruben(1983)].

The standardized time series methods use a functional central limit theorem to transform $X_1, X_2, \ldots, X_n$ into a process which is asymptotically distributed as a Brownian Bridge process. Suppose that we divide the replication into k>1 contiguous and nonoverlapping batches of size m. For large m, by using Brownian

Bridge properties, Schruben(1983) derived the following four estimators for the variance of the sample mean:-

I) <u>AREA METHOD</u>

$$\hat{\sigma}^2_{SM} = \frac{12}{nk(m^3-m)} \sum_{j=1}^{k} \hat{A}_j^2$$

where

$$\hat{A}_j = \sum_{\ell=1}^{m} \ell S_j(\ell))$$

$$S_j(\ell) = X_{j,m} - X_{j,\ell}$$

and

$$X_{j,\ell} = \frac{\sum_{t=1}^{\ell} X_{(j-1)m+t}}{\ell} \quad , \quad 1 \leqslant j \leqslant k$$

Here the degrees of freedom are $v_{SM}=k$.

II) <u>MAXIMUM METHOD</u>

$$\hat{\sigma}^2_{MX} = \frac{m}{3kn} \sum_{i=1}^{k} \left[\frac{\left[\hat{\ell}_j S_j(\hat{\ell}_j)\right]^2}{\hat{\ell}_j(m-\hat{\ell}_j)} \right]$$

where $\hat{\ell}_j$ is the location of the maximum of the process $\ell S_j(\ell)$ on the j^{th} batch. The degrees of freedom are $v_{MX}=3k$.

III) COMBINED AREA-NONOVERLAPPING BATCH MEAN METHOD

$$\hat{\sigma}^2_{CM} = \frac{\dfrac{12}{m^3-m}\sum_{j=1}^{k}\hat{A}_j^2 + m\sum_{j=1}^{k}\left[\bar{X}_{j,m} - \bar{X}_n\right]^2}{n(2k-1)}$$

The degrees of freedom are $v_{CM}=2k-1$.

IV) COMBINED MAXIMUM-NONOVERLAPPING BATCH MEANS METHOD

$$\hat{\sigma}^2_{CX} = \frac{m\left[\sum_{j=1}^{k}\left[\dfrac{\left[\hat{\ell}_j S_j(\hat{\ell}_j)\right]^2}{\hat{\ell}_j(m-\hat{\ell}_j)}\right] + \sum_{j=1}^{k}\left[\bar{X}_{j,m} - \bar{X}_n\right]^2\right]}{n(4k-1)}$$

For this method, the degrees of freedom are $v_{CX}=4k-1$.

Spectral method

The spectral method assumes that the process $\{X_t\}$ is covariance stationary. At zero frequency, the power spectrum of a covariance stationary process is given by

$$f(0) = \frac{\sum_{s=-\infty}^{\infty}\gamma_s}{2\pi} \tag{1.8}$$

where

$$\gamma_s = E\,(X_t - \mu)(X_{t+s} - \mu)$$

is the s^{th} lag theoretical autocovariance.

From (1.2),

$$\lim_{n\to\infty} \left[nV(\bar{X}_n) \right] = \sum_{s=-\infty}^{\infty} \gamma_s = 2\pi f(0) \qquad (1.9)$$

For large n, form (1.9) proposes an other way for estimating the variance of the sample mean in autocorrelated stationary processes; that is, by estimating the power spectrum at zero frequency.

In the simulation literature, two methods for estimating f(0) have been proposed. The first discussed by Fishman(1973b,1978b), Duket and Pritsker(1978) and Law and Kelton(1984) uses the Tukey spectral window

$$\lambda_w(s) = 0.5 \{ 1 + \cos(\pi s/w) \}$$

for estimating f(0) as

$$\hat{f}(0) = \frac{1}{2\pi} \left[\hat{\gamma}_0 + 2 \sum_{s=1}^{w-1} \lambda_w(s)\hat{\gamma}_s \right]$$

To reduce bias, these authors have proposed the following sample mean variance estimator:-

$$\hat{\sigma}^2_{SP} = \frac{1}{n-w} \left[\hat{\gamma}_0 + 2 \sum_{s=1}^{w-1} \lambda_w(s)\hat{\gamma}_s \right]$$

where

$$\hat{\gamma}_s = \frac{1}{n} \sum_{t=1}^{n-s} \left[X_t - \bar{X}_n \right] \left[X_{t+s} - \bar{X}_n \right]$$

Fishman(1973b,1978b) and Law and Kelton(1984) report that $v_{SP}=1.33n/w$.

Heidelberger and Welch(1981a) have proposed a different way for estimating f(0). This way is based on the periodogram coordinates

$$I(j/n) = \frac{1}{n} \left| \sum_{t=1}^{n} X_t e^{-2\pi i(i-1)j/n} \right|^2$$

where $i=\sqrt{-1}$.

Define K points $J(a_j)$

$$J(a_j) = \log\left[\tfrac{1}{2}\left[I((2j-1)/n) + I(2j/n)\right]\right]$$

with

$$a_j = (4j-1)/(2n) \ , \ j=1,2 \ ,\ldots,K$$

A polynomial of the form $\sum_{r=0}^{d} b_r a_r$ is fitted to $J(a_j)+270$ for $j=1,2,\ldots K$ using the least squares method. Let $\hat{b}_0$ be the least square estimator of b_0. Then an approximately unbiased estimator for f(0) is given by

$$\hat{f}_{HW}(0) = \frac{1}{2\pi} \exp\left[\frac{-\beta^2}{2}\right] \exp\left[\hat{b}_0\right]$$

with $\beta^2 = .645 s_{11}$ and s_{11} is the upper leftmost element of the product $(X'X)^{-1}$ where

$$X = \begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & a_2^3 & \cdots & a_2^d \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_k & a_k^2 & a_k^3 & & a_k^d \end{bmatrix}$$

For this method, the degrees of freedom are

$$\nu_{HW} = \frac{2}{e^{(\beta)^2} - 1}$$

<u>Autoregressive Method</u>

This method assumes that $\{X_t\}$ is a covariance stationary process and can be represented by the p^{th} order autoregressive process (AR(p))

$$\sum_{s=0}^{p} \varphi_{p,s}(X_{t-s} - \mu) = \epsilon_t \quad , \quad \varphi_{p,0} = 1$$

The ϵ_t's are independent, identical and normal random variables with mean 0 and common variance σ_p^2. We also assume that the autoregressive order is known and $\sum_{s=-\infty}^{\infty} |\gamma_s| < \infty$. For this process the power spectrum at zero frequency is given by

$$f(0) = \frac{\sigma_p^2}{2\pi \left[\sum_{s=0}^{p} \varphi_{p,s} \right]^2}$$

Durbin(1960) has developed a procedure for estimating the autoregressive coefficients of an AR(p) from the autoregressive coefficients of an AR(p-1) via the following recursive formulae

$$\hat{\varphi}_{0,0} = \hat{\varphi}_{p,0} = 1$$

$$\hat{\varphi}_{p,p} = \frac{\sum_{s=0}^{p-1} \hat{\varphi}_{p-1,s}\hat{\gamma}_{p-s}}{\sum_{s=0}^{p-1} \hat{\varphi}_{p-1,s}\hat{\gamma}_{s}}$$

$$\hat{\varphi}_{p,s} = \hat{\varphi}_{p-1,s} + \hat{\varphi}_{p,p}\hat{\varphi}_{p-1,p-s}\ , \quad s=1,2,\ldots,p-1$$

Moreover, the error variance σ_p^2 is estimated by

$$\hat{\sigma}_p^2 = \hat{\gamma}_0 + \hat{\varphi}_{p,1}\hat{\gamma}_1 + \hat{\varphi}_{p,2}\hat{\gamma}_2 + \cdots + \hat{\varphi}_{p,p}\hat{\gamma}_p$$

Fishman(1971,1973b,1978b) has proposed the order p of the AR(p) to be determined through the following test of hypothesis

H_0 : the order of the autoregressive scheme is p
H_1 : the order of the autoregressive scheme is q>p

As the sample size n increases, the statistic

$$n\left[1 - \frac{\hat{\sigma}_q^2}{\hat{\sigma}_p^2}\right]$$

converges to a χ^2 distribution with q-p degrees of freedom. Assuming that n is large enough, by setting p=1,2,3,..., the estimated order is the smallest p for which H_0 is accepted.

Providing that the autoregressive coefficients, the error variance and the autoregressive order have already been estimated, the power spectrum at zero frequency of the AR(p) is estimated by

$$\hat{f}(0) = \frac{\hat{\sigma}^2_p}{2\pi \left[\sum_{s=0}^{p} \hat{\varphi}_{p,s} \right]^2}$$

Then, the corresponding sample mean variance estimator and the degrees of freedom are given by

$$\hat{\sigma}^2_{AR} = \frac{\hat{\sigma}^2_p}{n \left[\sum_{s=0}^{p} \hat{\varphi}_{p,s} \right]^2}$$

and

$$v_{AR} = \frac{n \sum_{s=0}^{p} \hat{\varphi}_{p,s}}{2 \sum_{s=0}^{p} (p-2s)\hat{\varphi}_{p,s}}$$

For the derivation of the degrees of freedom see Fishman(1978b).

<u>Regenerative Method</u>

This method was developed simultaneously by Crane and Iglehart(1974a,b,c,1975). Its principle is based on the identification of random points where the process probabilistically starts over again. These points are called regeneration points. For example, for the delay in queue in the M/M/1 queueing model, the indices of customers who find the system completely empty could be considered as regeneration points. The amount of data between two regeneration points is called regeneration cycle.

Define now the random variables N_j, $Z_j(j=1,2,..)$ as

$$N_j = B_{j+1} - B_j \ , \ E(N_j)<\infty \ , \ j=1,2,\ldots$$

where $1 \le B_1 < B_2 < \ldots$ are regeneration points and

$$Z_j = \sum_{i=B_j}^{B_{j+1}-1} X_i$$

Providing that $E(N_j)<\infty$, the steady-state mean is defined as

$$\mu = E(Z)/E(N)$$

Two methods have been developed to estimate μ and produce confidence intervals for μ; the classical and Jacknife methods. A very good description of them is given in Law and Kelton(1982b). However, the major disadvantage of these methods is the identification of regeneration points, especially for complicated simulation models.

1.5 THESIS OBJECTIVE

This thesis presents new findings for the performance of the following five confidence interval methods:-

i) Nonoverlapping batch means method

ii) Standardized time series-area method

iii) Combined area-nonoverlapping batch means method

iv) Overlapping batch means method

v) Spectral method.

Given the sample size, each method achieves different actual confidence levels for different parameter values. The term "parameter value" indicates the number of batches for the first three methods, the batch size for the overlapping batch means method and the spectral window size for the spectral method.

For the case of small sample sizes, we compare the best actual confidence levels achieved by the above five methods. With respect to each method, the best actual confidence level is defined to be the one which is the closest to the nominal confidence level. Moreover, we consider the case where two or more methods attain approximately the same best actual confidence levels. Under such circumstances, we compare the precision and stability of confidence intervals produced by the five methods at the parameter values for which these confidence levels are attained.

Furthermore, for small sample sizes, we compare the performance of the five confidence interval methods at specific parameter values. These values are chosen in such a way that the minimum bias of the sample mean variance estimator of each method is observed. We call these values MB-parameter values. To determine the MB-parameter values, a family of functions is introduced. We call them "Bias Indicator functions". These functions are expressed in terms of the theoretical autocorrelation coefficients of the output process under study. This means that if the autocorrelation function of the process under study is known, exact analytical values for the minimum bias of each estimator and the MB-parameter values can be obtained.

On the other hand, for processes where the autocorrelation coefficients are not known, we propose two ways for estimating the minimum bias and the MB-parameter values. Based on the performance of the methods at the estimated MB-parameter values, we develop a procedure for applying the five confidence interval methods to approximately steady state simulation outputs displaying certain characteristics. These characteristics refer to the form of the autocorrelation function and the level of non-normality of the process.

To compare the performance of the five confidence interval methods in the above two contexts, we have created our own testing environment. In this environment, we have included almost all the output processes which have been used in other testing environments having been developed during the past two decades. Several statistical criteria have also been selected for studying the performance of the methods. For small sample sizes, these criteria have been estimated by using Monte Carlo methods.

Comparisons between the performance of the five confidence interval methods are also carried out when the sample size tends to infinity. The asymptotic forms of the Bias Indicator functions enable us to compute analytically the limiting coverages achieved by the above five methods, provided that the autocorrelation function of the output process under study is known. Three such processes are considered in this case; the AR(1), the AR(2) and the delay in queue in the M/M/1. For these processes, we study the limiting coverages of each method at different parameter values.

Provided that the simulation output process satisfies certain regularity conditions, as the batch size m tends to infinity, the nonoverlapping batch means, area and combined NOBM-AREA methods tend to achieve actual confidence levels equal to the nominal confidence level. For the spectral and overlapping batch means method, when the batch size m and the spectral window size w tend to infinity but in such a way that $(n/m)\to\infty$ and $(n/w)\to\infty$, these two methods tend to cover the true steady state mean with the nominal probability. Assuming such ideal cases, we compare the limiting precision and stability of the confidence intervals.

For the case of large sample sizes, all the statistical criteria considered are computed numerically i.e without using Monte Carlo methods.

1.6 STRUCTURE OF THE THESIS

Chapter two describes a survey on testing environments which have been used for evaluating the performance of confidence interval methods.

Chapter three introduces a family of functions which enable us to determine analytically both the minimum bias of each sample mean variance estimator under consideration and the MB-parameter values, provided that the autocorrelation function of the process under study is known. Exact values for the minimum bias and the MB-parameter values are obtained in the autoregressive process of order one.

Chapter four examines the asymptotic properties of the five confidence interval methods under consideration. Two issues are considered. The first concerns the computation of the limiting actual confidence levels the five confidence interval methods achieve. The second issue refers to the comparison of the limiting precision and stability of the confidence intervals produced by these methods

Chapter five describes the preparation stages for the simulation experiments which follow.

Chapter six examines the performance of the five methods at the MB-parameter values for the AR(1), AR(2) and the delay in queue in the M/M/1. Both true and estimated MB-parameter values are considered.

Chapter seven compares the best actual confidence levels achieved by the five confidence interval methods and provides several recommendations for applying these methods to approximately steady state simulation output processes.

Chapter eight summarizes the conclusions and suggests future areas of research.

C H A P T E R   TWO

A SURVEY ON TESTING ENVIRONMENTS OF CONFIDENCE INTERVAL METHODS

2.1 INTRODUCTION

In the introductory chapter, we have discussed several methods which can be used for constructing confidence intervals for steady-state means of simulation output processes. Although the evaluation of these methods has included analytic investigations[see Schmeiser(1982), Goldsman and Schruben(1984)], the main thrust of research has taken the form of empirical studies[see Law(1983)]. During the last two decades, testing environments have been developed for evaluating the performance of these confidence interval methods. These testing environments consist of the following three general components:-

i) Simulation models generating output processes on which the performance of the methods is tested

ii) Statistical criteria measuring the performance of the methods

iii) Necessary computer software including simulation languages and secondary computer programs for manipulating data.

In the present chapter we describe a survey of previous testing environments which have been used for evaluating empirically the performance of confidence interval methods. Our aim is to answer two very crucial questions. Firstly, do some testing environments reveal methods which attain acceptable performances in certain output processes? If this is the case, can the particular testing environments indicate ways for applying these methods to any process which displays similar characteristics to the output processes of these testing environments?

In the following section, we describe the simulation models which have been used in the previous testing environments. For each model, the output processes on which the confidence interval methods have been tested are specified.

In section 3.3 we discuss the statistical criteria that have been developed for measuring the performance of confidence interval methods. A theoretical definition is given and a methodology which produces estimates for these criteria is described.

In section 3.4, the components of each testing environment are described in detail. For each environment, we also summarize the conclusions drawn concerning the performance of particular confidence interval methods.

In the final section, we address the two questions stated above by comparing the structure of the testing environments.

2.2 SIMULATION MODELS AND OUTPUT PROCESSES IN THE PREVIOUS TESTING ENVIRONMENTS

2.2.1 Series Queues($M/M/N_1/M/N_2/.../M/N_c$)

It is the type of simulation models which is met in most of the previous testing environments. The operational rules of these models are very simple. A customer arriving at the system joins the queue of the first service station. After the service completion, he joins the queues of the remaining c-1 service stations successively where he is being served. Each station consists of $N_j (j=1,2,\ldots,c)$ number of servers. After the service completion at the last service station, he departs from the system. The interarrival times at the first station and the service times in each station are independent negative exponential random variables with means $(1/\lambda)$ and $(1/\mu_i)$ respectively.

The behaviour of this type of models depends on three factors; the queueing discipline(FIFO,LIFO), the number of servers and the traffic intensity $\tau_i=\lambda/(N_i\mu_i)$ in each service station. From this type of models, the following three output processes have been selected for testing the performance of confidence interval methods:-

i) the total delay of customers in the queues of the service stations,

ii) the time the customer spends into the system,

iii) the queue lengths in front of the service stations.

2.2.2 Time Shared Computer Model

This model was studied by Adiri and Avi-Itzak(1969) and is briefly described in Law and Carson(1978). Its entities are a Central Processing Unit(C.P.U) and N number of jobs which are submitted by N terminal users. A user *thinks* for an amount of time which is a negative exponential random variable with mean $1/\lambda$. Then he sends a job requiring a service time, say s. The service time is again a negative exponential random variable with mean $1/\mu$. Any job leaving the terminal joins the FIFO queue at the C.P.U. To each job, the C.P.U allocates a maximum service length, say q. Denote by $s_1 (s_1 \leq s)$ the remaining service time of a job and τ a fixed overhead setting-up time. If $s_1 \leq q$ then the C.P.U spends $s_1+\tau$ time processing the job which returns to the terminal after the service completion. On the other hand, if $s>q$ then the C.P.U spends $q+\tau$ time processing the job which rejoins the end of the queue of the C.P.U. after the service completion.

This model was used in Law and Kelton's(1984) testing environment with the parameter values being $N=35$, $(1/\lambda)=25$, $(1/\mu)=0.8$, $q=0.8$ and $\tau=0.015$. The output process on which the confidence interval methods were tested was the response time of the jobs. The response time is defined as the time from when a job departs a terminal until its next return to the terminal.

2.2.3 Interactive Multiprogrammed Computer Model

A brief description of this model is given by Heidelberger and Welch(1981a,b). Its entities are:-

- N jobs which are submitted by N terminal users
- one Central Processing Unit(C.P.U)
- M secondary storage devices(S.S.D).

A job, having been formulated at a terminal, joins the FIFO queue of the C.P.U. After the end of the processing work, the job returns to the terminal with probability p_c or it joins the FIFO queue of the i^{th} secondary storage device with probability $p_i(i=1,..,M)$. The time the job spends at each S.S.D is a negative exponential random variable with mean $1/\mu_i(i=1,...,M)$. After leaving each S.S.D, the job joins again the end of the C.P.U queue. The formulation process and the service time at the C.P.U are independent exponential random variables with means $1/\lambda_1$ and $1/\lambda_2$ respectively.

This model was used in Heidelberger and Welch's(1981a) testing environment. The authors selected two sets of parameter values; $N=25$, $m=4$, $p_1=p_2=0.36$, $p_3=p_4=0.04$, $(1/\lambda_1)=100$, $(1/\lambda_2)=1$, $(1/\mu_1)=(1/\mu_2)=1.39$, $(1/\mu_3)=(1/\mu_4)=12.5$ and $N=25$, $m=4$, $p_1=p_2=p_3=p_4=0.36$, $(1/\lambda_1)=100$, $(1/\lambda_2)=1$, $(1/\mu_1)=(1/\mu_2)=5.56$, $(1/\mu_3)=(1/\mu_4)=25$. The output processes under study were the folowing:-

i) response time of jobs under both sets of parameter values

ii) waiting time of jobs at the C.P.U under the first set of parameter values

iii) waiting time of jobs at the queue of the second secondary storage device under the second set of parameter values.

2.2.4 Inventory Model

Let X_i be the inventory amount of an item for a company at the start of period i. If $X_i<s$, an order of size $S-X_i$ takes place with cost $k+c(S-X_i)$ bringing the inventory level immediately to S. If $X_i>s$, no order is placed and the inventory amount remains at X_i. During the period, a demand Q_i occurs. If $(X_i-Q_i)>0$ or $(S-Q_i)>0$ then the company incurres a holding cost $h(X_i-Q_i)$ or $h(S-Q_i)$ respectively; otherwise, it incurres a shortage cost $v(Q_i-X_i)$.

From this model the output process on which the performance of confidence interval methods was tested was the cost per period.

2.2.5 Time Series Models

I) <u>Autoregressive Processes of Order p</u>

They have the general form

$$X_t = \varphi_0 + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \cdots + \varphi_p X_{t-p} + \epsilon_t \qquad (2.1)$$

where $\varphi_0 = \mu(1-\varphi_1-\varphi_2-\ldots-\varphi_{t-p})$, μ is the level of the process and ϵ_t's are independent and normal random variables with mean zero and common variance σ_p^2.

II) <u>MA(q) PROCESSES</u>

Their general form is given by

$$X_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_p \epsilon_{t-p} \qquad (2.2)$$

where μ is the level of the process and ϵ_t's are independent and normal random variables with mean zero and common variance σ_p^2.

III) <u>EAR(λ) PROCESSES</u>

They are linear autoregressive processes with the marginal distributions being exponential random variables with the same parameter λ. A detailed description of these processes is given by Lawrance and Lewis(1981,1982). The general form of EAR(λ) processes is given by

$$X_t = \begin{cases} \varphi X_{t-1} + 0 & \text{w.p} \quad \varphi \\ \varphi X_{t-1} + E_t & \text{w.p} \quad 1-\varphi \end{cases} \qquad 0<\varphi<1 \qquad (2.3)$$

where E_t's are independent exponential random variables with the same parameter λ. At lag k, this model gives autocorrelation coefficients φ^k and realizations where segments of large values alternate with segments of small values.

2.3 STATISTICAL CRITERIA FOR EVALUATING THE PERFORMANCE OF THE CONFIDENCE INTERVAL METHODS

The most common criterion is the probability with which the confidence intervals produced by different methods cover the steady-state mean. In the simulation literature, this probability is called coverage and has the general form

$$CVR_i = \Pr\left\{ \bar{X}_n - t_{v_i,\alpha/2}\hat{\sigma}_i \leq \mu \leq \bar{X}_n + t_{v_i,\alpha/2}\hat{\sigma}_i \right\} \tag{2.4}$$

where for the i^{th} confidence interval method, $\hat{\sigma}_i^2$ is the variance of the sample mean and v_i the degrees of freedom prescribed by the method.

A more complicated criterion, called coverage function, was introduced by Schruben(1981a). Given the sample size n, this function is defined by setting different values to the nominal confidence level $(1-\alpha)$, in (2.4). If this function is uniformly distributed in [0,1], the coverages will be equal to the nominal confidence levels.

In addition, the following criteria are being used for studying the precision and stability of the confidence intervals:-

a) <u>Expected values of confidence interval half lengths</u>

$$EHL_i = E\left[t_{v_i,\alpha/2}\hat{\sigma}_i \right] \tag{2.5}$$

b) <u>Variance of confidence interval half lengths</u>

$$VHL_i = E\left[t_{v_i,\alpha/2}\hat{\sigma}_i - E\left[t_{v_i,\alpha/2}\hat{\sigma}_i\right]\right]^2 \tag{2.6}$$

c) <u>Standard deviation of confidence interval half lengths</u>

$$SDHL_i = \sqrt{E\left[t_{v_i,\alpha/2}\hat{\sigma}_i - E\left[t_{v_i,\alpha/2}\hat{\sigma}_i\right]\right]^2} \tag{2.7}$$

The index i stands for the i^{th} confidence interval method. The statistical criteria (2.5), (2.6) and (2.7) should be used for comparing methods which attain approximately the same coverages.

Two additional criteria were proposed by Schmeiser(1982) and Schriber and Andrews(1981) respectively. These are:-

d) <u>Coefficient of variation of confidence interval half lengths</u>

$$CVHL_i = \frac{VHL_i}{EHL_i} \tag{2.8}$$

e) <u>Standard deviation of the variance estimators of the sample mean</u>

$$SD(\hat{\sigma}_i) = \sqrt{E\left[\hat{\sigma}_i^2 - E(\hat{\sigma}_i^2)\right]^2} \qquad (2.9)$$

For the above five criteria, analytical values cannot be obtained for finite sample sizes from the output processes cited in the previous section[see Goldsman et al.(1986)]. However, these criteria can be estimated by using Monte Carlo methods. But, before we describe the estimation procedure, let us discriminate between two types of experiments.

<u>SINGLE TYPE OF EXPERIMENTS:</u> This is the type of experiment which is used by simulation practitioners for studying the performance of real life discrete systems. It consists of a single run of the simulation program which produces a single replication of the output process under study. From this single replication, estimates are obtained for the steady state measures of performance.

<u>GENERAL TYPE OF EXPERIMENTS:</u> This type is used by the simulation researchers for evaluating the performance of confidence interval or truncation methods. It consists of several replications of the output process under study. These replications are produced by using independent streams of random numbers. The observations can be presented in the form of the data matrix

$$A = \begin{bmatrix} X_{11} & X_{12} & X_{13} & \cdots & X_{1r} \\ X_{21} & X_{22} & X_{23} & \cdots & X_{2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nr} \end{bmatrix}$$

where X_{tj} is the random variable for the t^{th} observation on the j^{th} replication $(1 \leq j \leq r)$.

The five statistical criteria defined above are estimated in the context of the general type of experiments. Let $\bar{X}_{nj}$ be the mean from a sample of size n on the j^{th} replication. Let also $\hat{\sigma}^2_{ij}$ be the variance of the sample mean according to the i^{th} method on the j^{th} replication. Based on this notation, we describe below how the criteria (2.4)-(2.9) are estimated:

a) <u>Coverage</u>

Define the random variable

$$\Omega_{ij} = \begin{cases} 1 \text{ if } \mu \epsilon [\bar{X}_{nj} \pm t_{v_i,\alpha/2} \hat{\sigma}_{ij}] \\ 0 \quad \text{elsewhere} \end{cases} \qquad j=1,2,\ldots,r \tag{2.10}$$

The coverage, the i^{th} confidence interval method attains, will be estimated by

$$\widehat{CVR}_i = \frac{\sum_{j=1}^{r} \Omega_{ij}}{r} \tag{2.11}$$

b) <u>Expected value of confidence interval half lengths</u>

$$\widehat{EHL}_i = \frac{\sum_{j=1}^{r} t_{v_i,\alpha/2} \hat{\sigma}_{ij}}{r} \tag{2.12}$$

c) <u>Variance of confidence interval half lengths</u>

$$\widehat{VHL}_i = \frac{\sum_{j=1}^{r} \left[t_{v_i,\alpha/2}\hat{\sigma}_{ij} - \left[\left[\sum_{j=1}^{r} t_{v_i,\alpha/2}\hat{\sigma}_{ij} \right] \Big/ r \right] \right]^2}{r-1} \tag{2.13}$$

d) <u>Standard deviation of confidence interval half lengths</u>

$$\widehat{SDHL}_i = \sqrt{\widehat{VHL}_i} \tag{2.14}$$

e) <u>Coefficient of variation of confidence interval half lengths</u>

$$\widehat{CV}(HL_i) = \frac{\widehat{VHL}_i}{\widehat{EHL}_i} \tag{2.15}$$

f) <u>Standard deviation of variance estimators of sample mean</u>

$$\widehat{SD(\hat{\sigma}_i)} = \sqrt{\frac{\sum_{j=1}^{r} \left[\hat{\sigma}_{ij}^2 - \left[\left[\sum_{j=1}^{r} \hat{\sigma}_{ij}^2 \right] \Big/ r \right] \right]^2}{r-1}} \tag{2.16}$$

2.4 A DESCRIPTION OF PREVIOUS TESTING ENVIRONMENTS

This section describes previous testing environments which, during the last two decades, have been used for evaluating the performance of confidence interval methods. For each environment, the simulation models, the output processes and the statistical criteria which have been used are reported. We also summarise the conclusions which have been drawn at the stage of testing the performance of the methods.

FISHMAN's(1973) TESTING ENVIRONMENT

Fishman(1973a) tested the performance of the classical regenerative method on two processes; the delay in queue and the queue length in the M/M/1 queueing model with queueing discipline FIFO and traffic intensity 0.80. For this method the regeneration cycles were 1000. Fishman used the following statistical criteria:-

i) the proportion of confidence intervals(coverages) which contained the true steady-state average delay,

ii) the proportion of confidence intervals which contained the true steady-state average queue length and

iii) the differences of the sample mean delay and queue length from the corresponding true steady-state values.

All criteria were estimated by Monte Carlo methods. The total number of replications and the nominal confidence level were 100 and 0.90 respectively.

The estimated coverage of the true steady state average delay was 0.86, while for the true steady state average queue length the coverage was 0.85. Furthermore, the estimates of the average delay and average queue length were 1.78 and 8.94 respectively. The corresponding true steady-state values are 1.80 and 9.

LAW's(1977) TESTING ENVIRONMENT

Law(1977) studied the performances of both the nonoverlapping batch means and replication methods on the following processes:-

i) Delay in queue in the M/M/1 with queueing discipline FIFO and traffic intensity 0.90 and

ii) Total cost of period t in the inventory model with parameter values s=17, S=57, k=32, c=3, h=1 and v=5.

For the second process the distribution of the demand in period t was Poisson with mean 25.

Law used two statistical criteria for studying the performance of the above two methods; the coverage, the confidence interval methods achieve, and the expected values of confidence interval half lengths. These criteria were estimated by using Monte Carlo methods. The nominal confidence level was 0.90. For each process, 400 realizations were generated. With respect to the first process, the statistical criteria were estimated at sample sizes 1600, 3200, 6400 and 12800. Different sample sizes were used for estimating the criteria in the second process. These sizes were 320, 640, 1280 and 2560. For both processes, the number of the replications per realization and nonoverlapping batch means was 5, 10, 20 and 40. For the delay in the M/M/1, the initial conditions were chosen to be empty and idle. For the total cost in the inventory model, the run of the simulation program started from S=52.

Neither of the two methods performed perfectly under the different combinations of sample sizes and number of nonoverlapping batch means or replications per realization. In fact, for the delay in the M/M/1 the estimated coverages were lower than 0.90 while for the total cost in the inventory model they were higher than 0.90. Law reported that for the replication method the differences between the estimated coverages and the nominal confidence level were caused mainly by the initialization bias. For the nonoverlapping batch means method the major cause for these differences was the autocorrelation between the batch means.

Law also made similar remarks for the following processes:-

i) the delay in queue in the M/M/1 with queueing discipline FIFO and traffic intensities 0.50 and 0.70,

ii) the delay in queue in the M/M/2 with queueing discipline FIFO and traffic intensity 0.90 and

iii) the total delay in the M/M/1/M/1 with queueing discipline for both queues FIFO and traffic intensities τ_1=0.90, τ_2=0.90.

FISHMAN's(1978) TESTING ENVIRONMENT

Based on Von-Neumann test, Fishman(1978) developed an algorithm for determining the batch size which guarantees approximately independent nonoverlapping batch means. The performance of this algorithm was evaluated on the total time a customer spends in the M/M/1 with queueing discipline FIFO. Three traffic intensities were considered; 0.50, 0.80 and 0.90. For each traffic intensity, the simulation program started by generating the delay of the first customer from the distribution of the steady-state delay. The following statistical criteria were estimated by Monte Carlo methods:

a) The number of confidence intervals that contained the true steady-state average total time a customer spends in the M/M/1

b) Proportions of runs that failed to determine a batch size

c) Average values for the degrees of freedom for each combination of sample size and traffic intensity.

The nominal confidence level was 0.90. For each traffic intensity, 60 replications were generated. The above statistical criteria were estimated for sample sizes 2048, 4096, 8192 and 16384. The conclusions concerning the performance of the algorithm are summarized as follows:

a) For high traffic intensities the estimated coverages were not close to 0.95

b) For high traffic intensities and small sample sizes some runs that failed to determine a batch size[in these cases, additional runs were performed so that the number of replications to be fixed at 60] were observed

c) Under high traffic intensities the average degrees of freedom were smaller than those under low traffic intensities

d) For a given traffic intensity, increasing the sample size leaded to higher degrees of freedom on average.

SCHRIBER AND ANDREWS's(1981) TESTING ENVIRONMENT

Schriber and Andrews(1981) compared the performance of the nonoverlapping batch means method with the performance of the autoregressive method on the following two processes:

i) a groupwise independent process consisting of trivariate observations which were generated from a trivariate normal distribution with correlation matrix

$$\begin{bmatrix} 1.0 & 0.1 & 0.8 \\ 0.1 & 1.0 & 0.1 \\ 0.8 & 0.1 & 1.0 \end{bmatrix}$$

ii) AR(2) with the autoregressive coefficients being φ_0=18000/99, φ_1=2/99, φ_2=79/99 and σ_p^2=356000/99.

For the nonoverlapping batch means method, the batch sizes were determined by a procedure described in Schriber and Andrews(1979).

The following statistical criteria were selected for evaluating the performance of the two methods:-

a) coverage

b) chi-square values for checking the qoodness of fit of the coverage function to a [0,1] uniform distribution

c) expected values of confidence interval half lengths

d) standard deviations of the sample mean variance estimators of the two methods.

The above criteria were estimated by Monte Carlo methods. The nominal confidence level was 95%. The number of replications and the sample sizes were 100 and 48, 96, 144, 192 respectively.

The performance of the nonoverlapping batch means method for the groupwise independent process was satisfactory. For all the sample sizes the estimated coverages were very close to 0.95 and the coverage functions fitted well to the [0,1] uniform distribution. Similar remarks were made for the performance of the autoregressive method on the AR(2).

The performance of the nonoverlapping batch means method was also evaluated on the AR(2). This was found to be rather bad. On the other hand, the performance of the autoregressive method on the groupwise independent process was found to be satisfactory.

HEIDELBERGER AND WELCH's(1981) TESTING ENVIRONMENT

Heidelberger and Welch(1981a) developed a new method for estimating the variance of the sample mean in covariance stationary output processes. This method was based on the estimation of the power spectrum at zero frequency via the periodogram coordinates. The performance of this method was evaluated on the four processes of the interactive multiprogrammed computer model[see section 2.2]. The following statistical criteria were selected:-

a) coverage,

b) expected values of confidence interval half lengths,

c) variance of confidence interval half lengths.

The latter two criteria were expressed in terms of the steady-state mean of each output process.

Estimates of the three criteria were obtained by using Monte Carlo methods. The number of replications and the nominal confidence level were 50 and 0.90 respectively. In order to eliminate the initialization bias effects, the authors removed 500 observations from each replication of each output process. Then, each criterion was estimated for different sample sizes being 500, 750, 1125, 1687, 2530, 3795, 5692, 8538, 12807, and 13500. For each combination of sample size and output process, a polynomial of degree two was fitted to both 25 and 50 nonoverlapping batch means. For these two numbers of batch means the batches, which were produced according to a batching procedure described in Heidelberger and Welch(1981a), were ranged from 100 to 200 and from 200 to 400 respectively.

For the processes of the response time, the estimated coverages ranged from 0.76 to 0.96. With respect to the processes of the waiting time in queues, the range of the estimated coverages was greater i.e. from 0.60 to 0.96. For small sample sizes the confidence interval half lengths had smaller expected values by using 25 rather than 50 nonoverlapping batch means. On the other hand, for large sample sizes and for both 25 and 50 batch means the estimated expected values of the confidence interval half lengths were equal. Furthermore, for all sample sizes, higher variances of the confidence interval half lengths were observed by using 25 rather than 50 batch means.

SCHRUBEN's(1983) TESTING ENVIRONMENT

Schruben(1983) tested the performance of the nonoverlapping batch means and the four standardized time series methods on the following processes:

i) delay in queue in the M/M/1 with queueing discipline FIFO and traffic intensities 0.20, 0.50, 0.80,

ii) total cost in the inventory model,

iii) EAR(1) model with the autoregressive parameters being 0, 0.2 and 0.8.

For the second process, two sets of parameter values were chosen; s=0, S=8, k=8, c=0, h=1, v=3 and s=16, S=22, k=16, c=0, h=1, v=27. For each set, the distribution of the demand was Poisson with mean 3 and 16 respectively.

The performance of the methods was evaluated by using three statistical criteria:-

a) coverage,

b) expected values of confidence interval half lengths,

c) standard deviation of confidence interval half lengths.

Monte Carlo methods were used for estimating the above three criteria. The nominal confidence level was 90%. For each process, 100 replications were generated. Estimates of the criteria were obtained for different combinations of sample sizes and number of batches. These sample sizes and number of batches are displayed in table (2.1).

T A B L E 2.1
Sample sizes and number of batches in Schruben's testing environment

Simulation Models	NUMBER OF BATCHES				
	1	2	5	10	20
M/M/1 τ=0.20	20000	10000	4000	2000	1000
M/M/1 τ=0.50	40000	20000	8000	4000	2000
M/M/1 τ=0.80	60000	30000	12000	6000	3000
Inventory(1)	2560	1280	512	256	128
Inventory(2)	10000	5000	2000	1000	500
EAR(1) φ= 0	2560	1280	512	256	128
EAR(1) φ= 0.2	2560	2560	512	256	128
EAR(10 φ= 0.8	10000	5000	2000	1000	500

The estimated coverages were ranged from 0.77 to 1.0. For large sample sizes and small number of batches, the coverages all the methods achieved were very close to the nominal confidence level 0.90. Furthermore, for most combinations of output processes, sample sizes and number of batches, the confidence intervals of the standardized time series methods were narrower and more stable than those of the nonoverlapping batch means method.

LAW AND KELTON's(1984) TESTING ENVIRONMENT

Law and Kelton(1984) studied the performance of the nonoverlapping batch means, autoregressive, spectral and regenerative methods on the following two processes:-

i) delay in queue in the M/M/1 with queueing discipline FIFO and traffic intensity 0.80 and

ii) response time in the time-shared computer model.

For both processes, the initial conditions were empty and idle.

Two statistical criteria were chosen for evaluating the performance of the four methods; the coverage and the expected values of confidence interval half lengths. These criteria were estimated by Monte Carlo methods. For the first process, 400 replications were generated, while for the second process the number of replications was 200. For both processes, the statistical criteria were estimated for the same sample sizes; 320, 640, 1280 and 2560. With regard to the nonoverlapping batch means method, the criteria were estimated for 5, 10, 20 and 40 batch means. For the spectral method, the size of the spectral window was determined in such a way that the degrees of freedom were the same with those of the nonoverlapping batch means method. Two versions of the regenerative method were tested; the classical and the jackknife.

In the M/M/1, the performance of the four confidence interval methods was not satisfactory. Although by increasing the sample size the four methods attained higher coverages, these coverages were smaller than the nominal confidence level 0.90. On the other hand, in the time-shared computer model and for sample sizes 1280 and 2560, the estimated coverages were very close to 0.90. Moreover, in the second model the confidence interval of the autoregressive method had the smallest expected half lengths. With regard to nonoverlapping batch means and spectral methods, Law and Kelton recommended simulation practitioners to use a small number of large batches or large spectral window sizes.

GOLDSMAN,KANG AND SARGENT's(1986) TESTING ENVIRONMENT

Goldsman et al.(1986) studied the performance of the nonoverlapping/overlapping batch means, area and combined area-nonoverlapping batch means methods on the stationary AR(1). Its parameter values were $\varphi_0=0$, $\varphi_1=0.9$ and $\sigma_p^2=1$. The authors selected two statistical criteria to evaluate the performance of the methods; the coverage and the expected values of confidence interval half lengths. These criteria were estimated by using Monte Carlo methods. The nominal confidence level and the number of replications were 90% and 1000 respectively.

Estimates of the above two criteria were reported for 2 and 16 batches. The size of the batches were $2^j (j=0,1,2,\ldots,10)$. The following conclusions were drawn:

i) When the number of batches was 2, the coverage of the nonoverlapping batch means method was approaching the nominal confidence level faster than were the coverages of the other methods. However, for large batch sizes, for which all the methods achieved similar coverages, the nonoverlapping batch means method produced confidence intervals with the largest expected values.

ii) When the number of batches was 16, the behaviour of the estimated coverages for the nonoverlapping and overlapping batch means methods was about the same. For small batch sizes, the confidence intervals of the nonoverlapping batch means method had the largest expected values. However, as the batch sizes become large, the confidence interval methods were producing intervals which on average had the same half lengths.

SARGENT,KANG AND GOLDSMAN's(1989) TESTING ENVIRONMENT

This is an expansion of Goldsman et al.'s(1986) testing environment. The performance of the nonoverlapping/ overlapping batch means area, and combined area-nonoverlapping batch means methods was tested on the following processes:-

i) AR(1) with the parameter values being $\varphi_0=0$, $\varphi_1= 0.0$, 0.9 and $\sigma_p^2=1$

ii) EAR(1) with $\varphi = 0.9$

iii) MA(1) with $\theta=\pm0.1$, ±0.9

iv) delay in queue in the M/M/1 with queueing discipline FIFO and traffic intensity 0.9.

Two statistical criteria were used for studying the performance of the above confidence interval methods; the coverage and the expected values of confidence interval half lengths. Each criterion was estimated by using Monte Carlo methods. The whole study was divided into two parts.

In the first part, the two criteria were estimated for all combinations of processes, batch sizes $m=2^j (j=0,1,2,...,10)$, number of batches $k=1,2,4,8,16$ and nominal confidence levels $(1-\alpha)=0.80$, 0.90, 0.95 and 0.99. For each process 1000 replications were generated. In each replication, the initial conditions were chosen from the appropriate steady-state distribution. For small m, all the methods attained coverages smaller than the nominal confidence levels. When both m and k were small, the nonoverlapping batch means method achieved the greatest coverages. For small m and large k the nonoverlapping and overlapping batch means methods attained about the same coverages; these were greater than the coverages the other two methods achieved. On the other hand, for large m and small k the estimated coverages of all the methods were close to the nominal confidence level. Furthermore, for each method, the expected values of confidence interval half lengths tended to decrease as the degrees of freedom increased. For large m and small k, the combined area-nonoverlapping batch means method on average produced the narrower confidence intervals.

In the second part of the study, the statistical criteria were estimated for each combination of sample sizes $n=2^j (j=4,5,6,...,14)$ and degrees of freedom df=3,15. For each process, the number of replications was 2000. For small n and df=3,15, the performance of the methods was not satisfactory in terms of the coverages. For small n and df=15, the overlapping batch means method seemed to produce the greatest coverages.

However, as n was increasing, the nonoverlapping/overlapping batch means and combined area-nonoverlapping batch means methods appeared to attain acceptable coverages at about the same n.

2.5 SUMMARY

From the detailed description of the testing environments, one general conclusion can be drawn; There appears to be no general agreement about the details of an appropriate testing environment. If there existed such an agreement, the confidence interval methods, which have been developed for the last two decades, could have been evaluated on the same simulation output processes and under a common range of combinations of parameter values, sample sizes and nominal confidence levels. In this way, the identification of the best method for different types of output process in terms of specific criteria would be a straightforward task.

Despite the lack of agreement in a single testing environment, let us check whether it is possible to compare confidence interval methods which have been tested in the different testing environment. Such a comparison will be feasible if for different testing environments these methods have been evaluated on the same processes and under the same combinations of parameter values, sample sizes and nominal confidence levels.

We start the analysis by identifying the simulation models which were common in two or more testing environments. From table (2.2) the M/M/1 was the common model in six testing environments; Fishman's(1973), Law's(1977), Fishman's(1978), Schruben's(1983), Law and kelton's(1984) and Sargent et al.'s(1989). Figure (2.1) illustrates the confidence interval methods which have been tested in the M/M/1.

From figure (2.1), the regenerative - spectral - autoregressive methods have never been compared in the same testing environment with the overlapping batch means and standardized time series methods. The issue which arises is whether we can compare these methods indirectly by comparing the results we have in Schruben's and Law and Kelton's or Law and Kelton's and Sargent et al.'s testing environments. This indirect comparison seems to be rather difficult. Different sample sizes were used in Schruben's and Law and Kelton's testing environments; for traffic intensity 0.80, in the first testing environment these sizes were 3000, 6000, 12000, 30000, 60000, while in the second environment they were 320, 640, 1280, 2560. Moreover, different sample sizes were selected in Law and Kelton's and Sargent et al.'s testing environments. For the second environment, these sizes were 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192.

Let us now consider the simulation models which were common in any two testing environments[see table (2.2)]. The inventory model was used both in Law's and Schruben's testing environments. In these environments different parameter values for this model have been selected. Moreover, the AR(1) and EAR(1) were the common models in Schruben's and Sargent et al.'s testing environments. However, the same confidence interval methods have been tested in these environments.

T A B L E 2.2

Simulation Models and output processes which have been used in the testing environments

Sim. Models	(F) 1973	(L)	(F) 1978	(S&A)	(H&W)	(Sc)	(L&K)	(G)	(Sa)
M/M/1	x	x	x			x	x		x
M/M/2		x							
M/M/1/M/1		x							
Time-shared							x		
Interactive					x				
Inventory		x				x			
Groupwise Ind. process				x					
AR(1)								x	x
AR(2)				x					
EAR(1)						x			x
MA(1)									x

F I G U R E 2.1

Confidence Interval methods which have been tested in the M/M/1

Fishman(1973)
|
V
Regenerative

Law(1977)
|
V
a) Independent Replications
b) Nonoverlapping Batch means

Fishman(1973)
|
V
Nonoverlapping Batch means

Schruben(1983)
|
V
a) Nonoverlapping Batch means
b) Standardized time series

Law & Kelton(1984)
|
V
a) Nonoverlapping Batch means
b) Spectral
c) Autoregressive
d) Regenerative

Sargent(1989)
|
V
a) Nonoverlapping Batch means
b) Overlapping Batch means
c) Area
d) Combined

All the other simulation models were used in only one testing environment. Therefore, for the performance of the confidence interval methods in the time-shared computer model, interactive multiprogrammed computer model and the AR(2) no comparative results can be extracted.

From the above analysis, the nonhomogeneity of the testing environments is evident. Different methods have been tested on different output processes under different combinations of sample sizes, parameter values and nominal confidence levels. Therefore, the best method in terms of specific statistical criteria and for certain types of output process cannot be identified.

Furthermore, recommendations concerning the application of confidence interval methods to output processes which have not been included in the testing environments are necessarily limited. For the nonoverlapping batch means and spectral methods Law and Kelton(1984) proposed to simulation researchers the use of a small number of large batches or large spectral window sizes. Schriber and Andrews(1981) recommended the autocorrelation and partial autocorrelation functions as links between real-life simulation output processes and processes on which the confidence interval methods have been tested. For the latter processes, these authors assumed that it is known a-priori for which parameter values the confidence interval methods attain acceptable performances.

The last two paragraphs indicate that no satisfactory answers exist for the two crucial questions stated in the introductory section. For this reason, in the subsequent chapters our objective is oriented in two domains. Firstly, the identification of some best method(s) in a well defined analytical and empirical context. Secondly, the provision of recommendations for applying certain confidence interval methods to output processes having specific characteristics.

C H A P T E R   THREE

STATISTICAL CRITERIA FOR EXPLORING THE BIAS OF VARIANCE ESTIMATORS OF THE SAMPLE MEAN

3.1 INTRODUCTION

In chapter one, we have discussed different estimators of the variance of the sample mean. These estimators produce alternative confidence interval methods for steady state means of simulation output processes. We have denoted the estimator of the i^{th} method by $\hat{\sigma}_i^2$. In the subsequent analysis i will be used as an index indicating methods 1,2,...,g. Based on this notation, we present the confidence interval produced by the i^{th} method as

$$\bar{X}_n - t_{v_i,\alpha/2}\hat{\sigma}_i < \mu < \bar{X}_n + t_{v_i,\alpha/2}\hat{\sigma}_i$$

where $\bar{X}_n$ is the mean from a sample of size n, v_i the degrees of freedom according to the i^{th} method and $(1-\alpha)$ the nominal confidence level.

Estimators of the variance of the sample mean with acceptable properties are more likely to give valid confidence intervals [see Law(1977), Goldsman et al.(1986)]. For instance, Law(1977) studied the performance of the nonoverlapping batch means method on the delay in queue in the M/M/1 by using Monte Carlo methods. The basic statistical criterion for evaluating the performance of the method was the coverage[see section 2.3]. For different combinations of sample sizes and number of batches the estimated coverages were lower than the nominal confidence level.

Law reported that the major cause for the differences between estimated coverages and nominal confidence levels was the bias of the corresponding estimator of the variance of the sample mean.

In this chapter, we study the bias of five sample mean variance estimators for small sample sizes. More specifically, we introduce a family of functions which enables us to compute analytically both the minimum bias of each estimator and its parameter values for which this minimum bias is attained, providing that the theoretical autocorrelation coefficients of the output process under study are known. We shall call these functions "Bias Indicator functions".

In the next section we report previous analytical results on the bias of sample mean variance estimators

In section 3.3, we derive the expected values of the five sample mean variance estimators. These expected values are expressed in terms of the theoretical autocorrelation coefficients of the output process under study.

In section 3.4, we obtain analytical forms of the Bias Indicator functions of the sample mean variance estimators. For small sample sizes, we also illustrate how to determine both the minimum bias of each estimator and its parameter values for which the minimum bias is attained.

In the final section, we compare the minimum bias of the five sample mean variance estimators under consideration in the AR(1) process under positive and negative autoregressive coefficients. We also state results obtained by Kevork and Balmer(1990) for the minimum bias of the above estimators in AR(2) processes and the delay in queue in the M/M/1.

3.2 PREVIOUS WORK ON THE BIAS OF SAMPLE MEAN VARIANCE ESTIMATORS

The following five confidence interval methods are considered in the subsequent chapters:-

i) nonoverlapping batch means denoted by NOBM,

ii) standardized time series-area denoted by AREA,

iii) combined area-nonoverlapping batch means denoted by NOBM-AREA,

iv) spectral denoted by SPEC, and

v) overlapping batch means denoted by OVBM

The corresponding sample mean variance estimators are defined as follows:

i) NOBM

$$\hat{\sigma}^2_{NB} = \frac{\hat{V}_{NB}}{n} \quad \text{where} \quad \hat{V}_{NB} = \frac{m}{k-1} \sum_{j=1}^{k} \left[\bar{X}_{j,m} - \bar{X}_n \right]^2 \tag{3.1}$$

$$\bar{X}_{j,m} = \frac{1}{m} \sum_{t=1}^{m} X_{(j-1)m+t} \quad , \quad j= 1,2,\ldots,k \tag{3.1a}$$

ii) AREA

$$\hat{\sigma}^2_{SM} = \frac{\hat{V}_{SM}}{n} \quad \text{where} \quad \hat{V}_{SM} = \frac{12}{k(m^3-m)} \sum_{j=1}^{k} \hat{A}^2_j \tag{3.2}$$

$$\hat{A}_j = \sum_{\ell=1}^{m} \ell \hat{S}_j(\ell) \tag{3.2a}$$

$$\hat{S}_j(\ell) = \bar{X}_{j,m} - \bar{X}_{j,\ell} \quad , \quad j= 1,2,\ldots,k \tag{3.2b}$$

iii) Combined NOBM-AREA

$$\hat{\sigma}^2_{CM} = \frac{\hat{V}_{CM}}{n} \quad \text{where } \hat{V}_{CM} = \frac{\frac{12}{m^3-m}\sum_{j=1}^{k}\hat{A}_j^2 + m\sum_{j=1}^{k}\left[\bar{X}_{j,m} - \bar{X}_n\right]^2}{(2k-1)} \tag{3.3}$$

iv) SPEC

$$\hat{\sigma}^2_{SP} = \frac{\hat{V}_{SP}}{n} \quad \text{where } \hat{V}_{SP} = \frac{n}{n-w}\left[\hat{\gamma}_0 + 2\sum_{s=1}^{w-1}\lambda_w(s)\hat{\gamma}_s\right] \tag{3.4}$$

$$\lambda_w(s) = 0.5(\ 1 + \cos(\pi s/w)) \tag{3.4a}$$

$$\hat{\gamma}_s = \frac{n-s}{n}\hat{\gamma}^*_s \tag{3.4b}$$

$$\hat{\gamma}^*_s = \frac{1}{n-s}\sum_{t=1}^{n-s}\left[X_t - \bar{X}_n\right]\left[X_{t+s} - \bar{X}_n\right] \tag{3.4c}$$

v) OVBM

$$\hat{\sigma}^2_{OB} = \frac{\hat{V}_{OB}}{n} \quad \text{where } \hat{V}_{OB} = \frac{m}{n-m+1}\sum_{j=1}^{n-m+1}\left[\bar{X}_j(m) - \bar{X}_n\right]^2 \tag{3.5}$$

$$\bar{X}_j(m) = \frac{1}{m}\sum_{t=0}^{m-1}X_{j+t} \tag{3.5a}$$

The evaluation of bias of sample mean variance estimators constitutes an active field of research in simulation output analysis. Goldsman and Meketon(1986) showed that as m and k becomes large

$$\text{Bias}(\hat{V}_{NB}) \simeq \text{Bias}(\hat{V}_{OB}) \simeq (\text{Bias}(\hat{V}_{CM}))/2 \simeq (\text{Bias}(\hat{V}_{CM}))/3$$

Besides, the authors reported that the bias of these four estimators is of order (1/m). Hence, the estimators are asymptotically unbiased as $m \to \infty$.

For the AR(1), Goldsman et al.(1986) provided exact forms for the $E(\hat{V}_{NB})$ and $E(\hat{V}_{SM})$ in terms of the autoregressive autoregressive coefficient. The authors verified that as $m \to \infty$ the bias of $\hat{V}_{SM}$ is three times more than that of V_{NB}.

Furthermore, for the AR(1), Sargent et al.(1989) obtained exact results for $E(\hat{V}_{NB})$, $E(\hat{V}_{OB})$, $E(\hat{V}_{SM})$, $E(\hat{V}_{CM})$ for k=2 and k=16. For k=2, the authors reported that

$$\text{Bias}(\hat{V}_{NB}) < \text{Bias}(\hat{V}_{OB}) < \text{Bias}(\hat{V}_{CM}) < \text{Bias}(\hat{V}_{SM})$$

while for k=16

$$\text{Bias}(\hat{V}_{NB}) \simeq \text{Bias}(\hat{V}_{OB}) < \text{Bias}(\hat{V}_{CM}) < \text{Bias}(\hat{V}_{SM})$$

From the above, we see that no exact results exist for the bias of the sample mean variance estimator of the spectral method.

3.3 EXPECTED VALUES FOR VARIOUS SAMPLE MEAN VARIANCE ESTIMATORS

Let $\{X_t\}$ be a stationary output process with

$$E(X_t) = \mu < \infty \qquad t \geq 1$$

and

$$Var(X_t) = \gamma_0 < \infty \quad t \geq 1$$

We derive below the expected values of the five sample mean variance estimators under consideration.

3.3.1 Nonoverlapping Batch Means

From Law(1977)

$$E\left[\hat{\sigma}^2_{NB}\right] = \gamma_{0,m}\left\{\frac{(k-1) - 2\sum_{j=1}^{k-1}\left[1 - \frac{j}{k}\right]\rho_{j,m}}{k(k-1)}\right\} \tag{3.6}$$

where $\rho_{j,m} = \gamma_{j,m}/\gamma_{0,m}$, and

$$\gamma_{j,m} = \frac{1}{m}\sum_{s=-(m-1)}^{m-1}\left[1 - \frac{|s|}{m}\right]\gamma_{jm+s} \tag{3.7}$$

is the covariance of $\bar{X}_{t,m}$ and $\bar{X}_{t+j,m}$ at any time point t.

Simplifying (3.6) we get

$$E\left[\hat{\sigma}^2_{NB}\right] = \frac{\gamma_{0,m}}{k} - \frac{2\sum_{j=1}^{k-1}\left[1 - \frac{j}{k}\right]\gamma_{j,m}}{k(k-1)} \tag{3.8}$$

Expanding the sum in (3.7) and recalling that $\gamma_i = \gamma_{-i}$, $\gamma_{j,m}$ is expressed in terms of the variance and the theoretical autocorrelation coefficients of the original output process $\{X_t\}$ as

$$\gamma_{j,m} = \frac{\gamma_0}{m} A_j(m) \qquad (3.9)$$

where

$$A_j(m) = \begin{cases} \rho_{jm} + \sum_{s=1}^{m-1}\left[1 - \frac{s}{m}\right]\left[\rho_{jm-s} + \rho_{jm+s}\right], & j \neq 0 \\ 1 + 2\sum_{s=1}^{m-1}\left[1 - \frac{s}{m}\right]\rho_s, & j = 0 \end{cases}$$

From (3.8) and (3.9), knowing γ_0 and $\rho_i (i=1,\ldots,n-1)$, exact analytic results for the expected value of the NOBM estimator can be obtained under different number of batches k and sample sizes n

3.3.2 Standardized Time Series-area

Taking expected values to both sides of (3.2), we get

$$E\left[\hat{\sigma}^2_{SM}\right] = \frac{12}{nk(m^3-m)} \sum_{j=1}^{k} E\left[\hat{A}_j^2\right] \qquad (3.10)$$

For the j_{th} batch, when the batch size m is even

$$\hat{A}_j = \frac{1}{2}\left\{ -(m-1)\tilde{X}_{(j-1)m+1} - (m-3)\tilde{X}_{(j-1)m+2} - \ldots - \tilde{X}_{(j-1)m+\frac{m}{2}} + \right.$$

$$\left. + \tilde{X}_{(j-1)m+\frac{m}{2}+1} + 3\tilde{X}_{(j-1)m+\frac{m}{2}+2} + \ldots + (m-3)\tilde{X}_{jm-1} + (m-1)\tilde{X}_{jm} \right\}$$

while when m is odd

$$\hat{A}_j = \frac{1}{2}\Big\{ -(m-1)\tilde{X}_{(j-1)m+1} - (m-3)\tilde{X}_{(j-1)m+2} - \ldots - 2\tilde{X}_{(j-1)m+\frac{m-1}{2}} +$$

$$+2\tilde{X}_{(j-1)m+\frac{m-1}{2}+2} + 4\tilde{X}_{(j-1)m+\frac{m-1}{2}+3} + \ldots + (m-3)\tilde{X}_{jm-1} + (m-1)\tilde{X}_{jm}\Big\}$$

where $\tilde{X}_{(j-1)m+s} = X_{(j-1)m+s} - \mu$

Squaring $\hat{A}_j$, taking expectations for each cross product term and recalling that

$$E\left[\tilde{X}_{(j-1)m+t}\tilde{X}_{(j-1)m+t+s}\right] = \gamma_s$$

we get

$$E\left[\hat{A}_j^2\right] = \frac{\gamma_0}{2}\sum_{r=0}^{m'}\sum_{s=0}^{m'}\delta_{rs} \qquad (3.11)$$

where $m' = [m/2] - 1$

and $\delta_{rs} = \{m-(1+2r)\}\{m-(1+2s)\}\{\rho_{r-s} - \rho_{m-(r+s+1)}\}$

The notation [m/2] stands for the greatest integer which is less than m/2.

Substituting (3.11) into (3.10), the expected value of area estimator is given by

$$E\left[\hat{\sigma}^2_{SM}\right] = \frac{6\gamma_0}{n(m^3-m)}\sum_{r=0}^{m'}\sum_{s=0}^{m'}\delta_{rs} \qquad m < n \qquad (3.12)$$

3.3.3 Combined Area-nonoverlapping Batch Means

From (3.3)

$$\hat{\sigma}^2_{CM} = \frac{\dfrac{12}{m^3 - m}\displaystyle\sum_{j=1}^{k} E\left[\hat{A}^2_j\right] + m\,E\left[\displaystyle\sum_{j=1}^{k} (\bar{X}_{j,m} - \bar{X}_n)^2\right]}{n(2k-1)} \qquad (3.13)$$

But

$$E\left[\sum_{j=1}^{k} \hat{A}^2_j\right] = \frac{nk(m^3-m)}{12}\, E\left[\hat{\sigma}^2_{SM}\right] \qquad (3.14)$$

and

$$E\left[\sum_{j=1}^{k} (\bar{X}_{j,m} - \bar{X}_n)^2\right] = k(k-1)E\left[\hat{\sigma}^2_{NB}\right] \qquad (3.15)$$

Substituting (3.14) and (3.15) into (3.13)

$$E\left[\hat{\sigma}^2_{CM}\right] = \frac{kE\left[\hat{\sigma}^2_{SM}\right] + (k-1)E\left[\hat{\sigma}^2_{NB}\right]}{2k-1} \qquad (3.16)$$

From (3.16), the expected value of the combined NOBM-AREA estimator is the weighted average of the expected values of the NOBM and AREA estimators. Moreover for $k=1$ the AREA and combined NOBM-AREA estimators have the same expected value.

