

# **Algebraic Graph Theory in the Analysis of Frequency Assignment Problems**

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the London School of Economics and Political Science  
for the degree of Doctor of Philosophy

London  
April, 2008

# Declaration

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# Abstract

Frequency Assignment Problems (FAPs) arise when transmitters need to be allocated frequencies with the aim of minimizing interference, whilst maintaining an efficient use of the radio spectrum. In this thesis FAPs are seen as generalised graph colouring problems, where transmitters are represented by vertices, and their interactions by weighted edges.

Solving FAPs often relies on known structural properties to facilitate algorithms. When no structural information is available explicitly, obtaining it from numerical data is difficult. This lack of structural information is a key underlying motivation for the research work in this thesis.

If there are  $N$  transmitters to be assigned, we assume as given an  $N \times N$  “influence matrix”  $W$  with entries  $w_{ij}$  representing influence between transmitters  $i$  and  $j$ . From this matrix we derive the Laplacian matrix  $L = D - W$ , where  $D$  is a diagonal matrix whose entries  $d_{ii}$  are the sum of all influences working in transmitter  $i$ .

The focus of this thesis is the study of mathematical properties of the matrix  $L$ . We generalise certain properties of the Laplacian eigenvalues and eigenvectors that hold for simple graphs. We also observe and discuss changes in the shape of the Laplacian eigenvalue spectrum due to modifications of a FAP. We include a number of computational experiments and generated simulated examples of FAPs for which we explicitly calculate eigenvalues and eigenvectors in order to test the developed theoretical results.

We find that the Laplacians prove useful in identifying certain types of problems, providing structured approach to reducing the original FAP to smaller size sub-problems, hence assisting existing heuristic algorithms for solving frequency assignments. In that sense we conclude that analysis of the Laplacians is a useful tool for better understanding of FAPs.

# Acknowledgements

It is my pleasure to acknowledge a number of people who contributed towards the successful completion of this dissertation.

First, I would like to express my sincerest gratitude to my supervisor, Jan van den Heuvel. I am grateful to Jan for introducing me to the field of spectral graph theory and for all the unreserved support and guidance he has offered me throughout the development of this thesis.

I am extremely grateful to the whole Mathematics Department of the London School of Economics, and in particular to Norman Biggs, Adam Ostaszewski, and Bernhard von Stengel, for their enormous academic and personal support, and to Jackie Everid, David Scott and Simon Jolly for encouragement and friendship.

I also want to thank Jim Finnie and Henrik Blank from the Radiocommunications Agency, with whom I cooperated in the development of this study, for their insightful questions, comments and suggestions, which significantly influenced, guided and motivated this work.

Throughout my education I received much encouragement from my family. I thank my parents for nurturing my interest and love of mathematics, and my sister Bojana for being a wonderful friend and support, even when across the ocean!

Finally, I would like to thank my lovely boys, Aleksa, Sergije and Veljko, for supporting my efforts, and for giving me their all important love that made this work possible and very worthwhile.

This work was funded by the U.K. Radio Communications Agency. I also give gratitude to the LSE Mathematics Department for providing an additional grant, supporting the completion of the research.

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# Chapter 1

## Introduction

### 1.1 Frequency Assignments

“There are very few things in the world of technology that are as financially important and frustratingly arcane as the allocation and use of radio spectrum.”

— The Times, 10th April 2007

The challenges in the field of radio spectrum application and use lie in optimising this essential but scarce resource between numerous modern industries that depend on it, whilst maintaining the technical quality and usability of each assigned frequency. The use of radio frequencies today is at the heart of the broadcast and satellite television industries, the mobile phone industry, the radio industry, numerous smaller businesses ranging from paging services to alarm monitoring companies, as well as many government services in fields such as health services, police, military, etc.

The reception quality of a signal is directly affected if the frequencies are assigned incorrectly. Spectrum allocation is constrained by the need to restrict interference between simultaneous transmissions to an acceptable level. For instance, if two services are close geographically, interference will occur if they are transmitting on frequencies which are close in the radio spectrum.

Rapid development of wireless networks in the 20th century was almost immediately accompanied by allocation and planning problems requiring special consideration by the governments controlling the radio spectrum and frequency ownership.

In the UK, as in the US, the government partially solved the demand side of this problem by deciding to begin auctioning off spectrum once it became available for public use. In the UK, the Office of Communications (Ofcom), an independent regulator and competition authority for the UK communications industry, is tasked to secure optimal use of the radio spectrum. A key means of achieving this aim is by releasing the available spectrum to the market, generally by means of auctions, or via the so called spectrum awards programme. The corresponding body in the US is the Federal Communications Commission.

Frequency sale has become an important source of government revenues, explaining government's active interest in the optimisation of frequency allocation. Contemporaneously, there is an unceasing interest in optimising the use of the spectrum from the users/operators point of view.

Auctioning, nevertheless, on its own does not solve the problems related to frequency assignment. To make possible further improvements in spectrum availability, optimisation approaches have been used and remain fundamental.

The optimisation problems were initially confined solely to radio operators, but were eventually presented to the wider community of researchers including mathematicians and computer scientists as well as radio engineers, who have all found this field attractive and fruitful for research within their own disciplines. As each application created a new model, each with its own objectives and instances, the problems requiring mathematical analysis were quickly in abundance.

The main body of research into optimal spectrum allocation started in early 1960's, and has become known as the Frequency Assignment Problem (FAP). Today, the FAP is an important area of interest both in theoretical analysis and practical applications related to the development of wireless communications networks.

In this thesis, frequency assignment problems are seen as problems occurring when a given set of transmitters or base stations are allocated frequencies or channels, with the aim of minimizing interference and optimising the use of the frequency spectrum.

To describe a frequency assignment problem, one must specify a set of transmitters and the nature of interference between them. Signal interference is measured by the *signal-to-noise ratio* at the receiving end of a connection. The 'noise' comes from other signals broadcasted at interfering frequencies. The actual signal-to-

noise ratio at a receiver depends on the choice of frequency, on the strength of the signal, the direction it is transmitted to, the shape of the environment, and even weather conditions. This is the reason why it is hard to obtain an accurate prediction of the signal-to-noise ratio at receivers. The smallest tolerable ratio of wanted to unwanted signal strength is known as the *protection ratio*. More details on how signal-to-noise ratio is modelled in different research studies can be found in [1].

In our work, a convenient representation of interference is by means of a constraint matrix  $W = [w_{ij}]$ . The quality of the service is deemed to be acceptable provided the channels assigned to each pair of transmitters  $i$  and  $j$  are  $w_{ij}$  channels apart. We define this formally later in the thesis.

The main modelling variations in the application of the FAP are as follows. Fixed Channel Assignment (FCA) is a model where the set of connections remains stable over time. Dynamic Channel Assignment (DCA) models allow for changes in demand for frequencies at an antenna. Hybrid Channel Assignment (HCA) problems combine the previous two: frequencies are assigned beforehand, with reserved allocation for new frequencies. In this thesis we concentrate on Fixed Channel Assignment problems.

There is a significant body of work completed in attempting to tackle all the FAP related issues, with major effort put in modelling, analysis and simplification, often in solving instances of a FAP. Two comprehensive surveys on models and solution techniques and Frequency Assignments Problems in general are provided in [1] and [41]. In order to tackle the problem of spectrum congestion, one can also concentrate efforts on carefully allocating and rearranging the frequencies on which receivers and transmitters broadcast. The branch of mathematics that is closest to this application is graph theory and in particular the area of graph colouring. The graph colouring problem is the exact field of our interest.

The first systematic graph theoretical approach was developed by Hale [24]. The work by Hale presented the FAPs as a generalisation of the graph colouring problem, an NP-hard problem. Consequently authors developed approximation algorithms to find feasible solutions that can be calculated in a reasonable amount of time, often providing upper bounds on the minimum solution value. A thorough comparison of several heuristic approaches can be found in [2] or [29]. Heuristic algorithms such as tabu search, genetic algorithms, simulated annealing and neu-

ral networks [11, 12, 16, 18, 33, 36] have been of interest to researchers, as well as several ad hoc heuristics [14, 31, 35].

The radio community has a strong interest in the use of approximation algorithms for solving frequency assignment problems. As most of the heuristics apply to some form of neighbourhood or hyperneighbourhood search, identifying useful underlying structures within the network of transmitters under consideration prior to applying the heuristic methods is valuable.

However, security and commercial reasons impose limits on information that can be revealed to the wider research community, including the information on spatial placements of transmitters. The question underlying this thesis is how much can be done in terms of analysis when no structural information is directly revealed.

## 1.2 Outline of the Thesis

Various methods for solving frequency assignment problems rely on the knowledge of certain structural properties to facilitate the algorithms. For instance, we may be given information about the geographical position of a set of transmitters, information about the power of the transmitters, and possibly some information about the terrain properties. If no such structural information is given explicitly, it may be difficult to obtain this information from the numerical data.

However this lack of existing structural information is also the central theme of this study—the required key assumption of our research work. The aim of this thesis is to investigate the relationship between algebraic graph theory and a frequency assignment problem as defined to us by the radio communications community.

In this thesis we discuss the possibility of using algebraic techniques to investigate the structure of frequency assignment problems. Our choice is influenced by the fact that algebraic objects have certain significant practical advantages over combinatorial ones. For instance, calculation with high numerical precision is fast and stable, even for large problems. Additionally, those objects usually do not depend on the actual representation of a problem, such as order of the nodes, etc. We implement spectral theory of Laplacians as a means of discovering and revealing as many structural properties of interest as possible.

We present our work as follows: In the first part of Chapter 2 we introduce the relevant notation and review all the elementary results from graph theory that are referred to in our work. Next, we introduce the necessary results and properties from Linear Algebra. We then formally define our version of Frequency Assignment Problems in terms of graph theory, and introduce the algebraic graph theory terms: Laplacian matrices, eigenvalues and eigenvectors of the FAP.

The theoretical results related to the application of spectral analysis of the frequency assignment problems are presented in Chapter 3. In the introductory section we recall the continuity of Laplacian eigenvalues which results from perturbation matrix theory, and explain why it provides strong support to our chosen approach to FAP analysis. Then we discuss various properties of graph Laplacian eigenvalues, focusing on extending the known results for simple graphs to the case of edge-weighted graphs.

Finally, in Chapter 4 we discuss the main findings of our work. The main findings include the inferences regarding the shape of the eigenvalue spectrum of certain types of edge-weighted graphs, along the corresponding Laplacian eigenvectors. It is important to emphasize that most of the theoretical work had to be done on graphs representing idealized frequency assignment problems. However, the perturbation theory result about the continuity of the Laplacian eigenvalues implies that the theoretical results can be applied in more realistic cases. We investigate this further by engaging in simulation work. Initially we generate simpler graphs, and observe changes occurring in the Laplacian eigenvalues and eigenvectors with a gradual change of parameters characterising the graphs.

# Chapter 2

## Preliminaries

In this chapter we present the mathematical terminology and notation used in our research. Initially we introduce graphs and concepts related to graphs, followed by the required concepts and basic results from matrix theory and linear algebra. Finally, we formally interpret the relation between Frequency Assignment Problems and graph theory as applied throughout our research.

### 2.1 Graph Theory Background

#### 2.1.1 Graphs in general; simple and weighted graphs

By a *graph*  $G$  we mean a pair  $(V(G), E(G))$ , with  $V(G)$  a nonempty finite set called the *vertices*, and  $E(G)$  a finite collection of unordered pairs of vertices, called the *edges*. An edge  $e = uv$  *connects* or *joins* the vertices  $u$  and  $v$ . The vertices  $u$  and  $v$  are called the *ends* of  $e$ . An edge  $e$  is said to be *incident* to  $v$  if  $v$  is an end of  $e$ . Vertices  $u$  and  $v$  are *adjacent* if there exists an edge that connects them. The number of edges of which  $v$  is an end is called the *degree* of  $v$  and is denoted by  $d(v)$ . An edge connecting a vertex to itself is called a *loop*. Distinct edges that connect the same pair of different vertices are called *multiple edges*. A graph with no loops and parallel edges is called a *simple graph*. In this thesis loops and parallel edges are not permitted. A *regular graph* is a graph where each vertex has the same degree. A regular graph with vertices of degree  $k$  is called a  $k$ -regular graph or regular graph of degree  $k$ . Two graphs  $G$  and  $H$  are said to be isomorphic, if there

is a one-to-one correspondence, called an isomorphism, between the vertices of the graph such that two vertices are adjacent in  $G$  if and only if their corresponding vertices are adjacent in  $H$ .

A *subgraph* of a graph  $G$  is a graph whose vertex and edge sets are subsets of those of  $G$ . We say a graph  $G$  *contains* another graph  $H$  if some subgraph of  $G$  is  $H$ , or is isomorphic to  $H$ . A subgraph  $H$  is a *spanning subgraph* of a graph  $G$  if it has the same vertex set as  $G$ . We say  $H$  *spans*  $G$ . A subgraph  $H$  of a graph  $G$  is said to be *induced* if, for any pair of vertices  $u$  and  $v$  of  $H$ ,  $uv$  is an edge of  $H$  if and only if  $uv$  is an edge of  $G$ . If the vertex set of  $H$  is the subset  $S$  of  $V(G)$ , then  $H$  can be written as  $G[S]$  and is said to be induced by  $S$ .

A *walk* is an alternating sequence of vertices and edges, beginning and ending with a vertex, in which each vertex is incident to the two edges that precede and follow it in the sequence, and the vertices that precede and follow an edge are the end vertices of that edge. A walk is closed if its first and last vertices are the same, and open if they are different. A walk in which all edges are different is called a *trail*. If, furthermore, the vertices in the walk are different then the sequence is called a *path*. Two vertices  $u$  and  $v$  are called *connected* if  $G$  contains a path from  $u$  to  $v$ . Otherwise, they are *disconnected*. A graph is called *connected* if every pair of distinct vertices in the graph is connected. A *connected component* is a maximal connected subgraph of  $G$ . Each vertex belongs to exactly one connected component, as does each edge. The *edge connectivity* of a simple graph  $G$  is the minimal number of edges whose removal would result in losing connectivity of the graph  $G$ . *Vertex connectivity* is defined analogously (vertices together with the adjacent edges are removed). In a graph, an *independent set* is a set of vertices no two of which are adjacent. A *clique* is a set of vertices such that for every two vertices in the set, there exists an edge connecting the two.

An *orientation* of a graph  $G$  is the assignment of a direction to each edge. A directed edge is an ordered pair  $(u, v)$  of end vertices  $u$  and  $v$  that can be represented graphically as an arrow drawn between the end vertices. In such an ordered pair the first vertex, vertex  $u$  is called the initial vertex or tail; the second one, vertex  $v$  is called the terminal vertex or head (because it appears at the arrow head). An *undirected* edge disregards any sense of direction and treats both end vertices interchangeably. An *oriented graph*, or a *digraph*,  $G^\sigma$  is a graph  $G$  with a particular orientation  $\sigma$ .

An oriented graph is analogous to an undirected graph except that it contains only directed edges. A *mixed* graph may contain both directed and undirected edges; it generalizes both directed and undirected graphs. An oriented graph is called simple if it has no loops and at most one directed edge between any pair of vertices.

When stated without any qualification, in this thesis, we assume graphs and diagraphs to be simple.

As a generalisation of these concepts *vertex-weighted* and/or *edge-weighted* graphs and diagraphs can be found in the literature. These are graphs with weights assigned to their vertices and/or edges, respectively.

### 2.1.2 Matrices associated with graphs

We describe several matrices that have been defined for simple graphs and diagraphs. These have all been of interest to algebraic graph theory. Considerable research has been devoted to trying to determine properties of graphs using algebraic properties of these matrices.

The matrix that has been considered most often is the adjacency matrix. The *adjacency matrix* of a graph  $G$  on  $n$  vertices is the  $n \times n$  matrix  $A = [a_{ij}]$  whose  $(i, j)$ -entry  $a_{ij}$  is equal to the number of edges with  $i$  and  $j$  as ends.

The *incidence matrix*  $Q$  of a graph  $G$  is the 01-matrix with rows and columns indexed by vertices and edges of  $G$ , respectively, such that the  $ve$ -entry of  $Q$  is equal to one if and only if the vertex  $v$  is an end of edge  $e$ . It is important to note that this matrix differs from the incidence matrix for oriented graphs.

The *incidence matrix*  $Q(G^\sigma)$  of an oriented graph  $G$  is the  $0, \pm 1$ -matrix with rows and columns indexed by the vertices and edges of  $G$ , respectively, such that the  $ve$ -entry of  $Q(G^\sigma)$  is equal to 1 if the vertex  $v$  is the tail of the edge  $e$ , equal to  $-1$  if the vertex  $v$  is in the head of the edge  $e$ , and 0 otherwise.

In addition to these matrices we introduce the Laplacian of a graph. The algebraic approach used in this study for the analysis of FAPs relies solely on the analysis of the Laplacians.

We list two different but equivalent definitions of the underlying meaning of “Laplacian matrix” of a graph, whilst all others we see as generalisations of the Laplacian.

If  $G$  is a graph on  $n$  vertices, then  $D(G)$  is the diagonal  $n \times n$  matrix with rows and columns indexed by  $V(G)$ , and each  $vv$ -entry equal to the degree of vertex  $v$ .

The *Laplacian* of a graph  $G$  is the matrix that represents a product of the incidence matrix and its transpose based on an arbitrary orientation of the graph  $G$ ; so

$$L(G) = Q(G^\sigma)Q^T(G^\sigma), \quad (2.1)$$

where  $\sigma$  is an arbitrary orientation of a graph  $G$  and  $Q(G^\sigma)$  is the incidence matrix of  $G^\sigma$ .

On the other hand one could also see the Laplacian as the difference between the diagonal matrix  $D(G)$  and the adjacency matrix  $A(G)$ ; so

$$L(G) = D(G) - A(G).$$

The following proposition proves that the two definitions are equivalent and that the Laplacian in the first definition is well-defined (i.e.,  $L$  is independent of the orientation given to  $G$ ). The proposition can be found in [9] and we include its proof for completeness.

**Proposition 2.1.1.** *If  $\sigma$  is an orientation of  $G$  and  $Q$  is the incidence matrix of  $G^\sigma$ , then  $QQ^T = D(G) - A(G)$ , where  $D(G)$  is the diagonal matrix of vertex degrees and  $A(G)$  is the adjacency matrix.*

**Proof.** The inner product of any two rows  $i$  and  $j$  of  $Q$  corresponds to the  $ij$ -entry of  $QQ^T$ . If  $i \neq j$  then these rows have a non-zero entry in the same column if and only if there is an edge  $e$  joining vertices  $v_i$  and  $v_j$ . In this case the two non-zero entries are  $+1$  and  $-1$  (the head and the tail of the edge  $e$ ), so  $(QQ^T)_{ij} = -1$ . Similarly,  $(QQ^T)_{ii}$  is equal to the inner product of the row  $i$  with itself and since the number of entries  $\pm 1$  in the row  $i$  is simply the degree of  $v_i$ , the result follows. ■

There are other “generalised Laplacians” associated with a graph. Throughout the thesis we will be referring to a version of such a generalised Laplacian for edge-weighted graphs. The formal definition is given in Section 2.3.

We list two other interesting generalisations of the Laplacian. Godsil and Royle [22] call a symmetric  $n \times n$  matrix a generalised Laplacian  $L$  of  $G$  if  $L_{uv} < 0$  when  $u$

and  $v$  are adjacent vertices of  $G$ , and  $L_{uv} = 0$  when  $u$  and  $v$  are distinct and not adjacent. There are no constraints on the diagonal entries of  $L$ ; in particular there is no requirement that  $L\mathbf{1} = 0$ . The ordinary Laplacian is a generalised Laplacian, as is  $-A(G)$ .

Chung's definition [13] refers to a "normalised" form of the ordinary Laplacian. It is denoted by  $\mathcal{L}$  and defined as:

$$\mathcal{L}_{uv} = \begin{cases} 1, & \text{if } u = v \text{ and } d(v) \neq 0, \\ -\frac{1}{\sqrt{d(u)d(v)}}, & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The ordinary and the normalized Laplacian are connected by the following expression:  $\mathcal{L} = D^{-1/2}LD^{-1/2}$ . Chung also defines a generalised normalized Laplacian for edge-weighted graphs.

## 2.2 Linear Algebra Background

### 2.2.1 Eigenvalues and eigenvectors of matrices

In this subsection we discuss a number of results from linear algebra used throughout the thesis. We begin with definitions of eigenvalues and eigenvectors for matrices in general. For completeness we first define Hermitian adjoint, Hermitian and normal matrices.

Let  $A = [a_{ij}]$  be a matrix over the field of complex numbers. The *transpose* of  $A$ , denoted by  $A^T$  is the matrix over the same field as  $A$  whose entries are  $a_{ji}$ ; that is, rows are exchanged for columns and vice versa.

The *Hermitian adjoint*  $A^*$  of  $A$  is defined by  $A^* = (\bar{A})^T$ , where  $\bar{A}$  is the entry-wise conjugate.

**Definition 2.2.1.** *A complex square matrix  $A$  is a Hermitian (or self-adjoint) matrix if it is equal to its own conjugate transpose, i.e.,  $A = (\bar{A})^T$ .*

Clearly, the entries on the main diagonal of any Hermitian matrix are necessarily real. A matrix that has only real entries is Hermitian if and only if it is a symmetric

matrix, i.e., if it is symmetric with respect to the main diagonal. A real and symmetric matrix is simply a special case of a Hermitian matrix.

**Definition 2.2.2.** *A complex square matrix  $A$  is a normal matrix if  $A^*A = AA^*$  where  $A^*$  is the conjugate transpose of  $A$ .*

It should be added that if  $A$  is a real matrix,  $A^* = A^T$  and so it is normal if  $A^T A = AA^T$ . Every Hermitian matrix, and hence every real symmetric matrix is normal.

**Definition 2.2.3.** *Let  $A$  be an  $n \times n$  matrix over the field of complex numbers. We consider the equation*

$$Ax = \lambda x, \quad x \neq 0,$$

*where  $x$  is an  $n$ -vector and  $\lambda$  is a scalar. If a scalar  $\lambda$  and a non-zero vector  $x$  happen to satisfy this equation, then  $\lambda$  is called an eigenvalue of  $A$  and  $x$  is called an eigenvector of  $A$  associated with  $\lambda$ .*

The set of all eigenvalues is called the *spectrum* of  $A$ . The space of eigenvectors of  $A$  associated with the eigenvalue  $\lambda$  together with the null vector is called the *eigenspace* belonging to  $\lambda$ . The dimension of this space is called the *geometric multiplicity* of  $\lambda$ . On the other hand, the *algebraic multiplicity* of  $\lambda$  is the multiplicity of  $\lambda$  as a root of the polynomial  $\det(A - \lambda I)$ . For Hermitian matrices the two multiplicities of  $\lambda$  are equal. As both the graph adjacency and Laplacian matrix are Hermitian we, in this work, need not distinguish between geometric and algebraic multiplicity and speak solely about the *multiplicity* of an eigenvalue.

Since in this thesis we only analyse matrices associated with graphs there is an additional refinement. Note that rows and columns of the adjacency and Laplacian matrix are indexed by the vertices of the considered graph  $G$ . This implies that we can view these matrices as linear mappings on  $\mathbb{R}^V(G)$ , the space of real functions on  $V(G)$ . If  $f \in \mathbb{R}^V(G)$  and  $A$  is the adjacency matrix of  $G$ , then the image  $Af$  of  $f$  under  $A$  is given by

$$(Af)(u) = \sum_{v \in V(G)} A_{uv} f(v),$$

and since  $A$  is just a 01-matrix we have:

$$(Af)(u) = \sum_{v \sim u} f(v).$$

If we suppose that  $f$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , then  $Af = \lambda f$ , and so

$$\lambda f(u) = \sum_{v \sim u} f(v).$$

Any function that satisfies this condition is an eigenvector of  $A$ .

Below we list several known facts for *symmetric* and *positive semidefinite* matrices. These are standard results from linear algebra (see, e.g., [28]).

**Proposition 2.2.4.** *Let  $A$  be a real  $n \times n$  symmetric matrix. Then*

- *two eigenvectors of  $A$  with different eigenvalues are orthogonal;*
- *all eigenvalues of  $A$  are real numbers;*
- *$\mathbb{R}^n$  has an orthonormal basis consisting of eigenvectors of  $A$ .*

A real symmetric matrix  $A$  is positive semidefinite if  $x^T A x \geq 0$  for all vectors  $x$ . If  $x$  is an eigenvector of  $A$  belonging to an eigenvalue  $\lambda$  then  $x^T A x = \lambda x^T x$ , which implies that a real symmetric matrix is positive semidefinite if and only if its eigenvalues are nonnegative. For this class of matrices we also know:

**Proposition 2.2.5.** *A real symmetric matrix  $A$  is positive semidefinite if and only if there is a matrix  $B$  such that  $A = BB^T$ .*

Moving our attention on to a particular matrix, the Laplacian matrix  $L$ , the previous two properties help determine that  $L$  is positive semidefinite (see equation (2.1) on page 14) implying that its eigenvalues are real and non-negative.

Next, we introduce the concepts of interlacing and tight interlacing of sequences. Consider two sequences of real numbers:  $\lambda_1 \geq \dots \geq \lambda_n$ , and  $\mu_1 \geq \dots \geq \mu_m$  with  $m < n$ . The second sequence is said to *interlace* the first one whenever

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for } i = 1, \dots, m.$$

The interlacing is called *tight* if there exists an integer  $k \in [0, m]$  such that

$$\lambda_i = \mu_i \quad \text{for } 0 \leq i \leq k \quad \text{and} \quad \lambda_{n-m+i} = \mu_i \quad \text{for } k+1 \leq i \leq m.$$

The interlacing inequalities become  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$ , when  $m = n - 1$ , which explains the name.

Now, assume that the sequence

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A),$$

represents the eigenvalues of a symmetric matrix  $A$  in decreasing order, and let the coordinates of the matrix and corresponding vectors be indexed by a set  $V$ , where  $|V| = n$ . A useful characterisation of the eigenvalues is given by the Rayleigh's and Courant-Fisher's formulae (see [39]):

$$\begin{aligned} \lambda_n(A) &= \min\{ \langle Ax, x \rangle \mid x \in \mathbb{R}^V, \|x\| = 1 \}, \\ \lambda_1(A) &= \max\{ \langle Ax, x \rangle \mid x \in \mathbb{R}^V, \|x\| = 1 \}, \\ \lambda_{n-k+1}(A) &= \min\{ \max\{ \langle Ax, x \rangle \mid x \in X, \|x\| = 1 \} \mid \\ &\quad X \text{ a } k\text{-dimensional subspace of } \mathbb{R}^V \}, \quad \text{for } k = 1, \dots, n. \end{aligned}$$

Also,

$$\lambda_{n-k+1}(A) = \min\{ \langle Ax, x \rangle \mid \|x\| = 1, x \perp x_i \text{ for } i = n - k + 2, \dots, n \},$$

where  $x_n, x_{n-1}, \dots, x_{n-k+2}$  are pairwise orthogonal eigenvectors corresponding to  $\lambda_n, \lambda_{n-1}, \dots, \lambda_{n-k+2}$ , respectively.

For the purposes of our work we single out one of the many important applications of this characterisation to eigenvalue interlacing between eigenvalues of two matrices.

**Proposition 2.2.6.** *Let  $A$  and  $B$  be Hermitian  $n \times n$  matrices. Assume that  $B$  is positive semidefinite and that the eigenvalues of  $A$  and  $A + B$  are arranged in decreasing order. Then*

$$\lambda_k(A) \leq \lambda_k(A + B),$$

for all  $k = 1, \dots, n$ .

Before we proceed further we include one more property that we refer to later, which does not involve interlacing, but describes another useful relation between eigenvalues in the case of commuting matrices.

**Proposition 2.2.7.** *Let  $A$  and  $B$  be  $n \times n$  real matrices with eigenvalues  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$ , respectively. If  $A$  and  $B$  commute (i.e.,  $AB = BA$ ), there is a*

permutation  $i_1, \dots, i_n$  of the indices  $1, \dots, n$ , such that the eigenvalues of  $A + B$  are  $\alpha_1 + \beta_{i_1}, \alpha_2 + \beta_{i_2}, \dots, \alpha_n + \beta_{i_n}$ .

In the following chapters we use these two properties. These, together with many other matrix analysis results, can be found in any standard textbook on matrix theory (e.g., [28]).

## 2.2.2 Linear equations

Consider a linear transformation  $L : V \rightarrow W$ , where  $V$  and  $W$  are vector spaces. One of the problems in linear algebra is that for a given vector  $b \in W$  one wants to find all vectors  $a \in V$  such that  $L(a) = b$ . Usually, this is written in the form

$$L(x) = b,$$

and is referred to as a *linear equation* (with functional part  $L$ , free member  $b$  and unknown  $x$ ).

Equivalently, consider a matrix equation with a coefficient  $A$ , a free member  $B$  and unknown  $X$ , where

$$AX = B,$$

and  $A$  and  $B$  are matrices over the same scalar field  $K$  that have the form:

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

It is obvious that the solutions of this matrix equation are the solutions of the system of linear equations with  $m$  linear equations and  $n$  unknowns over the scalar field  $K$ . In fact, the matrix equation can be seen as a linear equation

$$L_A(X) = B$$

with unknown  $X$ , where  $L_A : K^n \rightarrow K^m$  is the linear transformation defined by  $L_A(X) = AX$ . Consequently, if this system has at least one solution  $a \in K^n$ , then

the set of all its solutions is an affine subspace of  $K^n$

$$R(A, B) = a + R(A, \mathbf{0}),$$

where  $R(A, \mathbf{0})$  represents the vector space of all solutions of its corresponding homogenous system  $AX = \mathbf{0}$ . Furthermore, similarly to the previous case, we have  $\dim R(A, B) = n - \text{rank}(A)$ . Also:

**Proposition 2.2.8.** *A system of  $m$  linear equations with  $n$  unknowns over the scalar field  $K$  has at least one solution if and only if the matrix  $A$  and the augmented matrix  $[A|B]$  are of the same rank.*

This property implies that a system of linear equations over the scalar field  $K$  has exactly one solution if and only if  $\text{rank}([A|B]) = \text{rank}(A) = n$ , where  $n$  is the number of unknowns. In particular we have:

**Proposition 2.2.9.** *A homogenous system of linear equations, i.e., a matrix equation  $AX = \mathbf{0}$  has a non-trivial solution  $X \neq \mathbf{0}$  if and only if the rank of the matrix  $A$  is smaller than the number of unknowns  $n$ .*

### 2.2.3 Partitions of sets and matrices

A *partition*  $\pi$  of a set  $S$  is a set whose elements are themselves disjoint nonempty subsets of  $S$ , and whose union is  $S$ . The elements of the partition are called parts or cells and by  $\pi = (S_1, \dots, S_m)$  we mean a partition with  $m$  cells, the  $k$ -th of which is  $S_k$ . The partition with all cells containing exactly one element is called *discrete*, and the partition with one cell *trivial*.

For a partition  $\pi$  of  $S$  we construct vectors of size  $|S|$  whose entries correspond to elements of  $S$ , and hence to the entries of the cells. If entries of such a vector are equal to 1 on all elements of the cell  $S_k$  and equal to 0 otherwise, then we call such a vector the *characteristic vector* of the cell  $S_k$  of partition  $\pi$ .

Similarly, a partition  $\pi$  of  $S$  can be presented by its *characteristic matrix*  $P = [p_{ij}]$ , which is defined to be the  $|S| \times m$  matrix with characteristic vectors of the cells as its columns:

$$p_{ij} = \begin{cases} 1, & \text{if } i \in S_j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $P^T P$  is a diagonal matrix where  $(P^T P)_{kk} = |S_k|$ . Since the cells are nonempty, the matrix  $P^T P$  is invertible.

A *block matrix*, or a *partitioned matrix* is a partition of a matrix into rectangular smaller matrices called blocks. The matrix is written in terms of smaller matrices written side-by-side. A block matrix must conform to a consistent way of splitting up the rows, and the columns: we group the rows into some adjacent ‘bunches’, and the columns likewise. The partition is into the rectangles described by one bunch of adjacent rows crossing one bunch of adjacent columns. The matrix is split up by some horizontal and vertical lines that go all the way across.

Assuming that rows and columns of a square  $n \times n$  matrix  $A$  are both partitioned according to the partitioning  $\pi$ , partitioning of  $A$  can be presented as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where  $A_{ij}$  is a submatrix of  $A$ , whose rows are indexed by  $S_i$  and columns by  $S_j$ .

**Definition 2.2.10.** *The quotient matrix of a matrix  $A$  partitioned according to  $\pi$  is the matrix  $Q$  whose entries are the average row sums of the blocks of  $A$ . More precisely  $Q = [q_{ij}]$  is such that*

$$q_{ij} = \frac{1}{|S_i|} \mathbf{1}^T A_{ij} \mathbf{1} = \frac{1}{|S_i|} (P^T A P)_{ij}.$$

**Definition 2.2.11.** *The partition is called equitable if each block  $A_{ij}$  of  $A$  has constant row (and column) sum, that is,  $AP = PQ$ .*

Below we list interesting facts and results on interlacing of eigenvalues of matrices. We start with a classical and fundamental case of interlacing by Courant (see [15]).

**Proposition 2.2.12.** *Let  $A$  be a real symmetric  $n \times n$  matrix, and let  $S$  be a complex  $n \times m$  matrix ( $n \geq m$ ) such that  $S^* S = I_m$ . Set  $B$  equal to  $S^* A S$  and let  $\{y_1, \dots, y_m\}$  denote an orthogonal set of eigenvectors of  $B$ . Then*

- *The eigenvalues  $\{\mu_1, \dots, \mu_m\}$  of  $B$  interlace the eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$  of  $A$ .*

- If  $\mu_i = \lambda_i$  or  $\mu_i = \lambda_{n-m+i}$  for some  $i \in [1, m]$ , then there exists an eigenvector  $y$  of  $B$  with eigenvalue  $\mu_i$ , such that  $Sy$  is an eigenvector of  $A$  with eigenvalue  $\mu_i$ .
- If the interlacing is tight, then  $SB = AS$ .

A proof of the theorem and the corollaries that follow can be found in [22, 26]. We single out two interesting consequences.

**Corollary 2.2.13.** *Let  $A$  be a real  $n \times n$  symmetric matrix. The maximum of the quadratic form  $\text{tr}(X^T A X)$  on  $X^T X = I_k$  is the sum of the  $k$  largest eigenvalues of the matrix and the  $n \times k$  matrix  $X$  giving the maximum contains the corresponding eigenvectors in its columns.*

**Proposition 2.2.14.** *Let  $A$  be a Hermitian matrix partitioned as follows*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

such that  $A_{ii}$  is square for  $i = 1, \dots, m$ . Let  $b_{ij}$  be the average row sum of  $A_{ij}$ , for  $i, j = 1, \dots, m$ . Define the  $m \times m$  matrix  $B$  to be a matrix whose  $(i, j)$ -entry is equal to  $b_{ij}$  for all values of  $i$  and  $j$ . The following holds:

- The eigenvalues of  $B$  interlace the eigenvalues of  $A$ .
- If the interlacing is tight, then  $A$  is equitable.
- If, for  $i, j = 1, \dots, m$ ,  $A_{ij}$  has constant row and column sums, then any eigenvalue of  $B$  is also an eigenvalue of  $A$  with at least as large a multiplicity.

## 2.3 Graph Theory and FAPs

Graph theory, and in particular graph colouring play an important role in studying FAPs. In a lot of senses, a FAP can be seen as a generalisation of a graph colouring problem.

In this thesis we discuss the possibility of using algebraic techniques to investigate the structure of frequency assignment problems, using numerical data such as a constraint matrix only.

To explain our approach further, we first describe some of the underlying assumptions.

Frequency assignment problems can be described in different ways, using different collections of information. We will always assume that we are given a collection of  $N$  transmitters, numbered  $1, \dots, N$ .

What is further available may vary depending on the specific problem. For instance, we may have access to information about the geographic position of the set of transmitters, information about the power of the transmitters, or information about the terrain properties.

For our purposes we assume that all we have available is the *constraint matrix*:

**Definition 2.3.1.** A constraint matrix  $W = [w_{ij}]$  is an  $N \times N$  matrix such that if  $f_i$  denotes the frequency assigned to transmitter  $i$ , for  $i = 1, \dots, N$ , then in order to limit interference it is required that  $|f_i - f_j| \geq w_{ij}$ , for all  $i, j$ .

We use the term “transmitter” in a general sense, as a unit assigned with one channel. In a system where certain “actual transmitters” need to be assigned more than one channel, we will consider each channel as a separate “transmitter”. This makes sense because a transmitter with multiple channels is indistinguishable from a system where there is a number of transmitters all in more or less the same position.

We will also use the term *influence matrix* for the constraint matrix. After all, the constraint that needs to be imposed between two transmitters to manage the interference is directly related to the influence they have on one another.

**Definition 2.3.2.** A feasible assignment of a FAP with  $N$  transmitters, given the constraint matrix  $W$ , is a labelling  $f : \{1, \dots, N\} \rightarrow \{x \in \mathbb{R} \mid x \geq 0\}$  such that  $|f_i - f_j| \geq w_{ij}$ , for all  $i, j$ ,  $i \neq j$ .

The span of a FAP is the minimum over all feasible assignments of the largest label used. By solving a FAP we will mean determining its span and finding a feasible labelling that will give that span.

Accepting the hypotheses in the definitions above, the constraint matrix is the unique object needed to “solve the FAP”.

The main purpose of the methodology discussed here is to obtain further information about the FAP, which may be helpful to existing algorithms used.

For several reasons, including historical developments of solution methods for FAPs, we will usually discuss FAPs in the context of *weighted graphs*.

**Definition 2.3.3.** A weighted graph  $(G, w)$  is a triple  $(V(G), E(G), w)$ , where  $V(G)$  and  $E(G)$  form the vertex set and edge set, respectively, of a simple graph  $G$ , and  $w$  is a weight-function defined on  $E(G)$ . We will assume that all weights  $w(e)$ ,  $e \in E(G)$ , are non-negative real numbers.

In order to simplify notation we use  $G$  to denote a weighted graph.

Also, we treat the situation that there is no edge between two vertices  $u$  and  $v$  and the situation where there is an edge  $uv$  with  $w(uv) = 0$  as equivalent.

In an edge-weighted graph, an independent set would be a set of vertices no two of which are adjacent, or equivalently a vertex set where every two are connected with an edge of weight 0. A *clique* would then be a set of vertices such that for every two vertices in the set, there exists an edge of non-zero weight connecting the two.

If a FAP with the constraint matrix  $W = [w_{ij}]$  is given, then we can easily form a weighted graph  $G$  by setting  $V(G) = \{1, \dots, N\}$ , joining two vertices  $i, j$  whenever  $w_{ij} > 0$ , and setting the weight of an edge  $e = ij$  equal to  $w(e) = w_{ij}$ . Similarly, given a weighted graph, we can formulate a FAP on the vertex set in the reverse manner.

From now on we will mix both FAPs and weighted graphs, and hence we talk about the *span of a weighted graph*  $G$ , notation  $sp(G)$ , as the span of the related FAP. We also use a mixture of FAP-terms and graph theoretical terms. Therefore a vertex and a transmitter should be considered as being the same object.

The *chromatic number*  $\chi(G)$  of a graph  $G$  is the smallest number of labels needed for a labelling in which adjacent vertices must receive different labels. The span of a weighted graph  $(G, w)$  is equal to the one less than the chromatic number of  $G$  if  $w \equiv 1$ , but the two parameters can be completely different for general  $w$ .

As indicated previously, we always assume that a weighted graph  $G$  has an associated matrix  $W$ . We define the (*weighted*) *degree of a vertex*  $v$  as  $d(v) = \sum_{u \neq v} w(uv)$ .

**Definition 2.3.4.** *Given a weighted graph  $G$  with associated matrix  $W$ , denote by  $D$  the diagonal matrix with the degrees of the vertices on the diagonal. Then we define the Laplacian  $L$  of  $G$  as the matrix  $L = D - W$ ; hence*

$$L_{uv} = \begin{cases} -w(uv), & \text{if } u \neq v, \\ d(v), & \text{if } u = v \end{cases}$$

(here, and in the definition of degree above, we follow our convention that  $w(uv) = 0$  if  $uv \notin E(G)$ ).

If we want to emphasise the weighted graph  $G$  determining the Laplacian we denote the Laplacian as  $L(G)$ .

The matrix  $L$  can be seen as a generalisation of the *Laplacian matrix* from algebraic graph theory, in a similar way that  $W$  is a generalisation of the *adjacency matrix* of a graph.

Note that the information contained in the Laplacian is the same as that in the constraint matrix. In that sense, there seems to be no good reason to study algebraic properties of the Laplacian instead of the constraint matrix. However the Laplacian is more useful in obtaining structural information in relation to the underlying graph (or the FAP). It also has additional algebraic properties (for instance, it is positive semidefinite) which give us a headstart in the algebraic analysis.

There are, however, some (but quite specific in description) graphs for which there is a direct connection between the spectrums of the constraint matrix and the spectrum of the Laplacian. Those are generalised edge-weighted regular graphs with all edge weights equal and all vertices of constant degree  $\Delta$ . For those graphs we have:  $\lambda_i(L) = -\lambda_i(W) + \Delta$ .

**Definition 2.3.5.** *Given a weighted graph or a FAP with the Laplacian  $L$ , a Laplacian eigenvalue and a Laplacian eigenvector is an eigenvalue and eigenvector of  $L$ . So  $\lambda \in \mathbb{R}$  is a Laplacian eigenvalue with corresponding Laplacian eigenvector  $x \in \mathbb{R}^N$  if  $x \neq 0$  and  $Lx = \lambda x$ .*

Since  $L$  is a symmetric matrix ( $L_{uv} = L_{vu}$ ) we know it has  $N$  real eigenvalues

$\lambda_1, \dots, \lambda_N$ , and a collection of corresponding eigenvectors  $x_1, \dots, x_N$  that form a basis of  $\mathbb{R}^N$ . Because of the fairly nice structure of the Laplacian, determining the eigenvalues and eigenvectors is computationally fairly uncomplicated, even for larger values of  $N$ . Experiments with  $N \approx 10,000$  have been performed on a standard PC, where running times were in the order of minutes.

The guiding question in the remainder of this thesis is:

*What are the properties of the Laplacian eigenvalues and eigenvectors of a FAP, and what structural information regarding the FAP can be obtained from the eigenvalues and eigenvectors?*

# Chapter 3

## Spectral Analysis of FAPs

Modern spectral theory is widely used in all areas of science where a certain property can be described by connections and relations within a structure.

In theoretical chemistry the adjacency matrix eigenvalues are thoroughly investigated and used in revealing the properties and the structure of atoms, molecules and molecule groups [17].

Spectral theory in the past found a role in the theory of electrical networks, where the matrix of interest was the so called matrix of admittance or conductivity. From that time we have Kirchhoff's theorem (1847), a well-known theorem that shows that the number of spanning trees in a graph is equal to the determinant of the matrix obtained from the Laplacian by removing one row and column corresponding to the same vertex [32].

The difference Laplacian, a variant of the Kirchhoff matrix, is the matrix of a quadratic form expressing the energy of a discrete system. It also naturally describes the vibration of a membrane, or thermodynamical properties of crystalline lattices. The discrete analogue shares many important properties with its continuous version, undoubtedly the most important and best understood operator in mathematical physics.

Spectral techniques are applied in computer science in information retrieval and data mining (see, e.g., [4]). There they are used to cluster documents into areas of interest, and to determine the relative importance of documents based on some underlying structure. Web information retrieval is one of the main examples. The

web's hyperlink structure has been exploited by several of today's leading Web search engines, particularly Google [34].

Researchers also applied spectral techniques in observing the autonomous systems of the internet. An autonomous system is a collection of Internet Protocol (IP) networks and routers under the control of one entity (or sometimes more) that presents a common routing policy to the Internet. Their adjacency spectrum was investigated in [19], and the normalized Laplacians in [42].

Researchers also applied spectral techniques in observing the autonomous systems of the Internet. Informally, an autonomous system (AS) is a single network or a group of networks under a single administrative control. An example of an autonomous system might be the set of all computer networks owned by a university, or a company. Their adjacency spectrum was investigated in [19], and the normalized Laplacians in [42].

The research in graph theory in recent years has focused on the Laplacian spectrum in an attempt to design approximation algorithms for NP-hard problems, such as graph partitioning, graph colouring and finding the largest clique in a graph.

Fiedler [20] was among the first to study the properties of the second smallest Laplacian eigenvalue (called Fiedler eigenvalue), its corresponding eigenvector, and their relationship to the connectivity of a graph. He observed [21] that the differences in the components of this eigenvector are an approximate measure of the distance between the vertices. Alon and Millman [3] looked at the relationship between the Fiedler eigenvalue and the isoperimetric number which is related to the problem of computing good edge separators. Barnes [7] suggests an algorithm for partitioning the nodes of a graph. All these pieces of research closely relate to the research topic of clustering mentioned earlier.

Numerous other problems have been solved using Laplacians. Barnard, Pothen and Simon [6] use the Laplacian matrix for envelope reduction of sparse matrices, while Juvan and Mohar [30] use this eigenvector to compute bandwidth and  $p$ -sum reducing orderings. Mohar and Poljak [40] provide a comprehensive survey of the applications of the Laplacian spectra to combinatorial problems.

These papers cover only a small part of the research areas where the spectral approach has been applied.

Overall, it has been shown successfully that the Laplacian spectrum has many attractive theoretical properties which give it a fundamental role in spectral theory.

In this chapter we discuss the relevance of the theoretically derived results about the Laplacian eigenvalues (and in particular their eigenvectors) in algebraic graph theory and FAPs. The work presented includes, is motivated by, and at the same time is an extension of our published joint work [25].

### 3.1 Continuity of Laplacian Eigenvalues

The following properties from perturbation theory of matrices are important in our research. They support our idea that one can classify FAPs according to their similarities, i.e., slightly changed initial information in the influence matrix should not change the approach to solving a given problem. This becomes extremely important when we recognize that a new problem can be compared to an “almost similar problem” for which we have previously developed a method for solving.

