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BUBBLES AND CHAOTIC DYNAMICS IN ECONOMIES WITH EXTERNALITIES

Thesis Submitted for the Degree of PhD

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To Her Highness the Princess Brambilla

Et je vous dis que la vie est réellement obscurité
sauf là où il y a élan,
Et tout élan est aveugle sauf là où il y savoir,
Et tout savoir est vain sauf là où il y travail,
Et tout travail est vide sauf là où il y a amour ;
(...) Et qu'est-ce que travailler avec amour ?
(...) C'est mettre en toute chose que vous façonnez
un souffle de votre esprit,
Et savoir que tous les morts bienheureux se tiennent
auprès de vous et veillent.
Gibrane Khalil Gibrane, *Le prophète.*

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The person to whom this thesis is dedicated will easily reckognise herself. May she not feel insulted by the lack of beauty and the indelicate fragrance of this work.

Abstract

The present thesis deals with some consequences of the existence of external effects à la Romer, i.e. positive spillovers from the capital stock onto the efficiency of labour, and is mainly considering problems of discrete dynamics in the absence of any intrinsic (i.e. exogenous) shock. In the first chapter, using a one-sector three-period OLG model with borrowing constraints, it is shown that the standard result stating that, in the presence of externalities, any simple tax/subsidy policy undertaken to get rid of a bubble on an intrinsically useless asset creates an IOU which has exactly the same negative effects as the bubble itself, fails if there are agents who must borrow at some moment of their life. The other three chapters are mainly studying the problem of endogenous fluctuations in competitive equilibrium models. The second chapter looks at the possibility of Hopf bifurcations in the dynamical system characterizing a two-sector OLG economy meeting all neo-classical assumptions from the point of view of the private sector, and its ILA analogue : it demonstrates the existence of economies with stable closed orbits, derives some conditions on the parameters and compares the results to the continuous time modelization, concluding to a non robustness with regard to the time structure assumption. The third chapter is considering endogenous fluctuations in self-sustaining growth : using the same framework as previously, but under another assumption on the externalities, we establish that even if production inputs substitute perfectly and savings increase monotonically with the interest rate, cycles or even chaotic trajectories of the growth rate are possible. We show that this requires a strong externality in the consumption good sector in the absence of bubbles or sunspots, but not necessarily in their presence. Furthermore, we prove the existence of economies where, in the absence of any intrinsic uncertainty, the only possible equilibria involve bubbles or sunspots. The last and very short fourth chapter is a critical note on a recently published paper ; its main purpose is to show why current mathematical knowledge does not allow to sustain the claim of chaos in the proposed ILA framework.

TABLE OF CONTENTS

"Überzeugungen sind gefährlichere Feinde der Wahrheit als Lügen[1]."
Friedrich Nietzsche (1878),
Menschliches, Allzumenschliches.

Introduction

Pure economic theory, normative economics, applied economics, ideological economics : there exist many branches of economic investigation, each characterized by its specific goal and methodology. Some people consider theoretical economics as completely useless, and describe them as the frivolous activity of slightly degenerate minds. The critiques usually put forward are the utmost reductionism necessarily characterizing any mathematical modelisation of very complex phenomena implying human activity, the fact that most results of theoretical economics are apparently empirically refutated and that it is possible to prove anything and its contrary by the adequate choice of the model. Furthermore, the more and more intensive use of mathematics, and especially of 'high technology' tools of modern mathematics scares or irritates many people. However, even a brief and necessarily superficial reflection on the nature of economics clearly establishes the fundamental role pure theory has to play. Our ideas about the functioning of the economic

[1]Convictions are More Dangerous Ennemies of Truth than Lies.

mechanisms are the result of cultural and social a prioris and of simplifications, the official discourse on economics is highly ideological ; furthermore, the common use of language is imprecise or abusive, and the rhetorical use of sophisms to persuade other people of the truth of a claim is very common.

Theoretical economics constitute a critical activity in so far as their aim is to investigate the validity of pre-existing conceptions. The intention is to go beyond the appearances, to show that certain hidden mechanisms can account for observable facts, and to question well established convictions about what seems to be obvious. The methodology is hypothetico-deductive in the classical sense : the economist chooses a set of assumptions to build a model and studies the implications of his assumptions.

Economic theory seeks logical consistency and tries to highlight the role of assumptions. Obvious simplifications and idealizations, the existence of apparently ridiculous assumptions like rationality of agents, existence of a representative agent, perfect competition etc., lead some people to declare the uselessness of theoretical economics. The existence of simplifications is, of course, very often the result of the problem of tractability which forces economists to choose specific assumptions to be able to illustrate their ideas. But does the lack of realism imply futility ?

The answer is 'no' for different reasons. First of all, the reflection on models, even if these seem outrageously simplified, can develop our intuition about the involved mechanisms, can help us to isolate the fundamental determinants of certain phenomena and thus enhance our ability to understand real world events ; the use of different modelizations allows to specify the role of assumptions and thus develops our sense of rigour and critique. Leaving aside the auto-justification of all search for knowledge and internal consistency, we can immediately underline that idealization can give precious information about the implications of deviation from the idealized situation ; furthermore, even a very simple and unrealistic model can definitively invalidate ideas that are deep-rooted in our minds, contribute to a revolution in our conceptions, and have direct practical implications : an example of this is the introduction of the rational expectations hypothesis. In fact, a number of theories are developed simply in order to gainsay other theories, and this activity is of absolute importance not only from the point of view of pure science : since political decisions are often justified by arguments relying on economic theories in order to give them the stamp of truth, it is fundamental to know on which assumptions their recommandations are based and to dispose of counterarguments to open the debate.

Logical consistency is of extreme importance in all social sciences, including economics, simply because, unlike in physics for instance, theory cannot be sanctioned by experiments. A lot of, not to say most, physicists do not care about mathematical rigour, use approximative reasonings and practice mathematical acrobatics ; in their eyes, the really important thing is compatibility of the conclusions of their theories with the results of experiments. Since experiments cannot be performed in the field of economics and experience can only invalidate a theory, logical rigour is a fundamental requirement.

p.10 omitted

like 'we consider families, dynasties' etc. : the truth is simply that neither of these two modelisations is satisfactory[9].

Furthermore, it is not at all clear that OLG models with many periods cannot exhibit fluctuations : as a matter of fact, the argument to claim the impossibility relies on Aiyagari (1988), but its validity is easily gainsaid since Aiyagari considers very specific OLG models and cannot claim any generality. The possibility of endogenous fluctuations, even if not on a business cycle scale, is in itself a fundamental result. In our eyes, none of the two types of models can be considered as 'more adequate' or 'better'. In view of the imperfections characterising current modelization, we believe that the fact that endogenous fluctuations can occur in both types of models constitutes itself the most important point.

Finally, let us point out that we always consider extremely simple models : in the economies we deal with, agents are all characterised by the same tastes, and there is no intrinsic uncertainty, which means that we consider a world not submitted to any exogenous shock, neither on tastes nor on technologies. Indeed, we consider entirely deterministic models where the fundamentals are even assumed to be stationary (time independent). Furthermore, we consider one-sector models, or two-sector models with only one

[9] Blanchard's (1985) continuous time OLG model with uncertain life-time, which could appear, at first glance, as an improvement, is not more satisfactory since the probability of death is independent of the agent's age, and the agents are not, even after a million years, dead for sure. It is easy to see that this model is formally of the ILA type if we introduce insurance through a mutual fund.

ternalities has very important implications is well known by now. From basic game theory, for instance, we know that the presence of positive spillovers implies to non optimality of Nash equilibria and that, coupled with strategic complementarity, it can lead to multiplicity of symmetric Nash equilibria. Another standard result is that in dynamic equilibrium models, externalities constitute one source of non optimality of perfect competition equilibria which can be struggled against : for instance, in the case of an externality à la Romer (1986), i.e. of positive external effects operating in the production sectors from the aggregate capital stock onto the efficiency of labour (the standard interpretation is that of 'learning by doing'), then a public policy which aims to increase savings can lead to a Pareto improvement.

We shall consider here exclusively such externalities à la Romer. This has several explanations. There is, to be honest, the extreme formal tractability of models with this type of externality, which constitutes a non negligeable argument in our eyes : we are personally more interested in problems of existence than in questions of generality ; our aim and pleasure is to find simple models which can be used to illustrate the possibility of certain phenomena, and perhaps to gainsay well established ideas. Genericity within a given set of models, or structural stability of a given model[2], are of course important,

[2]Let us briefly recall the definitions. Let $\mathcal{M}$ be a set of models, and $\mathcal{K}$ the subset of models for which a given property holds. Genericity is often defined by the fact that $\mathcal{K}$ is dense in $\mathcal{M}$. The idea is that if we believe that the true model lies in $\mathcal{M}$, then the property will hold with high probability if genericity holds. We immediately see that there is a confusion between density and measure : rational numbers, for instance, are dense

but we made the choice not to investigate these aspects. Secondly, our aim here is not a, for sure highly interesting, comparison of the differing effects of several types of externalities[3], but rather the description of several important facts due to the presence of externalities ; in this perspective, it seems rather natural to use only one type of externality if the models we obtain allow interesting conclusions.

Externalities à la Romer can cause increasing returns to scale at the aggregate level, and thus allow to construct models where growth does take place in the long run : as we know, this is not the only possibility, since a standard result is that a linear production function can lead to self-sustaining growth in the ILA framework[4] ; this also holds in multi-sector OLG models where a concave production function in the investment good sector must be, at least asymptotically, linear in capital[5], but is not true in the one-sector OLG model where endogenous growth requires increasing returns to scale. Romer type externalities can give self-sustaining growth, and even balanced growth (i.e. with a constant growth rate), in an easy and tractable way. We shall see below why this is of extreme importance.

in $[0,1]$, but the probability that a randomly chosen number of this interval is rational is zero. The correct definition must therefore be : if we can define a measure on $\mathcal{M}$, then genericity means density and strictly positive measure of the subset $\mathcal{K}$. Structural stability means that the property holds for all models in the vicinity of a given model.

[3]An exercise of this type has been performed by Cazzavillan (1994) in his PhD thesis, which deals with the study of different types of externalities in a continuous time, two-sector ILA framework.

[4]See Jones and Manuelli (1990).

[5]See O'Neil Fisher (1992).

The models we shall study are all of one or the other of the two standard types of dynamic models, namely Infinitely Lived Agents (ILA) models and Overlapping Generations (OLG) models. Let us recall here one of the main differences between these two types of dynamic equilibrium models. Under strict neo-classical assumptions on utilities and production, thus in the absence of any externality, and assuming perfect markets and rational expectations, ILA models admit a unique perfect competition equilibrium which is Pareto optimal, whereas OLG models exhibit indeterminacy of perfect foresight equilibria and can, if there exist at least three sectors, admit perfect competition equilibria which are not Pareto optimal. This constitutes indeed a major difference, and finds its explanation in the incompleteness of markets in the OLG context (agents cannot trade with people who are not yet born). Purely neo-classical ILA models thus will not exhibit certain types of dynamics which can be easily obtained in the OLG framework, such as sunspot equilibria.

Sunspots are the archetype of situations with extrinsic uncertainty. The terminology intrinsic/extrinsic uncertainty, has been introduced to distinguish respectively uncertainty affecting the fundamentals of the economy from uncertainty having absolutely no influence on them. A sunspot is defined as a random phenomenon without any effect on usable technologies, available ressources or tastes of the economic agents. According to the objectivist conception, sunspots should not matter and have no influence on economic variables. Yet Azariadis (1981), Azariadis and Guesnerie (1983), Chiappori and Guesnerie (1987) showed that even if agents form totally unfounded expectations, these can be self-enforcing ('self-fulfilling prophecies'). If everybody believes in the sunspots' influence on prices, then the prices will indeed depend on the sunspot realisation. More complex situations where, for instance, one part of the population believes in sunspots and the other in moonspots, can easily be dealt with[6]. These illustrations of the direct action of representations on the economy are quite spectacular.

Cass and Shell's (1987) conjecture, which is a sort of 'folk theorem' of neo-classical growth theory, says that sunspots require some sort of frictions which violate the fundamental theorem of welfare economics, imperfection of financial markets or externalities, for instance. The first to note that an ILA economy with liquidity constraints can, in a certain sense, mimic an OLG model and thus admit sunspot equilibria, was Woodford (1986). Forward stability of perfect foresight equilibrium, a sufficient condition for the construction of sunspot equilibria, was obtained by Kehoe, Levine and Romer (1990) in an ILA model of finitely many agents with externalities (and indeed, distortionary taxes). Spear (1991) finally established, with a very special type of externality[7], the possibility of sunspots in the presence of a continuum of agents. The existence of externalities thus not only implies non Pareto optimality of perfect competition equilibria, but can lead to indeterminacy even in the ILA framework.

[6]See Azariadis and Guesnerie (1983).

[7]Spillover of the average savings of all agents...

A very classical critique addressed to the OLG modelisation concerns the length of the periods. Indeed, the most tractable and therefore most commonly used OLG framework is the two-period model à la Diamond (1965) where young agents work, consume and save for their old age, and old agents consume their savings. Imagine now that we want to build a model exhibiting endogenous fluctuations, probably in order to give an endogenous explanation to at least part of real world fluctuations. Until recently, most models of endogenous fluctuations were OLG models, for the reasons indicated above. However, the most recently presented models are mainly ILA models, simply because the introduction of externalities or the consideration of imperfect competition now allow to exhibit, relatively easily, tractable models ; the authors proposing these models all claim that ILA models with endogenous fluctuations are far more exhilarating because fluctuations do not occur on large time scales as in the OLG context (25-30 years in the case of a two-period OLG model...).

What is the value of their argumentation ? We are inclined to say 'naught', for both types of models lack realism, and none can be said better than the other. In the OLG modelisation, the length of the life of each agent is predetermined, the different phases of life are fixed and all are the same for everybody. In the ILA framework, the horizon of the agents is infinite[8], a very unappealing idea even if we try to justify it by scabrous arguments

[8] And this implies an extremely demanding assumption on the agents rationality, whereas in the OLG context the foresight is limited to a finite number of periods...

like 'we consider families, dynasties' etc. : the truth is simply that neither of these two modelisations is satisfactory[9].

Furthermore, it is not at all clear that OLG models with many periods cannot exhibit fluctuations : as a matter of fact, the argument to claim the impossibility relies on Aiyagari (1988), but its validity is easily gainsaid since Aiyagari considers very specific OLG models and cannot claim any generality. The possibility of endogenous fluctuations, even if not on a business cycle scale, is in itself a fundamental result. In our eyes, none of the two types of models can be considered as 'more adequate' or 'better'. In view of the imperfections characterising current modelisation, we believe that the fact that endogenous fluctuations can occur in both types of models constitutes itself the most important point.

Finally, let us point out that we always consider extremely simple models : in the economies we deal with, agents are all characterised by the same tastes, and there is no intrinsic uncertainty, which means that we consider a world not submitted to any exogenous shock, neither on tastes nor on technologies. Indeed, we consider entirely deterministic models where the fundamentals are even assumed to be stationary (time independent). Furthermore, we consider one-sector models, or two-sector models with only one

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type of capital. These assumptions, along with thoses on the form of utilities and production functions, will greatly simplify the calculus and allow to exhibit nice and strong results. Again, this kind of simplification can inspire some criticism which we shall try to argue against : quite standard results are that heterogeneity of agents or multiplicity of types of capital are sources of complexity of equilibrium dynamics ; the theory of nonlinear dynamical systems shows that the higher the dimension of a dynamical system, the larger the set of possible exotic dynamics. Furthermore, imperfect competition or imperfect markets[10] also constitue potential causes of endogenous fluctuations. To look for the most simple possible models therefore imposes more constraints, and does not mean to simplify basely ones task.

2 The Thesis' Content

Since the abstract gives a description (a very brief one, we must admit) of the content of each paper, we believe that, rather than giving a linear presentation of the papers, it is more interesting to talk here about the two big themes raised in this thesis : *welfare implications* (and the possibility of Pareto improvement through public policies) and the possibility of *endogenous fluctuations* generated by the presence of externalities. The first problem finds some answers in the first and the third paper, the second one is dealt with in the second, third and fourth paper.

[10]Of course, externalities can be interpreted as market failures...

2.1 Presence of Externalities, and Welfare

The fact that perfect competition equilibria are not Pareto optimal if externalities operate in the economy has been alluded to previously, and a policy to improve the social welfare has been exposed in the case which we are interested in here, namely externalities à la Romer : policies that lead to an increase of savings. But saying this, we implicitly assumed that we were considering what is traditionally called the 'fundamental' equilibrium of the economy. What is meant by that ?

Consider the following system of linear first order difference equations :

$$Y_t = aE_t(Y_{t+1}) + cX_t \qquad (*),$$

where Y_t is the vector of state variables, X_t a vector of exogenous variables, and where a and c are constants. If the constant a is strictly less than one, and if $\lim_{t \to +\infty} a^{T+t} E(Y_{T+t}) = 0$, then the simplest solution of $(*)$ is obviously

$$Y_t = \sum_{i=0}^{+\infty} a^{t+i} E(Y_{t+i}).$$

This is the 'fundamental' solution. It is not the only one, since we know that a solution of $(*)$ is the sum of the just exhibited particular solution of $(*)$ and a solution of the associated homogenous equation. Indeed, consider a sequence $(B_t)_{t \in \boldsymbol{N}}$ such that $B_t = aE(B_{t+1})$. $Y_t + B_t$ is obviously a solution of $(*)$. B_t sequences are traditionally called bubbles, the inspiration for this denomination coming from the finance area where the idea is that an asset share's price can be decomposed into the sum of the fundamental (reflecting the 'true' value of the asset) and the bubble component (due to self-fulfilling beliefs). There can exist ever-expanding bubbles, bubbles which have a certain probability to explode at each instant etc. The bubble component is not founded on the fundamental, and results from self-fulfilling prophecies.

Let us here note an important point, concerning the possibility of bubbles, which can be interpreted as degenerate sunspots. A still common view is that bubbles cannot occur in deterministic sequential market economies with a finite number of agents (see, for instance, Tirole (1982) still invoked in the latest edition of Blanchard and Fisher (1989...)). The reasoning leading to this assertion is incorrect : Kocherlakota (1992) showed that Tirole simply forgot to impose a no-Ponzi game condition, condition which is required for the existence of any equilibrium and which can support bubbles in an ILA model with a finite number of agents.

Traditional economic theory implicitly considers that, if a dynamic equilibrium model with rational expectations admits an equilibrium, than there exists a fundamental equilibrium (and, perhaps, some odd 'bubble' equilibria). Let us immediately criticize this attitude by using one of the results of chapter three : there exist economies admitting no fundamental equilibrium but equilibria with bubbles. This is illustrated in the context of a two-sector OLG model with production externalities ; a bubble in the OLG context can be interpreted as fiat money, for instance. The result is shown to be due the presence of a non convexity at the aggregate level in the investment good sector[11]. The standard terminology, stemming from the nearly exclusive focus on linear or linearised models, can thus be misleading.

An interesting problem is to study and compare the welfare effect of bubbles in strictly neo-classical models and in models with externalities. The seminal papers on bubbles on intrinsically useless assets in the neo-classical OLG framework have been written by Jean Tirole (1982), (1985) and (1990). It appears that in the neo-classical world, bubbles can occur only if the economy is inefficient (there is over-accumulation of capital) and their effect on the welfare is positive (the unique stationary bubble even completely eliminates the inefficiency). Thus, there is no reason for the government to intervene.

On the other hand, Grossman and Yanagawa (1992) have shown that in the context of a one-sector OLG model meeting all the neo-classical assumptions from the point of view of the private sector, but with externalities à la Romer in the production sector, bubbles can appear (remember that here there is always *underaccumulation* of capital), and they have always a negative effect on the welfare since they divert capital from productive investment. Thus, in this situation, there exists a reason for the government to attempt to get rid of the bubble. Unfortunately, as say Grossman and Yanagawa, any simple tax/subsidy policy intended to reduce the number of shares of the useless asset in the market necessarily creates an IOU which

[11] A related result, in another context, concerning the existence of economies with only sunspot equilibria has been established by Pietra (1990).

has exactly the same effects as the bubble. The problem is that the authors conclude on a very specific model, namely a model in which no agent ever has to borrow.

Our first chapter shows that there can exist the possibility of simple tax/subsidy schemes that progressively eliminate all the shares of the bubble asset and improve the welfare of all agents : for this, we use a one-sector three-period OLG model with production externalities and borrowing constraints, where young agents must borrow on their future income. Thus, in this type of world where the appearence of a bubble leads to a loss in welfare, the situation is not necessarily as hopeless as we would previously have thought : Pareto improvements through government policies can be achievable under some conditions.

Let us note here a strange consequence of the odd result of chapter three cited previously : if there exist economies with externalities where no 'fundamental' equilibrium exists, but equilibria with bubbles are possible, can we say that the existence of bubbles means a loss in welfare ? It would certainly be interesting, in such a situation, to compare the welfare loss due to the different possible bubbles.

Let us add some further general remarks : the research on speculation and bubbles has shown that prices do not necessarily reflect market fundamentals, multiple equilibria, and therefore indeterminacy, can be observed.

Speculation has been shown not to be necessarily stabilising[12], and cannot anymore be viewed as the force which brings back the price to its fundamental level. The following conception :

> "...People who argue that speculation is generally destabilizing seldom realise that this is largely equivalent to saying that speculators lose money, since speculation can be destabilising in general only if speculators on the average sell when the currency is low in price and buy when it is high..."
>
> Milton Friedman (1953), *Essays in Positive Economics*

is thus invalidated. Speculative bubbles, on the other hand, are no longer viewed as necessarily the effects of irrationality. An important result is also that government intervention might lead to a Pareto improvement compared to the traditional laisser-faire attitude.

In the financial and monetary spheres, price formation does not reflect exclusively a logic of rarity, but implies dynamics of self-validation of anticipations which are not necessarily linked to the fundamentals. To understand these dynamics of formation of expectations appears therefore to constitute a fundamental problem. Unfortunately, there exist very few papers which try to explain the emergence of financial bubbles, and their explosion[13], but the topic seems to have become more 'en vogue' most recently.

[12] See Hart and Kreps(1986), de Long et alii (1987).

[13] An early exception is Harrison and Kreps (1978)...

2.2 Endogenous Fluctuations in the Presence of Externalities

As we indicated previously, this problem constitutes the larger part of our work, and three out of our four papers deal with it. Surveys of the modern literature on endogenous fluctuations can be found in chapter two and three. We prefer to expose here the reasons of the renewal of interest in an area which becomes more and more fashionable if we judge by the number of publications on this topic in the major economic journals.

The modern literature on endogenous fluctuations tries to show that optimising behaviour, rational expectations and stationarity of the fundamentals of the economy, like tastes, technologies, institutional setup etc., do not rule out persistent, non explosive fluctuations. This is, of course, at the opposite of the classical view of the economy either converging to a nice steady state, or diverging on an 'explosive' path, in the absence of shocks. To practice research in the area of endogenous fluctuations does not, of course, mean to negate the effects of exogenous shocks on the economic path. Some interpret it as an attempt to show that fluctuations can find, at least partially, endogenous explanations. Woodford (1990) argues that the true point of the endogenous cycle literature is rather the suggestion that the determinacy theses of the orthodox business cycle theory might be too restrictive. This is indeed the probably fundamental point, since the knowledge of the possible forms of equilibrium paths in the limiting case of absence of any intrinsic uncertainty does not necessarily give any information on the behaviour of the economy in the case of presence of exogenous shocks. Let us note however that certain results have a direct implication for a theory of purely exogenous fluctuations, as is stressed in Guesnerie and Woodford (1994). Unfortunately, the mathematical difficulties encountered in nonlinear dynamic models do not allow, at this stage, to deal with nonlinear economies subject to extraneous uncertainty. Future developments in the concerned mathematical fields will hopefully enable economists to acquire one day an understanding of the properties of this kind of models.

The idea that internal mechanisms could be responsible for the observed variations in prices, employment, output was studied by von Hayek (1933), Shumpeter (1939), for instance, and endogenous cycles were obtained in Keynesian macroeconomic models[14] by Allais (1956), Goodwin (1951), Harrod (1936), Hicks (1950), Kaldor (1940) and many others. Nonlinearities and time lags constituted the source of persistent economic fluctuations. Many of these early models were brilliant in conception, their authors relying on an intuitive understanding of the problem at hand to build them. However, the problematic features of theses models were manifold : endogenous cycle models are essentially nonlinear, and this implied technical difficulties given the mathematical knowledge in the area of dynamical systems several decades ago (some very rich models could not be conveniently exploited), and

[14]...through the interaction of the consumption multiplier and several versions of the investment accelerator...

the type of dynamics which could then be described, namely *periodic* cycles, were easily empirically refutated. Optimizing behaviour was not incorporated in these models, and stability results obtained for many simple equilibrium models with explicit optimizing behaviour, like the Turnpike Theorems for ILA models, could easily lead to think that endogenous cycles were incompatible with optimization. Furthermore, econometric models were estimated that produced business cycle type data when submitted to repeated exogenous stochastic shocks, while converging to a steady state in the absence of shocks from the outside[15]. This type of considerations explains why the comforting vision of a self-stabilising market mechanism became so popular, leading to the overwhelming success of the so-called Slutsky-Frish-Tinbergen methodology.

The empirical arguments against the endogenous fluctuations theory nowadays appear to have lost of their strength : to-day, we know that an entirely deterministic system can generate erratic trajectories, qualified as *chaotic*, with autocorrelation functions and spectra which mimic those of a 'stable' linear stochastic model[16]. Furthermore, Blatt (1978) established that the linear autoregression fit to the data generated by the Hicks cycle model leads to conclude to a stable second order autoregressive process for output, the type of process which is obtained from autoregressions of actual GNP data. To distinguish stochastic fluctuations from data generated by a chaotic system,

[15]See Adelman and Adelman (1959).

[16]Sakai and Tokumaru (1980).

nonparametric tests for nonlinearity and instability are required ; some have been developed by Eckmann and Ruelle (1985) for the natural sciences, especially physics ; Brock(1986), Brock and Sayers (1988), Brock and Dechert (1987), Scheinkman and LeBaron (1986), (1987) and many other papers refine and improve these tests. Unfortunately, it seems that results cannot be obtained if we do not have quite large samples at our disposal, which means that there is little hope to come to a conclusion by this way in the field of economics.

Endogenous cycles in OLG models were obtained by Gale in 1973, but the very first general equilibrium models establishing the possibility of chaotic economic dynamics were Benhabib and Day (1982) and Grandmont (1985). As we noted previously, the indeterminacy of rational expectations equilibria in the OLG framework allowed to find rather easily different types of models exhibiting endogenous fluctuations : pure laisser-faire economies without or with production and fluctuations in fundamental or in sunspot equilibria, economies with government intervention[17]. But economist had to realise that unicity of the perfect foresight competitive equilibrium in the neo-classical ILA framework and existence of turnpike theorems for some classes of models do not imply that perpetuals oscillations are excluded. This was shown by Boldrin and Montrucchio (1986) ; unfortunately, their constructive proof implies very high discount rates, and Sorger (1991) has established that purely neo-classical (thus strictly concave) ILA models exhibiting chaotic dynamics

[17]Farmer (1986).

of logistic, tent or Henon map type as optimal equilibrium path necessarily imply very high discount rates (respectively 110, 100 and 80 %).

However, Sorger's results concern very specific dynamics and the assumption of small discounting does not rule out endogenous fluctuations in general (see Benhabib and Rustichini (1989)). Nonetheless, a general characteristic of neo-classical OLG or ILA models giving endogenous fluctuations is that some non-standard hypothesis concerning utilities (for instance negative interest rate elasticity of saving (IES) in Grandmont-type models, very high rate of impatience...) or production (Leontieff production function in at least one sector...) always exists. One aim of research is therefore to find models with more acceptable assumptions. Furthermore, all known neo-classical models with endogenous fluctuations are models whithout 'real', ongoing growth, and describe only closed orbits of the capital stock. It is thus important to build models of self-sustaining growth with endogenous fluctuations, and a natural way to achieve this seems to be the use of externalities.

The introduction of externalities or the consideration of imperfect, monopolistic competition constitute the two principal recent attempts to obtain richer and nicer models. For instance, we know that neo-classical ILA models admit a unique perfect foresight equilibrium ; the introduction of externalities can imply indeterminacy even in this framework, as has been shown by Howitt and McAfee (1988), Benhabib and Farmer (1991), Spear (1991) or Boldrin and Rustichini (1992, 1994). The same can occur when we introduce monopolistic competition[18]. The results obtained in the ILA framework with the assumption of externalities (or of monopolistic competition) differ less from those obtained in the analogous OLG framework. Research in the area of equilibrium dynamics has established the possibility of multiple equilibria, of cycles generated by flip bifurcations, closed orbits around a steady state generated by Hopf bifurcations and completely aperiodic, chaotic trajectories under laisser-faire in the fundamental equilibrium and, furthermore, of sunspot equilibria, in the ILA as well as in the OLG framework.

Let us briefly comment here our own findings. Chapter two is mainly interested in the consequence of the time structure choice, and studies the possibility of Hopf bifurcations in a discrete time two-sector OLG model, and its ILA analogue, in order to compare the results to those obtained for the continuous time ILA model by Cazzavillan (1992). The interesting results are that the dynamics are of exactly the same type for the OLG and the ILA model under the assumption of a high coefficient of intertemporal substitution, and that the discrete time assumption leads to less demanding conditions on the economies parameters than the continuous time hypothesis. Chapter three gives the most interesting results, because it deals with endogenous fluctuations in the context of self-sustaining growth. In this chapter, using the same OLG model as in the preceding chapter, but now under the necessary assumption on the externality operating in the investment good sector for the possibility of balanced growth, we establish the

[18]See Benhabib and Perli (1994) or Galí (1994).

possibility of flip-cycles[19] and of chaotic trajectories of the growth rate. This is a nice result because it allows to link the ideas of self-sustaining growth and endogenous fluctuations. Two preceding papers tried to do this, but both failed : Cazzavillan (1993) considered a discrete time one-sector ILA model with externalities onto the agents' utilities and onto production of a flow of public services financed through a lump sum tax ; unfortunately, his results require an assumption he uses but which is, at second glance, completely unacceptable : in his model, production requires public services as an input, but this input is financed through the current production, and logic would ask for a lag (public services which enter production as an externality are those financed through the taxes on previous period output), the introduction of which annihilates all the nice results. Boldrin and Rustichini[20] (1994) propose a two-sector ILA model with an externality à la Romer in the consumption good sector and a linear production function in the investment good sector. For this model, using results exposed in a paper by Boldrin and Persico (1993, 1994), they claim the possibility of endogenous growth with a chaotic growth rate. Unfortunately, Boldrin and Persico's paper contains some important errors and establishes the possibility of *observable* chaos in fact only in the case of complete depreciation of capital in each period, an

[19]Cycles generated through flip bifurcations are periodic, which is not necessarily the case for the closed orbits generated through Hopf bifurcations in chapter two. Flip-cycles are nonetheless of interest for applied economics if their period is high, since they will appear aperiodic over relatively short time intervals.

[20]An old version of their paper circulated since 1992 and concerned indeterminacy of equilibria, but the new 1994 version claims the possibility of chaos.

assumption which is unacceptable since the periods are supposed to be short in the ILA framework. To say it shortly, Boldrin and Persico completely forgot that the technique of Lagrange multipliers was not invented to annoy people ; they have a fundamental role to play in all optimisation problems where some constraints are binding from time to time, which is the case in the problem they consider. The critique of the two mentioned papers constitutes chapter four.

Let us finally emphasize a nice result obtained in chapter three : the possibility of Hopf bifurcations generated by bubbles, positive or negative, in the two-sector OLG framework with utilities and production function meeting all neo-classical assumptions from the point of view of the private sector. This result is important because Farmer (1986) proved that in one-sector OLG economies, only negative bubbles (which correspond to private debt toward the government) can generate Hopf bifurcations, and Reichlin (1986) exhibited a two-sector OLG model where positive bubbles could cause this type of bifurcation only if the IES was strictly negative.

A general remark on all the models obtained until now, including those presented in this thesis, is that the introduction of externalities allows to build models exhibiting endogenous fluctuations with very standard utilities (CRRA, for instance) or production functions (Cobb-Douglas, augmented with externalities), with more acceptable rates of time preference etc. ; nevertheless, they nearly all require very important, certainly unrealistic, external effects. The only exception is exposed in chapter three, where it is shown that endogenous fluctuations require less strong externalities, in some cases even none at all, in the consumption good sector if bubbles are present in the economy. However, the disappointment about the often very unappealing requirements for endogenous fluctuations in the fundamental equilibria remains.

Let us finally talk about the really problematic point of the theory : if perfect competition equilibria are indeterminate, then there exists obviously a problem of coordination. Agents do not have a firm basis on which to form their expectations, since the knowledge of the economy's fundamentals is insufficient (note that the assumption of perfect information about the fundamentals, even if common in the major part of economic theory, already makes us smile). Multiplicity of expectations-driven equilibria then poses the problem of knowledge of the others anticipations. This raises the question of the 'implementation' of a rational expectations equilibrium, i.e. of the process according to which the values of variables predicted by the equilibrium are actually reached. As a matter of fact, the problem of implementation is old (think about the tâtonnement process to justify the equilibrium in the standard general equilibrium framework), but the consideration of dynamic models adds the problem of formation of expectations. The claim that

> "...the rational expectations hypothesis is nothing else than the extension of the rationality hypothesis to expectations..."
>
> J. Muth (1961), "Rational Expectations and the Theory of Price Movements", *Econometrica* 39, pp 315-334

is clearly erroneous, and we must build a theory explaining how the rational expectations equilibrium is reached (and which r.e.e. is reached if there is multiplicity ; learning convergence can then be viewed as a selection procedure or a refinement device). Therefore, learning must be introduced, and learning processes must be modelized, specifying which information about current and past states of the economy is used by each agent and how forecasts are made. The implied complexity of the description of the economy's dynamics is easy to imagine, and explains the difficulties encountered in this area of research. We shall not give here a survey of the literature on learning, but restrain ourselves to the allusion to one rather spectacular result concerning stability under learning in the classical Grandmont (1985) model : Grandmont and Laroque[21] (1986) established that it may happen that the only equilibrium cycle that is stable under learning is unstable in the simple mathematical sense. This shows that we cannot conclude too quickly, for a given equilibrium path, from stability or instability, in the mathematical sense, to its economic relevance or irrelevance.

3 Some Features and Implications of 'Chaos'

The object of this section is to define the notion of 'chaos' and to expose some of the implications of the existence of deterministic dynamics generating erratic trajectories. The discovery of deterministic dynamics yielding trajectories that mimic the realisations of stochastic processes has raised

[21] See also Fuchs (1979).

a passionate and violent debate among scientists and philosophers on the problem of determinism. We shall talk about this 'quarrel', and insist on the confusion of concepts and errors of reasoning which sometimes explain the divergences.