3.3.4 Spectral

From (3.4)

$$E\left[\hat{\sigma}^2_{SP}\right] = \frac{1}{n-w}\left\{E\left[\hat{\gamma}_0\right] + 2\sum_{s=1}^{w-1}\lambda_w(s)E\left[\hat{\gamma}_s\right]\right\} \qquad (3.17)$$

where $$E\left[\hat{\gamma}_s\right] = \frac{n-s}{n}E\left[\hat{\gamma}^*_s\right]$$

From Anderson(1970,page 448)

$$E\left[\gamma^*_s\right] = \gamma_0\ g_s(n) \qquad (3.18)$$

with

$$g_s(n) = 1 - \frac{1}{n}\left\{1 + 2\sum_{r=1}^{n-1}\left[1 - \frac{r}{n}\right]\rho_r\right\} \qquad s=0$$

$$= \rho_s - \frac{1}{n}\left\{1 + 2\sum_{r=1}^{s}\left[1 - \frac{rs}{n(n-s)}\right]\rho_r + 2\sum_{r=n-s}^{n-1}\frac{(n-r)s}{n(n-s)}\rho_r + \right.$$

$$\left. + 2\sum_{r=s+1}^{n-s-1}\left[1 - \frac{rs}{n(n-s)} - \frac{r-s}{n-s}\right]\rho_r\right\} \qquad 1\leq s<n-s-1)$$

$$= \rho_s - \frac{1}{n}\left\{1+2\sum_{r=1}^{s}\left[1-\frac{rs}{n(n-s)}\right]\rho_r + 2\sum_{r=s+1}^{n-1}\frac{(n-r)s}{n(n-s)}\rho_r\right\}$$

$$1\leq s=n-s-1$$

$$=\rho_s - \frac{1}{n}\left\{1+2\sum_{r=1}^{n-s-1}\left[1-\frac{rs}{n(n-s)}\right]\rho_r + 2\sum_{r=n-s}^{s}\frac{r}{n}\rho_r + \right.$$

$$\left. + 2\sum_{r=s+1}^{n-1}\frac{(n-r)s}{n(n-s)}\rho_r\right\} \qquad n-s-1<s<n-1$$

$$=\rho_{n-1} - \frac{1}{n}\left\{1+2\sum_{r=1}^{n-1}\frac{r}{n}\rho_r\right\} \qquad s=n-1$$

3.3.5 Overlapping Batch Means

Substituting (3.5a) into the OVBM sample mean variance estimator

$$\hat{\sigma}^2_{OB} = \frac{m}{n(n-m+1)}\sum_{j=1}^{n-m+1}\left\{\frac{1}{m}\left[\sum_{t=0}^{m-1}X_{j+t}\right] - \bar{X}_n\right\}^2$$

$$= \frac{m}{n(n-m+1)}\sum_{j=1}^{n-m+1}\left\{\sum_{t=0}^{m-1}\left[\frac{X_{j+t} - \bar{X}_n}{m}\right]\right\}^2$$

$$= \frac{1}{mn(n-m+1)}\sum_{j=1}^{n-m+1}\left\{\sum_{t=0}^{m-1}(X_{j+t} - \bar{X}_n)\right\}^2$$

$$= \frac{1}{mn(n-m+1)}\sum_{j=1}^{n-m+1}\left\{\sum_{s=0}^{m-1}\sum_{\ell=0}^{m-1}(X_{j+s}-\bar{X}_n)(X_{j+\ell}-\bar{X}_n)\right\} \qquad (3.19)$$

We have found that it is rather difficult to derive the exact form of the expected value of the OVBM estimator from (3.19). Meketon and Schmeiser approximated (3.19) by

$$\hat{\sigma}^2_{OB} \simeq \frac{m \sum_{s=1}^{n} (X_i - \bar{X})^2 + 2 \sum_{s=1}^{m-1} (m-s) \sum_{j=1}^{n-s} (X_j - \bar{X}_n)(X_{j+s} - \bar{X}_n)}{mn(n-m+1)}$$

$$= \frac{1}{n-m+1} \left\{ \frac{\sum_{s=1}^{n} (X_s - \bar{X}_n)^2}{n} + 2 \sum_{s=1}^{m-1} \left[1 - \frac{s}{m} \right] \left[\frac{\sum_{j=1}^{n-s} (X_j - \bar{X}_n)(X_{j+s} - \bar{X}_n)}{n} \right] \right\}$$

$$= \frac{1}{n-m+1} \left\{ \hat{\gamma}_0 + 2 \sum_{s=1}^{m-1} \left[1 - \frac{s}{m} \right] \hat{\gamma}_s \right\} \tag{3.20}$$

We set $Y_i = X_i - \bar{X}_n$; From the following configuration we can investigate the accuracy of the approximation under different sample sizes. The denominators of (3.19) and (3.20) have been omitted.

n=5 , m=3

exact form	approximate form
$Y_1Y_1+ Y_1Y_2+ Y_1Y_3+$	$3Y_1Y_1+2Y_1Y_2+ Y_1Y_3+$
$Y_2Y_1+2Y_2Y_2+2Y_2Y_3+ Y_2Y_4+$	$2Y_2Y_1+3Y_2Y_2+2Y_2Y_3+ Y_2Y_4+$
$Y_3Y_1+2Y_3Y_2+3Y_3Y_3+2Y_3Y_4+Y_3Y_5+$	$Y_3Y_1+2Y_3Y_2+3Y_3Y_3+2Y_3Y_4+ Y_3Y_5$
$Y_4Y_2+2Y_4Y_3+2Y_4Y_4+Y_4Y_5+$	$Y_4Y_1+2Y_4Y_3+3Y_4Y_4+2Y_4Y_5$
$Y_5Y_3+ Y_5Y_4+Y_5Y_5$	$Y_5Y_3+2Y_5Y_4+3Y_5Y_5$

n=6 , m=3

exact form

$$
\begin{array}{l}
Y_1Y_1+ Y_1Y_2+ Y_1Y_3+ \\
Y_2Y_1+2Y_2Y_2+2Y_2Y_3+ Y_2Y_4+ \\
Y_3Y_1+2Y_3Y_2+3Y_3Y_3+2Y_3Y_4+Y_3Y_5+ \\
\quad Y_4Y_2+2Y_4Y_3+3Y_4Y_4+2Y_4Y_5+Y_4Y_6 \\
\quad\quad Y_5Y_3+2Y_5Y_4+2Y_5Y_5+Y_5Y_6 \\
\quad\quad\quad Y_6Y_4+ Y_6Y_5+Y_6Y_6
\end{array}
$$

approximate form

$$
\begin{array}{l}
3Y_1Y_1+2Y_1Y_2+ Y_1Y_3+ \\
2Y_2Y_1+3Y_2Y_2+2Y_2Y_3+ Y_2Y_4+ \\
Y_3Y_1+2Y_3Y_2+3Y_3Y_3+2Y_3Y_4+ Y_3Y_5+ \\
\quad Y_4Y_2+2Y_4Y_3+3Y_4Y_4+2Y_4Y_5+ Y_4Y_6 \\
\quad\quad Y_5Y_3+ 2Y_5Y_4+3Y_5Y_5+2Y_5Y_6 \\
\quad\quad\quad Y_6Y_4+2Y_6Y_5+ Y_6Y_6
\end{array}
$$

By using the approximate form (3.20), some early and late cross-product terms are counted more times than it should be. However, the approximation improves for a fixed batch size m as the sample size n increases. This is so because in (3.20) there are fewer terms with coefficients different from those defined by the exact form (3.19).

Taking expectations to both sides of (3.20)

$$
E\left[\hat{\sigma}^2_{OB}\right] \approx \frac{1}{n-m+1}\left\{ E\left[\hat{\gamma}_0\right] + 2\sum_{s=1}^{m-1}\left[1 - \frac{s}{m}\right]E\left[\hat{\gamma}_s\right]\right\} \qquad (3.21)
$$

where $E(\gamma_s)$ was defined in (3.18).

3.4 THE BIAS INDICATOR FUNCTIONS

Consider the following ratios:-

i) for the nonoverlapping batch means, area and combined NOBM-AREA methods

$$Bs(n,k)_i = \frac{E\left[\hat{\sigma}^2_i\right]}{V(\bar{X}_n)} \qquad i = NB\ ,\ SM\ ,\ CM \tag{3.22a}$$

ii) for the spectral method

$$Bs(n,w)_{SP} = \frac{E\left[\hat{\sigma}^2_{SP}\right]}{V(\bar{X}_n)} \tag{3.22b}$$

iii) for the overlapping batch means method

$$Bs(n,m)_{OB} = \frac{E\left[\hat{\sigma}^2_{OB}\right]}{V(\bar{X}_n)} \tag{3.22c}$$

where $V(\bar{X}_n)$ is the true variance of the sample mean.

For the five confidence interval methods, the ratios (3.22) measure the bias of the corresponding sample mean variance estimators. For this reason we shall call each ratio the "Bias Indicator function" of the corresponding estimator.

Exact analytic values for each Bias Indicator function can be obtained only when the theoretical autocorrelation coefficients of the process under study are known. Substituting the expected values of the sample mean variance estimators into (3.22) and recalling that

$$V(\bar{X}_n) = \frac{\gamma_0}{n} \left\{ 1 + 2\sum_{s=1}^{n-1} \left[1 - \frac{s}{n} \right] \rho_s \right\}$$

the Bias indicator functions take the following forms:

NONOVERLAPPING BATCH MEANS

$$Bs(n,k)_{NB} = \frac{A_0(m) - \dfrac{2\sum_{j=1}^{k-1}\left[1 - \frac{j}{k} \right] A_j(m)}{k-1}}{1 + 2\sum_{s=1}^{n-1}\left[1 - \frac{s}{n} \right]\rho_s} \tag{3.23a}$$

where $A_j(m)$ was defined in (3.9)

STANDARDIZED TIME SERIES METHOD-AREA

$$Bs(n,k)_{SM} = \frac{6}{m^3 - m} \left\{ \frac{\sum_{r=0}^{m'}\sum_{s=0}^{m'} \delta_{rs}}{1 + 2\sum_{s=1}^{n-1}\left[1 - \frac{s}{n} \right]\rho_s} \right\} \quad k>1 \, , \tag{3.23b}$$

where δ_{rs} and m', were defined in (3.11).

COMBINED AREA-NONOVERLAPPING BATCH MEANS

$$Bs(n,k)_{CM} = \frac{kBs(n,k)_{SM} + (k-1)Bs(n,k)_{NB}}{2k-1} \tag{3.22c}$$

SPECTRAL METHOD

$$Bs(n,w)_{SP} = \frac{1}{n-w}\left\{\frac{ng_0(n) + 2\sum_{s=1}^{w-1}(n-s)\lambda_w(s)g_S(n)}{1 + 2\sum_{s=1}^{n-1}\left[1 - \frac{s}{n}\right]\rho_s}\right\} \tag{3.22d}$$

where $\lambda_w(s)$ and $g_S(n)$ were defined in (3.4a) and (3.18) respectively.

OVERLAPPING BATCH MEANS

$$Bs(n,m)_{OB} \simeq \frac{1}{n-m+1}\left\{\frac{ng_0(n)+2\sum_{s=1}^{m-1}(n-s)\left[1-\frac{s}{m}\right]g_S(n)}{1 + 2\sum_{s=1}^{n-1}\left[1 - \frac{s}{n}\right]\rho_s}\right\} \tag{3.22e}$$

In the next chapter, we use the asymptotic forms of the Bias Indicator functions to compute analytically the limiting coverages of the five confidence interval methods under consideration. With respect to the nonoverlapping batch means, area and combined area-nonoverlapping batch means methods, for fixed m these coverages will be computed as $k\to\infty$ and also $n\to\infty$. For the spectral and overlapping batch means methods, for fixed m and w respectively, the limiting coverages will be obtained as $n\to\infty$.

Two criteria additional to the Bias Indicator functions are defined below. Given the sample size, these criteria are the minimum bias of each sample mean variance estimator and its parameter values for which this minimum bias is attained. We shall call these values "MB-parameter values".

Let k_{MB} be the MB-parameter values of the nonoverlapping batch means, area and combined area-nonoverlapping batch means estimators. Similarly, let us denote by w_{MB} and m_{MB} the MB-parameter values of the spectral and overlapping batch means estimators respectively. The above MB-parameter values will satisfy the following inequalities:-

i) for the NOBM, AREA and combined NOBM-AREA

$$| Bs(n,k_{MB})_i - 1 | \leqslant | Bs(n,k)_i - 1 |$$

for any $k \neq k_{MB}$

ii) for the spectral

$$| Bs(n,w_{MB})_{SP} - 1 | \leqslant | Bs(n,w)_{SP} - 1 |$$

for any $w \neq w_{MB}$

iii) for the overlapping batch means

$$| Bs(n,m_{MB})_{OB} - 1 | \leqslant | Bs(n,m)_{OB} - 1 |$$

for any $m \neq m_{MB}$

For each of the above five sample mean variance estimators and in terms of the true variance of the sample mean, the minimum bias will be given by

$$MB_i = \min \mid Bs(n,k)_i - 1 \mid \text{ for } i=NB, SM, CM$$

$$MB_{SP} = \min \mid Bs(n,w)_{SP} - 1 \mid$$

$$MB_{OB} = \min \mid Bs(n,m)_{OB} - 1 \mid$$

For the minimum bias and the MB-parameter values, exact analytic results can be obtained only when the theoretical autocorrelation coefficients of the process under study are known. Such a process is considered in the next section. For processes whose theoretical autocorrelation functions are not known estimation procedures for the above criteria are discussed in chapter six.

3.5 THE BIAS OF SAMPLE MEAN VARIANCE ESTIMATORS IN AR(1) PROCESSES

Three statistical criteria for studying the bias of each sample mean variance estimator were introduced in the previous section; the Bias Indicator function, the minimum bias and the MB-parameter values. Let us now compute the values of these criteria for the stationary AR(1) process which has the form

$$X_t = \mu + \varphi(X_{t-1} - \mu) + \epsilon_t \tag{3.23}$$

where the ϵ_t's are independent and normally distributed random variables with mean 0 and common variance σ_{ϵ}^2.

For this process, the s_{th} lag theoretical autocorrelation coefficient is $\varphi^{|s|}$ [see Harvey(1981)]. Therefore, when the autoregressive coefficient φ is positive, the autocorrelation function decays monotonically to zero. With negative φ the autocorrelation function converges to zero oscillating between positive and negative values.

For the AR(1), figure (3.1) illustrates the Bias Indicator functions of the five sample mean variance estimators which have been considered in the previous section. The autoregressive coefficients are .4074, -.4074, 0.963, and -.963. The choice of these particular values will be explained in the next chapter.

First, consider the nonoverlapping batch means(NOBM), area and combined area-nonoverlapping batch means sample mean variance estimators. For these three estimators, diagrams (a) and (b) display the shape and the relative position of the Bias Indicator functions for sample size 512. We have found that similar shapes hold for any other small sample. For positive φ, we observe that the three estimators underestimate the true variance of the sample mean, while for negative φ they overestimate it. For any number of batches $k>2$ the NOBM estimator has the smallest bias. For the same range of k, the combined NOBM-AREA estimator is less biased than the AREA estimator.

In diagrams (c) and (d), we have drawn the Bias Indicator functions of the NOBM and AREA estimators for φ=0.963, -0.963 and different sample sizes. For any other negative or positive value of φ, the shape of these functions is similar. For k>2, the Bias Indicator function of the combined NOBM-AREA estimator has shape similar to that of the corresponding functions of the NOBM and AREA estimators because this function of the combined NOBM-AREA estimator is the weighted average of the corresponding functions of the NOBM and AREA estimators[see form (3.22c)]. For k=1 the Bias Indicator functions of the AREA and combined NOBM-AREA estimators are identical. From the two diagrams, for any finite sample size the minimum bias of the NOBM estimator is attained for k=2 while the minimum bias of the other two estimators is achieved for k=1.

Let us now examine the properties of the Bias Indicator function of the spectral(SPEC) estimator. Diagrams (e) and (f) illustrate the form of this function under different values of sample size n, autoregressive coefficient φ and spectral window size w. When φ is positive, the Bias Indicator function is an increasing function of the spectral window size. For negative φ, the SPEC estimator overestimates the true variance of the sample mean for any combination of n,φ and w. By keeping fixed the sample size, higher autocorrelation levels move the whole function upwards.

The above properties of the Bias Indicator function of the SPEC estimator hold for the corresponding function of the overlapping batch means estimator(OVBM) [see diagrams (g) and (h)].

Figure 3.1
Bias Indicator functions for AR(1) processes

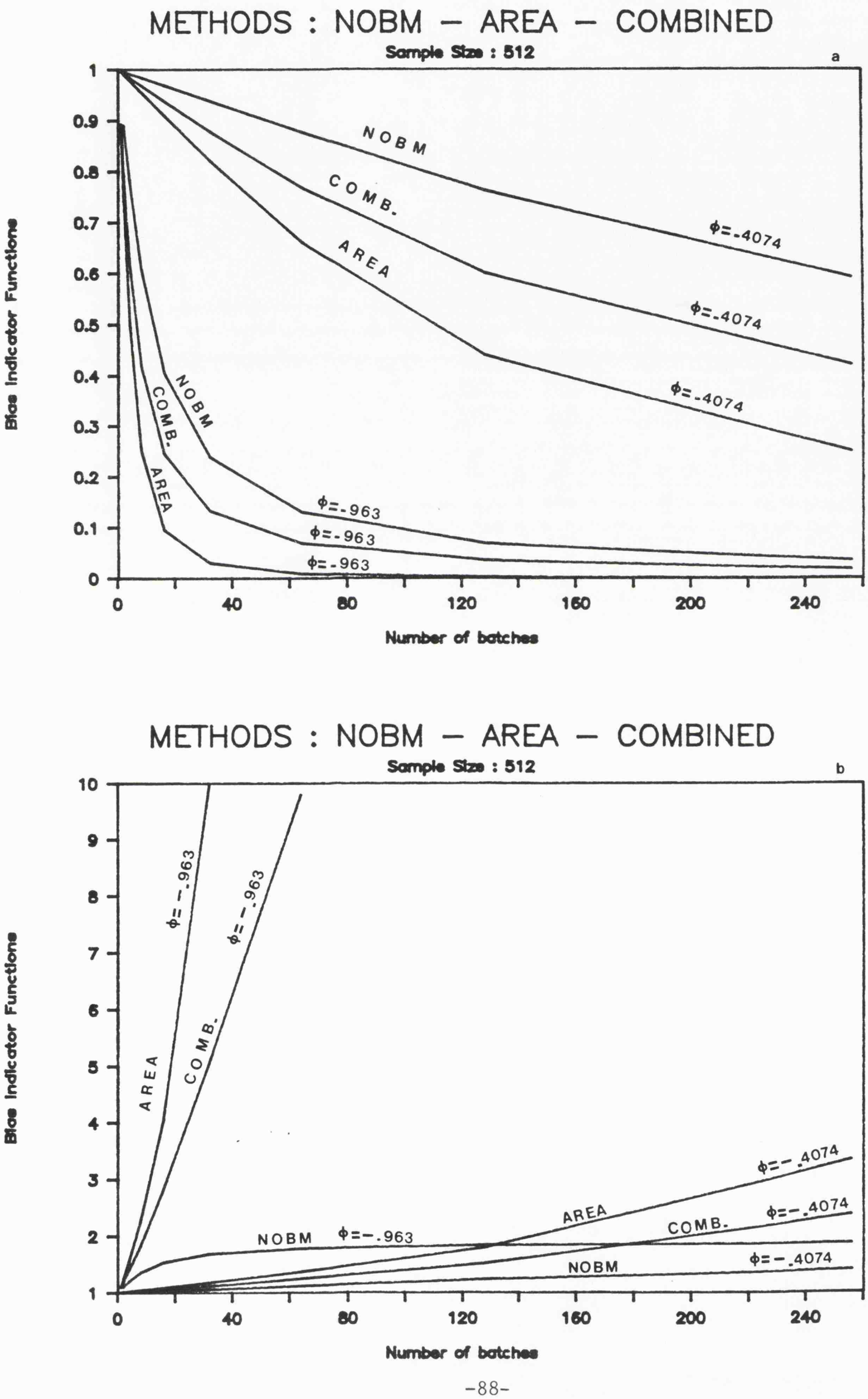
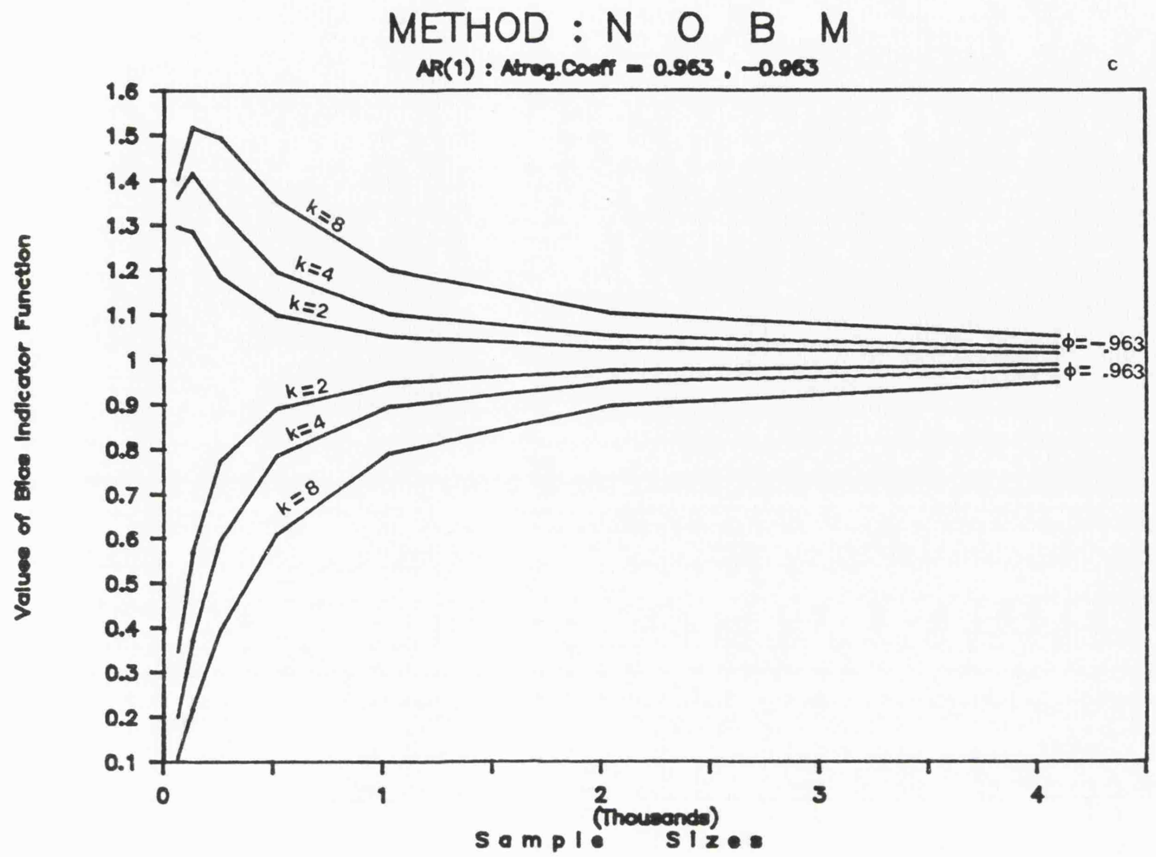
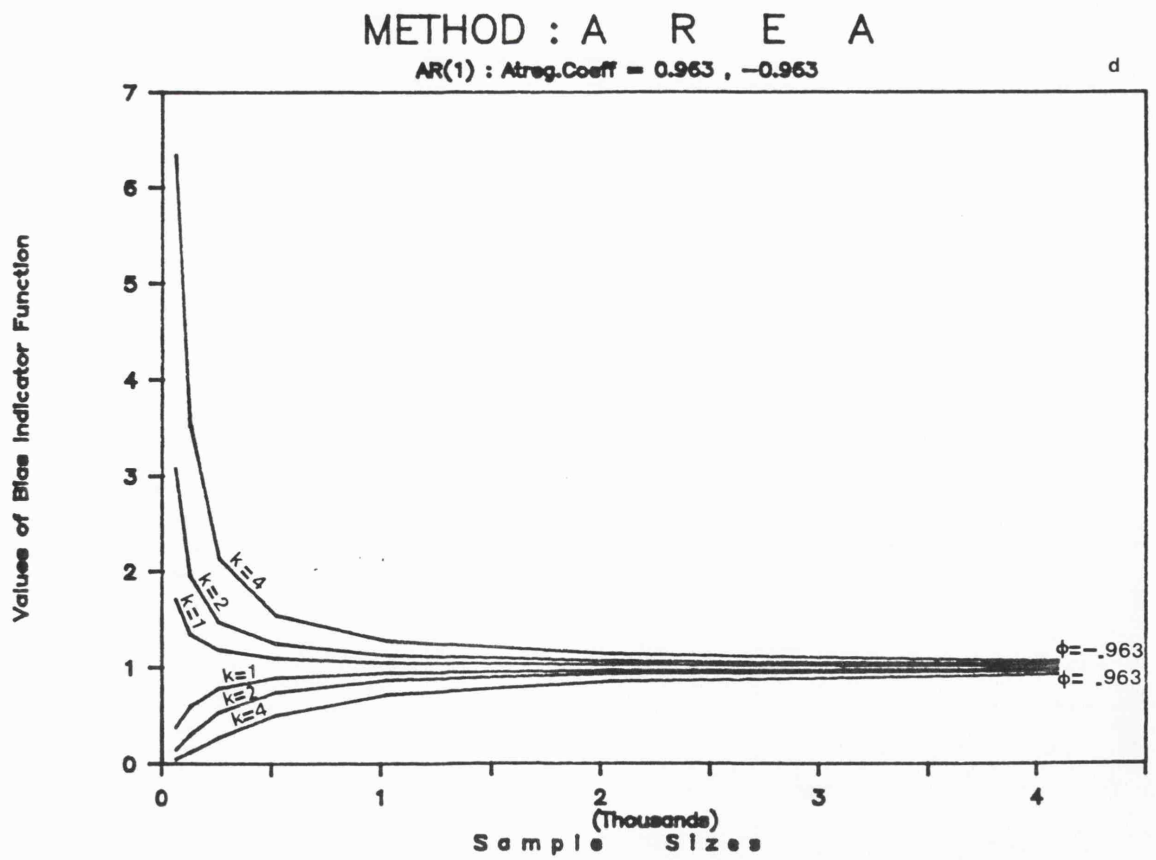
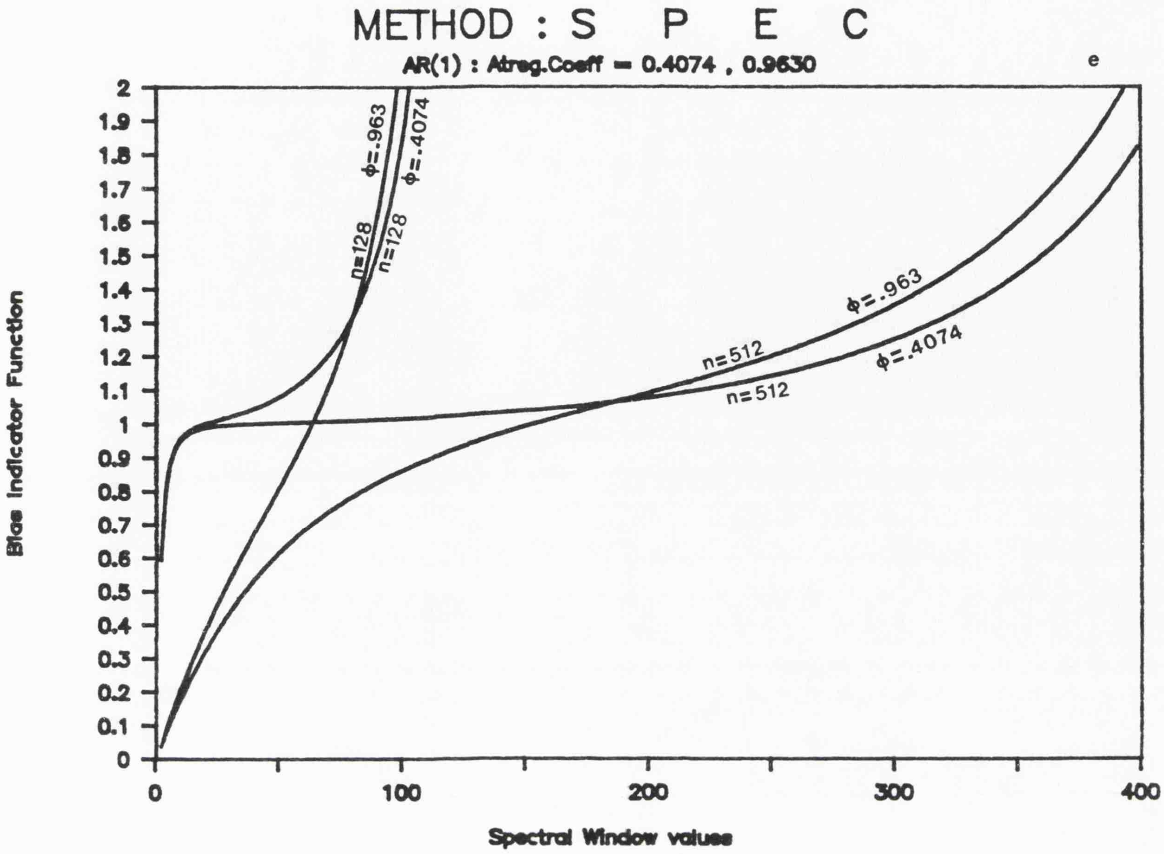


Figure 3.1 (Cont..)






METHOD : S P E C
AR(1) : Atreg.Coeff = 0.4074 , 0.9630
e
Bias Indicator Function
Spectral Window values
φ=.963
φ=.4074
n=128
n=128
φ=.963
n=512
φ=.4074
n=512

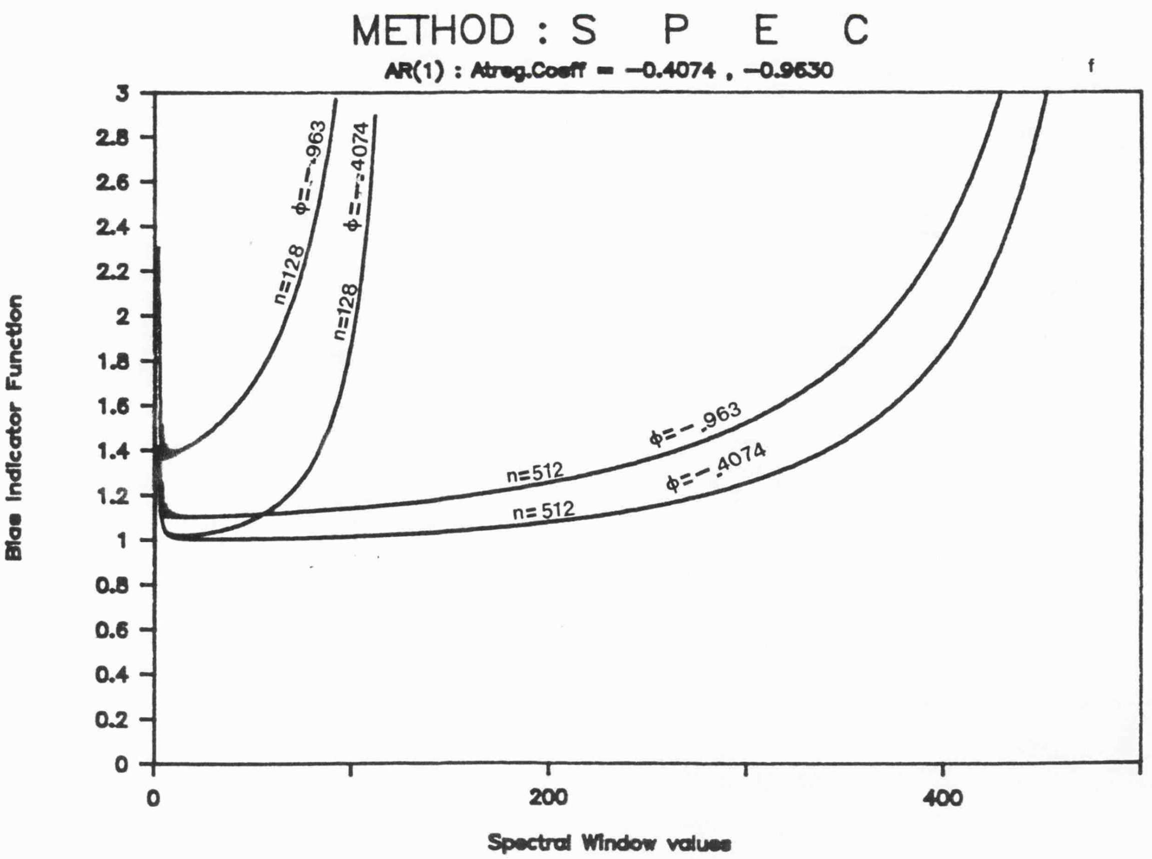
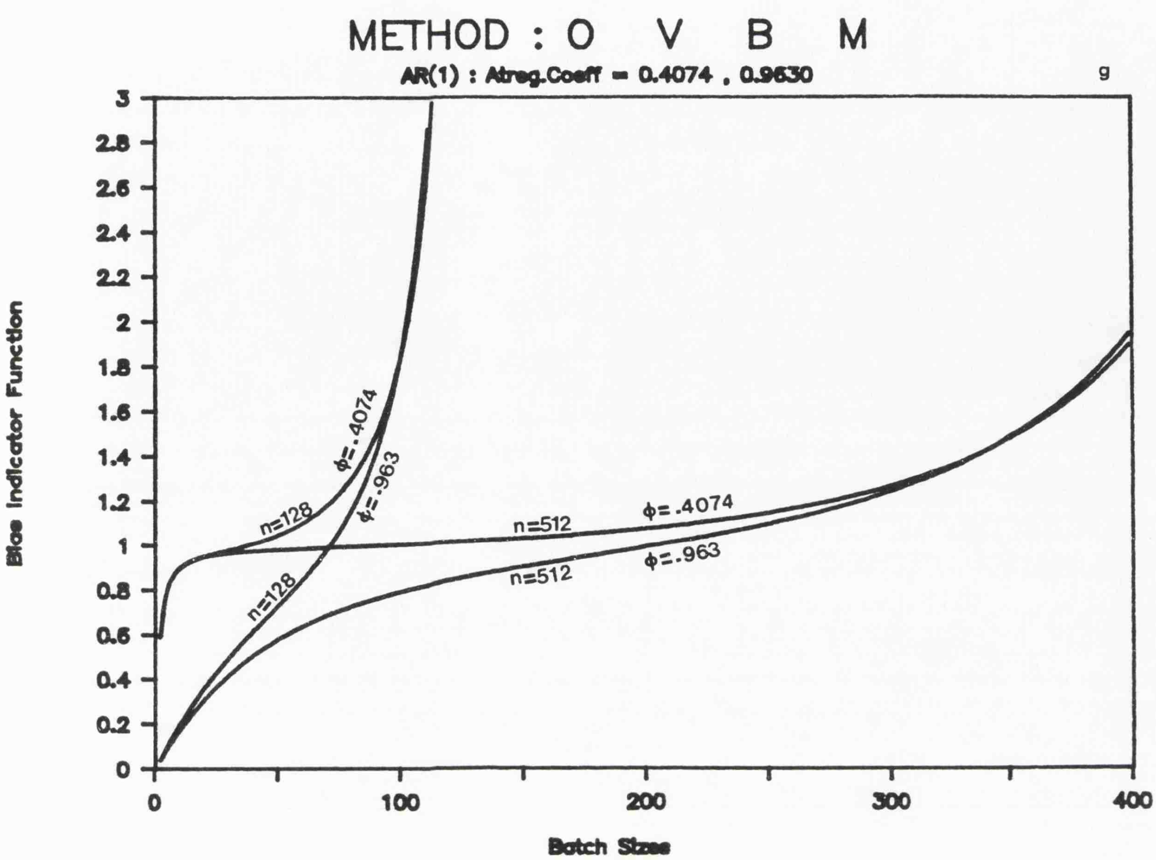
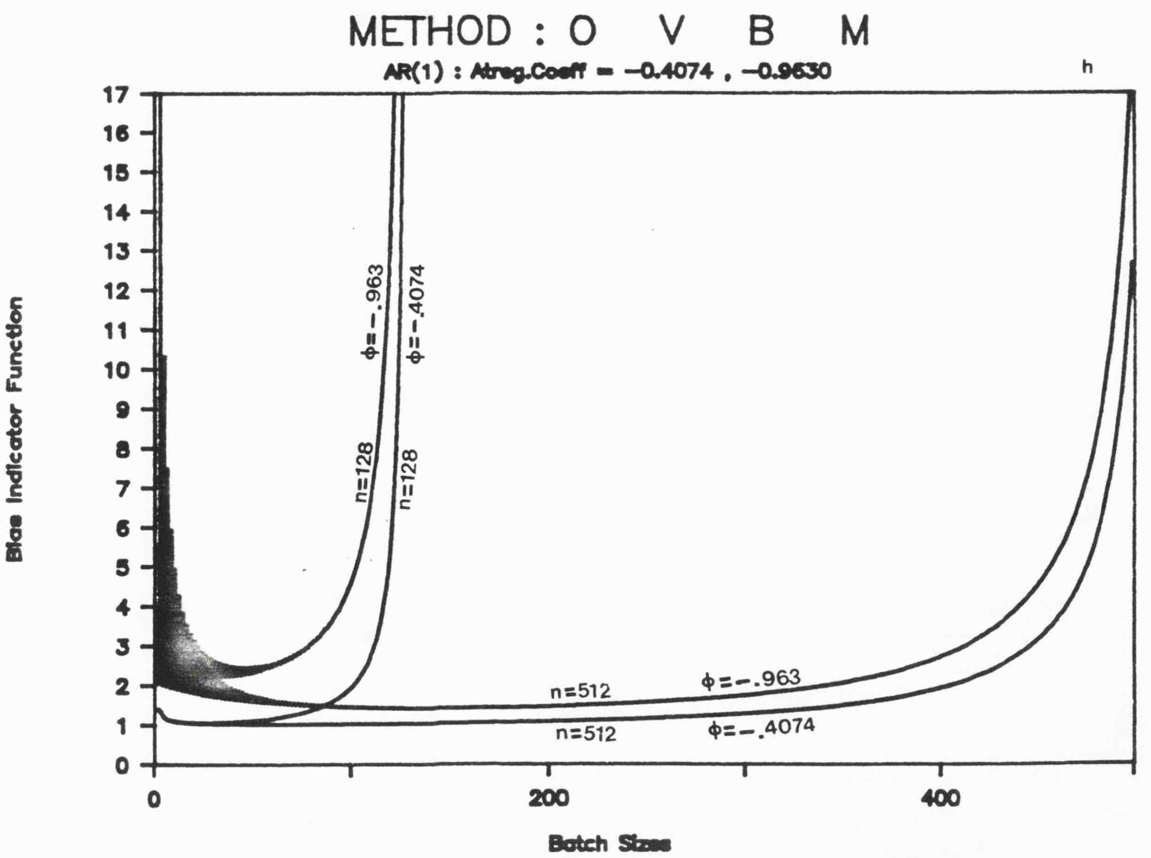

METHOD : S P E C
AR(1) : Atreg.Coeff = −0.4074 , −0.9630
f
Bias Indicator Function
Spectral Window values
φ=−.963
φ=−.4074
n=128
n=128
φ=−.963
φ=−.4074
n=512
n=512

Figure 3.1 (Cont..)





For different sample sizes, table (3.1) displays the minimum bias of the five sample mean variance estimators in the AR(1). The numbers in brackets are the parameter values for which the minimum bias is attained. First, consider the case where φ is positive. For high autocorrelation levels and small sample sizes the AREA estimator achieves smaller minimum bias than that of the NOBM estimator. This result contradicts that obtained by Sargent et al.(1989)[see section 3.2].

For any sample size, the SPEC estimator attaines smaller minimum bias than that of the NOBM and AREA. The same results hold for the OVBM estimator at large sample sizes. With regard to the SPEC and OVBM estimators the minimum bias is achieved at higher values of the spectral window size or the batch size as the sample size increases.

Examine now the minimum bias of the five estimators when φ is negative. For small sample sizes, the NOBM estimator achieves the smallest minimum bias. For high autocorrelation levels and small sample sizes, the minimum bias of the SPEC estimator is smaller than that of the AREA. For large sample sizes the OVBM estimator attaines the largest minimum bias.

TABLE 3.1
Minimum bias of sample mean variance estimators in AR(1)

$\varphi = 0.4074$

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
4	0.4466 (2)	0.4239 (1)	0.4239 (1)	0.1325 (3)	0.5825 (3)
8	0.2631 (2)	0.2481 (1)	0.2481 (1)	0.0530 (5)	0.1365 (7)
16	0.1299 (2)	0.1265 (1)	0.1265 (1)	0.0049 (7)	0.0074 (10)
32	0.0630 (2)	0.0626 (1)	0.0626 (1)	0.0051 (10)	0.0105 (17)
64	0.0310 (2)	0.0310 (1)	0.0310 (1)	0.0025 (15)	0.0021 (24)
128	0.0154 (2)	0.0154 (1)	0.0154 (1)	0.0007 (21)	0.0004 (38)
256	0.0077 (2)	0.0077 (1)	0.0077 (1)	0.0002 (29)	0.0005 (59)
512	0.0038 (2)	0.0038 (1)	0.0038 (1)	0.0001 (41)	0.0002 (94)
1024	0.0019 (2)	0.0019 (1)	0.0019 (1)	0.0000 (58)	0.0000 (148)
2048	0.0010 (2)	0.0010 (1)	0.0010 (1)	0.0000 (81)	0.0000 (234)

$\varphi = 0.9630$

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
4	0.9718 (2)	0.9682 (1)	0.9682 (1)	0.9565 (3)	0.9796 (3)
8	0.9488 (2)	0.9402 (1)	0.9402 (1)	0.2923 (6)	0.3544 (7)
16	0.9015 (2)	0.8850 (1)	0.8850 (1)	0.0960 (11)	0.0444 (13)
32	0.8110 (2)	0.7834 (1)	0.7834 (1)	0.0328 (21)	0.0387 (23)
64	0.6534 (2)	0.6168 (1)	0.6168 (1)	0.0030 (37)	0.0163 (41)
128	0.4323 (2)	0.4018 (1)	0.4018 (1)	0.0059 (64)	0.0002 (71)
256	0.2274 (2)	0.2163 (1)	0.2163 (1)	0.0013 (102)	0.0001 (122)
512	0.1092 (2)	0.1075 (1)	0.1075 (1)	0.0004 (155)	0.0001 (205)
1024	0.0532 (2)	0.0533 (1)	0.0533 (1)	0.0002 (227)	0.0003 (343)
2048	0.0262 (2)	0.0273 (1)	0.0273 (1)	0.0001 (327)	0.0001 (569)

T A B L E 3.1 (Cont..)

$\varphi = -0.4074$

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
4	0.2746	0.4539	0.4539	0.8720	0.2480
	(2)	(1)	(1)	(2)	(2)
8	0.2058	0.2223	0.2223	0.3926	0.3606
	(2)	(1)	(1)	(3)	(2)
16	0.1149	0.1156	0.1156	0.1801	0.2645
	(2)	(1)	(1)	(5)	(5)
32	0.0593	0.0593	0.0593	0.0805	0.1684
	(2)	(1)	(1)	(7)	(9)
64	0.0301	0.0301	0.0301	0.0371	0.1032
	(2)	(1)	(1)	(10)	(15)
128	0.0151	0.0152	0.0152	0.0175	0.0624
	(2)	(1)	(1)	(16)	(25)
256	0.0076	0.0076	0.0076	0.0084	0.0378
	(2)	(1)	(1)	(22)	(42)
512	0.0038	0.0038	0.0038	0.0040	0.0230
	(2)	(1)	(1)	(31)	(69)
1024	0.0019	0.0019	0.0019	0.0020	0.0141
	(2)	(1)	(1)	(50)	(114)
2048	0.0010	0.0010	0.0010	0.0010	0.0087
	(2)	(1)	(1)	(77)	(189)

$\varphi = -0.9630$

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
4	0.0363	20.0735	20.0735	13.6514	8.7676
	(2)	(1)	(1)	(2)	(2)
8	0.0697	8.9882	8.9882	6.4198	6.2364
	(2)	(1)	(1)	(3)	(2)
16	0.1284	3.9857	3.9857	2.7594	3.6549
	(2)	(1)	(1)	(3)	(2)
32	0.2151	1.6786	1.6786	1.3043	2.2390
	(2)	(1)	(1)	(3)	(2)
64	0.2955	0.7098	0.7098	0.6629	1.5761
	(2)	(1)	(1)	(3)	(2)
128	0.2849	0.3493	0.3493	0.3612	1.1824
	(2)	(1)	(1)	(5)	(24)
256	0.1847	0.1878	0.1878	0.1947	0.7478
	(2)	(1)	(1)	(7)	(70)
512	0.0985	0.0986	0.0986	0.1016	0.4224
	(2)	(1)	(1)	(13)	(140)
1024	0.0505	0.0506	0.0506	0.0518	0.2325
	(2)	(1)	(1)	(25)	(260)
2048	0.0256	0.0258	0.0258	0.0261	0.1289
	(2)	(1)	(1)	(47)	(431)

Kevork and Balmer(1990) compared the minimum bias of the five sample mean variance estimators on two additional processes:-

_ the steady state delay in queue in the M/M/1 and

_ the stationary AR(2) when its autocorrelation function shows a damped cyclical behaviour.

For the M/M/1, the spectral sample mean variance estimator achieved the smallest minimum bias for all the sample sizes that were considered. On the other hand, in the AR(2) for certain autoregressive coefficients and small sample sizes, the smallest bias was achieved by the combined estimator

3.6 SUMMARY

In this chapter, we introduced a family of functions for studying the bias of sample mean variance estimators in small samples. We have called these functions "Bias Indicator functions". Analytical forms of the Bias Indicator functions have been derived for five estimators; nonoverlapping batch means, overlapping batch means, area, combined area-nonoverlapping batch means and spectral. The above forms have been expressed in terms of the theoretical autocorrelation coefficients of the output process under study.

Moreover, for each sample mean variance estimator, we have defined the following two statistical criteria; the minimum bias and its parameter values for which the minimum bias is attained. These parameter values have been called "MB-parameter values". The latter two criteria are related to the Bias Indicator functions. That is, the values of these criteria are determined by the Bias Indicator functions. Therefore, for each estimator analytical values of the minimum bias and the "MB-parameter values" can be obtained only when the autocorrelation function of the process under study is known. The usefulness of the MB-parameter values will become evident in chapter six. There, we investigate the performance of the confidence interval methods at these parameter values.

Analytical values for the three statistical criteria under consideration have been obtained for the AR(1), AR(2) and the delay in queue in the M/M/1. For these processes, the theoretical autocorrelation coefficients at any lag are given by known difference equations. In the M/M/1 and the AR(1) with positive autoregressive coefficient, the spectral estimator achieves the smallest minimum bias. The nonoverlapping batch means estimator attains the smallest minimum bias in the AR(1) with negative autoregressive coefficient. In the AR(2), for certain autoregressive coefficients and small sample sizes, the smallest minimum bias is achieved by the combined estimator.

In the following chapter we derive the limiting forms of the Bias Indicator functions. These limiting forms are used for computing analytically the limiting coverages of the corresponding five confidence interval methods.

CHAPTER FOUR

ASYMPTOTIC COMPARISON OF CONFIDENCE INTERVAL METHODS

4.1 INTRODUCTION

For the past five years, the derivation of asymptotic properties of confidence interval methods has constituted one of the main object of research in the output analysis of steady-state simulations. Several criteria have been selected and used for measuring the asymptotic performance of each method. Such criteria are limiting coverages of steady-state means from confidence interval methods and limiting expected values and variances of confidence interval half lengths. Values of the above criteria are computed analytically i.e without using Monte Carlo methods.

In studying the asymptotic performance of confidence interval methods, two issues arise. The first refers to the numerical computation of the limiting coverages and the second to the limiting precision and stability of the confidence intervals, providing that these intervals cover the steady-state mean with the nominal probability.

In regard to the first issue, Goldsman et al.(1986) and Sargent et al.(1989) studied the limiting coverages of the nonoverlapping/overlapping batch means, area and combined NOBM-AREA methods on the AR(1) with the autoregressive coefficient being positive. With respect to the second issue, Schmeiser(1982) derived limiting forms for the expected values and variances of the confidence interval half lengths produced by the nonoverlapping batch means method. Goldsman and Schruben(1984) derived the corresponding limiting forms for the four standardized time series methods. The authors also compared the limiting expected values and variances of the confidence interval half lengths of the nonoverlapping batch means method with those of the four standardized time series methods. Goldsman et al.(1986) and Sargent et al.(1989) summarized the results of the previous two works.

In this chapter, we display some further results on the asymptotic properties of confidence interval methods. For the nonoverlapping/overlapping batch means, area, combined NOBM-AREA and spectral methods, the limiting coverages are computed numerically for different parameter values in the AR(1), AR(2) and the delay in queue in the M/M/1. Furthermore, for the spectral and overlapping batch means methods, we derive limiting forms of the expected values and variances of the confidence interval half lengths.

More specifically, in the following section we describe the way in which the limiting coverages of the above five methods can be computed analytically. This approach is general in that it can be applied to any process whose the theoretical autocorrelation coefficients are known. In section 4.3, we study the limiting coverages achieved by the five methods in the AR(1), AR(2) and the delay in queue in the M/M/1. In the final two sections, we discuss asymptotic comparisons of the limiting expected values and variances of the confidence interval half lengths produced by the five methods under consideration.

4.2 THE ANALYTICAL COMPUTATION OF LIMITING COVERAGES

Let $\{X_t, t=1,2,3...\}$ be a stationary output process with

$$E(X_t) = \mu < \infty \quad , \quad t \geq 1 \tag{4.1a}$$

$$Var(X_t) = \gamma_0 < \infty \; , \; t \geq 1 \tag{4.1b}$$

and

$$\sum_{s=-\infty}^{\infty} |\gamma_s| < \infty \tag{4.1c}$$

The last condition implies that the correlation between X_t and X_{t+s} is negligible when s is very large[see Law and Carson(1978)]. This property is satisfied by a wide class of processes including autoregressive processes, regenerative processes and m-dependent processes [see Law(1983), Schruben(1983)]. The term "m-dependence" means that X_t and X_{t+s} are autocorrelated only if $s \leq m$ [see Kleijnen(1975)].

For simulation output processes satisfying conditions (4.1), we illustrate the way in which the limiting coverages of the following confidence interval methods can be computed analytically:

i) Nonoverlapping batch means method(NOBM)

ii) Standardized time series-area method(AREA)

iii) Combined area-nonoverlapping batch means method(NOBM-AREA)

iv) Spectral method denoted(SPEC)

v) Overlapping batch means method(OVBM).

For the first three methods, the limiting coverages are obtained when the batch size m is fixed and the number of contiguous batches k tends to infinity. With regard to the overlapping batch means method the limitimg coverages are computed when the sample size n tends to infinity and the batch size m is fixed. For the spectral method the coverages under discussion are obtained when the sample size n tends to infinity keeping fixed the spectral window size.

In the subsequent analysis we use the notation which has been established by Goldsman and Schruben(1984) and Goldsman et al.(1986). Define the scalar quantities $\sigma_n^2 = nV(\bar{X}_n)$ and $\sigma^2 = \lim_{n\to\infty}[nV(\bar{X}_n)]$ where $V(\bar{X}_n)$ is the true variance of the sample mean. The above confidence interval methods propose the following estimators for σ_n^2:-

$$\hat{V}_{NB} = \frac{m}{k-1} \sum_{j=1}^{k} \left[\bar{X}_{j,m} - \bar{X}_n \right]^2$$

where $\bar{X}_{j,m}$ was defined in (3.1a)

$$\hat{V}_{SM} = \frac{12}{(m^3-m)k} \sum_{j=1}^{k} \hat{A}_j^2$$

where $\hat{A}_j$ was defined in (3.2a)

$$\hat{V}_{CM} = \frac{k\hat{V}_{SM} + (k-1)\hat{V}_{NB}}{2k-1}$$

$$\hat{V}_{SP} = \frac{n}{n-w} \left[\hat{\gamma}_0 + 2 \sum_{s=1}^{w-1} \lambda_w(s)\hat{\gamma}_s \right]$$

where $\lambda_w(s)$ and γ_s were defined in (3.4a) and (3.4b)

$$\hat{V}_{OB} = \frac{m}{n-m+1} \sum_{j=1}^{n-m+1} \left[\bar{X}_j(m) - \bar{X}_n \right]^2$$

where $\bar{X}_j(m)$ was defined in (3.5a).

We remind the reader that the initials NB, SM, CM, SP and OB stand for the NOBM, AREA, combined NOBM-AREA, SPEC and OVBM methods respectively.

For the estimators $\hat{V}_i$ of the NOBM, AREA and combined NOBM-AREA methods, Goldsman et al.(1986) and Sargent et al.(1989) report that as $k \to \infty$, $\hat{V}_i \to E(V_i)$ w.p 1. The same is true for $\hat{V}_i$ (i=SP,OB) as $n \to \infty$. Therefore, the following random variables:-

i) for fixed m and large k

$$T_1 = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\left[\hat{V}_{NB} \right]^{\frac{1}{2}}} \quad , \quad T_2 = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\left[\hat{V}_{SM} \right]^{\frac{1}{2}}} \quad , \quad T_3 = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\left[\hat{V}_{CM} \right]^{\frac{1}{2}}}$$

ii) for fixed w and large n

$$T_4 = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\left[\hat{V}_{SP} \right]^{\frac{1}{2}}}$$

iii) for fixed m and large n

$$T_5 = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\left[\hat{V}_{OB} \right]^{\frac{1}{2}}}$$

tend to be normal with mean zero and variances

$$\frac{1}{\left[\dfrac{E(\hat{V}_{NB})}{\sigma^2}\right]}, \quad \frac{1}{\left[\dfrac{E(\hat{V}_{SM})}{\sigma^2}\right]}, \quad \frac{1}{\left[\dfrac{E(\hat{V}_{CM})}{\sigma^2}\right]}, \quad \frac{1}{\left[\dfrac{E(\hat{V}_{SP})}{\sigma^2}\right]}, \quad \frac{1}{\left[\dfrac{E(\hat{V}_{OB})}{\sigma^2}\right]} \tag{4.2}$$

respectively.

Multiplying and dividing ratios (4.2) by n, we get the corresponding limiting Bias Indicator function at the denominator of each ratio i.e

$AsBs(m)_{NB} = \lim_{k\to\infty} Bs(n,k)_{NB}$ for the NOBM method

$AsBs(m)_{SM} = \lim_{k\to\infty} Bs(n,k)_{SM}$ for the AREA method

$AsBs(m)_{CM} = \lim_{k\to\infty} Bs(n,k)_{CM}$ for the combined NOBM-AREA method

$AsBs(w)_{SP} = \lim_{n\to\infty} Bs(n,w)_{SP}$ for the SPEC method

$AsBs(m)_{OB} = \lim_{n\to\infty} Bs(n,m)_{OB}$ for the OVBM method.

Standardizing T_1, T_2, T_3, T_4, T_5, the following new random variables

$$Z_1 = \frac{\bar{X}_n - \mu}{\hat{\sigma}_{NB}} \left\{ \frac{1}{\sqrt{AsBs(m)_{NB}}} \right\}$$

$$Z_2 = \frac{\bar{X}_n - \mu}{\hat{\sigma}_{SM}} \left[\frac{1}{\sqrt{AsBs(m)_{SM}}} \right]$$

$$Z_3 = \frac{\bar{X}_n - \mu}{\hat{\sigma}_{CM}} \left[\frac{1}{\sqrt{AsBs(m)_{CM}}} \right]$$

$$Z_4 = \frac{\bar{X}_n - \mu}{\hat{\sigma}_{SP}} \left[\frac{1}{\sqrt{AsBs(w)_{SP}}} \right]$$

$$Z_5 = \frac{\bar{X}_n - \mu}{\hat{\sigma}_{OB}} \left[\frac{1}{\sqrt{AsBs(m)_{OB}}} \right]$$

can be approximated by the standardized normal distribution Z with mean 0 and variance 1.

Therefore, for each confidence interval method the limiting coverages will be computed analytically by

$$AsCVR(m)_{NB} = 1 - 2\,\Phi\left[z_{\alpha/2}\sqrt{AsBs(m)_{NB}} \right] \tag{4.3a}$$

$$AsCVR(m)_{SM} = 1 - 2\,\Phi\left[z_{\alpha/2}\sqrt{AsBs(m)_{SM}} \right] \tag{4.3b}$$

$$AsCVR(m)_{CM} = 1 - 2\,\Phi\left[z_{\alpha/2}\sqrt{AsBs(m)_{CM}} \right] \tag{4.3c}$$

$$\mathrm{AsCVR}(w)_{SP} = 1 - 2\,\Phi\left[z_{\alpha/2}\sqrt{\mathrm{AsBs}(w)_{SP}} \right] \qquad (4.3d)$$

$$\mathrm{AsCVR}(m)_{OB} = 1 - 2\,\Phi\left[z_{\alpha/2}\sqrt{\mathrm{AsBs}(m)_{OB}} \right] \qquad (4.3e)$$

where $\Pr(Z>z_{\alpha/2}) = \alpha/2$

and $\Phi(z) = \displaystyle\int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x/2)}dx \quad (-\infty<x<+\infty).$

We derive below the asymptotic forms of the five Bias Indicator functions under consideration:

NONOVERLAPPING BATCH MEANS METHOD

Providing that $\sum_{s=-\infty}^{\infty} \rho_s<\infty$, when the batch size m is fixed then

$$\lim_{k\to\infty} \frac{2\sum_{j=1}^{k-1}\left[1 - \frac{j}{k} \right] A_j(m)}{k - 1} = 0 \qquad (4.4)$$

and

$$\lim_{k\to\infty}\left[\sum_{s=1}^{n-1}\left[1 - \frac{s}{n}\right]\rho_s\right] = \lim_{n\to\infty}\left[\sum_{s=1}^{n-1}\left[1 - \frac{s}{n}\right]\rho_s\right] = \sum_{s=1}^{\infty} \rho_s \qquad (4.5)$$

Taking limits to both sides of (3.22a) and using (4.4) and (4.5), the limiting form of the Bias Indicator function will be given by

$$AsBs(m)_{NB} = \lim_{k\to\infty} Bs(n,k)_{NB} = \frac{1 + 2\sum_{s=1}^{m-1}\left[1 - \frac{s}{m}\right]\rho_s}{1 + 2\sum_{s=1}^{\infty}\rho_s} \tag{4.6a}$$

STANDARDIZED TIME SERIES-AREA METHOD

Taking limits to both sides of (3.22b) and using (4.5)

$$AsBs(m)_{SM} = \lim_{k\to\infty} Bs(n,k)_{SM} = \frac{6\sum_{r=0}^{m'}\sum_{s=0}^{m'}\delta_{rs}}{(m-m^3)\left\{1 + 2\sum_{s=1}^{\infty}\rho_s\right\}} \tag{4.6b}$$

where m' and δ_{rs} were defined in (3.11).

COMBINED AREA-NONOVERLAPPING BATCH MEANS METHOD

From (3.22c)

$$Bs(n,k)_{CM} = \frac{kBs(n,k)_{SM} + (k-1)Bs(n,k)_{NB}}{2k-1}$$

Taking limits to both sides of this relationship

$$AsBs(m)_{CM} = \lim_{k\to\infty} Bs(n,k)_{CM} = \frac{\lim_{k\to\infty}\left[Bs(n,k)_{SM} + (1-(1/k))Bs(n,k)_{NB}\right]}{\lim_{k\to\infty}\left[2\left[1 - \frac{1}{k}\right]\right]}$$

$$= \frac{AsBs(m)_{SM} + AsBs(m)_{NB}}{2} \tag{4.6c}$$

From (4.6c), the limiting Bias Indicator function of the combined NOBM-AREA method is the mean of the corresponding functions of the NOBM and AREA methods.

