Approximating functions of integrals of Log-Gaussian Processes : Applications in Finance

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Abstract

This dissertation looks at various specific applications of stochastic processes in finance. The motivation for this work has been the work on the valuation of the price of an Asian option by Rogers and Shi (1995). Here, we look at functions of integrals of log - Gaussian processes to obtain approximations to the prices of various financial instruments.

We look at pricing of bonds and payments contingent on the interest rate. The interest rate is assumed to be log - Gaussian, thus ensuring that it does not go negative. Obtaining the exact price might not be easy in all cases - hence we use of a combination of a conditioning argument and Jensen's inequality to obtain the lower bound to the prices of the bond as well as payments contingent on interest rates. We look at single driver models as well as multi-driver models. We also look at bonds where default is possible.

We try to provide a mathematical justification for the choice of the conditioning factor used throughout the thesis to approximate the price of bonds and options. This is similar to the approach used by Rogers and Shi (1995) to valuing an Asian option; but they had provided no mathematical justification.

Another part of this dissertation deals with the problem of pricing European call options on stochastically volatile assets. Further, the price and the volatility processes are in general correlated amongst themselves. Obtaining an exact price is quite involved and computation intensive. Most of the previous work in this field has been based on the solution to a system of partial differential equations. As in the case of pricing bonds, here too, we use a conditioning argument to obtain an approximation to the prices. This method is much faster and less computation intensive. We look at the situations of fixed and stochastic interest rates separately and in each case, we look at the volatility process following a simple Brownian motion and an Ornstein Uhlenbeck process.

We also look at the value of stop - loss reinsurance contract for the case of a doubly stochastic Poisson process. Finally, we look at an alternative method of pricing bonds and Asian options. This is done by using a direct expansion and thus avoids the numerical integration that is used in the earlier chapters.

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Contents

Chapter 1

Introduction

1.1 General Introduction

Mathematical finance is a very interesting field and there exists a large number of application areas where statistical tools like probability and stochastic process could be extensively used. The problems examined in this thesis belong to this category - problems which could be explained based on the knowledge of stochastic processes and probability. Two of these problems are the following.

First, we look at the problem of pricing of bonds with non-negative interest rates and then in the second instance, we look at problems of pricing of options on assets with stochastic volatility. The first of the two problems is the one of pricing a bond with non-negative interest rates. We assume a log-normal model for the interest rate, thereby ensuring non-negative interest rates. Thus, the instantaneous rate of interest r_s is given by

$$r_s = be^{X_s},$$

where b is a scaling constant and

$$X_s = \mu_s + Y_s.$$

Here $\{Y_s; 0 \leq s \leq 1\}$ is a Gaussian process with zero mean and so μ_s is the drift of X_s.

In the course of this work, we have looked at pricing zero coupon bonds as well as bonds with coupon payments. In both cases, the interest rate is as defined above. We have also extended our study to situations where there is a possibility of default. Associated with this is also the problem of valuing a contingent payment on the interest rate. Now, the value of the "contingent" payment on the interest rate is the shortfall between *1* and the amount accumulated by an initial investment c. In the course of this dissertation, c is treated as the strike price at which the value of the contingent payment is calculated.

Here, we will outline the problem of valuing a zero coupon bond as also the problem of valuing a contingent payment on the interest rate. The price of a zero coupon bond is given by

$$E(e^{-b\int_0^1 e^{X_s}ds}); \tag{1.1}$$

and the value of a contingent payment to be made at the strike price c is given by

$$E(e^{-b\int_0^1 e^{X_s}ds} - c)^+, \tag{1.2}$$

where X_s is as defined earlier. The exponential nature of the model ensures that interest rates do not go negative since negative interest rates are unrealistic and could lead to undesirable consequences, as outlined by Rogers (1995). This can be put in the framework of the work set out by Heath, Jarrow and Morton (1992) and is also an extension of the work by Black and Karasinski (1991) and Black, Derman and Toy (1990).

The second problem is the one of valuing European call options on assets with stochastic volatility. Thus, we have

$$dX_t = rX_tdt + \sigma e^{\frac{kV_t}{2}}X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2}dB_t^{(2)}], \tag{1.3}$$

$$dV_t = \mu dt + dB_t^{(1)}, \tag{1.4}$$

$$\underline{\text{or}} \quad dV_t = -aV_tdt + dB_t^{(1)} \tag{1.5}$$

where X_t is the price process and V_t is the volatility process. Further, r is the rate of interest and $B_t^{(1)}$ and $B_t^{(2)}$ are two independent standard Brownian motions. When the volatility process V_t follows a simple Brownian motion (as defined by equation (1.4)), μ is the drift of the Brownian motion. In the case of V_t following an Ornstein - Uhlenbeck process (as defined by equation (1.5)), a is the mean reversion force of the Ornstein - Uhlenbeck process. Further, ρ is the correlation between V_t and the logarithm of X_t. In this situation r is treated as a constant, but later we also look into situations where it is stochastic in nature. Also, we let the volatility process, V_t to be either a simple Brownian motion (as in equation (1.4)) or an Ornstein - Uhlenbeck process (as in equation (1.5)). Here, we are interested in the price of a call option. Under an equivalent martingale measure (see Harrison and Kreps (1979) and Harrison and Pliska (1981)), this is given by

$$X_0\{e^{-r}E(e^{Y_1} - b)^+\}, \tag{1.6}$$

where b is the strike price at which the value of the option is calculated, X_0 is the current price of the asset and $Y_t = ln(\frac{X_t}{X_0})$. We shall also look into the situation of the interest rate being stochastic in the pricing of options on stochastically volatile assets.

The common strand is that both problems essentially involve the evaluation of functions of integrals of log-normal processes, although in the case of pricing of options on stochastically volatile assets the situation is more complicated.

Rogers and Shi (1995), in valuing an Asian option on a risky asset S_t, solved a somewhat similar problem. In fact, it was their work which served as the motivation for this thesis and is explained in detail in the next section.

1.2 Motivation for the work

Rogers and Shi assume that at time t, the price of a risky asset S_t is given by

$$S_t = S_0 exp(\sigma B_t - \frac{1}{2}\sigma^2 t + ct), \tag{1.7}$$

where, $\{B_t; 0 \leq t \leq 1\}$ is a standard Brownian motion, σ^2 is the instantaneous variance. Also, c is a constant. Another thing they also assume is that under an equivalent martingale measure, $c = r$, the riskless interest rate (see Harrison and Kreps (1979) and Harrison and Pliska (1981)). The problem that Rogers and Shi looked at is that of computing the value of an Asian (call) option with maturity T and the strike price K written on the risky asset

S_t. Rogers and Shi fix $T = 1$. Mathematically, this is the same as calculating

$$E(Y - K)^+ = E[max(Y - K, 0)], \quad (1.8)$$

where, Y is defined by

$$Y = \int_0^1 S_u du. \quad (1.9)$$

They calculate the price of the option for both fixed and floating strike prices.

Rogers and Shi used the following equation to obtain the lower bound to the price of the option.

$$E(f(Y)) = E(E(f(Y)|Z)) \geq E(f(E(Y|Z))). \quad (1.10)$$

In their case, the function f is defined as $f(x) = max(x\text{-}k, 0)$, where k is the strike price of the option. Further f is a convex function and Z is a suitably chosen conditioning factor. The second part of equation (1.10) is Jensen's Inequality. Hence, using this equation, Rogers and Shi obtain a lower bound to the price of an option. They also found an approximation to the upper bound and it turned out that the two bounds were very sharp. In fact, the bounds obtained were so sharp that the lower bound could indeed be treated as the true price of the option written on the risky asset.

Rogers and Shi have used the conditional factor Z to be a zero mean Gaussian variable. This ensures that the conditioned process, conditionally on Z still remains a Gaussian process. Their conditioning factor is of the form

$$Z = \int_0^1 B_s ds. \quad (1.11)$$

According to Rogers and Shi, they had investigated numerically several possible choices for Z, some of them bivariate. However, they found that for the fixed strike Asian option, the best choice was the one defined in equation (1.11).

Conditionally on Z, for $\{0 \leq t \leq 1\}$ and $\{0 \leq s \leq 1\}$, we have

$$E(B_t|Z) = m_t Z \quad \text{and} \quad \text{cov}(B_s, B_t|Z) = v_{st},$$

where

$$m_t = \frac{\text{Cov}(B_t, Z)}{\text{Var}(Z)} = \frac{E(B_t Z)}{E(Z^2)}$$

and

$$v_{st} = s \wedge t - \frac{E(B_s Z)E(B_t Z)}{E(Z^2)}.$$

We thus have

$$m_t = \frac{3t(2-t)}{2} \quad \text{and} \quad v_{st} = s \wedge t - \frac{3st(2-s)(2-t)}{4}.$$

Once the values of m_t and v_{st} are known, then one can easily find the value of $E(Y|Z)$, where Y is defined by equation (1.9). Finally, taking expectation over the distribution of Z, one gets the lower bound of the price of the option. The lower bound to the price of the option is thus given as

$$\int_{-\infty}^{\infty} \Omega(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \tag{1.12}$$

where

$$\Omega(z) = \left[\int_0^1 e^{zm_t + \frac{1}{2}v_{tt}} dt - k \right]^+ .$$

Now, the lower bound on equation (1.10) is not guaranteed to be good. However, the estimate of the error can be made using the following approach. We have, for any random variable U,

$$0 \le E(U^+) - E(U)^+$$

$$= \frac{1}{2}(E(|U|) - |E(U)|)$$

$$\le \frac{1}{2}E(|U - E(U)|)$$

$$\le \frac{1}{2}\text{Var}(U)^{\frac{1}{2}}.$$

Thus, in their case, Rogers and Shi had

$$0 \le E[E(Y^+|Z) - E(Y|Z)^+] \le \frac{1}{2}E(\text{Var}(Y|Z)^{\frac{1}{2}}).$$

Thus, using this, they found the upper bound to the price of the option. As already remarked, Rogers and Shi observed that the two bounds were so close to each other that it in fact represented the true price itself. We have used the same idea in finding the bounds to the prices of bonds or options on assets with stochastic volatility.

As a follow up to Rogers and Shi's work, Thompson (1999) has developed a method to refine the upper bound to the price of the Asian option.

1.3 Different Concepts Used

In the course of this thesis, the prime aim is to calculate the value of the price of an asset. We use the idea given by the inequality in equation (1.10), where the conditioning factor is suitably chosen. As it might not always be possible to obtain the price easily, we try to use Jensen's inequality to obtain a lower bound to the price. Now, in most cases, the lower bound obtained is so sharp that it can be regarded as a very close approximation to the true price. Thus, for any convex function f of a random variable X, we have

$$E(f(X)) = E[E(f(X|Z))] \geq E(f(E(Y|Z))),$$

where Z is the conditioning factor which is suitably chosen. This is similar to the approach used by Rogers and Shi (1995) to calculate the value of an Asian option. Here, we look at functions which are different from the one studied by Rogers and Shi. We continue to use the same technique as used by them to find the lower bound of the price and hope it works well. An important consideration is the appropriate choice of the conditioning factor Z. Various choices of Z have been tried and in all these situations,

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}$$

has been found to be the "optimal" choice in some respects which will be explained later in the thesis. $\{Y_s; 0 \leq s \leq 1\}$ represents the logarithm of the price process in the case of pricing of options, while in the case of bond pricing it is either a geometric Brownian motion or an exponential function of an Ornstein - Uhlenbeck process. We present a mathematical justification of the choice in chapter 3. Division by $\sqrt{\text{Var}(\int_0^1 Y_s ds)}$ ensures that the conditioning factor is suitably normalised and thus we have a standard normal distribution for the conditioning factor.

In the course of this thesis, the approach has been to make use of a suitable conditioning factor to calculate the price of bonds or options. This technique is particularly useful in pricing options, often there being no other way to calculate these. In the case of the bonds as well, this technique is quite useful, especially in the situations where the variance is relatively high and other more direct methods that we will also look at, fail.

In the case of pricing the bonds, one can alternatively make use of a direct expansion technique. This technique does not work for pricing of options, neither does it work for pricing of bonds when the variance is relatively high. In the case of bond pricing, for relatively lower values of the variance, this method can be used for comparison purposes.

In the calculation of options, we concentrate only on the European call option. However, having calculated the value of the European call option, the corresponding put option value can be easily calculated using the Put - Call Option parity concept. Also, for non-dividend paying stocks, one can easily calculate the value of an American call option from the European call option, as has been shown by Merton (1973).

1.4 Previous Work

A number of researchers other than Rogers and Shi have also made a significant contribution to the field of pricing derivative assets and options on such assets where the volatility is either constant or stochastic in nature. One of the earliest pioneering works in this field has been by Black and Scholes (1973) on pricing of assets and corporate liabilities. Merton's (1973) work on the theory of rational option pricing is also very important. This was followed by Rubenstein (1976) with his work on the pricing of options and valuing uncertain income streams. However, in all these three works mentioned, the volatility of the price is assumed to be constant.

However, constant volatility is not the most realistic of the situations - in fact, more often than not, the volatility present in the market is stochastic in nature. Due to this "stochastic nature", the price and the volatility processes can thus be represented as stochastic differential equations. Further, if the volatility process is stochastic in nature being driven by a Brownian motion (or a Wiener process), it can be represented as a simple Brownian motion process or an Ornstein - Uhlenbeck process. This general framework was introduced by Vasicek (1977) and is hence referred to as the Vasicek model. This framework has been modified by many researchers to model interest rates and price bonds and options. Quite a number of these modifications have been outlined by Baxter and Rennie (1996). The work of Harrison and Kreps (1979) and Harrison and Pliska (1981) on the use of martingales and stochastic integrals in financial applications, especially in the securities market and in continuous trading is also very important.

Notable work on modelling interest rates and pricing of bonds have been carried out by Black, Derman and Toy (1990), Black and Karasinski (1991), Hull and White (1990, 1993, Fall 1994, Winter 1994, 1996), Heath, Jarrow and Morton (1992) and Cox, Ingersoll and Ross (1985). Black, Derman and Toy as well as Black and Karasinski have used a binomial tree to model and hence calculate interest rates and thereby price bonds. Hull and White have used an idea similar to Black, Derman and Toy as well as Black and Karasinski, only using a trinomial tree rather than a binomial one to calculate interest rates. Furthermore, Black and Karasinski as well as Black, Derman and Toy use a log-normal model for the interest rate - this is similar to the model used by us in this thesis. The Hull and White model is quite similar to the Vasicek model and also takes into account the fact that the interest rate does not go negative. This is because negative interest rates have undesirable consequence as shown by Rogers (1995). Under the Gaussian set-up, Heath, Jarrow and Morton's contribution is also very important. Another idea to model the interest rate, as has been done later in this thesis, is to use a log-normal model for the interest rate as has been used by Goldys, Musiela and Sondermann (1994), Sandermann, Sondermann and Miltersen (1994) and Brace, Gatarek and Musiela (1997). The basis of research in this field has not been restricted only to the Gaussian set-up. In fact, a considerable amount of work has also been carried out under the assumption of a non-Gaussian set-up. Most significant among that is the contribution from Cox, Ingersoll and Ross which looks at the term structure of interest rates in a non- Gaussian framework. Most of the contributions referred above deal with the *one - factor* model. However, work has also been done on the *multi - factor* model. Prominent among them are Duffie and Kan (1994, 1996) and Longstaff and Schwartz (1992a, 1992b).

Research has also been done in the area of pricing of options. After all, options have become one of the most important financial instruments in recent years. The same set-up has been used to calculate the value of an option, European, American or Asian. Some of the most important work carried out in option pricing can be attributed to Hull and White (1987), Rogers and Shi (1995), Heston (1993), Jarrow and Rudd (1982), Stein and Stein (1991), Wiggins (1987), Willard (1996) and Romano and Touzi (1997). Hull and White have looked at the problem of pricing of European call options on assets with stochastic volatility where the volatility and the logarithm of the price process can be correlated. They solve the problem by making use of a set of partial differential equations based on the price and the volatility process and in their solution they further assume the correlation between the volatility and the logarithm of the price process to be zero. Another notable contribution has been the work by Rogers and Shi, which we have discussed in detail in the previous section. Heston, Jarrow and Rudd, Stein and Stein and Wiggins have looked at problems of option pricing and have used partial differential equations to find the value of the option. Romano and Touzi have also used a partial differential equation approach to look at contingent claims and market completeness in a stochastically volatile model with the price and the volatility process being correlated.

1.5 Order of work

Chapter 2 deals with modelling of interest rates and the calculation of bond prices for zero coupon bonds. The interest rate model is essentially taken to be log-normal. This ensures that the interest rate does not go negative. Here we are interested in finding the expected value of the integral of a log-normal distribution. We make use of the conditioning factor approach, it is similar to the one used by Rogers and Shi (1995). We also make use of a direct expansion technique to find the price, rather the bounds to the price, of the zero coupon bonds by a direct expansion method. Furthermore, for the zero coupon bonds, we also look at the value of a contingent payment at some strike price c.

In chapter 3, we try to provide a mathematical justification of the choice of the conditioning factor that we use to price zero coupon bonds in the previous chapter. We also justify the choice of the conditioning factor by Rogers and Shi (1995) for valuing the fixed strike Asian option. The conditioning factor is given by,

$$Z = \int_0^1 Y_u du,$$

where $\{Y_u, 0 \leq u \leq 1\}$ is a Gaussian process. Rogers and Shi had $\{Y_u, 0 \leq u \leq 1\}$ follow a simple Brownian motion. However, we not only present a detailed mathematical justification of this choice of the conditioning factor, but also extend it to the general case of $\{Y_u; 0 \leq u \leq 1\}$ following different Gaussian processes. We look at two situations - one where $\{Y_u; 0 \leq u \leq 1\}$ comprises of only one process and is referred to as the single driver case as well as the case of $\{Y_u; 0 \leq u \leq 1\}$ being a linear combination of a number of Gaussian processes and is referred to as the multi driver case.

Chapter 4 deals with the calculation of bond prices for bonds making coupon payments. We look at two situations - one where the bond has a zero probability of default and in the second case where the bond has a non - zero probability of default. The interest rate model is again taken to be log-normal. This ensures that the interest rate does not go negative. Here also, we are interested in finding the expected value of the integral of a log-normal distribution and make use of the conditioning factor approach.

Chapter 5 looks at the problem of pricing bonds where the interest rate process is governed by n Markov processes. These n Markov processes need not necessarily be independent of each other. We refer to these models as *multi - driver* models. In the chapter, we discuss two different ways of modelling the interest rate - we refer to them as model 1 and model 2. Model 1 looks at the situation when the interest rate process is still a log-normal process while model 2 is the situation when the interest rate process is a sum of log - Gaussian processes. In both cases, we use a conditioning factor based argument (similar to the one used in chapter 2) to obtain an approximation to the price of the bonds.

In chapter 6, we first look at the calculation of option prices (European call options) on assets with stochastic volatility. In the first section, we assume that the interest rates are fixed. The rationale of such an assumption - after all it is a "special" case of the problem, is that the mechanism is easier to understand in this case. In the second section of the chapter, we generalise to the situation of stochastic interest rates as well. The model in question here is the Hull and White (1987) model. Furthermore, the price and the volatility processes are stochastic in nature and in general the correlation co-efficient between the volatility process and the logarithm of the price process is assumed to be ρ. The volatility process follows either a Brownian motion (as given by equation (1.4)) or an Ornstein - Uhlenbeck process (as given by equation (1.5)). The objective here is also to find the expected value of the exponential of a log-normal process. The calculations are carried out by making use of the conditioning factor approach similar to Rogers and Shi (1995); the conditioning factor being of the form as described in chapter 3. In this part of the chapter, we also calculate the implied volatilities and comment on them. The second part of this chapter can be regarded in a way, as an extension of the first part. In this part, the results and concepts of the first part of this chapter, where we assume constant interest rates, are extended to the situation when the interest rate itself follows another stochastic process. Thus, we have

$$dX_t = r_t X_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\sqrt{1-\rho^2} dB_t^{(3)} + \rho_2 dB_t^{(2)} + \rho_1 dB_t^{(1)}] \tag{1.13}$$

$$dr_t = -b(r_t - r^*)dt + \phi[\sqrt{1-\gamma^2} dB_t^{(2)} + \gamma dB_t^{(1)}] \tag{1.14}$$

$$dV_t = \mu dt + dB_t^{(1)} \tag{1.15}$$

$$\underline{\text{or}} \quad dV_t = -aV_t dt + dB_t^{(1)} \tag{1.16}$$

As in the case of fixed interest rates defined by equations (1.3 - 1.5), here also, X_t is the price process and V_t is the volatility process. Similar to the situation of constant interest rates, the volatility process V_t follows either a Brownian Motion (equation (1.15)) or an Ornstein

- Uhlenbeck process (equation (1.16)). Further, r_t - the interest rate process, is taken to follow an Ornstein Uhlenbeck process. This ensures that the interest rate does not explode - a possibility if it followed a simple Brownian motion. Here, all the three processes are stochastic in nature and are in general correlated with each other. The calculations here are also based on the use of the conditioning factor approach, similar to the one used in the previous section. In this section, we do not calculate the implied volatilities. This is because of the fact that the picture provided by the implied volatilities is blurred whereas in the case of of constant interest rates, the picture is much clearer. Hence, this is another reason for us to look at the constant interest rate case separately, though it can be treated as a special case of the stochastic interest rate case.

Chapter 7 looks at other applications of the approximation technique used in the previous chapters. Instead of having a stochastic process defining the price of an asset, we have a Poisson process with the parameter λ itself following a stochastic process. Here also, we have a log-normal process and we are interested in the expected value of a function of the log-normal process. The log-Gaussian Cox process has been previously used mainly in the analysis of spatial data by Møller, Syversveen and Waagepetersen (1998) and Rathbun and Cressie (1994). Again, using the same conditioning technique approach, we try to find the price of an option in this set up. Work in this area has been done by Dassios (1987), Duffie (1996), Lando (1998) and Jang(1998) among others. In fact the idea of the doubly stochastic Poisson process - also known as the Cox process was the outcome of work in the related area by Cox (1955).

The first section of chapter 8 looks at an alternative way to value an Asian option. It is an alternative approach to the work of Rogers and Shi (1995). Here, we avoid the numerical integrations used by Rogers and Shi and replace them with an expansion of an exponential term and then look at exact integrals, a technique that can simplify calculations considerably. However, similar to the Rogers and Shi approach, we do make use of the conditioning factor approach as well. The second section of this chapter looks at the calculation of the prices of zero coupon bonds and contingent payments on them. Here also, we make use of the conditioning factor, as in chapter 2, but unlike chapter 2, we do not use any numerical integration. In fact, similar to the first section of this chapter, we replace all numerical integrations with an expansion of an exponential term and then calculate some integrals exactly. In both cases, this method without any numerical integration is very fast and easy to use.

Finally, we conclude by identifying a few problems where the technique discussed here could be put to use. We think these could be explored as future research areas and leave these as open problems.

The numerical results supporting our claims in each of the chapters are given in the form of tables, with self explanatory titles, at the end at the end of each of the chapters. The various program codes used are attached as an appendix at the end of the thesis. All these program codes are in Splus, although to obtain some of the codes, especially for expansion purposes, the algebraic package MAPLE has been used.

Chapter 2

Interest Rate Modelling and Bond Pricing : Zero Coupon Bonds

2.1 Introduction

Interest rate modelling is an interesting topic in mathematical finance. The fact that prices of bonds are dependent on the interest rate makes it more important to the finance industry. Now, a *bond*, is a certificate issued by the government or an institution promising to repay borrowed money at a fixed rate of interest at a fixed time in the future. With volatility in the market playing a very important role, the correct modelling of interest rates is of prime importance. In this chapter, we look at the pricing of zero coupon bonds. Zero coupon bonds are bonds which make only one payment - the payment is made at the end of the term of the bond. Generally, the time periods of the zero coupon bonds are far smaller than the bonds which make interim payments.

We adopt a log-normal model for interest rates. This is similar to the approach of Goldys, Musiela and Sondermann (1994), Sandermann, Sondermann and Miltersen (1994) and Brace, Gatarek and Musiela (1997). The log-normal model ensures that the interest rates cannot go negative. Negative interest rates are not practical and they have undesired consequences as explained by Rogers (1995).

Let the instantaneous rate of interest r_t be given by

$$r_t = be^{\mu_t + Y_t}$$

where Y_t is a Gaussian process with zero mean and a variance - covariance

$$\text{Cov}(Y_u, Y_v) = \sigma_{uv};$$

μ_t is the drift of Y_t and is deterministic in nature. Also, b is a scaling factor whose importance will become apparent in the next section. This can be put in the framework of the work set out by Heath, Jarrow and Morton (1992) as shown by Baxter and Rennie (1996) and is also an extension of the work by Black and Karasinski (1991), as well as Black, Derman and Toy (1990). In fact, Black and Karasinski as well as Black, Derman and Toy have used a log-normal model as has been used here. Black and Karasinski as well as Black, Derman and Toy have used a binomial tree approach to calculate the prices. A similar approach is used by Hull and White (1990) who essentially use a trinomial tree. The Hull and White model is quite similar to the Vasicek (1977) model. In the Hull and White model as in our case, the interest rate cannot become negative; this is a drawback of the Vasicek model.

In this chapter, we look into the problem of calculating bounds for the price of the zero coupon bond, in two ways - first by the use of a suitable conditioning factor as in Rogers and Shi (1995) and also alternatively by direct expansion. Further, we also look at the pricing of contingent payments on the paths of the interest rate for various strike prices which is the same as the contingent payments on the interest rate itself. A more general problem is the calculation of

$$E(f(\int_0^1 (Y_s + \mu_s)ds)),$$

where, f is a convex function. Thus, in particular the price of the bond ($f(x) = e^{-bx}$) is given by

$$E(e^{-b\int_0^1 e^{Y_s+\mu_s} ds}). \tag{2.1}$$

Now, the value of "contingent" payment on the interest rate is the shortfall between *1* and the amount accumulated by an initial investment c and is given as

$$E(e^{-\int_0^1 r_s ds}(1 - ce^{\int_0^1 r_s ds})^+) = E(e^{-\int_0^1 r_s ds} - c)^+.$$

Here c is treated as the strike price at which the contingent payment is calculated and we thus have $f(x) = (e^{-bx} - c)^+$. Hence, using the expression of the interest rate r_t as defined earlier, we have the value of the contingent payment, for a given strike price c, given as

$$E(e^{-b\int_0^1 e^{Y_s+\mu_s}ds} - c)^+. \tag{2.2}$$

To calculate the bounds of the price of a zero coupon we use a conditioning factor to obtain the values of the bounds and obtain both the lower and upper bounds explicitly.

To compare the results that we obtain by using the conditioning factor, we calculate the bounds to the price of a zero coupon bond directly - the method is explained in detail later in the chapter.

The conditioning factor technique that we use is similar to the one suggested by Rogers and Shi (1995) in valuing an Asian option. As stated earlier, we are in general interested in calculating the value of $E(f(X))$, where the function f is convex. Thus, making use of a suitable conditioning factor Z, we have

$$E(f(X)) = E(E(f(X)|Z)) \geq E(f(E(X|Z))).$$

The first part on the above statement is obvious; the second part being nothing but Jensen's inequality. Thus, in this way, we can obtain a lower bound of the price of the bond or the value of the contingent payment on the price of the bond. For the case of the zero coupon bonds, the upper bound to the price can be easily obtained as shown later. The conditioning factor we use for finding the bounds of the zero coupon bonds as well as the value of the contingent payments on it is given by

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}}.$$

Here Z follows an standard normal distribution. This is similar to the conditioning factor used by Rogers and Shi (1995). A detailed explanation for the choice of the conditioning factor is given in the next chapter.

2.2 Calculations using conditioning

As mentioned in the previous section, we condition on Z. Conditionally on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{2.3}$$

$$\text{where} \quad k_u = \text{Cov}(Y_u, Z) = \frac{\int_0^1 \text{Cov}(Y_u, Y_s)ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}} \tag{2.4}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \sigma_{uv} - k_u k_v = w_{uv} \quad \text{say}. \tag{2.5}$$

We are interested in calculating a lower bound (LB_1 in tables 1,2 and 3) and the corresponding upper bound (UB_1 in tables 1,2 and 3). We do that by considering the following argument. There exists some random variable ξ such that

$$E(f(X)) = E[f(E(X|Z))] + E[(X - E(X|Z))f'(E(X|Z))] + \frac{1}{2}E[(X - E(X|Z))^2 f''(\xi)],$$

$$\text{so,} \quad E(f(X)) = f(E(X|Z)) + \frac{1}{2}E[(X - E(X|Z))^2 f''(\xi)],$$

$$\Rightarrow E[f(E(X|Z))] \leq E(f(X)) \leq E[f(E(X|Z))] + \frac{1}{2}E(X - E(X|Z))^2 \sup_{x \geq 0} f''(x).$$

Thus, in the case where f(x) $= e^{-bx}$, a lower bound is given by

$$\text{LB}_1 = E[f(E(X|Z))] \tag{2.6}$$

and an upper bound is given by

$$\text{UB}_1 = \text{LB}_1 + \frac{1}{2}b^2 \text{E}(\text{Var}(X|Z)), \tag{2.7}$$

since $\sup_{x \geq 0} f''(x) = b^2$. Also, here $X = \int_0^1 e^{Y_s + \mu_s} ds$. Thus,

$$E[\text{Var}(\int_0^1 e^{Y_s+\mu_s} ds|Z)] = E[E(\int_0^1 \int_0^1 e^{Y_u} e^{Y_v} dudv|Z) - (E(\int_0^1 e^{Y_s} ds|Z))^2]$$

$$= \int_0^1 \int_0^1 \exp\left(\frac{1}{2}[k_u + k_v]^2 + \frac{1}{2}[w_{uu} + w_{vv}]\right)(e^{w_{uv}} - 1)\, dudv. \tag{2.8}$$

Let us define

$$h(z) = E(\int_0^1 e^{Y_s+\mu_s} ds | Z = z) = \int_0^1 e^{k_u z + \frac{1}{2} w_{uu}} du. \tag{2.9}$$

In the case of calculating the price of the bond,

$$E[e^{-b\int_0^1 e^{Y_s+\mu_s ds}}],$$

the lower bound is given by,

$$\text{LB}_1 = \int_{-\infty}^{\infty} e^{-bh(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{2.10}$$

and the corresponding upper bound is given by

$$\text{UB}_1 = \int_{-\infty}^{\infty} e^{-bh(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \frac{b^2}{2} \int_0^1 \int_0^1 \exp\left(\frac{1}{2}[k_u + k_v]^2 + \frac{1}{2}[w_{uu} + w_{vv}]\right) (e^{w_{uv}} - 1)\, dv du. \tag{2.11}$$

In the case of calculating the price of a contingent payment on the interest rate, we are interested in calculating

$$E[e^{-b\int_0^1 e^{Y_s+\mu_s} ds} - c]^+,$$

where c is the strike price at which the contingent payment is calculated. The lower bound is given by

$$\int_{-\infty}^{\infty} [e^{-bh(z)} - c]^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \tag{2.12}$$

We just present the lower bounds in this case (see Tables 4 - 6) as also the corresponding simulated values. We can employ a combination of the argument used above in the calculation of upper bounds for bond prices and a similar idea due to Rogers and Shi to calculate the upper bounds for the price of the contingent payment on the interest rate. But, as the calculated lower bounds are close to the simulated values, this was not deemed necessary.

To calculate $\int_{-\infty}^{\infty} h(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$, or anything similar, we make use of a numerical integration procedure. A transformation is used in the integration as it improves accuracy.

Alternatively, we can also look at it by first expanding $e^{k_u Z + \frac{1}{2} w_{uu}}$ in terms of σ and then obtaining a polynomial in σ and Z using an algebraic manipulation program such as MAPLE.

We can then calculate the value of the polynomial for various values of σ and finally perform one numerical integration to find out the final expectation. This method works very well for small values of σ. Moreover, when we actually applied it, results for larger values of σ were also very close to the results obtained by direct numerical integration. As a matter of fact, if we restrict our expansion to the 4^{th} power of σ, then we do not even need to use the numerical integration procedure - we can replace it with a set of exact integrals. This technique is explained in detail later in the thesis - in chapter 8, section 3.

2.2.1 Examples

In the following examples, we present the exact form of the bounds to the price of the bond as well as the value of the contingent payment for two special cases; first the Geometric Brownian Motion and then an exponential function of an Ornstein - Uhlenbeck process. Again, for the Ornstein - Uhlenbeck process, we look at two situations - first when the initial value is known and second when the initial value has a stationary distribution.

The Simple Brownian Motion case

In this case, we have,

$$r_t = be^{at+Y_t}.$$

Here,

$$Y_t = \sigma B_t, \tag{2.13}$$

where B_t is a standard Brownian motion, t = *1* and $b = r_0$ is the initial value of the interest rate. The one year bond price is

$$E\left[\exp\left(-b\int_0^1 \exp\{\sigma B_s + as\}ds\right)\right]. \tag{2.14}$$

The conditioning factor is $Z = \frac{\int_0^1 B_s ds}{\sqrt{\text{Var}(\int_0^1 B_s ds)}}$, where B_s is a standard Brownian Motion. Here, $\sigma_{uv} = \sigma^2(u \wedge v)$, $\Rightarrow \sigma_{uu} = \sigma^2 u$. Also, $\mu_u = au$.

Now,

$$\text{Var}(\int_0^1 B_s ds) = \int_0^1 (1-s)^2 ds = \frac{1}{3}.$$

Thus, we have, in this case

$$k_u = \text{Cov}(B_u, Z) = \sqrt{3}\sigma \int_0^u (1-s)ds = \sqrt{3}\sigma(u - \frac{u^2}{2}). \tag{2.15}$$

Conditioning on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = \mu u + k_u Z, \tag{2.16}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \sigma^2(u \wedge v) - k_u k_v = w_{uv}. \tag{2.17}$$

Once we have these values, we can then easily calculate the price of the bond by substituting (2.16) and (2.17) and as $\mu_t = at$ in equations (2.10) and (2.11) (results shown in Table 1) and the price of the contingent payment on the interest rate for various strike prices by substituting in equation (2.12) (results shown in Tables 4.1 - 4.3).

Ornstein - Uhlenbeck Process

Now, let us consider the case where the interest rate $\{r_s; 0 \leq s \leq 1\}$ is governed by an exponential function of the Ornstein Uhlenbeck process. First we will look at the situation where the initial value is known and then we will also look at the case where the initial value has a stationary distribution.

Initial value is known

Here the initial value of the process Y_0 is known and is assumed to take the value 0. The interest rate model is thus defined as

$$r_t = be^{Y_t}.$$

Y_t is the solution of the stochastic differential equation

$$dY_t = -aY_t dt + \sigma dB_t, \tag{2.18}$$

$$\text{i.e.} \quad Y_t = \sigma \int_0^t e^{-a(s-u)} dB_u,$$

where B_t is a standard Brownian motion and $t = 1$. Now,

$$r_t = be^{Y_t} = e^{lnb+Y_t}.$$

Thus, $ln\ b$ is the long term mean of the logarithm of the interest rate process. Hence, $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long term value of the interest rate. Also, $b = r_0$, the initial value of the interest rate. In this case, $\sigma_{uv} = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}]$. Further, $\mu_u = 0$. The conditioning factor is $Z = \frac{\int_0^1 Y_s ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}}$.

We thus have,

$$\int_0^1 Y_s ds = \sigma \int_0^1 \frac{1 - e^{-a(1-s)}}{a} dB_s.$$

$$\Rightarrow \text{Var}(\int_0^1 Y_s ds) = \sigma^2 \int_0^1 (\frac{1 - e^{-a(1-u)}}{a})^2 du = \frac{\sigma^2}{2a} \frac{2a + 4e^{-a} - e^{-2a} - 3}{a^2} = V, \quad \text{say} \tag{2.19}$$

and

$$k_u = \text{Cov}(Y_u, Z) = \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \left\{ \int_0^u (e^{a(s-u)} - e^{-a(u+s)}) ds + \int_u^t (e^{a(u-s)} - e^{-a(u+s)}) ds \right\}$$

$$= \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \left\{ \frac{1 - e^{-au}}{a} + \frac{1 - e^{-a(1-u)}}{a} - \frac{e^{-au} - e^{-a(1+u)}}{a} \right\}. \tag{2.20}$$

So, we then have that given Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{2.21}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] - k_u k_v = w_{uv}. \tag{2.22}$$

Once we have these values, we can then easily calculate the price of the bond by substituting equations (2.21) and (2.22) in equations (2.10) and (2.11) (results shown in Table 3) and the price of a contingent payment on the price of the bond for various strike prices by substituting in equation (2.12) (results shown in Table 6).

Initial value has a stationary distribution

The initial value of the process has a stationary distribution, the distribution being $N(0, \frac{\sigma^2}{2a})$. Here, Y_t is the solution of the stochastic differential equation

$$dY_t = -aY_t dt + \sigma dB_t, \tag{2.23}$$

$$\text{i.e.} \quad Y_t = \sigma \int_{-\infty}^{t} e^{-a(s-u)} dB_u,$$

where B_t is a standard Brownian motion and $t = 1$. Now,

$$r_t = be^{Y_t} = e^{lnb+Y_t}.$$

Thus, $ln\, b$ is the long term mean of the logarithm of the interest rate process. Hence, $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long term value of the interest rate. Also, $\sigma_{uv} = \frac{\sigma^2}{2a}e^{-a|u-v|}$ and $\mu_u = 0$. The conditioning factor is $Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}$.

Thus, we have

$$\int_0^1 Y_s ds = \sigma \int_0^1 e^{-as} \int_{-\infty}^{s} e^{au} dB_u ds = \sigma \int_0^1 e^{-as} \int_{-\infty}^{0} e^{au} dB_u ds + \sigma \int_0^1 e^{-as} \int_0^s e^{au} dB_u ds$$

$$= \sigma \frac{1-e^{-a}}{a} \int_{-\infty}^{0} e^{au} dB_u + \sigma \int_0^1 \frac{1-e^{-a(1-u)}}{a} dB_u.$$

$$\Rightarrow \mathrm{Var}(\int_0^1 Y_s ds) = \sigma^2 \left[\left(\frac{1-e^{a}}{a} \right)^2 \int_{-\infty}^{0} e^{2au} du + \int_0^1 \left(\frac{1-e^{-a(1-u)}}{a} \right)^2 du \right]$$

$$= \frac{\sigma^2}{2a} \left(\frac{1-e^{-a}}{a} \right)^2 + \frac{\sigma^2}{2a} \frac{2a+4e^{-a}-e^{-2a}-3}{a^2} = \frac{\sigma^2}{a} \frac{a+e^{-a}-1}{a^2} = V \quad \text{say}, \tag{2.24}$$

and

$$k_u = \mathrm{Cov}(Y_u, Z) = \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \int_0^1 e^{-a|u-s|} ds = \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} [\int_0^u e^{a(s-u)} ds + \int_u^1 e^{a(u-s)} ds$$

$$= \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} [\frac{1-e^{-au}}{a} + \frac{1-e^{-a(1-u)}}{a}]. \tag{2.25}$$

Once again, we have that given Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{2.26}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a} e^{-a|u-v|} - k_u k_v = w_{uv}, \quad \text{say.} \tag{2.27}$$

Once we have these values, we can then easily calculate the price of the bond by substituting equations (2.26) and (2.27) in equation(2.10) and (2.11) (results shown in Table 2) and the price of a contingent payment on the price of the bond for various strike prices by substituting in equation (2.12) (results shown in Table 5).

2.3 Calculation of Upper and Lower Bounds directly

Here, we employ a direct method for finding bounds for the one year bond price. This technique is used to calculate the bounds to the price of the bond, primarily for comparison purposes with the bounds obtained using the conditioning factor. Further, it should be noted that the direct expansion method as discussed here can only be used for the calculation of the bounds to the price of the bond. It cannot be used to calculate the value of the contingent payment on the bond, for which case we have to use a conditioning factor.

In the direct method of calculating the bounds to the price, we use a Taylor series expansion. We use the fact that for $x \geq 0$, $e^{-x} > 1 - x$, $e^{-x} < 1 - x + \frac{x^2}{2}$, $e^{-x} > 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$, and so on. We will use the last two inequalities as the bounds suggested are very close to each other. Here, we have,

$$1 - bI_1 + \frac{1}{2}b^2 I_2 - \frac{1}{6}b^3 I_3 \leq E[e^{-b\int_0^1 e^{Y_s+\mu_s} ds}] \leq 1 - bI_1 + \frac{1}{2}b^2 I_2, \tag{2.28}$$

where,

$$I_1 = E[\int_0^1 e^{Y_s+\mu_s} ds] = \int_0^1 \exp\left(\mu_s + \frac{1}{2}\sigma_{ss}\right) ds,$$

$$I_2 = E[\int_0^1 e^{Y_s+\mu_s} ds]^2 = \int_0^1 \int_0^1 \exp\left(\mu_u + \mu_s + \frac{1}{2}[\sigma_{uu} + \sigma_{ss} + 2\sigma_{su}]\right) dsdu,$$

and $I_3 = E[\int_0^1 e^{Y_s+\mu_s} ds]^3$

$$= \int_0^1 \int_0^1 \int_0^1 \exp\left(\mu_s + \mu_v + \mu_u + \frac{1}{2}\left[\sigma_{uu} + \sigma_{vv} + \sigma_{ss} + 2\sigma_{uv} + 2\sigma_{us} + 2\sigma_{vs}\right]\right) dsdvdu.$$

Thus the lower bound is given by

$$1 - bI_1 + \frac{1}{2}b^2 I_2 - \frac{1}{6}b^3 I_3 \tag{2.29}$$

and the corresponding upper bound is

$$1 - bI_1 + \frac{1}{2}b^2 I_2 \tag{2.30}$$

2.3.1 Examples

We use the same examples as used in the case of calculating the bounds to the price of the bond using a conditioning factor.

The Simple Brownian Motion case

In this case, $\sigma_{ss} = \sigma^2 s$ and $\sigma_{us} = \sigma^2(u \wedge s)$ and also $\mu_s = as$. Thus,

$$I_1 = \int_0^1 \exp\left(as + \frac{1}{2}\sigma^2 s\right) ds,$$

$$I_2 = 2\int_0^1 \int_0^u \exp\left(as + au + \frac{3}{2}\sigma^2 s + \frac{1}{2}\sigma^2 u\right) dsdu,$$

$$I_3 = 6\int_0^1 \int_0^u \int_0^v \exp\left(au + av + as + \frac{1}{2}\sigma^2 u + \frac{1}{2}\sigma^2 v + \frac{1}{2}\sigma^2 s + \sigma^2 v + 2\sigma^2 s\right) dsdvdu.$$

The upper bound is thus given by

$$\text{UB}_2 = 1 - b\int_0^1 \exp\left(as + \frac{1}{2}\sigma^2 s\right) ds + b^2 \int_0^1 \int_0^u \exp\left(as + au + \frac{3}{2}\sigma^2 s + \frac{1}{2}\sigma^2 u\right) dsdu$$

$$= 1 - b\frac{e^{a+\frac{1}{2}\sigma^2} - 1}{a + \frac{1}{2}\sigma^2} + b^2 \frac{1}{a + \frac{3}{2}\sigma^2}\left[\frac{e^{2(a+\sigma^2)} - 1}{2(a + \sigma^2)} - \frac{e^{a+\frac{1}{2}\sigma^2} - 1}{a + \frac{1}{2}\sigma^2}\right], \tag{2.31}$$

and the corresponding lower bound by

$$\mathrm{LB}_2 = \mathrm{UB}_2 - b^3 \int_0^1 \int_0^u \int_0^v \exp\left(au + av + as + \frac{1}{2}\sigma^2 u + \frac{1}{2}\sigma^2 v + \frac{1}{2}\sigma^2 s + \sigma^2 v + 2\sigma^2 s\right) dsdvdu$$

$$= \mathrm{UB}_2 - b^3 \left\{ \frac{e^{3(a+\frac{3}{2}\sigma^2)} - 1}{(a+\frac{5}{2}\sigma^2)(a+2\sigma^2)(a+\frac{3}{2}\sigma^2)} + \frac{3(e^{(a+\frac{1}{2}\sigma^2)} - 1)}{(a+\frac{5}{2}\sigma^2)(a+2\sigma^2)(a+\frac{1}{2}\sigma^2)} \right.$$

$$\left. + \frac{3(e^{2(a+\sigma^2)} - 1)}{(a+\frac{5}{2}\sigma^2)(a+\frac{3}{2}\sigma^2)(a+\sigma^2)} - \frac{6(e^{(a+\frac{1}{2}\sigma^2)} - 1)}{(a+\frac{5}{2}\sigma^2)(a+\frac{3}{2}\sigma^2)(a+\frac{1}{2}\sigma^2)} \right\}. \qquad (2.32)$$

Calculations are given in Table 1.