Since we are currently interested in the spectrum of Laplacian matrices, our intention is to draw conclusions about the behaviour of the spectrum when certain small changes are introduced.

We will be working with vector and matrix norms described below. Let  $x = [x_1, \dots, x_n]^T$  denote a vector of size  $n$ , and let  $A = [a_{ij}]$  denote an  $n \times n$  matrix.

**Definition 3.1.1.** *The vector  $l_2$ -norm (the Euclidian norm) on  $\mathbb{C}^n$  is defined by*

$$\|x\|_2 \equiv \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} .$$

**Definition 3.1.2.** *The matrix  $l_2$ -norm (the Frobenius norm) on  $\mathbb{C}^n$  is defined by*

$$\|A\|_2 \equiv \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2} .$$

We define a new term, the eigenvalue vector.

**Definition 3.1.3.** *The eigenvalue vector of a symmetric matrix is the vector whose entries are the eigenvalues of the matrix in increasing order.*

Our aim is to prove that slight changes in the Laplacian matrix cause small changes to its eigenvalue vector. In other words:

**Theorem 3.1.4.** *Let  $T$  be the operator that maps the set of symmetric matrices into the set of vectors, such that the image of a matrix is its eigenvalue vector. Then  $T$  is a continuous operator.*

The following theorem and its corollary are needed:

**Theorem 3.1.5.** *(Hoffman and Wielandt) Let  $A$  and  $E$  be  $n \times n$  matrices, such that  $A$  and  $A + E$  are both normal, let  $\{\lambda_1, \dots, \lambda_n\}$  be the eigenvalues of  $A$  in some given order, and let  $\{\lambda'_1, \dots, \lambda'_n\}$  be the eigenvalues of  $A + E$  in some order. Then there exists a permutation  $\sigma$  of the integers  $1, 2, \dots, n$  such that*

$$\left[ \sum_{i=1}^n |\lambda'_{\sigma(i)} - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2.$$

For Hermitian matrices, and therefore for the real symmetric ones that we are concerned with, this theorem is used to prove a stronger result stated below. The proof of the theorem and the corollary can be found in [28].

**Corollary 3.1.6.** *Let  $A$  and  $E$  be  $n \times n$  matrices, such that  $A$  is Hermitian and  $A + E$  is normal, let  $\{\lambda_1, \dots, \lambda_n\}$  be the eigenvalues of  $A$  arranged in increasing order ( $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ ), and let  $\{\lambda'_1, \dots, \lambda'_n\}$  be the eigenvalues of  $A + E$  ordered so that  $\operatorname{Re} \lambda'_1 \leq \operatorname{Re} \lambda'_2 \leq \dots \leq \operatorname{Re} \lambda'_n$ . Then we have*

$$\left[ \sum_{i=1}^n |\lambda'_i - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2.$$

**Proof.** According to Theorem 3.1.5, there is some permutation  $\sigma$  of the given order (increasing real parts) for the eigenvalues of  $A + E$  for which

$$\left[ \sum_{i=1}^n |\lambda'_{\sigma(i)} - \lambda_i|^2 \right]^{1/2} \leq \|E\|_2.$$

If the eigenvalues of  $A + E$  in the list  $\lambda'_{\sigma(1)}, \dots, \lambda'_{\sigma(n)}$  are already in increasing order of their real parts, there is nothing to prove. Otherwise, there are two successive

eigenvalues in the list that are not ordered in this way, so

$$\operatorname{Re} \lambda'_{\sigma(k)} > \operatorname{Re} \lambda'_{\sigma(k+1)} \quad \text{for some } k, 1 \leq k < n.$$

But since

$$\begin{aligned} & |\lambda'_{\sigma(k)} - \lambda_k|^2 + |\lambda'_{\sigma(k+1)} - \lambda_{k+1}|^2 = \\ & = |\lambda'_{\sigma(k+1)} - \lambda_k|^2 + |\lambda'_{\sigma(k)} - \lambda_{k+1}|^2 + 2(\lambda_k - \lambda_{k+1})(\operatorname{Re} \lambda'_{\sigma(k+1)} - \operatorname{Re} \lambda'_{\sigma(k)}), \end{aligned}$$

and since  $\lambda_k - \lambda_{k+1} \leq 0$  by assumption, we see that

$$|\lambda'_{\sigma(k)} - \lambda_k|^2 + |\lambda'_{\sigma(k+1)} - \lambda_{k+1}|^2 \geq |\lambda'_{\sigma(k+1)} - \lambda_k|^2 + |\lambda'_{\sigma(k)} - \lambda_{k+1}|^2.$$

Thus, the two eigenvalues  $\lambda'_{\sigma(k)}$  and  $\lambda'_{\sigma(k+1)}$  can be interchanged without increasing the sum of squared differences. By a finite sequence of such interchanges, the list of eigenvalues  $\lambda'_{\sigma(1)}, \dots, \lambda'_{\sigma(n)}$  can be transformed into the list  $\lambda'_1, \lambda'_2, \dots, \lambda'_n$  in which the real parts are increasing and the asserted bound holds. ■

For Hermitian matrices  $A$  and  $B$  and their eigenvalues arranged in increasing or decreasing order we have

$$\left[ \sum_{i=1}^n |\lambda_i(A) - \lambda_i(B)|^2 \right]^{1/2} \leq \|A - B\|_2,$$

and this is exactly the relation that proves Theorem 3.1.4.

## 3.2 Laplacian Eigenvalues of Weighted Graphs

In this section we talk about bounds for the Laplacian eigenvalues in terms of the number of vertices and the number of components of weighted graphs.

The results that follow are inspired by known theorems for simple graphs (see, e.g., [5, 23, 38]).

**Definition 3.2.1.** *The weighted complete graph  $K_N^{(c)}$ , where  $c \in \mathbb{R}$ ,  $c > 0$ , is the graph on  $N$  vertices with all vertices adjacent and all edges of constant weight  $c$ .*

**Lemma 3.2.2.** *The eigenvalues of the Laplacian  $L$  of a weighted complete graph  $K_N^{(c)}$  are 0 with multiplicity 1, and  $cN$  with multiplicity  $N - 1$ .*

**Proof.** It is easy to see that  $L\mathbf{1} = \mathbf{0}$ , (each row sum of  $L$  is 0), so we conclude that 0 is a Laplacian eigenvalue of the graph.

Let  $x$  be any vector orthogonal to  $\mathbf{1}$ . Then we easily obtain  $Lx = (cN)x$ . Since there are exactly  $N - 1$  linearly independent vectors orthogonal to  $\mathbf{1}$ ,  $cN$  is a Laplacian eigenvalue with multiplicity  $N - 1$ . ■

We recall the definition of weighted graph, Definition 2.3.3. There we note that we regard the situation of no edge between two vertices  $u$  and  $v$  and that there is an edge  $uv$  of weight 0 as identical. We are now ready to define the complement of an edge-weighted graph.

**Definition 3.2.3.** The complement  $(G, \bar{w})$  (or just  $\bar{G}$ ) of a weighted graph  $G$  is the weighted graph with the same vertex and edge set, and with weights  $\bar{w}(e) = C - w(e)$  for an edge  $e \in E(G)$ , where  $C = \max_{e \in E(G)} w(e)$ .

**Theorem 3.2.4.** If  $\lambda$  is a Laplacian eigenvalue of a weighted graph  $G$  on  $N$  vertices and maximal edge-weight  $C$ , then  $0 \leq \lambda \leq CN$ .

A vector orthogonal to  $\mathbf{1}$  is an eigenvector of the Laplacian  $L(G)$  corresponding to the eigenvalue  $\lambda$  if and only if it is an eigenvector of the Laplacian  $L(\bar{G})$  of the complement  $\bar{G}$  corresponding to the eigenvalue  $CN - \lambda$ .

**Proof.** Let  $G$  be a weighted graph with  $M$  edges. Choose an orientation of the edges of  $G$ . The vertex-edge incidence matrix  $Q$  of  $G$  is the  $N \times M$  matrix  $Q$  such that

$$Q_{ve} = \begin{cases} \sqrt{w(e)}, & \text{if edge } e \text{ points toward vertex } v, \\ -\sqrt{w(e)}, & \text{if edge } e \text{ points away from vertex } v, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this means that  $L(G) = QQ^T$ .

For an eigenvalue  $\lambda \neq 0$  of the Laplacian  $L$  there exists a vector  $x$ , with  $\|x\| = 1$ , such that  $Lx = \lambda x$ . Thus  $\lambda = \langle \lambda x, x \rangle = \langle Lx, x \rangle = \langle QQ^T x, x \rangle = \langle Q^T x, Q^T x \rangle = \|Q^T x\|^2$ . Therefore  $\lambda$  is real and non-negative.

Furthermore, according to the definition of the complement of the weighted graph  $G$  we have  $L(G) + L(\bar{G}) = L(K_N^{(C)})$ . If  $L(G)x = \lambda x$  for some vector  $x$  orthogonal

to  $\mathbf{1}$ , then according to Lemma 3.2.2 we have:

$$L(\overline{G})x = L(K_N^{(C)})x - L(G)x = (CN - \lambda)x. \quad (3.1)$$

Since the Laplacian eigenvalues of  $\overline{G}$  are also non-negative, we get  $\lambda \leq CN$ .

Expression (3.1) also proves the last part of the theorem.  $\blacksquare$

**Theorem 3.2.5.** *Let  $G$  be a weighted graph with  $N$  vertices and maximal edge-weight  $C$ . The multiplicity of the eigenvalue 0 equals the number of components of the graph and the multiplicity of the eigenvalue  $CN$  is equal to one less than the number of components of the complementary graph  $\overline{G}$ .*

**Proof.** We define the incidence matrix  $Q$  as in the previous proof, and without loss of generality we assume that the first  $m$  rows of the incidence matrix  $Q$  correspond to the vertices of a component of  $G$  of order  $m$ . The sum of these rows is  $\mathbf{0}^T$  and any  $m - 1$  rows are independent. This means that the rank of the matrix formed from these  $m$  rows is  $m - 1$  and hence its nullity is 1. Therefore, each component of the graph contributes 1 to the nullity of  $Q$ . Since the nullity of  $Q$  is the same as the nullity of  $QQ^T$  (and hence, of  $L$ ), we conclude that the nullity of  $L(G)$  is equal to the number of components of the graph. In other words, the multiplicity of the Laplacian eigenvalue 0 equals the number of components of the graph. This proves the first part of the theorem.

Furthermore, from Theorem 3.2.4, we know that for each eigenvalue  $CN$  of  $L(G)$  with associated eigenvector that is orthogonal to  $\mathbf{1}$ , there is one eigenvalue 0 of  $L(\overline{G})$  associated with the same eigenvector. Since in the Laplacian spectrum of any graph there is one eigenvalue 0 that is associated with all-1 vector  $\mathbf{1}$ , we deduce that the value  $CN$  is an eigenvalue of  $L(G)$  with multiplicity  $k$  if and only if 0 is an eigenvalue of  $L(\overline{G})$  of multiplicity  $k + 1$ . According to the first part of the theorem, the latter is true if and only if  $\overline{G}$  has  $k + 1$  components. This completes the proof.  $\blacksquare$

**Definition 3.2.6.** *The complete  $k^{(c)}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(c)}$  is the weighted graph isomorphic to  $K_{N_1, N_2, \dots, N_k}$  in which all edges have weight  $c$ . In other words, the vertices of  $K_{N_1, \dots, N_k}^{(c)}$  can be partitioned into  $k$  disjoint sets  $V_1, V_2, \dots, V_k$ , where each  $V_i$  contains  $N_i$  vertices, and two vertices are joined by an edge of weight  $c$  if and only if the vertices belong to different parts.*

**Theorem 3.2.7.** *Let  $G$  be a weighted graph on  $N$  vertices and maximal edge weight  $C$ . Then the multiplicity of  $CN$  as an eigenvalue of the Laplacian matrix  $L$  is equal to  $k - 1$  if and only if  $G$  contains a complete  $k^{(C)}$ -partite graph  $K_{N_1, \dots, N_k}^{(C)}$  as a spanning subgraph of  $G$ .*

**Proof.** According to Theorem 3.2.5 the multiplicity of  $CN$  is equal to  $k - 1$ , one less the number of components of the complementary graph  $\overline{G}$ . On the other hand  $\overline{G}$  has  $k$  components if and only if  $G$  has a complete  $k^{(C)}$ -partite graph  $K_{N_1, \dots, N_k}^{(C)}$  as a spanning subgraph. ■

### 3.2.1 Bounds for eigenvalues in terms of edge weights

In this subsection we prove certain results concerning bounds for the smallest non-zero and the largest Laplacian eigenvalue of a weighted graph. Throughout the section let  $G$  be a weighted graph on  $N$  vertices and suppose its Laplacian  $L$  has eigenvalues ordered as follows:

$$0 = \lambda_N(L) \leq \lambda_{N-1}(L) \leq \dots \leq \lambda_1(L).$$

We recall the Rayleigh's and Courant-Fisher's characterisation of the eigenvalues on page 18, which implies that for the second smallest eigenvalue of a weighted graph  $G$  with the Laplacian matrix  $L$  we can write:

$$\lambda_{N-1}(L) = \min\{ \langle Lx, x \rangle \mid \|x\| = 1, x \perp \mathbf{1} \}, \quad (3.2)$$

since  $\mathbf{1}$  is the eigenvector corresponding to the minimal eigenvalue  $\lambda_N(L) = 0$ .

**Theorem 3.2.8.** *Let  $G$  be a weighted graph on  $N$  vertices. Then we have*

$$\lambda_{N-1} \leq w(uv) + \frac{1}{2} (d(u) + d(v)),$$

for any two vertices  $u, v, u \neq v$ .

**Proof.** Choose two vertices  $u, v$ ,  $u \neq v$ , and define a vector with real entries  $x = [x_1, \dots, x_N]^T$  as follows:

$$x_i = \begin{cases} 1, & \text{if } i = u, \\ -1, & \text{if } i = v, \\ 0, & \text{otherwise.} \end{cases}$$

As vector  $x$  is orthogonal to  $\mathbf{1}$ , from expression (3.2), we have

$$\lambda_{N-1} \leq \frac{\langle Lx, x \rangle}{\langle x, x \rangle} = \frac{(Lx)^T x}{x^T x} = \frac{d(u) + d(v) + 2w(uv)}{2},$$

which proves the theorem. ■

From this theorem the following corollary may be obtained.

**Corollary 3.2.9.** *If a weighted graph has at least one pair of non-adjacent vertices  $u, v$ , then  $\lambda_{N-1} \leq \frac{1}{2}(d(u) + d(v)) \leq \frac{1}{2}(d_{\max} + d_{\max}) = d_{\max}$ , where  $d_{\max}$  denotes the maximum degree.*

Note that the value of  $\lambda_{N-1}$  can be greater than  $d_{\max}$  in the case of a weighted graph in which any two vertices are adjacent. For example,  $\lambda_{N-1}(K_N^{(c)}) = cN > c(N-1) = d_{\max}(K_N^{(c)})$ .

**Definition 3.2.10.** *The 2-degree  $d_2(v)$  of a vertex  $v$  of a weighted graph  $G$  is the sum of the degrees of its neighbours times the weight of the edge with which  $v$  is joined to that neighbour. So  $d_2(v) = \sum_{u \neq v} w(uv)d(u)$ .*

**Theorem 3.2.11.** *Let  $\lambda_1$  be the largest Laplacian eigenvalue of a connected weighted graph  $G$  with no isolated vertices. Then  $\lambda_1 \leq \max_{v \in V(G)} \left\{ d(v) + \frac{d_2(v)}{d(v)} \right\}$ .*

To prove Theorem 3.2.11, we use Geršgorin's Theorem and its corollaries. These are classical results regarding the location of eigenvalues of general matrices, and can be found in, e.g., [28].

**Theorem 3.2.12 (Geršgorin's Theorem).** *Let  $A = [a_{ij}]$  be an  $N \times N$  matrix, and let*

$$R'_i(A) = \sum_{j=1, j \neq i}^N |a_{ij}|, \quad i = 1, \dots, N,$$

denote the deleted absolute row sums of  $A$ . Then all the (complex) eigenvalues of  $A$  are located in the union of  $N$  discs

$$\bigcup_{i=1}^N \{z \in \mathbb{C} \mid |z - a_{ii}| \leq R'_i(A)\} = G(A).$$

Furthermore, if a union of  $k$  of these  $N$  discs forms a connected region in the complex plane that is disjoint from all other  $N - k$  discs, then there are precisely  $k$  eigenvalues of  $A$  in this region.

**Corollary 3.2.13.** Let  $A = [a_{ij}]$  be an  $N \times N$  matrix and let

$$C'_j(A) = \sum_{i=1, i \neq j}^N |a_{ij}|, \quad j = 1, \dots, N,$$

denote the deleted absolute column sums of  $A$ . Then all the (complex) eigenvalues of  $A$  are located in the union of  $N$  discs

$$\bigcup_{j=1}^N \{z \in \mathbb{C} \mid |z - a_{jj}| \leq C'_j(A)\} = G(A^T).$$

Furthermore, if a union of  $k$  of these  $N$  discs forms a connected region that is disjoint from all other  $N - k$  discs, then there are precisely  $k$  eigenvalues of  $A$  in this region.

Theorem 3.2.12 and Corollary 3.2.13 together show that all the eigenvalues of  $A$  lie in the intersection of the two regions, that is, in  $G(A) \cap G(A^T)$ .

**Corollary 3.2.14.** If  $A = [a_{ij}]$  is an  $N \times N$  matrix, then

$$\max\{|\lambda| \mid \lambda \text{ an eigenvalue of } A\} \leq \min\left\{\max_i \sum_{j=1}^N |a_{ij}|, \max_j \sum_{i=1}^N |a_{ij}|\right\}.$$

We are now ready to give a proof of Theorem 3.2.11.

**Proof.** In a graph with no edges both sides of the inequality are equal to 0. Furthermore, it is enough to consider connected weighted graphs only. Denote by  $D$  the diagonal matrix  $D = \text{diag}(d(1), d(2), \dots, d(N))$ . Then the matrix  $D^{-1}LD$  is

the matrix defined by

$$(D^{-1}LD)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -w(uv)d(v)/d(u), & \text{if there is an edge joining } u \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $D$  is invertible,  $D^{-1}LD$  has the same eigenvalues as  $L$ . Therefore, application of Corollary 3.2.14 to the rows of  $D^{-1}LD$  gives us

$$\lambda_1 \leq \max_u \left\{ d(u) + \frac{\sum_v w(uv)d(v)}{d(u)} \right\},$$

and hence the result follows. ■

Fiedler [20] looked to characterise simple graphs by means of the spectra of the Laplacian matrix. He is well-known for naming  $\lambda_{N-1}$  the *algebraic connectivity* of a simple graph, because of the relation with two common concepts from graph theory: edge and vertex connectivity of the graph. That work produced a couple of inequalities that easily generalise to edge-weighted graphs which we use later in the thesis. The following are the original Fiedler results rewritten for edge-weighted graphs.

**Theorem 3.2.15.** *Let  $G$  be a weighted graph on  $N$  vertices. Then*

$$\lambda_{N-1} \leq \frac{N\delta(G)}{N-1},$$

and

$$\lambda_1 \geq \frac{N\Delta(G)}{N-1},$$

where  $\delta(G)$  denotes the minimal, and  $\Delta(G)$  denotes the maximal generalised vertex degree.

**Proof.** For a weighted graph  $G$ , with the Laplacian matrix  $L$  (which is positive semidefinite) we know

$$\lambda_{N-1}(L) = \min\{ \langle Lx, x \rangle \mid \|x\| = 1, x \perp \mathbf{1} \},$$

which implies

$$x^T Lx - \lambda_{N-1} \geq 0, \text{ for all } x \perp \mathbf{1}.$$

Let

$$\tilde{L} = L - \lambda_{N-1}(I - N^{-1}J),$$

where  $J$  is the  $N \times N$  all-one matrix. We show that  $\tilde{L}$  is positive semidefinite.

Let  $y$  be any vector in  $\mathbb{R}^N$ . Then  $y$  can be written in the form  $y = \alpha \mathbf{1} + \beta x$ ,  $\alpha$  and  $\beta$  scalars,  $\|x\| = 1$ , and  $x \perp \mathbf{1}$ . Since  $\tilde{L}\mathbf{1} = 0$ , it follows that

$$y^T \tilde{L} y = \beta^2 x^T \tilde{L} x = \beta^2 (x^T L x - \lambda_{N-1}),$$

which is obviously non-negative according to earlier observations. Thus the minimum diagonal entry of  $\tilde{L}$  is nonnegative

$$\min_{1 \leq i \leq N} (L_{ii} - \lambda_{N-1}(1 - N^{-1})) \geq 0,$$

which in graph theory terms is exactly equivalent to

$$\delta(G) - \lambda_{N-1}(1 - N^{-1}) \geq 0.$$

Furthermore, according to Theorem 3.2.4 the following relation between the maximal Laplacian eigenvalue of  $G$  and the second minimal Laplacian eigenvalue of the complement holds

$$\lambda_1(G) = CN - \lambda_{N-1}(\bar{G}),$$

where  $C$  denotes the largest edge weight in  $G$ . We note from Definition 3.2.3 that  $\delta(\bar{G}) = C(N-1) - \Delta(G)$ , and hence, as in the previously proved inequality for  $\lambda_{N-1}(G)$  we have

$$\lambda_1(G) \geq CN - \frac{N\delta(\bar{G})}{N-1} = CN - \frac{N(C(N-1) - \Delta(G))}{N-1} = \frac{N\Delta(G)}{N-1}.$$

■

### 3.3 Laplacian Eigenvectors of Weighted Graphs

Since a matrix is not uniquely characterised by its collection of eigenvalues, one cannot expect all the important information on a FAP to be obtainable from the set of the Laplacian eigenvalues. However, when we combine the eigenvalues with

a full collection of eigenvectors, we can reconstruct the matrix. Therefore we can expect to obtain more detailed information about a FAP by considering both the eigenvalues and certain eigenvectors simultaneously. Eigenvectors themselves are sometimes observed through their entries and we note that each entry corresponds to exactly one vertex of the graph. Since the Laplacian matrix is symmetric, all the entries can be taken to be real numbers.

### 3.3.1 Recognition of specific induced subgraphs

Earlier we presented a couple of known results for eigenvalues of weighted graphs. For instance, the number of components is equal to the multiplicity of the Laplacian eigenvalue 0, and we saw that the multiplicity of  $CN$  is related to the existence of the induced complete  $k$ -partite graph. In our first interaction with eigenvectors we investigate if the entries of the eigenvectors that correspond to particular Laplacian eigenvalues can reveal more about the graph itself. The results follow.

**Theorem 3.3.1.** *Let  $G$  be a weighted graph on  $N$  vertices. The multiplicity  $k$  of the Laplacian eigenvalue 0 is equal to the number of components of  $G$ .*

*If each component has  $N_i$  vertices,  $i = 1, \dots, k$ , then the values of  $N_i$  and the vertices in each component can be recognised as follows: The  $N \times k$  matrix  $M$  whose columns are the  $k$  independent eigenvectors corresponding to the eigenvalue 0 contains  $k$  different rows  $m^{(i)}$ ,  $i = 1, \dots, k$ , where row  $m^{(i)}$  occurs  $N_i$  times. Moreover, the indices of the identical rows correspond to vertices belonging to the same component of  $G$ .*

**Proof.** Theorem 3.2.5 already explains that the multiplicity of the eigenvalue 0 represents the number of components of the corresponding weighted graph.

Let us, therefore, focus on the second part of the theorem that states the relation between coordinates of the eigenvectors and the components of the graph. Consider the matrix  $M$ , described in the statement of the theorem, where we propose that identical rows of  $M$  correspond to vertices of the same component.

The fact that  $G$  has  $k$  connected components of sizes  $N_1, \dots, N_k$  helps to reconstruct the Laplacian matrix  $L(G)$ . Without loss of generality we may assume that the first  $N_1$  rows of  $L$  correspond to the first component, the second  $N_2$  rows correspond

to the second component, etc. We define the vector  $X$  to be a vector that is constant on the entries of the  $k$  components of the graph. The Laplacian matrix  $L$  and the vector  $X$  can be represented as follows:

$$L = \left[ \begin{array}{c|c|c|c} L_{11} & 0 & \cdots & 0 \\ \hline 0 & L_{22} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & L_{kk} \end{array} \right], \quad X = \left[ \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_k \end{array} \right],$$

where  $L_{ii}$  is an  $N_i \times N_i$  submatrix that represents the Laplacian matrix of the  $i$ -th component of  $G$ , whilst  $X_i$  is a subvector of size  $N_i$  with entries corresponding to the  $i$ -th component of  $G$ .

It is easy to prove that  $LX = 0X$  (the row sum of each of the  $L_{ii}$  matrices is 0), and since there are exactly  $k$  linearly independent choices for  $X$ , we can conclude that any such choice gives a basis of the Laplacian eigenspace that corresponds to the eigenvalue 0. This also means that a matrix that has these  $k$  eigenvectors as its columns is such that it contains at most  $k$  different rows of which the first  $N_1$  are identical (equal to  $m^{(1)}$ , say), the second  $N_2$  are identical, (equal to  $m^{(2)}$ ), etc. for any choice of  $k$  independent eigenvectors. In fact, this matrix has exactly  $k$  distinct rows, for the rank of the matrix is  $k$ . This proves the existence of a bijection between the rows of the matrix  $M$  and the vertices of  $G$  so that indices of the identical rows are the indices of the vertices that belong to the same component of the graph  $G$ . ■

**Theorem 3.3.2.** *Let  $G$  be a weighted graph on  $N$  vertices and of maximal edge weight  $C$ . The graph  $G$  contains a complete  $k^{(C)}$ -partite graph  $K_{N_1, \dots, N_k}^{(C)}$  as a spanning subgraph if and only if the multiplicity of  $CN$  as an eigenvalue of the Laplacian matrix  $L$  is  $k - 1$ .*

*Furthermore, the  $N \times (k - 1)$  matrix  $M$  whose columns are the  $k - 1$  independent eigenvectors that correspond to the eigenvalue  $CN$  contains  $k$  different rows  $m^{(i)}$ ,  $i = 1, \dots, k$  where each row  $m^{(i)}$  occurs  $N_i$  times. Moreover, indices of identical rows correspond to the vertices belonging to the same part  $V_i$  of the  $k^{(C)}$ -partition of  $G$ .*

**Proof.** In Theorem 3.2.7 we have already proved that the multiplicity of  $CN$  is equal to  $k - 1$  if and only if  $G$  contains a complete  $k^{(C)}$ -partite graph  $K_{N_1, \dots, N_k}^{(C)}$  as

a spanning subgraph. The main point for discussion here is the remaining part of the theorem, describing the relationship between entries of the eigenvectors and the partition of vectors.

In order to assess which vertices of  $G$  belong to which part, we consider the  $k - 1$  independent eigenvectors that correspond to  $CN$ . More precisely, we look at the matrix  $M$  described in the statement of the theorem. Our claim is that identical rows of  $M$  correspond to vertices in the same part.

In order to do this, consider the complementary weighted graph  $\bar{G}$ . Following Theorem 3.2.4, we see that  $\bar{G}$  has a Laplacian eigenvalue 0 of dimension  $k$ , and all eigenvectors of  $L(G)$  for the eigenvalue  $CN$  are also eigenvectors of  $L(\bar{G})$  for the eigenvalue 0. Additionally, the all-1 vector  $\mathbf{1}$  is an eigenvector of  $L(\bar{G})$  corresponding to the eigenvalue 0. Using this information in Theorem 3.3.1 immediately reveals that  $\bar{G}$  has a structure consisting of  $k$  components, where the vertices of each component can be found by looking at identical rows in the matrix of eigenvectors (the extra vector  $\mathbf{1}$  makes no difference here). Going back to the original graph  $G$ , this means a structure for  $G$  as described in the theorem. ■

**Corollary 3.3.3.** *Let  $L$  be the Laplacian of a FAP with  $N$  channels, where the maximal interference between any pair is  $C$ . Then the two following statements are equivalent:*

1. *The system contains a partition  $S_1, S_2, \dots, S_k$  of the channels into  $k$  parts of size  $N_i$ ,  $i = 1, \dots, k$ , with the property that the interference between any pair of vertices from different parts is  $C$ ;*
2. *There is a Laplacian eigenvalue  $CN$  of multiplicity  $k - 1$  such that the  $k - 1$  corresponding eigenvectors have the following form:*
  - *they are orthogonal to  $\mathbf{1}$ ;*
  - *the  $N \times (k - 1)$  matrix  $M$  whose columns are the  $k - 1$  eigenvectors contains  $k$  different rows  $m^{(i)}$ ,  $i = 1, \dots, k$ , where each row  $m^{(i)}$  occurs  $N_i$  times.*

*Moreover, indices of rows in  $M$  that are identical correspond to channels belonging to the same part  $S_i$ .*

These first observations are useful to check if a given FAP can be divided into independent FAPs of smaller size, and whether a given set of transmitters can be split into subsets where the vertices of different subsets influence each another maximally.

The results instantly give hope that we should be able to use Laplacian eigenvalues and eigenvectors in the analysis of the underlying structure of a graph. Theorem 3.4.5 in the next section will show that the knowledge about the structure of a FAP along the lines of the corollary above, can be very useful when trying to find the span of the FAP. However, other connections are generally not so easily spotted. That is the reason why our research is initially directed to considering several graphs with specific structures and their Laplacians, and only then do we move to more realistic cases, where we explore ideas on how to use the Laplacians.

The following theorem explores whether a given set of transmitters can be separated into subsets where the vertices from different subsets influence one another with an average weight  $w$ . By this we mean that for the given two subsets of vertices the following holds: for any vertex the average of edge weights between the vertex and all the vertices in the other subset is  $w$ .

**Theorem 3.3.4.** *Let  $G$  be a weighted graph on  $N$  vertices. Suppose there is a non-zero Laplacian eigenvalue  $wN$  of multiplicity  $k-1$ , and if the  $N \times (k-1)$  matrix  $M$ , formed from  $k-1$  independent eigenvectors corresponding to the eigenvalue  $wN$ , contains  $k$  different rows  $m^{(i)}$ ,  $i = 1, \dots, k$  where each row  $m^{(i)}$  occurs  $N_i$  times, and where one of them have all entries equal to 0. Then  $G$  contains  $k$  vertex disjoint subgraphs that cover  $V(G)$  that influence one another with an average weight  $w$ .*

*Moreover, indices of rows that are identical correspond to vertices belonging to the same subgraphs of  $G$  described above.*

**Proof.** Without loss of generality we present the  $k-1$  eigenvalue-eigenvector equations in form of a matrix equation  $L(G)M = wN M$ , i.e.,

$$\begin{bmatrix} L^{(11)} & L^{(12)} & \dots & L^{(1k)} \\ L^{(21)} & L^{(22)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ L^{(k1)} & \dots & \dots & L^{(kk)} \end{bmatrix} \begin{bmatrix} M^{(1)} \\ M^{(2)} \\ \vdots \\ M^{(k)} \end{bmatrix} = wN \begin{bmatrix} M^{(1)} \\ M^{(2)} \\ \vdots \\ M^{(k)} \end{bmatrix}. \quad (3.3)$$

Here, by  $M^{(i)}$  we denote an  $N_i \times (k-1)$  submatrix of  $M$  with all rows equal to  $m^{(i)}$ , and by  $L^{(ij)}$  an  $N_i \times N_j$  submatrix of  $L$  containing information about the influence between  $N_i$  vertices corresponding to the rows equal to  $m^{(i)}$  and  $N_j$  vertices that correspond to rows equal to  $m^{(j)}$ . We present the entries of  $m^{(i)}$  as  $m^{(i)} = [m_1^{(i)}, \dots, m_{k-1}^{(i)}]$ , and we write  $R_i(A)$  when referring to the  $i$ -th row of some matrix  $A$ .

Our aim is to show that the average of the elements in the rows/columns of the matrices  $L^{(ij)}, i \neq j$  is equal to value  $-w$ . First, we prove it for the first row of  $L$  by considering the equation

$$R_1(L) M = wN R_1(M). \quad (3.4)$$

We denote the sum of entries of the  $r$ -th row of  $L^{(ij)}$  by  $x_r^{(ij)}$ . By taking the transpose of both sides of equation (3.4), using the property that the columns of  $M$  are orthogonal to  $\mathbf{1}$ , and simplifying the equation obtained we get:

$$\begin{aligned} - \left[ \begin{array}{ccc|ccc} m_1^{(1)} - m_1^{(2)} & \dots & m_1^{(1)} - m_1^{(2)} & \dots & m_1^{(1)} - m_1^{(k)} & \dots & m_1^{(1)} - m_1^{(k)} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ m_{k-1}^{(1)} - m_{k-1}^{(2)} & \dots & m_{k-1}^{(1)} - m_{k-1}^{(2)} & \dots & m_{k-1}^{(1)} - m_{k-1}^{(k)} & \dots & m_{k-1}^{(1)} - m_{k-1}^{(k)} \end{array} \right] \hat{X} \\ = wN \begin{bmatrix} m_1^{(1)} \\ \vdots \\ m_{k-1}^{(1)} \end{bmatrix}, \end{aligned}$$

where  $\hat{X}$  is a transpose of  $R_1(L)$ , the first row of  $L$ , with the first  $N_1$  entries omitted.

This equation is equivalent to

$$- \left[ \begin{array}{ccc|ccc} m_1^{(1)} - m_1^{(2)} & \dots & m_1^{(1)} - m_1^{(k)} \\ \vdots & \dots & \vdots \\ m_{k-1}^{(1)} - m_{k-1}^{(2)} & \dots & m_{k-1}^{(1)} - m_{k-1}^{(k)} \end{array} \right] \begin{bmatrix} x_1^{(12)} \\ \vdots \\ x_1^{(1k)} \end{bmatrix} = \begin{bmatrix} wNm_1^{(1)} \\ \vdots \\ wNm_{k-1}^{(1)} \end{bmatrix}.$$

As  $M$  is such that its columns are the  $k-1$  eigenvectors corresponding to eigenvalue  $wN$ , its rank must be  $k-1$ . Since one of the  $k$  different rows of  $M$  is the zero vector, that means that the remaining  $k-1$  different rows are linearly independent. Without loss of generality we can assume that the entries of  $m^{(1)}$  are all equal to 0.

Since the rank of

$$\left[ \begin{array}{c|ccc} m_1^{(1)} - m_1^{(2)} & \cdots & m_1^{(1)} - m_1^{(k)} & wNm_1^{(1)} \\ \vdots & \cdots & \cdots & \vdots \\ m_{k-1}^{(1)} - m_{k-1}^{(2)} & \cdots & m_{k-1}^{(1)} - m_{k-1}^{(k)} & wNm_{k-1}^{(1)} \end{array} \right]$$

is the same as the rank  $(k - 1)$  of

$$\left[ \begin{array}{c|ccc} m_1^{(1)} - m_1^{(2)} & \cdots & m_1^{(1)} - m_1^{(k)} \\ \vdots & \cdots & \vdots \\ m_{k-1}^{(1)} - m_{k-1}^{(2)} & \cdots & m_{k-1}^{(1)} - m_{k-1}^{(k)} \end{array} \right]$$

and since it is equal to the number of unknowns  $k - 1$ , we conclude that the equation has exactly one solution and that one solution implies that  $x_1^{(j)} = -wN_j$ , for all  $j = 2, \dots, k$ , as required.

A similar reasoning can be applied to each of the rows of  $L$ , and hence the required result for rows. The symmetry of  $L$  confirms the same for columns.  $\blacksquare$

If in addition to the information from the previous theorem we know more about the Laplacian spectrum, we would be able to go a step further in understanding the structure:

**Theorem 3.3.5.** *Let  $G$  be a weighted graph on  $N$  vertices. The graph  $G$  contains a complete  $k^{(w)}$ -partite graph  $K_{N_1, \dots, N_k}^{(w)}$  as a spanning subgraph if*

- *the multiplicity of  $wN$  as an eigenvalue of the Laplacian matrix  $L$  is  $k - 1$ ;*
- *the  $N \times (k - 1)$  matrix  $M$ , formed from the  $k - 1$  independent eigenvectors corresponding to the eigenvalue  $wN$ , contains  $k$  different rows  $m^{(i)}$ ,  $i = 1, \dots, k$ , where each row  $m^{(i)}$  occurs  $N_i$  times, and one of the rows  $m^{(i)}$  has all entries equal to 0;*
- *among the remaining  $N - k$  eigenvectors orthogonal to  $\mathbf{1}$ , for each  $i$  we find  $N_i - 1$  eigenvectors such that their entries indexed by the vertices that do not correspond to rows  $m^{(i)}$  of  $M$  are all 0.*

*Moreover, the indices of identical rows of  $M$  correspond to vertices belonging to the same part  $V_i$  of the  $k^{(w)}$ -partition of  $G$ .*

**Proof.** Since the description of the eigenvalue  $wN$  and its corresponding eigenvectors is the same as in Theorem 3.3.4 we start this proof similarly, by considering the matrix equation (3.3). Having the additional conditions in mind we want to prove that all the entries of matrix  $L^{(ij)}$ , for all  $i \neq j$  are equal to  $-w$ .

Before we proceed, we introduce the required notation. Let  $M_i$  denote an  $N \times (N_i - 1)$  matrix whose columns are the eigenvectors of the type described in the statement of the theorem. Clearly, the matrix product  $LM_i$  denoted by  $\hat{M}_i$  is an  $N \times (N_i - 1)$  matrix of a similar form as  $M_i$ , i.e., all entries on the positions different to the positions of the rows  $m^{(i)}$  in the matrix  $M$  are zero.

So for  $i = 1, \dots, k$  we have

$$\begin{bmatrix} & \vdots & & \\ & L^{(i-1,i)} & & \\ \dots & L^{(i,i-1)} & L^{(i,i)} & L^{(i,i+1)} \dots \\ & & L^{(i+1,i)} & \\ & & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{0} \\ M_i^{(i)} \\ \mathbf{0} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \mathbf{0} \\ \hat{M}_i^{(i)} \\ \mathbf{0} \\ \vdots \end{bmatrix}.$$

where the matrices  $M_i^{(i)}$  and  $\hat{M}_i^{(i)}$  are  $N_i \times (N_i - 1)$  submatrices of  $M_i$  and  $\hat{M}_i$ , respectively, representing the positions of entries that can have non-zero values. All other entries of  $M_i$  and  $\hat{M}_i$  are equal to zero.

Let us consider all the matrices  $L^{(ji)}$ , where  $j \neq i$ . From the matrix equation above we read that

$$R_p(L^{(ji)})M_i^{(i)} = \mathbf{0}, \quad \text{for any row } p \text{ of } L^{(ji)}.$$

For fixed values  $i$  and  $j$  this relation simply represents a system of  $N_i - 1$  linear equations with  $N_i$  unknowns. Since  $\text{rank}(M_i^{(i)}) = N_i - 1$  (where columns are linearly independent vectors orthogonal to  $\mathbf{1}$ ), we conclude that the dimension of the solution space of the system of linear equations is equal to 1. And since one solution is  $R_p(L^{(ji)}) = \mathbf{1}$ , the general solution is  $R_p(L^{(ji)}) = \alpha\mathbf{1}$ ,  $\alpha \in \mathbb{R}$ .

Therefore, for all  $i \neq j$  we have that the columns of  $L^{(ji)}$  are identical and the rows are multiples of  $\mathbf{1}$ .

However, as  $L^{(ji)}$  can also be seen as a transpose of  $L^{(ij)}$ , for which it also holds that its columns are identical and its rows multiples of  $\mathbf{1}$ , we can write that for all  $i \neq j$  the rows of  $L^{(ji)}$  are identical and the columns are multiples of  $\mathbf{1}$ .

This means that  $L^{(ij)}$  is a matrix with all rows and columns equal, where both the rows and the columns are multiples of  $\mathbf{1}$ . This confirms that for all  $i \neq j$  each matrix  $L^{(ij)}$  is a multiple of the ‘all-1 matrix’  $J$  of the appropriate size, i.e.,  $L^{(ij)} = l_{ij} J_{N_i \times N_j}$ , for some real value  $l_{ij}$ . We present matrix  $L$  as follows:

$$L = \left[ \begin{array}{c|c|c|c} L^{(11)} & -l_{12} J_{n_1 \times n_2} & \cdots & \\ \hline -l_{21} J_{n_2 \times n_1} & L^{(22)} & \ddots & \\ \hline \vdots & \ddots & \ddots & \vdots \\ \hline & & \cdots & L^{(kk)} \end{array} \right].$$

Returning to the matrix equation  $LM = wNM$  and specially focusing on the first row  $R_1(L)$  of the matrix  $L$  we have

$$R_1(L)M = wNR_1(M). \quad (3.5)$$

In the same way as in the proof of Theorem 3.3.4 we denote the sum of the entries of the  $r$ -th row of  $L^{(ij)}$ , where  $i \neq j$  by  $x_r^{(ij)}$ , consider the transpose of equation 3.5, and obtain the following matrix equation:

$$- \left[ \begin{array}{c|c|c} m_1^{(1)} - m_1^{(2)} & \cdots & m_1^{(1)} - m_1^{(k)} \\ \vdots & \cdots & \vdots \\ m_{k-1}^{(1)} - m_{k-1}^{(2)} & \cdots & m_{k-1}^{(1)} - m_{k-1}^{(k)} \end{array} \right] \left[ \begin{array}{c} x_1^{(12)} \\ \vdots \\ x_1^{(1k)} \end{array} \right] = \left[ \begin{array}{c} wNm_1^{(1)} \\ \vdots \\ wNm_{k-1}^{(1)} \end{array} \right],$$

where  $m_1^{(i)}, \dots, m_{k-1}^{(i)}$  are the entries of  $m^{(i)}$ .

As in the previous proof we deduce that the only solution of this system is  $x_1^{(1j)} = -wN_j$ , for all  $j = 2, \dots, k$ . Hence,  $l_{1j} = -w$ , for all  $j = 2, \dots, k$ , since the sum of entries of the first row of  $L^{(1j)}$  denoted by  $x_1^{(1j)}$  in this case is  $l_{1j}N_j$ .

A similar analysis can be applied to other rows of  $L$  and it leads to the final conclusion that  $L^{(ij)} = -wJ_{N_i \times N_j}$  for all  $i \neq j$ . ■

The converse of this theorem also holds. However, we choose not to discuss it here, as the converse is a special case of a result considered later. The more general result is elegantly derived from the equitable partitions results that we consider in the continuation of the chapter.

### 3.3.2 Equitable partitions

Another way of looking at the structure of edge-weighted graphs is via the characteristic matrix of a partition of the graph's Laplacian.

We recall the classical result about interlacing by Courant, outlined on page 21. In graph theory terms Proposition 2.2.14 can be related to specific partitions of particular types of weighted graphs.

Additionally, the two terms *quotient matrix* and *equitable partition*, as defined in the same chapter, have relevant interpretations in graph theory. The definitions we introduce here are slightly different from the ones found in the literature, since we need them adjusted to edge-weighted graphs.

In theory, we define quotient matrices with respect to a given partition. However, in this section we are mainly observing the quotient matrices obtained from the Laplacian matrix of a graph with respect to some partition, and in particular with respect to an equitable partition.

**Definition 3.3.6.** *A partition  $\pi = (V_1, \dots, V_k)$  of  $V(G)$  is equitable if the corresponding partition of the Laplacian matrix  $L$  is equitable.*

In the case of the equitable partitions we recall an interesting property, the relation between the eigenvectors of  $B$  and  $L$ . Since the partition is equitable,  $LP = PB$ , where  $P$  is the characteristic matrix of the partition  $\pi$ . One eigenvalue of  $B$  is 0 (equal to the sum of its rows). Suppose that  $v$  is an eigenvector of  $B$  with a simple eigenvalue  $\lambda \neq 0$ . Then  $Pv \neq 0$  and

$$LPv = PBv = \lambda Pv.$$

Therefore  $Pv$  is an eigenvector of  $L$  corresponding to its eigenvalue  $\lambda$ . Notice that the entries of  $Pv$  are constant on the cells of  $P$ .

The previously mentioned proposition attracted our attention and we investigate how useful it could be in the search for independent sets of an edge-weighted graph.

**Equitable partition with independent parts**

Let us consider a graph for which there exists a partition  $\pi$  of graph vertices such that each part represents an independent set, and that each vertex in a single part “feels” the same from the other parts. In other words, for any two parts  $i$  and  $j$ , the sum of weights of all edges that initiate from  $V_j$  and are incident to a vertex in  $V_i$ , is independent of the choice of a vertex in  $V_i$ . Note that in the case where  $i = j$  we have that no vertex in  $V_i$  influences other vertices in the same set, which is exactly what is meant by an independent set. Also vertices in the same part ( $V_i$ , say) have equal generalised degrees  $d_i$ .

Naturally it follows that, if  $L$  is partitioned with respect to  $\pi$ , then for all  $i$  the block matrix  $L_{ii}$  is a diagonal matrix  $d_i I_{N_i \times N_i}$ . On the other hand, for  $i \neq j$ , a matrix  $L_{ij}$  is an  $N_i \times N_j$  matrix that contains information on the relation between vertices in  $V_i$  and  $V_j$ , and in this case they are matrices with negative constant row and column sums (row sums are denoted by  $-c_{ij}$ ). Hence, the corresponding quotient matrix of  $L$  is of the form

$$B = \begin{bmatrix} d_1 & -c_{12} & \cdots & -c_{1m} \\ -c_{21} & d_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ -c_{m1} & \cdots & \cdots & d_m \end{bmatrix}.$$

It is obvious that this partition  $\pi$  is equitable and therefore every eigenvalue of  $B$  is an eigenvalue of  $L$  with not smaller multiplicity, and if these eigenvalues are of multiplicity one then their corresponding eigenvectors have entries that are constant on the parts of  $\pi$ .

However, this holds regardless of the type of cells (which are not necessarily independent sets). The result that follows gives us an idea of how to distinguish whether the parts of a given equitable partition are independent sets or not.