3.1 Description of the Idea of 'Chaos'

Let us note, first of all, that some very distinguished mathematicians like René Thom do not accept the now common terminology, and prefer to speak about 'sensitivity on initial conditions' rather than 'chaotic behaviour'. And this indeed was the original denomination used by those who built the first models exhibiting a property which we nowadays call 'chaos'. The denomination 'chaos' was in fact introduced by Li and Yorke (1975) and has led to many misuses and misinterpretations of the mathematical ideas. This is not really astonishing given the traditional analogy cosmos/chaos, order/disorder. Chaos is confusion, absence of structure :

> *...Ante mare et terras et, quod tegit omnia, caelum*
> *Vnus erat toto naturae uultus in orbe,*
> *Quem dixere chaos, rudis indigestaque moles*
> *Nec quicquam nisi pondus iners congestaque eodem*
> *Non bene iunctarum discordia semina rerum.*
> *Nullus adhuc mundo praebebat lumina Titan,*
> *Nec noua crescendo reparabat cornua Phoebe,*
> *Nec circumfuso pendebat in aere tellus*

Ponderibus librata suis, nec bracchia longo
Margine terrarum porrexerat Amphitrite.
Vtque erat et tellus illic et pontus et aer,
Sic erat instabilis tellus, innabilis unda,
Lucis egens aer ; nulli sua forma manebat
Obstabatque aliis aliud, quia corpore in uno
Frigida pugnabant calidis, umentia siccis,
Mollia cum duris, sine pondere habentia pondus...

P. Ovidii Nasonis, *Metamorphoseon*, Liber Primus.

Secondly, and this evolution seems significant of the great influence of the so-called 'school of complexity' (Nicolis, Prigogine, Stengers and many others), the terminology introduced by the american school in the seventies was 'deterministic chaos', but the qualificative 'deterministic' is usually omitted nowadays. Out of sheer laziness ? In general, certainly, and we ourselves have to plead guilty ; but it is important to remain conscious of the danger of semantic glide, and we leave it to the reader to judge the innocence of some authors fond of 'complexity'.

We shall give now an intuitive definition of 'chaos' in its most common acceptance[22].

[22]There exist two other concepts of chaos, namely *topological* and *ergodic* chaos. All the formal definitions can be found in chapter four. We consider here what is sometimes qualified as 'turbulent' chaos, and is the most commonly considered type of chaos.

We consider, for simplicity, discrete time models but the definitions are analogous for the continuous time case where we deal with differential equations instead of difference equations. The set of states of a system is represented in a space $\mathcal{L}$ called phase space. The evolution of a deterministic system is defined by a map $\Phi : \mathcal{L} \to \mathcal{L}$ such that, if X_t is the state at time t, then $X_{t+1} = \Phi(X_t)$. Suppose that there exists an Φ-invariant subset $\mathcal{K}$ of $\mathcal{L}$. Then the map Φ is said to be chaotic on $\mathcal{K}$ if there exists sensitivity on initial conditions, if the map admits a dense orbit[23] and if periodic points are dense in $\mathcal{K}$. Sensitivity on initial conditions means that, for all initial conditions, *any* arbitrarily small perturbation leads to an orbit that diverges from the initial one.

The first example of sensitivity on initial conditions can be found in a paper written by the french mathematician Jacques Hadamard and intitled *Les Surfaces à Courbures Opposées et leur Lignes Geodésiques* (1898), where Hadamard shows that if we interpret the geodesics of surfaces with negative curvature as trajectories of points moving on these surfaces, then any perturbation of the initial direction suffices to lead to a complete change in the form of the trajectory.

To say that the extreme importance of this finding was not generally reckognised until some decades ago constitutes a euphenism. However, some few people became aware earlier of the bearing of the discovery and, for in-

[23] A more general definition asks for 'topological transitivity', which is a consequence of the existence of a dense orbit.

stance, as soon as 1906, Duhem[24] insisted on the irreducible character of the distinction between mathematical determinism and physical prediction. Hadamard's example remained for a long time the only tractable example of a model with sensitivity on initial conditions ; furthermore no link to problems arising in the natural sciences was apparent, which explains why so few scientists took notice of it or, if they did, considered it at best as a mathematical curiosum.

The interest started to increase when the american meteorologist E.N. Lorenz (1963) showed, by using numerical integration procedures on computers, that a standard three dimensional system of differential equations describing the convection of a gas or a liquid placed between two horizontal isotherm plates and submitted to a vertical temperature gradient could exhibit sensitivity on initial conditions. Since then, and with the help of the mathematical theory of nonlinear differential systems[25], quite a lot of dynamics encountered in the areas of physics, astronomy, meteorology, chemistry, biology, economics have been shown to be exhibit this property.

[24]Pierre Duhem (1906), "Exemple de Déduction Mathématique à tout jamais inutilisable" in *La Théorie Physique. Son Objet et sa Structure.*

[25]The modern theory was developed, essentially on the basis of the work of Poincaré, by Andronov, Anosov, Arnold, Hopf, Kolmogorov, Moser, Sinaï, Smale and many others. The history of improvement in this field is absolutely fascinating.

The following constitutes probably the simplest example[26] of deterministic chaotic dynamics : consider a circle of center O, and let A be a given point on the circle ; any point M on the circle can be characterised by the angle $X = \widehat{AOM}$. We take 2π as unit. We then define the 'angle-doubling' dynamics by :

$$X_{t+1} = 2X_t \text{ mod } (1).$$

This seems really trivial, but we shall see that the dynamics are complex. Any perturbation of the initial condition is doubled after one period. To be able to make an even gross estimation, say with a precision of 0.5, after ten iterations, we must have a precision on the initial condition of 1/1000. After fifty iterations, we would need a precision of 10^{-15}, which is practically excluded by any physical experiment. The system exhibits sensitivity on initial conditions.

Consider now the dyadic development of an initial conditionX_0 : it is given by the sequence $(a_1, a_2, a_3, ..., a_n, ...)$, where

$$\begin{cases} X_0 = \sum_{i=0}^{+\infty} \frac{a_i}{2^i} \\ a_i = 0 \text{ or } 1. \end{cases}$$

The action of the angle-doubling map is then simple to describe :

$$(a_1, a_2, a_3, ..., a_n, ...) \mapsto (a_2, a_3, ..., a_n, ...)$$

[26]See Devaney (1987). The other standard example, the logistic map defined on [0, 1] by $X_{t+1} = \mu X_t(1 - X_t)$, is simply linearly topologically conjugate to the here presented map.

It is easy to see that the set of initial conditions having a dense orbit is of full measure. Not all do have a dense orbit : a rational X_0 has a periodic (if its denominator is odd) or a pre-periodic (if it is even) orbit, some irrational initial conditions have an orbit which is dense in a Cantor set[27]. Thus, the dynamics are chaotic in the sense defined previously.

If we introduce a very small noise :

$$X_{t+1} = 2X_t + \epsilon_t \bmod (1),$$

where ϵ_t is chosen at random in the interval $[-10^{-15}, +10^{-15}]$, then the noise is unmeasurable, but the system has become intrinsically 'non-deterministic[28]' in the sense of stochastic, uncertainty increasing in each period and filling the whole space after fifty iterations.

The 'Pursuit'-lemma states then the following : to any trajectory $(X_t^\epsilon)_{t\in N}$ of the model with noise corresponds an initial condition X_0 such that its trajectory $(X_t)_{t\in N}$ in the deterministic dynamics verifies $|X_t^\epsilon - X_t| \leq 10^{-15}$. This has important implications. First of all, if we cannot detect errors of the order of 10^{-15}, then the observation of trajectories does not allow to differentiate between the deterministic and the stochastic model. For each type of model, forecasts are excluded, in the stochastic case this is a tautology, and in the deterministic case it is due to the impossibility to obtain measures

[27] A definition of this notion can be found in chapter four.

[28] We shall come back later to this point of terminology.

with the necessary precision. Secondly, computer simulations can be used in the field of chaotic dynamical systems. At first glance, this seems quite amazing since one could be led to think that the existence of sensitivity on initial conditions necessarily excludes simulations which always imply truncation of intermediate results at each iteration. But the Pursuit lemma guarantees that any orbit calculated by a computer is a proxy of a true orbit (of course, the value of the initial condition of the true orbit is, and will always remain, unaccessible).

3.2 Implications for Economic Theory

Linear deterministic dynamics lead to the following types of equilibria : point equilibria, which are either completely stable (well), mixed (saddle-point) or completely unstable (source), or periodic cycles. Complex bounded orbits are excluded ; furthermore, the periodic solutions are structurally unstable : any arbitrary perturbation of the parameters of the model leads to completely different trajectories.

Nonlinear deterministic dynamics, on the other hand, can admit bounded, periodic or aperiodic, self-sustaining oscillations as solutions and can exhibit structural stability. The techniques used to detect the possibility of such trajectories are local bifurcation theory (flip, Hopf...), application of the Poincaré-Hopf Index Theorem or global bifurcation analysis. We have seen that nonlinearity of the characteristic dynamical system can lead to the existence of highly complex, 'chaotic' trajectories. The spectacular feature of chaotic dynamics is their sensitivity on initial conditions which has many implications, such as the impossibility to predict the future if the initial condition is not known with *absolute* precision, the impossibility to distinguish, by the use of standard linear econometrics, the data they generate from those given by the realisations of stochastic processes.

Nevertheless, as we briefly noted previously, it is possible to determine whether a given set of measures is generated by a chaotic, regular or noise-contaminated dynamical system. Beyond manifest irregularities, chaotic dynamics show hidden regularities. Techniques using the dimension of the attractor, the Kolmogorov-Sinaï entropy (Grassberger-Procaccia algorithm), or the Lyapounov exponents which measure the separation of trajectories with very close initial conditions, have been developed. Unfortunately, the convergence of the implied algorithms is often very slow, the techniques require quite large data sets and a careful attention from part of the researchers. This has sometimes led to wrong estimations and erroneous claims of chaos.

It is clear that in the field of natural sciences like physics, it is often easy to dispose of very large data sets that can be used to determine the chaotic or non chaotic nature of the dynamics of a given system. Unfortunately, this is not the true in economics where, except in the very special case of finance, large data sets are not available to the researcher. We shall not give here any review of the literature on tests of the chaos hypotheses, and cite only a few papers to give a flair of the results and the difficulties encountered. The search for evidence of low dimensional deterministic chaos has provided arguments in favour of nonlinearity in employment, unemployment, industrial production etc. (Brock and Sayers (1988)). An interesting point is that Neftçi and McNevin (1986) found some evidence of nonlinearity in disaggregated production series, whereas the aggregate real GNP appeared linear under the tests applied by Brock and Sayers. There is no evidence of the presence of a chaotic attractor, but the tests may reject the null hypotheses of deterministic chaos too often when it is true. Ramsey, Sayers and Rothman (1990) insist on the fact that the evidence is based on data sets which are minuscule compared to those used in natural sciences. Sheinkman and LeBaron (1989) studied the question of stock returns and established the inadequacy of the 'random walk' theory[29] that states that returns are independently and identically distributed over time, but they underlined the existence of a lot of technical difficulties that had made their investigations difficult and sometimes untractable. Even to-day, although a lot of progress has been made, strong conclusions cannot apparently be obtained in the area of economics, and chaos can neither be rejected nor claimed for.

In one of the preceding sections, we alluded to the fact that the higher the dimension of a dynamical system, the larger the possibility of complex dynamics. This result should make us think about one of the major ideas of 'economic wisdom', namely the idea that freedom of trade and suppression of economic barriers mean an improvement for everybody, simply because what is recommended is the creation of a complex economic system obtained by coupling several local economies. And this, as we have seen, bears the risk

[29]Granger and Morgenstern (1963), Fama (1970).

of leading to a complicated, chaotic temporal evolution instead of a pleasant equilibrium.

> "I shall say it in a more brutal manner. Economic textbooks discuss in detail the equilibrium situations between economic agents that are capable of predicting exactly the future. These treatises can give the impression that the role of legislators and responsible officials is to find and implement an equilibrium which is especially propitious to the community. The examples of chaos in physics teach us however that certain dynamic situations, instead of leading to an equilibrium, give rise to a chaotic and unpredictible temporal evolution. Legislators and responsible officials are thus confronted to the possibility that their decisions, instead of leading to a better equilibrium, will in fact generate violent and unpredictible oscillations, with perhaps desastruous effects."
>
> David Ruelle (1985), *Hasard et Chaos.*

There is certainly some truth in that statement, but we must be very careful in our deductions. Existence of endogenous fluctuations does not necessarily imply non optimality in Pareto's sense of the equilibrium path. Of course, the origin of the possibility of endogenous cycles lies quite often in the existence of a market failure (externalities, imperfect financial markets etc.) which implies non Pareto optimality of perfect competition equilibria. Endogenous fluctuations are thus very often encountered in situations of suboptimality. But there exist also many examples of models with Pareto optimal endogenous cycles (see, for instance, Boldrin and Montrucchio (1986)). Furthermore, it is not clear at all that the creation of a large economy with endogenous fluctuations implies a loss in welfare compared to a situation with small, local economies in stationary equilibrium. We are thus in a situation where we do not know anything a priori ; Ruelle's statement has therefore its importance as long as it has not been refutated, and we should be careful in our 'deductions' and claims about the effects of free trade. Unfortunately, the debate about freedom of trade is highly passionate and ideological ; conscious of their utter ignorance, economists should perhaps adopt an attitude of modesty and humility.

3.3 The End of 'Determinism'?

...Noi siam venuti al loco ov'i t'ho detto
Che tu vedrai le genti dolorose
C'hanno perduto il ben de l'intelletto...
Dante Alleghieri, *Inferno.*

Chaos theory has generated a tremendous excitement and is to-day a fashionable topic : one can find messiahs of chaos that assert that "it is everywhere", in the smoke of a cigar, the milk poured in our five o'clock tea, the functioning of our brain, the evolution of our universe, and that all this signifies the metamorphosis of science, if not its end. There even exist 'chaos-clubs' where mystics, under the cover of (pseudo-) science, celebrate the new religion. An approximated truth is often mingled with completely invented facts, chaos is proclaimed to exist in fields where, as a matter of fact, nobody has yet undertaken any serious research, the notion of chaos is often assimilated to hazard, theories are applied in contexts where their assumptions are not fulfilled, and a lot of confusion prevails.

Several years ago, meteorologists established that the atmospheric streams admit an attractor. From this, some simple-minded creatures immediately deduced the now (unbelievably) famous 'butterfly effect' : since the atmosphere's dynamics are sensitive on initial conditions, the fluttering of the wings of a butterfly can completely change the weather, not of to-morrow, but in the future. Why did these people forget that the poor butterfly is a part of the whole system ? The consequence of this simple fact is that the problem to consider is not sensitivity on initial conditions, but rather structural stability, and to reassure the reader, nothing has yet indicated any instability. We can thus rather confindently continue and swat the mosquitos which sting our tender skin and suck our precious blood.

The pseudo-scientific delirium even led some people to apply the second principle of thermodynamics to the whole universe and predict the ineluctable thermic death of our world. All the blabla and nonsense which can be heard makes us think about the rise of irrationality which is apparently characteristic of human behaviour at the end of each millenary (or... century ?). The role of the medias, always fond of catastrophic news that make people quiver with horror and delight, is obvious but some men of science, perhaps themselves subject to mystic attacks or simply consumed by the desire of fame, adopted a disgraceful attitude and presented, without the care and rigour required by scientifc honesty, 'spectacular' results on chaos theory in 'vulgarising' books, articles, colloquia and seminars throughout the world.

More seriously, the development of chaos theory and the empirical evidence of the existence of chaotic dynamics in the natural sciences has led to a debate on determinism. Some proclaim its death, others its survival, and others cannot understand why all this has led to a debate on determinism. Sometimes, there seems to exist a lack of precision which leads easily to confusions, and some actors of the debate rejoice in the use of abstruse sentences, mistaking hermetism and scientificity[30].

Our purpose is not and cannot be to cite all the arguments put forward pro or contra determinism ; we shall rather expose the problem and show why, in our eyes, the use of the existence of chaotic phenomena as an argument against determinism is erroneous. Note that the bibliography contains a list of works of interest on the topics of methodology in science, causality, determinism and finalism.

The term 'determinism' is often used, and even by scientists, in an approximative and ambiguous way. Definitions with differing connotations abound,

[30]See, for instance, Edgar Morin's attempts to be taken for an epistemologist : *La Méthode* (1977), (1980), (1986), or his mirth-provoking essay in *La Querelle du Déterminisme* (1984).

analytical philosophists, biologists, physicists and mathematicians have in general completely different concepts in mind when using the same word, and it is out of question to dress a list of all of them. Let us first consider here the simplest acceptance, shared by many, but not all mathematicians : they distinguish deterministic dynamical system from stochastic ones. A deterministic description of the evolution of a variable X of a phase space $\mathcal{M}$ is then simply given by the modelisation through a system of differential equations :

$$\begin{cases} \dfrac{dX(t)}{dt} = F(X(t), t), \\ X_0 \text{ given.} \end{cases}$$

Determinism in this acceptance applied to a modelisation of the world is then 'causalism' in the sense of Leibniz' doctrine of the principle of efficient causality, stating that everything has an antecedent, a 'cause' without which it could not exist. This doctrine thus claims that every event is *ontologically* determined. It is the doctrine most obviously inherent in the so often quoted passage of Laplace :

> "We must therefore consider the present state of the universe as the effect of its former state, and as the cause of the one which will follow..."
>
> Pierre Simon de Laplace (1814), *Essai philosophique sur les probabilités.*

The attempts to reconcile the notions of freedom and causalism[31] then may

[31] ...like the theory of 'pre-established harmony' of Leibniz himself...

appear as intellectual acrobatics that only underline the inherent contradictions.

Stochastic modelisation is intended to describe 'aleatory' events. We will come back to the problem of this notion in the following. Consider for simplicity a discrete time dynamic model. The idea is that, given the value of X at a date t, we 'know' that there exists a set, finite or infinite, of possible states at $t + 1$, and a law of probabilities defined of this set. The consequence is that we cannot describe the true state at $t + 1$, but only form expectations and describe an anticipated value. The existence of situations that we describe by the use of probability distributions finds several interpretations. There are, of course, all the situations where the obvious complexity of the system leads us to build statistical models even if we know that *in theory* we could write down a standard deterministic model. Furthermore, some people claim that 'hazard' exists and rules the world[32], others consider that stochastic modelisation often accounts for situations where there exist 'hidden' variables[33], and others share this conception and furthermore defend the attitude consisting in saying that stochastic models are simply deterministic models in higher dimensional spaces ; this is the position defended, amongst others, by René Thom[34]. Thom argues that even in classical mechanics, reality is not described in the 'natural' $\mathbb{R}^3$ space,

[32]Prigogine and Stengers (1979), *La Nouvelle Alliance.*

[33]For instance in quantum physics.

[34]See (1980), *Parabole e Catastrofi.*

but in the phase space $\mathbb{R}^6$ formed by the vectors (position, kinetic moments). Thus, to instaure determinism, the dimension of the space has been increased. But this is exactly what is done when we build a stochastic model. Instead of considering a classical deterministic system $(\mathcal{M}, F)$, we look at a stochastic model where a probability distribution $m(x)$ has an evolution governed by the associated Fokker-Planck equation

$$\frac{\delta m}{\delta t} = F(m),$$

F Lie derivative.

Doing this we change the phase space, substituting the space $C(\mathcal{M})$ of real smooth functions on $\mathcal{M}$ to the initial space $\mathcal{M}$. Determinism is thus reinstaured and

> "...on ne peut pas faire autrement."[35]
>
> René Thom (1984), "Halte au Hasard, silence au bruit" in
>
> *La Querelle du Déterminisme.*

In this sense, we can consider that quantum mechanics are as deterministic as classical mechanics, the physical reality at the quantum level being not the particle anymore but the wave function which follows Schrödinger's equation, an equation which is certainly more complicated although of the same type as the equations ruling classical mechanics.

[35] Approximately : "we cannot proceed differently".

The progress of science has shown that a same reality can admit several types of descriptions. In quantum physics, this could be seen in the duality of the respective conceptions of Heisenberg and Schrödinger. The results on sensitivity on initial conditions now illustrate the fact that a purely deterministic dynamical system as well as a stochastic model can both describe in a satisfactory manner an evolution for which we do not dispose of very large data sets. Unfortunately, imprecision and confusion has led some people to believe that chaos is then the consequence of hazard, of the existence of purely aleatory events in our world. This is a bad mistake.

What indeed is the signification of 'aleatory' in its strictest sense ? A process is aleatory if it can neither be simulated by any mechanism nor described by any formalism. This is why the theory of probabilities is not a theory of aleatory events in the strict sense, since we suppose that there is a certain regularity in the phenomenon that allows us to define a probability distribution and describe the evolution in a certain space. Hazard, on the contrary, is the unthinkable, the undescribable. To claim the existence of 'hazard' is simply tantamount to adopting the anti-scientific position considering that there exist natural phenomena that we will never be able to describe and understand. As Thom notes

> "...this means to renew the position of the famous *Ignorabimus* of Du Bois-Reymond, to resuscitate the wave of irrationality and anti-scientism of the years 1880-1890, the one of the apostles of the "crisis of science" : Boutroux, Le Roy..."
>
> René Thom (1984), *ibid.*

Is the world subject to determinism or do there exist aleatory events, irreducible to any description ? This question is of metaphysical nature. For pure consistency reasons, the position of a scientist must be optimistic and postulate that nothing, in the field of nature, is unknowable *a priori*. The glorification and hypostasis of 'hazard' by a Nobel prize winner like Ilya Prigogine must fill us with astonishment.

Furthermore, the 'logic' behind reasonings invoking the existence of chaotic phenomena like the Belousov-Zhabotinsky reaction[36] to conclude on the 'non deterministic' character of Nature itself leaves us amazed. First of all, no scientist, even in the field of mathematical physics, believes anymore that a mathematical model is the direct expression of reality. If the model is deterministic and gives a good description of reality, then this does not mean that reality obeys to causal laws, but simply that the deterministic description is efficacious. Of course, the converse also holds. So we may ask : why should the finding of situations well described by models with sensitivity on initial conditions have any implication on the 'true' character of Nature ? Secondly, the only possible discussion concerns what one could call, following Popper's definition, 'scientific determinism', that is the doctrine stating that

> "...any event can be rationally predicted with any degree of precision as soon as we dispose of a sufficiently precise description of past events, as well as of all laws of nature."
>
> Karl Popper (1945), *The Open Society and its Enemies.*

[36] Famous chemical reaction between sulphuric acid, malonic acid, natrium bromate and cerium sulphate.

To adhere to scientific determinism in this acceptance does not imply any pretention to give an answer to the problem of the 'true' character of Nature. But note : now we can say that chaos invalidates some notion of determinism : indeed, sensitivity on initial conditions exactly implies that there exist situations where absolute precision, of course completely excluded in practice, is necessary for predictions, and that any infinitesimal error on initial conditions can lead to completely different evolutions. This means the death of the myth of scientific determinism.

We would like to emphasize again the fact that Duhem, reflecting on Hadamard's results on the geodesics of surfaces with negative curvature, came to the conclusion :

> ..."l'idée de conférer un sens physique à la notion, mathématiquement bien définie, de déduction de l'évolution à partir des conditions initiales est un leurre."[37]
>
> Pierre Duhem, *op. cit.*

And this was in 1906...

Some people have claimed the end of determinism and interpreted chaos as the manifestation of hazard in our world. We have seen that many reasons lead to conclude that this type of deduction lacks any logical foundation. Non sequitur, modus ponens, false analogies and many other types of sophisms abound in the pseudo-epistemological litterature which seems to care more

[37]Approximately : "to confer a physical sense to the, mathematically well defined, notion of deduction of the evolution from the initial conditions means to delude one-self".

about sales than deontology. Scientific determinism is dead, but determinism as a possibility survives.

Let us conclude with a last quotation from the work of Thom concerning the essence of scientific investigation :

> "Rappelons cette trivialité : du fait même qu'elle vise à la constitution d'un savoir commun, la science est par essence déterministe. Qu'on le veuille ou non, la science est une entreprise dogmatique, puisqu'elle vise à susciter chez tout observateur la *même* réaction mentale en face d'un *même* donné scientifique, fait ou théorie. Tout modèle est "déterministe" puisqu'il vise à nous dire quelque chose, à spécifier, à déterminer en quelque manière notre connaissance[38]."
>
> René Thom (1984), "En guise de conclusion" in *La Querelle du Déterminisme.*

*

[38] Let us recall this triviality : given that her aim is to constitute a common knowledge, science is in her essence deterministic. Whether we want it or not, science is a dogmatic enterprise, since she aims at inspiring in every observer the *same* mental reaction in front of a *same* scientific 'given', fact or theory. Every model is 'deterministic' since its aim is to tell us something, to specify, to determine in some sense our knowledge.

ECONOMICS

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MATHEMATICS

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CHAPTER I

Tax/Subsidy Schemes in Bubbly Economies

TAX/SUBSIDY SCHEMES IN BUBBLY ECONOMIES

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Abstract

In a three-period OLG model with endogenous growth where young agents must borrow to consume, the condition for the possibility of bubbles does not only reflect parameters of taste and technology : bubbles might simply not be possible if markets are perfect. Constraining borrowings has a positive effect on the rate of growth of capital and can be used to improve the social welfare but, at the same time, increases the possibility of bubbles. We show here that when a bubble appears in the economy, the government can, under certain conditions, reduce the expected welfare loss by relatively simple tax/subsidy schemes. This result is in opposition with the classical results obtained in Diamond-type economies where simple tax/subsidy policies create an IOU which is formally equivalent to the bubble.

Introduction

The interaction between productive and non productive savings in a growing economy has been studied in the setting of neoclassical growth with overlapping generations by Tirole in his well-known papers [9], [10] and [11] ; a recent paper by Grossman and Yanagawa [4] studies the existence and the dynamics of positive bubbles ('fiat money') in an OLG model with endogenous growth à la Romer [8]. The conclusions of the latter are that positive bubbles can exist provided they are not too large and that the rate of growth in the equilibrium without bubbles is larger than the rate of interest. The existence condition reflects parameters of taste and production technology.

In the Tirole model, positive bubbles can exist only if the economy is inefficient, i.e. when there is overaccumulation of capital, and can improve the efficiency of the economy by diverting savings from productive investment. In the Grossman-Yanagawa framework on the contrary, such bubbles cannot have any positive effect for there is already under-accumulation of capital in the fundamental equilibrium, since agents do not internalise the positive externality of capital onto the efficiency of labour, and bubbles can thus only harm, exacerbating the existing distortion. In Grossman and Yanagawa's model, there is no possibility for a simple government intervention : simple tax/subsidy schemes that redistribute income across generations create an IOU equivalent to national debt and have exactly the same effect on capital accumulation as the bubble they are intended to struggle against.

Indeed, in a Diamond-Romer framework, the first agents to be harmed by a bubble, the young of date 1, are not yet born when the bubble appears in the economy. As a matter of fact, their actualised income loss is larger than the value of the bubble ; but there is no possibility of trading with those who sell the unproductive asset at date 0. Therefore, anti-bubble policies have to use an intermediary : the government must tax young agents of date 0 in order to buy the shares of the unproductive asset, then tax young agents of date 1 to subsidize the old agents of date 1, who were the young agents of date 0, etc. This shows why tax/subsidy schemes have the same effect on the economy as the bubble itself : the same amount of capital is diverted from productive investment.

The OLG model considered here is Jappelli and Pagano's [5], who used it to show the effect of borrowing constraints on the rate of growth of capital in an economy with endogenous growth. It is a model with three periods of life where young agents have to borrow to consume, their borrowings being constrained. In this type of model, the first agents to suffer from a bubble appearing in the economy are the young agents of date 0 ; thus, there might exist tax/subsidy policies that do not create an IOU with the same effects on the economy as the bubble. Furthermore, since the Grossman-Yanagawa model showed that the condition for the possibility of bubbles is a condition on the rate of growth of capital in the bubbleless equilibrium and the rate of interest, we shall obtain results about the effects of borrowing constraints on the possibility of bubbles, and their dynamics. The fact that we use here a slightly more general assumption about the form of the agents' utilities (CRRA instead of logarithmic) than Grossman-Yanagawa and Jappelli-

Pagano does not find its roots in the desire of obtaining (slightly) more complex equations, but has a simple explanation : the rate of intertemporal substitution plays an important role since, unless assuming an agents' preference for future consumption, bubbles cannot appear in the economy with perfect markets, i.e. without constraints on borrowing, if this rate is smaller than or equal to 1.

1 The Economy without Bubbles

1.1 The Model

The model we shall study here is the one Jappelli and Pagano [5] used to show the link between liquidity constraints on the consumers' side and the rate of growth in an economy with endogenous growth. There is one consumption good in the economy. People live for three periods : they borrow when young (from a mutual fund, for instance) to finance their consumption, work, repay their borrowing, consume and save for their old age in the second period of their life and consume their savings when old. We suppose that young agents are constrained in their borrowings : they can only borrow a fraction Γ of the present value of their lifetime income. The population is assumed to be stationary and the size of each generation is normalized to $L = 1$ without loss of generality. Labour is provided inelastically. There is no uncertainty in the model and we assume that agents have perfect foresight.

Utilities are assumed to be identical accross agents, additively separable and of the following form (the agents' discount coefficient for time preference is denoted by $\beta = 1/(1+\rho)$, where $\rho \in]-1, +\infty[$ is the rate of time preference.

$$U(C_{t,t}, C_{t,t+1}, C_{t,t+2}) = u(C_{t,t}) + \beta u(C_{t,t+1}) + \beta^2 u(C_{t,t+2}),$$

where $C_{t,t+i}$ is the consumption at time t+i of an agent born at time t, and where u is given by :

$$u(C) = \begin{cases} \dfrac{C^{1-\epsilon}}{1-\epsilon}, \ \epsilon \neq 1, \\ Ln(C), \ \epsilon = 1. \end{cases}$$

The utility chosen is characterized by a constant coefficient of intertemporal substitution $\sigma = 1/\epsilon$, thus slightly more general than in Jappelli-Pagano [5] for reasons that will become clear below.

The technology is a classical Romer (1986)-type one :

$$F(K_t, L_t) = A_t K_t{}^{\alpha} L_t{}^{1-\alpha} = A\bar{K_t}^{1-\alpha} K_t{}^{\alpha} L_t{}^{1-\alpha},$$

where K_t denotes the capital at time t, $\bar{K_t}$ the aggregate level of capital in the economy, L_t the labour input (here, $L_t = 1$). There is a spillover from capital on the efficiency of labour, which firms do not internalize and which gives, in equilibrium, increasing returns to scale at the aggregate level and thus endogenous growth. Firms are supposed to behave competitively, ignoring the externality operating in the production sector and maximising profits using the Cobb-Douglas function $A_t K_t{}^{\alpha} L_t{}^{1-\alpha}$.

The consumer's program is :

$$\begin{cases} maxV_t = \sum_{i=0}^{2} \beta^i u(C_{t,t+i}) \\ C_{t,t} + \frac{C_{t,t+1}}{R_{t+1}} + \frac{C_{t,t+2}}{R_{t+1}R_{t+2}} \leq \frac{e_{t+1}}{R_{t+1}} \\ C_{t,t} \leq \Gamma \frac{e_{t+1}}{R_{t+1}} \end{cases}$$

where $R_{t+1} = 1 + r_{t+1}$, r_{t+1} being the rate of interest on capital from t to $t+1$, and where e_{t+1} is real labour earnings of agents born at time t.

1.2 The Bubbleless Economy

Solving the program, we obtain :

$$C_{t,t} = \Phi \frac{e_{t+1}}{R_{t+1}},$$

$$C_{t,t+1} = \frac{1-\Phi}{1+\beta^{\sigma}(R_{t+2})^{\sigma-1}} e_{t+1},$$

where $\Phi = \Gamma$ if $\Gamma \leq \varphi$ and φ otherwise,

$$\varphi = \frac{1}{1+\beta^{\sigma}(R_{t+1})^{\sigma-1} + \beta^{2\sigma}(R_{t+1}R_{t+2})^{\sigma-1}}$$

being the fraction of actualised lifetime income a young agent would like to consume if there was no constraint on borrowing.

Net aggregate wealth W_t in this economy is income of middle-aged minus their actualised borrowings, their consumption and the borrowings of young people :

$$W_t = e_t - R_t C_{t-1,t-1} - C_{t-1,t} - C_{t,t}$$

$$\Leftrightarrow$$

$$W_t = \frac{\beta^\sigma (R_{t+2})^{\sigma-1}}{1+\beta^\sigma (R_{t+2})^{\sigma-1}}(1-\Phi)e_t - \Phi\frac{e_{t+1}}{R_{t+1}}.$$

If we write :

$$F(K_t, L_t) = A_t L_t F(\frac{K_t}{A_t L_t}, 1) = A_t L_t f(k_t),$$

then the competitive behaviour of the firms implies :

$$r_t = f'(k_t) = \alpha A {L_t}^{1-\alpha} = \alpha A = \rho,$$

$$e_t = f(k_t) - k_t f'(k_t) = (1-\alpha) A {L_t}^{\alpha} K_t = (1-\alpha) A K_t.$$

In the absence of bubbles in the economy, equilibrium in the capital market implies :

$$W_t - K_t = K_{t+1} - K_t.$$

Accumulation of capital is therefore given by the following equation :

$$K_{t+1} = (1+g_t)K_t = [1+g(\Phi)]K_t,$$

$$1 + g(\Phi) = \frac{\beta'}{1+\beta'}\frac{1-\Phi}{1+\Phi\dfrac{1-\alpha}{\alpha}}(1-\alpha)A,$$

where $\beta' = \beta^\sigma (1+\rho)^{\sigma-1}$. Since the rate of interest is constant over time, the expression of φ is simplified : $\varphi = 1/(1+\beta'+\beta'^2)$.

It is obvious that $g(\Phi)$ is decreasing in Φ ; we have therefore the result that the rate of growth of capital is an increasing function of the strength of the borrowing constraints. The parameter Γ could thus be considered as a policy instrument : in fact, a government could choose this parameter accordingly to a welfare criterion ; there is a trade-off between growth of capital and consumption of young agents, and Pareto-improvement constraints must be respected. We give in appendix A, for the special case $\epsilon = 1$, an example of calculus of the optimal Γ according to a welfare criterion, and show that fixing Γ in such a way may lead to the possibility of bubbles in an economy where bubbles cannot appear if young agents are not constrained.

2 Bubbles in the Economy

As there is only one consumption good and one productive capital good in this economy, the price of real capital cannot increase geometrically and therefore no bubble can exist on productive assets.

So let us consider an intrinsically useless asset, in the hands of old people at time 0, and in supply M. Under which conditions can a bubble exist on this asset ? If an agent is willing to buy shares of the unproductive asset, he must expect to get a rate of return at least equal to that of capital. The no-arbitrage condition yields :

$$B_{t+1} = R_{t+1} B_t = (1 + \rho) B_t,$$

where $B_t = p_t M$, p_t being the price of one share of the asset as of time t.

Let us suppose that all agents share the belief that the asset yields a rate of return equal to the rate of interest at each period. Who will buy the shares at time 0 ? Young agents have to borrow to consume. Without the assumption that these agents internalize the effect of the bubble on their future wages, which is an assumption difficult to defend in a price-taker economy, only middle-aged agents will be willing to buy shares of the unproductive asset. The transmission mechanism of a bubble in a price-taker economy should therefore be : at each time t, old agents sell the asset to middle-aged agents.