The Ornstein - Uhlenbeck Case

Initial value follows a stationary distribution

Now, for the stationary case, $\mu_s = 0$, $\mathrm{Var}(Y_s) = \frac{\sigma^2}{2a} = \sigma_{ss}$, and $\mathrm{Cov}(Y_u, Y_v) = \frac{\sigma^2}{2a} e^{-a|u-v|} = \sigma_{uv}$. Thus

$$I_1 = e^{\frac{1}{2}\frac{\sigma^2}{2a}},$$

$$I_2 = \int_0^1 \int_0^1 e^{\frac{1}{2}\frac{\sigma^2}{2a} + \frac{1}{2}\frac{\sigma^2}{2a} + \sigma_{uv}} dvdu = 2\int_0^1 \int_0^u e^{(\frac{\sigma^2}{2a} + \frac{\sigma^2}{2a} e^{-a(u-v)})} dvdu$$

$$= 2e^{\frac{\sigma^2}{2a}} \int_0^1 (1-w) e^{\frac{\sigma^2}{2a} e^{-aw}} dw,$$

$$I_3 = e^{\frac{3}{2}\frac{\sigma^2}{2a}} \int_0^1 \int_0^1 \int_0^1 e^{\sigma_{uv} + \sigma_{us} + \sigma_{vs}}$$

$$= 6e^{\frac{3}{2}\frac{\sigma^2}{2a}} \int_0^1 \int_0^u \int_0^v e^{\frac{\sigma^2}{2a}(e^{-a(u-s)} + e^{-a(u-v)} + e^{-a(v-s)})} dsdvdu.$$

and by using a suitable transformation and changing the order of integration,

$$I_3 = 6e^{\frac{3}{2}\frac{\sigma^2}{2a}} \int_0^1 (1-r) \int_0^r \exp\left(\frac{\sigma^2}{2a}\left[e^{-ar} + e^{-aw} + e^{-a(r-w)}\right]\right) dwdr.$$

Thus the upper bound is given by

$$\mathrm{UB}_2 = 1 - be^{\frac{1}{2}\frac{\sigma^2}{2a}} + b^2 e^{\frac{\sigma^2}{2a}} \int_0^1 (1-w) e^{\frac{\sigma^2}{2a} e^{-aw}} dw \tag{2.33}$$

and the corresponding lower bound is given by

$$\mathrm{LB}_2 = \mathrm{UB}_2 - b^3 e^{\frac{3}{2}\frac{\sigma^2}{2a}} \int_0^1 (1-r) \int_0^r \exp\left(\frac{\sigma^2}{2a}\left[e^{-ar} + e^{-aw} + e^{-a(r-w)}\right]\right) dw dr. \tag{2.34}$$

The results are presented in Table 2.

Initial value is known

For the non-stationary Ornstein - Uhlenbeck case, $\mu_s = 0$, $\mathrm{Var}(Y_s) = \frac{\sigma^2}{2a}(1 - e^{-2as}) = \sigma_{ss}$ and $\mathrm{Cov}(Y_u, Y_v) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] = \sigma_{uv}$. So,

$$I_1 = \int_0^1 e^{\frac{1}{2}\sigma^2 \frac{1-e^{-2au}}{2a}} du,$$

$$I_2 = \int_0^1 \int_0^1 e^{\frac{1}{2}\sigma^2\frac{1-e^{-2au}}{2a} + \frac{1}{2}\sigma^2\frac{1-e^{-2av}}{2a} + \sigma^2\frac{e^{a|u-v|} - e^{-a(u+v)}}{2a}} dv du$$

$$= 2\int_0^1 \int_0^u e^{\frac{1}{2}\sigma^2\frac{1-e^{-2au}}{2a} + \frac{1}{2}\sigma^2\frac{1-e^{-2av}}{2a} + \sigma^2\frac{e^{a(u-v)} - e^{-a(u+v)}}{2a}} dv du,$$

$$I_3 = \int_0^1 \int_0^1 \int_0^1 e^{\frac{1}{2}(\sigma_{ss}+\sigma_{vv}+\sigma_{uu}) + \sigma_{uv} + \sigma_{us} + \sigma_{vs}} ds dv du.$$

Thus the upper bound is given by

$$\mathrm{UB}_2 = 1 - b\int_0^1 \exp\left(\frac{1}{2}\sigma^2 \frac{1-e^{-2au}}{2a}\right) du$$

$$+ b^2 \int_0^1 \int_0^u \exp\left(\frac{1}{2}\sigma^2\frac{1-e^{-2au}}{2a} + \frac{1}{2}\sigma^2\frac{1-e^{-2av}}{2a} + \sigma^2 \frac{e^{a(u-v)} - e^{-a(u+v)}}{2a}\right) dv du, \tag{2.35}$$

and the corresponding lower bound is given by

$$\mathrm{LB}_2 = \mathrm{UB}_2 - b^3 \int_0^1 \int_0^1 \int_0^1 \exp\left(\frac{1}{2}\left[\sigma_{ss} + \sigma_{vv} + \sigma_{uu}\right] + \sigma_{uv} + \sigma_{us} + \sigma_{vs}\right) ds dv du. \tag{2.36}$$

The results are tabulated in Table 3.

Note :

Alternatively, we could make use of the following recursion relation to calculate I_1, I_2 and I_3 and thereby the bonds; especially for the case of the Non-stationary Ornstein - Uhlenbeck process. We have

$$dY_t = -aY_t dt + \sigma dB_t$$

and we define

$$Z_t = \int_0^t e^{Y_s} ds.$$

Further, let us also define

$$f(t, y, z) = Z_t^m e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha}e^{2\alpha t}}. \tag{2.37}$$

Let us apply Itô calculus on equation (2.37) to obtain $E(f(t, y, z))$. For that, we first need to calculate the relevant derivatives. They are

$$\frac{\partial f}{\partial t} = Z_t^m e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha}e^{2\alpha t}} (Y_t \gamma \alpha e^{\alpha t} - \frac{1}{2}\sigma^2\gamma^2 e^{2\alpha t}),$$

$$\frac{\partial f}{\partial y} = Z_t^m e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha}e^{2\alpha t}} (\gamma e^{\alpha t}),$$

$$\frac{\partial^2 f}{\partial y^2} = Z_t^m e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha}e^{2\alpha t}} (\gamma e^{\alpha t})^2$$

and

$$\frac{\partial f}{\partial z} = mZ_t^{m-1} e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha}e^{2\alpha t}}.$$

Thus, we have, using Ito's lemma,

$$\partial f = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2 + \frac{\partial f}{\partial}dZ_t$$

$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}(-\alpha Y_t dt + \sigma dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\sigma^2 dt + \frac{\partial f}{\partial z}dZ_t$$

$$\Rightarrow E(f(t, Y_t, Z_t)) = E\left[Z_t^m e^{Y_t \gamma e^{\alpha t} - \frac{1}{2}\frac{\sigma^2 \gamma^2}{2\alpha} e^{2\alpha t}}\right] = m \int_0^t E\left[Z_s^{m-1} e^{Y_s(\gamma e^{\alpha s}+1) - \frac{1}{2}\frac{\sigma^2\gamma^2}{2\alpha} e^{2\alpha s}}\right] ds.$$

Setting $\gamma = ke^{-\alpha t}$, we have

$$H_m(k,t) = E(Z_t^m e^{kY_t}) = m \int_0^t H_{m-1}(ke^{-\alpha(t-s)} + 1, s) e^{\frac{1}{2}\frac{\sigma^2}{2\alpha} k^2 (1-e^{-2\alpha(t-s)})} ds.$$

From this recursive relation, we can get $I_1 = H_1(0,1)$, $I_2 = H_2(0,1)$ and $I_3 = H_3(0,1)$. Thus the lower and upper bounds can be given as

$$\mathrm{UB}_2 = 1 - bH_1(0,1) + \frac{b^2}{2} H_2(0,1)$$

$$\mathrm{LB}_2 = 1 - bH_1(0,1) + \frac{b^2}{2} H_2(0,1) - \frac{b^3}{6} H_3(0,1).$$

The recursive procedure thus described is particularly useful in the calculation of triple integrals, and even higher order integrals, of the form as in equation (2.36).

It should be noted that the expansion technique described in this section is not guaranteed to work. Indeed, the expansion used might diverge making it impossible to improve accuracy by calculating more terms. For example, in the case of the Brownian motion we observed that the model completely breaks down for $\sigma \geq 1.5$. For the Ornstein Uhlenbeck case, the expansion starts to diverge for $\sigma \leq 1.5$. In our case, the method works because we mostly (but not always) consider small values of σ (σ ranges between 0.1 and 1).

2.4 Conclusion and Remarks

The lower bounds to the price of the bonds or even the approximation to the prices (in case of the contingent payments) calculated by using the conditioning factor seem to be so close to the actual price (in some cases, the simulated prices were lower than the lower bounds) that they can be regarded as a very good approximation to the true value. This is true for both the situations of a Brownian motion or an Ornstein - Uhlenbeck process being used as the driving force of the stochastic process.

One advantage of using a conditioning factor in the calculation of the one year bond prices is that the method works even for large values of σ. This is not the case when using the direct expansion method; here, for higher values of σ, the values start diverging quite fast, thereby causing the whole system to break down. Also, as shown earlier, the method using conditioning factors can be easily modified to calculate the value of a contingent payment defined on the price of a bond. Further, it is not possible to calculate the contingent payment on the interest rate of a bond using the direct expansion technique.

2.5 Tables

Next, we present a set of tables outlining the numerical results based on the formulae stated earlier in the chapter. Tables 1, 2 and 3 show the values of the upper and lower bounds of the price of a zero coupon bond. In these 3 tables, LB_1 and UB_1 refer to the values of the bond calculated using the conditioning factor while LB_2 and UB_2 refer to the directly calculated bounds.

The set of tables 4 - 6 show the values of the contingent payments on the interest rate for various strike prices, c. In all cases, the stochastic process governing the interest rate process is given in the title of the table. For comparison purposes, in these tables, the simulated values along with the standard errors of simulation are also presented.

In all cases, all the prices are multiplied by 100.

Table 1 : The interest rate follows a geometric Brownian motion.

a	σ	LB1	UB1	LB2	UB2
-0.5	0.1	94.657	94.657	94.629	94.636
	0.5	94.368	94.374	94.342	94.347
	0.75	93.965	93.979	93.943	93.951
	1	93.35	93.375	93.334	93.352
-0.2	0.1	93.86	93.86	93.839	93.843
	0.5	93.514	93.52	93.497	93.503
	0.75	93.034	93.047	93.021	93.033
	1	92.303	92.328	92.297	92.328
0	0.1	93.239	93.239	93.224	93.23
	0.5	92.849	92.855	92.838	92.847
	0.75	92.308	92.322	92.303	92.32
	1	91.49	91.514	91.491	91.538
0.2	0.1	92.534	92.534	92.526	92.534
	0.5	92.094	92.1	92.091	92.104
	0.75	91.486	91.5	91.489	91.513
	1	90.57	90.595	90.581	90.649
0.5	0.1	91.291	91.291	91.297	91.31
	0.5	90.765	90.771	90.777	90.798
	0.75	90.041	90.055	90.061	90.102
	1	88.962	88.986	88.987	89.11

Table 2 : The interest rate follows an exponential function of a stationary Ornstein - Uhlenbeck process with a = *1*.

σ	LB1	UB1	LB2	UB2
0.1	93.239	93.25	93.223	93.223
0.5	92.859	92.898	92.844	92.853
0.75	92.342	92.382	92.326	92.343
1	91.576	91.608	91.561	91.597

Table 3 : The interest rate follows an exponential function of a non-stationary Ornstein - Uhlenbeck process with a = *1*.

σ	LB1	UB1	LB2	UB2
0.1	93.245	93.246	92.227	93.233
0.5	93.029	93.031	92.939	92.948
0.75	92.736	92.749	92.557	92.575
1	92.308	92.331	92.001	92.043

Note : In some cases in tables 1,2 and 3, lower bounds calculated using one approach are slightly higher than the upper bounds calculated by the other method. This is due to small inaccuracies in the numerical integration procedures and indicates how close they are to the actual price.

Table 4.1 : The interest rate follows a geometric Brownian Motion with no drift.

σ	c	Calculated	Simulated	S.E.
0.1	93.1	0.223	0.222	0.00113
	93	0.292	0.290	0.00126
	92.9	0.368	0.367	0.00138
	92.8	0.452	0.4512	0.00147
	92.7	0.541	0.540	0.00154
	92.6	0.634	0.633	0.00159
0.5	93.5	0.475	0.470	0.00329
	93	0.715	0.71	0.0029
	92.5	1.005	1.0001	0.00489
	92	1.34	1.336	0.00397
	91.5	1.711	1.708	0.00627
	91	2.114	2.111	0.00483
0.75	93	0.913	0.905	0.00533
	92.5	1.183	1.175	0.00616
	92	1.484	1.475	0.00696
	91.5	1.812	1.803	0.00773
	91	2.164	2.154	0.00845
	90.5	2.537	2.527	0.00913
1	94	0.633	0.632	0.00461
	93	1.069	1.071	0.00632
	92	1.607	1.613	0.00801
	91	2.230	2.240	0.00963
	90	2.923	2.936	0.01113
	89	3.673	3.691	0.01251

Table 4.2 : The interest rate follows a geometric Brownian Motion with a drift of -0.5.

σ	c	Calculated	Simulated	S.E.
0.1	95	0.012	0.011	0.0002
	94.8	0.05	0.046	0.00045
	94.6	0.136	0.129	0.00076
	94.4	0.275	0.265	0.00228
	94.2	0.451	0.44	0.00263
	94	0.644	0.633	0.00277
0.5	95	0.295	0.288	0.00221
	94.5	0.522	0.512	0.00303
	94	0.816	0.805	0.00381
	93.5	1.168	1.156	0.0045
	93	1.564	1.552	0.00509
	92.5	1.993	1.981	0.00555
0.75	95	0.444	0.44	0.00313
	94	0.944	0.94	0.0048
	93	1.604	1.601	0.00631
	92	2.379	2.377	0.00757
	91	3.232	3.231	0.00858
	90	4.136	4.135	0.00935
1	95	0.564	0.56	0.00385
	94	1.049	1.043	0.00556
	93	1.656	1.651	0.00719
	92	2.357	2.354	0.00867
	91	3.131	3.129	0.00998
	90	3.958	3.959	0.01112

Table 4.3 : The interest rate follows a geometric Brownian Motion with a drift of 0.5.

σ	c	Calculated	Simulated	S.E.
0.1	91.6	0.079	0.083	0.00081
	91.4	0.147	0.154	0.00112
	91.2	0.246	0.256	0.00143
	91	0.375	0.389	0.0017
	90.8	0.529	0.546	0.00192
	90.6	0.703	0.721	0.00207
0.5	93	0.209	0.209	0.00234
	92	0.5	0.499	0.00387
	91	0.951	0.95	0.00551
	90	1.55	1.549	0.00707
	89	2.271	2.27	0.00842
	88	3.083	3.084	0.00952
0.75	93	0.423	0.425	0.00385
	92	0.766	0.796	0.00548
	91	1.218	1.223	0.00717
	90	1.771	1.778	0.00883
	89	2.41	2.421	0.01037
	88	3.121	3.137	0.01179
1	92	0.978	0.979	0.0067
	91	1.428	1.431	0.00843
	90	1.952	1.96	0.01014
	89	2.543	2.555	0.0118
	88	3.192	3.207	0.01338
	87	3.89	3.911	0.01487

Table 5 : The interest rate follows an exponential function of a stationary Ornstein - Uhlenbeck process with $a = 1$.

σ	c	Calculated	Simulated	S.E.
0.5	95	0.093	0.093	0.00134
	94	0.32	0.32	0.00275
	93	0.745	0.747	0.00437
	92	1.362	1.367	0.00589
	91	2.127	2.133	0.00711
	90	2.994	2.999	0.00801
0.75	95	0.232	0.234	0.00251
	94	0.524	0.528	0.00407
	93	0.955	0.962	0.00574
	92	1.517	1.525	0.00737
	91	2.187	2.197	0.00885
	90	2.944	2.956	0.02269
1	94	0.69	0.697	0.00512
	93	1.12	1.128	0.00687
	92	1.647	1.657	0.00852
	91	2.259	2.271	0.01012
	90	2.942	2.957	0.01163
	89	3.684	3.702	0.013

Table 6 : The interest rate follows an exponential function of a non-stationary Ornstein - Uhlenbeck process with $a = 1$.

σ	c	Calculated	Simulated	S.E.
0.5	95	0.02	0.02	0.00005
	94	0.16	0.159	0.00163
	93	0.56	0.559	0.0032
	92	1.235	1.234	0.0046
	91	2.093	2.091	0.00549
	90	3.04	3.038	0.00595
0.75	95	0.082	0.082	0.00122
	94	0.299	0.2979	0.002619
	93	0.713	0.709	0.00424
	92	1.315	1.312	0.00579
	91	2.065	2.062	0.00707
	90	2.916	2.913	0.00803
1	95	0.16	0.161	0.00192
	94	0.421	0.423	0.00343
	93	0.837	0.840	0.0051
	92	1.395	1.400	0.00673
	91	2.072	2.077	0.00821
	90	2.84	2.844	0.00949

Chapter 3

Choice of an appropriate conditioning factor

3.1 Introduction

In the previous chapter, we have used a conditioning factor argument to price zero coupon bonds and contingent payments on the interest rate. The motivation of using the conditioning factor approach was derived from the use of a similar technique by Rogers and Shi (1995) to value an Asian option (as discussed in chapter 1, section 2). Rogers and Shi have not given any mathematical justification for the choice of the conditioning factor - they just state that they tried a number of conditioning factors and the one used by them was found to perform the best. The objective of this chapter is to obtain an appropriate conditioning factor. Our aim is to provide a mathematical justification to the conditioning factor used - both of the one used by us in the previous chapter as well as the one used by Rogers and Shi. We also try to find a general form of the conditioning factor for a general Gaussian distribution.

In the previous chapter, when we looked at pricing of bonds and contingent payments, we were interested in calculating

$$E(e^{-bX} - K)^+ = E(\max[e^{-bX} - K, 0]) \tag{3.1}$$

where K was the strike price, b was a constant and X was the random variable.

To find the price of the bond, we took $K = 0$ while for the value of the contingent payment, we let K take the various values of the strike price at which the contingent payment was calculated.

Let us assume K such that $K = 1 - ab$, where $a \geq 0$ is a constant. Also, ab is of order b and b is small. We shall now prove the following *Lemma.*

Lemma 3.1 : *Assume that b is small and that X is a non-negative continuous random variable. Further assume that $g(x)$, the density function of X, is bounded and the second moment of X exists. Then, for b small enough, we have*

$$E\left[(e^{-bX} - (1-ab))^{+} - (1 - bX - (1-ab))^{+}\right] \leq b^2 C,$$

where C is a constant.

Proof : Now,

$$E\left[(e^{-bX} - (1-ab))^{+} - (1 - bX - (1-ab))^{+}\right]$$

$$= E(e^{-bX} - (1-ab))^{+} - E(1 - bX - (1-ab))^{+}$$

$$= \int_0^{\frac{-ln(1-ab)}{b}} (e^{-bx} - (1-ab))g(x)dx - \int_0^{a} (1 - bx - (1-ab))g(x)dx$$

$$= \int_0^{a} [e^{-bx} - (1-bx)]g(x)dx + \int_a^{\frac{-ln(1-ab)}{b}} [e^{-bx} - (1-ab)]g(x)dx$$

$$\leq \int_0^{a} \frac{b^2x^2}{2} g(x)dx - b\int_a^{\frac{-ln(1-ab)}{b}} (x-a)g(x)dx + \int_a^{\frac{-ln(1-ab)}{b}} \frac{b^2x^2}{2} g(x)dx.$$

Now, we know that the $g(x)$ is bounded and the second moment of X exists. Thus,

$$\int_0^{a} \frac{b^2x^2}{2} g(x)dx \leq b^2 C_1 \quad \text{and} \quad \int_a^{\frac{-ln(1-ab)}{b}} \frac{b^2x^2}{2} g(x)dx \leq b^2 C_3.$$

Also, for the term $b\int_a^{\frac{-ln(1-ab)}{b}} (x-a)g(x)dx$, the limits of the integral are very close to each other - in fact, the range of integration is

$$[-\frac{1}{b}(-ab - a^2b^2 - ....) - a] = b(1 + a^2 b + a^3 b^2 + ...) = O(b).$$

Further, since b is small, we have

$$b \int_{a}^{\frac{-ln(1-ab)}{b}} (x-a)g(x)dx \leq b.bC_2 = b^2C_2$$

i.e. this term is also of order b^2. Here, C_1, C_2 and C_3 are constants such that $C = C_1 + C_2 + C_3$.

Hence the result.

■

Now, using *Lemma 3.1* and assuming b to be small such that we can ignore terms of order b^2 and higher, equation (3.1) can be approximated by

$$E(1 - bx - K)^+ = E[\max[(1 - bx - K), 0]]. \tag{3.2}$$

Thus, to find the price of the bond, we calculate the expression given by equation (3.2) as that would give us an approximation to the price of the bond (for small b, the approximation is very accurate). Throughout the course of this chapter, we shall take

$$X = \int_0^1 e^{\sigma Y_s} ds \tag{3.3}$$

where $\{Y_s, 0 \leq s \leq 1\}$ is a stochastic process and σ is the instantaneous variance of the process Y_s.

Let us define

$$f(U) = [U - K]^+ = \max([U - K], 0) \tag{3.4}$$

where U is a random variable and K is a constant - in the case of pricing of contingent payments on the interest rate, K is the strike price at which the contingent payment is calculated. Note that f is convex. In general, we are interested in finding

$$E(f(U)].$$

Thus, in the case of pricing of bonds,

$$U = 1 - bX$$

where b is a constant whereas for the Rogers and Shi problem of valuing of Asian options,

$$U = X.$$

In both cases X is as defined in equation (3.3).

Using the fact that the unconditional expectation is the expected value of the conditional expectation and also Jensen's Inequality, we have

$$E[f(U)] = E[E\{f(U)|Z\}] \geq E[f(E\{U|Z\})] \tag{3.5}$$

where, Z is another suitably normalised random variable used for conditioning purposes.

The lower bound in the equation (3.5) is not guaranteed to be good. However, an estimate the error made using the following argument. For any random variable U, we have,

$$0 \leq E(U^+) - E(U)^+$$

$$= \frac{1}{2}(E(|U|) - |E(U)|)$$

$$\leq \frac{1}{2}E(|U - E(U)|)$$

$$\leq \frac{1}{2}\sqrt{\mathrm{Var}(U)}.$$

This implies that for the Rogers and Shi case, we have

$$0 \leq E\left[E([X-K]^+|Z) - E([X-K]|Z)^+\right] \leq \frac{1}{2}E\left[\sqrt{\mathrm{Var}([X-K]|Z)}\right]. \tag{3.6}$$

Further, using Cauchy - Schwarz inequality, we have from equation (3.6)

$$\frac{1}{2}E\left[\sqrt{\mathrm{Var}([X-K]|Z)}\right] \leq \frac{1}{2}\sqrt{E\left[\mathrm{Var}([X-K]|Z)\right]} = \frac{1}{2}\sqrt{E\left[\mathrm{Var}(X|Z)\right]}. \tag{3.7}$$

Similarly, for the problem of pricing of bonds and contingent payments on the interest rate (as discussed in the previous chapter),

$$0 \leq E\left[E([1-bX-K]^+|Z) - E([1-bX-K]|Z)^+\right]$$

$$\leq \frac{1}{2} E\left[\sqrt{\operatorname{Var}([1-bX-K] | Z)}\right]. \tag{3.8}$$

Again, using Cauchy - Schwarz inequality, we have from equation (3.8)

$$\frac{1}{2} E\left[\sqrt{\operatorname{Var}([1-bX-K] | Z)}\right] \leq \frac{1}{2} \sqrt{E\left[\operatorname{Var}([1-bX-K] | Z)\right]} = \frac{1}{2} \sqrt{b^2 E\left[\operatorname{Var}(X | Z)\right]}. \tag{3.9}$$

Thus, in order to minimise the error made by using the lower bound as an approximation to the true value as given in equation (3.5), we try to choose the conditioning factor Z such that

$$E\left[\operatorname{Var}(X | Z)\right] \tag{3.10}$$

is minimised.

This is true of both situation - the problem of pricing of bonds and contingent payments on the interest rate (equation 3.9) as well as the Rogers and Shi (1995) problem of valuing Asian options (equation 3.7). For pricing of European call options on assets with stochastic volatility, the situation is much more complicated. We propose to use a similar argument and minimise the same quantity as defined in equation (3.10).

In the following two sections, we look at the exact form of the conditioning factor that minimises the expected value of the conditional variance. We look at a general Gaussian process and try to obtain the conditioning factor that minimises the expected conditional variance. We look at two cases; first we look at the case when the Gaussian process is driven by just one stochastic process - this is what we call the *Single Driver* case and the the situation when the Gaussian process is driven by a linear combination of n stochastic processes - this is what we call the *Multi Driver* case. In both cases, for the Gaussian process following specific processes, the explicit form of the conditioning factor are shown as examples.

3.2 Single driver Model

Let $\{Y_s, 0 \leq s \leq 1\}$ be a general Gaussian process, where $Y_s = \int_{-\infty}^{\infty} L(s,u) dB_u$ subject to the constraint $\sup_s (\int_{-\infty}^{\infty} L^2(s,u) du \leq \infty)$. Also, let the conditioning variable, in general, be

Z, where

$$Z = \int_{-\infty}^{\infty} a(u) dB_u, \tag{3.11}$$

a(•) is so chosen that it satisfies the condition $\int_{-\infty}^{\infty} a^2(u)du = 1$. This condition ensures that the variance of the conditioning factor is 1. B_u is a standard Brownian motion. We are interested in finding

$$E(\int_0^t e^{\sigma Y_s} ds | Z) \tag{3.12}$$

$$\mathrm{Var}(\int_0^t e^{\sigma Y_s} ds | Z) \tag{3.13}$$

where σ^2 is the instantaneous variance of the process. For this, we require the following terms : $E(Y_s|Z)$, $\mathrm{Var}(Y_s|Z)$ and $\mathrm{Cov}(Y_s, Y_v|Z)$.

Our objective is to find Z such that the variance of Y_s conditionally on Z is minimised, that is $\mathrm{Var}(Y_s|Z)$ is minimum.

Now, for $0 \leq s \leq 1$, we have

$$E(Y_s|Z) = Z \int_{-\infty}^{\infty} L(s,u)a(u)du \tag{3.14}$$

$$\mathrm{Var}(Y_s|Z) = \int_{-\infty}^{\infty} L^2(s,u)du - \left(\int_{-\infty}^{\infty} L(s,u)a(u)duds\right)^2 \tag{3.15}$$

$$\mathrm{Cov}(Y_s, Y_v|Z) = \int_{-\infty}^{\infty} L(s,u)L(v,u)du - \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du \tag{3.16}$$

Therefore, we have,

$E(\int_0^1 e^{\sigma Y_s} ds|Z) =$

$$\int_0^1 exp\left\{\sigma Z \int_{-\infty}^{\infty} L(s,u)a(u)du + \frac{1}{2}\sigma^2 \int_{-\infty}^{\infty} L^2(s,u)du - \frac{1}{2}\sigma^2 \left[\int_{-\infty}^{\infty} L(s,u)a(u)du\right]^2\right\} ds \tag{3.17}$$

$E(\int_0^1 e^{\sigma Y_s} ds \int_0^1 e^{\sigma Y_v} dv|Z) =$

$$\left\{\int_0^t \int_0^t \left[\exp\left\{\sigma Z \left[\int_{-\infty}^{\infty} L(s,u)a(u)du + \int_{-\infty}^{\infty} L(v,u)a(u)du\right]\right\}\right] \times\right.$$

$$\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}L^2(s,u)du+\int_{-\infty}^{\infty}L^2(v,u)du\right]\right\}\times$$

$$\exp\left\{-\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}L(s,u)a(u)du)^2+\int_{-\infty}^{\infty}L(v,u)a(u)du\right]^2\right\}\times$$

$$\left[\exp\left\{\sigma^2\int_{-\infty}^{\infty}L(s,u)L(v,u)du-\sigma^2\int_{-\infty}^{\infty}L(s,u)a(u)du\int_{-\infty}^{\infty}L(v,u)a(u)du\right\}\right]dvds\right\} \tag{3.18}$$

$$\Rightarrow \mathrm{Var}(\int_0^1 e^{\sigma Y_s}ds|Z)=\left\{\int_0^1\int_0^1\left[\exp\left\{\sigma Z\int_{-\infty}^{\infty}L(s,u)a(u)du+\int_{-\infty}^{\infty}L(v,u)a(u)du\right\}\right]\times\right.$$

$$\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}L^2(s,u)du+\int_{-\infty}^{\infty}L^2(v,u)du\right]\right\}\times$$

$$\exp\left\{-\frac{1}{2}\sigma^2\left[\left(\int_{-\infty}^{\infty}L(s,u)a(u)du\right)^2+\left(\int_{-\infty}^{\infty}L(v,u)a(u)du\right)^2\right]\right\}\times$$

$$\left[\exp\left\{\sigma^2\left[\int_{-\infty}^{\infty}L(s,u)L(v,u)du-\int_{-\infty}^{\infty}L(s,u)a(u)du\int_{-\infty}^{\infty}L(v,u)a(u)du\right]\right\}-1\right]dvds\right\} \tag{3.19}$$

Therefore,

$$E(Var(\int_0^1 e^{\sigma Y_s}ds|Z))=\left\{\int_0^1\int_0^1\left[\exp\left\{\frac{1}{2}\left(\int_{-\infty}^{\infty}L(s,u)du+\int_{-\infty}^{\infty}L(v,u)du\right)^2\right\}\right]\times\right.$$

$$\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}L^2(s,u)du+\int_{-\infty}^{\infty}L^2(v,u)du\right]\right\}\times$$

$$\exp\left\{-\frac{1}{2}\sigma^2\left[\left(\int_{-\infty}^{\infty}L(s,u)a(u)du\right)^2-\left(\int_{-\infty}^{\infty}L(v,u)a(u)du\right)^2\right]\right\}\times$$

$$\left[\exp\left\{\sigma^2\left[\int_{-\infty}^{\infty}L(s,u)L(v,u)du-\int_{-\infty}^{\infty}L(s,u)a(u)du\int_{-\infty}^{\infty}L(v,u)a(u)du\right]\right\}-1\right]dvds\right\} \tag{3.20}$$

$$= \left[\left\{\int_0^1 \int_0^1 \left\{exp\left(\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right]\right) \times \right.\right.\right.$$

$$\left.\left. exp\left(\sigma^2 \int_{-\infty}^{\infty} L(s,u)L(v,u)du\right)\right\} dvds\right\}$$

$$- \left\{\int_0^1 \int_0^1 \left[exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right]\right\}\right] \times \right.$$

$$\left.\left.\left[exp\left\{\sigma^2 \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du\right\}\right] dvds\right\}\right] \qquad (3.21)$$

Now to minimise the expected value of the conditional variance, we need to maximise the second term of equation (3.21), given by

$$\int_0^1 \int_0^1 \left\{\exp\left(\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right]\right) \times \right.$$

$$\left. \exp\left(\sigma^2\left[\int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du\right]\right)\right\} dvds. \qquad (3.22)$$

This is because the other part of equation (3.21) does not involve any $a(u)$ and hence is fixed for fixed values of σ. Further σ is assumed to be small, thereby allowing the linearisation to be carried out. On linearisation of the integrand in equation (3.22), we have

$$\int_0^1 \int_0^1 \left\{1 + \frac{\sigma^2}{2}\left(\int_{-\infty}^{\infty} L^2(s,u)du + \int_{-\infty}^{\infty} L^2(v,u)du\right)\right.$$

$$\left. +\sigma^2 \int_{-\infty}^{\infty} L(s,u)a(u)du \int_{-\infty}^{\infty} L(v,u)a(u)du + O(\sigma^4)\right\} dvds. \qquad (3.23)$$

Now, equation (3.23) contains some terms independent of $a(s)$. These terms are fixed and hence equation (3.22) is maximised by maximising the terms involving $a(u)$ in equation (3.23). This is the same as maximising

$$\left(\int_0^1 \int_{-\infty}^{\infty} L(s,u)a(u)duds\right)^2,$$

which is the same as maximising

$$\int_0^1 \int_{-\infty}^{\infty} L(s,u)a(u)duds$$

subject to the constraint $\int_{-\infty}^{\infty} a^2(s)ds = 1$, i.e. the variance of the conditioning factor is 1. On changing the order of integration of the function to be maximised, we are required to maximise $\int_{-\infty}^{\infty}\int_0^1 L(s,u)dsa(u)du$ subject to the constraint $\int_{-\infty}^{\infty} a^2(s)ds = 1$. This implies that the optimal

$$a(u) \propto \int_0^1 L(s,u)ds \quad : \quad u \le 1$$

$$\Rightarrow Z = \int_{-\infty}^{\infty} a(u)dB_u = \int_{-\infty}^{\infty}\int_0^1 L(s,u)dsdB_u \quad : \quad u \le t.$$

3.3 Multi Driver Model

As in the case of the single driver model, here also we try to find the general form of the conditioning factor. We have again assumed $\{Y_s, 0 \le s \le 1\}$ to be a Gaussian process where

$$Y_s = \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)dB_u^{(i)}$$

subject to the constraint $\sup_s(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du \le \infty)$. Let the conditioning factor in general be Z, where

$$Z = \int_{-\infty}^{\infty}\sum_{i=1}^{n} a_i(u)dB_u^{(i)}, \tag{3.24}$$

$a_i(\bullet)$ is so chosen that it satisfies the condition $\int_{-\infty}^{\infty}\sum_{i=1}^{n} a_i^2(u)du = 1$. This condition ensures that the variance of the conditioning factor is 1. $B_u^{(i)}$ for i = 1, 2,...n is a standard Brownian motion and are independent of each other. We are interested in finding

$$E(\int_0^t e^{\sigma Y_s}ds|Z) \tag{3.25}$$

$$\mathrm{Var}(\int_0^t e^{\sigma Y_s}ds|Z) \tag{3.26}$$

where σ^2 is the instantaneous variance of the process. For this, we require the following terms : $E(Y_s|Z)$, $\mathrm{Var}(Y_s|ds)$ and $\mathrm{Cov}(Y_s, Y_v|Z)$.

Our objective, as in the case of the single driver, is to find Z such that the variance of Y_s conditionally on Z is minimised, that is $\mathrm{Var}(Y_s|Z)$ is minimum.

Now, for $0 \leq s \leq 1$, we have

$$E(Y_s|Z) = Z\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du \tag{3.27}$$

$$\mathrm{Var}(Y_s|Z) = \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - (\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)duds)^2 \tag{3.28}$$

$$\mathrm{Cov}(Y_s,Y_u|Z) = \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)L_i(v,u)du - \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du \tag{3.29}$$

Therefore, we have,

$E(\int_0^1 e^{\sigma Y_s}ds|Z) =$

$$\int_0^1 exp\left\{\sigma Z\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du + \frac{1}{2}\sigma^2\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - \frac{1}{2}\sigma^2(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du)^2\right\}ds \tag{3.30}$$

Also $E(\int_0^1 e^{\sigma Y_s}ds\int_0^1 e^{\sigma Y_v}dv|Z) =$

$$\left\{\int_0^1\int_0^1\left[\exp\left\{\sigma Z\left[\int_{-\infty}^{\infty}\sum_{i-1}^{n} L_i(s,u)a_i(u)du + \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right]\right\}\right]\times\right.$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - \frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\right]^2\right\}\right]\times$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(v,u)du - \frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\right]^2\right\}\right]\times$$

$$\left.\left[\exp\left\{\sigma^2\left(\int_{-\infty}^{\infty} L(s,u)L(v,u)du - \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right)\right\}\right]dvds\right\} \tag{3.31}$$

$\Rightarrow \mathrm{Var}(\int_0^1 e^{\sigma Y_s} ds|Z) =$

$$\left\{\int_0^1\int_0^1 \left[\exp\left\{\sigma Z\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du + \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right]\right\}\right] \times\right.$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - \sigma^2\left(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\right)^2\right]\right\}\right] \times$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - \sigma^2\left(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\right)^2\right]\right\}\right] \times$$

$$\left[\exp\left\{\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)L_i(v,u)du - \right.\right.\right.$$

$$\left.\left.\left.\left.\left(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right)\right]\right\} - 1\right] dvds\right\} \qquad (3.32)$$

Therefore,

$$E(\mathrm{Var}(\int_0^1 e^{\sigma Y_s} ds|Z)) = \left\{\int_0^1\int_0^1 \left[\left(\exp\left\{\frac{1}{2}\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)du + \int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)du\right]^2\right\}\right) \times\right.\right.$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(s,u)du - \left(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right)^2\right]\right\}\right] \times$$

$$\left[\exp\left\{\frac{1}{2}\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i^2(v,u)du - \left(\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right)^2\right]\right\}\right] \times$$

$$\left(\exp\left\{\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)L_i(v,u)duright\right]\right\}\right] \times$$

$$\left.\left.\exp\left\{-\sigma^2\left[\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(s,u)a_i(u)du\int_{-\infty}^{\infty}\sum_{i=1}^{n} L_i(v,u)a_i(u)du\right]\right\} - 1\right)\right] dvds\right\} \qquad (3.33)$$

$$= \left[\int_0^1 \int_0^1 exp \left\{ \frac{1}{2}\sigma^2 \left[\int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(s,u)du + \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(v,u)du \right] dvds \right\} \times \right.$$

$$\int_0^1 \int_0^1 exp \left\{ \sigma^2 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)L_i(v,u)du \right\} dvds$$

$$- \left\{ \int_0^1 \int_0^1 \left[exp \left[\frac{1}{2}\sigma^2 \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(s,u)du + \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(v,u)du \right) \right] \times \right. \right.$$

$$\left. \left. exp \left(\sigma^2 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)a_i(u)du \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(v,u)a_i(u)du \right) \right] dvds \right\} \right] \tag{3.34}$$

Now to minimise the expected value of the conditional variance, we need to maximise the second term of equation (3.34) given by

$$\left\{ \int_0^1 \int_0^1 \left[exp \left[\frac{1}{2}\sigma^2 \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(s,u)du + \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(v,u)du \right) \right] \times \right. \right.$$

$$\left. \left. exp \left(\sigma^2 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)a_i(u)du \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(v,u)a_i(u)du \right) \right] dvds \right\} \tag{3.35}$$

This is due to the fact that the first integral in equation (3.34) does not involve any $a_i(u)$ and hence is fixed for fixed values of σ. Further σ is assumed to be small, thereby allowing the linearisation to be carried out in equation (3.35). Thus, on linearisation, we have

$$\int_0^1 \int_0^1 \left[1 + \frac{\sigma^2}{2} \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(s,u)du + \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i^2(v,u)du \right) \right.$$

$$\left. + \sigma^2 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)a_i(u)du \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(v,u)a_i(u)du + O(\sigma^4) \right] dvds. \tag{3.36}$$

Now, equation (3.36) contains some terms independent of $a_i(u)$. These terms are fixed and hence equation (3.35) is maximised by maximising the terms involving $a_i(u)$ in equation (3.36). This is the same as maximising

$$\left(\int_0^1 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)a_i(u)duds \right)^2,$$

which is the same as maximising $\int_0^1 \int_{-\infty}^{\infty} \sum_{i=1}^{n} L_i(s,u)a_i(u)duds$ subject to the constraint $\int_{-\infty}^{\infty} \sum_{i=1}^{n} a_i^2(s)ds = 1$, i.e. variance of the conditioning factor is 1. Changing the order of integration, we need to maximise $\int_{-\infty}^{\infty} \int_0^1 \sum_{i=1}^{n} L_i(s,u)dsa_i(u)du$ subject to the constraint $\int_{-\infty}^{\infty} \sum_{i=1}^{n} a_i^2(s)ds = 1$. This implies that

$$a_i(u) \propto \int_0^1 L_i(s,u)ds \quad : \quad u \leq 1$$

$$\Rightarrow Z = \int_{-\infty}^{\infty} \sum_{i=1}^{n} a_i(u)dB_u^{(i)} = \int_{-\infty}^{\infty} \int_0^1 \sum_{i=1}^{n} L_i(s,u)dsdB_u^{(i)} \quad : \quad u \leq t.$$

3.4 Examples

3.4.1 Single Driver Models

In this section, we obtain the exact form of the conditioning factor Z for certain specific forms of the Gaussian process. In particular, we look at the situations when *L(s,u)* follows a Brownian Motion, an Ornstein - Uhlenbeck process and a Brownian Bridge.

Brownian Motion

$$L(s,u) = \begin{cases} 1 & : \quad 0 \leq u \leq s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

Thus, we have, $Z = \int_0^1 \int_u^1 dsdB_u = \int_0^1 (1-u)dB_u = \int_0^1 Y_u du$, where $\{Y_u, 0 \leq u \leq 1\}$ is a Brownian Motion.

Ornstein - Uhlenbeck : Non-stationary $(Y_0 = 0)$

$$L(s,u) = \begin{cases} e^{-\alpha(s-u)} & : \quad 0 \leq u \leq s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

Here, we have,

$$Z = \int_0^1 \int_u^1 e^{-\alpha(s-u)}dsdB_u = \int_0^1 \frac{1-e^{-\alpha(1-u)}}{\alpha}dB_u = \int_0^1 Y_u du,$$

where $\{Y_u, 0 \leq u \leq 1\}$ is a non-stationary Ornstein - Uhlenbeck process.

Ornstein - Uhlenbeck : Y_0 has the stationary distribution

$$L(s,u) = \begin{cases} e^{-\alpha(s-u)} & : \quad u \leq s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

In the case of the stationary Ornstein - Uhlenbeck case, we have,

$$Z = \int_{-\infty}^{1} \int_{u}^{1} e^{-\alpha(s-u)} ds dB_u = \int_{-\infty}^{1} \frac{1-e^{-\alpha(1-u)}}{\alpha} dB_u = \int_{-\infty}^{1} Y_u du,$$

$\{Y_u, 0 \leq u \leq 1\}$ being a stationary Ornstein - Uhlenbeck process.

Brownian Bridge

$\{Y_u, 0 \leq u \leq 1\}$ is a Brownian Bridge and is represented as $\{B_u - uB_1\}$ where B_u is a simple Brownian Motion and B_1 is the value of the Brownian Motion at time *1*.

$$L(s,u) = \begin{cases} 0 & : \quad u \leq 0 \\ & : \\ 1-s & : \quad 0 \leq u \leq s \\ & : \\ -s & : \quad s \leq u \leq 1 \\ & : \\ 0 & : \quad u \geq 1 \end{cases}$$

For the Brownian Bridge, the situation is slightly different than from the ones discussed earlier. Here we also take into account some amount of the information of the future, in fact, the final value. Thus

$$Z = \int_0^1 \{\int_0^u -sds + \int_u^1 (1-s)ds\} dB_u = \int_0^1 \int_u^1 ds dB_u - \int_0^1 \int_0^1 s ds dB_u = \int_0^1 Y_u du.$$

3.4.2 Multi-driver models

In this section, we look at situations where $\{Y_u, 0 \leq u \leq 1\}$, the Gaussian process, follows a linear combination of Ornstein Uhlenbeck processes only. The situation of $\{Y_u, 0 \leq u \leq 1\}$, the Gaussian process, following a linear combination of Brownian motions is quite trivial as the linear combination of Brownian motions result in another Brownian motion and can thus be treated as in the single driver case.

Ornstein - Uhlenbeck : Non-stationary ($Y_0 = 0$)

$$L_i(s,u) = \begin{cases} e^{-\alpha_i(s-u)} & : \quad 0 \le u \le s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

Here, we have

$$Z = \int_0^1 \int_u^1 \sum_{i=1}^n e^{-\alpha_i(s-u)} ds dB_u^{(i)} = \int_0^1 \sum_{i=1}^n \frac{1-e^{-\alpha_i(1-u)}}{\alpha_i} dB_u^{(i)} = \int_0^1 \sum_{i=1}^n Y_u^{(i)} du,$$

where $\{Y_u^i, 0 \le u \le 1\}$ is a non-stationary Ornstein Uhlenbeck process.

Ornstein - Uhlenbeck : Y_0 has a stationary distribution

$$L_i(s,u) = \begin{cases} e^{-\alpha_i(s-u)} & : \quad u \le s \\ & : \\ 0 & : \quad \text{otherwise} \end{cases}$$

In the case of the Stationary Ornstein - Uhlenbeck case, we have,

$$Z = \int_{-\infty}^1 \int_u^1 \sum_{i=1}^n e^{-\alpha_i(s-u)} ds dB_u^{(i)} = \int_{-\infty}^1 \sum_{i=1}^n \frac{1-e^{-\alpha_i(1-u)}}{\alpha_i} dB_u^{(i)} = \int_{-\infty}^1 \sum_{i=1}^n Y_u^{(i)} du,$$

$\{Y_u^i, 0 \le u \le 1\}$ being a stationary Ornstein - Uhlenbeck process.