SPECTRAL METHOD

For the function $g_s(n)$ defined in (3.18)

$$\lim_{n\to\infty} g_s(n) = \begin{cases} 1 & \text{when } s=0 \\ 0 & \text{when } s\neq 0 \end{cases}$$

Using (4.5), the limiting form of the Bias Indicator function will be given by

$$AsBs(w)_{SP} = \lim_{n\to\infty} Bs(n,w)_{SP} = \frac{1 + 2\sum_{s=1}^{w-1} \lambda_w(s)\rho_s}{1 + 2\sum_{s=1}^{\infty} \rho_s} \tag{4.6d}$$

where $\lambda_w(s) = 0.5(1+\cos(\pi s/w))$.

OVERLAPPING BATCH MEANS METHOD

From (3.22e)

$$AsBs(m)_{OB} = \lim_{n\to\infty} Bs(n,m)_{OB} = \frac{1 + 2\sum_{s=1}^{m-1} \left[1 - \frac{s}{m} \right]\rho_s}{1 + 2\sum_{s=1}^{\infty} \rho_s} \tag{4.6e}$$

For the NOBM method, since n=mk, when m is fixed and $k\to\infty$ then $n\to\infty$. Therefore, for the NOBM and OVBM methods the limiting forms of the Bias Indicator functions are exactly the same[compare (4.6a) with (4.6e)]. This means that when the sample size tends to infinity these two methods produce the same limiting coverages for equal batch sizes.

By substituting (4.6) into (4.3), the limiting coverages, that the five confidence interval methods achieve, can be computed exactly, providing that the theoretical autocorrelation coefficients of the output process under study are known.

4.3 STUDYING THE LIMITING COVERAGES IN DIFFERENT STATIONARY OUTPUT PROCESSES

Five confidence interval methods have been considered in the previous section. For these methods, the limiting coverages can be computed analytically only when the theoretical autocorrelation coefficients of the output process under study are known. Three processes, whose autocorrelation functions are known, are considered in this section; the AR(1), AR(2) and the delay in queue in the M/M/1 queueing system. The limiting coverages of the five methods are studied on these processes.

4.3.1 AR(1) processes

Table (4.1) displays the limiting coverages, the nonoverlapping/overlapping batch means, area, combined NOBM-AREA and spectral methods achieve in the AR(1). For the AR(1) process defined in (3.23), the s^{th} lag theoretical autocorrelation coefficient is $\varphi^{|s|}$.

TABLE 4.1
Limiting Coverages of Confidence Interval Methods in the AR(1)

φ	m	NOBM , OVBM	AREA	Combined NOBM-AREA
.4074	2	.7946	.5888	.7142
	4	.8942	.7244	.7978
	8	.8768	.8165	.8508
	16	.8890	.8638	.8770
	32	.8948	.8832	.8890
	64	.8974	.8920	.8948
	...	...	...	...
	∞	.9000	.9000	.9000
.7778	2	.5618	.2162	.4392
	4	.6868	.3482	.5654
	8	.7872	.5342	.6926
	16	.8480	.7102	.7922
	32	.8766	.8164	.8498
	64	.8890	.8632	.8768
	128	.8946	.8830	.8890
	256	.8974	.8928	.8946
	...	...	...	...
	∞	.9000	.9000	.9000
.9630	2	.2480	.0346	.1784
	4	.3410	.0628	.2486
	8	.4572	.1184	.3424
	16	.5886	.2196	.4604
	32	.7132	.3798	.5940
	64	.8058	.5784	.7196
	128	.8574	.7432	.8102
	256	.8806	.8320	.8586
	512	.8908	.8698	.8808
	1024	.8956	.8858	.8908
	2048	.8978	.8932	.8966
	...	...	...	...
	∞	.9000	.9000	.9000

Spectral Method	φ		
w	0.4074	0.7778	0.9630
2	.7946	.5618	.2480
4	.8568	.6918	.3416
8	.8864	.7970	.4592
16	.8964	.8608	.5930
32	.8990	.8882	.7214
64	.8998	.8968	.8178
128	...	.8992	.8780
256	...	...	.8978
512	...	...	.8994
...	...	...	...
∞	.9000	.9000	.9000

Goldsman et al.(1986) and Sargent et al.(1989) report that the limiting coverages of the NOBM, OVBM methods tend to achieve the nominal confidence level more quikly than the limiting coverages of the AREA and combined NOBM-AREA methods. This can be verified from the first part of table (4.1). Among these four methods, for equal small batch sizes m, the NOBM and OVBM methods achieve limiting coverages which are the nearest to the nominal confidence level. We can also observe that as the spectral window size w increases the limiting coverages of the spectral method tend to attain the nominal confidence level rather fast.

4.3.2 M/M/1 queueing model

The process under study is the delay of the j^{th} customer in queue. Two forms exist for computing analytically the theoretical autocorrelation function of this process. The first one has been given by Blomquist(1967) and the second by Daley(1968). Let λ and τ be the arrival rate and the traffic intensity respectively. According to Daley, the s^{th} lag theoretical autocorrelation coefficient is given by the difference equation

$$\rho_{s+1} = \rho_s - \frac{\mu(1-\tau) - C_{s+1}}{\lambda\sigma^2} \tag{4.7}$$

where

$$C_s = \frac{1-\tau^3}{2\lambda}\left\{ \frac{2\tau s}{1+\tau} - \sum_{r=1}^{s}\left[(s+1-r)\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)\Gamma(\frac{1}{2})} z^r \right] \right\}$$

and

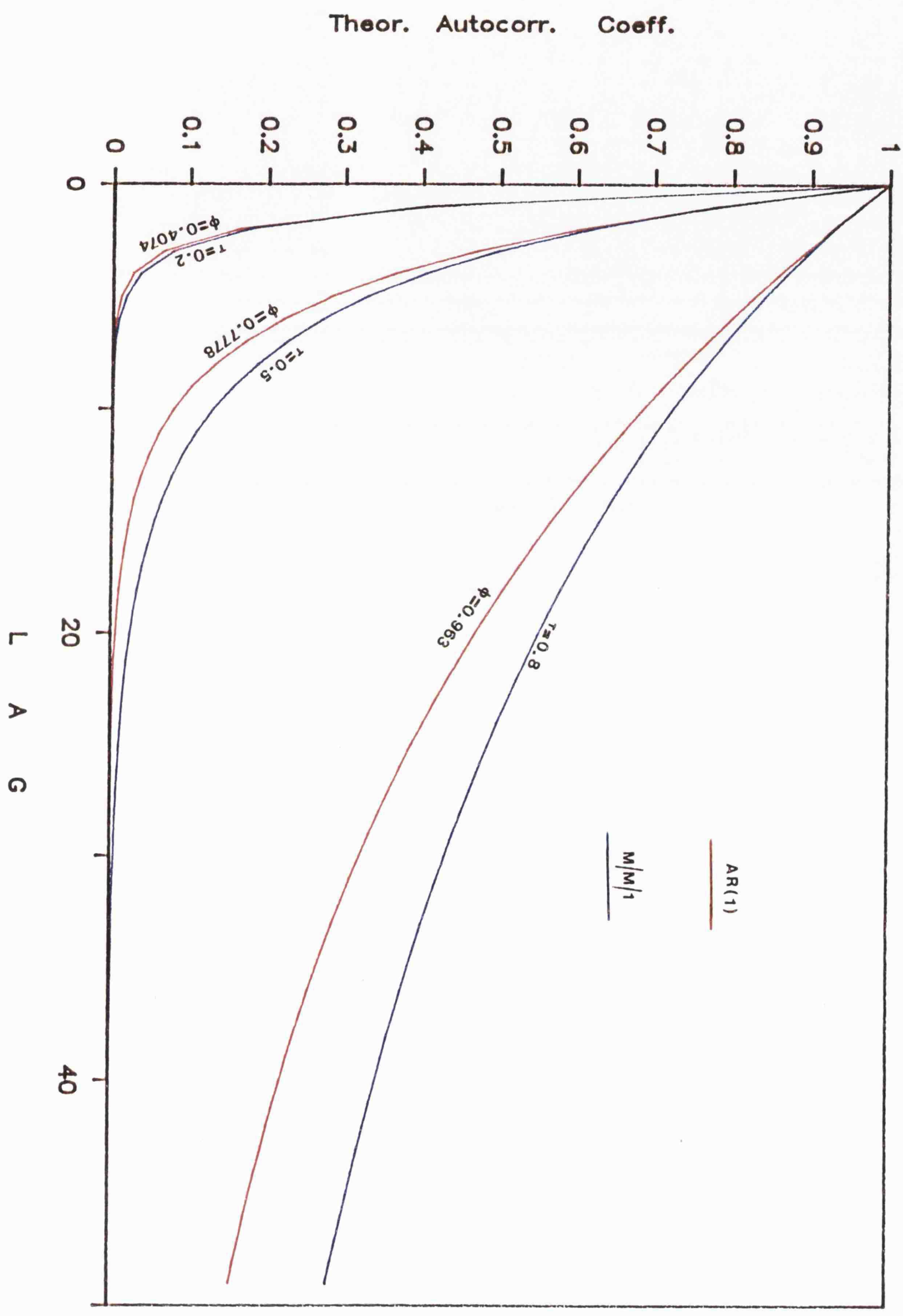
$$z = \frac{4\tau}{(1+\tau)^2} \quad , \quad \mu = \frac{\tau^2}{\lambda(1-\tau)} \quad , \quad \sigma^2 = \frac{\tau^3(2-\tau)}{(\lambda(1-\tau))^2}$$

From Daley, we can also see that the autocorrelation function of the AR(1) decays faster than the autocorrelation function of the delay in the M/M/1, providing that the two processes have the same first lag theoretical autocorrelation coefficient. To compare the performance of the confidence interval methods between the above two processes, the values of the autoregressive coefficient φ and the traffic intensity τ were chosen in a way such that these processes have the same first lag theoretical autocorrelation coefficient. We have selected the following values for τ; 0.20, 0.50 and 0.80. The corresponding values for φ are 0.4074, 0.7778 and 0.963. For these values of φ and τ, figure (4.1) illustrates the autocorrelation functions of the AR(1) and M/M/1.

Table (4.2) displays the limiting coverages achieved by the five confidence interval methods under consideration. The infinite sum of the autocorrelation coefficients at the denominators of the limiting Bias Indicator functions was computed by

$$1 + 2\sum_{s=1}^{\infty} \rho_s = \frac{1+\tau}{1-\tau} + \frac{2\tau(3-\tau)}{(2-\tau)(1-\tau)^2} \quad \text{[see Daley(1968)]}$$

Figure 4.1
Autocorrelation functions of the AR(1) and the delay in queue in the M/M/1



T A B L E 4.2
Limiting Coverages of Confidence Interval Methods in the M/M/1

τ	m	NOBM , OVBM	AREA	Combined NOBM-AREA
0.2	2	.7854	.5794	.7046
	4	.8428	.7106	.7886
	8	.8736	.8084	.8448
	16	.8876	.8588	.8742
	32	.8940	.8810	.8876
	64	.8970	.8910	.8950
	...	...	...	...
	∞	.9000	.9000	.9000
0.5	2	.5190	.1970	.4032
	4	.6432	.3108	.5224
	8	.7512	.4756	.6484
	16	.8256	.6506	.7566
	32	.8658	.7790	.8286
	64	.8842	.8460	.8666
	128	.8924	.8754	.8842
	256	.8962	.8884	.8924
	...	...	...	...
	∞	.9000	.9000	.9000
0.8	2	.2004	.0278	.1436
	4	.2768	.0492	.2008
	8	.3758	.0908	.2780
	16	.4944	.1638	.3780
	32	.6208	.2814	.4982
	64	.7334	.4442	.6262
	128	.8144	.6216	.7390
	256	.8600	.7600	.8178
	512	.8816	.8368	.8590
	1024	.8912	.8714	.8818
	2048	.8958	.8862	.8912
	...	...	...	...
	∞	.9000	.9000	.9000

Spectral method	τ		
w	0.2	0.5	0.8
2	.7854	.5190	.2004
4	.8504	.6476	.2774
8	.8832	.7604	.3774
16	.8954	.8378	.4978
32	.8988	.8778	.6270
64	.8998	.8936	.7432
128	...	.8984	.8266
256	...	.8996	.8724
512	...	...	.8916
...	...	...	...
∞	.9000	.9000	.9000

As in the case of AR(1), the limiting coverages of the NOBM, OVBM methods tend to attain the nominal confidence level faster than the corresponding coverages of the AREA and combined NOBM-AREA methods. We can also observe how quikly the limiting coverages of the SPEC method tend to achieve the nominal confidence level.

Considering each method separately, its limiting coverages tend to achieve the nominal confidence level faster in the AR(1) than in the M/M/1, providing that the two processes have the same first lag theoretical autocorrelation coefficient. This happens because the autocorrelation function of the AR(1) decays faster to zero.

4.3.3 AR(2) processes

The form of this process is

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \epsilon_t$$

The ϵ_t's are independent and normally distributed random variables with mean 0 and common variance σ_ϵ^2. The s^{th} lag theoretical autocorrelation coefficient is given by the difference equation

$$\rho_s = \varphi_1 \rho_{s-1} + \varphi_2 \rho_{s-2}$$

with initial values $\rho_0 = 1$ and $\rho_1 = \varphi_1/(1-\varphi_2)$.

The following two AR(2) processes are considered in this section

$$X_t = 0.75X_{t-1} - 0.50X_{t-2} + \epsilon_t$$

$$X_t = 0.99X_{t-1} - 0.90X_{t-2} + \epsilon_t$$

Figure 4.2
Autocorrelation functions of AR(2) processes

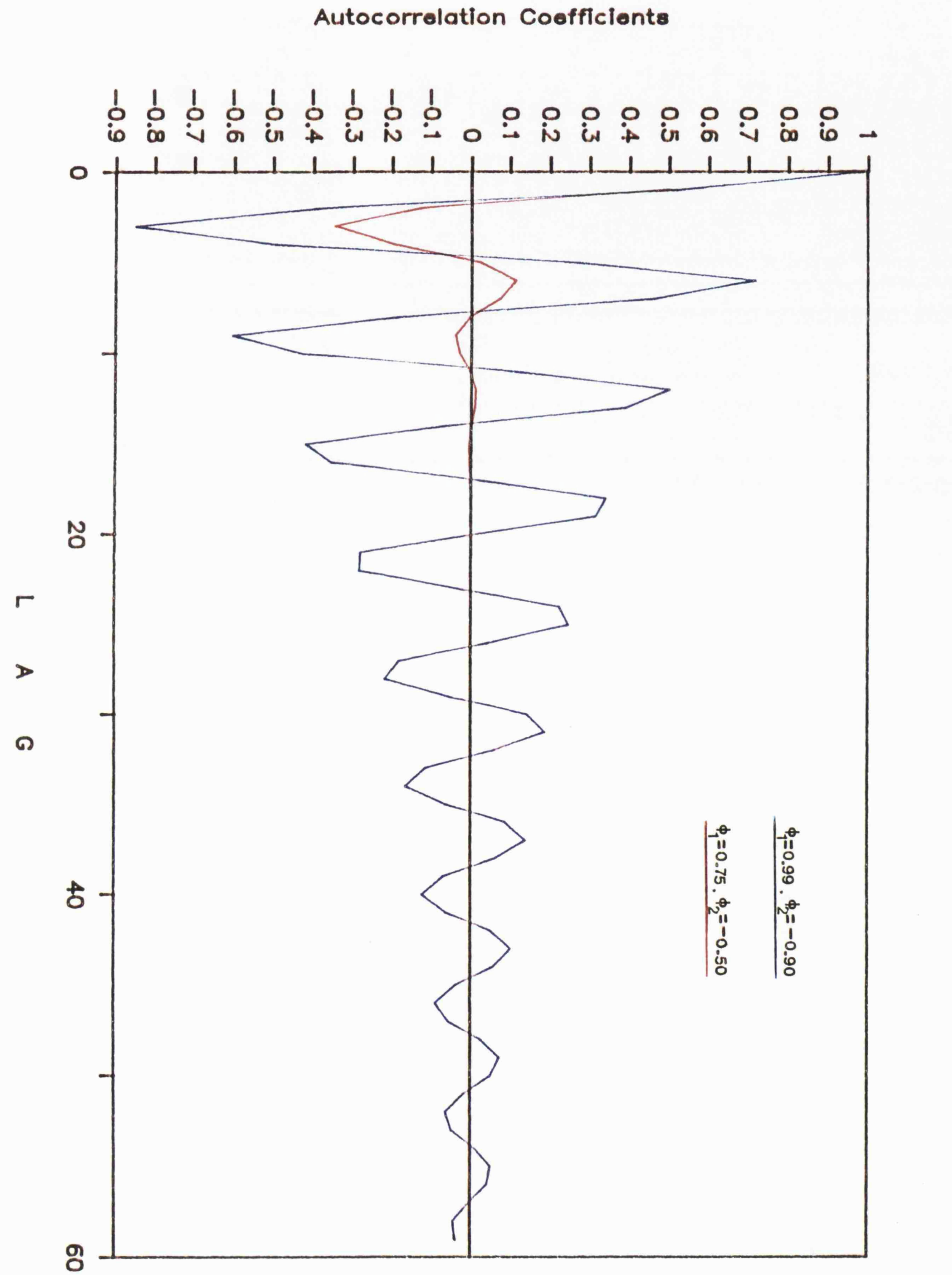


TABLE 4.3
Limiting Coverages of Confidence Interval Methods in the AR(2)

φ_1	m	NOBM , OVBM	AREA	Combined NOBM-AREA
.75	2	.9556	.7552	.8999
	4	.9526	.9646	.9586
	8	.9234	.9574	.9426
	16	.9132	.9340	.9250
	32	.9068	.9190	.9128
	64	.9036	.9100	.9070
	...	...	...	...
	∞	.9000	.9000	.9000
.99	2	1.0000	.9946	1.0000
	4	1.0000	1.0000	1.0000
	8	.9936	.9990	.9974
	16	.9820	.9918	.9880
	32	.9448	.9860	.9724
	64	.9282	.9604	.9468
	128	.9150	.9384	.9276
	256	.9078	.9216	.9150
	512	.9040	.9116	.9078
	...	...	...	...
	∞	.9000	.9000	.9000

Figure (4.2) presents the theoretical autocorrelation functions of the two processes. These functions display a damped cyclical behaviour. To obtain the limiting values of the Bias Indicator functions, the infinite sum of the autocorrelation coefficients in forms (4.6) was replaced by the finite sum of the first r autocorrelation coefficients such that $|\rho_r|<10^{-9}$.

Table (4.3) contains the limiting coverages, the NOBM, OVBM, AREA and combined NOBM-AREA methods achieve. As in the cases of AR(1) and M/M/1, the limiting coverages of the first two methods tend to achieve the nominal confidence level more quikly than the other two. For φ_1=0.75 and m=2, notice that the limiting coverage of the combined method is very close to 0.90. In other words, for this method and for m=2 the limiting Bias Indicator function is close to 1. This happens because the infinite sum of the autocorrelation coefficients is quite close to 0. More specifically, for m=2 the limiting Bias Indicator functions of the AREA and NOBM methods are

$$\frac{1 - \rho_1}{1 + 2\sum_{s=1}^{\infty} \rho_s} \quad , \quad \frac{1 + \rho_1}{1 + 2\sum_{s=1}^{\infty} \rho_s} \quad \text{respectively.}$$

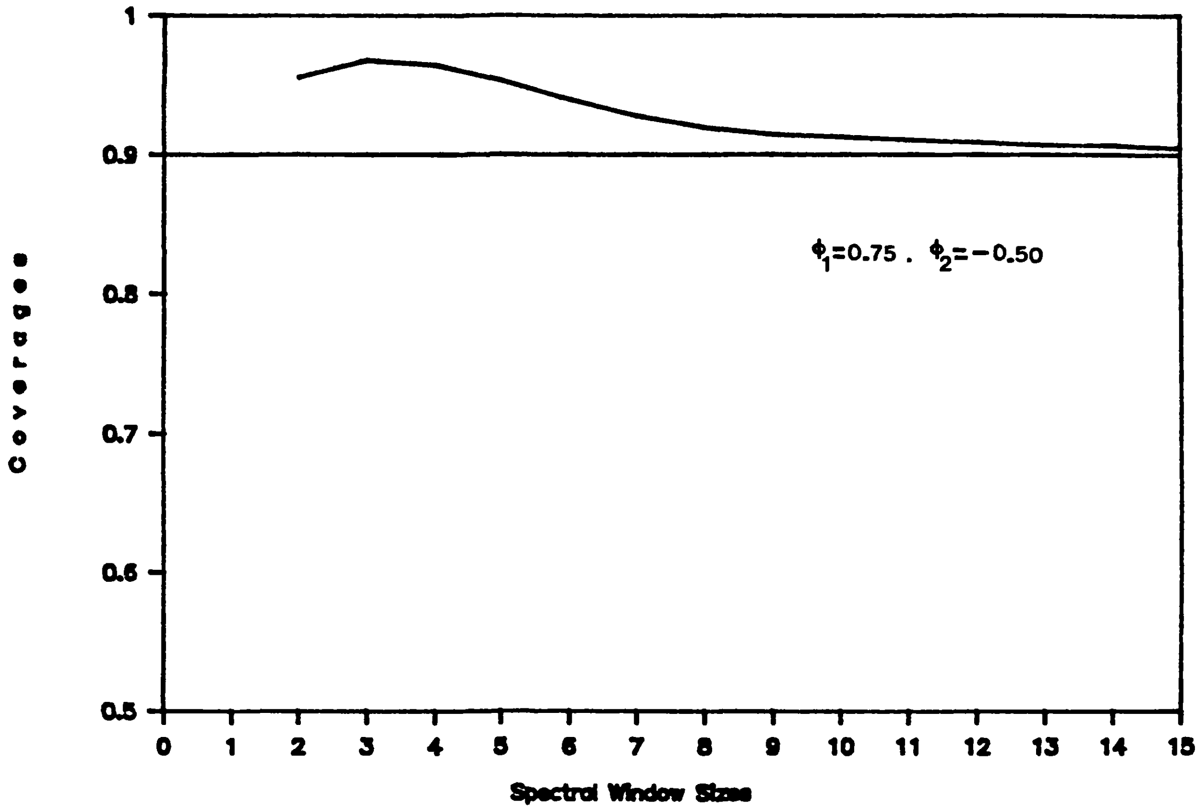
From (4.6c), for m=2 the limiting form of the Bias Indicator function of the combined method will be $1/(1+2\sum_{s=1}^{\infty} \rho_s)$. This function will be close to one only if the infinite sum of the autocorrelation coefficients tends to be close to zero.

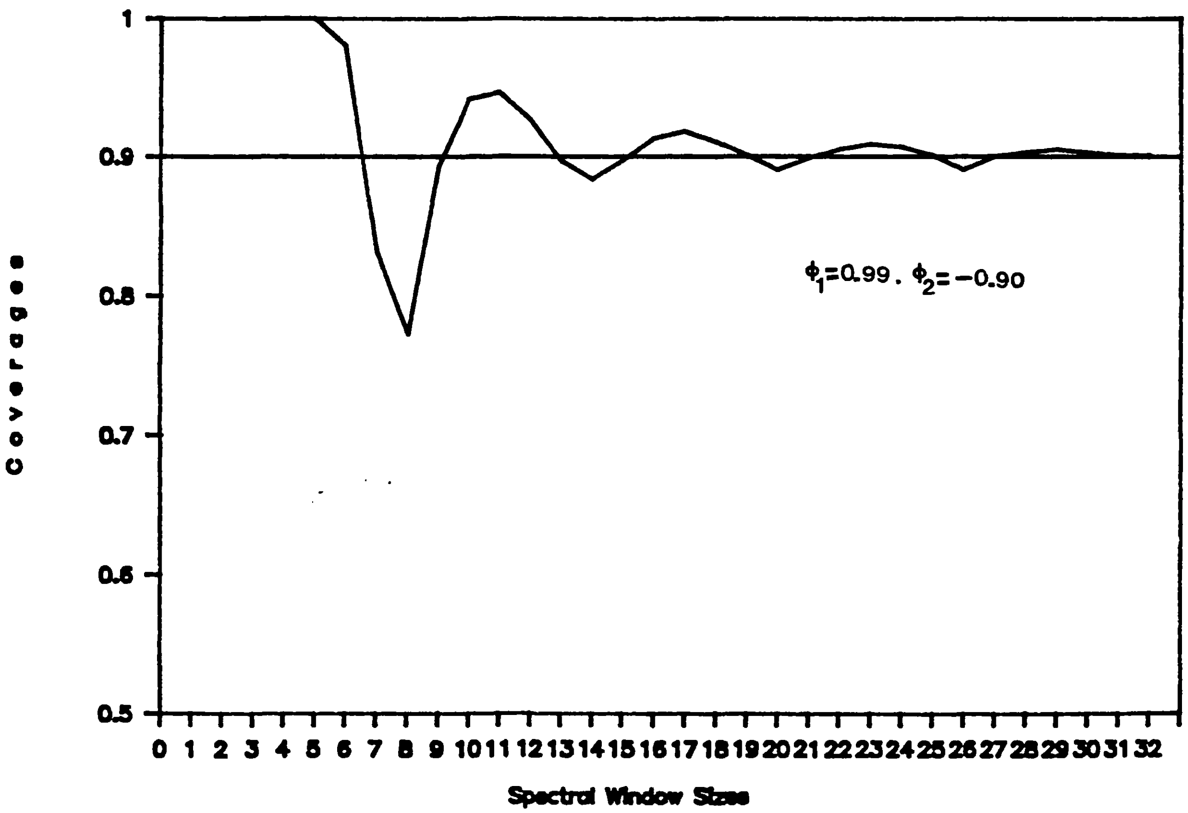
Figure (4.3) illustrates the limiting coverages the spectral method achieves in the above two processes. For φ_1=0.99 we see that the limiting coverages converge fluctuating around the nominal confidence level.

Let us now discuss some interesting empirical findings. Define a small number e. For the AR(1), AR(2) and M/M/1 and different e's, table (4.4) provides the parameter values m^0_{NB}, m^0_{SM}, m^0_{CM}, m^0_{OB} and w^0 for which the limiting Bias Indicator functions of the five methods lie in the interval [1-e,1+e]. We will call these parameter values optimum parameter values. For the above three models, when $0.001 < e < 0.15$

$$\frac{m^0_{OB}}{m^0_{NB}} = 1 \quad , \quad \frac{m^0_{CM}}{m^0_{NB}} \simeq 2 \quad , \quad \frac{m^0_{SM}}{m^0_{NB}} \simeq 3$$

Figure 4.3
Limiting coverages achieved by the spectral method in AR(2) processes





T A B L E 4.4

Optimum parameter values for confidence interval methods when $n \to \infty$

AR(1)

φ	e	NOBM	AREA	Combined NOBM-AREA	SPEC
0.4074	.10	10	30	20	7
	.01	98	294	196	24
	.001	977	2932	1954	76
0.9630	.10	266	792	527	167
	.01	2652	7955	5304	585
	.001	26518	79554	53036	1862

AR(2)

φ_1	e	NOBM	AREA	Combined NOBM-AREA	SPEC
0.75	.15	10	27	18	8
	.10	14	41	27	10
	.01	134	402	268	28
0.99	.15	84	166	248	13
	.10	113	248	372	18

M/M/1

τ	e	NOBM	AREA	Combined NOBM-AREA	SPEC
0.2	.15	8	22	15	6
	.10	11	33	22	8
	.01	110	330	220	27
	.001	1107	2214	3321	87
0.5	.15	37	109	72	27
	.01	56	165	110	36
	.01	554	1689	1116	128
0.8	.15	336	659	999	246
	.10	506	1002	1509	326

For above values of e, the limiting coverages of the five confidence interval methods range from 0.8714 to 0.8998.

When e is very small, the computing time we need to determine m^0_{NB}, m^0_{OB}, m^0_{SM}, m^0_{CM} and w^0 is very large. For this reason we have not considered values for e smaller than 0.001. On the other hand, we have found that it is rather difficult to prove mathematically that the above ratios converge to some specific values.

4.4 ASYMPTOTIC COMPARISONS OF EXPECTED HALF LENGTHS OF CONFIDENCE INTERVAL METHODS

For fixed number of batches k=n/m and providing that the simulation output process satisfies conditions (4.1), Goldsman and Schruben(1984) derived the following forms for the limiting expected half lengths of the confidence intervals produced by the NOBM, AREA and combined NOBM-AREA methods:-

$$\lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{NB}\right] = \sigma t_{k-1,\alpha/2}\sqrt{\frac{2}{k-1}}\ \frac{\Gamma\left[\frac{k}{2}\right]}{\Gamma\left[\frac{k-1}{2}\right]}$$

$$\lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{SM}\right] = \sigma t_{k,\alpha/2}\sqrt{\frac{2}{k}}\ \frac{\Gamma\left[\frac{k+1}{2}\right]}{\Gamma\left[\frac{k}{2}\right]}$$

$$\lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{CM}\right] = \sigma t_{2k-1,\alpha/2}\sqrt{\frac{2}{2k-1}}\ \frac{\Gamma\left[k\right]}{\Gamma\left[\frac{2k-1}{2}\right]}$$

TABLE 4.5
Ratios of limiting expected half lengths of confidence intervals produced by the NOBM, AREA and combined NOBM-AREA methods

k	$\lim_{m\to\infty} \frac{(mk)^{\frac{1}{2}}EHL_{CM}}{(mk)^{\frac{1}{2}}EHL_{NB}}$ $\alpha=.10$	$\alpha=.05$	$\lim_{m\to\infty} \frac{(mk)^{\frac{1}{2}}EHL_{CM}}{(mk)^{\frac{1}{2}}EHL_{SM}}$ $\alpha=.10$	$\alpha=.05$	$\lim_{m\to\infty} \frac{(mk)^{\frac{1}{2}}EHL_{SM}}{(mk)^{\frac{1}{2}}EHL_{NB}}$ $\alpha=.10$	$\alpha=.05$
2	.4303	.2891	.8379	.7688	.5135	.3760
3	.7411	.6417	.8846	.8347	.8378	.7688
4	.8435	.7785	.9127	.8748	.9242	.8898
5	.8899	.8434	.9297	.8991	.9572	.9380
6	.9158	.8793	.9422	.9165	.9720	.9594
7	.9320	.9026	.9500	.9284	.9810	.9721
8	.9427	.9183	.9562	.9376	.9859	.9794
9	.9510	.9297	.9616	.9450	.9890	.9838
10	.9571	.9389	.9657	.9507	.9911	.9876
....	...	...	...	...	...	...
20	.9811	.9730	.9829	.9757	.9982	.9973
....	...	...	...	...	...	...
30	.9889	.9828	.9898	.9840	.9991	.9988
....	...	...	...	...	...	...
∞	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

where $\sigma^2 = \gamma_0 \left\{ 1 + 2 \sum_{s=1}^{\infty} \rho_s \right]$

For fixed k, Goldsman and Schruben(1984) showed diagramatically that

$$\lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{NB}\right] > \lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{SM}\right] > \lim_{m\to\infty}\left[\sqrt{mk}\ EHL_{CM}\right] \tag{4.8}$$

Inequality (4.8) can be verified by comparing the ratios displayed in table (4.5).

Consider now the case where m_i and k_i are proportional to n^{β_i} and n^{θ_i}[$m_i \propto n^{\beta_i}$, $k_i \propto n^{\theta_i}$, $0<\delta_i, \theta_i<1$, $i = NB, SM, CM$] respectively such that $n=m_i k_i$. Under these values of m_i, k_i and $n \to \infty$, Goldsman and Schruben(1984), Goldsman et al.(1986) and Sargent et al.(1989) have reported that

$$\lim_{n\to\infty}\left[\sqrt{n}\ EHL_{NB}\right] = \lim_{n\to\infty}\left[\sqrt{n}\ EHL_{SM}\right] = \lim\left[\sqrt{n}\ EHL_{CM}\right] = \sigma z_{a/2} \tag{4.9}$$

In the remaining part of this section we shall derive the limiting form of the expected half length of the confidence intervals produced by the spectral method, assuming that the simulation output process satisfies conditions (4.1). The limiting form of the expected half length of the overlapping batch means method can be derived in a similar way.

Let f(0) and w be the spectral density at zero frequency and the spectral window size respectively. Consider the case where $w \propto n^a (0<a<1)$. For different a's, the values of w ensure the asymptotic situation that if $n \to \infty$ then $w \to \infty$ but in such a way that $(n/w) \to \infty$[see Chatfield(1984)]. By using Tukey's spectral window, the degrees of freedom v are proportional to n^{1-a}[see Law and Kelton(1984)]. Under these asymptotic conditions, for large sample size n, the random variable

$$T_1 = \frac{v\hat{f}(0)}{f(0)} \tag{4.10}$$

is approximately distributed as χ^2 with v degrees of freedom [see Jenkins and Watts(1968), Fuller(1976), Chatfield(1984)].

For large n, the random variable $Y=\sqrt{T_1}$ follows the Weibul distribution which has density function

$$g(y) = y^{v-1} e^{-y^2/2} 2^{1-(v/2)} \Gamma(v/2)$$

and expected value

$$E(Y) = \frac{\sqrt{2}\,\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]} \tag{4.11}$$

Combining (4.10) with (4.11), for large n

$$E\left[\sqrt{\hat{f}(0)}\right] = \sqrt{\frac{2f(0)}{v}}\left[\frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]}\right] \tag{4.12}$$

Multiplying both sides of (4.12) by $t_{v,(\alpha/2)}\sqrt{2\pi}$ we get

$$E\left[t_{v,\alpha/2}\sqrt{2\pi\,\hat{f}(0)}\right] = t_{v,\alpha/2}\sqrt{2\pi f(0)\left\{\frac{2}{v}\right\}}\left[\frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]}\right] =$$

$$=\sqrt{n}\,E\left[t_{v,\alpha/2}\sqrt{\frac{2\pi\hat{f}(0)}{n}}\right]=t_{v,\alpha/2}\sqrt{2\pi f(0)\left\{\frac{2}{v}\right\}}\left[\frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]}\right]$$

Recalling that $\lim_{n\to\infty} f(0) = \dfrac{\sigma^2}{2\pi}$ and $t_{v,\alpha/2}\sqrt{\dfrac{2\pi\hat{f}(0)}{n}}$ is the half length of the spectral method, for large n

$$\left[\sqrt{n}\ EHL_{SP}\right] = \sigma t_{v,\alpha/2}\sqrt{\frac{2}{v}}\ \frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]}$$

But $\lim_{v\to\infty} t_{v,\alpha/2} = z_{\alpha/2}$ and

$$\lim_{v\to\infty}\frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]} = \sqrt{\frac{v}{2}}$$

[see Goldsman and Schruben(1984)]

Therefore

$$\lim_{n\to\infty}\left[\sqrt{n}\ EHL_{SP}\right] = \sigma z_{\alpha/2} \tag{4.13}$$

In regard to the overlapping batch means method, consider the case where $m_i \propto n^{\beta_i}(0<\beta_i<1,\ i=OB)$. This value of m_{OB} ensures the asymptotic situation that if $n \to \infty$ then $m_{OB} \to \infty$ but in such a way that $(n/m_{OB}) \to \infty$. Then, for large n the random variable

$$T_2 = \frac{v \hat{\sigma}^2_{OB}}{V(\bar{X}_n)}$$

is approximately distributed as χ^2 with degrees of freedom v_{OB} proportional to $n^{1-\beta_{OB}}$[see Meketon and Schmeiser(1984)].

For the overlapping batch means method, we can show that

$$\lim_{n \to \infty} \left[\sqrt{n}\ EHL_{OB}\right] = \sigma z_{\alpha/2} \tag{4.14}$$

From (4.9), (4.13) and (4.14), as the sample size n tends to infinity the five confidence interval methods under consideration produce confidence intervals with equal expected half lengths.

4.5 ASYMPTOTIC COMPARISONS OF VARIANCE OF HALF LENGTHS OF CONFIDENCE INTERVAL METHODS

For small k, Goldsman and Schruben(1984), Goldsman et al.(1986) and Sargent et al.(1989) reported the following forms for the limiting variance of the half lengths of the confidence intervals produced by the NOBM, AREA and combined NOBM-AREA methods:

$$\lim_{m\to\infty}\left[(mk)VHL_{NB}\right]=\left[\sigma t_{k-1,\alpha/2}\right]^2\left\{1-\frac{2}{k-1}\left[\frac{\Gamma\left[\frac{k}{2}\right]}{\Gamma\left[\frac{k-1}{2}\right]}\right]^2\right\}$$

$$\lim_{m\to\infty}\left[(mk)VHL_{SM}\right]=\left[\sigma t_{k,\alpha/2}\right]^2\left\{1-\frac{2}{k}\left[\frac{\Gamma\left[\frac{k+1}{2}\right]}{\Gamma\left[\frac{k}{2}\right]}\right]^2\right\}$$

$$\lim_{m\to\infty}\left[(mk)VHL_{CM}\right]=\left[\sigma t_{2k-1,\alpha/2}\right]^2\left\{1-\frac{2}{2k-1}\left[\frac{\Gamma\left[k\right]}{\Gamma\left[\frac{2k-1}{2}\right]}\right]^2\right\}$$

For small number of batches k, Goldsman and Schruben(1984) showed diagramatically that

$$\lim_{m\to\infty}\left[(mk)VHL_{NB}\right] > \lim_{m\to\infty}\left[(mk)VHL_{SM}\right] > \lim_{m\to\infty}\left[(mk)VHL_{CM}\right]$$

This inequality can be verified from table (4.6).

Let us now derive the limiting form for the variance of the half lengths of the spectral method. For large v

$$\lim_{v\to\infty}\left[\sqrt{2\chi^2_v}-\sqrt{2v-1}\right] \xrightarrow{D} N(0,1) \qquad (4.15)$$

T A B L E 4.6

Ratios of limiting variances of confidence interval half lengths produced by the NOBM, AREA and combined NOBM-AREA methods

k	$\lim_{m\to\infty} \frac{(mk)\ VHL_{CM}}{(mk)\ VHL_{NB}}$ $\alpha=.10$	$\alpha=.05$	$\lim_{m\to\infty} \frac{(mk)\ VHL_{CM}}{(mk)\ VHL_{SM}}$ $\alpha=.10$	$\alpha=.05$	$\lim_{m\to\infty} \frac{(mk)\ VHL_{SM}}{(mk)\ VHL_{NB}}$ $\alpha=.10$	$\alpha=.05$
2	.0577	.0261	.4572	.3855	.1263	.0677
3	.2088	.1565	.4567	.4065	.4572	.3850
4	.2949	.2512	.4643	.4266	.6352	.5889
5	.3408	.3061	.4740	.4434	.7190	.6904
6	.3748	.3458	.4766	.4513	.7864	.7662
7	.3917	.3680	.4779	.4564	.8196	.8063
8	.4074	.3864	.4802	.4616	.8484	.8371
9	.4181	.4001	.4832	.4667	.8653	.8573
10	.4274	.4115	.4841	.4695	.8829	.8764
....	...	...	...	...	...	...
20	.4664	.4581	.4912	.4837	.9496	.9471
....	...	...	...	...	...	...
30	.4859	.4798	.4916	.4882	.9885	.9829
....	...	...	...	...	...	...
∞	.5000	.5000	.5000	.5000	1.0000	1.0000

From (4.15), for large v, $\mathrm{Var}\left[\sqrt{\chi^2_v}\right] = \frac{1}{2}$

In regard to the spectral method, for $w \propto n^a$ the degrees of freedom v are proportional to n^{1-a}. Hence for large n

$$\mathrm{Var}\left[\sqrt{\frac{v\hat{f}(0)}{f(0)}}\right] = \frac{1}{2} \quad \Rightarrow$$

$$\mathrm{Var}\left[\sqrt{\frac{2\pi v\hat{f}(0)}{\sigma^2}}\right] = \frac{1}{2} \tag{4.16}$$

Multiplying both sides of (4.16) by $t^2_{v,\alpha/2}$

$$\mathrm{Var}\left[\sqrt{n}\; t_{v,\alpha/2}\sqrt{\frac{2\pi\hat{f}(0)}{n}}\right] = \frac{\left[\sigma t_{v,\alpha/2}\right]^2}{2v} \tag{4.17}$$

For large n, $\lim_{v\to\infty} t_{v,\alpha/2} = z_{\alpha/2}$. Also $t_{v,\alpha/2}\sqrt{\left[2\pi\hat{f}(0)\right]/n}$ is the is the half length of the spectral method. Hence

$$\lim_{n\to\infty}\left[n\ (VHL_{SP})\right] = \frac{\left[\ \sigma z_{\alpha/2}\ \right]^2}{2v} \tag{4.18}$$

For the nonoverlapping batch means, area, and combined NOBM-AREA methods, consider the case where $m_i \propto n^{\beta_i}$ and $k \propto n^{\theta_i}$ ($i=$ NB, SM, CM) such that $n = m_i k_i$. In respect to the overlapping batch means method consider $m_{OB} \propto n^{1-\beta_{OB}}$. Under the above values, of m_i, for the four methods, the degrees of freedom v_i($i=$NB, SM, CM, OB) are proportional to $n^{1-\beta_i}$. Hence, for large n, we can show that

$$\lim_{n\to\infty}\left[n\ (VHL_i)\right] = \frac{\left[\ \sigma z_{\alpha/2}\ \right]^2}{2v_i} \tag{4.19}$$

From (4.18) and (4.19), comparisons of the limiting variances of the confidence interval half lengths can be made by setting different values to β_i, θ_i, a and the constants of proportionality. An obvious choice is $\beta_i = \beta = w$ (i=NB, SM, CM, OB) and $\theta_i = \theta = 1 - \beta$. For such values

$$\ell im \frac{VHL_i}{VHL_j} = \frac{v_j}{v_i} = \frac{c_j}{c_i} \tag{4.20}$$

where the indices i and j stand for the i^{th} and j^{th} confidence interval methods respectively and c_i, c_j are the constants of proportionality. Ratio (4.20) implies that, different parameterization for the constants of proportionality c_i's (i=NB, SM, CM, OB, SP) can result in different ranking of the confidence interval methods in terms of the limiting variances of the half lengths.

4.6 SUMMARY

In this chapter, we have reported further results on the asymptotic properties of confidence interval methods. Two issues have been discussed.

The first issue concerns the numerical computation of the limiting coverages achieved by the confidence interval methods for different parameter values. Provided that the output process satisfies certain regularity conditions and its autocorrelation function is known, we have provided the necessary methodology for the analytical computation of the limiting coverages of five methods; nonoverlapping/overlapping batch means, area, combined area-nonoverlapping batch means and spectral methods. These coverages have been computed numerically in three output processes; the AR(1), the AR(2) and the delay in queue in the M/M/1. For the AR(2), the autoregressive coefficients were chosen such that its autocorrelation function displayed a damped cyclical behaviour.

The following remarks have been made:-

_ In the AR(1) and M/M/1, for equal small batch sizes m, the nonoverlapping/overlapping batch means methods achieve limiting coverages which are greater than those of the area and combined NOBM-AREA methods but lower than the nominal confidence level.

_ In the AR(2), under certain autoregressive coefficients, the spectral and combined NOBM-AREA methods can achieve acceptable limiting coverages for small batch sizes and spectral window sizes respectively.

_ In the AR(1), M/M/1 and AR(2), the limiting coverages of the spectral method tend to achieve the nominal confidence level rather fast.

Furthermore, we have computed numerically the batch size for which the nonoverlapping batch means, area and combined NOBM-AREA methods achieve limiting coverages which differ to the nominal confidence level by a small positive number e. For the three processes under study and different e's, we have found that the area method requires this batch size to be approximately three times more than that of nonoverlapping batch means method. On the other hand, the batch size of the combined method should be approximately two times more than the batch size of the nonoverlapping batch means method.

The second issue refers to the limiting precision and stability of the confidence intervals produced by the five methods. In this chapter, we have derived limiting forms for the expected values and variances of the confidence interval half lengths produced by the spectral and overlapping batch means methods. Let n, k, m, w be the sample size, the number of contiguous batches, the batch size and spectral window size respectively. For the nonoverlapping batch means, area and combined NOBM-AREA methods we have considered $k_i \propto n^{\theta_i}$; $m_i \propto n^{\beta_i}$ such that $n = m_i k_i$ $(0<\theta_i,\beta_i<0)$. The index i stands for the nonoverlapping batch means, area and combined NOBM-AREA methods. In regard to the spectral and overlapping batch means methods, we have taken taken $w \propto n^a$ and $m_{OB} \propto n^{\beta_{OB}}$ $(0<a,\beta_{OB}<1)$.

Under the above values of k, m, w and providing that the simulation output process satisfies certain regularity conditions, as $n \to \infty$, the methods tend to cover the true steady state mean with the nominal probability. In this case, for any values of the parameters β_i, θ_i, a and constants of proportionality, we have shown that all the methods produce confidence intervals with the same limiting expected half length. On the other hand, by setting different values to these parameters, we can result in different ranking of the confidence interval methods in regard to the limiting variances of the half lengths.

CHAPTER FIVE

PREPARATION STAGES FOR SIMULATION EXPERIMENTS

5.1 INTRODUCTION

In the present chapter, we describe the testing environment we have created for carrying out Monte Carlo experiments. The output processes, we have included in our environment, are specified in the next section. In section 5.3, we discuss the test we have applied to each replication of the output processes for eliminating the initialization bias. An algorithm for determining student-t values under both integer and fractional degrees of freedom is described in section 5.4. In section 5.5, we discuss the computing software we have developed and used as a tool for estimating the statistical criteria, which are described in the next two chapters. In the final section, we describe how we have implemented and tested the random number generator we have used to produce replications of the simulation model output processes.

5.2 SIMULATION MODELS AND OUTPUT PROCESSES

In our testing environment, we have included most of the simulation models and output processes that have been used in the environments described in chapter two. We classify the output processes into two categories; The pilot and the studied processes.

<u>PILOT PROCESSES</u>: Their general characteristic is that the theoretical autocorrelation coefficients are given by known difference equations. Therefore, for these processes analytical values for the minimum bias and the MB-parameter values can be obtained. Furthermore, the limiting coverages achieved by the five confidence interval methods, which are considered in this thesis, can be computed analytically. In the next two chapters, for small sample sizes, we shall make several recommendations for applying the five methods to approximately steady state simulation output processes displaying certain characteristics. These recommendations have been based on a study of the performance of the methods on the pilot processes. The AR(1), AR(2) and the delay in queue in the M/M/1 belong to this category of processes.

<u>STUDIED PROCESSES</u>: These come from more complicated simulation models. For these processes, the theoretical autocorrelation functions are not known. In the seventh chapter, we compare the behaviour of the performance of the five confidence interval methods between the pilot and studied processes. Furthermore, the validity of the recommendations, which have been extracted on the pilot processes, are tested on the studied processes. The processes from the inventory model, time shared computer model and interactive multiprogrammed computer model are classified in the second category.

Table 5.1 displays the simulation models and the output processes which we have used in our testing environment. For each process, the true steady state mean is provided. For each model, the meaning of the parameter values was given in chapter two.

To generate simulation programs for the simulation models

T A B L E 5.1

Simulation models and output processes in the testing environment of the present research

Simulation models	Output processes	True steady state means (μ)
AR(1)		
φ=0.4074, σ_ϵ^2=1		1
φ=0.7778, σ_ϵ^2=1	$X_t=\mu(1-\varphi)+\varphi X_{t-1}+\epsilon_t$	1
φ=0.9630, σ_ϵ^2=1		1
φ=0.99 , σ_ϵ^2=1		1
AR(2)		
φ_1=0.75, φ_2=-0.50, σ_ϵ^2=1	$X_t=\mu(1-\varphi_1-\varphi_2)+$ $+\varphi_1 X_{t-1}+\varphi_2 X_{t-2}+\epsilon_t$	1
φ_2=0.99, φ_2=-0.90, σ_ϵ^2=1		1
M/M/1 Queueing Discipline:FIFO		
τ=0.2	Delay in queue	0.05
τ=0.5	Delay in queue	0.50
τ=0.8	Delay in queue	3.20
τ=0.9	Delay in queue	18.05
Inventory model s=17, S=57, k=32, c=3, h=1, v=5 Orders→Poisson(25)	Total cost at period t	112.108
Time shared computer model N=35, $1/\lambda$=25, $1/\mu$=0.8, q=0.8, τ=0.015	Response time of the jobs	8.246
Interactive computer model N=25, M=4, $p_1=p_2$=0.36, $p_3=p_4$=0.04, $1/\lambda_1$=100, $1/\lambda_2$=1, $1/\mu_1=1/\mu_2$=1.39 $1/\mu_3=1/\mu_4$=12.5	Waiting time at the C.P.U	3.770

of table (5.1), we have used the Extended Lancaster Simulation Environment system(eLSE). The description of this system is given in Balmer and Paul(1985). Appendix A provides the program listings of the above models.

5.3 AN OPTIMAL TEST FOR THE PRESENCE OF INITIALIZATION BIAS

During the initialization process of simulation programs, the researcher cannot choose initial conditions identical to the "representative conditions" held under steady-state behaviour of the simulated systems. Furthermore, when the initial conditions are different from the representative ones, early observations are distributed with mean different from the steady-state mean. This problem is known in the simulation literature as the initialization bias problem. In the introductory chapter, we have discussed the effects of the initialization bias on the construction of confidence intervals for steady-state means.

The usual approach to overcome the initialization bias problem is to initialize the simulation program in some convenient fashion and ignore in the subsequent analysis a number of observations from the initial part of the simulation run. If this number of observations is large enough, then the remaining series can be considered as approximately stationary [see Law(1983)].

Schruben et al.(1983) developed a procedure for testing whether, after the deletion of the initial part of observations in the simulation run, the initialization bias has been eliminated in the remaining series. This procedure and its implementation in our simulation experiments are described below.

Consider the simulation output process

$$X_1 , X_2 , X_3 , \ldots , X_n$$

with $\mu = \lim_{t\to\infty} E(X_t)$ and $E(X_t) = \mu(1-a_t)$ for $t<\infty$.

The function $E(X_t)$ is called initial transient mean function. The null hypothesis that no initialization bias presents in the series is stated as follows:

H_0 : a_t=0 for all t, against the alternative
H_1 : a_t is an arbitrary(specified) function of t.

The derivation of the most powerful test of the null hypothesis was based on the assumption that the output series satisfies some regularity conditions which essentially require that two observations far apart in time are approximately independent. Under this assumption, as $n \to \infty$ and by using the Neymann-Pearson lemma, Schruben et al.(1983) derived that the most powerful test of the null hypothesis against any specified alternative has the form

$$T = \sum_{j=1}^{n} c_j j S_j \qquad , \qquad c_j = a_j - a_{j+1}$$

and $$S_j = \bar{X}_n - \bar{X}_j$$

In practice a_t's are unknown, and therefore the optimal weights cannot be used. However, it is known that in simulations which converge to a steady state the weights $c_j \to 0$ since $E(X_t) \to \mu$. In addition, if bias of a particular sign is suspected, then the signs of the weights can be appropriately chosen. Schruben et al.(1983) deduced that any weighting function with decreasing magnitude and appropriate sign, which is optimal against some initial transient mean function, should perform reasonably well against similar transients. In several examples of simulation output processes, they used the weighting function $c_j = 1-(j/n)$ which is optimal against a simple quadratic transient mean function with $a_t = 1/\{(2n)(t^2 - t(2n+1))\}$ plus a constant.

Under the previous weighting function, as n tends to be large, the test statistic T has an approximate normal distribution with mean 0 and variance $n^4 V(\bar{X}_n)/45$, where $V(\bar{X}_n)$ is the true variance of the sample mean.

A remaining problem in applying the test in practice is that the true variance of the sample mean is usually unknown. Different sample mean variance estimators $\hat{\sigma}_i^2$ were discussed in chapter three. Under certain regularity conditions (see section 4.2), as $n \to \infty$, the ratio $v\hat{\sigma}_i^2/V(\bar{X}_n)$ is assumed to have an asymptotic χ^2 distribution with v degrees of freedom. Hence, dividing the test statistic T by the square root of the previous ratio over the degrees of freedom, the random variable

$$\hat{T} = \frac{\sqrt{45}}{n^2 \hat{\sigma}_i} \sum_{j=1}^{n} \left[1 - \frac{j}{n} \right] j \left[\bar{X}_n - \bar{X}_j \right] \qquad (5.1)$$

can be assumed to have a student-t distribution with v degrees of freedom.

In the examples of simulation output processes, Schruben et al.(1983) used the sample mean variance estimators of the autoregressive and batch means methods. For both estimators, the test performed well in all the examples. Particularly, for each process, when the initial conditions were chosen according to the steady-state distribution, the proportion of replications for which the null hypothesis was rejected was close to the level of significance. On the other hand, when initialization bias was present, the test rejected the null hypothesis for a large proportion of the output replications. Besides, the test performed well for fairly small values of the sample size even though it was based on an asymptotic result($n \to \infty$). No run exceeded 500 observations.

Most of the processes where the stationarity test was evaluated were included in our testing environment. We also selected output processes which were not included in Schruben et al.'s testing environment. However, pilot experiments showed that the latter processes had similar transient mean functions to the processes of Schruben et al.'s(1983) testing environment.

Schruben et al.(1983) also pointed out that when better estimators are developed for the true variance of the sample mean, these estimators should be used for the stationarity test. For AR(1), AR(2) and the delay in the M/M/1 queueing system, we have found that the sample mean variance estimator of the spectral method is the least biased in small samples. Therefore, for these processes and not very small sample sizes, the test statistic

$$\hat{T} = \frac{\sqrt{45}}{n^2 \hat{\sigma}_{SP}} \sum_{j=1}^{n} \left[1 - \frac{j}{n} \right] j \left[\bar{X}_n - \bar{X}_j \right] \tag{5.2}$$

could be treated approximately as a student-t distribution. The degrees of freedom are $1.33n/w^*$ and w^* is the spectral window size for which the sample mean variance estimator attains its minimum bias.

For the above three processes, after deleting an early part of data in each replication, the test (5.2) was applied to sample sizes 512 and 1024. While the null hypothesis was being rejected, we were increasing the amount of deleted data by 250 and applying the test again to the same sample sizes. This procedure was being repeated until the acceptance of the null hypothesis. To avoid large number of iterations, we chose the initial amount of deleted data to be rather large. For example, for the AR(1) with $X_0=0$, it is known theoretically that the conditional mean μ_t at time t is $\mu_t=\mu(1-\varphi^t)$[see Fishman(1972)]. Therefore, the initial transient period, say n^*, can be determined from $(\mu_{n^*}/\mu)<1-e$ where e is a very small positive number. That is $n^*=\log_{10}e/\log_{10}\varphi$. We have chosen $e=10^{-9}$. To avoid large number of iterations, the final transient period was selected to be approximately $10n^*$.

Table 5.2 displays the initial amount of data removed from each replication of the processes of the simulation models of our testing environment. For the inventory model, time shared computer model and interactive multiprogrammed computer model, an initial number of replications was generated for determining the least biased estimator. Estimation procedures for both the minimum bias and the parameter values for which the minimum bias is attained are discussed in the next chapter. For the latter three models, from the initial replications we have found that the spectral estimator was the least biased estimator. Thus, the test statistic (5.2) was applied to any other replication of these models at sample size 512 and 1024 by using the estimated spectral window size for which the minimum bias was attained.

T A B L E 5.2

Initial amount ℓ of deleted data in each replication of the output processes before the application of Schruben et al.'s test

Simulation models	Output processes	ℓ
AR(1) $\varphi=0.4074,\ \sigma_\epsilon^2=1$ $\varphi=0.7778,\ \sigma_\epsilon^2=1$ $\varphi=0.9630,\ \sigma_\epsilon^2=1$ $\varphi=0.99\ ,\ \sigma_\epsilon^2=1$	$X_t=\mu(1-\varphi)+\varphi X_{t-1}+\epsilon_t$	 240 830 5500 20620
AR(2) $\varphi_1=0.75,\ \varphi_2=-0.50,\ \sigma_\epsilon^2=1$ $\varphi_2=0.99,\ \varphi_2=-0.90,\ \sigma_\epsilon^2=1$	$X_t=\mu(1-\varphi_1-\varphi_2)+$ $+\varphi_1 X_{t-1}+\varphi_2 X_{t-2}+\epsilon_t$	 501 501
M/M/1 Queueing discipline:FIFO $\tau=0.2$ $\tau=0.5$ $\tau=0.8$ $\tau=0.9$	 Delay in queue Delay in queue Delay in queue Delay in queue	 5000 10000 15000 25000
Inventory model s=17, S=57, k=32, c=3, h=1, v=5 Orders→Poisson(25)	Total cost at period t	1500
Time shared computer model N=35, $1/\lambda=25$, $1/\mu=0.8$, q=0.8, $\tau=0.015$	Response time of the jobs	201
Interactive computer model N=25, M=4, $p_1=p_2=0.36$, $p_3=p_4=0.04$, $1/\lambda_1=100$, $1/\lambda_2=1$, $1/\mu_1=1/\mu_2=1.39$ $1/\mu_3=1/\mu_4=12.5$	Waiting time at the C.P.U	3001

5.4 STUDENT-t VALUES UNDER BOTH INTEGER AND FRACTIONAL DEGREES OF FREEDOM

In constructing confidence intervals according to the spectral and overlapping batch means methods, we face the problem of determining student-t values at fractional degrees of freedom.

For this reason, we have developed a computing program which calculates the following quantities:

i) the right-tail area of the student-t distribution at a given student-t value under both integer and fractional degrees of freedom

ii) the student-t value at a given right-tail area under both integer and fractional degrees of freedom.

The computing algorithm for the previous quantities was based on a set of forms given by Bracken and Shleifer(1964). Let the right tail area of the student-t distribution be

$$G(t/v) = \int_t^\infty f(x/v)dx \qquad 0<v<\infty$$

$$f(x/v) = \begin{cases} \dfrac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]\Gamma\left[\frac{1}{2}\right]}\left[v + x^2\right]^{-\frac{1}{2}(v+1)} & \begin{matrix}0<v<\infty\\ -\infty<x<+\infty\end{matrix} \\ (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2} & \begin{matrix}v=\infty\\ -\infty<x<+\infty\end{matrix} \end{cases}$$

By repeated integration by parts, Bracken and Schleifer showed that when v is integer and even

$$G(t/v) = \frac{1}{2}\left[1 - \frac{t}{\sqrt{v+t^2}}\sum_{i=1}^{\frac{1}{2}v} u_i\right]$$

where

$$u_1 = 1 \ , \ u_i = \frac{(2i-3)v}{(2i-2)(v+t^2)}u_{i-1} \quad , \ i\geq 2$$

while when v is integer and odd

$$G(t/v) = \frac{1}{\pi}\left\{\tan^{-1}\left[\frac{\sqrt{v}}{t}\right] - \frac{t}{\sqrt{v}}\sum_{i=0}^{\frac{1}{2}(v-1)} u_i\right\}$$

where

$$u_0 = 0\ ,\ u_1 = \frac{v}{v+t^2}\ ,\ u_i = \frac{(2i-2)v}{(2i-1)(v+t^2)}\,u_{i-1}\quad ,\ i \geq 2$$

Under fractional degrees of freedom, when $v<t^2$

$$G(t/v) = \frac{\Gamma\left[\frac{v+1}{2}\right]}{\Gamma\left[\frac{v}{2}\right]\Gamma\left[\frac{1}{2}\right]}\left[\frac{v}{v+t^2}\right]^{\frac{1}{2}v}\sum_{i=1}^{\infty} u_i \qquad (5.3)$$

$$u_1 = \frac{1}{v}\ ,\ u_i = \frac{(2i-3)(v+2i-4)v}{(2i-2)(v+2i-2)(v+t^2)}\,u_{i-1}\quad ,\ i \geq 2$$

otherwise

$$G(t/v) = \frac{1}{2} - \frac{\Gamma\left[\frac{v+1}{2}\right]\frac{t}{\sqrt{v}}}{\Gamma\left[\frac{v}{2}\right]\Gamma\left[\frac{1}{2}\right]}\sum_{i=1}^{\infty} u_i \qquad (5.4)$$

$$u_1 = 1\ ,\ u_i = -\frac{(2i-3)(v+2i-3)t^2}{(2i-2)(2i-1)v}\,u_{i-1}\quad ,\quad i \geq 2$$

Bracken and Scleifer pointed out that the infinite sums in forms (5.3) and (5.4) converge faster than a power series.