2.1 Dynamics of the Economy with Bubbles

The equilibrium of the capital market gives now :

$$W_t = K_{t+1} + B_t.$$

If we denote by b_t the value of the bubble per unit of capital ($b_t = B_t/K_t$), which is straightforwardly related to the value of the bubble per unit of productivity of labor, then we can write :

$$b_{t+1} = \frac{1+\rho}{1+g_t} b_t$$

$$\left(1 + \Phi\frac{1-\alpha}{\alpha}\right)(1+g_t) = \frac{\beta'}{1+\beta'}(1-\Phi)(1-\alpha)A - b_t$$

which we rewrite under the following form :

$$\frac{u(\Phi)}{1+\rho}(1+g_t) = v(\Phi) - b_t.$$

Eliminating g_t, we obtain the dynamics of the reduced bubble (b_t) :

$$b_{t+1} = \Psi(b_t, \Phi)$$

$$\Psi(x, y) = \frac{u(y)x}{v(y) - x}.$$

The necessary condition for the possibility of bubbles is that a portion of the curve Ψ lies beneath the 45 degree line, condition which we can write here, since $\Psi(0) = 0$ and Ψ is strictly convex in x :

$$\frac{\partial \Psi}{\partial x}(0, y) = \frac{u(y)}{v(y)} < 1$$

$$\Leftrightarrow \quad g(\Phi) > \rho.$$

This condition, which says that, for bubbles to be possible, the rate of increase in capital of the bubbleless economy must be greater than the rate of interest, is of course similar to the one in the Grossman-Yanagawa model. But here, we are working with a three period OLG model where young agents must borrow, which implies that the condition for the possibility of bubbles reflects not only parameters of taste and technology, but also market imperfections through the parameter Γ. It is easy to see that, unless we assume strict preference for future consumption, bubbles cannot exist in our model when markets are perfect and σ, the rate of intertemporal substitution, is smaller than or equal to 1 : the rate of increase in capital is always to low to allow any bubbly belief to be consistent :

$$\beta \leq 1 \quad \text{and} \quad \sigma \leq 1 \Rightarrow g(\varphi) = g\left(\frac{1}{1 + \beta' + \beta'^2}\right) < \rho,$$

whatever are the parameters α, A and β. Indeed, as we have seen previously, g is decreasing in Φ. Therefore, a necessary condition for the possibility of bubbles is :

$$\begin{cases} g(0) > \rho, \\ \Gamma \in]0, \bar{\Gamma}[, \quad g(\bar{\Gamma}) = \rho. \end{cases}$$

Thus, if $\beta \leq 1$ and $\sigma \leq 1$, bubbles can only exist in constrained economies ($\Phi = \Gamma$). Furthermore we see that by strengthening the constraint on borrowing (i.e. by reducing Γ), bubbles may become possible in a world where bubbles could not exist before (see Appendix A). The situation is a bit different if $\sigma > 1$. As a matter of fact, under this condition and even when agents prefer future consumption, $\bar{\Gamma}$ might be larger than φ, which means that bubbles can appear even if young agents are not constrained. In Appendix B, we give the formal proof of this result.

The only admissible values for b_0 are those which are smaller than the value of the stationary reduced bubble :

$$b^*(\Gamma) = v(\Gamma) - u(\Gamma).$$

Indeed, a belief such that $b_o > b^*(\Gamma)$ would be inconsistent because, at some date, B_t would exceed the capital stock of the economy. Hence, the possible bubbles are the stationary one and the vanishing bubbles starting from below the stationary value. The value of the stationary bubble is decreasing in Γ ; therefore, the more constrained the economy, the larger the range of possible bubbles.

In the presence of the stationary bubble, the economy's rate of growth of capital is always $g_t = \rho$. If a bubble starts from below the stationary value, then this rate is always greater than the rate of interest, strictly increasing and asymptotically equal to $g(\Gamma)$, the rate in the bubbleless economy. Furthermore, the speed of convergence is decreasing in Γ, in the sense that a bubble that is admissible in two economies vanishes quicker in the more constrained one, since :

$$\left(\frac{\partial b_{t+1}}{\partial \Gamma}\right)_{b_t} > 0$$

and

$$\left(\frac{\partial b_{t+1}}{\partial b_t}\right)_\Gamma > 0.$$

If $\Gamma' < \Gamma$, then the range of possible bubbles is larger in the Γ'-economy. We know that in the bubbleless equilibrium, the rate of growth of capital is larger in this economy. It is clear that a bubble which is admissible in the two economies does not alter the relation $g_t(\Gamma) < g_t(\Gamma')$; yet bubbles which are not consistent in the Γ-economy ($b_0 > b^*(\Gamma)$) can appear in the Γ'-economy. Such bubbles can lower the rate of increase in capital beneath $g(\Gamma)$, the rate of growth in the Γ-economy, for ever in the case of a stationary bubble, for a finite lapse of time otherwise. Thus, even if the effect of constraining young people is positive in the case of bubbleless equilibria, it is ambiguous if we admit the possibility of bubbles.

2.2 Effects on the Welfare

The effect of bubbles on the welfare of generations born after or at time 0 is obviously negative. Indeed, in an economy with endogenous growth, there is under-accumulation of capital in the bubbleless situation ; bubbles divert capital from saving, and can therefore only have a negative effect. This point has been insisted upon by Grossman and Yanagawa [4] : whereas a stationary bubble has a positive effect in a Diamond economy (see Tirole [10], [11]) by removing the intertemporal inefficiency, the effect of bubbles in a Romer-OLG model is always an exacerbation of the existing distortion.

Let us suppose that a bubble of value B_0 appears at time 0. The generation that is old at t=0 sales a new asset and therefore increases its welfare.

Middle-aged agents, which are those who buy the unproductive asset, are not affected because their labour income is already determined and the rate of interest fixed. But young agents born at time 0, and all subsequent generations will suffer from the existence of the bubble, labour income being reduced (the growth of labour productivity being lowered by the bubble...). The actualised income loss of the young agents of time 0 (in fact, the result holds for all generations born after time 0) is large enough to compensate the old of time 0 for their gain from the unproductive asset ($\hat{e}_1$ denotes the income in the bubbleless equilibrium) :

$$\frac{\hat{e}_1 - e_1}{1+\rho} = \frac{(1-\alpha)A}{(1+\rho)\left(1+\Gamma\frac{1-\alpha}{\alpha}\right)} B_0 > B_0.$$

But, first of all, young agents are constrained in their borrowings and, secondly, young agents would have to internalize the negative effect of the bubble on their life-time income, and this constitutes an unacceptable assumption in a pure price-taker economy in which agents are supposed not to internalize the effects of their consumption choices on the level of capital, the rate of interest etc.

Nevertheless, we could imagine a public intervention to get rid of the bubble or, at least, to lower its effects. Indeed, in our framework, there might exist situations where tax/subsidy policies will not necessarily have the same effects on welfare as the bubble : in that, a three generations OLG model differs fundamentally from a Diamond model where the IOU created by a simple tax/subsidy scheme has exactly the same effects as the bubble, diverting the same amount of capital from productive investment as the bubble (see Grossman-Yanagawa [4]).

3 Government Intervention

In the preceding sections, we have studied the different equilibria of the economy without considering government intervention. But it is clear that the type of economy considered here can offer possibilities for Pareto-improving policies in the bubbleless equilibrium since the rate of growth of capital is an increasing function of the strength of the borrowing constraint imposed on young agents. We can imagine two simple ways of improving the social welfare : the government could fix Γ, that means regulate borrowings, or it could intervene indirectly through a tax/subsidy system, taxing young agents and increasing investment (injecting the whole amount in the productive capital or only a part, giving some fractions to middle-aged and old agents..., according to the welfare criterion the government is working with...). In Appendix A, we give an example of optimal regulation, according to a classical welfare criterion, for the special case $\sigma = 1$.

As we shall see in the following part, and as is easy to understand, if the government already uses an optimal regulation policy, then there is no possibility left for Pareto-improving intervention through simple tax/subsidy schemes in the case of a bubble appearing in the economy : the borrowing constraint has then been totally exhausted in a certain sense. The same is true of course when there is an optimal tax/subsidy policy in place before a bubble appears. But if we think about government intervention, we realise that it is never chosen optimally according to a welfare criterion, for many reasons (one very simple amongst others is probably the desire of being re-elected : $\Gamma e_{t+1}(\Gamma)/R_{t+1}$ decreases with Γ decreasing, and the agents are supposed myopic...). Thus, we believe that it is not too shocking an assumption to suppose that the policy undertaken by the government in the bubbleless equilibrium is not necessarily optimal. The results which follow are obtained under the assumption of an already, but non optimally, constrained economy. The existence of a Pareto-improving public policy obtained under this assumption obviously guarantees the possibility of Pareto-improvement in unconstrained economies.

In this section, we shall consider the possibility of government intervention in the case of a bubble under the assumption of a policy of borrowing-regulation before the bubble appears. We do not treat explicitly the assumption of tax/subsidy schemes in the bubbleless economy because the principal result remains unchanged : if the policy undertaken is Pareto-improving, but not 'radical', then there can exist possibilities for government intervention when a bubble appears. So let us consider now the economy with $\Gamma \in]0, \varphi]$ (if there is regulation before a bubble appears, then $\Gamma \leq \varphi$; the question of possibilities for government intervention is trivial otherwise). We shall look for 'anti-bubble' policies that are optimal according to some criterion. This could give the impression of inconsistence with what we've said before, but consider that when b_0 is small, then an optimal intervention seems to be feasable ; for 'large' values of b_0, if there exists an optimal policy, then there exist non-optimal but feasable policies.

In an economy where young agents have to borrow to consume, there might exist ways of lowering the effects of a bubble appearing at date 0. Indeed, if young agents of time 0 internalized the negative effect of the bubble on their income, they could try to buy some shares of the unproductive asset, increasing their future income by diverting part of the bubble from the investment market. Of course, there is a trade-off between increasing future income and constraining consumption when young. Young agents therefore would choose to buy an optimal amount of shares according to this trade-off. This would occur at each subsequent date t, and agents would solve the following program :

$$\begin{cases} max \sum_{i=0}^{2} \beta^i u(C_{t,t+i}) \\ (1)\ C_{t,t} + \dfrac{C_{t,t+1}}{R_{t+1}} + \dfrac{C_{t,t+2}}{R_{t+1}R_{t+2}} \leq \dfrac{e_{t+1}(\theta_t)}{R_{t+1}} \\ (2)\ C_{t,t} + \theta_t B_t \leq \Gamma \dfrac{e_{t+1}(\theta_t)}{R_{t+1}}. \end{cases}$$

Since we work with the assumption of price-taking agents, young agents are supposed not to do this : they remain passive. But let us suppose that the government wants to intervene without taking strong centralised decisions as in the sense of a central planer who would choose the consumption paths ; the government is supposed to let agents choose themselves their consumption at each time t. The government could try to buy progressively the unproductive asset to make it disappear from the investment market. To finance this policy, the government taxes young agents and assures them to give, in the future, subsidies equal to the actualised amount of the taxes.

Let us formalize this : let us suppose that at time t, the government levies taxes $T_{t,t}$ of value $T_{t,t} = \theta_t B_t$ on young agents and pays subsidies of value $S_{t-1,t} = (1+\rho)T_{t-1,t-1}$ to middle-aged agents of time t ($B_t = (1+\rho)^t B_0$ being the actualised value of the initial amount of non productive asset). We suppose that the sequence (θ_t) is chosen increasing. With the surplus, the government buys an amount $T_{t,t} - S_{t-1,t} = (\theta_t - \theta_{t-1})B_t$ of shares of the non productive asset.

But how does the government make the choice of the sequence (θ_t) ? A conceptually very simple way would consist in choosing (θ_t) according to the program young agents of time t would solve if they internalized the effects of the bubble (and only its effects) and knew the capital accumulation equation. The condition to make this possible is that the sequence (θ_t) thus obtained is increasing (taxes must finance subsidies..). If this is the case, and if the government chooses the sequence according to this scheme, it will obviously achieve a Pareto-improvement compared to non-intervention. Of course, this kind of behavior puts all the weight on young agents of date t, as it does not take into account the effects of intervention at time t on subsequent generations. To choose a sequence whilst considering the consequences on not yet born agents, the government needs a choice criterion. Such a criterion could be of the following form :

$$\begin{cases} max \sum_{t=\tau}^{\Delta+\tau} R^t V_t(\theta_\tau, ..., \theta_{t-1}, \theta_t) \\ (P) \;\; V_t(\theta_\tau, ..., \theta_{t-1}, \theta_t) \geq V_t(0, ..., 0) \;\; \forall t \in [\tau, \Delta + \tau]. \end{cases}$$

$$(0 \leq R)$$

where V_t is the indirect utility function of an agent born at time t, agents solving the program :

$$\begin{cases} max \sum_{i=0}^{2} \beta^i u(C_{t,t+i}) \\ (1) \; C_{t,t} + \dfrac{C_{t,t+1}}{R_{t+1}} + \dfrac{C_{t,t+2}}{R_{t+1}R_{t+2}} \leq \dfrac{e_{t+1}}{R_{t+1}} \\ (2) \; C_{t,t} + T_{t,t} \leq \Gamma \dfrac{e_{t+1}}{R_{t+1}}. \end{cases}$$

(P) is the set of Pareto-improvement constraints. Taxes and subsidies do not appear in the budget constraint (1) since the government announces that subsidies received when middle-aged equal actualised taxes paid when young. Δ is a decision horizon and R a social discount factor ; if $R = 0$, then the government cares only about the current generations and is absolutely indifferent to not yet born agents : this is of course the decision criterion we first described. We shall see that the case $R = 0$ is interesting not only because it gives a simple way of achieving a Pareto-improvement, but also because it allows us to characterize the type of dynamics of the economy in the case $R \neq 0$, without solving formally the optimisation problem, which looks rather tedious if Δ is large. We shall see that if the tax/subsidy scheme allows an improvement in social welfare in the case $R = 0$, then the optimal policy is a finite sequence in the sense of θ_t equalling 1 after a finite lapse of time, the policy becoming a pure tax/subsidy scheme once the entire bubbly asset eliminated from the market, and necessarily the policy chosen with a $R \neq 0$-criterion will be finite too.

3.1 A Simple Tax/Subsidy Policy

We shall give here the results for the optimisation problem under the assumption $R = 0$, i.e. a government which, at each date, cares only about the present generation. The solution of the problem of the choice of (θ_t) is the following :

Proposition : *If $\hat{\phi}(\bar{\Gamma}) \geq \bar{\Gamma}$, where $\hat{\phi}$ is given by :*

$$[\hat{\phi}(\Gamma]^{-1} = 1 + (\beta' + \beta'^2)\Big(1 + \frac{(1-\alpha)A}{(1+\rho)(1+\Gamma\frac{1-\alpha}{\alpha}) - \Gamma(1-\alpha)A}\Big),$$

then the optimal choice of θ_0 is 0, and it is easy to verify that at each date t, the government will choose $\theta_t = 0$, because the condition for this is structural (only related to the parameters of taste and technology and not to the size of the bubble...).

Otherwise, since the function $\hat{\phi}$ is strictly increasing and concave in its variable Γ, there exists a value $\hat{\Gamma}$ such that :

$$\{(1)\ \Gamma \in]0, \hat{\Gamma}]\} \ \Rightarrow\ \theta_t = 0\ \ \forall t.$$

$$\{(2)\ \Gamma \in]\hat{\Gamma}, \bar{\Gamma}[\} \ \Rightarrow\ \theta_t > 0\ \ \forall t.$$

Furthermore, if (2) *holds, then the sequence* (θ_t) *is strictly increasing until* $\theta_t = 1$*, which always occurs after a finite lapse of time* T^0.

The proof is given in Appendix C, where we exhibit parameter values which show the existence of economies such that $\hat{\phi}(\bar{\Gamma}) < \bar{\Gamma}$. Appendixes A and C show us furthermore the following results for the case $\epsilon = 1$: if we denote by $\Gamma_{R=0}$ the value of Γ which is optimal for young agents of date 0 given K_0, then we have an upper bound for all Γ's that are optimal according to classical welfare criteria. But $\hat{\phi}(\Gamma_{R=0}) > \Gamma_{R=0}$, which implies that there are no possibilities for tax/subsidy schemes if Γ has been fixed optimally, whatever is the welfare-criterion the central-planner is working with. Another consequence of the strict inequality is that $\Gamma > \Gamma_{R=0}$ does not imply the possibility of Pareto-improving intervention through a policy of the type considered here. It is even possible to exhibit parameters such that $\Gamma_{R=0} < \bar{\Gamma} < \hat{\Gamma}$; this means that there exist economies in which policies of the type considered here cannot be undertaken, even if the parameter Γ has not been chosen very close to $\Gamma_{R=0}$.

We suppose now that the condition holds, which means that the government policy can achieve a Pareto-improvement by converting progressively, and in a finite lapse of time, the bubble into a sort of public debt forced onto young agents ; formally, the variables of the economy evolve as if the bubble, which would have been transmitted from old to middle-aged agents, was now progressively bought by young agents, the economy reaching, after a finite lapse of time, a regime with the bubble transmitted from middle-aged to young agents. When the government applies the $(R = 0)$-optimal policy, the capital accumulation is given by (we pose $\theta_{-1} = 0$) :

$$\left(1 + \Phi\frac{1-\alpha}{\alpha}\right)(1 + g_t) = \frac{\beta'}{1+\beta'}(1 - \Phi)(1 - \alpha)A - (1 - \theta_t)b_t + \frac{\beta'}{1+\beta'}\theta_{t-1}b_t$$

where $b_t = B_t/K_t$. For $T \geq T^0 + 1$, $S_{t-1,t} = B_t$ and therefore we have :

$$\left(1 + \Phi\frac{1-\alpha}{\alpha}\right)(1 + g_t) = \frac{\beta'}{1+\beta'}(1 - \Phi)(1 - \alpha)A + \frac{\beta'}{1+\beta'}s_t,$$

where $s_t = S_{t-1,t}/K_t$. This shows us that the rate of growth of capital is increased, at least in the short run, by the action of the government, and even beyond the rate of growth of capital in the bubbleles economy. This is not surprising : to achieve a Pareto-superior outcome, the government uses a policy that is formally equivalent to an increased constraint on borrowing for young agents, which implies increased savings, and we know that more saving means increased rate of growth of capital. The effect on the growth-rate is asymptotically null : s_t vanishes to 0 as is rather obvious and nevertheless shown in the appendix.

The dynamics of b_t and s_t, equal to b_t after time $T^0 + 1$, are given in the appendix. As we have said before, we prove in appendixes A and C, for the case $\epsilon = 1$ (this is the only case which can be solved formally ; nonetheless, it seems quite reasonable to conjecture that the results hold in general), that if Γ has been fixed by a central planner, which internalizes the effect of this parameter on the rate of growth of capital, according to a social welfare criterion, then $\hat{\phi}(\Gamma) \geq \Gamma$ and there is no possibility for a

Pareto-superior outcome by simple tax/subsidy schemes. This is easy to understand : a central planner chooses a Γ that is less or equal to the Γ that maximises the welfare of the current young generation (he chooses a smaller value if he cares about future generations...). If Γ has been fixed by a central planner, then the increase in future consumption resulting from the constraint on first period consumption and from the elimination of a part of the bubble does not exceed the loss in consumption of young agents. Thus, if Γ has been fixed optimally, any attempt to struggle against a bubble using a tax/subsidy policy necessarily diminishes the welfare of early generations. Furthermore, as we have seen before, the fact that there exists a possibility for Pareto-improvement in the bubbleless equilibrium ($\Gamma > \Gamma_{R=0}$) does not imply the possibility of tax/subsidy schemes when a bubble appears, and there even exist economies $(\sigma, \beta, A, \alpha)$ in which tax/subsidy policies never lead to a Pareto-superior outcome in the case of a bubble, whatever be the parameter Γ.

3.2 Caring about Future Generations

If the government cares about future generations ($R \neq 0$), the policy described above yields a Pareto-improvement, but is not optimal in the sense of the choice criterion. As the problem of finding the (R, Δ)-optimal policy is rather difficult and tedious, we shall give qualitative characterizations of the optimal sequence $(\theta_t^{R,\Delta})$.

In the case $\Gamma \in]0, \hat{\Gamma}]$, the government cannot apply the tax/subsidy scheme described here and achieve a Pareto-superior outcome since agents of early generations would incur a welfare loss, gains in lifetime income not counterbalancing the loss in consumption when agents are young, at least in the first periods.

So let us now consider the case ($\Gamma \in]\hat{\Gamma}, \bar{\Gamma}[$) where Pareto-superior outcomes are possible. If $R \neq 0$, the government cares about the effect of a choice θ_t on generations born after time t ; intuitively, this should imply that $\theta_t^{R,\Delta}$ is bigger than $\theta_t^{R=0}$ or $\theta_t^{R,\Delta} = \theta_t^{R=0} = 1$. The larger R or Δ, the more the government cares about the increase in capital and the less it is concerned with the constrained consumption of the current young. Therefore $\theta_t^{R,\Delta}$ should be increasing in R and in Δ.

We give the formal proof of the result of $\theta_0^{R,\Delta}$ increasing (in a large sense) in R. The dependence in Δ can be established in a similar but, because Δ is a discrete parameter, more difficult manner. Considering the structure of the economy and of the choice problem, it is immediate to deduce : $[\theta_0^{R,\Delta}$ increasing in R and $\Delta] \Rightarrow [\theta_t^{R,\Delta}$ increasing in R and $\Delta\ \forall t]$.

- If $\theta_0^{R=0} = 1$ (which occurs when the value of the bubble is small), then of course $\theta_t^{R,\Delta} = 1$.
- If $\theta_0^{R=0} \neq 1$, then there are two possibilities : either (a) $V_0(1) < V_0(0)$ or (b) $V_0(1) \geq V_0(0)$. V_0 is a strictly concave function of θ_0, and under the assumption on Γ, admits a maximum on the interval $]0,1[$. If (a) holds, then there is a θ^0 such that $V_0(\theta^0) = V_0(0)$. This is the maximum value the government can choose at date 0 if it tries to achieve a Pareto-superior outcome. If (b) holds, then 1 is the largest possible value for θ_0.

As long as the solution of the unconstrained maximisation problem is smaller than the highest possible θ_0-value, we have, at the optimum :

$$\frac{\partial V_\tau}{\partial \theta_\tau} + \sum_{t=\tau+1}^{\Delta} R^{t-\tau} \frac{\partial V_t}{\partial \theta_\tau} = 0,$$

since V_τ is independent of θ_t, $t \geq \tau + 1$. But the effect of an increase of θ_τ on the welfare of generations born after or at time $\tau + 1$ is unambiguously strictly positive. Thus

$$\frac{\partial V_0}{\partial \theta_0} < 0$$

at the optimum. This implies $\theta_0^{R,\Delta} > \theta_0^{R=0}$. But we can show more : the function

$$F(R,\theta) = (\frac{\partial V_\tau}{\partial \theta_\tau} + \sum_{t=\tau+1}^{\Delta} R^{t-\tau} \frac{\partial V_t}{\partial \theta_\tau}, \tau = 0...\Delta)$$

is obviously C^∞ in its variables, and we have $\partial_\theta F$ inversible since each V_τ is strictly concave in θ_t, $t \leq \tau$. From the implicit function theorem we deduce :

$$\frac{\partial \theta_0^{R,\Delta}}{\partial R} = \frac{-\sum_{t=1}^{\Delta} t R^{t-1} \frac{\partial V_t}{\partial \theta_0}}{\sum_{t=0}^{\Delta} R^t \frac{\partial^2 V_t}{\partial {\theta_0}^2}}.$$

But

$$\sum_{t=1}^{\Delta} t R^{t-1} \frac{\partial V_t}{\partial \theta_0} > 0$$

as a sum of strictly positive terms. Thus we have :

$$\frac{\partial \theta_0^{R,\Delta}}{\partial R} > 0.$$

If the $P(0)$ constraint is binding, then $\theta_t^{R,\Delta} = \theta^0$ (case (a)) or 1 (case (b)), which is always larger than $\theta_t^{R=0}$.

The general and rather obvious result is therefore : the more the government cares about future generations, the more shares of the unproductive asset it will buy at each date, but there is a limiting sequence (θ^t) which the government must respect to achieve a Pareto-superior outcome. Thus, $T^{R,\Delta}$ is decreasing in its arguments but has a Pareto-limiting lower bound $\underline{T}$. The dynamics of b_t and s_t are of course of the same type as those in the case $R = 0$, the convergence being accelerated.

CONCLUSION

In the OLG framework considered here, we have seen that there exist economies in which bubbles can appear only because markets are imperfect : for the parameters of taste and technology that characterize the economy, if markets are perfect and agents prefer, in a large sense, present consumption, then the rate of growth of capital is always too low to allow bubbly beliefs to be consistent. On the other hand, we have shown that there exist economies where young agents are not constrained in their borrowings and where bubbles can appear. If we rule out preference for future consumption, then a necessary condition for this is, in our framework, that the rate of intertemporal substitution is large ($\sigma > 1$). The more constrained an economy with given parameters of taste and technology, the larger the range of possible bubbles.

Many of the conclusions of Grossman and Yanagawa's model hold, of course, in the three-period OLG model considered here. This is not very astonishing, for young agents cannot internalize the effects on their welfare of a bubble and therefore do not have the least incentive to intervene in the market for the unproductive asset, which means that the transmission mechanism of the bubble is similar to that in the Grossman-Yanagawa model : the bubble is sold by the old agents that live on their savings and is bought by the agents that save (the young in the Diamond model, the middle-aged in the present one). Thus, the effect of bubbles on the rate of growth of capital is negative and asymptotically zero except in the case of the stationary bubble.

But the main object of our model was to show that although it is true in a Diamond model that eliminating a bubble by a simple tax/subsidy scheme does not yield any improvement, this might be false in a three period OLG model where young agents must borrow and face market imperfections, here in the form of a simple, linear constraint on borrowing. Indeed, the government might be able to improve the welfare of all agents by a policy, financed by a tax/subsidy scheme, of progressive elimination of the unproductive asset. When total elimination of the bubbly asset is achieved, the rate of growth of capital is larger than it would have been in the absence of the bubble, because the policy consists in restraining the consumption of young agents, thus increasing the savings. This effect is, of course, asymptotically zero since the ratios of taxes and subsidies over capital decrease to zero. Of course, the possibility for Pareto-improving policies of the type described here is given only if the bubbleless economy allows an improvement by stronger constraints on borrowing (see Appendixes A and C). This means that in an economy where, in the absence of bubbles, Γ has been fixed optimally by a central planner or where the central planner uses an optimal tax/subsidy scheme to maximise the social welfare, the Pareto-improvement possibilities by increased borrowing constraints have been totally exhausted : in such an economy, when a bubble appears, no tax/subsidy scheme of the type described above can lead to a Pareto-superior equilibrium. Yet, though non-optimality of the policy undertaken by the government before a bubble appears is a necessary condition for the possibility of tax/subsidy schemes in the case of a bubble, it is not sufficient. There exist economies in which borrowing constraints are rather far from being 'optimally' exploited and where, nevertheless, no policy of the form considered here can achieve a Pareto-superior outcome.

APPENDIX A :

Optimal Constraining

The object of this appendix is to show that if, for social welfare reasons holding in the bubbleless economy, the government fixes Γ, then it can happen that it chooses a value such that bubbles become possible whereas they could not exist in the unconstrained economy. We suppose $\epsilon = 1$ to enable us to exhibit formal solutions (this cannot be done otherwise) ; we then have $\beta' = \beta$.

We suppose that at date 0, the economy is characterized by Φ, this parameter being equal to φ if there has been no constraint on borrowing until date 0, and equal to some $\Gamma_0 < \varphi$ otherwise. We consider the optimal choice of Γ in an economy that is not subject to central planing (a central planner fixes Γ, but the agents choose their consumption path according to their own desire). We treat a formally simple case where the central planner has a criterion of the following classical form :

$$\max \bar{W}_\Delta = \sum_{t=0}^{\Delta} R^t[Ln(C_{t,t}) + \beta Ln(C_{t,t+1}) + \beta^2 Ln(C_{t,t+2})],$$

and we make the assumption that the social discount factor R is in $[0,1[$ for simplicity (the result holds in the general case $R \geq 0$, the restriction is done here for pure calculus reasons in order to obtain nice formulas) ; the horizon Δ can be finite or infinite (we have made the asumption $R < 1$). As the choice of Γ at t=0 does not have any effect on the welfare of people born before this date, we do not have to be preoccupied by them. To achieve an outcome that Pareto-dominates the unconstrained economy, the central planner must solve the following problem :

$$\begin{cases} \max \sum_{t=0}^{\Delta} R^t V_t(\Gamma) \\ (P) \;\; V_t(\Gamma) \geq V_t(\Phi) \;\; \forall t \in [o, \Delta], \end{cases}$$

where V_t is the indirect utility function of an agent born at time t, agents solving the program :

$$\begin{cases} \max \sum_{i=0}^{2} \beta^i Ln(C_{t,t+i}) \\ C_{t,t} + \dfrac{C_{t,t+1}}{R_{t+1}} + \dfrac{C_{t,t+2}}{R_{t+1}R_{t+2}} \leq \dfrac{e_{t+1}}{R_{t+1}} \\ C_{t,t} \leq \Gamma \dfrac{e_{t+1}}{R_{t+1}}. \end{cases}$$

We have :

$$\Big(1 + \Gamma\frac{1-\alpha}{\alpha}\Big)(1+g_1) = \frac{\beta}{1+\beta}(1-\Phi)(1-\alpha)A,$$

and

$$\Big(1 + \Gamma\frac{1-\alpha}{\alpha}\Big)(1+g_t) = \frac{\beta}{1+\beta}(1-\Gamma)(1-\alpha)A.$$

Let us solve first the unconstrained problem. The objective function is the following :

$$\max W_\Delta = \sum_{t=0}^{\Delta} R^t \Big[Ln(\Gamma) + (\beta+\beta^2)Ln(1-\Gamma)$$
$$+ (1+\beta+\beta^2)\Big(tLn(\Gamma) - (t+1)Ln(1+\Gamma\frac{1-\alpha}{\alpha})\Big)\Big].$$

Maximising this function, we get :

$$\Gamma_{R,\Delta} = \frac{\alpha}{\alpha+\beta+\beta^2+(1+\beta+\beta^2)\dfrac{1-R}{1-R^{\Delta+1}}\sum_0^{\Delta} tR^t},$$

for Δ finite and :

$$\Gamma_{R,+\infty} = \frac{\alpha}{\alpha+\beta+\beta^2+(1+\beta+\beta^2)\dfrac{R}{1-R}}.$$

Now, if the welfare of young agents of period 0 is increased by the choice of Γ, then all subsequent generations welfare will be increased. The constraints therefore remain unbinding as long as the solution of the unconstrained maximisation problem is larger than the value $\Gamma^0 \leq \Gamma_{R=0}$ defined by $V_0(\Gamma^0) = V_0(\Phi)$ (remember that V_0 is strictly concave in Γ, admits a maximum at $\Gamma_{R=0}$ and tends to $-\infty$ when Γ tends to zero ; $\Gamma^0 = \Phi$ of course if $\Phi \leq \Gamma_{R=0}$). If this is not the case, then the optimal choice is of course Γ^0. In all cases, $\Gamma_{R,\Delta}$ is smaller than $\Gamma_{R=0}$ and decreasing in R and Δ. It is obvious that once Γ optimally chosen at date 0, the central planner does not have to reconsider the choice at a subsequent date if its choice criterion has remained unchanged.

To prove that an optimal choice of Γ can lead to the possibility of bubbles, it is therefore sufficient to exhibit an example where $\Gamma_{R=0}$ is strictly smaller than $\bar{\Gamma}$ (the upper strict bound for the possibility of bubbles in a Γ-economy). This is not difficult :

$$\Gamma_{R=0} = \frac{\alpha}{\alpha + \beta + \beta^2}.$$

$$\bar{\Gamma} = \frac{\frac{\beta}{1+\beta}(1-\alpha)A - (1+\alpha A)}{\frac{\beta}{1+\beta}(1-\alpha)A + \frac{1-\alpha}{\alpha}(1+\alpha A)}.$$

Taking β=1, α=1/8 and A=12, we get :

$$\Gamma_{R=0} = 1/17 < 1/9.$$

$$\bar{\Gamma} = 0.1208... > 1/9,$$

Thus, the choice of Γ can lead to the possibility of bubbles in this economy where bubbles cannot appear if borrowing isn't constrained.

APPENDIX B :

Conditions for the Possibility of Bubbles

We want to show that if $\{\epsilon < 1 \Leftrightarrow \sigma > 1\}$, then bubbles might be possible in an unconstrained economy even if agents prefer present consumption. The upper bound for Γ allowing bubbles to appear is :

$$\bar{\Gamma} = \frac{\frac{\beta'}{1+\beta'}(1-\alpha)A - (1+\alpha A)}{\frac{\beta'}{1+\beta'}(1-\alpha)A + \frac{1-\alpha}{\alpha}(1+\alpha A)},$$

with

$$\beta' = \beta^{\sigma}(1+\rho)^{\sigma-1},$$

If $\beta(1+\rho) > 1$, then :

$$\lim_{\epsilon \to 0^+}(\bar{\Gamma}) = \frac{(1-\alpha)A - (1+\alpha A)}{(1-\alpha)A + \frac{1-\alpha}{\alpha}(1+\alpha A)},$$

$$\lim_{\epsilon \to 0^+}(\varphi) = 0.$$

This proves that there exist economies such that $\bar{\Gamma} \geq \varphi$.

If we now suppose $\{\epsilon \geq 1 \Leftrightarrow \sigma \leq 1\}$, then :

$$1 + g(\varphi) = \frac{\beta'^2}{1+\beta'+\beta'^2}\frac{(1-\alpha)A}{1+\varphi\dfrac{1-\alpha}{\alpha}}.$$

It is easy to see that this is smaller than $1+\rho = 1+\alpha A$ under the assumption $\beta \leq 1$, whatever are the other parameters, because then $\epsilon \geq 1 \Rightarrow \beta' < 1$. This means that $\bar{\Gamma}$ is always smaller than φ if we do not assume strict preference for future consumption. Thus we have the following : in an unconstrained economy, and assuming that the rate of intertemporal substitution is smaller than or equal to 1, bubbles are incompatible with the standard hypothesis on β.