3.5 Alternative forms of the Conditioning Factor

The conditioning factor used in all the above examples is based on only one term. In all cases, we take the conditioning factor Z, to be

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}}, \tag{3.37}$$

where, $\{Y_s, 0 \le s \le 1\}$ is a stochastic process - following a Brownian motion or an Ornstein - Uhlenbeck process. The denominator term in the conditioning factor ensures that the conditioning factor is suitably normalised. It is essentially the numerator that we are interested in, while obtaining the conditioning factor. We tried using an extra term in the numerator of conditioning factor. Thus, for the situation where $\{Y_s, 0 \le s \le 1\}$ is a Brownian motion, we had the conditioning factor as

$$Z' = \frac{\sigma\left\{\int_0^1 (1-u) dB_u + \gamma \int_0^1 (1-u)^2 dB_u\right\}}{\sqrt{\text{Var}\left(\sigma\left\{\int_0^1 (1-u) dB_u + \gamma \int_0^1 (1-u)^2 dB_u\right\}\right)}},$$

where γ is a constant. We performed the same calculations with this conditioning factor Z' for different values of γ. The case of $\gamma = 0$ is the case where $Z' = Z$. When $\gamma \neq 0$, we found that the results obtained using the new conditioning factor Z' was a very slight improvement on the ones obtained by using Z - in fact less than the order of 0.01. Hence, we concluded that by using the conditioning factor Z in preference to Z', we would not be making any substantial error. On the other hand, using Z in preference to Z' would allow us to gain in terms of speed since calculations involving Z' take considerably longer time than those involving Z. The results were the same for the case where $\{Y_s, 0 \leq s \leq 1\}$ followed an Ornstein Uhlenbeck process.

We also looked at other forms of the conditioning factor. Among them were forms that maximised the correlation between the conditioning factor Z and the stochastic process $\{Y_s, 0 \leq s \leq 1\}$. We also tried a conditioning factor of the form

$$Z'' = \frac{\int_0^1 \left[1 - e^{-\delta(1-u)}\right] dY_u}{\sqrt{\mathrm{Var}(\int_0^1 [1 - e^{-\delta(1-u)}]\, dY_u)}},$$

where the stochastic process $\{Y_u, 0 \leq u \leq 1\}$ could be either a Brownian motion or an Ornstein - Uhlenbeck process and δ is a constant. However, here also, the gain in accuracy was negligible, hence we decided to continue with the conditioning factor of the form given by equation (3.37).

3.6 Conclusion and Remarks

The conditioning factor used in all cases discussed above is of the form $Z = \int_0^1 Y_s ds$. This is the one that we obtained in the situations for the Brownian motion and the Ornstein Uhlenbeck process where the initial value is known. However, in the case of the Ornstein Uhlenbeck process where the initial value has a stationary distribution, the correct expression should have been $Z = \int_{-\infty}^1 Y_s ds$. However, we still use $Z = \int_0^1 Y_s ds$. This means that we lose the information between $-\infty$ to *1* - in practice it has been observed that conditioning on information available from time 0 to time *1* yields results very close to the ones obtained on conditioning on all the information available from $-\infty$ to time *1*. This justifies our conditioning on the information available between *0* and *1*.

For the single driver, the reason for not using the conditioning factors based on two or more processes was that the gain in accuracy was negligible when compared to the results by using the conditioning factor based on one process. Also, using more than one process as a conditioning factor meant increased computing time.

We have dealt with the *Single Driver* case separately, even though it can be treated as a special case of the *Multi Driver* situation. This is because understanding the mechanism for the *Single Driver* case is much easier than the *Multi Driver* case, thus making it much simpler to generalise to the *Multi Driver* situation.

Finally, the choice of the conditioning factor discussed here has been based on the problem of pricing bonds and contingent payments on interest rates of the bonds. As stated earlier, for the problem of pricing of European call options on assets with stochastic volatility, the situation is more complicated. However, we still propose to use the conditioning factor Z as defined in equation (3.37) to obtain an approximation to the price of the call option. As will be seen later in the thesis, the results obtained by using the conditioning factor are quite accurate.

Chapter 4

Valuation of coupon carrying bonds : Non-defaultable and Defaultable

4.1 Introduction

In this chapter, we look at pricing of bonds which make coupon payments and have a longer time to maturity. As a matter of fact, coupon paying bonds are quite common in practice. The only difference from the zero coupon bonds is that coupon paying bonds are generally of a longer term. There can be two situations : one when there is a zero probability of default occurring and the second situation is when there is a non - zero probability of default. The approach of calculations of the prices is the same as in the case of a zero coupon bonds as discussed in chapter 2. We make use of an appropriate conditioning factor to find a lower bound of the price of the bond. As before, we assume the interest rate to be governed by a stochastic process. Here, we assume the stochastic process to be an Ornstein - Uhlenbeck process where the initial value is known. The choice of such a process governing interest rate is based on the fact that in practice, rarely does one come across a situation of the interest rate being governed by a Brownian motion or an Ornstein - Uhlenbeck process with the initial value having a stationary distribution. Furthermore, both these situations can be treated as special cases of the Ornstein - Uhlenbeck process where the initial value is known. For example, when the mean reversion force in the Ornstein - Uhlenbeck process goes to zero, we have a Brownian motion. Also for the case of the Ornstein - Uhlenbeck process with the initial value having a stationary distribution, we have the initial value distributed as a normal variable with zero mean and variance $\frac{\sigma^2}{2a}$; σ^2 is the instantaneous variance of the Ornstein - Uhlenbeck process and a is the mean reversion force. However, in both the cases, the formulae need to be slightly adjusted.

4.2 Non-defaultable bonds

In this section, we look at the situation of the bond making coupon payments during the life of the bond. This is in some sense a generalisation of the zero coupon bond situation. Note that the coupon is payable at a continuous rate.

Here we want to calculate,

$$E\left[C\int_0^T e^{-\int_0^s r_u du}ds + e^{-\int_0^T r_u du}\right], \tag{4.1}$$

where, the value of the coupon is given by

$$E\left[C\int_0^T e^{-\int_0^s r_u du}ds\right]$$

and the value the principal is given by

$$E\left[e^{-\int_0^T r_u du}\right].$$

As before,

$$r_t = be^{\sigma Y_t}$$

$$\text{and} \quad Y_t = \int_0^t e^{-a(t-s)}dB_s,$$

where, r_t is the instantaneous rate of interest, σ the instantaneous variance and Y_t is a stochastic process - in this case, it is an Ornstein - Uhlenbeck process where the initial value is known and is assumed to be 0. b is a scaling constant. Also, $b = r_0$, the initial value of the interest rate and $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long-term value of the interest rate. Further, C is the coupon rate and b is the discount factor.

4.2.1 Choosing a conditioning factor

Here, there are two quantities that we want to calculate; one is the value of the coupon payments and the other is the principal. The calculation of the value of the principal is exactly the same as calculating the value of a zero coupon bond, the details of which are given in the previous section. To calculate the value of the coupon payments we again make use of a suitable conditioning factor. The conditioning factor used for calculating the value of the coupon payments is slightly different from the one used in the calculation of the price of the zero coupon bonds, but is still based on the same principles as outlined in chapter 3. The function that we are interested in is

$$\int_0^T e^{-b\int_0^t e^{\sigma Y_s}ds}dt. \tag{4.2}$$

As in chapter 3, we shall use a linearisation argument to obtain the conditioning factor. For this purpose, we assume σ and b to be small. First, we look at

$$\int_0^t e^{\sigma Y_s}ds.$$

Expanding the exponential term with respect to σ, we have

$$\int_0^t (1+\sigma\int_0^t Y_s ds + O(\sigma^2))ds.$$

Thus, equation (4.2) can be rewritten as

$$\int_0^T e^{-b(t+\sigma\int_0^t Y_s ds + O(\sigma^2))}dt.$$

Now, we expand the exponential term again with respect to b. We thus have,

$$\int_0^T (1-b(t+\sigma\int_0^t Y_s ds + O(\sigma^2)) + O(b^2))dt.$$

$$= T - \frac{bT^2}{2} - b\sigma\int_0^T\int_0^t Y_s ds dt - bO(\sigma^2) + O(b^2).$$

Since we assume σ and b to be small, we ignore terms involving σ^2 and higher as well as those involving b^2 and above. Thus the only stochastic part in the integral is the integral involving Y_s.

Thus the conditioning factor in this case is proportional to $\int_0^T \int_0^t Y_s ds dt$, i.e. the conditioning factor Z_1 is given as

$$Z_1 = \frac{\int_0^T \int_0^t Y_s ds dt}{\sqrt{\text{Var}(\int_0^1 \int_0^t Y_s ds dt)}}.$$

The conditioning factor obtained based on the expansion techniques described above is expected to work for small values of σ and b. However, in practice, the method works for even relatively high values of σ and b, as will be evident from the results in tables 7.1 and 8.1.

Here, as stated earlier, we take the stochastic process $\{Y_s; 0 \leq s \leq 1\}$ as an Ornstein - Uhlenbeck process with the initial value $Y_0 = 0$.

4.2.2 Calculation of interest payments

Once we have obtained the conditioning factor as above, we can then easily calculate the value of the coupon payment. In general, we take $T = 1$ and adjust the other parameters accordingly to account for the longer time period of the bond. Thus, with $T = 1$, we have

$$Z_1 = \frac{\int_0^1 \int_0^s Y_u du ds}{\sqrt{\text{Var}(\int_0^1 \int_0^s Y_u du ds)}}.$$

Now, using the fact that $\{Y_s; 0 \leq s \leq 1\}$ here follows an Ornstein - Uhlenbeck process with $Y_0 = 0$, where Y_0 is the initial value, we have,

$$\int_0^1 \int_0^s Y_u du ds = \sigma \int_0^1 e^{as} \left[\int_s^1 (1-u) e^{-au} du \right] dB_s = \sigma \int_0^1 \frac{e^{-a(1-s)} + a(1-s) - 1}{a^2} dB_s.$$

$$\Rightarrow \text{Var}(\int_0^1 \int_0^s Y_u du ds) = \frac{\sigma^2}{a^4} \int_0^1 (e^{-a(1-s)} + a(1-s) - 1)^2 ds$$

$$= \frac{\sigma^2}{6a^5} \left[3 - 3e^{-2a} - 12ae^{-a} - 6a^2 + 6a \right] = V \quad say.$$

Further, Z_1 is distributed as a standard normal variable.

Conditionally on Z_1, Y_u is a Gaussian process with

$$E(Y_u | Z_1) = k_u Z_1 \tag{4.3}$$

where $k_u = \text{Cov}(Y_u, Z_1) = \frac{\sigma^2}{\sqrt{V}} \int_0^u \left\{ e^{-a(u-s)} \frac{e^{-a(1-s)} + a(1-s) - 1}{a^2} \right\} ds$

$$= \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \frac{e^{-a(1-u)} + 2a(1-u) - e^{-a(1+u)} - 2ae^{-au}}{a^2}. \tag{4.4}$$

Also,

$$\text{Cov}(Y_u, Y_v | Z_1) = \text{Cov}(Y_u, Y_v) - k_u k_v = w_{uv} \quad \text{say.} \tag{4.5}$$

Here, $\text{Cov}(Y_u, Y_v) = \frac{\sigma^2}{2a} \left[e^{a|u-v|} - e^{-a(u+v)} \right]$, as $\{Y_u; 0 \leq u \leq 1\}$ follows an Ornstein Uhlenbeck process. Once we have these values, then we can easily calculate the the value of the coupon payments.

So, conditionally on Z_1, we have the lower bound of the value of the intermediate payment given as

$$\int_0^1 \exp\left(-b \int_0^u \exp\left[k_s Z_1 + \frac{1}{2} w_{ss} \right] ds \right) du = h_1(Z_1) \quad \text{say.} \tag{4.6}$$

Thus, to get the lower bound to the value of the intermediate interest payments, we take the expectation of $h_1(Z_1)$ with respect to Z_1; that is we calculate

$$H_1 = \int_{-\infty}^{\infty} h_1(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

The value obtained thus is inflated by the coupon rate. The results are for a short term bond is shown in table 7.1 while for a long term bond is shown in table 8.1. The coupon rate assumed here for the numerical results is 5%, i.e. $C = 0.05$.

Also, the value of the final payment, H_2, is calculated exactly the same way as the zero coupon bonds and hence the detailed calculation of that is not shown in this section. Finally, the value of the long term bond is obtained by discounting the intermediate payment by the coupon rate and then adding the final payment to it, i.e., the value of the coupon paying bond is given as

$$CH_1 + H_2.$$

We have calculated an approximation to the lower bound to price of a coupon paying bond for a number of values of σ, b and a - the mean reversion force of the Ornstein - Uhlenbeck process. In general, we take T =*1*, that is the term of the bond to be one year. However, we account for long term bonds by adjusting the values of the other parameters. Thus for a bond with a life of t years, σ^2 changes to $\sigma^2 t$, a changes to at and b changes to bt and the we are able to keep T = *1*. Here the long period bond is taken to have a life of 10 years.

The results for different values of the parameters are given in tables 7.1 and 8.1. Table 7.1 shows the results for the short term bond while table 8.1 shows the results for the long term bonds. In each case, for comparison purposes, the simulated values along with their standard errors are given in the same table.

4.3 Defaultable Bonds

In this section, we look at situations where there is a non-zero probability of default taking place. However, as is observed in practice, the probability of default is quite small. Work in this area has been done by, among others, Lando (1997) and Duffie and Singleton (1995). Here it is assumed that in case of a default all payments cease (including coupons) and a certain percentage of the value of the bond at maturity (known in advance) is paid out, else the full value is paid on maturity. The analysis here has been based on coupon paying bonds. The reason being that a zero coupon bond cannot default at any other time but the final maturity time and hence is of little interest. It is the coupon paying bonds which are of considerable interest as they might default at any time prior to maturity and thus the coupon payments would stop.

Here, we are interested in calculating

$$E\left[D\int_0^T (e^{-rs-\int_0^s \lambda_u du}\lambda_s)ds + (e^{-r}e^{-b\int_0^T \lambda_s ds}) + C\int_0^T (\int_0^s (e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds)\right.$$

$$\left. +C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right], \tag{4.7}$$

where

$$\lambda_t = be^{\sigma Y_t}$$

$$\text{and} \quad Y_t = \int_0^t e^{-a(t-s)} dB_s.$$

Here, λ_t is the rate of default and Y_t is a stochastic process - in this case, it is the non-stationary Ornstein - Uhlenbeck process. r is the interest rate which is assumed to be constant. Also, σ is the instantaneous variance. Further, D is the percentage paid out in case default occurs and C is the rate of coupon payments during the life of the bond. b is a scaling factor, representing the discount rate. The terms in equation (4.7) represent the following.

$E\left[D\int_0^T e^{-rs-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Payment at default.

$E\left[e^{-r}e^{-b\int_0^T \lambda_s ds}\right]$ = Final payment on maturity, when no default takes place.

$E\left[C\int_0^T (\int_0^s e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Coupon payments in case of default.

$E\left[C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right]$ = Coupon payments in case no default occurs.

As in the case of coupon paying bond discussed in the previous section, we take $T = 1$ and adjust the other parameters for a long period bond. Now, equation (4.7) can be rewritten as

$$E\left[(D-C)\int_0^1 e^{-rs}e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds + \frac{C}{r}\int_0^1 e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds\right.$$

$$\left. + (1-\frac{C}{r})e^{-r}e^{-b\int_0^1 e^{\sigma Y_u}du} + \frac{C}{r}e^{-b\int_0^1 e^{\sigma Y_u}du}\right]. \tag{4.8}$$

Now,

$$\frac{C}{r}\int_0^1 e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds = \frac{C}{r}\left(1 - e^{-b\int_0^1 e^{\sigma Y_u}du}\right).$$

Substituting this in equation (4.8), we have

$$E\left[(D-C)\int_0^1 e^{-rs}e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds + (1-\frac{C}{r})e^{-r}e^{-b\int_0^1 e^{\sigma Y_u}du} + \frac{C}{r}\right]. \tag{4.9}$$

This is what we are interested in calculating.

4.3.1 Choosing a Conditioning Factor

In this section, we are really interested in calculating the first term of equation (4.9). This term gives us the value of the payment that is made in case of default. The second term of equation (4.9) gives the value of the bond, assuming no default. To calculate the second term - rather the integral involved, we use the same approach as used earlier in the case of the non-defaultable bonds without any coupon payments. However, the parameters are suitably adjusted to account for the long period of the bond. In case of the bond being a short term one, then the parameters remain exactly the same as in the case of the zero coupon bonds. Now, to calculate the value of the payment if default occurs, we need to calculate the first integral of equation (4.9). As in all the cases stated earlier, we make use of a suitable conditioning factor.

As in the case of the bonds making coupon payments with zero probability of default, we use a different conditioning factor for each of the two integrals. For the second integral, the conditioning factor is exactly the same as that in the zero coupon case. This is given by

$$Z_1 = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

The first integral in equation (4.9) is

$$\int_0^1 e^{-rt-b\int_0^s e^{\sigma Y_u}du} b e^{\sigma Y_t} dt = b\int_0^1 e^{-rt-b\int_0^s e^{\sigma Y_u}du} e^{\sigma Y_t} dt.$$

As in the previous section, here also, we assume σ and b to be small, thereby allowing us to expand the exponential term with respect to σ first and then b. Thus we have, on expansion,

$$b\int_0^1 \left[e^{-rt-b\int_0^s (1+\sigma Y_u)du}(1+\sigma Y_t)\right] dt + O(b\sigma)$$

$$= b\int_0^1 \left[e^{-rt}\left(1+\sigma Y_t + bs + bs\sigma Y_t + b\sigma\int_0^s Y_u du + b\sigma^2 Y_t \int_0^s Y_u du\right)\right] dt + O(b\sigma)$$

$$= \int_0^1 (e^{-rs}b + b\sigma e^{-rs} Y_s) ds + O(b\sigma). \tag{4.10}$$

Again, the only stochastic term in equation (4.10) is $\int_0^1 Y_s ds$. Thus, the conditioning factor, Z_2, we use is proportional to $\int_0^1 Y_s ds$, i.e.,

$$Z_2 = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

Here, we observe that Z_1 and Z_2 are exactly the same and thus we use the same conditioning factor for both the integrals. Let us call this conditioning factor Z. So,

$$Z = Z_1 = Z_2.$$

As in the situation of coupon paying bonds, the expansion is supposed to be valid for small values of σ and b. However, as is noted in practice, it works well for even not so small values of σ and b.

4.3.2 Calculations for defaultable bonds

Once we have obtained the conditioning factor as above, we can then easily calculate the value of the interim payment. We take $T = 1$ and adjust the other parameters accordingly to account for the longer time period of the bond. The conditioning factor Z is exactly the same as the one in the zero coupon case. Now, conditionally on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z \tag{4.11}$$

$$k_u = \mathrm{Cov}(Y_s, Z) = \frac{1}{\sqrt{V}}\frac{\sigma^2}{2a}\{\int_0^u (e^{a(s+u)} - e^{-a(s+u)})ds + \int_u^t (e^{a(u-s)} - e^{-a(u+s)})ds$$

$$= \frac{1}{\sqrt{V}}\frac{\sigma^2}{2a}\{\frac{1-e^{-au}}{a} + \frac{1-e^{-a(1-u)}}{a} - \frac{e^{-au} - e^{-a(1+u)}}{a}\}, \tag{4.12}$$

$$\text{and} \quad \mathrm{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] - k_u k_v = w_{uv}. \tag{4.13}$$

Once we have these values, then we can easily calculate the the value of the first integral.

So, conditionally on Z, we have

$$\int_0^1 e^{-ru}\left\{\exp\left(-b\int_0^u \exp\left[k_s Z+\frac{1}{2}w_{ss}\right]ds\right)\right\} b\left\{\exp\left(k_u Z+\frac{1}{2}w_{uu}\right)\right\} du = h_1(Z) \quad \text{say.} \tag{4.14}$$

Finally, we take the expectation of $h_1(Z)$ with respect to Z; that is we calculate

$$H_1 = \int_{-\infty}^{\infty} h_1(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

Once we have obtained the value of H_1, we multiply it with the difference between the percentage paid out in case of default and the ratio of the percentage of coupon payments to the interest rate, that is we calculate

$$(D-\frac{C}{r})H_1.$$

Here, we assume that the amount paid it in case of default is 50%, i.e. $D = 0.5$, the percentage of coupon payments is 4%, i.e. $C = 0.04$ and the interest rate is 5%, i.e. $r = 0.05$. Now, the term

$$(D-\frac{C}{r})$$

can go negative depending on different choices of D, C and r. That is why the price obtained using this term will not be a lower bound to the price - but just an approximation to the price. However, as is evident from the results the approximation is a very accurate one.

Next, the second integral is calculated exactly similarly to the zero coupon bond case. Let us denote that by H_2. Since we have discussed the calculation of H_2 earlier (section 2.2.2 and the examples in that section), we do not go into it here. Thus, the second term of equation (4.8) is calculated as

$$(1-\frac{C}{r})H_2.$$

Finally, the value of the bond with a non-zero probability of default is given as

$$(D-\frac{C}{r})H_1+(1-\frac{C}{r})H_2+\frac{C}{r}.$$

We have calculated an approximation to the price of the bond with a non-zero probability of default for a number of values of σ, b and a - the mean reversion force of the Ornstein Uhlenbeck process. In general, we take T =*1*, that is the life of the bond to be one year. However, we account for long term bonds by adjusting the values of the other parameters. Thus for a bond with a life of t years, σ^2 changes to $\sigma^2 t$, a changes to at and b changes to bt and the we can be able to keep T = *1*. Here the long period bond is taken to have a life of 10 years. This is exactly the same as discussed in the case of coupon paying bonds with zero probability of default.

The results for different values of the parameters are given in tables 7.2 and 8.2. Table 7.2 shows the results for the short term bond while table 8.2 shows the results for the long term bonds. In each case, for comparison purposes, the simulated values along with their standard errors are given in the same table.

4.4 Conclusion and Remarks

The lower bounds to the price of the bonds or even the approximation to the prices calculated by using the conditioning factor seem to be so close to the actual price (in some cases, the simulated prices were lower than the lower bounds) that they can be regarded as a very good approximation to the true value. This is true of both the situations discussed - that of coupon paying bonds with a zero probability of default as well as coupon paying bonds with a non-zero probability of default having a payout at default using an Ornstein - Uhlenbeck process as the driving force of the stochastic process.

The conditioning factor approach is also useful in calculating prices of coupon paying bonds. These values could not be calculated by a direct expansion.

4.5 Tables

Next, we present a set of tables outlining the numerical results based on the formulae stated earlier in the chapter. Table 7.1 shows the results for the case of a coupon paying short term (1 year) bond while table 8.1 shows the results for the case of a coupon paying long term (10 year) bond. Table 7.2 shows the results for a short term (1 year) bond which has a non-zero probability of default and table 8.2 shows the results for a long term (10 year) bond which has a non-zero probability of default. In both cases of a coupon paying bond as well as a bond with a non-zero probability of default (short term and long term), the stochastic process governing the interest rate process is assumed to be an Ornstein - Uhlenbeck process where the initial value is known. Again, for comparison purposes the simulated values along with their standard error are presented.

In all cases, all the prices are multiplied by 100.

Table 7.1 : Table showing the calculated values of the total payments of coupon paying bonds along with the simulated values and their standard errors where the term of the bond is 1 year and the coupon rate is 5%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	98.07985	98.05825	0.0027
0.5	1	0.07	97.68948	97.82111	0.0145
0.75	1	0.07	97.16662	97.54738	0.023

Table 7.2 : Table showing the calculated values of the payments of bonds at default along with the simulated values and their standard errors where the term of the bond is 1 year and and the amount paid out in case of default is 50%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	95.7805	95.7549	0.0015
0.5	1	0.07	95.6761	95.652	0.0078
0.75	1	0.07	95.5354	95.4768	0.01208

Table 8.1 : Table showing the calculated values of the total payments of coupon paying bonds along with the simulated values and their standard errors where the term of the bond is 10 years and the coupon rate is 5%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	53.25209	53.17027	0.0104
$\sqrt{0.1}$	1	0.07	52.58027	52.4876	0.0334
0.5	1	0.07	51.46158	51.37964	0.0537
0.75	1	0.07	49.1404	49.1597	0.081

Table 8.2 : Table showing the calculated values of the payments of bonds at default along with the simulated values and their standard errors where the term of the bond is 10 years and and the amount paid out in case of default is 50%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	74.6547	74.65141	0.0057
$\sqrt{0.1}$	1	0.07	74.3201	74.3066	0.0179
0.5	1	0.07	73.795	73.8483	0.0288
0.75	1	0.07	72.705	72.7103	0.044

Note : To calculate the prices of the long - term (10 year) bonds, we use the same formulae as in the case of 1 year bonds. However, as stated earlier in sections 4.2.2 and 4.3.2, for calculation purposes, we take the term of the bond $T = 1$ but adjust the other parameters accordingly to represent a $T = t$ year bond. Thus, for a bond with a life of t years, σ^2 changes to $\sigma^2 t$, a changes to at and b changes to bt. In our case, $t = 10$.

Chapter 5

Pricing of Bonds based on Multi-driver Models

5.1 Introduction

In this chapter, we look at the situation where the interest rate process is a linear combination of n Markov processes which need not necessarily be independent of each other. Such models are referred to as *multi-driver* models and are used quite commonly in practice. Work on *multi Driver* models have been done by a number of researchers, prominent among them are the works by Heath, Jarrow and Morton (1992), Duffie and Kan (1994, 1996) and Longstaff and Schwartz (1992a, 1992b).

We have the instantaneous rate of interest given by r_t. As before, we are interested in the calculation of

$$E(e^{-b\int_0^1 r_t dt}), \tag{5.1}$$

where b is a scaling factor.

Continuing with the idea of the use of a suitable conditioning factor as employed in the previous chapter and using Jensen's inequality, we approximate the price of the bond by the lower bound of the price of the bond. Here, as in chapter 2, the function that we look at is $f(x) = e^{-bx}$ and we are interested in finding the expected value of f. Also, the function f is convex and hence Jensen's inequality holds. Thus, we have

$$E(f(X)) = E[f(X)|Z] \geq E[f\{E(X|Z)\}]$$

and we want to find $E(f(E(X|Z)))$ as it is a lower bound to the true price. Here, Z is the conditioning factor. This is similar to the approach of Rogers and Shi (1995). The choice of the conditioning factor Z, is based on the explanations given in chapter 3.

Here, in general, r_t is governed by n stochastic processes - say $\{Y_t^{(i)}, 0 \leq t \leq 1, i = 1, 2, ..., n\}$. Further, the stochastic processes $\{Y_t^{(i)}, 0 \leq t \leq 1\}$ and $\{Y_t^{(j)}, 0 \leq t \leq 1\}$ could be correlated amongst themselves with a correlation coefficient ρ. Now, we can have two situations. One is when r_t is just the sum of the stochastic processes. That is, we have

$$r_t = exp(\sum_{i=1}^{n} \beta_i Y_t^{(i)}), \tag{5.2}$$

where $\beta_i, i = 1, 2, ..., n$ is a constant. In this situation, r_t is still a log-normal process and hence cannot go negative. We will refer to this as *model 1.*

A slightly different model which we will look at is when r_t is based on n drivers directly. This is given by

$$r_t = \sum_{i=1}^{n} \gamma_i e^{\beta_i Y_t^{(i)}}, \tag{5.3}$$

where γ_i and β_i are constants for $i = 1, 2, ..., n$. Here r_t is a sum of n log-normal processes. We will refer to this as *model 2.*

In particular, for reasons of simplicity, we shall take $n = 2$ throughout this chapter. Thus, we have the two stochastic processes $\{Y_t^{(1)}, 0 \leq t \leq 1\}$ and $\{Y_t^{(2)}, 0 \leq t \leq 1\}$. Also, we assume $\beta_1 = 1$, $\beta_2 = \beta$, $\gamma_1 = 1$ and $\gamma_2 = \gamma$. Thus, equation (5.2) (model 1) becomes

$$r_t = e^{Y_t^{(1)} + \beta Y_t^{(2)}} \tag{5.4}$$

and equation(5.3) (model 2) becomes

$$r_t = e^{Y_t^{(i)}} + \gamma e^{\beta Y_t^{(2)}}. \tag{5.5}$$

Both model 1 and model 2 can be regarded as special cases of the Heath, Jarrow and Morton (1992) model. We shall discuss these two cases separately in the following two sections. We have $Y_t^{(1)}$ and $Y_t^{(2)}$ given by

$$dY_t^{(1)} = -a_1 Y_t^{(1)} dt + \sigma dB_t^{(1)} \tag{5.6}$$

$$\text{and} \quad dY_t^{(2)} = -a_2 Y_t^{(2)} dt + \sigma[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}] \tag{5.7}$$

where, $B_t^{(1)}$ and $B_t^{(2)}$ are two independent Brownian motions driving the two processes $Y_t^{(1)}$ and $Y_t^{(2)}$. Also, a_1 is the mean reversion force of $Y_t^{(1)}$ and a_2 is the mean reversion force of $Y_t^{(2)}$. Further, ρ is the correlation between $Y_t^{(1)}$ and $Y_t^{(2)}$ and σ is the instantaneous variance.

We shall now look at the two situations defined by equations (5.4) and (5.5) separately.

5.2 Model 1

Here, we have r_t given by

$$r_t = e^{Y_t^{(1)} + \beta Y_t^{(2)}},$$

where, β is a constant. Here, we have the interest rate following an exponential of a linear combination of two Gaussian processes which is a Gaussian process itself. We are interested in finding

$$E(e^{-b\int_0^1 r_t dt}) = E\left[e^{-b\int_0^1 (e^{Y_t^{(1)} + \beta Y_t^{(2)}})dt}\right].$$

Here $Y_t^{(1)}$ and $Y_t^{(2)}$ are as defined by equations (5.6) and (5.7). The conditioning factor Z used here is given, as explained in chapter 3 section 3, by

$$Z = \frac{\int_0^1 Y_s^{(1)} ds + \beta \int_0^1 Y_s^{(2)} ds}{\sqrt{\text{Var}(\int_0^1 Y_s^{(1)} ds + \beta \int_0^1 Y_s^{(2)} ds)}}. \tag{5.8}$$

Further, Z has a standard normal distribution.

Now, $\text{Var}(\int_0^1 Y_s^{(1)} ds + \beta \int_0^1 Y_s^{(2)} ds) = \text{Var}(\int_0^1 Y_s^{(1)} ds) + \beta^2 \text{Var}(\int_0^1 Y_s^{(2)} ds)$

$$+2\beta \text{Cov}(\int_0^1 Y_s^{(1)} ds, \int_0^1 Y_s^{(2)} ds) = V \quad \text{say}, \tag{5.9}$$

where

$$\text{Var}(\int_0^1 Y_s^{(1)} ds) = \frac{\sigma^2}{2a_1^2} \frac{2a_1 + 4e^{-a_1} - e^{-2a_1} - 3}{a_1},$$

$$\text{and} \quad \text{Var}(\int_0^1 Y_s^{(2)} ds) = \frac{\sigma^2}{2a_2^2} \frac{2a_2 + 4e^{-a_2} - e^{-2a_2} - 3}{a_2}.$$

Also, $\mathrm{Cov}(\int_0^1 Y_s^{(1)}ds, \int_0^1 Y_s^{(2)}ds) = \int_0^1\int_0^1 \mathrm{Cov}(Y_u^{(1)}, Y_v^{(2)})dudv$

$$= \sigma^2\rho\int_0^1\left[\int_0^u\int_0^v e^{-a_1(u-s)}e^{-a_2(v-s)}dsdv + \int_u^1\int_0^v e^{-a_1(u-s)}e^{-a_2(v-s)}dsdv\right]du$$

$$= \frac{\sigma^2\rho}{a_2(a_1+a_2)}\left[a_2 - 1 + \frac{(a_2+1)(e^{-a_1}-1)}{a_1} - \frac{2(e^{-(a_1+a_2)}-1)}{a_1+a_2} + \frac{1-e^{-a_2}}{a_2} + \frac{e^{-(a_1+a_2)}-e^{-a_2}}{a_1}\right].$$

Thus,

$$V = \frac{\sigma^2}{2a_1^2}\frac{2a_1 + 4e^{-a_1} - e^{-2a_1} - 3}{a_1} + \beta^2\frac{\sigma^2}{2a_2^2}\frac{2a_2 + 4e^{-a_2} - e^{-2a_2} - 3}{a_2}$$

$$+\frac{2\beta\sigma^2\rho}{a_2(a_1+a_2)}\left[a_2 - 1 + \frac{(a_2+1)(e^{-a_1}-1)}{a_1} - \frac{2(e^{-(a_1+a_2)}-1)}{a_1+a_2} + \frac{1-e^{-a_2}}{a_2} + \frac{e^{-(a_1+a_2)}-e^{-a_2}}{a_1}\right].$$

Now, we have

$$E(Y_u^{(1)} + \beta Y_u^{(2)}|Z) = k_u Z, \tag{5.10}$$

where $k_u = \mathrm{Cov}(Y_u^{(1)} + \beta Y_u^{(1)}, Z)$

$$= \frac{1}{\sqrt{V}}\left\{\int_0^u \mathrm{Cov}(Y_u^{(1)}, Y_s^{(1)})ds + \beta^2\int_0^u \mathrm{Cov}(Y_u^{(2)}, Y_s^{(2)})ds + 2\beta\int_0^u \mathrm{Cov}(Y_u^{(1)}, Y_s^{(2)})ds\right\}$$

$$= \frac{1}{\sqrt{V}}\left[\frac{\sigma^2}{2a_1}\left\{\frac{1-e^{-a_1u}}{a_1} + \frac{1-e^{-a_1(1-u)}}{a_1} - \frac{e^{-a_1u}-e^{-a_1(1-u)}}{a_1}\right\}\right.$$

$$+\frac{\beta^2\sigma^2}{2a_2}\left\{\frac{1-e^{-a_2u}}{a_2} + \frac{1-e^{-a_2(1-u)}}{a_2} - \frac{e^{-a_2u}-e^{-a_2(1-u)}}{a_2}\right\}$$

$$\left.+2\beta\sigma^2\rho\left\{\frac{a_2 - e^{-a_1u}(a_2+1) + 2e^{-(a_1+a_2)u} + e^{(u-1)a_2} - e^{-(a_1u+a_2)} - 1}{a_2(a_1+a_2)}\right\}\right]. \tag{5.11}$$

Also, we have

$\mathrm{Var}(Y_u^{(1)} + \beta Y_u^{(2)}|Z) = \mathrm{Var}(Y_u^{(1)}) + \beta^2\mathrm{Var}(Y_u^{(2)}) + 2\beta\mathrm{Cov}(Y_u^{(1)}, Y_u^{(2)}) - k_u^2$

$$= \frac{\sigma^2}{2a_1}(1-e^{-2a_1u}) + \beta^2\frac{\sigma^2}{2a_2}(1-e^{-2a_2u}) + 2\beta\sigma^2\rho\frac{1-e^{-(a_1+a_2)u}}{a_1+a_2} - k_u^2 = v_u. \tag{5.12}$$

Once we have these vales of k_u and v_u as given by equations (5.11) and (5.12), we can then easily find the lower bound to the price of the bond conditionally on Z, by using equation (5.10). Thus, the lower bound conditionally on Z is given by

$$\Omega(Z) = \exp\left(-b\int_0^1 \exp\left[k_u Z + \frac{1}{2}v_u\right] du\right). \tag{5.13}$$

Finally, we find the lower bound to the price of the bond by taking expectation over Z, i.e.

$$\int_{-\infty}^{\infty} \Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz. \tag{5.14}$$

For comparison purposes, we use a simulated set of values based on the same values of the parameters as used in the calculation of the lower bound to the price or the contingent payment on the price of the bond. The results showing the approximations to the lower bound to the price of the bond are given in table 9.

Further, as in the situation of the zero coupon bond governed by one driver as described in chapter 2, we can calculate the price of a contingent payment on the price of a bond in this case as well. In that case, we have the function f defined as

$$f(x) = (e^{-bx} - c)^+ = \max((e^{-bx} - c), 0),$$

where c is the strike price at which the contingent payment is made. Thus, once we have obtained $\Omega(Z)$ as defined in equation (5.11), all we need to do is to take the expectation over Z in the appropriate region, i.e. perform the following integration :

$$\int_{-\infty}^{\infty} \max((\Omega(z) - c), 0)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{5.15}$$

5.3 Model 2

Here, we have r_t given by

$$r_t = e^{Y_t^{(1)}} + \gamma e^{\beta Y_t^{(2)}},$$

where β and γ are constants. Here the interest rate process follows a linear combination of two log-Gaussian processes. This situation is thus significantly different from the one discussed in the previous section. We are interested in finding

$$E(e^{-b\int_0^1 r_t dt}) = E\left(e^{-b\int_0^1 (e^{Y_t^{(1)}} + \gamma e^{\beta Y_t^{(2)}})dt}\right).$$

Here, $Y_t^{(1)}$ and $Y_t^{(2)}$ are as defined by equations (5.6) and (5.7). The conditioning factor Z used here is given, as explained in chapter 3 section 3, by

$$Z = \frac{\int_0^1 Y_s^{(1)} ds + \gamma \int_0^1 \beta Y_s^{(2)} ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s^{(1)} ds + \gamma \int_0^1 \beta Y_s^{(2)} ds)}}. \tag{5.16}$$

Further, Z has a standard normal distribution.

Now,

$$\mathrm{Var}(\int_0^1 Y_s^{(1)} ds) + \gamma^2\beta^2 \mathrm{Var}(\int_0^1 Y_s^{(2)} ds) + 2\gamma\beta \mathrm{Cov}(\int_0^1 Y_s^{(1)} ds, \int_0^1 Y_s^{(2)} ds) = V, \quad \text{say}, \tag{5.17}$$

where

$$\mathrm{Var}(\int_0^1 Y_s^{(1)} ds) = \frac{\sigma^2}{2a_1}\frac{2a_1 + 4e^{-a_1} - e^{-2a_1} - 3}{a_1^2},$$

$$\mathrm{Var}(\int_0^1 Y_s^{(2)} ds) = \frac{\sigma^2}{2a_2}\frac{2a_2 + 4e^{-a_2} - e^{-2a_2} - 3}{a_2^2}$$

and

$$\mathrm{Cov}(\int_0^1 Y_s^{(1)} ds, \int_0^1 Y_s^{(2)} ds) = \int_0^1 \int_0^1 \mathrm{Cov}(Y_u^{(1)}, Y_v^{(2)}) dv du$$

$$= \int_0^1 \int_0^1 \left[\sigma^2 \rho \int_0^{u\wedge v} e^{-a_1(u-s)} e^{-a_2(v-s)} ds\right] dv du$$

$$= \sigma^2 \rho \int_0^1 \left[\int_0^u \int_0^v e^{-a_1(u-s)} e^{-a_2(v-s)} ds dv + \int_u^1 \int_0^v e^{-a_1(u-s)} e^{-a_2(v-s)} ds dv\right] du$$

$$= \sigma^2 \rho \int_0^1 \left[\frac{a_2 - e^{-a_1 u}(a_2+1) + 2e^{-(a_1+a_2)u} + e^{(u-1)a_2} - e^{-(a_1 u + a_2)}}{a_2(a_1+a_2)}\right] du$$

$$= \frac{\sigma^2\rho}{a_2(a_1+a_2)}\left[a_2 - 1 + \frac{(a_2+1)(e^{-a_1}-1)}{a_1} - 2\frac{e^{-(a_1+a_2)}-1}{a_1+a_2} + \frac{1-e^{-a_2}}{a_2} + \frac{e^{-(a_1+a_2)}-e^{-a_2}}{a_1}\right].$$

Now, we have

$$E(Y_u^{(i)}|Z) = k_u^{(i)} Z \quad i = 1,2 \tag{5.18}$$

where

$$k_u^{(i)} = \mathrm{Cov}(Y_u^{(i)}, Z) = \frac{1}{\sqrt{V}}\mathrm{Cov}(Y_u^{(i)}, \int_0^1 Y_s^{(1)} ds + \gamma \int_0^1 \beta Y_s^{(2)} ds)$$

$$\text{and} \quad \mathrm{Var}(Y_u^{(i)}|Z) = \mathrm{Var}(Y_u^{(i)}) - (k_u^{(i)})^2 = v_u^{(i)} \quad \text{say.} \tag{5.19}$$

Here,

$$\mathrm{Var}(Y_u^{(i)}) = \frac{\sigma^2}{2a_i}(1 - e^{-2a_i u}). \tag{5.20}$$

Further,

$k_u^{(1)} = \frac{1}{\sqrt{V}}\mathrm{Cov}(Y_u^{(1)}, \int_0^1 Y_s^{(1)} ds + \gamma \int_0^1 \beta Y_s^{(2)} ds)$

$$= \frac{1}{\sqrt{V}}\left\{\int_0^u \mathrm{Cov}(Y_u^{(1)}, Y_s^{(1)}) ds + \gamma\beta \int_0^u \mathrm{Cov}(Y_u^{(1)}, Y_s^{(2)}) ds\right\}$$

$$= \frac{1}{\sqrt{V}}\left[\frac{\sigma^2}{2a_1}\left\{\frac{1-e^{-a_1 u}}{a_1} + \frac{1-e^{-a_1(1-u)}}{a_1} - \frac{e^{-a_1 u}-e^{-a_1(1+u)}}{a_1}\right\}\right.$$

$$\left. + \gamma\beta\sigma^2\rho\left\{\frac{a_2 - e^{-a_1 u}(a_2+1) + 2e^{-(a_1+a_2)u} + e^{(u-1)a_2} - e^{-(a_1 u + a_2)}}{a_2(a_1+a_2)}\right\}\right] \tag{5.21}$$

and

$k_u^{(2)} = \frac{1}{\sqrt{V}}\mathrm{Cov}(\beta Y_u^{(2)}, \int_0^1 Y_s^{(1)} ds + \gamma \int_0^1 \beta Y_s^{(2)} ds)$

$$= \frac{1}{\sqrt{V}}\left\{\gamma\beta^2 \int_0^u \mathrm{Cov}(Y_u^{(2)}, Y_s^{(2)}) ds + \beta \int_0^u \mathrm{Cov}(Y_u^{(1)}, Y_s^{(2)}) ds\right\}$$

$$= \frac{1}{\sqrt{V}}\left[\frac{\gamma\beta^2\sigma^2}{2a_2}\left\{\frac{1-e^{-a_2 u}}{a_2} + \frac{1-e^{-a_2(1-u)}}{a_2} - \frac{e^{-a_2 u}-e^{-a_2(1+u)}}{a_2}\right\}\right.$$

$$\left. + \beta\sigma^2\rho\left\{\frac{a_2 - e^{-a_1 u}(a_2+1) + 2e^{-(a_1+a_2)u} + e^{(u-1)a_2} - e^{-(a_1 u + a_2)}}{a_2(a_1+a_2)}\right\}\right]. \tag{5.22}$$

Once we have these values of $k_u^{(1)}$ and $k_u^{2)}$ as given by equations (5.21) and (5.22), we can find $v_u^{(1)}$ and $v_u^{(2)}$ by using these values of $k_u^{(1)}$ and $k_u^{2)}$ and the unconditional variance of $Y_u^{(i)}$ given by equation (5.20). Thus, the lower bound to the price of the bond conditionally on Z is given by

$$\Omega(Z) = \exp\left(-b\left\{\int_0^1 \exp\left[k_u^{(1)}Z + \frac{1}{2}v_u^{(1)}\right]du + \gamma\int_0^1 \exp\left[k_u^{(2)}Z + \frac{1}{2}v_u^{(2)}\right]du\right\}\right). \quad (5.23)$$

Finally, we can find the lower bound to the price of the bond by taking expectation over Z, i.e.

$$\int_{-\infty}^{\infty} \Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \quad (5.24)$$

For comparison purposes, we use a simulated set of values based on the same values of the parameters as used in the calculation of the lower bound to the price or the contingent payment on the price of the bond. The results showing the approximations to the lower bound to the price of the bond are given in table 10.