**Theorem 3.3.7.** *Suppose that  $B$  is the quotient matrix of the Laplacian of an edge-weighted graph with respect to an equitable partition  $\pi = (V_1, \dots, V_m)$  with  $|V_i| = N_i$ . Then, every cell is an independent set if and only if  $\sum_{i=1}^m b_{ii} N_i$  is equal to the sum of all eigenvalues of  $L$ .*

**Proof.** If the parts are independent sets it is trivial to prove the suggested equality.

Assume that  $\sum_{i=1}^m b_{ii}N_i$  is equal to the sum of all eigenvalues of  $L$  and that not all cells are independent sets. Due to the conditions of the theorem the entries on the main diagonal of  $L$  are constant on each of the cells and equal to the generalised degree of the corresponding vertices. Without loss of generality we can assume that the cell  $V_1$  is not an independent set. The principal  $N_1 \times N_1$  submatrix of  $L$  corresponding to the vertices of  $V_1$  has value  $d_1$  (the degree of any vertex in  $V_1$ ) for each of its diagonal entries and for all off-diagonal entries that are non-positive (at least one negative). This means that the entry  $b_{11}$  of  $B$  must be strictly less than  $d_1$ . For all other cells  $V_i$ , where  $i = 2, \dots, m$ , we have a less rigid inequality  $b_{ii} \leq d_i$ . Therefore

$$\sum_{i=1}^N \lambda_i(L) = \text{tr}(L) = \sum_{i=1}^m d_i N_i > \sum_{i=1}^m b_{ii} N_i,$$

i.e., the sum of all eigenvalues of  $L$  is strictly greater than the given sum which contradicts our assumption, and thus proves the theorem. ■

### 3.3.3 Local properties

In all earlier examples we have looked at the relationship between overall global regularity of graphs and the Laplacians. On the other hand, small changes applied on a small subset of vertices within a graph also happen to be recorded in the entries of a Laplacian eigenvector. The next two results presented, motivated our research in this direction, and the simple graph versions can be found in Merris [37]. For consistency, we restate them and give proofs for edge-weighted graphs.

**Theorem 3.3.8 (Edge Principle).** *Let  $\lambda$  be an eigenvalue of a weighted graph  $G$  on  $N$  vertices corresponding to eigenvector  $x$ . Let  $x_p$  and  $x_q$  be the entries of  $x$  corresponding to the vertices  $p$  and  $q$ , respectively. If  $x_p = x_q$ , then  $\lambda$  is also an eigenvalue of  $G'$  corresponding to  $x$ , where  $G'$  is the weighted graph obtained from  $G$  by changing the weight of  $e = (p, q)$  to any new value.*

**Proof.** For a Laplacian eigenvalue-eigenvector pair  $(\lambda, x)$  and any entry  $i$  of  $x$ , we can rewrite  $Lx = \lambda x$  as

$$\sum_{j=1, j \neq i}^N -w_{ij}x_j + d_i x_i = \lambda x_i,$$

or

$$(d_i - \lambda)x_i = \sum_{j \sim i, j \neq i} w_{ij}x_j, \quad (3.6)$$

where  $d_i = \sum_{j \sim i, j \neq i} w_{ij}$  denotes the weighted degree of the vertex  $i$ .

Let us consider the described entries  $p$  and  $q$  of  $x$ .

Setting  $i = p$  and adding  $wx_p$  to both sides of the equality condition (3.6) we get

$$(d_p + w - \lambda)x_p = \sum_{j \neq p, j \sim p} w_{pj}x_j + wx_p. \quad (3.7)$$

Since  $x_p = x_q$ , the equality (3.7) becomes equivalent to

$$(d_p + w - \lambda)x_p = \sum_{j \neq p, j \sim p} w_{pj}x_j + wx_q. \quad (3.8)$$

Similarly, when  $i$  evaluated at  $q$ , from (3.6) we obtain another equality

$$(d_q + w - \lambda)x_q = \sum_{j \neq q, j \sim p} w_{qj}x_j + wx_p. \quad (3.9)$$

On the other hand, for the other entries, i.e., for  $i \neq p$  and  $i \neq q$  we remain to consider (3.6) in its original form

$$(d_i - \lambda)x_i = \sum_{j \neq i, j \sim i} w_{ij}x_j. \quad (3.10)$$

It is now not difficult to see that the expressions (3.8), (3.9) and (3.10), respectively, are exactly the conditions that should be met at  $G'$  vertices  $p$ ,  $q$ , and all the other different to  $p$  and  $q$ , for the vector  $x$  to be an eigenvector of  $G'$  corresponding to the eigenvalue  $\lambda$ . This proves the theorem.  $\blacksquare$

The Edge Principle theorem can easily be extended to changes in weights in more than one edge. We state its general form in the following corollary.

**Corollary 3.3.9.** *Let  $\lambda$  be an eigenvalue of a weighted graph  $G$  on  $N$  vertices corresponding to eigenvector  $x$ . Let  $x_{p_1}, \dots, x_{p_k}$  be the entries of  $x$  corresponding to the  $k < N$  vertices  $p_1, \dots, p_k$ , for which  $x_{p_1} = \dots = x_{p_k}$ , then  $\lambda$  is also an eigenvalue of  $G'$  corresponding to  $x$ , where  $G'$  is the weighted graph obtained from  $G$  by changing the weight of any of the edges  $(p_i, p_j)$  for all  $1 \leq i, j \leq k$ .*

Beside generalising the Edge Principle result, we observe changes in the spectrum of a graph when some additional particular conditions are imposed. As above, let  $x_i$  denote the entry of  $x$  corresponding to vertex  $i$  of a graph.

**Theorem 3.3.10.** *Let  $\lambda$  be an eigenvalue of a connected weighted graph  $G$  on  $N$  vertices corresponding to eigenvector  $x$ . Let  $G'$  be a weighted graph obtained from  $G$  by increasing the weight of an edge  $e = (p, q)$ . If the entries  $x_p$  and  $x_q$  of  $x$  are equal, and if  $\lambda$  is the smallest positive eigenvalue of  $G$  corresponding to  $x$ , then  $\lambda$  is also the smallest positive eigenvalue of  $G'$  with corresponding eigenvector  $x$ .*

**Proof.** Since  $x_p = x_q$ , according to Theorem 3.3.8, the eigenvalue pair  $(\lambda, x)$  of  $G$  is also an eigenvalue pair of  $G'$ .

Let us order the eigenvalues of  $G$  in decreasing order

$$\lambda_1(G) \geq \dots \geq \lambda_{N-1}(G) \geq \lambda_N(G).$$

Since  $G$  is connected, it has exactly one zero eigenvalue, i.e.,  $\lambda_N(G) = 0$ . Therefore, the next smallest eigenvalue is positive, and it must be that the value of  $\lambda_{N-1}(G)$  is  $\lambda$ .

Furthermore, as graph  $G'$  is obtained by increasing weight of an edge in  $G$ , we have that  $G'$  is also connected, hence  $\lambda_N(G') = 0$  and  $\lambda_{N-1}(G') > 0$ .

We now recall a result on eigenvalues interlacing, Proposition 2.2.6. In graph theory terms this property confirms that if a weight of an edge in a graph increases, the eigenvalues in the spectrum either increase in value, or remain unchanged. Or, in this case, in terms of our two considered graphs  $G$  and  $G'$  we have

$$\lambda_i(G) \leq \lambda_i(G'), \quad \text{for all } 1 \leq i \leq N.$$

This implies that the value of  $\lambda$ , being the eigenvalue of both  $G$  and  $G'$  is the only value that the smallest positive eigenvalue of  $G'$  can take, i.e.,  $\lambda_{N-1}(G) = \lambda_{N-1}(G') = \lambda$ , as required.

Note that if the multiplicity of the eigenvalue  $\lambda$  is higher than one, the same argument holds. ■

**Theorem 3.3.11 (Principle of Reduction and Extension).** *Let  $G$  be a weighted graph and  $W$  a nonempty subset of  $V(G)$ . Let  $G \setminus W$  denote the graph obtained*

by deleting all the vertices in  $V \setminus W$  that are not adjacent to any vertex in  $W$  and by removing any remaining edges that are not incident to any vertex of  $W$ . If  $x$  is an eigenvector of  $G \{W\}$  with an eigenvalue  $\lambda$  such that only the entries that correspond to the vertices of  $W$  are non-zero, then  $x$  extends to an eigenvector of  $G$  with the same eigenvalue  $\lambda$ .

**Proof.** We construct a vector such that the entries indexed by vertices in  $W$  are equal to the corresponding entries of  $x$ , whilst the remaining entries are equal to 0. According to Theorem 3.3.8 this vector is an eigenvector of  $G$  corresponding to  $\lambda$ , as required. ■

The results that follow focus on a subgraph  $S'$  of  $G$  which is of a particular predetermined type. In each of the cases the underlying assumption is that the subgraph equally interacts with the remaining part of the graph, while we do not have any information about the mutual interaction of the vertices outside  $S$ .

**Theorem 3.3.12.** *If a weighted graph  $G$  on  $N$  vertices contains a collection  $S$  of  $m > 1$  vertices with the following property:*

- $w(uv) = w_v$  (i.e.,  $w(uv)$  is only depending on  $v$ ) for all  $u \in S$  and  $v \in V(G) \setminus S$ ;

*then there is a collection of  $m - 1$  Laplacian eigenvalues such that the corresponding eigenvectors have the following form:*

- *there exist  $N - m$  coordinates for which the  $m - 1$  eigenvectors are all 0.*

*Moreover, the  $m$  coordinates for which not all  $m - 1$  eigenvectors are 0 represent the vertices that belong to the set  $S$ .*

**Proof.** Without loss of generality we can assume that the first  $m$  rows of the Laplacian matrix  $L(G)$  correspond to the  $m$  vertices in  $S$ . Therefore we have the following form of  $L$ :

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

We now consider a vector with the property that its last  $N - m$  entries are zeros

$$X = \begin{bmatrix} X_1 \\ \mathbf{0} \end{bmatrix},$$

where  $X_1$  is a vector of size  $m$  and  $\mathbf{0}$  is zero vector of size  $N - m$ .

If the vector  $X$  is an eigenvector of  $L$ , then relying on the above described presentation of  $L$  we can write the corresponding eigenvalue equation

$$L_{11} X_1 + L_{12} \mathbf{0} = \lambda X_1, \quad (3.11)$$

$$L_{21} X_1 + L_{22} \mathbf{0} = \lambda \mathbf{0}. \quad (3.12)$$

Let us consider the first equation  $L_{11} X_1 = \lambda X_1$ . It is obvious that  $L_{11}$  can be seen as the sum of  $L(G_S)$ , the Laplacian matrix of the induced subgraph on  $S$ , and the diagonal matrix  $tI$ . Hence we obtain the following equation

$$(L(G_S) + tI) X_1 = \lambda X_1, \quad (3.13)$$

which is equivalent to

$$L(G_S) X_1 = (\lambda - t) X_1. \quad (3.14)$$

The last equation presents the eigenvalue equation for the Laplacian matrix  $L(G_S)$ . From the properties of the Laplacian eigenvalues we deduce that this equation has exactly  $m$  solutions. It remains to check whether these  $m$  choices for  $X_1$  satisfy equation (3.12).

According to the condition of the theorem, matrix  $L_{21}$  is such that each row has all entries equal, i.e., the rows are multiples of  $\mathbf{1}$ .

First, notice that  $X_1 = \mathbf{1}$  is an eigenvector of  $L(G_S)$  that is associated with eigenvalue  $\lambda - t = 0$  and that this choice of  $X_1$  does not satisfy (3.12). On the other hand, according to the characterisation of Laplacian eigenvectors, every other eigenvector  $X_1$  of  $L(G_S)$  is orthogonal to  $\mathbf{1}$ , which implies that they satisfy equation (3.12).

We deduce that there are exactly  $m - 1$  linearly independent vectors of type  $\begin{bmatrix} X_1 \\ \mathbf{0} \end{bmatrix}$  that are eigenvectors of  $L$ , as stated in the theorem. ■

In the following theorem we assume an additional condition imposed on two chosen vertices in the induced subgraph.

**Theorem 3.3.13.** *Let  $G$  be a graph as described in Theorem 3.3.12. Let  $\mathcal{L}$  be the collection of  $m - 1$  eigenvalues according to that theorem. Suppose, also, that the two vertices  $i$  and  $j$  in  $S$  have identical weights on their incident edges, i.e.,  $w(ik) = w(jk)$  for any vertex  $k \neq i, k \neq j$  of the graph  $G$ . Then:*

- the entries of  $i$  and  $j$  are equal in all eigenvectors that correspond to the Laplacian eigenvalues in  $\mathcal{L}$ , except in the case of the Laplacian eigenvalue  $\lambda = d + w(ij)$ , where  $d$  represents the weighted degrees of the two vertices.

**Proof.** Without loss of generality we may assume that the two vertices correspond to the first two rows of  $L$ . Again, let  $X$  denote the form of an eigenvector corresponding to any eigenvalue from  $\mathcal{L}$ .

Writing out the relevant coordinates derived from the equation  $LX = \lambda X$ , and focusing on the first two rows of the matrix equation, we get

$$dx_1 - w(12)x_2 - \lambda x_1 = -w(12)x_1 + dx_2 - \lambda x_2,$$

where  $d$  represents the weighted degrees of the two vertices,  $x_i$  is the  $i$ -th entry of  $X$ , and  $w(ij)$  is the weight of the edge  $(i, j)$ .

This gives the equivalent equation

$$(x_1 - x_2)(d - \lambda + w(12)) = 0,$$

which proves the theorem. ■

To the graph in Theorem 3.3.12 we add yet another condition.

**Theorem 3.3.14.** *Let  $G$  be a graph as described in Theorem 3.3.12 and let  $\mathcal{L}$  be as described in Theorem 3.3.13. Let  $t$  denote the total influence each vertex in  $S$  feels from the vertices in  $V(G) \setminus S$ , i.e.,  $t = \sum_{v \in V(G) \setminus S} w(uv)$  for each vertex  $u \in S$ . Suppose that there is a connected induced subgraph on  $S'$ , ( $S' \subset S$ ) that is not connected to any other vertex in  $S \setminus S'$ . Then:*

- the entries in eigenvectors corresponding to  $S'$  sum up to 0 for any of the Laplacian eigenvalues in  $\mathcal{L}$ , except in the case of the Laplacian eigenvalue  $\lambda = t$ , when the entries are all equal.

**Proof.** We can assume that the first few rows of the Laplacian matrix  $L(G)$  correspond to the vertices in  $S'$ . Therefore we can see  $L$  in the following form

$$L = \left[ \begin{array}{cc|cc} L_{11}^{(1)} & 0 & & \\ 0 & L_{11}^{(2)} & L_{12} & \\ \hline & L_{21} & & L_{22} \end{array} \right],$$

where the entries off diagonal in  $L_{11}^{(1)}$  and  $L_{11}^{(2)}$  describe the influence between the vertices in  $S'$  and  $S \setminus S'$ , respectively. We consider the eigenvalue requirement  $LX = \lambda X$ , where, for convenience, we present  $X$  in the form

$$X = \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ \mathbf{0} \end{bmatrix},$$

where  $X_1^{(1)}$  and  $X_1^{(2)}$  are vectors of size  $|S'|$  and  $|S \setminus S'|$ , respectively.

We obtain the equation

$$L_{11}^{(1)} X_1^{(1)} = \lambda X_1^{(1)}.$$

Since we can see  $L_{11}^{(1)}$  as the sum of the Laplacian matrix of the induced subgraph on  $S'$  (denoted by  $L(G_{S'})$ ) and the diagonal matrix  $tI$ , we write

$$(L(G_{S'}) + tI) X_1^{(1)} = \lambda X_1^{(1)},$$

which is equivalent to

$$L(G_{S'}) X_1^{(1)} = (\lambda - t) X_1^{(1)}.$$

This equation however, is simply the eigenvalue condition for the Laplacian matrix  $L(G_{S'})$ , and since  $S'$  is connected, we deduce that all eigenvectors are orthogonal to  $\mathbf{1}$ , except the eigenvector corresponding to the eigenvalue  $0 = \lambda - t$ , which is a multiple of  $\mathbf{1}$ . This completes the proof of the theorem.  $\blacksquare$

After careful consideration of previously presented graphs and their eigenvector properties, we observe the converses of the listed theorems. Here we present and prove the converse of Theorem 3.3.12.

**Theorem 3.3.15.** *Let  $G$  be a weighted graph on  $N$  vertices. Suppose that the set of the Laplacian eigenvectors of the graph  $G$  is such that there is a collection  $\mathcal{L}$  of  $m - 1$  Laplacian eigenvalues such that the eigenvectors have the following form:*

- *there exist  $N - m$  coordinates for which the eigenvectors are all 0;*

*then there is an induced subgraph  $S$  of  $G$  of a specific structure described below:*

- *the subgraph  $S$  is of order  $m$  and its vertex set contains all the vertices of  $G$  that are **not** the vertices corresponding to the  $N - m$  zero positions in the eigenvectors described above;*
- *the subgraph  $S$  feels the same weight from each of the surrounding vertices, i.e.,  $w(uv) = w_v$  (i.e.,  $w_v$  is only depending on  $v$ ) for all  $u \in S$  and  $v \in V(G) \setminus S$ ;*

**Proof.** Without loss of generality we can assume that the described  $m - 1$  Laplacian eigenvectors have a 0 for their last  $N - m$  entries, i.e., the  $m - 1$  eigenvectors are of the form

$$\begin{bmatrix} X_1^{(i)} \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \dots, m - 1,$$

where  $X_1^{(i)}$  is a vector of size  $m$  and  $\mathbf{0}$  is a zero vector of size  $N - m$ . Similarly, we present  $L$  as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

where  $L_{11}$  and  $L_{22}$  are  $m \times m$  and  $(N - m) \times (N - m)$  matrices, respectively.

For  $i = 1, \dots, m - 1$  the following holds:

$$L_{11} X_1^{(i)} = \lambda^{(i)} X_1^{(i)},$$

$$L_{21} X_1^{(i)} = \mathbf{0}.$$

The equality  $L_{21} X_1^{(i)} = \mathbf{0}$  for all  $i = 1, \dots, m - 1$  implies that the nullity of  $L_{21}$  is at least  $m - 1$ . On the other hand, the rank-nullity theorem for linear mappings gives:  $\text{rank}(L_{21}) + \text{nullity}(L_{21}) = m$ , and hence  $\text{rank}(L_{21}) \leq 1$ .

The case where  $\text{rank}(L_{21}) = 0$  means that  $L_{21}$  is a zero matrix and the result follows. On the other hand, if  $\text{rank}(L_{21}) = 1$  that implies that each of the rows of

$L_{21}$  can be seen as a multiple of some non-zero vector  $y$  presented as

$$y = [y_1, \dots, y_m]^T.$$

Beside this requirement vector  $y$  must satisfy the equations

$$y^T X_1^{(i)} = 0, \quad i = 1, \dots, m - 1.$$

When considered as a system of  $m - 1$  linear equations we notice that there are  $m$  unknowns  $y_1, \dots, y_m$ . Also, the rank of the matrix  $[X_1^{(1)} \mid \dots \mid X_1^{(m-1)}]$  is  $m - 1$ , since the set of Laplacian eigenvectors considered is a linearly independent set. Hence, the dimension of the solution space of the system of linear equations is  $m - (m - 1) = 1$ . Since the Laplacian eigenvectors that correspond to non-zero eigenvalue are orthogonal to  $\mathbf{1}$ , we conclude that one solution of the system is  $y_1 = \dots = y_m = 1$ , and therefore the general solution is  $y_1 = \dots = y_m = \alpha$ ,  $\alpha \in \mathbb{R}$ , as required. ■

A natural next step in our research is to observe a special case where the considered induced subgraph is a complete graph. The following theorem describes this:

**Theorem 3.3.16.** *A weighted graph  $G$  on  $N$  vertices contains a collection  $S$  of  $m > 1$  vertices with the following properties:*

- $w(uv) = w$ , for some constant  $w$ , for all  $u, v \in S$ ,  $u \neq v$ ;
- $w(uv) = w_v$  (i.e.,  $w_v$  is only depending on  $v$ ) for all  $u \in S$  and  $v \in V(G) \setminus S$ ;

*if and only if there is a Laplacian eigenvalue of multiplicity  $m - 1$  such that the  $m - 1$  corresponding eigenvectors have the following form:*

- *they are orthogonal to  $\mathbf{1}$ ;*
- *there exist  $N - m$  coordinates for which the  $m - 1$  eigenvectors are all 0.*

*Moreover, the  $m$  coordinates for which not all  $m - 1$  eigenvectors are 0 represent the vertices that belong to the set  $S$ .*

**Proof.** First suppose that a collection  $S$  as described in the theorem exists. Without loss of generality we can assume that the first  $m$  rows of the Laplacian matrix  $L(G)$  correspond to the  $m$  vertices in  $S$ .

Therefore we have the following form of  $L$ :

$$L = \left[ \begin{array}{cccc|ccc} d & -w & \cdots & -w & \cdots & -w_v & \cdots \\ -w & d & \ddots & \vdots & & \vdots & \\ \vdots & \ddots & \ddots & -w & & \vdots & \\ -w & \cdots & -w & d & \cdots & -w_v & \cdots \\ \hline \vdots & & & \vdots & & & \\ -w_v & \cdots & \cdots & -w_v & & & \\ \vdots & & & \vdots & & & \end{array} \right],$$

where  $d$  is the (weighted) degree of a channel in  $S$  (which is constant), and  $v$  is any vertex outside  $S$ .

It is not difficult to prove that any vector of the form  $x = [x_1, x_2, \dots, x_m, 0, \dots, 0]^T$ , where  $\sum_i x_i = 0$  (i.e.,  $x$  is orthogonal to  $\mathbf{1}$ ), is an eigenvector of  $L$  with eigenvalue  $d + w$ . There are  $m - 1$  linearly independent vectors of this form. This proves the “only if” part of the theorem.

Next suppose that there is an eigenvalue  $\lambda$  of multiplicity  $m - 1$ . Without loss of generality the corresponding  $m - 1$  eigenvectors described in the statement of the theorem can be presented in the form

$$\begin{bmatrix} X_1^{(i)} \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \dots, m - 1.$$

where  $X_1^{(i)}$  is a vector of size  $m$  and  $\mathbf{0}$  is a zero vector of size  $N - m$ . With this assumption we write  $L$  as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

where  $L_{11}$  and  $L_{22}$  are  $m \times m$  and  $(N - m) \times (N - m)$  matrices, respectively.

Writing the eigenvalue equations for  $i = 1, \dots, m - 1$  we have

$$L_{11} X_1^{(i)} + L_{12} \mathbf{0} = \lambda X_1^{(i)},$$

$$L_{21} X_1^{(i)} + L_{22} \mathbf{0} = \mathbf{0}.$$

Similarly to the observation in Theorem 3.3.3 we consider the second equation and

obtain that each row of the matrix  $L_{21}$  is a multiple of  $\mathbf{1}$  which proves the first part of the theorem. On the other hand, this implies that the matrix  $L_{12}$  has constant row sums, say,  $-s$ , where  $s \geq 0$ .

Therefore, the first equation  $L_{11} X_1^{(i)} = \lambda X_1^{(i)}$  can be seen as

$$(L(S) + sI) X_1^{(i)} = \lambda X_1^{(i)},$$

since  $L_{11}$  can be represented as a sum of  $L(S)$ , the Laplacian matrix that corresponds to the subgraph  $S$ , and a diagonal matrix  $sI_{m \times m}$ . The equation considered is equivalent to

$$L(S) X_1^{(i)} = (\lambda - s) X_1^{(i)}.$$

In words, we have a subgraph  $S$  of order  $m$  with a Laplacian  $\lambda - s$  of multiplicity  $m - 1$ , and  $m - 1$  corresponding eigenvectors  $X_1^{(i)}$  orthogonal to  $\mathbf{1}$ . This is possible if and only if  $S$  is a complete weighted graph  $K_m^{((\lambda - s)/m)}$  (see special case of Theorem 3.2.7 where all parts are of size 1). This completes the proof of the theorem. ■

The corollary that follows expresses Theorem 3.3.16 in FAP terminology.

**Corollary 3.3.17.** *Let  $L$  be the Laplacian matrix of a FAP with  $N$  channels. Then the following two statements are equivalent:*

1. *The system contains a collection  $S$  of  $m > 1$  channels with the following properties:*
  - *the  $m$  channels with high mutual influence  $w$*
  - *the influence between any of the channels in  $S$  and a channel outside  $S$  does not depend on the channel in  $S$ ;*
2. *There is a Laplacian eigenvalue of multiplicity  $m - 1$  such that the  $m - 1$  eigenvectors have the following form:*
  - *they are orthogonal to  $\mathbf{1}$ ;*
  - *there exist  $N - m$  coordinates that are equal to 0 in each of the  $m - 1$  eigenvectors.*

*Moreover, the  $m$  coordinates that are not equal to 0 in each of  $m - 1$  eigenvectors index the channels that belong to the set  $S$ .*

An equally interesting topic regarding the Laplacian eigenvectors relates to the sign of its entries. Good research examples in this field are work inspired by Courant's nodal theorem [27], and various graph partitioning heuristics algorithms. Our intention was to go in that direction and to try to connect the supports of the Laplacian eigenvectors to the structure of  $k$ -partite edge-weighted graphs. However our contribution is only related to the eigenvectors of edge-weighted bipartite graphs.

**Definition 3.3.18.** *The support of a vector  $x = [x_1, \dots, x_n]^T$  is the set of all indices  $i$  with  $x_i \neq 0$ . It is denoted by  $\text{supp}(x)$ . The positive support,  $\text{supp}_+(x)$ , is the set of all indices  $i$  with  $x_i > 0$ . The negative support,  $\text{supp}_-(x)$ , is the set of all  $i$  with  $x_i < 0$ .*

The following lemma is well-known for simple graphs (see, e.g. [22]). It is also valid for edge-weighted graphs.

**Lemma 3.3.19.** *Let  $L$  denote the Laplacian matrix of a weighted graph  $G$  and let  $x$  be a Laplacian eigenvector. Then any vertex not in  $\text{supp}(x)$  either has no neighbours in  $\text{supp}(x)$  or has neighbours in both  $\text{supp}_+(x)$  and  $\text{supp}_-(x)$ .*

**Proof.** Suppose that  $i \notin \text{supp}(x)$ , so  $x_i = 0$ . Then

$$0 = (Lx)_i = L_{ii}x_i + \sum_{i \sim j} L_{ij}x_j = \sum_{i \sim j} L_{ij}x_j.$$

Since  $L_{ij} < 0$  when  $i$  is adjacent to  $j$ , either  $x_j = 0$  for all vertices  $j$  adjacent to  $i$ , or the sum has both positive and negative terms. In the former case  $j$  is not adjacent to any vertex in  $\text{supp}(x)$ ; in the latter it is adjacent to vertices in both  $\text{supp}_+(x)$  and  $\text{supp}_-(x)$ . ■

We prove that the maximal Laplacian eigenvalue of multiplicity one reveals whether a given graph  $G$  is bipartite.

**Theorem 3.3.20.** *Let  $G$  be a connected edge-weighted bipartite graph and let by  $x$  denote the eigenvector that is associated with the largest Laplacian eigenvalue  $\lambda_1$ . Then the following holds:*

- the support of the eigenvector  $x$  contains all vertices of  $G$ ;

- the positive and negative support of  $x$  contain the vertices that belong to the first and the second part of  $G$  respectively.

**Proof.** Recall that the eigenvalue  $\lambda_1$  can be characterised as

$$\lambda_1(L) = \max\{ \langle Lx, x \rangle \mid x \in \mathbb{R}^N, \|x\| = 1 \},$$

which is for the Laplacian  $L$  equivalent to the following quadratic form

$$\lambda_1 = \max_{\|x\|=1} \sum_{j \sim i} w_{ij} (x_i - x_j)^2,$$

where  $x_i$  is the  $i$ -th entry of the vector  $x$ . The maximum is reached at the Laplacian eigenvector  $x$  that corresponds to  $\lambda_1$ .

We consider a number of situations which would contradict the theorem, and show why none of them holds.

Let  $G_1$  and  $G_2$  denote the parts of the bipartition of  $G$ .

In the first instance let us assume the entries of eigenvector  $x$  corresponding to the vertices of part  $G_2$  are equal to 0. Then for a vertex  $i$  in  $G_1$  we construct the sum

$$s_i = \sum_{j \in G_2} (x_i - 0)^2 w_{ij} = x_i^2 d_i,$$

where  $d_i$  is the generalised degree of vertex  $i$ .

Hence

$$\lambda_1 = \max_{\|x\|=1} \sum_{i \in G_1} s_i = \max_{\|x\|=1} \sum_{i \in G_1} x_i^2 d_i \leq \max_{\|x\|=1} \sum_{i \in G_1} x_i^2 \Delta(G) \leq \Delta(G) \sum_{i \in G_1} x_i^2 = \Delta(G),$$

since  $\sum_{i \in G_1} x_i^2 = 1$ . This is a contradiction to the Fiedler's inequality for  $\lambda_1$  in Theorem 3.2.15.

As  $\lambda_1$  is the maximum eigenvalue, the corresponding eigenvector is also orthogonal to  $\mathbf{1}$ , and hence contains both positive and negative entries.

Next, without loss of generality we can assume that there are positive entries in both of the parts.

We observe the value of the quadratic form

$$\sum_{j \sim i} w_{ij} (y_i - y_j)^2,$$

where

$$y_i = \begin{cases} |x_i|, & \text{if } i \in G_1, \\ -|x_i|, & \text{if } i \in G_2. \end{cases}$$

Note that for all values  $x_i$  and  $x_j$  of the same sign, the strict inequality holds

$$|x_i - x_j| < ||x_i| - (|x_j|)|,$$

which is equivalent to

$$|x_i - x_j| < |y_i - y_j|,$$

where  $i \neq j$ .

Hence  $\sum_{j \sim i} w_{ij} (y_i - y_j)^2 > \sum_{j \sim i} w_{ij} (x_i - x_j)^2$ . However  $\|y\| = 1$  and so we have contradicted the fact that  $x$  maximizes the quadratic form.

Finally, we solve the first part of the theorem using Lemma 3.3.19. Assume that  $x$  does contain some entries with value 0. Each of the vertices that correspond to those entries are, according to the lemma, either connected to both  $\text{supp}_+(x)$  and  $\text{supp}_-(x)$  or not connected to any of the two. The former would mean that such a vertex is connected to both first and second part of the bipartite graph  $G$ , which is impossible. This, together with the latter would mean that all such vertices are connected among themselves, but disconnected from the rest of the graph, which contradicts our assumption that  $G$  is connected. ■

### 3.4 The Chromatic Number of a Weighted Graph

**Definition 3.4.1.** *The chromatic number  $\chi(G)$  of a graph  $G$  (weighted or not) is the smallest number of labels needed for a labelling in which adjacent vertices must receive different labels.*

*An alternative definition is to say that the chromatic number of a weighted graph*

with all weights positive is the smallest number of labels needed in a feasible assignment.

Note that the chromatic number carries less information for frequency assignment than the span does, but it is certainly a useful parameter providing ideas about the order of the span.

The following theorem gives bounds on the chromatic number of weighted graphs.

**Theorem 3.4.2.** *Let  $G$  be a connected weighted graph on  $N$  vertices, with at least one pair of non-adjacent vertices. Let  $\lambda_1, \dots, \lambda_N$  be the Laplacian eigenvalues of  $G$  in decreasing order and denote by  $\Delta$  the maximal vertex degree of  $G$ . Then we have:*

- If  $\lambda_{N-1} < \Delta$ , then the chromatic number  $\chi(G)$  satisfies

$$\chi(G) \geq \frac{\lambda_1 - 2\lambda_{N-1}}{\Delta - \lambda_{N-1}}.$$

- If  $\lambda_{N-1} = \lambda_{N-2} = \dots = \lambda_{N-k+1} = \Delta$  and  $\lambda_{N-k} > \Delta$ , then

$$\chi(G) \leq \frac{(k+1)\lambda_{N-k} - \lambda_1 - (k-1)\Delta}{\lambda_{N-k} - \Delta}.$$

Recall from Corollary 3.2.9 that for a weighted graph with at least one pair of non-adjacent vertices, we have that the second smallest Laplacian eigenvalue  $\lambda_{N-1}$  is smaller than or equal to the maximum vertex degree. Note that the condition that  $G$  must have at least one non-adjacent pair of vertices is not a strong restriction. Any graph that fails that restriction has all vertices adjacent, meaning that the chromatic number is equal to the number of vertices.

The following result from matrix analysis is used in the proof of Theorem 3.4.2. Its proof can be found in [26].

**Lemma 3.4.3.** *Let  $A$  be a real symmetric matrix of order  $N$ , and let  $S_1, \dots, S_t$ ,  $t \geq 2$ , be a partition of  $\{1, \dots, N\}$  into nonempty subsets. Denote by  $A_{kk}$  the submatrix of  $A$  with row and column indices from  $S_k$ . If  $0 \leq i_k < |S_k|$ ,  $k = 1, \dots, t$ , then*

$$\lambda_{i_1+i_2+\dots+i_t+1}(A) + \sum_{i=1}^{t-1} \lambda_{N-i+1}(A) \leq \sum_{k=1}^t \lambda_{i_k+1}(A_{kk}),$$

where  $\lambda_i(X)$ ,  $i = 1, 2, \dots$  are the eigenvalues of the matrix  $X$  in decreasing order.

**Proof of Theorem 3.4.2.** Set  $t = \chi(G)$ . There exists a partition  $S_1, \dots, S_t$  of the vertices of  $G$  such that each of the subgraphs of  $G$  induced by  $S_i$  contains no edges. With  $i_1 = i_2 = \dots = i_t = 0$ , Lemma 3.4.3 gives

$$\lambda_1 + \sum_{i=1}^{t-1} \lambda_{N-i+1} \leq \sum_{k=1}^t \lambda_1(L_{kk}). \quad (3.15)$$

Furthermore, since  $\lambda_N = 0$ , we get

$$\sum_{i=1}^{t-1} \lambda_{N-i+1} = 0 + \lambda_{N-1} + \sum_{i=3}^{t-1} \lambda_{N-i+1} \geq (t-2) \lambda_{N-1}. \quad (3.16)$$

Since the sum of the eigenvalues of a symmetric matrix is equal to the sum of its diagonal elements, and the diagonal elements of  $L$  are the degrees of the vertices, for each  $k$  we get that

$$\sum_{i=1}^{|S_k|} \lambda_i(L_{kk}) = \sum_{v \in S_k} d(v) \leq |S_k| d_{max}.$$

Since the  $L_{kk}$  are diagonal matrices, their eigenvalues are just the elements on the diagonal. So for all  $k$  we have  $\lambda_1(L_{kk}) = \max_{v \in S_k} d(v) \leq d_{max}$ , and therefore:

$$\sum_{k=1}^t \lambda_1(L_{kk}) \leq t d_{max}. \quad (3.17)$$

The last two inequalities (3.16) and (3.17) together with (3.15) give

$$\lambda_1 + (t-2) \lambda_{N-1} \leq t d_{max}. \quad (3.18)$$

If  $\lambda_{N-1} < d_{max}$ , then after simplifying expression (3.18) we get

$$\chi(G) = t \geq \frac{\lambda_1 - 2 \lambda_{N-1}}{d_{max} - \lambda_{N-1}}.$$

In the second case if  $\lambda_{N-1} = d_{max}$ , then there is a  $k$  such that  $\lambda_{N-2} = \dots = \lambda_{N-k+1} = d_{max}$  and  $\lambda_{N-k} > d_{max}$ .

Then we can write

$$\begin{aligned} \sum_{i=1}^{t-1} \lambda_{N-i+1} &= \mathbf{0} + \lambda_{N-1} + \cdots + \lambda_{N-k+1} + \sum_{i=k+1}^{t-1} \lambda_{N-i+1} \\ &\geq (k-1) d_{max} + (t-k-1) \lambda_{N-k}. \end{aligned} \quad (3.19)$$

From inequalities (3.15), (3.17) and (3.19) we obtain

$$\lambda_1 + (k-1) d_{max} + (t-k-1) \lambda_{N-k} \leq t d_{max},$$

which is equivalent to

$$\chi(G) = t \leq \frac{(k+1) \lambda_{N-k} - \lambda_1 - (k-1) d_{max}}{\lambda_{N-k} - d_{max}},$$

as required. ■

The following theorem provides another way of finding an upper bound for the chromatic number.

**Theorem 3.4.4.** *Let  $G$  be a connected weighted graph on  $N$  vertices. Let  $\lambda_1, \dots, \lambda_N$  be the Laplacian eigenvalues of  $G$  in decreasing order and denote by  $d_{total}$  the sum of the degrees of all vertices of  $G$ . Then we have:*

$$\chi(G) \leq 2 + \frac{d_{total} - \lambda_1}{\lambda_{N-1}}.$$

**Proof.** We follow the proof of Theorem 3.4.2 up to equation (3.16). Then we continue with the approximation  $\sum_{k=1}^t \lambda_1(L_{kk}) \leq d_{total}$ , and we get  $t \lambda_{N-1} \leq d_{total} - \lambda_1 + 2 \lambda_{N-1}$ . The result follows, since  $\lambda_{N-1}$  is always a positive number for connected graphs. ■

### 3.4.1 The span of weighted graphs with specific structure

In the introduction we mentioned the search for useful structures, where by useful we mean structural information that can assist us in determining the span of a FAP. Many existing algorithms use properties such as the existence of cliques and ‘almost cliques’ to assist them in the search for good solutions. But finding these

structural properties may be equally hard as solving the FAP. Algebraic methods have at least the useful property that they are easy to compute.

We can in fact provide an example of a structural property that can be determined by algebraic means and provides information about the span of a weighted graph. The following is a consequence of the Theorem 3.2.7.

**Theorem 3.4.5.** *Let  $G$  be a weighted graph on  $N$  vertices and maximal edge weight  $C$ . If the graph has a Laplacian eigenvalue  $CN$  with multiplicity  $k - 1$ , then  $G$  contains a complete  $k^{(C)}$ -partite graph  $K_{N_1, \dots, N_k}^{(C)}$  as a spanning subgraph.*

For  $i = 1, \dots, k$ , let  $S_i$  denote the weighted subgraph of  $G$  spanned by the vertices belonging to part  $i$  of  $K_{N_1, \dots, N_k}^{(C)}$  and let  $s_i$  denote the span of  $S_i$ . Then the span  $sp(G)$  of the whole graph  $G$  is equal to  $\sum_{i=1}^k s_i + (k - 1)C$ .

**Proof.** We first show that we can find a feasible labelling for  $G$  using labels from 0 to  $\sum_{i=1}^k s_i + (k - 1)C$ . Namely, we find an optimal assignment, using labels in  $[0, s_i]$  for each of the subgraphs  $S_i$ , independently. Then we combine these “independent” assignments in the following way.

Keep the labels of  $S_1$  unchanged. Increase each label of  $S_2$  by  $s_1 + C$ ; increase each label of  $S_3$  by  $s_1 + s_2 + 2C$ , etc. In other words, any label assigned to a vertex in  $S_j$ , for some  $j$ , will be increased by  $\sum_{i=1}^{j-1} s_i + (j - 1)C$ . It is easy to see that the resulting labelling is feasible and that the maximum label used is  $\sum_{i=1}^k s_i + (k - 1)C$ , which implies

$$sp(G) \leq \sum_{i=1}^k s_i + (k - 1)C. \quad (3.20)$$

We still have to prove that it is not possible to do any better, so that the span is at least the sum in the right hand side of (3.20). We prove this by induction on  $k$ .

If there is only one part ( $k = 1$ ), then there is nothing to prove.

Suppose the theorem is true for  $k - 1$ , ( $k \geq 2$ ), i.e., if a weighted graph with maximum weight  $C$  contains a complete  $(k - 1)^{(C)}$ -partite graph  $K_{N_1, \dots, N_{k-1}}^{(C)}$  as a spanning subgraph, then the span is at least  $\sum_{i=1}^{k-1} s_i + (k - 2)C$ . We prove that under this assumption the statement is true for  $k$ .

Consider a feasible labelling of the weighted graph  $G$  with maximum weight  $C$  for which the maximum label is as small as possible. We refer to this particular labelling as  $\mathcal{L}$ .

Now consider the subgraph  $G'$  spanned by the vertices of the parts  $S_1, S_2, \dots, S_{k-1}$ . Suppose that the labelling  $\mathcal{L}$  labels the vertices of  $G'$  using labels from  $[p, p']$ , for some  $p, p' \geq 0$ , where at least one vertex has label  $p$  and another one has label  $p'$ . Since  $G'$  contains  $K_{N_1, \dots, N_{k-1}}^{(C)}$  as a spanning subgraph, by induction this means that the span of  $G'$  is at least  $\sum_{i=1}^{k-1} s_i + (k-2)C$ .

So we certainly must have  $p' - p \geq \sum_{i=1}^{k-1} s_i + (k-2)C$ .

Next we look at the labels that  $\mathcal{L}$  assigns to the vertices of  $S_k$ . Suppose these labels are in the range  $[q, q']$ , for some  $q, q' \geq 0$ , where again the labels  $q$  and  $q'$  actually occur. If  $p' \leq q$ , then we must in fact have  $p' + C \leq q$ . Since the span of the graph spanned by  $S_k$  is  $s_k$ , we get that  $q' - q \geq s_k$ . For the largest label in  $\mathcal{L}$  (label  $q'$  in this case) this implies the following:

$$q' \geq q + s_k \geq p' + C + s_k \geq \sum_{i=1}^{k-1} s_i + (k-2)C + p + C + s_k = \sum_{i=1}^k s_i + (k-1)C + p,$$

as required. If  $q' \leq p$ , then we use a similar argument to complete the proof.

We are left to consider the cases  $p < q < p'$  or  $q < p < q'$ . Since they are similar, we only consider the first one. This means that the labelling  $\mathcal{L}$  uses some labels  $t$  and  $t'$  for  $G'$  such that in the interval  $(t, t')$  no label is used for  $G'$ , but there are vertices in  $S_k$  that get labels from  $(t, t')$ . Let  $a$  be the smallest and  $a'$  the largest label in  $(t, t')$  assigned to some vertex in  $S_k$ . Then we in fact have  $t + C \leq a \leq a' \leq t' - C$ . Suppose that the largest label in  $\mathcal{L}$  is  $m$ .

Now we transform the labelling  $\mathcal{L}$  as follows:

- All labels in the range  $[0, t]$  remain the same.
- Every vertex which has a label  $b$  in the range  $(t, t')$  (all these vertices are in  $S_k$ ) is relabelled to  $b + (m + C - t')$ .
- Vertices that have a label  $b'$  larger than or equal to  $t'$  are relabelled with  $b' - (t' - a)$ .

It is straightforward to check that this transformation gives a new feasible labelling  $\bar{\mathcal{L}}$  of  $G$ . Moreover, the largest label  $\bar{m}$  of  $\bar{\mathcal{L}}$  is  $\bar{m} = a' + (m + C - t') \leq m$ , since  $a' \leq t' - C$ . Since  $\mathcal{L}$  is a feasible labelling with minimal largest label, we in fact must have  $\bar{m} = m$ .

The largest label that a vertex in  $G'$  receives in  $\bar{\mathcal{L}}$  is  $\bar{p}' = p' - (t' - a) \leq p' - C$ . The smallest label that a vertex in  $S_k$  receives in  $\bar{\mathcal{L}}$  remains either  $q$  (if  $q < t$ ) or is increased to  $\bar{q} = q + (m + C - t') \geq q + C$  ( $q \geq t$ ). We conclude that the described transformation gives a new feasible labelling  $\bar{\mathcal{L}}$  with the same smallest and largest label in which the number of labels of  $S_k$  occurring between labels of  $G'$  is smaller than in  $\mathcal{L}$ .

Repeating the transformation, if necessary, gives as an end result a feasible labelling of the type  $p' \leq q$  with the same smallest and largest label. We have shown earlier in the proof that the existence of such a labelling proves the theorem. ■

### 3.5 Distribution of the Eigenvalues of FAPs

In this section we determine eigenvalues and eigenvectors of the Laplacian matrix for the following FAPs:

- A system of one transmitter with multiple channels.
- FAPs formed by regularly placed transmitters on a square and triangular grid, with influence based on a collection of constraints of the same type for each transmitter.
- Radio networks described in terms of disjoint subsets of transmitters; in particular graphs with a  $k$ -part underlying structure.

#### 3.5.1 FAP with one multiple-channel transmitter

When defining the FAPs in our thesis, we asserted that a multiple-channel transmitter with  $N$  channels would be considered to be a set of several  $N$  transmitters whose influence matrix is an  $N \times N$  matrix with zeros on the diagonal and a constant  $C$  on all other positions, where  $C$  is the influence between any two channels

(see page 23). The spectrum for the Laplacian matrix  $L$  of such system has an eigenvalue 0 with multiplicity 1, and an eigenvalue  $CN$  with multiplicity  $N - 1$ . In particular, apart from one eigenvalue 0, all other eigenvalues are concentrated in one large number.

### 3.5.2 The Laplacian eigenvalues of infinite grids

The approach in this section is taken from [10]. In order to discuss the Laplacian eigenvalues of FAPs that are defined on infinite grids, we represent them as weighted graphs defined on two-dimensional integer lattices. The vertices of the graphs are the points in  $\mathbb{Z}^2$ , i.e., points of the form  $v_x = (x_1, x_2)$ , where  $x_1$  and  $x_2$  are integers. The edges are determined by an ‘edge-defining multi-set’  $E'$  such that any two vertices  $v_x$  and  $v_y$  are joined if and only if the ordered pair  $v_y - v_x = (y_1 - x_1, y_2 - x_2)$  belongs to  $E'$ .