APPENDIX C :

A Simple Tax/Subsidy Scheme

C.1. Resolution of the case $R = 0$

The Lagrangian L_t of the problem at time t is :

$$L_t = \sum_{i=0}^{2}\beta^i u(C_{t,t+i}) + \lambda_t\Big(\frac{e_{t+1}}{R_{t+1}} - (C_{t,t} + \frac{C_{t,t+1}}{R_{t+1}} + \frac{C_{t,t+2}}{R_{t+1}R_{t+2}})\Big)$$
$$+ \mu_t\Big(\Gamma\frac{e_{t+1}}{R_{t+1}} - (C_{t,t} + T_{t,t})\Big).$$

Agents do not internalize the effects of their actions on the rate of interest or the labour income, therefore we have :

$$(C_{t,t})^{-\epsilon} = \lambda_t + \mu_t,$$

$$\beta(1+\rho)(C_{t,t+1})^{-\epsilon} = \beta^2(1+\rho)^2(C_{t,t+2})^{-\epsilon} = \lambda_t.$$

Since we suppose that agents are already constrained in the bubbleless economy, we can write $(T_{t,t} = \theta_t B_t)$:

$$C_{t,t} = \Gamma \frac{e_{t+1}}{R_{t+1}} - \theta_t B_t.$$

The budget constraint gives :

$$C_{t,t+1} = \frac{1-\Gamma}{1+\beta'} e_{t+1} + \frac{1}{1+\beta'}(1+\rho)\theta_t B_t.$$

Thus, writing the equilibrium of the capital market $(T_{-1,-1} = S_{-1,0} = \theta_{-1} = 0)$:

$$K_{t+1} + (1-\theta_t)B_t = e_t + S_{t-1,t} - (1+\rho)(\Gamma\frac{e_t}{1+\rho}) - \Big(\frac{1-\Gamma}{1+\beta'}e_t + \frac{1}{1+\beta'}(1+\rho)T_{t-1,t-1}\Big) - \Gamma\frac{e_{t+1}}{1+\rho},$$

we get the following equation for capital accumulation :

$$(1+\Gamma\frac{1-\alpha}{\alpha})(1+g_t) = \frac{\beta'}{1+\beta'}(1-\Gamma)(1-\alpha)A - (1-\theta_t)b_t + \frac{\beta'}{1+\beta'}\theta_{t-1}b_t.$$

The government chooses θ_t. If $\theta_t \in]0,1[$, the optimality condition is :

$$\frac{\partial L_t}{\partial \theta_t} = 0,$$

$$\Leftrightarrow$$

$$\frac{\lambda_t}{1+\rho}\frac{\partial e_{t+1}}{\partial \theta_t} + \mu_t(\frac{\Gamma}{1+\rho}\frac{\partial e_{t+1}}{\partial \theta_t} - 1) = 0,$$

$$\Leftrightarrow$$

$$\mu_t = \frac{\bar{A}(\Gamma)}{1 - \Gamma\bar{A}(\Gamma)}\lambda_t,$$

where

$$\bar{A}(\Gamma) = \frac{(1-\alpha)A}{(1+\rho)(1+\Gamma\frac{1-\alpha}{\alpha})}.$$

Writing :

$$C_{t,t}\big[1 + (\beta' + \beta'^2)(1 + \frac{\mu_t}{\lambda_t})\big] = [\hat{\phi}(\Gamma)]^{-1}\, C_{t,t} = \frac{e_{t+1}}{1+\rho},$$

we get :

$$\theta_t B_t = \Pi(\Gamma)\, e_{t+1}(\theta_t = 0),$$

where :

$$\Pi(\Gamma) = \frac{1}{1+\rho}\frac{\Gamma - \hat{\phi}(\Gamma)}{\bar{A}(\Gamma)\hat{\phi}(\Gamma) + 1 - \Gamma\bar{A}(\Gamma)}.$$

The conditions for consistency of the hypothesis '$\theta_t \in]0,1[$' are therefore :

$$\begin{cases} (1). \quad \Gamma > \hat{\phi}(\Gamma), \\ (2_t). \quad \Pi(\Gamma)\dfrac{e_{t+1}(0)}{B_t} < 1. \end{cases}$$

Condition (1) is purely structural, so if (1) is not true, then for all t, $\theta_t = 0$ because, as is shown below, $\hat{\phi}$ is increasing in its variable. Condition (2_t) on the contrary is not purely structural : it is time-dependent ; if (1) holds and (2_t) is not true, then necessarily $\theta_t = 1$ as is shown below. $\theta_0 = 1$ can be optimal when b_0 is small as is easy to verify.

C.2. Existence of Economies with Pareto-Superior Outcomes by Tax/Subsidy schemes

We have to show that there exist economies for which condition (1) holds. Since we are preoccupied by existence, it is sufficient to exhibit one example. Nonetheless, we shall prove the existence in the two cases $\epsilon \geq 1$ and $\epsilon < 1$ because of the difference of the possibility condition for bubbles (remember : if $\beta \leq 1$ the existence of bubbles requires a rather strong constraint on borrowing if $\epsilon \geq 1$).

Let us therefore first take $\beta = \epsilon = 1$, $A = 12$ and $\alpha = 0.125$. We get :

$$\bar{\Gamma} = 0.1208.. > 1/9.$$

$$[\hat{\phi}(\bar{\Gamma})]^{-1} = 9.275 > 9.$$

As $\hat{\phi}$ is obviously increasing in Γ, we have the existence of $\bar{\Gamma}$ such that $\hat{\Gamma} < \bar{\Gamma}$, where $\hat{\phi}(\hat{\Gamma}) = \hat{\Gamma}$.

If $\epsilon < 1$, then Appendix B has shown that we can have $\bar{\Gamma} > \varphi$; but $\hat{\phi}(\varphi) < \varphi$, and $\hat{\phi}$ is strictly increasing and concave. □

This proves, for both cases $\epsilon \geq 1$ or < 1, the existence of economies where a $R = 0$-policy can achieve a Pareto-improvement by eliminating the bubbly asset by a tax/subsidy financed policy. If we reflect on what the policy considered here consists in, we realise that Γ must be such that it allows for a Pareto-improvement by stronger constraints on borrowing ; indeed, for the case $\epsilon = 1$, we verify that $\hat{\phi}(\Gamma_{R=0}) \geq \Gamma_{R=0}$:

$$[\hat{\phi}(\Gamma_{R=0})]^{-1} = 1 + (\beta + \beta^2)\frac{1 + \beta + \beta^2 + A(\alpha + \beta + \beta^2)}{1 + \beta + \beta^2 + \alpha A(\alpha + \beta + \beta^2)}$$
$$< 1 + (\beta + \beta^2)\frac{1}{\alpha} = [\Gamma_{R=0}]^{-1},$$

since $\alpha \in]0,1[$. Furthermore, we can see that choosing Γ according to a welfare criterion does not necessarily lead to the possibility of bubbles ($\epsilon = 1$ for the calculus) :

$$\Gamma_{R=0} \leq \bar{\Gamma}$$
$$\Leftrightarrow$$
$$g(\Gamma_{R=0}) \geq g(\bar{\Gamma})$$

since the rate of growth of capital is decreasing in Γ.

$$1 + g(\Gamma_{R=0}) = \frac{\beta^2}{1 + \beta + \beta^2}(1 - \alpha)A.$$

Thus, the condition $g(0) > \rho$ does not imply $g(\Gamma_{R=0}) \geq \rho = g(\bar{\Gamma})$. This enables us to ascertain that there exist parameter values such that $\hat{\phi}(\bar{\Gamma}) \geq \bar{\Gamma}$, which means that there are parameters β, α and A such that, whatever is the value of Γ, there do not exist Pareto-improving tax/subsidy policies of the type considered here when a bubble appears in the economy.

C.3. Dynamics

Capital accumulation is given by ($\theta_{-1} = 0$) :

$$(1 + \Gamma\frac{1 - \alpha}{\alpha})K_{t+1} = \frac{\beta'}{1 + \beta'}(1 - \Gamma)(1 - \alpha)AK_t - (1 - \theta_t)B_t$$
$$+ \frac{\beta'}{1 + \beta'}(1 + \rho)\theta_{t-1}B_{t-1},$$

$$B_{t+1} = (1+\rho)B_t.$$

Another way of writing the accumulation equation gives us, for $t \geq 1$ and under the assumption of $\theta_t \in]0,1[$:

$$b_{t+1} = \Psi((1-\theta_t)b_t, \quad \hat{\phi}(\Gamma)) < \Psi(b_t, \hat{\phi}(\Gamma)) < b_t,$$

where Ψ is the function relating b_{t+1} to b_t in the bubbly economy without government intervention. Thus, if $\Gamma \in]\hat{\Gamma}, \bar{\Gamma}[$, as long as $\theta_t < 1$, the sequence is strictly increasing : indeed, writing

$$\theta_t B_t = \Pi(\Gamma) e_{t+1}(\theta_t = 0),$$

we get

$$\frac{\theta_{t+1}}{\theta_t} = \frac{b_t}{b_{t+1}} \frac{1 - ab_{t+1}}{1 - ab_t} > 1$$

$a > 0$, $b_t = B_t/K_t$. But as necessarily $lim_{t \to +\infty} b_t = 0$ (with non intervention, a non stationary reduced bubble is always vanishing ; the government policy converts progressively the bubble into an IOU equivalent to public debt ('forced loans' to the government here), but $B_t = (1+\rho)^t B_0$ whilst the rate of growth of capital is enhanced compared to the non-intervention case ; if the bubble starts with the stationary value, a consequence of the government's intervention at time 0 is that B_1 has a non stationary value...), we see that there must exist a finite lapse of time T^0 after which the bubbly asset has been entirely eliminated from the investment market (i.e. $\theta_t = 1$

for $t \geq T^0$).

Proposition : *After a finite lapse of time, the government has bought the entire stock of the non-productive asset, and the policy becomes a pure redistribution scheme.*

• For $t \geq T^0 + 1$, the rate of growth of capital is given by :

$$g_t = g(\Gamma) + \frac{\beta'}{1+\beta'} \frac{1}{1 + \Gamma \frac{1-\alpha}{\alpha}} s_t.$$

The mechanism works as a sort of supplementary restraint on borrowing of B_t for young agents at time t. Therefore savings must increase by $[\beta'/(1+\beta')]B_{t+1}$ at t+1.

• For $t \geq T^0 + 1$, the dynamics of $s_t = b_t$ are given by :

$$s_{t+1} = \Psi(- \frac{\beta'}{1+\beta'} s_t, \Gamma).$$

Since this function is concave and lies always beneath the 45 degree line, we verify that $lim_{t \to +\infty} s_t = 0$.

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CHAPTER II

Externalities and Dynamics

EXTERNALITIES AND DYNAMICS

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Abstract

We consider a neo-classical OLG Model with two sectors, one for a consumption good and one for an investment good. We show that the presence of externalities can enlarge the dynamical system with the possibility of Hopf bifurcations. Applying Bifurcation Theory, we establish the possibility of attractive closed orbits around the steady state of the economy. We derive some results on the questions of existence, multiplicity and stability of balanced growth paths. Furthermore, we show that some of the results obtained in the framework of Overlapping Generations also hold for the discrete-time version of the Infinitely Lived Agents Model ; as a consequence, we conclude to non robustness with regard to the time structure assumption.

Introduction

The question whether economic cycles always originate in exogenous shocks or can be, at least partly, explained endogenously is a fundamental question of modern economics, not only from a theoretical point of view but also for the purpose of policy. The idea of endogenous cycles is old, since we can find it in the work of Hicks, Kaldor or Goodwin but, for a long time, the argument of irregularity of the real-world business cycles forbade serious consideration of such theories which were predicting periodic recessions. The main part of the economics profession therefore concentrated on the study of exogenous shock models of economic fluctuations, which have become so familiar that many textbooks refer only to this type of explanation. However, nothing induces to believe that the so popular exogenous shock theory provides the only possible relevant explanation for business cycles or other fluctuations : the internal mechanics of a market economy might bear at least a part of responsability.

The fact that irregular cycles and even a certain type of chaos can have their roots in an entirely deterministic dynamic is well known by the mathematicians since the foundations of bifurcation and chaos theory have been elaborated. Very simple dynamics like the so often used 'tent map' or the unidimensional discrete logistic map have been shown to lead to cycles and to different types of deterministic chaos. Furthermore, the research on multidimensional dynamic systems has had a considerable impact on several fields such as physics, biology, chemistry and ecology, spheres in which several examples of chaotic behaviour originating in determinism have been found in recent years, and some economists have started to investigate the possibility of endogenous cycles and of chaos in the framework of modern economics.

Research has shown that competitive equilibrium models may, in the absence of any exogenous shock, generate endogenous fluctuations which are entirely consistent with complete markets and perfect foresight. The fact that the literature on endogenous fluctuations almost exclusively concentrates on models with no intrinsic uncertainty does not mean that its attempt is to explain business cycles etc exclusively by the effects of market mechanisms ; to look at extreme situations in which only extrinsic uncertainty may matter constitutes a methodological choice which aims at exposing the direct effects of the market and to avoid, at this stage of mathematical knowledge about non-linear dynamical systems, unextricable technical complications. The literature can be divided into two parts : on the one hand, there is the research on Optimal Growth Models, on the other, the work on Overlapping Generations Models. The latter constitutes the main part, fact which has quite simple an explanation : it is the only neo-classical model which requires sequential trading in the absence of market imperfections.

Endogenous equilibrium cycles in the framework of Overlapping Generations Models have been, in fact, a fashionable topic since the mid-eighties, when Jean-Michel Grandmont, in his seminal paper [8], studied and proved the possibility of endogenous cycles in a pure exchange, perfectly competitive economy. In his model however, endogenous cycles are possible only if saving is a decreasing function of the interest rate and, to be more precise, no cycle can exist if the IES (interest rate elasticity of saving at the Golden

Rule stationary state) is larger than -0.5. Farmer's paper [6] was the first to consider an OLG economy with production, but in his model the existence of cycles depends on the presence of financial debt of the private sector toward the government (negative outside money). The very first paper to study the problem in the framework of a OLG economy with production and no intervention of a public authority, thus in a world of "laisser-faire", was Reichlin [11]. In his one-sector model, production is characterised by a CES function with two production inputs, capital and labour, the latter being supplied wage-elastically. Reichlin showed that cycles are possible in his framework with a positive IES and a low elasticity of substitution between the factors (enough complementarity is required). In another paper [12], now in a two-sector model, Reichlin proved that cycles and chaotic dynamics with positive IES do not necessarily require wage-elastic labour supply. Jullien [10] examined an OLG economy with one sector, where production is made through a neo-classical CRS technology (capital and labour are supplementary), and labour supplied inelastically ; the existence of a nominal asset (a bubble) is required to relax the link between investment and aggregate saving and generate cycles through self-fulfilling expectations on returns. However, a drawback to his model is that again Grandmont's condition on the saving function is required, and thus makes the model empirically unlikely. Furthermore, the example given for the existence of cycles of order three involves a negative rate of time preference (agents prefer future consumption...).

A topic which appears to be very interesting is the study of endogenous cycles in an economy with externalities. In his paper "Dynamic Externalities, Multiple Equilibria, and Growth", Boldrin [1] considered a one-sector OLG model with perfect markets, neo-classical production with a dynamic externality of Romer [13] type, a rate of depreciation of capital equal to one per period and established the possibility of multiple equilibria and trapping regions. In the case of a time-separable utility of CRRA type with the same coefficient of relative risk aversion in each period, a rate of time-preference equal to zero and Cobb-Douglas production with externality, Boldrin showed that the overspill-effect has to be rather important for poverty traps to occur. His example proves cycles to be possible with positive IES and supplementarity of the production inputs.

The present paper deals with the least possible level of disaggregation of an OLG model satisfying all the neo-classical assumptions from the point of view of the private sector, namely a two-sector model with one consumption good and one investment good, only one type of capital and one type of labour. We study, under the conditions insuring existence of a stationary state of the economy, the local dynamics around this point and conclude to the impossibility of Hopf bifurcations. We then consider the effects of positive spillovers from total capital stock onto the efficiency of labour in each sector and show how local dynamics are changed, allowing now Hopf bifurcations to occur. We establish the existence of attractive closed orbits around the steady state by using the Hopf Bifurcation Theorem for the discrete-time setting. Furthermore, we look at the existence and stability problem of balanced growth paths in the two-sector OLG economy and show the difference with regard to one-sector models. Finally, we prove that some of the results extend to the Infinitely Lived Agent problem in discrete-time setting and establish therefore the link between the results obtained by Cazzavillan [4] and the setting of continuous time.

1 The Model

We take the framework of Galor [7] augmented by spillover from the total capital stock onto the efficiency of labour in each sector. Time is discrete. At each date t, a new generation of agents is born ; the size of each generation is assumed to remain constant over time and will be normalised to one. Agents live for two periods ; in the first period of their life, they work, consume and save for their old age, and in the second, they consume their actualised savings. Bequests are not allowed. There are two goods in the economy, one homogeneous perishable consumption good and one homogeneous investment good. Both production sectors combine two factors, capital and labour, the latter being provided inelastically. There exists only one type of capital and one type of labour which can be costlessly allocated between sectors. Firms are owned by the old people ; the number of firms is supposed large enough to have perfect competition and thus profit maximisation in both sectors.

Agents are characterised by their utility, which is supposed time-separable and of CRRA type :

$$u(c_{t,t}, c_{t,t+1}) = \begin{cases} \dfrac{c_{t,t}^{1-\sigma}}{1-\sigma} + \Theta\dfrac{c_{t,t+1}^{1-\sigma}}{1-\sigma}, & \sigma \neq 1 \\ Ln(c_{t,t}) + \Theta Ln(c_{t,t+1}) & \end{cases}$$

where $c_{t,t}$, $c_{t,t+1}$ denote respectively consumption when young and when old of an agent of generation t, $\sigma > 0$ is the coefficient of relative risk aversion (equal to the inverse of the elasticity of substitution between consumption at any two points in time) and $\Theta = 1/(1+r)$, where $r \in]-1, +\infty]$ is the rate of time preference.

The production of each good is of the following type :

$$C_t = K_{1,t}^{\alpha} L_{1,t}^{1-\alpha} K_t^{\nu},$$

$$I_t = K_{2,t}^{\beta} L_{2,t}^{1-\beta} K_t^{\psi},$$

$(\alpha, \beta) \in]0,1[^2$ and $(\nu, \psi) \in I\!R_+ \times [0,1[$, $K_{i,t}$ being the level of capital at time t in sector i, K_t the aggregate level of capital at time t. The rate of depreciation per period of the capital is δ. We suppose $\alpha \neq \beta$ (two sectors) and assume full employment.

$$\begin{cases} K_{1,t} + K_{2,t} = K_t \\ L_{1,t} + L_{2,t} = L_t. \end{cases}$$

We reformulate the model in per capita terms ; denoting $l_{i,t}$ the proportion of the labour force allocated to sector i and $k_{i,t}$ the per capita capital stock in sector i, we get :

$$c_t = k_{1,t}^{\alpha} l_{1,t} K_t^{\nu},$$

$$i_t = k_{2,t}^{\beta} l_{2,t} K_t^{\psi}.$$

Since we have

$$\begin{cases} l_{1,t} + l_{2,t} = 1 \\ l_{1,t} k_{1,t} + l_{2,t} k_{2,t} = k_t \end{cases}$$

we can eliminate $l_{i,t}$ and obtain :

$$c_t = \frac{k_t - k_{2,t}}{k_{1,t} - k_{2,t}} k_{1,t}^{\alpha} K_t^{\nu},$$
$$i_t = \frac{k_{1,t} - k_t}{k_{1,t} - k_{2,t}} k_{2,t}^{\beta} K_t^{\psi}.$$

Firms behave competitively and maximise their profits in each period, without taking into account the externalities. Let e_t be the competitive wage rate and r_t the competitive rate of return on capital and p_t the relative price of the consumption good in terms of investment good (the numeraire here). Profit maximisation and constancy of labour force imply :

$$e_t = p_t(1-\alpha)k_{1,t}^{\alpha} k_t^{\nu} = (1-\beta)k_{2,t}^{\beta} k_t^{\psi},$$

$$r_t = p_t \alpha k_{1,t}^{\alpha-1} k_t^{\nu} = \beta k_{2,t}^{\beta-1} k_t^{\psi},$$

If we denote by ω_t the wage-interest rate ratio, we see that

$$k_{1,t} = a\omega_t, \; k_{2,t} = b\omega_t,$$

where $a = \alpha/(1-\alpha)$ and $b = \beta/(1-\beta)$. Thus price, wage and interest rate can be expressed as functions of ω_t and k_t, which will be chosen as state variables. Notice that only k_t is predetermined.

Maximisation of utility

$$\begin{cases} max \; u(c_{t,t}, c_{t,t+1}) \\ s.t. \; p_t c_{t,t} + \dfrac{p_{t+1} c_{t,t+1}}{1-\delta+r_{t+1}} \leq e_t, \end{cases}$$

yields the Euler equation :

$$c_{t,t}^{-\sigma} = \frac{p_t}{p_{t+1}} \Theta(1-\delta+r_{t+1}) c_{t,t+1}^{-\sigma}.$$

The equilibrium conditions

$$s_t = e_t - p_t c_{t,t} = (1-\delta)k_t + (k_{t+1} - (1-\delta)k_t),$$

$$k_{t+1} = (1-\delta)k_t + i_t,$$

give the dynamics of the economy :

$$k_{t+1} = \frac{a\omega_t - k_t}{a-b} b^\beta \omega_t^{\beta-1} k_t^\psi + (1-\delta)k_t, \tag{1}$$

$$\omega_{t+1}^{\beta-\alpha} k_{t+1}^{\psi-\nu} \left[\frac{1}{\Theta}\frac{k_{t+1}}{e_t - k_{t+1}}\right]^{\frac{\sigma}{1-\sigma}} = \omega_t^{\beta-\alpha} k_t^{\psi-\nu}\Theta[(1-\delta) + \beta b^{\beta-1}\omega_{t+1}^{\beta-1}k_{t+1}^\psi]. \tag{2}$$

If $\sigma = 1$, equation (2) reduces to :

$$k_{t+1} = \frac{\Theta}{1+\Theta} e_t. \tag{2'}$$

It is easy to see that a steady state $(\bar{k}, \bar{\omega})$ can only exist if $\beta + \psi \neq 1$. If this condition does not hold, then the economy does not admit any stationary state for the variables k and ω, but there is a possibility for balanced growth to occur. We shall proceed as follows : in section 2, under the assumption $\beta + \psi \neq 1$, we look at the existence of a stationary state and study the nature of the local dynamics around this point, proving the possibility of attractive closed orbits. In section 3, we consider the case $\beta + \psi = 1$ and examine the questions of existence, uniqueness, stability of balanced growth paths.

2 Local Dynamics around the Steady State

Contemplating (1), (2), we realise that we are confronted to a highly complex system of non-linear dynamic equations which cannot be formally studied in general : except for the cases $\sigma = 0 \vee 1$, a formal and relatively 'useful'[1] expression for the steady state cannot be determined. We shall therefore adopt the following strategy : we consider the limit case of risk neutral agents ($\sigma = 0$) and study the behaviour of the variables in a neighbourhood of the steady state. All the results obtained for this extreme case will also hold for σ small enough. The argument we use here is continuity of the operators trace and determinant of a matrix and is exposed in Appendix A.

2.1 Steady State, and Linearization

Fixing σ equal to 0 we get :

$$\bar{\omega} = \left[\frac{\delta(1-\beta) + r(1-\alpha)}{\alpha(r+\delta)}\right]\bar{k},$$
$$\bar{k} = \left\{\frac{r+\delta}{\beta}\left[\frac{\delta(1-\beta) + r(1-\alpha)}{\alpha(r+\delta)}\right]^{1-\beta} b^{1-\beta}\right\}^{\frac{1}{\beta+\psi-1}},$$

under the assumptions

$$\begin{cases} (C_0) \ : \ \beta + \psi \neq 1. \\ (C_1) \ : \ r + \delta > 0. \\ (C_2) \ : \ \delta(1-\beta) + r(1-\alpha) > 0. \end{cases}$$

Condition (C_0) rules out, as we shall see below, the case where balanced growth may be possible. Condition (C_1) imposes a restriction on agents'

[1] It is clear that the equation giving the steady state can be solved formally for more values of σ, namely for all those which yield algebraic equations of first, second or third order. Nevertheless, only the cases 0 and 1 ensure relatively simple discussions on local dynamics.

preference for future consumption ; thus, whenever agents are supposed to prefer strictly present consumption ($r > 0$), this condition is automatically met. Finally, (C_2) is automatically met whenever agents prefer, in a large sense, present consumption[2] ($r \geq 0$) ; if this condition does not hold, then (C_2) imposes an investment good production not too much more capital intensive than the consumption good production. Conditions C_1 and C_2 can be summarised by : $r + \delta > max(0, \delta(\beta - \alpha)/(1 - \alpha))$.

Since we want to study the local behaviour around the steady state, we linearize the system around Ω $(\bar{k}, \bar{\omega})$. We write :

$$\begin{cases} k_t = \bar{k}(1 + \epsilon_t), \\ \omega_t = \bar{\omega}(1 + \eta_t). \end{cases}$$

Thus, the jacobian of the system (1),(2) evaluated at the steady state has the same roots as the matrix J defined by :

$$\begin{pmatrix} \epsilon_{t+1} \\ \eta_{t+1} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix},$$

which has a slightly simpler expression. We obtain :

$$J_{11} = 1 - \delta(1 - \psi) - (r + \delta)\frac{1 - \alpha}{\alpha - \beta}.$$

$$J_{12} = \delta\beta + (r + \delta)\frac{1 - \alpha}{\alpha - \beta}.$$

$$J_{21} = \frac{[(1 + r)(\psi - \nu) + [(1 + r)\nu - (1 - \delta)\psi][1 - \delta(1 - \psi) - (r + \delta)\frac{1 - \alpha}{\alpha - \beta}]]}{[(1 + r)(\beta - \alpha) + (r + \delta)(1 - \beta)]},$$

$$J_{22} = \frac{[(1 + r)(\beta - \alpha) + [(1 + r)\nu - (1 - \delta)\psi][\delta\beta + (r + \delta)\frac{1 - \alpha}{\alpha - \beta}]]}{[(1 + r)(\beta - \alpha) + (r + \delta)(1 - \beta)]},$$

[2]Preference for actual consumption is often taken as an assumption...

if $[(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)] \neq 0$. If this denominator happens to be equal to zero, then the system is locally degenerate and reduces, in a neighbourhood of the stationary state, to :

$$\begin{cases} \epsilon_{t+1} = \lambda_k \epsilon_t, \\ \eta_t = \Xi \epsilon_t, \end{cases}$$

situation which is of no interest for us here since our purpose is to look at parameter configurations which give complex eigenvalues for the Jacobian.

For convenience, we define :

$$(C_3) \ : \ [(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)] > 0.$$

and symmetrically :

$$(C_3') \ : \ [(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)] < 0.$$

None of these conditions does, a priori, reflect any theoretical economical idea. But we can notice that if $\delta = 1$, then (C_3) reduces to $(1+r)(1-\alpha) > 0$, which is always true. Thus, this condition imposes a restriction on the capital intensity of the consumption good sector compared to the one characterising the investment good sector if and only if capital does not depreciate in one period[3].

For the rest of section 2, we suppose that conditions (C_0), (C_1) and (C_2) hold and show which results can be derived from the supplementary assumption (C_3) or (C_3').

[3]As a matter of fact, most OLG models fix $\delta = 1$ ... for a question of tractability.

If we consider the slope of the locus : $k_{t+1} - k_t = 0$, we see that it is always positive under assumption (C_2). We have indeed :

$$[\delta(1-\psi) + (r+\delta)\frac{1-\alpha}{\alpha-\beta}]\frac{k_t - \bar{k}}{\bar{k}} = [\delta\beta + (r+\delta)\frac{1-\alpha}{\alpha-\beta}]\frac{\omega_t - \bar{\omega}}{\bar{\omega}}.$$

We have thus consistency with the theory's prediction saying that the lower the interest rate for a given wage (i.e. the higher ω), the higher the demand for the investment good and the higher the total per capita capital stock[4]. Furthermore, we see that the slope of the locus : $\omega_{t+1} - \omega_t = 0$ has a sign depending upon the relative magnitude of the parameters of the spillover effects ψ and ν.

2.2 Absence of Spillovers

This corresponds to the standard neo-classical model. But caution is required here : saddle-point stability is not, in general, the only possibility under this assumption, as Galor [7] has shown. Indeed, we shall see below that even with our extremely nice utility and production functions, saddle-point stability is guaranteed only if the consumption good sector is less capital intensive than the investment good sector (i.e. when $\alpha < \beta$). If this condition does not hold, then the steady state can be a node, a saddle point or a source.

When $\nu = \psi = 0$, the dynamical system $(1),(2)$ is of the following form :

$$k_{t+1} = \frac{a\omega_t - k_t}{a-b} b^{\beta}\omega_t^{\beta-1} + (1-\delta)k_t, \tag{1'}$$

$$\omega_{t+1}^{\beta-\alpha} = \omega_t^{\beta-\alpha}\Theta[(1-\delta) + \beta b^{\beta-1}\omega_{t+1}^{\beta-1}], \tag{2'}$$

[4]Now the assumption $\psi \in [0,1[$ becomes clearer...

which means that the dynamic of ω_t is completely independent of k_t, which implies a matrix J of upper triangular form :

$$J = \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix},$$

where

$$J_{11} = (1-\delta) - (r+\delta)\frac{1-\alpha}{\alpha-\beta}$$
$$J_{22} = \frac{(1+r)(\beta-\alpha)}{(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)}.$$

But this means that the eigenvalues of J are always real, and the dynamics are thus limited to non-spiral behaviour, and Hopf bifurcations[5] cannot occur in the dynamical system. Condition (C_2) implies that J_{11} is strictly larger than 1 if $\alpha < \beta$ and condition (C_3) implies that it is strictly negative if $\beta < \alpha$. J_{22} is positive and strictly less than 1 if $\alpha < \beta$, and, under assumption (C_3), strictly negative otherwise. As a matter of fact, to assume (C_3) appears to be rather useless here since it does not rule out any possibility for the nature of Ω : it can be a sink, a saddle-point or a node, just as under assumption (C_3') (but under the latter assumption Ω is monotone, whereas it is oscillating under (C_3)). Therefore, we can conclude to :

Proposition : *In the absence of spillovers, there is no possibility for Ω to be a spiral sink or a spiral source, and Hopf bifurcations are excluded for the dynamical system. Under the assumptions necessary for existence, the steady state equilibrium Ω is always a saddle-point if the consumption good*

[5] A Hopf bifurcation occurs in a dynamical system when the eigenvalues of the Jacobian cross the unit circle in $\boldsymbol{C}^2$.

sector is less capital intensive than the investment good sector ($\alpha < \beta$). If the reverse is true, then the steady state can be either a sink, a saddle point or a source.

2.3 Presence of spillovers

In a discrete-time setting, complete stability is guaranteed whenever both roots of the characteristic polynomial of J are strictly smaller in modulus than 1. This offers, of course, far more possibilities than the continuous-time setting where the real part of both roots must be negative to have complete stability. As a matter of fact, four situations can arise in our framework :

$$\begin{cases} \text{(i) } 0 < Det(J) < 1 \text{ and } \Delta < 0 \\ \text{(ii) } 0 \leq Det(J) < 1 \text{ and } \Delta \geq 0 \text{ and } 0 < Tr(J) < 1 + Det(J) \\ \text{(iii) } 0 \leq Det(J) < 1 \text{ and } \Delta \geq 0 \text{ and } -[1 + Det(J)] < Tr(J) < 0 \\ \text{(iv) } -1 < Det(J) < 0 \text{ and } |Tr(J)| < 1 + Det(J) \end{cases}$$

where Δ denotes the discriminant of the characteristic equation ($\Delta = Tr(J)^2 - 4Det(J)$). The first case corresponds to two complex conjugate roots : this is the case which gives a spiral sink. The second case corresponds to two positive roots that are smaller than 1 : Ω is then a monotone node. The third case gives two negative roots larger than -1, and the fourth leads to one negative root larger than -1 and one positive root smaller than one, and convergence to the node Ω is oscillating in these situations.

A first consequence of the presence of spillovers is that the steady state Ω is no longer necessarily a saddle point when $\alpha < \beta$. It is indeed very easy to exhibit numerical examples showing that all four cases cited above can occur, and to show that the dynamical system bears the possibility of three types of interesting bifurcations : saddle-point, flip and Hopf bifurcations. In the following, we shall nevertheless focus our attention on the possibility of Hopf bifurcations.

It is (nearly) straightforward to verify that we have :

$$\begin{aligned} Tr(J) =& 1 + Det(J) + (\beta + \psi - 1)\frac{r+\delta}{\alpha - \beta}\frac{\delta(1-\beta) + r(1-\alpha)}{(1+r)(\beta-\alpha) + (r+\delta)(1-\beta)} \\ &+ (1-\delta)\frac{(1+r)(1-\alpha)}{(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)}. \end{aligned}$$

We know that if there exist values of the parameters such that $Det(J) = +1$ and $\Delta < 0$, then there is a possibility of obtaining Hopf bifurcations. These conditions impose :

$$\begin{aligned} -4 <& (\beta + \psi - 1)\frac{r+\delta}{\alpha - \beta}\frac{\delta(1-\beta) + r(1-\alpha)}{(1+r)(\beta-\alpha) + (r+\delta)(1-\beta)} \\ &+ (1-\delta)\frac{(1+r)(1-\alpha)}{(1+r)(\beta-\alpha)+(r+\delta)(1-\beta)} < 0. \end{aligned}$$

We see that if $\alpha < \beta$, then we need $\beta + \psi > 1$ to have the possibility of Hopf bifurcations. Furthermore, $Det(J) = +1$ then implies that necessarily $\nu > \psi$. If we suppose $\beta < \alpha$, the discussion is more complex : under assumption (C_3), we need $\beta + \psi < 1$, but if we suppose (C_3'), then there is no general condition on $\beta + \psi$; furthermore, $Det(J) = +1$ does not impose any general condition on the relative magnitude of the sector specific spillovers under assumption (C_3), but it is easy to verify that ψ has to be smaller than ν under assumption (C_3'). We can summarise :

$$\begin{cases} (\alpha < \beta) \ \wedge \ (\beta + \psi > 1) \ \wedge \ (\nu > \psi) \\ (\beta < \alpha) \ \wedge \{ \ [(C_3) \ \wedge \ (\beta + \psi < 1)] \vee [(C_3') \ \wedge \ (\nu > \psi)]\} \,. \end{cases}$$

Notice the interesting fact that in the case of a more, but not too much more, capital intensive consumption good sector, Hopf bifurcations can only occur when the externality in the investment good production is not too strong ($\beta + \psi < 1$), a strong externality in the investment good sector enhancing on the other hand the possibility of Hopf bifurcations in the situation where the investment good sector is the more capital intensive. Numerical examples given below will establish that Hopf bifurcations do indeed occur in both situations, and that it is possible to exhibit parameters such that the bifurcation is supercritical. But first, a remark : following Grandmont [9], we should avoid cases of large resonance, which are not well understood until now and correspond to the cases where, for the value of the bifurcation parameter η_0, the argument $\Theta(\eta)$[6] is of the form $2\pi/q$, $q = 1, 2, 3$ or 4. Thus, we should look for $Det(J) = +1$ and $|Tr(J)| < 2$ and $Tr(J) \notin \{-1, 0\}$.

2.3.1 Existence of Stable Closed Orbits in the Case ($\alpha < \beta$)

Let us take the following values for the parameters : $\delta = 1$, $r = 0.1$, $\beta = 0.7$, $\alpha = 0.5$. We have :

$$Det(J) = 1.1 + (4.5)\psi - (4.1)\nu,$$

$$Tr(J) = 3.15 + \psi - (4.1)\nu.$$

When $\psi = \psi_0 = 41/70$ and $\nu = 383/574$, we have $Tr(J) = +1$ and $Det(J) = +1$. Therefore, the eigenvalues are $\lambda_1 = cos(\pi/3) + i\ sin(\pi/3)$ and $\lambda_2 = \bar{\lambda}_1$.