Further, as in the situation of the zero coupon bond governed by one driver as discussed in chapter 2, here also, we can calculate the price of a contingent payment on the price of a bond. In that case, we have the function f defined as

$$f(x) = (e^{-bx} - c)^+ = \max((e^{-bx} - c), 0),$$

where c is the strike price at which the contingent payment is made. Thus, once we have obtained $\Omega(Z)$ as defined in equation (5.23), all we need to do is to take the expectation over Z in the appropriate region, i.e. perform the following integration :

$$\int_{-\infty}^{\infty} \max((\Omega(z) - c), 0)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \quad (5.25)$$

5.4 Calculations

We obtain the approximations to the lower bound to the price of the bond in both case - that is when the interest rate r_t remains a log-normal process as well as when r_t is the sum of two log-normal processes. In each case, the results are presented in tables 9 and 10 respectively. We take different values for the parameters. The values taken are $\gamma = 1$, $\beta = 1$, $a_1 = 1$, $a_2 = 2$. Also, b takes the value 0.07 in the first case where the interest rate process is still a log-normal process. In the second situation, when the interest rate process is a sum of log-normal distributions, $b = 0.03$. The values of ρ considered are -0.5, 0 and 0.5 and for each value of ρ, we take σ to take values 0.1, 0.5 and 0.75.

5.5 Conclusion and Remarks

An important point to note in the situation of *multi-driver* models is the dependence of the conditioning factor on the value of the constants. In both the cases, model 1 and model 2, as discussed earlier, the conditioning factor is dependent on the value of the constants β_i and γ_i, for $i = 1, 2, ..., n$. Thus, in choosing a conditioning factor that is appropriate, we cannot ignore the presence of these constants without compromising on the accuracy of the calculations.

In some cases, the calculated lower bounds that we obtain by using the conditioning factor approach, gives us results which are very slightly above the simulated values. This is primarily due to small inaccuracies in the numerical integration procedure used - in effect the lower bounds thus calculated are so accurate that even a slight inaccuracy pushes the calculated values to be above the simulated values. This problem can be avoided by using finer sub-divisions of the interval while performing the numerical integration.

5.6 Tables

In the following section, we present two tables. The calculated value refers to the value of the lower bound to the price obtained by using the conditioning factor. For comparison purposes, in each table we also present a simulated set of values along with their standard errors. We present the results for prices of bonds for both the situations of the interest rate r_t as defined in sections 5.2 and 5.3.

The following two tables show the lower bound of the calculated prices and the simulated values with the standard errors of simulation for different values of ρ and σ. Here $a_1 = 1$, $a_2 = 2$ and $\gamma = 1$.

Table 9 : Here the interest rate follows a log-normal process and b = *0.07.*

ρ	σ	Calculated Price	Simulated Price	Standard Error
0	0.1	93.239	93.224	0.0015
	0.5	92.874	92.864	0.0081
	0.75	92.37	92.362	0.0134
0.5	0.1	93.232	93.218	0.0018
	0.5	92.672	92.656	0.0102
	0.75	91.876	91.893	0.0176
-0.5	0.1	93.247	93.232	0.0011
	0.5	93.066	93.055	0.0057
	0.75	92.824	92.792	0.009

Table 10 : Here the interest rate follows a sum of two log-normal processes. Also, b = *0.03* and $\beta = 1$.

ρ	σ	Calculated Price	Simulated Price	Standard Error
0	0.1	94.185	94.172	0.0002
	0.5	94.024	94.015	0.0034
	0.75	93.806	93.796	0.0054
0.5	0.1	94.185	94.167	0.0008
	0.5	94.024	94.012	0.0041
	0.75	93.807	93.798	0.0065
-0.5	0.1	94.185	94.17	0.0005
	0.5	94.023	94.003	0.0025
	0.75	93.806	93.796	0.0041

Chapter 6

The Pricing of Options on Assets with Stochastic Volatility

6.1 Introduction

An interesting problem in mathematical finance with a widespread applicability is the pricing of European call options on assets with stochastic volatility. Problems of this nature were addressed by Hull and White (1987). They observed that using a simple log - normal model, as used by Black - Scholes (1973), frequently overprices the price of the asset. Hull and White looked at the pricing of European call options on assets with stochastic volatility. The price of an asset according to Hull and White, under an equivalent martingale measure [see Harrison and Krepps (1979) and Harrison and Pliska (1981)] follows the following stochastic process :

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}], \tag{6.1}$$

$$dV_t = \mu dt + dB_t^{(1)}, \tag{6.2}$$

where X_t is the price process, σ is the instantaneous variance of the price process and r is the rate of interest, which is a constant. Equation (6.2) describes the volatility process and μ is the drift of the Brownian motion defining the volatility process. A slight variation from the Hull and White set up is the volatility process following an Ornstein - Uhlenbeck process (as used by Stein and Stein (1991)) and is given by

$$dV_t = -aV_t dt + dB_t^{(1)}, \tag{6.3}$$

where a is the mean reversion force of the Ornstein - Uhlenbeck process.

Further, $B_t^{(1)}$ and $B_t^{(2)}$ are two independent standard Brownian motions. The two processes, the price process, X_t, and the volatility processes, V_t, are in general correlated with ρ being the correlation between V_t and the logarithm of X_t.

Work in the area of option pricing has also been done by Williard (1996). He has used a conditional Monte Carlo technique to calculate the prices of the derivatives. Wiggins (1987) has numerically solved the call option valuation problem for a fairly general continuous stochastic process for return volatility. He has obtained the estimators for the volatility process parameters, the estimates being obtained for several individual stocks and indices. He has also looked at the relative implied volatilities in the sample from which he obtains the estimates. Various other models have also been used in this field. Stein and Stein (1991) have used an Ornstein - Uhlenbeck process as the volatility process as given by equation (6.3). The price process is similar to the one given by equation (6.1). However, they do not assume an exponential link between the price and the volatility processes as shown in the equations (6.1) and (6.3). Work in this area has also been done by Romano and Touzi (1997) in which they consider the price and the volatility process to be correlated with each other. They use the solutions to a set of partial differential equations to solve the problem, but offer no closed form solutions to the problem.

Here we want to calculate the prices of European call options on assets with stochastic volatility. Mathematically, it is given by

$$X_0\{e^{-r}E(e^{Y_1} - b)^+\} = f(Y_1) \quad \text{say,}$$

where b is the strike price at which the value of the option is calculated, r is rate of interest and X_0 is the current price. Also, $Y_t = ln(\frac{X_t}{X_0})$ where X_t is the price process and is given as

$$Y_t = Y_0 + rt + \sigma \int_0^t \rho e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)} + \sigma \int_0^t \sqrt{1-\rho^2} e^{\frac{kB_s^{(1)}}{2}} dB_s^{(2)} - \frac{1}{2}\sigma^2 \int_0^t e^{kB_s^{(1)}} ds.$$

This is the same as equation (6.7) and the way we obtain it is described in the next section. Further, Y_1 is the value of Y_t at time t = *1*. The exact form of Y_1 is given later in equation (6.8).

To calculate the price of the call option, we will use a conditioning factor approach. This approach is similar to the Rogers and Shi (1995) technique of valuing an Asian option as well as the method of pricing bonds we have discussed in the previous two chapters. The only difference here being that in this case the function f as defined above is not convex and hence Jensen's inequality cannot be used to obtain a lower bound to the price. We will thus try to find an approximation to the price of the call option itself, rather than try to find a lower bound to the price of the option.

Now, the interest rate r defined in equation (6.1) could be a constant, as in the Hull and White model. However, it could be stochastic in nature and thus, we could have a stochastic process $\{r_t, 0 \leq t \leq 1\}$ defining the interest rate process (see equation (6.41)). We look at the two situations separately.

6.2 Constant Interest Rate

6.2.1 The Simple One Dimensional Brownian Motion Problem

The situation when the volatility process follows a standard Brownian motion is exactly similar to the Hull and White model, with the drift in the volatility process being 0. Thus the stochastic volatility process and the price process is explicitly defined as

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}], \tag{6.4}$$

$$dV_t = dB_t^{(1)}. \tag{6.5}$$

We are interested in finding

$$X_0\{e^{-r} E(e^{Y_1} - b)^+\}, \tag{6.6}$$

where b is the strike price at which the value of the option is calculated and X_0 is the current price. Also, let $Y_t = ln(\frac{X_t}{X_0})$. Taking logarithm of equation (6.4) and then integrating it, we get

$$Y_t = Y_0 + rt + \sigma \int_0^t \rho e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)} + \sigma \int_0^t \sqrt{1-\rho^2} e^{\frac{kB_s^{(1)}}{2}} dB_s^{(2)} - \frac{1}{2}\sigma^2 \int_0^t e^{kB_s^{(1)}} ds. \tag{6.7}$$

We set $t = 1$ and $Y_0 = 0$. Thus equation (6.5) gives us

$$Y_1 = r + \sigma \int_0^1 \rho e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)} + \sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kB_s(1)}{2}} dB_s^{(2)} - \frac{1}{2}\sigma^2 \int_0^1 e^{kB_s^{(1)}} ds. \tag{6.8}$$

Here, conditionally on the paths of $\{B_s^{(1)}, 0 \leq s \leq 1\}$, we have $\sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kB_s^{(1)}}{2}} dB_s^{(2)}$ following a normal distribution with zero mean and variance $\sigma^2(1-\rho^2)\int_0^1 e^{kB_s^{(1)}} ds$.

Also, conditioned on the path $\{B_s^{(1)}, 0 \leq s \leq 1\}$, Y_1 follows a normal distribution with mean $(r - \frac{1}{2}\sigma^2 P + \rho\sigma Q)$ and variance $\sigma^2(1-\rho^2)P$, where

$$P = \int_0^1 e^{kB_s^{(1)}} ds \quad \text{and} \quad Q = \int_0^1 e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)}.$$

Now, Q consists of a stochastic integral and to calculate the stochastic integral we need to express it terms of time integrals. Using Itô calculus, we have

$$d\left[\exp\left(\frac{k}{2}B_t^{(1)}\right)\right] = \frac{k}{2}\exp\left(\frac{k}{2}B_t^{(1)}\right) dB_t^{(1)} + \frac{1}{2}\left(\frac{k}{2}\right)^2 \exp\left(\frac{k}{2}B_t^{(1)}\right) dt.$$

Now, integrating both sides in the range [0,1], we have,

$$\exp\left(\frac{k}{2}B_t\right) - 1 = \frac{k}{2}\int_0^1 \exp\left(\frac{k}{2}B_t^{(1)}\right) dB_t^{(1)} + \frac{(\frac{k}{2})^2}{2}\int_0^1 \exp\left(\frac{k}{2}B_t^{(1)}\right) dt$$

$$\Rightarrow Q = \int_0^1 \exp\left(\frac{kB_s^{(1)}}{2}\right) dB_s^{(1)} = \left\{\frac{\exp\left(\frac{kB_1}{2}\right) - 1}{\frac{k}{2}} - \frac{1}{2}\frac{k}{2}\int_0^1 \exp\left(\frac{kB_s^{(1)}}{2}\right) ds\right\}. \tag{6.9}$$

The second term of equation (6.9) is similar to P, the only difference being that in the exponent, instead of having a k as in P, it now has a $\frac{k}{2}$. So it can be calculated exactly the same way as P, replacing k by $\frac{k}{2}$.

We suggest an approximation approach as given by the following lemma :

Lemma 6.1 : *Let P, Q and Z be random variables. Also, let σ and ρ be constants. Then, assuming*

1. *σ is small*

2. *$\Psi(\sigma^2 P, \rho\sigma Q)$ is a function such that it is at least twice differentiable and piecewise continuous*

3. Z is used as a conditioning factor and is suitably normalised

we have

$$E(\Psi(\sigma^2 P, \rho\sigma Q)) = E[\Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))]$$

$$+\frac{1}{2}\rho\sigma^2 E\{\Psi_{22}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\mathrm{Var}(Q|Z)\} + O(\sigma^3). \tag{6.10}$$

Proof : Expanding $\Psi(\sigma^2 P, \rho\sigma Q)$ in a Taylor series expansion (in terms of σ) conditioned on a suitable random variable Z, appropriately normalised, we have,

$$E\left[\Psi(\sigma^2 P, \rho\sigma Q)\right] = E\{E[\Psi(\sigma^2 P, \rho\sigma Q)|Z]\} = E[\Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))]$$

$$+E\{\sigma^2\Psi_1(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))[P - E(P|Z)] + \rho\sigma\Psi_2(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))[Q - E(Q|Z)]\}$$

$$+E\{\frac{1}{2}\rho^2\sigma^2\Psi_{22}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))[Q - E(Q|Z)]^2\} + O(\sigma^3)$$

$$= E[\Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))] + \frac{1}{2}\rho^2\sigma^2 E\{\Psi_{22}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\mathrm{Var}(Q|Z)\} + O(\sigma^3).$$

■

Note here that Ψ_1 indicates the first derivative with respect to the first argument of Ψ, Ψ_2 indicates the first derivative with respect to the second argument of Ψ and Ψ_{22} indicates the second derivative with respect to the second argument of Ψ. It is easy to see that all other terms will be of order σ^3 or higher.

In this case, let us define

$$\Psi(\sigma^2 P, \rho\sigma Q) = (e^{Y_1} - b)^+ = \max[(e^{Y_1} - b), 0],$$

where Y_1 is given by equation (6.8). Further, P and Q are of the form as described earlier. Also, as stated previously, σ is the instantaneous variance of the price process and ρ is the covariance between the volatility process and the logarithm of the price process. $\Psi(\sigma^2 P, \rho\sigma Q)$ is piecewise continuous and differentiable and hence the second derivative of $\Psi(\sigma^2 P, \rho\sigma Q)$

exists. We are interested in finding

$$E\left[\Psi(\sigma^2 P, \rho\sigma Q)\right] = E(e^{Y_1} - b)^+ = E[\max((e^{Y_1} - b), 0)]$$

$$= \exp\left(r - \frac{1}{2}\sigma^2\rho^2 P + \rho\sigma Q\right)\Phi\left(\frac{r + \frac{1}{2}\sigma^2 P(1-2\rho^2) + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right)$$

$$-b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 P + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right). \tag{6.11}$$

Also, the second derivative of $\Psi(\sigma^2 P, \rho\sigma Q)$ with respect to Q is given by

$$\Psi_{QQ}(\sigma^2 P, \rho\sigma Q) = \left\{\exp\left(r + \rho\sigma Q - \frac{1}{2}\sigma^2\rho^2 P\right)\Phi\left(\frac{r + \rho\sigma Q + \frac{1}{2}\sigma^2(1-2\rho^2)P - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right)\right.$$

$$\left. + \frac{r + \rho\sigma Q - \frac{1}{2}\sigma^2\rho^2 P}{\sqrt{2\sigma^2\pi(1-\rho^2)P}}\exp\left(-\frac{(r + \rho\sigma Q + \frac{1}{2}\sigma^2(1-2\rho^2)P - lnb)^2}{2\sigma^2(1-\rho^2)P}\right)\right\}. \tag{6.12}$$

Equation (6.11) represents the first term approximation to the price of the call option. However, the first term alone does not approximate the price well enough as is evident from the tables given later. So, we need the second term in *Lemma 6.1.* We shall call the second term in *Lemma 6.1* the *Correction Factor.*

To calculate $E[\Psi(\sigma^2 P, \rho\sigma Q)]$, we make use of *Lemma 6.1.* Thus, we first calculate $\Omega(Z)$, where

$$\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z)).$$

Here, Z is a suitably chosen conditioning factor and has a standard normal distribution. Finally, to get the unconditional value of the first term approximation to the price, we take the expectation of $\Omega(Z)$ with respect to Z.

Similarly, to obtain the correction factor we define $\Theta(Z)$ as

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\text{Var}(Q|Z).$$

This is exactly the same as the second term in *Lemma 6.1.* To get the unconditional value of the correction factor, we take the expectation of $\Theta(Z)$ with respect to Z.

Now, to calculate the required prices, we need to calculate $E(P|Z)$ and $E(Q|Z)$ as well as $\text{Var}(Q|Z)$. To do this, we first need to find out a value j_u such that the conditional expectation is independent of the conditioning factor. We also need to get the conditional variance and covariance of the standard Brownian motion conditioned on the conditioning factor Z, i.e. we need to obtain $\text{Var}(\int_0^1 B_u du|Z)$ and $\text{Cov}(\int_0^1 B_u du, \int_0^1 B_v dv|Z)$. As has been shown in chapter 3, the conditioning factor used here is

$$Z = \frac{\int_0^1 B_s ds}{\sqrt{\text{Var}(\int_0^1 B_s ds)}} \tag{6.13}$$

where $\text{Var}(\int_0^1 B_s ds) = \frac{1}{3}$. This is the same as obtained in chapter 3. Thus, we have

$$E(B_u|Z) = j_u Z \tag{6.14}$$

where $j_u = \text{Cov}(B_u, Z) = \frac{1}{\sqrt{\text{Var}(\int_0^1 B_s ds)}} \text{Cov}(B_u, \int_0^1 B_s ds)$

$$= \sqrt{3} \int_0^u (1-s)ds = \sqrt{3}(u - \frac{u^2}{2}). \tag{6.15}$$

Also,

$$\text{Cov}(B_u, B_v|Z) = (u \wedge v) - j_u j_v = s_{uv}. \tag{6.16}$$

Moreover, B_u conditioned on Z is a Gaussian process.

Once we have these values, we can then easily get the expected values of P and Q. We do that by first getting the expected values conditionally on Z and finally taking the expectations over Z to yield the unconditional expectations. Thus we have

$$E(P|Z) = \int_0^1 \exp\left(k j_u Z + \frac{k^2}{2} s_{uu}\right) du \tag{6.17}$$

$$E(Q|Z) = \left\{ \frac{\exp\left(\frac{k}{2}\frac{\sqrt{3}}{2} Z + \frac{k^2}{4}\frac{1}{8}\right) - 1}{\frac{k}{2}} - \frac{k}{4} \int_0^1 \exp\left(\frac{k}{2} j_u Z + \frac{k^2}{8} s_{uu}\right) du \right\} \tag{6.18}$$

$\Rightarrow$ Conditionally on Z

$\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \exp\left(r - \frac{1}{2}\sigma^2\rho^2 E(P|Z) + \rho\sigma E(Q|Z)\right)\Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)$$

$$-b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 E(P|Z) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right). \tag{6.19}$$

Thus the first term approximation to the price of the call option is obtained by taking the expectation of $\Omega(Z)$ with respect to Z, i.e.

$$H_1 = \int_{-\infty}^{\infty} \Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz. \tag{6.20}$$

As noted earlier, the first term alone does not approximate the lower bound of the price of the option accurately enough. Thus, we have to use the second term of *Lemma 6.1* - the correction factor. To calculate the correction factor, we need to calculate Var($Q|Z$) and $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$.

Now, continuing from equation (6.12), we have

$\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \left\{\exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right)\Phi\left(\frac{r + \rho\sigma E(Q|Z) + \frac{1}{2}\sigma^2(1-2\rho^2)E(P|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)\right.$$

$$\left. + \frac{r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)}{\sqrt{2\sigma^2\pi(1-\rho^2)E(P|Z)}}\exp\left(-\frac{(r + \rho\sigma E(Q|Z) + \frac{1}{2}\sigma^2(1-2\rho^2)E(P|Z) - lnb)^2}{2\sigma^2(1-\rho^2)E(P|Z)}\right)\right\}.$$

Also,

$$\text{Var}(Q|Z) = \left\{\text{Var}(\frac{e^{\frac{kB_1}{2}} - 1}{\frac{k}{2}}|Z) + \frac{k^2}{16}\text{Var}(\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z) - \frac{k}{2}\text{Cov}(\frac{e^{\frac{kB_1}{2}} - 1}{\frac{k}{2}}, \int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z)\right\}.$$

We thus have,

$$\text{Var}(\frac{e^{\frac{kB_1}{2}} - 1}{\frac{k}{2}}|Z) = \frac{1}{\frac{k^2}{4}}\left[e^{\frac{\sqrt{3}Zk}{2}}e^{\frac{k^2}{8}} - e^{\frac{\sqrt{3}Zk}{2}}e^{\frac{k^2}{16}}\right] = \frac{4}{k^2}\left[e^{\frac{\sqrt{3}Zk}{2}}(e^{\frac{k^2}{8}} - e^{\frac{k^2}{16}})\right],$$

$$\mathrm{Var}(\int_0^1 e^{\frac{kB_s}{2}}ds|Z) = \int_0^1\int_0^1 \left[\left(e^{\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv}+2s_{uv})}\right) - \left(e^{\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv})}\right)\right] dudv$$

$$= \int_0^1\int_0^1 \exp\left(\frac{k}{2}(j_u+j_v)Z + \frac{k^2}{8}(s_{uu}+s_{vv})\right)\left[\exp\left(\frac{k^2}{4}s_{uv}\right) - 1\right] dudv$$

and

$$\mathrm{Cov}(e^{\frac{kB_1}{2}}, \int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z) = \int_0^1 \exp\left(\frac{k}{2}(j_u+j_1)Z + \frac{k^2}{8}(s_{uu}+s_{11}) + \frac{k^2}{4}s_{1u}\right) du$$

$$-\left[\exp\left(\frac{\sqrt{3}Zk}{2} + \frac{k^2}{32}\right)\int_0^1 \exp\left(\frac{k}{2}j_u Z + \frac{k^2}{8}s_{uu}\right) du\right].$$

Having obtained these values, we can easily find the value of $\mathrm{Var}(Q|Z)$ and using $\mathrm{Var}(Q|Z)$ and $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$, we can find the correction factor, conditionally on Z, given by

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\mathrm{Var}(Q|Z).$$

To calculate the correction factor, we take the expectation of $\Theta(Z)$ i.e. we calculate

$$H_2 = \int_{-\infty}^{\infty} \Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \tag{6.21}$$

Finally, we just add up the calculated values of the prices and the corresponding correction factor to get an approximation to price of the option. Thus, the Corrected Calculated Price is given by

$$100e^{-r}(H_1 + H_2) \tag{6.22}$$

where H_1 is the first term approximation to the price, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

6.2.2 The Ornstein - Uhlenbeck Case

In this case, we have the volatility process following an Ornstein Uhlenbeck process. The tendency of an Ornstein Uhlenbeck process to move towards a long - term average value (see Stein and Stein (1991)) makes it a more realistic model for the volatility process. As in the situation of the volatility process following a Brownian motion, here also, we have two independent standard Brownian motions, $B_t^{(1)}$ and $B_t^{(2)}$. Further, the volatility process and the logarithm of the price are correlated with correlation co-efficient ρ. Also, r, σ and k denote exactly the same thing as earlier, the only extra term being a - the mean reversion force of the Ornstein - Uhlenbeck process. Thus the stochastic process defining this set up under an equivalent martingale measure (Harrison & Kreps (1979) and Harrison & Pliska (1981)) is given by

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}], \tag{6.23}$$

$$dV_t = -aV_t dt + dB_t^{(1)}. \tag{6.24}$$

As before, we are interested in finding

$$X_0\{e^{-r}E(e^{Y_1} - b)^+\},$$

where b is the strike price, X_0 is the current price of the asset and Y_1, the price of the asset at time t = *1*, is as defined in equation (6.25). Here also, we take $Y_t = ln(\frac{X_t}{X_0})$, and assume t=1 and $Y_0 = 0$. Proceeding in a similar manner as before, we get

$$Y_1 = r + \sigma \int_0^1 \rho e^{\frac{kV_s}{2}} dB_s^{(1)} + \sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kV_s}{2}} dB_s^{(2)} - \frac{1}{2}\sigma^2 \int_0^1 e^{kV_s} ds. \tag{6.25}$$

Again, conditionally on the paths of $\{B_s^{(1)}, 0 \le s \le 1\}$, we have $\sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kV_s}{2}} dB_s^{(2)}$ following a normal distribution with zero mean and variance $\sigma^2(1-\rho^2)\int_0^1 e^{kV_s} ds$.

Also, Y_1 conditioned on the path $\{B_t^{(1)}; 0 \le t \le 1\}$ follows a normal distribution with mean A and variance Σ^2, where

$$A = r - \frac{1}{2}\sigma^2 \int_0^1 e^{kV_t} dt + \sigma \int_0^1 \rho e^{\frac{kV_t}{2}} dB_t^{(1)} \tag{6.26}$$

$$\Sigma^2 = \sigma^2(1-\rho^2) \int_0^1 e^{kV_t} dt \tag{6.27}$$

Let us, as in the case of the Brownian motion, define

$$P = \int_0^1 e^{kV_t} dt \quad \text{and} \quad Q = \int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)}.$$

Thus,

$$A = r - \frac{1}{2}\sigma^2 P + \rho\sigma Q$$
$$\text{and} \quad \Sigma^2 = \sigma^2(1-\rho^2)P.$$

We use the argument similar to the one used in the case of the volatility following a simple Brownian motion to express $\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)}$ in terms of time integrable terms. Thus, using Itô calculus

$$d\left[\exp\left(\frac{k}{2}V_t\right)\right] = \frac{k}{2}\exp\left(\frac{k}{2}V_t\right)\left[-aV_t dt + dB_t^{(1)}\right] + \frac{(\frac{k}{2})^2}{2}\exp\left(\frac{k}{2}V_t\right) dt.$$

Integrating both sides over the range [0,1], as T is assumed to be 1, we have

$$\exp\left(\frac{k}{2}V_1\right) - 1 = -\frac{k}{2}\int_0^1 aV_t\exp\left(\frac{k}{2}V_t\right) dt + \frac{k}{2}\int_0^1 \exp\left(\frac{k}{2}V_t\right) dB_t^{(1)} + \frac{(\frac{k}{2})^2}{2}\int_0^1 \exp\left(\frac{k}{2}V_t\right) dt$$

$$\Rightarrow Q = \int_0^1 \exp\left(\frac{kV_t}{2}\right) dB_t^{(1)} = \left\{\frac{\exp\left(\frac{kV_1}{2}\right) - 1}{\frac{k}{2}} - \frac{k}{4}\int_0^1 \exp\left(\frac{kV_t}{2}\right) dt + a\int_0^1 V_t\exp\left(\frac{kV_t}{2}\right) dt\right\}. \tag{6.28}$$

As before, let us define

$$\Psi(\sigma^2 P, \rho\sigma Q) = (e^{Y_1} - b)^+ = \max[(e^{Y_1} - b), 0]$$

where Y_1 is given by equation (6.25). Again, we are interested in finding

$$E[\Psi(\sigma^2 P, \rho\sigma Q)] = E(e^{Y_1} - b)^+ = \exp\left(A + \frac{\Sigma^2}{2}\right)\Phi\left(\frac{A + \Sigma^2 - lnb}{\sqrt{\Sigma^2}}\right) - b\Phi\left(\frac{A - lnb}{\sqrt{\Sigma^2}}\right)$$

$$= \exp\left(r - \rho^2 P + Q\right)\right)\Phi\left(\frac{r + \frac{1}{2}\sigma^2 P(1-2\rho^2) + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right) - b\Phi\left(\frac{r - \frac{1}{2}P + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right). \tag{6.29}$$

Equation (6.29) represents the first term approximation to the price of the option. However, as stated earlier, the first term alone does not approximate the price well enough - a fact reflected in the tables given later. Thus, we also need the second term of *Lemma 6.1* - in effect the *Correction Factor.*

To calculate $E[\Psi(\sigma^2 P, \rho\sigma Q)]$, we make use of *Lemma 6.1.* Thus, we first calculate $\Omega(Z)$, where

$$\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z)).$$

Here, Z is a suitably chosen conditioning factor and has a standard normal distribution. To get the unconditional value of the first term approximation to the price, we take the expectation of $\Omega(Z)$ with respect to Z.

Similarly, to obtain the correction factor we define $\Theta(Z)$ as

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\text{Var}(Q|Z).$$

This is exactly the same as the second term in *Lemma 6.1.* Finally, to get the unconditional value of the correction factor, we take the expectation of $\Theta(Z)$ with respect to Z.

Thus to calculate the price, we make use of the same conditioning argument as earlier. The conditioning factor used here is

$$Z = \frac{\int_0^1 V_s ds}{\sqrt{\text{Var}(\int_0^1 V_s ds)}} \tag{6.30}$$

where

$$\text{Var}(\int_0^1 V_s ds) = \int_0^1 \left\{\frac{1 - e^{-a(1-s)}}{a}\right\}^2 ds = \frac{2a - (1 - e^{-a})(3 - e^{-a})}{2a^3}.$$

The justification of using the above form of Z as a conditioning factor has been shown in chapter 3.

To calculate $\Omega(Z)$ and $\Theta(Z)$, we first find out a j_u such that the conditional expectation is independent of the conditioning factor. We also need to find the conditional variance and covariance of the Ornstein - Uhlenbeck process $\{V_t, 0 \leq t \leq 1\}$. Thus, we have,

$$E(V_u|Z) = j_u Z, \tag{6.31}$$

where

$$j_u = \text{Cov}(V_u, Z) = \frac{1}{\sqrt{\text{Var}(\int_0^1 V_s ds)}}\text{Cov}(V_u, \int_0^1 V_s ds)$$

$$= \frac{1}{\sqrt{\mathrm{Var}(\int_0^1 V_s ds)}} \left[\int_0^u E(V_u V_t)dt + \int_u^1 E(V_u V_t)dt\right]$$

$$= \frac{1}{\sqrt{\mathrm{Var}(\int_0^1 V_s ds)}} \left[\int_0^u \frac{e^{-au} sinh(at)}{a} dt + \int_u^1 \frac{e^{-at} sinh(au)}{a} dt\right]$$

$$= \frac{1}{\sqrt{\mathrm{Var}(\int_0^1 V_s ds)}} \frac{e^{-au}(cosh(au) - 1) + sinh(au)\{e^{-au} - e^{-a}\}}{a^2}$$

$$= \sqrt{\frac{2}{a}}\left[\frac{e^{-au}\{\cosh(au) + \sinh(au)\} - e^{-au} - e^{-a}\sinh(au)}{\sqrt{2a - (1 - e^{-a})(3 - e^{-a})}}\right]. \qquad (6.32)$$

Also,

$$\mathrm{Cov}(V_u, V_v|Z) = \mathrm{Cov}(V_u, V_v) - j_u j_v = \left(\frac{e^{a|u-v|} - e^{-a(u+v)}}{2}\right) - j_u j_v = s_{uv} \qquad (6.33)$$

Moreover, V_u conditioned on Z is a Gaussian process.

Once we have these values, we can easily calculate the values of $E(P|Z)$ and $E(Q|Z)$.

$$E(P|Z) = \int_0^1 \exp\left(k j_u Z + \frac{k^2}{2} s_{uu}\right) du, \qquad (6.34)$$

$$E(Q|Z) = \left\{ \frac{\exp\left(\frac{kLZ}{2} + \frac{k^2}{8}\{\frac{1-e^{-2a}}{2a} - L^2\}\right) - 1}{\frac{k}{2}} - \int_0^1 \frac{k}{4}\left[\exp\left(\frac{k}{2} j_u Z + \frac{k^2}{8} s_{uu}\right)\right] du \right.$$

$$\left. + a\int_0^1 \left[j_u Z + \frac{1}{2}\left(\frac{1 - e^{-2au}}{2a} - j_u^2\right)\right] \exp\left(\frac{j_u Z}{2} + \frac{1}{8}\left[\frac{1 - e^{-2au}}{2a} - j_u^2\right]\right) du \right\} \qquad (6.35)$$

where $L = \frac{(1-e^{-a})^2}{2a^2 B}$ and $B = \sqrt{\frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}}$.

Thus, conditionally on Z, we have

$$\Omega(Z) = \exp\left(r - \frac{1}{2}\sigma^2\rho^2 E(P|Z) + \rho\sigma E(Q|Z)\right) \Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1 - 2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1 - \rho^2)E(P|Z)}}\right)$$

$$-b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 E(P|Z) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1 - \rho^2)E(P|Z)}}\right). \qquad (6.36)$$

To obtain the first term in the approximation to the price of the call option, as given by *Lemma 6.1*, we take the expectation of $\Omega(Z)$ with respect to Z. Thus, the first term approximation to the price of the option is

$$H_1 = \int_{-\infty}^{\infty} \Omega(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \tag{6.37}$$

To obtain the correction factor, the second term in *Lemma 6.1*, we proceed exactly the same way as in case of the volatility process following a Brownian motion. We need the terms $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$ and $\text{Var}(Q|Z)$.

Now,

$\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \Bigg[\exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right) \Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)$$

$$+ \frac{\exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right)}{\sqrt{2\sigma^2\pi(1-\rho^2)E(P|Z)}} \exp\left(-\frac{(r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb)^2}{2\sigma^2(1-\rho^2)E(P|Z)}\right)\Bigg]$$

and

$$\text{Var}(Q|Z) = \Bigg\{\text{Var}(\frac{e^{\frac{kV_1}{2}} - 1}{\frac{k^2}{2}}|Z) + \frac{k^2}{16}\text{Var}(\int_0^1 e^{\frac{kV_t}{2}} dt|Z) + a^2\text{Var}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z)$$

$$+2a\text{Cov}(\frac{e^{\frac{kV_1}{2}} - 1}{\frac{k^2}{2}}, \int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z) - \frac{k}{2}\text{Cov}(\frac{e^{\frac{kV_1}{2}} - 1}{\frac{k^2}{2}}, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$$

$$-\frac{ak}{2}\text{Cov}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)\Bigg\}$$

$$= I_1 + \frac{k^2}{16} I_2 + a^2 I_3 + 2aI_4 - \frac{k}{2} I_5 - \frac{ak}{2} I_6$$

where,

$I_1 = [\exp(kLz)\{\exp(\frac{k^2}{2}(\frac{1-e^{-2a}}{2a} - L^2)) - \exp(\frac{k^2}{4}[\frac{1-e^{-2a}}{2a} - L^2])\}]$,

$I_2 = \text{Var}(\int_0^1 \exp(\frac{kV_t}{2})dt|Z) = \int_0^1\int_0^1 \exp(\frac{k}{2}(j_u + j_v)Z + \frac{k^2}{8}[s_{uu} + s_{vv}])\{\exp(\frac{k^2}{4}s_{uv}) - 1\}dudv$,

$I_3 = \text{Var}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z) = \int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}])[\exp(\frac{k^2}{4}s_{tu})s_{tu} - 1]dtdu$

$$+ \int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}] + \frac{k^2}{4}s_{tu})\{j_t Z + \frac{k}{2}(s_{tt} + s_{tu})\}\{j_u Z + \frac{k}{2}(s_{uu} + s_{tu})\}dtdu,$$

$I_4 = \text{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z)$

$$= \int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{(j_t Z + \frac{k}{2}(s_{tt} + s_{1t}))\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1t}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}d$$

$I_5 = \text{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$

$$= \frac{1}{\frac{k^2}{2}} \int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1u}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}dt,$$

$I_6 = \text{Cov}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$

$$= \int_0^1 \int_0^1 \exp(\frac{k}{2}[j_s + j_t]Z + \frac{k^2}{8}[s_{ss} + s_{tt}])\{[\exp(\frac{k^2}{4}s_{st})(j_t Z + \frac{k}{2}\{s_{ss} + s_{ts}\})] - [j_t Z + \frac{k}{2}s_{tt}]\}dsdt.$$

Here $L = \frac{(1-e^{-a})^2}{2a^2 M}$ and $M = \sqrt{\text{Var}(\int_0^1 V_s ds)} = \sqrt{\frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}}$.

Once we have all the values of I_1, I_2, I_3, I_4, I_5 and I_6, we can easily have the value of $\text{Var}(Q|Z)$. Knowing $E(P|Z)$ and $E(Q|Z)$, as given by equation (6.34) and (6.35), we can find $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$. The correction factor $\Theta(Z)$, conditionally on Z, is given as

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\text{Var}(Q|Z).$$

Thus, the correction factor is obtained by taking the expectation of $\Theta(Z)$ with respect to Z and is given by

$$H_2 = \int_{-\infty}^{\infty} \Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{6.38}$$

Finally, we just add up the calculated values of the prices and the corresponding correction factor to get an approximation to the price of the option. Thus, the Corrected Calculated Price is given by

$$100e^{-r}(H_1 + H_2) \tag{6.39}$$

where H_1 is the first term approximation to the price, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

6.3 Stochastic Interest Rates

The validity of a constant interest rate, as discussed in the previous section, might not hold in all circumstances. In this section, we look at the situation where the interest rate process $\{r_t, 0 \leq t \leq 1\}$ is itself stochastic in nature. Empirically it has been observed that interest rates have a tendency to move towards a long - term average value. Using this empirical knowledge, we shall model the interest rate process as an Ornstein - Uhlenbeck process. In fact, we shall take the interest rate to follow the Vasicek (1977) model. Now, generalising from equations (6.1) and (6.2) and the Hull and White approach, the price of a derivative asset with stochastic interest rate under an equivalent martingale measure [see Harrison and Krepps (1979) and Harrison and Pliska (1981)] follows the following stochastic process :

$$dX_t = r_t X_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\sqrt{1-\rho^2} dB_t^{(3)} + \rho_2 dB_t^{(2)} + \rho_1 dB_t^{(1)}] \tag{6.40}$$

$$dr_t = -b(r_t - r^*)dt + \phi[\sqrt{1-\gamma^2} dB_t^{(2)} + \gamma dB_t^{(1)}] \tag{6.41}$$

$$dV_t = \mu dt + dB_t^{(1)} \tag{6.42}$$

where $\{V_t, 0 \leq t \leq 1\}$ is the volatility process, μ is the drift of the Brownian motion defining the volatility process, b is the mean reversion force of the Ornstein - Uhlenbeck process defining the interest rate process and r^* is the long term interest rate value. As in the case of constant interest rates, the volatility process could follow an Ornstein - Uhlenbeck process as well and is given by

$$dV_t = -aV_t dt + dB_t^{(1)} \tag{6.43}$$

where a is the mean reversion force of the Ornstein - Uhlenbeck process defining the volatility process. $B_t^{(1)}$, $B_t^{(2)}$ and $B_t^{(3)}$ are three independent Brownian motions. Also, $\{r_t, 0 \leq t \leq 1\}$ is the interest rate process and X_t is the price process. Further, the volatility process, the interest rate process and the logarithm of the price process are correlated amongst themselves. We thus have, γ as the correlation between the volatility process and the interest rate process, ρ_1 as the correlation between the volatility process and the logarithm of the price process and ρ_2 as the correlation between the price and the interest rate processes. Also ρ_1 and ρ_2 are such that $\rho^2 + \rho_1^2 + \rho_2^2 = 1$.

As before, we look at the situations of the volatility process following a simple Brownian motion and that of it following an Ornstein Uhlenbeck process separately.

6.3.1 The Simple One Dimensional Brownian Motion Problem

We first discuss the situation of the volatility process following a standard Brownian motion, with the drift of the Brownian motion being 0. Thus, the price process $\{X_t, 0 \leq t \leq 1\}$, interest rate process $\{r_t, 0 \leq t \leq 1\}$ and the volatility process $\{V_t, 0 \leq t \leq 1\}$ are defined as

$$dX_t = r_t X_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\sqrt{1-\rho^2} dB_t^{(3)} + \rho_2 dB_t^{(2)} + \rho_1 dB_t^{(1)}], \tag{6.44}$$

$$dr_t = -b(r_t - r^*)dt + \phi[\sqrt{1-\gamma^2} dB_t^{(2)} + \gamma dB_t^{(1)}], \tag{6.45}$$

$$dV_t = dB_t^{(1)}. \tag{6.46}$$

We are interested in finding

$$X_0 E[e^{-\int_0^1 r_s ds}(e^{Y_1} - c)^+], \tag{6.47}$$

where c is the strike price at which the value of the option is calculated and X_0 is the current price. The difference in this situation from the one in section 6.2.1., equation (6.6) is that here the interest rate is not a constant and hence cannot be taken out of the expectation. Now, let $Y_t = ln(\frac{X_t}{X_0})$ and $t = 1$. Then taking logarithm and then integrating equation (6.44), we have

$$Y_1 = Y_0 + \int_0^1 r_t dt + \sigma\sqrt{1-\rho^2} \int_0^1 e^{\frac{kB_t^{(1)}}{2}} dB_t^{(3)} + \sigma\rho_1 \int_0^1 e^{\frac{kB_t^{(1)}}{2}} dB_t^{(1)}$$

$$+\sigma\rho_2 \int_0^1 e^{kB_t^{(1)}} dB_t^{(1)} - \frac{1}{2}\sigma^2 \int_0^1 e^{kB_t^{(1)}}. \tag{6.48}$$

Further, on integrating equation (6.45), we have

$$r_u = r^* + (r_0 - r^*)e^{-bu} + \phi\gamma \int_0^u e^{-b(u-s)} dB_s^{(1)} + \phi\sqrt{1-\gamma^2} \int_0^u e^{-b(u-s)} dB_s^{(2)} \tag{6.49}$$

$$\Rightarrow R_1 = \int_0^1 r_u du = r^* + (r_0 - r^*)\frac{1-e^{-b}}{b}$$

$$+\phi\gamma \int_0^1 \int_0^u e^{-b(u-s)} dB_s^{(1)} du + \phi\sqrt{1-\gamma^2} \int_0^1 \int_0^u e^{-b(u-s)} dB_s^{(2)} du \tag{6.50}$$

$$= r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(1)} + \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(2)}. \quad (6.51)$$

Let us assume $Y_0 = 0$ and on replacing $\int_0^1 r_t dt$ in equation (6.48) by the expression of $\int_0^1 r_t dt$ as given by equation (6.51), we have

$$Y_1 = r^+(r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(1)} + \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(2)}$$

$$+\sigma\sqrt{1-\rho^2}\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(3)} + \sigma\rho_1\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(1)} + \sigma\rho_2\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(2)} - \frac{1}{2}\sigma^2\int_0^1 e^{kB_s^{(1)}}ds. \quad (6.52)$$

Let

$$C_1 = \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(2)} + \sigma\rho_2\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(2)}.$$

Now, conditionally on the paths of $\{B_s^{(1)}, 0 \le s \le 1\}$, Y_1 and R_1 have a bivariate normal distribution with means

$$r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(1)} + \sigma\rho_1\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(1)} - \frac{1}{2}\sigma^2\int_0^1 e^{kB_s^{(1)}}ds$$

and

$$r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b}dB_s^{(1)},$$

variances

$\sigma^2(1-\rho^2)\int_0^1 e^{kB_s^{(1)}}ds + \phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3} + \sigma^2\rho_2^2\int_0^1 e^{kB_s^{(1)}}ds$

$$+\phi\sigma\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kB_s^{(1)}}{2}}\frac{1-e^{b(s-1)}}{b}ds$$

and

$$\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}$$

respectively. Further, the covariance is given by

$$\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3} + \sigma\phi\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kB_s^{(1)}}{2}}\frac{1-e^{b(s-1)}}{b}ds.$$

Writing R_1 for $\int_0^1 r_s ds$ in equation (6.47), we are thus interested in

$$E[e^{-R_1}(e^{Y_1} - c)^+]. \tag{6.53}$$

Let us define R_1' such that

$$R_1' = R_1 - pY_1$$

where

$$\text{Cov}(R_1', Y_1) = \text{Cov}(R_1, Y_1) - p\text{Var}(Y_1) = 0$$

$$\Rightarrow p = \frac{\text{Cov}(R_1, Y_1)}{\text{Var}(Y_1)}.$$

Substituting in equation (6.53), we have

$$e^{-R_1}(e^{Y_1} - c)^+ = e^{-(R_1'+pY_1)}(e^{Y_1} - c)^+ = e^{-R_1'}(e^{-pY_1}(e^{Y_1} - c)^+) = \Lambda(Y_1, R_1) \quad \text{say.}$$

We are interested in finding

$$E[\Lambda(Y_1, R_1)] = E[e^{-R_1'}]E[e^{-pY_1}(e^{Y_1} - c)^+]. \tag{6.54}$$

As before, we use a conditioning factor to obtain an approximation to the price of the call option. The volatility process follows a Brownian motion. Thus, following the explanation given in chapter 3, the conditioning factor Z is given as

$$Z = \frac{\int_0^1 B_s^{(1)} ds}{\sqrt{\text{Var}(\int_0^1 B_s^{(1)} ds)}},$$

where $\text{Var}(\int_0^1 B_s^{(1)} ds) = \frac{1}{3}$ and Z has a standard normal distribution. This conditioning factor is similar to the one used by Rogers and Shi (1995) in valuing an Asian option. Thus, we have

$$E(B_u|Z) = j_u Z \tag{6.55}$$

where $j_u = \text{Cov}(B_u, Z) = \frac{1}{\sqrt{\text{Var}(\int_0^1 B_s ds)}} \text{Cov}(B_u, \int_0^1 B_s ds)$

$$= \sqrt{3} \int_0^u (1 - s) ds = \sqrt{3}(u - \frac{u^2}{2}). \tag{6.56}$$

Also,

$$\mathrm{Cov}(B_u, B_v|Z) = (u \wedge v) - j_u j_v = s_{uv}. \tag{6.57}$$

Note here that the three stochastic integrals that we need to calculate to evaluate $E(Y_1)$ and $E(R_1)$ are

$$\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)},$$

$$\int_0^1 e^{kB_s^{(1)}} ds$$

and

$$\int_0^1 e^{\frac{k}{2}B_s^{(1)}} dB_s^{(1)}.$$

Now, the last two are exactly the same as P and Q as defined in section 6.2.1. However, to calculate $\int_0^1 e^{\frac{k}{2}B_s^{(1)}} dB_s^{(1)}$, we need to find a g_1 such that conditioning on Z, $\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)} - g_1 Z$ is independent of Z. Here g_1 is given by

$$g_1 = Cov(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)}, Z) = \int_0^1 \frac{1 - e^{-b(1-s)}}{b}(1 - s)ds$$

$$= \sqrt{3}\{\frac{1}{2b} + e^{-b}\frac{b+1}{b^3} - \frac{1}{b^3}\}. \tag{6.58}$$

Also,

$$\mathrm{Var}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s - g_1 Z) = \mathrm{Var}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s) - 2g_1 \mathrm{Cov}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s, Z) + g_1^2$$

$$= \int_0^1 (\frac{1 - e^{-b(1-s)}}{b})^2 ds - g_1^2 = \frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} - g_1^2. \tag{6.59}$$

This implies, conditionally on the paths of $\{B_s^{(1)}, 0 \le s \le 1\}$, R_1 follows a normal distribution with mean

$$r^* + (r_0 - r^*)\frac{1 - e^{-b}}{b} + \phi\gamma g_1 Z$$

and variance

$$\phi^2\left(\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}\right)-\phi^2\gamma^2g_1^2.$$

Let us define the L = $Cov(Y_1, R_1|Z)$, $A_1 = E(Y_1|Z)$, $\Sigma_1^2 = Var(Y_1|z)$, $A_2 = E(R_1|Z)$ and $\Sigma_2^2 = Var(R_1|Z)$. We thus have,

$$L=\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}+\sigma\phi\rho_2\sqrt{1-\gamma^2}E\left[\int_0^1 e^{\frac{kB_s^{(1)}}{2}}\left(\frac{1-e^{-b(1-s)}}{b}\right)ds|Z\right]$$

$$=\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}+\sigma\phi\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{(\frac{kk_uZ}{2}+\frac{1}{2}s_{uu})}\left(\frac{1-e^{-b(1-u)}}{b}\right)du, \tag{6.60}$$

$$A_1=r^*+(r_0-r^*)\frac{1-e^{-b}}{b}+\phi\gamma g_1Z+\sigma\rho_1E\left[\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(1)}|Z\right]-E\left[\frac{1}{2}\sigma^2\int_0^1 e^{kB_s^{(1)}}ds|Z\right]$$

$$=r^*+(r_0-r^*)\frac{1-e^{-b}}{b}+\phi\gamma g_1Z+\sigma\rho_1E[Q|Z]-\frac{1}{2}\sigma^2E[P|Z],$$

where

$$P=\int_0^1 e^{kB_s^{(1)}}ds$$

$$Q=\int_0^1 e^{\frac{kB_s^{(1)}}{2}}dB_s^{(1)}=\frac{e^{\frac{kB_1}{2}}-1}{\frac{k}{2}}-\frac{k}{4}\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds \quad \text{(using Ito calculus (section 6.2.1))}$$

and $E(P|Z)$ and $E(Q|Z)$ are given by equations (6.17) and (6.18) respectively.Also,

$$\Sigma_1^2=\sigma^2(1-\rho^2)\int_0^1 e^{kk_uZ+\frac{1}{2}s_{uu}}du+\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}+\sigma^2\rho_2^2\int_0^1 e^{kk_uZ+\frac{1}{2}s_{uu}}du$$

$$+\sigma\phi\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kk_uZ}{2}+\frac{1}{2}s_{uu}}\frac{1-e^{-b(1-u)}}{b}du-\phi^2\gamma^2g_1^2, \tag{6.61}$$

$$A_2=r^*+(r_0-r^*)\frac{1-e^{-b}}{b}+\phi\gamma g_1Z, \tag{6.62}$$

$$\Sigma_2^2=\phi^2\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}-\phi^2\gamma^2g_1^2. \tag{6.63}$$

Now, conditionally on Z, Y_1 follows a normal distribution with mean A_1 and variance Σ_1^2 and R_1 follows a normal distribution with mean A_2 and variance Σ_2^2.