Define now a small positive number e. Given the right tail area α of the student-t distribution, on the i^{th} iteration the final student-t value t_i will satisfy the inequality

$$| G(t_i/v) - \alpha | < e \qquad (5.5)$$

If inequality (5.5) does not hold, on the $i^{th}+1$ iteration

$$t_{i+1} = t_i - k_r \quad \text{if} \quad G(t_{i+1}/v)<\alpha$$

or

$$t_{i+1} = t_i + k_r \quad \text{if} \quad G(t_{i+1}/v)>\alpha$$

with

$$k_r = k_{r-1} m^r$$

and r stands for the number of times where

$$G(t_k/v)>\alpha \quad \text{changes to} \quad G(t_{k+1}/v)<\alpha$$

or

$$G(t_k/v)<\alpha \quad \text{changes to} \quad G(t_{k+1}/v)>\alpha.$$

For the above iterative algorithm, we have used $e=0.00001$, $t_0=2$, $k_0=1$ and $m=0.10$.

5.5 STATISTICAL ROUTINES FOR THE ANALYSIS OF SIMULATION OUTPUTS

In this section, we describe a set of statistical routines we have developed for carrying out certain types of statistical analysis in simulation outputs. These routines constituted the basic tool for the generation of the simulation results which are displayed in the next two chapters.

Since the establishment of the CASM research group, the research activities have been orientated mainly in the design and production of simulation languages[see Chew(1986), El Sheich(1987)]. Little work has been done at this stage on developing routines for the statistical analysis of simulation outputs. Histograms of queue sizes, entity attributes and time series plots only are available in the simulation languages which exist at the L.S.E. From histograms, only basic statistical measures such as mean and variances can be computed.

These statistical measures, however, are not adequate to produce confidence intervals according to the methodology which was discussed in the introductory chapter. On the other hand, standard statistical packages cannot offer the appropriate procedures and functions for applying this methodology on simulation outputs without further programming effort. So, we have decided to develop the statistical routines, which are described below, independently of any statistical package or simulation language. We coded them into standard Pascal and embodied them in a separate module which was linked with the eLSE system, one of the available simulation languages[see Balmer and Paul(1985)]. The new module was called SIM_STAT_LIB.

Three categories of routines were included in the SIM_STAT_LIB. The description of the input and output variables for each routine of the SIM_STAT_LIB is given in Appendix C

FIRST STAGE ROUTINES

First stage routines are useful for extracting initial information for the characteristics of the simulation output process under study. The following procedures and functions are included in this category:

Procedure PICK_UP is useful when any statistical analysis on simulation output process replications is applied after the deletion of an early number of observations.

Function MEAN_EST calculates the sample mean.

Function VAR_EST calculates the sample variance.

Procedure CL_VAR_MEAN estimates the variance of the sample mean according to the classical interval estimator.

Function ACV_EST calculates the sample autocovariance for a given lag.

Procedure ACV_SET_EST calculates the sample autocovariances for a given range of lags.

Function ACR_EST calculates the sample autocorrelations for a given lag.

Procedure ACR_SET_EST calculates the sample autocorrelations for a given range of lags.

Procedure NOVBATCHED_MEANS calculates the nonoverlapping batch means for given batch size and number of batches.

Procedure SERIES_PARTIAL_MEANS calculates the differences $S_j=X_n-X_j$ where X_j is the j_{th} cumulative mean.

Function CHI_SQUARE gives the critical values of the chi-square distribution at right tail area 0.01 and 0.05.

Function STUDENT_T gives the critical values of the student-t distribution computed by the forms which were displayed in the previous section.

SECOND STAGE ROUTINES

Second stage routines cover three aspects of the statistical analysis of steady-state simulation outputs

- the fit of p^{th} order autoregressive processes to simulation output process replications according to Fishman's procedure(1973b,1978b).
- the determination of the batch size which guarantees approximately independent nonoverlapping batch means according to the iterative algorithm which is given in Fishman(1978a).
- the test for the elimination of the initiliazation bias after the deletion of an early part of the data.

The following procedures were included in the second stage routines:

Procedure AR_PARAM_EST estimates the autoregressive coefficients and the error variance of AR(p) processes which are fitted to simulation output replications, provided that the autoregressive order p is already fixed.

Procedure AR_SCHEME_FIT estimates the autoregressive order, the autoregressive coefficients and the error variance of AR(p) processes which are fitted to simulation output replications as well as the variance of the output process according to the autoregressive method.

Procedure NMN_TEST_STAT calculates the Von-Neumann ratio as it is defined in Fishman(1978a).

Procedure FISHMAN_NUM_BATCH gives the number of approximately independent nonoverlapping batch means according to Fishman's(1978a) iterative algorithm.

Procedure TEST1_INIT_BIAS calculates the test statistic which was developed by Schruben(1982) for testing the elimination of the initialization bias after the deletion of an early number of observations.

Procedure TEST2_INIT_BIAS calculates the statistic which was developed by Schruben et al.(1983) for testing the elimination of the initialization bias after the deletion of an early part of data.

FINAL STAGE ROUTINES

Final stage routines produce different estimates for the variance of the sample mean in stationary autocorrelated simulation output processes. From these estimates, confidence intervals for the steady-state mean can be automatically generated. The following procedures were included in this stage:

Procedure NOBM_VAR_MEAN estimates the variance of the sample mean according to the nonoverlapping batch means method.

Procedure OVBM_VAR_MEAN estimates the variance of the sample mean according to the overlapping batch means method.

Procedure AREA_VAR_MEAN estimates the variance of the sample mean according to the method of area which is based on the theory of standardized time series.

Procedure COMB_AREA_NOBM_VAR_MEAN estimates the variance of the sample mean combining the methods of the area- nonoverlapping batch means.

Procedure MAX_VAR_MEAN estimates the variance of the sample mean according to the method of the standardized maximum.

Procedure COMB_MAX_NOBM_VAR_MEAN estimates the variance of the sample mean combining the methods of maximum- nonoverlapping batch means.

Procedure SPEC_VAR_MEAN estimates the variance of the sample mean according to the spectral method.

Procedure AR_VAR_MEAN estimates the variance of the sample mean according to the autoregressive method.

5.6 THE RANDOM NUMBER GENERATOR

In simulation experiments, random numbers are generated by pseudo-random number generators. The validity of the results of simulation experiments depends upon the quality of such generators. Qood quality generators satisfy certain properties in regard to the values generated on the interval (0,1)[for more about these properties see Pidd(1984), Law and Kelton(1982b)]. For example, two very important properties are the following:-

i) The generator must have a long period, that is, the number of values on the interval (0,1) produced before the cycle repeats must be as large as possible.

ii) The generator must pass certain statistical tests. These tests are for checking for the statistical properties that would be expected in a truly random sequence.

In section 5.2, we mentioned that the programs of simulation models of our testing environment were developed by using the eLSE system[see Balmer and Paul(1985)]. Several questions arose when we tested its random number generator. The most serious ones were the following:-

i) There were only 20 streams for generating values on the interval (0,1). Furthermore, the period of each stream was rather short.

ii) It was rather difficult to check quickly whether the streams were non-overlapping.

iii) After applying the appropriate statistical test, there were enough evidence that in each stream the generated values were not uniformly distributed on the interval (0,1).

The above questions made clear the need for updating the random generator of the eLSE system.

From the different types of pseudo-number generators, we selected the multiplicative congruential which is given by the form

$$X_{t+1} = aX_t + (\text{mod } m)$$

Appropriate values for the multiplier a and the modulus m can guarantee that this generator attains its maximum period m-1. We chose a=16807 and $m=2^{31}-1=2147483647$. For these values of a and m, it has been proved that the above generator attains its maximum period $2^{31}-2$ [see Pidd(1984), Law and Kelton(1982b)].

The next issue was the implementation of this generator on the VAX 6330. Since a=16307 and X_t any number between 1 and $2^{31}-2$, the product aX_t is likely to exceed the maximum word length of VAX which is 2^{32}. However, VAX 6330 offers certain system procedures with which we can overcome this problem. These procedures are:-

LIB$EMUL for multiplications where the product is greater than 2^{32} but smaller that 2^{64}.

LIB$EDIV for divisions aX_t/m where the product aX_t is greater than 2^{32} but smaller than 2^{64}.
Furthermore, VAX 6330 offers certain functions for checking whether both the extended multiplications and divisions are carried out correctly. We applied these functions to the whole range of the generator.

The whole period of the generator was subdivided into segments of 500,000 observations. Each segment constituted a separate stream and therefore, any two streams were completely independent each other. The first value of each stream was stored and constituted the initial seed of the particular stream. By storing the initial seed, we had guaranteed the reproduction of any stream at any time.

The next stage was to test in each stream whether the sequence of the numbers U_1, $U_2, \ldots$, U_n on the interval (0,1) could be considered as a truly random sequence. Three empirical tests were considered. We applied each test to several sample sizes.

RUN-UP TEST

By this test, we are testing the independence assumption of the numbers U_1, $U_2, \ldots, U_n$. More specifically, we examine the U_i's sequence for unbroken subsequences of maximum length within which the U_i's monotonically increase. Any such subsequence is called run up. The test statistic is given by

$$R_1 = \frac{1}{n}\sum_{i=1}^{6}\sum_{j=1}^{6} a_{ij}(r_i - nb_i)(r_j - nb_j)$$

where

$$r_i = \begin{cases} \text{number of runs up of length } i \text{ for } i=1,2,\ldots,5 \\ \text{number of runs up of length greater than six for } i=6 \end{cases}$$

a_{ij} is the $(i,j)^{th}$ element of the matrix

$$\begin{bmatrix} 4529.4 & 9044.9 & 13568 & 18091 & 22615 & 27892 \\ 9044.9 & 18097 & 27139 & 36187 & 45234 & 55789 \\ 13568 & 27139 & 40721 & 54281 & 67852 & 83685 \\ 18091 & 36187 & 54281 & 72414 & 90470 & 111580 \\ 22615 & 45234 & 67852 & 90470 & 113262 & 139476 \\ 27892 & 55789 & 83685 & 111580 & 139476 & 172860 \end{bmatrix}$$

and $(b_1,b_2,b_3,b_4,b_5,b_6)=\left[\frac{1}{6}, \frac{5}{24}, \frac{11}{120}, \frac{19}{720}, \frac{29}{5040}, \frac{1}{840}\right]$

For large n, R_1 will be approximately distributed as chi-squared distribution with 6 degrees of freedom under the null hypothesis that the U_i's are independent random variables. Knuth(1969) recommends that n>4000.

In each stream of the generator we implemented on VAX, we applied this test to eight sample sizes; 6400, 12800, 20480, 61440, 102400, 143360, 184320, 225280.

ONE DIMENSION UNIFORMITY TEST

This test is designed to check whether the sequence U_1, $U_2, \ldots, U_n$ appear to be uniformly distributed on the interval (0,1). In particular, the interval [0,1] is subdivided into k equal subintervals. Then the test statistic is given by

$$R_2 = \frac{k}{n} \sum_{j=1}^{k} \left[f_j - \frac{n}{k} \right]^2$$

where f_j is the number of the U_i's in the j^{th} subinterval. For large n, R_2 is approximately distributed as chi-square distribution with k-1 degrees of freedom. Thus, we accept the null hypothesis that the sequence $U_1, U_2, \ldots, U_n$ is uniformly distributed in the interval (0,1) when $R_2<\chi^2_{k-1,\alpha/2}$ where $\chi^2_{k-1,\alpha/2}$

is the right tail α critical value. For large k

$$\chi^2_{k-1,\alpha/2} \approx (k-1)\left\{1 - \frac{2}{9(k-1)} + z_{\alpha/2}\left[\frac{2}{9(k-1)}\right]^{\frac{1}{2}}\right\}^3$$

Law and Kelton(1982b) suggests $n/k > 5$.

In each stream of the generator we implemented on VAX, we applied this test to eleven sample sizes n; 800, 1600, 3200, 6400, 12800, 20480, 61440, 102400, 143360, 184320, 225280. For n<20480 k=100 otherwise k=4096.

TWO DIMENSIONS UNIFORMITY TEST

Generating n pairs of the sequence U_i of values on the interval (0,1), i.e

$$U_1=(U_1,U_2),\ U_2=(U_3,U_4),\ U_3=(U_5,U_6),\ldots.$$

and partitioning the interval [0,1] into k subintervals of equal size, the test statistic is given by

$$R_3 = \frac{k^2}{n}\sum_{i=1}^{k}\sum_{j=1}^{k}\left[f_{ij} - \frac{n}{k^2}\right]^2$$

where f_{ij} is the number of pairs U's having the first number in subinterval i and the second in subinterval j. For large n, R_3 is approximately distributed as chi-square distribution with k^2-1 degrees of freedom. Law and Kelton(1982b) suggests $n/k^2 > 5$.

For each stream of the generator we implenented on VAX, we applied this test to different samples n of pairs U. The sizes of these samples were 800, 1600, 3200, 6400, 12800, 20480, 61440, 102400, 143360, 184320 and 225280. For $n<20480$, $k=10$ otherwise $k=64$.

Initially, when we subdivided the whole period of the generator into segments of 500,000 observations, we had over 4000 streams. Eventually, 1436 streams passed all three tests at any of the sample sizes we considered. For each test, the level og significance we used was 5%. Appendix D displays the initial seeds of several such streams. The values of the test statistics of the three tests are also displayed at different sample sizes.

C H A P T E R SIX

RELATING THE MINIMUM BIAS OF SAMPLE MEAN VARIANCE ESTIMATORS TO THE PERFORMANCE OF CONFIDENCE INTERVAL METHODS

6.1 INTRODUCTION

In chapter three, statistical criteria were introduced for studying the bias of different sample mean variance estimators in small sample sizes. The most basic criterion was the "Bias Indicator" function of each estimator. This function was expressed in terms of the theoretical autocorrelation coefficients of the output process under study.

The other criteria, related to the bias indicator functions, for each estimator were the minimum bias and the parameter values, for which the minimum bias is attained. These values were called MB-parameter values. Analytical values for the above criteria can be obtained only when the autocorrelation coefficients can be computed exactly.

However, in the majority of simulation outputs the theoretical autocorrelation function is not known. One way for estimating the Bias Indicator function of each estimator, the minimum bias and the MB-parameter values is to estimate the autocorrelation coefficients first.

In the next section, two methods are proposed for estimating autocorrelation functions of simulation output processes. First, we use these methods to estimate the autocorrelation coefficients of three specific processes whose theoretical autocorrelation functions are known. We called these processes pilot processes. Then, we evaluate the two methods by examining the discrepancies between true and estimated values of the minimum bias and the MB-parameter values for the sample mean variance estimators of the following five confidence interval methods:

_ Nonoverlapping batch means method

_ Standardized time series-area method

_ Combined area-nonoverlapping batch means method

_ Overlapping batch means method

_ Spectral method.

For the pilot processes, we study in section (6.3) the performance of the above five methods at the MB-parameter values. We consider both true and estimated MB-parameter values. In addition, several recommendations are provided for applying the five methods to approximately steady-state simulation output processes when these processes have similar autocorrelation functions to those of the pilot processes.

In any replication of the pilot processes, the initialization bias was removed using Schruben et al.'s test(1983) discussed in section (5.3). The true spectral window sizes for which the minimum bias of the corresponding sample mean variance estimator is attained were used in the test statistic defined in (5.2).

6.1 ESTIMATION OF THE STATISTICAL CRITERIA FOR STUDYING THE BIAS OF THE SAMPLE MEAN VARIANCE ESTIMATORS

In chapter three, we derived the "Bias Indicator" functions of different sample mean variance estimators. These functions can be expressed in the general form

$$Bs(n,b)_i = f_i(\rho_0 \, , \, \rho_1 \, , \, \rho_2 \, , \ldots , \, \rho_{n-1}) \tag{6.1}$$

The sample size and the s^{th} lag theoretical autocorrelation coefficient are denoted by n and $\rho_s(0 \leq s \leq n-1)$ respectively. The qualitative index i indicates the i^{th} sample mean variance estimator.

The parameter b stands for:-

i) the number of batches, k, for the nonoverlapping batch means(NOBM), area(AREA) and combined area-nonoverlapping batch means(NOBM-AREA) methods

ii) the spectral window size, w, for the spectral method(SPEC)

iii) the batch size, m, for the overlapping batch means method(OVBM)

Let $\hat{\rho}_{sj}(n)$ be, on the j^{th} replication $(1 \leq j \leq r)$, the estimated s^{th} lag autocorrelation coefficient from a sample of size n. The Bias Indicator function of the i^{th} sample mean variance estimator will be estimated on the j^{th} replication by

$$\widehat{Bs(n,b)}_{ij} = f_i(\rho_0, \hat{\rho}_{1j}(n), \hat{\rho}_{2j}(n), \ldots, \hat{\rho}_{(n-1)j}(n)) \tag{6.2}$$

Let $\hat{b}^j_{MB}$ be the estimator of the MB-parameter value on the j^{th} replication. This estimator will satisfy the inequality

$$\left| \widehat{Bs(n,\hat{b}^j_{MB})}_{ij} - 1 \right| < \left| \widehat{Bs(n,b)}_{ij} - 1 \right| \tag{6.3}$$

for any other b in the range where the Bias Indicator function of the i^{th} estimator is defined.

Generating r replications of the output process under study, for the i^{th} estimator, the MB-parameter value b_{MB}, will be estimated by

$$\hat{b}_{MB} = \frac{1}{r} \sum_{j=1}^{r} \hat{b}^{j}_{MB} \tag{6.4}$$

Moreover, the following form can be used for estimating the minimum bias of the i^{th} estimator

$$\widehat{MB} = \frac{1}{r} \sum_{j=1}^{r} \left| \widehat{Bs(n,\hat{b}^{j}_{MB})_{ij}} - 1 \right| \tag{6.5}$$

Let $\{X_{tj}, 1<t\leq n, 1\leq j\leq r\}$ be a given replication of the output process under study. Consider the following estimator of the theoretical autocorrelation coefficients

$$\hat{\rho}_{sj}(n) = \frac{\sum_{t=1}^{n-s} (X_{tj} - \bar{X}_{nj})(X_{(t+s)j} - \bar{X}_{nj})}{\sum_{t=1}^{n} (X_{tj} - \bar{X}_{nj})^{2}} \tag{6.6}$$

where $\bar{X}_{nj}$ is the sample mean on the j^{th} replication. For s near n $\hat{\rho}_{sj}(n)$'s will be based on only few observations. Hence, they will have poor statistical properties [see Law and Kelton(1984), Box and Jenkins(1976)]. To overcome this problem, we shall suggest the modified form

$$\hat{\rho}_{sj}(n^{*}) = \frac{\sum_{t=1}^{n^{*}-s} (X_{tj} - \bar{X}_{n^{*}j})(X_{(t+s)j} - \bar{X}_{n^{*}j})}{\sum_{t=1}^{n^{*}} (X_{tj} - \bar{X}_{n^{*}j})^{2}} \tag{6.7}$$

where $n^* = c.n$, c is a positive integer greater than one and $\bar{X}_{n^*j}$ is the estimated mean from a sample of size n^*. The updated estimator for the Bias Indicator function of the i^{th} sample mean variance estimator on the j^{th} replication will be

$$\widehat{Bs(n,b)_{ij}} = f_i(\rho_0, \hat{\rho}_{1j}, (n^*), \hat{\rho}_{2j}(n^*), \ldots, \hat{\rho}_{(n-1)j}(n^*)) \qquad (6.8)$$

The autocorrelation coefficients can be estimated from (6.8) only when the value of the constant c is known a-priori.

We used (6.7) to estimate the first n (n = 64, 128, 256, 512) autocorrelation coefficients of the AR(1)[see table 5.1] under different values of c and the autoregressive coefficient φ. For each φ, we generated 400 replications with initial value $X_0 = 0$. In each replication, we removed the initialization bias by using Schruben et al.'s(1983) test[see sections 5.3 or 6.1]. Then, the minimum bias and the MB-parameter values were estimated from (6.4) and (6.5). Since for the AR(1) with positive autoregressive coefficient φ the s^{th} lag autocorrelation coefficient is φ^s, the true values for both the minimum bias and the MB-parameter values can be computed as well.

For the five sample mean variance estimators, table (6.1) displays estimated values of both the minimum bias and the MB-parameter values for the AR(1). The estimates of the MB-parameter values have been rounded to the nearest integer. The corresponding true values of the criteria are given in parentheses.

TABLE 6.1

AR(1):True and estimated values for the minimum bias and the MB-parameter values of the sample mean variance estimators

$$\widehat{Bs(n,b)}_{ij} = f_i(\rho_0, \hat{\rho}_{1j}(n^*), \hat{\rho}_{2j}(n^*), \ldots, \hat{\rho}_{(n-1)j}(n^*))$$

$$\hat{\rho}_{sj}(n^*) = \frac{\sum_{t=1}^{n^*-s} (X_{tj} - \bar{X}_{n^*j})(X_{(t+s)j} - \bar{X}_{n^*j})}{\sum_{t=1}^{n^*} (X_{tj} - \bar{X}_{n^*j})^2}$$

$n^* = c.n \quad , \quad j=1,2,\ldots,400$

$\varphi = 0.4074$

n,c	Stat. Crit.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.0628 (.0310)	.0619 (.0310)	.0596 (.0310)	.0057 (.0025)	.0037 (.0021)
c=80	$\hat{b}_{MB}$	5 (2)	2 (1)	2 (1)	15 (15)	21 (24)
128	$\widehat{MB}$	.0618 (.0154)	.0589 (.0154)	.0580 (.0154)	.0079 (.0007)	.0050 (.0004)
c=40	$\hat{b}_{MB}$	11 (2)	4 (1)	6 (1)	24 (21)	30 (38)
256	$\widehat{MB}$	.0888 (.0077)	.0658 (.0077)	.0665 (.0077)	.0152 (.0002)	.0123 (.0005)
c=20	$\hat{b}_{MB}$	24 (2)	11 (1)	17 (1)	48 (29)	50 (59)
512	$\widehat{MB}$	.1548 (.0038)	.0809 (.0038)	.1017 (.0038)	.0331 (.0001)	.0410 (.0002)
c=10	$\hat{b}_{MB}$	51 (2)	34 (1)	45 (1)	106 (41)	88 (96)

$\varphi = 0.99$

n,c	Stat. Crit.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.8869 (. 8960)	.8678 (.8784)	.8678 (.8784)	.0139 (.0198)	.0117 (.0079)
c=80	$\hat{b}_{MB}$	2 (2)	1 (1)	1 (1)	41 (41)	44 (44)
128	$\widehat{MB}$	.7768 (.7999)	.7452 (.7712)	.7452 (.7712)	.0055 (.0089)	.0056 (.0030)
c=40	$\hat{b}_{MB}$	2 (2)	1 (1)	1 (1)	77 (78)	82 (83)
256	$\widehat{MB}$	.5723 (.6352)	.5313 (.5983)	.5313 (.5983)	.0024 (.0046)	.0024 (.0030)
c=20	$\hat{b}_{MB}$	2 (2)	1 (1)	1 (1)	137 (142)	149 (153)
512	$\widehat{MB}$	.3035 (.4110)	.2940 (.3821)	.2799 (.3821)	.0011 (.0018)	.0009 (.0018)
c=10	$\hat{b}_{MB}$	2 (2)	1 (1)	1 (1)	215 (246)	241 (274)

First, observe the estimates for $\varphi=0.4074$. The minimum bias of the five estimators has been overestimated. The same is true for the MB-parameter values of the NOBM, AREA, combined NOBM-AREA and SPEC estimators. On the other hand, the MB-parameter values of the OVBM estimator have been underestimated.

Examine now the estimates of both the minimum bias and the MB–parameter values for $\varphi=0.99$. First, consider the NOBM, AREA and combined NOBM-AREA estimators. For $c \geq 40$, the estimated values of the minimum bias are close to the true values. Also, notice for all the combinations of n and c that the estimates of the MB-parameter values are identical to the corresponding true values. Regarding now the SPEC and OVBM estimators, both the estimated and true values for the minimum bias are very small for $c \leq 20$. However, for $c \leq 20$, the MB-parameter values of the latter two estimators have been underestimated.

Summarizing the remarks made in the previous two paragraphs, we can conclude the following. First, consider autocorrelation functions converging exponentially to zero which have high positive autocorrelation coefficients at low lags. Acceptable estimates for both the minimum bias and the MB-parameter values can be obtained only when the value of c is large. Setting $c>40$ might be a successful choice although this also depends on how fast the autocorrelation function decays to zero.

On the other hand, for autocorrelation functions which decay exponentially to zero very fast the value of c must be extremely large. At this point, we should also take into account the trade off between a very large sample size which is required for achieving acceptable estimates for both the minimum bias and the MB-parameter values and the time we need to collect such a sample. For this reason, the modified form (6.7) should not be recommended for estimating the autocorrelation coefficients of an output process replication.

Let us now propose a more complicated way to estimate the autocorrelation function of a steady state simulation output process. It is based on fitting the following autoregressive scheme

$$\sum_{s=0}^{p} \varphi_{p,s}(X_{t-s} - \mu) = \epsilon_t \quad , \quad \varphi_{p,0} = 1$$

to an output process replication and estimating the order p and the coefficients $\varphi_{p,s}(s=0,1,\ldots,p)$ according to a procedure given by Fishman(1973b,1978b). The ϵ_t's are independent, identical and normal random variables with mean zero and common variance σ_p^2. It is also assumed in each replication that the initialization bias has already been removed.

In each replication, the first n theoretical autocorrelation coefficients of the fitted AR(p) will replace the first n sample autocorrelations estimated either from (6.6) or (6.7). Provided that the autoregressive order is finite and

$$\sum_{s=-\infty}^{\infty} |\gamma_s| < \infty$$

Fishman's procedure can be described by the following iterative algorithm:

STEP 1: For a sample of size n, set the maximum order of the fitted autoregressive scheme equal to q and estimate the first q autocovariances by

$$\hat{\gamma}_s(n^*) = \frac{1}{n^*} \sum_{t=1}^{n^*-s} (X_t - \bar{X}_{n^*})(X_{t+s} - \bar{X}_{n^*}) \qquad 0 \leq s \leq q$$

$$n^* = c.n$$

where c is a positive integer greater than one.

STEP 2: For $p=0,1,2,\ldots,q$, estimate $\varphi_{p,s}(s=0,1,\ldots,q)$ and σ_q^2 from the following forms

$$\hat{\varphi}_{0,0} = 1 \ , \ \hat{\varphi}_{p,0} = 1$$

$$\hat{\varphi}_{p,p} = \frac{\sum_{s=0}^{p-1} \hat{\varphi}_{p-1,s}\hat{\gamma}_{p-s}(n^*)}{\sum_{s=0}^{p-1} \hat{\varphi}_{p-1,s}\hat{\gamma}_s(n^*)}$$

$$\hat{\varphi}_{p,s} = \hat{\varphi}_{p-1,s} + \hat{\varphi}_{p,p}\hat{\varphi}_{p-1,p-s} \qquad s=1,2,\ldots,p-1$$

$$\hat{\sigma}_q^2 = \sum_{s=0}^{q} \hat{\varphi}_{p,s}\hat{\gamma}_s(n^*)$$

STEP 3: Set the autoregressive order $p=1$

STEP 4: Estimate σ_p^2 by

$$\hat{\sigma}_p^2 = \sum_{s=0}^{p} \hat{\varphi}_{p,s}\hat{\gamma}_s(n^*)$$

STEP 5: Compute the test statistic

$$V = n^* \left(1 - (\hat{\sigma}_q^2/\hat{\sigma}_p^2) \right)$$

If : i) $V > \chi^2_{q-p,\alpha}$ then set p=p+1 and go to step 4

ii) $V < \chi^2_{q-p,\alpha}$ then go to step 6

STEP 6: Give p, $\hat{\varphi}_{p,0}, \hat{\varphi}_{p,1}, \ldots, \hat{\varphi}_{p,p}$ and $\hat{\sigma}_p^2$

The theoretical autocorrelation coefficients of the fitted AR(p) are given by the difference equation

$$\rho_k = \hat{\varphi}_{p,1}\rho_{k-1} + \hat{\varphi}_{p,2}\rho_{k-2} + \cdot \cdot \cdot + \hat{\varphi}_{p,p}\rho_{k-p}$$

where $\hat{\varphi}_{p,s}$ $(s=1,\ldots,p)$ are the estimated autoregressive coefficients of the fitted AR(p) to each replication. The first p-1 autocorrelation coefficients are determined from the following matrix equation

$$\rho_{p-1} = R^{-1} . \Phi$$

with $\rho_{p-1} = [\ \rho_1\ \rho_2\ \cdots\ \rho_{p-1}]'$

$$\Phi = [\ \hat{\varphi}_{p,1}\ \hat{\varphi}_{p,2}\ \cdots\ \hat{\varphi}_{p,p}\ ]'$$

and $R = [\ r_{s\ell}\ ]$.

The elements of matrix R have the following structure

$$r_{s\ell} = \begin{cases} \hat{\varphi}_{p,s-\ell} - \hat{\varphi}_{p,s+\ell} & s = \ell \\ -\hat{\varphi}_{p,s-\ell} & s \neq \ell \;,\; s+\ell \leq p \\ -(\hat{\varphi}_{p,s-\ell} + \hat{\varphi}_{p,s+\ell}) & s \neq \ell \;,\; s+\ell > p \end{cases}$$

and

$$\hat{\varphi}_{p,z} = \begin{cases} 1 & z = 0 \\ 0 & z<0 \;,\; z>p-1 \end{cases}$$

Consider again the statistical criteria under study, i.e. the minimum bias of the five sample mean variance estimators and their MB-parameter values. Table (6.2) displays estimates of these criteria for the AR(1), AR(2) and the delay in queue in the M/M/1. The estimated values were obtained from forms (6.4) and (6.5). For each process, 400 replications were generated. The initial conditions were the following; $X_0=0$ for the AR(1), $X_0=0$, $X_{-1}=0$ for the AR(2), and empty and idle conditions for the M/M/1. The initialization bias was removed by using Schruben et al.'s(1983) test. In each replication, we estimated each Bias Indicator function, by using the theoretical autocorrelation coefficients of the fitted AR(p) instead of the corresponding sample autocorrelations in form (6.2). Different values for both n and c were used. The estimates of the MB-parameter values have been rounded to the nearest integer.

First, observe the estimates of both the minimum bias and the MB-parameter values in the AR(1) for $\varphi=0.4074$, M/M/1 for $\tau=0.2$ and the two AR(2) for $\varphi_1=0.75$, $\varphi_1=0.99$. For all the combinations of n and c, the estimated values of the minimum bias of the five estimators are identical or very close to the true values. The same is true for the MB-parameter values. The only exception is the estimated values of the criteria for the combined NOBM-AREA estimator in the AR(2) for φ_1=0.75. Here, the minimum bias has been overestimated and the MB-parameter values have been badly estimated.

Let us now examine the estimates of the above criteria in the AR(1) for φ=0.99 and M/M/1 for τ=0.80. The minimum bias for the NOBM, AREA and combined NOBM-AREA estimators has been underestimated for all the combinations of n and c. On the other hand, the estimates of the MB-parameter values of these three estimators are identical to the corresponding true values. Regarding the OVBM and SPEC estimators, both the estimated and true values of the minimum bias are very small for $c \leq 8$. However, the MB-parameter values of the latter two estimators have been underestimated for all cases of n and c.

We draw the following conclusions. First, consider autocorrelation functions which decay to zero fast and have low positive autocorrelation coefficients or autocorrelation functions which display a damped cyclical behaviour. The above analysis showed that the use of the second method[based on fitting an AR(p) to each replication of the output process] for estimating such autocorrelation functions improved the quality of the estimates for both the minimum bias and the MB-parameter values. Also, notice that the values of the constant c we used in the second method were smaller than those in the first method.

TABLE 6.2

AR(1) , M/M/1 , AR(2) : True and estimated values for both the minimum bias and the MB-parameter values of the sample mean variance estimators when in the Bias indicator functions the first n sample autocorrelations of simulation output replications are replaced by the first n theoretical autocorrelation coefficients of fitted AR(p) processes

AR(1)

$\varphi = 0.4074$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.0300	.0300	.0300	.0019	.0022
		(.0310)	(.0310)	(.0310)	(.0025)	(.0021)
c=16	$\hat{b}_{MB}$	2	1	1	14	24
		(2)	(1)	(1)	(15)	(24)
128	$\widehat{MB}$	.0149	.0149	.0149	.0008	.0008
		(.0154)	(.0154)	(.0154)	(.0007)	(.0004)
c= 8	$\hat{b}_{MB}$	2	1	1	20	37
		(2)	(1)	(1)	(21)	(38)
256	$\widehat{MB}$	.0074	.0074	.0074	.0003	.0003
		(.0077)	(.0077)	(.0077)	(.0002)	(.0005)
c= 4	$\hat{b}_{MB}$	2	1	1	29	59
		(2)	(1)	(1)	(29)	(59)
512	$\widehat{MB}$	.0037	.0037	.0037	.0001	.0002
		(.0038)	(.0038)	(.0038)	(.0001)	(.0002)
c= 2	$\hat{b}_{MB}$	2	1	1	41	92
		(2)	(1)	(1)	(41)	(96)

$\varphi = 0.99$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.8491	.8262	.8262	.0124	.0120
		(.8960)	(.8784)	(.8784)	(.0198)	(.0079)
c=16	$\hat{b}_{MB}$	2	1	1	41	43
		(2)	(1)	(1)	(41)	(44)
128	$\widehat{MB}$	.7205	.6883	.6883	.0052	.0055
		(.7999)	(.7712)	(.7712)	(.0089)	(.0030)
c= 8	$\hat{b}_{MB}$	2	1	1	75	80
		(2)	(1)	(1)	(78)	(83)
256	$\widehat{MB}$	.5292	.4964	.4964	.0022	.0021
		(.6352)	(.5983)	(.5983)	(.0046)	(.0030)
c= 4	$\hat{b}_{MB}$	2	1	1	133	146
		(2)	(1)	(1)	(142)	(153)
512	$\widehat{MB}$	.3191	.2994	.2994	.0008	.0008
		(.4110)	(.3821)	(.3821)	(.0018)	(.0018)
c= 2	$\hat{b}_{MB}$	2	1	1	223	256
		(2)	(1)	(1)	(246)	(274)

TABLE 6.2 (Cont...)

M/M/1

$\tau = 0.20$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64 c=16	$\widehat{MB}$ $\hat{b}_{MB}$	.0349 (.0349) 2 (2)	.0346 (.0348) 1 (1)	.0346 (.0348) 1 (1)	.0018 (.0020) 15 (15)	.0023 (.0031) 25 (25)
128 c= 8	$\widehat{MB}$ $\hat{b}_{MB}$	.0172 (.0173) 2 (2)	.0172 (.0173) 1 (1)	.0172 (.0172) 1 (1)	.0006 (.0008) 21 (22)	.0007 (.0010) 38 (39)
256 c= 4	$\widehat{MB}$ $\hat{b}_{MB}$	.0086 (.0086) 2 (2)	.0086 (.0086) 1 (1)	.0086 (.0086) 1 (1)	.0002 (.0003) 30 (31)	.0003 (.0001) 59 (61)
512 c= 2	$\widehat{MB}$ $\hat{b}_{MB}$	.0043 (.0043) 2 (2)	.0043 (.0043) 1 (1)	.0043 (.0043) 1 (1)	.0000 (.0001) 42 (43)	.0001 (.0001) 94 (96)

$\tau = 0.80$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64 c=16	$\widehat{MB}$ $\hat{b}_{MB}$	.5570 (.7479) 2 (2)	.5259 (.7187) 1 (1)	.5259 (.7187) 1 (1)	.0092 (.0056) 35 (39)	.0089 (.0034) 40 (42)
128 c= 8	$\widehat{MB}$ $\hat{b}_{MB}$	.3580 (.5902) 2 (2)	.3368 (.5570) 1 (1)	.3368 (.5570) 1 (1)	.0035 (.0081) 59 (70)	.0034 (.0040) 68 (77)
256 c= 4	$\widehat{MB}$ $\hat{b}_{MB}$	.1980 (.3923) 2 (2)	.1889 (.3664) 1 (1)	.1889 (.3664) 1 (1)	.0012 (.0011) 93 (122)	.0012 (.0011) 115 (137)
512 c= 2	$\widehat{MB}$ $\hat{b}_{MB}$	.1003 (.2130) 2 (2)	.0978 (.2025) 1 (1)	.0978 (.2025) 1 (1)	.0005 (.0005) 141 (199)	.0005 (.0005) 194 (237)

T A B L E 6.2 (Cont...)

AR(2)

$\varphi_1 = 0.75$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.0416	.0417	.0347	.0153	.0633
		(.0408)	(.0408)	(.0207)	(.0153)	(.0633)
c=16	$\hat{b}_{MB}$	2	1	11	17	18
		(2)	(1)	(32)	(17)	(18)
128	$\widehat{MB}$	.0211	.0211	.0195	.0059	.0462
		(.0206)	(.0206)	(.0104)	(.0059)	(.0458)
c= 8	$\hat{b}_{MB}$	2	1	11	25	29
		(2)	(1)	(64)	(25)	(29)
256	$\widehat{MB}$	.0106	.0106	.0103	.0024	.0318
		(.0104)	(.0104)	(.0052)	(.0024)	(.0316)
c= 4	$\hat{b}_{MB}$	2	1	(37)	36	47
		(2)	(1)	(128)	(36)	(47)
512	$\widehat{MB}$	.0053	.0053	.0053	.0010	.0212
		(.0052)	(.0052)	(.0026)	(.0010)	(.0211)
c= 2	$\hat{b}_{MB}$	2	1	9	51	75
		(2)	(1)	(1)	(50)	(75)

$\varphi_1 = 0.99$

n,c	Statis. Criter.	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	$\widehat{MB}$	.2660	.2997	.2996	.0047	.3514
		(.2646)	(.3039)	(.3039)	(.0077)	(.3566)
c=80	$\hat{b}_{MB}$	2	1	1	39	23
		(2)	(1)	(1)	(42)	(25)
128	$\widehat{MB}$	.1821	.1731	.1731	.0011	.2763
		(.1803)	(.1717)	(.1717)	(.0009)	(.2810)
c=40	$\hat{b}_{MB}$	2	1	1	60	43
		(2)	(1)	(1)	(64)	(49)
256	$\widehat{MB}$	.0904	.0905	.0905	.0004	.1848
		(.0840)	(.0849)	(.0849)	(.0003)	(.1815)
c=20	$\hat{b}_{MB}$	2	1	1	91	81
		(2)	(1)	(1)	(98)	(80)
512	$\widehat{MB}$	.0463	.0463	.0463	.0001	.1161
		(.0422)	(.0425)	(.0425)	(.0002)	(.1113)
c=10	$\hat{b}_{MB}$	2	1	1	138	140
		(2)	(1)	(1)	(141)	(136)

Now, consider positive autocorrelation functions which decay slowly to zero and have high autocorrelation coefficients at early lags. By using the second method for estimating such autocorrelation functions, we underestimated the minimum bias of the NOBM, AREA and combined NOBM-AREA estimators. Furthermore, we underestimated the MB-parameter values of the SPEC and OVBM estimators.

However, in the next section we shall investigate whether the performance of the latter two methods at the estimated MB-parameter values differentiates from the performance at the true MB-parameter values.

6.3 PERFORMANCE OF CONFIDENCE INTERVAL METHODS FOR THE MB-PARAMETER

The confidence intervals produced by the five methods under consideration at the MB-parameter values, i.e the values for which the minimum bias of the sample mean variance estimators is achieved, cover the true steady-state mean with probability

$$\Pr\left[\bar{X}_n - t_{v_i, \alpha/2} \widehat{\sigma_i(n, b_{MB})} \leq \mu \leq \bar{X}_n + t_{v_i, \alpha/2} \widehat{\sigma_i(n, b_{MB})} \right] \qquad (6.9)$$

where $\widehat{\sigma_i(n, b_{MB})}$ is the standard deviation of the sample mean according to the i^{th} confidence interval method for the parameter values n and b_{MB},

b_{MB} is the value for which the minimum bias of the i^{th} estimator is attained

and v_i are the degrees of freedom for the i^{th} method for the values n and b_{MB}.

In the present section, we check in specific output processes, how close the probabilities given by (6.9) are to the nominal confidence level for each confidence interval method. These processes, whose the theoretical autocorrelation functions are known, were called pilot processes.

Three pilot processes have been selected. These are the delay of the j^{th} customer in queue in the M/M/1 queueing system, the AR(1) and the AR(2). For each process, the initial conditions were the following; $X_0=0$ for the AR(1), $X_0=0$, $X_{-1}=0$ for the AR(2), and empty and idle conditions for the M/M/1. In each replication, the initialization bias was removed using Schruben et al.'s(1983) test[see section 5.3]. The forms of the theoretical autocorrelation functions of the first two pilot processes are similar. In particular, when the two processes have the same first lag theoretical autocorrelation coefficient, the autocorrelation function of the AR(1) decays faster to zero than does the autocorrelation function of the M/M/1. For the AR(2) processes, we have selected the autoregressive coefficients such that the autocorrelation functions display damped cyclical behaviour.

For the three pilot processes, table (6.3) displays the performance of the five confidence interval methods at the true MB-parameter values. The nominal confidence level is 90%. For the same processes, the corresponding performance at 95% nominal confidence level is displayed in tables D1,D2,D3 of appendix D. The performance of the methods at the MB-parameter values is measured by three statistical criteria; the coverage, given by (6.8), of true steady-state means from the confidence interval methods and the expected values and variances of the confidence interval half lengths. Estimates of these criteria were obtained via Monte Carlo experiments.

Define the random variable

$$\Omega_{ij} = \begin{cases} 1 & \text{if} \quad \mu \epsilon [\, \bar{X}_n \pm t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,b_{MB})}\,] \\ 0 & \text{otherwise} \end{cases}$$

For the j^{th} replication, $\widehat{\sigma_{ij}(n,b_{MB})}$ is the standard deviation of the sample mean according to the i^{th} confidence interval method for the parameter values n,b_{MB}
and v_i are the degrees of freedom for the i^{th} method on the j^{th} replication for the values n,b_{MB}.

Generating r replications of the output process under study, the coverage of the true steady-state mean from the i^{th} confidence interval method at the parameter values n,b_{MB} will be estimated from

$$\widehat{CVR}_i = \frac{1}{r}\sum_{j=1}^{r} \Omega_{ij} \quad ,$$

For the i^{th} method, the estimators of the expected values and variances of the confidence interval half lengths at the parameter values n,b_{MB} will be

$$\widehat{EHL}_i = \frac{1}{r}\sum^{r} t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,b_{MB})}$$

$$\widehat{VHL}_i = \frac{1}{r-1}\sum_{j=1}^{r}\left\{ t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,b_{MB})} - \left[\frac{1}{r}\sum_{j=1}^{r} t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,b_{MB})}\right]\right\}^2$$

TABLE 6.3
Performance of confidence interval methods for the true MB-parameter values

Number of replications : 400
Nominal confidence level : 0.90

AR(1)

$\varphi = 0.4074$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9150 (1.0608) (.6497)	.9200 (1.0498) (.6115)	.9200 (1.0498) (.6115)	.9400 (.3940) (.0125)	.8925 (.3418) (.0177)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9275 (.7356) (.3220)	.9150 (.7346) (.3030)	.9150 (.7346) (.3030)	.9350 (.2663) (.0050)	.8950 (.2495) (.0080)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8975 (.5019) (.1628)	.8950 (.4756) (.1223)	.8950 (.4756) (.1223)	.9125 (.1817) (.0012)	.8675 (.1715) (.0030)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9100 (.3589) (.0700)	.9050 (.3349) (.0637)	.9050 (.3349) (.0637)	.9225 (.1249) (.0005)	.8950 (.1208) (.0010)

$\varphi = 0.99$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7750 (11.595) (72.847)	.8050 (12.729) (83.030)	.8050 (12.729) (83.030)	.5975 (5.9967) (13.140)	.2250 (1.8443) (.8772)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8275 (14.613) (111.05)	.8300 (15.872) (130.59)	.8300 (15.872) (130.59)	.7025 (7.3906) (17.225)	.3800 (2.7240) (1.7915)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8625 (16.261) (135.36)	.8725 (16.741) (153.29)	.8725 (16.741) (153.29)	.8100 (7.5452) (15.848)	.5250 (3.4133) (2.8193)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8600 (15.266) (140.54)	.8725 (15.314) (135.89)	.8725 (15.314) (135.89)	.8500 (6.7270) (10.898)	.6400 (3.8530) (3.4193)

TABLE 6.3 (Cont...)

M/M/1

$\tau = 0.20$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8525 (.1170) (.0140)	.8475 (.1045) (.0152)	.8475 (.1045) (.0152)	.7875 (.0418) (.0010)	.7475 (.0368) (.0009)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8825 (.0975) (.0082)	.8675 (.0909) (.0081)	.8675 (.0909) (.0081)	.8325 (.0328) (.0004)	.8050 (.0305) (.0004)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8725 (.0656) (.0034)	.8625 (.0647) (.0033)	.8625 (.0647) (.0033)	.8325 (.0230) (.0001)	.8125 (.0223) (.0002)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8950 (.0520) (.0018)	.8850 (.0506) (.0019)	.8850 (.0506) (.0019)	.8650 (.0171) (.0000)	.8500 (.0168) (.0000)

$\tau = 0.80$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7775 (7.2177) (50.591)	.7750 (7.5301) (58.780)	.7750 (7.5301) (58.780)	.6800 (3.6884) (9.2454)	.4325 (1.3974) (1.0328)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7950 (6.9881) (56.468)	.7950 (7.2163) (59.893)	.7950 (7.2163) (59.893)	.7300 (3.3971) (8.4363)	.5350 (1.6326) (1.6804)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8175 (6.9300) (74.401)	.8200 (6.9176) (71.063)	.8200 (6.9176) (71.063)	.7400 (2.9565) (8.1166)	.5575 (1.7040) (2.4997)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8500 (5.6436) (35.912)	.8750 (5.6728) (40.319)	.8750 (5.6728) (40.319)	.7775 (2.2853) (3.4267)	.6600 (1.5747) (1.6663)

TABLE 6.3 (Cont...)

AR(2)

$\varphi_1=0.75$, $\varphi_2=-0.50$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8875 (.8346) (.4334)	.9150 (.8844) (.4562)	.8950 (.2756) (.0011)	.9025 (.3330) (.0119)	.8950 (.2992) (.0082)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9100 (.5801) (.2044)	.9100 (.5945) (.2028)	.9200 (.1939) (.0003)	.9250 (.2185) (.0039)	.9175 (.2032) (.0032)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (.3981) (.0830)	.8975 (.4194) (.0994)	.9200 (.1368) (.0001)	.9200 (.1454) (.0012)	.9125 (.1373) (.0012)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9075 (.2678) (.0420)	.9050 (.2813) (.0425)	.9050 (.2813) (.0425)	.9000 (.1002) (.0004)	.8975 (.0961) (.0004)

$\varphi_1=0.99$, $\varphi_2=-0.90$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8750 (.8291) (.4366)	.8900 (.8663) (.4602)	.8900 (.8663) (.4602)	.9350 (.4913) (.0606)	.9200 (.3114) (.0058)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9300 (.5563) (.1592)	.9075 (.5516) (.1817)	.9075 (.5516) (.1817)	.9200 (.2488) (.0127)	.9125 (.1962) (.0028)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (.3754) (.0765)	.9200 (.4033) (.0879)	.9200 (.4033) (.0879)	.9325 (.1544) (.0034)	.9225 (.1307) (.0012)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9125 (.2357) (.0313)	.8725 (.2455) (.0380)	.8725 (.2455) (.0380)	.9150 (.0944) (.0009)	.8950 (.0862) (.0005)

Let us examine first the performance of the confidence interval methods in the AR(1) for φ=0.4074 and the two AR(2) processes. The estimated coverages, the methods achieve for the MB-parameter values, are close to the nominal confidence levels for all the sample sizes [see tables (6.3),(D1),(D3)]. Regarding the precision of the confidence intervals, the overlapping batch means and spectral methods produce the narrowest and most stable intervals. Comparing the latter two methods, the overlapping batch means method seems to produce narrower intervals, especially for very small sample sizes.

Compare now the performance that the confidence interval methods have in the M/M/1 for τ=0.2 with the performance in the AR(1) for φ=0.4074. Although the forms of the theoretical autocorrelation functions of the two processes are similar, the performance of the methods at the MB-parameter values does not display similar behaviour. For all the sample sizes we considered in the AR(1), the five methods achieved coverages which were very close to the corresponding nominal confidence levels. On the other hand, in the M/M/1 ,the five methods produce coverages which approach the corresponding nominal confidence levels when the sample size is greater than 512. This could be attributed to two reasons. First, the theoretical autocorrelation function of the AR(1) for φ=0.4074 decays faster to zero. Second, the marginal distributions of the two output processes under discussion are different

Under steady-state conditions, the marginal distribution of the random variables in the AR(1) is normal. On the other hand, the steady-state marginal distribution for the delay $\{X_t\}$ in queue in the M/M/1 is

$$f(x) = \begin{cases} 0 & \text{w.p} \quad 1-\tau \\ v(1-\tau)\, e^{-v(1-\tau)x} & \text{w.p} \quad \tau \end{cases}$$

where v is the service rate and τ the traffic intensity [see Law and Kelton(1984)]. The r^{th} moment around zero for X is

$$E\left[X^r\right] = \tau\left[\frac{1}{v(1-\tau)}\right]^r \Gamma(r+1) \tag{6.10}$$

Define the coefficients of skewness β_1 and kyrtosis β_2 as

$$\beta_1 = \frac{E(X-\mu)^3}{\left[E(X-\mu)^2\right]^{1.5}} \tag{6.11}$$

$$\beta_2 = \frac{E(X-\mu)^4}{\left[E(X-\mu)^2\right]^2} \tag{6.12}$$

Setting r=1,2,3,4 in (6.10) we have

$$E(X-\mu)^2 = E(X^2) - \{E(X)\}^2 = (\tau(2-\tau))/\{v(1-\tau)\}^2 \tag{6.13}$$

$$E(X-\mu)^3 = E(X^3) - E(X)[\ 3E(X^2) - 2\{E(X)\}^2\ ] =$$

$$= (2\tau(\tau^2 - 3\tau + 3))/\{v(1-\tau)\}^3 \tag{6.14}$$

$$E(X-\mu)^4 = E(X^4) - E(X)[\ 4E(X^3) - 6E(X)E(X^2) + 3\{E(X)\}^3\ ] =$$

$$= (3\tau(8-\tau^3+4\tau^2-8\tau))/\{v(1-\tau)\}^4 \qquad (6.15)$$

Substituting (6.13),(6.14) into (6.11) and (6.13),(6.15) into (6.12) and simplifying, we get

$$\beta_1 = \frac{2\tau(\tau^2-3\tau+3)}{\{\tau(2-\tau)\}^{1.5}}$$

$$\beta_2 = 3\ \frac{8-8\tau+4\tau^2-\tau^3}{\{\tau(2-\tau)\}^2}$$

The values for β_1 and β_2 under different traffic intensities are given below

τ	0.2	0.5	0.8	0.9
β_1	4.518	2.964	2.109	2.028
β_2	30.333	13.000	9.500	9.121

As τ tends to 1, β_1 and β_2 tend to 2 and 9 respectively.

Consider now the performance of the confidence interval methods at the MB-parameter values in the AR(1) for φ=0.99 and M/M/1 for τ=0.8 [see tables 6.3 and D1,D2]. The NOBM, AREA and combined NOBM-AREA methods produce greater coverages than do the SPEC and OVBM methods. However, the coverages, the five methods achieve, are smaller than the corresponding nominal confidence levels for small sample sizes.

Under approximately steady-state conditions, we have observed that the performance of the five confidence interval methods is satisfactory for the MB-parameter values in the following output processes:

- AR(1) for φ=0.4074
- delay in queue in M/M/1 with τ=0.2 when the sample size is greater than 512
- AR(2) whose theoretical autocorrelation functions display damped cyclical behaviour.

In the majority of simulation outputs, the theoretical autocorrelation coefficients cannot be computed exactly. Consequently, we can determine neither the true Bias indicator functions nor the true MB-parameter values. Therefore, it is worthwhile to investigate the performance of the methods at the estimated MB-parameter values. In the previous section we saw that the estimation of the MB-parameter values depends upon the method we adopt for estimating the autocorrelation function. Two such methods were discussed there. For the above processes, acceptable estimates for the MB-parameter values are obtained when we use the second method. Namely, in the Bias Indicator functions, we use the theoretical autocorrelation coefficients of fitted AR(p)'s to output process replications instead of the sample autocorrelations.

Define the new random variable

$$\Omega'_{ij} = \begin{cases} 1 & \text{if} \quad \mu \in [\ \bar{X}_n \pm t_{v_i,\alpha/2} \widehat{\sigma_{ij}(n, \hat{b}^j_{MB})}\ ] \\ 0 & \text{otherwise} \end{cases}$$

where $\hat{b}^j_{MB}$ is the estimator of the MB-parameter value for the i^{th} sample mean variance estimator on the j^{th} replication. The

Bias indicator function of the i^{th} sample mean variance estimator is obtained on the j^{th} replication, by replacing the $s^{th}(0<s<n-1)$ sample autocorrelation by the s^{th} lag theoretical autocorrelation coefficient of the fitted AR(p) in form (6.2).

Table (6.4) displays the performance of the five confidence interval methods for the estimated MB-parameter values. The nominal confidence level is 90%. The corresponding performance at 95% is given in tables (D4),(D5) and (D6) of appendix E. The coverages of the true steady-state means from the methods and the expected values and variances of the confidence interval half lengths were estimated from

$$\widehat{CVR}_i = \frac{1}{r}\sum_{j=1}^{r}\Omega'_{ij} \quad ,$$

$$\widehat{EHL}_i = \frac{1}{r}\sum_{j=1}^{r} t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,\hat{b}^j_{MB})}$$

$$\widehat{VHL}_i = \frac{1}{r-1}\sum_{j=1}^{r}\left\{t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,\hat{b}^j_{bt})} - \left[\frac{1}{r}\sum_{j=1}^{r} t_{v_i,\alpha/2}\widehat{\sigma_{ij}(n,\hat{b}^j_{bt})}\right]\right\}^2$$

First, consider the estimated coverages that the methods achieve in the AR(1) for $\varphi=0.4074$, M/M/1 for $\tau=0.2$ and the two AR(2) processes. We can observe no significant differences in these coverages between tables (6.3) and (6.4). Moreover, the OVBM and SPEC methods produce the narrowest and most stable confidence intervals even for the estimated MB-parameter values.

T A B L E 6.4

Performance of confidence interval methods for the estimated MB-parameter values

Number of replications : 400
Nominal confidence level : 0.90

AR(1)

$\varphi = 0.4074$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.9175	.9200	.9200	.9400	.8950
	$\widehat{EHL}_i$	(1.0582)	(1.0434)	(1.0422)	(.3901)	(.3416)
	$\widehat{VHL}_i$	(.6481)	(.6038)	(.6063)	(.0130)	(.0176)
128	$\widehat{CVR}_i$	.9275	.9150	.9125	.9375	.8925
	$\widehat{EHL}_i$	(.7318)	(.7847)	(.7304)	(.2651)	(.2495)
	$\widehat{VHL}_i$	(.3228)	(.3028)	(.3021)	(.0047)	(.0079)
256	$\widehat{CVR}_i$	.8975	.8950	.8950	.9100	.8675
	$\widehat{EHL}_i$	(.4990)	(.4756)	(.4740)	(.1813)	(.1711)
	$\widehat{VHL}_i$	(.1613)	(.1223)	(.1222)	(.0014)	(.0030)
512	$\widehat{CVR}_i$	.9100	.9050	.9050	.9250	.8950
	$\widehat{EHL}_i$	(.3590)	(.3349)	(.3349)	(.1248)	(.1207)
	$\widehat{VHL}_i$	(.0699)	(.0637)	(.0637)	(.0005)	(.0010)

$\varphi = 0.99$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.7750	.8050	.8050	.5900	.2350
	$\widehat{EHL}_i$	(11.595)	(12.729)	(12.729)	(5.9098)	(1.8800)
	$\widehat{VHL}_i$	(72.847)	(83.030)	(83.030)	(12.901)	(.9183)
128	$\widehat{CVR}_i$	.8275	.8300	.8300	.6900	.4000
	$\widehat{EHL}_i$	(14.613)	(15.872)	(15.872)	(7.0727)	(2.8049)
	$\widehat{VHL}_i$	(111.05)	(130.59)	(130.59)	(15.907)	(1.9011)
256	$\widehat{CVR}_i$	.8625	.8725	.8725	.7900	.5325
	$\widehat{EHL}_i$	(16.741)	(16.741)	(16.741)	(7.1409)	(3.5273)
	$\widehat{VHL}_i$	(135.36)	(153.29)	(153.29)	(14.193)	(2.9711)
512	$\widehat{CVR}_i$	.8600	.8725	.8725	.8250	.6375
	$\widehat{EHL}_i$	(15.266)	(15.314)	(15.314)	(6.3189)	(3.9385)
	$\widehat{VHL}_i$	(140.54)	(135.89)	(135.89)	(9.9001)	(3.5855)

TABLE 6.4 (Cont...)

M/M/1

$\tau = 0.20$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.8525	.8475	.8475	.7925	.7475
	$\widehat{EHL}_i$	(.1170)	(.1044)	(.1045)	(.0418)	(.0369)
	$\widehat{VHL}_i$	(.0139)	(.0152)	(.0152)	(.0012)	(.0009)
128	$\widehat{CVR}_i$	.8825	.8675	.8675	.8350	.8125
	$\widehat{EHL}_i$	(.0975)	(.0909)	(.0909)	(.0328)	(.0305)
	$\widehat{VHL}_i$	(.0082)	(.0081)	(.0081)	(.0004)	(.0004)
256	$\widehat{CVR}_i$	.8725	.8625	.8625	.8325	.8100
	$\widehat{EHL}_i$	(.0656)	(.0647)	(.0647)	(.0230)	(.0223)
	$\widehat{VHL}_i$	(.0034)	(.0033)	(.0033)	(.0001)	(.0002)
512	$\widehat{CVR}_i$	.8950	.8850	.8850	.8675	.8525
	$\widehat{EHL}_i$	(.0520)	(.0506)	(.0506)	(.0171)	(.0168)
	$\widehat{VHL}_i$	(.0018)	(.0018)	(.0018)	(.0000)	(.0000)

$\tau = 0.80$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.7775	.7750	.7750	.6525	.4450
	$\widehat{EHL}_i$	(7.2177)	(7.5301)	(7.5301)	(3.3968)	(1.4832)
	$\widehat{VHL}_i$	(50.591)	(58.780)	(58.780)	(8.2029)	(1.1489)
128	$\widehat{CVR}_i$	.7950	.7950	.7950	.6950	.5475
	$\widehat{EHL}_i$	(6.9881)	(7.2163)	(7.2163)	(3.0796)	(1.7264)
	$\widehat{VHL}_i$	(56.468)	(59.893)	(7.2163)	(7.5424)	(1.8320)
256	$\widehat{CVR}_i$	.8175	.8200	.8200	.7075	.5825
	$\widehat{EHL}_i$	(6.9300)	(6.9176)	(6.9176)	(2.7135)	(1.7845)
	$\widehat{VHL}_i$	(74.401)	(71.063)	(71.063)	(8.2802)	(2.4937)
512	$\widehat{CVR}_i$	.8500	.8750	.8750	.7750	.6750
	$\widehat{EHL}_i$	(5.6436)	(5.6728)	(5.6728)	(2.1154)	(1.6197)
	$\widehat{VHL}_i$	(35.912)	(40.319)	(40.319)	(3.9897)	(1.5882)

TABLE 6.4 (Cont...)

AR(2)

φ_1=0.75 , φ_2=-0.50

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8875 (.8331) (.4341)	.9150 (.8844) (.4562)	.9200 (.7013) (.3974)	.9025 (.3349) (.0123)	.8950 (.2986) (.0082)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9100 (.5797) (.2046)	.9100 (.5945) (.2028)	.9100 (.4097) (.0986)	.9250 (.2185) (.0038)	.9175 (.2030) (.0032)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (.3975) (.0832)	.8975 (.4194) (.0994)	.9075 (.2766) (.0425)	.9200 (.1456) (.0012)	.9125 (.1376) (.0013)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9075 (.2674) (.0421)	.9050 (.2813) (.0425)	.9050 (.2813) (.0425)	.9000 (.1002) (.0004)	.8950 (.0961) (.0005)

φ_1=0.99 , φ_2=-0.90

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8750 (.8296) (.4364)	.8900 (.8663) (.4602)	.8900 (.8663) (.4602)	.9300 (.4678) (.0567)	.9225 (.3111) (.0058)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9300 (.5563) (.1592)	.9075 (.5516) (.1817)	.9075 (.5516) (.1817)	.9250 (.2420) (.0110)	.9150 (.1993) (.0025)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (.3754) (.0765)	.9200 (.4033) (.0879)	.9200 (.4033) (.0879)	.9300 (.1502) (.0031)	.9225 (.1305) (.0012)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9125 (.2357) (.0313)	.8725 (.2455) (.0380)	.8725 (.2455) (.0380)	.9150 (.0931) (.0008)	.8950 (.0860) (.0005)

Let us now examine the estimated coverages in the AR(1) for φ=0.99 and M/M/1 for τ=0.8. The NOBM, AREA and combined NOBM-AREA methods achieve coverages which are identical in tables (6.3) and (6.4). On the contrary, the OVBM and SPEC methods produce coverages which differ in the two tables. Notice, however, that the coverages of the latter two methods for both the true and estimated MB-parameter values differ significantly from the nominal confidence levels.