Since we are now working with infinite graphs, we extend our algebraic methods. Here we give an informal description. For an infinite graph with finite degrees it is possible to define the *Laplacian operator*  $L$ . If  $f$  is a function defined on the set of vertices  $V$ , then the function  $Lf$  is defined by

$$(Lf)(v) = \sum_{uv \in E} [f(v) - f(u)],$$

where  $E$  denotes the set of edges of the graph. Generalising the finite case, we call a number  $\lambda$  an eigenvalue of  $L$  if there exists a non-zero function  $f$  such that  $Lf = \lambda f$ , i.e., so that for all  $v \in V$  we have

$$\lambda f(v) = \sum_{uv \in E} [f(v) - f(u)].$$

For the grids defined as above, this becomes equivalent to

$$\lambda f(x_1, x_2) = \sum_{\substack{(y_1, y_2) \in \mathbb{Z}^2 \\ (y_1 - x_1, y_2 - x_2) \in E'}} [f(x_1, x_2) - f(y_1, y_2)], \quad \text{for all } (x_1, x_2) \in \mathbb{Z}^2.$$

In order to overcome the problem that a suitable function  $f$  must satisfy an infinite number of conditions, we use the translational symmetry of lattice graphs. Specifically we define a function  $f$  by assigning a value  $f_0$  to the vertex  $(0, 0)$ , and

write it as  $f(0, 0) = f_0$ .

Then we assign values to the remaining vertices  $(x_1, x_2)$  as follows:

$$f(x_1, x_2) = h_1^{x_1} h_2^{x_2} f_0, \quad \text{for all } (x_1, x_2) \in \mathbb{Z}^2,$$

for certain  $h_1, h_2$ . It can be shown that this construction gives enough eigenvalues and eigenvectors, enabling us to give detailed information on the algebraic properties of infinite grid FAPs.

### The infinite square grid

As stated before, we assume the vertices of the two-dimensional square grid to be the elements of  $\mathbb{Z}^2$ . The edge-defining multi-set  $E'$  is

$$E' = \{(0, 1), (0, -1), (1, 0), (-1, 0)\},$$

and we also assume that all edges have weight one, i.e., there is a constraint one between any two “adjacent” transmitters.

As described above, in order to find the eigenvectors we consider function  $f$  on the vertex set with  $f((0, 0)) = f_0$ . According to the information given on the edge-defining multi-set  $E'$  and constraints, it follows that the neighbouring vertices to  $(0, 0)$  are  $(0, 1)$ ,  $(-1, 0)$ ,  $(1, 0)$  and  $(0, -1)$  and therefore the  $\lambda f = L f$  condition at  $(0, 0)$  is

$$\lambda f_0 = [f(0, 0) - f(1, 0)] + [f(0, 0) - f(-1, 0)] + [f(0, 0) - f(0, 1)] + [f(0, 0) - f(0, -1)].$$

Substituting  $f(x_1, x_2) = h_1^{x_1} h_2^{x_2} f_0$  for these points gives the equation

$$\lambda f_0 = ([1 - h_1] + [1 - h_1^{-1}] + [1 - h_2] + [1 - h_2^{-1}]) f_0,$$

which can be simplified to

$$\lambda = 4 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2}.$$

In particular every choice for  $h_1, h_2$  different from 0 gives an eigenvalue  $\lambda$  and corresponding eigenfunction  $f$ .

### The infinite triangular grid

Again, in this case the set of vertices is the set of all two-dimensional integer pairs. This time the edge-defining multi-set is

$$E' = \{(0, 1), (0, -1), (1, 0), (-1, 0), (1, 1), (-1, -1)\}.$$

All edges are assumed to have weight one.

Following calculations similar to the square grid, we get the eigenvalues

$$\lambda = 6 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2} - h_1 h_2 - \frac{1}{h_1 h_2},$$

for each non-zero choice of  $h_1$  and  $h_2$ .

### 3.5.3 The Laplacian eigenvalues of finite parts of grids

The calculations in the previous subsection provide an overview of eigenvalues of regular infinite grids. In this subsection we look only at a finite part of the grids under consideration.

Our first approach, which allows analytical analysis, is to consider the finite part as being a finite grid embedded on a torus. In this way we avoid boundary problems. In other words, this gives us a finite problem with ‘infinite-like’ properties.

Following the arguments above, we see the grids as lying on a torus curved in two directions. We assume that after going for  $m$  steps in the first direction we return to the original position. Similarly, we reach the original position after going  $n$  steps in the second direction.

To compute the eigenvalues in this case we make an appropriate choice of the multipliers  $h_1$  and  $h_2$ . In each coordinate direction we must make sure that the graph does not repeat indefinitely, but “folds” to itself after  $m$  repetitions with respect to the first coordinate and  $n$  repetitions with respect to the second coordinate. Hence we can no longer use any values for  $h_1$  and  $h_2$ , but we must have  $h_1^m = h_1^0 = 1$  and  $h_2^n = h_2^0 = 1$ .

These equations have the solutions

$$\begin{aligned} h_1 &= \exp\left(2\pi i \frac{r}{m}\right), & \text{for some } r, 0 \leq r \leq m-1; \\ h_2 &= \exp\left(2\pi i \frac{s}{n}\right), & \text{for some } s, 0 \leq s \leq n-1, \end{aligned}$$

for each possible value of  $h_1$  and  $h_2$ . Hence, for each possible value of  $r$  and  $s$ , we get a Laplacian eigenvalue according to the formulae for the infinite grids.

- For the square grid we have  $\lambda = 4 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2}$ , which after simplifying, gives the following eigenvalues

$$\lambda_{rs} = 4 - 2 \cos\left(2\pi \frac{r}{m}\right) - 2 \cos\left(2\pi \frac{s}{n}\right), \quad r = 0, 1, \dots, m-1, s = 0, 1, \dots, n-1.$$

A graphical representation of these eigenvalues, for  $m = n = 30$ , is provided below.

- For the triangular grid we derive  $\lambda = 6 - h_1 - \frac{1}{h_1} - h_2 - \frac{1}{h_2} - h_1 h_2 - \frac{1}{h_1 h_2}$ , which results in the following  $m \times n$  eigenvalues

$$\begin{aligned} \lambda_{rs} = 6 - 2 \cos\left(2\pi \frac{r}{m}\right) - 2 \cos\left(2\pi \frac{s}{n}\right) - 2 \cos\left(2\pi \left(\frac{r}{m} + \frac{s}{n}\right)\right), & r = 0, 1, \dots, m-1, \\ & s = 0, 1, \dots, n-1. \end{aligned}$$

Although we can give the eigenvalues of grids embedded on a torus explicitly, it is not clear how close they are to the eigenvalues of ‘real’ finite parts, lying in a plane. It would appear hard to give exact analytical results for eigenvalues in those cases. Numerical analysis however can provide a result here. Using the computer program Maple, we calculate the eigenvalues of the Laplacian associated to finite parts of the infinite square and triangular grid with 900 points.

For the square grid we took a  $30 \times 30$  square part of the infinite square grid embedded in the plane. The distribution of the eigenvalues of this system is given in Figure 3.1, on the right plot. For comparison, the distribution for the  $30 \times 30$  square grid on the torus is shown in the same picture on the left. Note that the symmetry from the part on the torus is broken on the right side because of boundary effects.

A similar calculation was done for a part of the triangular lattice with 900 points embedded in the plane. Since this lattice is not of the same form, in  $x$  and  $y$

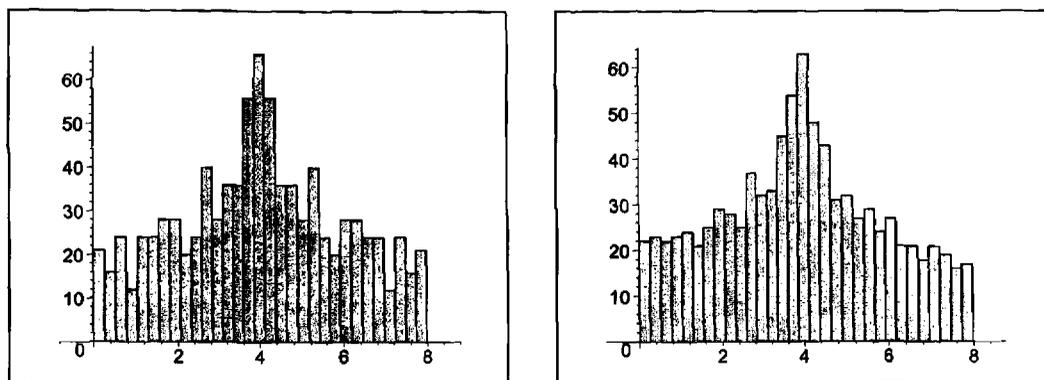
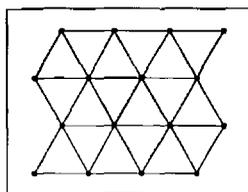


Figure 3.1: Distribution of the Laplacian eigenvalues for  $30 \times 30$  finite parts of the square lattice. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

directions, we take the points lying in a rectangular box similar to the shape of the  $4 \times 4$  part below.



The distribution of the eigenvalues for both the triangular grid with 900 points embedded on the torus and in the plane are given in Figure 3.2.

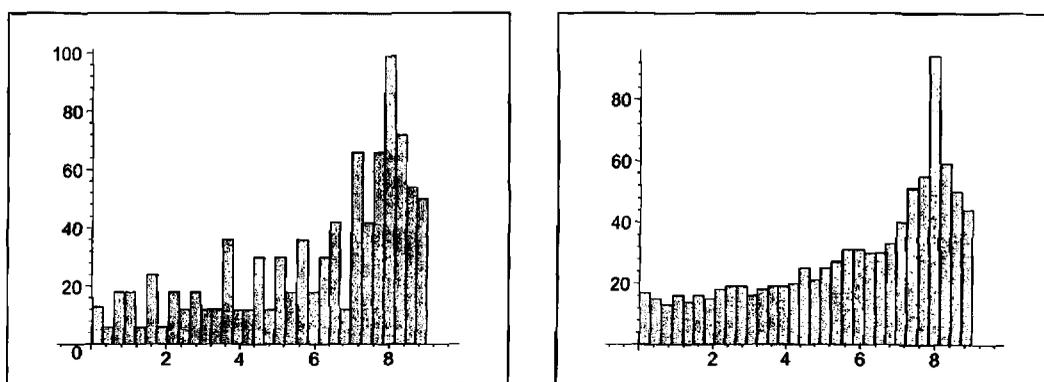


Figure 3.2: Distribution of the Laplacian eigenvalues for  $30 \times 30$  finite parts of the triangular lattice. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

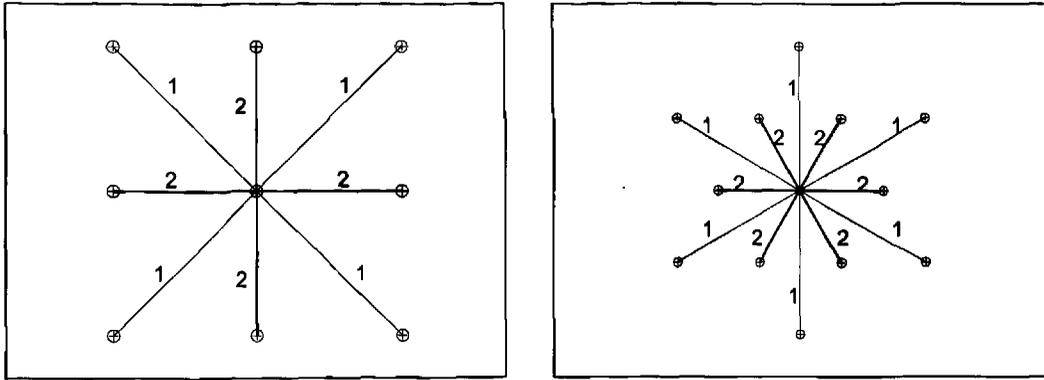


Figure 3.3: More complicated constraint patterns for grids. The plots indicate the constraints experienced by one point related to the position of its neighbouring points, for the square grid (left) and the triangular grid (right).

### 3.5.4 Square and triangular grids with different constraints

The calculations above (both analytical and numerical) can also be done for more involved situations. As an example of this we consider both the square and triangular grid with constraints given in Figure 3.3.

For the purposes of analytical analysis, we use the same technique as before. In order to take into account edges with weight of more than one, we increase the multiplicity in the edge-defining multi-set. So for the square grid we again use the set of all two-dimensional integer pairs as the set of vertices. For the edge-defining multi-set giving the constraints as illustrated in the left of Figure 3.3 we use

$$E' = \{(1,0), (1,0), (0,1), (0,1), (-1,0), (-1,0), (0,-1), (0,-1), (1,1), (-1,1), (-1,-1), (1,-1)\}.$$

When we consider the grid embedded on the torus, using a similar procedure as before gives the form of the eigenvalues:

$$\lambda_{rs} = 12 - 4 \cos\left(2\pi \frac{r}{m}\right) - 4 \cos\left(2\pi \frac{s}{n}\right) - 2 \cos\left(2\pi \left(\frac{r}{m} + \frac{s}{n}\right)\right) - 2 \cos\left(2\pi \left(\frac{r}{m} - \frac{s}{n}\right)\right),$$

$$r = 0, 1, \dots, m-1, s = 0, 1, \dots, n-1.$$

The distribution of the eigenvalues for the case  $m = n = 30$ , hence with 900 vertices, is given on the left side of Figure 3.4.

Again, for comparison, we numerically calculated the Laplacian eigenvalues of a

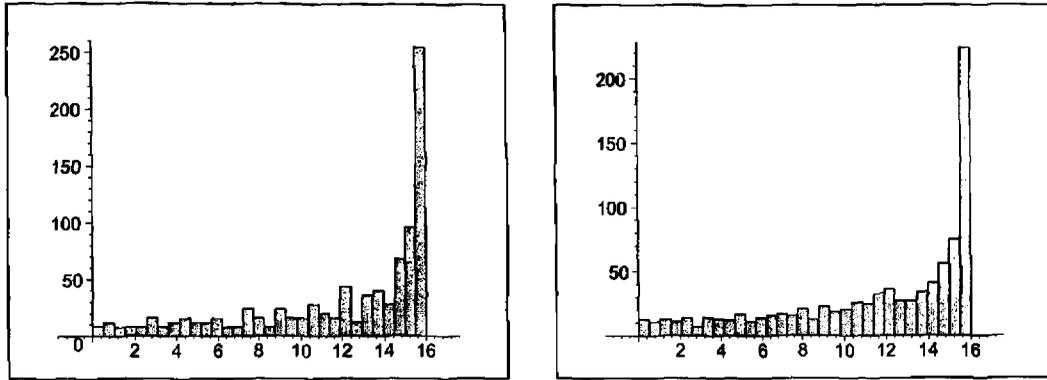


Figure 3.4: Distribution of the Laplacian eigenvalues for  $30 \times 30$  finite parts of the square lattice, with constraints determined by the left picture in Figure 3.3. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

$30 \times 30$  part of the square grid in the plane, with similar constraints. This distribution can be found in Figure 3.4 as well.

Finally, for the triangular lattice with constraints according to Figure 3.3, we can analytically determine the Laplacian eigenvalues for the part embedded on the torus as follows. Take as the edge-defining multi-set

$$E' = \{(1, 1), (1, 1), (0, 1), (0, 1), (-1, 0), (-1, 0), (-1, -1), (-1, -1), (0, -1), (0, -1), (1, 0), (1, 0), (1, 2), (-1, 1), (-2, -1), (-1, -2), (1, -1), (2, 1)\},$$

where multiple entries correspond to edges with weight 2. Note that this means that the picture on the right of Figure 3.3 corresponds to  $E'$  by identifying  $(1, 0)$  with the vector labelled 2 pointing to the right, and  $(0, 1)$  corresponds to the vector labelled 2 pointing upwards and to the left. This gives the following form of the eigenvalues

$$\begin{aligned} \lambda_{rs} = & 18 - 4 \cos\left(2\pi \frac{r}{m}\right) - 4 \cos\left(2\pi \frac{s}{n}\right) - 4 \cos\left(2\pi \left(\frac{r}{m} + \frac{s}{n}\right)\right) - 2 \cos\left(2\pi \left(\frac{r}{m} - \frac{s}{n}\right)\right) \\ & - 2 \cos\left(2\pi \left(\frac{2r}{m} + \frac{s}{n}\right)\right) - 2 \cos\left(2\pi \left(\frac{r}{m} + \frac{2s}{n}\right)\right), \end{aligned} \quad \begin{aligned} r = & 0, 1, \dots, m-1, \\ s = & 0, 1, \dots, n-1. \end{aligned}$$

And, finally, for the same case we calculate the eigenvalues for  $m = n = 30$ . We complete numerical calculations for this part of the triangular grid with the same constraints as for a similar finite part, however now embedded in the plane. A graphical representation of these outcomes is given in Figure 3.5.

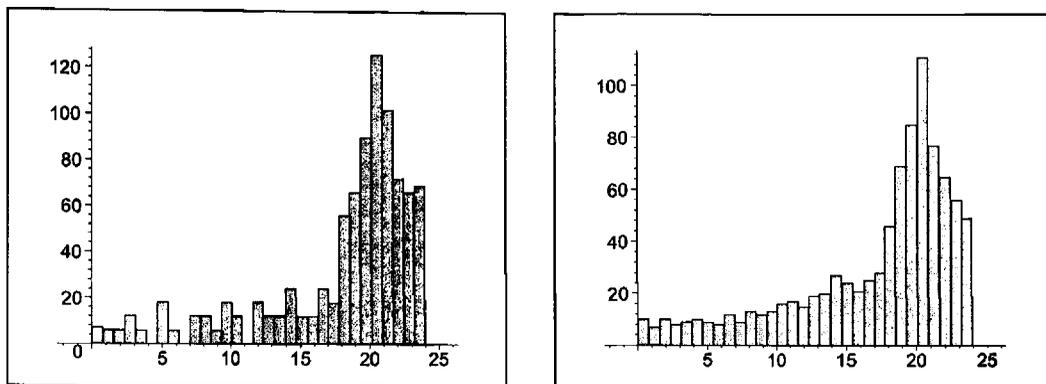


Figure 3.5: Distribution of the Laplacian eigenvalues for  $30 \times 30$  finite parts of the triangular lattice, with constraints determined by the right picture in Figure 3.3. The left picture shows the part embedded on the torus; the right picture is of a similar part in the plane.

### 3.5.5 The Laplacian eigenvalues and specific structures

The goal of this study is to explore the limits of algebraic graph techniques that rely on the Laplacians. In this section we describe a way to analyse and understand the distribution of the Laplacian eigenvalue spectrum of certain graphs, using a known result stated below as the main guidance. For clarity, we present our findings by describing them in steps.

#### Analysis of two-part graphs

Throughout this section we will be referring to a known result for commutative matrices (see, e.g., [28]):

**Theorem 3.5.1.** *Let  $A$  and  $B$  be  $n \times n$  real matrices with eigenvalues  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$ , respectively. If  $A$  and  $B$  commute, there is a permutation  $i_1, \dots, i_n$  of the indices  $1, \dots, n$ , such that the eigenvalues of  $A+B$  are  $\alpha_1 + \beta_{i_1}, \alpha_2 + \beta_{i_2}, \dots, \alpha_n + \beta_{i_n}$ .*

The nature of the theorem and its condition that it is applicable to commutative matrices implies that this approach will only apply to a limited class of graphs. Nevertheless, we find this piece of research valuable as it provides the starting point and the necessary explanation for changes in the spectrum of non-ideal graphs.

We have previously looked into the spectrum of a complete edge-weighted graph  $K_N^{(w)}$ . It consists of one eigenvalue 0 and  $N - 1$  eigenvalues of value  $wN$ . As the next step we consider bipartite graphs.

**Lemma 3.5.2.** *Let  $G$  be a complete bipartite graph  $K_{N_1, N_2}^{(w)}$ , and let  $N = N_1 + N_2$ . Then its Laplacian spectrum consists of:*

- *a single eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;*
- *eigenvalue  $wN_2$  of multiplicity  $N_1 - 1$  with corresponding eigenvectors of type  $\begin{bmatrix} X \\ \mathbf{0} \end{bmatrix}$ , where  $X$  is a non-zero vector of size  $N_1$  orthogonal to  $\mathbf{1}$  whose entries correspond to the vertices of the first part;*
- *eigenvalue  $wN_1$  of multiplicity  $N_2 - 1$  with corresponding eigenvectors of type  $\begin{bmatrix} \mathbf{0} \\ Y \end{bmatrix}$ , where  $Y$  is a non-zero vector of size  $N_2$  orthogonal to  $\mathbf{1}$  whose entries correspond to the vertices of the second part;*
- *a single eigenvalue  $wN$  with eigenvector of type  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ , i.e., whose entries are constant on the entries of the parts of the graph.*

**Proof.** Recall the equitable partition properties from Subsection 3.3.2, page 47. Let  $\pi$  denote the described bipartition of  $G$ . As  $\pi$  is equitable we have that the eigenvalues of its corresponding quotient matrix  $B$ ,

$$B = \begin{bmatrix} wN_2 & -wN_2 \\ -wN_1 & wN_1 \end{bmatrix}$$

are eigenvalues of the Laplacian of  $G$ . The first eigenvalue of  $B$  is 0, which is also an eigenvalue of  $L(G)$  with corresponding eigenvector  $\mathbf{1}$ . On the other hand, its other (largest) eigenvalue  $\lambda_1(B) = w(N_1 + N_2)$  for the corresponding eigenvector has a vector that is a multiple of the vector  $[\mathbf{1}, -\frac{N_1}{N_2}]^T$ . This eigenvalue is also an eigenvalue of  $L(G)$  associated with a vector of type  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ , a vector that is constant on the cells of  $\pi$ , i.e., constant on the entries of the two parts.

We also find that a vector of type  $\begin{bmatrix} X \\ \mathbf{0} \end{bmatrix}$ , where  $X$  is a non-zero vector of size  $N_1$  orthogonal to  $\mathbf{1}$ , is an eigenvector of  $L(G)$  with eigenvalue  $wN_2$ , and that there are exactly  $N_1 - 1$  of them since that is the total number of linearly independent vectors orthogonal to  $\mathbf{1}$ . Similarly,  $N_2 - 1$  vectors of type  $\begin{bmatrix} \mathbf{0} \\ Y \end{bmatrix}$ , where  $Y$  is a non-zero vector of size  $N_2 - 1$  orthogonal to  $\mathbf{1}$ , are eigenvectors of  $L(G)$  corresponding to  $wN_1$ .

■

Furthermore, in general, if a graph  $G$  is such that there exists an equitable partition  $\pi$  of its vertices in two parts of, say, sizes  $N_1$  and  $N_2$ , where the corresponding quotient matrix is of the form

$$B = \begin{bmatrix} cN_2 & -cN_2 \\ -cN_1 & cN_1 \end{bmatrix}.$$

Using the same argument as in the proof of Lemma 3.5.2 we can show that such graph has an eigenvalue of value  $c(N_1 + N_2)$  with corresponding eigenvector of type  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ , a vector that is constant on the entries of the parts.

**Theorem 3.5.3.** *Let  $G$  be an edge-weighted graph on  $N$  vertices that consists of two connected subgraphs  $G_1$  and  $G_2$ , where each vertex of graph  $G_1$  is connected by an edge of weight  $w$  to every vertex in  $G_2$ . Then its Laplacian spectrum consists of: one eigenvalue 0; a number of  $N_1 - 1$  eigenvalues  $\{wN_2 + \lambda_i(G_1)\}_{i=1, \dots, N_1-1}$ ; a number of  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_{i=1, \dots, N_2-1}$ ; one eigenvalue  $wN$ . The corresponding eigenvectors are of the same type as vectors presented in Lemma 3.5.2.*

**Proof.** The Laplacian matrix  $L(G)$  can be written as a sum of three matrices

$$L(G) = \begin{bmatrix} wN_2 I_{N_1 \times N_1} & -wJ_{N_1 \times N_2} \\ -wJ_{N_2 \times N_1} & wN_1 I_{N_2 \times N_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & L(G_2) \end{bmatrix} + \begin{bmatrix} L(G_1) & 0 \\ 0 & 0 \end{bmatrix}$$

As the three matrices in the summation commute, we show using Theorem 3.5.1, how the understanding of the Laplacian spectrums of the three matrices can help us understand the Laplacian eigenvalue spectrum of  $G$ .

The first matrix in the sum is the Laplacian of a complete bipartite graph. The properties of its Laplacian eigenvalues and eigenvectors have been looked at in Lemma 3.5.2. Its spectrum is as follows:

- one eigenvalue 0 and corresponding eigenvector  $\mathbf{1}$ ;
- eigenvalue  $wN_2$  of multiplicity  $N_1 - 1$  and corresponding eigenvectors of type  $\begin{bmatrix} X \\ \mathbf{0} \end{bmatrix}$ , where  $X$  is orthogonal to  $\mathbf{1}$ ;
- eigenvalue  $wN_1$  of multiplicity  $N_2 - 1$  and corresponding eigenvectors of type  $\begin{bmatrix} \mathbf{0} \\ Y \end{bmatrix}$ , where  $Y$  is orthogonal to  $\mathbf{1}$ ;

- one eigenvalue  $w(N_1 + N_2)$  with corresponding eigenvector  $\begin{bmatrix} 1 \\ Z \end{bmatrix}$ , where  $Z$  is a multiple of  $\mathbf{1}$ , more precisely  $Z = -\frac{N_1}{N_2} \mathbf{1}$ .

The second matrix is simply the Laplacian of the induced subgraph that contains the graph  $G_2$  surrounded by  $N_1$  isolated vertices. All the eigenvalues  $\beta_j = \lambda_j(L(G_2))$ ,  $j = 1, \dots, N_2 - 1$ , of  $L(G_2)$  are also the eigenvalues of second matrix, as each eigenvector  $\tilde{Y}$  of  $L(G_2)$  orthogonal to  $\mathbf{1}$  can be extended to the eigenvector  $\begin{bmatrix} 0 \\ \tilde{Y} \end{bmatrix}$  of the second matrix with the same eigenvalue. Each such eigenvector  $\tilde{Y}$  is also eifenvector of the first matrix with eigenvalue  $wN_1$ .

Since the second matrix can be viewed as a Laplacian of a graph containing  $N_1 + 1$  disconnected components, we have that 0 is an eigenvalue of multiplicity  $N_1 + 1$ . It is easy to show that earlier listed vectors:  $\mathbf{1}$ ,  $\begin{bmatrix} X \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$  satisfy the eigenvalue equation and are hence the eigenvectors that correspond to the eigenvalue 0.

The spectrum of the sum of the first two matrices is as follows:

- one eigenvalue 0 and corresponding eigenvector  $\mathbf{1}$ ;
- a number of  $N_1 - 1$  eigenvalues  $wN_2 + 0$  and corresponding eigenvectors of type  $\begin{bmatrix} X \\ 0 \end{bmatrix}$ , where  $X$  is orthogonal to  $\mathbf{1}$ ;
- a number of  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_{i=1, \dots, N_2-1}$ , and corresponding eigenvectors of type  $\begin{bmatrix} 0 \\ \tilde{Y} \end{bmatrix}$ , where  $\tilde{Y}$  is orthogonal to  $\mathbf{1}$ ;
- eigenvalue  $w(N_1 + N_2) + 0$  and corresponding eigenvector  $\begin{bmatrix} 1 \\ Z \end{bmatrix}$ , where  $Z$  is a multiple of  $\mathbf{1}$ , more precisely  $Z = -\frac{N_1}{N_2} \mathbf{1}$ .

The same procedure can be applied further, since the sum of the first two matrices commutes with the third one. Therefore the introduction of the structure of  $G_1$  to the vertices of  $G$  in the same manner as was done with graph  $G_2$ , produces similar changes. We conclude with the spectrum of  $G$ :

- one eigenvalue 0 and corresponding eigenvector  $\mathbf{1}$ ;
- a number of  $N_1 - 1$  eigenvalues  $\{wN_2 + \lambda_i(G_1)\}_{i=1, \dots, N_1-1}$ , and corresponding eigenvectors of type  $\begin{bmatrix} \tilde{X} \\ 0 \end{bmatrix}$ , where  $\tilde{X}$  is orthogonal to  $\mathbf{1}$ ;

- a number of  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_{i=1, \dots, N_2-1}$ , and corresponding eigenvectors of type  $\begin{bmatrix} \mathbf{0} \\ \tilde{Y} \end{bmatrix}$ , where  $\tilde{Y}$  is orthogonal to  $\mathbf{1}$ ;
- one eigenvalue  $w(N_1 + N_2) + 0$  and corresponding eigenvector  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z$  is a multiple of  $\mathbf{1}$ , more precisely  $Z = -\frac{N_1}{N_2} \mathbf{1}$ .

■

In analysing the eigenvalue spectrum of this two-part type of graph, without loss of generality, we assume that the order of the first part is not larger than the order of the second.

In such a case, the order of the eigenvalues in the spectrum of the bipartite graph  $K_{N_1, N_2}^{(w)}$  is as follows:  $0 < wN_1 \leq wN_2 < wN$ . As shown above, the eigenvalues can be split into groups according to the type of an eigenvector they are associated with: the first type is an eigenvalue associated with a vector constant on all vertices, all-1 vector  $\mathbf{1}$ ; the second type of eigenvalues is associated with a collection of vectors with entries 0 on the vertices of the first part  $\begin{bmatrix} \mathbf{0} \\ \tilde{Y} \end{bmatrix}$ , where  $\tilde{Y} \perp \mathbf{1}$  is a non-zero vector of size  $N_2$ ; the third with a collection of vectors with entries 0 on the vertices of the second part  $\begin{bmatrix} \tilde{X} \\ \mathbf{0} \end{bmatrix}$ , where  $\tilde{X} \perp \mathbf{1}$  is a non-zero vector of size  $N_1$ ; finally the fourth type corresponds to a vector that is constant on the vertices belonging to the same part  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ .

Unlike in the case of a complete bipartite graph, observing the sizes of the parts is no longer sufficient in predicting the shape of the histogram of the graphs. Now we need to take into consideration the influence of the subgraphs  $G_1$  and  $G_2$ . Both the number of edges introduced and the changes in the edge weights in the parts  $G_i$  will affect the spectrum of the considered graph  $G$ .

Starting with the structure of a complete bipartite graph, the other graphs are formed by gradually introducing edges to the parts.

1. If  $wN_1 + \lambda_{max}(G_2) < wN_2 + \lambda_{min}(G_1)$  and  $wN_2 + \lambda_{max}(G_1) < w(N_1 + N_2)$  we know that the order of the eigenvalues with respect to their type remains the same as in the previous case: 0, then  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_i$ ,  $N_1 - 1$  eigenvalues  $\{wN_2 + \lambda_i(G_1)\}_i$ , and finally  $wN$ . The associated eigenvectors follow the same order as in the case of the complete bipartite graph: one  $\mathbf{1}$ ,

$N_2 - 1$  eigenvectors of the form  $\begin{bmatrix} \mathbf{0} \\ \bar{Y} \end{bmatrix}$ ,  $N_1 - 1$  eigenvectors of the form  $\begin{bmatrix} \bar{X} \\ \mathbf{0} \end{bmatrix}$ , and one  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ .

A possible interpretation is that this is an example where the mutual influence of vertices within the subgraphs  $G_1$  and  $G_2$  is still weak in comparison to the structure of the induced complete bipartite graph  $K_{N_1, N_2}^{(w)}$ .

2. If  $wN_1 + \lambda_{max}(G_2) < wN_2 + \lambda_{min}(G_1)$  and  $wN_2 + \lambda_{max}(G_1) > w(N_1 + N_2)$ , then the order of the eigenvalues with respect to type of corresponding eigenvectors changes partially in comparison to the previous example. As before we have 0, then  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_i$ , but some or all of the  $N_1 - 1$  eigenvalues  $\{wN_2 + \lambda_i(G_2)\}_i$  are no longer smaller than  $wN$  – so far always the largest eigenvalue. The associated eigenvectors are of the same type as before, with the important change that the form of the largest eigenvalue of the graph is not  $\begin{bmatrix} \mathbf{1} \\ Z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ , but  $\begin{bmatrix} \bar{X} \\ \mathbf{0} \end{bmatrix}$ , as  $wN < wN_2 + \lambda_{max}(G_1)$ .

In other words, under these conditions, one could note that the structure of the subgraph  $G_1$  is starting to dominate the structure of the induced bipartite graph  $K_{N_1, N_2}^{(w)}$ . We give an example of when that happens, i.e., we find a condition under which graph  $G_1$  is such that  $wN_2 + \lambda_{max}(G_1) > wN$  or  $\lambda_{max}(G_1) \geq wN_1$ . From Theorem 3.2.15 we know that  $\lambda_{max}(G_1) \geq \frac{N_1 \Delta(G_1)}{N_1 - 1}$ , where  $\Delta(G_1)$  represents the maximal weighted degree in  $G_1$ . Hence it is sufficient to have  $\frac{N_1 \Delta}{N_1 - 1} > wN_1$ , or equivalently  $\Delta(G_1) > w(N_1 - 1)$ , in order to get  $G_1$  to influence changes in the eigenvalue histogram of  $G$  as described.

A case of particular interest is when  $G_1$  is a complete graph  $K_{N_1}^{(w')}$ . The question arising is for what values of  $w'$  this clique changes the eigenvalue histogram and dominates the observed edge-weighted graph  $G$ . We look back at the values  $wN_2 + \lambda_{max}(G_1)$  and see that they become one eigenvalue  $wN_2 + w'N_1$  of multiplicity  $N_1 - 1$ . They will dominate the spectrum for all values of  $w'$  for which  $wN_2 + w'N_1 > wN$ , or  $w'N_1 > wN_1$  i.e.,  $w' > w$ .

3. Similarly, by changing the structure of  $G_2$ , one can get the eigenvalue spectrum moving in different directions so that it is dominated by the interaction between the vertices in  $G_2$ .
4. In a case where the part  $G_2$  (recall  $N_1 < N_2$ ) is an independent set and  $G_1$  is not, we have that the eigenvalues of the first type  $wN_1$  will never be larger than the eigenvalues of the second type  $\{wN_2 + \lambda_i(G_1)\}_i$ , since that

requirement is equivalent to  $\lambda_{max}(G_1) < (N_1 - N_2)w$ , i.e.,  $\lambda_{max}(G_1) < 0$ , which is not possible. This implies that when  $G_2$  is independent set a part's edge weights play an important role in the ordering of eigenvalues according to the type of corresponding eigenvectors, and that a part that is independent set will always be dominated by the the part that is not an independent set, no matter how much the size of the independent set part is larger than the other. This is not necessarily the case if  $N_2 < N_1$ .

5. If neither of the parts are independent sets, then at least one of the eigenvalues of the first type  $\{wN_1 + \lambda_i(G_2)\}_i$  is larger than all of the eigenvalues of the second type  $\{wN_2 + \lambda_i(G_1)\}_i$  when  $\lambda_{max}(G_2) + (N_1 - N_2)w > \lambda_{max}(G_1)$ , for any  $i$  which implies that sufficiently high edge weights could make either of the parts dominant over the other, with respect to the ordering of the corresponding eigenvalues.
6. The final remark relates to the obstacles we can find when looking at the eigenvectors. In the previous examples we were careful with inequalities, always insisting on distinct eigenvalues. The problems arise when changes in spectrum mean that eigenvalues from different groups have the same value, i.e., some of  $N_1 - 1$  eigenvalues  $\{wN_2 + \lambda_i(G_1)\}_i$ , become equal to some of  $N_2 - 1$  eigenvalues  $\{wN_1 + \lambda_i(G_2)\}_i$  or to  $wN$ . There are no changes in the spectrum analysis presented earlier, but there may be changes in the associated eigenvectors. While we know that the first group of eigenvectors is of the form  $\begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}$ , the second  $\begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix}$  and third (one eigenvector in this case)  $\begin{bmatrix} 1 \\ z \end{bmatrix}$ , where  $Z = -\frac{N_1}{N_2} \mathbf{1}$ , regardless of the multiplicities of eigenvalues within the groups, we cannot be certain in general. For two identical eigenvalues that come from different groups it is not certain that a numerical algorithm will return the associated eigenvectors of the familiar type, as any linear combination of the familiar type eigenvectors is a valid eigenvector that may be of a completely different form. For example, let us consider the case where  $wN_2 + \lambda_i(G_1) = wN_1 + \lambda_{max}(G_2)$  and  $wN_1 + \lambda_{max}(G_2) > w(N_1 + N_2)$ . We see that now there are at least two eigenvalues of maximal value whose associated eigenvalues are at least a linear combination of vectors of the type  $\begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix}$ . A 'wrong' choice of the scalars in the linear combination would give us an eigenvector with entries that cannot directly lead us back to their origin, and therefore the information that we have two dominant subgraphs  $G_1$  and

$G_2$  joined completely with edges of high weight would remain hidden.

### Analysis of $k$ -part graphs

From the results and observations presented above for bipartite structure we see that there is a possibility for extension and generalisation. That is the intention of this subsection: we focus on graphs with the  $k$ -partite regular graph as the basis and use the bipartite case to guide us.

To be in line with the bipartite case we could consider researching complete  $k^{(w)}$ -partite graphs (see Definition 3.2.6). In fact as more general results hold we introduce another weighted  $k$ -partite graph structure, with the complete  $k^{(w)}$ -partite graph being a special case. We define  $k^{(w_{ij})}$ -partite graph.

**Definition 3.5.4.** *The complete  $k^{(w_{ij})}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$  is the weighted graph whose vertices can be partitioned into  $k$  disjoint sets  $V_1, V_2, \dots, V_k$ , where each  $V_i$  contains  $N_i$  vertices, and two vertices are joined by an edge of weight  $w_{ij}$  if and only if the vertices belong to different parts  $i$  and  $j$ .*

In the case where  $w_{ij} = w$  for all pairs  $i$  and  $j$ , we have the complete  $k^{(w)}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w)}$ , and if  $w_{ij} = w = 1$  we then have the simple complete  $k$ -partite graph  $K_{N_1, N_2, \dots, N_k}$ . We have already indirectly seen the Laplacian eigenvalue spectrum of the latter two examples in Theorems 3.3.2 and 3.3.5. However, with the introduction of the  $k^{(w_{ij})}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$  we can formally present and discuss their Laplacian spectrum.

**Theorem 3.5.5.** *The Laplacian spectrum of a complete  $k^{(w_{ij})}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$  consists of:*

- an eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;
- $k$  eigenvalues  $\alpha_j = \sum_{i=1, i \neq j}^k w_{ji} N_i$  ( $j = 1, \dots, k$ ), each of multiplicity  $N_j - 1$ , with corresponding eigenvectors as follows: the entries on the parts different to part  $j$  are equal to 0; the entries on part  $j$  are not all 0, but do sum to 0;
- $k - 1$  non-zero eigenvalues  $\beta_i$  ( $i = 1, \dots, k - 1$ ), with eigenvectors whose entries are constant on the entries of the parts of the graph.

In addition, the  $N_j - 1$  eigenvalues  $\alpha_j$  are equal to generalised degrees of vertices in the corresponding part  $j$  of the graph.

**Proof.** We can define the partition  $\pi$  on the parts of graph  $K = K_{N_1, \dots, N_k}^{(w_{ij})}$ . Let  $B$  denote the quotient matrix of its Laplacian matrix with respect to the partition  $\pi$

$$B = \begin{bmatrix} d_1 & -w_{12}N_2 & \cdots & -w_{1k}N_k \\ -w_{12}N_1 & d_2 & \cdots & -w_{2k}N_k \\ \vdots & \vdots & \ddots & \vdots \\ -w_{1k}N_1 & -w_{2k}N_2 & \cdots & d_k \end{bmatrix}, \quad (3.21)$$

where the diagonal entries  $d_j$  are equal to  $\sum_{i \neq j} w_{ji}N_i$ .

As the partition is equitable the eigenvalues of  $B$  are also the eigenvalues of the Laplacian of the graph. Moreover, these eigenvalues are such that their eigenvectors are constant on the entries that correspond to the parts of  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$ . One of them is the eigenvalue 0 with associated eigenvector  $\mathbf{1}$ . Note that these give the eigenvalues  $\beta_1, \dots, \beta_{k-1}$ .

Similarly to the bipartite case we consider the vectors of type  $\begin{bmatrix} \mathbf{0} \\ X_j \\ \mathbf{0} \end{bmatrix}$ , where  $X_j \perp \mathbf{1}$  is a subvector positioned in the vector so that it is indexed by the vertices of the part  $j$ .

We find that they are the eigenvectors of  $L(K)$  associated with the eigenvalue  $\sum_{i \neq j} w_{ji}N_i$ . As there are exactly  $N_j - 1$  non-zero vectors of size  $N_j$  orthogonal to  $\mathbf{1}$ , that proves the theorem.

In addition, since for any  $j$  we have  $\sum_{i \neq j} w_{ji}N_i = \alpha_j = d_j$ , that means the described  $N_j - 1$  eigenvalues are directly related to the values of the generalised degrees  $d_j$  of vertices that belong to the part  $j$ . ■

Like with bipartite graphs, we point out that in general, if a graph  $G$  is such that it has an equitable partition  $\pi$  of its vertices in  $k$  parts of sizes  $N_i$ , for  $i = 1, \dots, k$ , where the corresponding quotient matrix is of the form as in (3.21), then its Laplacian eigenvalues are just as described in Theorem 3.5.5. The same holds for any of the results that follow in which the equitable partitions are used.

An interesting question is how the Laplacian spectrum of this graph changes with changes in the weights  $w_{ij}$ . We focus on a change in weights between just two

parts. As some of the eigenvalues are directly related to the generalised degrees of the graph the change in the spectrum is accordingly predictable. Without loss of generality we can observe the change in the edge-weights between parts 1 and 2.

**Theorem 3.5.6.** *An increase of the value  $w_{12}$  by  $w$  brings the following changes to the Laplacian spectrum of  $k^{(w_{ij})}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$ :*

- *the eigenvalue  $\sum_{i \neq 1} w_{i1} N_i$  of multiplicity  $N_1 - 1$  and the eigenvalue  $\sum_{i \neq 2} w_{i2} N_i$  of multiplicity  $N_2 - 1$  change their values by  $wN_2$  and  $wN_1$ , respectively.*
- *the  $k - 1$  eigenvalues with eigenvectors whose entries are constant on the entries of the parts of the graph change and are equal to the eigenvalues of the quotient of the new Laplacian matrix.*

*On the other hand all the other eigenvalues remain unchanged: eigenvalue 0, as well as all the eigenvalues  $\sum_{i \neq j} w_{ij} N_i$  each of multiplicity  $N_j - 1$  where  $j \neq 1$  and  $j \neq 2$ . Their corresponding eigenvectors remain of the same form as described in Theorem 3.5.5.*

**Proof.** The key point is that even with the increase of the value  $w_{12}$  the graph remains to be a  $k^{(w_{ij})}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w_{ij})}$  whose Laplacian eigenvalue spectrum can be calculated exactly as shown in Theorem 3.5.5. It is now easy to see that the increase in the value  $w_{12}$  brings only a couple of obvious changes.

If we use the notation adopted in the previous theorem, the first type of change in the Laplacian spectrum is seen in the eigenvalues  $\alpha_1$  and  $\alpha_2$ . The sums  $\sum_{i \neq 1} w_{i1} N_i$  and  $\sum_{i \neq 2} w_{i2} N_i$  are the only two that contain value  $w_{12}$  and are affected by its increase. Therefore, the eigenvalue  $\alpha_1$  increases by  $wN_2$ , and  $\alpha_2$  increases by  $wN_1$ . A different way of proving this change is through establishing the direct relation between these groups of eigenvalues and the generalised degrees of vertices that belong to parts 1 and 2.

The second type of change is found by noticing that the graph quotient matrix changes too. The  $k - 1$  eigenvalues  $\beta_i$  of the graph Laplacian change accordingly. ■

If, instead of one, there were changes in many values  $w_{ij}$ , we would reason in the same manner. The conclusion is that certain changes in eigenvalue spectrum

directly relate to the changes in generalised degrees of vertices, while the remaining eigenvalues change according to the changes in the quotient matrix.

Next we look at the eigenvalue spectrum of  $K_{N_1, N_2, \dots, N_k}^{(w)}$ . The eigenvalue spectrum can easily be derived from the previous result as it is the special case where the value  $w_{ij}$  equals  $w$  for all parts  $i$  and  $j$ , with  $i \neq j$ .

**Lemma 3.5.7.** *The Laplacian spectrum of a complete  $k^{(w)}$ -partite graph  $K_{N_1, N_2, \dots, N_k}^{(w)}$  on  $N$  vertices consists of:*

- *an eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;*
- *a number of  $k$  eigenvalues  $w(N - N_j)$ , each of multiplicity  $N_j - 1$ , whose corresponding eigenvectors are such that the entries on all parts  $i$ ,  $i \neq j$ , are equal to 0. While the eigenvalues' entries on the part  $j$  are not all 0, they do sum to 0.*
- *an eigenvalue  $wN$  of multiplicity  $k - 1$  with eigenvectors whose entries are constant on the entries of the parts of the graph.*

**Proof.** All the results from Theorem 3.5.5 can be applied to  $K_{N_1, N_2, \dots, N_k}^{(w)}$  directly. As  $w_{ij} = w$  and  $\sum_i N_i = N$  we have that the sum  $\sum_{i \neq j} wN_i$  equals  $w(N - N_j)$  which confirms the second point. Similarly, the quotient matrix of the Laplacian matrix simplifies to

$$B = \begin{bmatrix} w(N - N_1) & -wN_2 & \cdots & -wN_k \\ -wN_1 & w(N - N_2) & \cdots & -wN_k \\ \vdots & \vdots & \ddots & \vdots \\ -wN_1 & -wN_2 & \cdots & w(N - N_k) \end{bmatrix}.$$

It is easy to check that all non-zero vectors of size  $k$  orthogonal to  $[N_1, N_2, \dots, N_k]^T$  are eigenvectors of  $B$ . There are  $k - 1$  such linearly independent vectors. We find that they correspond to the eigenvalue  $wN$ . This completes the proof of the theorem. ■

All the results gathered so far regarding regular  $k$ -partite structure are presented in detail, as they are useful in delivering the following general result:

**Theorem 3.5.8.** *Let  $G$  be an edge-weighted graph that consists of  $k$  connected subgraphs  $G_1, \dots, G_k$ , where each vertex of a subgraph  $G_i$  is connected by an edge of weight  $w_{ij}$  to every vertex in  $G_j$ . Then its Laplacian spectrum can be directly derived from the eigenvalues of the induced  $k$ -partite graph with interconnecting edges as described, and the Laplacian eigenvalue spectrums of the  $G_i$ 's.*

**Proof.** The Laplacian matrix  $L(G)$  can be written as follows:

$$L(G) = L(K_{N_1, \dots, N_k}^{(w_{ij})}) + \begin{bmatrix} L(G_1) & 0 & \cdots & 0 \\ 0 & L(G_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L(G_k) \end{bmatrix}$$

The two matrices in the sum commute and through Theorem 3.5.1 we know that for a certain permutation of eigenvalues of the first matrix each eigenvalue of  $L(G)$  is equal to a sum of an eigenvalue from the first and an eigenvalue of the second matrix. A natural partition of the induced subgraph  $K = K_{N_1, \dots, N_k}^{(w_{ij})}$  is the partition  $\pi$  defined on the graph's  $k$  parts. The partition  $\pi(K)$  is equitable and the eigenvalues of its quotient matrix  $B(K)$  are also the eigenvalues of the graph  $K$ . The eigenvectors corresponding to these  $k$  eigenvalues of  $K$  are such that they are constant on the entries corresponding to the parts.