To calculate $E[\Lambda(Y_1, R_1)]$, we use the technique used in the previous section. We first find the expected value of $\Lambda(Y_1, R_1)$ conditionally on Z and then take expectation over Z to obtain the unconditional value. Now, conditionally on Z, we have

$$E[\Lambda(Y_1, R_1|Z)] = E[R_1'|Z]E[e^{-pY_1}(e^{Y_1} - c)^+|Z] = \Delta(Z) \quad \text{say.} \tag{6.64}$$

Now, $E[\Delta(Z)]$ is exactly the same as the first term in *Lemma 6.1.* To calculate $E[\Delta(Z)]$, we need to calculate $E[R_1'|Z]$and $E[e^{-pY_1}(e^{Y_1} - c)^+|Z]$. Here

$$E[R_1'|Z] = \exp\left(-[A_2 + pA_1] + \frac{1}{2}\left[\Sigma_1^2 - \frac{L}{\Sigma_2^2}\right]\right). \tag{6.65}$$

Next, we need to find

$$E[e^{-pY_1}(e^{Y_1} - c)^+|Z] = \int_{\ln c}^{\infty} e^{-pY_1}(e^{Y_1} - c)\frac{1}{\sqrt{2\pi}\Sigma_1}e^{-\frac{(Y_1-A_1)^2}{2\Sigma_1^2}}dY_1$$

$$= \int_{\ln c}^{\infty} e^{(1-p)Y_1}\frac{1}{\sqrt{2\pi}\Sigma_1}e^{-\frac{(Y_1-A_1)^2}{2\Sigma_1^2}}dY_1 - c\int_{\ln c}^{\infty} e^{-pY_1}\frac{1}{\sqrt{2\pi}\Sigma_1}e^{-\frac{(Y_1-A_1)^2}{2\Sigma_1^2}}dY_1$$

$$= I_1 - cI_2 \quad \text{say.}$$

Let us first prove the following lemma which we will then use to obtain the values of I_1 and I_2.

Lemma 6.2: *Let Y have a normal distribution with mean A and variance Σ^2, then*

$$\int_G^{\infty} e^{\phi Y}\frac{1}{\sqrt{2\pi}\Sigma}exp\left(-\frac{(Y-A)^2}{2\Sigma^2}\right)dY = exp\left(A\phi + \frac{\Sigma^2\phi^2}{2}\right)\Phi\left(\frac{A + \Sigma^2\phi - G}{\Sigma}\right).$$

Proof : We have,

$$\int_G^{\infty} e^{\phi Y}\frac{1}{\sqrt{2\pi}\Sigma}\exp\left(-\frac{(Y-A)^2}{2\Sigma^2}\right)dY$$

$$= \int_G^{\infty} \exp\left(-\frac{1}{2\Sigma^2}(Y - [A + \phi\Sigma^2])^2 + \frac{\Sigma^2\phi^2}{2} + A\phi\right)\frac{1}{\sqrt{2\pi}\Sigma}dY$$

$$= \exp\left(A\phi + \frac{\Sigma^2\phi^2}{2}\right)\left\{1 - \Phi\left(\frac{G - [A + \phi\Sigma^2]}{\Sigma}\right)\right\}$$

$$= \exp\left(A\phi + \frac{\Sigma^2\phi^2}{2}\right)\Phi\left(\frac{A + \Sigma^2\phi - G}{\Sigma}\right).$$

■

To obtain I_1, we replace ϕ by *(1 - p)*, to obtain I_2, we replace ϕ by $-p$. In both cases, G takes the value of *ln c*. Thus, conditionally on Z, the first term approximation to the price of the call option given by

$$\Delta(Z) = \left[\exp\left(-A_2 + \frac{1}{2}[\Sigma_2^2 + \Sigma_1^2 - 2L]\right)\exp\left(A_1(1 - \frac{2L}{\Sigma_1^2})\right)\Phi\left(\frac{A_1 + \Sigma_1^2 - L - lnc}{\Sigma_1^2}\right)\right]$$

$$-c\left[\exp\left(-A_2 + \frac{\Sigma_2^2}{2}\right)\exp\left(-A_1\frac{2L}{\Sigma_1^2}\right)\Phi\left(\frac{A_1 - L - lnc}{\Sigma_1^2}\right)\right]. \quad (6.66)$$

This implies that the first order approximation to the price of the call option is obtained by taking the expectation of $\Psi(Z)$ with respect to Z, i.e.

$$H_1 = \int_{-\infty}^{\infty}\Delta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \quad (6.67)$$

As in the case of constant interest rate, the first term on *Lemma 6.1* alone does not approximate the price accurately enough. So, we need the second term of the *Lemma 6.1* - the correction factor. The correction factor is calculated exactly in the same way as for constant interest rate. As before, we look at the second derivative of $\Delta(Z)$ with respect to A_1 (A_1 is as defined earlier in the section) and multiply it with the variance of Q conditionally on Z. The reason for looking only at the derivative with respect to A_1 is because the approximation error arises from A_1 only; it is in A_1 that we approximate a stochastic integral by the final value of the stochastic process and a time integral. Also, Q is as defined earlier in this section as well as by equation (6.9).

Thus, we have

$$\frac{\partial^2}{\partial A_1^2}\Delta(z) = \Delta''(z) = \exp\left(-A_2 + \tfrac{1}{2}[\Sigma_2^2 + \Sigma_1^2 - 2L] + A_1[1 - \tfrac{2L}{\Sigma_1^2}]\right)\Phi\left(\tfrac{A_1 + \Sigma_1^2 - L - ln\,c}{\Sigma_1}\right)\left(1 - \tfrac{2L}{\Sigma_1^2}\right)^2$$

$$-\frac{4L^2}{\Sigma_1^4}\exp\left(-A_1 + \frac{\Sigma_2^2}{2} + lnc - \frac{2LA_1}{\Sigma_1^2}\right)\Phi\left(\frac{A_1 - L - lnc}{\Sigma_1}\right)$$

$$+\frac{1}{\sqrt{2\pi\Sigma_1^2}}\exp\left(-A_2+\frac{\Sigma_2^2}{2}+lnc-\frac{2LA_1}{\Sigma_1^2}\right)\exp\left(-\frac{(A_1-L-lnc)^2}{2\Sigma_1^2}\right). \tag{6.68}$$

Also,

$$\mathrm{Var}(Q|Z)=\left\{\mathrm{Var}(\frac{e^{\frac{kB_1}{2}}-1}{\frac{k}{2}}|Z)+\frac{k^2}{16}\mathrm{Var}(\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z)-\frac{k}{2}\mathrm{Cov}(\frac{e^{\frac{kB_1}{2}}-1}{\frac{k}{2}},\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z)\right\},$$

where

$$\mathrm{Var}(\frac{e^{\frac{kB_1}{2}}-1}{\frac{k}{2}}|Z)=\frac{1}{\frac{k^2}{4}}\left[\exp\left(\frac{\sqrt{3}Zk}{2}\right)\exp\left(\frac{k^2}{8}\right)-\exp\left(\frac{\sqrt{3}Zk}{2}\right)\exp\left(\frac{k^2}{16}\right)\right]$$

$$=\frac{4}{k^2}\left[\exp\left(\frac{\sqrt{3}Zk}{2}\right)\left\{\exp\left(\frac{k^2}{8}\right)-\exp\left(\frac{k^2}{16}\right)\right\}\right],$$

$$\mathrm{Var}(\int_0^1 e^{\frac{kB_s}{2}}ds|Z)=\int_0^1\int_0^1\left[\left(e^{\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv}+2s_{uv})}\right)-\left(e^{\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv})}\right)\right]dudv$$

$$=\int_0^1\int_0^1\exp\left(\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv})\right)\left[\exp\left(\frac{k^2}{4}s_{uv}\right)-1\right]dudv$$

and

$$\mathrm{Cov}(e^{\frac{kB_1}{2}},\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z)=\int_0^1\exp\left(\frac{k}{2}(j_u+j_1)Z+\frac{k^2}{8}(s_{uu}+s_{11})+\frac{k^2}{4}s_{1u}\right)du$$

$$-\left[\exp\left(\frac{\sqrt{3}Zk}{2}+\frac{k^2}{32}\right)\int_0^1\exp\left(\frac{k}{2}j_uZ+\frac{k^2}{8}s_{uu}\right)du\right].$$

Having obtained these values, we have the correction factor, conditionally on Z, is given by

$$\Theta(Z)=\frac{1}{2}\rho_1^2\sigma^2\Delta''(z)\mathrm{Var}(Q|Z). \tag{6.69}$$

Finally, to calculate the correction factor, we take the expectation of $\Theta(Z)$ with respect to Z, i.e., we calculate

$$H_2=\int_{-\infty}^{\infty}\Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{6.70}$$

Once we obtain the calculated value of the price and the corresponding correction factor all that is left to do is to add up the two values, the calculated price and the correction factor, to get the value of the price of the option comparable to the actual price. Thus, the Corrected Calculated Price is given by

$$X_0(H_1 + H_2), \tag{6.71}$$

where H_1 is the first order approximation to the price of the call option, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

6.3.2 The Ornstein - Uhlenbeck Case

Here, we take the volatility process to follow an Ornstein - Uhlenbeck process. As discussed in the situation of constant interest rates, the Ornstein - Uhlenbeck process is a more realistic model for the volatility process. Furthermore, the Brownian Motion can be regarded as a special case of the Ornstein - Uhlenbeck process with the mean reversion force *a = 0.* Thus, we have the price process $\{X_t, 0 \leq t \leq 1\}$, interest rate process $\{r_t, 0 \leq t \leq 1\}$ and the volatility process $\{V_t, 0 \leq t \leq 1\}$ given as

$$dX_t = r_t X_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho_1 dB_t^{(1)} + \rho_2 dB_t^{(2)} + \sqrt{1-\rho^2} dB_t^{(3)}], \tag{6.72}$$

$$dr_t = -b(r_t - r^*)dt + \phi[\gamma dB_t^{(1)} + \sqrt{1-\gamma^2} dB_t^{(2)}], \tag{6.73}$$

$$dV_t = -aV_t dt + dB_t^{(1)}. \tag{6.74}$$

Here all the parameters are the same as in the case of the volatility following a Brownian motion and a is the mean reversion force of the Ornstein - Uhlenbeck process defining the volatility process. We also assume the initial value of the Ornstein - Uhlenbeck process V_0 to be zero. We are interested in finding

$$X_0 E[e^{-\int_0^1 r_s ds}(e^{Y_1} - c)^+] \tag{6.75}$$

where c is the strike price at which the value of the option is calculated and X_0 is the current price.

As in the earlier case, let us define $Y_t = ln\frac{X_t}{X_0}$. Further, let us also assume $Y_0 = 0$ and $t = 1$. Thus, on integration of equation (6.72) we have

$$Y_1 = \int_0^1 r_t dt + \sigma\sqrt{1-\rho^2}\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(3)} + \sigma\rho_1\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)}$$

$$+\sigma\rho_2\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)} - \frac{1}{2}\sigma^2\int_0^1 e^{kV_t}dt. \tag{6.76}$$

Also, let $R_1 = \int_0^1 r_t dt$

$$= r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(1)} + \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(2)}. \tag{6.77}$$

Equations (6.76) and (6.77) imply

$$Y_1 = r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(1)} + \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(2)}$$

$$+\sigma\sqrt{1-\rho^2}\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(3)} + \sigma\rho_1\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)} + \sigma\rho_2\int_0^1 e^{\frac{kV_t}{2}} dB_t^{(2)} - \frac{1}{2}\sigma^2\int_0^1 e^{kV_t}dt. \tag{6.78}$$

Further, R_1 is normally distributed with mean

$$r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(1)}$$

and variance

$$\phi^2\gamma^2\frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3}.$$

As in the case of the volatility process following a Brownian motion, let us define

$$C_1 = \phi\sqrt{1-\gamma^2}\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(2)} + \sigma\rho_2\int_0^1 e^{\frac{kV_t}{2}} dB_s^{(2)}. \tag{6.79}$$

Now, conditionally on the paths of $\{V_s, 0 \le s \le 1\}$ i.e. on the paths of $\{B_s^{(1)}, 0 \le s \le 1\}$, Y_1 and R_1 have a bivariate normal distribution with means

$$r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{(b(s-1)}}{b} dB_s^{(1)} + \sigma\rho_1\int_0^1 e^{\frac{kV_t}{2}} dB_s^{(1)} - \frac{1}{2}\sigma^2\int_0^1 e^{kV_s}ds$$

and

$$r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma\int_0^1 \frac{1-e^{b(s-1)}}{b} dB_s^{(1)},$$

variances

$$\sigma^2(1-\rho^2)\int_0^1 e^{kV_s}ds + \phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3} + \sigma^2\rho_2^2\int_0^1 e^{kV_s}ds$$

$$+\phi\sigma\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kV_s}{2}}\frac{1-e^{b(s-1)}}{b}ds$$

and

$$\phi^2\gamma^2\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}$$

respectively. Further, the covariance is given by

$$\phi^2(1-\gamma^2)\frac{2b+4e^{-b}-e^{-2b}-3}{2b^3}+\sigma\phi\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kV_s}{2}}\frac{1-e^{b(s-1)}}{b}ds.$$

Writing R_1 for $\int_0^1 r_s ds$ in equation (6.75), we are thus interested in

$$E[e^{-R_1}(e^{Y_1}-c)^+]. \tag{6.80}$$

Let us define R_1' such that

$$R_1' = R_1 - pY_1$$

where

$$\text{Cov}(R_1', Y_1) = \text{Cov}(R_1, Y_1) - p\text{Var}(Y_1) = 0$$

$$\Rightarrow p = \frac{\text{Cov}(R_1, Y_1)}{\text{Var}(Y_1)}.$$

Thus, we have

$$e^{-R_1}(e^{Y_1}-c)^+ = e^{-(R_1'+pY_1)}(e^{Y_1}-c)^+ = e^{-R_1'}[e^{-pY_1}(e^{Y_1}-c)^+] = \Lambda(Y_1, R_1).$$

We are interested in finding

$$E[\Lambda(Y_1, R_1)] = E[e^{-R_1'}]E[e^{-pY_1}(e^{Y_1}-c)^+]. \tag{6.81}$$

This is exactly similar to equation (6.54).

As before, we use a conditioning factor to obtain a lower bound to the price of the call option. The volatility process follows an Ornstein - Uhlenbeck process. Thus, following the explanation given in chapter 3, the conditioning factor Z is given as

$$Z = \frac{\int_0^1 V_s ds}{\sqrt{\text{Var}(\int_0^1 V_s ds)}},$$

where

$$\text{Var}(\int_0^1 V_s^{(1)} ds) = \int_0^1 \left\{ \frac{1 - e^{-a(1-s)}}{a} \right\}^2 ds = \frac{2a - (1 - e^{-a})(3 - e^{-a})}{2a^3} = V \quad \text{say.}$$

Further, Z has a standard normal distribution. Thus, we have

$$E(V_u|Z) = j_u Z, \tag{6.82}$$

where $j_u = \frac{1}{\sqrt{V}}\text{Cov}(V_u, \int_0^1 V_s ds)$

$$= \sqrt{\frac{2}{a}} \left[\frac{e^{-au}\left\{cosh(au) + sinh(au)\right\} - e^{-au} - e^{-a} sinh(au)}{\sqrt{2a - (1 - e^{-a})(3 - e^{-a})}} \right]. \tag{6.83}$$

Also,

$$\text{Cov}(V_u, V_v|Z) = \text{Cov}(V_u, V_v) - j_u j_v = \frac{e^{a|u-v|} - e^{-a(u+v)}}{2} = s_{uv}. \tag{6.84}$$

Note here that the three stochastic integrals that we need to calculate to evaluate $E(Y_1)$ and $E(R_1)$ are

$$\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)},$$

$$\int_0^1 e^{kV_s^{(1)}} ds$$

and

$$\int_0^1 e^{\frac{k}{2}V_s^{(1)}} dB_s^{(1)}.$$

Now, the last two are exactly the same as P and Q as defined in section 6.2.2. to calculate $\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)}$, we need to find a g_1 such that conditioning on Z, $\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)} - g_1 Z$ is independent of Z. Here g_1 is given by

$$g_1 = \text{Cov}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)}, Z) = \frac{1}{\sqrt{V}}\text{Cov}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)}, \int_0^1 e^{-as} \int_0^u e^{au} dB_u^{(1)} ds)$$

$$= \frac{1}{\sqrt{V}}\text{Cov}(\int_0^1 \frac{1 - e^{-b(1-s)}}{b} dB_s^{(1)}, \int_0^1 \int_u^1 e^{-as} ds e^{au} dB_u^{(1)})$$

$$= \frac{1}{\sqrt{V}} \mathrm{Cov}(\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)}, \int_0^1 \frac{e^{-au}-e^{-a}}{a} e^{au} dB_u^{(1)}$$

$$= \frac{1}{\sqrt{V}} \left[\int_0^1 \frac{1-e^{-b(1-s)}}{b} \frac{1-e^{-a(1-s)}}{a} ds \right]$$

$$= \frac{1 - \frac{1-e^{-b}}{b} - \frac{1-e^{-a}}{a} + \frac{1-e^{-(a+b)}}{a+b}}{\sqrt{b^2(1 - \frac{(1-e^{-a})(3-e^{-a})}{2a})}}. \tag{6.85}$$

Also, $\mathrm{Var}(\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)} - g_1 Z)$

$$= \mathrm{Var}(\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)}) - 2g_1 \mathrm{Cov}(\int_0^1 \frac{1-e^{-b(1-s)}}{b} dB_s^{(1)}, Z) + g_1^2$$

$$= \int_0^1 \left\{ \frac{1-e^{-b(1-s)}}{b} \right\}^2 ds - g_1^2 = \frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} - g_1^2.$$

This implies, conditionally on the paths of $\{V_s; 0 \le s \le 1)$ that is the same as the paths of $\{B_s^{(1)}; 0 \le s \le 1)$, R_1 follows a normal distribution with mean

$$r^* + (r_0 - r^*) \frac{1-e^{-b}}{b} + \phi\gamma g_1 Z$$

and variance

$$\phi^2 \left(\frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} \right) - \phi^2 \gamma^2 g_1^2.$$

Let us define L $= \mathrm{Cov}(Y_1, R_1|Z)$, $A_1 = E(Y_1|Z)$, $A_2 = E(R_1|Z)$, $\Sigma_1^2 = \mathrm{Var}(Y_1|Z)$ and $\Sigma_2^2 = \mathrm{Var}(R_1|Z)$. We thus have

$$L = \phi^2(1-\gamma^2) \frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} + \sigma\phi\rho_2\sqrt{1-\gamma^2} E \left[\int_0^1 e^{\frac{kV_s}{2}} \left(\frac{1-e^{-b(1-s)}}{b} \right) ds | Z \right]$$

$$= \phi^2(1-\gamma^2) \frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} + \sigma\phi\rho_2\sqrt{1-\gamma^2} \int_0^1 e^{(\frac{kj_u Z}{2} + \frac{1}{2} s_{uu})} \left(\frac{1-e^{-b(1-u)}}{b} \right) du, \tag{6.86}$$

$$A_1 = r^* + (r_0 - r^*) \frac{1-e^{-b}}{b} + \phi\gamma g_1 z + \sigma\rho_1 E \left[\int_0^1 e^{\frac{kV_s}{2}} dB_s^{(1)} | Z \right] - \frac{1}{2}\sigma^2 E \left[\int_0^1 e^{kV_s} ds | Z \right]$$

$$= r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma g_1 z + \sigma\rho_1 E[Q|Z] - \frac{1}{2}\sigma^2 E[P|Z], \tag{6.87}$$

where, as defined earlier in section 6.2.2,

$$P = \int_0^1 e^{kV_s} ds$$

and

$$Q = \int_0^1 e^{\frac{k}{2}V_s} dB_s^{(1)} = \frac{e^{\frac{kV_1}{2}} - 1}{\frac{k}{2}} - \frac{k}{4}\int_0^1 e^{\frac{kV_s}{2}} ds + a\int_0^1 V_t e^{\frac{k}{2}V_t} dt \quad \text{(using Itô calculus)},$$

and $E(P|Z)$ and $E(Q|Z)$ are given by equation (6.34) and (6.35) respectively. Also,

$$\Sigma_1^2 = \sigma^2(1-\rho^2)\int_0^1 e^{kj_u Z + \frac{1}{2}s_{uu}} du + \phi^2(1-\gamma^2)\frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} + \sigma^2\rho_2^2\int_0^1 e^{kj_u Z + \frac{1}{2}s_{uu}} du$$

$$+\sigma\phi\rho_2\sqrt{1-\gamma^2}\int_0^1 e^{\frac{kj_u Z}{2} + \frac{1}{2}s_{uu}} \frac{1 - e^{-b(1-u)}}{b} du - \phi^2\gamma^2 g_1^2, \tag{6.88}$$

$$A_2 = r^* + (r_0 - r^*)\frac{1-e^{-b}}{b} + \phi\gamma g_1 Z, \tag{6.89}$$

$$\Sigma_2^2 = \phi^2\frac{2b + 4e^{-b} - e^{-2b} - 3}{2b^3} - \phi^2\gamma^2 g_1^2. \tag{6.90}$$

Now, conditionally on Z, Y_1 follows a normal distribution with mean A_1 and variance Σ_1^2 and R_1 follows a normal distribution with mean A_2 and variance Σ_2^2.

Once we have found all these values, we then proceed to find an approximation to the price of the call option. To do this we use a conditional approach as in all previous situations. Thus, we first look at

$$E[\Lambda(Y_1, R_1|Z)] = E[R_1''|Z]E[e^{-pY_1}(e^{Y_1} - c)^+|Z] = \Delta(Z) \quad \text{say.}$$

This is exactly similar to equation (6.64) when the volatility process is a simple one dimensional Brownian motion. Proceeding exactly in the same manner as in the case of the volatility following a simple Brownian motion, we have using equation (6.65) and *Lemma 6.2*, the first order approximation to the price of the option, conditionally on Z, given by

$$\Delta(z) = \exp\left(-A_2 + \frac{1}{2}[\Sigma_2^2 + \Sigma_1^2 - 2L] + A_1[1 - \frac{2L}{\Sigma_1^2}]\right)\Phi\left(\frac{A_1 + \Sigma_1^2 - L - lnc}{\Sigma_1}\right)$$

$$-c\left[\exp\left(-A_2 + \frac{\Sigma_2^2}{2} - \frac{2LA_1}{\Sigma_1^2}\right)\Phi\left(\frac{A_1 - L - lnc}{\Sigma_1}\right)\right]. \tag{6.91}$$

This is the same as the first term of *Lemma 6.1.* Thus, the first order approximation t to the price of the call option is obtained by taking the expectation of $\Psi(Z)$ with respect to Z; i.e.

$$H_1 = \int_{-\infty}^{\infty} \Delta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{6.92}$$

As noted in all the situations discussed earlier, the first term of *Lemma 6.1* alone does not approximate the price accurately enough. So, we need the second term of the *Lemma 6.1* - the correction factor. To calculate the correction factor we take the product of the second derivative of $\Delta(Z)$ with respect to A_1 and the variance of Q conditionally on Z where Q is as defined earlier in this section and is the same as given by equation (6.28) and A_1 is defined in equation (6.87). The reason for looking only at the derivative with respect to A_1 is because the approximation error arises from A_1 only; it is in A_1 that we approximate a stochastic integral by the final value of the stochastic process and a time integral.

Thus, we have

$$\frac{\partial^2}{\partial A_1^2}\Delta(z) = \Delta''(z) = CF1 - CF2 + CF3, \tag{6.93}$$

where,

$$CF1 = \exp\left(-A_2 + \frac{1}{2}[\Sigma_2^2 + \Sigma_1^2 - 2L] + A_1[1 - \frac{2L}{\Sigma_1^2}]\right)\Phi\left(\frac{A_1 + \Sigma_1^2 - L - lnc}{\Sigma_1}\right)\left(1 - \frac{2L}{\Sigma_1^2}\right)^2,$$

$$CF2 = \frac{4L^2}{\Sigma_1^4}\exp\left(-A_2 + \frac{\Sigma_2^2}{2} + lnc - \frac{2LA_1}{\Sigma_1^2}\right)\Phi\left(\frac{A_1 - L - ln\,c}{\Sigma_1}\right)$$

and $$CF3 = \exp\left(-A_2 + \frac{\Sigma_2^2}{2} + lnc - \frac{2LA_1}{\Sigma_1^2}\right)\frac{1}{\sqrt{2\pi\Sigma_1^2}}\exp\left(-\frac{(A_1 - L - lnc)^2}{2\Sigma_1^2}\right).$$

Now,

$$\mathrm{Var}(Q|Z) = \left\{\mathrm{Var}(\frac{e^{\frac{kV_1}{2}} - 1}{\frac{k^2}{2}}|Z) + \frac{k^2}{16}\mathrm{Var}(\int_0^1 e^{\frac{kV_t}{2}}dt|Z) + a^2\mathrm{Var}(\int_0^1 V_t e^{\frac{kV_t}{2}}dt|Z)\right.$$

$$+2a\mathrm{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z) - \frac{k}{2}\mathrm{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$$

$$-\frac{ak}{2}\mathrm{Cov}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)\Big\}$$

$$= I_1 + \frac{k^2}{16} I_2 + a^2 I_3 + 2aI_4 - \frac{k}{2} I_5 - \frac{ak}{2} I_6$$

where,

$$I_1 = \left[\exp\left(kDZ\right)\left\{\exp\left(\frac{k^2}{2}(\frac{1-e^{-2a}}{2a} - D^2)\right) - \exp\left(\frac{k^2}{4}\left[\frac{1-e^{-2a}}{2a} - D^2\right]\right)\right\}\right],$$

$$I_2 = \mathrm{Var}(\int_0^1 e^{\frac{kV_t}{2}} dt|Z) = \int_0^1\int_0^1 \exp\left(\frac{k}{2}[j_u + j_v]Z + \frac{k^2}{8}[s_{uu} + s_{vv}]\right)\left[\exp\left(\frac{k^2}{4}s_{uv}\right) - 1\right] dudv,$$

$$I_3 = \mathrm{Var}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z) = \int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}])[\exp(\frac{k^2}{4}s_{tu})s_{tu} - 1]dtdu$$

$$+\int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}] + \frac{k^2}{4}s_{tu})\{j_t Z + \frac{k}{2}(s_{tt} + s_{tu})\}\{j_u Z + \frac{k}{2}(s_{uu} + s_{tu})\}dtdu,$$

$I_4 = \mathrm{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 V_t e^{\frac{kV_t}{2}} dt|Z)$

$$= \int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{(j_t Z + \frac{k}{2}(s_{tt} + s_{1t}))\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1t}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}dt,$$

$I_5 = \mathrm{Cov}(\frac{e^{\frac{kV_1}{2}}-1}{\frac{k^2}{2}}, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$

$$= \frac{1}{\frac{k^2}{2}}\int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1u}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}dt,$$

$I_6 = \mathrm{Cov}(\int_0^1 V_t e^{\frac{kV_t}{2}} dt, \int_0^1 e^{\frac{kV_t}{2}} dt|Z)$

$$= \int_0^1\int_0^1 \exp(\frac{k}{2}[j_s + j_t]Z + \frac{k^2}{8}[s_{ss} + s_{tt}])\{[\exp(\frac{k^2}{4}s_{st})(j_t Z + \frac{k}{2}\{s_{ss} + s_{ts}\})] - [j_t Z + \frac{k}{2}s_{tt}]\}dsdt.$$

Here $D = \frac{(1-e^{-a})^2}{2a^2 M}$ and $M = \sqrt{\mathrm{Var}(\int_0^1 V_s ds)} = \sqrt{\frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}}$.

Once we have all the values of I_1, I_2, I_3, I_4, I_5 and I_6, we can easily have the value of $\text{Var}(Q|Z)$. Having obtained $\text{Var}(Q|Z)$, we can easily find the correction factor, conditionally on Z, is given by

$$\Theta(Z) = \frac{1}{2}\rho_1^2\sigma^2[CF1 - CF2 + CF3]\text{Var}(Q|Z). \tag{6.94}$$

Finally, to calculate the correction factor, we take the expectation of $\Theta(Z)$ with respect to Z, i.e., we calculate

$$H_2 = \int_{-\infty}^{\infty} \Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{6.95}$$

Once we have obtained the first order approximation to the price and the corresponding correction factor all that is left to do is to add up the two values to get the approximate price of the option comparable to the actual price. The Corrected Calculated Price is given by

$$X_0(H_1 + H_2), \tag{6.96}$$

where H_1 is the first order approximation to the price of the call option, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

6.4 Calculations

For all cases, we look at various values of the strike price; in fact, we let the strike price vary between 110 and 90 in steps of 5. Further, in the case of the volatility process following an Ornstein - Uhlenbeck process, we let the mean reversion force, a to take values between 0.1 and 10. Also, k = 1 and $\sigma = 0.1$.

For the constant interest rate situation, we allow ρ, the correlation factor, to take any value between ± 1. The case of $\rho = 0$ is included as a specific case of this general framework. The value of the interest rate is taken to be 5%, i.e. r = 0.05.

For the stochastic interest rate situation, we allow ρ_1, ρ_2 and thereby ρ as well as γ to take any value between ± 1. The long term value of the interest rate is taken to be 5 %, i.e.

$r^* = 0.05$ and $\gamma = 0.025$. We also let b, the mean reversion force in the process defining the interest rate to take different values in the range of 2 to 100.

For comparison purposes, we calculate the prices from simulated values for both the volatility processes. Also included are the standard errors of the simulated values.

The results in the case of the volatility process following a Brownian motion with constant interest rate is given in Table 11; while for the volatility process following an Ornstein - Uhlenbeck process with constant interest rate, the results are given in Tables 12.1 to 12.4. The results in the case of the volatility process following a Brownian motion with stochastic interest rates are given in Tables 13.1 and 13.2; while for the volatility process following an Ornstein - Uhlenbeck process with stochastic interest rates, the results are given in Tables 14.1 and 14.2.

6.5 Implied Volatilities

We first discuss the situation of constant interest rate. Here, we have two stochastic processes - the stochastic volatility process and the price process. Tables 15.1 and 15.2 show the values of implied volatilities for different values of ρ and the strike price. The 3-dimensional plots (Figure 1 - 5) show how implied volatilities behave with changes in the correlation between the two stochastic processes as well as the strike price. In attempting to explain these changes more accurately, we also look at two sets of 2-dimensional plots. Figures 6 -10 are plots of implied volatilities against the strike price for fixed values of ρ while figures 11 - 15 are plots of implied volatilities against ρ for fixed values of the strike price.

It is well known, and as is observed in practice, volatility is usually higher when prices fall than when prices rise. This is evident from the fact that crashes occur in a very short time, while for prices to rise, it takes a considerably longer time.

The plots of the implied volatility, that is assuming the volatility is constant, against the strike price for fixed values of the level of correlation between the two processes, show an expected pattern of behaviour. The strike price here is taken to vary between 110 to 90. The exercise is repeated for different values of ρ between -0.95 and 0.95.

In the Brownian Motion situation, we have the effect of ρ and the stochastic volatility on the system. Here, we observe a "smile" - a "smile" is a situation where values at the extremes are higher than the values at the centre. However, lower values of the strike price have higher implied volatility compared to higher values of the strike price, same distance away from the centre. When ρ is non-zero, it plays a more important role in the system. For negative values of ρ, the implied volatility is high for low strike prices and it decreases with increasing strike prices. However, for positive values of ρ, the implied volatility increases with increasing strike prices. Furthermore, in most cases, for the strike price 100, the implied volatility is slightly higher than the initial value.

In the Ornstein - Uhlenbeck process, the presence of the mean reversion force, complicates matters slightly. The behaviour is similar to the Brownian motion, only that the picture gets slightly blurred. The "smile" is no longer at $\rho = 0$, it is shifted to the right - to positive values of ρ. The extent of the shift depends on the value of the mean reversion force. Other than that the pattern of behaviour of the implied volatilities is the same as the Brownian motion.

The justification in both cases (Brownian motion as well as the Ornstein - Uhlenbeck process) is due to the fact that negative values of ρ are associated with falling prices and thereby higher volatility; whereas positive values of ρ indicate comparatively lower volatility. The plots, (Figures 6 to 10) illustrate these facts.

The other set of 2-dimensional plots - figures 11 to 15 are plots of the implied volatilities against the correlation ρ for fixed values of the strike price. We repeat this exercise for all the five values of the strike price that we use - namely 90, 95, 100, 105 and 110 for both the situations of the volatility process following a Brownian motion and an Ornstein - Uhlenbeck process. In these plots we observe that for low strike prices, in fact for strike prices less than or equal to 100 the implied volatility decrease as the correlation increases from -1 to 1. However, for higher values of the strike price, namely above 100, we see a reversal in this trend of the implied volatilities. For those situations, we have the implied volatilities increasing from -1 to 1. As in the case of the plots of the implied volatility against the strike prices for fixed correlations, here also, the picture becomes slightly blurred for the Ornstein - Uhlenbeck case with a high value of the mean reversion force.

For the case of the stochastic interest rate, calculation of the implied volatilities is not that simple. Also, when calculated, the plots are not too informative even for the Brownian motion. This is due to the fact that the interaction of the three correlation co-efficients blurs the picture. Thus, we do not go into the detailed analysis of the implied volatility in the case of stochastic interest rates.

6.6 Conclusions and Remarks

The tables show the values of the calculated price along with the simulated values. A look at the tables show that in all the cases, the calculated value of the option, including the correction factor, is very close to the simulated value. When the volatility process follows a Brownian motion, any difference in the values is within one standard error of the simulated set. This is true for both constant and stochastic interest rates. When the volatility process follows an Ornstein - Uhlenbeck process, the lower the value a the closer agreement of the calculated values with the simulated values; again, any differences are within standard errors. Even for higher values of a (as high as $a = 10$) the calculated and simulated values agree quite closely, any differences are within two standard errors of the simulated values.

Another fact to be noticed is that higher the value of ρ, i.e. the closer ρ is to ± 1 the greater the contribution of the correction factor to the corrected calculated price. In fact, for $\rho = 0$, the value of the correction factor is 0. This is true for all cases that we have considered.

The biggest advantage of this method is that one does not need to make restrictive assumptions such as independence of the price and the volatility processes as has been done by Hull and White. In fact, in practice, rarely do price and volatility act independent of each other - price fluctuations affect volatility; price falls are associated with higher volatilities whereas price rises are associated with low volatility. The method is quite fast to use for different values of the strike price.

Another justification of use of the correction factor is in the approximation carried out during conditioning. In the case of the volatility process following a Brownian motion, conditioning

$\int_0^1 e^{kB_s} ds$ on $\int_0^1 B_s ds$ works well for k relatively small or not too large. But, conditioning B_1 on $\int_0^1 B_s ds$ does not work so well and leads to an error. Probably one reason for this is the fact that B_1 and $\int_0^1 B_s ds$ are rather more closely correlated than $\int_0^1 e^{kB_s} ds$ and $\int_0^1 B_s ds$. Similar is the situation when the volatility process follows an Ornstein - Uhlenbeck process. In that case, conditioning $\int_0^1 e^{kV_s} ds$ on $\int_0^1 V_s ds$ works well for k relatively small or not too large, but conditioning V_1 on $\int_0^1 V_s ds$ does not work so well and leads to an error. Here also, V_1 and $\int_0^1 V_s ds$ are more closely correlated than $\int_0^1 e^{kV_s} ds$ and $\int_0^1 V_s ds$. Thus, in both cases, the correction factor is needed to rectify that error.

6.7 Tables

In the following tables, we present the Calculated prices, the associated Correction Factors (C.F.), the Corrected Calculated prices along with the simulated prices and the standard errors of simulation (S.E.). We present the results for both the constant and stochastic interest rates with the volatility process being either a Brownian motion or an Ornstein - Uhlenbeck process. For each table, the volatility process and the values of the parameters are stated in the table headings. We also present the values of the implied volatilities for the constant interest rate situation.

6.7.1 Constant Interest Rates

Table 11 : The volatility process here is a Simple Brownian Motion with $\sigma = 0.1$, r = 0.05 and k = 1.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.044508	0.779841	2.824346	2.990192	0.142218
	105	3.497231	0.816586	4.313817	4.453916	0.15819
	100	5.910315	0.672918	6.583233	6.720349	0.174477
	95	9.546253	0.370436	9.916688	10.067917	0.187047
	90	14.09038	0.200528	14.290908	14.421779	0.19195
-0.95	110	1.449652	0.515492	1.965143	1.979887	0.046055
	105	3.61698	0.664221	4.281493	4.32144	0.070859
	100	6.730394	0.656562	7.386937	7.478487	0.093149
	95	10.522814	0.560182	11.082996	11.252283	0.110801
	90	14.743085	0.443986	15.187071	15.427257	0.123809
0.75	110	2.321014	0.440063	2.761077	2.79501	0.109881
	105	3.857499	0.475741	4.33324	4.370133	0.128513
	100	6.310175	0.42701	6.737185	6.803907	0.146796
	95	9.854223	0.291273	10.145325	10.274714	0.160558
	90	14.235053	0.167333	14.402386	14.549206	0.167553
-0.75	110	1.764733	0.351344	2.116077	2.080526	0.054295
	105	3.897485	0.411583	4.309067	4.276188	0.077837
	100	6.94123	0.393316	7.334547	7.277515	0.099984
	95	10.673079	0.331616	11.004534	10.929826	0.118031
	90	14.852886	0.136765	14.989651	15.024697	0.131557
0.5	110	2.51401	0.1796	2.69361	2.677158	0.099293
	105	4.162327	0.195099	4.357426	4.349432	0.118616
	100	6.703056	0.18134	6.884301	6.887848	0.137491
	95	10.214073	0.138677	10.35275	10.389251	0.151995
	90	14.457174	0.093119	14.550293	14.596035	0.160504
-0.5	110	2.106229	0.15975	2.265979	2.249182	0.063796
	105	4.159931	0.15833	4.31826	4.309344	0.086421
	100	7.104181	0.168977	7.273158	7.214149	0.107748
	95	10.766899	0.141742	10.908641	10.812996	0.125257
	90	14.91327	0.111661	15.024931	14.920499	0.137719

Table 11. Continued..........

Table 11. Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.573018	0.042315	2.614529	2.578021	0.089701
	105	4.329345	0.046211	4.375555	4.340218	0.109824
	100	6.968613	0.043416	7.012029	6.983028	0.129222
	95	10.490538	0.037539	10.528077	10.506015	0.144528
	90	14.660471	0.016196	14.676667	14.660233	0.154393
-0.25	110	2.359796	0.040272	2.400067	2.38458	0.072381
	105	4.320381	0.044348	4.364726	4.322913	0.094257
	100	7.162415	0.041809	7.204224	7.133907	0.114915
	95	10.766043	0.034854	10.800897	10.711205	0.131551
	90	14.897422	0.027139	14.924561	14.835279	0.143077
0	110	2.517138			2.486435	0.080825
	105	4.378393			4.326996	0.101893
	100	7.118547			7.062116	0.121822
	95	10.674126			10.61299	0.137816
	90	14.811251			14.743523	0.148634

Table 12.1 : The volatility process follows an Ornstein - Uhlenbeck process with a=0.1, k=1, r=0.05, $\sigma = 0.1$, $X_0 = 100$ and $V_0 = 0$.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.052779	0.7262169	2.778996	3.010594	0.1194083
	105	3.516785	0.7696407	4.286425	4.521037	0.1380072
	100	5.938746	0.6435333	6.582279	6.827903	0.1563951
	95	9.570367	0.3594557	9.929823	10.20763	0.1701949
	90	14.10503	0.1884444	14.29347	14.57596	0.1754179
-0.95	110	1.465225	0.5031988	1.968424	2.009105	0.04744766
	105	3.625432	0.6366818	4.262114	4.322088	0.07209224
	100	6.730491	0.6212323	7.351724	7.436452	0.09447091
	95	10.51867	0.5240718	11.04274	11.16606	0.1127158
	90	14.73891	0.4112558	15.15017	15.28758	0.1265338
0.75	110	2.312387	0.4122456	2.724633	2.963393	0.1109139
	105	3.858761	0.4486269	4.307388	4.549214	0.1302168
	100	6.321038	0.4044438	6.728094	6.978501	0.1488933
	95	9.86696	0.2757934	10.14276	10.39694	0.1632714
	90	14.24454	0.1568302	14.40137	14.67709	0.1701071
-0.75	110	1.769973	0.3372281	2.107201	2.154169	0.05585269
	105	3.89501	0.392482	4.287492	4.369176	0.07926726
	100	6.931021	0.3723157	7.303337	7.424602	0.1009169
	95	10.65958	0.3112112	10.97079	11.11974	0.118411
	90	14.84905	0.25439755	15.08492	15.24523	0.13155
0.5	110	2.49434	0.1686056	2.662946	2.8703595	0.1017094
	105	4.149523	0.1846999	4.334223	4.549074	0.1216292
	100	6.69702	0.1717737	6.868794	7086679	0.1407383
	95	10.21131	0.1307052	10.34202	10.56835	0.1554402
	90	14.45612	0.08677706	14.5429	14.79708	0.1635396
-0.5	110	2.098748	0.1524248	2.251173	2.309457	0.06623995
	105	4.147055	0.1704263	4.317481	4.397041	0.0884599
	100	7.085885	0.1602199	7.246105	7.371216	0.1090454
	95	10.74705	0.1333718	10.88042	11.04136	0.1257021
	90	14.89608	0.1041142	15.0002	15.18395	0.1376807

Table 12.1 Continued.........