[6] The two complex conjugate eigenvalues are written under the form $\lambda = \rho(\eta)e^{i\Theta(\eta)}$ and $\bar{\lambda} = \rho(\eta)e^{-i\Theta(\eta)}$.

Since

$$(\frac{\partial \rho}{\partial \psi})_{\psi_0} = \frac{1}{2}(\frac{\partial Tr(J)}{\partial \psi})_{\psi_0} = \frac{1}{2} > 0,$$

the real part of the eigenvalues is not stationary with respect to the parameter ψ at the point considered here and we have a Hopf bifurcation. For $\psi < \psi_0$, Ω is completely stable, whereas for $\psi_0 < \psi$, Ω is completely unstable. The Hopf bifurcation Theorem applies : in the neighbourhood of the bifurcation value ψ_0, there exist values such that the economy exhibits closed orbits around the steady state. The stability properties of these orbits depend on whether the bifurcation is subcritical or supercritical[7]. If the bifurcation is *supercritical*, then we can conclude to the existence of a whole family of parameters such that a *stable closed orbit* around the steady state exists. A *subcritical* bifurcation only gives *unstable closed orbits* and seems therefore *a priori* less interesting. We shall give here, without any justification, the methodology to apply to determine whether or not the bifurcation is supercritical :

Proposition : *Suppose* $\lambda = \cos(\Theta(\psi_0)) + i\sin(\Theta(\psi_0))$ *and* $\bar{\lambda}$ *are the eigenvalues of the jacobian at* $\psi = \psi_0$. *There always exists a basis where the local dynamical system can be written under the form :*

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \cos(\Theta(\psi_0)) & -\sin(\Theta(\psi_0)) \\ \sin(\Theta(\psi_0)) & \cos(\Theta(\psi_0)) \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} f(x_t, y_t) \\ g(x_t, y_t) \end{pmatrix}.$$

Then the bifurcation is supercritical if $a(\psi_0) > 0$ *and subcritical if* $a(\psi_0) < 0$,

[7]For details on the theory of bifurcation in discrete time settings see, for instance, Grandmont [9].

$]\psi_0, \psi_2[$, *all the orbits are aperiodic and even dense in the closed curve* C_ψ.

2.3.2 Existence of closed orbits in the case ($\beta < \alpha$)

We consider now a case where the consumption good sector is more capital intensive than the investment good sector. We know from a preceding discussion that if a Hopf bifurcation occurs in this situation when (C_3) holds, then necessarily the parameters β and ψ satisfy the condition $\beta + \psi < 1$ at the bifurcation point.

Let us take $\alpha = 0.7$, $\beta = 0.5$, $\delta = 1$ and $r = 0.1$. This choice of parameter values leads to saddle-point stability in the absence of externalities. We have :

$$Det(J) = (0.3)^{-1}[2.15\nu - 2.35\psi + 0.33],$$
$$Tr(J) = 1 + Det(J) + \frac{53}{6}(\psi - \frac{1}{2}).$$

We choose to look for a Hopf bifurcation such that $\Theta(\psi_0) = 3\pi/4$, which requires $Det(J) = +1$ and $Tr(J) = -\sqrt{2}$. This leads to the choice of :

$$\psi_0 = \frac{29 - 12\sqrt{2}}{106},$$
$$\nu_0 = \frac{6497 - 2820\sqrt{2}}{22790}.$$
$$\left(\frac{\partial \rho}{\partial \psi}\right)_{\psi_0} = \frac{1}{2}\left(\frac{\partial Tr(J)}{\partial \psi}\right)_{\psi_0} = \frac{1}{2}\left(\frac{53}{12} - \frac{2.35}{0.3}\right) < 0.$$

Again, the Hopf bifurcation theorem applies. Notice that $\beta + \psi_0 < 1$, and $\alpha + \nu_0 < 1$. To determine the nature of the bifurcation, we apply exactly the same method as in the preceding example[8] and thus get :

$$a(\psi_0) \approx -1274.25 < 0.$$

[8]For the calculus, see Appendix B.

Therefore, the bifurcation is subcritical and we can only proclaim the following : there exist $\psi_1 < \psi_0 < \psi_2$ and an open neighbourhood V of Ω such that :

• If $\psi_1 < \psi \leq \psi_0$, then Ω is asymptotically unstable and there exists no invariant closed curve in V.

• If $\psi_0 < \psi < \psi_2$, then Ω is stable and there exists a unique, asymptotically unstable closed curve C_ψ in V.

We see that closed orbits exist even if the externalities are not very strong. Models with externalities like Boldrin [1], Cazzavillan [4], [5]... exhibit closed orbits only when the spillover is strong. But as a matter of fact, the result is not really astonishing since Galor [7] showed that cycles are possible in a neo-classical two-sector model in the absence of spillovers for an adequate choice of the utility function characterising the agents preferences and of the production functions. Nonetheless, it is remarkable that, in our model, strong externalities are required if the investment good sector is the more capital intensive, but not if it is the consumption good sector which is the more capital intensive.

The question whether supercritical bifurcations can occur when the consumption good sector is the more capital intensive remains to be investigated. We shall now turn our attention to the problem of balanced paths in the two-sector economy.

3 Balanced Growth Paths

We address here the question of existence of a balanced growth path, and study, under the assumption of existence, the stability of the path. It is sufficient to show that these questions are non trivial in the case $\sigma = 0$, which we shall assume in this section.

Balanced behaviour takes place whenever :

$$\frac{\omega_{t+1}}{\omega_t} = \frac{k_{t+1}}{k_t} = \lambda > 0.$$

There is balanced growth if $\lambda > 1$. Injecting the assumption of balanced behaviour into the dynamic equations (1) and (2), we get immediately the following condition :

$$\frac{\omega_t}{k_t} \quad \text{and} \quad \omega_t^{\beta-1}k_t^{\psi} \quad \text{independent of t} \quad \Rightarrow \quad \beta + \psi = 1.$$

Notice that, at this stage, no condition on $\alpha + \nu$ appears. If we pose $\omega_t = \mu k_t$, we get the following equations for the existence of a balanced path :

$$\begin{cases} \lambda^{1-(\alpha+\nu)} = \Theta[(1-\delta) + \beta b^{\beta-1}\mu^{\beta-1}] \\ \lambda(a-b) = [a\mu - 1]b^{\beta}\mu^{\beta-1} + (1-\delta). \end{cases}$$

Not only is it impossible to exhibit a formal solution for this system except for the special case $\alpha + \nu = 1$, but it may well be that the system does not have any solution at all for $\alpha + \nu \neq 1$: then existence is guaranteed only if $(\alpha + \nu < 1) \ \wedge \ (\alpha > \beta)$ or $(\alpha + \nu > 1) \ \wedge \ (\alpha < \beta)$. Furthermore, even in the case $\alpha + \nu = 1$, existence of a balanced path does not necessarily imply growth since the condition $\lambda > 1$ may not hold, and this in both cases $\alpha < \beta$

and $\beta < \alpha$.

Proof : If $\alpha + \nu = 1$, then $(r + \delta) = \beta b^{\beta-1}\mu^{\beta-1}$ and $\lambda(a - b) = [a\mu - 1]b^{\beta}\mu^{\beta-1} + (1-\delta)$. If the parameters of the economy verify $r+\delta = \beta b^{\beta-1}(\alpha/(1-\alpha))^{1-\beta}$, then $\mu = (1-\alpha)/\alpha = 1/a$ and $\lambda = (1-\delta)$. Thus in this situation, which can occur in both cases $\alpha < \beta$ and $\alpha > \beta$, the only balanced path is the non-production path[9] which for sure does not correspond to growth. □

Let us suppose now that a balanced growth path does exist. Is it stable or unstable ? To see which situation occurs, we pose $Z_t = \omega_t/k_t$ and study the stability of the steady state of :

$$Z_{t+1}^{\beta-\alpha}\left[\frac{aZ_t - 1}{a-b}b^{\beta}Z_t^{\beta-1} + (1-\delta)\right]^{1-(\alpha+\nu)} = Z_t^{\beta-\alpha}\Theta\Big[(1-\delta) + \beta b^{\beta-1}Z_{t+1}^{\beta-1}\Big].$$

There is no general answer to give concerning stability of the path. To illustrate the complexity, let us consider the very simple case $\alpha + \nu = 1$. We see that, at a fixed point μ :

$$\left(\frac{\partial Z_{t+1}}{\partial Z_t}\right)_{\mu} = \frac{1}{1 + \dfrac{1-\beta}{\beta-\alpha}\Theta\beta b^{\beta-1}\mu^{\beta-1}}.$$

We see that if a balanced growth path exists, then it is :

- stable if $\alpha < \beta$.
- either stable or unstable if $\beta < \alpha$.

[9] $\mu = 1/a \;\Rightarrow\; l_{1,t} = 1$ and there is therefore no investment...

Thus, there exists a great difference with regard to the one-sector model where balanced growth is taking place for all initial conditions of the economy if the condition for the existence of a balanced growth path, which is simply $u + v = 1$ if the production is of the form $F(K, L) = K^u L^{1-u} \bar{K}^v$, is met. In the framework of a two-sector model, existence is not guaranteed and stability or non-stability are both possible. But this is not the end of the story : in a forthcoming paper, we shall prove that there is the possibility of cycles of the growth rate around a steady state value and that the model considered here allows even the dynamics of λ_t to be chaotic.

4 Conclusion

We used a two-sector OLG model meeting all standard neo-classical assumptions from the point of view of the private sector to show the role of externalities in the determination of the dynamics of the economy. We established first that, in the model without externalities, the nature of the steady state depends on the relative magnitude of the capital intensivities in the different sectors of the economy, showing that saddle-point stability is guaranteed only in the case of a more capital intensive investment good sector, the stationary state being either a sink, a saddle point or a source if this condition is not met. Spiral behaviour is excluded under our assumptions when externalities do not operate. But when there exist spillovers from the total capital stock onto the efficiency of labour in each sector, then Hopf bifurcations can occur in the dynamical system, without any specific condition on the relative magnitude of the external effects. Existence of supercritical Hopf bifurcations, and therefore existence of attractive closed orbits around the steady state for whole continua of the parameters, has been proven for the case of a more capital intensive investment good sector. Whether supercritical Hopf bifurcations can appear in the alternative situation remains to be investigated. Finally, the difference with the one-sector model with regard to the existence and the stability of balanced growth paths has been emphasized.

In Appendix C, we show that the local dynamics around the steady state of an ILA model in discrete-time setting are of the same nature as for the OLG model when the coefficient of relative risk aversion is sufficiently small. From this we conclude to the non-robustness of the conditions for Hopf bifurcations or the existence and the stability of balanced growth paths with regard to the specification of continuous or discrete time, for the results obtained here differ strongly from those found by Cazzavillan [4] in the continuous-time setting. For instance, in the continuous-time setting, Hopf bifurcations can occur only if the spillover in the investment good sector is strong ($\beta + \psi > 1$) and if ν, the spillover in the consumption good sector, is larger than ψ. We have seen that in the discrete-time setting, neither of these conditions is necessary. We therefore have to give this sad conclusion of non-robustness.

The problem of existence of parameter configurations such that the economy exhibits self-sustained growth with a cyclic, or even chaotic, growth rate is clearly appealing since most existing models deal with economies where no long run growth is possible. Until now, this type of dynamics has been obtained in only two models, namely Cazzavillan [5] and Boldrin and Rustichini [3]. Unfortunately, these two papers contain important errors which invalidate the claims of their authors. Cazzavillan's paper studies a discrete-time one-sector ILA model with an externality, onto the productivity of labour in the production of the consumption good and onto the utilities of the agents, of a flow of public services which are financed through a proportional tax on income. The use of immediate feedback, which seems already abusive in a continuous-time setting à la Barro, appears to be absolutely unacceptable when time is discrete, and the whole result of Cazzavillan relies on this strange assumption. Therefore we cannot accept this model. Boldrin and Rustichini's model, a two-sector infinitely lived agents model with externalities in the production sector, leads to dynamics which cannot, except in the case of total capital depreciation in each period, be studied with the standard mathematical tools used to detect cycles or chaos ; the authors' claim of chaos relies on results 'established' by Boldrin and Persico [4] who, unfortunaly, consider an incorrect representation of the dynamics. Thus, both existing models have to be rejected, and the challenge still remains.

Further research could be undertaken to study the question of what is called sunspots. In OLG models, the rational expectations equilibria are not necessarily deterministic, even in the absence of exogenous shocks. Perfect foresight equilibria are only one class of possible equilibria as the research on sunspots has shown, and the fact that fluctuations can also originate in self-fulfilling beliefs should not be underestimated. It could be interesting to look at the possibility of sunspot equilibria in our model and to establish conditions for their existence, conditions that have to be compared to those necessary for deterministic cycles, which are often more constraining. Furthermore, the non Pareto-optimality of competitive equilibria when externalities do operate in the economy leaves open a whole field of research concerning the welfare improvement through government intervention and the effects of public policies on the possibility of endogenous fluctuations. And last, but not least, there remains the problem of implementation of rational expectations equilibria, i.e. a possible process according to which the values of variables predicted by the equilibrium are reached. The problem of stability under learning of perfect foresight equilibrium trajectories has been examined, for instance, by Grandmont and Laroque [10], in the context of backward[10] equilibrium dynamics ; in our framework, the dynamics happen to be forward, and the problem should be rather easy to study.

[10] The dynamics of a system are backward if the equations are under the form $x_t = F(x_{t+1})$, where F is not bijective. A forward dynamic has the form $y_{t+1} = G(y_t)$.

APPENDIX A : The Continuity Argument

From basic mathematics, we know that $(\sigma, \xi) = (\sigma, \alpha, \beta, \nu, \psi, r, \delta) \rightarrow Jac(F)$, where $F = (\Phi, \psi)$, is continuous. Furthermore Det and Tr are continuous operators. By composition of continuous applications, we deduce continuity of the eigenvalues with respect to the parameter vector. This continuity and the intermediate value theorem imply that if a Hopf bifurcation occurs for $(0, \bar{\xi})$, then there exists a neighbourhood V of $\sigma = 0$ such that for all $\sigma \in V$, there exist ξ_σ such that a Hopf bifurcation occurs at the point (σ, ξ_σ).

As a consequence of this, in a sufficiently small neighbourhood of $\sigma = 0$, the analysis undertaken in this paper is still valid.

APPENDIX B : Hopf Bifurcation

A glance at equation (2) induces us to simplify the tedious calculus of partial derivatives of an implicitly defined function by assuming $\delta = 1$. Under this assumption, ω_{t+1} is explicitly defined.

1.1.) Derivatives of F defined by : $k_{t+1} = F(k_t, \omega_t)$.

These partial derivatives are easily determined :

$$F_k = \frac{1}{a-b} b^\beta \omega^{\beta-1} k^{\psi-1} \Big[\psi(a\omega - k) - k\Big].$$

$$F_\omega = \frac{1}{a-b} b^\beta \omega^{\beta-2} k^{\psi} \Big[\beta(a\omega - k) + k\Big].$$

$$F_{kk} = \frac{\psi}{a-b} b^\beta \omega^{\beta-1} k^{\psi-2} \Big[(\psi - 1)(a\omega - k) - 2k\Big].$$

$$F_{\omega\omega} = \frac{(\beta-1)}{a-b} b^{\beta}\omega^{\beta-3}k^{\psi}\Big[2a\omega + (\beta-2)(a\omega - k)\Big].$$
$$F_{k\omega} = \frac{1}{a-b} b^{\beta}\omega^{\beta-2}k^{\psi-1}\Big[a\psi\omega + \psi(\beta-1)(a\omega-k) + (1-\beta)k\Big].$$
$$F_{kkk} = \frac{\psi(\psi-1)}{a-b} b^{\beta}\omega^{\beta-1}k^{\psi-3}\Big[(\psi-2)(a\omega-k) - 3k\Big].$$
$$F_{k\omega\omega} = \frac{(\beta-1)}{a-b} b^{\beta}\omega^{\beta-3}k^{\psi-1}\Big[2a\psi\omega + (\beta-2)(a\omega-k) - (\beta-2)k\Big].$$
$$F_{kk\omega} = \frac{\psi}{a-b} b^{\beta}\omega^{\beta-2}k^{\psi-2}\Big[a\beta(\psi-1)\omega - (\psi+1)(\beta-1)k\Big].$$
$$F_{\omega\omega\omega} = \frac{(\beta-1)(\beta-2)}{a-b} b^{\beta}\omega^{\beta-4}k^{\psi}\Big[3a\omega + (\beta-3)(a\omega-k)\Big].$$

1.2.) Derivatives of G defined by $\omega_{t+1} = G(k_t, \omega_t)$.

To get not too boring expressions, it is preferable to write :

$$G = \lambda \omega^x k^y F^z,$$

where $\lambda = \Big[\Theta\beta b^{\beta-1}\Big]^{\frac{1}{1-\alpha}}$, $x = (\beta-\alpha)/(1-\alpha)$, $y = (\psi-\nu)/(1-\alpha)$ and $z = \nu/(1-\alpha)$. We obtain thus the following expressions for the partial derivatives of G :

$$G_{kk} = \lambda\omega^x k^{y+z-2}[y(y-1) + 2yzF_k + z(z-1)F_k^2 + zkF_{kk}].$$
$$G_{\omega\omega} = \lambda\omega^{x-2}k^{y+z-2}[x(x-1)k^2 + 2xz\omega kF_\omega + z(z-1)\omega^2F_\omega^2 + z\omega^2kF_{\omega\omega}].$$
$$G_{k\omega} = \lambda\omega^{x-1}k^{y+z-2}[xyk + xzkF_k + yz\omega F_\omega + z(z-1)\omega F_kF_\omega + z\omega kF_{k\omega}].$$
$$\begin{aligned} G_{\omega\omega\omega} = {} & \lambda\omega^{x-3}k^{y+z-3}[x(x-1)(x-2)k^3 + 3x(x-1)z\omega k^2F_\omega \\ & + 3xz(z-1)\omega^2kF_\omega^2 + 3xz\omega^2k^2F_{\omega\omega} + 3z(z-1)\omega^3kF_\omega F_{\omega\omega} \\ & + z(z-1)(z-2)\omega^3F_\omega^3 + z\omega^3k^2F_{\omega\omega\omega}]. \end{aligned}$$

$$
\begin{aligned}
G_{kk\omega} =& \lambda\omega^{x-1}k^{y+z-3}\Big\{xk\big[y(y-1)+2yzF_k+z(z-1)F_k^2+zk^2F_{kk}\big] \\
&+z\omega\big[y(y-1)F_\omega+2y(z-1)F_kF_\omega+2ykF_{k\omega}+(z-1)k \\
&\times(F_{kk}F_\omega+2F_kF_{k\omega})+(z-1)(z-2)F_k^2F_\omega+k^2F_{kk\omega}\big]\Big\}. \\
G_{kkk} =& \lambda\omega^{x-2}k^{y+z-3}[y(y-1)(y-2)+3y(y-1)zF_k+3yz(z-1)F_k^2 \\
&+3yzkF_{kk}+3z(z-1)kF_kF_{kk}+z(z-1)(z-2)F_k^3+zk^2F_{kkk}]. \\
G_{k\omega\omega} =& \lambda\omega^{x-2}k^{y+z-3}\Big\{x(x-1)k^2[y+zF_k]+2xz\omega k[yF_\omega+(z-1)F_\omega F_k] \\
&+z\omega^2[x(z-1)F_\omega^2+xF_{\omega\omega}+(z-1)(z-2)F_\omega^2F_k \\
&+(z-1)k(F_{\omega\omega}F_k+2F_\omega F_{\omega k})+k^2F_{\omega\omega k}]\Big\}.
\end{aligned}
$$

We give now the calculus of the two numerical examples of 2.3.1 and 2.3.2, thus showing the procedure used to calculate $a(\psi_0)$.

2.1.) Case $\alpha < \beta$ and $\beta + \psi > 1$

Take $\alpha = 0.5$, $\beta = 0.7$, $r = 0.1$, $\delta = 1$, $\psi_0 = 41/70$ and $\nu = 383/574$. We have seen that this corresponds to a Hopf bifurcation, the unit circle of C^2 being crossed at $e^{i\pi/3}$. First, we have to find a basis in which the Jacobian takes the desired form

$$
\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.
$$

This is achieved by finding an eigenvector in C for the eigenvalue $\lambda = 1/2 + i\sqrt{3}/2$ of the Jacobian in the old basis. If $z = a + i\ b$ is an eigenvector corresponding to λ, then (b, a) is an adequate basis.

Proof: if we develop $J(a+i\ b) = (u+i\ v)(a+i\ b)$, we get : $Ja = u\ a - v\ b$, $Jb = v\ a + u\ b$. This shows that the Jacobian takes the right form in the basis (b, a). $\square$

The Jacobian is :

$$\begin{pmatrix} J_{11} & J_{12}(\bar{k}/\bar{\omega}) \\ \\ J_{21}(\bar{\omega}/\bar{k}) & J_{22} \end{pmatrix} = \begin{pmatrix} 467/140 & -(205/100)(11/7) \\ \\ (172309/40180)(7/11) & -13407/5740 \end{pmatrix}.$$

An eigenvector of this matrix for $1/2 + i\sqrt{3}/2$ is :

$$z = \begin{pmatrix} 1 \\ (2779 - i490\sqrt{2})/3157 \end{pmatrix},$$

and an adequate basis is therefore given by the matrix

$$P = \begin{pmatrix} 0 & 1 \\ -490\sqrt{3}/3157 & 2779/3157 \end{pmatrix}.$$

Thus, we have :

$$\begin{pmatrix} f(k,\omega) \\ g(k,\omega) \end{pmatrix} = P^{-1} \begin{pmatrix} F(k,\omega) \\ G(k,\omega) \end{pmatrix},$$

P^{-1} being the inverse of the matrix P and thus equal to :

$$P^{-1} = \begin{pmatrix} 2779/(490\sqrt{3}) & -3157/(490\sqrt{3}) \\ 1 & 0 \end{pmatrix}.$$

The extremely tedious calculus yields

$$F_k = +3.3357142857.$$

$$F_\omega = -3.2214285714.$$

$$F_{kk} = g_{kk} = +0.4043340068.$$

$$F_{k\omega} = g_{k\omega} = -0.4320911097.$$

$$F_{\omega\omega} = g_{\omega\omega} = +0.4826744131.$$

$$F_{kkk} = g_{kkk} = -0.03056115.$$

$$F_{kk\omega} = g_{kk\omega} = -0.0135788512.$$

$$F_{k\omega\omega} = g_{k\omega\omega} = +0.0945225245.$$

$$F_{\omega\omega\omega} = g_{\omega\omega\omega} = -0.9391088488.$$

$$G_{kk} = +0.6633836309.$$

$$G_{k\omega} = -0.55750081862.$$

$$G_{\omega\omega} = -0.51800217818.$$

$$G_{kkk} = -0.0256236692.$$

$$G_{kk\omega} = +0.063696038.$$

$$G_{k\omega\omega} = +0.2201104268.$$

$$G_{\omega\omega\omega} = -0.5826399932.$$

$$f_{kk} = -1.14369308191.$$

$$f_{k\omega} = +0.6589712779.$$

$$f_{\omega\omega} = +3.5073993024.$$

$$f_{kkk} = -0.0047560525.$$

$$f_{kk\omega} = -0.8632261326.$$

$$f_{k\omega\omega} = +0.0725690451.$$

$$f_{\omega\omega\omega} = -0.9077230925.$$

We calculate now the characteristic complex numbers which determine whether the bifurcation is supercritical or not :

$$C_{20} = -0.6894 - i\,0.174528.$$

$$C_{11} = 0.590907 + i\,0.221752.$$

$$C_{02} = -0.473345 + i\,0.154943.$$

$$C_{21} = -0.091668 + i\,0.0783174.$$

We get $a(\psi_0) = -0.03737 + 0.44723 - 0.02199 \approx +0.387$ and conclude to a supercritical Hopf bifurcation.

2.2.) Case $\alpha > \beta$ and $\beta + \psi < 1$

Take $\alpha = 0.7$, $\beta = 0.5$, $r = 0.1$, $\delta = 1$, $\psi_0 = (29 - 12\sqrt{2})/106$ and $\nu = (6497 - 2820\sqrt{2})/22790$. This corresponds to a Hopf bifurcation in the case of a more capital intensive consumption good sector, the unit circle of C^2 being crossed at $e^{i3\pi/4}$. Again, we have to find a basis in which the Jacobian takes the desired form

$$\begin{pmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

The Jacobian in the old basis is :

$$\begin{pmatrix} -1.536514 & +3.123585 \\ -0.380306 & +0.122301 \end{pmatrix}.$$

An adequate basis is given by the matrix

$$P = \begin{pmatrix} 0 & 1 \\ 0.226377 & 0.265531 \end{pmatrix}, \qquad \Rightarrow$$

$$P^{-1} = \begin{pmatrix} -1.172960 & +4.417420 \\ 1 & 0 \end{pmatrix}.$$

The values of the partial derivatives evaluated at the steady state are :

$$F_k = -1.53651.$$

$$F_\omega = +3.12358.$$

$$F_{kk} = g_{kk} = -2.25358.$$

$$F_{k\omega} = g_{k\omega} = +7.36666.$$

$$F_{\omega\omega} = g_{\omega\omega} = -19.02235.$$

$$F_{kkk} = g_{kkk} = +15.47469.$$

$$F_{kk\omega} = g_{kk\omega} = -0.94966.$$

$$F_{k\omega\omega} = g_{k\omega\omega} = -37.43241.$$

$$F_{\omega\omega\omega} = g_{\omega\omega\omega} = +282.00981.$$

$$G_{kk} = -0.108362.$$

$$G_{k\omega} = +7.290299.$$

$$G_{\omega\omega} = -11.796961.$$

$$G_{kkk} = -1188.320761.$$

$$G_{kk\omega} = +65.983671.$$

$$G_{k\omega\omega} = -113.032485.$$

$$G_{\omega\omega\omega} = -8734.614381.$$

$$f_{kk} = +2.096398.$$

$$f_{k\omega} = +23.784486.$$

$$f_{\omega\omega} = -29.799625.$$

$$f_{kkk} = -5267.458502.$$

$$f_{kk\omega} = +292.591244.$$

$$f_{k\omega\omega} = -455.404802.$$

$$f_{\omega\omega\omega} = -38915.212744.$$

The characteristic complex numbers are now :

$$C_{20} = 5.8286 - i\,3.8499.$$

$$C_{11} = -6.9258 - i\,5.3189.$$

$$C_{02} = 2.1453 + i\,8.0422.$$

$$C_{21} = -340.1127 + i\,2412.5415.$$

We get $a(\psi_0) \approx -1274.25$ and conclude therefore to a subcritical Hopf bifurcation.

APPENDIX C : The ILA Model

We keep all the assumptions concerning production and form of the one period utility, assume preference for the present ($r > 0$), and consider the problem of infinitely lived agents. The agents' problem is :

$$max \ \sum_{0}^{+\infty} \Theta^t \frac{c_{a,t}^{1-\sigma}}{1-\sigma},$$

$$s.t. \ \ k_{t+1} = (1-\delta+r_t)k_t + e_t - p_t c_{a,t}.$$

If we let the coefficient of relative risk aversion σ tend to zero, we see that the dynamics of the ILA model are given by :

$$k_{t+1} = i_t + (1-\delta)k_t,$$

$$p_{t+1} = p_t \Theta\Big((1-\delta) + r_{t+1}\Big),$$

which can be written under the following form :

$$k_{t+1} = \frac{a\omega_t - k_t}{a-b} b^{\beta} \omega_t^{\beta-1} k_t^{\psi} + (1-\delta)k_t, \tag{1''}$$

$$\omega_{t+1}^{\beta-\alpha} k_{t+1}^{\psi-\nu} = \omega_t^{\beta-\alpha} k_t^{\psi-\nu} \Theta[(1-\delta) + \beta b^{\beta-1} \omega_{t+1}^{\beta-1} k_{t+1}^{\psi}]. \tag{2''}$$

These are exactly the equations characterising the dynamics of the OLG model. In particular, Hopf bifurcations can occur in both situations ($\beta+\psi > 1$) and ($\beta + \psi < 1$), which is at the opposite of the results established by Cazzavillan [4] in the continuous time setting[11]. We must even see that

[11] Our numerical examples given for the OLG Model assumed $\delta = 1$, which, of course, does not make much sense in an ILA Model with spillover from capital under the form of learning by doing. But it is easy to verify that Hopf bifurcations are possible with $\delta << 1$. We took $\delta = 1$ in order to simplify the calculus necessary to determine the nature of the bifurcation...

many other results (concerning for instance the problem of balanced growth path) are linked to the continuous-time specification and do not hold in the discrete-time version of the model. This is of course a sad conclusion, since results are not robust with regard to time specification.

APPENDIX D : The Case $\sigma = 1$

Let us look at the particular case $\sigma = 1$ where savings are independent of the interest rate. We know[12] that the dynamical system $(1), (2')$ collapses into one single equation since $(2')$ is of the form $k_{t+1} = F(k_t, \omega_t)$. Indeed, we find :

$$k_{t+1} = \frac{a\omega_t - k_t}{a-b} b^\beta \omega_t^{\beta-1} k_t^\psi + (1-\delta)k_t, \tag{1}$$

$$k_{t+1} = \frac{\Theta}{1+\Theta}(1-\beta)b^\beta {\omega_t}^\beta k_t^\psi. \tag{2'}$$

It is immediate to verify that

$$\frac{\partial k_t}{\partial \omega_t} \neq 0,$$

and we can apply the implicit function theorem to write : $(1), (2') \Leftrightarrow$

$$\begin{cases} k_{t+1} = \bar{F}(k_t), \\ \omega_t = \bar{G}(k_t). \end{cases}$$

A steady state $\Omega = (\bar{k}, \bar{\omega})$ exists if and only if $\beta + \psi \neq 1$. Under this assumption, we get :

$$\bar{k} = \left[\frac{\delta(\beta - \alpha) + (2+r)\alpha}{(1-\alpha)(2+r)}\right]\bar{\omega},$$

[12] Galor [7].

$$\bar{k} = \left\{ \frac{2+r}{(1-\beta)b^{\beta}} \left[\frac{\delta(\beta-\alpha)+(2+r)\alpha}{(1-\alpha)(2+r)} \right]^{\beta} \right\}^{\frac{1}{\beta+\psi-1}}.$$

There are no conditions imposed on the parameters. If we want to study the stability of the stationary state, we linearize the dynamical system to obtain :

$$k_{t+1} - \bar{k} = \lambda_k (k_t - \bar{k}),$$

$$\lambda_k = \frac{\beta + \delta\psi - (\beta+\psi)(2+r)\alpha/(\alpha-\beta)}{\delta + (1-\beta) - (2+r)\alpha/(\alpha-\beta)}.$$

We have[13] :

- Complete stability of $\Omega \Leftrightarrow \beta + \psi < 1$,
- Complete unstability of $\Omega \Leftrightarrow \beta + \psi > 1$.

Let us suppose now $\beta + \psi = 1$. Equations $(1), (2)$ imply :

$$\begin{aligned} \frac{k_{t+1}}{k_t} &= \frac{aZ_t - 1}{a-b} b^{\beta} Z_t^{\beta-1} + (1-\delta) \\ &= \frac{\Theta}{1+\Theta}(1-\beta) b^{\beta} Z_t^{\beta}, \end{aligned}$$

where $Z_t = \omega_t / k_t$. Therefore Z_t is constant[14]. Thus, if the coefficient of intertemporal substitution is equal to one, the dynamics are of a very simple type and do not bear any possibility of endogenous cycles.

[13]Equation $(2')$ clearly shows that, in the neighbourhood of Ω, $(\omega_t)_t$ is of the same nature as $(k_t)_t$ since we made the assumption $G \in [0, 1[$.

[14]Existence of a solution for the equation is guaranteed if $\delta = 1$. But this implies existence for all δ...

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CHAPTER III

Chaos and Bubble-Fluctuations in Endogenous Growth

CHAOS AND BUBBLE-FLUCTUATIONS IN ENDOGENOUS GROWTH

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Abstract

We establish that a two-sector OLG economy where production inputs substitute perfectly and savings monotonically increase with the interest rate can exhibit endogenous growth with endogenous cyclic or even chaotic fluctuations in the growth rate if externalities operate in the production sectors. We show that this requires a strong externality in the consumption good sector in the absence of bubbles or sunspots, but not necessarily in their presence. We furthermore prove that there exist production economies where, in the absence of any intrinsic uncertainty, the only possible equilibria involve bubbles or sunspots, a result which makes appear questionable the notion of 'fundamental' equilibrium trajectory.

Keywords : Externalities, Bubbles, Cycles and Chaos.

Introduction

Business Cycle Theory explains the fluctuations displayed by economic time series as deviation from trends in consequence of exogenous shocks originating in variations of private sector behaviour, technological changes, stochastic shifts in government policy etc. In the standard models of this theory, the equilibrium is, in the absence of shocks, defined and at least locally unique (determinate and stable), and the economy converges to this steady state. This conception is so widespread that most textbooks refer only to this type of explanation of economic fluctuations. Nothing induces us, however, to believe in this idealized story of a world that would be nicely monotone in the absence of shocks from the outside. Fluctuations may well have, at least partly, endogenous origins. The methodology that is adopted by researchers interested in this area of 'endogenous business cycles' is generally the following : they consider models without any intrinsic uncertainty (no exogenous shocks), and show that, under certain conditions on tastes, technologies and beliefs of the agents, cycles, closed orbits or even chaos can occur. This does not mean, of course, that these economists do not take seriously the importance of exogenous shocks, but is simply the result of two considerations : first of all it is, from a purely epistemological point of view and to gainsay the traditional conception, interesting to show that the standard assumptions on the agents' behaviour, on production etc. do not rule out fluctuations in the absence of extraneous uncertainty, and that at least a part of real world fluctuations may be explained endogenously ; secondly, the mathematical tools available nowadays simply do not allow to deal formally with non-linear models subject to exogenous shocks.

The question of endogenous cycles or chaos in economic models without any intrinsic uncertainty has now been studied for several years. This does not mean that the topic has been exhausted : in fact, when we consider the main part of the literature on endogenous fluctuations, one fact strikes our attention : nearly all models, Overlapping Generations Models (OLG) as well as Infinitely Lived Agents Models (ILA), need, in order to establish existence results for cycles or chaotic trajectories, and especially for the latter, assumptions on the utilities of the agents or on the production functions (when production takes place) that are not empirically defendable. For instance, one often met assumption is the negative interest rate elasticity of savings at the Golden Rule steady state (see Grandmont [20] ; Jullien [24]...) ; another is that of complementarity, or not too important substitutability, of the production factors (see Reichlin [27], [28]). A recently published paper by Boldrin and Rustichini [8] tries to give an example of a two-sector ILA economy with externalities where, under the assumption of linear utility, endogenous growth with chaotic growth rate can take place ; unfortunately, the dynamics exhibited by the authors do not correspond to the optimal path. Two preceding papers have established, one in the ILA framework with continuous time (Cazzavillan [9]), the other in the framework of OLG and ILA in discrete-time setting (v. Coester [11]), that a model meeting all standard neo-classical assumptions from the point of view of the private sector, but with externalities of the total capital stock onto the efficiency of labour in each productive sector of the economy, can exhibit closed orbits of the economy's state variables.