We recall the eigenvalue spectrum of  $K$  (see Theorem 3.5.5):

- an eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;
- a number of  $k$  eigenvalues  $\sum_{i \neq j} w_{ij} N_i$ , each of multiplicity  $N_j - 1$ , with corresponding eigenvectors of type  $\begin{bmatrix} \mathbf{0} \\ X_j \\ \mathbf{0} \end{bmatrix}$ , where  $X_j \perp \mathbf{1}$  is a subvector positioned in the vector so that it is indexed by the vertices of the part  $j$ ;
- a number of  $k - 1$  eigenvalues of the quotient matrix  $B(K)$  with eigenvectors whose entries are constant on the entries of the parts of  $K$ .

Following the procedure presented in the proof of Theorem 3.5.3, we conclude that the eigenvalue spectrum of  $L(G)$  is as follows:

- an eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;

- a number of  $k$  sets of eigenvalues, each set containing  $N_j - 1$  values of form  $\sum_{i \neq j} w_{ij} N_i + \lambda_l(L(G_j))$ , with corresponding eigenvectors of type  $\begin{bmatrix} \mathbf{0} \\ X_j \\ \mathbf{0} \end{bmatrix}$ , where  $X_j \perp \mathbf{1}$  is a subvector positioned in the vector so that it is indexed by the vertices of the part  $j$ ;
- a number of  $k - 1$  eigenvalues of the quotient matrix  $B(K)$ , with eigenvectors whose entries are constant on the entries of the parts of  $K$ .

■

Finally, we consider the eigenvalues of a special case of a graph that we call the generalised regular weighted  $k$ -part graph.

**Definition 3.5.9.** *The generalised regular weighted  $k^{(w_p, w_q)}$ -part graph  $K_{N_1, N_2, \dots, N_k}^{(w_p, w_q)}$  is the graph that consists of  $k$  complete weighted subgraphs  $K_{N_i}^{(w_p)}$ , where any two vertices from different parts are connected with edges of weight  $w_q$ .*

It is easy to see that obtaining the eigenvalues of such a graph is simple, as the result directly follows from Lemma 3.2.2 and Theorem 3.5.8.

**Theorem 3.5.10.** *The Laplacian spectrum of  $k^{(w_p, w_q)}$ -part graph  $K_{N_1, N_2, \dots, N_k}^{(w_p, w_q)}$  consists of:*

- an eigenvalue 0 with corresponding eigenvector  $\mathbf{1}$ ;
- a number of  $k$  eigenvalues  $w_q N - (w_q - w_p) N_j$ , each of multiplicity  $N_j - 1$ , with corresponding eigenvectors as follows. The entries on all parts different to part  $j$  are equal to 0. The entries on the part  $j$  are not all 0, but do sum to 0.
- an eigenvalue  $w_q N$  of multiplicity  $k - 1$  with eigenvectors whose entries are constant on the entries of the parts of the graph.

Although the theorems of this section provide a good starting point for understanding the underlying structure of an observed edge-weighted graph, there are limits on how much can we apply these results. Ideally, we would like to identify familiar shapes of the spectrum and familiar forms of the eigenvectors and to use them to recognise an underlying structure of a graph. However, without additional

constraints, it is not possible to prove the converse of Theorem 3.5.8. Problems can be found in the eigenvalues of the same value. Any linear combination of their corresponding eigenvectors is also a valid eigenvector. Therefore the ‘familiar’ useful format of an eigenvector can remain disguised in a linear combination resultant vector. Even if the eigenvalues are different but closely grouped, this still does not directly imply that the group corresponds to a ‘familiar’ shape described in the proved theorems. One has to investigate in detail the shape of the spectrum and draw conclusions in relation to the considered graph.

Through the changes in the partition and the structure of the parts we can observe the dynamic of the eigenvalues in the spectrum.

In the case of a regular  $k$ -partite  $K_{N_1, N_2, \dots, N_k}^{(w)}$  graph we had a straightforward relation. The ordering of  $N_i$ 's uniquely describes the order of the eigenvalues in the spectrum. Without loss of generality we assume  $N_1 < \dots < N_k$ . That implies the order of eigenvalues  $wN > w(N - N_1) > \dots > w(N - N_k) > 0$ , with multiplicities  $k - 1, N_1 - 1, \dots, N_k - 1, 1$ , respectively.

However, a structural change within any of the parts modifies the eigenvalue spectrum. Adding edges to part  $i$  changes  $L(G_i)$ . Hence the  $N_i - 1$  eigenvalues  $w(N - N_i)$  with associated eigenvectors of type  $\begin{bmatrix} 0 \\ X_i \\ 0 \end{bmatrix}$  described earlier, increase in value to  $w(N - N_i) + \lambda_l(L(G_i))$ , where  $l = 1, \dots, N_i - 1$ .

In order to receive useful information about an edge-weighted graph we go in reverse order, and start from its eigenvalue spectrum and corresponding eigenvectors. Therefore we very much depend on the familiar shapes of the spectrum and the familiar forms of the eigenvectors. As long as we spot those we can be comfortable in going further in our analysis of the graph structure. However, one must be aware of the difficulties that may arise, as occasionally the eigenvalues of higher multiplicities can distort the picture of the spectrum and the format of the eigenvectors so significantly that those would not be of any use to us any more.

While in this chapter we included examples where the theoretical results could be applied directly, we next observe weighted graphs in general and simulate examples of interest that extend work done on considered ideal examples. Empirically derived results are presented and discussed in Chapter 4.

# Chapter 4

## Simulated Examples of FAPs

### 4.1 Overview of the Chapter and Introduction to the Simulations

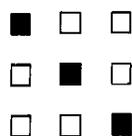
#### 4.1.1 Presentation of FAPs using random graph structures

A natural way to express Frequency Assignment Problems without directly including information about the geographical positions of transmitters is numerically, using “quantified” relations between transmitters. A common problem presentation is to list all necessary constraints required for achieving satisfactory signal-to-noise ratio. When constraints are defined in terms of relationship between two transmitters we talk about a binary constraint presentation. An example of a binary constraint is a value that represents the minimal channel separation between any two transmitters. Some research favours a non-binary description, arguing that the standard binary model fails to capture certain important aspects of real FAPs, such as multiple interference, hence resulting in non-optimal frequency assignments (see, e.g., [8]). However, we in this study remain supportive of the binary presentation, and we see it as a useful approach to describe Frequency Assignment Problems. This means that any problem can be and is assumed to be fully specified by a so called influence matrix.

In this chapter we present a number of simulations that relate to the theory discussed in previous chapters. Often the properties observed through simulations are an important guidance for the direction of our research. As stated earlier, our

primary interest is in Laplacians. We observe specifically constructed graphs, so as to gain better understanding of the information that Laplacians hold. Most attention is given to two apparently distinct analysis directions, the first being ‘clique’ and the second being ‘independent set’ recognition. However, as the theory of Laplacians confirms, the first is simply the complementary problem of the second. Both structures are equally interesting. In the case of cliques we are reducing the original problem to one, or a number of problems of considerably smaller size relatively independent from one another. Identifying such underlying structures leads to a straightforward way to reuse frequency channels. In the latter case we partition a problem into a number of independent sets. For independent sets, only one or at most a few frequencies are needed for an optimal assignment. However the strong connections between independent sets may force us to be careful in channel assignment, which in turn creates the need to keep necessary distances to retain a satisfactory signal-to-noise requirement.

In each of the examples of simulation we generate a random graph of a particular order. We start with a set of  $N$  vertices and add edges between them at random. During this process we keep two things in mind: the probability of an edge occurring between two different vertices, and the probability that the generated edge is of a certain weight. These additional specifications are here to help us describe and construct the type of the graph structure we wish to observe. Intuitively, a subgraph of a graph that is described with a high probability of an edge occurring between any two of its vertices would resemble a dense, clique-like part, whereas a very low probability of an edge occurring within the subgraph would be nearer to resembling an independent set. In addition, one must be careful in describing the subgraph considered and its position with respect to the remaining vertices of the graph. Similarly, using the edge occurring probability terminology we can define the connection of the subgraph considered to the rest of the graph. The higher the weight of an edge, the stronger the mutual influence between the vertices. In an informal manner we could ‘draw’ dense parts or strong connections in a certain graph with a black square symbol ‘■’ and less dense parts or weak connections with a white square symbol ‘□’. For example



represents a graph with three dense parts of not necessarily the same size that do not influence each other strongly.

### 4.1.2 Introduction to the simulations

In this subsection, by observing two examples of generated graphs, we present how we intend to illustrate graphically the algebraically derived properties of graphs, the Laplacian eigenvalues and eigenvectors. We specify types of plots we tend to observe throughout this chapter. We also note that the generation of graph structures, numerical computation, and plots in this study are all done in the computer program Maple.

As an introductory simulation we consider an edge-weighted graph with one ‘hidden’ clique. Since in realistic FAP examples any two nearby transmitters are required to be five channels apart we tend to perform simulations on graphs assuming that edges are most of the time assigned integer weights  $w = 1, \dots, 5$  (see, e.g., [1]).

**Example 4.1.1.** In a randomly generated graph  $G$  of order  $N = 100$  an ‘almost clique’ of size  $m = 18$  is embedded. The clique is such that it contains an edge between any two of the vertices, where 80% of edges are of weight 5 and 20% of weight 4. The probability of an edge occurring outside the clique is 0.4. Each of the edges outside of the clique are of weights 1, 2, 3, 4 or 5, with probabilities 0.44, 0.3, 0.2, 0.05 and 0.01, respectively.

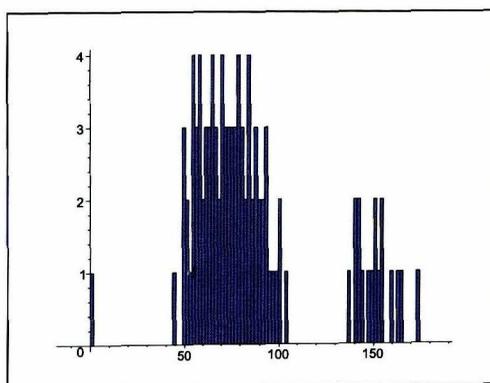


Figure 4.1: Distribution of the Laplacian eigenvalues described in Example 4.1.1.

First, we focus on observing the distribution of the Laplacian eigenvalues that is presented in the form of an eigenvalue histogram in Figure 4.1. The histogram

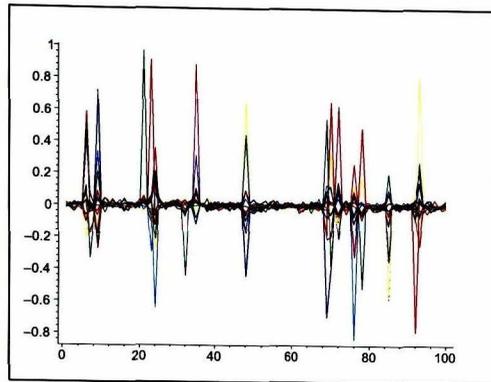


Figure 4.2: Presentation of entries of the eigenvectors corresponding to the seven largest eigenvalues noted in Figure 4.1.

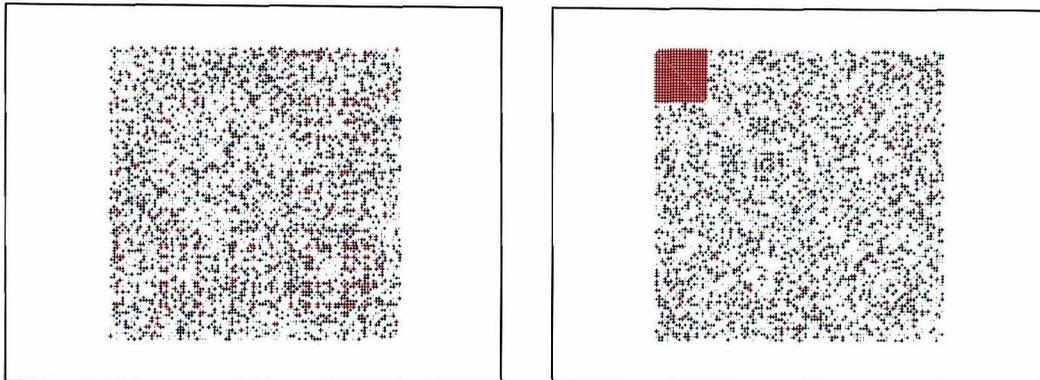


Figure 4.3: Two influence matrices corresponding to the graph described in Example 4.1.1: the first describing the graph with the initial ordering of vertices and the second the ordering of vertices following the analysis of the eigenvectors. Different coloured entries correspond to the different edge-weights (red: 4 and 5; blue: 3; green: 2; gray: 1; white if there is no edge between two vertices).

plots are produced as follows. The interval between the smallest and the largest eigenvalue is divided into  $N$  equal small intervals. The height of a bar is equal to the number of eigenvalues in that interval.

We next study a group of large eigenvalues, by looking at their eigenvectors. A way to present entries of the eigenvectors is by drawing them in one plot where along the  $x$ -axis we read the order of an entry, and on the  $y$ -axis we read the entry's value. The points of the same colour correspond to the same eigenvector. This is depicted in Figure 4.2.

The final step in the analysis is identifying the 18 vertices singled out in Figure 4.2, by looking at high absolute value entries of gathered eigenvectors. Figure 4.3 contains images representing two matrices. Both matrices represent the same graph

the difference being that the first is the adjacency matrix with the original ordering of vertices, whilst the second matrix has its rows permuted so that the first 18 rows now correspond exactly to the 18 vertices singled out in Figure 4.2. From the second matrix we see that these vertices form the almost-clique embedded in  $G$ .

As another example we present here a graph that consists of three independent sets of similar size.

**Example 4.1.2.** Let  $G$  be an edge-weighted graph on  $N = 100$  vertices that consists of three independent disjoint sets that influence each other equally,  $N_1 = 29$ ,  $N_2 = 33$ ,  $N_3 = 38$ . Also, we assume that an edge occurs between any two parts with probability 0.8 and that weight of an existing edge is chosen to be 4, 3, 2, 1, with probabilities 0.1, 0.4, 0.3, 0.2, respectively.

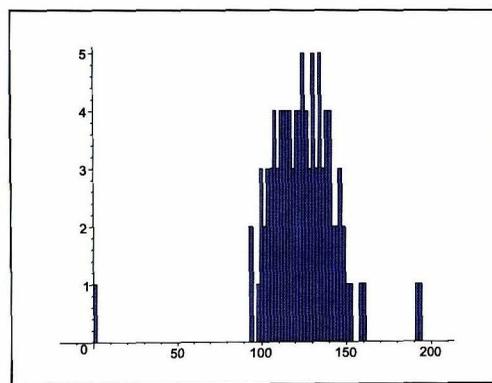


Figure 4.4: Distribution of the Laplacian eigenvalues presented in Example 4.1.2.

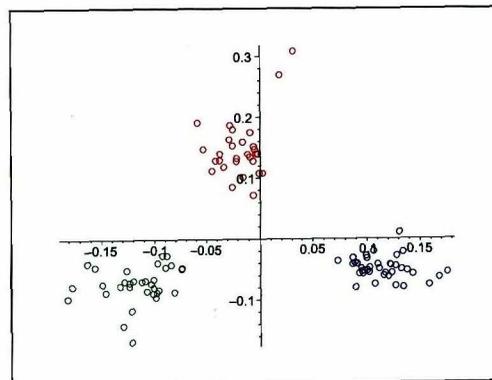


Figure 4.5: Presentation of vertices in the plane using the entries of the two maximal eigenvectors noted in Figure 4.4.

Figure 4.4 is associated with this example. In the histogram plot we clearly see two maximal Laplacian eigenvalues that stand out from the remaining eigenvalues. The entries of the two corresponding eigenvectors are presented in Figure 4.5. This

is different to the presentation shown in the previous example. Here, each point in the plot has, for its first coordinate, an entry of the first vector, and for its second coordinate the value of the corresponding entry in the second vector. Therefore, each vertex of the graph is drawn in the plane with coordinates defined by the corresponding entries of the two considered eigenvectors. This way of presenting vertices is of interest since it graphically suggests the division of the vertices of the graph into three groups. Figure 4.6 shows that the suggested partition into three groups reveals the three independent sets.

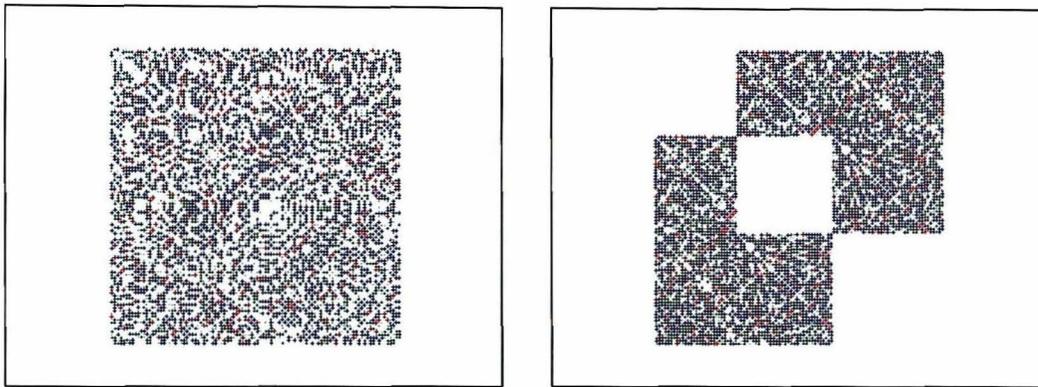


Figure 4.6: Two influence matrices corresponding to the graph described in Example 4.1.2: the first describing the graph with the initial ordering of vertices and the second the ordering of vertices following the analysis of the eigenvectors. Different coloured entries correspond to the different edge-weights (red: 4 and 5; blue: 3; green: 2; gray: 1; white if there is no edge between two vertices).

While we here avoided discussing and analysing discovered algebraic properties of the simulated graphs, we note that the number of large eigenvalues was as expected and the corresponding vertices were clearly (empirically) identified. We analyse the algebraic properties in detail in the proceeding sections.

### 4.1.3 Contents of the chapter

One of the objectives of undertaking simulations and analysis similar to two examples presented in the previous section, and in the examples that follow, is to gain understanding about desirable features of the Laplacian eigenvalue spectra. In order to present our findings we have chosen to show a number of simulated graph examples. Separate sections will consider different types of graph structures. Within each section we will present qualitatively similar examples of graphs, that will help us to observe in greater detail the effects that minor changes have on the

eigenvalue spectra. In Section 4.2 we first focus on simplified examples—simple graphs with regular  $k$ -part underlying structure. These examples may be seen as simplified modelling of the real problems, however they are very important for exploring and understanding the features of the Laplacian spectra which we hope to use in the analysis of more complex graph structures. We look at more general simulation examples in the second part of the same section, where we allow weights to be assigned to the edges of the graphs. We gradually relax the regularity requirements in Section 4.3, and observe so called irregular  $k$ -part graphs, focusing on graphs with almost cliques and independent sets. In Section 4.4 we give a possible intuitive interpretation of the results analysed in simulations, and finally in Section 4.5 present two generated graphs that closely resemble realistic FAPs and study their algebraic properties.

In addition to observing the eigenvalue spectrum shapes, the following sections also explore eigenvectors. Our intention is to empirically acquire understanding of the usefulness of the Laplacian analysis approach through structured simulations.

## 4.2 Graphs with ‘Regular $k$ -part’ Underlying Structure

Understanding the macro structure of a graph is as important as understanding its subsets’ properties. The graph may seem large or disorganised in the sense of the ordering of vertices and the distribution of the edge weights. The aim of the simulations presented in this section is to show that if a certain regularity exists in the graph structure, this may be identified by the Laplacians.

Before we describe what we mean by graphs with ‘regular  $k$ -part’ underlying structure, we introduce some notation. A way to present an edge-weighted graph with ‘ $k$ -part’ underlying structure is to give information on the order and type of all subgraphs that correspond to the  $k$  parts of the structure and describe their mutual connections within the graph. In case any of the subgraphs is randomly generated we specify the probabilities of an occurrence of an edge of a certain weight for any two parts  $i$  and  $j$  of the  $k$ -part underlying structure. We denote by  $p_i$  the probability of an edge occurring in part  $i$ , and by  $q_{ij}$  the probability of an edge occurring between parts  $i$  and  $j$ . Sometimes we refer to these parameters as densities.

We recall Definition 2.3.3 where we note that no edge between two vertices is the same as saying that the two vertices are connected with an edge of weight 0. As the edges can be of different weights, we approximate proportions of considered edges and their specific weights: we denote by  $p_i^{(w)}$  the probability that weight  $w$  is assigned to an edge in part  $i$ . All the probabilities over all edge weights in the part sum to 1, or  $\sum_{w \in W} p_i^{(w)} = 1$  for any part  $i$ , where  $W$  represents the set of all edge weights  $w$  contained in the graph. Similarly, by  $q_{ij}^{(w)}$  we denote the probability that weight  $w$  is assigned to an edge that is connecting part  $i$  with part  $j$ . As above, these probabilities must satisfy  $\sum_{w \in W} q_{ij}^{(w)} = 1$  for any two different parts  $i$  and  $j$ .

Following the assumptions from above it is clear that  $p_i^{(0)} = 1$  means that part  $i$  is an independent set. Similarly  $q_{ij}^{(0)} = 1$  describes a situation where there are no edges that directly connect vertices belonging to two different parts  $i$  and  $j$ .

**Definition 4.2.1.** *A random graph with regular  $k$ -part underlying structure is a graph in which parameters  $p_i$ ,  $p_i^{(w)}$ ,  $q_{ij}$  and  $q_{ij}^{(w)}$  are constant for any two parts  $i$  and  $j$  ( $i \neq j; i, j = 1 \dots k$ ).*

Therefore, we can write  $p_i := p$  and  $q_{ij} := q$ , for all different parts  $i$  and  $j$ , where  $p$  and  $q$  are constants that need not be the same. Similarly,  $p_i^{(w)} := p^{(w)}$  and  $q_{ij}^{(w)} := q^{(w)}$ .

For the sake of simplicity, we start with the graphs where all edges are of weight  $w = 1$ , limit ourselves to presenting the graphs with no more than  $k = 4$  parts, and most of the time look at the examples in which  $p \leq q$ . Generally, the initial choice for probabilities is as follows:  $p = 0.1$ , and  $q = 0.9$ . The proposed constraints are useful in determining a family of graphs that act as reference cases.

Once we get familiar with the use of the Laplacians in the simplified cases of simulations, we move on to observe generalised simulations. We construct graphs of different structures by gradually introducing changes to the parameters, and analyse the new features by focusing on one parameter change at the time.

### 4.2.1 Two-part simulation examples with $p_i < q_{12}$

The first observations are done on the simplest form of simulation, a class of two-part type of graphs. The parts share the following characteristics: they have the

same edge density  $p$ , that is smaller than  $q$  the density of the edges connecting the parts, and all the edges are of weight one (i.e.,  $p^{(1)} = q^{(1)} = 1$ ).

We consider a graph with  $N = 200$  vertices, where the parts are of equal sizes  $N_1 = N_2$ . We set probabilities  $p$  and  $q$  to be 0.1 and 0.9, respectively. This arrangement describes a graph that consists of two parts with strong mutual connections:



The spectrum of the graph is plotted in Figure 4.7(a). It shows that almost all non-zero eigenvalues are grouped at one place, the exception being the largest eigenvalue. Its corresponding eigenvector's entries, pictured in Figure 4.8(a), suggests a clear distinction between the two parts. The entries of a positive sign ( $supp_+$ ) correspond to the vertices of one part, and the negative ones ( $supp_-$ ) correspond to the other part.

We next look at the behaviour of the eigenvalue spectrum with respect to the gradual changes in the sizes of the parts and/or changes in 'densities' of edges within and between the parts.

### The spectrum as a function of the part sizes

In the following collection of examples we assume graph parameters the same as above:  $N = 200$ ,  $p = 0.1$ ,  $q = 0.9$ ,  $p^{(1)} = q^{(1)} = 1$ , while we let the sizes of the parts vary. In terms of the ratios in sizes between the parts, the changes implemented are 1 : 2, 1 : 3, 1 : 4, 2 : 3 and 3 : 4, respectively. The six histogram plots are presented in Figure 4.7.

We observe how these changes in part sizes influence the shape of the eigenvalue histogram.

The first noteworthy feature is that a 'detached' maximal eigenvalue ( $\sim N$ ) is common to all presented examples. An additional property, that the corresponding eigenvector's entries of the opposite sign belong to the vertices in different parts, as we previously had in Figure 4.7(a) ( $N_1 : N_2 = 1 : 1$ ), holds in all the cases. The plots corresponding to the maximal eigenvalue eigenvector for all graphs are shown in Figure 4.8.

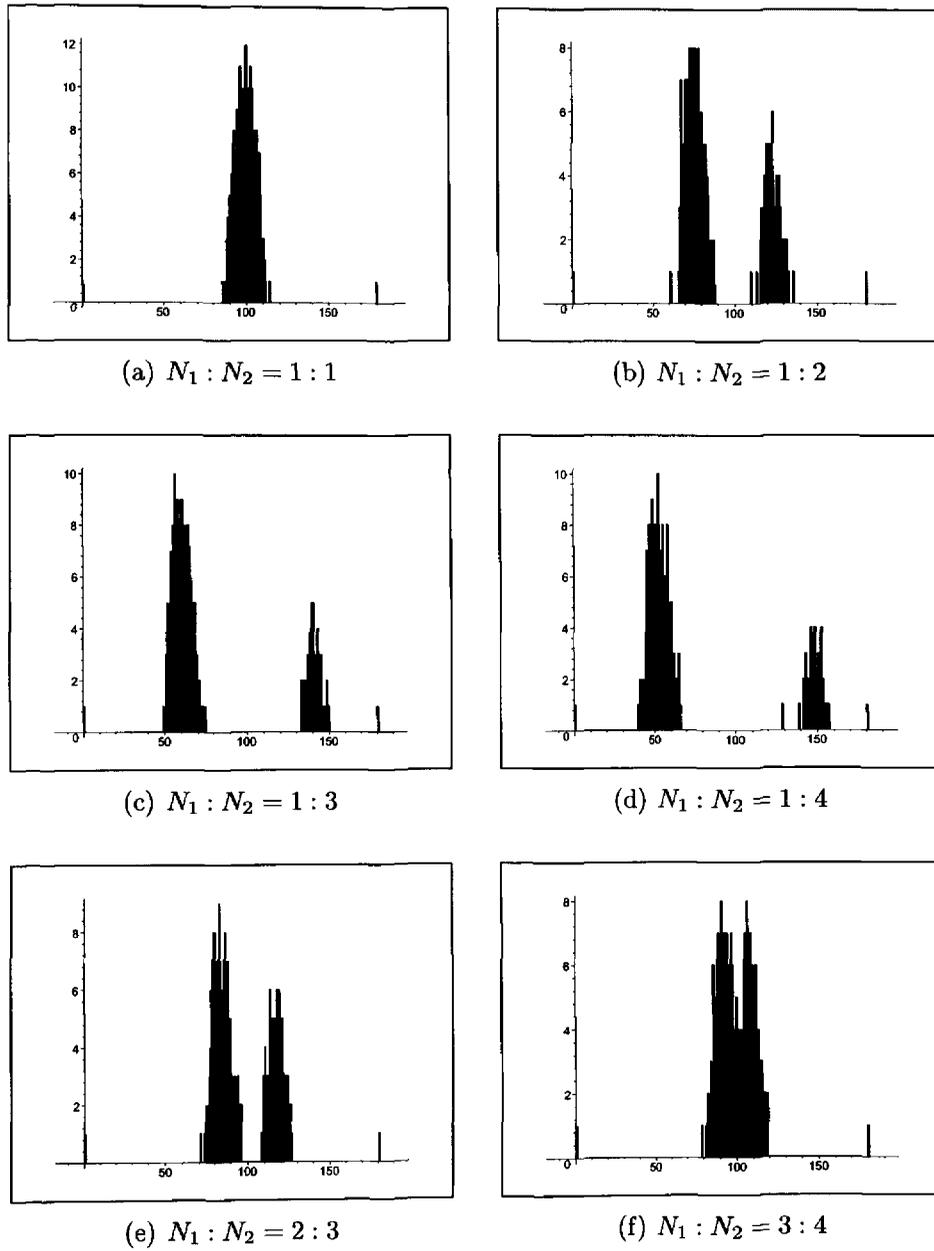


Figure 4.7: Distribution of eigenvalues of the two-part examples produced when  $N = 200$ ,  $p = 0.1$ ,  $q = 0.9$  and all edges are of weight 1.

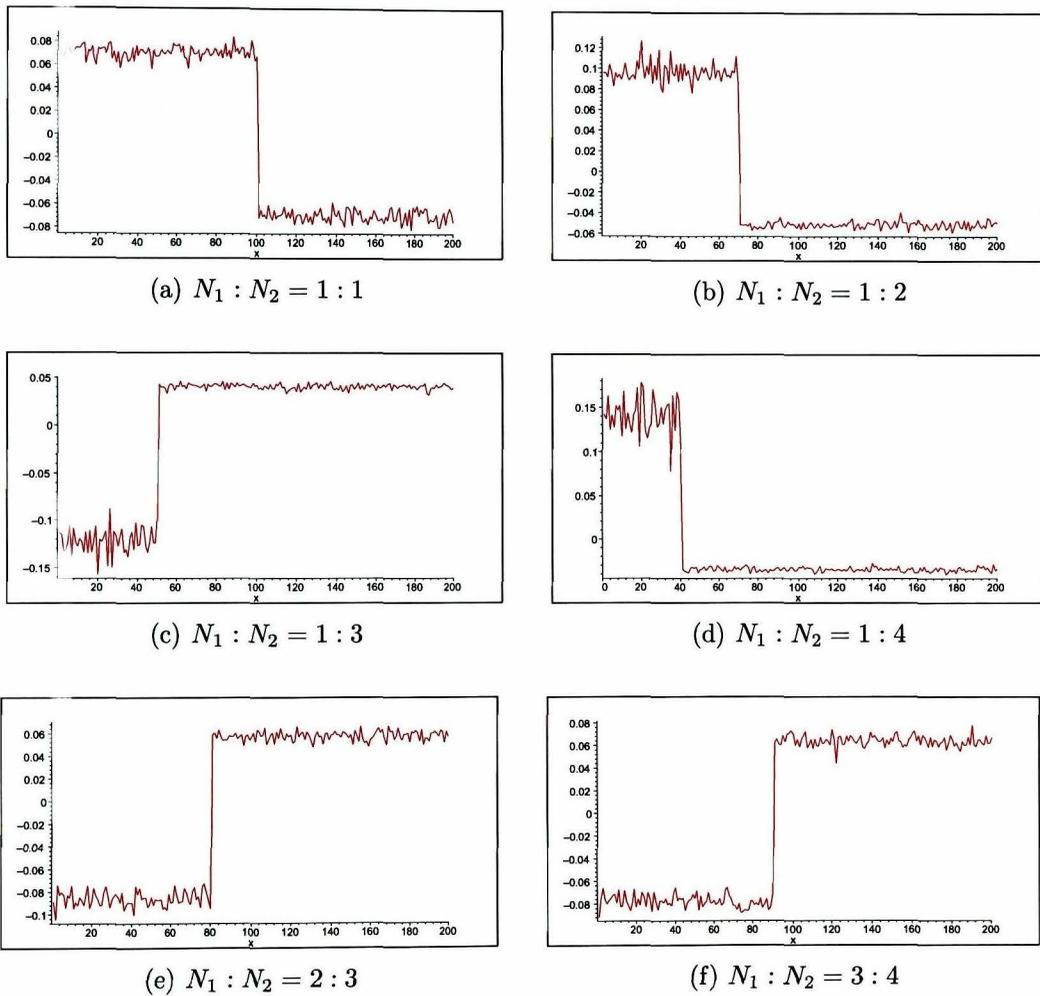


Figure 4.8: Presentation of the entries of the maximal eigenvalue eigenvector corresponding to the examples in Figure 4.7.

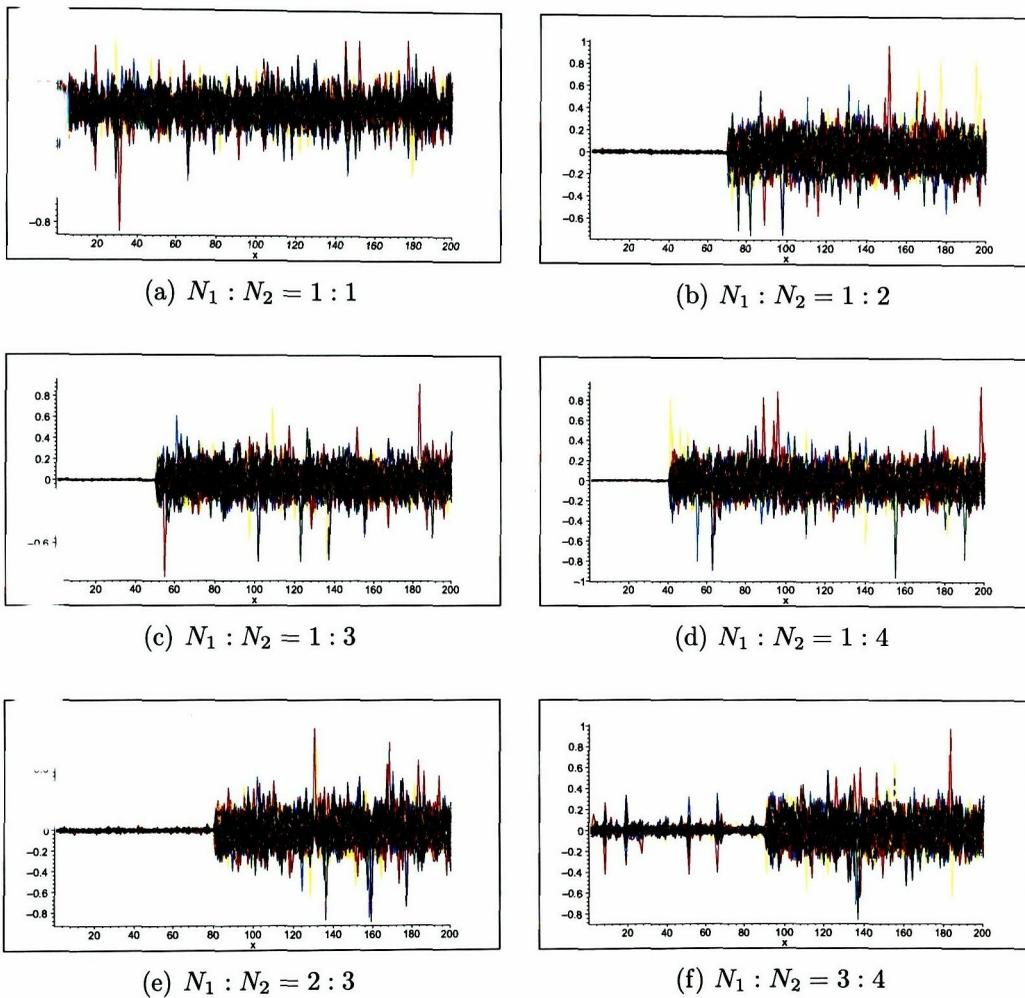


Figure 4.9: Entries of the eigenvectors corresponding to the smaller in value (left) group of eigenvalues corresponding to the examples in Figure 4.7.

The rest of the eigenvalue spectrum, by which we mean the spectrum excluding the minimal eigenvalue 0 and the discussed lone maximal eigenvalue, itself has interesting properties. We recognise that the eigenvalues tend to be grouped either around one or two points. In the case of two distinctly formed groups, we see that they are necessarily of different sizes. We find that both of these properties are in relation to the part sizes,  $N_1$  and  $N_2$ , or their ratio,  $N_1/N_2$ . If the parts are of the same size the eigenvalues are concentrated around the same, in this case mid point (i.e., approximately  $N/2$ ). A graph with the same number of vertices  $N$ , but with parts of significantly different sizes, has an eigenvalue histogram in which groups are around two points both moved away from  $N/2$  and in different directions, approximately close to values  $N_1$  and  $N_2$ . We further see that as the difference in the sizes of the parts grows not only the distance between the groups increases, but the sizes of the groups change too. The groups sizes are always  $N_1 - 1$  and  $N_2 - 1$ .

We then focus on the grouped eigenvalues that are of smaller value (the left group of the spectrum), whilst remembering that in the simulations presented we have  $N_1 < N_2$ . One could look at the corresponding eigenvalues independently, but it is the joint picture of all those eigenvectors that attracts our attention. The plots in Figure 4.9 present information contained in the grouped considered eigenvalues. The eigenvectors' entries have values close to 0 on the vertices of the part of size  $N_1$ , and in absolute terms large values vary on the vertices of the other part, of size  $N_2$ . The same holds for the second group of eigenvalues, suggesting that each group is 'responsible' for describing one part within the context of the graph in general.

The simulations allow us to test the extent to which the eigenvalue spectrum of a randomly produced graph with certain features differs from the spectrum of a similar but idealized situation.

Recalling Theorem 3.5.3 and the observations that follow the theorem in Chapter 3 we find that all the analysis is in line with the theoretical work done. For instance, an example of an ideal graph with properties similar to the simulations presented above could be a graph with the same number of vertices  $N = 200$  with all edges of weight  $w = 1$ , and probabilities  $p_i^{(1)} = 0$  and  $q_{12}^{(1)} = 1$ . This is a simple complete bipartite graph  $K_{N_1, N_2}$ , whose spectrum consists of one eigenvalue 0,  $N_2 - 1$  eigenvalues of value  $N_1$ ,  $N_1 - 1$  eigenvalues of value  $N_2$  and one eigenvalue  $N$ , as shown

in Lemma 3.5.2.

It is clear that the shape of the spectrum and the order of the eigenvalues, resemble the shape of the spectrum and the order of the eigenvalues in the corresponding ideal examples. Moreover, the properties we observe in the associated eigenvectors in simulated graphs agree in their structure with the properties observed in the ideal case. For instance, the eigenvalue 0 is associated with the all-1 vector  $\mathbf{1}$ . The smaller value group is associated with vectors of type close to vector  $\begin{bmatrix} \mathbf{0} \\ Y \end{bmatrix}$ , where  $Y$  is a non-zero vector of size  $N_2$ , with the difference that in the simulated examples eigenvectors' entries take values close, but not exactly equal to 0 on the vertices of the part of size  $N_1$ . We have a similar situation for the larger value eigenvalues group. Finally, the entries of the eigenvector that corresponds to the maximal eigenvalue are such that they are almost constant on the parts of the graph. With regards to the entries of the maximal eigenvalue they are also in line with a result about signs of eigenvectors entries' that holds for bipartite graphs, Theorem 3.3.20.

### The spectrum as a function of the density parameters $p$ and $q$

We next produce simulations of graphs in which the sizes of the two parts are kept fixed while other parameters are allowed to vary. We are interested in observing how the eigenvalue spectrum is affected by changes in the density of the edges within the subgraphs (parameter  $p$ ), and how it is affected by changes in the density of the edges between the parts (parameter  $q$ ).

Plots in Figure 4.10 display the effect of changes in the probabilities  $p$  and  $q$  on the Laplacian eigenvalues when the sizes of the two subgraphs are equal.

The first feature of interest is the detached maximal eigenvalue. We note that with the decrease in the value  $q$ , the maximal eigenvalue decreases. We look for the explanation in an ideal example for which we know precisely the properties of the eigenvalue spectrum.

An ideal example similar in its characteristics to the simulated examples could be a graph that has an equitable two-partition with parts of the sizes  $N_1$  and  $N_2$ . Its quotient matrix is of the form

$$B = \begin{bmatrix} qN_2 & -qN_2 \\ -qN_1 & qN_1 \end{bmatrix}.$$

As we have already seen in an observation made on page 78 one of the eigenvalues of such a graph is  $q(N_1 + N_2)$  and the corresponding eigenvector is of the form  $[1, \dots, 1, -\frac{N_1}{N_2}, \dots, -\frac{N_1}{N_2}]^T$  with  $N_1$  entries of value 1, and  $N_2$  entries of value  $-N_1/N_2$ .

The continuity of the Laplacian eigenvalues implies that the approximate value of the maximal eigenvalue in the simulated example is  $qN$ , and hence the confirmation of the direct relationship between the value  $q$  and the maximal eigenvalue.

As for the rest of the non-zero eigenvalue spectrum, our analysis defines two cases. In one we fix the value  $p$ , and in the other we keep the value  $q$  fixed.

In the first four plots (a)-(d) in Figure 4.10 the value  $p$  remains unchanged at 0.1, while  $q$  takes values 0.9, 0.8, 0.5 and 0.3, respectively. With no changes in other parameters, and a decrease in  $q$ , we see a general shift of the spectrum to the left, i.e., we have obtained an overall decrease in the values of eigenvalues (including the maximal eigenvalue).

We also consider graphs where the density of edges between parts remains unchanged, i.e., the value  $q$  is fixed, while we allow  $p$  to change. Plots (a), (e), (f) and (g) in Figure 4.10 show these examples. The value  $q$  is set to be 0.9. With no changes in other parameters, and with an increase in  $p$ , we see that there seem to be no changes in the maximal eigenvalue, while the rest of the spectrum moves to the right towards the maximal value.

Next we compare all simulations in Figure 4.10 and discuss the eigenvalue spectrum changes with respect to the mutual relationship between  $p$  and  $q$  parameters. As previously noted, the maximal eigenvalue position is approximately equal to  $qN$ . The position of the rest of the eigenvalue spectrum with respect to the position of the maximal eigenvalue provides an interesting observation: as  $p$  gets closer in value to  $q$ , the distance between the largest eigenvalue and the grouped eigenvalues reduces. When this difference is large, as in the reference example in Figure 4.10(a), we clearly see the maximal eigenvalue positioned away from the rest of the eigenvalues, whereas in the case of a small difference (see plots (d), (g) and (h)) the maximal eigenvalue remains close to the rest of the spectrum.

As the plots show, the effect of the changes in probability parameters  $p$  and  $q$  is significant in cases where the sizes of the parts are equal. We test whether the same holds in cases where the sizes of the subgraphs are different. In order to analyse this

in the next set of plots (Figures 4.11-4.13) we refer to the same set-up, but apply different ratios between the subgraph sizes (1 : 2, 1 : 3 and 1 : 4, respectively).

For fixed values  $p$  and  $q$ , the value of the maximal eigenvalue does not move around even if it appears that its distance from the rest of the spectrum decreases. Instead, we note a pattern in ‘movement’ of the remaining non-zero eigenvalues.

Let us consider the non-zero eigenvalues without the maximal eigenvalue as a group of  $N - 2$  eigenvalues. We notice that the increase in the difference of the sizes of the parts (with all the other parameters unchanged) ‘widens’ this group, that is simultaneously approaching the maximal eigenvalue.

We recall that when the sizes of the parts are not similar it is possible to have two clearly formed groups. The more pronounced the difference in the sizes of the parts is, the larger the distance between the two groups of eigenvalues becomes. As we choose smaller values for  $N_1$  and accordingly larger ones for  $N_2$ , the smaller in value group of  $N_2 - 1$  eigenvalues moves to the left (the eigenvalues decrease), while the other group of  $N_1 - 1$  eigenvalues moves to the right (the eigenvalues increase) approaching the maximal eigenvalue.

A possible interpretation of this is that the decline in the ‘detachment’ of the maximal eigenvalue occurs with the increase in ‘dominance’ of one of the two parts within the graph.

### 4.2.2 Two-part simulation examples with $p_i > q_{12}$

Beside the presented relations between the considered parts, where we had probability of an edge occurring in the parts  $p_i$  lower than the probability  $q_{12}$  of an edge occurring between the parts, we look into instances where the parts are dense in comparison to the subset of edges that connect them:



In a similar manner to the previous analysis we could run a number of simulations of graphs of order  $N$ , with different choices for the sizes of the parts  $N_1$  and  $N_2$  and probability parameters  $p_i$  and  $q_{12}$ . However, we find it sufficient to present one only (see Figure 4.14).