Table 12.1 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.548681	0.04003625	2.588718	2.755464	0.09290375
	105	4.309619	0.04386708	4.353486	4.534631	0.1133543
	100	6.952337	0.04116863	6.993506	7.176826	0.1328749
	95	10.47678	0.03287068	10.50966	10.71232	0.1479818
	90	14.65015	0.02367886	14.67383	14.90507	0.1571716
-0.25	110	2.342994	0.03831577	2.381316	2.472027	0.07555303
	105	4.301222	0.04222176	4.343443	4.436763	0.09709313
	100	7.140569	0.0396711	7.18024	7.314232	0.1170629
	95	10.74409	0.03283458	10.77692	10.95539	0.13306
	90	14.8789	0.02532481	14.90423	15.1089	0.1440347
0	100	2.494862			2.447907	0.08430024
	105	4.356859			4.48986	0.1052663
	100	7.097289			7.257128	0.1248408
	95	10.65413			10.83947	0.1405096
	90	14.79503			15.01577	0.1505679

Table 12.2 : The volatility process follows an Ornstein - Uhlenbeck process with a=1, k=1, $\sigma = 0.1$, r=0.05, $X_0 = 100$ and $V_0 = 0$

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.09989	0.447118	2.547003	2.548632	0.097955
	105	3.646248	0.5089709	4.220439	4.112885	0.117202
	100	6.1333845	0.464807	6.598191	6.525032	0.1362929
	95	9.73706	0.293513	10.03057	9.963072	0.1508313
	90	14.195358	0.1382265	14.333585	14.279104	0.157423
-0.95	110	1.5899428	0.4052	1.995117	1.991513	0.0498523
	105	3.69556	0.4622925	4.157852	4.236061	0.073901
	100	6.735551	0.418772	7.154322	7.33981	0.095757
	95	10.489035	0.330147	10.819182	11.06061	0.113573
	90	14.706939	0.243942	14.950881	15.240569	0.125994
0.75	110	2.262628	0.264496	2.527124	2.584736	0.095141
	105	3.879919	0.300598	4.180517	4.223171	0.114804
	100	6.408653	0.280251	6.688904	6.717862	0.134083
	95	9.963557	0.196094	10.159651	10.184946	0.148958
	90	14.30883	0.107945	14.416775	14.43912	0.156727
-0.75	110	1.826288	0.251618	2.077907	2.110221	0.055864
	105	3.895454	0.280364	4.175818	4.276621	0.079287
	100	6.873095	0.254095	7.127265	7.318380	0.100699
	95	10.575978	0.201038	10.777024	11.03507	0.117759
	90	14.765211	0.148761	14.913972	15.205012	0.12995
0.5	110	2.380043	0.112509	2.492552	2.580589	0.089893
	105	4.213388	0.126909	4.340297	4.316617	0.110125
	100	6.679581	0.118985	6.798567	6.896327	0.129877
	95	10.213332	0.088932	10.302254	10.389594	0.145253
	90	14.461569	0.055906	14.517475	14.610227	0.153949
-0.5	110	2.073016	0.110172	2.183188	2.256874	0.063475
	105	4.081688	0.121574	4.203263	4.319746	0.086171
	100	6.979388	0.110921	7.090308	7.271171	0.107133
	95	10.626602	0.087954	10.714556	10.945949	0.123765
	90	14.792283	0.064725	14.857008	15.122616	0.133774

Table 12.2 Continued ....

Table 12.2 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.413203	0.027436	2.440639	2.541466	0.083981
	105	4.201249	0.030675	4.231924	4.356218	0.104821
	100	6.868059	0.028669	6.896728	7.029882	0.125354
	95	10.407819	0.022154	10.429973	10.563992	0.140587
	90	14.599412	0.015009	14.61442	14.760626	0.150066
-0.25	100	2.255778	0.078588	2.334366	2.380508	0.070676
	105	4.195983	0.03005	4.226033	4.348916	0.089182
	100	7.014515	0.02766	7.042175	7.205699	0.113402
	95	10.614835	0.021898	10.636734	10.840324	0.129624
	90	14.771185	0.016628	14.787079	15.024725	0.140343
0	110	2.370209			2.47521	0.077528
	105	4.236678			4.364865	0.099009
	100	6.978257			7.132334	0.119306
	95	10.541857			10.71344	0.135304
	90	14.705379			14.900923	0.145439

Table 12.3 : The volatility process follows an Ornstein - Uhlenbeck process with a = 3, k = 1, r = 0.05, $V_0 = 0$, $X_0 = 100$ and $\sigma = 0.1$.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E
0.95	110	2.137067	0.237643	2.37471	2.561357	0.861595
	105	3.794151	0.281266	4.075416	4.291533	0.107003
	100	6.370623	0.270195	6.640818	6.865626	0.127188
	95	9.952418	0.192543	10.144961	10.384923	0.14248
	90	14.306623	0.102126	14.408749	14.674882	0.150144
-0.95	110	1.776722	0.253205	2.029927	2.017369	0.055157
	105	3.803327	0.272852	4.07618	4.046229	0.078814
	100	6.747577	0.237855	6.985433	6.948214	0.10078
	95	10.447001	0.17913	10.62613	10.584016	0.118278
	90	14.659225	0.125902	14.785127	14.714235	0.130749
0.75	110	2.220781	0.146139	2.36692	2.531267	0.082694
	105	3.928297	0.170457	4.093047	4.269829	0.104039
	100	6.537557	0.162546	6.704859	6.858439	0.124502
	95	10.100211	0.119333	10.219544	10.375287	0.140108
	90	14.393528	0.069114	14.462642	14.648838	0.14817
-0.75	110	1.931459	0.153773	2.085232	2.062847	0.057827
	105	3.928801	0.16661	4.095411	4.064735	0.081134
	100	6.826811	0.147206	6.974016	6.941111	0.102902
	95	10.490471	0.111763	10.602281	10.551997	0.120401
	90	14.686249	0.078441	14.76469	14.703153	0.132249
0.5	110	2.284024	0.064315	2.348339	2.477775	0.078619
	105	4.050093	0.073812	4.123905	4.236122	0.100423
	100	6.702199	0.069637	6.771836	6.860935	0.121084
	95	10.255014	0.052321	10.307335	10.40566	0.136807
	90	14.492846	0.032667	14.525512	14.633903	0.145889
-0.5	110	2.088548	0.066547	2.155095	2.142708	0.061576
	105	4.046267	0.07275	4.119017	4.098209	0.08456
	100	6.889473	0.065271	6.954744	6.93823	0.106045
	95	10.51469	0.049876	10.564564	10.384923	0.14248
	90	14.696808	0.034769	14.731577	14.684328	0.134854

Table 12.3 Continued ....

Table 12.3 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.301905	0.016037	2.317942	2.391139	0.745337
	105	4.122086	0.018038	4.140224	4.200953	0.096481
	100	6.818457	0.016922	6.835379	6.892887	0.117254
	95	10.374146	0.012888	10.387034	10.434283	0.133539
	90	14.576525	0.008442	14.584967	14.636719	0.143274
-0.25	110	2.203506	0.016312	2.219818	2.220772	0.065822
	105	4.118925	0.018011	4.136936	4.133687	0.088377
	100	6.909927	0.016391	6.926318	6.929604	0.109585
	95	10.50377	0.012568	10.516338	10.514062	0.126455
	90	14.681065	0.008657	14.689722	14.659044	0.137839
0	110	2.278312			2.302384	0.070203
	105	4.144716			4.169655	0.092405
	100	6.887281			6.913079	0.113403
	95	10.457036			10.474615	0.130023
	90	14.640267			14.643454	0.140643

Table 12.4 : The volatility process follows an Ornstein - Uhlenbeck process with a = 10, k = 1, r = 0.05, $V_0 = 0$, $X_0 = 100$ and $\sigma = 0.1$.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.145946	0.085411	2.231357	2.308187	0.073264
	105	3.928759	0.098936	4.027696	4.066879	0.095204
	100	6.610466	0.093486	6.703952	6.723736	0.115949
	95	10.192842	0.069991	10.262832	10.29781	0.131655
	90	14.454283	0.04308	14.497363	14.543108	0.140696
-0.95	110	1.991431	0.090238	2.081669	2.078354	0.060331
	105	3.922141	0.098474	4.020615	3.999647	0.083325
	100	6.751405	0.087625	6.83903	6.853382	0.104518
	95	10.386169	0.065568	10.473794	10.507584	0.121012
	90	14.596767	0.044268	14.641035	14.695058	0.132083
0.75	110	2.180094	0.053217	2.23332	2.299371	0.072509
	105	3.983075	0.061118	4.044764	4.094231	0.094412
	100	6.678632	0.057421	6.736053	6.773408	0.115251
	95	10.256393	0.04322	10.299613	10.334223	0.131357
	90	14.499057	0.027137	14.526194	14.5503	0.140927
-0.75	110	2.058914	0.055518	2.114432	2.099698	0.061669
	105	3.977592	0.060891	4.038484	4.022176	0.084489
	100	6.787376	0.054626	6.842002	6.826005	0.106001
	95	10.406935	0.041081	10.448016	10.438221	0.12277
	90	14.611664	0.027673	14.639337	14.625447	0.133612
0.5	110	2.208241	0.023676	2.231917	2.279903	0.07128
	105	4.035839	0.026927	4.062766	4.123707	0.093146
	100	6.74748	0.025112	6.772597	6.803292	0.1202188
	95	10.322142	0.018987	10.341129	10.364329	0.130605
	90	14.544878	0.012187	14.557065	14.555246	0.140691
-0.5	110	2.128028	0.024329	2.152356	2.152001	0.064217
	105	4.031225	0.026861	4.058086	4.051619	0.086781
	100	6.818265	0.024318	6.842583	6.809635	0.108321
	95	10.421298	0.018368	10.439667	10.399439	0.125018
	90	14.620672	0.012319	14.632991	14.596948	0.13546

Table 12.4 Continued ....

Table 12.4 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E
0.25	110	2.218443	0.005933	2.224377	2.2814	0.069658
	105	4.068162	0.006689	4.074851	4.128462	0.091961
	100	6.796636	0.006197	6.802833	6.807212	0.113399
	95	10.371454	0.0047	10.376154	10.373525	0.12969
	90	14.579721	0.00307	14.582791	14.56239	0.139915
-0.25	110	2.178563	0.006012	2.184576	2.209668	0.664102
	105	4.06566	0.006681	4.072341	4.075665	0.088959
	100	6.831474	0.0061	6.837573	6.810703	0.110319
	95	10.420737	0.004623	10.42536	10.37738	0.12704
	90	14.617989	0.003084	14.621073	1.4585143	0.137122
0	110	2.208792			2.255537	0.068118
	105	4.078353			4.107981	0.090602
	100	6.824676			6.811075	0.112042
	95	10.404252			10.372034	0.128603
	90	14.604139			14.571978	0.138699

6.7.2 Stochastic Interest Rates

For the following two tables, the volatility process is a Simple Brownian motion.

Table 13.1 : $r^* = r_0 = 0.05, \rho_1 = -0.5, \rho_2 = 0, \rho = 0.5, \phi = 0.025, \gamma = 0.5, k = 1, \sigma = 0.1.$

b	c	Calculated	CF	CCP	Simulated	SE
2	110	2.064465	0.1611706	2.2256356	2.310308	0.0660562
	105	4.095731	0.1804528	4.2761838	4.382218	0.08842041
	100	7.034479	0.1695461	7.2040251	7.331723	0.1091908
	95	10.70558	0.1409024	10.8464824	11.02632	0.1254156
	90	14.86396	0.1099334	14.9738934	15.19951	0.1368878
10	110	2.093385	0.1602639	2.2536489	2.259263	0.06468547
	105	4.141573	0.1794597	4.3210327	4.328258	0.08711183
	100	7.085474	0.1692707	7.2547447	7.246289	0.1082364
	95	10.75171	0.141638	10.893348	10.8837	0.1252286
	90	14.90246	0.111301	15.013761	15.01425	0.1373567
100	110	2.104869	0.1598071	2.2646761	2.211097	0.0641309
	105	4.158042	0.1789455	4.3369875	4.295031	0.08627042
	100	7.102307	0.1690128	7.2713198	7.252563	0.1070702
	95	10.76544	0.1417372	10.9071772	10.91719	0.1238452
	90	14.9123	0.1116313	15.0239313	15.04275	0.1362203

Table 13.2 : $r^* = r_0 = 0.05, \rho_1 = \rho_2 = -0.5, \rho = \sqrt{0.5}, \phi = 0.025, \gamma = 0.5, k = 1, \sigma = 0.1.$

b	c	Calculated	CF	CCP	Simulated	SE
2	110	2.017732	0.1656024	2.1833344	2.206077	0.06216576
	105	4.065671	0.1863177	4.2519887	4.325182	0.08479317
	100	7.0414	0.1752696	7.2166696	7.35174	0.1055256
	95	10.75224	0.1461538	10.8983938	11.09033	0.1218745
	90	14.94813	0.1150628	15.0631928	15.28329	0.1335367
10	110	2.078971	0.1616269	2.2405979	2.258156	0.06211812
	105	4.132393	0.1812652	4.3136582	4.383701	0.08494235
	100	7.087163	0.1710576	7.2582206	7.394359	0.1059289
	95	10.76594	0.1432922	10.9092322	11.07711	0.1229537
	90	14.92828	0.1129232	15.0412032	15.24218	0.1348843
100	110	2.103928	0.1599549	2.2638829	2.182876	0.06147877
	105	4.157038	0.1791416	4.3361796	4.294288	0.08400627
	100	7.102475	0.1692079	7.2716829	7.26275	0.1052165
	95	10.76696	0.1419183	11.9088783	10.91344	0.1224794
	90	14.91508	0.1118078	15.0268878	15.01305	0.1354558

For the following two tables, the volatility process follows an Ornstein - Uhlenbeck process with a = *10*.

Table 14.1 : $r^* = r_0 = 0.05, \rho_1 = \rho_2 = -0.5, \rho = \sqrt{0.5}, \phi = 0.025, \gamma = 0, k = 1, \sigma = 0.1.$

b	c	Calculated	CF	CCP	Simulated	SE
2	110	2.06303	0.1639484	2.2269784	2.102134	0.06165597
	105	4.133098	0.1844626	4.3175606	4.153669	0.08408443
	100	7.110789	0.1745349	7.2853239	7.125634	0.1048683
	95	10.81206	0.1469173	10.9589773	10.79806	0.1218218
	90	14.99419	0.116723	15.110913	14.93185	0.1342896
10	110	2.094306	0.1709928	2.2652988	2.185812	0.06353131
	105	4.189908	0.1910147	4.3809227	4.268511	0.0857572
	100	7.210147	0.1788339	7.3889809	7.23119	0.1065925
	95	10.96168	0.1485368	11.1102168	10.87745	0.1237045
	90	15.19063	0.1165081	15.3071381	14.99816	0.1362137
100	110	2.054383	0.1648769	2.2192599	2.19794	0.06428884
	105	4.129651	0.185607	4.315258	4.296375	0.08629285
	100	7.116331	0.1756074	7.2919384	7.20394	0.107596
	95	10.82718	0.147904	10.975084	10.84065	0.1245641
	90	15.01792	0.1177059	15.1356259	14.99894	0.1361814

Table 14.2 : $r^* = r_0 = 0.05, \rho_1 = \rho_2 = -0.5, \rho = \sqrt{0.5}, \phi = 0.025, \gamma = -0.5, k = 1, \sigma = 0.1.$

b	c	Calculated	CF	CCP	Simulated	SE
2	110	2.116959	0.1613869	2.2783459	2.149737	0.0611461
	105	4.204124	0.1814496	4.3855736	4.233522	0.08382539
	100	7.177411	0.1725708	7.3529818	7.182837	0.105094
	95	10.86007	0.146436	11.006506	10.84428	0.1221353
	90	15.021022	0.117141	15.138361	14.99599	0.134167
10	110	2.075828	0.1635142	2.2393422	2.204069	0.06165472
	105	4.153163	0.1839162	4.3370892	4.310886	0.08433407
	100	7.133133	0.1742366	7.3073696	7.312911	0.1051778
	95	10.83257	0.1470139	10.9795839	10.98678	0.1222365
	90	15.01177	0.1170782	15.1288482	15.11348	0.1348204
100	110	2.062773	0.1641028	2.2268758	2.21406	0.06285299
	105	4.135536	0.1845996	4.3201356	4.325516	0.08524463
	100	7.116246	0.174644	7.29089	7.284714	0.1063538
	95	10.81996	0.1470448	10.9670048	10.95219	0.1232499
	90	15.00403	0.1168847	15.1209147	15.07071	0.1358424

Table 15.1 : Table showing the values of Implied volatilities for different values of ρ and strike price b when the volatility process follows a simple Brownian motion.

ρ	b	Implied volatility
-0.95	110	0.0947
	105	0.1069
	100	0.1192
	95	0.1329
	90	0.1481
-0.75	110	0.0975
	100	0.1058
	100	0.1136
	95	0.1211
	90	0.1274
-0.5	110	0.1021
	100	0.1066
	100	0.1118
	95	0.1167
	90	0.1213
-0.25	110	0.1057
	100	0.107
	100	0.1095
	95	0.1127
	90	0.1158
0	110	0.1084
	100	0.1071
	100	0.1075
	95	0.1088
	90	0.1094
0.25	110	0.1109
	100	0.1074
	100	0.1052
	95	0.1044
	90	0.1028

Table 15.1. Continued ....

Table 15.1. Continued

ρ	b	Implied Volatility
0.5	110	0.1135
	100	0.1076
	100	0.1024
	95	0.0993
	90	0.097
0.75	110	0.1167
	100	0.1081
	100	0.0999
	95	0.0941
	90	0.092
0.95	110	0.1218
	100	0.1102
	100	0.0975
	95	0.0836
	90	0.0708

Table 15.2 : Table showing the values of Implied volatilities (I.V.) for different values of ρ and strike price b when the volatility process follows an Ornstein - Uhlenbeck process with mean reversion force a.

ρ	b	I.V. (a=0.1)	I.V. (a=1)	I.V. (a=3)	I.V. (a=10)
-0.95	110	0.0955	0.095	0.0979	0.0974
	105	0.1069	0.1048	0.1001	0.0988
	100	0.1181	0.1153	0.1042	0.1014
	95	0.1298	0.126	0.1094	0.1044
	90	0.1414	0.139	0.1072	0.1057
-0.75	110	0.0995	0.0983	0.097	0.0979
	100	0.1081	0.1058	0.1005	0.0994
	100	0.1177	0.1147	0.104	0.1006
	95	0.1281	0.1251	0.1084	0.1015
	90	0.1393	0.1374	0.1062	0.0997
-0.5	110	0.1036	0.1022	0.0991	0.0994
	100	0.1088	0.1068	0.1013	0.1002
	100	0.1162	0.1134	0.1039	0.1001
	95	0.1253	0.1217	0.1068	0.0997
	90	0.1361	0.1329	0.1048	0.097
-0.25	110	0.108	0.1056	0.1013	0.1009
	100	0.1098	0.1076	0.1022	0.1008
	100	0.1146	0.1116	0.1036	0.1002
	95	0.1221	0.1178	0.1047	0.0988
	90	0.1321	0.1275	0.1027	0.0959
0	110	0.1117	0.1082	0.1035	0.1022
	100	0.1111	0.1081	0.1031	0.1016
	100	0.113	0.1095	0.1032	0.1002
	95	0.1177	0.1128	0.1031	0.0986
	90	0.1269	0.1201	0.1013	0.0945
0.25	110	0.1156	0.1099	0.1059	0.1027
	100	0.1123	0.1078	0.1039	0.1021
	100	0.1107	0.1065	0.1025	0.1001
	95	0.1128	0.1068	0.1013	0.0986
	90	0.1204	0.1106	0.1007	0.0935

Table 15.2. Continued ....

Table 15.2. Continued

ρ	b	I.V. (a=0.1)	I.V. (a=1)	I.V. (a=3)	I.V. (a=10)
0.5	110	0.1187	0.1109	0.1082	0.1029
	100	0.1126	0.1068	0.1048	0.102
	100	0.1081	0.1027	0.1017	0.1
	95	0.1069	0.0994	0.1001	0.0982
	90	0.1132	0.0983	0.1004	0.0927
0.75	110	0.1211	0.111	0.1096	0.1034
	100	0.1126	0.1044	0.1056	0.1012
	100	0.105	0.0974	0.1016	0.0991
	95	0.0996	0.0897	0.0987	0.0968
	90	0.1042	0.0753	0.1018	0.0921
0.95	110	0.1223	0.1101	0.1105	0.1037
	100	0.1119	0.1017	0.1062	0.1005
	100	0.1007	0.0918	0.1018	0.0976
	95	0.0908	0.0775	0.0991	0.0951
	90	0.095	0.0678	0.1039	0.0913

6.8 Figures

Here we present a set of figures showing plots of implied volatilities against the correlation co-efficient and strike price for the constant interest rate case. These are the figures referred to earlier in section 5.5. As stated earlier, the first five figures (Figure 1 - 5) are three dimensional plots showing changes of implied volatility with changes in correlation between the two stochastic processes as well as the strike price. Figures 6 - 10 show plots of implied volatility against different values of the strike price but for fixed values of correlation. Finally, figures 11 - 15 show plots of implied volatility against correlation co-efficients for fixed values of the strike price.