6.4 SUMMARY

Five confidence interval methods were considered in this chapter; the nonoverlapping/overlapping batch means, area, combined area-nonoverlapping batch means and spectral methods. We have studied their performance at the parameter values for which the minimum bias of the corresponding sample mean variance estimators is attained. These parameter values were called MB-parameter values. The MB-parameter values can be determined only when the theoretical autocorrelation function of the output process under study is known.

We have found that the performance of the five methods at the MB–paramater values was satisfactory in the following types of process:

_ Normal processes characterized by autocorrelation functions which have low positive coefficients and decay exponentially to zero fast.

_ Normal processes characterized by autocorrelation functions which display a damped cyclical behaviour.

_ Non-normal processes, characterized by autocorrelation functions which have low positive coefficients and decay to zero fast, when the sample size is greater than 512.

For these types of process, the overlapping batch means and spectral methods produced confidence intervals whose half lengths had the smallest expected values and variances.

In the majority of steady-state simulation outputs, the theoretical autocorrelation functions are not known. Therefore, the MB-parameter values cannot be determined exactly. In this chapter, we proposed a method for estimating autocorrelation coefficients of approximately steady state simulation output processes. In each replication after removing the initialization bias by applying an appropriate test, the sample autocorrelations are replaced by the theoretical autocorrelation coefficients of the fitted AR(p). Having estimated the autocorrelation coefficients, we can determine the bias indicator functions and the MB-parameter values. For the above three types of process, we have found no significant differences in the performances of the methods both at the true and estimated MB-parameter values.

The performance of the methods at the MB-parameter values was also studied on processes characterized by positive autocorrelation functions, which have high coefficients in the early lags and decay slowly to zero. All the methods produced coverages smaller than the nominal confidence levels for small sample sizes.

In the next chapter, for such autocorrelation functions, we shall investigate if the coverage which each method achieves for the MB-parameter values is the nearest to the nominal confidence level.

CHAPTER SEVEN

OPTIMUM PERFORMANCE OF CONFIDENCE INTERVAL METHODS

7.1 INTRODUCTION

In the previous chapter, we studied through Monte Carlo experiments the performance of five confidence interval methods for specific values of their parameters. These values, called MB-parameter values, were chosen in such a way that the minimum bias of the corresponding sample mean variance estimators was achieved. The testing environment consisted of processes whose the theoretical autocorrelation coefficients for any lag can be computed exactly. We have called these processes pilot processes.

All the methods for the MB-parameter values achieved acceptable coverages in the following pilot processes:

- AR(1) for φ=0.4074
- delay in queue in the M/M/1 for τ=0.2, when the sample size was greater than 512
- AR(2) whose autocorrelation function displayed damped cyclical behaviour.

On the other hand, in the AR(1) for φ=0.99 and M/M/1 for τ=0.8, for small sample sizes all the methods produced coverages smaller than the corresponding nominal confidence levels.

In the present chapter, three processes additional to the pilot processes are considered. For different sample sizes, we determine the parameter values for which each confidence interval method achieves the best coverage i.e. the coverage which is the nearest to the nominal confidence level. The performance of the methods for the parameter values for which the best coverage is attained will be called optimum performance. For the above processes, the initial conditions were chosen to be different to the 'representative' steady state conditions. However, in each replication we removed the initialization bias by using the Schruben et al.'s test(1983) discussed in section 5.3.

In the next section, for each confidence interval method three types of function are introduced. They correspond to each of the statistical criteria usually used to study the general performance of such methods; the coverages of the true steady-state means from these methods and the expected values and variances of the confidence interval half lengths. New statistical criteria, related to these functions, are defined. We use the new criteria for studying and comparing the optimum performance of the methods.

In section 7.3, we discuss the optimum performance of the methods in the pilot processes. This will be compared with the performance the methods had for the parameter values, for which the minimum bias of the corresponding sample mean variance estimators is attained.

In section 7.4, the optimum performance of the five methods is studied in three additional processes. These processes come from three simulation models; the inventory model, the interactive computer model and the time-shared computer model.

The last section provides general recommendations for applying confidence interval methods to approximately steady-state simulation output processes.

7.2 STATISTICAL CRITERIA FOR STUDYING THE OPTIMUM PERFORMANCE OF CONFIDENCE INTERVAL METHODS

Three functions are introduced in this section. We shall use them to define new statistical criteria. These criteria will be used for measuring and comparing the optimum performance of confidence interval methods in small sample sizes.

Keeping fixed the sample size n and the nominal confidence level $(1-\alpha)$, the first function will provide the coverage, each confidence interval method achieves, for different parameter values. We shall call the function coverage curve. This will have the general form

$$CVR_i(\theta)=Pr\left[\bar{X}_n-t_{v_i,\alpha/2}\widehat{\sigma_i(n,\theta)} \leq \mu \leq \bar{X}_n+t_{v_i,\alpha/2}\widehat{\sigma_i(n,\theta)}\right] \qquad (7.1)$$

where $\widehat{\sigma_i(n,\theta)}$ is the standard deviation of the sample mean according to the i^{th} confidence interval method. The parameter θ denotes either the number of contiguous batches or the batch size or the size of spectral window. The following coverage curves are defined for the five methods under consideration:

_ NB-coverage curve for the nonoverlapping batch mean method(NOBM) denoted by $CVR_{NB}(k)$

_ SM-coverage curve for the standardized time series-area method(AREA) denoted by $CVR_{SM}(k)$

_ CM-coverage curve for the combined area-nonoverlapping batch means method(NOBM-AREA) denoted by $CVR_{CM}(k)$

_ SP-coverage curve for the spectral method(SPEC) denoted by $CVR_{SP}(w)$ and

_ OB-coverage curve for the overlapping batch means method(OVBM) denoted by $CVR_{OB}(m)$.

Similar functions to the coverage curve can also be defined for the other two statistical criteria; the expected values and variances of the confidence interval half lengths. These functions will have the general form

$$EHL_i(\theta) = E\left[\, t_{v_i,\alpha/2}\widehat{\sigma_i(n,\theta)} \,\right] \tag{7.2}$$

and

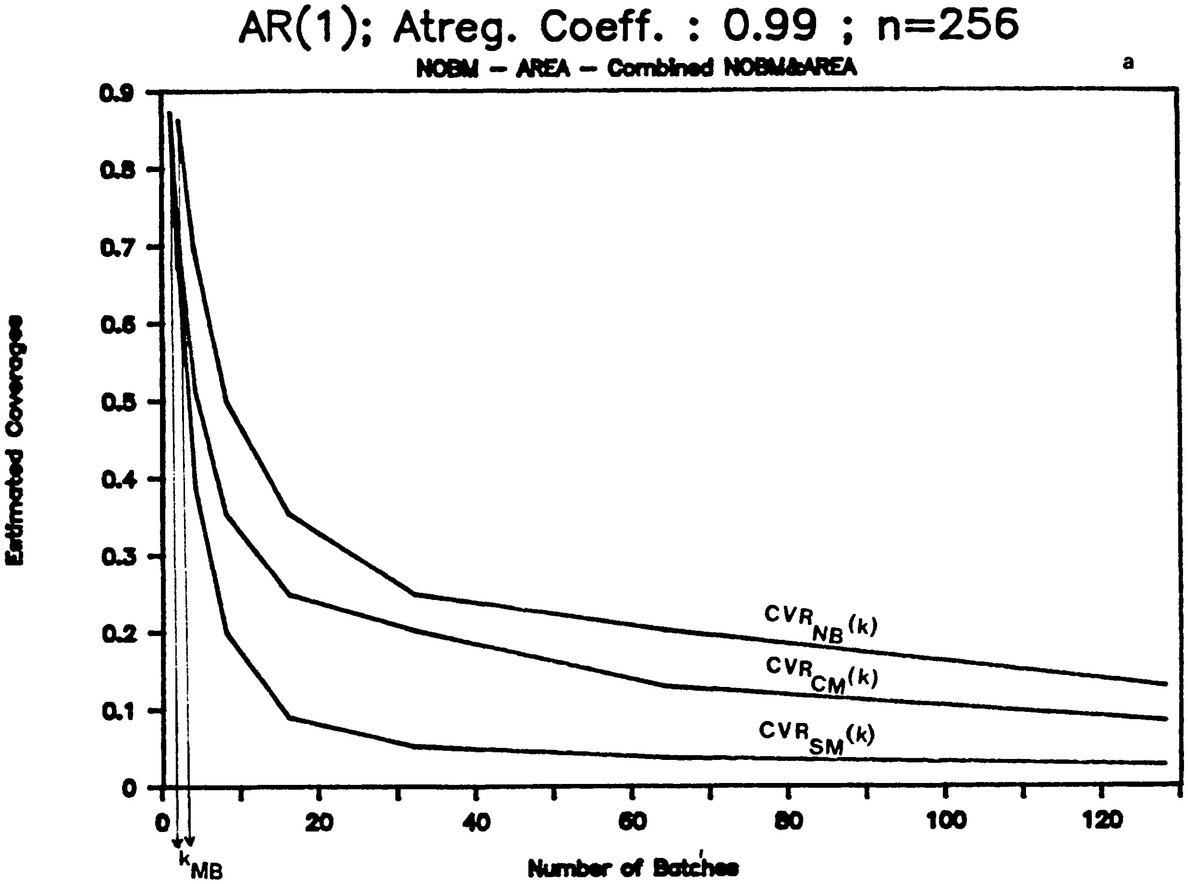
$$VHL_i(\theta) = E\left[\, t_{v_i,\alpha/2}\widehat{\sigma_i(n,\theta)} - E\left[t_{v_i,\alpha/2}\widehat{\sigma_i(n,\theta)}\right] \,\right]^2 \tag{7.3}$$

where i stands for the i^{th} confidence interval method.

As it has been pointed out in the simulation literature, analytical values for the coverage of steady-state means from confidence interval methods cannot be obtained in small sample sizes [see for example Goldsman et al(1986)]. The same is true for the expected values and variances of the confidence interval half lengths. The alternative solution, therefore, is the estimation of these criteria through the Monte Carlo experiments. That is, given the sample size, the nominal confidence level and the parameter values of each method, we can produce estimates for the criteria using the general estimation procedure which was described in chapter two [see section 2.3].

Figure (7.1) illustrates estimated coverage curves of the five methods for the AR(1) and the delay in queue in the M/M/1 for large values of φ and τ respectively. The nominal confidence level is 90%. The parameter values k_{MB}, m_{MB} and w_{MB}, for which the minimum bias of the sample mean variance estimators is attained, are marked as well.

Figure 7.1
Estimated coverage curves of five confidence interval methods



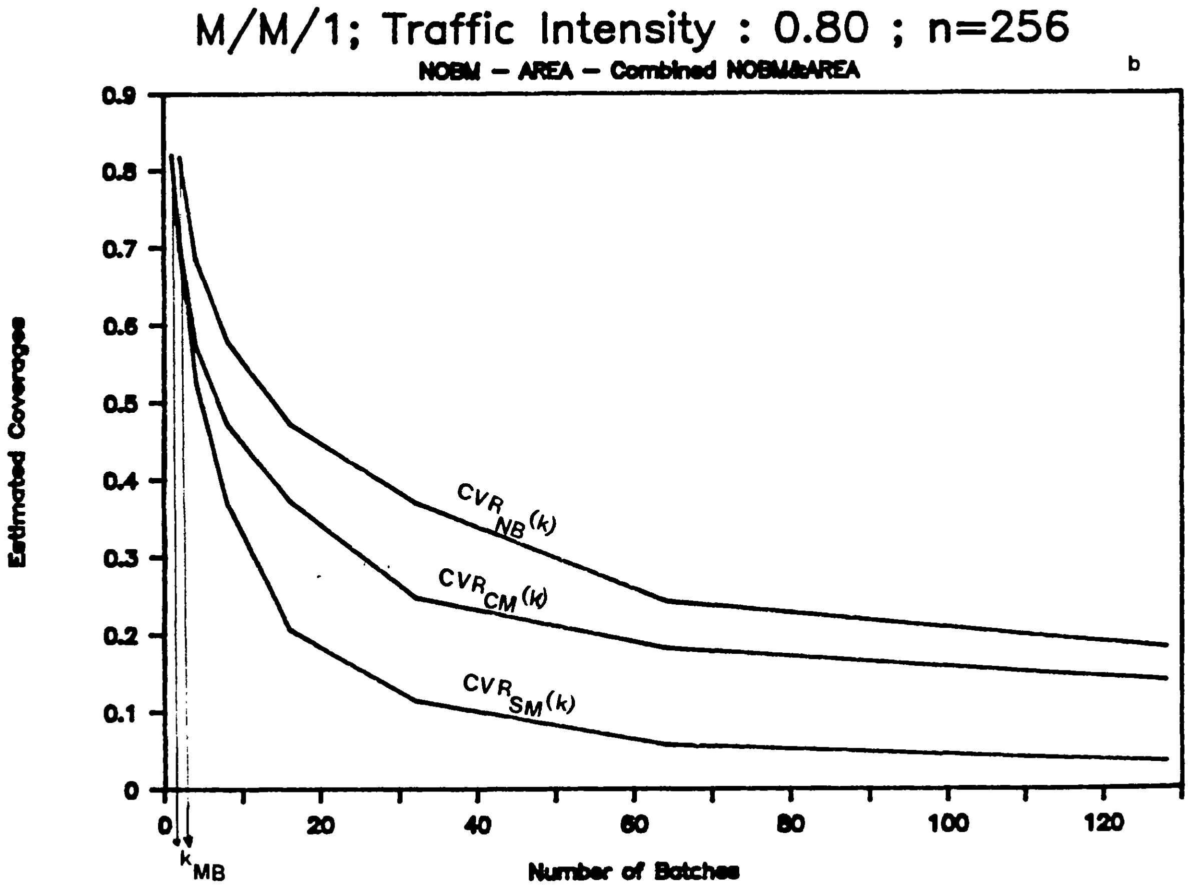
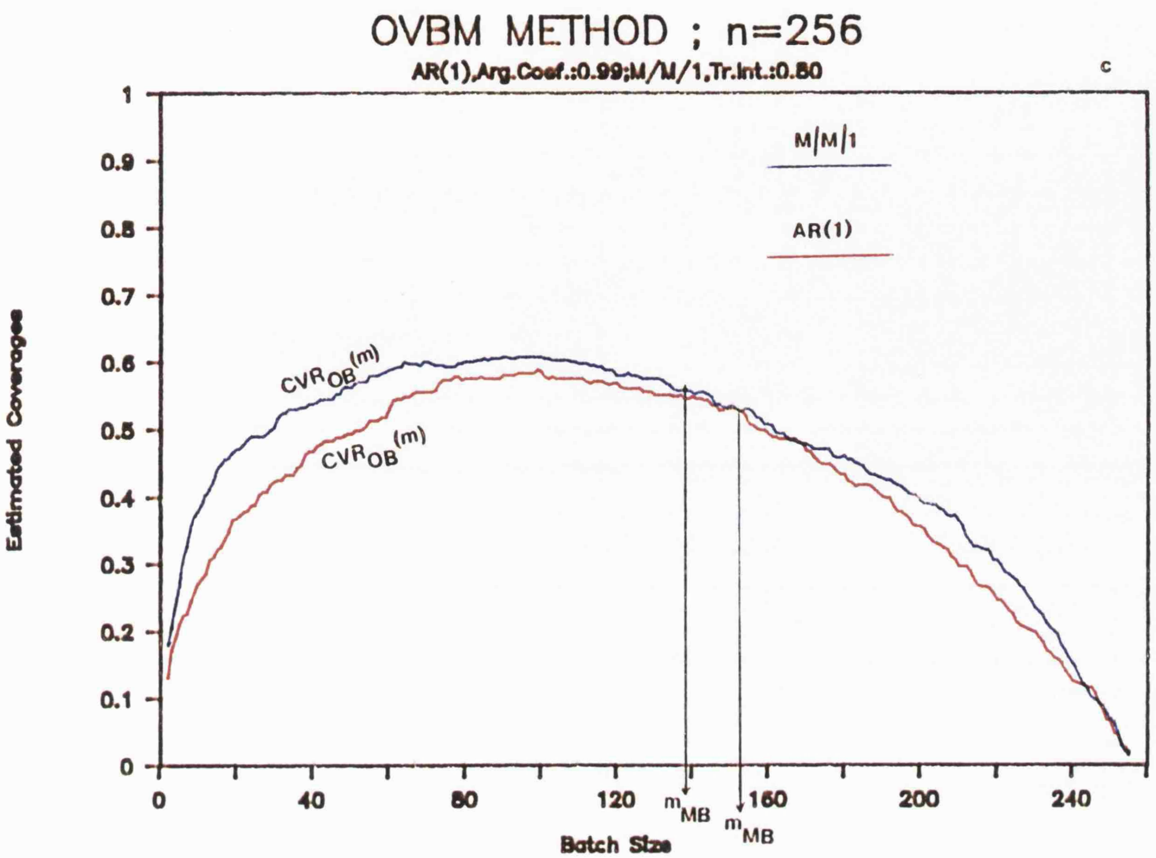
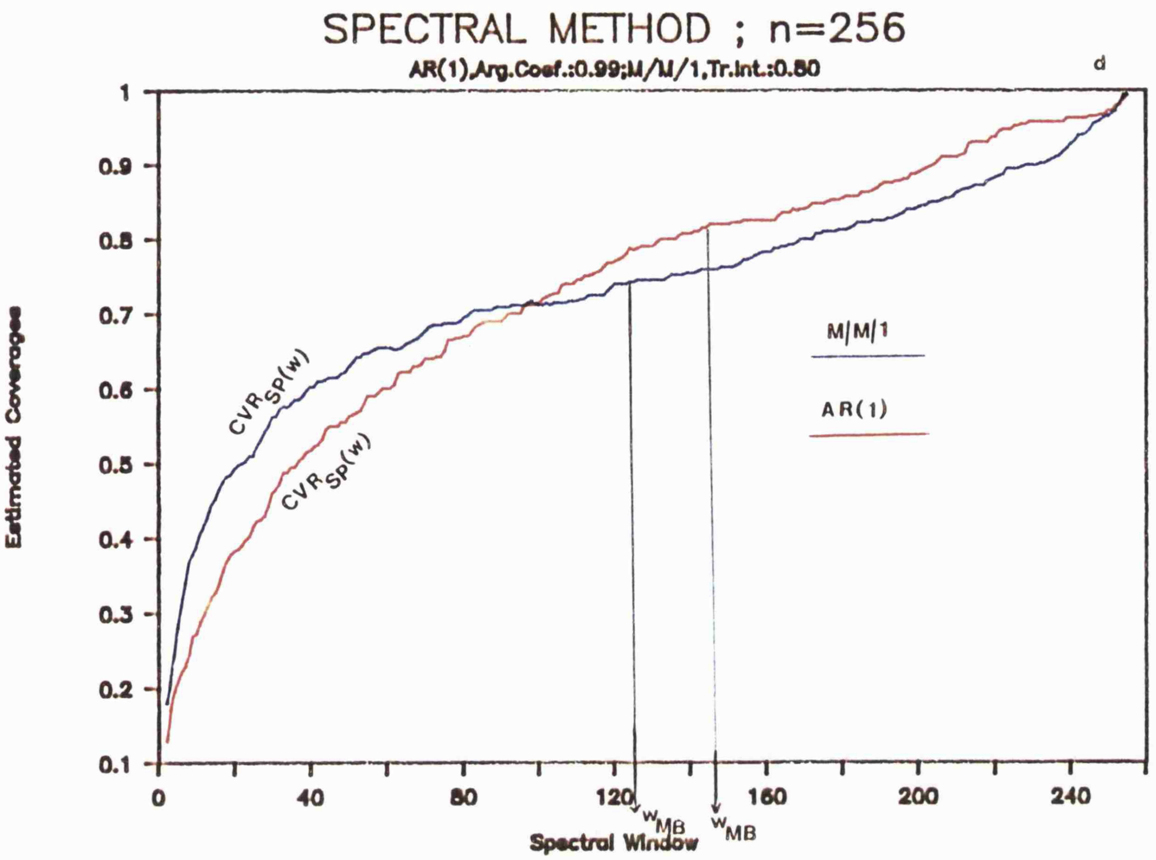


Figure 7.1 (Cont..)





First, we observe that for a certain range of spectral window sizes, the curves of the spectral method lie very close to the nominal confidence level. Therefore, the coverages this method achieves for the parameter values, for which the minimum bias occurs, are not the nearest to the nominal confidence level. On the other hand, the best coverages the other four methods achieve do not seem to be differentiated from the coverages on the minimum bias.

The analysis of figure (7.1) shows the usefulness of the estimated coverage curves in determining the optimum performance of confidence interval methods. In each estimated coverage curve, consider those values which are the nearest to the corresponding nominal confidence level. We shall call these values "best coverages" and they will constitute the basic criterion for measuring the optimum performance.

When several confidence interval methods achieve similar best coverage, the following two criteria will be used for comparing their optimum performance:

i) BEST MEAN HALF LENGTH OF CONFIDENCE INTERVALS (BMHL)

It is defined as the mean of the confidence interval mean half lengths for those parameter values for which the best coverage is attained

ii) BEST AVERAGE VARIANCE OF CONFIDENCE INTERVAL HALF LENGTHS (BVHL)

It is defined as the mean of the estimated values of the variance of the confidence interval half lengths for those parameter values for which the best coverage is attained.

Define now a small positive number ε. Also, denote the nominal confidence level by $(1-\alpha)$. When the best coverage of a confidence interval method lies within the range $(1-\alpha)\pm\epsilon$, we shall say that this method achieves an ϵ-ideal performance. The ϵ-ideal performance constitute a special case of the optimum performance. Furthermore, the ϵ-ideal performance will be attained not only for the parameter values for which the best coverage is achieved but also for any other parameter value for which the estimated coverage lies within the range $(1-\alpha)\pm\epsilon$. In the next sections, we discriminate the ϵ-ideal performance from the optimum performance when the last one is characterized by best coverages which do not lie within the range $(1-\alpha)\pm\epsilon$.

The following criteria will be used for comparing ϵ-ideal performances of different confidence interval methods:

i) IDEAL MEAN HALF LENGTH OF CONFIDENCE INTERVALS (IMHL)

It is defined as the mean of the confidence interval mean half lengths for those parameter values for which the estimated coverages lie within the range $(1-\alpha)\pm\epsilon$

ii) IDEAL AVERAGE VARIANCE OF CONFIDENCE INTERVAL HALF LENGTHS (IAVHL)

It is defined as the mean of the estimated values of the variance of the confidence interval half lengths for those parameter values for which the estimated coverages lie within the range $(1-\alpha)\pm\epsilon$.

7.3 OPTIMUM PERFORMANCE OF CONFIDENCE INTERVAL METHODS IN THE PILOT PROCESSES

First, we study the optimum performance of the five confidence interval methods in the pilot processes. We use the new statistical criteria which were introduced in the previous section and whose values as presented here have been estimated at 90%

nominal confidence level. Their corresponding values at 95% are displayed in appendix E, tables E1,E3 and E5. In the same appendix, tables E2,E4 and E6 contain the parameter values for which the optimum performance of the methods is observed for the two nominal confidence levels under consideration.

For the pilot processes, the initial conditions were the following; $X_0=0$ for the AR(1), $X_0=0$, $X_{-1}=0$ for the AR(2) and empty and idle conditions for the M/M/1. In each replication of the pilot processes, we have removed the initialization bias using the Schruben et al.'s(1983) test defined in (5.2).

7.3.1 AR(1)

Tables (7.1) and (E1) display the optimum performance of the five confidence interval methods in the AR(1). First, consider the values of the statistical criteria for $\varphi=0.4074$ and $\varphi=0.7778$. All the methods achieve ε-ideal performance for $\varepsilon=0.025$ and $n\geq128$. However, the spectral(SPEC) and overlapping batch means(OVBM) methods seem to be superior. For the range of parameter values, where the five methods attain the ε-ideal performance, the SPEC and OVBM methods produce on average narrower and more stable confidence intervals.

The superiority of the latter two methods was also recognized when we investigated their performance for the parameter values, for which the minimum bias of the sample mean variance estimators is attained. Therefore, these two methods should be preferred for constructing confidence intervals in normal output processes for which the autocorrelation functions decay exponentially to zero fast. Given the sample size in any replication, we recommend the confidence interval to be built for the batch size or spectral window size where the minimum bias of the corresponding sample mean variance estimator is attained. These sizes are determined by the Bias Indicator functions[see section 3.4]. These functions are estimated in each replication, by using the theoretical autocorrelation coefficients of the fitted AR(p) instead of the corresponding sample autocorrelations in form (6.2).

Examine now the values of the statistical criteria for φ=0.963 and φ=0.99. First, consider the nonoverlapping/ overlapping batch means(NOBM,OVBM), area and combined area-nonoverlapping batch means methods. From tables (7.1) and (E1), we see that the lower the nominal confidence level we select, the larger the sample size we need so that these four methods achieve ε-ideal performance for ε=0.025. Comparing the optimum performance of these four methods, we see that the OVBM method produces the smallest best coverages which lie far from the two nominal confidence levels under consideration.

For the above values of φ, let us compare the optimum performance of the four methods with their performance at the MB-parameter values. For small sample sizes, the optimum performance of the NOBM and AREA, combined NOBM-AREA methods is achieved for k=2 and k=1 respectively [see table E2 of appendix E]. However, for these number of batches, the minimum bias of the corresponding sample mean variance estimators is attained. Therefore, the optimum performance and the performance at the MB-parameter values are exactly the same.

T A B L E 7.1

AR(1) : Optimum performance of confidence interval methods at 90% nominal confidence level

Number of Replications : 400 , $\varepsilon = 0.025$

$\varphi = 0.4074$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.6830	.3264			
	AREA	.5625	.1760			
	NOBM-AREA	.5410	.1640			
	SPEC	.3224	.0320			
	OVBM	.3328	.0123			
128	NOBM	.2683	.0085			
	AREA	.3703	.0717			
	NOBM-AREA	.4056	.1031			
	SPEC	.3191	.0245			
	OVBM	.2408	.0053			
256	NOBM	.2496	.0353			
	AREA	.2566	.0297			
	NOBM-AREA	.2286	.0225			
	SPEC	.2125	.0232			
	OVBM	.1677	.0014			
512	NOBM	.1593	.0120			
	AREA	.1716	.0128			
	NOBM-AREA	.1554	.0115			
	SPEC	.1358	.0108			
	OVBM	.1207	.0012			

$\varphi = 0.7778$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	2.601	4.242			
	AREA			.9300	2.704	3.577
	NOBM-AREA			.9300	2.704	3.577
	SPEC	1.218	.320			
	OVBM			.7850	.790	.119
128	NOBM	1.163	.781			
	AREA	1.451	1.141			
	NOBM-AREA	1.391	1.085			
	SPEC	.674	.035			
	OVBM	.616	.043			
256	NOBM	.762	.355			
	AREA	1.031	.563			
	NOBM-AREA	.751	.338			
	SPEC	.525	.031			
	OVBM	.441	.017			
512	NOBM	.539	.155			
	AREA	.607	.186			
	NOBM-AREA	.568	.169			
	SPEC	.432	.032			
	OVBM	.307	.013			

T A B L E 7.1 (Cont..)

$\varphi = 0.9630$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.8550	8.695	38.882
	AREA			.8150	8.961	44.714
	NOBM-AREA			.8150	8.961	44.714
	SPEC	8.853	30.737			
	OVBM			.4700	2.039	1.098
128	NOBM			.8500	8.022	39.841
	AREA			.8650	8.179	42.825
	NOBM-AREA			.8650	8.179	42.825
	SPEC	6.124	14.051			
	OVBM			.6175	2.127	1.109
256	NOBM			.8650	7.155	33.471
	AREA			.8700	7.576	34.109
	NOBM-AREA			.8700	7.576	34.109
	SPEC	4.042	4.850			
	OVBM			.7225	2.078	.929
512	NOBM	4.145	10.320			
	AREA	2.782	2.122			
	NOBM-AREA	5.852	19.125			
	SPEC	2.450	1.042			
	OVBM			.8175	1.773	.476

$\varphi = 0.99$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.7750	11.59	72.85
	AREA			.8050	12.73	83.03
	NOBM-AREA			.8050	12.73	83.03
	SPEC	20.85	188.47			
	OVBM			.3300	2.44	1.84
128	NOBM			.8275	14.61	111.05
	AREA			.8300	15.87	130.59
	NOBM-AREA			.8300	15.87	130.59
	SPEC	17.00	111.31			
	OVBM			.4775	3.19	2.60
256	NOBM			.8625	16.26	135.36
	AREA			.8725	16.74	153.29
	NOBM-AREA			.8725	16.74	153.29
	SPEC	12.39	53.66			
	OVBM			.5875	3.97	3.63
512	NOBM			.8600	15.27	140.54
	AREA			.8725	15.31	135.89
	NOBM-AREA			.8725	15.31	135.89
	SPEC	9.37	27.66			
	OVBM			.6725	4.10	3.96

Furthermore, the OVBM method attains best coverages which do not differ significantly from those produced at the MB-parameter values.

Now, consider the optimum performance of the spectral method in the AR(1) for φ=0.9630 and φ=0.99. In chapter six we have seen that the performance of this method in the two processes, for the spectral window sizes for which the minimum bias occurs, was not satisfactory. On the other hand, we see in tables (7.1) and (E1) that the spectral method achieves ε-ideal performance(ε=0.025) for all the combinations of n and φ. How much we should have increased the average spectral window size, for which the minimum bias occured, so that the spectral method achieves ε-ideal performance(ε=0.025) is given below.

		SAMPLE SIZES			
(1-α)	φ	64	128	256	512
90%	0.9630	1.48-1.54	1.53-1.68	1.34-1.90	0.97-1.53
	0.99	1.46-1.51	1.44-1.53	1.39-1.55	1.36-1.84
95%	0.9630	1.32-1.54	1.30-1.69	1.25-1.96	0.76-2.22
	0.99	1.44-1.51	1.38-1.52	1.22-1.77	1.21-1.84

Observe that by increasing the average size of the spectral window by 1.50, the spectral method achieves ε-ideal performances(ε=0.025) for all the combinations of n and φ.

7.3.2 M/M/1

Denote the traffic intensity by τ and the service rate by v. The marginal distribution of the steady-state delay in queue in the M/M/1 is 0 with probability 1-τ and exponential with mean

$(1/(v(1-\tau))$ with probability τ [see Law and Kelton(1984)]. Regarding this process, in the previous chapter we have seen that the greater the traffic intensity we observe, the less skew the marginal distribution we take. Furthermore, in chapter four we have shown diagramatically, that the theoretical autocorrelation functions of both the AR(1) with positive autoregressive coefficient and the delay in queue in the M/M/1 had similar shapes. Observe in figure (4.1) that when the two processes have the same first lag theoretical autocorrelation coefficient, the autocorrelation function of the delay in the M/M/1 decays slower to zero.

Tables (7.2) and (E3) display the optimum performance of the five confidence interval methods in the M/M/1. First, observe the values of the statistical criteria for the NOBM, OVBM, AREA and combined NOBM-AREA methods. The greater the traffic intensity we observe, the larger the sample size we need so that these four methods achieve ε-ideal performance for $\varepsilon=0.025$. For every sample size, the OVBM method has the worst optimum performance as it produces the lowest coverages.

Let us compare the optimum performance of the previous four methods with their performance at the MB-parameter values. From table (E4)[see appendix E], we see for small sample sizes that the NOBM and AREA, combined NOBM-AREA methods achieve their optimum performances for $k=2$ and $k=1$ respectively. However, for these values, the minimum bias of the corresponding sample mean variance estimators occurs. Hence, as in the case of the AR(1) for large φ, the optimum performance is not differentiated from the performance at the MB-parameter values.

Furthermore, the best coverages achieved by the OVBM method do not differ significantly from those produced on the minimum bias [see tables (6.3) and (D2) of appendix D].

Consider now the optimum performances of the four methods in the AR(1) for φ=0.4074 and the delay in queue in the M/M/1 for τ=0.20. The two processes have the same first lag theoretical autocorrelation coefficient. We have found for all combinations of sample sizes and nominal confidence levels we considered that the five methods achieve ϵ-ideal performances for ϵ=0.025 in the AR(1). However, the same is not true in the M/M/1. For instance, refering to 90% nominal confidence level, the four methods fail to achieve the ϵ-ideal performance for n<256. The difference in the behaviour of the optimum performance of these methods in the two processes can be attributed to two reasons. First, to the non-normality of the process in the M/M/1. Second, to the fact that the autocorrelation function of the AR(1) for φ=0.4074 decays faster to zero.

For all combinations of traffic intensities, sample sizes and nominal confidence levels, the spectral method achieves ϵ-ideal performance for ϵ=0.025. From tables (7.2) and (E3), we see in large sample sizes that not only the spectral method but also the NOBM, AREA and combined NOBM-AREA methods achieve ϵ-ideal performance(ϵ=0.025).. However, for these sample sizes, the spectral method produces confidence intervals whose half lengths have on average the smallest expected values and variances.

T A B L E 7.2

M/M/1 : Optimum performance of confidence interval methods at 90% nominal confidence level

Number of Replications : 400 , $\epsilon = 0.025$

$\tau = 0.20$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.8525	.1170	.0140
	AREA			.8475	.1045	.0152
	NOBM-AREA			.8475	.1045	.0152
	SPEC	.1016	.0097			
	OVBM			.7725	.0356	.0007
128	NOBM	.0975	.0082			
	AREA			.8675	.0909	.0081
	NOBM-AREA			.8675	.0909	.0081
	SPEC	.0596	.0022			
	OVBM			.8175	.0294	.0004
256	NOBM			.8725	.0656	.0034
	AREA			.8700	.0342	.0005
	NOBM-AREA			.8625	.0647	.0033
	SPEC	.0111	.0008			
	OVBM			.8225	.0216	.0001
512	NOBM	.0520	.0018			
	AREA	.0506	.0019			
	NOBM-AREA	.0360	.0010			
	SPEC	.0250	.0002			
	OVBM			.8600	.0161	.0000

$\tau = 0.50$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.8250	1.235	1.830
	AREA			.8400	1.197	1.848
	NOBM-AREA			.8400	1.197	1.848
	SPEC	1.223	1.109			
	OVBM			.6825	.362	.092
128	NOBM			.8625	1.044	1.032
	AREA	1.025	.934			
	NOBM-AREA	1.025	.934			
	SPEC	.804	.349			
	OVBM			.7550	.312	.055
256	NOBM			.8550	.550	.318
	AREA			.8450	.764	.524
	NOBM-AREA			.8450	.764	.524
	SPEC	.559	.210			
	OVBM			.8125	.233	.023
512	NOBM			.8725	.525	.272
	AREA			.8600	.389	.129
	NOBM-AREA			.8700	.231	.022
	SPEC	.306	.093			
	OVBM			.8200	.176	.012

T A B L E 7.2 (Cont...)

$\tau = 0.80$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.7775	7.218	50.591
	AREA			.7750	7.530	58.780
64	NOBM-AREA			.7750	7.530	58.780
	SPEC	11.002	100.36			
	OVBM			.5025	1.771	1.606
	NOBM			.7950	6.988	56.469
	AREA			.7950	7.216	59.893
128	NOBM-AREA			.7950	7.216	59.893
	SPEC	9.952	89.301			
	OVBM			.5600	1.857	2.201
	NOBM			.8175	6.930	74.401
	AREA			.8200	6.918	71.063
256	NOBM-AREA			.8200	6.918	71.063
	SPEC	7.856	75.810			
	OVBM			.6075	1.885	3.058
	NOBM			.8500	5.644	35.512
	AREA	5.673	40.319			
512	NOBM-AREA	5.673	40.319			
	SPEC	4.919	22.261			
	OVBM			.6925	1.587	1.494

$\tau = 0.90$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.6575	10.82	90.49
	AREA			.6450	11.59	103.30
64	NOBM-AREA			.6450	11.59	103.30
	SPEC			.8725	23.42	337.11
	OVBM			.2500	2.45	2.17
	NOBM			.6725	12.98	165.05
	AREA			.6875	13.90	192.45
128	NOBM-AREA			.6875	13.90	192.45
	SPEC	30.31	756.93			
	OVBM			.3525	3.01	4.65
	NOBM			.7625	15.11	205.46
	AREA			.7525	15.64	231.43
256	NOBM-AREA			.7525	15.64	231.43
	SPEC	27.32	559.43			
	OVBM			.4425	3.76	7.28
	NOBM			.8225	16.48	280.63
	AREA			.7725	16.16	295.60
512	NOBM-AREA			.7725	16.16	295.60
	SPEC	24.94	569.21			
	OVBM			.5600	4.27	11.09

We have also studied the optimum performance of the spectral method in the M/M/1 for ε=0.005 and ε=0.015. The sample sizes, n, we used were 64,128,256 and 512. For any combination of the above n and ε, the optimum performance of this method was ideal. The spectral window sizes for which the ε-ideal performance was observed are given below. The sample size is 128.

		τ			
ε	1-α	0.2	0.5	0.8	0.9
0.005	90%	90- 98	100-103	119	125
	95%	88- 93	107-109	118	125
0.015	90%	79-100	95-106	117-120	124-125
	95%	81- 95	99-113	114-123	124-125

Decreasing ε, the range of spectral window sizes, where the ε-ideal performance is attained, is reduced. We have also found that this remark is true for all the other sample sizes we have considered.

Moreover, we display below how much we should have increased the average size of the spectral window, for which the minimum bias occurs, so that the spectral method achieves ε-ideal performance(ε=0.025).

		SAMPLE SIZES			
(1-a)	τ	64	128	256	512
90%	0.2	3.20-3.87	3.29-5.00	4.73-6.70	2.88-8.83
	0.5	2.35-2.48	2.76-3.30	3.62-4.59	3.64-5.91
	0.8	1.68-1.74	1.97-2.05	2.34-2.57	2.72-3.22
	0.9	1.59	1.77-1.79	1.99-2.08	2.40-2.53
95%	0.2	3.40-4.00	3.19-4.76	4.97-7.43	2.50-9.74
	0.5	2.22-2.57	2.79-3.58	3.53-5.04	3.96-6.15
	0.8	1.66-1.74	1.90-2.14	2.31-2.63	2.59-3.23
	0.9	1.56-1.59	1.76-1.80	1.98-2.08	2.37-2.53

We have found that the coverage curve of the spectral method in the M/M/1 is an increasing function of the spectral window size [see for example figure (7.1)]. Increasing the average spectral window size, for which the minimum bias is attained, by

$$c\log n - \log\tau \quad , \quad c = \begin{array}{l} 0.80 \text{ for } \tau\leq 0.5 \\ 0.40 \text{ for } \tau>0.5, \end{array}$$

the spectral method will achieve coverages either within $(1-\alpha)\pm\epsilon$ or greater than $(1-\alpha)+\epsilon$.

7.3.3 AR(2)

The theoretical autocorrelation functions of the two AR(2) processes display damped cyclical behaviour[see figure 4.2]. The optimum performance of the five confidence interval methods is given in tables (7.3) and (E5). This is similar to the optimum performance the methods had in the AR(1) for low positive autoregressive coefficients. Although all the methods achieve ϵ-ideal performance for $\epsilon=0.025$, the overlapping batch means and spectral methods seem to be superior. The latter two methods produce on average the narrowest and most stable confidence intervals for the parameter values for which the ϵ-ideal performance is achieved.

T A B L E 7.3

AR(2) : Optimum performance of confidence interval methods at 90% nominal confidence level

Number of Replications : 400 , $\varepsilon = 0.025$

$\varphi_1 = 0.75$, $\varphi_2 = -0.50$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.5146	.1558			
	AREA	.5773	.1776			
	NOBM-AREA	.4693	.1232			
	SPEC	.3752	.0085			
	OVBM	.2975	.0069			
128	NOBM	.4182	.1084			
	AREA	.4532	.1155			
	NOBM-AREA	.3942	.1016			
	SPEC	.2156	.0063			
	OVBM	.2004	.0038			
256	NOBM	.2858	.0445			
	AREA	.2682	.0394			
	NOBM-AREA	.2213	.0269			
	SPEC	.1695	.0044			
	OVBM	.1380	.0019			
512	NOBM	.1648	.0151			
	AREA	.1653	.0129			
	NOBM-AREA	.1420	.0093			
	SPEC	.1176	.0021			
	OVBM	.0967	.0004			

$\varphi_1 = 0.99$, $\varphi_2 = -0.90$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.8291	.4366			
	AREA	.8663	.4602			
	NOBM-AREA	.8663	.4602			
	SPEC	.3416	.0163			
	OVBM	.3007	.0068			
128	NOBM	.2513	.0110			
	AREA	.4317	.1051			
	NOBM-AREA	.4021	.0970			
	SPEC	.2071	.0057			
	OVBM	.1913	.0027			
256	NOBM	.3754	.0765			
	AREA	.4033	.0879			
	NOBM-AREA	.4033	.0879			
	SPEC	.1287	.0027			
	OVBM	.1243	.0014			
512	NOBM	.1725	.0166			
	AREA	.1416	.0049			
	NOBM-AREA	.1128	.0021			
	SPEC	.0969	.0012			
	OVBM	.0855	.0005			

TABLE 7.4

Estimated coverages that the spectral method achieves when its confidence intervals have approximately equal mean half lengths with the confidence intervals of the nonoverlapping batch means method for k=2.

Models	Nominal Confidence Level 90% Sample Sizes 64	128	256	512	Nominal Confidence Level 95% Sample Sizes 64	128	256	512
AR(1)								
φ=.963	.9050 (56)	.9450 (111)	.9625 (218)	.9750 (442)	.9800 (59)	.9800 (114)	.9900 (230)	.9900 (464)
φ=.99	.8325 (56)	.8775 (111)	.9475 (223)	.9525 (443)	.9300 (59)	.9700 (117)	.9675 (233)	.9900 (465)
M/M/1								
τ=0.50	.8950 (56)	.9300 (112)	.9350 (223)	.9525 (443)	.9850 (62)	.9725 (117)	.9725 (234)	.9875 (465)
τ=0.80	.8375 (56)	.8475 (111)	.8950 (223)	.9225 (443)	.9475 (59)	.9400 (116)	.9575 (234)	.9775 (465)
τ=0.90	.6750 (56)	.7275 (112)	.8150 (222)	.8425 (448)	.8550 (62)	.8850 (117)	.9100 (233)	.9375 (468)
AR(2)								
φ_1=.75	.9525 (55)	.9750 (110)	.9700 (218)	.9775 (438)	.9825 (58)	.9900 (116)	.9875 (230)	.9975 (462)
φ_1=.99	.9700 (55)	.9775 (112)	.9775 (219)	.9650 (438)	.9875 (58)	.9825 (117)	1.000 (246)	.9900 (462)

Table (7.4) for different processes provides the coverages, that the spectral method achieves for specific spectral window sizes. These sizes, which are given in parantheses, were chosen in such a way that the confidence intervals of the spectral method have approximately the same mean half length with the confidence intervals of the nonoverlapping batch means method for k=2. We can see that these coverages are greater than the coverages the other four methods produce at their optimum performance. Besides, for each sample size the particular spectral window sizes have approximately the same value in all the processes. In terms of the sample size n, w≃0.87n for 90% nominal confidence level and w≃0.91n for 95%.

7.4 THE PERFORMANCE OF CONFIDENCE INTERVAL METHODS IN SEMI REAL-LIFE SIMULATION MODELS

In this section, three additional processes are considered. They come from the following simulation models; inventory model, interactive multiprogrammed computer model and time-shared computer model. First, we study the optimum performance of the five confidence interval methods under consideration. Then, we compare this performance with the one the methods have for the estimated MB-parameter values. In each replication, these parameter values are determined from the Bias Indicator functions which are estimated by using in form (6.2) the theoretical autocorrelation coefficients of the fitted AR(p)'s.

For the above processes, the initial conditions were different to the steady state conditions. However, in each replication of each process, we removed the initialization bias by using the Schruben et al.'s test. From the first twenty replications after deleting a number of early observations[see table 5.2], we found that the spectral sample mean variance estimator was the least biased estimator. Then the mean spectral window size for which the minimum bias occured was used for computing both the test statistic (5.2) and the degrees of freedom for the student-t distribution[see section 5.3].

7.4.1 Inventory Model

The operational rules of this model were described in chapter two. The output process under study is the cost at period i. The initial condition is S=52.

Table (7.5) displays the optimum performance of the five confidence interval methods. The parameter values for which this performance is attained are given in table (E7) of appendix E. In the inventory model, the optimum performance of the methods is similar to that in the following two pilot processes:-

_ AR(1) when the autoregressive coefficient has a low positive value and

_ AR(2) when its autocorrelation function shows damped cyclical behaviour.

Although all the methods achieve ε-ideal performance for ε=0.025, the OVBM and SPEC methods produce on average the narrowest and most stable confidence intervals for the parameter values where the ε-ideal performance is attained. By comparing these two methods, we can see that the half lengths of the confidence intervals produced by the OVBM method seem to have on average smaller expected values and variances.

In figure (7.2), we have drawn the theoretical autocorrelation functions of two AR(p)'s fitted to two replications of the process under study. In each replication, the order and the autoregressive coefficients of the fitted AR(p) were estimated from the iterative algorithm discussed in chapter six. The sample size n^* was 1024. We observe that both autocorrelation functions damp down to zero oscillating between positive and negative values.

The performance of the methods for the estimated MB-parameter values is displayed in table (7.6). In each replication, these parameter values were determined by the Bias Indicator functions which were estimated by using in form (6.2) the theoretical autocorrelation coefficient of the fitted AR(p). The numbers in parentheses are the average parameter values where the minimum bias occurs.

TABLE 7.5

Optimum performance of confidence interval methods in the inventory model

Number of Replications : 400 , $\varepsilon = 0.025$

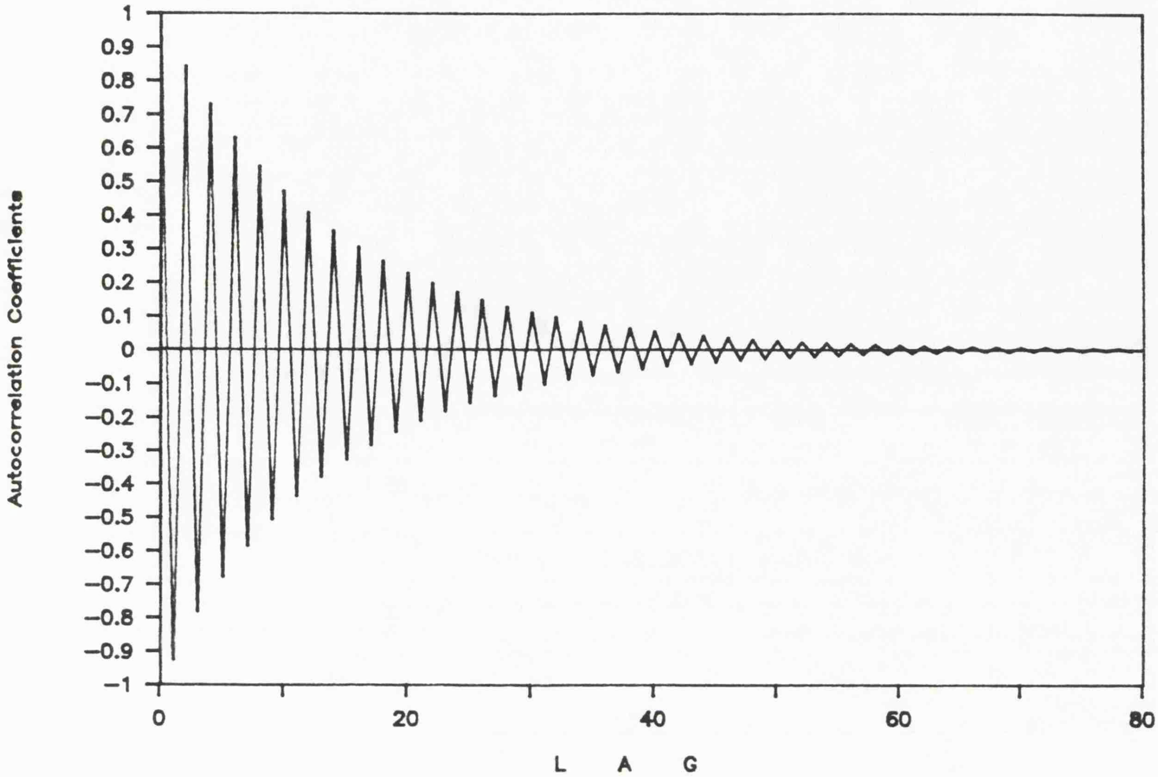
Nominal Confidence Level : 90%

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	14.322	111.904			
	AREA	14.602	113.432			
	NOBM-AREA	14.602	113.432			
	SPEC	4.740	1.953			
	OVBM	4.534	2.074			
128	NOBM	9.057	49.507			
	AREA	8.744	46.642			
	NOBM-AREA	8.744	46.642			
	SPEC	3.094	.857			
	OVBM	2.815	.661			
256	NOBM	5.977	18.335			
	AREA	5.789	18.551			
	NOBM-AREA	5.789	18.551			
	SPEC	2.247	.657			
	OVBM	1.775	.339			
512	NOBM	2.739	4.684			
	AREA	2.971	4.984			
	NOBM-AREA	2.780	4.637			
	SPEC	1.535	.346			
	OVBM	1.278	.177			

Nominal Confidence Level : 95%

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	28.825	453.306			
	AREA	29.388	459.494			
	NOBM-AREA	29.388	459.494			
	SPEC	6.034	3.245			
	OVBM	6.481	4.765			
128	NOBM	18.228	200.545			
	AREA	12.710	102.860			
	NOBM-AREA	17.598	188.940			
	SPEC	4.017	1.665			
	OVBM	3.683	1.216			
256	NOBM	12.031	74.272			
	AREA	8.197	40.634			
	NOBM-AREA	7.611	38.704			
	SPEC	3.646	2.940			
	OVBM	2.246	.574			
512	NOBM	4.940	18.359			
	AREA	5.415	18.992			
	NOBM-AREA	5.030	18.138			
	SPEC	2.172	.936			
	OVBM	1.701	.283			

Figure 7.2
Inventory model: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the cost per period



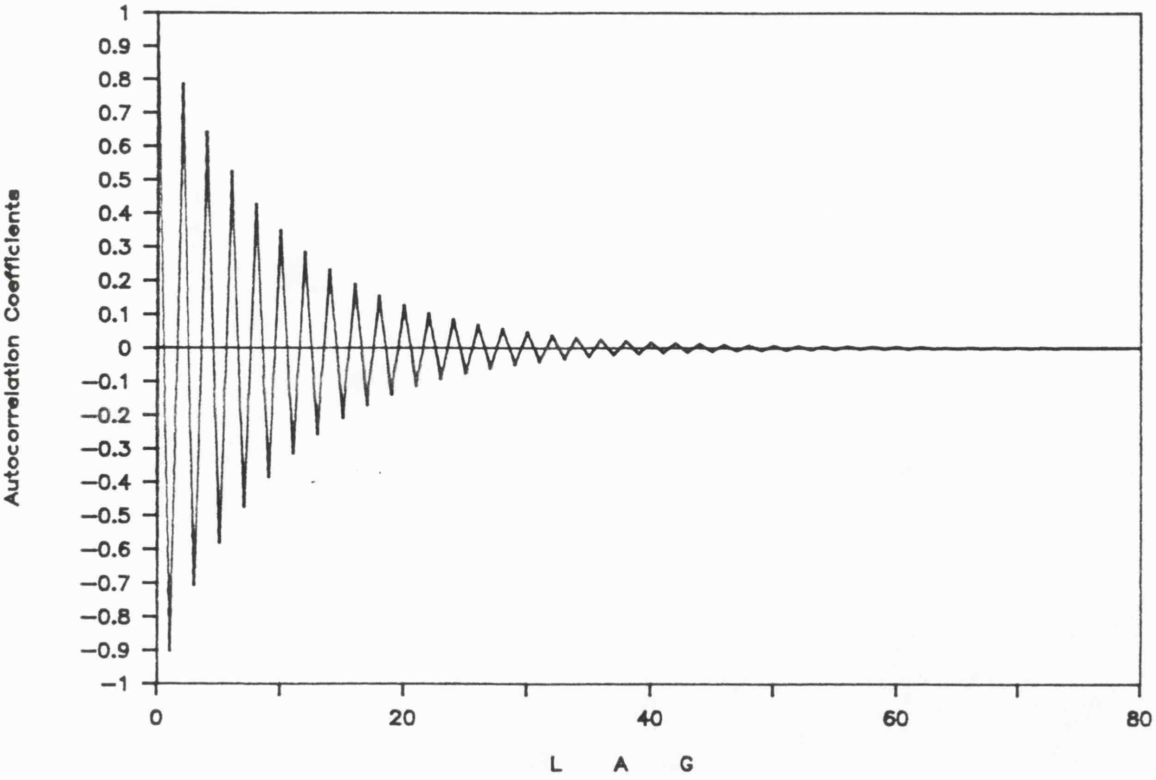


TABLE 7.6

Inventory Model : Performance of Confidence Interval Methods for the parameter values for which the minimum bias of the sample mean variance estimators is attained

Number of Replications : 400

Nominal Confidence Level : 0.90

n	Statist. Criteria	NOBM	AREA	NOBM&AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.9250	.9225	.9225	.9325	.9775
	$\widehat{EHL}_i$	14.322	14.602	14.602	4.367	5.751
	$\widehat{VHL}_i$	111.904	113.432	113.432	.973	1.695
	$\hat{b}$	(2)	(1)	(1)	(7)	(17)
128	$\widehat{CVR}_i$	.9025	.9100	.9100	.9225	.9525
	$\widehat{EHL}_i$	9.057	8.744	8.744	2.808	3.495
	$\widehat{VHL}_i$	49.507	46.642	46.642	.310	.520
	$\hat{b}$	(2)	(1)	(1)	(11)	(39)
256	$\widehat{CVR}_i$	.9225	.9250	.9250	.9400	.9625
	$\widehat{EHL}_i$	5.977	5.789	5.789	1.866	2.178
	$\widehat{VHL}_i$	18.335	18.551	18.551	.118	.274
	$\hat{b}$	(2)	(1)	(1)	(17)	(75)
512	$\widehat{CVR}_i$	.9000	.9075	.9075	.9250	.9500
	$\widehat{EHL}_i$	3.745	3.849	3.849	1.278	1.391
	$\widehat{VHL}_i$	8.814	8.693	8.693	.044	.118
	$\hat{b}$	(2)	(1)	(1)	(29)	(135)

Nominal Confidence Level : 0.95

n	Statist. Criteria	NOBM	AREA	NOBM&AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.9675	.9700	.9700	.9675	.9875
	$\widehat{EHL}_i$	28.825	29.388	29.388	5.333	7.432
	$\widehat{VHL}_i$	453.306	459.494	459.494	1.452	2.817
128	$\widehat{CVR}_i$	.9650	.9575	.9575	.9500	.9900
	$\widehat{EHL}_i$	18.228	17.598	17.598	3.409	4.569
	$\widehat{VHL}_i$	200.545	188.940	188.940	.458	.887
256	$\widehat{CVR}_i$	.9575	.9575	.9575	.9750	.9850
	$\widehat{EHL}_i$	12.031	11.652	11.652	2.258	2.834
	$\widehat{VHL}_i$	74.272	75.146	75.146	.173	.464
512	$\widehat{CVR}_i$	.9500	.9600	.9600	.9700	.9750
	$\widehat{EHL}_i$	7.537	7.748	7.748	1.278	1.792
	$\widehat{VHL}_i$	35.703	35.215	35.215	1.542	.196

We observe in table (7.6) that the OVBM and SPEC methods achieve greater coverages than the other three methods. On the other hand, the coverages of the NOBM, AREA and combined NOBM-AREA methods lie within the range $(1-\alpha)\pm\epsilon$ for $\epsilon=0.025$. However, the confidence intervals of the first two methods are narrower and more stable at the estimated MB-parameter values.

The performance of the classical method is given in table (E10) of appendix E. For each combination of sample size and nominal confidence level, the estimated coverages are equal to unity. Furthermore, the confidence intervals of the classical method are on average wider than those the other five methods produce at the estimated MB-parameter values.

7.4.2 Interactive Multiprogrammed Computer Model

This model was described in chapter two. Different output processes can be defined in it. We have selected the waiting time at the CPU as the process under study. In each replication, we have initialized the simulation program from empty and idle conditions.

Table (7.7) displays the optimum performance of the five confidence interval methods in the process under study. This performance is similar to the one the methods had in the AR(1) and the delay in queue in the M/M/1 for high values of φ and τ respectively. For small sample sizes, the NOBM, OVBM, AREA and combined NOBM-AREA methods fail to achieve ϵ-ideal performances for $\epsilon=0.025$. On the other hand, the spectral method attains ϵ-ideal performance($\epsilon=0.025$) for any combination of sample size and nominal confidence level. The spectral window sizes, for which the ϵ-ideal performance is achieved, are given in table (E8) of appendix E.

The performance of the methods, for the parameter values for which the minimum bias occurs, is given in table (D7). No method achieves satisfactory performance for small sample sizes as the produced coverages are smaller than the nominal confidence levels.

Now, let us examine the performance of the spectral method for specific sizes of the spectral window. These sizes were chosen in such a way that the confidence intervals of the NOBM for k=2 and spectral methods have approximately the same mean half lengths. For different sample sizes, we give below the coverages the spectral method achieves for these spectral window sizes. These sizes are given in parentheses.

	SAMPLE SIZES			
$1-\alpha$	64	128	256	512
90%	.6000 (56)	.7725 (112)	.8675 (223)	.9125 (438)
95%	.8150 (58)	.9125 (117)	.9425 (233)	.9675 (462)

For n>512, the coverages are greater than the corresponding nominal confidence levels. In regard to the spectral window sizes, $w \simeq 0.87n$ for 90% nominal confidence level and $w \simeq 0.91n$ for 95%. We remind the reader that in the AR(1) and M/M/1 for the same spectral window sizes, the NOBM for k=2 and spectral methods had also produced confidence intervals with approximately the same half lengths.