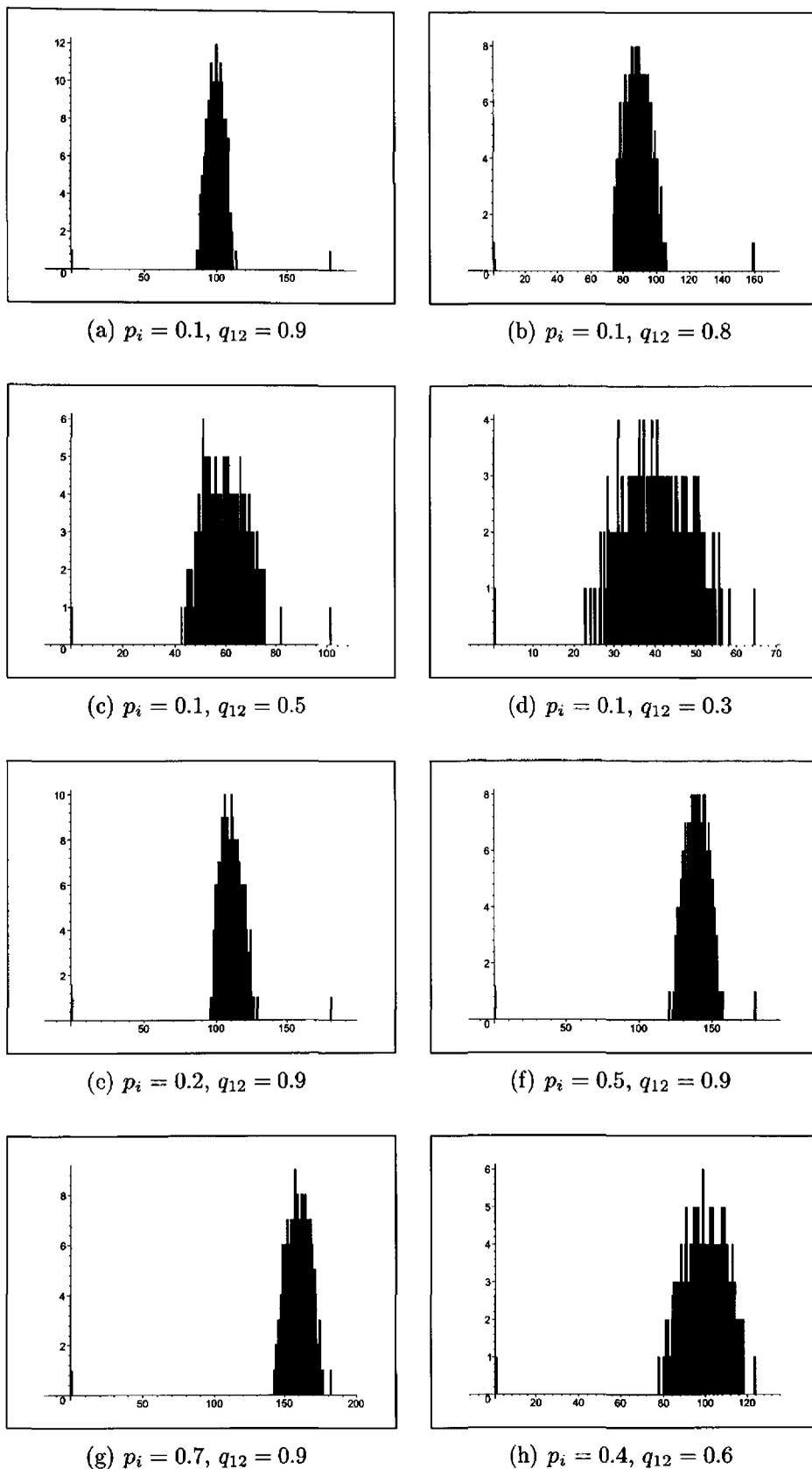
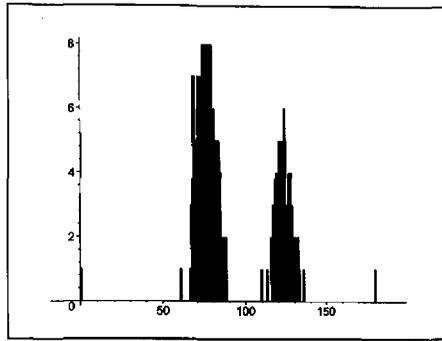
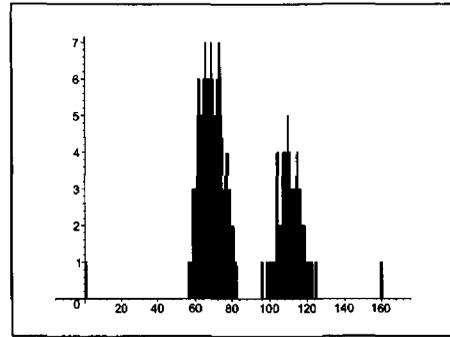


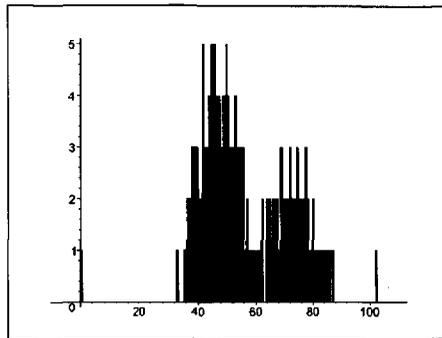
Figure 4.10: Distribution of eigenvalues of the two-part examples with  $N_1 = N_2$ .



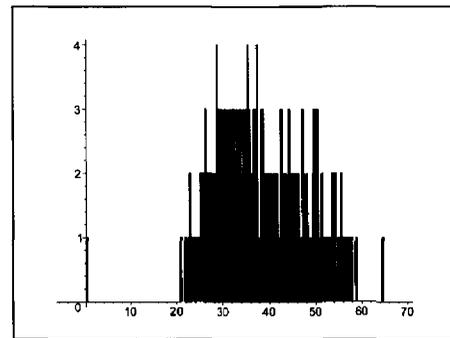
(a)  $p_i = 0.1, q_{12} = 0.9$



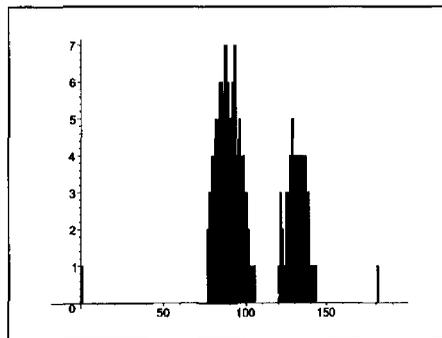
(b)  $p_i = 0.1, q_{12} = 0.8$



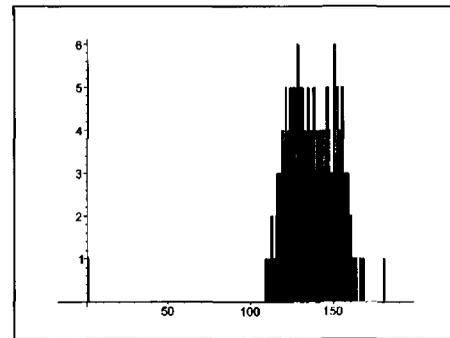
(c)  $p_i = 0.1, q_{12} = 0.5$



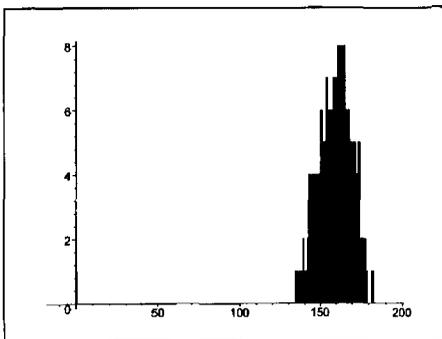
(d)  $p_i = 0.1, q_{12} = 0.3$



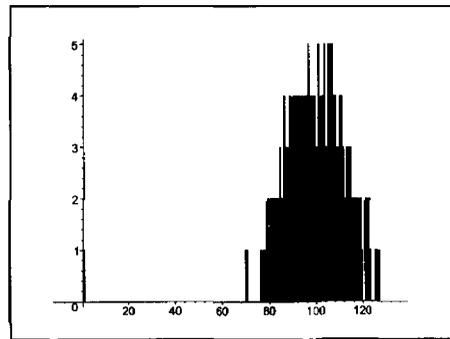
(e)  $p_i = 0.2, q_{12} = 0.9$



(f)  $p_i = 0.5, q_{12} = 0.9$

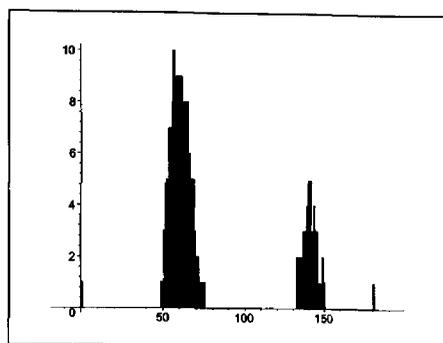


(g)  $p_i = 0.7, q_{12} = 0.9$

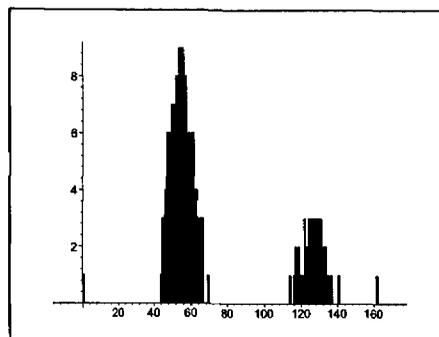


(h)  $p_i = 0.4, q_{12} = 0.6$

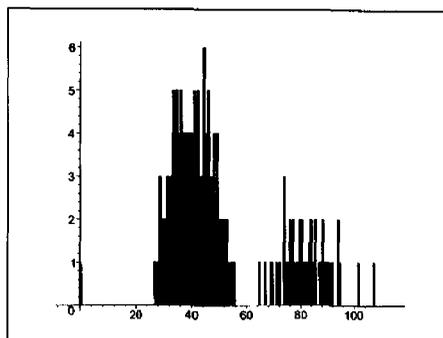
Figure 4.11: Distribution of eigenvalues of the two-part examples with  $N_1 : N_2 = 1 : 2$ .



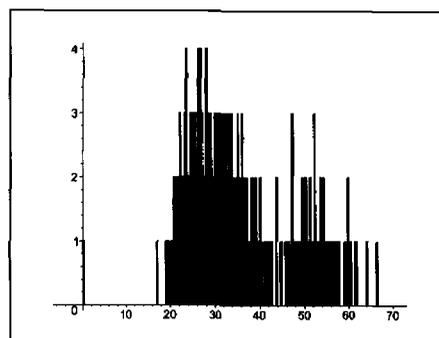
(a)  $p_i = 0.1, q_{12} = 0.9$



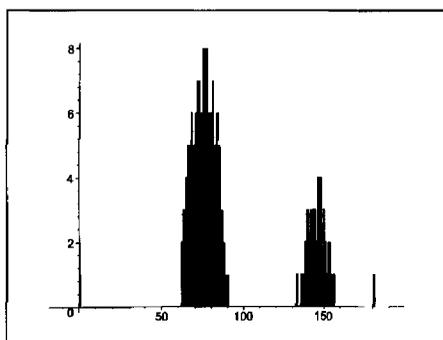
(b)  $p_i = 0.1, q_{12} = 0.8$



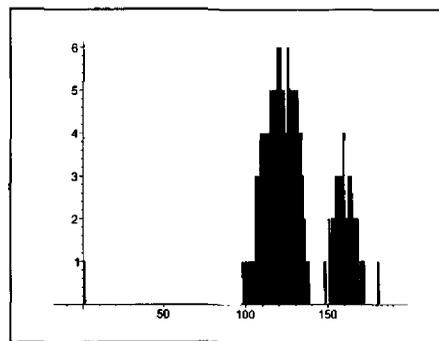
(c)  $p_i = 0.1, q_{12} = 0.5$



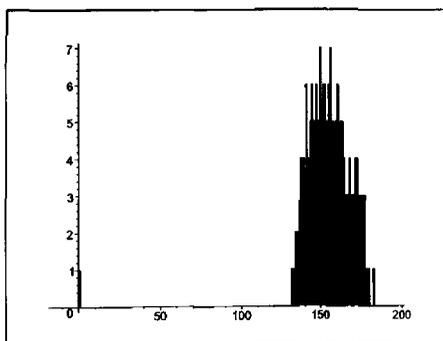
(d)  $p_i = 0.1, q_{12} = 0.3$



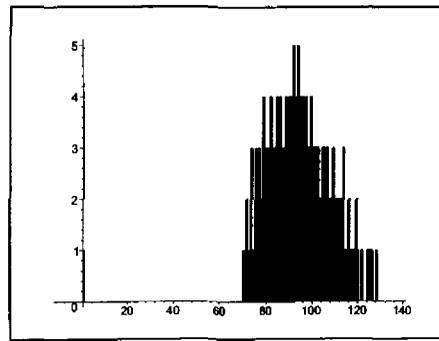
(e)  $p_i = 0.2, q_{12} = 0.9$



(f)  $p_i = 0.5, q_{12} = 0.9$

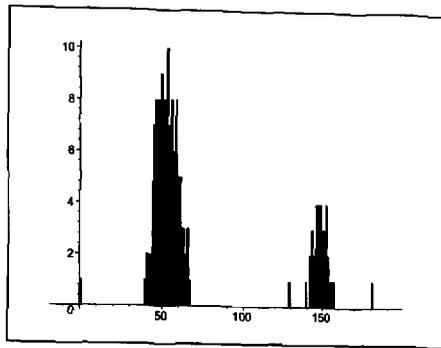


(g)  $p_i = 0.7, q_{12} = 0.9$

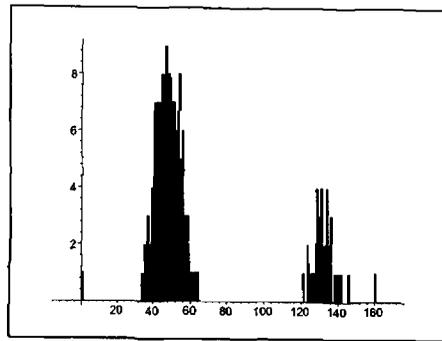


(h)  $p_i = 0.4, q_{12} = 0.6$

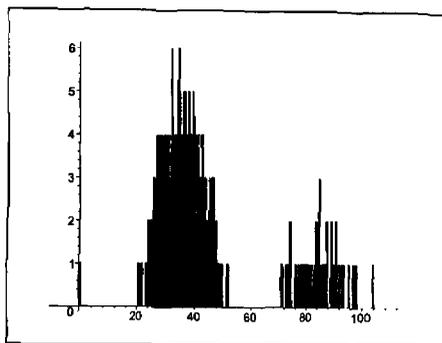
Figure 4.12: Distribution of eigenvalues of the two-part examples with  $N_1 : N_2 = 1 : 3$ .



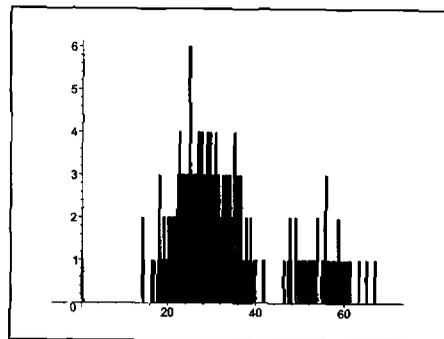
(a)  $p_i = 0.1, q_{12} = 0.9$



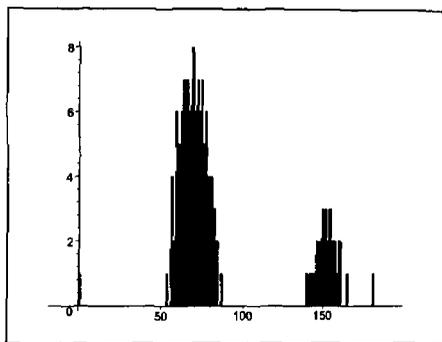
(b)  $p_i = 0.1, q_{12} = 0.8$



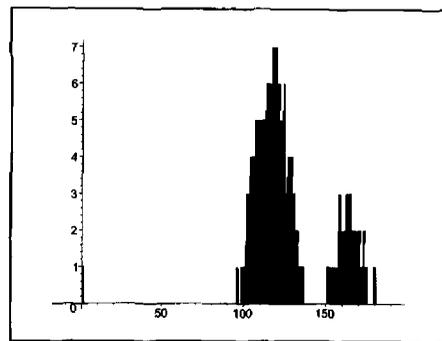
(c)  $p_i = 0.1, q_{12} = 0.5$



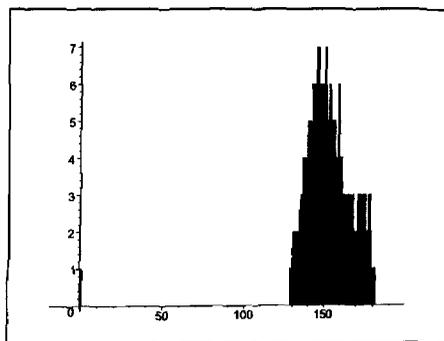
(d)  $p_i = 0.1, q_{12} = 0.3$



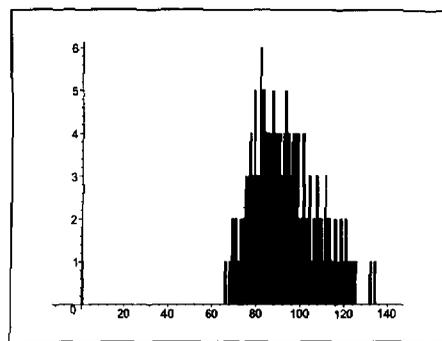
(e)  $p_i = 0.2, q_{12} = 0.9$



(f)  $p_i = 0.5, q_{12} = 0.9$



(g)  $p_i = 0.7, q_{12} = 0.9$



(h)  $p_i = 0.4, q_{12} = 0.6$

Figure 4.13: Distribution of eigenvalues of the two-part examples with  $N_1 : N_2 = 1 : 4$ .

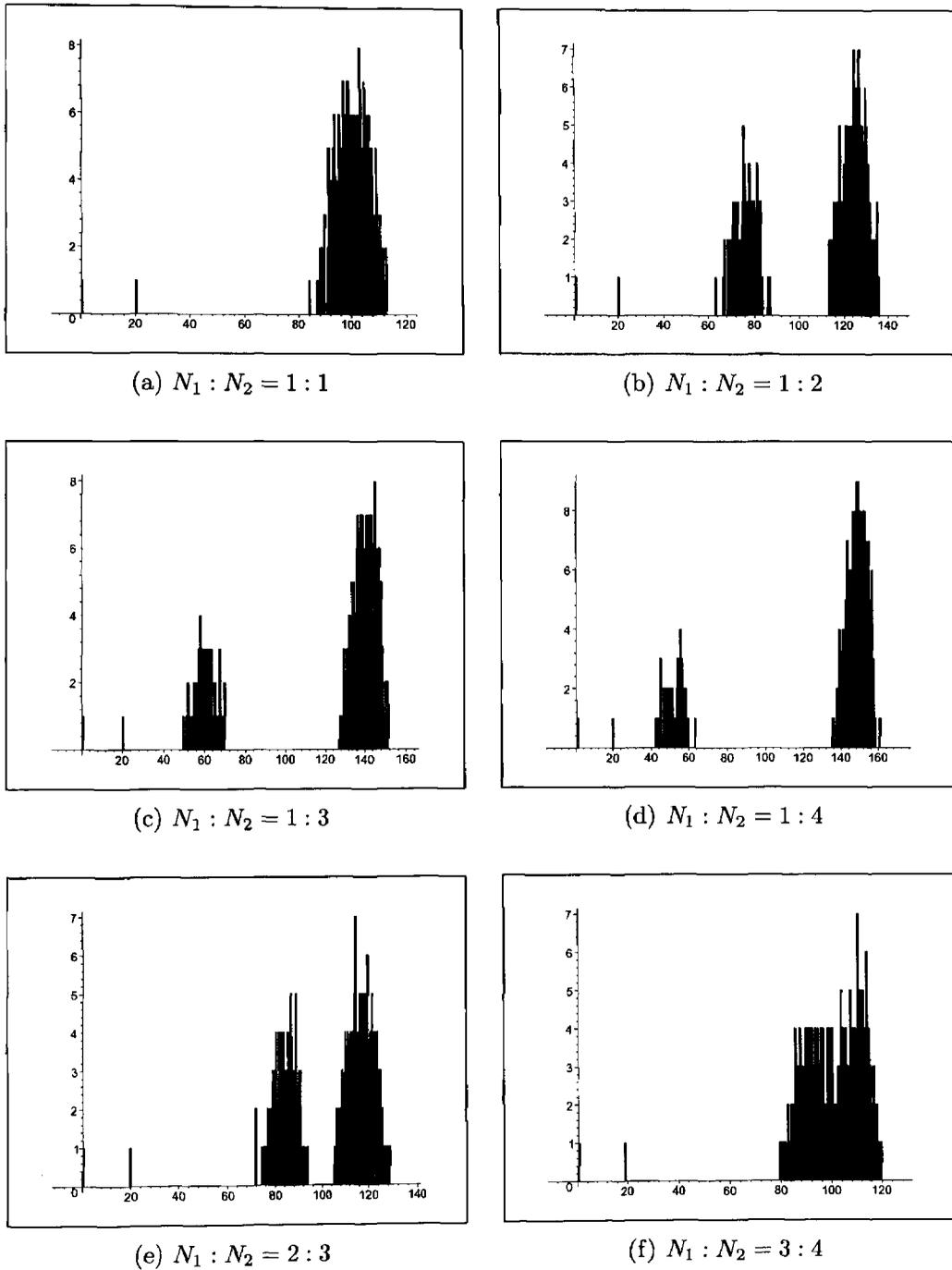


Figure 4.14: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.9$  and  $q_{12} = 0.1$ , where the sizes of the parts vary.

In the simulations presented earlier the eigenvalue of interest was the maximal eigenvalue. In the case where the parts are dense and the connections weak, we find that it is the smallest non-zero eigenvalue that has the noteworthy feature of being detached from the rest of the spectrum. The signs of eigenvector entries easily distinguish between the vertices that belong to the two different parts. Why would that be the case? The observation is not surprising, as it is in line with our knowledge about the relationship between eigenvalues of the ‘complementary’ edge-weighted graphs. Recalling Definition 3.2.3 and Theorem 3.2.4 we see that a random graph where the parameter  $p_i$  is greater than the parameter  $q_{12}$  is simply a complement to a graph  $G$ , presented in the previous subsection, where the parameter  $p$  was less than  $q$ . And as these are the graphs with all edges of weight 1 we have

$$p_i(\bar{G}) = 1 - p_i(G),$$

$$q_{12}(\bar{G}) = 1 - q_{12}(G).$$

We know the following relation between the eigenvalues: for each eigenvalue-eigenvector pair  $(\lambda, x)$  of  $G$  there is an eigenvalue  $N - \lambda$  in  $\bar{G}$  with the same corresponding eigenvector  $x$ . Hence, if in  $G$  there exists a lone maximal eigenvalue  $\lambda$  that is approximately equal to  $q_{12}(G)N$  with properties found in the previous section, then we must have a lone minimal eigenvalue  $N - q_{12}(G)N$ , which simplifies to  $q_{12}(\bar{G})N$ . Interestingly, the smallest positive eigenvalue is related to the probability parameter  $q$  in the same way as is the maximal eigenvalue in the simulations presented earlier, and is approximately equal to  $q_{12}(\bar{G})N$ .

Similarly, we see that the behaviour of the remaining eigenvalues shows the same pattern to the observations made in the previous section (compare Figures 4.7 and 4.14).

A special case in this group of simulations would be an example where  $p_i = 1$  and  $q_{12} < 1$ , representing a graph that consists of two cliques with edges between that describe their mutual influence.

### 4.2.3 Three-part examples with $p_i < q_{ij}$

Naturally we are interested to assess whether three-part graphs produced in the same manner as two-part graphs follow similar patterns. Analogously, the proba-

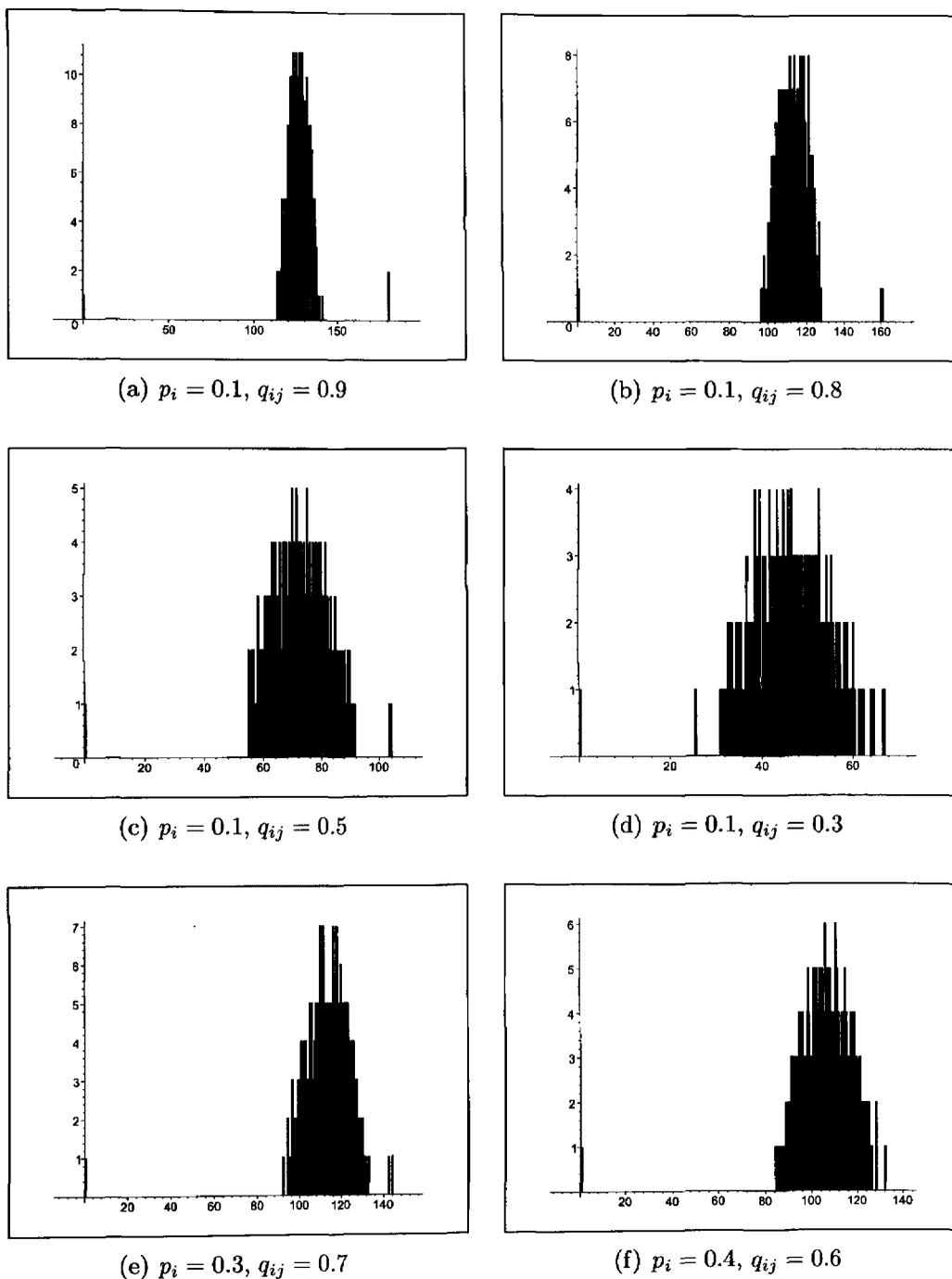


Figure 4.15: Distribution of eigenvalues of the three-part examples with variable edge occurrence probabilities and the parts of equal sizes  $N_1 = N_2 = N_3$ .

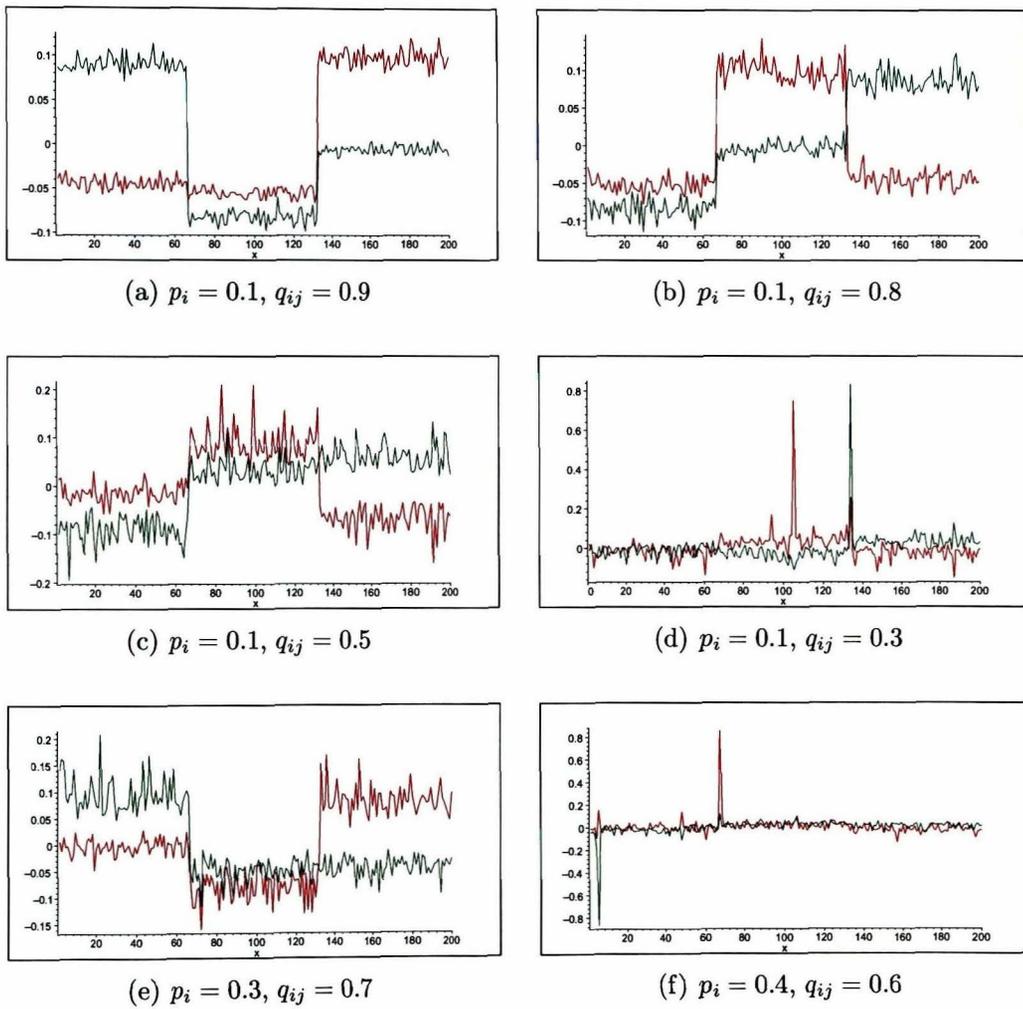


Figure 4.16: Eigenvectors' entries corresponding to the grouped two maximal eigenvalues depicted in Figure 4.15.

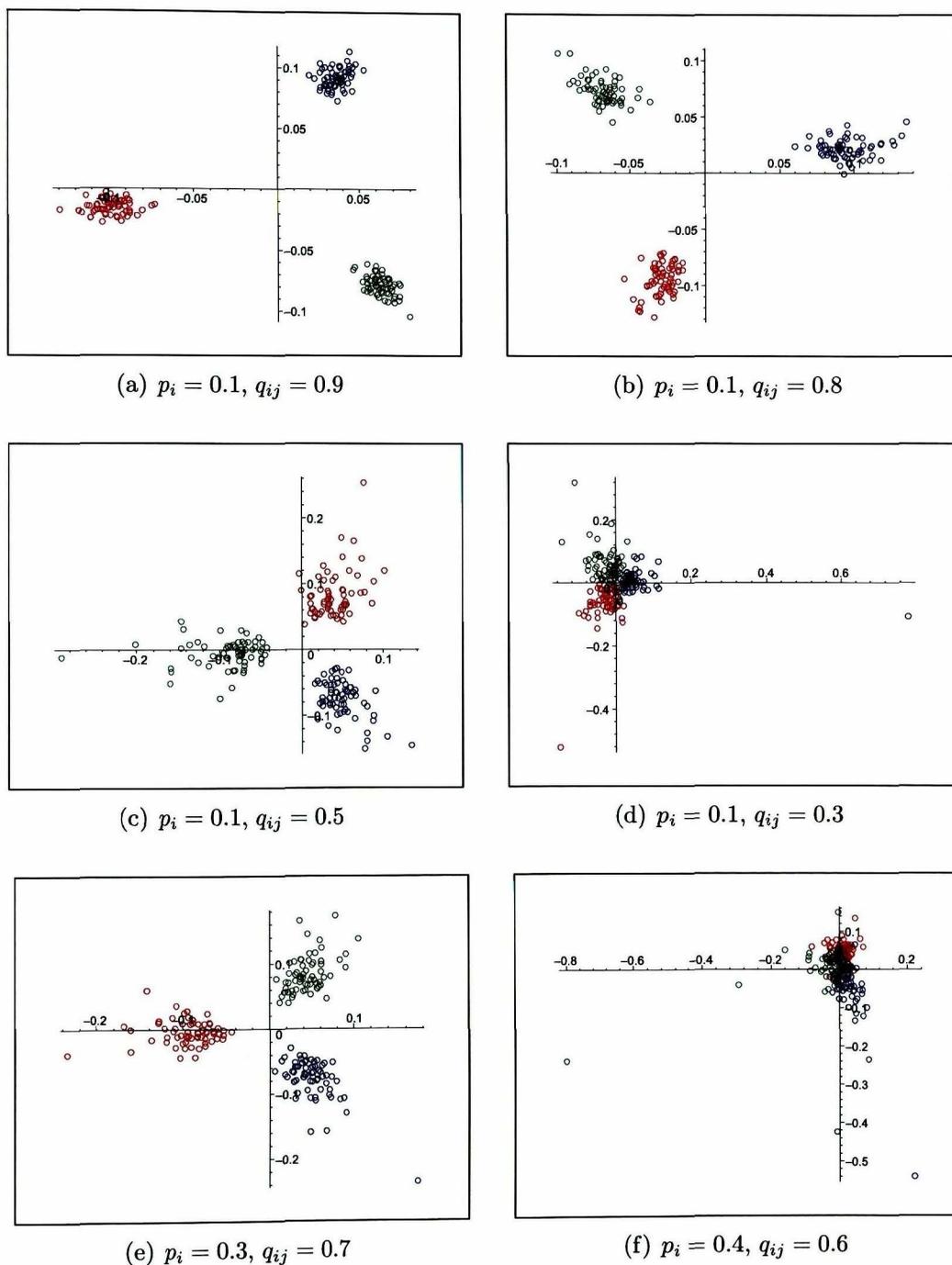
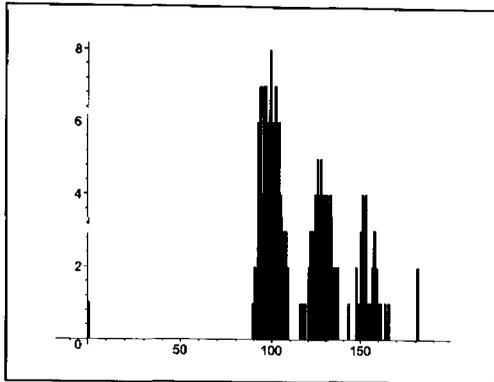
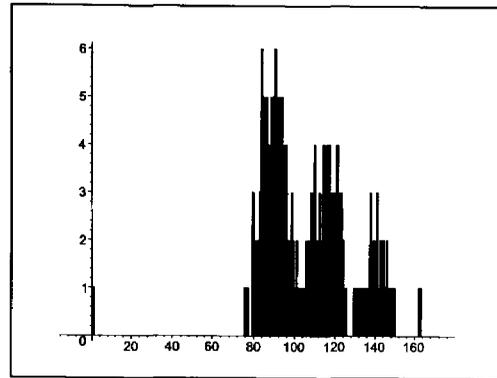


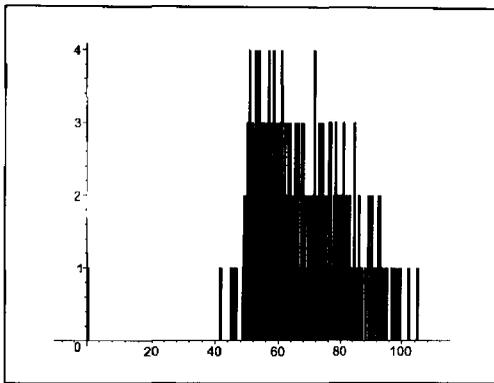
Figure 4.17: Eigenvectors' entries corresponding to the two grouped maximal eigenvalues depicted in Figure 4.15 presented in the plane.



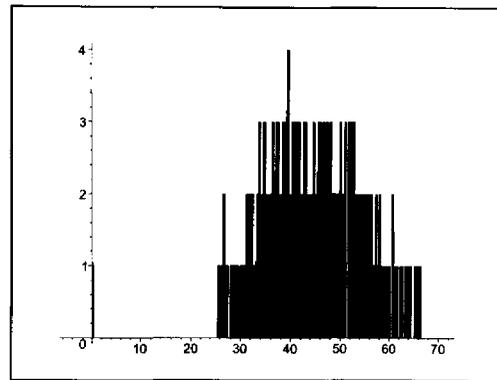
(a)  $p_i = 0.1, q_{ij} = 0.9$



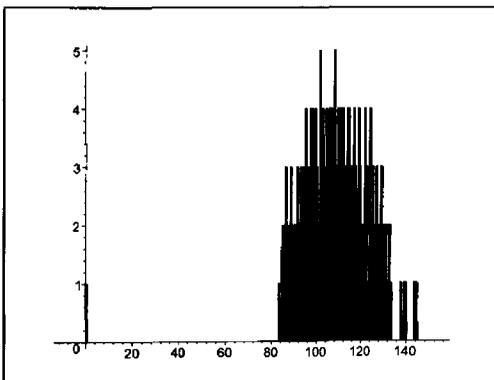
(b)  $p_i = 0.1, q_{ij} = 0.8$



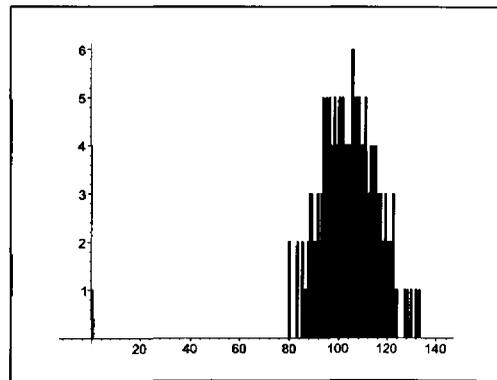
(c)  $p_i = 0.1, q_{ij} = 0.5$



(d)  $p_i = 0.1, q_{ij} = 0.3$



(e)  $p_i = 0.3, q_{ij} = 0.7$

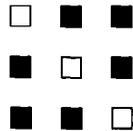


(f)  $p_i = 0.4, q_{ij} = 0.6$

Figure 4.18: Distribution of eigenvalues of the three-part examples with variable edge occurrence probabilities and  $N_1 : N_2 : N_3 = 1 : 2 : 3$ .

bility that an edge occurs in part  $i$  is here denoted by  $p_i$ , and the probability that an edge occurs between two distinct parts  $i$  and  $j$  is denoted by  $q_{ij}$  ( $i, j = 1, 2, 3$ ).

In this subsection we consider simulated graphs that consist of three parts whose parameters  $p$  and  $q$  are constant, i.e.,  $p_1 = p_2 = p_3 = p$  and  $q_{12} = q_{13} = q_{23} = q$ , and all generated edges are of weight 1. Furthermore, we assume that the graphs consist of three ‘weak’ parts of sizes  $N_1, N_2$  and  $N_3$ , connected by strong links:



Thus, we have  $p_i < q_{ij}$  for all values of  $i$  and  $j$ ,  $i \neq j$ . It is apparent from Figure 4.15 that we now have two maximal eigenvalues that stand apart from the rest of the eigenvalues. As the the difference between the values  $p$  and  $q$  decreases, the distance between these two maximal eigenvalues and the rest of the spectrum reduces. Despite the qualitative similarity in the presented simulations, in Figure 4.15(a) we can observe that the two largest eigenvalues are distant from the rest of the spectrum, whereas in plots (d) and (f) of Figure 4.15 there is no such separation.

When the two maximal eigenvalues are distinguishable from the rest of the spectrum, the eigenvectors play an important role in recognising the vertices of the parts. It is no longer enough to rely on the sign of the entries of one of the eigenvectors only. It appears that the specific clustering of the ordered pairs made from the corresponding entries produces the partitioning of the graph into exactly the three parts considered. This is empirically confirmed for the examples in Figure 4.16. An alternative way of revealing the three parts is given in the plots in Figure 4.17.

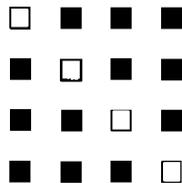
As expected, the two maximal eigenvalues in plots (d) and (f) cannot be used to reconstruct the partition of the considered graph into three predefined parts. A logical explanation is that with the decrease in the difference between the values of  $p$  and  $q$ , the graph itself cannot be seen *any longer* as a valid example of a three-part graph formed of weak parts with strong interconnecting links. Rather, it turns into a graph whose global properties resemble the properties of the individual parts.

As the changes in the sizes of the parts are implemented we note the changes in

the spectrum. The more pronounced the difference between the sizes is, the larger the widening of the spectrum. In the case of very weak parts and very strong connections we can see three clearly different grouped subsets of the Laplacian eigenvalues. Figure 4.18 shows eigenvalue histograms of graphs with different edge probability parameters and ratio between the parts  $N_1 : N_2 : N_3 = 1 : 2 : 3$ .

#### 4.2.4 Four-part examples with $p_i < q_{ij}$

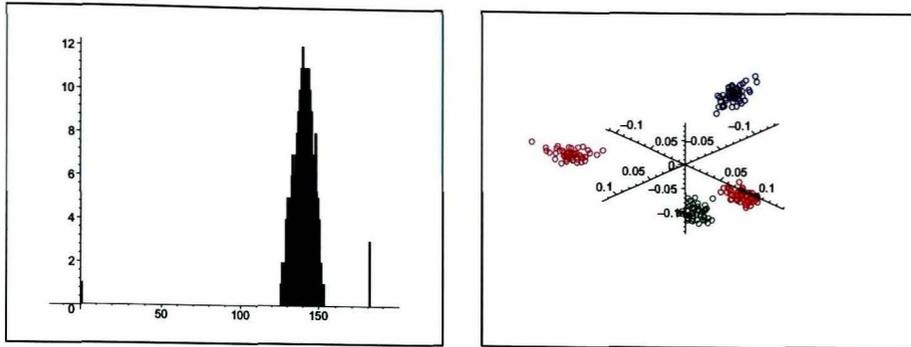
Similarly to two and three-part graphs, we assume that here the graph consists of four weak parts of sizes  $N_1, N_2, N_3$  and  $N_4$ , connected by strong relations. Again, we keep the parts and connecting parameters constant:  $p_i := p$  and  $q_{ij} := q$ , for all  $i$ 's and  $j$ 's,  $i \neq j$ , and assume that all generated edges are of weight 1.



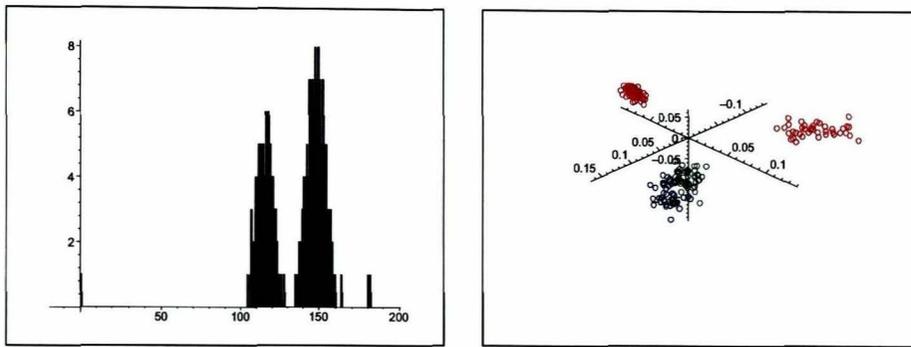
The most noteworthy features of two and three part simulated examples are evident here as well. We show them by presenting a few examples where values  $p$  and  $q$  are fixed, while we let the sizes of the four parts vary.

Figure 4.19 shows that for a graph where the weak parts ( $p = 0.1$ ) strongly influence each other ( $q = 0.9$ ), regardless of the chosen sizes of the parts, we can see the three largest eigenvalues grouped and away from the other eigenvalues. Interestingly, if we represent each vertex  $i$  of  $G$  by an ordered triple, with entries that correspond to the  $i$ -th entries of the three eigenvectors, we see the clear division of the vertices (shown in 3-dimensional plots in Figure 4.19). The different point clusters correspond to different weak parts of graph  $G$ .

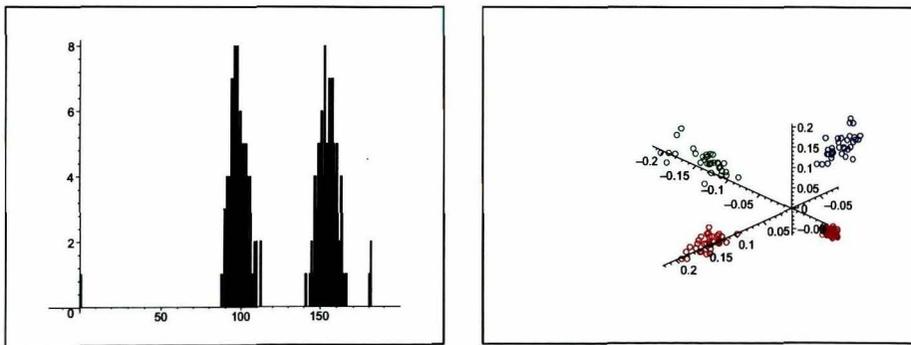
Although not included in this section we also find, as expected, that as the difference between the values  $p$  and  $q$  decreases so does the distance of the three maximal eigenvalues from the rest of the spectrum. Likewise, by observing the rest of the spectrum we find more properties similar to those found in simulations for two and three part regular graphs.