All these papers show the possibility of cycles or of chaotic, but bounded, trajectories of the capital stock, and therefore deal with economies where there is no real growth, except Boldrin and Rustichini's model. In a critical note on their paper, we have shown that the authors' claim of chaotic trajectories for the growth rate must be considerably attenuated since *observable* chaos cannot be established in their model unless assuming that capital depreciates entirely in each period, an unacceptable assumption in the ILA framework given the period's length. Furthermore, we proved that even cycles or topological chaos cannot be claimed for without the assumption of a very important capital depreciation per period.

We exhibit here a two-sector OLG economy with standard utility, where savings increase monotonically with the interest rate (positive IES), and where production inputs substitute perfectly, for which endogenous growth with endogenous fluctuations of observable chaotic type is a possible issue. We use again here the two-sector OLG model of v. Coester [11] under the assumption of an externality in the investment good sector such that balanced growth may take place. A question arises : why consider a two-sector model with externalities since we know from Galor [18] that endogenous cycles may occur in a neoclassical two-sector OLG model without externalities and from O'N. Fisher[1] [26] that it is not necessary to assume any externality to get self-sustaining growth in such a framework ? The answer to this is, first, that standard utility and production functions give neither cycles nor self-sustaining growth in a two-sector model, secondly that we wish to have the possibility of balanced growth paths, a property which can be obtained in a simple manner only by assuming externalities or an AK investment good production function.

The fact that bubbles on an intrinsically useless asset can generate flip or Hopf bifurcations and thus cycles or closed orbits is well known by now. Most models show this with negative bubbles, called 'negative outside money', resulting from a 'constant zero budget deficit' constraint of the government (Benhabib and Day [3], Farmer [15]...) ; Farmer's model is particularly interesting because it establishes the fact that in a one-sector OLG model with production and inelastic labour supply, Hopf bifurcations can be generated only by negative bubbles ; the examples he exhibits to show that negative bubbles can generate cycles all involve production with complementarity of inputs. Jullien [24] shows, in a very elegant manner, that cycles can be generated by positive bubbles in a one-sector economy with production, where production inputs substitute perfectly, if the savings function is non mono-

[1] O'N. Fisher establishes the very important fact that any growing convex OLG economy with at least two sectors must exhibit an investment sector with asymptotically linear technology. That one-sector convex OLG economies cannot exhibit sustained growth is a well known fact.

function already appeared in Grandmont's [20] seminal paper on a pure endowment economy with endogenous fluctuations generated by fiat money (positive bubble). Jullien's example of a period three cycle furthermore requires a preference for future consumption. Reichlin [29] shows that, in a two-sector model, positive bubbles can lead to closed orbits through Hopf bifurcations, but he needs not only a negative IES but also complementarity of the production inputs to get his result.

The paper "Growth, Externalities, and Sunspots" by Spear [31] analyses the existence problem of sunspots in an ILA model of neo-classical capital accumulation with production externalities. His conclusion is that, in the presence of the externalities, sunspots can exist, and thus cyclic trajectories of the state variable. Spear's externality is really non-standard : the spillover comes from the anticipated average savings of all agents, i.e. from the anticipated per capita stock of capital of the following period ; furthermore, cycles in Spear's model are cycles of the capital stock, and therefore the title of his paper is misleading since there is no real growth taking place.

We show that, in our framework with positive IES and perfect substitutability between the production factors, we can have positive as well as negative bubbles generating Hopf bifurcations. This can lead to a situation where we have endogenous growth with endogenous fluctuations. In the presence of bubbles, we can have self-sustaining growth with a fluctuating growth rate even if the externality in the consumption good sector is not very strong, and even in the absence of any externality in this sector in the case of negand even in the absence of any externality in this sector in the case of negative bubbles. This is an important result since Boldrin and Rustichini [8] (if we take their model seriously and accept to take the unappealing parameter configurations which are required to guarantee the existence of cycles or topological chaos in their framework) need a strong externality in the consumption good sector for the possibility of cycles or chaotic trajectories[2]. We also show that if the consumption good sector is more capital intensive than the investment good sector, then we cannot obtain endogenous fluctuations through Hopf bifurcations in the presence of a positive bubble. Furthermore, we exhibit an economy having no non-bubbly equilibrium, but equilibria with bubbles. This is due to the non-convexity at the aggregate level of the production function of the investment good. A similar result of economies with sunspot equilibria but no equilibrium without extrinsic uncertainty has been established by Pietra [27] in the context of a pure endowment economy with a finite set of agents and a finite horizon.

1 The OLG Model

Time is discrete. At each date t, a new generation of agents is born ; the size of each generation is assumed to remain constant over time and will be normalised to one. Agents live for two periods ; in the first period of their life, they work, consume and save for their old age, and in the second, they consume their actualised savings. Bequests are not allowed. Firms are

[2] In our model too, a strong externality must operate in the consumption good sector in the absence of bubbles.

owned by the old people ; the number of firms is supposed large enough to have perfect competition and thus profit maximisation in both sectors.

Agents are characterised by their utility, which is supposed to be time-separable and of CRRA type :

$$u(c_{t,t}, c_{t,t+1}) = \begin{cases} \dfrac{c_{t,t}^{1-\sigma}}{1-\sigma} + \Theta \dfrac{c_{t,t+1}^{1-\sigma}}{1-\sigma}, & \sigma \neq 1 \\ Ln(c_{t,t}) + \Theta Ln(c_{t,t+1}) \end{cases}$$

where $c_{t,t}$, $c_{t,t+1}$ denote respectively consumption when young and consumption when old of an agent of generation t, $\sigma > 0$ is the coefficient of relative risk aversion (equal to the inverse of the elasticity of substitution between consumption at any two points in time) and $\Theta = 1/(1+r)$, where $r \in]-1, +\infty]$ is the rate of time preference.

The production of each good is of the following type :

$$\begin{aligned} C_t &= C K_{1,t}^{\alpha} L_{1,t}^{1-\alpha} K_t^{\nu}, \\ I_t &= I K_{2,t}^{\beta} L_{2,t}^{1-\beta} K_t^{1-\beta}, \end{aligned}$$

$(\alpha, \beta) \in]0,1[^2$ and $\nu \in \mathbb{R}_+$, $K_{i,t}$ being the level of capital at time t in sector i, K_t the aggregate level of capital at time t. As is easy to verify, the externality in the investment good sector is such that the necessary condition for the existence of a (not necessarily unique) balanced growth path is met. Without any loss of generality, we take $C = 1$; the parameter I will not be taken equal to one since endogenous growth in a context of total capital depreciation, an assumption which simplifies the study of the dynamics, requires $I > 1$. The rate of depreciation per period of the capital is $\delta > 0$. We suppose $\alpha \neq \beta$ (two sectors) and assume full employment.

$$\begin{cases} K_{1,t} + K_{2,t} = K_t \\ L_{1,t} + L_{2,t} = L_t. \end{cases}$$

We reformulate the model in per capita terms ; denoting $l_{i,t}$ the proportion of the labour force allocated to sector i and $k_{i,t}$ the per capita capital stock in sector i, we get :

$$c_t = k_{1,t}^{\alpha} l_{1,t} K_t^{\nu},$$

$$i_t = I k_{2,t}^{\beta} l_{2,t} K_t^{1-\beta}.$$

Since we have

$$\begin{cases} l_{1,t} + l_{2,t} = 1 \\ l_{1,t} k_{1,t} + l_{2,t} k_{2,t} = k_t \end{cases}$$

we can eliminate $l_{i,t}$ and obtain :

$$c_t = \frac{k_t - k_{2,t}}{k_{1,t} - k_{2,t}} k_{1,t}^{\alpha} K_t^{\nu},$$

$$i_t = I \frac{k_{1,t} - k_t}{k_{1,t} - k_{2,t}} k_{2,t}^{\beta} K_t^{1-\beta}.$$

Firms behave competitively and maximise their profits in each period, without taking the externalities into account. Let e_t be the competitive wage rate and r_t the competitive rate of return on capital and p_t the relative price of the consumption good in terms of investment good (the numeraire here). Profit maximisation and constancy of labour force imply :

$$e_t = p_t(1-\alpha) k_{1,t}^{\alpha} k_t^{\nu} = (1-\beta) I k_{2,t}^{\beta} k_t^{1-\beta},$$

$$r_t = p_t \alpha k_{1,t}^{\alpha-1} k_t^{\nu} = \beta I k_{2,t}^{\beta-1} k_t^{1-\beta},$$

If we denote by ω_t the wage-interest rate ratio, we see that

$$k_{1,t} = a\omega_t, \; k_{2,t} = b\omega_t,$$

where $a = \alpha/(1-\alpha)$ and $b = \beta/(1-\beta)$. Thus price, wage and interest rate can be expressed as functions of ω_t and k_t, which will be chosen as state variables. Notice that only k_t is predetermined.

In the next section, we study the competitive equilibria in the absence of any intrinsic uncertainty (no exogenous shocks), bubbles or sunspots, the so-called 'fundamental' equilibria. We show that flip bifurcations may occur, and that for adequate parameter configurations, topological, ergodic or even turbulent chaos can exist, erratic trajectories requiring a strong externality in the consumption good sector ($\alpha + \nu > 1$...). In the third section, we consider the equilibria in the presence of bubbles and show that Hopf bifurcations may occur and thus generate closed orbits of the growth rate. We show that a strong externality ν is no longer required to obtain endogenous growth with endogenous fluctuations. The last and very short fourth section is devoted to the problem of economies that do not exhibit any 'fundamental' equilibrium but do have equilibria in the presence of bubbles, a result which shows that the notion of 'fundamental' equilibrium does not make much sense in a non-linear world.

2 Fluctuations in the Bubbleless Economy

In the absence of any bubble, all the savings s_t are used for productive investment purposes, one part being used to buy the capital $(1-\delta)k_t$ stock from the old generation, the rest being used to produce the investment good. The optimal behaviour of the agents and the equilibrium conditions imply the following :

Maximisation of utility

$$\begin{cases} max\ u(c_{t,t}, c_{t,t+1}) \\ s.t.\ p_t c_{t,t} + \dfrac{p_{t+1}c_{t,t+1}}{1-\delta+r_{t+1}} \leq e_t, \end{cases}$$

yields the Euler equation :

$$c_{t,t}^{-\sigma} = \frac{p_t}{p_{t+1}}\Theta(1-\delta+r_{t+1})c_{t,t+1}^{-\sigma}.$$

The equilibrium conditions

$$s_t = e_t - p_t c_{t,t} = (1-\delta)k_t + (k_{t+1} - (1-\delta)k_t),$$

$$k_{t+1} = (1-\delta)k_t + i_t,$$

give the dynamics of the economy :

$$k_{t+1} = I\frac{a\omega_t - k_t}{a-b}b^{\beta}\omega_t^{\beta-1}k_t^{1-\beta} + (1-\delta)k_t, \tag{1}$$

$$\omega_{t+1}^{\beta-\alpha}k_{t+1}^{1-\beta-\nu}\left[\frac{1}{\Theta}\frac{k_{t+1}}{e_t - k_{t+1}}\right]^{\frac{\sigma}{1-\sigma}} = \omega_t^{\beta-\alpha}k_t^{1-\beta-\nu}\Theta[1-\delta+\beta b^{\beta-1}I\omega_{t+1}^{\beta-1}k_{t+1}^{1-\beta}]. \tag{2}$$

If $\sigma = 1$, equation (2) reduces to :

$$k_{t+1} = \frac{\Theta}{1+\Theta}e_t. \tag{2'}$$

For simplicity, we consider the case of total per period depreciation of capital. This could seem to be a rather strange assumption to take after having criticised Boldrin and Rustichini by showing that it is not possible to establish, in their framework, existence of observable chaos unless assuming entire depreciation of the capital in each period. But remember : in an ILA framework, periods are short, whereas in the OLG context they are long. Furthermore, for our model, $\delta < 1$ does not imply impossibility of proving that *observable* chaos can occur, but simply increases the mathematical complexity (see Appendix A).

We rule out the case $\sigma = 1$ since we have established, in a preceding paper[3], that logarithmic per period utility does not lead to any type of erratic dynamics in the absence of bubbles or extrinsic uncertainty. It is immediate to verify that, under the assumption of $\sigma \neq 1$, the dynamical system in (k_t, ω_t) is equivalent to the following system, giving the dynamics of (λ_t, Z_t), where $\lambda_t = k_{t+1}/k_t$, $Z_t = \omega_t/k_t$:

$$\lambda_t = I\frac{aZ_t - 1}{a-b} b^{\beta} Z_t{}^{\beta-1},$$

$$Z_{t+1}^{1-\alpha}\left[I\frac{aZ_t - 1}{a-b} b^{\beta} Z_t^{\beta-1}\right]^{1-(\alpha+\nu)} = \Theta\beta b^{\beta-1} I Z_t^{\beta-\alpha}\left[\Theta\frac{(1-\alpha)-\beta Z_t}{\alpha Z_t - (1-\alpha)}\right]^{\frac{\sigma}{1-\sigma}}.$$

It is therefore sufficient to study the dynamics of Z_t which give those of the growth rate $\lambda_t - 1$. Notice that the dynamics are forward. We can rewrite our second equation under one of the following forms :

- If $\alpha < \beta$,

$$U_{t+1} = KU_t^m(1-U_t)^n(cU_t - 1)^p,$$

[3]See v. Coester [11].

- If $\alpha > \beta$,

$$U_{t+1} = KU_t^m(U_t - 1)^n(1 - cU_t)^p,$$

where $U_t = aZ_t$, $c = \beta/\alpha$,

$$K = \Big(\frac{\alpha}{1-\alpha}\Big)^{\frac{(1-\beta)(\alpha+\nu)}{1-\alpha}} \left[\Theta^{\frac{1}{1-\sigma}} \left(\frac{I\beta}{b^{1-\beta}}\right)^{\alpha+\nu} \left(\frac{1-\alpha}{|\beta-\alpha|}\right)^{\alpha+\nu-1}\right]^{\frac{1}{1-\alpha}},$$

$$m = 1 - \frac{1-\beta}{1-\alpha}(\alpha+\nu),$$

$$n = \frac{1}{1-\alpha}\Big(\alpha+\nu-\frac{1}{1-\sigma}\Big),$$

$$p = \frac{\sigma}{(1-\sigma)(1-\alpha)}.$$

Linear conjugacy does not, of course, affect the nature of the dynamics. The dynamics are defined on a subinterval of $[1/c, 1]$ if $(\alpha < \beta)$ and of $[1, 1/c]$ otherwise. Since our aim is to give an example establishing our claims, we refrain from exposing a general study of the dynamics. It can be shown that if $n < 0$ or $p < 0$, then a steady state always exists and is necessarily unstable, which is of no interest. For our purpose here, we need a hump-shaped mapping Υ, which requires $n > 0$ and $p > 0$, and therefore a strong externality in the consumption good sector $(\alpha + \nu > 1)$ and a coefficient of intertemporal substitution larger than one $(1/\sigma > 1)$.

Since our aim is to establish an existence result, we suppose now that $\alpha < \beta$, and choose $m = 0$, $n = 1$ and $p = 1$ by taking :

$$\sigma = \frac{1-\alpha}{2-\alpha},$$

$$\nu = 3(1-\alpha),$$

$$\beta = \frac{2-\alpha}{3-2\alpha}.$$

α, Θ and I remain parameters. The mapping $\Upsilon = \Upsilon_{\alpha,K(\Theta,I)}$ is hump-shaped and of the simplest possible form : a well chosen restriction of it is linearly topologically conjugate to the standard logistic map Γ_μ defined by : $X_{t+1} = \mu X_t(1-X_t)$.

Proposition : $\forall \alpha \in]0,1[$, $\forall B > 2$, *there exists a closed interval* $\Lambda = [K_2, K_B]$ *such that :* $K \in \Lambda \Rightarrow \exists J \subset]\alpha/\beta, 1[$, $\exists \mu \in [2, B]$ *such that the restriction to* J *of* Υ *is a linear topological conjugate of* Γ_μ *:* $\Upsilon_{\alpha,K/J} \overset{\text{lin. top.}}{\equiv} \Gamma_\mu$.

Proof : This is obvious. If K is large enough, then there exist two fixed points, $U_1 < U_2$, the first being always unstable, the stability of the second one depending on K. The mapping $\Upsilon_{\alpha,K(\Theta,I)}$ admits a maximum at

$$U^* = \frac{1}{2}\Big(1+\frac{1}{c}\Big), \quad U^* \in]U_1, U_2[.$$

If we write

$$X_t = \frac{U_t - U_1}{2(U^* - U_1)},$$

then we get :

$$\begin{cases} X_{t+1} = \mu X_t(1-X_t), \\ \mu = \mu_{\alpha,K} = \dfrac{cK}{2(U^*-U_1)}. \end{cases}$$

It is easy to see that there exists $\Lambda = [K_2, K_B]$ such that when K increases from K_2 to K_B, μ increases from 2 to B. □

If we consider our family of maps Υ, we see that for each α, we can choose K such that $\Upsilon_{\alpha,K}$ is conjugate to Γ_μ, $\mu \geq 2$. We can thus invoke all the classical or less classical results about the maps Γ_μ : from μ increasing from 2 to 4, the family undergoes a flip bifurcation cascade. First, there appears a cycle of period 2 for the value $\mu = 3$. A two-cycle exists and is stable for all $\mu \in [3, 3.449499[$. For $\mu = 3.449499$, a 4-cycle appears ; at $\mu = 3.549090$, a cycle of period 8 emerges etc. The periods of the emerging cycles follow Šarkovskii's ordering which is defined as follows : $3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2 \cdot 5 \succ \cdots \succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ \cdots \succ 2^m \succ \cdots \succ 4 \succ 2 \succ 1$. For μ comprised between two bifurcation values, the existing cycle is stable. Šarkovskii's theorem says that for any *continuous* hump-shaped map from an interval $[a, b]$ into itself, if a cycle of order k exists, then there exists a cycle of period k' for every $k \succ k'$. But for μ large enough, there exists a cycle of period three. From Šarkovskii's theorem, we can thus deduce that for μ large enough, there exist cycles of every imaginable period. An important question is then stability of these cycles.

As a consequence of a theorem established by Li and Yorke[4] in 1975, for all values of μ large enough to allow the existence a cycle of period three, the map Υ exhibits *topological* chaos. We know that the notion of topological chaos is not exhilarating because nothing prevents the set of initial conditions giving chaos to be of zero Lebesgue measure : this happens when there exists a stable cycle, for instance when $\mu = 3.839$ where the map admits a stable

[4]The famous and often misinterpreted "period 3 implies chaos"...

cycle of period three, but where chaos as well as the infinity of cycles of other periods than three are simply unobservable.

Fortunately, sufficient conditions to guarantee a non zero measure of the set S of initial conditions for chaos have been established, and we know that Γ_4 exhibits *ergodic* as well as *turbulent* chaos on $[0,1]$, the set S being even of full measure. Furthermore, a less standard result is that, for $\mu > 4$, Γ_μ is defined and chaotic on a Cantor set included in $[0,1]$, the set S being of full Cantor measure, and that $\mu > 2+\sqrt{5}$, guarantees structural stability. Unfortunately, these last 'exotic' dynamics are of no interest for economists since they require an assumption on the degree of calculation power of the agents that cannot be accepted. However, we have very nice properties for the dynamics in this model, since we can claim existence of parameter configurations such that the spectrum of a trajectory a.s. resembles the spectrum of a random noise.

For a given K, we can choose I such that $\lambda_t > 1$ on the whole interval $[U_1(K), U^1(K)]$, where $U^1(K)$ is such that $\Upsilon_{\alpha,K}(U^1(K)) = U_1(K)$, and then take Θ such that $K(I,\Theta) = K$, which is always possible[5]. If we do this, we obtain, of course, a very unrealistic type of endogenous growth with a chaotic rate of growth since no recessions can take place. But using the fact that, in the case of the logistic map, we know the ergodic distribution, we can choose our parameters in order to have a positive average growth rate of a reasonable value and possibility of recessions.

[5]We do not rule out here preference for future consumption...

Notice that to obtain endogenous growth under the assumption of total capital depreciation per period, we need a parameter I larger than one :

$$\lambda_t = (1-\delta) + I\frac{\alpha}{\alpha-\beta}(1-\beta)^{1-\beta}(1-\alpha)^{\beta}\left(\frac{\beta}{\alpha}\right)^{\beta}\frac{U_t-1}{U_t^{1-\beta}}.$$

It is immediate to realise that $\lambda_t < \lambda(1/c)$. But then we have :

$$\lambda_t < (1-\delta) + I(1-\beta)^{1-\beta}(1-\alpha)^{\beta}.$$

We see that if $\delta > I(1-\beta)^{1-\beta}(1-\alpha)^{\beta}$, then the trend is negative and the economy collapses in the long term. This is especially true when capital depreciation is one per period and $I \leq 1$. $\square$

Proposition : *In our framework, self-sustaining growth with a chaotic, and in the average positive, growth rate is a possible observable issue. Thus, even if the neo-classical assumptions on utilities or production are met from the point of view of the private sector, we cannot rule out endogenous fluctuations in growth in the absence of exogenous shocks. In the absence of bubbles or sunspots, erratic dynamics require a strong externality in the consumption good sector and a coefficient of intertemporal substitution larger than one.*

3 Bubbles as Generators of Fluctuations

We consider the situation where there exists an intrinsically useless asset in which the agents can invest. This type of asset is traditionally called a bubble. What changes compared to the situation encountered in the first section is that now productive capital stock of next period is no longer necessarily equal to actual savings : a positive bubble for instance diverts capital from productive investment. The traditional interpretation of a positive bubble is fiat money, and a negative bubble can result, for instance, from a 'constant zero budget deficit' constraint of the Government[6] : the private sector is then a net debtor. It is clear that positive bubbles are far more appealing since a negative bubble's interpretation lies in a public policy, and a very specific one, whereas positive bubbles can exist in a pure 'laisser-faire' economy.

We show here that the introduction of a bubble can generate cycles in an economy where cycles do not occur in a bubbleless equilibrium. This idea is not new, as we have explained in the introduction, since it has been exploited by Benhabib and Day [3] and Farmer [15] etc. in the case of negative bubbles, Grandmont [20], Jullien [24] and Reichlin [29] and others in the case of positive bubbles. But the latter only prove that endogenous cycles can occur in the presence of positive bubbles under the following assumptions : Grandmont, in a pure endowment economy, and Jullien, in a one-sector economy with production, need an interest rate elasticity of savings at the Golden Rule state smaller than -0.5, condition needed again by Reichlin, in his two-sector model, in addition to his assumption of production with complementary factors. Complementarity of production inputs appears also in the two examples given by Farmer of negative bubbles generating Hopf bifurcations in a one

[6] See Farmer [15].

sector economy with production.

We present here a model which not only meets all standard neo-classical assumptions from the point of view of the private sector and where positive as well as negative bubbles can generate endogenous cycles, but better : in our framework, cycles are cycles in the growth rate and not in the capital stock, and we can thus exhibit again a model of endogenous growth with a fluctuating rate of growth. As we shall see hereafter, the conditions for endogenous fluctuations generated by bubbles differ much from those required for cycles or chaotic trajectories to be possible 'fundamental' equilibria.

We consider the general case $\delta \in]0,1]$. We want to study the dynamics of competitive equilibria with bubbles. Let B_t be the per capita value of the bubble at date t. The equilibrium condition 'investment equals savings' is :

$$s_t = e_t - p_t c_{t,t} = (1-\delta)k_t + (k_{t+1} - (1-\delta)k_t) + B_t.$$

The no arbitrage condition concerning the useless asset which yields no dividends (or the Government's 'constant zero budget deficit' constraint in the case of a negative bubble) imposes :

$$B_{t+1} = (1 - \delta + r_{t+1})B_t.$$

Maximisation of utility, perfect competition and equilibrium in the markets imply, if we define $\lambda_t = k_{t+1}/k_t$, $Z_t = \omega_t/k_t$, and $b_t = B_t/k_t$ (b_t is the 'reduced' bubble[7]), a dynamic system of the following form :

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[7]By pure laziness, we shall hereafter use the term 'bubble' instead of 'reduced bubble'...

$$\lambda_t = I\frac{aZ_t - 1}{a-b}b^\beta Z_t{}^{\beta-1} + (1-\delta),$$

$$\frac{Z_{t+1}^{\beta-\alpha}}{\Theta[1-\delta+I\beta b^{\beta-1}Z_{t+1}^{\beta-1}]} = \lambda_t^{(\alpha+\nu)-1}Z_t^{\beta-\alpha}\left[\Theta\left(\frac{(1-\beta)b^\beta IZ_t^\beta - \lambda_t - b_t}{\lambda_t}\right)\right]^{\frac{\sigma}{1-\sigma}},$$

$$b_{t+1} = \frac{1-\delta+\beta b^{\beta-1}IZ_{t+1}^{\beta-1}}{\lambda_t}b_t.$$

In the very special case of $\sigma = 1$ we must, of course, write :

$$\lambda_t + b_t = \frac{\Theta}{1+\Theta}(1-\beta)b^\beta IZ_t^\beta.$$

These equations show clearly that λ_t is uniquely determined by Z_t ; we can therefore substitute and study the dynamics of (Z_t, b_t) and characterise the dynamics of the whole economy.

The equations obtained here do not allow a global characterisation of the dynamics ; we shall therefore restrain our study to the local behaviour in the vicinity of a stationary point $\Omega^* = (Z^*, b^*)$. The technique employed is the following : we look at the conditions for the existence of a bubbly steady state ; under the assumption of existence, and if the dynamical system is of dimension two, we linearise the dynamic equations in the vicinity of the stationary state Ω^* in order to study the possibility of Hopf bifurcations[8]. If the dynamical system is degenerate of dimension one, which happens here,

[8]In a two-state dynamic, a Hopf bifurcation happens when the eigenvalues of the Jacobian at the steady state are complex and cross the unit circle. Hopf bifurcations are more satisfying than flip bifurcations since they are more robust to changes in the period's length.

as we shall see below, when the agents' one-period utility is logarithmic, we look for the possibility of flip bifurcations[9] or even try to characterise globally the dynamics. A very readable introduction to the mathematical method invoked here can be found in Grandmont [21].

3.1 Case $\sigma = 1$: Impossibility Results

Let us suppose here that the agents' per period utility is logarithmic. Under this assumption, we determine the bubbly steady state by writing $Z_t = Z^*$, $b_t = b^* \neq 0$. We see that a stationary state with a bubble always exist unless the solution b^* is equal to 0. We get :

$$Z^* = \frac{1-\beta}{\alpha},$$

$$\lambda^* = (1-\delta) + I\beta^\beta\alpha^{1-\beta},$$

$$b^* = I\beta^\beta\alpha^{1-\beta}\left[\frac{\Theta}{1+\Theta}\frac{1-\beta}{\alpha} - 1\right] - (1-\delta).$$

We see that the steady state corresponds to self-sustaining growth ($\lambda^* > 1$) if and only if $\delta < I\beta^\beta\alpha^{1-\beta}$, which is *not* met, for instance, in the case of total capital depreciation ($\delta = 1$) when I is too small (for instance, smaller than or equal to one). Furthermore, it appears clearly that positive bubbles are possible only if $\alpha + \beta < 1$. The dynamic system given is degenerate and of order 1 : we can express λ_t, b_t and Z_{t+1} as functions of Z_t. It is therefore enough to study the dynamics of $(Z_t)_{t\in\boldsymbol{N}}$. We distinguish two cases :

[9] A flip bifurcation correponds, in a one-dimensional system, to the eigenvalue crossing -1. Furthermore, a condition involving the first three derivatives evaluated at the steady state has to be met.

3.1.1 Case of Total Capital Depreciation

It is easy to see that, under this assumption, it is possible to write :

$$\left[\frac{\Theta}{1+\Theta} - \frac{\alpha}{\alpha-\beta}\right] Z_{t+1} = -\frac{1-\beta}{\alpha-\beta} + (\alpha-\beta)\frac{\Theta}{1+\Theta}\frac{Z_t}{\alpha Z_t - (1-\alpha)}.$$

The function H defined by $Z_{t+1} = H(Z_t)$ is therefore homographic. The slope at the steady state is :

$$H'(Z^*) = \left[\frac{\Theta}{1+\Theta} - \frac{\alpha}{\alpha-\beta}\right]^{-1} \times \left[\frac{1-\alpha}{\beta-\alpha}\frac{\Theta}{1+\Theta}\right].$$

It is immediate to check that the steady state is always unstable with $H'(Z^*) > +1$ if ($b^* > 0$ and $\alpha < \beta$) or ($b^* < 0$ and $\alpha > \beta$), and always stable with $0 < H'(Z^*) < 1$ if ($b^* < 0$ and $\alpha < \beta$) or ($b^* > 0$ and $\alpha > \beta$). Local cycles are thus excluded[10]. But we can even exclude global cycles by considering a further argument invoking the shape of the function H[11].

3.1.2 Case of Partial Capital Depreciation

The case where capital is supposed to depreciate only partially in each period ($0 < \delta < 1$) is far more difficult to deal with, and we cannot obtain very strong results. We can eliminate the emergence of local cycles through flip bifurcations, but cannot say anything about the possibility of global

[10] Another argument to rule out cycles through flip bifurcations consists in invoking the fact that a homographic function has a zero Schwarzian derivative.

[11] H is strictly convex if $\alpha < \beta$, strictly concave if $\alpha > \beta$; this fact implies, with the sign of the derivative at the steady state, that even global cycles are impossible.

cycles. The analysis remains local because Z_{t+1} remains defined implicitly. We determine the slope at the steady state ; it is given by :

$$J = 1 + \frac{1}{\beta - \alpha} \frac{b^*}{b^* + C},$$

where b^* is given above and :

$$C = \lambda^* \left[1 + \frac{\Theta}{1+\Theta} + \frac{1}{\beta - \alpha} \right].$$

A necessary condition for the emergence of local cycles through a flip bifurcation is that, for some parameter configurations, the slope J is equal to -1. Ad absurdum : let us suppose that this is possible. It implies :

$$\text{(0)} \qquad b^* = -2\lambda^* + 2\lambda^* \frac{\beta - \alpha}{1 + 2(\beta - \alpha)} \left[1 - \frac{\Theta}{1+\Theta} \frac{\beta}{\alpha} \right].$$

Notice that the expression of b^* implies $b^* > -\lambda^*$, whatever is the parameter configuration. Suppose now that the investment good sector is the more capital intensive ($\alpha < \beta$). Then (0) implies $b^* < -2\lambda^* + \lambda^*$, which is absurd. Flip bifuractions are therefore excluded, for both positive and negative bubbles, if $\alpha < \beta$. Suppose now that $\alpha > \beta$. Obviously, we have

$$0 < \left[1 - \frac{\Theta}{1+\Theta} \frac{\beta}{\alpha} \right] < 1$$

and

$$\frac{2(\beta - \alpha)}{1 + 2(\beta - \alpha)} < 0 \quad \text{if} \quad 1 + 2(\beta - \alpha) > 0,$$

and thus positive as well as negative bubbles are excluded in this case, for (0) implies $b^* < -2\lambda^*$. Let us consider the case $1 + 2(\beta - \alpha) < 0$. Now we have

$$\frac{2(\beta - \alpha)}{1 + 2(\beta - \alpha)} > 2.$$

It is not possible to conclude directly ; but let us replace b^* by its formal expression. Noticing that $\Theta/(1+\Theta) < 1$, we obtain the following necessary condition :

$$(0') \qquad 0 < I\beta^\beta\alpha^{1-\beta}[\alpha-\beta-1] + (1-\delta)[\alpha(2\beta-1)-2\beta^2].$$

But $\alpha-\beta-1$ is obviously negative, and $1+2(\beta-\alpha)<0$ implies $2\beta-1<0$. Therefore, $(0')$ cannot be met and the assumption $J=-1$ is absurd. Again, flip bifurcations are not possible. We can proclaim the following :

Proposition : *In our framework, if the agents' coefficient of intertemporal substitution is equal to one, the dynamical system collapses to dimension one and local cycles through flip bifurcations generated by bubbles are excluded. Furthermore, if capital depreciates entirely in each period, even global cycles cannot exist.*

3.2 Case $\sigma \neq 1$: Existence Results

Bubbly steady state : under the assumption $Z_t = Z^*$, $b_t = b^* \neq 0$, we get :

$$Z^* = \frac{1-\beta}{\alpha},$$

$$\lambda^* = (1-\delta) + I\beta^\beta\alpha^{1-\beta},$$

$$b^* = \lambda^*\left(\frac{1-\beta}{\alpha}\frac{I\beta^\beta\alpha^{1-\beta}}{(1-\delta)+I\beta^\beta\alpha^{1-\beta}} - 1 - \frac{1}{\Theta^{1/\sigma}\lambda^{*(\alpha+\nu)(1-\sigma)/\sigma}}\right).$$

Hereof we deduce that the steady state (Z^*, b^*) cannot correspond to growth if $\delta \geq I\beta^\beta\alpha^{1-\beta}$ which, in particular, is met if capital depreciates entirely in one period ($\delta = 1$) and $I \leq 1$. Furthermore, we see that a positive steady state value of the bubble cannot exist, and therefore that Hopf bifurcations generated by positive bubbles cannot occur in our model, if $\alpha + \beta \geq 1$. If the consumption good sector is the more capital intensive ($\alpha > \beta$), then positive bubbles require even $\alpha + \beta < 1 - (1-\delta)/I < 1$ unless $\delta = 1$. Notice that the externality operating in the consumption good sector does affect neither Z^* nor λ^*, but only the stationary value of the bubble b^*.

Linearization of the dynamics (Z_t, b_t) around the steady state (Z^*, b^*) gives, if we write $Z_t = Z^*(1+u_t)$ and $b_t = b^*(1+v_t)$, the following linear first order system :

$$\begin{pmatrix} Au_{t+1} \\ A'u_{t+1} + v_{t+1} \end{pmatrix} = \begin{pmatrix} B & C \\ B' & 1 \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

with

$$\bullet A =(\beta-\alpha) + (1-\beta)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}},$$

$$\begin{aligned}\bullet B =&(\beta-\alpha) + (1-\beta)\Big(1+\frac{1}{\beta-\alpha}\Big)\Big(\frac{1}{1-\sigma} - (\alpha+\nu)\Big)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} \\ &+ \frac{\sigma}{1-\sigma}(1-\beta)\Big(1+\frac{\beta}{\alpha}+\frac{1}{\beta-\alpha}\Big)\Theta^{1/\sigma}\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} \\ &\times [1-\delta+I\beta^\beta\alpha^{1-\beta}]^{(\alpha+\nu)\frac{1-\sigma}{\sigma}},\end{aligned}$$

$$\bullet C = -\frac{\sigma}{1-\sigma}\Theta^{1/\sigma}[1-\delta+I\beta^\beta\alpha^{1-\beta}]^{(\alpha+\nu)\frac{1-\sigma}{\sigma}}\left(\frac{1-\beta}{\alpha}\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} - 1\right),$$

$$\bullet A' =(1-\beta)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}},$$

$$\bullet B' =(1-\beta)\Big(1+\frac{1}{\beta-\alpha}\Big)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}.$$

The Jacobian of our dynamical system has then the same eigenvalues as the matrix J defined by :

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

and we have :

$$Det(J) = \frac{B - B' \cdot C}{A},$$

$$Tr(J) = 1 + Det(J) + (B' - A') \cdot \frac{C}{A}.$$

Total stability of the stationary state corresponds, for the Jacobian of the dynamic system, to two eigenvalues of modulus strictly smaller than one. There exist several configurations, corresponding to a monotone, an oscillating or a spiral convergence to the steady state. It is easy to see that all configurations can occur in our framework. What is of real interest for us is the possibility of two complex (conjugate) eigenvalues of modulus one : if this situation is possible, and if the real part of the eigenvalues is not stationary with respect to the chosen bifurcation parameter, then a Hopf bifurcation occurs in the dynamical system, and the Hopf bifurcation theorem ascertains the existence of closed orbits in our economy.