T A B L E 7.7
Optimum performance of confidence interval methods in the interactive computer model

Number of Replications : 400 , $\varepsilon = 0.025$

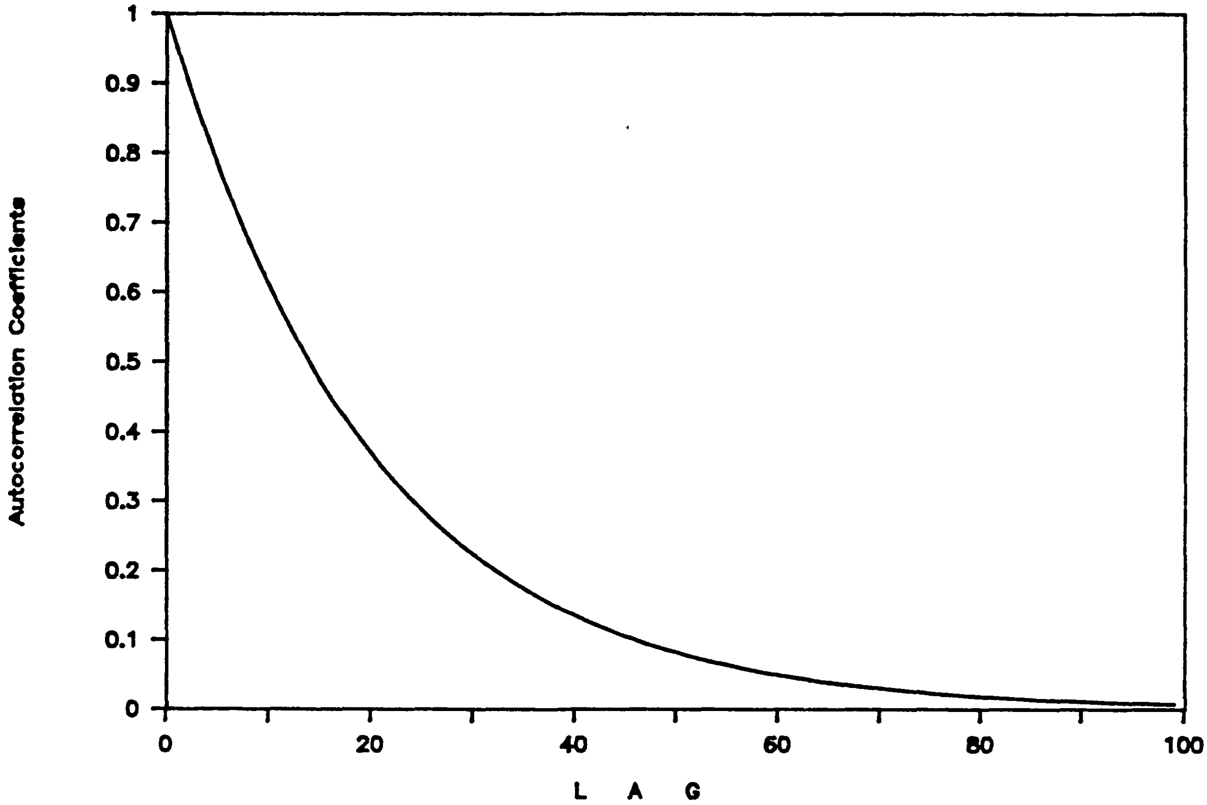
Nominal Confidence Level : 90%

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.5700	4.312	18.357
	AREA			.5875	4.796	21.907
64	NOBM-AREA			.5875	4.796	21.907
	SPEC	13.799	150.060			
	OVBM			.2300	1.053	.670
	NOBM			.7200	5.409	23.853
	AREA			.7325	5.644	25.204
128	NOBM-AREA			.7325	5.644	25.204
	SPEC	9.993	64.500			
	OVBM			.4175	1.346	.887
	NOBM			.7925	5.856	27.263
	AREA			.7975	6.021	28.168
256	NOBM-AREA			.7975	6.021	28.168
	SPEC	7.022	28.876			
	OVBM			.5475	1.504	.899
	NOBM			.8400	4.746	16.897
	AREA			.8050	4.787	18.029
512	NOBM-AREA			.8050	4.787	18.029
	SPEC	4.524	9.677			
	OVBM			.6800	1.429	.627

Nominal Confidence Level : 95%

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.7425	8.678	74.362
	AREA			.7725	9.652	88.741
64	NOBM-AREA			.7725	9.652	88.741
	SPEC	23.453	433.510			
	OVBM			.2925	1.417	1.210
	NOBM			.8375	10.887	96.624
	AREA			.8400	11.359	102.100
128	NOBM-AREA			.8400	11.359	102.100
	SPEC	17.211	195.630			
	OVBM			.5000	1.777	1.473
	NOBM			.8850	11.786	110.440
	AREA			.8875	12.118	114.100
256	NOBM-AREA			.8875	12.118	114.100
	SPEC	11.257	78.319			
	OVBM			.6350	1.886	1.246
	NOBM			.8950	9.551	68.448
	AREA			.8975	9.635	73.032
512	NOBM-AREA			.8975	9.635	73.032
	SPEC	6.962	23.606			
	OVBM			.7350	1.852	1.005

Figure 7.3
Interactive computer model: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the waiting time at the central processing unit



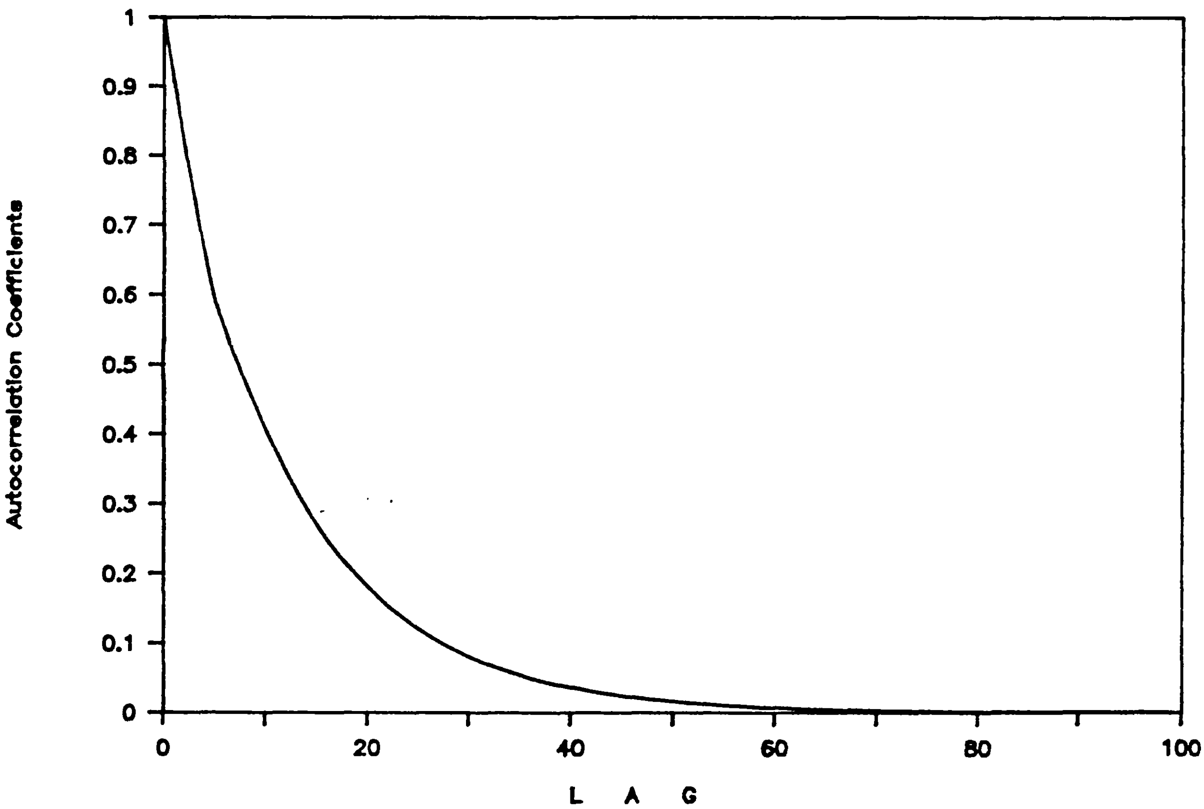


Figure (7.3) illustrates the theoretical autocorrelation functions of two AR(p)'s fitted to two replications of the output process under study. The iterative algorith described in chapter six was used for estimating the order and the autoregressive coefficients of each AR(p). The sample size n^* was 1024. We observe that both autocorrelation functions are similar to those of the AR(1) or the M/M/1 for high values of φ and τ respectively.

7.4.3 Time-shared Computer Model

The process under study was the response time of the i^{th} job. In each replication, the initial conditions were empty and idle.

The optimum performance of the methods is displayed in table (7.8). This is almost the same with the optimum performance the methods had in the M/M/1 for τ=0.2. For all the combinations of sample size and nominal confidence level, only the spectral method achieves ϵ-ideal performance for ϵ=0.025. The spectral window sizes, for which the ϵ-ideal performance is attained, are given in table (E9).

On the other hand, the performance of the methods for the estimated MB-parameter values is not satisfactory. This can be verified from table (D8) of appendix D.

Given below are the coverages, the spectral method achieves, for the spectral window sizes w=0.87n at 90% nominal confidence level and w=0.91n at 95%. For these spectral window sizes in the pilot processes, the NOBM for k=2 and spectral methods produce confidence intervals with approximately the same mean half length.

TABLE 7.8

Optimum performance of confidence interval methods in the time-shared computer model

Number of Replications : 400 , $\epsilon = 0.025$

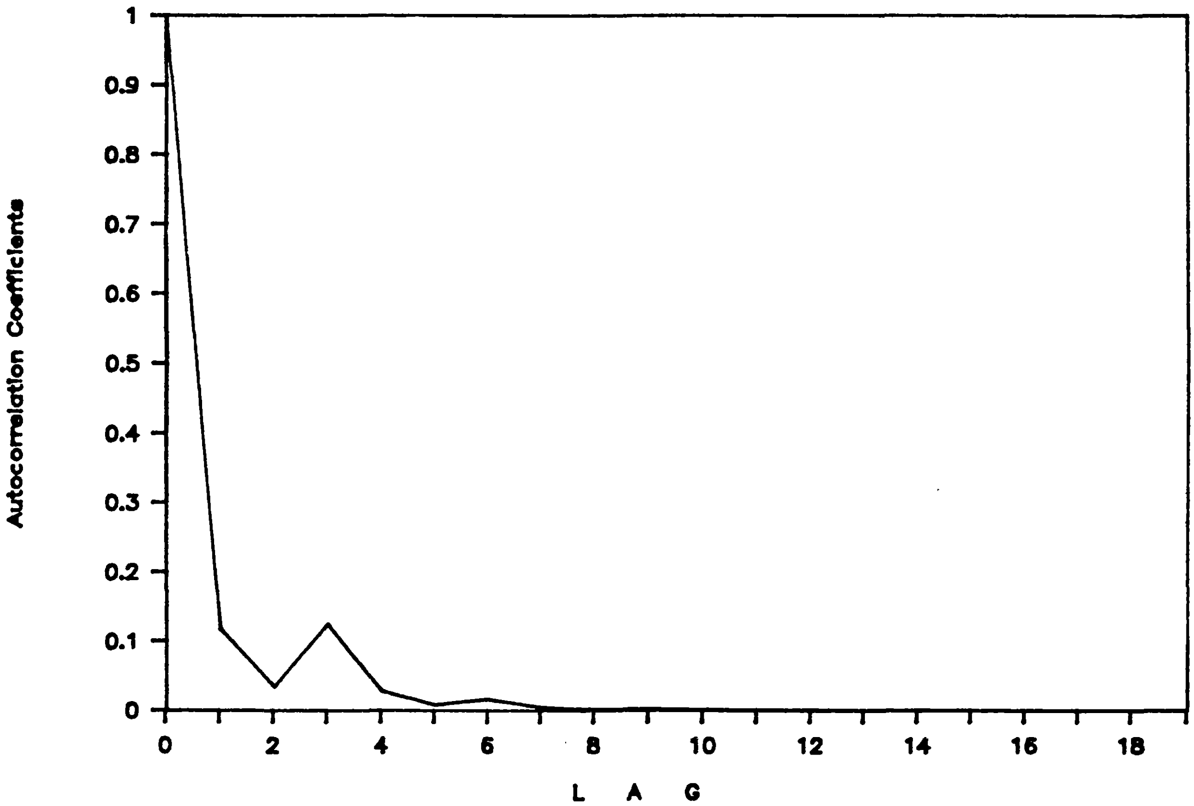
Nominal Confidence Level : 90%

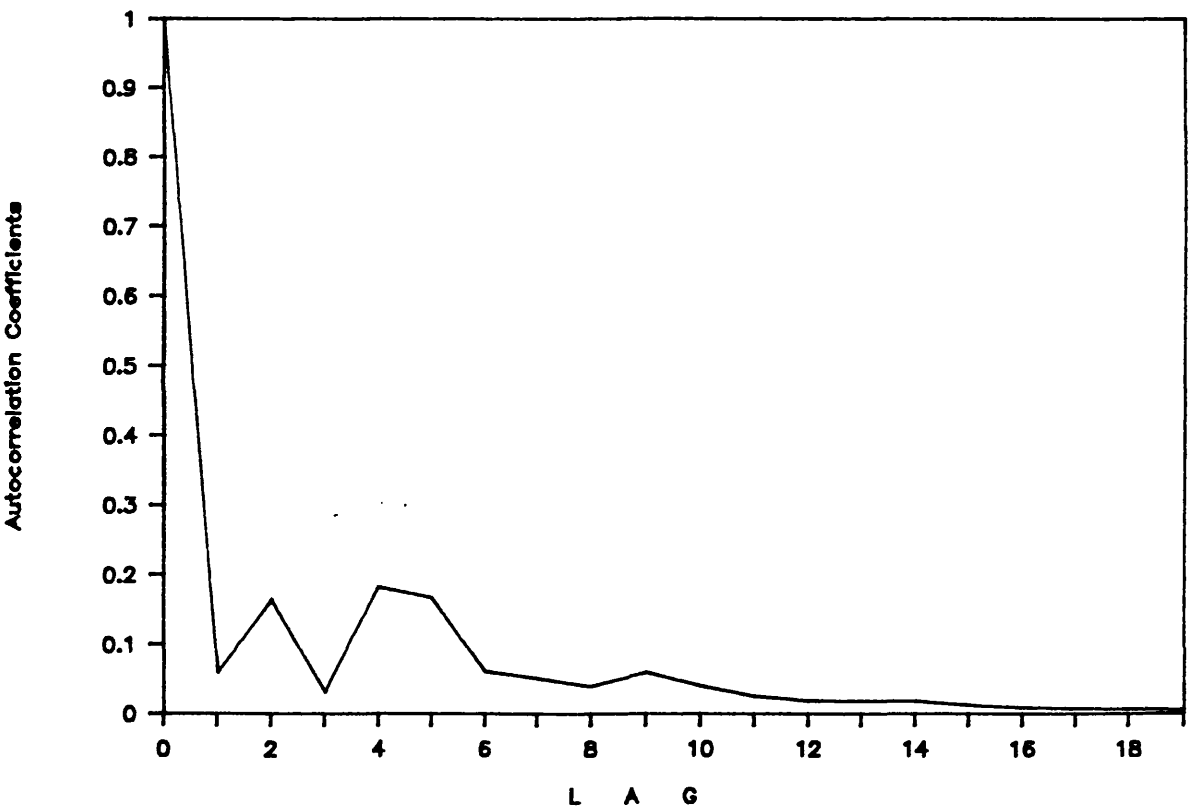
n	Methods	ϵ-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.7733	7.563	43.315
	AREA			.8300	8.384	42.497
	NOBM-AREA			.8300	8.384	42.497
	SPEC	8.459	33.575			
	OVBM			.5367	2.047	.949
128	NOBM			.8400	8.178	42.226
	AREA			.8600	8.423	45.781
	NOBM-AREA			.8600	8.423	45.781
	SPEC	7.262	24.562			
	OVBM			.5900	2.080	1.330
256	NOBM			.8567	6.943	27.787
	AREA			.8633	6.751	27.059
	NOBM-AREA			.8633	6.751	27.059
	SPEC	5.160	9.084			
	OVBM			.7000	2.100	.995
512	NOBM	5.268	16.796			
	AREA	5.365	15.654			
	NOBM-AREA	5.365	15.654			
	SPEC	4.178	5.475			
	OVBM			.7300	1.734	.529

Nominal Confidence Level : 95%

n	Methods	ϵ-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.9000	15.223	175.460
	AREA			.9100	16.873	172.150
	NOBM-AREA			.9100	16.873	172.150
	SPEC	12.418	72.851			
	OVBM			.6500	2.789	2.124
128	NOBM			.9133	16.460	171.050
	AREA			.9200	16.953	185.450
	NOBM-AREA			.9200	16.953	185.450
	SPEC	11.439	61.588			
	OVBM			.6967	2.763	2.330
256	NOBM	27.787	112.560			
	AREA			.9067	13.588	109.610
	NOBM-AREA			.9067	13.588	109.610
	SPEC	6.951	15.983			
	OVBM			.7933	2.796	1.842
512	NOBM	10.602	68.039			
	AREA	10.798	63.410			
	NOBM-AREA	10.798	63.410			
	SPEC	6.298	13.761			
	OVBM			.8033	2.132	.660

Figure 7.4
Time shared computer model: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the response time of the jobs





1-a	64	128	256	512
		SAMPLE	SIZES	
90%	.9033	.9167	.9500	.9367
95%	.9633	.9800	.9867	.9733

For any combination of n and $(1-\alpha)$, these coverages are greater than the best coverages achieved by the NOBM, OVBM, AREA and combined NOBM-AREA methods.

Figure (7.4) illustrates the theoretical autocorrelation functions of two AR(p)'s fitted to two replications of the output process under study.

7.5 APPLYING THE CONFIDENCE INTERVAL METHODS TO STEADY-STATE SIMULATION OUTPUTS

In this section, we shall make several recommendations for applying confidence interval methods to approximately steady-state simulation output processes displaying certain characteristics.

Schruben et al.'s test(1983) can be used for the elimination of initialization bias after deleting a number of early observations. This test is based on the assumption that the correlation of two observations which lie far apart in time is negligible. To apply this test, first the least biased sample mean variance estimator must be determined. This is achieved as follows:

1) Generate 5 or 10 replications of the process under study. In each replication, collect a total sample size, n^*, and delete

an arbitrary number of early observations. The remaining series must contain at least 500 observations.

2) Fit an AR(p) process to the remaining series of each replication. The autoregressive coefficients and the order of the AR(p) will be estimated using Fishman's iterative algorithm; this was discussed in chapter six.

3) In each replication, estimate first the Bias Indicator functions by using in form (6.2) the theoretical autocorrelation coefficients of the fitted AR(p). Then determine the minimum bias of each sample mean variance estimator and the parameter value for which the minimum bias is attained.

4) Compute the average absolute minimum bias of each estimator using (6.5). Determine the least biased estimator and the average parameter value where the minimum bias occurs.

Now, consider a single replication. Remove an arbitrary early part of the data such that the number of observations ,n, in the remaining series is greater than 500. Then compute the test statistic

$$\hat{T} = \sqrt{\frac{45}{n^4 \hat{\sigma}_i^2}} \sum_{j=1}^{n} \left[1 - \frac{j}{n} \right] j (X_n - X_j) \qquad (7.4)$$

where $\hat{\sigma}_i^2$ is the least biased sample mean variance estimator at the average parameter value for which the minimum bias is achieved. The test statistic value must be compared with the student-t value $t_{v,\alpha/2}$. The degrees of freedom depend on the size of n and the average parameter value. If $\hat{T} < t_{v,\alpha/2}$ then we accept the null hypothesis that the initialization bias has been removed.

Otherwise, we increase the number of observations we remove from the analysis and the total sample size, n^*, and apply the same test again. This is repeated until we achieve the acceptance of the null hypothesis.

In the last two chapters we have seen that the performance of the methods, for the parameter values for which the minimum bias of the sample mean variance estimators occurs, is satisfactory when the autocorrelation function of the process under study

_ shows a damped cyclical behaviour

_ damps down oscillating between positive and negative values

_ has small positive autocorrelation coefficients and decays to zero very fast.

The theoretical autocorrelation functions of the fitted AR(p)'s to 5 or 10 replications can be used as a tool for studying the autocorrelation structure of the process under consideration.

Let us assume that the autocorrelation function of the process under study displays one of the above three forms. For such cases, we have found that the spectral and the overlapping batch means methods are superior to the nonoverlapping batch means, area and combined area-nonoverlapping batch means methods. For each replication, the confidence interval will be built using the estimated spectral window sizes or the batch sizes for which the minimum bias of the corresponding sample mean variance estimator is achieved. These sizes will be determined in each replication from form (6.3). The Bias Indicator functions will be estimated, by using the theoretical autocorrelation coefficients of the fitted AR(p) instead of the corresponding sample autocorrelations in form (6.2).

Now, consider autocorrelation functions which cannot be classified into one of the above three forms. In such cases, the spectral method must be chosen. The confidence intervals will be built for spectral window sizes w=0.87n if we use 90% nominal confidence level or w=0.91n if we use 95%. We have seen such autocorrelation functions in the output processes of the M/M/1 for high τ, the interactive computer model and the time shared-computer model. In these models, we have seen that the spectral method for w=0.87n or w=0.91n achieves greater coverages than the best coverages produced by the other four methods.

Steelworks: A case study

Steelworks is a real-life simulation model which was developed in the Computer Aided Simulation Modelling(CASM) environment. The operational rules of this model are described in Balmer and Paul(1985). The listing of the simulation program is given in appendix A. Three processes have been selected; the total wastage, the waiting time in queue in front of the crane and the response time of the j^{th} torpedoe. The system consists of the following entities; two blastfurnaces, ten torpedoes, one crane and five steelfurnaces.

Ten replications were generated for each process. The first ℓ observations were deleted in each replication. ℓ was 2000 for the total wastage and waiting time and 500 for the response time. An AR(p) process was fitted to the remaining series of each replication. The size of the remaining series was 512 for all the replications. The average minimum bias and the average parameter value, for which the minimum bias of each estimator occured in each process, were the following:

Output Process	NOBM	AREA	NOBM-AREA	SPEC	OVBM
Wastage	0.0107 (2)	0.0107 (1)	0.0107 (1)	0.0039 (50)	0.0423 (87)
Waiting Time	0.0187 (2)	0.0187 (1)	0.0187 (1)	0.0034 (56)	0.0661 (104)
Response Time	0.0258 (2)	0.0259 (1)	0.0259 (1)	0.00000 (84)	0.0001 (146)

We can see that the estimator of the spectral method is the least biased estimator.

Ten additional replications were generated for each process. An early part of data in each replication was removed from the analysis. We applied Schruben et al.'s test to the remaining series of each replication. The size of the remaining series was 512 for all the replications. The estimator of the spectral method was used in form (7.4). The degrees of freedom, v, were (1.33x512)/w where w was the estimated average spectral window for which the minimum bias of the spectral estimator occured in each process. For each replication, we give below the number of the early observations, ℓ, which was removed from the analysis, the test statistic values and the critical values at 0.025 level of significance.

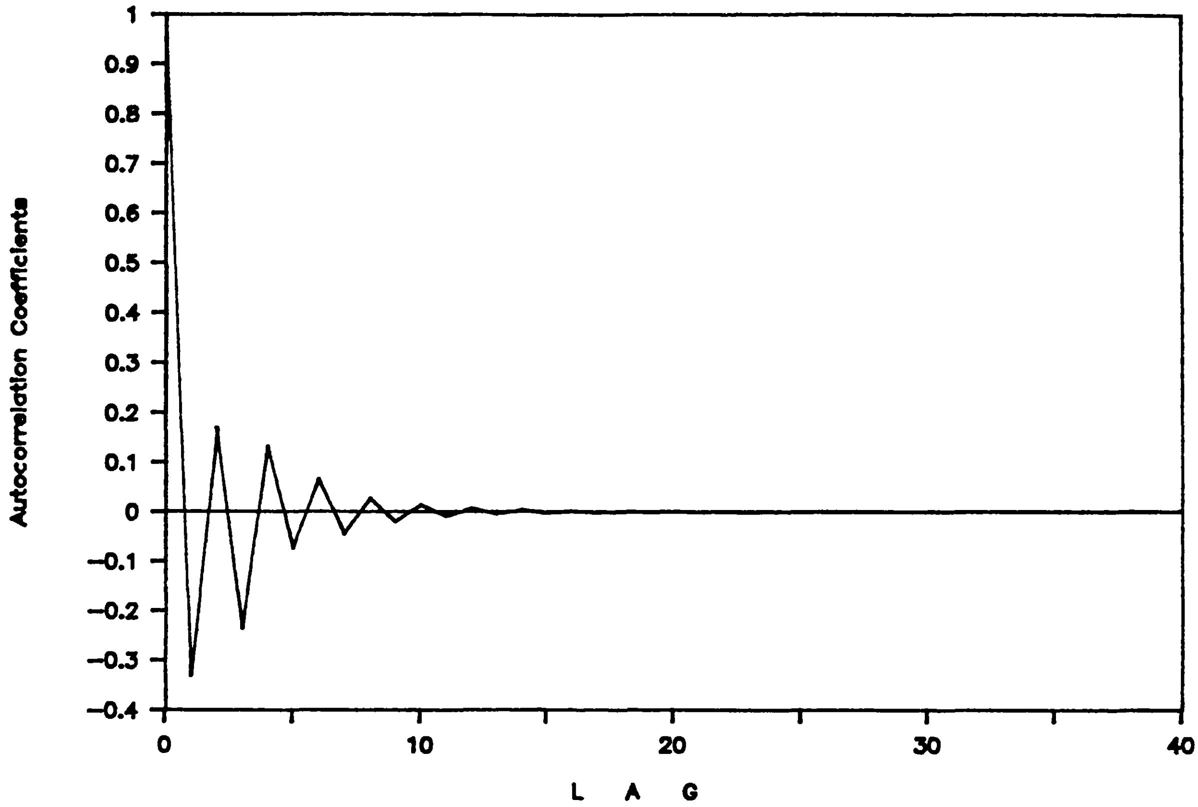
Wastage			Waiting Time			Response time		
ℓ	$\hat{T}$	$t_{v,\alpha/2}$	ℓ	$\hat{T}$	$t_{v,\alpha/2}$	ℓ	$\hat{T}$	$t_{v,\alpha/2}$
2000	1.856	2.150	2000	1.334	2.192	500	1.314	2.314
2000	0.021	2.150	2000	2.572	2.192	500	0.062	2.314
2000	0.474	2.150	2000	1.936	2.192	500	0.440	2.314
2000	1.470	2.150	2000	0.490	2.192	500	1.972	2.314
2000	1.946	2.150	2000	0.092	2.192	500	1.090	2.314
2000	0.626	2.150	2000	0.330	2.192	500	1.332	2.314
2000	0.348	2.150	2000	0.960	2.192	500	1.126	2.314
2000	0.294	2.150	2000	0.634	2.192	500	1.403	2.314
2000	0.120	2.150	2000	0.303	2.192	500	1.436	2.314
2000	0.537	2.150	2000	0.091	2.192	500	1.232	2.314

The null hypothesis is rejected only in the second replication of the waiting time.

Figures (7.5), (7.6) and (7.7) illustrate theoretical autocorrelation functions of AR(p)'s fitted to replications of the three output processes. The autocorrelation function for the total wastage damps down oscillating between positive and negative values. On the other hand, negative autocorrelation coefficients dominate in the autocorrelation function of the waiting time. Table (7.9) contains the lower and upper limits of the five confidence intervals for the true steady-state average wastage and average waiting time. In each replication, the confidence intervals were build for the estimated parameter values for which the minimum bias occured. The nominal confidence level is 90%.

Table (7.10) presents the lower and upper limits of confidence intervals for the steady-state average response time of torpedoes. The spectral and nonoverlapping batch means methods were used. The confidence intervals of the spectral method were constructed for spectral window sizes w=0.87x512. For the nonoverlapping batch means method, we used two batches in each replication. The nominal confidence level was 90% for both methods.

Figure 7.5
Steelworks: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the total wastage



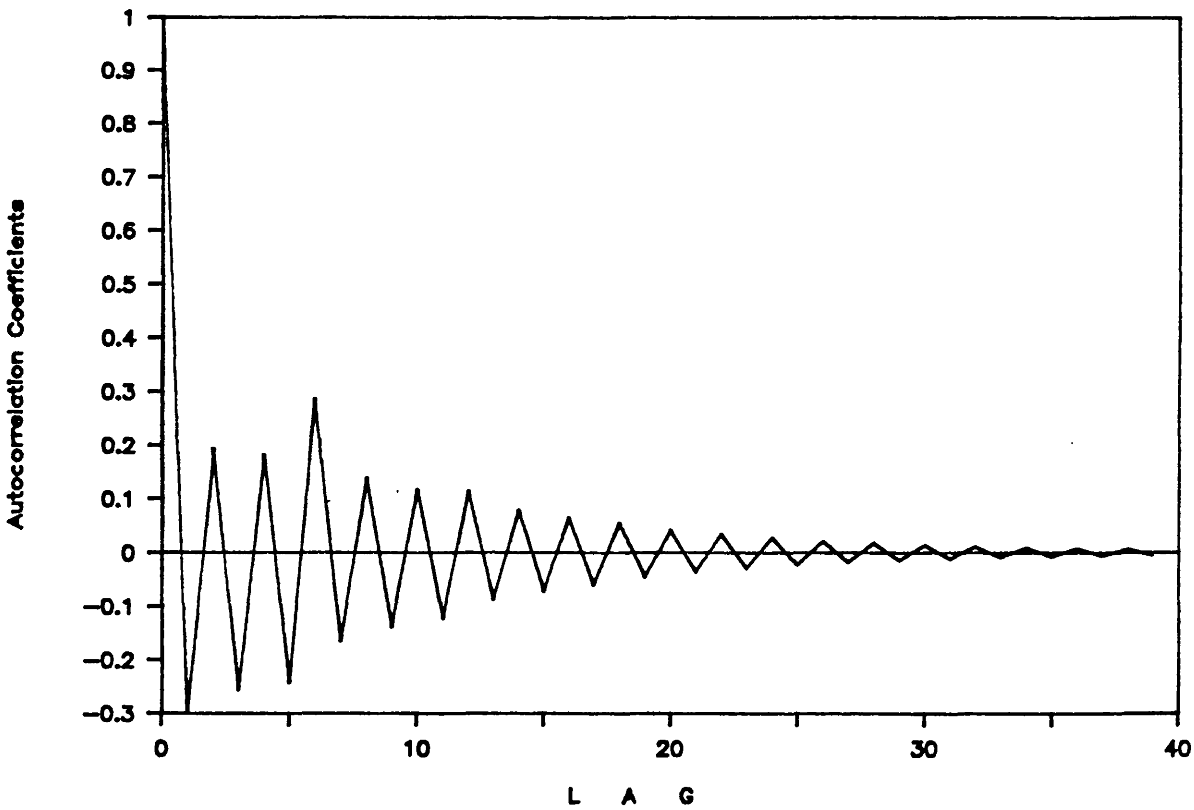
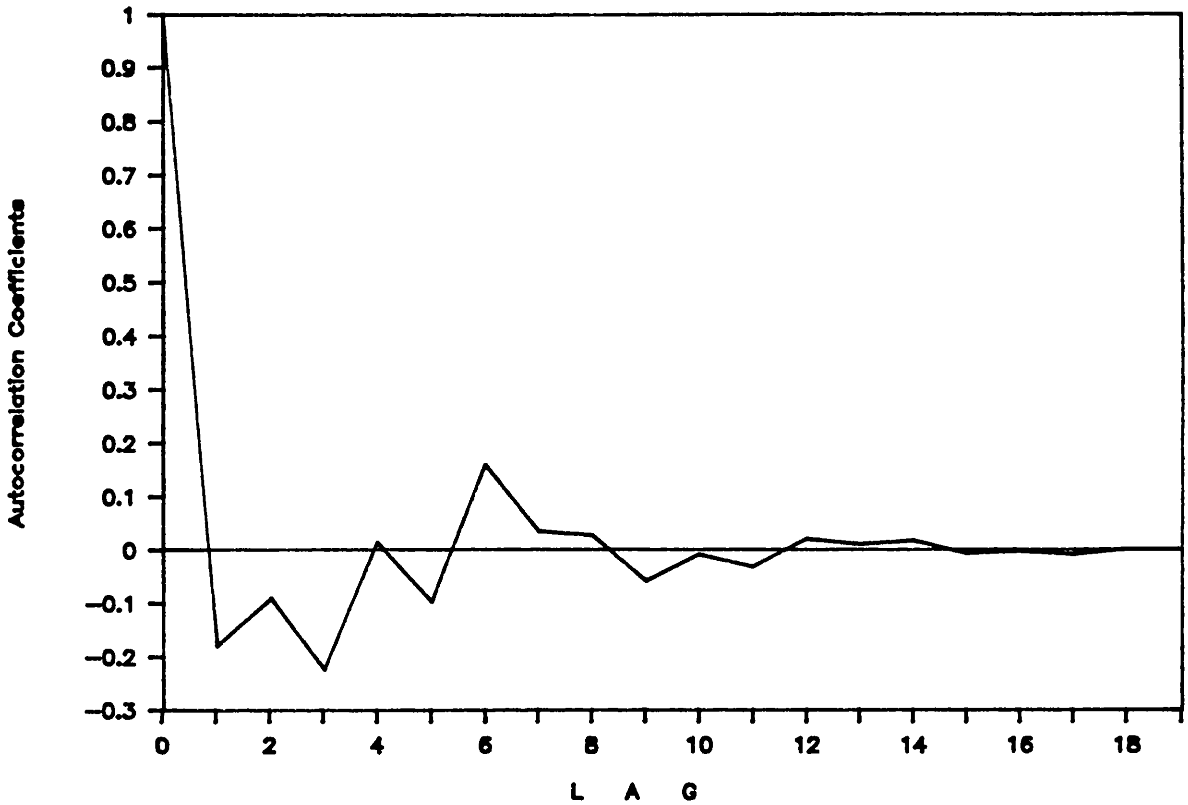


Figure 7.6
Steelworks: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the waiting time in front of the crane



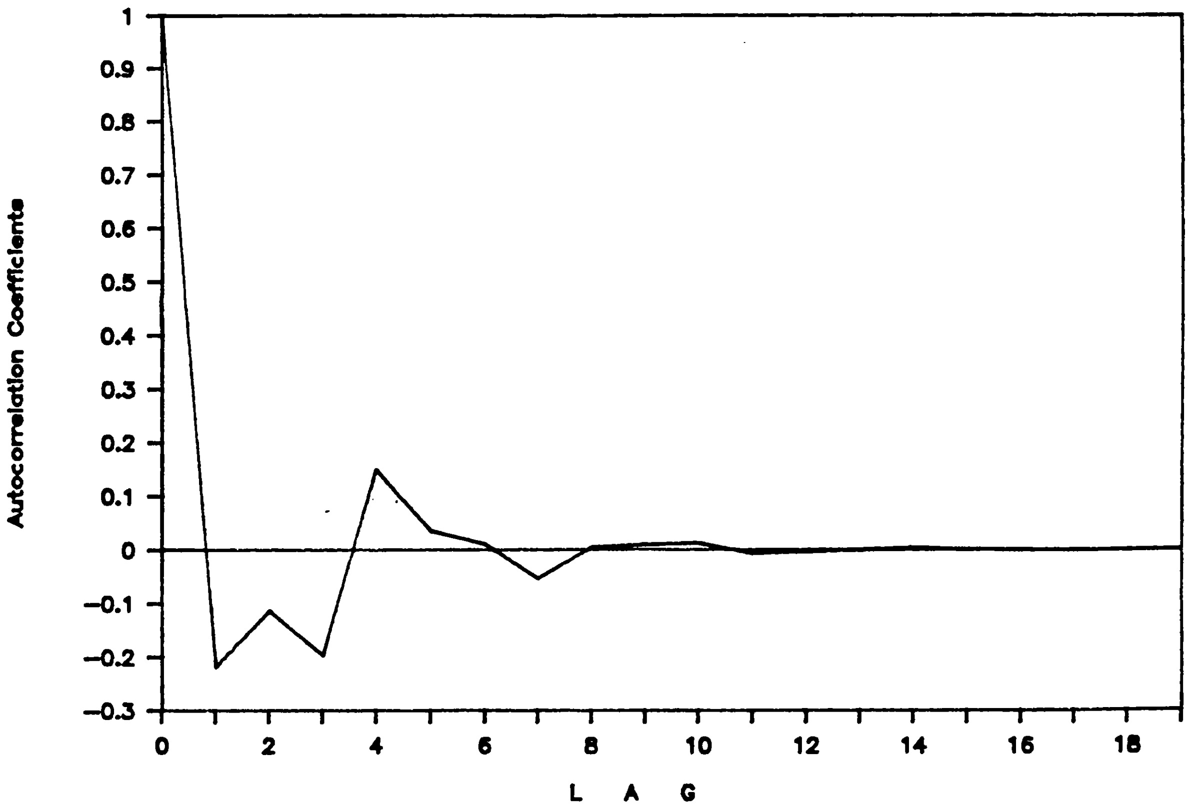
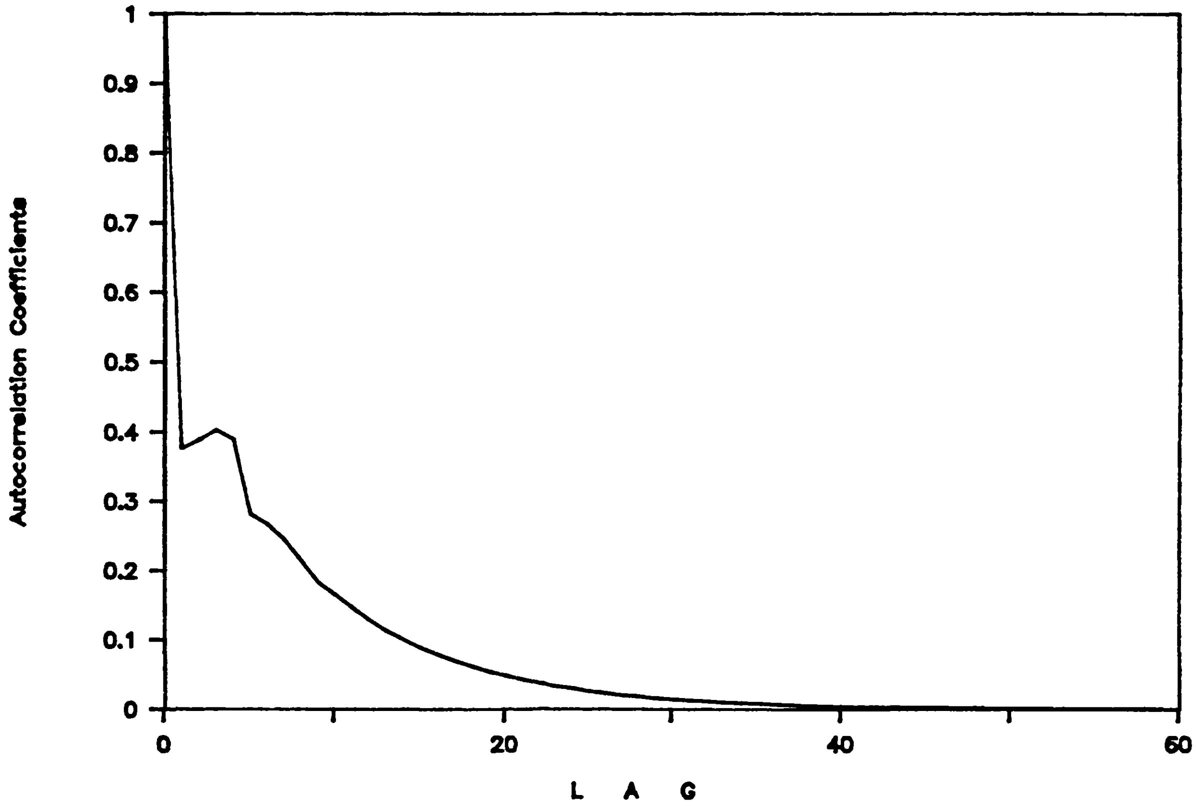


Figure 7.7
Steelworks: Theoretical autocorrelation functions of fitted AR(p)'s to replications of the response time of torpedoes



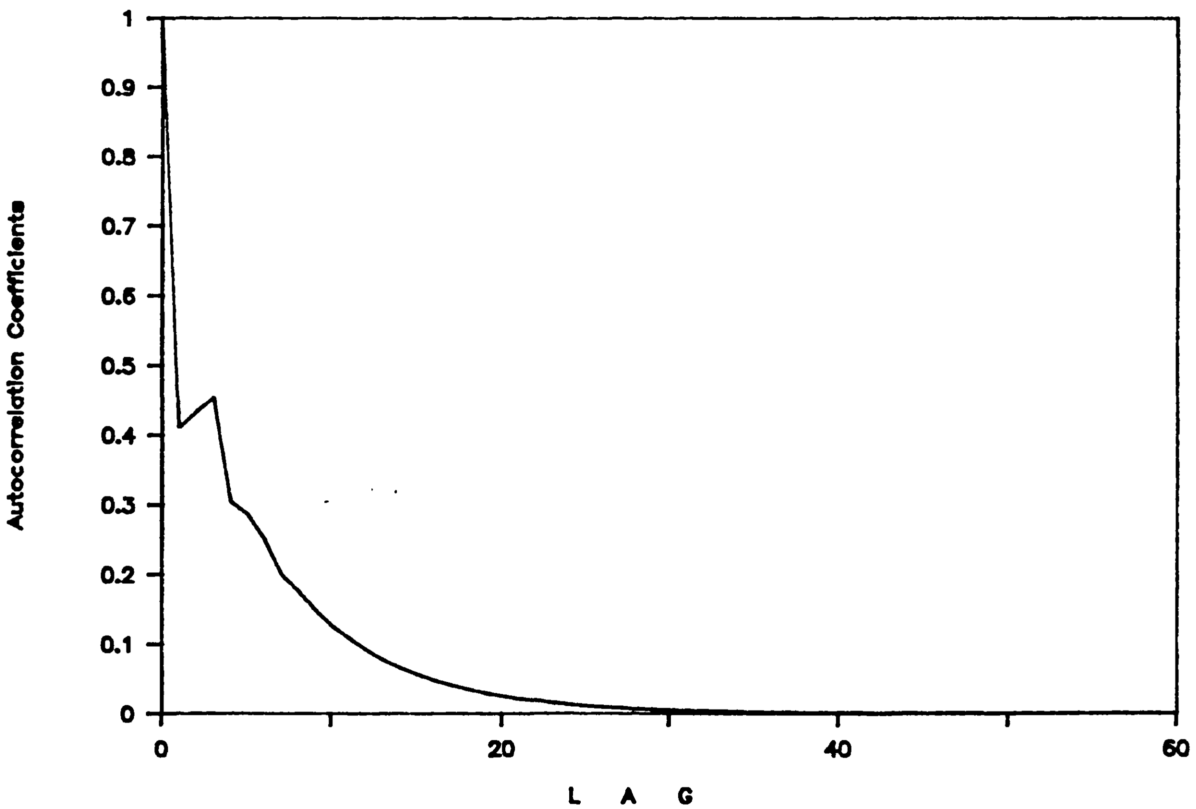


TABLE 7.9
Steelworks: Confidence intervals for the true steady-state average wastage and waiting time in front of the crane

Output Process : Wastage

NOBM	AREA	SPEC	OVBM
[49.87,98.42]	[40.86,107.4]	[69.46,78.84]	[69.22,79.07]
[74.57,75.14]	[74.27,75.44]	[72.10,77.62]	[72.28,77.43]
[70.06,82.90]	[66.53,86.43]	[70.71,82.24]	[71.00,81.96]
[57.03,100.2]	[36.61,120.6]	[72.49,84.77]	[72.32,84.94]
[56.47,103.2]	[41.63,118.0]	[73.86,85.79]	[74.12,85.53]
[65.69,88.11]	[65.23,88.58]	[73.16,80.65]	[72.33,81.13]
[76.76,78.61]	[70.55,84.82]	[73.72,81.65]	[73.57,81.80]
[73.01,73.17]	[68.82,77.36]	[68.83,77.34]	[68.33,77.84]
[67.02,84.78]	[73.79,78.01]	[71.53,80.27]	[72.21,79.59]
[62.09,93.12]	[74.20,81.01]	[72.69,82.52]	[72.87,82.35]

Output Process : Waiting time infront of crane

NOBM	AREA	SPEC	OVBM
[64.25,95.58]	[65.64,94.18]	[76.92,82.90]	[76.34,83.48]
[62.86,138.7]	[63.06,138.5]	[95.89,105.7]	[95.16,106.4]
[79.72,130.1]	[79.78,130.1]	[101.1,108.8]	[101.0,108.9]
[93.89,115.7]	[99.21,110.4]	[98.70,110.9]	[98.60,111.0]
[96.21,116.4]	[96.67,115.9]	[99.84,112.7]	[99.57,113.0]
[105.2,115.9]	[99.87,121.2]	[105.9,115.2]	[106.3,114.8]
[72.12,145.6]	[91.47,126.2]	[100.5,117.2]	[100.4,117.3]
[91.04,123.5]	[82.53,132.0]	[100.3,114.2]	[99.27,115.3]
[88.74,121.0]	[95.38,114.4]	[99.39,110.4]	[98.04,111.8]
[93.65,118.1]	[103.6,108.1]	[101.5,110.2]	[102.0,109.7]

TABLE 7.10
Steelworks: Confidence intervals for the true steady-state average response time of the torpedoes

NOBM	SPEC
[313.9,441.8]	[333.1,422.7]
[381.2,381.6]	[372.5,390.4]
[316.3,427.7]	[325.7,418.3]
[316.6,452.6]	[335.3,433.9]
[336.5,396.5]	[330.3,402.6]
[379.3,385.8]	[359.8,405.4]
[351.6,394.4]	[348.1,397.1]
[313.4,441.9]	[328.9,426.4]
[290.9,456.9]	[316.6,431.2]
[352.7,401.3]	[328.5,425.5]

CHAPTER EIGHT

GENERAL CONCLUSIONS - FUTURE RESEARCH

In steady state simulation output processes, autocorrelations at different lags is a very common phenomenon. In this case, the classical confidence interval estimator for true steady state means is not valid. The actual confidence levels(coverages) this estimator achieves are different to the target or nominal confidence levels. Particularly, for the AR(1) we have shown that for a given sample size n the greater the positive value of the autoregressive coefficient φ the lower the coverage achieved. Furthermore, for a given φ, the coverage decreases as n increases.

For the last two decades alternative estimators have been developed for the variance of the sample mean. These estimators are intended for stationary autocorrelated processes. The sample mean variance estimators produce corresponding confidence interval estimators for steady state means. A sample mean variance estimator together with a confidence interval estimator constitute a confidence interval method. The evaluation of these methods takes place in the context of appropriate testing environments. In chapter two, we have described the structure of such environments.

Two crucial questions arise at the stage of testing the performance of the confidence interval methods. Firstly, can some testing environments identify best methods in regard to certain contexts? Secondly, do the testing environments provide specific rules for applying the methods to simulation output processes displaying certain characteristics? We have shown in chapter two that in the context of the past testing environments for the above two questions the answers are limited. The main cause is the nonhomogeneity of the testing environments. "Nonhomogeneity" of the environments means that different methods have been tested on different processes under different combinations of sample sizes, parameter values and nominal confidence levels.

Five confidence interval methods have been considered in our research; Nonoverlapping batch means(NOBM), overlapping batch means(OVBM), area, combined NOBM-AREA and spectral(SPEC). We have compared the performance of these methods in regard to the following contexts:-

i) the minimum bias of the corresponding sample mean variance estimators for small sample sizes

ii) the asymptotic coverages the methods attain for different parameter values

iii) the asymptotic expected values and variances of the confidence interval half lengths, providing that the methods cover the steady state mean with the nominal probability

iv) the performance of the methods for the parameter values for which the minimum bias of the corresponding sample mean variance estimators occurs

v) the optimum performance of the methods, that is, the performance for the parameter values for which each method achieves the nearest coverage to the nominal confidence level.

Furthermore, we have provided specific recommendations for applying the five methods to steady state simulation output processes displaying certain characteristics. These characteristics refer to the level of non-normality and the shape of the autocorrelation function. In the remaining part of this chapter, we summarize these recommendations as well as the conclusions we have drawn for the performance of the methods in the above five contexts.

In chapter three we have derived analytic forms for specific functions called "Bias Indicator" functions. They enable us to compute analytically both the minimum bias of each sample mean variance estimator and the parameter values, called MB-parameter values, for which the minimum bias occurs. These functions have been expressed in terms of the theoretical autocorrelation coefficients of the output process under study. When these coefficients are known, exact analytical results can be obtained for the minimum bias and the MB-parameter values.

We have compared the minimum bias of the five sample mean variance estimators in the stationary AR(1). Both positive and negative autoregressive coefficients φ have been considered. For φ positive, the autocorrelation function decays exponentially to zero. For negative φ, this function damps down oscillating between positive and negative values. When φ is positive, we have found that the SPEC estimator has the smallest minimum bias in small samples. On the other hand, when φ is negative, the smallest minimum bias is achieved by the NOBM estimator for small sample sizes.

We think that the minimum bias of the sample mean variance estimators should be explored on further stationary processes whose autocorrelation functions are known. There is a wide variety of time series processes for which the theoretical autocorrelation coefficients can be computed analytically. These processes could be classified into several categories according to the shape of their autocorrelation functions. Then the minimum bias of the estimators would be studied and compared in the different categories. For example, in a recent paper, Kevork and Balmer(1990) have studied the minimum bias in two other stationary processes additional to the AR(1); the delay in queue in the M/M/1 and the AR(2). For the AR(2), the autocorrelation function displayed a damped cyclical behaviour. For both the M/M/1 and AR(2), the authors have found that the SPEC estimator achieves the smallest minimum bias in small sample sizes.

The asymptotic forms of the Bias Indicator functions enable us to compute analytically the limiting coverages the five confidence interval methods achieve for different parameter values on processes satisfying certain regularity conditions. The limiting coverages are obtained when the sample size n tends to infinity. We have shown that for equal batch sizes, the NOBM and OVBM methods attain the same limiting coverages. For the AR(1), AR(2) and delay in queue in the M/M/1, we have reported that by increasing the batch size m, the limiting coverages of the NOBM method tend to achieve the nominal confidence level faster than those of the AREA method. Additional to this, we state a few other interesting asymptotic results.

i) In the AR(1), AR(2) and M/M/1, we have observed that the limiting coverages of the SPEC method tend to attain the nominal confidence level rather fast.

ii) For the AR(2) under specific autoregressive coefficients, we have seen that the combined NOBM-AREA and SPEC methods attain acceptable limiting coverages for small batch sizes and spectral window sizes respectively.

iii) For any process considered from the AR(1), AR(2) and M/M/1, we have found that the AREA method requires a batch size approximately three times more than the NOBM such that the limiting coverages of the two methods differ to the nominal confidence level by a small positive number ϵ. On the other hand, the combined NOBM-AREA requires a batch size approximately two times more than the NOBM.

Assuming that the simulation output process satisfies certain regularity conditions, as the batch size tends to infinity, the NOBM, AREA and combined NOBM-AREA methods tend to achieve coverages equal to the nominal confidence level. Moreover, for the SPEC and OVBM methods, as the batch size m and the spectral window size w tend to infinity but in such a way that $n/m \to \infty$ and $n/w \to \infty$, these two methods tend to attain the desired coverages. Under these asymptotic situations, for the latter two methods we have derived limiting forms for the expected values and variances of the confidence interval half lengths. As n, k(number of batches), m, w tend to infinity but in such a way that $(n/k) \to \infty$, $(n/m) \to \infty$, $(n/w) \to \infty$, we have shown that the five confidence interval methods produce confidence intervals with the same half length. On the other hand, we have seen that such a general conclusion cannot be stated for the limiting variance of the confidence interval half lengths.

For the case of small sample sizes, firstly we have compared the performance of the five confidence interval methods at the MB-parameter values i.e the parameter values for which the minimum bias of the corresponding sample mean variance estimators occurs. Processes from several simulation models have been selected. The coverage and the expected values and variances of the confidence interval half lengths each method achieves have been estimated by using Monte Carlo methods. For the output processes for which the autocorrelation functions were unknown, the true MB-parameter could not be determined. We estimated these values by following a certain estimation procedure. That is, the MB-parameter values were determined by the Bias Indicator functions. These functions had been expressed in terms of the theoretical autocorrelation coefficients of the output process under study. In each replication from the processes we selected, these coefficients were replaced by the theoretical autocorrelation coefficients of the fitted AR(p). The algorithm for fitting AR(p) processes to approximately steady state simulation outputs was given in chapter six.

At the MB-parameter values, all the methods have achieved acceptable coverages in the processes whose autocorrelation functions

i) have low positive coefficients and decay to zero fast

ii) damp down oscillating between positive and negative values

iii) display a damped cyclical behaviour.

However, we have found that the SPEC and OVBM are superior to the other three methods in terms of the expected values and variances of the confidence interval half lengths. This means that the SPEC and OVBM methods should be prefered for constructing confidence intervals in processes characterized by autocorrelation functions displaying the above three characteristics. More specifically, given the sample size in any replication, we recommend the confidence interval to be built at the estimated batch sizes or spectral window sizes for which the minimum bias of the sample mean variance estimators is attained. In section 7.5, we have provided an example illustrating how the confidence intervals can be constructed at the estimated MB-parameter values. Furthermore, for eliminating the effects of non-normality, we suggest the analysis to be carried out in sample sizes greater than 300 or even 500 observations.

On the other hand, none of the methods have performed satisfactorily on simulation output processes characterized by autocorrelation functions which have high coefficients at the early lags and decay to zero slowly. Particularly, for small sample sizes, all the methods have achieved coverages significantly lower than the desired nominal confidence levels.

The performance of the methods on the minimum bias has been compared with their corresponding optimum performance. The optimum performance is attained for those parameter values for which each method achieves the nearest coverages to the nominal confidence level. We called these coverages "Best coverages". First, we considered the processes for which all the methods achieved satisfactory performance on the minimum bias. For these processes, the SPEC and OVBM methods attained better optimum performance than the other three methods in terms of the expected values and variances of the confidence interval half lengths.

On the other hand, interesting remarks have been made for the processes whose autocorrelation functions have high positive coefficients at the early lags and decay slowly to zero. For any small sample size n considered, there were spectral window sizes w for which the SPEC method achieved coverages which lied very close to the nominal confidence levels $(1-\alpha)$. At the same sample sizes, all the other methods produced coverages which were significantly lower than $(1-\alpha)$. Empirically, we have found that by choosing $w=0.87n$ for $(1-\alpha)=0.90$ or $w=0.91n$ for $(1-\alpha)=0.95$, the SPEC method achieves the greatest coverages of all the five methods.

For the above specific spectral window sizes, if the confidence intervals have larger width than the desired one, we can reduce it by increasing the sample size. Based on these values of w, we think that a sequential method can be developed for generating confidence intervals in processes whose autocorrelation functions have high positive coefficients in the early lags and decay slowly to zero. On the other hand, for the processes for which the performance of the methods on the minimum bias is satisfactory, any sequential method could be based on the estimated MB-parameter values. For large n, fast Fourier transforms can reduce the total computing time of the sample autocorrelations. However, the question is how large n must be relatively to the required n of other sequential methods such that we obtain an acceptable confidence interval.

In the introductory chapter, we have also discussed two other methods which have been tested on a very limited number of simulation processes; the autoregressive and the spectral based on the periodogram coordinates. We believe that the performance of these methods should be studied in regard to the five contexts described above and compared with the performance of the five methods considered in this research.

Finally, we think that a computer software support system should be developed for helping the simulation researcher in selecting the right method for steady state simulation output processes displaying specific characteristics.