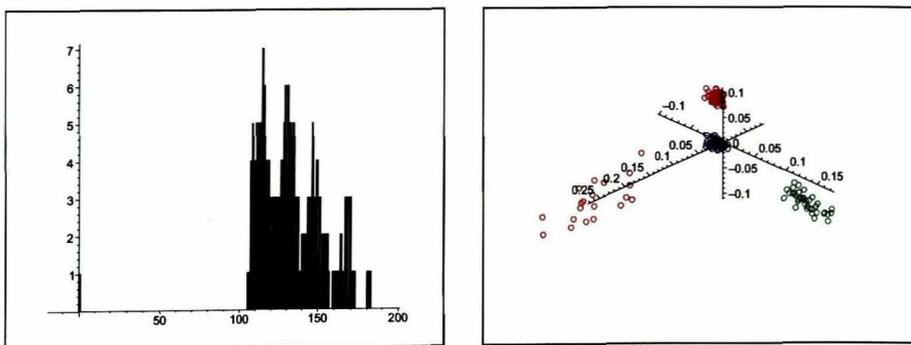
(a)  $N_1 : N_2 : N_3 : N_4 = 1 : 1 : 1 : 1$



(b)  $N_1 : N_2 : N_3 : N_4 = 1 : 1 : 1 : 2$



(c)  $N_1 : N_2 : N_3 : N_4 = 1 : 1 : 1 : 3$



(d)  $N_1 : N_2 : N_3 : N_4 = 1 : 2 : 3 : 4$

Figure 4.19: Four-part examples with variable sizes of the parts: the distribution of their eigenvalues and eigenvectors' entries of the three grouped maximal eigenvalues in three dimensions.

### 4.2.5 Edge weighted graphs

So far we have looked at examples where all edges in a graph are of the same normalised weight  $w = 1$ , where we are able to easily follow changes in the spectrum related to the implementation of parameter changes (changes in part sizes, and changes in edge densities of subgraphs and connections).

Undoubtedly, the next parameter change of major interest to us is allowing for varied edge weights. As higher weight allocated to an edge enhances its ‘influence’, we expect to see more complex behaviour of the eigenvalue spectrum. We can argue that an increase of edge weights in a subgraph would affect the influence or the importance of the subgraph within a graph. Previously less pronounced parts could become ‘dominant’ ones, with edge weighting as an additional parameter that attributes to and determines their importance. Some aspects of this proposition are explored in this section.

In order to carry out observations on edge-weighted graphs referred to above, we use the same framework as described in previously presented simulations. We assume that the underlying structure of a graph is such that the density of the edges within the different parts (parameter  $p$ ) is equal for each one of them, and that the same holds for the occurrence of edges between the parts (parameter  $q$ ). The two values, however, can be different.

#### Two-part graphs

We recall the reference examples of the simple random two-part graph, where we had assigned  $p = 0.1$ ,  $q = 0.9$ , while the ratio between the sizes of the parts varied (see Figure 4.7).

The aim of simulations presented in this section is to observe how the features of the shape of the eigenvalue spectrum change as varied edge weights are assigned in a graph. For the purpose of easier comparison, we look at the changes of all parameters, but as previously allow only one change at the time. We focus on two distinct cases: What changes to the eigenvalue spectrum are created with the introduction of varied edge weights within graph parts? What changes are created with the introduction of varied weights to the edges connecting the parts?

Even though it is possible to apply complicated combinations of parameters, we

believe that the relatively simple analysis presented in this section is useful to understand eigenvalue histogram shapes.

We start with the set up as in the reference two-part examples of simple graphs, where  $p = 0.1, q = 0.9$  (see Figure 4.10). Next, in Figures 4.20 and 4.21 we show examples where all existing edges in the parts are of weights  $w = 2$  and  $w = 3$ , respectively. The presence of the maximal eigenvalue is apparent in all plots regardless of the size of parts. Interestingly, the maximal eigenvalue is in all cases approximately the same, and similar to the value of the maximal eigenvalue in the simple graph examples.

The maximal eigenvalue which is approximately equal to  $qN$  is clearly independent from changes applied to the edge weights of the parts. However, when we set all edges in the parts in these simulations to be of higher weight  $w = 5$ , say, we no longer see a detached maximal eigenvalue. Instead, we notice an overall grouping and a shift of all the eigenvalues in the spectrum to the right, with a number of eigenvalues moving beyond the value  $qN$  (Figure 4.22).

Next, we consider the changes in the parameters of the connecting edges. The eigenvalue histogram plots showing the effect of edge weights increase to  $w = 2$  and  $w = 3$  are presented in Figures 4.24 and 4.25, respectively.

Comparing these plots to the simple two-part graph examples shown in Figure 4.7 we note an evident shift of the non-zero part of the eigenvalues spectrum to the right.

Let us concentrate on a simulated graph with given parameters, where  $N_1 : N_2 = 1 : 4$ . Beside the largest eigenvalue, we see two grouped sets of eigenvalues. The smaller in value group counts  $N_2 - 1$  eigenvalues, and the larger counts  $N_1 - 1$  of them. We seek to assess how these two groups of eigenvalues behave with the change in the weights of all the edges connecting the two considered parts. We keep two cases separate: in one we have the connecting edges change their edge weight from 1 to 2, and in the other the edge weight changes from 1 to 3. In absolute terms, in the first instance we have the edge weights increased by one unit and in the second by two units.

With no other changes implemented the eigenvalues in the former group increase by approximately  $qN_2$  in the first instance and  $2qN_2$  in the second, while the ones in the latter group increase by approximately  $qN_1$  in the first instance and  $2qN_1$

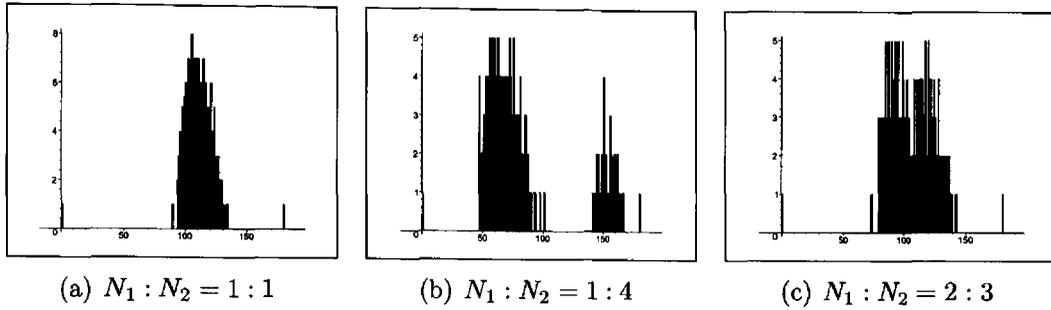


Figure 4.20: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.1$ ,  $p_1^{(2)} = p_2^{(2)} = 1$ , and  $q_{12} = 0.9$ , where the sizes of the parts vary.

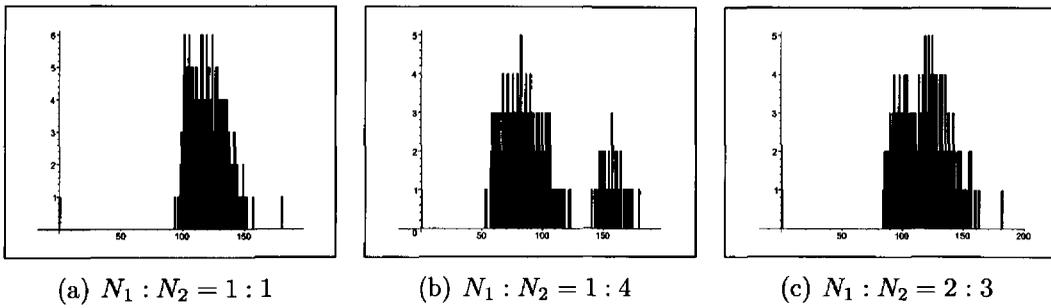


Figure 4.21: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.1$ ,  $p_1^{(3)} = p_2^{(3)} = 1$ , and  $q_{12} = 0.9$ , where the sizes of the parts vary.

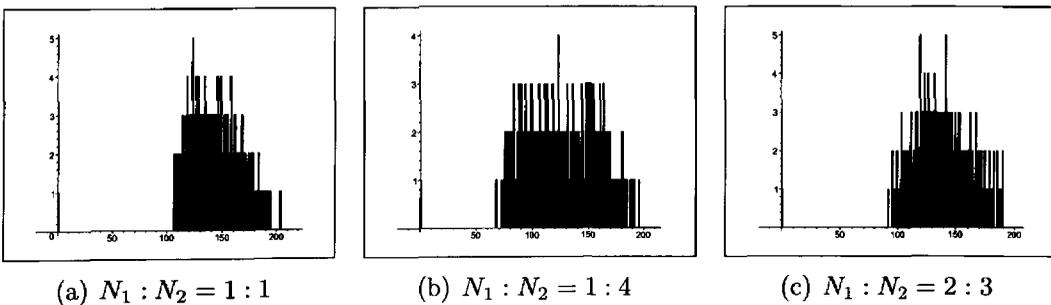


Figure 4.22: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.1$ ,  $p_1^{(5)} = p_2^{(5)} = 1$  and  $q_{12} = 0.9$ , where the sizes of the parts vary.

in the second. Additionally, the change in the value of the largest eigenvalue is around  $qN$  and  $2qN$ , respectively (see Figures 4.23, 4.24 and 4.25).

Recalling Theorem 3.5.6 and considering an ideal case of a complete bipartite graph  $K_{N_1, N_2}^{(w_{12})}$ , we note that a similar situation occurs here. An increase of weights of connecting edges by  $w$  shifts the eigenvalue spectrum to the right as follows: the small non-zero eigenvalue of multiplicity  $N_1 - 1$  increases by  $wN_2$ , the one of multiplicity  $N_2 - 1$  increases by  $wN_1$ , while the maximal eigenvalue increases by  $wN$ .

With the examples of simulations presented we have confirmed the understanding of effects of changes in the weights of the parts and connecting edges on the shape of the eigenvalue histogram.

We include one more set of simulations where we focus on changes in parameters of the parts, whilst remembering their relation to the parameters in connecting edges. In particular, we start with an example described earlier, where there are few edges, all of high weight in the parts ( $p = 0.1$ ,  $p^{(5)} = 1$ ), connected strongly with edges of lower weight ( $q = 0.9$ ,  $q^{(5)} = 1$ ). We observe what happens to the spectrum when the density of the weighted edges within the parts increases, i.e., all parameters remain unchanged apart from the value  $p$ . In Figures 4.26-4.28 we show the spectrum changes as  $p$  takes values 0.3, 0.5 and 0.9, respectively. We now have the smallest positive eigenvalue as the one that stands out, a result which is both intuitive and in line with the theoretical work presented.

### Three, four and $k$ -part edge-weighted graphs

Analogously, we can consider simulated graphs with more parts. We show examples with three and four parts. The parameters  $p$  and  $q$  are chosen to be as above:  $p_i = 0.1$  for all parts  $i$  and  $q_{ij} = 0.9$  for any two different parts  $i$  and  $j$ . The first three-part and four-part graphs of this type we looked at were simple graphs and their algebraic properties are presented in Figure 4.15(a) and Figure 4.19(a), respectively. The plots of regular varied edge-weighted simulations are shown in Figures 4.29-4.30 for three-part and Figures 4.31-4.32 for four-part graphs.

As in the case of two-part edge-weighted graphs, we find that Theorem 3.5.6 and the more general case, a complete  $k$ -partite graph  $K_{N_1, \dots, N_k}^{(w_{ij})}$ , are useful in understanding the features of the shape of the spectrum.

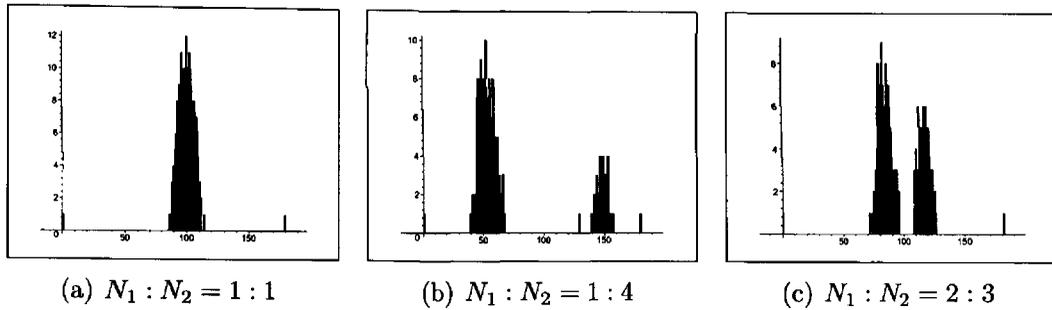


Figure 4.23: Distribution of eigenvalues of the two-part examples with parameters  $p_i^{(1)} = 0.1$  and  $q_{12}^{(1)} = 0.9$ .

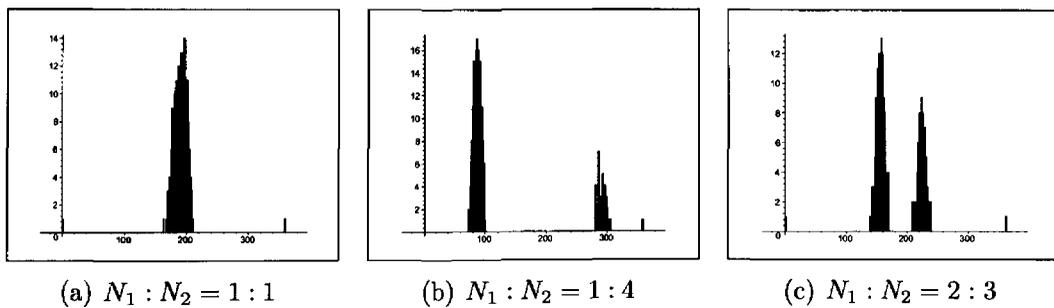


Figure 4.24: Distribution of eigenvalues of the two-part examples with parameters  $p_i^{(1)} = 0.1$  and  $q_{12}^{(2)} = 0.9$ .

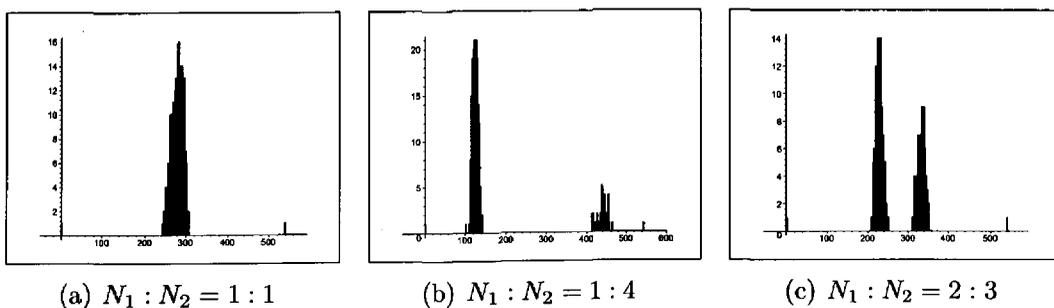


Figure 4.25: Distribution of eigenvalues of the two-part examples with parameters  $p_i^{(1)} = 0.1$  and  $q_{12}^{(3)} = 0.9$ .

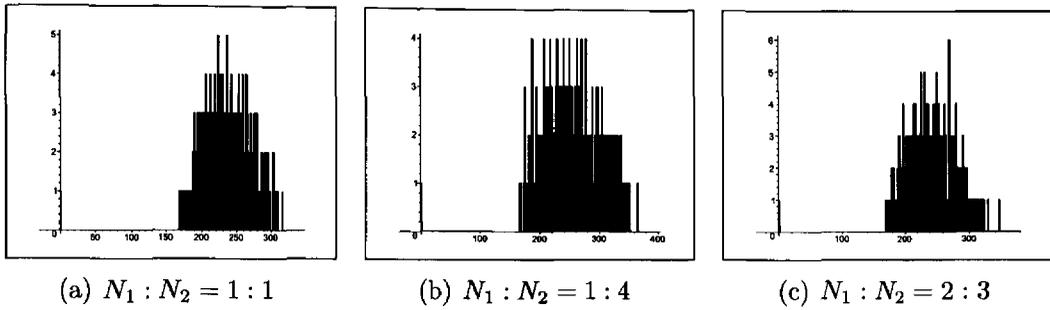


Figure 4.26: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.3$ ,  $p_1^{(5)} = p_2^{(5)} = 1$  and  $q_{12} = 0.9$ , where the sizes of the parts vary.

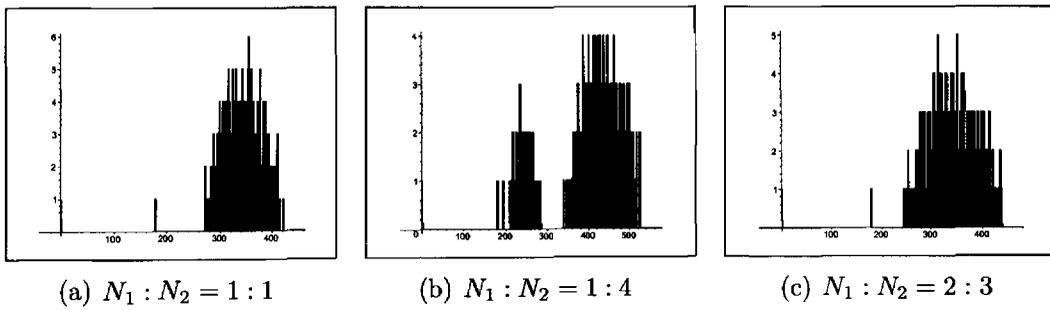


Figure 4.27: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.5$ ,  $p_1^{(5)} = p_2^{(5)} = 1$  and  $q_{12} = 0.9$ , where the sizes of the parts vary.

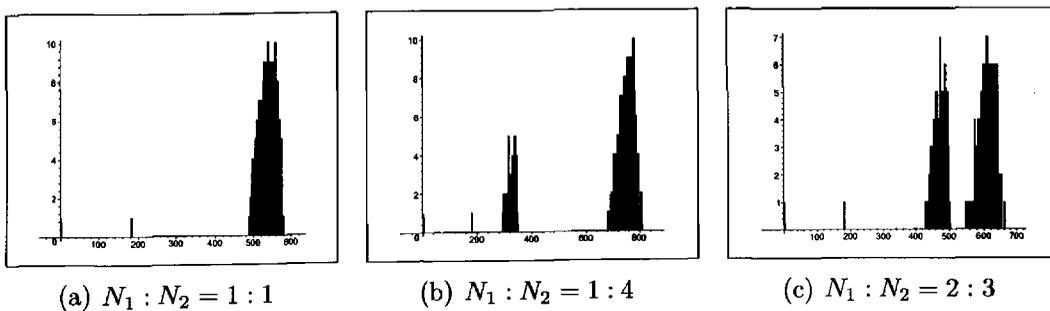


Figure 4.28: Distribution of eigenvalues of the two-part examples with parameters  $p_1 = p_2 = 0.9$ ,  $p_1^{(5)} = p_2^{(5)} = 1$  and  $q_{12} = 0.9$ , where the sizes of the parts vary.

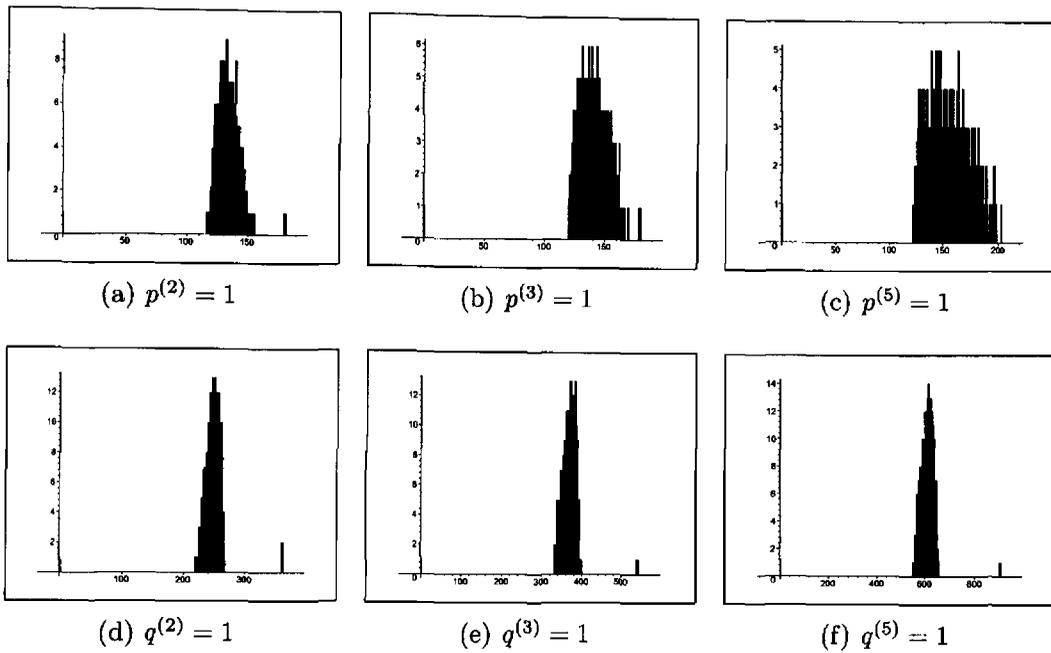


Figure 4.29: Distribution of eigenvalues of the three-part examples with parameters  $p = 0.9$ ,  $q = 0.1$ , equal sizes of the parts, and varied edge weights in parts or connecting edges. Unless specified otherwise  $p^{(1)}$  and  $q^{(1)}$  are assumed to be 1.

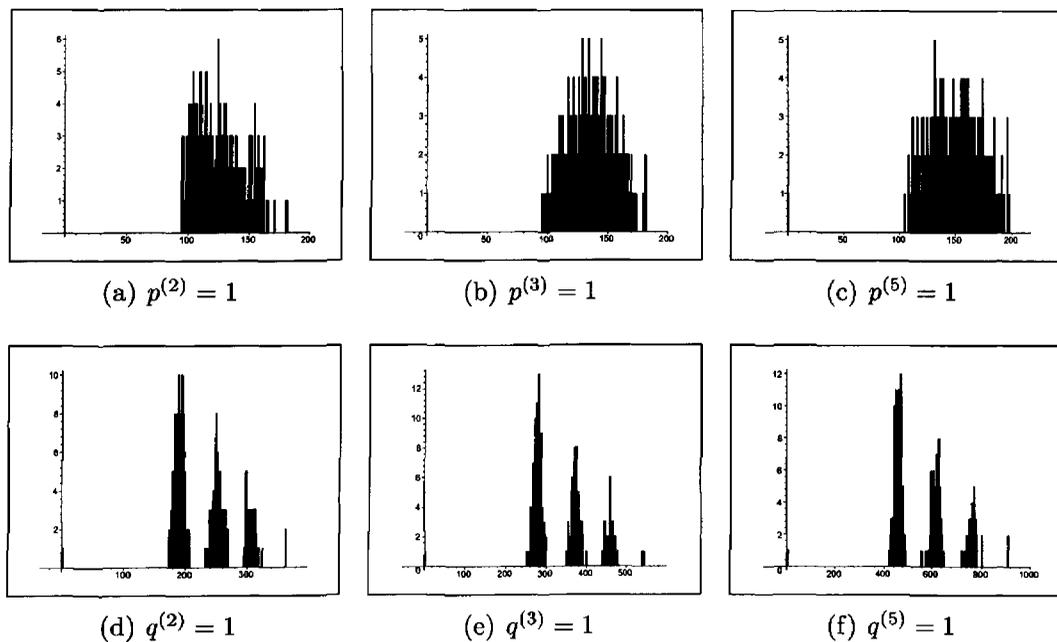


Figure 4.30: Distribution of eigenvalues of the three-part examples with parameters  $p = 0.9$ ,  $q = 0.1$ ,  $N_1 : N_2 : N_3 = 1 : 2 : 3$ , and varied edge weights in parts or connecting edges. Unless specified otherwise  $p^{(1)}$  and  $q^{(1)}$  are assumed to be 1.

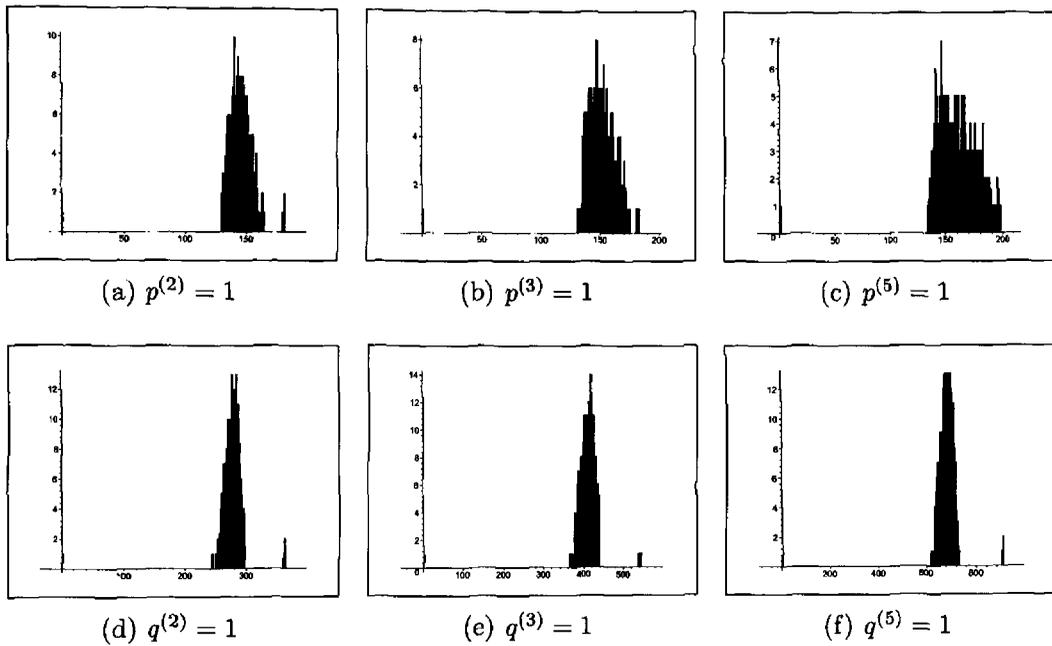


Figure 4.31: Distribution of eigenvalues of the four-part examples with parameters  $p = 0.9$ ,  $q = 0.1$ , equal sizes of the parts, and varied edge weights in parts or connecting edges. Unless specified otherwise  $p^{(1)}$  and  $q^{(1)}$  are assumed to be 1.

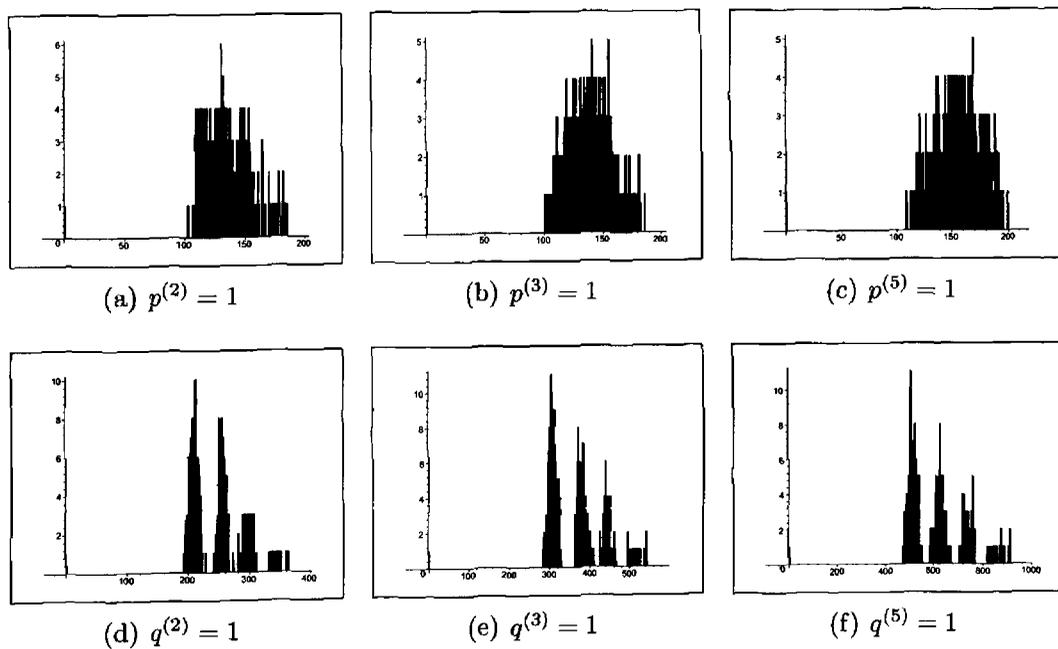


Figure 4.32: Distribution of eigenvalues of the four-part examples with parameters  $p = 0.9$ ,  $q = 0.1$ ,  $N_1 : N_2 : N_3 : N_4 = 1 : 3 : 5 : 7$ , and varied edge weights in parts or connecting edges. Unless specified otherwise  $p^{(1)}$  and  $q^{(1)}$  are assumed to be 1.

### 4.3 Graphs with ‘Irregular $k$ -part’ Underlying Structure

As shown in the previous section, a way to present an edge-weighted graph with ‘ $k$ -part’ underlying structure is to provide information on the order and the type of all subgraphs that correspond to the  $k$  parts of the structure, and to describe their mutual connections within the graph. We defined the parameters  $p_i^{(w)}$  and  $q_{ij}^{(w)}$  that correspond to the parts and the connecting edges, respectively. While we held these parameters constant and referred to such graphs as ‘regular generated  $k$ -part graphs’, we now move away from regularity by relaxing that requirement. In other words, we no longer insist on the parts being of the same type, and hence we describe each of the parts independently. In the same manner we treat the connections between the parts.

Relaxing the regularity constraint signifies that we have freedom to go in a number of directions in generating interesting examples. However, we show only a few, remain focused on parts’ ‘importance’ and the ‘strength’ of the inter-parts connections, and observe what effect these properties have on the shape of the graph’s eigenvalue spectrum. For easier presentation, within the same graph we denote dense subgraphs by ‘■’, ones of medium density by ‘⊕’, and the least dense by ‘□’.

#### 4.3.1 Graphs with two parts

**Example 4.3.1.** For convenience we start with the reference examples defined while analysing regular  $k$ -part simulated examples. We recall that the parameters in the reference examples are as follows:  $p_i = 0.1$  for all parts  $i$ , and  $q_{ij} = 0.9$  for any two different parts  $i$  and  $j$ . There is no requirement regarding the parts’ size. We start moving towards an irregular  $k$ -part simulated graph in steps, by initially keeping the connecting parameters constant, while relaxing the condition for the parts’ parameters. We look at the simulated graphs with two-part underlying structure, where one dense part is strongly connected with a non-dense part of equal size. Informally we represent examples of such graphs pictorially as shown

below. The first graph is a regular reference example.



The histogram of eigenvalues and the entries of the significant eigenvectors are shown in Figures 4.33 and 4.34, respectively. In the first two examples we have two relatively weak parts strongly connected, captured by the existence of a detached maximal eigenvalue (of value close to  $q_{12}N$ ) whose eigenvector entry signs clearly distinguish between the parts' vertices. On the other hand, the two histograms differ in the positions of the remaining non-zero eigenvalues. Both examples consist of two parts of equal size, where in the first simulation the parts are of the same type, and in the second simulation one part is of higher density. The former, therefore, has all of the eigenvalues grouped at one place, while the latter has half of the eigenvalues grouped at a different, larger value. As the density of a part grows, the corresponding grouped eigenvalues move even further to the right, and the previously detached maximal eigenvalue feature disappears (see Figure 4.33(c)-(d)). The larger the difference between the values  $p_1$  and  $p_2$  the larger the distance between the two groups. Observing the eigenvectors that correspond to the two distinct groups of eigenvalues, we find that they point to the different parts: the group on the right points to the part of smaller edge density, whilst the group on the left points to the other part. The smaller in value group is depicted in Figure 4.34(c)-(d).

**Example 4.3.2.** The next set of examples of the two-parts graphs of equal size focuses on observing the eigenvalue spectrum as we implement changes in properties of connecting edges. As previously we begin with one dense part strongly connected with a non-dense part, and then gradually “weaken” the connecting edges by decreasing the value of the parameter  $q_{12}$ . We consider two such sets of examples:



In the former three simulated graphs we keep the parameters  $p_1 = 0.9$  and  $p_2 = 0.1$  unchanged, but have the parameter  $q_{12}$  take three different values in decreasing order: 0.9, 0.5 and 0.1.

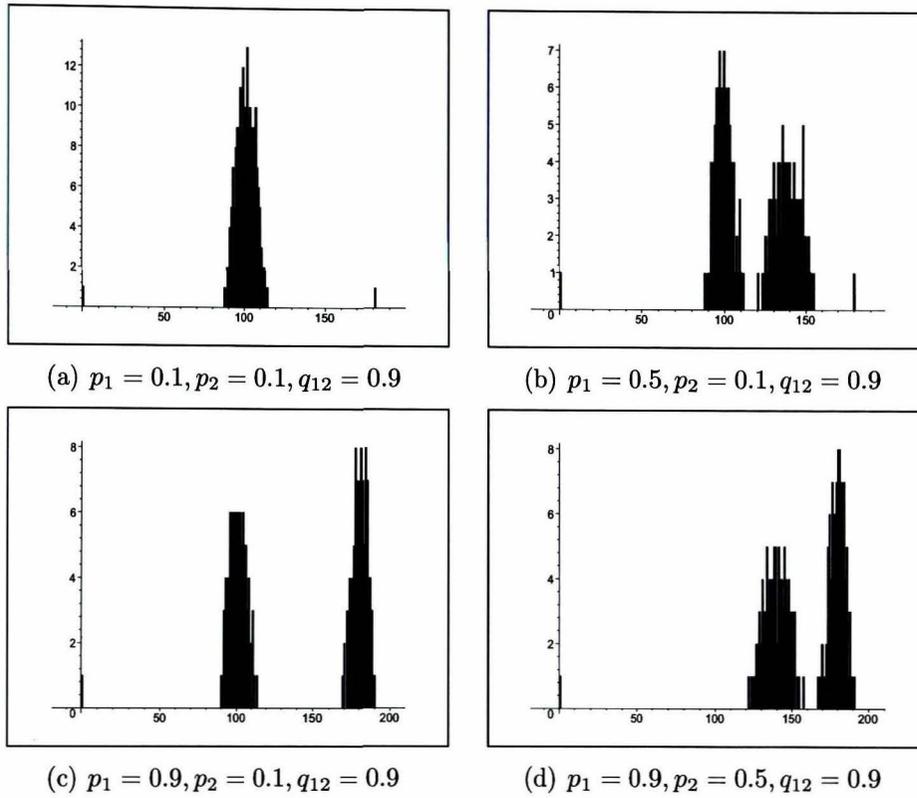


Figure 4.33: The simulated graphs with irregular two-part underlying structure described in Example 4.3.1.

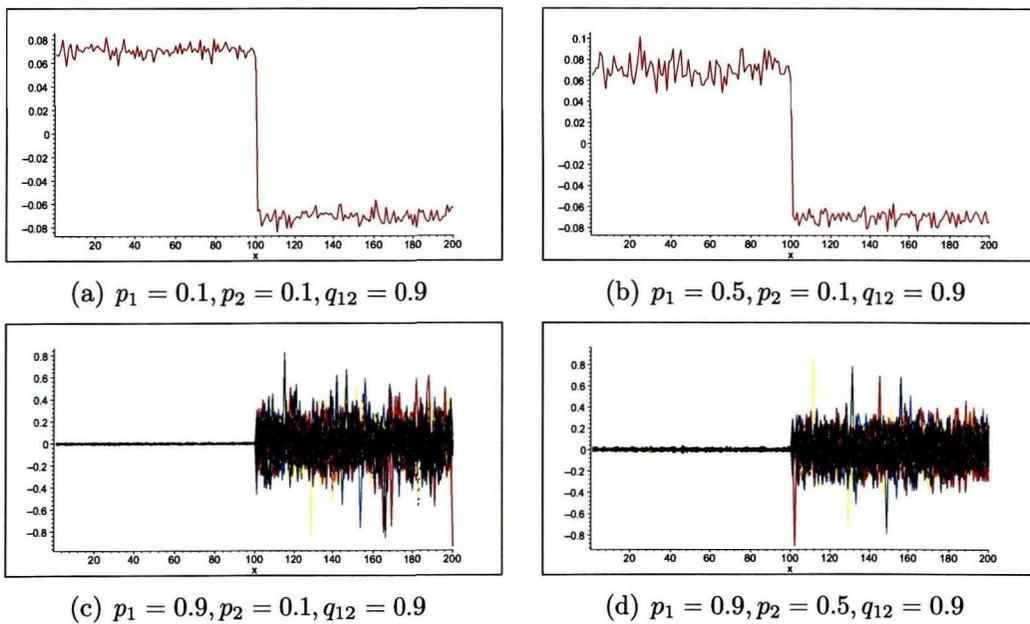


Figure 4.34: Entries of the interesting eigenvectors noted in the eigenvalue spectrum in Figure 4.33: Plots (a) and (b) correspond to the maximal eigenvalue, whilst plots (c) and (d) correspond to the left group of eigenvalues in the spectrum.

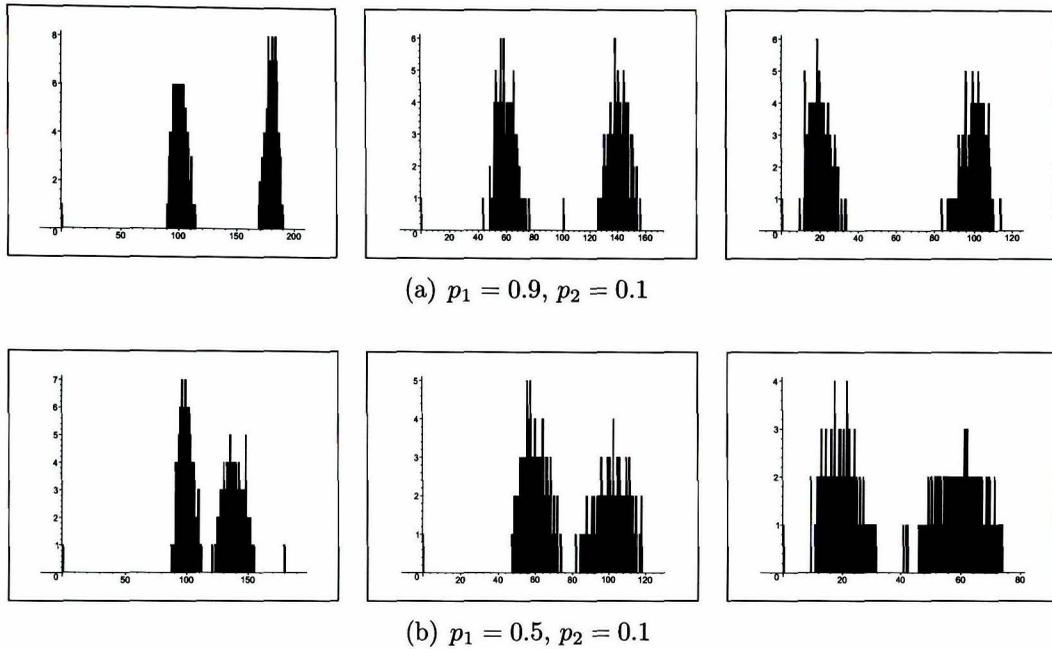
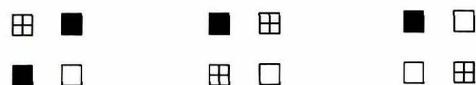


Figure 4.35: Irregular two-part graphs described in Example 4.3.2 with varying connecting parameter  $q_{12}$  that takes values 0.9, 0.5 and 0.1, respectively.

We do the same in the latter three examples, where we set the parameters  $p$  to be as follows:  $p_1 = 0.5$  and  $p_2 = 0.1$ . As discussed earlier, when two parts of equal size are of different edge density (parameter  $p$ ) the eigenvalues tend to be split in two groups. That remains true regardless of the density of the connecting edges. However, we note that there is an obvious shift of all the eigenvalues in the spectrum to the left as the parameter  $q_{12}$  reduces, while the distance between the groups remains unaffected (see the corresponding plots in Figure 4.35).

**Example 4.3.3.** Beside the already discussed properties in the presented plots we notice a couple of examples where a stand-alone eigenvalue occurs. Among the examples we note that each has all three parameters of different value 0.1, 0.5 or 0.9, where the relation among the parameters is as follows. In the first graph we have  $q_{12} > p_1 > p_2$ , in the second  $p_1 > q_{12} > p_2$ . Without a particular importance in their order, we display them together with one more example, in which  $p_1 > p_2 > q_{12}$  (the parameters  $p$  and  $q$  are, for consistency, chosen to be such that the first part is more dense than the other). They can be sketched as follows:



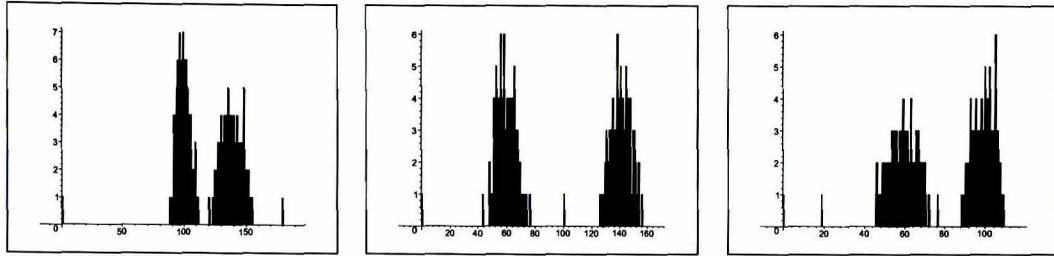


Figure 4.36: Distribution of eigenvalues of irregular two-part graphs with varying parts and connecting parameters  $p_i$  and  $q_{12}$ .

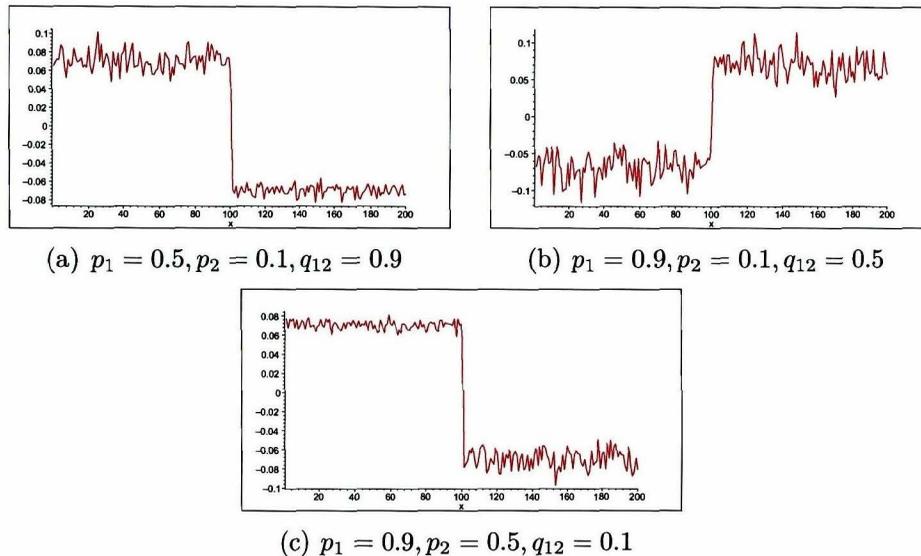


Figure 4.37: Eigenvectors corresponding to the detached eigenvalues in histograms presented in Figure 4.36.

Their eigenvalue histograms are presented in Figure 4.36. We immediately notice that beside the two sets of grouped eigenvalues we have a non-zero stand-alone eigenvalue of the approximate value  $q_{12}N$  in each three of the cases. In the first example where the connecting parameter is the dominant one, the eigenvalue is maximal. The second example is such that the connections between the strong and the weak part are of medium strength. The stand-alone eigenvalue is in this case between the two groups of eigenvalues. Finally, in the third case, where the connecting parameter is the smallest of the three, we see that the stand-alone eigenvalue is minimal. To show that the detached eigenvalues in the examples are of the same type we show the entries of the corresponding eigenvectors in Figure 4.37.

From the presented simulations in this section we learn that similarly to the regular

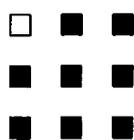
two-part structures, the spectrum featuring ‘grouped eigenvalues’ or a detached, stand-alone eigenvalue, often holds some information regarding the structure of a graph with two-part underlying structure. We discover those properties by looking at the corresponding eigenvectors.

In graphs with larger number of inputs we expect to observe similar relation. However, the difficulty is that with larger  $k$  the number of different describing parameters  $p$  and  $q$  increases. As the graphs eigenvalues depend on all of the parameters, we may not find the shape of the eigenvalue spectrum as informative. We now look further at the three-part irregular graphs.

### 4.3.2 Graphs with three parts

Initially avoiding simulating complicated graphs helped us to focus on investigating useful features of the spectrum shape. In this section we choose to work with simple graphs with parts of similar size. Analogously to examples considered previously, we present dense parts and strong connections by ‘■’ and less dense by ‘□’, and here we assume that these symbols correspond to probabilities of 0.9 and 0.1 respectively. We list a number of examples and for each present the shape of the spectrum and the eigenvectors’ of interest. We comment on the possible relationship between noted features of the spectrum and the underlying structure of the graph. For instance, the fact that certain eigenvectors’ entries are of small values, approximately equal to zero, we see this as similar to temporary omission of the corresponding vertices and therefore focus on the relationship among the remaining vertices within the graph. What we mean by this becomes more obvious as we go through the examples of graphs.