Let us consider now the following problem : do parameter configurations exist such that $Det(J) = +1$, $\Delta = Tr(J)^2 - 4 < 0$ and $Tr(J) \notin \{-1, 0\}$? The latter condition is needed to rule out cases of large resonance which can be very complex[12].

[12]See Guckenheimer and Holmes [23].

Proposition : *In the framework of our model, positive as well as negative bubbles can be at the origin of endogenous fluctuations in the growth rate. Neither a negative IES nor complementarity of the production inputs are required for the possibility of endogenous cycles, even in the case of a positive bubble. There exist parameter configurations giving endogenous growth with endogenous fluctuations.*

Proof : To establish this proposition, it is sufficient to exhibit adequate parameter configurations. Let us take $I = 1$, $\alpha = 0.1$, $\beta = 0.7$, $\sigma = 0.5$ and $\delta = 0.38$; we see that the steady state value of the growth rate $(\lambda^* - 1)$ is approximately one per cent. It is easy to see that if ν is large enough[13], then there exists Θ such that $b^* > 0$, $Det(J) = +1$ and $\Delta < 0$. Thus, this example gives endogenous growth with endogenous fluctuations generated by a positive bubble (money...). But it is immediate to check that if we take ν small but such that :

$$1 - \Big(1 + \frac{1}{\beta - \alpha}\Big)\Big[\frac{1}{1-\sigma} - (\alpha + \nu)\Big] > 0,$$

then we can obtain Hopf bifurcations with a negative bubble (to require a positive steady state value for the bubble means just to impose an additional constraint for the externality parameter ν). Taking $\nu = 0.7$, we can choose Θ in order to have a configuration where the economy exhibits endogenous growth with endogenous fluctuations generated by a negative bubble. □

[13]Existence is established by taking $\alpha + \nu = 3$, for instance...

We shall now look more precisely at the constraints imposed upon the parameters, for our purpose is to establish some technical lemmas and propositions on general necessary conditions for Hopf bifurcations in our framework, on the role of the externality in the consumption good sector, or of the relative capital intensity in the two sectors. The first lemma concerns the possibility of Hopf bifurcations, without imposing any condition on λ^* or b^*.

Lemma : *In our framework and under the assumption $\sigma \neq 1$, if the investment good sector is more capital intensive than the consumption good sector ($\alpha < \beta$), then Hopf bifurcations require the following : $\alpha + \beta < 1$, $0 < \sigma < 1$ and $\nu > 0.5$. If the consumption good sector is the more capital intensive, then there are no such general conditions, and endogenous fluctuations can be generated even in the case $\sigma > 1$, but only by negative bubbles.*

The proof of this lemma is not difficult but rather long and tedious, therefore it has been banished to Appendix B. A consequence of this lemma, and of numerical examples given below establishing the possibility of Hopf bifurcations with $\nu = 0$ if $\alpha > \beta$, is that increasing returns to scale at the aggregate level are not necessary in the consumption good sector if this sector is the more capital intensive, whereas strong increasing returns to scale are required if the converse is true. We have exposed previously an example of Hopf bifurcation giving endogenous growth with endogenous fluctuations in the presence of a positive bubble for the case of a more capital intensive investment sector. What can be said if $\alpha > \beta$?

Proposition : *In our framework, if the consumption good sector is more capital intensive than the investment good sector, then positive bubbles cannot generate endogenous fluctuations through Hopf bifurcations, but negative bubbles can give endogenous growth with endogenous fluctuations.*

Proof: This is immediate. If $b^* > 0$, then, in order to have $(B' - A')C/A < 0$, we need $\sigma > 1$. From the formal expression of b^*, we know that :

$$\{(b^* > 0) \wedge (\alpha > \beta)\} \quad \Rightarrow$$
$$\left(\frac{1-\beta}{\alpha}\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} - 1\right)\Theta^{1/\sigma}[1-\delta+I\beta^\beta\alpha^{1-\beta}]^{(\alpha+\nu)\frac{1-\sigma}{\sigma}} > 1,$$

Thus, we have

$$\begin{aligned}(A' - B')\cdot\frac{C}{A} &> \frac{\sigma}{\sigma-1}\frac{1}{(\alpha-\beta)(1-\alpha)}\frac{(1-\alpha)I\beta^\beta\alpha^{1-\beta}}{(1-\alpha)I\beta^\beta\alpha^{1-\beta}-(\alpha-\beta)(1-\delta)} \\ &> \frac{\sigma}{\sigma-1}\frac{1}{(\alpha-\beta)(1-\alpha)}.\end{aligned}$$

But

$$\frac{\sigma}{\sigma-1}\frac{1}{(\alpha-\beta)(1-\alpha)} > \frac{\sigma}{\sigma-1}\frac{4}{(1-\beta)^2} > 4.$$

As a consequence, $\Delta > 0$ and Hopf bifurcations cannot occur. The fact that negative bubbles can give fluctuations in the context of self-sustaining growth is illustrated by numerical examples given below. □

We have seen above, in the technical lemma, that Hopf bifurcations always imply an externality parameter ν strictly larger than 0.5 in the case of a more capital intensive investment good sector, and that ($\alpha > \beta$) did not seem to imply any general condition for ν. An interesting question is whether the additional requirement of growth ($\lambda^* > 1$) imposes stronger restrictions on the parameters. The answer is no :

Proposition : *In our framework, if the investment good sector is the more capital intensive, then Hopf bifurcations in the context of growth require, in the presence of positive as well as in the presence of negative bubbles, increasing returns to scale at the aggregate level in the consumption good sector. If the consumption good sector is the more capital intensive, Hopf bifurcations can occur even if no spillover operates in the consumption good sector, and it is even possible, but only with negative bubbles, to obtain endogenous growth with endogenous fluctuations when constant returns to scale operate in the consumption good sector.*

Proof: Let us consider first the case $\alpha < \beta$. If we take $\alpha = 0.01$, $\beta = 0.95$, $\delta = 0.74$, $\sigma = 0.01$ and $\nu = 0.7$, then $\alpha + \nu < 1$, $\lambda^* > 1$, $b^* > 0$ and there exists a Θ such that the conditions for a Hopf bifurcation are met. Notice that, in this case, the initial condition resulting from the equalisation of $Det(J)$ to one gives $\nu > 0.515$. The lower bound 0.5 seems therefore rather correct. Consider now the case $\alpha > \beta$. We want to give examples of Hopf bifurcations in the context of self-sustaining growth and absence of spillovers in the consumption good sector. Let us take $I = 1$, $\alpha = 0.9$, $\beta = 0.1$, $\delta = 0.71$, $\sigma = 5$ and $\nu = 0$. Then $\lambda^* > 1$ and $b^* < 0$ and it is easy to check that there exist a Θ such that a Hopf bifurcation occurs. □

4 Necessary Self-Fulfilling Beliefs

This short section deals with the following problem : can there exist a world with agents characterised by rational expectations, where the rationally unfounded, but self-fulfilling belief in the realisation of some event is *necessary* ? In other words, can there exist an economy under rational expectations with no equilibrium path in the absence of bubbles or sunspots, but exhibiting bubble or sunspot equilibria ? The traditional answer to this is "no", since standard economic theory defines the notion of 'fundamental' equilibrium, i.e. an equilibrium in the absence of any bubble or sunspot, and thus implicitly defends the point of view that an economy either exhibits no equilibrium path or admits a fundamental equilibrium and perhaps, for instance in the OLG framework, some other, 'odd' equilibrium paths with bubbles or sunspots due to shocks on beliefs. But the traditional point of view simply results from the abusive use of linear models in the past. As a matter of fact, non-linearities allow far more complex situations to arise as we shall see here.

An interesting paper on the existence of economies with sunspot equilibria and no non-sunspot equilibrium has been written by Tito Pietra [27]. The author shows, in the context of a pure endowment economy where intertemporal transactions and transactions across states of nature between a finite set of agents take place through trade in assets, that there are economies for which there are no equilibria if there is no extrinsic uncertainty, while there are sunspot equilibria. Pietra gives two examples : the first one concerns an economy where non-existence of equilibria in the economy without sunspots is due to the collapse of rank of the return matrix ; the second one is an economy with non-convex preferences for one consumer.

Let us consider our model : there is a large, or even not finite, set of agents, all identical, and production of two goods can take place. At the private level, the model is neo-classical, there is no externality operating on the utilities, but there are spillovers of Romer type at the aggregate level in the production of the goods. In this framework, it is possible to exhibit economies with no 'fundamental' equilibrium path but with equilibria in the presence of bubbles. This can occur with positive or with negative bubbles, depending of course on the parameters of the economy.

Proposition : *There exist economies with production where the only possible equilibrium outcomes are equilibria with bubbles. The economy exhibited here is such that the result is due to a non-convexity, at the aggregate level, in the production of the investment good.*

Proof: The role of the externality in the investment good sector is established by the following consideration : if no externality operates at all in the investment good sector, then there always exist a stationary state (k^*, ω^*) if the investment good sector is more capital intensive ($\alpha < \beta$), and the dynamics are therefore defined, at least in a neighbourhood of the steady state, in this case. Existence of a steady state is established as follows :

$$\psi = 0 \quad \Rightarrow$$
$$\delta k^* = I\frac{a\omega^* - k^*}{a-b}b^\beta \omega^{*\beta-1}$$
$$\Theta\Big[1 - \delta + I\beta b^{\beta-1}\omega^{*\beta-1}\Big] = \left[\frac{1}{\Theta}\frac{k^*}{(1-\beta)b^\beta I\omega^{*\beta} - k^*}\right]^{\frac{\sigma}{1-\sigma}}.$$

If we write $Z^* = \omega^*/k^*$, then we have :

$$\Theta\Big[1 - \delta + \frac{\delta(\beta-\alpha)}{(1-\alpha) - \alpha Z^*}\Big] = \left[\frac{1}{\Theta}\frac{(1-\alpha) - \alpha Z^*}{[(1-\delta)\alpha + \beta]Z^* - (1-\alpha)}\right]^{\frac{\sigma}{1-\sigma}}.$$

Since $\alpha < \beta$, we have :

$$\frac{1-\alpha}{(1-\delta)\alpha + \beta} < \frac{1-\alpha}{\alpha}.$$

The two curves obviously intersect in the case $0 < \sigma < 1$. In the case $\sigma > 1$, existence of a solution Z^* is guaranteed by the fact that $\sigma > 1 \Rightarrow \sigma/(\sigma-1) > 1$ and $\lim_{x\to+\infty} x^{1+\epsilon}/x = +\infty$. Thus, Z^*, and therefore k^* and ω^* always exist in the absence of spillovers in the investment good sector.

We have to consider the dynamics of the bubbleless economy. Simply writing $b_t = 0$ in the dynamic equations of the previous section, we get the following degenerate system :

$$\lambda_t = I\frac{aZ_t - 1}{a-b}b^\beta Z_t^{\beta-1} + (1-\delta),$$
$$\Psi(Z_{t+1}) = \Phi(Z_t),$$

with

$$\Psi(Z) = \frac{Z^{\beta-\alpha}}{(1-\delta) + I\beta b^{\beta-1}Z^{\beta-1}},$$

$$\Phi(Z) = \Theta^{1/(1-\sigma)} \cdot Z^{m'} \cdot \left[\frac{(1-\alpha)-\alpha Z}{\beta-\alpha} I\beta b^{\beta-1} + (1-\delta)Z^{1-\beta}\right]^{n'}$$

$$\cdot \left[I\beta b^{\beta-1}Z - \left(\frac{(1-\alpha)-\alpha Z}{\beta-\alpha} I\beta b^{\beta-1} + (1-\delta)Z^{1-\beta}\right)\right]^{p'},$$

where

$$m' = (1-\alpha)m = (1-\alpha) - (1-\beta)(\alpha+\nu)$$

$$n' = (1-\alpha)n = (\alpha+\nu) - \frac{1}{1-\sigma}$$

$$p' = (1-\alpha)p = \frac{\sigma}{1-\sigma}.$$

We want to give here an example where the equations above show that the economy does not exhibit any dynamic in the absence of bubbles, but does so if a bubble appears near enough to the steady state value determined by the parameters of the economy. Let us take the following numerical values used above to establish the possibility of endogenous growth with endogenous fluctuations resulting from the existence of a positive bubble : $I = 1$, $\alpha = 0.1$, $\beta = 0.7$, $\sigma = 0.5$, $\delta = 0.38$ and $\alpha + \nu = 5$. The condition $i_t \geq 0$ implies $Z_t \leq (1-\alpha)/\alpha = 9$, and $e_t - k_{t+1} \geq 0$ implies $Z_t \geq \underline{Z} \approx 2.5877$. To obtain a Hopf bifurcation, we need $\Theta \approx 0.9201$. With these values, we get : $((\Psi^{-1}) \circ \Phi)' > 0$, on the interval $[\underline{Z}, 9]$, and $(\Psi^{-1}) \circ \Phi(9) \approx 2.0188 < \underline{Z} \Rightarrow \forall Z \in [\underline{Z}, 9]$, $(\Psi^{-1}) \circ \Phi(Z) < \underline{Z}$. Therefore, the dynamics are not defined in the absence of bubbles. □

Conclusion

In a context of inelastic labour supply, without requiring savings to be non monotone in the interest rate or the production to exhibit complementarity or not too much substitutability, but with standard CRRA utility and production with perfect substitutability between inputs, we showed that, in the absence of any intrinsic uncertainty, bubbles or sunspots, self-sustaining growth with cyclic or even chaotic trajectories of the growth rate is possible in the framework of overlapping generations. For this, we needed externalities à Romer in the two production sectors, and a necessarily strong one in the consumption good sector. Boldrin and Rustichini [8] exhibited, in the framework of infinitely lived agents, an example of endogenous growth with chaotic growth rate in the case of a linear utility function ; they too required a strong externality in the consumption good sector. But we showed that observable chaos cannot be established in their model under the (in the ILA framework) standard assumption on capital depreciation, and that even the proof of the existence of cycles or topological chaos requires a very high rate of capital depreciation. Furthermore, we showed that in the presence of bubbles, we can have endogenous growth with a fluctuating growth rate with a weaker, and in the case of negative bubbles, even with a zero externality in the consumption good sector. Endogenous growth with endogenous fluctuations generated by positive bubbles through Hopf bifurcations can be obtained, in our framework, only if the investment good sector is the more capital intensive.

We also established the existence of economies with no non-bubbly equilibrium but with equilibria in the presence of a bubble, a result which makes us reflect on the term 'fundamental' used traditionally to qualify equilibria in economies without intrinsic shocks, bubbles or sunspots. Furthermore, we proved in Appendix C that with the same type of utility and production functions the assumption of multiple sectors is crucial : in a one-sector economy with CRRA one-period utility and production function of Cobb-Douglas type with externality à la Romer [30], Hopf bifurcations can be generated by neither positive nor negative bubbles. In one-sector economies, non neo-classical assumptions at the private level seem to be necessary to obtain endogenous cycles, even in the presence of bubbles. Notice also that in the one-sector economy with externalities à la Romer, in the absence of exogenous shocks, bubbles or sunspots, there always exists a unique balanced growth path on which the economy starts immediately, whatever is the initial capital stock, whereas in the case of the two-sector economy with externalities, the dynamic is not necessarily defined, a balanced growth path does not necessarily exist, is not necessarily unique, can be stable or unstable etc.

It would have been nice to characterize the stability properties of the closed orbits generated by the Hopf bifurcations. To achieve this, it is necessary to determine whether a given bifurcation is supercritical or subcritical : a supercritical Hopf bifurcation gives the existence of attractive, a subcritical only existence of unstable closed orbits. The latter seems clearly less exhilerating since non-stability means zero probability to have an economy on or remaining close to such an orbit[14]. But studies of the problem of learning perfect foresight equilibria have shown (see Fuchs [16] or Grandmont and Laroque [22]) that the notion of stability of an equilibrium trajectory can be entirely reversed under learning : in Grandmont's model, for instance, the only equilibrium trajectory that is stable under a certain learning process is an unstable equilibrium orbit. Therefore, unstability of a given orbit does not necessarily imply it's economic insignificance. This is a nice excuse for avoiding the utmost tedious calculus necessary to determine the nature of a given Hopf bifurcation. The nevertheless interested reader may find the methodology in Grandmont [21].

APPENDIX A :

Fundamental Equilibria, Case $\delta \neq 1$

The equations giving the dynamics of (Z_t, λ_t) in the case $0 < \delta < 1$ can be found in section 4. We suppose here $\alpha < \beta$; the other case is similar. Let us consider a parameter configuration $\xi_0 = (\alpha_0, \beta_0, \nu_0, \sigma_0)$,$(\Theta_0, I_0)$, $\delta = 1$ such that the map $\Upsilon_{\xi_0,(\Theta_0,I_0),1}$ has a negative Schwarzian derivative, satisfies $\Upsilon^2_{\xi_0,(\Theta_0,I_0),1}(Z^*) \leq Z_{(\Theta_0,I_0),1}$ and thus exhibits observable chaos. Arguments invoking the functional form and regularity insure that, at least for δ close to one, the map $\Upsilon_{\xi_0,(\Theta,I),\delta}$ has a negative Schwarzian derivative at every non critical point (the parameters Θ and I are completely neutral with regard to the Schwarzian derivative). The conditions $i_t \geq 0$ and $e_t - k_{t+1} \geq 0$

[14] A curve has zero Lebesgue-measure in $\mathbb{R}^2$...

yield $Z_t \in [\underline{Z}_{(\Theta,I),\delta}, \overline{Z}_{(\Theta,I),\delta}]$, where $\underline{Z}_{(\Theta,I),\delta} \geq 1/ac$ and $\overline{Z}_{(\Theta,I),\delta} \geq 1/a$ are C^∞ functions of δ (implicit function theorem) such that :

$$\lim_{\delta\to 1^-} \underline{Z}_{(\Theta,I),\delta} = \frac{1}{ac}^+, \quad \lim_{\delta\to 1^-} \overline{Z}_{(\Theta,I),\delta} = \frac{1}{a}^+.$$

It is easy to see that, by continuity, for δ close enough to one, it is possible to choose $(\Theta_\delta, I_\delta)$ such that :

$$Z^1/\Upsilon_{\xi_0,(\Theta_\delta,I_\delta),\delta}(Z^1) = Z_1 \text{ in } [\underline{Z}_\delta, \overline{Z}_\delta],$$

$$\Upsilon^2_{\xi_0,(\Theta_\delta,I_\delta),\delta}(Z^*) \leq Z_1,$$

where we have omissed to indicate the dependence in (Θ, I) etc.

APPENDIX B :

Proof of the Lemma, and Examples

<u>*Proof*</u> : First of all, notice that $\sigma = 0$ implies $Tr(J) = 1 + Det(J)$. Therefore, a linear utility rules out any possibility of Hopf bifurcations. Let us assume $\sigma > 0$. The determinant of J is given by :

$$\begin{aligned}
Det(J) =& \left[(\beta-\alpha) + (1-\beta)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}\right]^{-1} \\
&\times \left[(\beta-\alpha) + (1-\beta)\Big(1+\frac{1}{\beta-\alpha}\Big)\Big(\frac{1}{1-\sigma} - (\alpha+\nu)\Big)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}\right. \\
&+ \frac{\sigma}{1-\sigma}\frac{1-\beta}{\alpha}\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}\Theta^{\frac{1}{\sigma}}[1-\delta+I\beta^\beta\alpha^{1-\beta}]^{(\alpha+\nu)\frac{1-\sigma}{\sigma}} \\
&\times \left.\left(\beta + (1-\beta)\Big(1+\frac{1}{\beta-\alpha}\Big)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}\right)\right].
\end{aligned}$$

Let us denote by R the term

$$R = \left(\beta + (1-\beta)\Big(1+\frac{1}{\beta-\alpha}\Big)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}}\right).$$

We must distinguish the two cases $\alpha < \beta$ and $\alpha > \beta$.

1. Case ($\alpha < \beta$). Under this assumption, $B' - A'$, A and R are all positive. If we suppose $\sigma > 1$, then $A \cdot Det(J) < \beta - \alpha$, with $A > \beta - \alpha$, which implies $Det(J) < 1$ and excludes the possibility of Hopf bifurcations. Therefore, $0 < \sigma < 1$, which implies $\alpha + \beta < 1$ since to allow $\Delta < 0$, we must have $C < 0$. Notice that all this does not imply anything general for λ^* or b^*.

If we return to the equation '$Det(J) = 1$', we see that necessarily :

$$\left(1 + \frac{1}{\beta - \alpha}\right)\left[\frac{1}{1-\sigma} - (\alpha + \nu)\right] < 1,$$

which imposes, for given (α, β, σ), a lower bound to the externality parameter ν. Let us try to find the best general lower bound for ν. Consider the following function :

$$F_\beta(\alpha) = \alpha + \frac{\beta - \alpha}{1 + \beta - \alpha}.$$

This function is strictly increasing since

$$\frac{dF_\beta}{d\alpha}(\alpha) = (\beta - \alpha)\frac{2 + \beta - \alpha}{1 + \beta - \alpha} > 0 \quad \forall \alpha < \beta.$$

Furthermore, we know that $\alpha + \beta < 1$. Therefore, if $\beta < 1 - \beta$, then

$$F_\beta(\alpha) < \lim_{\alpha \to \beta^-} F_\beta(\alpha) = \beta,$$

which implies

$$\nu > \frac{1}{1-\sigma} - \beta > \frac{1}{2}.$$

If $\beta \geq 1-\beta$, then

$$F_\beta(\alpha) < F_\beta(1-\beta) = 2 - \beta - \frac{1}{2\beta},$$

and thus

$$\nu > \frac{1}{1-\sigma} - 2 + \beta + \frac{1}{2\beta} > \frac{1}{2}$$

since here $0.5 \leq \beta < 1$.

2. Case $(\alpha > \beta)$. Let us denote

$$S = \frac{1-\beta}{\alpha} \frac{I\beta^\beta \alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} - 1.$$

We distinguish two subcases :

2.1. $S > 0$. This implies, if $I \leq 1$, $\alpha + \beta < \delta$ and therefore, as is easy to check, $\lambda^* < 1$, but in the case of a large I, we can have $\lambda^* > 1$. The sign of b^* is indeterminate, but we know from a proof given in the paper that positive bubbles cannot give Hopf bifurcations, therefore they are of no interest for us here. We have $A > 0$, and therefore must have $\sigma > 1$. But $A > 0$ implies

$$(1-\beta)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} > (\alpha - \beta)$$

$$\Rightarrow$$

$$\left(1 + \frac{1}{\beta-\alpha}\right)(1-\beta)\frac{I\beta^\beta\alpha^{1-\beta}}{1-\delta+I\beta^\beta\alpha^{1-\beta}} < \alpha - \beta - 1,$$

and thus $R < -(1-\alpha) < 0$. If we now look at the equation $Det(J) = 1$ which previously yielded a lower bound for the parameter ν, we realise that now nothing general can be deduced concerning ν :

$$\Big(\frac{1}{\alpha-\beta}-1\Big)\Big[\alpha+\nu+\frac{1}{\sigma-1}\Big]<1$$

imposes simply $\beta<\alpha^2/(1+\alpha)$, but we see that if $\delta=1$, $\alpha\to 1^-$ and $\beta\to 0^+$, then there results no constraint on ν.

2.2. $S<0$. Then $b^*<0$, but nothing can be said concerning λ^*. As numerical examples will establish, we can have $A>0$, which implies $0<\sigma<1$ and $R<0$, but also $A<0$, which implies $\sigma>1$ but does not give the sign of R. In this latter case, the condition $Det(J)=1$ does not impose a general condition on ν, but in the first case ($A>0$), it is easy to check that the externality must be strictly positive :

$$\nu>\frac{1}{1-\sigma}-F_\beta(\alpha),$$

where $F_\beta(\alpha)$ has been defined previously. Notice that now F_β is decreasing in α since $\alpha>\beta$. Thus we know that necessarily

$$\nu>\frac{1}{1-\sigma}-F_\beta(1)=\frac{1}{1-\sigma}-2+\frac{1}{\beta}>0.$$

Furthermore, the condition $S<0$ implies $\alpha>(1-\beta)/(1+\beta)$. Therefore, if $0<\beta<\sqrt{2}-1$,

$$F_\beta(\alpha)\leq F_\beta\Big(\frac{1-\beta}{1+\beta}\Big)=\frac{\beta^2+4\beta-1}{\beta(1+\beta)(3+\beta^2)},$$

and we have

$$\sup_{\beta\in]0,\sqrt{2}-1[} F_\beta\Big(\frac{1-\beta}{1+\beta}\Big)=F_\beta(\sqrt{2}-1)\approx 0.446.$$

and if $\sqrt{2}-1 \leq \beta < 1$,

$$F_\beta(\alpha) \leq F_\beta(\beta) = \beta.$$

We conclude therefore only to $\nu > 0$, and $\nu > 0.554$ if $\beta < \sqrt{2}-1 \approx 0.414$. □

It is easy to verify that all the situations described here can occur, parameter configurations illustrating this statement can be found in the paper when we prove the proposition on the role of the externality ν, two cases remaining to be illustrated here. To obtain $S < 0$, $A < 0$ and $R > 0$, take $I = 1$, $\alpha = 0.9$, $\beta = 0.1$, $\delta = 0.05$, $\sigma = 1.3215$ and $\nu = 0$. $\lambda^* > 1$ and $b^* < 0$ and an adequate Θ exists. If we want $S < 0$ and $A > 0$, then from what precedes, we know that necessarily $R < 0$ and $\nu > 1 - 0.5 = 0.5$: take $I = 1$, $\alpha = 0.51$, $\beta = 0.5$, $\delta = 0.49$ and $\sigma = 0.1$. Then $\lambda^* > 1$ and $b^* < 0$. If we take ν small, but larger than the lower bound $(1/99 + 1/0.9 - 0.51 \approx 0.612)$, then the conditions for the Hopf bifurcations can be met.

APPENDIX C :

The One-Sector Model. A Negative Result

We consider the standard one-sector Diamond OLG model augmented by an externality of Romer [29]-type in the production of the consumption good, and look at the problem of the possibility of Hopf bifurcations in the dynamic system when a bubble exists in the economy. We keep notations similar to those used in our two-sector model (in particular, α denotes the capital intensity in the consumption good sector[15], and ν the externality parameter).

The equations giving the dynamics of the economy are :

$$(1) \qquad k_{t+1} + B_t = \frac{\Theta^{1/\sigma}[1-\delta+\alpha k_{t+1}^{\alpha+\nu-1}]^{(1-\sigma)/\sigma}}{1+\Theta^{1/\sigma}[1-\delta+\alpha k_{t+1}^{\alpha+\nu-1}]^{(1-\sigma)/\sigma}}(1-\alpha)k_t^{\alpha+\nu},$$

$$(2) \qquad B_{t+1} = [1-\delta+\alpha k_{t+1}^{\alpha+\nu-1}]B_t.$$

If $\alpha + \nu = 1$, then the two-state system collapses into one characteristic dynamic equation :

$$b_{t+1} = \frac{(1-\delta+\alpha)b_t}{\Theta'/(1+\Theta')-b_t},$$

where $b_t = B_t/k_t$ is the reduced bubble and $\Theta' = \Theta^{1/\sigma}[1-\delta+\alpha]^{(1-\sigma)/\sigma}$. The steady state is given by :

$$b^* = \frac{\Theta'}{1+\Theta'} - (1-\delta+\alpha).$$

If we write $b_{t+1} = \Psi(b_t)$, it is easy to verify that Ψ is always strictly increasing and convex. The slope at the stationary state b^* is :

$$\frac{d\Psi}{db}(b^*) = \frac{1}{1-\delta+\alpha}\frac{\Theta'}{1+\Theta'}.$$

Therefore, if the parameters of the economy are such that the steady state value of the bubble is positive, then $\Psi'(b^*) > 1$ and the steady state is unstable. If $b^* < 0$, then $\Psi'(b^*) < 1$ and the steady state is stable. In both cases, local cycles are excluded, and global cycles are excluded because of the shape

[15] Notice that, again, we can take $C = 1$ without any loss of generality.

of the mapping ψ.

Let us suppose $\alpha + \nu \neq 1$ in the following. Under this assumption, a stationary state necessarily corresponds to a constant level of capital[16] $k_t = k^*$. We get :

$$k^* = \Big[\frac{\delta}{\alpha}\Big]^{\frac{1}{\alpha+\nu-1}},$$

$$B^* = \Big[\frac{\delta}{\alpha}\Big]^{\frac{1}{\alpha+\nu-1}}\Big\{\frac{\delta\Theta^{1/\sigma}}{1+\Theta^{1/\sigma}}\frac{1-\alpha}{\alpha} - 1\Big\}.$$

We employ the standard technique : linearization around the steady state (k^*, B^*) and study of the possibility of Hopf bifurcations by looking at the determinant and the trace of the dynamic's Jacobian. If we write :

$$\begin{cases} B_t = B^*(1+\epsilon_t) \\ k_t = k^*(1+\eta_t). \end{cases}$$

we get :

$$\begin{aligned} Det(J) &= \frac{C}{A} \\ &= \frac{\Big[\delta\Theta^{\frac{1}{\sigma}}/(1+\Theta^{\frac{1}{\sigma}})\Big]\Big(\frac{1-\alpha}{\alpha}\Big)(\alpha+\nu)}{1-(\alpha+\nu-1)\Big(\frac{1-\alpha}{\alpha}\Big)\Big(\frac{1-\sigma}{\sigma}\Big)\delta^2\Theta^{\frac{1}{\sigma}}/(1+\Theta^{\frac{1}{\sigma}})^2}, \end{aligned}$$

and

$$\begin{aligned} Tr(J) &= 1 + Det(J) - (\alpha+\nu-1)\frac{B}{A} \\ &= 1 + Det(J) - (\alpha+\nu-1)\frac{\Big[\delta\Theta^{\frac{1}{\sigma}}/(1+\Theta^{\frac{1}{\sigma}})\Big]\Big(\frac{1-\alpha}{\alpha}\Big) - 1}{1-(\alpha+\nu-1)\Big(\frac{1-\alpha}{\alpha}\Big)\Big(\frac{1-\sigma}{\sigma}\Big)\delta^2\Theta^{\frac{1}{\sigma}}/(1+\Theta^{\frac{1}{\sigma}})^2}. \end{aligned}$$

[16]...whereas a steady state corresponds to a constant growth rate of capital if $\alpha + \nu = 1$.

Necessary conditions for a Hopf bifurcation are, as usual :

$$Det(J) = +1$$

and

$$\Delta = Tr(J)^2 - 4 < 0.$$

The first condition implies that necessarily $A > 0$, the second condition gives

$$0 < (\alpha + \nu - 1) \cdot \frac{B}{A} < 4.$$

Let us show now that Hopf bifurcations can occur neither with positive nor with negative bubbles in this one-sector model :

Notice that $B^* = k^* \times B$. As a consequence, to obtain a Hopf bifurcation with $\{B^* > 0 \Leftrightarrow B > 0\}$, we must have $\alpha + \nu - 1 > 0$. Suppose $B^* > 0$. Then, since $A > 0$, we have $Det(J) > (\alpha + \nu)/A > 1/A$. Now, if σ is smaller than or equal to one, $A \leq 1$ and thus $Det(J) > 1$. If σ is strictly larger than one, then we have, if we write :

$$U = \frac{\delta \Theta^{1/\sigma}}{1 + \Theta^{1/\sigma}} \frac{1 - \alpha}{\alpha} (\alpha + \nu) > 1,$$

$$\zeta = \text{sign}\left(\frac{\partial Det(J)(U)}{\partial U} \right) = \text{sign}\left[\left(\frac{1 - \sigma}{\sigma} \right) \left(\frac{1 - \alpha}{\alpha} \right) \frac{\delta^2 \Theta^{1/\sigma}}{(1 + \Theta^{1/\sigma})^2} + 1 \right].$$

The sign is indeterminate if $B^* > 0$. But

$$\zeta > 0 \quad \Rightarrow \quad Det(J) > Det(J)_{\alpha + \nu = 1} = 1 + B > 1,$$

$$\zeta < 0 \quad \Rightarrow \quad Det(J) > \lim_{\alpha + \nu \to +\infty} Det(J) = \frac{\sigma}{\sigma - 1} \frac{1 + \Theta^{1/\sigma}}{\delta} > 1.$$

Therefore, in the presence of a positive bubble, Hopf bifurcations are excluded.

The case $B^* < 0$ is nearly symmetric : necessarily $B < 0$ and $\alpha + \nu < 1$. But then we have $Det(J) < 1$ if $\sigma \leq 1$, and under the assumption of σ larger than one, we get $U < 1$ and $\zeta > 0$, where U and ζ are defined as previously. Indeed, the sign is clearly positive since $B < 0$ and

$$0 < \frac{\sigma - 1}{\sigma} \frac{\delta}{1 + \Theta^{1/\sigma}} < 1.$$

Therefore, $Det(J) < Det(J)_{\alpha+\nu=1} = 1 + B < 1$, and we can again conclude to the impossibility of Hopf bifurcations.

We conclude to the fact that the one-sector model with CRRA utility and Cobb-Douglas production function with Romer externality cannot give balanced growth with local or global cycles generated by bubbles (case $\alpha+\nu = 1$), and cannot give local cycles of the capital stock through Hopf bifurcations generated by bubbles (case $\alpha + \nu \neq 1$). □

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CHAPTER IV

A Critical Note on "Growth and Indeterminacy in Dynamic Models with Externalities" and on "A Chaotic Map Arising in the Theory of Endogenous Growth"

A CRITICAL NOTE ON "GROWTH AND INDETERMINACY IN DYNAMIC MODELS WITH EXTERNALITIES" AND ON "A CHAOTIC MAP ARISING IN THE THEORY OF ENDOGENOUS GROWTH"

By Sorbas von Cœster[1]

Abstract

In a recently published paper, Boldrin and Rustichini present a two-sector infinitely lived agents model with production externalities and, invoking results exposed in a paper written by Boldrin and Persico, claim that their model can exhibit endogenous fluctuations in the growth rate under the form of chaotic trajectories. Unfortunately, Boldrin and Persico are mistaken in their conclusions, for the dynamics they propose do not correspond to the optimal path. We describe here qualitatively the possible dynamics and show why it seems difficult to conclude to chaos.

Keywords : Topological, ergodic and turbulent chaos.