A P P E N D I X   A

PROGRAM LISTINGS FOR SIMULATION MODELS

PROGRAM LISTING FOR THE M/M/1 QUEUEING MODEL

```
[INHERIT('DISKB:[KEVORK]ENTITIES1.PEN','DISKB:[KEVORK]
QUEUES1.PEN','DISKB:[KEVORK]NEW_SAMPLING.PEN',
'DISKB:[KEVORK]STAT_LIBRARY.PEN')]

PROGRAM  MM1(INPUT,OUTPUT);
VAR
  DUR, COUNT, COUNT1, FIRST_SEED, SECOND_SEED      : INTEGER;
  RUN_IN_PERIOD, SERVICE_TIME, ARRIVAL_TIME        : REAL;
  INTARRIV_RATE, ARV_RATE, INTSERV_TIME, SERV_RATE : REAL;
  RESTART, SERVER, SPARE_SERVER, CUSTOMER     : ENAME;
  SPARE_CUSTOMER, DOOR    : ENAME;
  CWAIT, CQUEUE, SIDLE    : QUEUE;
  DEL, START_TIME : RGENER_ARRAY;


PROCEDURE GO_THRU_C_EVENTS;FORWARD;



PROCEDURE CREATE_RECORDING;
BEGIN
   COUNT:=1;
   COUNT1:=1;
END;



PROCEDURE STARTUP;
BEGIN
   CREATE_RECORDING;
   THEREARE(200,CUSTOMER,'CUSTOMER');
   THEREARE(1,DOOR,'DOOR');
   THEREARE(1,SERVER,'SERVER');
   MAKEQ(SIDLE,'SIDLE');
   MAKEQ(CWAIT,'CWAIT');
   MAKEQ(CQUEUE,'CQUEUE');
   SPARE_SERVER := SERVER;
   FOR J := 1 TO 1 DO
   BEGIN
     ADDTO(BACK,SIDLE,SPARE_SERVER);
     SPARE_SERVER := SPARE_SERVER^.NEXT;
   END;
   SPARE_CUSTOMER := CUSTOMER;
   FOR J := 1 TO 200 DO
   BEGIN
     ADDTO(BACK,CWAIT,SPARE_CUSTOMER);
     SPARE_CUSTOMER := SPARE_CUSTOMER^.NEXT;
   END;
   GO_THRU_C_EVENTS;
END;
```

```
PROCEDURE C1;  (* ARRIVAL *)
BEGIN
  IF QSIZE(CWAIT)=0 THEN
  WRITELN('WARNING===> CWAIT QUEUE IS EMPTY');
  WHILE (QSIZE(CWAIT) >= 1)
    AND (DOOR^.AVAIL)
  DO BEGIN
       SPARE_CUSTOMER:=HEAD(CWAIT);
       SPARE_CUSTOMER^.ATTR:=COUNT;
       INTARRIV_TIME:=1/ARV_RATE;
       ARRIVAL_TIME := NEGEXP(INTARRIV_TIME,FIRST_SEED);
       CAUSE( 1,BEHEAD(CWAIT),ARRIVAL_TIME);
       CAUSE( 2,DOOR,ARRIVAL_TIME);
       COUNT:=COUNT+1;
     END;     (* of while loop *)
END;   (* of procedure c1 *)


PROCEDURE C2;  (* SERVICE *)
BEGIN
  WHILE (QSIZE(CQUEUE) >= 1)
    AND (QSIZE(SIDLE) >= 1)
  DO BEGIN
       SPARE_CUSTOMER:=HEAD(CQUEUE);
       SPARE_SERVER:=HEAD(SIDLE);
       DEL[SPARE_CUSTOMER^.ATTR]:=TIM-
       START_TIME[SPARE_CUSTOMER^.ATTR];
       INTSERV_TIME:=1/SERV_RATE;
       SERVICE_TIME := NEGEXP(INTSERV_TIME,SECOND_SEED);
       CAUSE( 3,BEHEAD(CQUEUE),SERVICE_TIME);
       CAUSE( 4,BEHEAD(SIDLE),SERVICE_TIME);
     END;     (* of while loop *)
END;   (* of procedure c2 *)


PROCEDURE B1;          (* CUSTOMER ENDS ARRIVAL *)
BEGIN
  START_TIME[CUR_NO_ENT]:=TIM;
  ADDTO(BACK,CQUEUE,CURRENT);
END;


PROCEDURE B2;          (* DOOR ENDS ARRIVAL *)
BEGIN
END;


PROCEDURE B3;          (* CUSTOMER ENDS SERVICE *)
BEGIN
  ADDTO(BACK,CWAIT,CURRENT);
  COUNT1:=COUNT1+1;
END;
```

```
PROCEDURE B4;            (* SERVER ENDS SERVICE *)
BEGIN
  ADDTO(BACK,SIDLE,CURRENT);
END;

PROCEDURE EndRunin;
BEGIN
  CREATE_RECORDING;
END;

PROCEDURE CALL_FOR_NEXT_B_EVENT;
BEGIN
  CASE NO_NEXTB OF
        1 : B1;
        2 : B2;
        3 : B3;
        4 : B4;
      127 : EndRunin;
  END;
END;

PROCEDURE GO_THRU_C_EVENTS;
VAR CFLAG,CNUM : INTEGER;
BEGIN
  CNUM := 2;
  FOR CFLAG:=1 TO CNUM DO
       CASE CFLAG OF
         1 : C1;
         2 : C2;
       END;
END;

PROCEDURE EXECUTE(DUR:INTEGER);
BEGIN
  STARTUP;
  RUNNING := TRUE;
  WHILE RUNNING DO
  BEGIN
     TIM := NEXT_TIME ;
     IF DUR>=COUNT1 THEN
     BEGIN
       WHILE (TIM = NEXT_TIME) AND (RUNNING) DO
       BEGIN
         GET_NEXT_ENTITY;
         CALL_FOR_NEXT_B_EVENT;
       END;
       IF RUNNING THEN
           GO_THRU_C_EVENTS
     END ELSE
          BEGIN
              RUNNING:=FALSE;
          END;
   END;
   INITCALL;
END;
```

```
BEGIN    (* main program *)
  INITCALL;
  WRITE('SELECT AN INTEGER FROM 1 UP TO 1436 =====>');
  READLN(FIRST_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('THE ONE YOU HAVE SELECTED BEFORE =====>');
  READLN(SECOND_SEED);
  WRITE('GIVE THE ARRIVAL RATE =====>');READLN(ARV_RATE);
  WRITE('GIVE THE SERVICE RATE =====>');READLN(SERV_RATE);
  WRITE('GIVE THE NUMBER OF CUSTOMERS COMPLETING
         SERVICE =====>');READLN(DUR);
  EXECUTE(DUR);
END.
```

PROGRAM LISTING FOR THE INVENTORY MODEL

```
[INHERIT('DISKB:[KEVORK]ENTITIES1.PEN','DISKB:[KEVORK]
QUEUES1.PEN','DISKB:[KEVORK]NEW_SAMPLING.PEN',
'DISKB:[KEVORK]STAT_LIBRARY.PEN')]


   PROGRAM INVENTORY(INPUT,OUTPUT);


   VAR
     FIRST_SEED, SS, SM, K, C, H, PI, II, DUR : INTEGER;
     XX, YY, COST : REAL_ARRAY;
     DEMAND,ORDER : REAL;


   BEGIN
     INITCALL;
     WRITE('SELECT AN INTEGER FROM 1 UP TO 1436 =====>');
     READLN(FIRST_SEED);
     WRITELN('GIVE THE NUMBER OF DAYS FOR WHICH THE');
     WRITE('TOTAL COST IS BEING COMPUTED =====>');
     READLN(DUR);
     SS:=57;
     SM:=17;
     K:=32;
     C:=3;
     H:=1;
     PI:=5;
     II:=1;
     XX[II]:=SS;
     REPEAT
       IF XX[II]<SM THEN
         BEGIN
           ORDER:=SS-XX[II];
           COST[II]:=COST[II]+K+C*ORDER;
           YY[II]:=XX[II]+ORDER;
         END
       ELSE
           YY[II]:=XX[II];
       DEMAND:=POISSON(25,FIRST_SEED);
       IF (YY[II]-DEMAND)>=0 THEN
         COST[II]:=COST[II]+H*(YY[II]-DEMAND)
       ELSE
         COST[II]:=COST[II]+PI*(DEMAND-YY[II]);
       XX[II+1]:=YY[II]-DEMAND;
       II:=II+1;
     UNTIL (II=DUR+1);
   END.
```

PROGRAM LISTING FOR THE INTERACTIVE COMPUTER MODEL

```
[INHERIT('DISKB:[KEVORK]ENTITIES1.PEN','DISKB:[KEVORK]
QUEUES1.PEN','DISKB:[KEVORK]NEW_SAMPLING.PEN',
'DISKB:[KEVORK]STAT_LIBRARY.PEN')]

PROGRAM INTER(INPUT,OUTPUT);
VAR
    K, DUR, COUNT1, COUNT2, COUNT3, COUNT4, COUNT5  : INTEGER;
    COUNT6, COUNT7, COUNT8, COUNT9, COUNT10, COUNT  : INTEGER;
    FIRST_SEED, SECOND_SEED, THIRD_SEED, FOURTH_SEED: INTEGER;
    FIFTH_SEED, SIXTH_SEED, SEVENTH_SEED            : INTEGER;
    STORE1_TIME, STORE2_TIME, STORE3_TIME, STORE4_TIME : REAL;
    RUN_IN_PERIOD, FORM_TIME, PROCESS_TIME : REAL;
    TEST : ARRAY [1..99999] OF BOOLEAN; CONT_RUN : BOOLEAN;
    START_TIME, CHOICE, RESPONSE_TIME, WCPU, W1 : REAL_ARRAY;
    W2, W3, W4, WCPU_START, W1_START, W2_START  : REAL_ARRAY;
    W3_START, W4_START : REAL_ARRAY;
    RESTART, JOB,  SPARE_JOB, CPU, SD1, SD2, SD3, SD4 : ENAME;
    JWAIT, QPROCESS, JSTORE1, JSTORE2, JSTORE3, JSTORE4:QUEUE;

PROCEDURE GO_THRU_C_EVENTS;FORWARD;

PROCEDURE CREATE_RECORDING;
BEGIN
   FOR I:=1 TO 9999
   DO TEST[I]:=FALSE;
   COUNT:=1;
   COUNT1:=1; COUNT2:=1; COUNT3:=1; COUNT4:=1;
   COUNT5:=1; COUNT6:=1; COUNT7:=1; COUNT8:=1;
   COUNT9:=1;COUNT10:=1;
END;

PROCEDURE STARTUP;
BEGIN
   CREATE_RECORDING;
   THEREARE(25,JOB,'JOB');
   THEREARE(1,CPU,'CPU');
   THEREARE(1,SD1,'SD1');
   THEREARE(1,SD2,'SD2');
   THEREARE(1,SD3,'SD3');
   THEREARE(1,SD4,'SD4');
   MAKEQ(JWAIT,'JWAIT');
   MAKEQ(QPROCESS,'QPROCESS');
   MAKEQ(JSTORE1,'JSTORE1');
   MAKEQ(JSTORE2,'JSTORE2');
   MAKEQ(JSTORE3,'JSTORE3');
   MAKEQ(JSTORE4,'JSTORE4');
   SPARE_JOB := JOB;
   FOR J := 1 TO 25 DO
   BEGIN
     ADDTO(BACK,JWAIT,SPARE_JOB);
     SPARE_JOB := SPARE_JOB^.NEXT;
   END;
   GO_THRU_C_EVENTS;
END;
```

```
PROCEDURE C1;  (* FORM *)
BEGIN
  WHILE (QSIZE(JWAIT) >= 1)
  DO BEGIN
       SPARE_JOB:=HEAD(JWAIT);
       IF COUNT>25 THEN
       SPARE_JOB^.ATTR:=COUNT;
       FORM_TIME := NEGEXP(100,FIRST_SEED);
       CAUSE( 1,BEHEAD(JWAIT),FORM_TIME);
       COUNT:=COUNT+1;
    END;     (* of while loop *)
END;   (* of procedure c1 *)

PROCEDURE C2;  (* PROCESS *)
BEGIN
  WHILE (QSIZE(QPROCESS) >= 1)
    AND (CPU^.AVAIL)
  DO BEGIN
       WCPU[COUNT10]:=TIM-WCPU_START[COUNT10];
       SPARE_JOB:=HEAD(QPROCESS);
       PROCESS_TIME := NEGEXP(1,SECOND_SEED);
       CAUSE( 2,BEHEAD(QPROCESS),PROCESS_TIME);
       CAUSE( 3,CPU,PROCESS_TIME);
       COUNT10:=COUNT10+1;
    END;     (* of while loop *)
END;   (* of procedure c2 *)

PROCEDURE C3;  (* STORE1 *)
BEGIN
  WHILE (QSIZE(JSTORE1) >= 1)
    AND (SD1^.AVAIL)
  DO BEGIN
       W1[COUNT2]:=TIM-W1_START[COUNT2];
       SPARE_JOB:=HEAD(JSTORE1);
       STORE1_TIME := NEGEXP(1.39,THIRD_SEED);
       CAUSE( 4,BEHEAD(JSTORE1),STORE1_TIME);
       CAUSE( 5,SD1,STORE1_TIME);
       COUNT2:=COUNT2+1;
    END;     (* of while loop *)
END;   (* of procedure c3 *)

PROCEDURE C4;  (* STORE2 *)
BEGIN
  WHILE (QSIZE(JSTORE2) >= 1)
    AND (SD2^.AVAIL)
  DO BEGIN
       W2[COUNT4]:=TIM-W2_START[COUNT4];
       SPARE_JOB:=HEAD(JSTORE2);
       STORE2_TIME := NEGEXP(1.39,FOURTH_SEED);
       CAUSE( 6,BEHEAD(JSTORE2),STORE2_TIME);
       CAUSE( 7,SD2,STORE2_TIME);
       COUNT4:=COUNT4+1;
    END;     (* of while loop *)
END;   (* of procedure c4 *)
```

```
PROCEDURE C5;  (* STORE3 *)
BEGIN
  WHILE (QSIZE(JSTORE3) >= 1)
    AND (SD3^.AVAIL)
  DO BEGIN
       W3[COUNT6]:=TIM-W3_START[COUNT6];
       SPARE_JOB:=HEAD(JSTORE3);
       STORE3_TIME := NEGEXP(12.5,FIFTH_SEED);
       CAUSE( 8,BEHEAD(JSTORE3),STORE3_TIME);
       CAUSE( 9,SD3,STORE3_TIME);
       COUNT6:=COUNT6+1;
     END;     (* of while loop *)
END;   (* of procedure c5 *)
```

```
PROCEDURE C6;  (* STORE4 *)
BEGIN
  WHILE (QSIZE(JSTORE4) >= 1)
    AND (SD4^.AVAIL)
  DO BEGIN
       W4[COUNT8]:=TIM-W4_START[COUNT8];
       SPARE_JOB:=HEAD(JSTORE4);
       STORE4_TIME := NEGEXP(12.5,SIXTH_SEED);
       CAUSE(10,BEHEAD(JSTORE4),STORE4_TIME);
       CAUSE(11,SD4,STORE4_TIME);
       COUNT8:=COUNT8+1;
     END;     (* of while loop *)
END;   (* of procedure c6 *)
```

```
PROCEDURE B1;           (* JOB ENDS FORM *)
BEGIN
  WCPU_START[COUNT9]:=TIM;
  START_TIME[CUR_NO_ENT]:=TIM;
  ADDTO(BACK,QPROCESS,CURRENT);
  COUNT9:=COUNT9+1;
END;
```

```
PROCEDURE B2;            (* JOB ENDS PROCESS *)
VAR
  TEST1,TEST2,TEST3,TEST4,TEST5,TEST6,TEST7,TEST8:BOOLEAN;
BEGIN
  CHOICE[CUR_NO_ENT]:=RND(SEVENTH_SEED);
  TEST1:=CHOICE[CUR_NO_ENT]>0.20;
  TEST2:=CHOICE[CUR_NO_ENT]<=0.56;
  TEST3:=CHOICE[CUR_NO_ENT]>0.56;
  TEST4:=CHOICE[CUR_NO_ENT]<=0.92;
  TEST5:=CHOICE[CUR_NO_ENT]>0.92;
  TEST6:=CHOICE[CUR_NO_ENT]<=0.96;
  TEST7:=CHOICE[CUR_NO_ENT]>0.96;
  TEST8:=CHOICE[CUR_NO_ENT]<=1.00;
  IF CHOICE[CUR_NO_ENT]<=0.20
  THEN BEGIN
         RESPONSE_TIME[CUR_NO_ENT]:=TIM-
                                        START_TIME[CUR_NO_ENT];
         TEST[CUR_NO_ENT]:=TRUE;
         ADDTO(BACK,JWAIT,CURRENT);
       END;
  IF (TEST1 AND TEST2)
  THEN BEGIN
         W1_START[COUNT1]:=TIM;
         ADDTO(BACK,JSTORE1,CURRENT);
         COUNT1:=COUNT1+1;
       END;
  IF (TEST3 AND TEST4)
  THEN BEGIN
         W2_START[COUNT3]:=TIM;
         ADDTO(BACK,JSTORE2,CURRENT);
         COUNT3:=COUNT3+1;
       END;
  IF (TEST5 AND TEST6)
  THEN BEGIN
         W3_START[COUNT5]:=TIM;
         ADDTO(BACK,JSTORE3,CURRENT);
         COUNT5:=COUNT5+1;
       END;
  IF (TEST7 AND TEST8)
  THEN BEGIN
         W4_START[COUNT7]:=TIM;
         ADDTO(BACK,JSTORE4,CURRENT);
         COUNT7:=COUNT7+1;
       END;
END;


PROCEDURE B3;            (* CPU ENDS PROCESS *)
BEGIN
END;
```

```
PROCEDURE B4;            (* JOB ENDS STORE1 *)
BEGIN
  ADDTO(BACK,QPROCESS,CURRENT);
  WCPU_START[COUNT9]:=TIM;
  COUNT9:=COUNT9+1;
END;
PROCEDURE B5;            (* SD1 ENDS STORE1 *)
BEGIN
END;


PROCEDURE B6;            (* JOB ENDS STORE2 *)
BEGIN
  ADDTO(BACK,QPROCESS,CURRENT);
  WCPU_START[COUNT9]:=TIM;
  COUNT9:=COUNT9+1;
END;


PROCEDURE B7;            (* SD2 ENDS STORE2 *)
BEGIN
END;


PROCEDURE B8;            (* JOB ENDS STORE3 *)
BEGIN
  ADDTO(BACK,QPROCESS,CURRENT);
  WCPU_START[COUNT9]:=TIM;
  COUNT9:=COUNT9+1;
END;


PROCEDURE B9;            (* SD3 ENDS STORE3 *)
BEGIN
END;


PROCEDURE B10;            (* JOB ENDS STORE4 *)
BEGIN
  ADDTO(BACK,QPROCESS,CURRENT);
  WCPU_START[COUNT9]:=TIM;
  COUNT9:=COUNT9+1;
END;


PROCEDURE B11;            (* SD4 ENDS STORE4 *)
BEGIN
END;
```

```
PROCEDURE EndRunin;
BEGIN
  CREATE_RECORDING;
END;
```

```
PROCEDURE CALL_FOR_NEXT_B_EVENT;
BEGIN
  CASE NO_NEXTB OF
        1 : B1;
        2 : B2;
        3 : B3;
        4 : B4;
        5 : B5;
        6 : B6;
        7 : B7;
        8 : B8;
        9 : B9;
        10 : B10;
        11 : B11;
      127 : EndRunin;
  END;
END;
```

```
PROCEDURE GO_THRU_C_EVENTS;
VAR CFLAG,CNUM : INTEGER;
BEGIN
  CNUM := 6;
  FOR CFLAG:=1 TO CNUM DO
       CASE CFLAG OF
         1 : C1;
         2 : C2;
         3 : C3;
         4 : C4;
         5 : C5;
         6 : C6;
       END;
END;
```

```
PROCEDURE EXECUTE(DUR:INTEGER);
BEGIN
  STARTUP;
  RUNNING := TRUE;
  WHILE RUNNING DO
  BEGIN
     TIM := NEXT_TIME ;
     IF DUR>=COUNT10 THEN
     BEGIN
       WHILE (TIM = NEXT_TIME) AND (RUNNING) DO
       BEGIN
         GET_NEXT_ENTITY;
         CALL_FOR_NEXT_B_EVENT;
       END;
       IF RUNNING THEN
           GO_THRU_C_EVENTS
     END ELSE
          BEGIN
              RUNNING:=FALSE;
          END;
   END;
   INITCALL;
END;



BEGIN    (* main program *)
  INITCALL;
  WRITE('SELECT AN INTEGER FROM 1 UP TO 1436 =====>');
  READLN(FIRST_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONE YOU HAVE SELECTED BEFORE =====>');
  READLN(SECOND_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(THIRD_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(FOURTH_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(FIFTH_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(SIXTH_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(SEVENTH_SEED);
  WRITELN('GIVE THE NUMBER OF JOBS COMPLETING SERVICE AT');
  WRITE('THE CENTRAL PROCESSING UNIT =====>');
  READLN(DUR);
  EXECUTE(DUR);
END.
```

Appendix A

PROGRAM LISTING FOR THE TIME SHARED COMPUTER MODEL

```
[INHERIT('DISKB:[KEVORK]ENTITIES1.PEN','DISKB:[KEVORK]
QUEUES1.PEN','DISKB:[KEVORK]NEW_SAMPLING.PEN',
'DISKB:[KEVORK]STAT_LIBRARY.PEN')]


PROGRAM SHARE(INPUT,OUTPUT);


VAR
   K, COUNT1, FIRST_SEED, SECOND_SEED, DUR : INTEGER;
   RUN_IN_PERIOD, PROCESS_TIME, FORM_TIME  : REAL;
   RESPONSE_TIME, REMAIN_TIME     : REAL_ARRAY
   START_TIME, END_TIME, SERVICE : REAL_ARRAY;
   TEST : ARRAY [1..9999] OF BOOLEAN;
   CONT_RUN : BOOLEAN;
   RESTART,JOB, SPARE_JOB, CPU, SPARE_CPU : ENAME;
   JWAIT,PWAIT,QCPU : QUEUE;


PROCEDURE GO_THRU_C_EVENTS;FORWARD;


PROCEDURE CREATE_RECORDING;
BEGIN
   COUNT1:=1;
   FOR I:=1 TO 9999
   DO TEST[I]:=FALSE;
END;


PROCEDURE STARTUP;
BEGIN
   CREATE_RECORDING;
   THEREARE(35,JOB,'JOB');
   THEREARE(1,CPU,'CPU');
   MAKEQ(PWAIT,'PWAIT');
   MAKEQ(JWAIT,'JWAIT');
   MAKEQ(QCPU,'QCPU');
   SPARE_JOB := JOB;
   FOR J := 1 TO 35 DO
   BEGIN
     ADDTO(BACK,JWAIT,SPARE_JOB);
     SPARE_JOB := SPARE_JOB^.NEXT;
   END;
   ADDTO(BACK,QCPU,CPU);
   GO_THRU_C_EVENTS;
END;
```

```
PROCEDURE C1;  (* FORM *)
BEGIN
  WHILE (QSIZE(JWAIT) >= 1)
  DO BEGIN
       SPARE_JOB:=HEAD(JWAIT);
       IF (COUNT1>35) THEN
       SPARE_JOB^.ATTR:=COUNT1;
       FORM_TIME := NEGEXP(25,FIRST_SEED);
       CAUSE( 1,BEHEAD(JWAIT),FORM_TIME);
       COUNT1:=COUNT1+1;
     END;     (* of while loop *)
END;   (* of procedure c1 *)


PROCEDURE C2;  (* PROCESS *)
BEGIN
  WHILE (QSIZE(PWAIT) >= 1)
  AND  (QSIZE(QCPU) >= 1)
  DO BEGIN
       SPARE_JOB:=HEAD(PWAIT);
       REMAIN_TIME[SPARE_JOB^.ATTR]:=SERVICE[SPARE_JOB^.ATTR]-
       0.1;
       IF REMAIN_TIME[SPARE_JOB^.ATTR] <= 0
         THEN PROCESS_TIME:=SERVICE[SPARE_JOB^.ATTR]+0.015
       ELSE
         PROCESS_TIME :=0.1+0.015;
       CAUSE(2,BEHEAD(PWAIT),PROCESS_TIME);
       CAUSE(3,BEHEAD(QCPU),PROCESS_TIME);
     END;     (* of while loop *)
END;   (* of procedure c2 *)


PROCEDURE B1;           (* JOB ENDS FORM *)
BEGIN
  SERVICE[CUR_NO_ENT] := NEGEXP(0.8,SECOND_SEED);
  START_TIME[CUR_NO_ENT]:=TIM;
  ADDTO(BACK,PWAIT,CURRENT);
END;


PROCEDURE B2;           (* JOB ENDS PROCESS *)
BEGIN
  SERVICE[CUR_NO_ENT] := SERVICE[CUR_NO_ENT]-0.1;
  IF SERVICE[CUR_NO_ENT] > 0
    THEN ADDTO(BACK,PWAIT,CURRENT)
  ELSE
    BEGIN
      ADDTO(BACK,JWAIT,CURRENT);
      TEST[CUR_NO_ENT]:=TRUE;
      END_TIME[CUR_NO_ENT]:=TIM;
      RESPONSE_TIME[CUR_NO_ENT]:=END_TIME[CUR_NO_ENT]-
      START_TIME[CUR_NO_ENT];
    END;
END;
```

```
PROCEDURE B3;        (*  CPU  ENDS  PROCESS*)
BEGIN
  ADDTO(BACK,QCPU,CURRENT);
END;
PROCEDURE EndRunin;
BEGIN
  CREATE_RECORDING;
END;
```

```
PROCEDURE CALL_FOR_NEXT_B_EVENT;
BEGIN
  CASE NO_NEXTB OF
        1 : B1;
        2 : B2;
        3 : B3;
      127 : EndRunin;
  END;
END;
```

```
PROCEDURE GO_THRU_C_EVENTS;
VAR CFLAG,CNUM : INTEGER;
BEGIN
  CNUM := 2;
  FOR CFLAG:=1 TO CNUM DO
       CASE CFLAG OF
         1 : C1;
         2 : C2;
       END;
END;
```

```
PROCEDURE EXECUTE(DUR:INTEGER);
BEGIN
  STARTUP;
  RUNNING := TRUE;
  WHILE RUNNING DO
  BEGIN
     TIM := NEXT_TIME ;
     CONT_RUN:=FALSE;
     K:=1;
     REPEAT
       IF TEST[K]=FALSE THEN
       CONT_RUN:=TRUE;
       K:=K+1;
     UNTIL (K=DUR) OR (CONT_RUN=TRUE);
     IF (CONT_RUN=TRUE) THEN
     BEGIN
       WHILE (TIM = NEXT_TIME) AND (RUNNING) DO
       BEGIN
         GET_NEXT_ENTITY;
         CALL_FOR_NEXT_B_EVENT;
       END;
       IF RUNNING THEN
           GO_THRU_C_EVENTS
     END ELSE
          BEGIN
              RUNNING:=FALSE;
          END;
   END;
   INITCALL;
END;

BEGIN    (* main program *)
  INITCALL;
  WRITE('SELECT AN INTEGER FROM 1 UP TO 1436 =====>');
  READLN(FIRST_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONE YOU HAVE SELECTED BEFORE =====>');
  READLN(SECOND_SEED);
  WRITE('GIVE THE NUMBER OF JOBS COMPLETING SERVICE =====>);
  READLN(DUR);
  EXECUTE(DUR);
END.
```

PROGRAM LISTING FOR THE STEELWORKS

```
[INHERIT('DISKB:[KEVORK]ENTITIES1.PEN','DISKB:[KEVORK]
QUEUES1.PEN','DISKB:[KEVORK]NEW_SAMPLING.PEN',
'DISKB:[KEVORK]STAT_LIBRARY')]



PROGRAM SWS (INPUT,OUTPUT);



VAR
   K, DUR, FIRST_SEED, SECOND_SEED, THIRD_SEED      : INTEGER;
   FOURTH_SEED, NUM_OF_TOR, COUNT1, COUNT2, COUNT3 : INTEGER;
   FT_TRTIME, WORK_TIME, MELT_TIME, BUNL_TIME, LTIME : REAL;
   TTRAVEL_TIME, TUNLC_TIME, SD_TRTIME, TRETURN_TIME : REAL;
   CTRAVEL_TIME, CUNLS_TIME, CRETURN_TIME : REAL;
   RUN_IN_PERIOD : REAL;
   WEIGHT, CARGO, LADLE, WASTE, WAIT_TIME : REAL_ARRAY;
   START_TIME, RESPONSE_TIME, ENTER_TIME  : REAL_ARRAY;
   RESTART, STEELFURN, SPARE_STEELFURN, BLASTFURN   : ENAME;
   SPARE_BLASTFURN, TORPEDEO, SPARE_TORPEDEO, CRANE : ENAME ;
   SPARE_CRANE,PIT : ENAME;
   BIDLE, BWAIT, TIDLE, CFULL, TWAIT, DT2, CIDLE : QUEUE
   CWAIT, SIDLE : QUEUE;
   CONT_RUN:BOOLEAN;
   TEST : ARRAY [1..3000] OF BOOLEAN;



PROCEDURE GO_THRU_C_EVENTS;FORWARD;



PROCEDURE CREATE_RECORDING;
BEGIN
  COUNT1:=1;
  COUNT2:=1;
  COUNT3:=1;
  FOR II:=1 TO 5000
  DO TEST[II]:=FALSE;
END;
```

```
PROCEDURE STARTUP;
BEGIN
   CREATE_RECORDING;
   THEREARE(2,BLASTFURN,'BLASTFURN');
   THEREARE(NUM_OF_TOR,TORPEDEO,'TORPEDEO');
   THEREARE(1,CRANE,'CRANE');
   THEREARE(1,PIT,'PIT');
   THEREARE(5,STEELFURN,'STEELFURN');
   MAKEQ(BIDLE,'BIDLE');
   MAKEQ(CFULL,'CFULL');
   MAKEQ(BWAIT,'BWAIT');
   MAKEQ(TIDLE,'TIDLE');
   MAKEQ(TWAIT,'TWAIT');
   MAKEQ(DT2,'DT2');
   MAKEQ(CIDLE,'CIDLE');
   MAKEQ(CWAIT,'CWAIT');
   MAKEQ(SIDLE,'SIDLE');
   SPARE_BLASTFURN := BLASTFURN;
   FOR JJ:=1 TO 2 DO
   BEGIN
     ADDTO(BACK,BIDLE,SPARE_BLASTFURN);
     SPARE_BLASTFURN:=SPARE_BLASTFURN^.NEXT;
   END;
   SPARE_STEELFURN := STEELFURN;
   FOR JJ := 1 TO 5 DO
   BEGIN
     ADDTO(BACK,SIDLE,SPARE_STEELFURN);
     SPARE_STEELFURN := SPARE_STEELFURN^.NEXT;
   END;
   SPARE_TORPEDEO := TORPEDEO;
   FOR JJ := 1 TO NUM_OF_TOR DO
   BEGIN
     ADDTO(BACK,TIDLE,SPARE_TORPEDEO);
     SPARE_TORPEDEO := SPARE_TORPEDEO^.NEXT;
   END;
   SPARE_CRANE := CRANE;
   FOR JJ := 1 TO 1  DO
   BEGIN
     ADDTO(BACK,CIDLE,SPARE_CRANE);
     SPARE_CRANE := SPARE_CRANE^.NEXT;
   END;
   GO_THRU_C_EVENTS;
END;


PROCEDURE C1;  (* MELT *)
BEGIN
  WHILE (QSIZE(BIDLE) >= 1)
  DO BEGIN
         SPARE_BLASTFURN := HEAD(BIDLE);
         MELT_TIME := NORMAL(110,15,FIRST_SEED);
         CAUSE( 1,BEHEAD(BIDLE),MELT_TIME);
     END;
END;   (* of procedure c1 *)
```

```
PROCEDURE C2;  (* BUNL *)
BEGIN
  WHILE (QSIZE(BWAIT) >= 1)
  DO BEGIN
         SPARE_BLASTFURN := HEAD(BWAIT);
         IF QSIZE(TIDLE) =0 THEN
         BEGIN
          WASTE[COUNT1] := WEIGHT[SPARE_BLASTFURN^.ATTR];
          CAUSE(3,BEHEAD(BWAIT),BUNL_TIME);
          COUNT1:=COUNT1+1;
         END;
         IF QSIZE(TIDLE) =1 THEN
         BEGIN
           BUNL_TIME:=10;
           TTRAVEL_TIME := POISSON(10,SECOND_SEED);
           SPARE_TORPEDEO := HEAD(TIDLE);
           IF COUNT3>NUM_OF_TOR THEN
           SPARE_TORPEDEO^.ATTR:=COUNT3;
           ENTER_TIME[SPARE_TORPEDEO^.ATTR]:=TIM+10;
           CARGO[SPARE_TORPEDEO^.ATTR]:=300;
           WASTE[COUNT1]:=WEIGHT[SPARE_BLASTFURN^.ATTR]-300;
           CAUSE( 2,BEHEAD(TIDLE),BUNL_TIME + TTRAVEL_TIME);
           CAUSE( 3,BEHEAD(BWAIT),BUNL_TIME);
           COUNT1:=COUNT1+1;
           COUNT3:=COUNT3+1;
         END;
         IF QSIZE(TIDLE) >=2 THEN
         BEGIN
            BUNL_TIME:=10;
            FT_TRTIME:=POISSON(10,SECOND_SEED);
            TTRAVEL_TIME := FT_TRTIME;
            WASTE[COUNT1]:=0;
            SPARE_TORPEDEO := HEAD(TIDLE);
            IF COUNT3>NUM_OF_TOR THEN
            SPARE_TORPEDEO^.ATTR:=COUNT3;
            ENTER_TIME[SPARE_TORPEDEO^.ATTR]:=TIM+10;
            COUNT3:=COUNT3+1;
            CARGO[SPARE_TORPEDEO^.ATTR]:=300;
            CAUSE(2,BEHEAD(TIDLE),BUNL_TIME+TTRAVEL_TIME);
            SPARE_TORPEDEO := HEAD(TIDLE);
            IF COUNT3>NUM_OF_TOR THEN
            SPARE_TORPEDEO^.ATTR:=COUNT3;
            ENTER_TIME[SPARE_TORPEDEO^.ATTR]:=TIM+10;
            COUNT3:=COUNT3+1;
            SD_TRTIME := POISSON(10,SECOND_SEED);
            IF SD_TRTIME>FT_TRTIME
            THEN TTRAVEL_TIME:=FT_TRTIME
            ELSE TTRAVEL_TIME:=SD_TRTIME;
            CARGO[SPARE_TORPEDEO^.ATTR]:=
                             WEIGHT[SPARE_BLASTFURN^.ATTR]-300;
            CAUSE(2,BEHEAD(TIDLE),BUNL_TIME + TTRAVEL_TIME);
            CAUSE(3,BEHEAD(BWAIT),BUNL_TIME);
            COUNT1:=COUNT1+1;
         END;
     END;
END;   (* of procedure c2 *)
```

```
PROCEDURE C3;  (* TUNLC *)
BEGIN
  WHILE (PIT^.AVAIL)
   AND (QSIZE(TWAIT) >= 1)
    AND (QSIZE(CIDLE) >= 1)
  DO BEGIN
         TUNLC_TIME := 5;
         SPARE_CRANE := HEAD(CIDLE);
         SPARE_TORPEDEO := HEAD(TWAIT);
         WAIT_TIME[COUNT2]:=TIM-
                             START_TIME[SPARE_TORPEDEO^.ATTR];
         LOADAM:=MINOF(CARGO[SPARE_TORPEDEO^.ATTR],
                                100-LADLE[SPARE_CRANE^.ATTR]);
         LADLE[SPARE_CRANE^.ATTR]:=
                              LADLE[SPARE_CRANE^.ATTR]+LOADAM;
         CARGO[SPARE_TORPEDEO^.ATTR]:=
                            CARGO[SPARE_TORPEDEO^.ATTR]-LOADAM;
         CAUSE( 4,PIT,TUNLC_TIME);
         CAUSE( 5,BEHEAD(TWAIT),TUNLC_TIME);
         CAUSE( 6,BEHEAD(CIDLE),TUNLC_TIME );
         COUNT2:=COUNT2+1;
     END;
END;   (* of procedure c3 *)


PROCEDURE C4; (* TRETURN*)
BEGIN
  WHILE (QSIZE(DT2) >=1)
  DO BEGIN
         TRETURN_TIME := 4;
         CAUSE( 7,BEHEAD(DT2),TRETURN_TIME);
     END;
END;      (* OF PROCEDURE C4*)

PROCEDURE C5;  (* CTRAVEL *)
BEGIN
  WHILE (QSIZE(CFULL) >= 1 )
  DO BEGIN
         CTRAVEL_TIME := 5;
         CAUSE(8,BEHEAD(CFULL),CTRAVEL_TIME);
     END;
END;   (* OF PROCEDURE   C5 *)

PROCEDURE C6;  (* CUNLS *)
BEGIN
  WHILE (QSIZE(SIDLE) >= 1)
   AND (QSIZE(CWAIT) >= 1)
  DO BEGIN
         CUNLS_TIME := 5;
         WORK_TIME := 50+NEGEXP(10,THIRD_SEED);
         CAUSE( 9,BEHEAD(SIDLE),CUNLS_TIME + WORK_TIME);
         CRETURN_TIME := 2;
         CAUSE( 10,BEHEAD(CWAIT),CUNLS_TIME + CRETURN_TIME);
     END;
END;   (* of procedure c6 *)
```

```
PROCEDURE B1;            (* BLASTFURN ENDS MELT *)
BEGIN
   REPEAT
   WEIGHT[CUR_NO_ENT] := NORMAL(380,50,FOURTH_SEED);
   UNTIL(WEIGHT[CUR_NO_ENT]>=320)  AND
   (WEIGHT[CUR_NO_ENT]<=480);
   ADDTO(BACK,BWAIT,CURRENT);
END;

PROCEDURE B2;            (* TORPEDEO ENDS TTRAVEL *)
BEGIN
  ADDTO(BACK,TWAIT,CURRENT);
  START_TIME[CUR_NO_ENT]:=TIM;
  COUNT2:=COUNT2+1;
END;

 PROCEDURE B3;           (* BLASTFURN ENDS BUNLT *)
BEGIN
  ADDTO(BACK,BIDLE,CURRENT);
END;


PROCEDURE B4;            (* PIT ENDS TUNLC *)
BEGIN
END;

PROCEDURE B5;            (* TORPEDEO ENDS TUNLC *)
BEGIN
  IF CARGO[CUR_NO_ENT] >  0  THEN
    BEGIN
      ADDTO(FRONT,TWAIT,CURRENT);
      START_TIME[CUR_NO_ENT]:=TIM;
    END
  ELSE
    BEGIN
      ADDTO(BACK,DT2,CURRENT);
    END;
END;

PROCEDURE B6;           (* CRANE ENDS TUNLC *)
BEGIN
  IF LADLE[CUR_NO_ENT] <100  THEN
  ADDTO(FRONT,CIDLE,CURRENT)
  ELSE BEGIN
  ADDTO(BACK,CFULL,CURRENT);
  END;
END;


PROCEDURE B7;           (* TORPEDEO ENDS TRETURN *)
BEGIN
  ADDTO(BACK,TIDLE,CURRENT);
  TEST[CUR_NO_ENT]:=TRUE;
  RESPONSE_TIME[CUR_NO_ENT]:=TIM-ENTER_TIME[CUR_NO_ENT]
END;
```

```
PROCEDURE B8;           (* CRANE ENDS CTRAVEL*)
BEGIN
  ADDTO(BACK,CWAIT,CURRENT);
END;

PROCEDURE B9;           (* STEELFURN ENDS WORK *)
BEGIN
  ADDTO(BACK,SIDLE,CURRENT);
END;


PROCEDURE B10;           (* CRANE ENDS CRETURN *)
BEGIN
  LADLE[CUR_NO_ENT] := 0;
  ADDTO(BACK,CIDLE,CURRENT);
END;


PROCEDURE EndRunin;
BEGIN
  CREATE_RECORDING;
END;


PROCEDURE CALL_FOR_NEXT_B_EVENT;
BEGIN
  CASE NO_NEXTB OF
       1 : B1;
       2 : B2;
       3 : B3;
       4 : B4;
       5 : B5;
       6 : B6;
       7 : B7;
       8 : B8;
       9 : B9;
      10 : B10;
     127 : EndRunin;
  END;
END;


PROCEDURE GO_THRU_C_EVENTS;
BEGIN
  CNUM :=  6;
   FOR CFLAG:=1 TO CNUM DO
       CASE CFLAG OF
          1: C1;
          2: C2;
          3: C3;
          4: C4;
          5: C5;
          6: C6;
   END;
END;
```

```
PROCEDURE EXECUTE(DUR:INTEGER);
BEGIN
  STARTUP;
  RUNNING:=TRUE;
  WHILE RUNNING DO
  BEGIN
    TIM:=NEXT_TIME;
    CONT_RUN:=FALSE;
    K:=1;
    REPEAT
      IF TEST[K]=FALSE THEN
      CONT_RUN:=TRUE;
      K:=K+1;
    UNTIL (K=DUR) OR (CONT_RUN=TRUE);
    IF (CONT_RUN=TRUE) THEN
    BEGIN
      WHILE (TIM=NEXT_TIME) AND (RUNNING) DO
      BEGIN
        GET_NEXT_ENTITY;
        CALL_FOR_NEXT_B_EVENT;
      END;
      IF RUNNING THEN
           GO_THRU_C_EVENTS
    END ELSE
           BEGIN
             RUNNING:=FALSE;
           END;
     END;
     INITCALL;
END;


BEGIN    (* main program *)
  INITCALL;
  WRITE('SELECT AN INTEGER FROM 1 UP TO 1436 =====>');
  READLN(FIRST_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONE YOU HAVE ALREADY SELECTED =====>');
  READLN(SECOND_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(THIRD_SEED);
  WRITELN('SELECT AN INTEGER FROM 1 UP TO 1436 DIFFERENT');
  WRITE('FROM THE ONES YOU HAVE ALREADY SELECTED =====>');
  READLN(FOURTH_SEED);
  WRITE('GIVE THE NUMBER OF TORPEDOES =====>');
  READLN(NUM_OF_TOR);
  WRITELN('GIVE THE NUMBER OF TORPEDOES COMPLETING THE');
  WRITE('CYCLE INTO THE SYSTEM =====>');READLN(DUR);
  EXECUTE(DUR);
END.
```

A P P E N D I X   B

DESCRIPTION OF THE SIM_STAT_LIB

DICTIONERY OF SIM_STAT_LIB
THE SIMULATION STATISTICAL LIBRARY

ACR_EST

```
function ACR_EST(n,s:INTEGER;X:REAL_ARRAY):REAL;
```
calculates the sample autocorrelation at lag s of the output sequence $X_1, X_2, \ldots, X_n$.

ACR_SET_EST

```
procedure ACR_SET_EST(n,ls,us:INTEGER;
                      X,set_acr:REAL_ARRAY);
```
calculates the sample autocorrelations from lag ls up to lag us of the output sequence $X_1, X_2, \ldots, X_n$.

ACV_EST

```
function ACV_EST(n,s:INTEGER;X:REAL_ARRAY):REAL;
```
calculates the sample autocovariance at lag s of the output sequence $X_1, X_2, \ldots, X_n$.

ACV_SET_EST

```
procedure ACV_SET_EST(n,ls,us:INTEGER;
                      X,set_acv:REAL_ARRAY);
```
calculates the sample autocovariances from laf ls to lag us of the output sequence $X_1, X_2, \ldots, X_n$.

AR_PARAM_EST

```
procedure AR_PARAM_EST(n,p:INTEGER;
                       X,atreg_coeff:REAL_ARRAY;
                       error_var:REAL);
```
provides estimates for the autoregressive coefficients and the error variance of the AR(p) fitted to the output sequence $X_1, X_2, \ldots, X_n$.

AR_VAR_MEAN

```
procedure AR_VAR_MEAN(n:INTEGER; X:REAL_ARRAY;
                      atreg_mean_var,df_atreg:REAL);
```
estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the autoregressive method.

AR_SCHEME_FIT

procedure AR_SCHEME_FIT(n,lev_sig:INTEGER;
X,atreg_coeff:REAL_ARRAY;
p:INTEGER; error_var:REAL);
estimates the autoregressive order p, the autoregressive coefficients and the error variance of AR(p) fitted to the output sequence $X_1, X_2, \ldots, X_n$.

AREA_VAR_MEAN

procedure AREA_VAR_MEAN(n,k:INTEGER; X:REAL_ARRAY;
area_mean_var:REAL;
df_area:INTEGER);
estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the standardized time series-area method for k batches.

CHI_SQUARE

function CHI_SQUARE(df,lev_sig:INTEGER):REAL;
provides the values of the χ^2 distribution with df degrees of freedom at 1(1%) and 5(5%) right tail areas.

CL_VAR_MEAN

procedure CL_VAR_MEAN(n:INTEGER;X:REAL_ARRAY;
cl_mean_var:REAL);
estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ according to the classical method.

COMB_AREA_NOBM_VAR_MEAN

procedure COMB_AREA_NOBM_VAR_MEAN(n,k:INTEGER;
X:REAL_ARRAY;cm_mean_var:REAL;
df_cm:INTEGER);
estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the combined NOBM-AREA method for k number of batches.

COMB_MAX_NOBM_VAR_MEAN

procedure COMB_MAX_NOBM_VAR_MEAN(n,k:INTEGER;
X:REAL_ARRAY;cx_mean_var:REAL;
df_cx:INTEGER);
estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the combined NOBM–MAXIMUM method for k number of batches.

FISHMAN_NUM_BATCH

```
procedure  FISHMAN_NUM_BATCH(n:INTEGER;lev_sig:REAL;
                             X:REAL_ARRAY;
                             est_num_batch:INTEGER);
```

provides the number of approximately independent batched-means in the output sequence $X_1, X_2, \ldots, X_n$ according to a procedure developed by Fishman(1978a).

INDEX_MAX_VALUE

```
procedure  INDEX_MAX_VALUE(n:INTEGER;X:REAL_ARRAY;
                           max_value:REAL;
                           max_index:INT_ARRAY);
```

provides the locations of the maximum value of the output sequence $X_1, X_2, \ldots, X_n$.

INT_ARRAY

```
type  INT_ARRAY : ARRAY[1..maxsize] of INTEGER;
```

MAX_VAR_MEAN

```
procedure  MAX_VAR_MEAN(n,k:INTEGER;X:REAL_ARRAY;
                        max_mean_var:REAL;
                        df_max:INTEGER);
```

estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the standardized time series-maximum method for k number of batches.

MAXIMUM

```
function MAXIMUM(n:INTEGER;X:REAL_ARRAY):REAL;
```

provides the maximum value of the output sequence $X_1, X_2, \ldots, X_n$.

MEAN_EST

```
function  MEAN_EST(n:INTEGER;X:REAL_ARRAY):REAL;
```

calculates the sample mean of the output sequence $X_1, X_2, \ldots, X_n$.

NMN_TEST_STAT

```
function NMN_TEST_STAT(n,k:INTEGER;X:REAL_ARRAY):REAL;
```

calculates the Neumman test statistic used by Fishman's(1978a) procedure for determining the number of approximately independent batched-means in the output sequence $X_1, X_2, \ldots, X_n$.

NOBM_VAR_MEAN

procedure NOBM_VAR_MEAN(n,k:INTEGER;X:REAL_ARRAY;
nobm_mean_var:REAL;
df_nb:INTEGER);

estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the nonoverlapping batched-means method.

NOVBATCHED_MEAN

procedure NOVBATCHED_MEAN(n,k:INTEGER;X:REAL_ARRAY;
batch_means:REAL_ARRAY);

provides the k nonoverlapping batched-means for the output sequence $X_1, X_2, \ldots, X_n$.

PICK_UP

procedure PICK_UP(ℓm,um:INTEGER;var X:REAL_ARRAY);

transforms the indices of the output sequence $X_{\ell m+1}, \ldots, X_{um}$ into $X_1, X_2, \ldots, X_{um-\ell m+1}$.

OVBM_VAR_MEAN

procedure OVBM_VAR_MEAN(n,m:INTEGER;X:REAL_ARRAY;
ovbm_mean_var, df_ob:REAL);

estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals according to the overlapping batched-means method for batch size m.

SERIES_PARTIAL_MEANS

procedure SERIES_PARTIAL_MEANS(n:INTEGER;X:REAL_ARRAY;
partial_means:REAL_ARRAY);

calculates the differences $\bar{X}_n-\bar{X}_j (j=1,2,\ldots,n)$ for the output sequence $X_1, X_2, \ldots, X_n$ where $\bar{X}_j$ is the mean of the first j observations.

REAL_ARRAY

type REAL_ARRAY : ARRAY[1..maxsize] of REAL;

SPEC_VAR_MEAN

procedure SPEC_VAR_MEAN(n,w:INTEGER;X:REAL_ARRAY;
spec_mean_var,df_sp:REAL);

estimates the variance of the sample mean of the output sequence $X_1, X_2, \ldots, X_n$ and provides the degrees of freedom of student-t distribution for constructing confidence intervals, according to the spectral method for spectral window size w.

STUDENT_T

function student_t(df,rta:REAL):REAL;
provides the student-t values for df degrees of freedom and rta rigth tail area.

TEST1_INIT_BIAS

procedure TEST1_INIT_BIAS(n:INTEGER;X:REAL_ARRAY;
sch1_stat:REAL);
calculates the test statistic developed by Schruben(1982) for testing if the output sequence X_1, X_2,..., X_n is stationary.

TEST2_INIT_BIAS

procedure TEST2_INIT_BIAS(n:INTEGER;X:REAL_ARRAY;
sch2_stat:REAL);
calculates the test statistic developed by Schruben et al.(1983) for testing if the output sequence X_1, X_2,..., X_n is stationary.

VAR_EST

function VAR_EST(n:INTEGER;X:REAL_ARRAY):REAL;
estimates the variance of the output sequence X_1, X_2,...,X_n.

A P P E N D I X   C

INITIAL SEEDS FOR THE RANDOM NUMBER GENERATOR

CRITICAL VALUES

I) RUNUP TEST ———————————> 12.59

II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ———> n<12800 : 123.23
———> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1943572824	R_1				10.50	11.13	8.60
	R_2	110.75	97.88	104.38	87.72	73.05	4019.20
	R_3	75.75	86.88	95.19	97.94	101.00	4044.80
2143428239	R_1				5.59	2.56	4.40
	R_2	79.25	90.25	90.63	90.75	95.17	4162.40
	R_3	78.50	76.88	68.56	65.59	85.40	4131.20
620995539	R_1				1.10	4.05	2.70
	R_2	72.25	95.50	84.13	90.91	96.89	4004.00
	R_3	84.25	83.63	108.44	121.53	107.19	4155.60
189203242	R_1				10.45	7.94	4.25
	R_2	86.75	80.13	75.00	106.97	115.58	4036.40
	R_3	80.75	91.50	93.88	92.97	104.52	4082.40
1828038570	R_1				6.21	4.65	5.65
	R_2	95.50	108.13	86.63	74.71	104.88	3973.00
	R_3	97.75	94.00	112.00	108.59	101.83	4088.00

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1943572824	R_1	3.31	6.82	5.43	9.48	9.85
	R_2	3948.27	3998.48	4120.80	4157.78	4179.64
	R_3	4099.73	4194.48	4096.06	4093.64	4194.91
2143428239	R_1	4.83	8.18	6.70	3.34	5.08
	R_2	4092.13	4078.24	4169.71	4123.78	4113.31
	R_3	4052.27	4104.96	4076.17	4114.18	4180.36
620995539	R_1	7.29	6.44	7.37	6.82	4.92
	R_2	4215.60	4021.20	4085.31	4127.82	4172.65
	R_3	3996.13	4162.80	4054.00	4077.00	4052.18
189203242	R_1	2.31	5.51	4.29	5.14	4.52
	R_2	3977.33	3993.36	4061.66	4018.84	4011.16
	R_3	4029.87	4186.00	4113.43	4176.40	4080.22
1828038570	R_1	6.21	7.03	6.77	8.79	6.60
	R_2	3977.87	3970.08	3976.34	3980.18	3968.58
	R_3	4002.40	3997.92	4022.80	4085.91	4190.91

CRITICAL VALUES

I) RUNUP TEST ———————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1739631983	R_1				7.45	2.69	3.84
	R_2	85.75	80.00	75.75	78.34	87.73	3950.40
	R_3	93.75	95.38	96.19	90.56	101.81	4040.40
794716743	R_1				7.95	8.62	7.39
	R_2	77.75	97.63	92.13	85.66	88.38	3962.00
	R_3	100.00	91.25	78.06	85.84	71.95	4132.80
157895578	R_1				5.30	3.63	0.39
	R_2	105.50	95.00	98.63	79.53	96.44	3988.80
	R_3	91.75	116.88	94.19	73.53	85.55	4078.80
1450029478	R_1				8.56	6.01	6.40
	R_2	78.75	89.38	94.88	92.47	76.75	4088.00
	R_3	95.75	101.63	100.56	93.09	96.64	4049.20
1910805443	R_1				3.37	5.23	4.28
	R_2	68.00	85.75	75.50	91.50	92.61	4154.00
	R_3	96.50	92.50	86.25	96.94	82.13	4036.40

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1739631983	R_1	9.62	11.88	7.62	5.96	4.14
	R_2	4030.27	3954.56	4066.80	4082.36	4057.93
	R_3	4179.07	4053.20	4040.69	4084.93	4168.22
794716743	R_1	2.43	2.59	4.68	3.58	5.88
	R_2	3926.40	4131.44	4179.60	4123.69	4186.91
	R_3	4102.00	3997.52	4022.00	4143.96	4058.98
1578951578	R_1	4.74	7.22	8.98	5.79	7.35
	R_2	4114.67	4028.48	4007.43	4047.69	4082.98
	R_3	4102.00	4190.88	4126.63	4163.16	4120.29
1450029478	R_1	2.79	6.21	5.15	7.91	2.00
	R_2	4038.67	4058.88	4055.94	3999.16	3978.15
	R_3	4127.47	4094.72	4106.40	4124.53	4141.64
1910805443	R_1	1.90	3.12	5.03	4.25	6.33
	R_2	3901.07	4006.96	3971.77	3986.22	3903.78
	R_3	4144.13	4215.20	4105.37	4049.56	4024.29

CRITICAL VALUES

I) RUNUP TEST → 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS → $n<12800$: 123.23
→ $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
236023503	R_1				10.03	7.10	6.82
	R_2	88.50	96.38	107.38	90.72	75.94	3974.80
	R_3	80.25	103.88	96.50	89.38	101.17	4004.80
2091686145	R_1				4.12	4.70	8.27
	R_2	90.00	102.00	76.44	98.56	89.67	3973.60
	R_3	114.25	98.25	81.38	76.13	99.64	4048.40
1459566082	R_1				5.60	2.10	5.44
	R_2	74.50	86.63	81.75	91.03	83.78	4116.40
	R_3	79.75	81.88	89.06	85.25	82.39	4121.60
1175435042	R_1				5.60	5.66	3.74
	R_2	77.75	90.75	103.13	91.03	107.25	4011.60
	R_3	111.00	107.63	99.38	85.25	109.48	4078.40
2026915971	R_1				4.70	1.27	1.84
	R_2	74.00	81.63	104.69	92.50	90.67	4181.20
	R_3	101.75	118.13	112.25	120.91	107.36	4065.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
236023503	R_1	5.87	4.91	4.18	3.40	4.54
	R_2	3994.67	4121.28	4086.23	4201.91	4115.38
	R_3	4198.80	4179.84	4187.71	4193.82	4182.84
2091686145	R_1	2.29	5.95	2.76	4.84	4.04
	R_2	3962.93	4001.60	3989.71	4038.80	4095.96
	R_3	4026.00	3985.20	3984.69	3997.51	4037.42
1459566082	R_1	9.22	4.34	6.32	2.86	2.55
	R_2	4060.53	4113.12	4130.29	4184.44	4130.55
	R_3	4114.67	4100.48	4243.89	4180.71	4147.35
1175435042	R_1	2.50	3.98	5.20	5.87	4.59
	R_2	3996.40	3932.00	3967.37	3988.98	3932.11
	R_3	4196.27	4116.72	4045.03	4087.42	4110.73
2026915971	R_1	4.57	4.55	5.47	6.01	7.67
	R_2	4095.47	3928.00	3985.66	3994.18	4040.22
	R_3	4003.87	4014.24	3984.29	4032.71	4000.65

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59

II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1674167085	R_1				6.18	5.42	1.92
	R_2	97.25	102.00	85.06	82.63	85.47	4039.60
	R_3	105.75	116.50	98.63	87.59	85.67	3987.20
547023411	R_1				5.43	3.63	2.72
	R_2	86.00	85.38	90.94	94.97	87.72	4043.60
	R_3	94.25	113.00	104.25	111.72	110.31	4008.00
985719448	R_1				7.51	5.39	5.82
	R_2	78.25	91.63	100.63	78.50	82.67	3977.60
	R_3	83.25	92.13	103.19	86.59	104.16	4210.40
143381419	R_1				5.02	9.36	7.55
	R_2	96.00	93.00	73.38	90.50	97.20	4094.40
	R_3	101.00	114.25	99.06	85.03	86.75	4122.80
30951768	R_1				6.65	6.02	5.69
	R_2	86.25	57.63	75.00	92.06	106.25	4197.60
	R_3	75.75	82.63	98.00	72.13	107.06	4116.00

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1674167085	R_1	6.31	6.01	2.08	1.86	2.03
	R_2	3956.53	3981.04	4000.17	4043.56	4043.96
	R_3	4069.87	3988.08	4072.97	4067.64	4012.36
547023411	R_1	5.78	4.19	3.37	5.55	6.01
	R_2	4168.13	4059.12	4054.11	4053.73	4029.49
	R_3	4080.80	3939.76	3928.34	4044.80	4045.82
985719448	R_1	7.61	3.52	4.55	4.12	2.46
	R_2	3966.33	3984.00	3869.09	3924.18	3996.87
	R_3	4157.73	4166.00	3935.14	4017.51	4033.67
143381419	R_1	3.94	2.10	4.06	3.56	4.42
	R_2	4183.60	3992.16	3964.86	3889.82	3970.15
	R_3	4121.73	4160.80	4148.91	4129.24	4097.13
30951768	R_1	6.73	6.18	5.81	5.19	3.93
	R_2	4028.67	4100.56	4056.63	3984.49	4130.54
	R_3	4046.67	4027.76	4042.34	4028.62	4056.76

CRITICAL VALUES

I) RUNUP TEST ———> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ———> $n<12800$: 123.23
———> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1289898067	R_1				7.38	8.27	2.31
	R_2	84.25	85.50	87.63	99.00	123.17	4206.80
	R_3	73.75	82.75	90.25	100.00	102.08	3994.80
959997498	R_1				3.40	5.59	8.29
	R_2	120.75	85.88	87.56	91.88	115.08	4039.60
	R_3	94.50	87.88	104.50	97.94	108.34	4121.20
370543480	R_1				6.31	10.03	6.94
	R_2	90.00	98.13	86.88	93.59	95.98	4238.00
	R_3	95.75	84.75	94.81	87.59	83.51	4051.20
1263795324	R_1				2.94	7.65	12.37
	R_2	117.75	105.88	95.38	91.00	81.27	3980.80
	R_3	96.50	79.25	77.88	91.97	83.30	3978.00
954561845	R_1				3.91	4.91	6.56
	R_2	81.00	84.38	96.88	111.84	99.58	3968.40
	R_3	76.25	97.38	100.56	106.94	95.47	3967.60

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1289898067	R_1	6.87	6.86	4.70	5.06	4.39
	R_2	4093.47	4020.72	4147.94	4184.13	4189.60
	R_3	4125.20	4185.92	4152.06	4123.69	4038.51
959997498	R_1	5.07	2.03	3.53	3.55	2.61
	R_2	4131.20	4045.20	4080.91	4141.16	4207.82
	R_3	4147.20	4062.32	4066.51	4090.89	4128.22
370543480	R_1	5.16	3.48	5.14	3.11	3.46
	R_2	4094.67	4041.04	4008.97	3948.67	3987.24
	R_3	4008.00	3985.20	3968.40	3960.40	3987.56
1263795324	R_1	7.68	9.42	10.25	7.87	8.27
	R_2	3902.40	3893.76	3886.11	3920.71	3908.91
	R_3	4040.80	3983.84	3992.29	3987.51	4005.82
954561845	R_1	5.65	2.59	5.74	4.14	4.27
	R_2	4030.13	4024.96	4072.69	4086.93	4040.36
	R_3	3930.80	4008.88	4035.89	3989.96	3972.69

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
	R_1				3.44	2.15	4.39
1702824678	R_2	93.50	94.50	83.56	88.56	115.56	3998.00
	R_3	86.25	79.25	81.81	93.81	85.63	3971.00
	R_1				6.73	6.59	9.04
1277220766	R_2	96.25	99.00	84.88	82.72	90.63	4046.00
	R_3	84.50	92.75	82.13	90.06	85.58	4050.00
	R_1				7.63	3.60	2.31
1489356369	R_2	96.25	91.50	94.56	91.59	84.56	4158.80
	R_3	92.00	95.00	87.25	89.88	84.11	4099.20
	R_1				2.56	2.72	8.20
1868098207	R_2	85.00	76.50	92.25	100.84	73.92	4157.60
	R_3	79.75	77.50	99.56	98.25	90.52	4094.80
	R_1				3.67	6.04	7.60
59329321	R_2	72.75	94.50	88.44	85.69	91.13	4095.20
	R_3	88.25	90.50	88.94	77.81	113.05	4026.40

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
	R_1	4.52	5.65	8.75	5.93	6.29
1702824678	R_2	4019.33	4090.24	4094.00	4062.98	4058.95
	R_3	4106.67	4159.52	4129.49	4062.80	4041.67
	R_1	5.20	5.79	6.44	3.91	1.68
1277220766	R_2	4106.40	3948.24	3938.29	3905.42	3965.67
	R_3	4111.67	4046.24	4180.17	4073.60	4175.82
	R_1	1.72	4.86	6.04	7.97	4.81
1489356369	R_2	4101.47	4232.88	4187.09	4182.84	4200.51
	R_3	4042.67	4051.92	4040.63	3920.31	4003.35
	R_1	12.05	8.29	8.09	9.14	7.84
1868098207	R_2	4041.07	4096.16	4083.17	4128.53	4127.35
	R_3	4008.00	4044.40	3981.49	3868.84	3931.78
	R_1	3.79	1.90	2.30	3.41	3.24
59329321	R_2	4051.47	3885.84	3950.23	4028.93	4013.53
	R_3	4093.07	3995.52	3908.00	3979.02	3961.74

CRITICAL VALUES

I) RUNUP TEST ——————————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
└——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1446878011	R_1				12.39	6.83	8.45
	R_2	83.05	90.00	96.50	82.91	112.28	4074.80
	R_3	85.75	94.00	86.75	100.28	80.33	4072.80
384075273	R_1				3.50	4.32	5.03
	R_2	82.25	90.75	78.19	72.97	78.63	4218.80
	R_3	88.25	84.88	98.88	104.53	102.67	3982.40
188220181	R_1				3.69	3.12	4.37
	R_2	87.00	90.38	77.38	96.94	86.59	3987.20
	R_3	109.25	82.25	112.69	91.50	90.47	4104.00
1974130508	R_1				5.22	10.55	9.28
	R_2	109.75	99.25	97.25	98.28	79.83	3958.40
	R_3	84.00	99.50	90.63	85.38	98.97	4046.40
1360328911	R_1				5.48	8.00	1.31
	R_2	78.75	78.63	90.94	102.22	91.31	4080.80
	R_3	79.00	97.50	76.50	96.50	105.16	4140.80

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1446878011	R_1	2.19	5.38	4.97	7.70	7.31
	R_2	4028.93	3984.80	4060.17	4129.78	4120.04
	R_3	4115.20	4102.00	4198.63	4145.11	4096.11
384075273	R_1	6.49	4.92	4.02	4.42	5.82
	R_2	4201.60	4105.84	4025.37	4047.47	3950.00
	R_3	4158.53	4120.08	4036.63	4071.47	4086.87
188220181	R_1	4.57	3.60	4.48	2.68	5.74
	R_2	4008.93	4063.68	4120.97	4071.73	3992.62
	R_3	4069.47	4038.88	4043.09	4019.64	4067.56
1974130508	R_1	11.03	12.32	6.28	5.13	5.23
	R_2	3944.40	3997.12	3965.71	3974.49	3916.04
	R_3	4074.53	4049.44	4095.94	4104.31	4125.93
1360328911	R_1	1.49	1.77	0.90	2.45	2.79
	R_2	3992.13	3985.92	3947.77	3941.42	3895.60
	R_3	3997.20	4047.68	4116.46	4112.18	4091.27

CRITICAL VALUES

I) RUNUP TEST ——————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> n<12800 : 123.23
——> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1600322510	R_1				3.26	5.09	5.31
	R_2	93.25	94.00	77.00	82.78	122.11	3938.40
	R_3	90.00	103.38	102.88	96.31	94.64	4048.00
799507543	R_1				4.72	4.77	4.45
	R_2	98.75	87.00	74.94	82.56	73.75	4046.40
	R_3	104.25	85.75	90.38	72.00	92.09	4096.00
2039198411	R_1				5.74	3.26	8.27
	R_2	98.75	93.50	84.81	72.84	89.15	4010.40
	R_3	72.75	90.75	106.13	89.31	95.28	4068.00
1758564474	R_1				12.43	7.19	8.66
	R_2	77.50	86.38	83.44	82.50	71.45	4058.40
	R_3	82.25	84.88	86.75	110.63	96.58	4047.20
853287183	R_1				2.14	2.65	1.95
	R_2	111.75	99.63	91.56	91.28	99.73	3868.80
	R_3	86.75	95.00	93.25	85.75	104.73	4162.00

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1600322510	R_1	2.95	3.01	3.32	5.96	7.03
	R_2	4008.53	4018.32	4033.37	4046.00	4118.95
	R_3	4092.40	4169.28	3971.89	3970.89	3919.13
799507543	R_1	7.43	6.98	8.44	4.20	5.24
	R_2	3996.13	4099.28	4018.17	4040.13	4057.85
	R_3	4067.87	4092.72	4095.31	4065.42	4012.84
2039198411	R_1	12.16	9.70	10.26	9.57	5.73
	R_2	3968.00	4020.80	3952.69	3967.56	3914.11
	R_3	4109.20	4043.04	4069.09	4061.64	4145.75
1758564474	R_1	3.75	3.14	3.41	2.42	1.25
	R_2	4159.60	4090.00	4078.17	4145.91	4124.44
	R_3	4056.27	4051.92	4074.97	4123.20	4107.64
853287183	R_1	3.33	6.10	3.13	3.99	2.18
	R_2	3989.73	3974.24	3940.97	3960.84	3981.09
	R_3	4121.07	4022.40	4090.51	4064.89	4091.89