Figure 1 : The Simple One Dimensional Brownian Motion

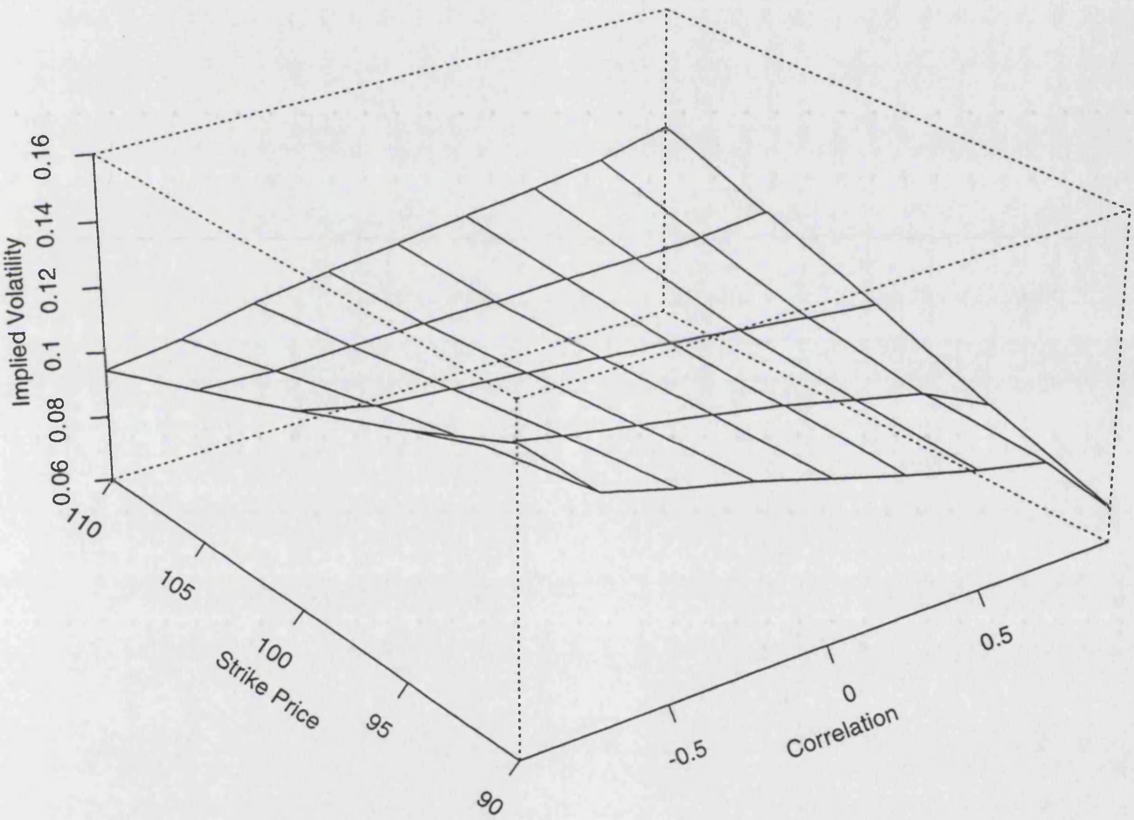


Figure 2 : The Ornstein - Uhlenbeck Case; a = 0.1

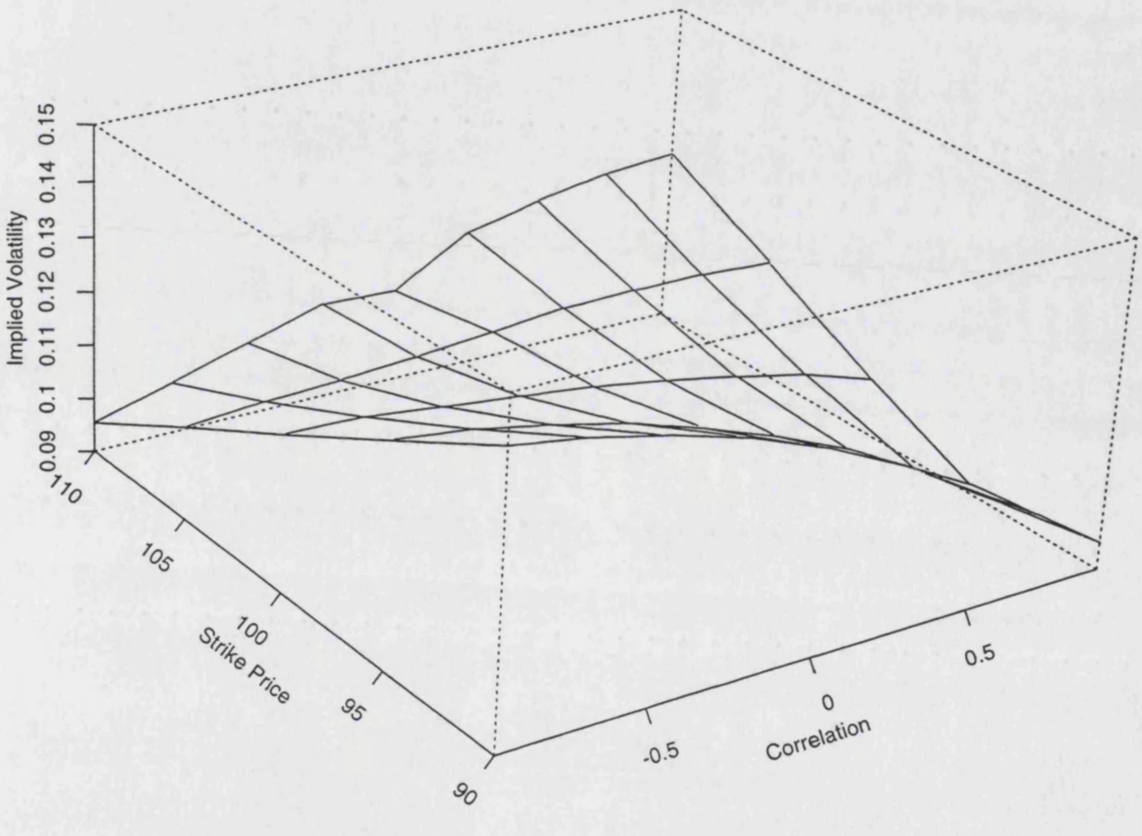


Figure 3 : The Ornstein - Uhlenbeck Case; a = 1

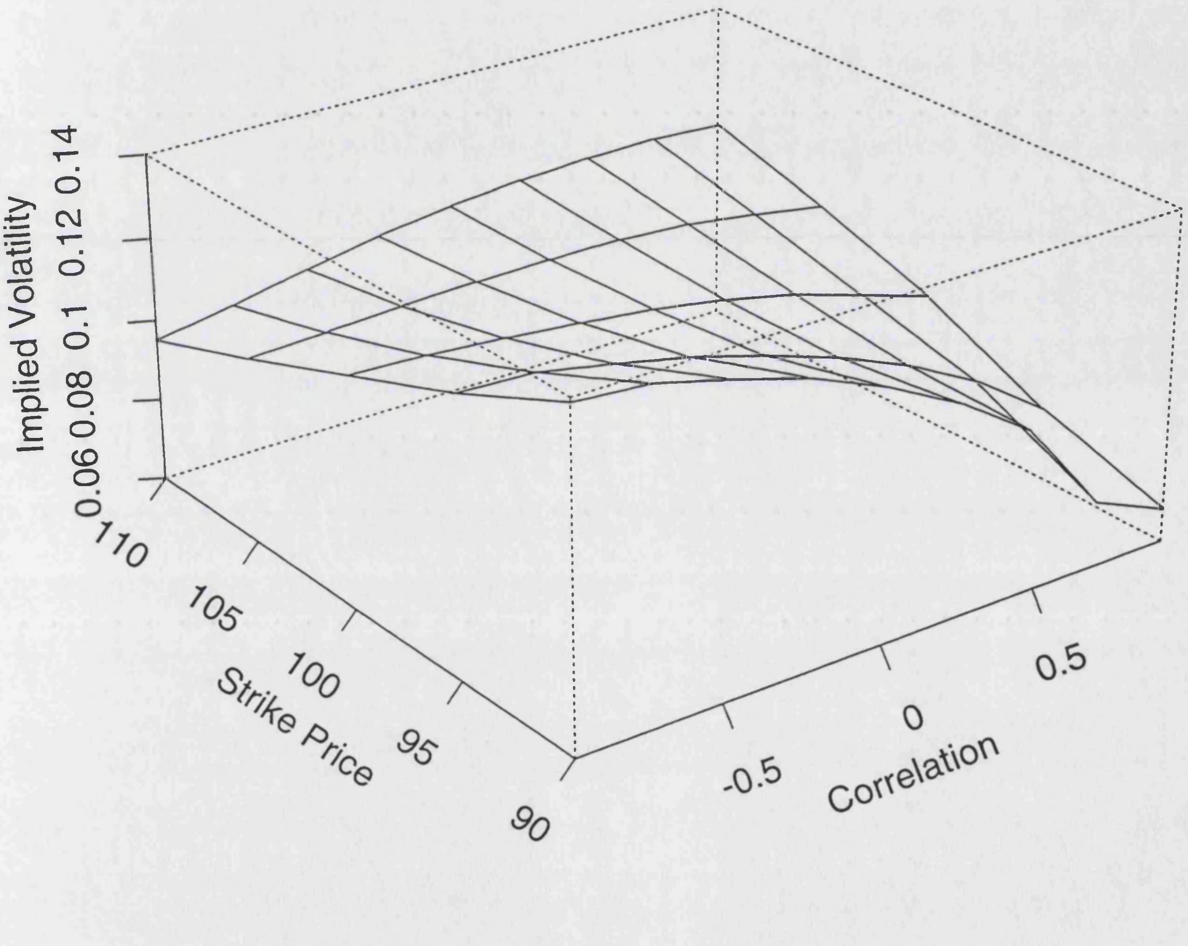


Figure 4 : The Ornstein - Uhlenbeck Case; a = 3

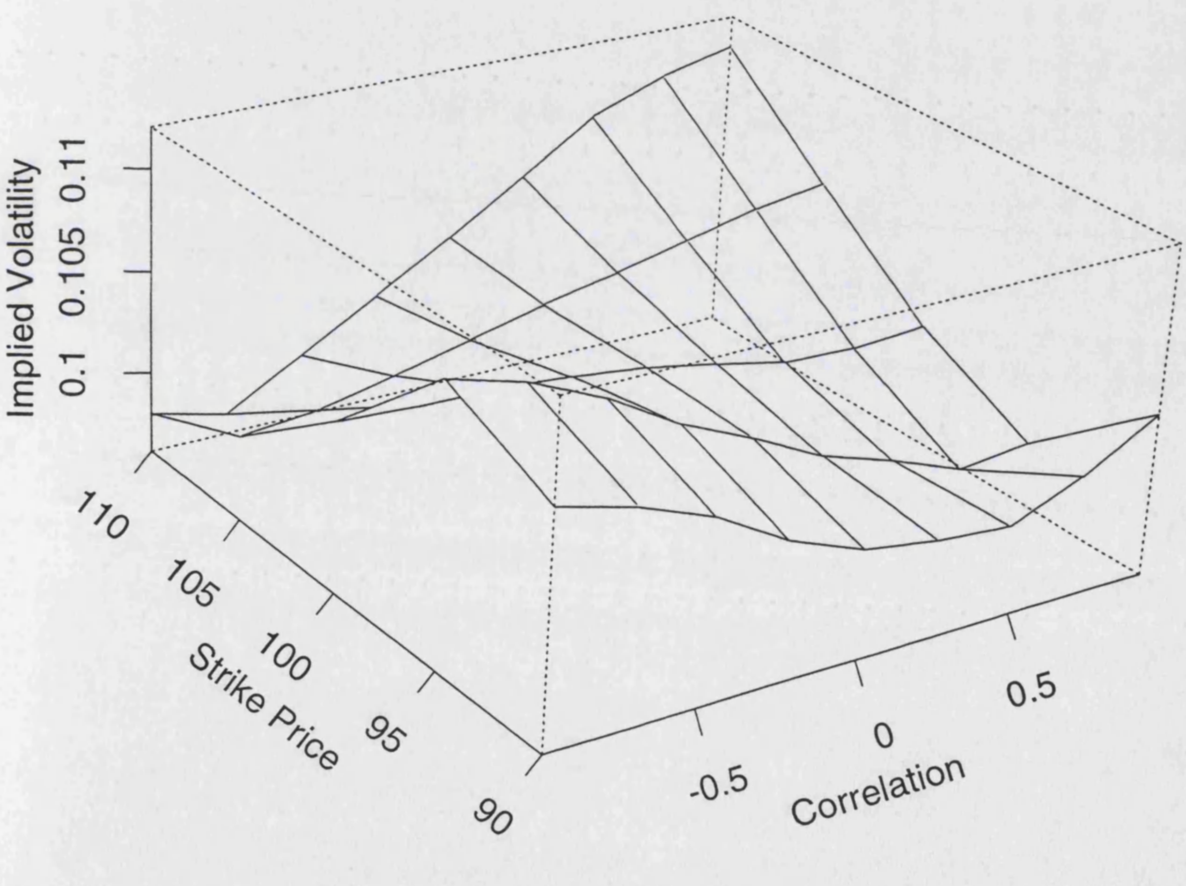


Figure 5 : The Ornstein - Uhlenbeck Case; a = 10

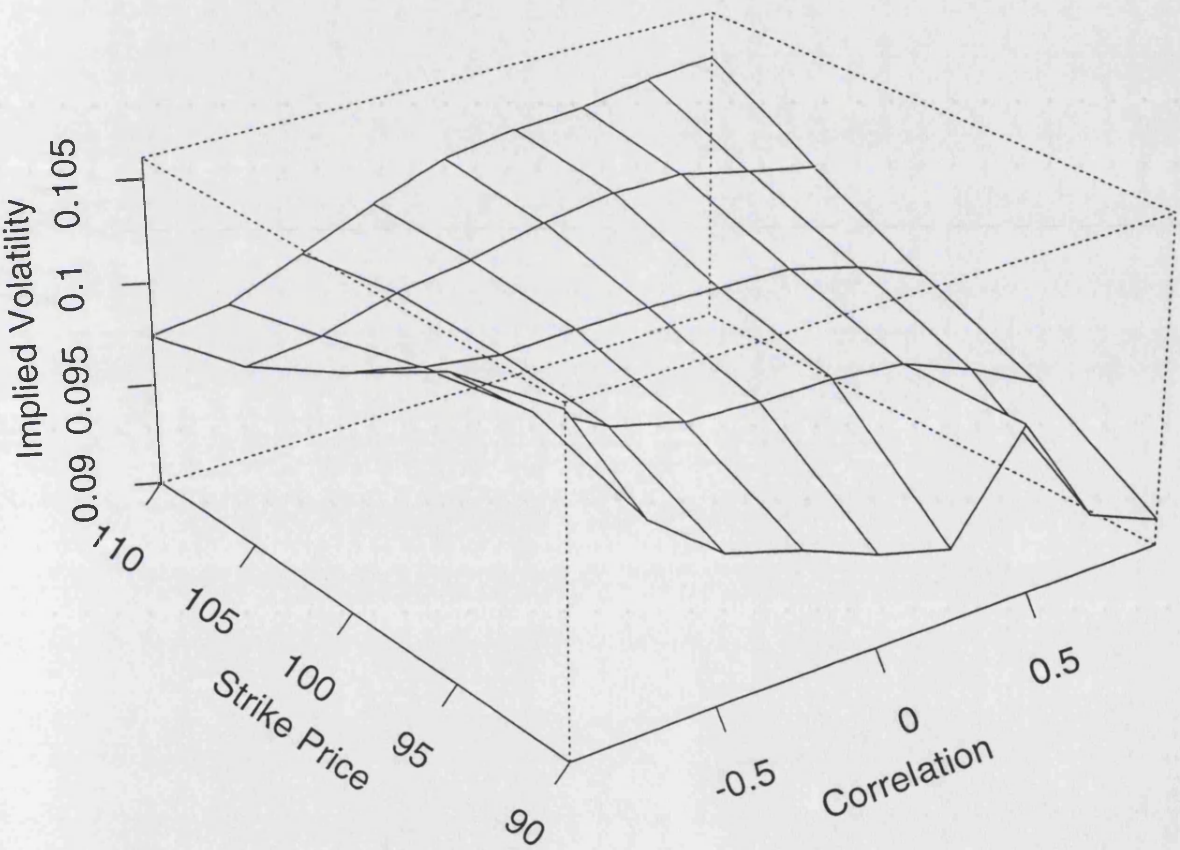


Figure 6 : The Simple Brownian Motion

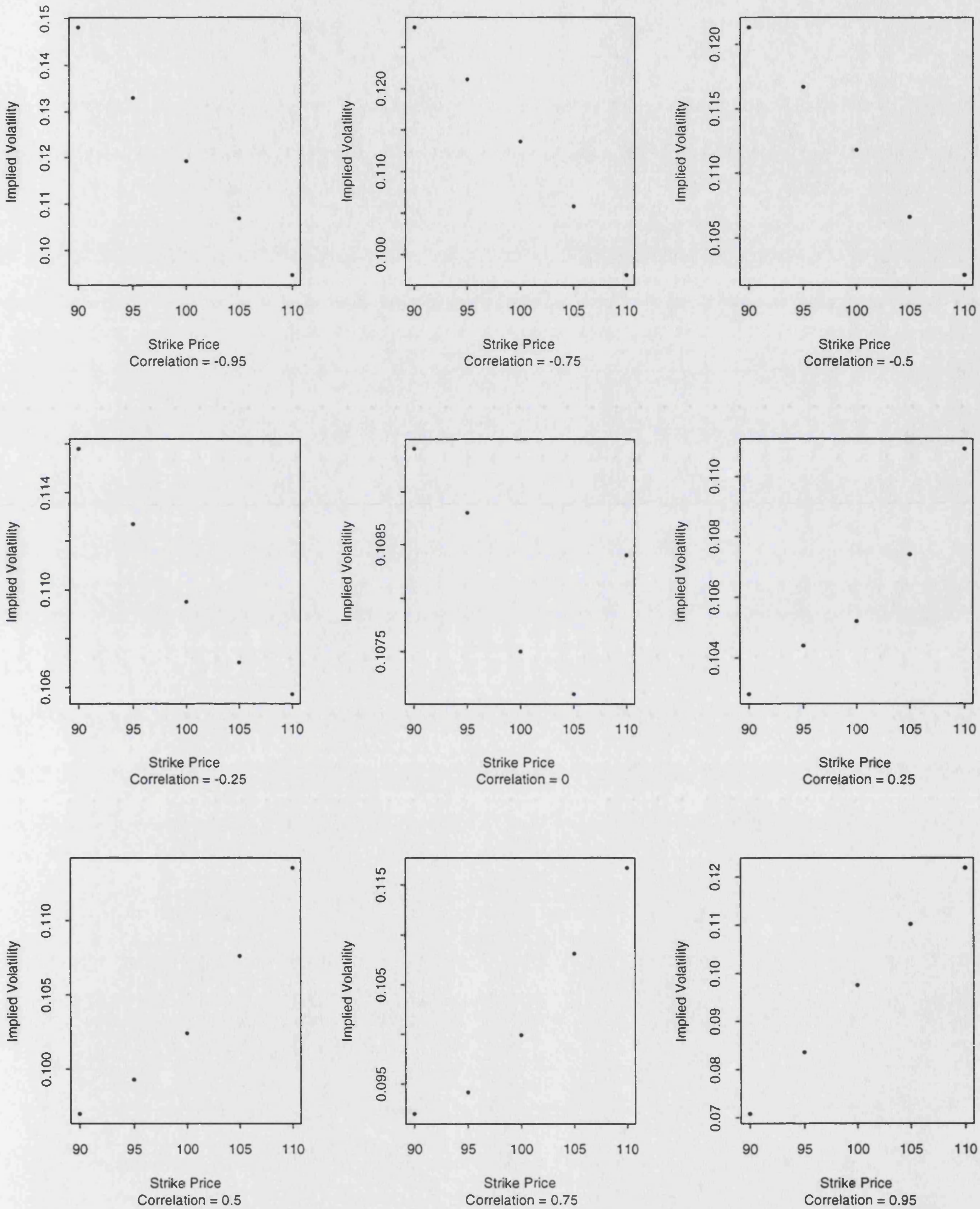


Figure 7 : The Ornstein - Uhlenbeck Process; a = 0.1

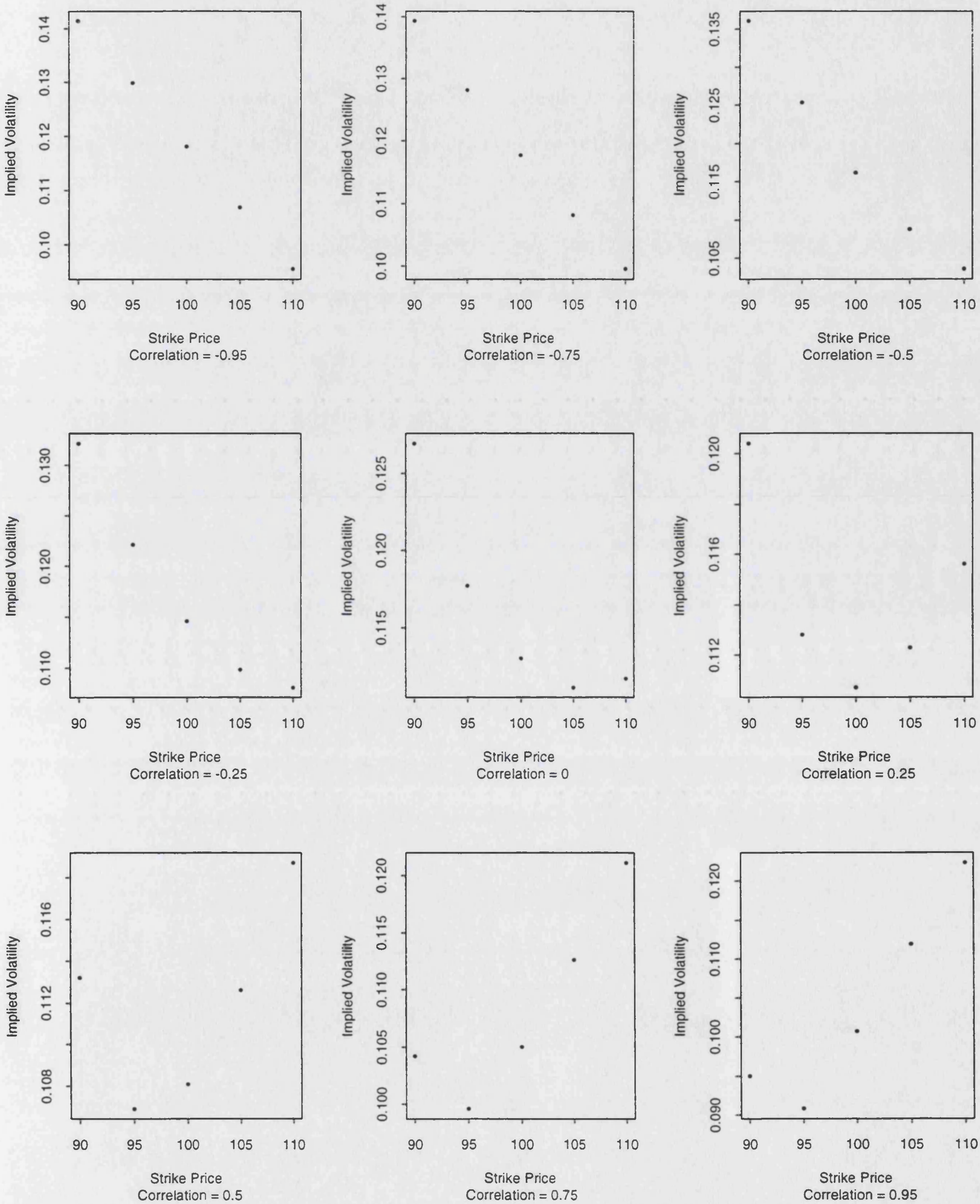


Figure 8 : The Ornstein - Uhlenbeck Process; a = 1

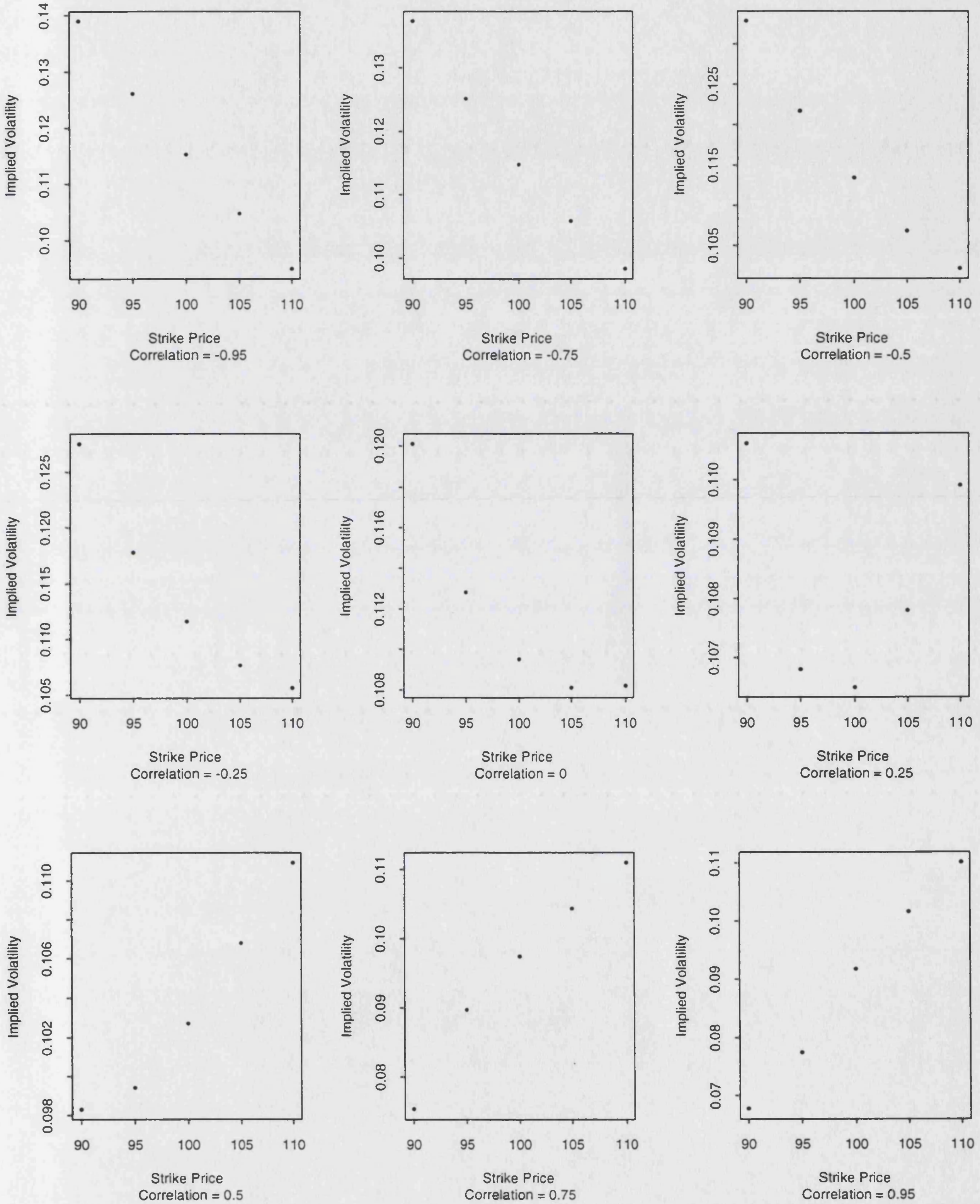


Figure 9 : The Ornstein - Uhlenbeck Process; a = 3

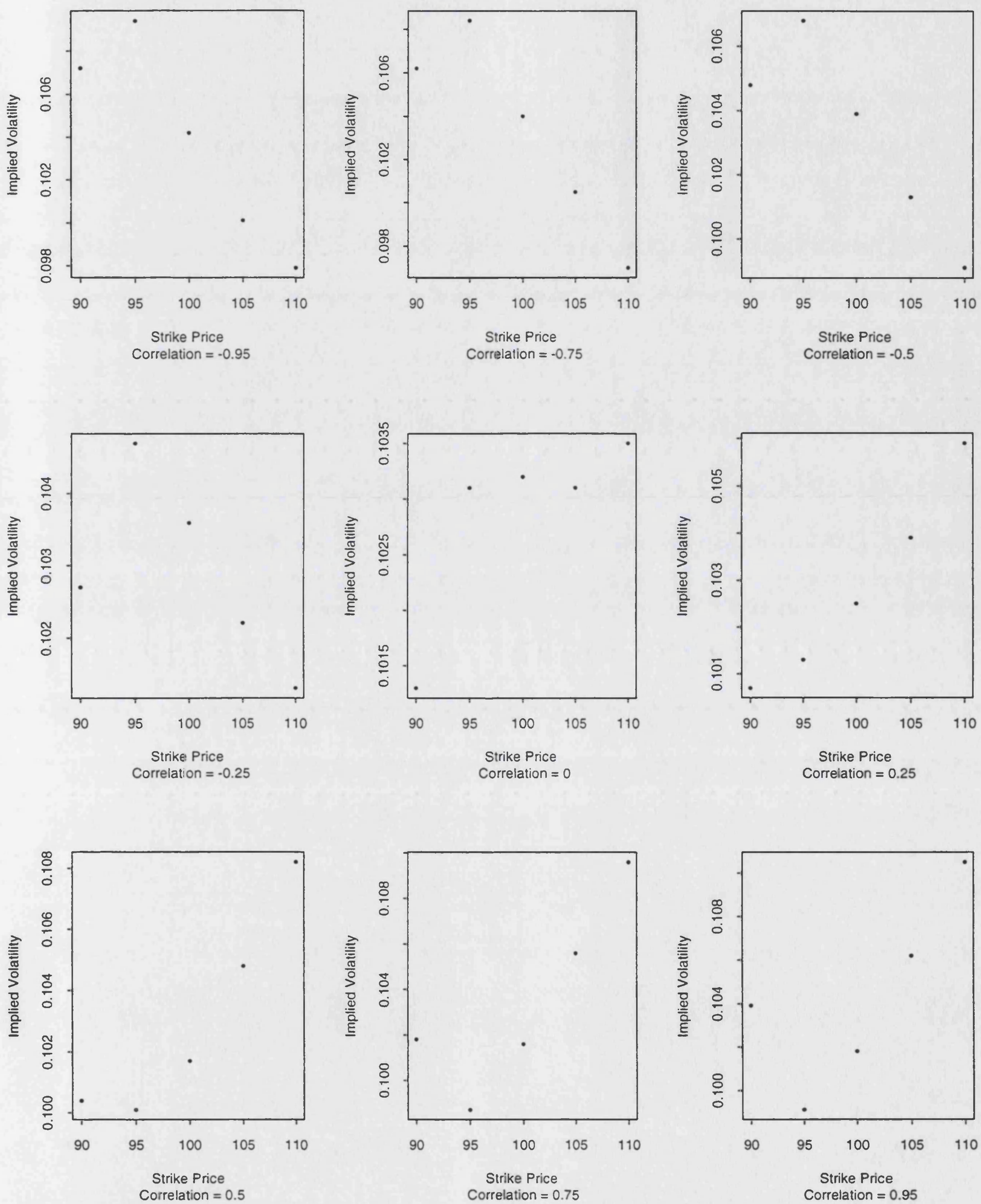


Figure 10 : The Ornstein - Uhlenbeck Process; a = 10

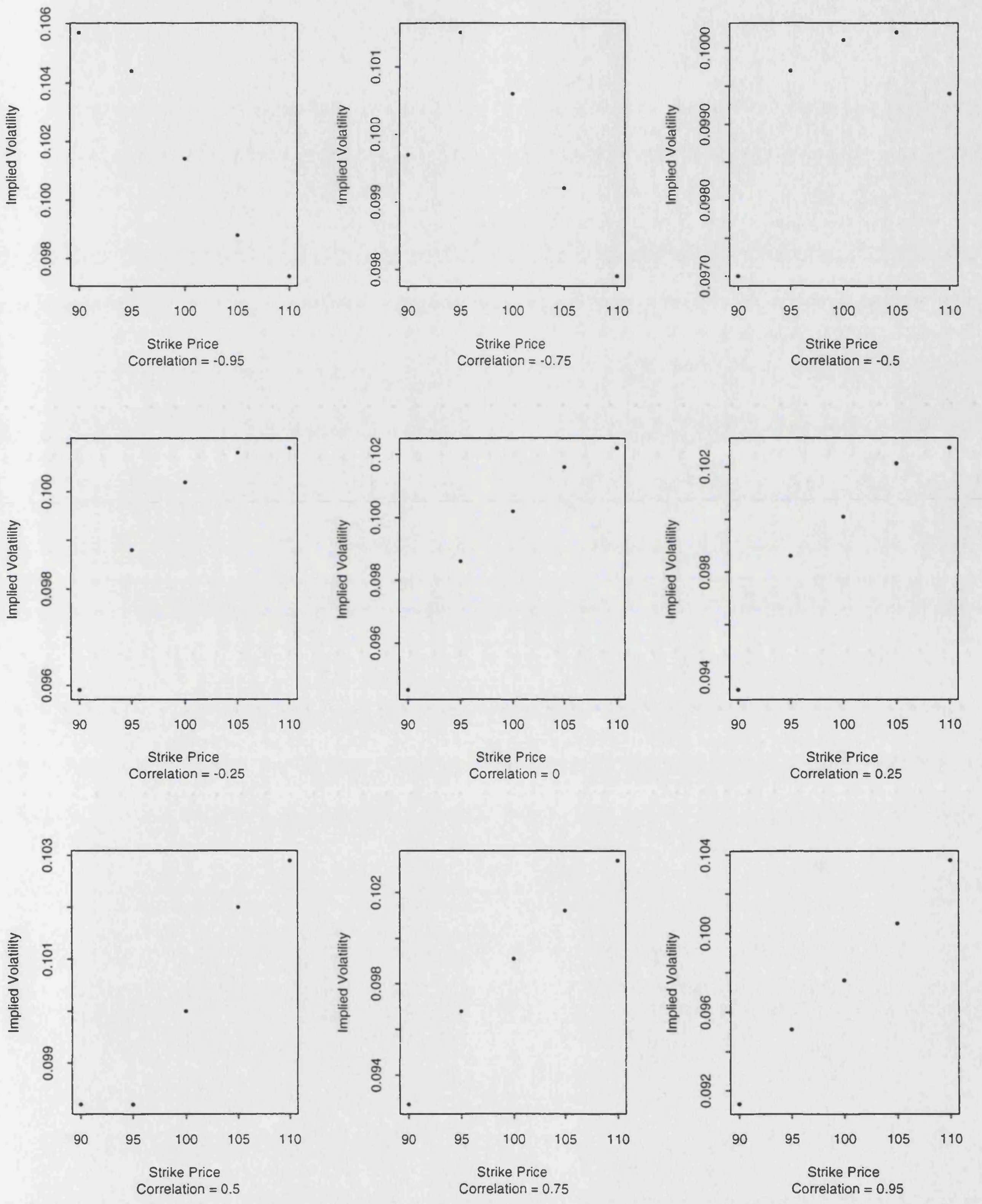


Figure 11 : The Simple Brownian Motion : Plots of Implied Volatility against Correlation

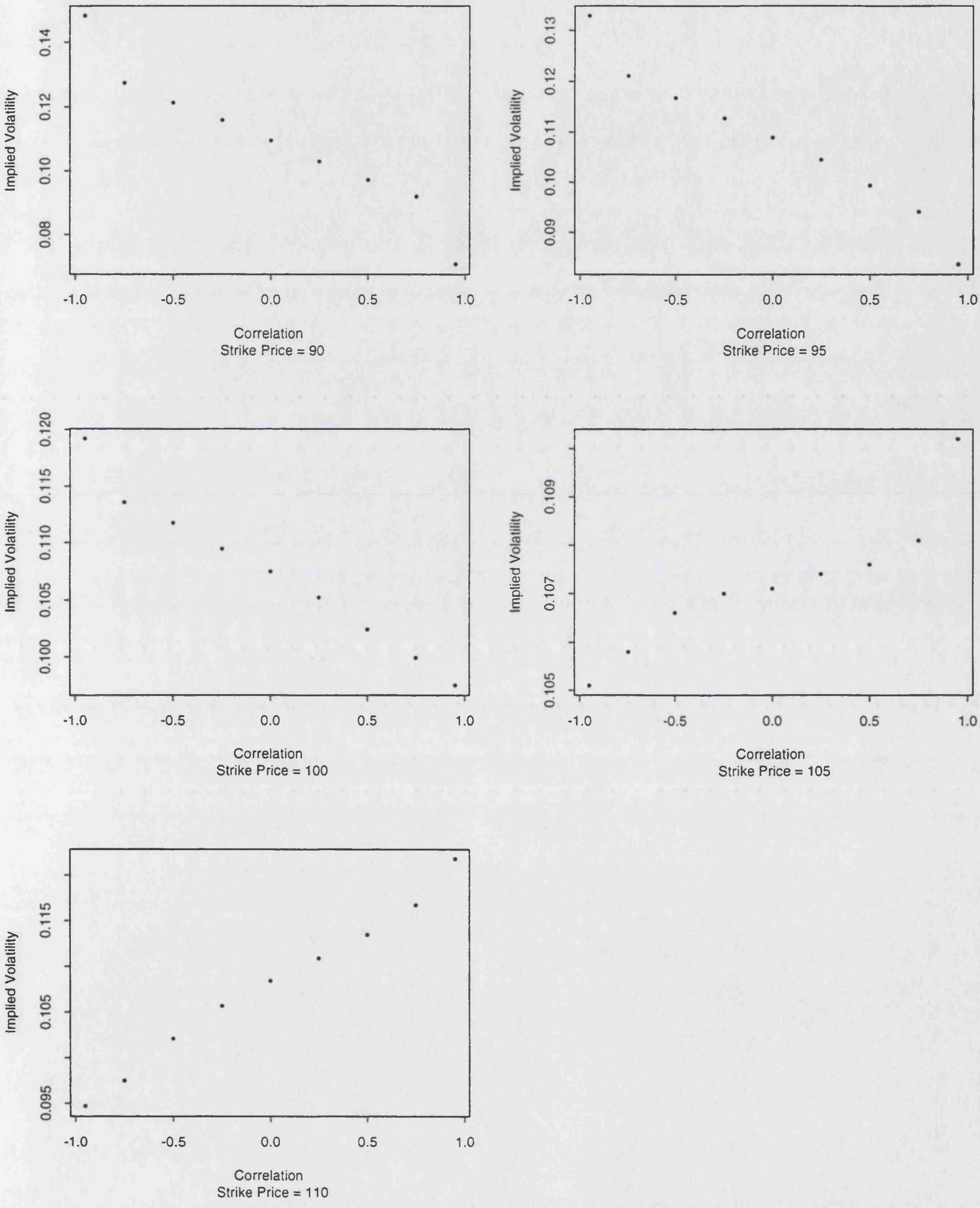


Figure 12 : The Ornstein - Uhlenbeck Process with a = 0.1 : Plots of Implied Volatility against Correlation

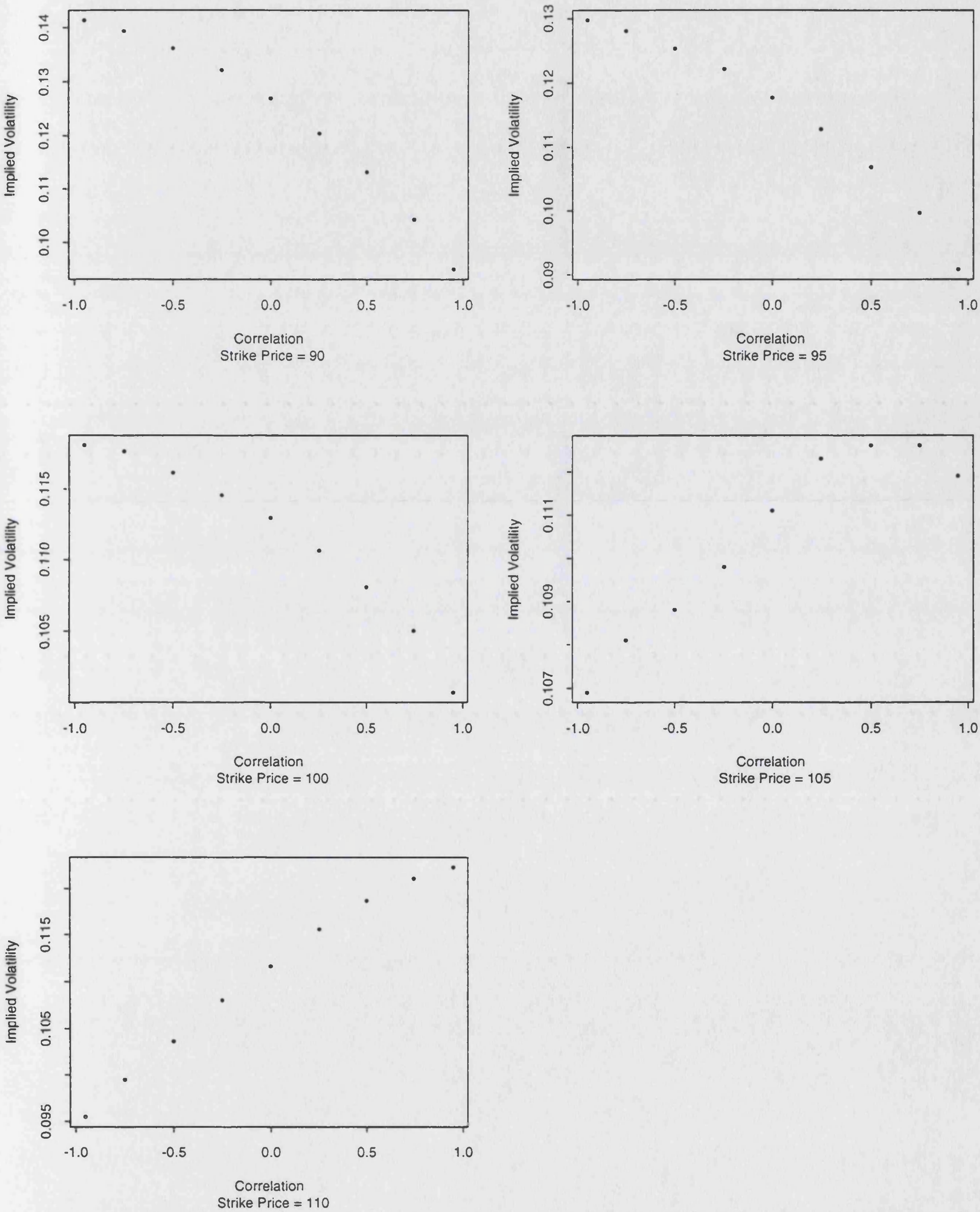


Figure 13 : The Ornstein - Uhlenbeck Process with a = 1 : Plots of Implied Volatility against Correlation

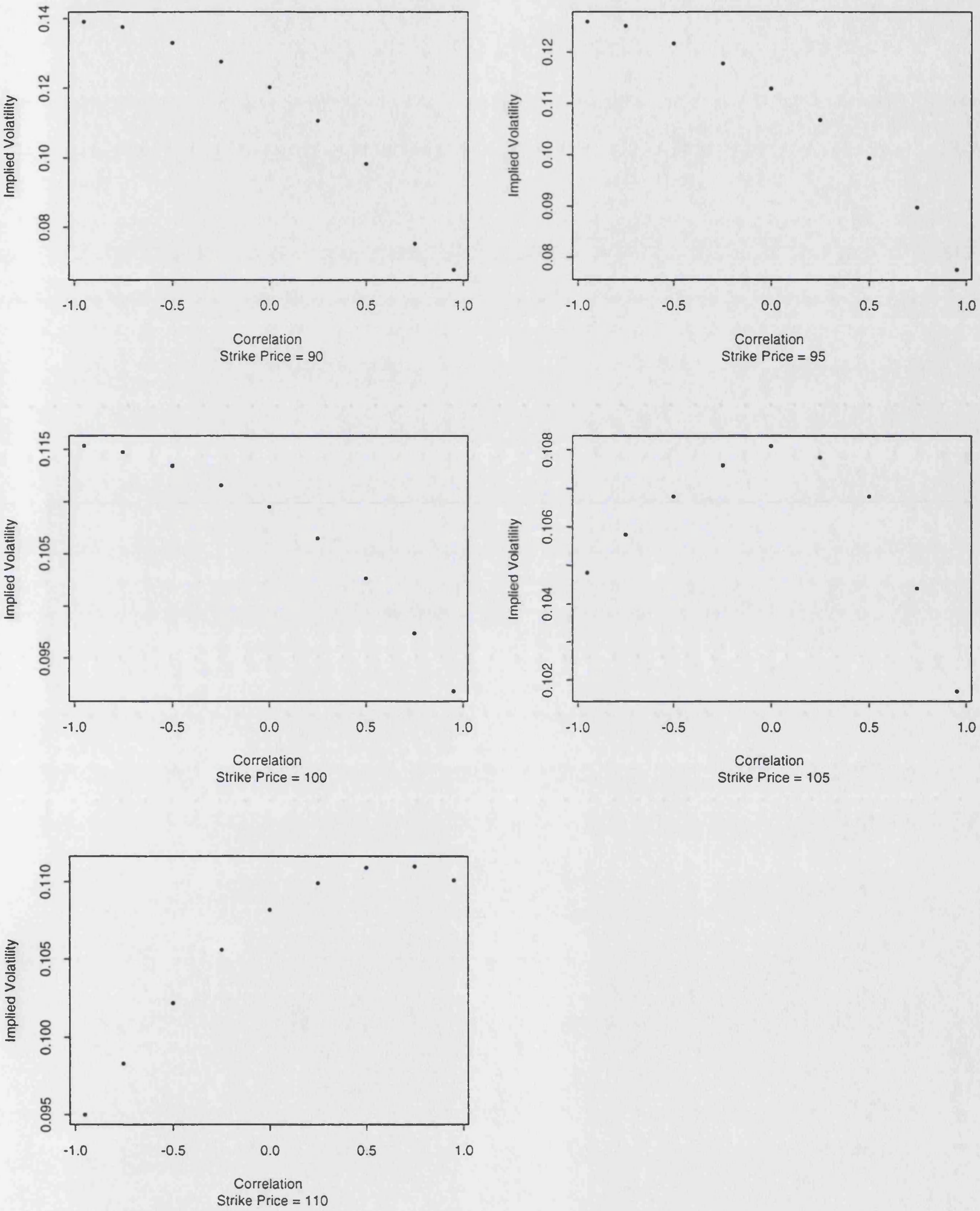


Figure 14 : The Ornstein - Uhlenbeck Process with a = 3 : Plots of Implied Volatility against Correlation

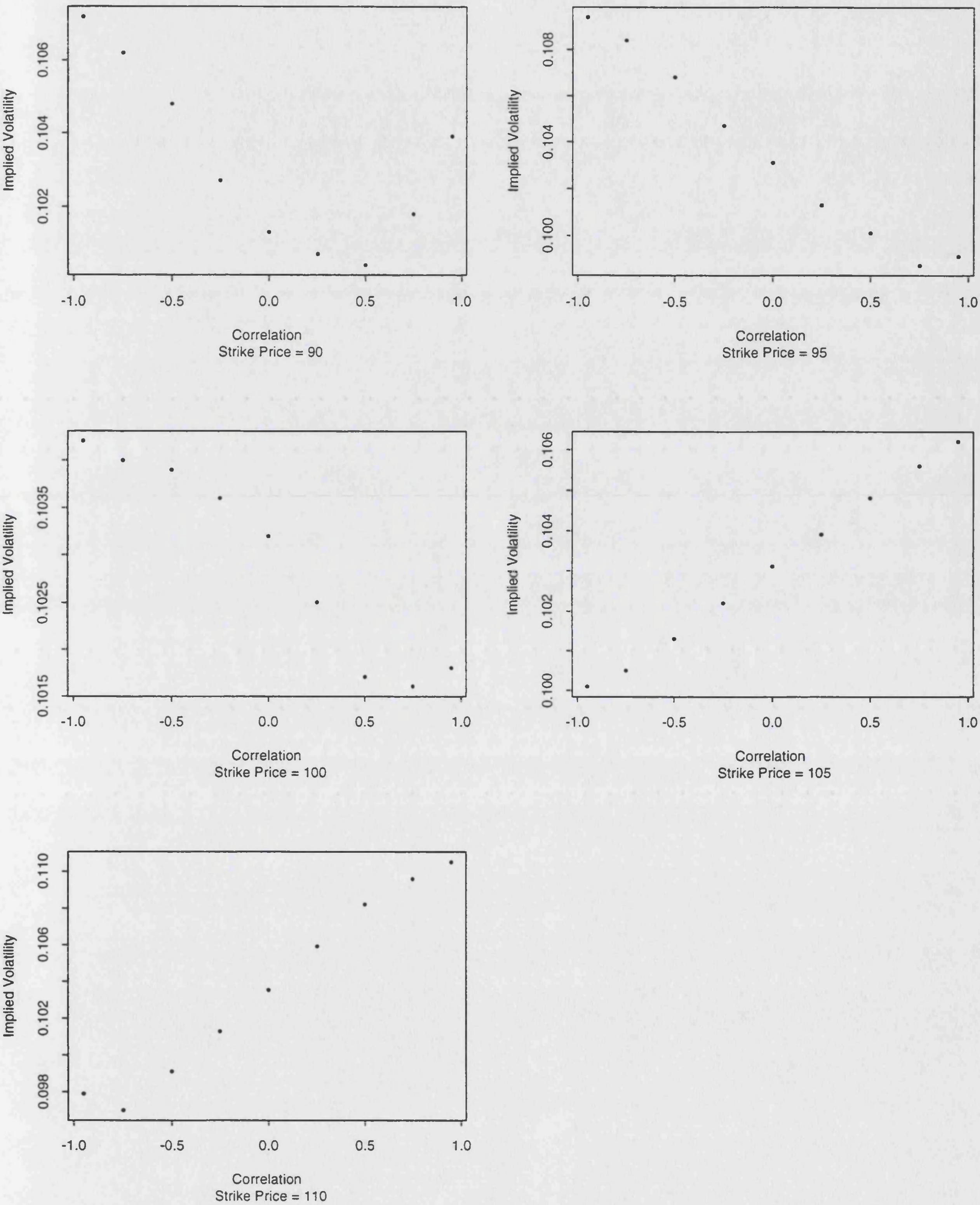
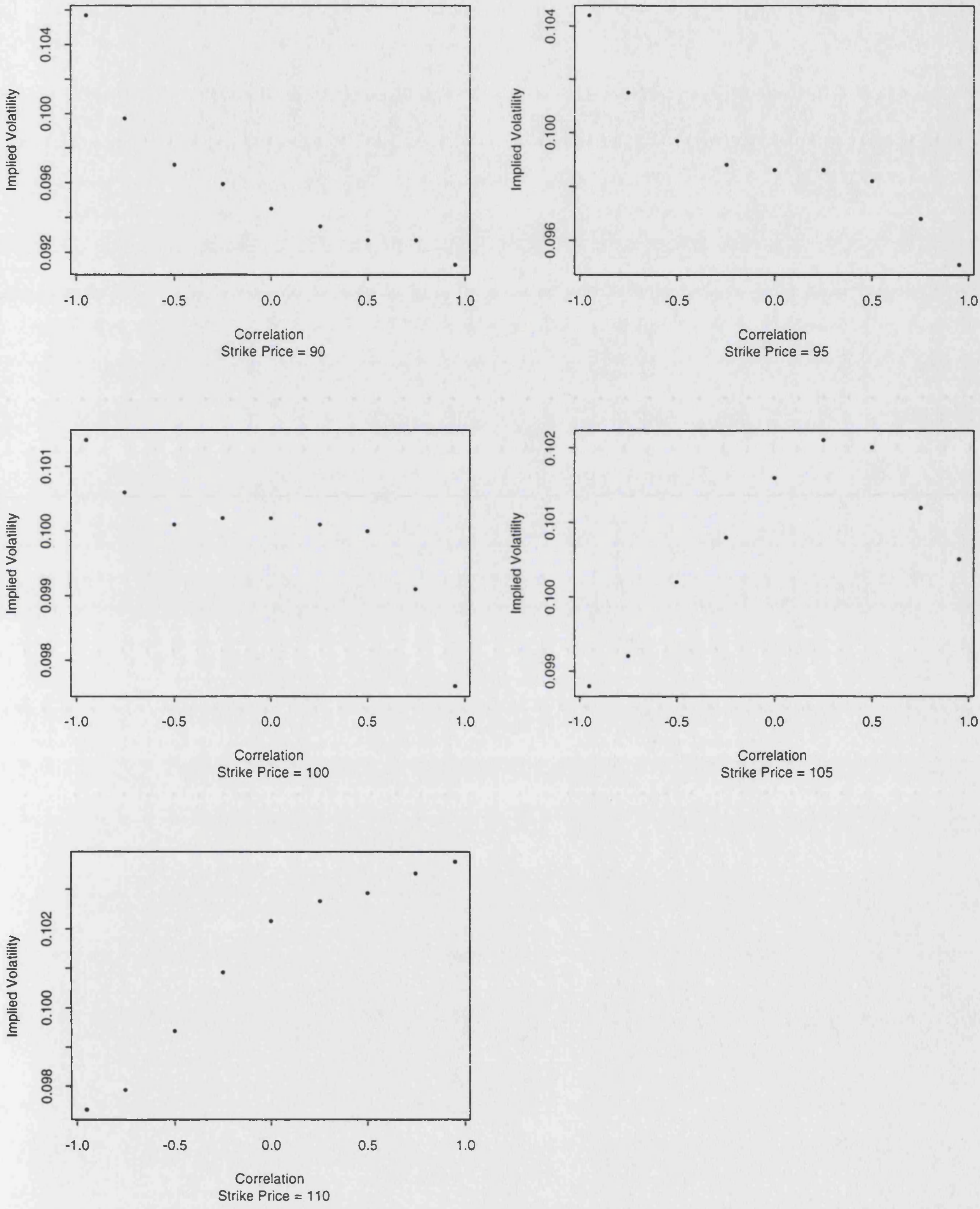


Figure 15 : The Ornstein - Uhlenbeck Process with a = 10 : Plots of Implied Volatility against Correlation



Chapter 7

Doubly Stochastic (Cox) Poisson Process

7.1 Introduction

In this chapter, we extend the approximation technique used to price bonds and options to the case of the Cox process (for details see Daley and Vere - Jones (1988) and Kallenberg (1997))- also known as the doubly stochastic Poisson process. The approximation technique used is the same as in the earlier chapters. The Cox process provides us with a very useful framework for modelling prices of financial instruments in which credit risk is a significant factor. Examples of such instruments are bonds, insurance policies, reinsurance policies among other. Work in this area has been done by a number of people; notable among them are Duffie (1996), Lando (1998), Dassios (1987) and Jang (1998). Most of Dassios' and Jang's work has been to look at the application of the Cox process in valuing insurance and reinsurance claims. On the other hand, Duffie and Lando have looked at the applications of the Cox process in pricing of bonds and valuing contingent payments to be made on bonds.

Claims arising from catastrophic events depend on the intensity of such natural disasters. Therefore the intensity means the frequency of claims arising from the natural disaster.

In order to calculate the price for catastrophe reinsurance contracts and insurance derivatives, the *claim arrival process* needs to determined. A homogeneous Poisson process can be used as a claim arrival process. Under this approach, the *claim intensity function* is assumed to be constant. Another approach is to use a non-homogeneous Poisson process where the claim intensity is assumed to be a non-random function of time. However, both these processes do not adequately explain the phenomena of catastrophes.

Under a *doubly stochastic Poisson process,* or a *Cox process,* the claim intensity function is assumed to be stochastic. The Cox process is more appropriately used as a claim arrival process as it can allow for the assumption that catastrophic events occur periodically.

A doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. This means that N_t is a Poisson process conditional on λ_t (if λ_t is deterministic, then N_t is simply a Poisson process). The term "doubly stochastic" was introduced by Cox (1955).

Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Bremaud (1981).

Definition : Let N_t be a point process adopted to a history $\mathcal{F}_t$ and let λ_t be a non-negative process. Suppose that λ_t is $\mathcal{F}_t$-measurable, $t \geq 0$ and that

$$\int_0^t \lambda_s ds < \infty \quad \text{almost surely (no explosions).}$$

If for all $0 \leq t_1 \leq t_2$ and $u \in \mathcal{R}$

$$E\left\{e^{iu(N_{t_2}-N_{t_1})}|\mathcal{F}_{t_1}\right\} = \exp\left((e^{iu}-1)\int_{t_1}^{t_2} \lambda_s ds\right) \tag{7.1}$$

then N_t is called a $\mathcal{F}_t$-doubly stochastic Poisson process with intensity λ_t.

In this dissertation, we will take $\mathcal{F}_t$ to be the natural filtration of the probability space.

Equation (7.1) gives us

$$Pr\left\{N_{t_2} - N_{t_1} = k|\lambda_s; t_1 \leq s \leq t_2\right\} = \frac{e^{-\int_{t_1}^{t_2}\lambda_s ds}\left(\int_{t_1}^{t_2}\lambda_s ds\right)^k}{k!} \tag{7.2}$$

and

$$E\left\{\theta^{N_{t_2}-N_{t_1}}|\lambda_s; t_1 \leq s \leq t_2\right\} = \exp\left(-(1-\theta)\int_{t_1}^{t_2}\lambda_s ds\right) \tag{7.3}$$

so

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{E\left(\theta^{N_{t_2}-N_{t_1}}|\lambda_s; t_1 \leq s \leq t_2\right)\right\} = E\left\{e^{-(1-\theta)\int_{t_1}^{t_2}\lambda_s ds}\right\} \tag{7.4}$$

$$\Rightarrow E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\right\} \tag{7.5}$$

where

$$X_t = \int_0^t \lambda_s ds \quad \text{the aggregated process.}$$

Thus, it is easy to note that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of X_t, the aggregated process.

The log-normal Cox process, rather the log-Gaussian Cox process, has also been used in the past in studying spatial data by Møller, Syversveen and Waagepetersen (1998) as well Rathbun and Cressie (1994).

7.2 Calculations

Here, we are again interested in finding the value of a stop-loss reinsurance contract. We assume $t = 1$. Thus, the value of the stop - loss reinsurance contract is given by

$$E(N_1 - k)^+, \tag{7.6}$$

where, N_1 is conditionally a Poisson random variable with a random parameter M and k is the strike price at which the contract is calculated. Also, let us assume

$$\lambda_t = ce^{\sigma Y_t}$$

where $\{Y_t, 0 \leq t \leq 1\}$ is a Gaussian process. Also, c is a constant and $c = \lambda_0$, where λ_0 is the initial value of the process λ_t. Now, in this case, define

$$M = c\int_0^1 e^{\sigma Y_s} ds,$$

In continuation of the examples used earlier, Y_t could represent either a Brownian motion or a stationary Ornstein - Uhlenbeck process or a non-stationary Ornstein - Uhlenbeck process.

Now, as we can see from *Lemma 7.1*, given later

$$E[(N_1 - k)^+|M] = MG(M,k) - kG(M,k+1) = f(M) \quad \text{say.} \tag{7.7}$$

Here $G(a, b)$ is the distribution function of a Gamma distribution with parameters *(a, b)*, $a > 0$, $b > 0$ and is given as

$$G(a,b) = \int_0^x \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1} dx.$$

Further, for convenience, we assume k to be an integer.

Lemma 7.1 : *Let N be a Poisson random variable with parameter t. Then,*

$$E(N-k)^+ = tG(t,k) - kG(t,k+1).$$

Proof : Suppose $\{\tilde{N}_t, t \geq 0\}$ is a Poisson process with parameter 1. Then, $\tilde{N}_t$ is a Poisson random variable with parameter t. Then, we have,

$$E(N-k)^+ = E(\tilde{N}_t - k)^+ = \sum_{j=k+1}^{\infty} (j-k) Pr(\tilde{N}_t = j) = \sum_{j=k+1}^{\infty} \sum_{i=k+1}^{j} Pr(\tilde{N}_t = j)$$

$$= \sum_{i=k+1}^{\infty} \sum_{j=i}^{\infty} Pr(\tilde{N}_t = j) = \sum_{i=k+1}^{\infty} Pr(\tilde{N}_t \geq i) = \sum_{i=k}^{\infty} Pr(\tilde{N}_t \geq i+1). \tag{7.8}$$

Now, $Pr(\tilde{N}_t \geq i+1) = Pr(T_{i+1} \leq t) = \int_0^t \frac{v^i e^{-v}}{i!} dv$, where T_i is the time of the i^{th} jump. Thus, we have using equation (7.8),

$$E(\tilde{N}_t - k)^+ = \sum_{i=k}^{\infty} Pr(T_{i+1} \leq t) = \sum_{i=k}^{\infty} \int_0^t \frac{v^i e^{-v}}{i!} dv$$

$$= \int_0^t \sum_{i=k}^{\infty} \frac{v^i e^{-v}}{i!} dv = \int_0^t Pr(\tilde{N}_v \geq k) dv$$

$$= \int_0^t \int_0^v \frac{u^{k-1} e^{-u}}{(k-1)!} du dv = \int_0^t (t-u) \frac{u^{k-1} e^{-u}}{(k-1)!} du$$

$$= tG(t,k) - kG(t,k+1). \tag{7.9}$$

■

Further, the function f is exactly the same as defined in equation (7.7) and is given by

$$f(M) = E[(N_1 - k)^+|M] = MG(M,k) - kG(M,k+1).$$

Now, f is convex. This is obvious from the fact that f can be written as

$$\int_0^t \int_0^v \frac{u^{k-1}e^{-u}}{(k-1)!} du dv.$$

Now, the second differential of this expression with respect to t is positive and hence the function f is convex.

As stated earlier, we are interested in obtaining

$$E[(N_1 - k)^+] = E[E(N_1 - k)^+|M] = E[f(M)].$$

Now, since f is convex, we have using a suitable conditioning factor Z and Jensen's inequality,

$$E[f(M)] = E(E[f(M)|Z]) \geq E(f(E(M|Z))).$$

The choice of the conditioning factor Z is based on the same principle as explained in chapter 3 and is given by

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}}. \tag{7.10}$$

Now, conditionally on on Z, Y_t has a Gaussian distribution. Furthermore, Z, itself has a standard normal distribution. Also,

$$E(Y_u|Z) = k_u Z,$$

$$\text{where} \quad k_u = \text{Cov}(Y_u, Z)$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \text{Cov}(Y_u, Y_v) - k_u k_v = v_u \quad \text{say.}$$

Thus,

$$E(M|Z = z) = E(\lambda_0 \int_0^1 e^{\sigma Y_s} ds = \lambda_0 \int_0^1 e^{\sigma k_u z + \frac{\sigma^2}{2} v_u} du = h(z) \quad \text{say.} \tag{7.11}$$

Now, once we have obtained the value of $h(z)$, we then obtain the lower bound to the value of the stop-loss reinsurance contact, conditionally on the conditioning factor Z. This is obtained by using equation (7.7) and *Lemma 7.1* and is given by

$$\int_0^{h(z)}\int_0^{v}\frac{u^k e^{-u}}{k!}dudv=\int_0^{h(z)}\int_u^{h(z)}dv\frac{u^k e^{-u}}{k!}du=\int_0^{h(z)}(h(z)-u)\frac{u^k e^{-u}}{k!}du \tag{7.12}$$

$$=h(z)G(h(z),k)-kG(h(z),k+1)=\Omega(z). \tag{7.13}$$

Finally, the lower bound to the unconditional price of the stop-loss reinsurance contract is obtained by taking the expectation of $\Omega(z)$ with respect to Z, where Z has a standard Normal distribution. Thus, we finally calculate

$$\int_{-\infty}^{\infty}\Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \tag{7.14}$$

to obtain the unconditional price of the stop-loss reinsurance contract.

Here, as an example, we assume that the process $\{Y_s, 0\le s\le 1\}$ follows an Ornstein - Uhlenbeck process. We give the explicit forms of Z, k_u and v_u in that case. Having these values, using equation (7.11) it is easy to obtain *h(z)* and having obtained *h(z)*, we can easily find the lower bound to the value of the stop-loss reinsurance contract, conditionally on Z, by using equation (7.13). Once we have that, we then use equation (7.14) to obtain the unconditional value of the lower bound of the stop-loss reinsurance contract.

Thus, here we have

$$dY_t=-aY_tdt+\sigma dB_t$$

$$\text{i.e.}\quad Y_t=\sigma\int_0^t e^{-a(t-u)}dB_u.$$

Here, Y_0, the initial value is zero. The conditioning factor, Z, is then given by

$$Z=\frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

We observe that

$$\mathrm{Var}(\int_0^1 Y_s ds) = \sigma^2 \int_0^1 \int_0^s (e^{-a(s-u)} dB_u)^2 ds = \frac{\sigma^2}{2a} \frac{2a + 4e^{-a} - e^{-2a} - 3}{a^2} = V, \quad \text{say.}$$

Thus,

$$k_u = \mathrm{Cov}(Y_u, Z) = \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \{ \int_0^u (e^{a(s+u)} - e^{-a(s+u)}) ds + \int_u^1 (e^{a(u-s)} - e^{-a(u+s)}) ds$$

$$= \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \{ \frac{1 - e^{-au}}{a} + \frac{1 - e^{-a(1-u)}}{a} - \frac{e^{-au} - e^{-a(1+u)}}{a} \}.$$

Also,

$$\mathrm{Cov}(Y_u, Y_v | Z) = \frac{\sigma^2}{2a} [e^{a|u-v|} - e^{-a(u+v)}] - k_u k_v = v_u.$$

Once we have this, then using equations (7.11), (7.13) and (7.14), we can easily find the lower bound to the value of the stop-loss reinsurance contract. The numerical results based on these calculations are given in tables 16.1 and 16.2.

7.3 Conclusion and Remarks

Using the conditioning factor in the Cox process situation, we can thus very easily calculate the price of the option. Evaluation of $h(z)$ is similar to the situation in the earlier chapters (2, 4 and 5) as also discussed by Rogers and Shi (1995). Once M, rather $E(M|Z)$, is evaluated, given the strike price, k, the calculation of the price of the option is just looking up the Gamma distribution tables - in fact, all statistical software would return the values. It is time saving as well as very efficient. Furthermore, the use of the conditioning factor approach means that we can account for all values of the instantaneous variance of the stochastic process driving λ, the parameter.

7.4 Tables

The following two tables show the comparison of the values obtained by using the conditioning factor approach (Calculated Value) contrasted against simulated values. Also included are standard errors of simulation.

Table 16.1 : $c = \lambda_0 = 10$

σ	Strike Price	Calculated Value	Simulated Value	Standard Error
0.5	8	2.924	2.911	0.0144
	10	1.706	1.698	0.0117
	12	0.901	0.898	0.0088
	15	0.292	0.292	0.005
	20	0.031	0.03	0.0015
0.75	8	3.49	3.507	0.0184
	10	2.268	2.285	0.0158
	12	1.401	1.416	0.013
	15	0.631	0.642	0.009
	20	0.147	0.152	0.0044
1	8	4.293	4.278	0.0244
	10	3.067	3.065	0.0219
	12	2.143	2.147	0.0192
	15	1.22	1.229	0.0152
	20	0.466	0.47	0.0099

Table 16.2 : $c = \lambda_0 = 100$

σ	Strike Price	Calculated Price	Simulated Value	Standard Error
0.5	80	25.053	25.001	0.0986
	100	11.198	11.162	0.0754
	120	3.948	3.925	0.0473
0.75	80	31.238	31.194	0.1491
	100	18.053	18.047	0.1258
	110	9.678	9.706	0.0982
	120	3.496	3.549	0.0625
1	80	39.771	39.767	0.2173
	100	26.909	29.956	0.1947
	120	17.783	17.856	0.1684
	150	9.385	9.545	0.1303
	200	3.251	3.41	0.0818

Chapter 8

Alternative Numerical Techniques

8.1 Introduction

In this chapter, we make use of certain alternative numerical techniques to solve the problem of pricing the Asian option. Rogers and Shi (1995) has also looked at this problem and obtained bounds to the price. In fact, the lower bounds that they obtain are so close that it can be regarded as the true price itself. However, they make use of a numerical integration technique to solve the problem. Now, this can be time consuming and also might require sophisticated machines and programs. In this chapter, we make use of a simple expansion technique to solve the problem and avoid the numerical integration by replacing it with a set of exact integrations. For the expansions, we use the algebraic package MAPLE.