**Example 4.3.4.** The simulated graph in this example consists of three parts of equal size, two of which have high edge occurrence ( $p_1 = 0.1$ ,  $p_2 = p_3 = 0.9$ ), that influence each other strongly ( $q_{ij} = 0.9$  for all  $i, j = 1, 2, 3$ ).



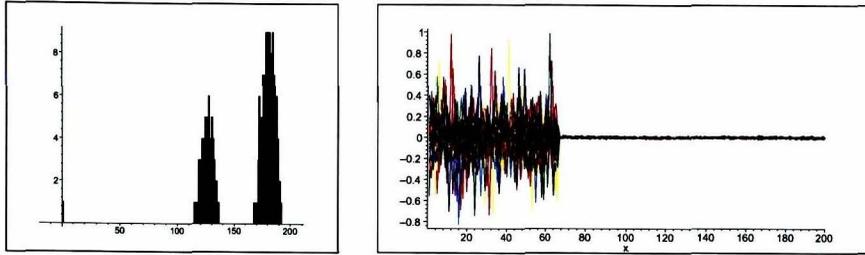


Figure 4.38: Eigenvalue spectrum of the graph from Example 4.3.4. Presented eigenvectors correspond to the first group of eigenvalues, and are describing the non-dense ('weak') part of the graph.

**Example 4.3.5.** In this instance we again have three parts of equal size that strongly influence each other ( $q_{ij} = 0.9$  for all  $i, j = 1, 2, 3$ ), but only one has high edge density ( $p_1 = p_2 = 0.1$  and  $p_3 = 0.9$ ).

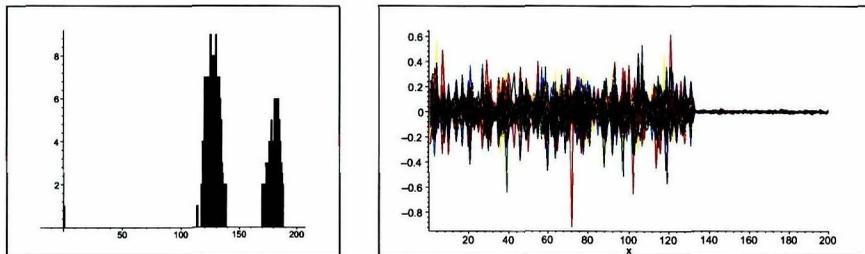
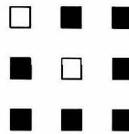
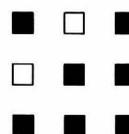


Figure 4.39: Eigenvalue spectrum of the graph from Example 4.3.5. Presented eigenvectors correspond to the first group of eigenvalues, but unlike the previous example they are describing the first and the second part of the graph (the 'weak' parts).

**Example 4.3.6.** Graph with three parts of high edge density ( $p_i = 0.9$ , for all  $i = 1, 2, 3$ ), where one is strongly connected with two that do not influence each other strongly ( $q_{12} = 0.1$ , and  $q_{13} = q_{23} = 0.9$ ).



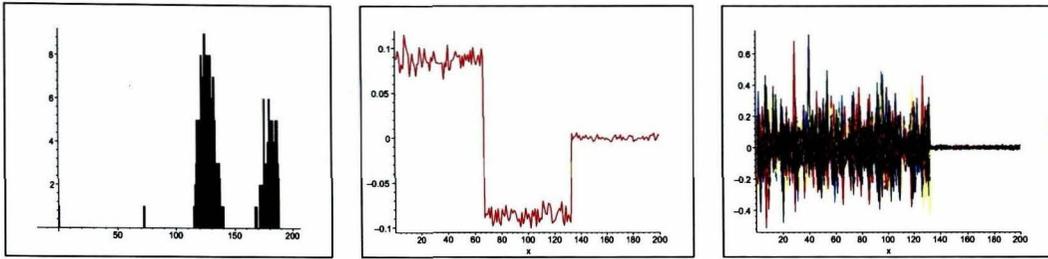


Figure 4.40: Eigenvalue spectrum of the graph from Example 4.3.6. We present two plots of selected eigenvectors: the first contains one eigenvector corresponding to the smallest stand-alone eigenvalue; the second corresponds to the first group of eigenvalues. Clearly, both plots describe the entries of the eigenvectors associated with the two weakly connected parts.

**Example 4.3.7.** In this three-part graph we have two weakly connected parts of low edge density ( $p_1 = p_2 = q_{12} = 0.1$ ) that are strongly connected to the third part which has high edge density ( $p_3 = q_{13} = q_{23} = 0.9$ ).

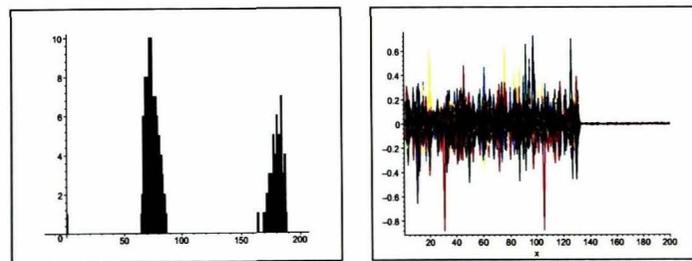
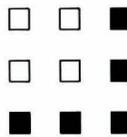


Figure 4.41: Eigenvalue spectrum of the graph from Example 4.3.7. The presented eigenvectors' entries correspond to the smaller in value eigenvalue group, and as expected are pointing to the vertices of the two weak parts.

**Example 4.3.8.** Here we generate a graph in which the equally sized parts have their own specific characteristics different to the others. Without loss of generality we assume that the first part is the only one of low edge density ( $p_1 = 0.1$ ), and is connected weakly to the strong second part and strongly to the strong third part ( $q_{12} = 0.1$  and  $q_{13} = 0.9$ ). The second and the third part are connected strongly

$(p_2 = p_3 = q_{23} = 0.9)$ .

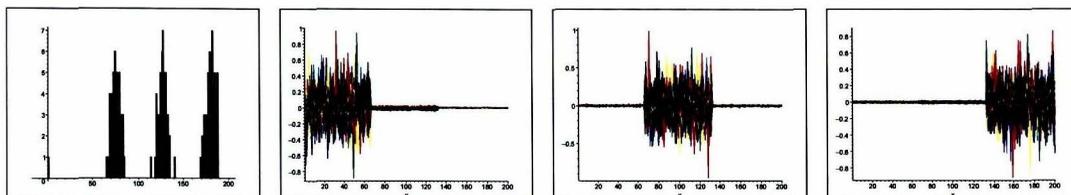
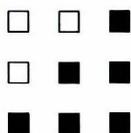


Figure 4.42: Eigenvalue spectrum of the graph from Example 4.3.8. We present three plots of selected eigenvectors each corresponding to the three eigenvalue groups. We find that the smallest in value group describes the least influential part, while the largest describes the most important one.

**Example 4.3.9.** It is also possible to generate a graph with three parts of equal sizes where neither of the parts is obviously more important or influential than the other. We consider an example where a weak part is strongly connected to two dense parts with weak mutual influence. The parameters are as follows:  $p_1 = q_{23} = 0.1$  and  $p_2 = p_3 = q_{12} = q_{13} = 0.9$ .

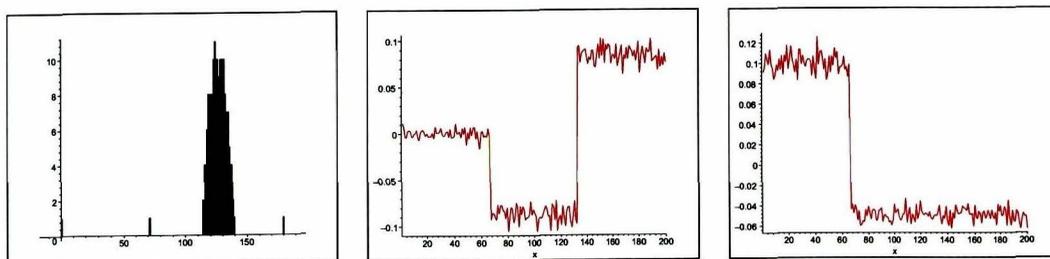
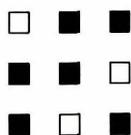


Figure 4.43: Eigenvalue spectrum of the graph from Example 4.3.9. We note the two detached lone eigenvalues and depict the associated eigenvectors: it appears that the eigenvector corresponding to the smaller eigenvalue points to the weak connection between the second and third part, while the other highlights the strong influence the two parts have on the first.

**Example 4.3.10.** The following example consists of two strong parts connected with few edges. The third part is non dense part such that it is strongly influenced by one and weakly by the other part. After observing the eigenvalue histogram and the eigenvectors of interest we find that this is equivalent to saying that the graph consists of two dense parts where one part is double the size of the other, where they weakly influence each other (see the entries of the eigenvector associated to the smallest non-zero eigenvalue). Permuting the positions of the vertices in the adjacency matrix or, more precisely, swapping the positions of the vertices of the second and the third part, we clearly see the weak connections described above.

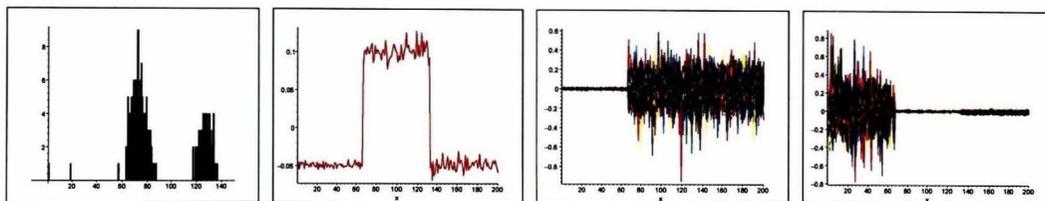
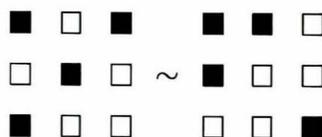


Figure 4.44: Eigenvalue spectrum of the graph from Example 4.3.10 (first sketch). The detached smallest non-zero eigenvalue points to the weak connection between the subgraph induced by the vertices of the first and second part and the subgraph corresponding to the dense third part. On the other hand, the second and third plots are seen as similar by the smaller in value group of eigenvalues, while the first part is marked as the most important by the larger in value group of eigenvalues.

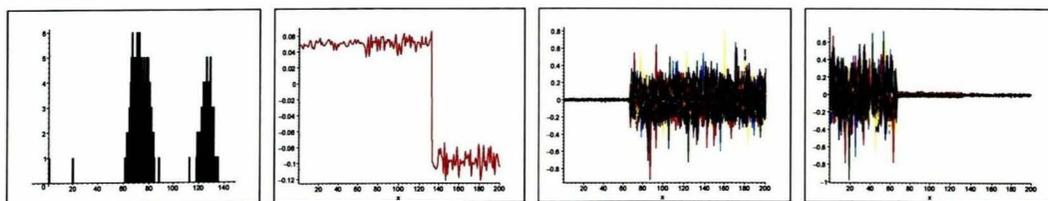


Figure 4.45: Eigenvalue spectrum of the graph from Example 4.3.10 (second sketch).

**Example 4.3.11.** In this example we present three non dense parts connected to each other with some strong and some weak connections. We observe the shape of the eigenvalue histogram. The largest eigenvalue eigenvector entries show that the second part is different to the other two. Permuting the relevant vertices in

the adjacency matrix brings us to the second picture that says that this is simply a graph of two non-dense parts of different sizes that influence each other strongly.

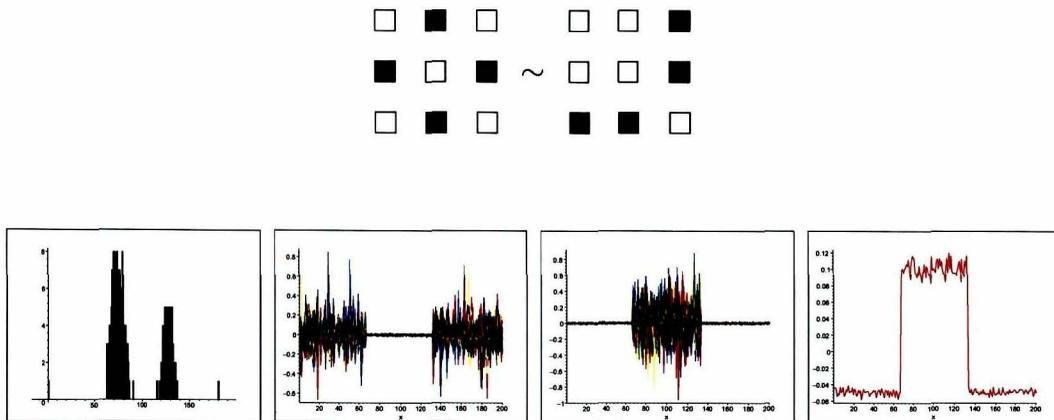


Figure 4.46: Eigenvalue spectrum of the graph from Example 4.3.11. The detached largest eigenvalue points to the strong connection between the subgraph induced by the vertices of the first and third part and the subgraph corresponding to the subgraph corresponding to the third part. On the other hand, the first and the third parts are seen as similar by the smaller in value group of eigenvalues, while the second is marked as the most important by the larger in value group of eigenvalues.

## 4.4 Possible Interpretation of the Algebraic Properties

As expected, the analysis of qualitative features of the spectrum shape is complex, because of the abundance of simulations considering interactions between various parameters characterising the observed graphs. Here we present possible interpretation of the observed algebraically derived spectral properties.

Starting with the simple graphs with regular  $k$ -part underlying structure helped us to focus on the interplay between few important parameters: the number and the size of the parts, the probability of an edge occurrence within parts (one value,  $p$ , for all parts), and the probability of an edge occurrence between parts (one value,  $q$ , for connecting edges between any two parts). Although changes to the parameters are applied gradually, it is important to note that the parameter values are at all times seen within the context of all other parameters. In this context we look

for the relationship of the regular  $k$ -part graph structure and the corresponding Laplacian eigenvalue spectrum, and its fair interpretation.

One of the first observations we make is in relation to the extreme eigenvalues. We find that they are in close relation to the type of the parts and connections between them.

As our reference  $k$ -part example, we take a graph with ‘weak’ parts that are strongly connected. By this we mean a graph whose parameter  $p$  is not only of low value, but also lower than that of the parameter  $q$ . If the values  $p$  and  $q$  are chosen so that  $p$  is considerably smaller, then we find that the spectrum of such a graph has  $k - 1$  largest eigenvalues grouped and separated from the rest of the spectrum. Moreover, the graph’s parts are recognised in the entries of the eigenvectors that correspond to those eigenvalues.

On the other hand,  $k$  dense parts with few edges between them (i.e.,  $p$  larger than  $q$ ) are recognised using the  $k - 1$  smallest non-zero Laplacian eigenvalues and their eigenvectors.

We find that the value of the eigenvalues that uncover the parts in either case is in direct relation with the value of the parameter  $q$ . The grouped eigenvalues are approximately of the value  $qN$ . The position of the remaining eigenvalues in the spectrum depends on the relation between the values  $p$  and  $q$ . The larger the difference between  $p$  and  $q$ , the further away they are from the value  $qN$ .

Next we look at the edge-weighted graphs. Allowing for varied edge weights in case of regular  $k$ -part edge-weighted graphs means that the proportion of the edge weights assigned is held constant over all parts, i.e., the parameter  $p^{(w)}$  is the same for all parts and edges in the graph. The same is assumed for the connecting edges and the corresponding parameter  $q^{(w)}$ .

As seen in simple regular  $k$ -part generated graphs, the strong influence of parts to one another in the Laplacian eigenvalue spectrum is described by the  $k - 1$  maximal eigenvalues. Introducing the weights in the connecting edges can be interpreted as further strengthening in connections between the parts. As a result we note the increase of all eigenvalues with the largest increase in the grouped maximal eigenvalues. They become approximately equal to the value  $wq^{(w)}N$ , and remain detached from the rest of the spectrum.

On the other hand, when increasing the parts' edge weights in the reference example the shape of the eigenvalue spectrum can change significantly. We note that the distance between the rest of the spectrum and the maximal eigenvalue group reduces. A sufficient increase in parts' weights can reduce this distance so much that we no longer see detached maximal eigenvalues, but all non-zero eigenvalues grouped together around the value  $qN$ , suggesting that the connections between the parts are as 'strong and influential' as the parts themselves are. Further increase in the parts' edge weights pushes the observed eigenvalues further to the right. Therefore we obtain a situation where the previously maximal eigenvalues in the reference example remain at the same place and thus became a group of the smallest non-zero eigenvalues, since the rest of the spectrum increases in value and 'passes' the value  $qN$ . This can be interpreted as follows. Edge weights of parts combined with the parts' density define and add to the importance of the parts within a graph.

Observing edge-weighted irregular  $k$ -part graphs is more challenging because neither the parts nor the connecting edges need to be of the same type. The number of combinations of the parameters give many more possibilities for generating graphs. The conclusions are similar. It is worth paying attention to the detached eigenvalues, as well as to groups of eigenvalues within the spectrum. As we noted earlier, the importance of a part depends on its size, the density of edges occurring in the part, the weight of the edges, but also how it is related to the remaining vertices of the graph. It seems as if all these factors contribute to the part's dominance and play a role in the shape of the eigenvalue spectrum. If the overall strong links between the parts are dominating the graph it shows in the spectrum as a few large grouped eigenvalues, just as we see in the case of  $k$ -part regular graphs. If, on the other hand, there are parts of high importance within a graph that dominate the other graph parts and the links between the parts then, instead, the large grouped eigenvalues are describing that feature of the graph (e.g., an existence of a clique-like subgraph).

Empirical analysis shows that the difference between grouped maximal eigenvalues in the case of a graph that can be divided into parts that are connected with strong overall links, and the eigenvalues in the case of the graphs with a clique, is in the type of the corresponding eigenvectors.

In the first case, the entries associated with the vertices belonging to the same

part are of a similar value. These may not all be significantly different for any two different parts within the same eigenvector, but could be for two different eigenvectors (see, e.g., Figure 4.16, plots (a)-(c)). Assuming this, sorting the entries of these eigenvectors in increasing order helps re-index the vertices so that the ones corresponding to the same part are positioned near each other in the influence matrix.

In the second case, in the analysis of the eigenvectors' entries we focus on identifying entries that are significantly different to 0 and separate them from the entries that are in value close to 0. To do that we order the entries of an eigenvector in increasing order of the absolute value of the entries. The largest among the entries correspond to the vertices that are contained in the dense part.

Similarly we argue in the case of the the grouped minimal eigenvalues (see, e.g., Figure 4.40), where we expect to identify from the eigenvectors' entries either a weak part, or a partition of a graph into dense parts weakly connected.

## 4.5 Examples with Geographic Position of Transmitters

We conclude this chapter by testing the effectiveness of the previously shown spectral analysis methods and observations on a simulated 'realistic' FAP. We consider a small radio network of  $N = 200$  transmitters randomly placed in a square area. We assume that each transmitter operates with uniform effective radiated power and that free-space loss is the only cause of signal attenuation. In other words, all transmitting powers are equal, the propagation is omnidirectional and the larger the distance between the transmitters the weaker the interference. While the radio engineers refer to the last condition as the 'inverse fourth power propagation law', in this particular example we present it in simpler terms by defining the influence constraints as follows. Closely placed transmitters are assigned frequencies that are at least 3 channels apart, 1 channel apart if they are sufficiently away from each other, and 0 if the distance between the transmitters is beyond the influence range. Figure 4.47 depicts the described FAP.

As before, we are interested to see if the shape of the distribution of Laplacian eigenvalues can tell us about this edge-weighted graph. We present the influence

matrix, and the Laplacian histogram in plots (a) and (b) in Figure 4.48.

Considering that there are no obvious stand-alone groups of eigenvalues that could be of particular interest, in the first phase of the analysis we observe the maximal eigenvalue and the smallest non-zero eigenvalue.

When studying the eigenvector that corresponds to the maximal eigenvalue we are focusing our efforts on finding a clique-like structure. We do not look into identifying a partition of the graph into weak parts that are strongly connected. The reason being that in this case the largest eigenvalue is comparatively low, much smaller than the value  $w_{max}N$ .

If a clique within a graph exists it would be revealed by finding the entries of the largest eigenvector that are significantly different from 0. As noted before a way to capture vertices that correspond to such entries is by ordering the absolute values of entries of the eigenvector in increasing order and considering the largest few.

On the other hand, we can focus on the entries of the smallest non-zero eigenvalue in the belief that they could reveal a partition of the graph into subsets of strongly related transmitters that mutually weakly influence each other. If such a partition exists, we expect to start seeing it after ordering the entries of this eigenvalue in increasing order.

We start with the analysis of the smallest non-zero eigenvalue. Sorting its eigenvector entries in ascending order, and permuting the rows of the influence matrix accordingly gives us a new form of the FAP influence matrix shown in Figure 4.48(c).

We next consider the maximal eigenvalue. After sorting the absolute values of the eigenvector entries in ascending order, and permuting the entries of the initial influence matrix accordingly, we get a different representation of the influence matrix. This is depicted in Figure 4.48(d).

We highlight the placement of the vertices that correspond to the 20 largest entries of the eigenvector associated with the minimal non-zero eigenvalue in Figure 4.49. Similarly Figure 4.50 shows the placement of the vertices corresponding to a few largest in absolute value entries of the eigenvector associated with the maximal eigenvalue. We note that in the first plot highlighted are 20 vertices of a dense part, whilst in the second case the highlighted 20 vertices are such that they belong to two dense parts that weakly influence the rest of the transmitters in the network.

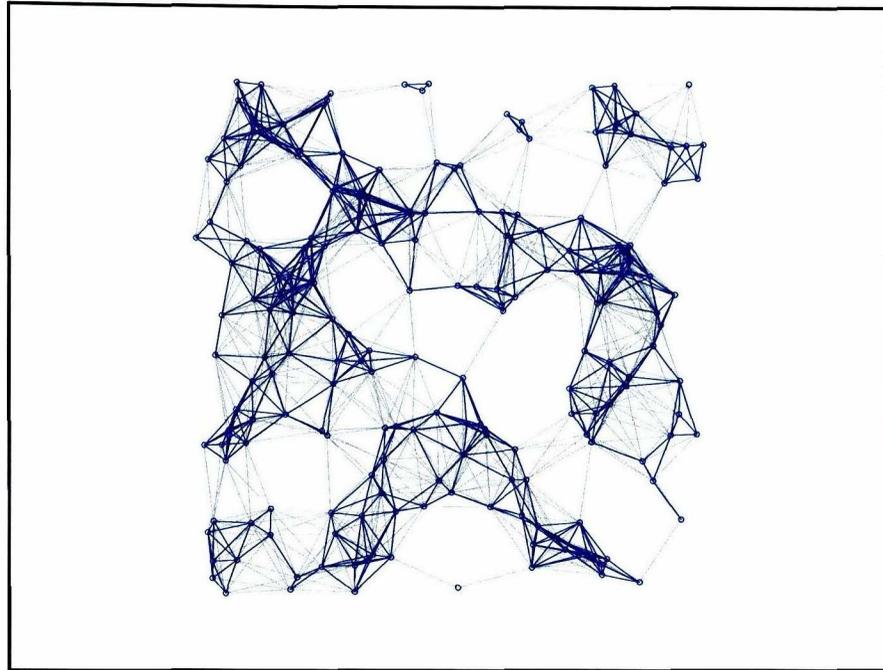


Figure 4.47: Geographical position of randomly placed transmitters; blue: edge-weight 3, gray: edge-weight 1; no edge: edge-weight 0.

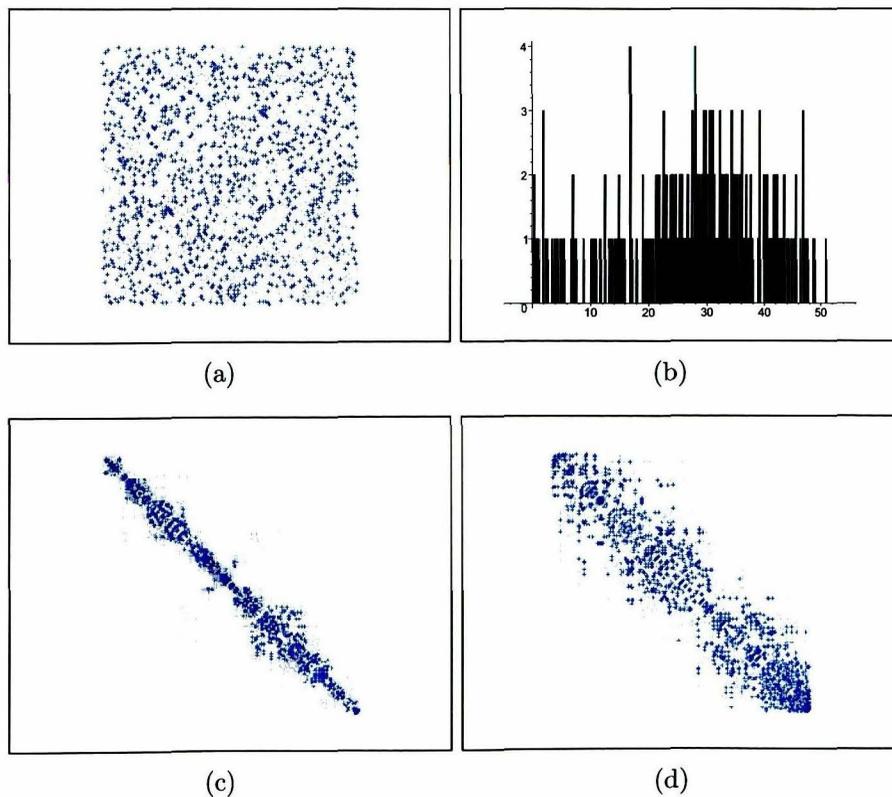


Figure 4.48: (a) the influence matrix; (b) the eigenvalue spectrum; (c) permuted influence matrix with respect to increasing order of the values of entries of the eigenvector of the minimal eigenvalue; (d) permuted influence matrix with respect to increasing order of absolute value of entries of the eigenvector of the maximal eigenvalue.

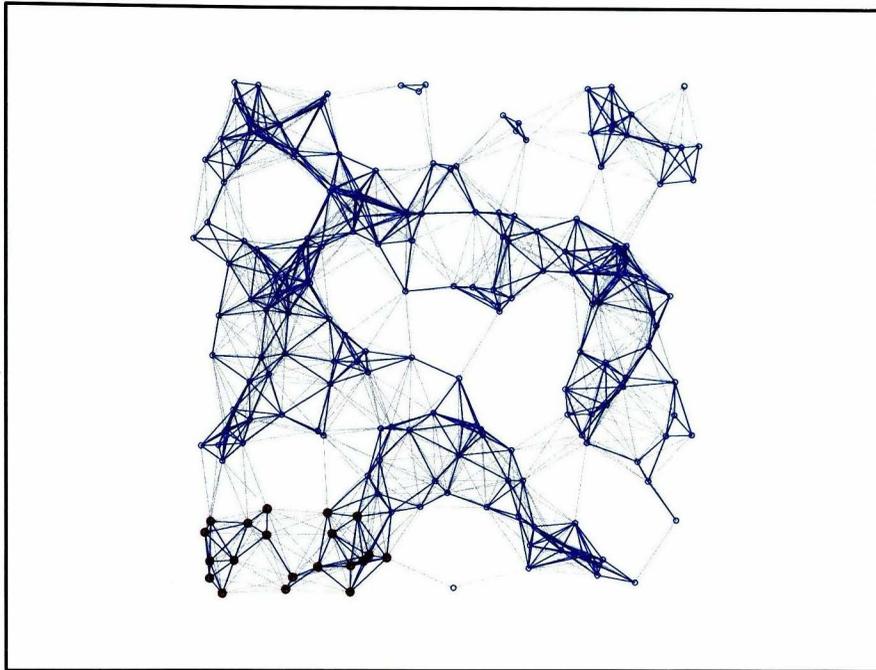


Figure 4.49: Geographical position of randomly placed transmitters shown in Figure 4.47 highlighting the first 20 largest entries of the eigenvector associated with the minimal non-zero eigenvalue.

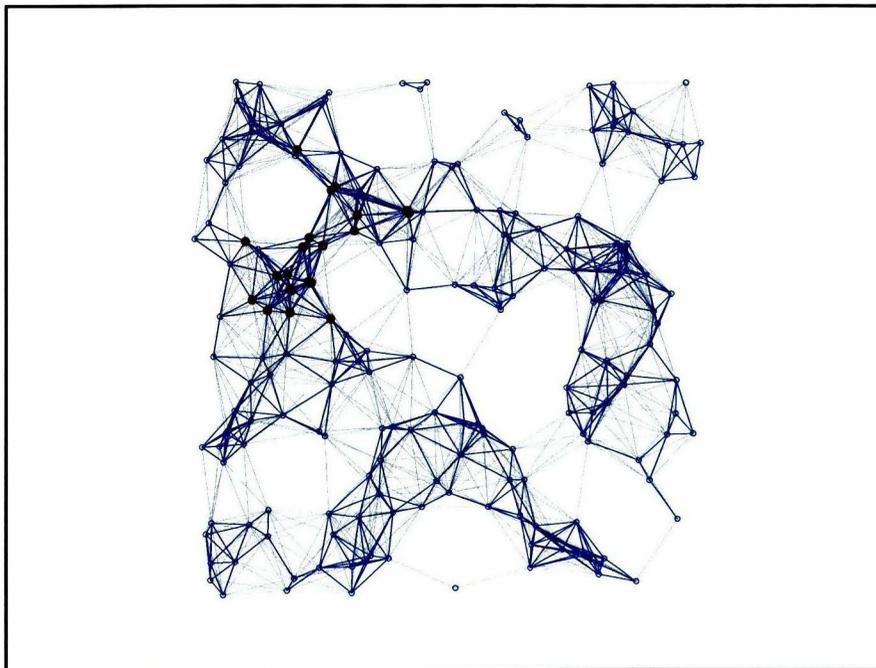


Figure 4.50: Geographical position of randomly placed transmitters shown in Figure 4.47 highlighting the first 20 in absolute value largest entries of the eigenvector associated with the maximal eigenvalue.

A further example we show is a simulated network of the same number of  $N = 200$  transmitters, with two large regions that have transmitters located closely to each other. This is illustrated in Figure 4.51.

In comparison to the previous example we notice that the maximal eigenvalue is larger and that the eigenvalue spectrum of the new graph is wider. Nevertheless, we again consider both the minimal non-zero and maximal eigenvalue and their eigenvectors. The reason being that the largest eigenvalue remains comparatively low, which as before provides a signal that the graph cannot be partitioned into weak parts that are strongly connected.

The corresponding representations of the influence matrix are shown in plots (c) and (d) in Figures 4.52, whilst the placement of the vertices of interest are highlighted in Figures 4.53 and 4.54.

From the combined analysis of the two examples we observe the following. As expected, the entries of the eigenvector of the maximal eigenvalue that are largest in absolute value point to a dense part. Similarly, the largest entries of the eigenvector associated with the smallest non-zero eigenvalue do suggest a partition into strong parts that are weakly connected.

In addition, we see that the more explicit the feature is, the easier it seems for the eigenvalues and their eigenvectors to identify it.

We also find that the usefulness of this analysis is in that while the suggested partitioning into a number of parts that influence each other weakly, say, may not solve the FAP, it may present a significant first step in an attempt to fully understand the graphs' underlying structure. We could go for the most convenient partitions in two or three parts dividing the original FAP into few independent subproblems of smaller size. We then apply the algebraic spectral analysis to these. This process could be continued recursively.

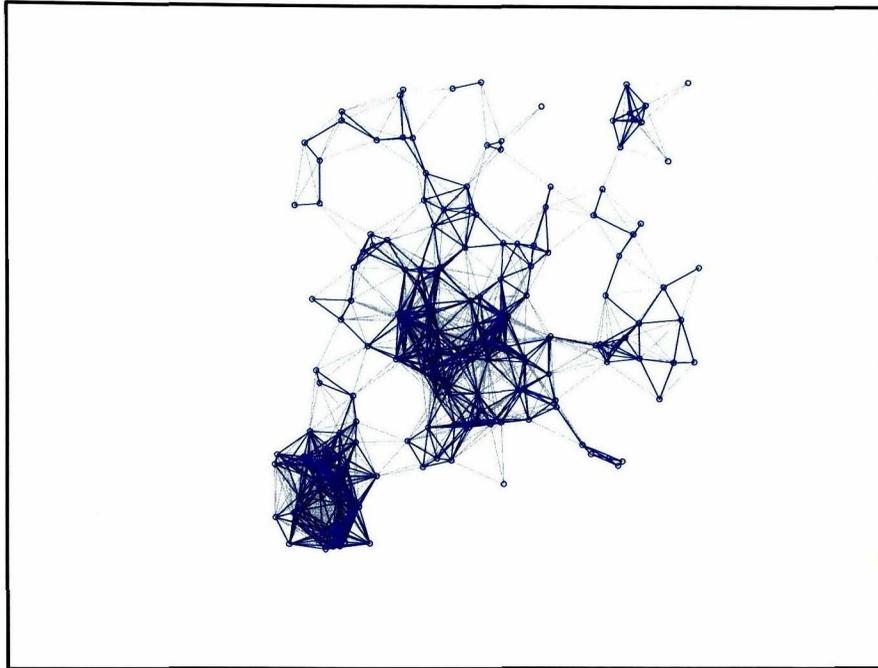


Figure 4.51: Geographical position of randomly placed transmitters with some dense areas; blue: edge-weight 3, gray: edge-weight 1; no edge: edge-weight 0.

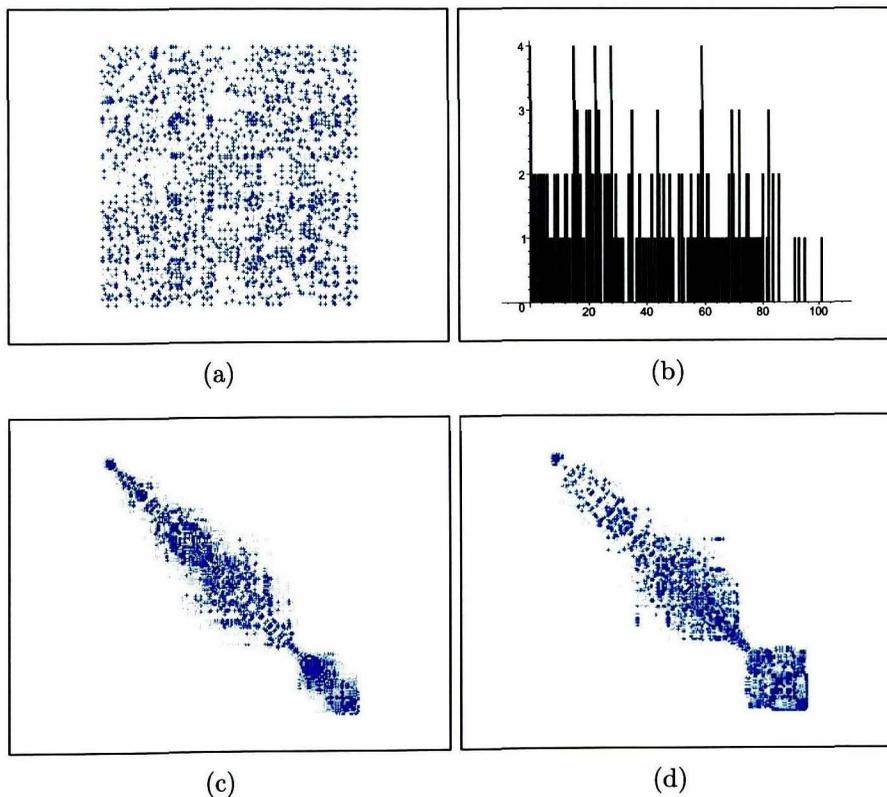


Figure 4.52: (a) the influence matrix; (b) the eigenvalue spectrum; (c) permuted influence matrix with respect to increasing order of the values of entries of the eigenvector of the minimal eigenvalue; (d) permuted influence matrix with respect to increasing order of absolute value of entries of the eigenvector of the maximal eigenvalue.

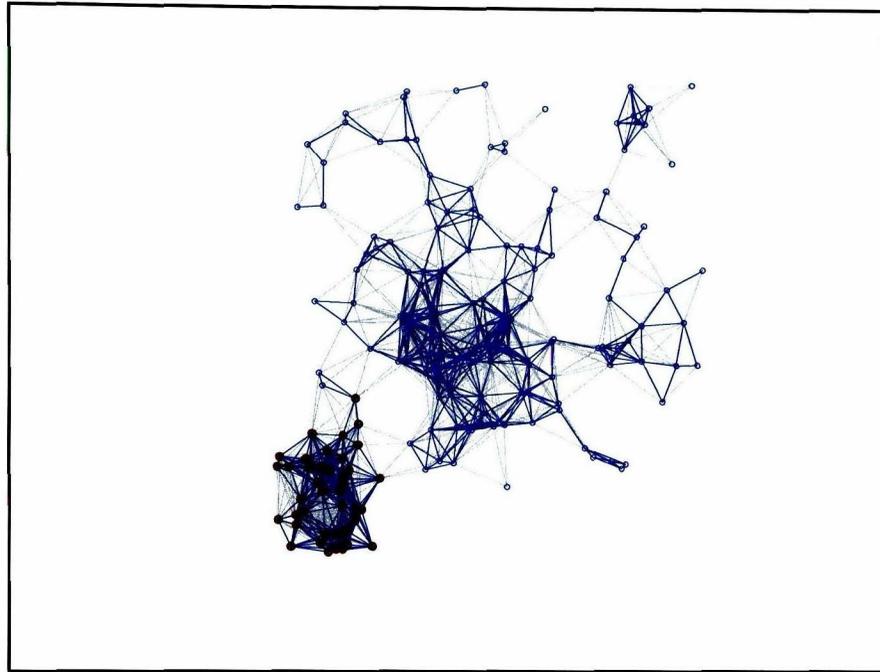


Figure 4.53: Geographical position of randomly placed transmitters shown in Figure 4.51 highlighting the first 50 largest entries of the eigenvector associated with the minimal non-zero eigenvalue.

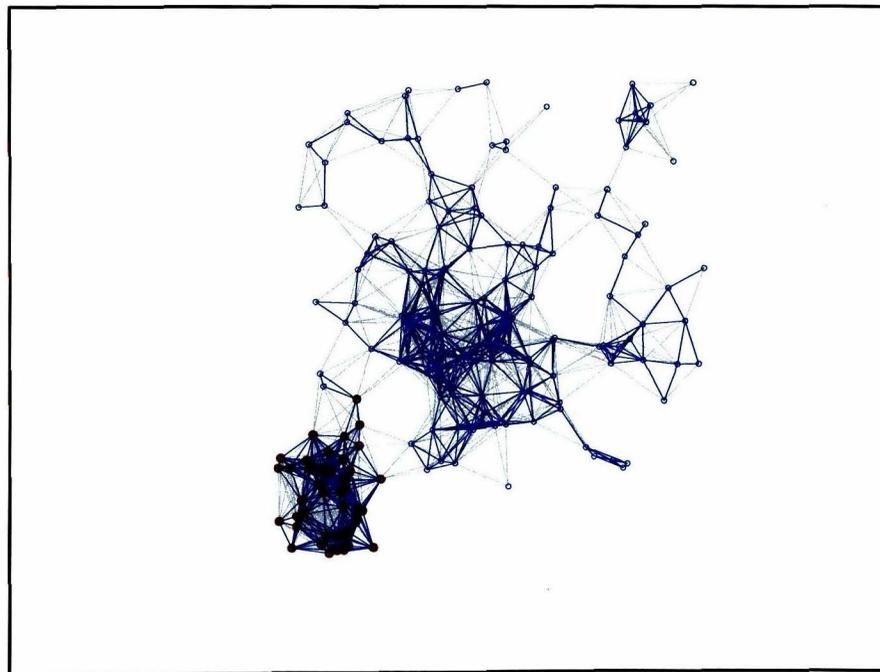


Figure 4.54: Geographical position of randomly placed transmitters shown in Figure 4.51 highlighting placement the first 50 in absolute value largest entries of the eigenvector associated with the maximal eigenvalue.

# Conclusion

In this chapter, we summarize our work and indicate how our contribution fits within the area of Frequency Assignment Problems. Then we provide an overview of interesting remaining questions open for future research.

Frequency Assignment Problems are a class of problems related to assigning frequencies to transmitters. The main purpose of our work is to assist the numerous existing heuristics for solving FAPs that rely on structural properties to facilitate algorithms, rather than to solve the defined problems. Gaining additional knowledge about the properties of a FAP can also lead to an improved understanding of the assignment problem and support development of new algorithms and applications. The focus of this thesis is on the study of mathematical properties for the Laplacian matrix that can provide structural information on the underlying FAP. We investigate the use of Laplacian eigenvalues and eigenvectors as possible tools in this analysis.

The main findings of this thesis are related to the Laplacian eigenvalue histogram shapes and the entries of the Laplacian eigenvectors corresponding to particular types of Laplacian eigenvalues. *Contrary to the approaches found in the spectral theory literature, we study the whole spectrum instead of solely the maximal or minimal eigenvalue, with particular preference for the Laplacians instead of the more common adjacency matrix.*

For an edge-weighted graph of order  $N$ , where the largest edge weight is of value  $w$ , we know that the spectrum lies in the interval between 0 and  $wN$ . Having this in mind we can scan and arbitrarily split the spectrum looking for subsets of eigenvalues of low and high value. Broadly, we find that the occurrence of groups

of large eigenvalues is related to the existence of a clique-like substructure within a graph or to overall strong links within the graph. Small eigenvalues, on the other hand, imply weak connections between denser graph parts. In order to identify the vertices that belong to certain types of parts, we analyse eigenvectors that correspond to eigenvalues of interest. We find that it is the combination of these two steps that helps reveal certain underlying structures. We also find that these observations are valid and possible to prove when dealing with ‘ideal’ graphs as described in the theory chapter. We expect that these results can be extended to graphs that closely resemble the structure of ideal examples because of the continuity property of eigenvalues.

Using simulations we find that eigenvectors in certain types of graphs (graphs with particular underlying  $k$ -part structure ( $k \leq 4$ ) discussed in the thesis) can help partition the original problem into smaller versions of FAPs. We find that working with a small number of parts can be sufficient, since subproblems of the original FAP can be further analysed and classified in the same way, and the process could be continued recursively. This could be interpreted as a form of similarity in FAPs. However, since the quantitative measures are different and in different scales, the similarity applies only to the structural view of the graph, since ‘continuity’ of eigenvectors does not hold in general.

Although a lot of our work relies on empirically confirmed observations, they provide several immediate leads for future work. The insights we obtain through the analysis of the eigenvalue distribution can be used to explore the potential for characterising FAPs by assessing the shape of the eigenvalue distribution, making this observation a potential first step in an approach to solving the assignment problem. While we conclude that the entries of the eigenvectors are informative in the observed simulations, we also find that, to our knowledge, no extensive work has been done on Laplacian eigenvectors. We are aware of difficulties that arise when observing the vectors associated with eigenvalues of high multiplicities, but we believe that more attention could be given to the research of Laplacian eigenvectors in algebraic graph theory.

Further work could also be done in developing other methods for analysing the eigenvector entries of graphs in cases where we suspect that there are more than four parts ( $k > 4$ ): for example identifying ‘grouped’ vertices in  $k$ -dimensional space analogously to what we did for the case with smaller dimensions.

Besides focusing on identifying useful structural information aiming to assist heuristics, we also looked to use the structural properties obtained in determining the span of a FAP. We only touched upon this subject by deriving certain bounds on the chromatic number in terms of Laplacian eigenvalues, and presenting an example of a FAP where the span can be directly calculated from the spans of the subproblems. The stated results, unfortunately, are either too weak or can be applied to simplified FAPs only. An important task in this area would be identifying all different edge-weighted graphs that can be partitioned into subgraphs so that the graph span (or a bound) can be derived directly from the spans of the partitions.

With regards to our other theoretical work we produce upper and lower bounds for eigenvalues and attempt to give bounds on the span of FAPs. While many of these results are generalisation of the properties that hold for simple graphs, we nevertheless think that the bounds obtained could be improved. An open question is how do the algebraically derived bounds in this thesis compare to the eigenvalue and span bounds already known for graphs observed in the literature.

In our simulations work, we analysed changes in the spectra of graphs by starting with a predetermined particular underlying graph structure, and then introducing changes simulated graph parameters. The continuity insured that the gradual changes in parameters brought gradual changes to the eigenvalue spectrum. However these observations in our work were only empirical. We would like to see some research done on quantifying these observation. In addition, when a change in parameter indicates a 'phase transition' in the behaviour of the eigenvalues/eigenvectors, it would be useful to know when this occurs and weather it occurs very suddenly.

As a further note in relation to other potential next steps in the research work in the field, we comment on a few areas that may be of interest.

Tests on proposed spectral theory methods in analysing real life FAPs, while simultaneously obtaining quantitative data will provide further insights into structural characteristics of properties of real FAPs. In particular, a research in this direction would be able to identify which random graph simulations presented in our work are most closely related to the real-life FAPs.

Joint work with heuristic algorithm research teams may provide valuable detailed

information on useful underlying structures that will assist the algorithms. In the same way, collaboration with the heuristic researchers can produce methods for classifying FAPs according to the distribution of Laplacian eigenvalues with regards to their underlying structures, or with respect to the level of solvability when applying existing heuristics.

Lastly, specific FAP case studies would help identify those graph properties that have substantial effects on improving solving algorithms, as opposed to the ones that are only of theoretical interest.

Our conclusion is that Laplacians have proven themselves to be useful in identifying certain types of problems, as well as in providing a structured approach to reducing the original FAP to smaller size subproblems, and hence in assisting the existing heuristic algorithms for solving frequency assignments. Overall, we feel that analysis of the Laplacian spectrum has the potential to become an important tool to better understand the structure of FAPs and their underlying graphs. But we also must admit that much further research in this direction will be required.

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