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Introduction

In section 3.2 of their paper, Boldrin and Rustichini (1994) present a two-sector one capital good ILA model and, refering to some research work undertaken by Boldrin and Persico (1993), claim for it the possibility of chaos. Both papers unfortunately contain several errors, some minor, others apparently not. We take here leave to address several critiques to the three authors. To enable non specialists to understand the main critique, we shall briefly expose, in section 1, the different notions of chaos and their most important characteristics ; our argument to criticize the claim of chaos will then be developed in section 2. The conclusion will allude to two minor points of methodology concerning Boldrin and Rustichini's paper and to a secondary, but highly incorrect statement made by Boldrin and Persico.

1. Some Definitions and Standard Results

In the literature on erratic dynamics, we can find several notions of chaos : *topological*, *ergodic* and *turbulent* chaos. Mathematicians do not accept the notion of topological chaos, for it is too weak and does not allow nice conclusions. Chaos in the ergodic sense has several interesting implications, but the most important is *observability* of erratic trajectories. Finally, turbulent chaos is chaos in its strict mathematical acceptance ; it is a very strong notion with rather spectacular implications. In this section, we shall present the definitions of these notions and briefly discuss the features of the different types of chaos. We consider here continuous functions which map a real closed interval $J = [a, b]$ into itself.

1.1. Topological Chaos

Its abstract definition is :

Definition : *$H : J \rightarrow J$ exhibits topological chaos if :*

- *$\forall N$, $\exists x_N$ such that $H^N(x_N) = x_N$.*
- *$\exists S \subset J$ non denumerable and $\exists \epsilon > 0$ such that : $\forall (x, y) \in S^2$, $x \neq y \Rightarrow$ $\limsup_{n \rightarrow +\infty} |H^n(x) - H^n(y)| \geq \epsilon$ and $\liminf_{n \rightarrow +\infty} |H^n(x) - H^n(y)| = 0$.*

An intuitive interpretation is that the orbits of x and y become infinitely close an infinite number of times and again separate. Topological chaos is, in general, established by invoking Li and Yorke's (1975) theorem which states that if a unimodal map exhibits a cycle of period three, then there exists a non empty set S of initial conditions whose orbits are chaotic in the topological sense. A unimodal map is a continuous function H from J into J for which there exists $x^* \in]a, b[$ such that H is strictly increasing for $x < x^*$ and strictly decreasing for $x > x^*$ (i.e. H is 'hump-shaped'). Notice that the existence of a period three cycle has another very strong consequence : cycles of every imaginable period can exist. Indeed, Šarkovskii's (1964) theorem says that for any *continuous* map from an interval J into itself, if a cycle of order k exists, then there exists a cycle of period k' for every $k \succ k'$, where the ordering is defined as follows : $3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ \cdots \succ 2^n \cdot 3 \succ$

$2^n \cdot 5 \succ \cdots \succ 2^m \succ \cdots \succ 4 \succ 2 \succ 1.$

The problem with the notion of topological chaos is that the Lebesgue-measure of the subset S may well be equal to zero, and chaos therefore unobservable : this is the case, for instance, for the logistic map Γ_Φ defined by : $X_{t+1} = \Phi X_t(1 - X_t)$, when $\Phi = 3.828427$ or $\Phi = 3.839$, where almost all initial conditions lead asymptotically to a period three cycle. The problem thus is to find conditions that ensure $m(S) > 0$.

1.2. Ergodic Chaos

Definition : *H exhibits ergodic chaos if :*

- $m(S) > 0$.
- *asymptotically, the sequence $(x_t)_t$ approximates an ergodic and absolutely continuous distribution which is invariant under H and which summarises the limiting statistical properties of the (deterministic) chaotic trajectories.*

This is a far more attractive notion than topological chaos. For instance, in the case of the logistic map Γ_Φ, when $\Phi = 4$, the set S is of full measure and the absolutely continuous ergodic invariant distribution is $f(x) = 1/\left(\pi\sqrt{x(1-x)}\right)$. Ergodic chaos is observable : if we choose at random an initial condition x_0, there is a positive probability of generating a chaotic trajectory. To establish the existence of an invariant, ergodic and absolutely continuous distribution, the standard method is to invoke Singer's (1978) theorem which states that the number of stable orbits of an arbitrary C^3 map H with negative Schwarzian derivative is bounded above by the number of its critical points. The Schwarzian derivative of a function H is :

$$SH(U) = \frac{H'''(U)}{H'(U)} - \frac{3}{2}\left(\frac{H''(U)}{H'(U)}\right)^2,$$

at a non-critical point U. The Schwarzian derivative is therefore negative if and only if $|H'|^{-1/2}$ is convex on each interval of monotonicity of H, and sufficient (but not necessary) conditions are : $|H'|$ or $\log |H'|$ convex on each of these intervals. If the Schwarzian derivative is not negative on the whole interval, then we cannot rule out little waves for H or its iteratives, the attractor is not necessarily unique and it is often very difficult, if not impossible, to establish strong results.

Let us now assume the following :

1. H is unimodal and C^3,
2. $H'(x) = 0 \Rightarrow x = x^*$,
3. $H(x) > x$ if $x < x^*$ and $H'(a) > 1$ if $H(a) = a$,
4. $H''(x^*) < 0$,
5. $SH < 0$ on the whole interval J.

Under these assumptions, the orbit of the sole critical point is fully informative of the dynamics defined by H : if there exists a stable cycle, then it attracts the critical point ; thus, if the trajectory of the critical point is unstable, then there does not exist any stable cycle and the map exhibits ergodic chaos (see, for instance, Grandmont (1988)). In the case of the map

$\Gamma_{3.839}$, for instance, there exists one stable cycle, of period three, which attracts very quickly the critical point, and the map thus exhibits an infinity of unobservable cycles ; the map Γ_4 admits cycles of all possible periods, but none of them is observable since the set of initial conditions giving chaos is of full measure ! Ergodic chaos exists for $\Phi = 4$ because $\Gamma_4^2(1/2) = 0$ and the origin is an unstable fixed point, and thus the critical point $1/2$ has an unstable orbit.

1.3. Turbulent Chaos

This notion corresponds to what most mathematicians consider as the *true* chaos ; if a map exhibits turbulent chaos, not only are there erratic orbits but there exists strong sensitivity on initial conditions. We adopt here the definition given in Devaney (1987), which finds its inspiration in Guckenheimer (1979). Let I be a set.

Definition : *$H : I \longrightarrow I$ is said to be topologically transitive if for any pair of open subsets U and V of I, there exists $k > 0$ such that $H^k(U) \cap V \neq \emptyset$.*

Intuitively, a topologically transitive map has points which eventually move under iteration from one arbitrary small neighbourhood to any other. Therefore, the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map. The existence of a dense orbit thus implies topological transitivity.

Definition : *$H : I \to I$ has sensitive dependence on initial conditions if : $\exists \epsilon > 0$ such that, $\forall x \in I$, $\forall N$ neighbourhood of x, $\exists y \in N$ and $n \geq 0$ such that $|H^n(x) - H^n(y)| \geq \epsilon$.*

Notice that the existence of a stable cycle is incompatible with sensitive dependence on initial conditions. In the case of an aperiodic, i.e. without any stable cycle, and sensitive map, to be able to describe, even approximately, a trajectory, it is not at all sufficient to know the law of motion of the dynamic system and to have a proxy of the initial state ; this could lead to believe that computer simulations do not make any sense in an area where the characteristic dynamics exhibit chaos but, fortunately, the situation is not as desperate as one might believe at first glance[2].

Definition : *H exhibits turbulent chaos on the set I if :*

- *H has sensitive dependence on initial conditions,*
- *H is topologically transitive,*
- *periodic points are dense in I.*

Unpredictability, indecomposability and an element of regularity thus characterise a map which is chaotic in the turbulent sense. The fact that periodic points are dense in I does not, of course, imply anything on the measure of the subset these points constitute in I : consider $Q \cap [0,1]$, for

[2] We invoke here the so-called 'Pursuit'-lemma.

instance ; this is a dense subset of $[0,1]$ which has a zero Lebesgue measure. And indeed here, for a map exhibiting turbulent chaos, there exists a set of initial conditions, with positive Lebesgue measure defined on I, such that any trajectory starting from it looks 'chaotic' in the sense that its spectrum closely resembles the spectrum of a random noise. Other definitions are possible : Ruelle (1979), for instance, has given a definition involving the so-called Lyapounov exponent ; he looks for maps with an absolutely invariant measure for which $(1/n)\log|Df^n|$ tends to a strictly positive constant as $n \to +\infty$ almost surely with respect to the invariant measure. For the logistic map Γ_4, $I = J = [0,1]$ and $L = \ln(2) > 0$. Notice that turbulent chaos is implied neither by topological nor by ergodic chaos.

2. Boldrin and Rustichini's Model

We invite the reader to consult Boldrin and Rustichini (1994). We only summarise here the assumptions and results exposed by these authors in section 3.2. of their paper. Time is discrete. The economy is composed of a continuum of identical infinitely lived agents, indexed by $i \in [0,1]$. There are two goods in the economy, one homogeneous perishable consumption good and one homogeneous investment good. Both production sectors can combine two factors, capital and labour, the latter being provided inelastically. There exists only one type of capital and one type of labour which can be costlessly allocated between sectors. Labour supply is normalised to one and there is full employment. The agents' coefficient of time-preference is denoted by $\delta \in]0,1[$, their one-period utility is assumed to be linear and the production technologies of the consumption and the investment good are respectively :

$$\begin{aligned} c_t &= x_{1,t}^{\alpha} l_{1,t}^{1-\alpha} k_t^{\eta}, \\ I_t &= b x_{2,t}, \end{aligned}$$

$\alpha \in]0,1[$ and $\eta \in I\!R_+$, $x_{i,t}$ being the level of capital at time t in sector i, k_t the aggregate level of capital at time t (equilibrium implies $k_t = x_t = x_{1,t} + x_{2,t}$). Capital depreciates at rate $\mu > 0$ (in an economy with production of an investment good, we must suppose $\mu > 0$, for otherwise the capital stock necessarily increases at each date) per period.

With these assumptions, Boldrin and Rustichini come to the conclusion that the dynamics of $\lambda_t = x_{t+1}/x_t$ are given by :

$$\lambda_{t+1} = \tau(\lambda_t) = \Theta - (\delta\Theta)^{1/(1-\alpha)} \lambda_t^{\beta} (\Theta - \lambda_t), \tag{3.10}$$

where $\Theta = b + (1-\mu) > 1$ is necessary to make persistent growth possible, and $\beta = (\alpha + \eta - 1)/(1-\alpha)$. This is an extremely nice and rich dynamic.

The problem is that (3.9), from which (3.10) is derived, does not correspond to the optimisation problem we have to consider here, for one constraint, $(1-\mu)x_t \leq x_{t+1}$, implied by the capital accumulation equation, has been omitted. The correct optimisation problem, under the assumption of a CRRA utility with coefficient of intertemporal substitution σ, is:

$$\begin{cases} \max \displaystyle\sum_{t=0}^{+\infty} \delta^t (1-\sigma)^{-1} \left(k_t^{\eta} (\gamma x_t - a x_{t+1})^{\alpha} \right)^{1-\sigma} \\ (1-\mu) x_t \leq x_{t+1} \leq \Theta x_t \end{cases}$$

This is the optimisation problem considered by Boldrin and Persico (1993). Unfortunately, Boldrin and Persico, while writing down their first order condition (EE), completely forget the constraints and the use of Lagrange multipliers, and thus propose a dynamic which is not the solution of the here considered program, unless assuming that capital depreciates entirely in each period, a very unappealing assumption in the ILA framework where periods are supposed to be short. Indeed, they propose the following first order difference equation (2.2) :

$$\lambda_{t+1} = \begin{cases} \tau(\lambda_t) = \Theta - (\delta\Theta)^{1/(1-\alpha(1-\sigma))}\lambda_t^{\beta'}(\Theta - \lambda_t) & \text{if} \quad \tau(\lambda_t) \geq (1-\mu), \\ (1-\mu) \quad \text{otherwise,} \end{cases}$$

where $\beta' = ((\alpha+\eta)(1-\sigma)-1)/(1-\alpha(1-\sigma))$. Of course $\lambda_{t+1} = 1-\mu$ if $\lambda_t \geq 1-\mu$ and $\tau(\lambda_t) < 1-\mu$, but the intuitively obvious thing is that here, because of the presence of the floor value $(1-\mu)$, the dynamics cannot be written under the classical form $\lambda_{t+1} = \Psi(\lambda_t)$. Indeed, let us suppose that $(1-\mu)$ is strictly smaller than λ_2. This implies $\tau(1-\mu) > 1-\mu$. If we believed in (2.2), then we would think that $(1-\mu)$ can be mapped onto only one possible point, namely $\tau(1-\mu)$, but this would be highly surprising : imagine, for instance, that $\lambda_t > 1-\mu$ and $\tau(\lambda_t)$ is very small compared to $1-\mu$; intuitively, agents would like to disinvest actively, but here, unlike in standard one-sector models, investment is irreversible, therefore there can only exist passive disinvestment under the form of waiting until enough capital has vanished through obsolescence, and thus λ_{t+i} will remain equal to $(1-\mu)$ for more than a period. When the dynamics start again, the optimal choice obviously must take into account the *history* of capital accumulation, in the sense that $(1-\mu)$ will be mapped onto a point which depends on λ_t. Therefore the dynamics cannot be written under the form $\lambda_{t+1} = \Psi(\lambda_t)$ which allows standard cycle or chaos detection. This intuition is confirmed by the very basic calculus exposed in the appendix.

Boldrin and Persico thus expose a very nice and thorough study of the case $\mu = 1$, but their treatment of the more natural case $0 < \mu < 1$ is incorrect. In order to give a clear understanding of what can be concluded to, we shall proceed step by step, often invoking results which can be found in Boldrin and Persico (1993), but which we deem prudent to recall here. The fact that we take $\sigma = 0$ is absolutely innocuous since the unconstrained dynamics (2.2) remain of the same type as (3.10) and it is easy to check that the results established in the appendix hold for every $\sigma < 1$.

We have to distinguish the two cases $\lambda_2 \leq 1-\mu$ and $\lambda_2 > 1-\mu$. The first case is simpler for there cannot exist orbits lying in $]1-\mu, \Theta[$ (remember that the fixed point $\lambda_1 = \Theta$ is always unstable under the assumption of existence of two fixed points, and is furthermore ruled out as an equilibrium by the transversality condition), and there exist only two possible types of trajectories. As we said previously, the results invoked here are established in the appendix.

Thus, let us assume $\lambda_2 \leq 1-\mu$. We then have $\tau(1-\mu) \leq 1-\mu$. For every admissible initial condition λ_0 ($\in [1-\mu, \Theta[$), there exists an infinite sequence $(t_j, i_j)_{j \in \mathbb{N}}$, $i_j \geq 0$, such that $\lambda_t = 1-\mu$, $\forall t \in [t_j, t_j + i_j]$, and

$\lambda_t \in]1-\mu, \Theta[$ otherwise (in the very special case $\lambda_2 = 1-\mu$, one of the i_j may well be infinite, i.e. the 'growth' rate remains constant and equal to $(1-\mu)$ after some finite date ; otherwise, all i_j are obviously finite). As we said previously, there exist periods where the optimal choice leads agents to 'disinvest passively'. In periods where this happens, the 'growth' rate is equal to $1-\mu$. When investment starts again, the optimal choice takes into account the capital accumulation preceding the investment stop ; the point onto which $(1-\mu)$ is mapped depends on the orbit described by the growth rate before taking the value $(1-\mu)$. It seems hardly possible to determine whether cycles or erratic trajectories are possible or not.

Let us suppose now that $1-\mu < \lambda_2$. We know that (3.10) holds for $\lambda_t \in [1-\mu, \Theta]$ such that $\tau(\lambda_t) \geq (1-\mu)$. Considering the map τ, we see that there may well exist τ-orbits remaining in $[1-\mu, \Theta]$, especially when $(1-\mu)$ is small. From a purely theoretical point of view, we can thus have cycles and topological chaos. Notice however that to be able to ascertain the existence of cycles or of topological chaos, we need $1-\mu < \lambda_2$ and $\tau(\lambda^*)$ small enough, and for topological chaos the additional condition $\tau(\lambda^*) \geq (1-\mu)$ must also hold ; all this has a bad consequence : classical parameter values do not fulfill all the requirements imposed by these conditions. Thus, unless assuming a very important capital depreciation per period, it is not even possible to deduce the possibility of cycles or topological chaos in the *true* dynamics from their existence in the τ-dynamic. Furthermore, even if we take non appealing, but adequate parameter configurations, the claim of topological chaos is not especially exhilarating as the previous section has shown. It must be established that chaos can be observable, and we must therefore show that the set S of initial conditions giving chaos can be of strictly positive measure. Our methodology is the following : we first consider the τ-dynamics and show under which assumptions chaos can be shown to be observable. The fact that 0 is then an accumulation point of every chaotic trajectory implies that none of these chaotic orbits can lie in $[1-\mu, \Theta]$, even if $(1-\mu)$ is very small. A first consequence of this is that the existence of observable chaos in the τ-dynamics does not imply anything for the true dynamics.

Thus let us consider first the dynamics without the constraint $\lambda_t \geq (1-\mu)$. By a simple change of variables, the family of maps $\tau_{\beta,\delta,\Theta}$ can be transformed into the family $\Gamma_{\beta,\Phi}$ defined by $X_{t+1} = \Phi X_t(1-X_t)^\beta$, where Φ has the following expression :

$$\Phi = \delta^{1/(1-\alpha)}\Theta^\beta,$$

with $\beta = (\alpha+\eta)/(1-\alpha)$ and $\Theta = b + (1-\mu)$. It is immediate to verify that for every fixed β, the parameters of the economy can be chosen such that Φ describes the whole interval $[0, +\infty[$. Some properties of the family $\Gamma_{\beta,\Phi}$ can be derived very easily, others are more demanding. For every given $\beta > 0$, for Φ large enough, there exists a cycle of period three ; therefore topological chaos can occur. As we have seen in the previous section, this is not very satisfactory since stable cycles, perhaps of very long period and thus difficult to observe in a simulation, may exist and the set S may well be of zero Lebesgue measure. Therefore, it is necessary to study the possibility of ergodic or turbulent chaos. For the family of maps considered here, establishing $m(S) > 0$ requires the following conditions :

$$\beta \geq 1 \quad \text{and} \quad \Gamma_{\beta,\Phi}(X^*) \geq 1, \tag{$*$}$$

where X^* is the point at which the map reaches its maximum. The first condition is necessary to guarantee a negative Schwarzian derivative on the whole interval of definition and existence of at most one (weakly) stable cycle. Notice that if this condition holds, the classical flip bifurcation cascade occurs as Φ increases from 0 to the value where a three cycle appears. As long as there does not exist any period three cycle, a standard result is that the cycle generated by the last flip bifurcation is stable (and it is the only stable cycle). In the case where a period three cycle exists and $\Gamma_{\beta,\Phi}(X^*) < 1$, stability of some cycle, and thus zero measure of S, has been established for some values of Φ, but nothing general could be obtained until now. If we assume a negative Schwarzian derivative, then the second condition is needed to establish that the measure of S is not equal to zero. We can distinguish two cases : $\Gamma(X^*) = 1$ and $\Gamma(X^*) > 1$.

The first case is relatively simple : if $\Gamma_{\beta,\Phi}(X^*) = 1$, then the second iterative is equal to 0, which is an unstable fixed point. Since the here considered maps are C^∞ and meet all the required assumptions, we can apply the same reasoning as in the case of $\Gamma_4 = \Gamma_{1,4}$ and conclude to the existence of ergodic chaos, which implies a strictly positive Lebesgue measure of S. It is even possible to show that the map is aperiodic and sensitive and exhibits turbulent chaos on $[0,1]$. Thus, the orbit of any point in S is dense in $[0,1]$.

The second case is more exotic and more difficult to deal with, for the set of definition of the dynamic is no longer the whole interval $[0,1]$, but a subset Λ which is a Cantor set, i.e. a fractal. A Cantor set is a set that is closed, totally disconnected (it does not contain any interval) and perfect (every point in it is an accumulation point). The definition set Λ thus has a zero Lebesgue measure in $[0,1]$. The map exhibits turbulent chaos on Λ, the set of initial conditions S giving chaos has a strictly positive Cantor measure in Λ and the trajectory of any point in S is dense in Λ. In the case of the logistic maps $\Gamma_{1,\Phi}$, we are in this situation if $\Phi > 4$, and the dynamics are even structurally stable at least for $\Phi > 2+\sqrt{5}$ (see Devaney (1987)). Furthermore notice that $(0,1) \in \Lambda^2$, which implies that these are accumulation points of any chaotic trajectory. The fact that the definition set is a Cantor set is rather simple to establish in the case of the logistic map when $\Phi > 2+\sqrt{5}$ and uses the property of expansiveness of the map ; this property cannot, obviously, be invoked when $\Phi \in]4, 2+\sqrt{5}]$ or in the case $\beta > 1$, and the proof is much more elaborate (see, for instance, Boldrin and Persico (1993), relying on a theorem proven by Nusse (1987)). It is clear that the case of a Cantor definition set constitutes a mathematical curiosum, without much economic significance.

Let us turn back now to the correct dynamics. We do not treat the case $\tau(\lambda^*) < 0$ which is similar to the case $\tau(\lambda^*) = 0$ except for the fact that the optimal choice has to be done such that λ_t always lies in the Cantor set Λ. The results exposed here rely on those established in the appendix. Since $1-\mu < \lambda_2$, there can exist τ-cycles which lie in $[1-\mu, \Theta]$, especially when

$(1-\mu)$ is small, and these are possible cycles in the correct dynamics. Notice however that the set of initial conditions giving chaos in the unconstrained dynamics is of full measure, which implies that τ-cycles which do not lie in $[1-\mu, \tau(1-\mu)]$ are unobservable. Now, if the initial condition corresponds to chaos (this is almost sure), then there exists a finite date t_0 at which agents decide not to invest. If in the τ-dynamics $\lambda_{t_0} = 1-\mu$, then for sure $\lambda_{t_0} = 1-\mu$ and $\lambda_{t_0+1} = \tau(1-\mu)$ in the true dynamics. But when we have $\lambda_{t_0-1} > 1-\mu$ and $\tau(\lambda_{t_0-1}) < 1-\mu$, then there exists a finite $i_0 \geq 0$ such that $\lambda_{t_0} = ... = \lambda_{t_o+i_0} = 1-\mu$ and $\lambda_{t_0+i_0+1} \in]1-\mu, \tau(1-\mu)[$ as is shown in the appendix. Thus, after a finite lapse of time, the trajectory a.s. lies in $[1-\mu, \tau(1-\mu)]$. There are two possibilities : $\lambda_{t_0+i_0+1}$ can be on a τ-orbit $\subset [1-\mu, \tau(1-\mu)]$ or not. Remember that a random choice has a zero probability to be on a cycle, but $\lambda_{t_0+i_0+1}$ is predetermined and does not fall 'at random' into the interval. If $\lambda_{t_0+i_0+1}$ is not on a cycle $\subset [1-\mu, \tau(1-\mu)]$, then necessarily there exists a finite date t_1 at which $\tau(\lambda_t)$ hits the lower boundary $(1-\mu)$, and we have again the phenomenon described previously.

We can thus conclude to the following : either there exists a finite date after which the orbit is a τ-cycle $\subset [1-\mu, \tau(1-\mu)]$, or there exists a finite date after which the orbit is a cycle which is not a τ-cycle (because in the τ-dynamics there exist dates at which $\tau(\lambda_t) < 1-\mu$), or the orbit is not cyclical and there exists an infinite sequence $(t_j, i_j)_{j\in N}$ such that $\lambda_t = 1-\mu$, $\forall t \in [t_j, t_j + i_j]$, and $\lambda_t \in]1-\mu, \tau(1-\mu)[$ otherwise for $t > t_0 + i_0$. Thus, the only thing we can say is that cycles may well exist, but to establish their existence (or, perhaps, the impossibility of their occurence) seems difficult, if not impossible because of the fact we mentioned previously for the case $\lambda_2 \leq 1 - \mu$ and which also holds here : the point onto which $(1 - \mu)$ is mapped depends on the initial condition, and therefore standard cycle and chaos analysis, which considers classical first order difference equations, cannot give answers to the questions we want to address here. The only thing we know for sure is that if we choose an initial condition at random, then the dynamics lie, with probability one, in $[1 - \mu, \tau(1 - \mu)]$ after a finite lapse of time. □

The dynamics are thus quite different from those exhibited by Boldrin and Persico, and unlike for the dynamics given by (3.10) in Boldrin and Rustichini (1994) or (2.2) in Boldrin and Persico (1993), current mathematical knowledge does not seem to provide any 'simple' argument in favour of observable chaos if $0 < \mu < 1$ in the case of the optimal solution. Furthermore, as we mentioned earlier, we need non classical parameter values to be able to claim the existence of cycles or of topological chaos.

Conclusion

What are we to conclude on Boldrin and Rustichini's model ? It establishes the important fact that even in the framework of infinitely lived agents, indeterminacy of the steady state and endogenous fluctuations cannot be ruled out under the assumption of perfect markets and perfect foresight. Unfortunately, the authors' claim of endogenous growth with a chaotic growth rate is indefendable since the *correct* solution of the *right* optimisation program does not yield a solution allowing to characterise thoroughly the trajectories. The different types of possible orbits can be described qualitatively, but it does not seem possible to establish the existence of cycles or of chaos (even topological) unless assuming a very high depreciation rate of capital. Chaos cannot be shown to be observable in Boldrin and Rustichini's framework unless assuming entire capital depreciation in each period, an unappealing assumption which we should elude in an ILA context where periods are short. Therefore, we deem prudent to conclude only to the possibility of self-sustaining growth with a *fluctuating* growth rate. Two further critiques : first of all, there is an evident lack of consistency in taking a linear investment good production function since such a function does not meet assumption 2.2 (which implies, in particular, strict concavity in $x_{2,t}$) imposed in the rest of the paper ; secondly, the assumption of linear utility certainly simplifies the formal analysis but is not *necessarily* innocuous in the context of global analysis : local analysis, like Hopf bifurcation detection, easily allows reasonings 'by continuity' (see, for instance, Cazzavillan (1992) or v. Cœster (1993)), but this is not true in general when we practice global analysis (structural stability, for instance, is not automatically guaranteed). It is true that in Boldrin and Rustichini's model the assumption is innocuous, but this fact has to be established and especially emphasized. Notice also that in certain models the assumption of linear utility is simply desastrous : there are cases where it is possible, with well known mathematical tools, to establish the existence of cycles or even chaos under any assumption on a finite coefficient of intertemporal substitution while it is impossible to characterise in a satisfactory manner the orbits in the case of a linear utility, which corresponds to an infinite coefficient of intertemporal substitution (see, for instance, v. Cœster (1994)). We should therefore handle such an assumption with great care.

Endogenous growth with (observable) chaotic trajectories of the growth rate has been established in a two-sector OLG economy meeting all neoclassical assumptions from the point of view of the private sector (CRRA per period utility and Cobb-Douglas production functions with externalities à la Romer (1986)) in v. Cœster (1994). There it is shown that endogenous fluctuations may require strong external effects in the absence of bubbles or sunspots, but not necessarily in their presence. Similar results, and especially the possibility of chaotic orbits of the growth rate, still remain to be established in the ILA framework.

Let us add some further critiques concerning Boldrin and Persico's (1993) paper. Subsection 2.3., intitled 'A More Complicated Example', claims to "dispel the impression that the example given above may be special...". With assumptions that can be found in the paper, Boldrin and Persico obtain *backward* dynamics of basically the following type :

$$\xi_t = \frac{\mu \xi_{t+1}}{(1+\xi_{t+1})^{\beta}},$$

with $\mu > 0$ and $\beta > 1$. The authors conclude that by the same methods as applied in section 3. of their paper, cycles and chaos can be shown to exist for adequate parameter configurations. Let us say that the example is very badly chosen, for it is immediate to check that the map does not have a Schwarzian derivative that is always negative, which already implies that Singer's theorem cannot be applied and unicity of attractors therefore appears to be questionable ; furthermore, no finite point is ever mapped onto the origin, and we do not understand how Boldrin and Persico can see any 'obvious' analogy with the $\Gamma_{\beta,\Phi}$ family. Maybe the family of maps obtained in 2.3. can exhibit ergodic chaos under the assumption $\mu = 1$, but standard arguments cannot be used to give an answer. Unless giving a specific proof, only cycles and topological chaos can be claimed for (and this even only in the case $\mu = 1$ unless exhibiting a proof), and thus the most interesting results of section 3. cannot be invoked here.

Subsection 3.2. contains the proof that the set of admissible initial conditions has the structure of a Cantor set in the case of maps of the $\Gamma_{\beta,\Phi}$ family when $\beta > 1$ and $\Gamma_{\beta,\Phi} > 1$; Boldrin and Persico insist on the fact that their case is non standard because expansiveness does not hold on the entire definition set, which implies that the method exposed in Devaney (1987) cannot be applied here. The proof given is quite nice, but a rather standard result is the fact that the definition set of $\Gamma_{1,\Phi}$ is a Cantor set when $\Phi > 4$, which includes the case $4 < \Phi < 2 + \sqrt{5}$ where we certainly do not have expansiveness. Even if Devaney (1987) does not give the proof, a thorough study of his book reveals that he nevertheless invokes several times this fact. Boldrin and Persico therefore have to show that their case differs fundamentally from the case $\Gamma_{1,\Phi}$, $4 < \Phi < 2 + \sqrt{5}$.

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APPENDIX : The Optimization

We suppose here $\sigma = 0$; it is immediate to verify that this assumption has no effect on the result, which holds for all $\sigma < 1$. We assume to be in the case where two fixed points exist. Under this assumption, the higher steady state $\lambda_1 = \Theta$ is unstable and is furthermore ruled out by the transversality condition. Our purpose is not to solve completely the optimisation problem, but to give a 'dirty', intuitive description of the solution. The program to solve is :

$$\begin{cases} \max \sum_{t=0}^{+\infty} \delta^t k_t^\eta (\gamma x_t - a x_{t+1})^\alpha \\ (1-\mu) x_t \leq x_{t+1} \leq \Theta x_t \end{cases}$$

It is easy to see that the right hand inequality is never strictly binding (in the sense that the τ-dynamics are such that for any admissible initial condition λ, $\tau^t(\lambda)$ is strictly less than $\Theta, \forall t$) , and therefore we can omit it. Let B_t denote the Lagrange multiplier corresponding to the remaining constraint. Consider the first order and equilibrium conditions corresponding to our program :

$$\begin{cases} -\alpha a \delta^t k_t^\eta (\gamma x_t - a x_{t+1})^{\alpha-1} + \alpha \gamma \delta^{t+1} k_{t+1}^\eta (\gamma x_{t+1} - a x_{t+2})^{\alpha-1} + B_t \\ -(1-\mu) B_{t+1} = 0, \quad \forall t \\ B_t (x_{t+1} - (1-\mu) x_t) = 0, \quad \forall t \\ k_t = x_t, \quad \forall t. \end{cases}$$

Let us write

$$C_t = \frac{B_t}{\alpha\delta^t x_t^{\alpha+\eta-1}}.$$

Then we have

$$-a(\gamma - a\lambda_t)^{\alpha-1} + \gamma\delta\lambda_t^{\alpha+\eta-1}(\gamma - a\lambda_{t+1})^{\alpha-1} + C_t - (1-\mu)\delta\lambda_t^{\alpha+\eta-1}C_{t+1} = 0.$$

$\lambda_t > 1-\mu \Rightarrow C_t = 0$. Therefore (BR) holds for every $\lambda_t > 1-\mu$ such that $\tau(\lambda_t) > 1-\mu$, but it also holds for $\tau^{-1}(1-\mu)$, and for $(1-\mu)$ if $\tau(1-\mu) > 1-\mu$. Let us consider now a date t_0 such that $\lambda_{t_0-1} > 1-\mu$ and $\tau(\lambda_{t_0-1}) < 1-\mu$; then we have $C_{t_0-1} = 0$ and $\lambda_{t_0} = 1-\mu$, and there exists a finite $i_0 \geq 0$ such that $\lambda_t = 1-\mu$, $\forall t \in [t_0, t_0+i_0]$ and $\lambda_{t_0+i_0+1} > 1-\mu$, the latter implying $C_{t_0+i_0+1} = 0$. To determine the value of $\lambda_{t_0+i_0+1}$, we have thus to calculate $C_{t_0+i_0}$. We are interested here only in the sign of $C_{t_0+i_0}$. We have :

$$(1-\mu)\delta\lambda_{t_0-1}^{\alpha+\eta-1}C_{t_0} = \delta\gamma\lambda_{t_0-1}^{\alpha+\eta-1} - a(\gamma - a\lambda_{t_0-1})^{\alpha-1}.$$

The function $f(x) = \delta\gamma(\gamma - ax){\lambda_{t_0-1}}^{\alpha+\eta-1} - a(\gamma - a\lambda_{t_0-1})^{\alpha-1}$ is strictly increasing in x and $f(\tau(\lambda_{t_0-1})) = 0$. Since, by assumption, $\tau(\lambda_{t_0-1}) < 1-\mu$, we have thus $f(1-\mu) > 0$ and therefore $C_{t_0} > 0$. Suppose $i_0 \geq 1$; the link between C_t and C_{t+1}, for $t \in [t_0, t_0+i_0-1]$ is given by :

$$\delta(1-\mu)^{\alpha+\eta}C_{t+1} = C_t + \delta\gamma(1-\mu)^{\alpha+\eta-1} - a.$$

If $\lambda_2 > 1-\mu$, then the sequence (C_t) is increasing and therefore $C_{t_0+i_0} > 0$, which implies $\lambda_{t_0+i_0+1} < \tau(1-\mu)$. Thus, under the assumption $\lambda_2 > 1-\mu$, the dynamics lie, after a finite lapse of time, in the interval $[1-\mu, \tau(1-\mu)]$. As a consequence, we have the following possibilities :

•(i) either λ_0 is on a τ-cycle $\subset [1-\mu, \Theta]$,

•(ii) or, after a finite lapse of time, λ_t is on a τ-cycle $\subset [1-\mu, \tau(1-\mu)]$,

•(iii) or, after a finite lapse of time, λ_t is on an orbit $\subset [1-\mu, \tau(1-\mu)]$ and there exists an infinite sequence $(t_j, i_j)_{j \in \mathbb{N}}$ such that $\lambda_t = 1-\mu$, $\forall t \in [t_j, t_j + i_j]$ and $\lambda_t > 1-\mu$ otherwise. Notice that in the case where observable chaos exists in the unconstrained dynamics, there is a positive probability of $\lambda_t \in [1-\mu, \tau(1-\mu)]$ after a finite lapse of time, and even probability one in the case of $\tau(\lambda^*) = 0$, for instance.

If, on the contrary, $\lambda_2 \leq 1-\mu$, then the sequence (C_t) is decreasing, which is rather intuitive since to start again, we need $C_{t_0+i_0}$ such that $\lambda_{t_0+i_0+1} > 1-\mu \geq \tau(1-\mu)$, which requires a strictly negative $C_{t_0+i_0}$. Under our present assumption on λ_2, the only possible type of trajectory is (iii) but lying, of course, in $[1-\mu, \Theta[$. □

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