CRITICAL VALUES

I) RUNUP TEST ——————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
967370914	R_1				2.26	4.35	4.04
	R_2	81.00	89.13	94.75	97.63	114.48	3948.00
	R_3	89.25	99.75	93.00	100.88	106.88	4129.20
1305541621	R_1				7.03	6.96	3.95
	R_2	99.25	91.88	84.00	75.09	79.63	4187.60
	R_3	72.00	79.75	101.56	99.63	118.09	4129.60
23448094	R_1				2.04	8.07	6.25
	R_2	96.25	71.38	77.88	85.59	91.20	4201.20
	R_3	108.25	112.25	101.00	101.50	95.91	3846.40
2003527858	R_1				5.64	2.51	1.42
	R_2	70.25	82.13	94.06	92.31	84.67	4120.40
	R_3	95.50	97.88	74.63	86.00	101.64	4204.80
202835952	R_1				2.22	2.73	3.23
	R_2	90.50	73.13	89.19	95.53	93.41	4063.60
	R_3	119.00	120.75	116.69	109.13	80.84	4067.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
967370914	R_1	7.37	7.41	4.87	5.83	9.38
	R_2	4000.40	4053.12	4044.63	4038.58	4125.16
	R_3	4146.27	3922.24	3958.57	4048.98	4098.34
1305541621	R_1	5.43	5.42	2.55	5.51	6.39
	R_2	4058.67	4182.48	4150.00	4123.02	4171.02
	R_3	4085.60	4231.76	4132.97	4094.67	4130.07
23448094	R_1	1.55	1.11	1.63	1.93	2.62
	R_2	4022.93	3964.88	3914.80	4036.53	4014.95
	R_3	3972.40	4165.84	4001.49	4058.04	4018.65
2003527858	R_1	4.04	5.97	5.25	4.81	6.11
	R_2	3963.73	4027.84	4045.83	4000.44	4005.78
	R_3	3960.67	3914.08	3931.37	4072.22	4016.95
202835952	R_1	4.30	4.51	5.21	7.05	4.33
	R_2	4124.13	4156.48	4052.11	3998.62	3978.54
	R_3	4138.00	4101.76	4102.97	4130.44	4056.65

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> n<12800 : 123.23
——> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1578282090	R_1				1.11	2.38	2.04
	R_2	73.25	81.38	79.81	82.53	93.97	4101.20
	R_3	107.50	101.38	108.00	97.41	103.28	4116.40
2067032889	R_1				2.23	7.34	5.54
	R_2	71.75	70.25	89.31	99.06	104.12	4149.20
	R_3	78.00	76.38	96.63	111.72	90.92	4216.00
1251948670	R_1				3.32	4.65	8.14
	R_2	70.75	83.63	84.31	75.56	96.73	4141.20
	R_3	86.75	97.50	114.81	98.78	93.58	4035.60
1756937875	R_1				7.43	10.60	5.52
	R_2	93.50	81.75	97.64	81.47	79.95	3958.00
	R_3	117.75	102.50	79.06	83.97	98.20	4091.00
1627347439	R_1				1.99	3.19	3.97
	R_2	122.00	98.38	75.13	91.97	99.61	4171.60
	R_3	90.50	110.13	98.06	99.75	81.66	4045.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1578282090	R_1	4.96	4.01	5.00	5.21	6.37
	R_2	4048.40	4070.16	4034.11	4106.84	4010.18
	R_3	3995.60	3922.04	3915.77	3973.64	4083.96
2067032889	R_1	4.19	7.01	6.18	4.88	6.35
	R_2	4003.87	4012.56	4068.63	4094.71	4036.54
	R_3	4086.27	4157.84	4200.23	4143.29	4097.82
1251948670	R_1	10.15	9.38	6.58	3.81	3.37
	R_2	4024.27	4104.80	4017.49	3958.40	3991.56
	R_3	4126.27	4084.24	4102.69	3923.91	3933.75
1756937875	R_1	3.89	2.69	3.58	4.46	6.09
	R_2	4002.53	3998.00	4000.23	3932.84	3983.13
	R_3	4087.87	4166.24	4193.09	4210.27	4115.13
1627347439	R_1	6.80	5.86	4.11	4.22	6.77
	R_2	3954.53	3886.40	3963.66	3936.71	3843.56
	R_3	3921.47	4031.92	4001.60	3999.91	4053.85

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1650412949	R_1				4.69	7.63	3.01
	R_2	86.75	88.88	72.31	85.75	93.03	4038.00
	R_3	104.50	112.75	106.75	92.81	106.17	4021.60
215061405	R_1				6.84	10.27	9.98
	R_2	76.00	92.50	92.38	108.16	83.27	4114.80
	R_3	77.25	52.25	63.75	77.72	89.83	4043.60
1530738151	R_1				2.30	6.03	6.07
	R_2	107.50	78.25	94.75	93.56	82.97	4079.60
	R_3	106.00	98.38	94.25	101.56	87.28	3980.40
2053545054	R_1				5.95	2.28	2.99
	R_2	106.50	100.13	97.69	103.69	98.23	3950.80
	R_3	79.00	80.13	99.00	79.81	102.27	4194.00
989891314	R_1				9.27	5.82	3.85
	R_2	92.00	110.88	102.19	87.88	102.30	3903.20
	R_3	75.50	101.00	113.69	119.22	85.02	4026.80

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1650412949	R_1	2.10	4.27	6.44	6.58	4.35
	R_2	4092.53	4057.36	4025.94	4049.82	4068.95
	R_3	3893.87	3991.04	3923.83	3903.33	4001.42
215061405	R_1	6.91	6.13	6.42	8.57	10.89
	R_2	3941.73	3959.68	4097.31	4050.89	4079.91
	R_3	4080.13	4116.56	4119.49	4131.07	4172.07
1530738151	R_1	5.69	8.07	4.62	5.88	3.42
	R_2	4039.07	3978.88	3961.09	4037.24	3943.53
	R_3	4108.67	3978.88	3942.74	3971.38	3970.76
2053545054	R_1	5.05	3.29	3.35	5.01	3.90
	R_2	3925.87	3931.36	3907.43	3907.33	3933.35
	R_3	4087.60	4017.52	4011.60	3966.00	4024.98
989891314	R_1	3.31	8.01	5.31	2.81	1.54
	R_2	4073.07	4034.40	3999.20	4014.84	4096.65
	R_3	3998.00	4023.28	4023.60	4046.80	3997.64

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
——> $n>20480$: 4245.00

Initial Seeds		800	1600	3200	6400	12800	20480
		SAMPLE SIZES					
991176826	R_1				4.58	6.71	7.24
	R_2	93.75	97.13	90.81	104.22	101.59	4046.00
	R_3	96.75	85.13	100.63	100.31	105.11	4100.40
1037740495	R_1				7.85	2.61	2.15
	R_2	88.00	97.00	94.56	91.22	98.63	4174.40
	R_3	94.25	89.63	102.31	111.72	106.97	4150.00
1927746234	R_1				1.14	7.69	8.08
	R_2	98.25	78.88	73.13	98.13	111.95	4128.00
	R_3	90.75	89.25	98.94	98.28	116.69	4200.00
368472391	R_1				1.51	6.75	8.27
	R_2	78.25	80.88	90.44	105.69	95.31	4132.00
	R_3	103.75	113.63	119.06	100.97	113.38	4131.60
1227740495	R_1				4.45	6.02	3.67
	R_2	75.50	96.25	66.00	86.63	90.20	4026.00
	R_3	97.50	84.38	77.19	70.53	81.84	4010.80

Initial Seeds		61440	102400	143360	184320	225280
		SAMPLE SIZES				
991176826	R_1	4.14	4.28	5.45	4.25	7.22
	R_2	4186.67	4070.88	4035.66	4090.84	4056.80
	R_3	4150.93	4208.08	4169.54	4196.80	4129.93
1037740495	R_1	3.76	5.78	5.11	6.70	9.30
	R_2	4198.27	4018.80	4128.51	4165.56	4134.87
	R_3	4162.00	3970.24	4053.94	4047.33	4020.84
1927746234	R_1	11.69	7.47	9.39	7.63	4.77
	R_2	4051.47	4089.20	3941.71	4006.27	4004.55
	R_3	4204.93	4241.12	4243.77	4176.80	4211.75
368472391	R_1	10.84	7.20	6.45	9.66	4.90
	R_2	3974.80	4165.20	4149.37	4031.96	4032.87
	R_3	4057.73	4057.76	3992.06	4015.56	4045.13
1227740495	R_1	1.68	4.58	5.55	4.01	3.91
	R_2	4044.00	4161.52	4242.29	4142.98	4147.93
	R_3	4147.60	3974.16	3936.51	3942.00	4017.16

CRITICAL VALUES

I) RUNUP TEST ——————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23 ——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1529370012	R_1				5.57	6.28	8.58
	R_2	77.50	82.50	91.25	90.81	97.17	4119.60
	R_3	88.75	89.38	97.31	98.25	103.41	4138.40
283210790	R_1				8.52	9.54	10.36
	R_2	71.50	82.13	91.25	78.63	82.80	4050.80
	R_3	72.00	84.50	99.31	97.41	94.86	4006.40
1670071181	R_1				1.73	2.12	5.64
	R_2	118.00	95.75	90.63	92.66	80.08	3912.80
	R_3	109.25	96.13	105.81	114.19	106.45	4105.60
157805309	R_1				7.30	7.94	8.00
	R_2	100.00	87.00	80.88	78.69	67.47	4179.20
	R_3	86.25	91.50	104.38	106.03	116.77	4152.00
981222908	R_1				11.23	6.37	5.79
	R_2	85.00	86.38	85.69	74.69	86.41	4022.40
	R_3	100.25	115.88	98.94	99.50	73.58	4106.80

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1529370012	R_1	7.93	5.67	5.20	4.86	5.02
	R_2	4178.93	4111.36	4060.91	4053.07	3994.98
	R_3	4197.47	4202.32	4132.57	4042.31	4003.02
283210790	R_1	9.11	2.75	1.81	1.15	1.17
	R_2	4076.13	3994.08	3991.60	4051.73	3971.71
	R_3	4159.07	4128.88	4080.74	4094.84	4069.78
1670071181	R_1	2.34	4.42	3.45	2.33	2.74
	R_2	3904.40	3827.36	3855.26	3911.78	3989.20
	R_3	4071.60	4075.68	4043.89	3992.93	3980.62
157805309	R_1	5.91	7.81	9.38	12.59	9.46
	R_2	4098.93	4193.84	4132.06	3949.02	3932.22
	R_3	4120.40	4222.48	4215.89	4148.31	4156.73
981222908	R_1	2.40	2.87	3.93	4.84	4.49
	R_2	4123.60	4153.20	4151.31	4173.38	4143.20
	R_3	4102.67	4010.64	4105.20	4159.07	4105.05

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS ——> n<12800 : 123.23

——> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1914200114	R_1				3.27	2.83	6.11
	R_2	88.25	90.00	76.25	102.69	94.88	4174.80
	R_3	102.75	80.88	62.81	77.41	104.42	4071.60
2029045855	R_1				6.87	10.91	4.82
	R_2	86.50	90.13	71.13	103.72	113.06	4118.40
	R_3	74.25	88.00	96.44	119.66	115.48	4145.20
132406470	R_1				1.32	4.10	3.71
	R_2	85.00	76.00	98.75	118.88	122.42	4026.40
	R_3	92.25	98.75	106.00	99.75	106.32	4046.80
593840624	R_1				6.12	5.09	5.05
	R_2	99.50	93.25	102.00	109.69	89.25	3977.60
	R_3	105.75	101.75	96.50	99.69	85.28	3984.40
827995463	R_1				2.65	6.48	3.25
	R_2	71.25	75.13	80.94	83.91	83.52	4050.80
	R_3	92.50	98.75	111.38	105.47	92.03	4041.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1914200114	R_1	0.75	1.38	2.78	2.29	5.06
	R_2	4079.87	4109.68	4075.31	4135.16	4039.78
	R_3	4094.00	3941.68	4000.91	4001.11	4062.69
2029045855	R_1	4.28	1.18	2.09	2.11	1.61
	R_2	4125.20	4181.36	4057.54	3990.00	4075.60
	R_3	4225.20	4211.28	4189.94	4212.31	4223.53
132406470	R_1	1.41	1.01	1.31	2.16	1.13
	R_2	4030.67	4047.04	4055.49	3994.44	4039.53
	R_3	4148.93	4053.76	4069.14	4093.20	4064.25
5933840624	R_1	4.21	3.20	1.05	1.09	0.59
	R_2	3980.00	4069.44	4116.29	3988.67	3934.76
	R_3	4084.40	4056.64	3955.31	4001.20	3942.22
827995463	R_1	1.19	6.10	4.66	3.11	2.33
	R_2	4038.53	4061.04	4077.66	4156.13	4100.22
	R_3	3927.33	3949.44	3978.51	4034.62	4013.56

CRITICAL VALUES

I) RUNUP TEST ——> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23

——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1109160874	R_1				9.55	4.29	6.24
	R_2	105.50	84.25	77.88	79.34	108.39	3993.20
	R_3	84.50	75.50	87.38	100.28	86.30	4064.80
1715455718	R_1				4.04	1.11	3.61
	R_2	80.75	86.63	88.25	82.66	87.50	4188.00
	R_3	103.25	85.25	100.69	102.31	113.64	4080.40
1132396986	R_1				8.26	4.20	1.05
	R_2	78.00	89.25	108.50	85.41	121.98	3995.00
	R_3	107.75	101.75	92.63	112.66	109.50	4068.00
1713250490	R_1				0.67	1.79	6.24
	R_2	82.50	79.75	89.69	90.09	87.47	4003.20
	R_3	88.25	91.50	102.56	103.19	84.64	4018.00
654138949	R_1				2.65	2.94	3.14
	R_2	86.00	101.00	111.63	107.22	98.81	4091.20
	R_3	118.00	120.75	101.06	103.50	97.00	4175.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1109160874	R_1	3.74	3.39	4.43	5.19	3.37
	R_2	4064.80	3991.52	4030.86	4076.44	4143.60
	R_3	4031.07	4072.40	4008.17	4007.38	4067.35
1715455718	R_1	2.01	4.51	3.23	2.28	5.00
	R_2	4048.13	3957.52	4079.54	4114.67	4163.24
	R_3	4036.13	4240.72	4202.69	3984.58	4037.20
1132396986	R_1	1.96	4.88	5.30	8.64	7.95
	R_2	4090.13	4025.04	3985.94	3993.56	4043.89
	R_3	4146.40	4123.76	4052.17	3997.38	3988.04
1713250490	R_1	3.11	4.02	3.02	2.25	2.11
	R_2	4117.73	4128.72	4084.69	3984.58	4037.20
	R_3	3963.47	3938.80	4047.89	4130.93	4033.64
654138949	R_1	3.22	4.14	9.38	9.66	11.71
	R_2	3988.13	3975.12	4020.51	4092.67	4093.24
	R_3	3918.40	4053.52	4158.74	4097.38	4129.64

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS

→ $n < 12800$: 123.23

→ $n > 20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1711135126	R_1				2.00	2.84	3.58
	R_2	106.00	92.75	94.00	104.13	122.09	4068.00
	R_3	89.50	88.50	81.00	88.84	102.63	4068.40
374236500	R_1				8.65	11.20	8.62
	R_2	74.75	88.75	76.81	109.75	87.95	3956.80
	R_3	102.25	77.25	102.81	84.25	89.61	4151.20
145026474	R_1				4.38	3.39	2.40
	R_2	105.75	88.75	106.31	98.41	77.98	3980.00
	R_3	110.75	117.50	96.56	105.25	94.45	4145.20
986267275	R_1				1.87	4.34	11.55
	R_2	76.50	81.63	109.69	111.75	110.94	4021.60
	R_3	92.50	94.50	95.50	99.50	110.95	4016.00
1595040427	R_1				4.96	4.34	2.16
	R_2	95.00	79.50	95.06	99.47	113.70	4160.80
	R_3	100.00	102.88	85.69	115.06	103.78	4225.60

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1711135126	R_1	5.38	3.41	5.05	5.16	4.70
	R_2	4106.87	3992.08	4031.77	4091.33	3981.89
	R_3	3945.60	3901.20	3963.37	4006.09	4065.02
374236500	R_1	0.80	6.60	9.32	7.28	5.44
	R_2	4028.53	4136.16	4146.06	4120.58	4129.24
	R_3	4105.20	4085.52	4061.26	4006.40	3970.36
145026474	R_1	6.13	4.21	5.00	4.14	7.76
	R_2	4002.00	4179.20	4114.29	4163.96	4208.25
	R_3	4043.73	3943.04	3872.11	4003.47	4009.65
986267275	R_1	11.38	8.11	6.45	3.87	2.86
	R_2	3980.53	3862.16	4014.23	3955.56	4025.02
	R_3	3997.47	4066.40	4034.51	4054.49	4095.42
1595040427	R_1	7.99	4.93	7.44	4.49	4.45
	R_2	3992.00	3875.44	3811.77	3906.62	3952.87
	R_3	4180.67	4088.88	4121.31	4096.13	4121.93

CRITICAL VALUES

I) RUNUP TEST ————————> 12.59

II) UNIFORMITY TEST IN ONE AND TWO DIMENSIONS → n<12800 : 123.23 ; n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
378061762	R_1				9.69	11.27	7.89
	R_2	88.50	93.13	108.31	84.25	77.34	4072.80
	R_3	111.75	94.75	99.56	108.63	86.56	4100.40
2033344783	R_1				1.73	0.41	1.93
	R_2	75.75	89.00	97.00	88.78	90.41	4146.80
	R_3	112.75	100.38	104.69	114.06	117.84	4094.80
523936387	R_1				4.10	5.67	8.73
	R_2	95.25	94.13	97.56	87.13	82.66	4196.40
	R_3	85.50	101.00	78.00	76.75	86.80	4063.60
278609684	R_1				3.02	2.80	6.06
	R_2	79.75	76.75	90.50	98.03	95.22	4208.80
	R_3	81.25	75.38	106.63	101.78	86.30	4181.60
832602364	R_1				2.99	5.87	3.00
	R_2	74.50	88.38	96.44	66.78	80.88	4100.80
	R_3	81.00	88.50	74.06	81.38	91.86	4177.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
378061762	R_1	7.64	3.22	3.88	3.12	3.37
	R_2	4241.07	4095.60	4179.89	4190.80	4121.71
	R_3	4148.93	4203.12	4080.69	4064.40	4120.98
2033344783	R_1	9.49	4.34	6.45	5.94	4.53
	R_2	4103.20	4105.36	4119.37	4102.00	4170.62
	R_3	4104.67	4107.04	4048.29	4087.51	4068.87
523936387	R_1	11.50	11.35	11.21	10.40	9.34
	R_2	4013.33	4021.52	4018.69	4150.62	4108.47
	R_3	4167.47	4127.28	4094.00	4069.02	4050.87
278609684	R_1	4.22	6.10	5.11	4.03	3.91
	R_2	4133.73	4103.20	3988.06	3984.93	3959.49
	R_3	4051.33	3994.80	4009.43	4076.76	4072.40
832602364	R_1	4.87	3.81	5.99	3.80	4.96
	R_2	4059.73	4120.16	4107.49	4123.16	4110.84
	R_3	4104.27	4090.64	4116.06	4158.98	4148.15

CRITICAL VALUES

I) RUNUP TEST ——————————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> n<12800 : 123.23
——> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
382388249	R_1				1.34	5.44	2.07
	R_2	71.75	73.63	95.38	112.47	83.29	4169.60
	R_3	89.00	89.25	102.69	80.75	71.27	4076.80
1042192142	R_1				2.77	3.39	4.58
	R_2	78.75	89.38	76.25	105.84	104.95	4230.00
	R_3	87.25	97.25	104.56	98.94	93.97	4128.00
1988091511	R_1				4.58	5.43	7.33
	R_2	69.75	98.75	105.13	108.00	110.06	4011.20
	R_3	93.00	100.38	92.56	91.38	63.34	4139.60
1835767197	R_1				9.11	5.65	5.90
	R_2	87.75	88.88	83.57	85.38	94.81	4033.60
	R_3	89.00	92.75	108.88	84.63	93.59	4144.80
522512559	R_1				4.97	2.48	1.85
	R_2	73.25	91.13	91.69	94.44	89.25	3997.60
	R_3	96.00	94.38	81.31	81.19	89.97	3975.60

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
382388249	R_1	3.66	2.35	5.66	4.48	5.65
	R_2	3934.27	3928.16	4043.09	4047.38	4022.00
	R_3	4061.20	4100.88	4117.14	4092.22	4129.96
1042192142	R_1	7.27	7.32	5.31	5.89	6.06
	R_2	4103.73	4135.28	4059.83	4003.64	4114.95
	R_3	4092.27	4155.36	4042.63	4013.38	3995.75
1988091511	R_1	11.54	11.13	6.96	6.69	9.33
	R_2	4071.20	4114.48	4114.23	4037.16	3991.78
	R_3	4115.07	4147.68	4186.69	4162.31	4111.67
1835767197	R_1	1.69	2.31	3.80	6.21	10.92
	R_2	4071.73	4057.12	4062.34	3956.71	3941.75
	R_3	4221.20	4168.40	4133.54	4119.20	4144.58
522512559	R_1	10.59	4.83	3.53	3.04	3.49
	R_2	4007.07	4008.72	3936.06	4009.33	4079.75
	R_3	4215.87	4133.20	4210.80	4120.40	4166.25

CRITICAL VALUES

I) RUNUP TEST ——————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> n<12800 : 123.23
——> n>20480 : 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
870576586	R_1				5.90	9.67	5.38
	R_2	79.25	86.63	90.06	99.31	93.70	4087.20
	R_3	109.75	103.75	109.25	105.84	106.67	4038.00
1564597822	R_1				8.36	8.18	6.55
	R_2	76.50	93.88	108.00	95.63	88.81	4032.00
	R_3	101.00	87.00	82.56	93.03	97.03	4134.00
1546676231	R_1				3.26	2.00	4.78
	R_2	81.25	106.25	97.25	91.25	118.53	4196.00
	R_3	83.25	97.25	78.00	96.63	80.05	4016.00
555492368	R_1				4.28	5.32	2.49
	R_2	70.75	82.50	98.88	95.13	99.75	4199.20
	R_3	91.25	85.75	97.81	121.69	112.41	4083.60
1737119562	R_1				1.57	1.40	3.21
	R_2	71.50	88.25	72.00	92.66	95.92	4010.80
	R_3	110.00	99.75	95.25	99.78	84.73	3985.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
870576586	R_1	9.32	7.36	7.96	6.48	7.27
	R_2	3955.73	4026.80	3956.91	4003.33	3994.76
	R_3	4157.20	4138.40	4042.40	4016.04	4107.42
1564597822	R_1	1.75	0.75	3.73	5.54	5.14
	R_2	4009.33	4178.00	4063.37	4089.51	4100.11
	R_3	4197.07	4075.36	4010.11	4030.98	4056.51
1546676231	R_1	5.10	1.20	5.50	6.31	6.30
	R_2	4065.33	4046.16	4001.54	3988.40	4151.13
	R_3	3964.67	3990.40	4084.46	4193.33	4205.56
555492368	R_1	1.61	3.52	6.36	5.66	8.61
	R_2	4058.27	3951.68	3955.49	4029.11	3934.80
	R_3	4159.47	4037.76	4032.97	4067.87	4068.91
1737119562	R_1	2.33	7.90	11.21	10.96	7.16
	R_2	4060.40	4043.04	3905.26	3920.13	3914.58
	R_3	3900.00	4073.92	4033.09	3976.04	3961.89

CRITICAL VALUES

I) RUNUP TEST ——————————> 12.59
II) UNIFORMITY TEST IN
ONE AND TWO DIMENSIONS ——> $n<12800$: 123.23
└——> $n>20480$: 4245.00

Initial Seeds		SAMPLE SIZES					
		800	1600	3200	6400	12800	20480
1357014479	R_1				7.97	6.94	7.97
	R_2	90.75	99.00	92.19	78.31	94.50	3954.40
	R_3	86.25	101.75	97.00	87.94	107.82	3997.20
1457702813	R_1				7.53	4.14	5.18
	R_2	83.00	83.63	88.31	98.97	98.09	3967.20
	R_3	69.50	86.13	92.56	84.78	83.25	4064.00
1435644507	R_1				4.29	3.44	2.95
	R_2	88.75	80.63	90.63	74.47	99.50	4004.40
	R_3	84.50	86.38	92.19	83.94	89.63	4104.80
1818969274	R_1				7.68	5.91	6.03
	R_2	115.00	113.13	101.56	74.88	80.33	3986.00
	R_3	92.00	113.00	109.50	87.47	94.30	4158.80
814509814	R_1				11.67	2.69	3.31
	R_2	86.25	78.50	91.81	98.09	90.06	4059.60
	R_3	114.50	118.25	101.38	107.19	116.14	4015.20

Initial Seeds		SAMPLE SIZES				
		61440	102400	143360	184320	225280
1357014479	R_1	5.16	10.61	8.46	7.24	7.46
	R_2	4054.40	4106.80	4013.77	4126.22	4213.24
	R_3	4083.33	4148.56	4070.57	3952.13	3979.16
1457702813	R_1	2.14	1.20	1.80	1.58	1.57
	R_2	3824.67	3972.96	4202.57	4136.36	4123.45
	R_3	4137.73	4122.72	4138.57	4106.27	4132.80
1435644507	R_1	9.20	10.54	10.51	11.27	6.59
	R_2	4105.07	4147.92	4168.11	4224.80	4221.09
	R_3	4201.07	4196.32	4218.17	4215.11	4228.40
1818969274	R_1	4.20	3.12	3.77	1.90	3.24
	R_2	4071.60	4035.04	4013.54	4017.02	4109.63
	R_3	4232.67	4112.00	4138.57	4018.13	4002.25
814509814	R_1	1.99	1.31	3.35	3.08	2.56
	R_2	3993.73	4003.60	3981.26	4056.53	4158.00
	R_3	4092.67	4102.32	4024.80	4182.40	4172.98

APPENDIX D

PERFORMANCE OF CONFIDENCE INTERVAL METHODS FOR THE MB-PARAMETER VALUES : THE CASE OF 95% NOMINAL CONFIDENCE LEVEL

T A B L E   D1

AR(1) : Performance of confidence interval methods at the true MB-parameter values

Number of replications : 400

φ=0.4074

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.9450	.9500	.9500	.9725	.9400
	$\widehat{EHL}_i$	(2.1350)	(2.1129)	(2.1129)	(.4936)	(.4548)
	$\widehat{VHL}_i$	(2.6318)	(2.4770)	(2.4770)	(.0200)	(.0314)
128	$\widehat{CVR}_i$	.9675	.9525	.9525	.9675	.9550
	$\widehat{EHL}_i$	(1.4805)	(1.4785)	(1.4785)	(.3328)	(.3253)
	$\widehat{VHL}_i$	(1.3043)	(1.2273)	(1.2273)	(.0077)	(.0136)
256	$\widehat{CVR}_i$	.9400	.9475	.9475	.9650	.9325
	$\widehat{EHL}_i$	(1.0101)	(.9572)	(.9572)	(.2218)	(.2183)
	$\widehat{VHL}_i$	(.6595)	(.4953)	(.4953)	(.0018)	(.0049)
512	$\widehat{CVR}_i$	.9525	.9225	.9225	.9475	.9550
	$\widehat{EHL}_i$	(.7223)	(.6741)	(.6741)	(.1503)	(.1511)
	$\widehat{VHL}_i$	(.2835)	(.2582)	(.2582)	(.0007)	(.0016)

φ=0.99

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$	.8800	.8950	.8950	.7200	.3125
	$\widehat{EHL}_i$	(23.336)	(25.618)	(25.618)	(8.7486)	(2.5406)
	$\widehat{VHL}_i$	(295.09)	(336.34)	(336.34)	(27.968)	(1.6646)
128	$\widehat{CVR}_i$	.9075	.9050	.9050	.8300	.5150
	$\widehat{EHL}_i$	(29.411)	(31.945)	(31.945)	(10.647)	(3.7358)
	$\widehat{VHL}_i$	(449.84)	(528.99)	(528.99)	(35.746)	(3.3696)
256	$\widehat{CVR}_i$	.9425	.9400	.9400	.8825	.6475
	$\widehat{EHL}_i$	(32.727)	(33.693)	(33.693)	(10.635)	(4.6579)
	$\widehat{VHL}_i$	(548.32)	(620.93)	(620.93)	(31.488)	(5.2501)
512	$\widehat{CVR}_i$	.9375	.9450	.9450	.9150	.7525
	$\widehat{EHL}_i$	(30.726)	(30.822)	(30.822)	(9.2207)	(5.2266)
	$\widehat{VHL}_i$	(569.32)	(550.47)	(550.47)	(20.457)	(6.2919)

TABLE D2

M/M/1 : Performance of confidence interval methods at the true MB-parameter values

Number of replications : 400

τ=0.20

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9275 (.2354) (.0565)	.9125 (.2102) (.0617)	.9125 (.2102) (.0617)	.8325 (.0528) (.0017)	.7700 (.0416) (.0012)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9500 (.1963) (.0331)	.9400 (.1829) (.0329)	.9400 (.1829) (.0329)	.8775 (.0408) (.0007)	.8625 (.0399) (.0008)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9350 (.1320) (.0138)	.9250 (.1301) (.0133)	.9250 (.1301) (.0133)	.8850 (.0283) (.0002)	.8725 (.0285) (.0002)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9375 (.1046) (.0074)	.9350 (.1019) (.0075)	.9350 (.1019) (.0075)	.9100 (.0208) (.0001)	.8850 (.0211) (.0001)

τ=0.80

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8950 (14.527) (204.93)	.8950 (15.158) (238.11)	.8950 (15.158) (238.11)	.7725 (5.3134) (19.187)	.5225 (1.9183) (1.9463)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8875 (14.065) (228.74)	.8775 (14.524) (242.62)	.8775 (14.524) (242.62)	.7975 (4.7743) (16.663)	.6275 (2.2287) (3.1316)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8975 (13.948) (301.39)	.9000 (13.923) (287.86)	.9000 (13.923) (287.86)	.8200 (4.0468) (15.207)	.6600 (2.3115) (4.5997)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (11.359) (145.47)	.9350 (11.417) (163.33)	.9350 (11.417) (163.33)	.8375 (3.0321) (6.0322)	.7425 (2.1213) (3.0236)

TABLE D3

AR(2) : Performance of confidence interval methods at the true MB-parameter values

Number of replications : 400

$\varphi_1=0.75$, $\varphi_2=-0.50$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9350 (1.6798) (1.7556)	.9650 (1.7801) (1.8481)	.9425 (.3299) (.0016)	.9475 (.4248) (.0193)	.9325 (.3880) (.0137)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9500 (1.1675) (.8281)	.9575 (1.1966) (.8215)	.9600 (.2316) (.0004)	.9725 (.2732) (.0060)	.9650 (.2584) (.0052)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9600 (.8012) (.3363)	.9500 (.8442) (.4025)	.9625 (.1632) (.0001)	.9600 (.1795) (.0018)	.9550 (.1721) (.0020)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9675 (.5391) (.1703)	.9650 (.5662) (.1723)	.9650 (.5662) (.1723)	.9600 (.1221) (.0005)	.9400 (.1188) (.0007)

$\varphi_1=0.99$, $\varphi_2=-0.90$

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9400 (1.6687) (1.7685)	.9175 (1.7436) (1.8640)	.9175 (1.7436) (1.8640)	.9725 (.7213) (.1307)	.9825 (.4154) (.0103)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9650 (1.1197) (.6449)	.9500 (1.1102) (.7359)	.9500 (1.1102) (.7359)	.9650 (.3435) (.0242)	.9750 (.2614) (.0050)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9625 (.7555) (.3101)	.9675 (.8116) (.3562)	.9675 (.8116) (.3562)	.9700 (.2044) (.0060)	.9725 (.1712) (.0020)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9600 (.4745) (.1267)	.9475 (.4941) (.1540)	.9475 (.4941) (.1540)	.9550 (.1208) (.0015)	.9600 (.1110) (.0009)

T A B L E   D4

AR(1) : Performance of confidence interval methods at the estimated MB-parameter values

Number of replications : 400

φ=0.4074

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9475 (2.1282) (2.6297)	.9500 (2.0993) (2.4481)	.9500 (2.0962) (2.4602)	.9675 (.4916) (.0209)	.9400 (.4543) (.0313)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9675 (1.4719) (1.3097)	.9525 (1.4784) (1.2276)	.9525 (1.4692) (1.2256)	.9650 (.3282) (.0073)	.9550 (.3247) (.0135)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9400 (1.0039) (.6539)	.9475 (.9572) (.4953)	.9475 (.9536) (.4954)	.9675 (.2218) (.0020)	.9325 (.2178) (.0049)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9525 (.7223) (.2835)	.9225 (.6741) (.2582)	.9225 (.6741) (.2582)	.9625 (.1514) (.0007)	.9525 (.1509) (.0016)

φ=0.99

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8800 (23.336) (295.09)	.8950 (25.618) (336.34)	.8950 (25.618) (336.34)	.7100 (8.5993) (27.408)	.3175 (2.5867) (1.7382)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9075 (29.411) (449.84)	.9050 (31.945) (528.99)	.9050 (31.945) (528.99)	.8175 (10.103) (32.710)	.5200 (3.8392) (3.5615)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9425 (32.727) (548.32)	.9400 (33.693) (620.93)	.9400 (33.693) (620.93)	.8750 (9.9538) (28.011)	.6700 (4.8010) (5.5013)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9375 (30.726) (569.32)	.9450 (30.822) (550.47)	.9450 (30.822) (550.47)	.8900 (8.5467) (18.633)	.7400 (5.3260) (6.5624)

T A B L E D5

M/M/1 : Performance of confidence interval methods at the estimated MB-parameter values

Number of replications : 400

τ=0.20

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9275 (.2354) (.0565)	.9125 (.2102) (.0617)	.9125 (.2102) (.0617)	.8375 (.0528) (.0019)	.8100 (.0491) (.0016)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9500 (.1963) (.0331)	.9400 (.1829) (.0329)	.9400 (.1829) (.0329)	.8800 (.0407) (.0007)	.8625 (.0397) (.0008)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9350 (.1320) (.0137)	.9250 (.1301) (.0133)	.9250 (.1301) (.0133)	.8825 (.0282) (.0002)	.8775 (.0285) (.0003)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9375 (.1046) (.0074)	.9350 (.1019) (.0075)	.9350 (.1019) (.0075)	.9025 (.0208) (.0001)	.8850 (.0211) (.0001)

τ=0.80

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8950 (14.527) (204.93)	.8950 (15.158) (238.11)	.8950 (15.158) (238.11)	.7450 (4.8010) (16.767)	.5375 (2.0271) (2.1502)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8875 (14.065) (228.74)	.8775 (14.524) (242.62)	.8775 (14.524) (242.62)	.7750 (4.2268) (14.728)	.6375 (2.3432) (3.3924)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8975 (13.948) (301.39)	.9000 (13.923) (287.86)	.9000 (13.923) (287.86)	.7825 (3.6261) (15.774)	.6775 (2.4054) (4.5855)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9200 (11.359) (145.47)	.9350 (11.417) (163.33)	.9350 (11.417) (163.33)	.8150 (2.7441) (7.3729)	.7575 (2.1615) (2.8973)

T A B L E D6

AR(2) : Performance of confidence interval methods at the estimated MB-parameter values

Number of replications : 400

φ_1=0.75 , φ_2=-0.50

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9350 (1.6763) (1.7600)	.9650 (1.7801) (1.8481)	.9625 (1.3363) (1.7483)	.9475 (.4281) (.0201)	.9325 (.3871) (.0138)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9500 (1.1663) (.8297)	.9575 (1.1966) (.8215)	.9650 (1.0509) (.7968)	.9725 (.2733) (.0060)	.9675 (.2580) (.0051)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9600 (.7997) (.3372)	.9500 (.8442) (.4025)	.9550 (.8174) (.4080)	.9600 (.1793) (.0018)	.9550 (.1721) (.0020)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9675 (.5381) (.1707)	.9650 (.5662) (.1723)	.9650 (.5544) (.1739)	.9575 (.1222) (.0006)	.9375 (.1188) (.0007)

φ_1=0.99 , φ_2=-0.90

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9400 (1.6683) (1.7691)	.9175 (1.7436) (1.8640)	.9175 (1.7395) (1.8645)	.9525 (.6790) (.1259)	.9775 (.4117) (.0102)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9650 (1.1197) (.6449)	.9500 (1.1103) (.7359)	.9500 (1.1103) (.7359)	.9625 (.3316) (.0213)	.9750 (.2629) (.0025)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9625 (.7555) (.3101)	.9675 (.8116) (.3562)	.9675 (.8116) (.3562)	.9650 (.1973) (.0054)	.9725 (.1712) (.0021)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9600 (.4775) (.1267)	.9475 (.4941) (.1540)	.9475 (.4941) (.1540)	.9425 (.1187) (.0014)	.9575 (.1111) (.0009)

T A B L E D7
Interactive Computer Model : Performance of confidence interval methods at the estimated MB-parameter values

Number of replications : 400

Nominal Confidence Level : 90%

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.5700 (4.3116) (18.357)	.5875 (4.7957) (21.907)	.5875 (4.7957) (21.907)	.3850 (1.8702) (2.1998)	.2125 (.9560) (.5297)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7200 (5.4093) (23.853)	.7325 (5.6435) (25.204)	.7325 (5.6435) (25.204)	.5275 (1.9360) (1.7323)	.3825 (1.2921) (.8138)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7925 (5.8557) (27.263)	.7975 (6.0207) (28.168)	.7975 (6.0207) (28.168)	.6300 (1.8500) (1.1678)	.5425 (1.4949) (.9163)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8400 (4.7457) (16.897)	.8050 (4.7872) (18.029)	.8050 (4.7872) (18.029)	.7000 (1.5491) (.5532)	.6800 (1.4265) (.6294)

Nominal Confidence Level : 95%

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7425 (8.6779) (74.362)	.7725 (9.6520) (88.741)	.7725 (9.6520) (88.741)	.4700 (2.5820) (4.2291)	.2725 (1.3011) (.9809)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8375 (10.887) (96.625)	.8400 (11.359) (102.10)	.8400 (11.359) (102.10)	.6000 (2.5751) (3.1063)	.4750 (1.7438) (1.4824)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8850 (11.786) (110.44)	.8875 (12.118) (114.04)	.8875 (12.118) (114.04)	.7150 (2.3842) (1.9607)	.6250 (1.9968) (1.6378)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8950 (9.5514) (68.448)	.8975 (9.6351) (73.032)	.8975 (9.6351) (73.032)	.7675 (1.9491) (.8837)	.7325 (1.8789) (1.0957)

T A B L E D8

Time-shared Computer Model : Performance of confidence interval methods at the estimated MB-parameter values

Number of replications : 300

Nominal Confidence Level : 90%

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.7733 (7.5635) (43.315)	.8300 (8.3835) (42.497)	.8300 (8.3835) (42.497)	.6100 (2.6602) (2.4630)	.5100 (1.0307) (1.0884)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8400 (8.1783) (42.226)	.8600 (8.4234) (45.781)	.8600 (8.4234) (45.781)	.6033 (2.2315) (1.7993)	.5867 (2.0179) (1.3023)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8567 (6.9432) (27.787)	.8633 (6.7511) (27.059)	.8633 (6.7511) (27.059)	.6200 (1.9025) (.8015)	.6467 (1.9643) (.9071)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.8767 (5.2678) (16.796)	.8867 (5.3650) (15.653)	.8867 (5.3650) (15.653)	.6500 (1.5394) (.3886)	.7000 (1.6417) (.4805)

Nominal Confidence Level : 95%

n	Statist. Criteria	NOBM	AREA	Combined NOBM-AREA	SPEC	OVBM
64	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9000 (15.223) (175.46)	.9100 (16.873) (172.15)	.9100 (16.873) (172.15)	.6933 (3.4727) (4.5516)	.6433 (2.7006) (1.9701)
128	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9133 (16.460) (171.05)	.9200 (16.954) (185.45)	.9200 (16.954) (185.45)	.6767 (2.8384) (3.1052)	.6900 (2.6469) (2.3369)
256	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9300 (13.974) (112.56)	.9067 (13.588) (109.61)	.9067 (13.588) (109.61)	.7067 (2.3712) (1.3170)	.7533 (2.5359) (1.5962)
512	$\widehat{CVR}_i$ $\widehat{EHL}_i$ $\widehat{VHL}_i$	.9133 (10.602) (68.039)	.9433 (10.798) (63.410)	.9433 (10.798) (63.410)	.7233 (1.8912) (.6155)	.7600 (2.0842) (.8272)

APPENDIX E

OPTIMUM PERFORMANCE OF CONFIDENCE INTERVAL METHODS: THE CASE OF 95% NOMINAL CONFIDENCE LEVEL

T A B L E E1

AR(1) : Optimum performance of confidence interval methods

Number of Replications : 400 , $\varepsilon = 0.025$

$\varphi = 0.4074$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.8923	.6778			
	AREA	.9364	.6662			
	NOBM-AREA	.8848	.6387			
	SPEC	.5835	.0788			
	OVBM	.4253	.0209			
128	NOBM	.5603	.2703			
	AREA	.6658	.3334			
	NOBM-AREA	.5595	.2545			
	SPEC	.4139	.0366			
	OVBM	.3055	.0114			
256	NOBM	.3555	.1138			
	AREA	.4056	.1095			
	NOBM-AREA	.3463	.0862			
	SPEC	.3166	.0232			
	OVBM	.2094	.0031			
512	NOBM	.2367	.0422			
	AREA	.2633	.0474			
	NOBM-AREA	.2298	.0384			
	SPEC	.2143	.0108			
	OVBM	.1534	.0012			

$\varphi = 0.7778$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.9200	3.302	8.757
	AREA	5.442	14.488			
	NOBM-AREA	3.415	7.418			
	SPEC	1.514	.450			
	OVBM			.8850	1.042	.222
128	NOBM	.938	.120			
	AREA	2.667	4.422			
	NOBM-AREA	1.943	2.870			
	SPEC	.972	.119			
	OVBM	.790	.077			
256	NOBM	1.180	1.092			
	AREA	1.232	1.094			
	NOBM-AREA	1.160	1.038			
	SPEC	.776	.107			
	OVBM	.572	.040			
512	NOBM	.826	.476			
	AREA	1.011	.688			
	NOBM-AREA	.962	.656			
	SPEC	.597	.071			
	OVBM	.387	.009			

T A B L E E1 (Cont..)

$\varphi = 0.9630$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	17.50	157.50			
	AREA	18.04	181.13			
	NOBM-AREA	18.04	181.13			
	SPEC	11.95	55.96			
	OVBM			.6325	2.72	1.99
128	NOBM	16.15	161.39			
	AREA	16.46	173.48			
	NOBM-AREA	16.46	173.48			
	SPEC	8.35	25.71			
	OVBM			.7300	2.83	1.98
256	NOBM	14.40	135.59			
	AREA	15.25	138.17			
	NOBM-AREA	15.25	138.17			
	SPEC	6.22	12.08			
	OVBM			.8150	2.79	1.85
512	NOBM	7.56	40.66			
	AREA	7.94	41.04			
	NOBM-AREA	7.51	39.69			
	SPEC	3.68	2.98			
	OVBM			.9025	2.42	1.14

$\varphi = 0.99$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.8800	23.34	295.09
	AREA			.8950	25.62	336.34
	NOBM-AREA			.8950	25.62	336.34
	SPEC	32.57	465.46			
	OVBM			.3850	3.27	3.33
128	NOBM			.9075	29.41	449.84
	AREA			.9050	31.95	528.99
	NOBM-AREA			.9050	31.95	528.99
	SPEC	24.71	234.80			
	OVBM			.5675	4.43	5.09
256	NOBM	32.73	548.32			
	AREA	33.69	620.93			
	NOBM-AREA	33.69	620.93			
	SPEC	22.35	203.37			
	OVBM			.7075	5.26	6.63
512	NOBM	30.73	569.32			
	AREA	30.82	550.47			
	NOBM-AREA	30.82	550.47			
	SPEC	12.99	52.80			
	OVBM			.7875	5.57	7.10

TABLE E2

AR(1) : Parameter values for optimum performance of the confidence interval methods

Number of Replications : 400 , ε=0.025

φ = 0.4074 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,16	1,2,4,8	1,2,4,8	3-7	4-31
128	4,8,32	1,2,4,8, 16	1,4,16	3,4,53,56,58, 59-61,66,67	4-48
256	2,4,8,16, 32	1,2,4,8, 16	1,2,4,8, 16,32	5-163	6-50, 63-65
512	2,8,16, 32,64,128	1,2,4,8, 16,32	1,2,4,8, 16,32,64	4-23,32-45,63-94, 113-126,132-155, 166-170,211, 216-232	4-216

φ = 0.7778 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	21-39,41	22
128	2,4,8	1,2	1,2	15-32	22-30,33,36-39
256	2,8,16	1,2	1,4,8	13-24,50-54, 62-95	20-59,71-74, 79,80
512	2,8,16	1,2,4	1,2,4	21-352	32,33,37-45,52

φ = 0.9630 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	55-57	24,25
128	2	1	1	95-104	55
256	2	1	1	133-188	78-84,88
512	2,4	2	1	146-231	118-125

φ = 0.99 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	60-62	24,25
128	2	1	1	111-118	59,67,68
256	2	1	1	191-212	99
512	2	1	1	294-396	234-239

T A B L E E2 (Cont...)

$\varphi = 0.4074$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,4,8,16	1,2,4,8	1,2,4,8	3-24,26-40	3-30,32-34
128	2,4,8, 16,32	1,2,4,16	1,2,4,8, 16	3-82	3-71,74
256	2,4,8,16, 32,64	1,2,4,8, 16	1,2,4,8, 16,32	3-198	4-67,70,73, 74,76-80
512	2,4,8,16 32,64,128	1,2,4,8, 16,32	1,2,4,8, 16,32,64	4-343,360 363-384	5-243, 250-258

$\varphi = 0.7778$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,4	1	1,2	17-35,38	25
128	4,8	1,2	1,2,4	11-60	18-45
256	2,4,8,16	1,2,4,8	1,2,4,8	11-24,26-160	13-121,143
512	2,4,8,16	1,2,8	1,2,4	20-369	28-46,71-95, 97,98,173

$\varphi = 0.9630$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	49-57	27,28
128	2	1	1	81-105	55-60
256	2	1	1	124-194	97
512	2,4	1,2	1,2	115-335	161-165,170-173

$\varphi = 0.99$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	59-62	27,28
128	2	1	1	106-117	55-58
256	2	1	1	168-242	107,114-119
512	2	1	1	260-395	184-190

TABLE E3

M/M/1 : Optimum performance of confidence interval methods

Number of Replications : 400 , $\varepsilon = 0.025$

$\tau = 0.20$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.2354	.0565			
	AREA			.9125	.2102	.0617
	NOBM-AREA			.9125	.2102	.0617
	SPEC	.1917	.0085			
	OVBM			.8150	.0472	.0014
128	NOBM	.1963	.0331			
	AREA	.1829	.0329			
	NOBM-AREA	.1829	.0329			
	SPEC	.0829	.0043			
	OVBM			.8675	.0394	.0007
256	NOBM	.1320	.0138			
	AREA	.1301	.0133			
	NOBM-AREA	.1301	.0133			
	SPEC	.0738	.0027			
	OVBM			.8825	.0293	.0003
512	NOBM	.0522	.0026			
	AREA	.0693	.0041			
	NOBM-AREA	.0654	.0038			
	SPEC	.0370	.0006			
	OVBM	.0227	.0001			

$\tau = 0.50$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM			.9200	2.485	7.415
	AREA			.9200	2.409	7.488
	NOBM-AREA			.9200	2.409	7.487
	SPEC	1.966	2.643			
	OVBM			.7525	.468	.149
128	NOBM	2.101	4.179			
	AREA	2.063	3.781			
	NOBM-AREA	2.063	3.782			
	SPEC	1.481	.890			
	OVBM			.8225	.416	.103
256	NOBM	1.571	2.346			
	AREA	1.537	2.122			
	NOBM-AREA	1.537	2.122			
	SPEC	1.005	.504			
	OVBM			.8625	.299	.040
512	NOBM	1.056	1.101			
	AREA	.707	.487			
	NOBM-AREA	1.004	.890			
	SPEC	.481	.164			
	OVBM			.8900	.235	.024

T A B L E E3 (Cont...)

$\tau = 0.80$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.8950	14.53	204.93
	AREA			.8950	15.16	238.11
64	NOBM-AREA			.8950	15.16	238.11
	SPEC	17.29	249.76			
	OVBM			.5850	2.36	2.93
	NOBM			.8875	14.07	228.74
	AREA			.8775	14.52	242.62
128	NOBM-AREA			.8775	14.52	242.52
	SPEC	19.16	374.89			
	OVBM			.6525	2.46	3.80
	NOBM			.8975	13.95	301.39
	AREA			.9000	13.92	287.86
256	NOBM-AREA			.9000	13.92	287.86
	SPEC	13.70	239.54			
	OVBM			.6975	2.47	4.99
	NOBM			.9200	11.36	145.47
	AREA	11.42	163.33			
512	NOBM-AREA	11.42	163.33			
	SPEC	7.39	150.70			
	OVBM			.7700	2.15	2.94

$\tau = 0.90$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
	NOBM			.7975	21.79	366.6
	AREA			.8050	23.33	418.5
64	NOBM-AREA			.8050	23.33	418.5
	SPEC	35.64	788.1			
	OVBM			.3425	3.35	4.3
	NOBM			.8100	26.14	668.6
	AREA			.8325	27.98	779.6
128	NOBM-AREA			.8325	27.98	779.6
	SPEC	39.27	2393.6			
	OVBM			.4400	4.05	8.3
	NOBM			.8775	30.41	832.3
	AREA			.8475	31.47	937.5
256	NOBM-AREA			.8475	31.47	937.5
	SPEC	46.43	1637.4			
	OVBM			.5275	4.99	12.5
	NOBM	33.16	1136.8			
	AREA			.8850	32.52	1197.4
512	NOBM-AREA			.8850	32.52	1197.4
	SPEC	39.75	1455.5			
	OVBM			.5600	4.27	20.21

T A B L E E4
M/M/1 : Parameter values for optimum performance of the confidence interval methods

Number of Replications : 400 , $\epsilon = 0.025$

$\tau = 0.20$, **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	48-58	10
128	2	1	1	69-105	18,19
256	2	2	1	142-201	13,47,48,50
512	2	1	1,2	121-371	27

$\tau = 0.50$, **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	54-57	20
128	2	1	1	91-109	35-37
256	2,4	1	1	170-216	32,33,46-48
512	2	1,2	2	244-396	71-75,149-154

$\tau = 0.80$, **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	59-61	22
128	2	1	1	116-121	36,37,42,51-54
256	2	1	1	218-239	90,91,94-100
512	2	1	1	384-454	127,128

$\tau = 0.90$, **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	62	18-20,23,24
128	2	1	1	124,125	56
256	2	1	1	239-249	100,108-112
512	2	1	1	469-493	176-189

TABLE E4 (Cont...)

$\tau = 0.20$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	51-60	15,16
128	2	1	1	67-100	34,35
256	2	1	1	149,153-223	70-72
512	2,4,8	1,2	1,2	105-409	156-165,167-171, 173,174

$\tau = 0.50$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	51-59	18-19
128	2	1	1	92-118	45,46
256	2	1	1	166-237	38-40,47,77,78
512	2	1,2	1	265-412	162,163

$\tau = 0.80$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	58-61	23,24
128	2	1	1	112-126	42,43
256	2	1	1	215-245	74,77,78,82-90
512	2	1	1	365-455	162,165

$\tau = 0.90$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	61,62	23,25-27
128	2	1	1	123-126	48-51
256	2	1	1	238-250	83,87-90,94-96, 112-115
512	2	1	1	463-493	185,186,201,202, 214-218

TABLE E5

AR(2) : Optimum performance of confidence interval methods

Number of replications : 400 , $\varepsilon = 0.025$

$\varphi_1 = 0.75$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	.8683	.6053			
	AREA	.9885	.6686			
	NOBM-AREA	.6907	.3836			
	SPEC	.5801	.0731			
	OVBM	.3824	.0112			
128	NOBM	.7570	.4253			
	AREA	.6644	.2987			
	NOBM-AREA	.5150	.2129			
	SPEC	.3578	.0309			
	OVBM	.2558	.0074			
256	NOBM	.3513	.0877			
	AREA	.4610	.1470			
	NOBM-AREA	.3238	.0836			
	SPEC	.2622	.0166			
	OVBM	.1760	.0030			
512	NOBM	.2166	.0354			
	AREA	.2400	.0382			
	NOBM-AREA	.2049	.0299			
	SPEC	.1598	.0045			
	OVBM	.1211	.0009			

$\varphi_1 = 0.99$

n	Methods	ε-ideal perfor.		best performance		
		IMHL	IAVHL	BCVR	BMHL	BVHL
64	NOBM	1.1390	.9183			
	AREA	.7903	.1618			
	NOBM-AREA	.5704	.0564			
	SPEC	.5460	.0672			
	OVBM	.3894	.0129			
128	NOBM	.7297	.3325			
	AREA	.7849	.3988			
	NOBM-AREA	.7259	.3792			
	SPEC	.3061	.0181			
	OVBM	.2445	.0046			
256	NOBM	.4884	.1597			
	AREA	.4433	.1298			
	NOBM-AREA	.4074	.1227			
	SPEC	.1742	.0031			
	OVBM	.1621	.0021			
512	NOBM	.2475	.0438			
	AREA	.3514	.0824			
	NOBM-AREA	.2550	.0528			
	SPEC	.1475	.0049			
	OVBM	.1075	.0009			

T A B L E E6

AR(2) : Parameter values for optimum performance of the confidence interval methods

Number of replications : 400 , ϵ=0.025

φ_1 = 0.75 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,4,8	1,2,4	1,2,4,32	7-40	5-27
128	2,4	1,2	1,64	20-24	12-61
256	2,4	1,2,4	1,2,4,128	33-36,40,51-80, 85,87-99, 118-133,139	36-109
512	2,4,8	1,2,4, 8	1,2,4,8, 256	15-316	16-102,108, 109-125,134, 135-137

φ_1 = 0.75 , **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,4,8	1,2,4	1,2,4,8, 32	2,7-50	2,4-28
128	2,4	1,2,4	1,2,4,64	13,18-26,28-30, 47-50,87-92	16,17,20-76
256	2,4,8,16	1,2,4	1,2,4,8, 128	9-196	11-131, 139-141,143
512	2,4,8,16 32	1,2,4, 8,16	1,2,4,8, 16,256	8-340	6-174

φ_1 = 0.99 , **Nominal Confidence Level : 90%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	10,12,16-33	6,24,25,29-32, 34-41
128	4	1,2	1,2	9,13-64	24,30,36,37, 42-45,47-72,75
256	2	1	1	9,13-15,19-21, 26,29,31,32, 45-47,54-65, 68-73	74,79,80,84,86, 87,90-139, 141-145
512	2,4	2	2	9,13-15,19-75, 78-273,275-277	55,71-74, 76-200,202,203

TABLE E6 (Cont...)

$\varphi_1 = 0.99$, **Nominal Confidence Level : 95%**

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2,4	2	2	10-12,15-35, 38-40,42-48	6,12,13,19, 30-33,35-47,51
128	2,4	1,2	1,2	9-86	6,12,18,19,24, 25,30-32,37,52, 53-88,92,93
256	2,4	1,2,4	1,2,4	9,13-112	25,31,32,36, 37-173,177,178, 183,184,186, 187,189,191-192
512	2,4,8	1,2	1,2,8	9,12-328,358, 359,367-374, 399-417	49,55-57, 60-278,284,285, 290-298,300, 301,304-319, 321,322,328,329

T A B L E E7

Inventory Model : Parameter values for optimum performance of the confidence interval methods

Number of replications : 400 , $\epsilon = 0.025$

Nominal Confidence Level : 90%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	9,11-20	46,48,50,52,55, 59,63
128	2	1	1	8-50	78-95,97,99,101
256	2	1	1	81-88	147,149,150,152, 153-169,171,173, 175,179
512	2,4	1,2	1,2	95-245	196,198,200,202, 203-286,288,290, 292

Nominal Confidence Level : 95%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	5-24	40,42,44,46-54
128	2	1,2	1	5-52,54, 59-61	77,79-113,119,121, 123,125,127
256	2	1,2	1,2	5,8-10,13, 14-16,73, 74-90,95, 96-177	141,149,154,155, 157,159-211,213, 215,217,219,,221, 223
512	2,4	1,2	1,2	7,9,17-346	80,82,84,86,88,90, 92,94,96-109,111, 112-140,150,152, 154,156,158,160, 162-164,166,168, 170-358,360

TABLE E8
Interactive Computer Model:Parameter values for optimum performance of the confidence interval methods

Number of replications : 400 , ϵ=0.025

Nominal Confidence Level : 90%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	63	24,26-29
128	2	1	1	121-124	50-52
256	2	1	1	226-237	93,94
512	2	1	1	397-459	163-170,175-178

Nominal Confidence Level : 95%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	63	26
128	2	1	1	120-125	42-44
256	2	1	1	211-245	64,65,68-74, 78-80
512	2	1	1	370-469	153-156

T A B L E E9

Time-shared Computer Model:Parameter values for optimum performance of the confidence interval methods

Number of replications : 400 , ε=0.025

Nominal Confidence Level : 90%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	55-58	15
128	2	1	1	100-113	42,43,55
256	2	1	1	176-212	84
512	2	1	1	350-430	140-146

Nominal Confidence Level : 95%

n	NOBM	AREA	NOBM-AREA	SPEC	OVBM
64	2	1	1	52-58	21
128	2	1	1	98-114	44,45
256	2	1	1	154-206	86-98
512	2	1	1	268-470	113

T A B L E E10

The performance of Classical Confidence Interval estimator in the Inventory Model, the Interactive Computer Model and the Time–shared Computer Model

Nominal Confidence Level : 90%

n	Criteria	Inventory Model	Interactive Computer Model	Time-shared Computer Model
64	$\widehat{CVR}_{CL}$	1.0000	0.2500	0.4667
	$\widehat{EHL}_{CL}$	(21.7504)	(1.0314)	(1.6870)
	$\widehat{VHL}_{CL}$	(0.0919)	(1.3458)	(0.3125)
128	$\widehat{CVR}_{CL}$	1.0000	0.0475	0.3500
	$\widehat{EHL}_{CL}$	(15.3235)	(0.2634)	(1.2256)
	$\widehat{VHL}_{CL}$	(0.0238)	(0.0357)	(0.1124)
256	$\widehat{CVR}_{CL}$	1.0000	0.0400	0.2367
	$\widehat{EHL}_{CL}$	(10.8128)	(0.2225)	(0.8977)
	$\widehat{VHL}_{CL}$	(0.0063)	(0.0228)	(0.0344)
512	$\widehat{CVR}_{CL}$	0.9967	0.0250	0.1533
	$\widehat{EHL}_{CL}$	(7.6407)	(0.1724)	(0.6421)
	$\widehat{VHL}_{CL}$	(0.0014)	(0.0122)	(0.0116)

Nominal Confidence Level : 95%

n	Criteria	Inventory Model	Interactive Computer Model	Time-shared Computer Model
64	$\widehat{CVR}_{CL}$	1.0000	0.2950	0.5333
	$\widehat{EHL}_{CL}$	(25.9944)	(1.2359)	(2.0162)
	$\widehat{VHL}_{CL}$	(0.1306)	(1.9223)	(0.4464)
128	$\widehat{CVR}_{CL}$	1.0000	0.0550	0.4133
	$\widehat{EHL}_{CL}$	(18.3134)	(0.3148)	(1.4647)
	$\widehat{VHL}_{CL}$	(0.0341)	(0.0511)	(0.1606)
256	$\widehat{CVR}_{CL}$	1.0000	0.0450	0.2933
	$\widehat{EHL}_{CL}$	(12.9226)	(0.2659)	(1.0729)
	$\widehat{VHL}_{CL}$	(0.0087)	(0.0326)	(0.0491)
512	$\widehat{CVR}_{CL}$	1.0000	0.0350	0.1900
	$\widehat{EHL}_{CL}$	(9.1316)	(0.2061)	(0.7674)
	$\widehat{VHL}_{CL}$	(0.0020)	(0.0174)	(0.0166)

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