We also extend this same idea of using a simple expansion technique to obtaining the price of the zero coupon bonds. Here also, this method allows us to avoid the numerical integration and use exact integrals in there place.

8.2 The Asian Option

Rogers and Shi assumes that at time t, the price of a risky asset S_t is given by

$$S_t = S_0 \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t + ct\right), \tag{8.1}$$

where, B_t is a standard Brownian motion, σ^2 is the instantaneous variance. Also, c is a constant. They assume also that under an equivalent martingale measure $c = r$, the riskless interest rate (see Harrison and Kreps (1979) and Harrison and Pliska (1981)). The problem that Rogers and Shi looked at is that of computing the value of an Asian (call) option with maturity T and the strike price F written on the risky asset S_t. Mathematically, this is the same as calculating

$$E(Y - F)^+, \tag{8.2}$$

where, Y is defined by

$$Y = \int_0^T S_u du. \tag{8.3}$$

Without loss of generality, we take t = 1. We make use of the Rogers and Shi idea of using Jensen's inequality to obtain a lower bound to the price. Thus, like Rogers and Shi, we are interested in finding $E(f(E(Y|Z)))$, where the function f is convex in nature and is defined exactly the same way as is done by Rogers and Shi. Thus, we have $f(x) = max([x - F], 0)$. Z is the conditioning factor used and is suitably normalised. This is similar to the one used by Rogers and Shi with $t = 1$ and is given by

$$Z = \frac{\int_0^1 B_s ds}{\sqrt{\text{Var}(\int_0^1 B_s ds)}},$$

where $\text{Var}(\int_0^1 B_s ds) = \frac{1}{3}$.

Thus, like Rogers and Shi, we are interested in finding

$$E\left[\int_0^1 \exp\left(\sigma B_u - \frac{1}{2}\sigma^2 u + ru\right) du|Z\right]. \tag{8.4}$$

This is similar to the lower bound of the price as found by Rogers and Shi.

Now, to find the expectation as defined by equation (8.4), we first find the following.

$$E(B_u|Z) = k_u Z, \tag{8.5}$$

$$k_u = \text{Cov}(B_u, Z) = \frac{\text{Cov}(B_u, \int_0^1 B_s ds)}{\sqrt{\text{Var}(\int_0^1 B_s ds}} = \sqrt{3}(u - \frac{u^2}{2}), \tag{8.6}$$

$$\text{Var}(B_u|Z) = \sigma^2(u - k_u^2). \tag{8.7}$$

Once we have these values, we are then interested in finding, conditionally on Z, the expected value of

$$E[\int_0^1 e^{\sigma B_u - \frac{1}{2}\sigma^2 u + ru} du|Z] = \int_0^1 \exp\left(ru + \sigma k_u Z - \frac{3}{2}\sigma^2(u - \frac{u^2}{2})^2\right) du$$

$$= \int_0^1 \left\{ exp\left(ru + \sigma\sqrt{3}\left(u - \frac{u^2}{2}\right) Z - \frac{3}{2}\sigma^2 \left(u - \frac{u^2}{2}\right)^2 \right) \right\} du.$$

Writing $k = \frac{r}{\sigma}$, we have the lower bound to the price of the asset, conditionally on Z, as

$$\int_0^1 \left\{ exp\left(k\sigma u + \sigma\sqrt{3}\left(u - \frac{u^2}{2}\right) Z - \frac{3}{2}\sigma^2 \left(u - \frac{u^2}{2}\right)^2 \right) \right\} du = \int_0^1 g(k,\sigma,u,z) du \quad \text{say.} \tag{8.8}$$

Rogers and Shi performed a numerical integration at this stage in order to obtain the price of the option conditionally on Z and then finally the expectation is taken over Z to obtain the final price of the option. However, at this stage that we make use of an expansion argument and differ from the approach of Rogers and Shi. This is done so as to allow us to avoid the numerical integrations involved.

We expand the exponential term, $g(k,\sigma,u,z)$, in equation (8.8) in terms of σ, and retain terms up to the fourth power of σ. Thus, we have, conditionally on Z = z,

$$g(k,\sigma,u,z) = g_1(k,\sigma,u,z) + O(\sigma^5),$$

where,

$$g_1(k,\sigma,u,z) = 1 + \left(ku + \sqrt{3}zu - \frac{1}{2}\sqrt{3}zu^2\right)\sigma$$

$$+ \left\{ -\frac{3}{2}u^2 + \frac{3}{2}u^3 - \frac{3}{8}u^4 + \frac{1}{2}k^2u^2 + ku^2\sqrt{3}z - \frac{1}{2}ku^3\sqrt{3}z + \frac{3}{2}z^2u^2 - \frac{3}{2}z^2u^3 + \frac{3}{8}z^2u^4 \right\} \sigma^2$$

$$+ \left\{ -\frac{1}{16}z^3u^6\sqrt{3} + \frac{3}{8}z^2u^5k - \frac{3}{2}z^2u^4k + \frac{3}{8}z^3u^5\sqrt{3} + \frac{1}{2}k^2u^3\sqrt{3}z - \frac{1}{4}k^2u^4\sqrt{3}z \right.$$

$$+\frac{3}{2}z^2u^3k + \frac{1}{2}z^3u^3\sqrt{3} - \frac{3}{4}z^3u^4\sqrt{3} + \frac{1}{6}k^3u^3 - \frac{3}{8}ku^5 + \frac{3}{16}\sqrt{3}zu^6$$

$$-\frac{3}{2}ku^3+\frac{3}{2}ku^4-\frac{3}{2}\sqrt{3}zu^3+\frac{9}{4}\sqrt{3}zu^4-\frac{9}{8}\sqrt{3}zu^5\Big\}\sigma^3$$

$$+\Big\{-\frac{9}{4}z^2u^4+\frac{9}{8}u^4-\frac{9}{4}u^5+\frac{27}{16}u^6-\frac{3}{4}k^2u^4+\frac{3}{4}k^2u^5-\frac{3}{16}k^2u^6+\frac{9}{2}z^2u^5+\frac{3}{16}z^2u^6k^2-\frac{3}{4}z^2u^5k^2$$

$$+\frac{3}{16}ku^7\sqrt{3}z-\frac{3}{2}ku^4\sqrt{3}z+\frac{9}{4}ku^5\sqrt{3}z-\frac{3}{16}z^4u^7-\frac{1}{16}z^3u^7\sqrt{3}k-\frac{9}{8}ku^6\sqrt{3}z-\frac{3}{4}z^4u^5+$$

$$\frac{9}{16}z^4u^6+\frac{3}{8}z^4u^4-\frac{3}{4}z^3u^5k\sqrt{3}+\frac{3}{8}z^3u^6k\sqrt{3}-\frac{1}{12}k^3u^5\sqrt{3}z+\frac{1}{2}z^3u^4k\sqrt{3}+\frac{1}{6}k^3u^4\sqrt{3}z$$

$$+\frac{3}{128}z^4u^8+\frac{3}{4}z^2u^4k^2-\frac{27}{8}z^2u^6+\frac{9}{8}z^2u^7-\frac{9}{64}z^2u^8+\frac{1}{24}k^4u^4+\frac{9}{128}u^8-\frac{9}{16}u^7\Big\}\sigma^4. \quad (8.9)$$

Next we integrate out u from (8.9) and re-arrange the equation so that we have a polynomial in z. Thus, we have

$$\int_0^1 g_1(k,\sigma,u,z)du = \frac{1}{105}\sigma^4z^4+\left(\sqrt{3}[\frac{93}{4480}\sigma^4k+\frac{1}{35}\sigma^3]\right)z^3$$

$$+\left(-\frac{2}{35}\sigma^4+\frac{29}{560}\sigma^4k^2+\frac{11}{80}\sigma^3k+\frac{1}{5}\sigma^2\right)z^2$$

$$+\left(\sqrt{3}[\frac{5}{24}\sigma^2k+\frac{1}{3}\sigma+\frac{3}{40}\sigma^3k^2-\frac{279}{4480}\sigma^4k-\frac{3}{35}\sigma^3+\frac{7}{360}\sigma^4k^3]\right)z$$

$$+\left(1-\frac{1}{5}\sigma^2+\frac{1}{35}\sigma^4-\frac{29}{560}\sigma^4k^2+\frac{1}{24}\sigma^3k^3-\frac{11}{80}\sigma^3k+\frac{1}{6}\sigma^2k^2+\frac{1}{120}\sigma^4k^4+\frac{1}{2}k\right)=g_2(k,\sigma,z) \quad \text{say.}$$

We are thus left with expressions in terms of k, σ and z. Treating k and σ as constants, or known values, we thus have a 4^{th} degree polynomial in z.

Like Rogers and Shi, we are also interested in finding the lower bound to price of the option given by

$$E(E(Y-F)^+|Z) = E[g_2(k,\sigma,z)-F]^+,$$

where F is the strike price of the option. Now, the strike price value is grouped with the coefficient of z^0 in the polynomial $g_2(k,\sigma,z)$. The next thing that we need to do is to find the roots of this 4^{th} degree polynomial in z. Now, being a 4^{th} degree polynomial, it can have at most 4 real roots. Let these be ρ_1, ρ_2, ρ_3 and ρ_4. Without loss of generality, let us assume that

$$\rho_1 \le \rho_2 \le \rho_3 \le \rho_4.$$

Our objective is to calculate the price of the option in the region where the function $E(Y - F|Z)$ is positive. This is the area defined by the intervals $(-\infty, \rho_1)$, (ρ_2, ρ_3) and (ρ_4, ∞). In case, the polynomial has some imaginary roots, we ignore them and concentrate on the real roots only.

Let us define the coefficient of z^j by a_j for j = 0, 1, 2, 3, 4. Thus, we have,

$a_0 = 1 - \frac{1}{5}\sigma^2 + \frac{1}{35}\sigma^4 - \frac{29}{560}\sigma^4 k^2 + \frac{1}{24}\sigma^3 k^3 - \frac{11}{80}\sigma^3 k + \frac{1}{6}\sigma^2 k^2 + \frac{1}{120}\sigma^4 k^4 + \frac{1}{2}k - F,$

$a_1 = \sqrt{3}[\frac{5}{24}\sigma^2 k + \frac{1}{3}\sigma + \frac{3}{40}\sigma^3 k^2 - \frac{279}{4480}\sigma^4 k - \frac{3}{35}\sigma^3 + \frac{7}{360}\sigma^4 k^3],$

$a_2 = -\frac{2}{35}\sigma^4 + \frac{29}{560}\sigma^4 k^2 + \frac{11}{80}\sigma^3 k + \frac{1}{5}\sigma^2,$

$a_3 = \sqrt{3}[\frac{93}{4480}\sigma^4 k + \frac{1}{35}\sigma^3],$

$a_4 = \frac{1}{105}\sigma^4.$

Knowing the values of r, σ and F, we know $k = \frac{r}{\sigma}$. Once we know the values of k, σ and F, we can easily find the roots of the polynomial in z. Having obtained the value of ρ_1, ρ_2, ρ_3 and ρ_4 to calculate the value of the option, we then need to calculate

$$\sum_{j=0}^{4}\int_{-\infty}^{\rho_1} a_j z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=0}^{4}\int_{\rho_2}^{\rho_3} a_j z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=0}^{4}\int_{\rho_4}^{\infty} a_j z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \sum_{j=0}^{4} a_j \int_{-\infty}^{\rho_1} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=0}^{4} a_j \int_{\rho_2}^{\rho_3} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

$$\ge \sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \tag{8.10}$$

Here, a_j is the co-efficient of z^j and being independent of z can be taken outside the integral.

Since we are interested in the lower bound to the price, we look at

$$\sum_{j=0}^{4} a_j \int_{\rho_4}^{\infty} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \tag{8.11}$$

where ρ_4 is the largest of the real roots. Furthermore, in practice the contribution from

$$\sum_{j=0}^{4} a_j \int_{-\infty}^{\rho_1} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=0}^{4} a_j \int_{\rho_2}^{\rho_3} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,$$

is negligible and hence can be ignored. This fact is also reflected in the results obtained, as shown in the tables (Tables 17.1 - 17.4).

Being interested in the lower bound of the price as given by equation (8.11), we are thus interested in the following integrals;

$$\int_{\rho}^{\infty} z^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 - \Phi(\rho),$$

$$\int_{\rho}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{\rho^2}{2}},$$

$$\int_{\rho}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{\rho e^{-\frac{\rho^2}{2}} + \sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}},$$

$$\int_{\rho}^{\infty} z^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{2 + \rho^2}{\sqrt{2\pi}} e^{-\frac{\rho^2}{2}},$$

$$\int_{\rho}^{\infty} z^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{\rho^3 e^{-\frac{\rho^2}{2}} + 3\rho e^{-\frac{\rho^2}{2}} + 3\sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}}.$$

Finally, the only thing that remains to be done to calculate the value of the lower bound of the option is to multiply the appropriate coefficients of z, a_j (j = 0, 1,2,3,4) with the corresponding values of the integrals and add them up. To obtain an approximation to the price of the asset, all that needs to be done is to put the strike price of the option at 0. Thus, in effect one needs to calculate

$$a_0[1 - \Phi(\rho_4)] + a_1 \frac{1}{\sqrt{2\pi}} e^{-\frac{\rho_4^2}{2}} + a_2[\frac{\rho e^{-\frac{\rho_4^2}{2}} + \sqrt{2\pi}(1 - \Phi(\rho_4))}{\sqrt{2\pi}}] + a_3[\frac{2 + \rho_4^2}{\sqrt{2\pi}} e^{-\frac{\rho_4^2}{2}}]$$

$$+a_4[\frac{\rho_4^3 e^{-\frac{\rho_4^2}{2}} + 3\rho_4^{-\frac{\rho_4^2}{2}} + 3\sqrt{2\pi}(1 - \Phi(\rho_4))}{\sqrt{2\pi}}] = \Omega(r, \sigma, F) \quad \text{say.} \qquad (8.12)$$

This is because $k = \frac{r}{\sigma}$ and ρ is a function of k, σ and F.

The values obtained using this method is given in Tables 17.1 to 17.4. The values of σ and r as well as the strike price b are exactly the same as the ones used by Rogers and Shi (1995). In fact, we also give the values of the Asian option obtained by Rogers and Shi. We give the values which they denote by LB_2 - according to them, it is the closest approximation to the true price.

8.3 Bond Pricing : Zero Coupon Bonds

As in the case of the Asian option, here also, we employ a technique which combines the use of the conditioning factor Z to obtain the prices and the expansion technique. The fact that we use the expansion technique in conjunction with the conditioning factor ensures that the method does not collapse for relatively high values of σ ($\sigma > 1.5$; see also chapter 2, section 3). This method reduces the number of numerical integrations performed when only the conditioning factor is used and replaces these by simple integrations. The way it works is that first it performs a linearisation of the exponential term and then evaluates the integral exactly. In fact, no integration is required as long as the exponent is expanded up to 4 terms. This is because, in this case, the final integration to calculate the expected value is over Z, which is raised to various powers, and the co-efficients are dependent on certain fixed values of the parameters, but independent of Z. The expressions are multiplied by the standard normal density function, as we are interested in finding the expected value.

We have the situation

$$r_t = be^{X_t}$$

$$\text{and} \quad X_t = \mu_t + \sigma Y_t,$$

where r_t is the instantaneous rate of interest, μ_t is the drift and Y_t is a is a Gaussian process with zero mean and a variance - covariance $\text{Cov}(Y_u, Y_v) = l_{uv}$ and b is a constant which takes different values in different situations. In our examples, we take Y_s to be either a Brownian motion with a drift μ or an Ornstein - Uhlenbeck process. In the last two cases, $\mu_t = 0$. We are interested in finding

$$E(f(\int_0^1 (\sigma Y_s + \mu_s)ds)),$$

where f is a convex function. Thus, in particular the price of the bond (f(x) = e^{-bx}) is given by

$$E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds}) \tag{8.13}$$

and the value of the contingent payment on the bond (f(x) = $[e^{-bx} - c]^+$) is given by

$$E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds} - c)^+. \tag{8.14}$$

We make use of the conditioning factor approach as discussed in chapter 3 to calculate the price of the bond as well as the contingent payment on it.

$$E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds}|Z) \tag{8.15}$$

and to calculate the contingent payment on the bond, we look at

$$E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds} - c|Z)^+, \tag{8.16}$$

where Z, the conditioning factor, is chosen as explained in chapter 2 and is given by

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}}. \tag{8.17}$$

Now, Y_s is either a Brownian motion or an Ornstein - Uhlenbeck process, depending on what the volatility process is. Conditionally on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{8.18}$$

$$\text{where} \quad k_u = \text{Cov}(Y_s, Z) = \frac{\int_0^1 \text{Cov}(Y_u, Y_s)ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}} \tag{8.19}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = l_{uv} - k_u k_v = w_{uv} \quad \text{say.} \tag{8.20}$$

What we want to find out is an approximation to the price of the bond. That is given by

$$E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds}) = E[E(e^{-b\int_0^1 e^{\sigma Y_s + \mu_s} ds})|Z]. \tag{8.21}$$

Also, the price of the contingent payment is given by

$$E(e^{-b\int_0^1 e^{\sigma Y_s+\mu_s ds}} - c)^+ = E[E(e^{-b\int_0^1 e^{\sigma Y_s+\mu_s ds}} - c|Z)^+]. \qquad (8.22)$$

Now, in both cases, we first need to find out

$$E(e^{-b\int_0^1 e^{\sigma Y_u+\mu_u du}}|Z) = \exp\left(-b\int_0^1 \exp\{\mu_u + \sigma k_u z + \frac{1}{2}\sigma^2 w_{uu}\}du\right) = g(\mu_u, a, \sigma, b, Z) = \quad \text{say.}$$

The expressions for k_u are different for Y_u taking different forms and hence $g(\mu_u, a, \sigma, b, z, b)$ takes a different form for different forms of Y_u. When Y_u represents a Brownian motion, a $=0$ and μ_u represents the drift, while in the case of Y_u representing an Ornstein - Uhlenbeck process, $\mu_u = 0$ and a represents the mean reversion force of the Ornstein - Uhlenbeck process.

At this point, this method diverges from the one used in chapter 2 in valuing zero coupon bonds. In that case, we had made use of numerical integrations to obtain the approximation to the lower bound to the price of the bond as well as the lower bound of the value of the contingent payment on the bond. However, here, we do not use any numerical integrations and look at the expansion of $g(\mu_u, a, \sigma, b, z)$ in terms of σ and retain terms up to the 4^{th} order. We thus have a a polynomial in σ, b, Z, k and a, where $k = \frac{\mu_s}{\sigma}$. μ_u is the drift in case of the Brownian motion and $\mu_u = 0$ in the case on the Ornstein Uhlenbeck process. Let this polynomial be denoted by $g_1(k, \sigma, u, z, b)$. Thus, we have

$$g(\mu_u, a, \sigma, b, z) = g_1(k, a, \sigma, b, z) + O(\sigma^5),$$

where,

$$g_1(k, a, \sigma, b, z) = b_0 + b_1(k, a, z, b)\sigma + b_2(k, a, z, b)\sigma^2 + b_3(k, a, z, b)\sigma^3 + b_4(k, a, z, b)\sigma^4. \qquad (8.23)$$

Now, the first term in the expansion is a constant and thus, $b_0 = e^{-b}$ and $b_j(k, u, z, b)$ is a function of k, u, z and b for j = 1, 2, 3, 4. The exact form of $g_1(k, \sigma, u, z, b)$ for the case of Y_u following a Brownian Motion and a Stationary Ornstein - Uhlenbeck process are given in the appendix to this chapter.

Next we re-arrange the terms of equation(8.23) in terms of z, we have, $g_1(k,a,\sigma,b,z)$ a polynomial in z, since we treat σ, b, k and a as constants. Thus, we have,

$$g_1(k,a,\sigma,b,z) = \sum_{j=0}^{4} a_j(k,a,\sigma,b)z^j. \tag{8.24}$$

Here, a_j's are the co-efficients of z^j; j = 0, 1, 2, 3, 4. Further, $a_0 = b_0 = e^{-b}$. Also, $a_j(k,a,\sigma,b)$ are the co-efficients of z^j; j = 1, 2, 3, 4 and are functions of k, σ and b.

To calculate the price of the bond, we need to calculate

$$E[E(e^{-b\int_0^1 e^{\sigma Y_s+\mu_s}ds}|Z)] = E[g_1(k,a,\sigma,b,z)], \tag{8.25}$$

while to calculate the contingent payment on the price of the bond we need to calculate

$$E[E(e^{-b\int_0^1 e^{\sigma Y_s+\mu_s}ds} - c|Z)^+] = E[\{g_1(k,a,\sigma,b,z) - c\}^+], \tag{8.26}$$

where c is the strike price at which the contingent payment on the bond is calculated.

From the above two equations, equations (8.25) and (8.26), it is obvious that by equating c to *0*, in equation (8.26), we obtain the approximation to the price of the bond. Since this is the more general set up, we will use this equation to value both the bond as well as the contingent payment on it. For approximating the price of the bond, we will just equate c, the strike price, to *0*.

Now, to calculate an approximation to the price of the contingent payment on the bond (as well as an approximation to the price of the bond), we group the value of the strike price at which the contingent payment on the bond is calculated, c, with the coefficient of z^0 in the polynomial $g_1(k,a,\sigma,b,z)$. The next thing that we need to do is to find the roots of this roots of this 4^{th} degree polynomial in z and choose the largest of the real roots, say ρ. This is because while taking the expectation over Z to calculate the price of the option we are only interested in the region where the function $E(e^{-b\int_0^1 e^{Y_s+\mu_s}ds} - c|Z)$ is positive.

Knowing the values of μ_u, σ, b, a and c, we know $k = \frac{\mu_u}{\sigma}$. Once we know the values of k, σ b, a and c, we can easily find the roots of the polynomial in z. Having obtained the value of ρ, to calculate the value of the contingent payment, we then need to calculate

$$\int_{-\infty}^{\rho}(a_0 - c)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \sum_{j=1}^{4}\int_{-\infty}^{\rho} a_j \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$= (a_0 - c)\int_{-\infty}^{\rho} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sum_{j=1}^{4} a_j \int_{-\infty}^{\rho} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \qquad (8.27)$$

Here, a_j is the co-efficient of z^j and being independent of z can be taken outside the integral.

Now, we have,

$$\int_{-\infty}^{\rho} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi(\rho),$$

$$\int_{-\infty}^{\rho} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{\rho^2}{2}},$$

$$\int_{-\infty}^{\rho} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 - \int_{\rho}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 - \frac{\rho e^{-\frac{\rho^2}{2}} + \sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}},$$

$$\int_{-\infty}^{\rho} z^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -\frac{2 + \rho^2}{\sqrt{2}} e^{-\frac{\rho^2}{2}},$$

$$\int_{-\infty}^{\rho} z^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 3 - \int_{\rho}^{\infty} z^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 3 - \frac{\rho^3 e^{-\frac{\rho^2}{2}} + 3\rho e^{-\frac{\rho^2}{2}} + 3\sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}}.$$

Finally, the only thing that remains to be done to calculate the value of the contingent payment on the price of the bond is to multiply the appropriate coefficients of z, a_j; j = 0, 1, 2, 3, 4 with the corresponding values of the integrals and add them up. To obtain an approximation to the price of the asset, all that needs to be done is to put the strike price of the option at *0*. Thus, re-writing equation (8.26) explicitly with the form of $\int_{-\infty}^{\rho} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$, j = 0, 1, 2, 3, 4, one needs to calculate

$$(a_0 - c)\Phi(\rho) + a_1[\frac{1}{\sqrt{2\pi}} e^{-\frac{\rho^2}{2}}] + a_2[1 - \frac{\rho e^{-\frac{\rho^2}{2}} + \sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}}] + a_3[-\frac{2 + \rho^2}{\sqrt{2}} e^{-\frac{\rho^2}{2}}]$$

$$+ a_4[3 - \frac{\rho^3 e^{-\frac{\rho^2}{2}} + 3\rho e^{-\frac{\rho^2}{2}} + 3\sqrt{2\pi}(1 - \Phi(\rho))}{\sqrt{2\pi}}] = \Omega(r, \sigma, b, c) \quad \text{say.} \qquad (8.28)$$

This is because $k = \frac{\mu_u}{\sigma}$ and ρ is a function of k, σ, b and c.

Thus, to calculate the price of the bond, we put $c = 0$ in $\Omega(r, \sigma, b, c)$, whereas to calculate the value of the contingent payment on the bond, we let c take the value of the strike price at which the payment is to be made.

8.4 Conclusion and Remarks

The prices calculated in using this approach for both the Asian options as well as the bonds are exactly similar to the ones calculated by using the conditioning factor approach. This method has a few distinct advantages. First of all, it is very fast and can provide output in real time and does not need to perform any numerical integration. Secondly, and more importantly, all calculation in this approach can be carried out on such simple machines as a programmable calculator. The only care that needs to taken is to ensure that it has the facility to calculate the roots of a polynomial. Though the method involves the calculation of the roots of a 4^{th} degree polynomial, packages exist for it and can be done very easily. Further, the alternative would be to make use of two numerical integrations and thus obtaining the roots of the polynomial in z seems to be much better option.

The method works well for fairly large values of σ as well; for σ taking values up to 1, the value of the option calculated by this method is very close to the simulated values.

The exact Splus codes used to calculate the values of the Asian Option as well as the bonds are attached in the appendix to this chapter.

8.5 Tables

The following tables show the comparison of the values obtained by the alternative method of valuing an Asian option as contrasted to the values obtained by Rogers and Shi (1995). The values obtained by the alternative method is given as in the calculated price.

Table 17.1 : $\sigma = 0.05$

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	95	7.178	7.178
	100	2.716	2.716
	105	0.337	0.337
0.09	95	8.809	8.809
	100	4.308	4.308
	105	0.958	0.958
0.15	95	11.094	11.094
	100	6.794	6.794
	105	2.744	2.744

Table 17.2 : $\sigma = 0.1$

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	11.951	11.951
	100	3.641	3.641
	110	0.331	0.331
0.09	90	13.385	13.385
	100	4.915	4.915
	110	0.630	0.630
0.15	90	15.399	15.399
	100	7.028	7.028
	110	1.413	1.413

Table 17.3 : $\sigma = 0.2$

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	12.596	12.595
	100	5.763	5.762
	110	1.989	1.989
0.09	90	13.831	13.831
	100	6.777	6.777
	110	2.545	2.545
0.15	90	15.642	15.641
	100	8.408	8.408
	110	3.555	3.554

Table 17.4 : $\sigma = 0.3$

r	Strike Price	Calculated Price	Rogers & Shi Price
0.05	90	13.952	13.952
	100	7.944	7.944
	110	4.070	4.070
0.09	90	14.983	14.983
	100	8.827	8.827
	110	4.695	4.695
0.15	90	16.512	16.512
	100	10.208	10.208
	110	5.728	5.728

8.6 Appendix

In this appendix, we give the exact form of $g_1(k, \sigma, u, z, b)$ as defined by equation (6.23) for Y_u following a Brownian motion or a Stationary Ornstein - Uhlenbeck process.

8.6.1 Brownian Motion with drift

We rewrite $k = \frac{r}{\sigma}$. We, thus have,

$$\exp\left(-b\int_0^1 \exp\left(k\sigma u + \sigma z\sqrt{3}(u - \tfrac{u^2}{2}) + \tfrac{\sigma^2}{2}\left(u - 3(u - \tfrac{u^2}{2})^2\right)\right)\right)$$

$$= g_1(k, \sigma, u, z, b) + O(\sigma^5) = e^{-b} - \left(\tfrac{1}{6}e^{-b}b\tfrac{3k^2+5k\sqrt{3}z+6z^2}{k+\sqrt{3}z}\right)\sigma$$

$$+\left(-20k^4 - 65\sqrt{3}zk^3 - 234z^2k^2 - 123z^3\sqrt{3}k - 72z^4 - 6k^2 - 12k\sqrt{3}z\right.$$
$$\left.-18z^2 + 15bk^4 + 50b\sqrt{3}zk^3 + 185bz^2k^2 + 100bz^3k\sqrt{3} + 60bz^4\right)\tfrac{\sigma^2}{120}\tfrac{e^{-b}b}{(k+\sqrt{3}z)^2}$$

$$-\left(-1386b\sqrt{3}zk^3 - 3402bz^3k\sqrt{3} + 3888z^6 + 630k^6 + 19530b^2z^4k^2 + 9765b^2z^2k^4 - 9072bz^6\right.$$
$$+2511z^4 - 37737bz^2k^4 + 17955z^2k^4 - 1260bk^6 - 72765bz^4k^2 + 32805z^4k^2 + 2520b^2z^6$$
$$+315b^2k^6 - 40509bk^3\sqrt{3}z^3 + 10125z^5\sqrt{3}k - 6195bk^5\sqrt{3}z - 23058bz^5\sqrt{3}k + 1575b^2k^5\sqrt{3}z$$
$$+3024k^5\sqrt{3}z + 10675b^2z^3\sqrt{3}k^3 + 6300b^2z^5\sqrt{3}k + 18765z^3\sqrt{3}k^3 + 6480z^2k^2$$
$$\left.+3834z^3\sqrt{3}k + 1602\sqrt{3}zk^3 + 441k^4 - 2268bz^4 - 378bk^4 - 5670bz^2k^2\right)\tfrac{\sigma^3}{15120}\tfrac{e^{-b}b}{(k+\sqrt{3}z)^3}$$

$$+\left(-16200z^2k^2 - 10800z^3k\sqrt{3} - 900k^4 - 8100z^4 + 756bk^4\right.$$
$$+6804bz^4 + 13608bz^2k^2 + 3024bk^3z\sqrt{3} + 9072bz^3k\sqrt{3} - 33615k^5z\sqrt{3} + 479808bz^2k^4$$
$$+154872bz^6 + 1073844bz^4k^2 - 133560b^2z^2k^4 - 306180b^2z^4k^2 - 6480k^6 - 63180z^6$$
$$-216000z^2k^4 - 461700z^4k^2 + 13860bk^6 - 45360b^2z^6 - 3780b^2k^6 + 5468526bz^4k^4$$
$$+1149435bz^2k^6 + 4063203bz^6k^2 - 1188900z^4k^4 - 263160z^2k^6 + 21000bk^8 - 5040k^8$$
$$+554832bk^3z^3\sqrt{3} - 244350k^3z^3\sqrt{3} - 153495z^5\sqrt{3}k - 20160b^2k^5z\sqrt{3} - 105840b^2z^5k\sqrt{3}$$
$$-156240b^2z^3\sqrt{3}k^3 + 366336bz^5k\sqrt{3} + 73200bk^5z\sqrt{3} - 31920k^7z\sqrt{3} + 10500b^3k^7z\sqrt{3}$$
$$+264384bz^8 - 181440b^2z^8 + 25200b^3z^8 + 1575b^3k^8 - 51840z^8 + 462175b^3z^4k^4$$
$$+91350b^3z^2k^6 + 365400b^3z^6k^2 - 3507840b^2z^4k^4 - 712320b^2z^2k^6 - 2698920b^2z^6k^2$$
$$-837540z^6k^2 - 12600b^2k^8 - 1157940b^2k^5z^3\sqrt{3} - 612360b^2z^7k\sqrt{3} + 3452004bk^3z^5\sqrt{3}$$
$$+1836900bk^5z^3\sqrt{3} + 301000b^3z^5k^3\sqrt{3} + 150500b^3z^3k^5\sqrt{3} + 84000b^3z^7k\sqrt{3}$$
$$-82950b^2k^7z\sqrt{3} - 2253510b^2z^5k^3\sqrt{3} + 906552bz^7k\sqrt{3}$$
$$\left.+136080bk^7z\sqrt{3} - 182115z^7k\sqrt{3} - 730710k^3z^5\sqrt{3} - 409995k^5z^3\sqrt{3}\right)\tfrac{\sigma^4}{604800}\tfrac{e^{-b}b}{(k+\sqrt{3}z)^r} + O(\sigma^5).$$

8.6.2 Stationary Ornstein - Uhlenbeck Process

Here, we assume that the process has a mean reversion force of a. We, thus have,

$$\exp\left(-b\int_0^1 \exp\left(\sigma z\frac{2-e^{-au}-e^{-a(1-u)}}{2\sqrt{a}\sqrt{a+e^{-a}-1}}+\frac{\sigma^2}{2}\left(\frac{1}{2a}-\left(\frac{2-e^{-au}-e^{-a(1-u)}}{2\sqrt{a}\sqrt{a+e^{-a}-1}}\right)^2\right)\right)du\right)$$

$$= g_1(k,\sigma,u,z,b)+O(\sigma^5)$$

$$= e^{-b}-\left(e^{-b}bz\frac{\sqrt{a+e^{-a}-1}}{(\sqrt{a})^3}\right)\sigma$$

$$-\left(2a^3+7a+4a^2z^2-6a^2+8z^2ae^{-a}-8ae^{-a}-7z^2a-z^2ae^{-2a}+ae^{-2a}\right.$$

$$\left.+2z^2a^2e^{-a}-4bz^2a^2-8bz^2ae^{-a}+8bz^2a-4bz^2e^{-2a}+8bz^2e^{-a}-4bz^2\right)\frac{\sigma^2}{8}\frac{e^{-b}b}{a^3(a+e^{-a}-1}$$

$$+\left(171e^{-a}a^3+55e^{-a}a^2z^2-28a^2z^2-18e^{-a}a^4z^2-12a^4z^2+40a^3z^2-174a^3-21e^{-a}a^3z^2\right.$$

$$-21e^{-3a}a^2+3e^{-3a}a^3-27e^{-2a}a^2z^2+e^{-3a}a^2z^2-219e^{-a}a^2+135e^{-2a}a^2+102a^2-z^2a^2e^{-4a}$$

$$-e^{-3a}a^3z^2-18e^{-2a}a^3z^2-18a^5+90a^4+3e^{-4a}a^2+48b^2z^2a^3-72b^2z^2a^2-12b^2z^2a^4$$

$$+48b^2z^2a-135bz^2a^3-180ba^2+18ba^5-90ba^4+189ba^3-12b^2z^2e^{-4a}-72b^2z^2e^{-2a}-48b^2z^2ae^{-3a}$$

$$+48b^2z^2e^{-a}+48b^2z^2e^{-3a}+18ba^2e^{-3a}+9bae^{-4a}+9bae^{-2a}+18bz^2a^4e^{-a}-18bae^{-3a}+18bz^2a^2e^{-3a}$$

$$+36bz^2a^3e^{-2a}+9ba^3e^{-2a}+144bz^2a^2e^{-2a}+108bz^2a^3e^{-a}+18ba^2z^2e^{-2a}-9baz^2e^{-2a}-18ba^2z^2e^{-3a}$$

$$-9ba^3z^2e^{-2a}+18baz^2e^{-3a}+72bz^2ae^{-3a}-9baz^2e^{-4a}+207bae^{-2a}-18ba^2e^{-2a}-72bae^{-3a}$$

$$-198bae^{-a}-144b^2z^2ae^{-2a}+144b^2z^2a^2e^{-a}-324bz^2a^2e^{-a}+144b^2z^2ae^{-2a}-207bz^2ae^{-2a}$$

$$-48b^2z^2a^3e^{-a}-216ba^3e^{-a}+378ba^2e^{-a}+36ba^4e^{-a}+18ba^3e^{-2a}-198ba^2e^{-2a}+36bz^2a^4-12b^2z^2$$

$$\left.+63ba-72b^2z^2a^2e^{-2a}-63bz^2a+162bz^2a^2+198bz^2ae^{-a}\right)\frac{\sigma^3}{72}\frac{e^{-b}bz}{(\sqrt{a})^9(\sqrt{a+e^{-a}-1})^5}$$

$$+A(z,b,a)\sigma^4+O(\sigma^5),$$

where $A(z,b,a)$ is a function of z, b and a. Essentially it is the co-efficient of σ^4 in the expansion. The term being too long, is written in this form and not explicitly as in the other cases.

Chapter 9

Further Comments and Open Problems

We conclude the thesis by outlining three problems - problems which are similar to the ones discussed in this thesis. Note that we do not attempt to solve these problems but just provide an outline of them and leave them as open problems for future work. The first two problems are extensions to the problem of pricing of options on stochastically volatile assets and the problem of pricing of bonds. We believe that both these problems can be solved using the approximation technique discussed throughout this thesis. As a matter of fact, the approximation technique discussed throughout the thesis can be used in any situation where there is a log - Gaussian process. The third problem is to find another justification of the conditioning factor used throughout this thesis. We shall now briefly define each of the problems.

The first problem is of pricing an European call option on a basket or portfolio of stochastically volatile assets. The idea used to price bonds based on *multi - driver* models (as discussed in chapter 5) could possibly be extended to price the call option on the portfolio of stochastically volatile assets. As in the case of pricing of options on just one stochastically volatile asset (as discussed in chapter 6), in this case also, we might not obtain a lower bound. However, we could try to obtain an approximation to the price of the option instead. We believe that as long as the log - normality of the model is not violated very close approximations to the actual price of the option can be obtained using the approximation technique.

Another problem is of using a different model for the interest rate process in pricing of options on stochastically volatile assets with stochastic interest rates. In our analysis, we have taken the interest rate process to follow an Ornstein - Uhlenbeck process. This is very similar to the Vasicek (1977) model. However, one could try to model the interest rate as a log - Gaussian process; this is similar to the model we have used in chapters 2, 4 and 5 when modelling interest rates and pricing bonds. Again, we believe that the approximation technique used in chapter 6 would still work though the calculations could become very complicated and involved.

The third problem is of providing another explanation for the choice of the conditioning factor. Throughout this thesis we have used a conditioning argument to obtain the approximations - lower bounds in the case of bond prices and approximate prices in case of option prices. In chapter 3, we have provided one justification to the choice of the conditioning factor starting from a general Gaussian distribution. However, an attempt could be made to explain the choice of the conditioning factor by using factor analysis or principal component analysis techniques. The idea is to try to obtain the conditioning factor Z which explains the majority of the variation. We believe that the form of the conditioning factor would still remain as defined in chapter 3 - only that there would be one more justification for using this form of the conditioning factor.

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Appendix I : Programs

In this section, we present the code written in Splus to carry out the various calculations throughout the thesis. Before each program, or the group of programs, there is a short note describing what the program is supposed to do.

Valuing bonds using conditioning factors.

The following set of programs calculates the value of bonds using the appropriate conditioning factor. The first set of programs calculate the value of the bond when the underlying stochastic process follows a Brownian motion, the second set assumes that the underlying stochastic process follows a Non-stationary Ornstein Uhlenbeck process and in the third situation, the underlying stochastic process follows a Stationary Ornstein Uhlenbeck process.

Brownian Motion with a drift.

Here, the drift of the Brownian motion is a0, the instantaneous variance is si and the discounting factor is b. The first program, kkug, calculates the value of the conditioning factor. The second program, kdb, calculates the value of the integral for individual values of z and u, while the third program actually ensures the integration with respect to u, over the range [0,1].

```
kkug <- function(u) {
ak <- si * (3^0.5) * (u - ((u^2)/2))
return(ak)
}

kdb <- function(z, u) {
a1 <- kkug(u) * z
a2 <- u - (3 * (u^2)) + (3 * (u^3)) - ((3 * (u^4))/4)
a2 <- a2 * (si^2)
a3 <- ((a0 - ((si^2)/2)) * u) + y0 + a1 + (a2/2)
return(exp(a3))
}

kdbb <- function(z) {
x <- 1:100
for(i in 1:100) {
x[i] <- kdb(z, i/100)
}
return(exp(-b * mean(x)))
}
```

Finally, we evaluate the price by integrating with respect to z, after having multiplied with the standard normal density function over the entire range of z, that $(-\infty, \infty)$.

Non-stationary Ornstein Uhlenbeck Process

Here, the mean reversion force of the Non-stationary Ornstein Uhlenbeck process is a0, the instantaneous variance is si and the discounting factor is b. The first subroutine, vz, calculates the variance of the conditioning factor and the subroutine kug calculates the value of the conditioning factor. The subroutine db calculates the value of the integral for individual values of z and u, while the subroutine dbb actually ensures the integration with respect to u, over the range [0,1].

```
vz <- function(si) {
q <- (2 * a * t) + (4 * exp( - a * t)) - exp(-2 * a * t) - 3
qq <- (si^2) * (q/(2 * (a^3)))
return(qq)
}

kug <- function(u) {
b1 <- 1 - exp( - a * u)
b2 <- 1 - exp( - a * (t - u))
b3 <- exp( - a * u) - exp( - a * (t + u))
b <- (b1 + b2 - b3)/a
bb <- (si^2) * b * (1/(2 * a)) * (1/(vz(si)^0.5))
```

```
return(bb)
}

db <- function(z, u) {
a1 <- kug(u) * z
a2 <- 1 - exp(-2 * a * u)
a2 <- (si^2) * a2 * (1/(2 * a))
a3 <- kug(u)^2
a4 <- a1 + ((a2 - a3)/2)
return(exp(a4))
}

dbb <- function(z) {
x <- 1:1000
for(i in 1:1000) {
x[i] <- db(z, i/1000)
}
return(exp(-b * mean(x)))
}
```

Finally, we evaluate the price by integrating with respect to z, after having multiplied with the standard normal density function over the entire range of z, that (- ∞, ∞).

Stationary Ornstein Uhlenbeck Process.

Here, the mean reversion force of the Stationary Ornstein Uhlenbeck process is a0, the instantaneous variance is si and the discounting factor is b. The first subroutine, vzs, calculates the variance of the conditioning factor and the subroutine kus calculates the value of the conditioning factor. The subroutine dbs calculates the value of the integral for individual values of z and u, while the subroutine dbbs actually ensures the integration with respect to u, over the range [0,1].

```
vzs <-function(si) {
q1 <- a + exp( - a) - 1
q2 <- q1 * (si^2) * (1/(a^3))
return(q2)
}

kus <- function(u) {
b1 <- 1 - exp( - a * u)
b2 <- 1 - exp( - a * (1 - u))
b <- (b1 + b2)/a
bb <- (si^2) * b * (1/(2 * a)) * (1/(vzs(si)^0.5))
return(bb)
}

dbs <- function(z, u) {
a1 <- kus(u) * z
a2 <- (si^2)/(2 * a)
a3 <- kus(u)^2
a4 <- a1 + ((a2 - a3)/2)
return(exp(a4))
}

dbbs <- function(z) {
x <- 1:1000
for(i in 1:1000) {
x[i] <- dbs(z, i/1000)
}
return(exp(-0.07 * mean(x)))
}
```

Finally, we evaluate the price by integrating with respect to z, after having multiplied with the standard normal density function over the entire range of z, that (- ∞, ∞).

Subroutines to calculate the prices, using the alternative method.

The following set of subroutines calculates the value of bonds using the alternative method as described in section 3.6. The first set of subroutines calculate the value of the bond when the underlying stochastic process follows a Brownian motion, the second set assumes that the underlying stochastic process follows a Non-stationary Ornstein Uhlenbeck process and in the third situation, the underlying stochastic process follows a Stationary Ornstein Uhlenbeck process.

Brownian Motion with a drift

The following subroutines calculate the value of the bond as well as the value of a European option priced on the bond. The subroutines coeffb0, coeffb1, coeffb2, coefb3 and coeffb4 calculates the values of the co-efficients of the polynomial in "z". The subroutine rootb evaluates the roots of the polynomial and returns that largest real root as its output. The subroutines intb0, intb1, intb2, intb3 and intb4 calculate the value of the integrals. In fact, here the value of the integrals can be calculated exactly. Finally, the subroutine valueb combines the outputs obtained from the subroutines coeffb0, coeffb1, coeffb2, coeffb3, coeffb4 and intb0, intb1, intb2, intb3, intb4 to return the value of the bond or the value of the option priced on the bond. The user of this program only sees the subroutine valueb and all the user has to do is input the values of variance (si), the drift of the Brownian motion (a), the discount factor of the bond (b) and the strike price of the option of the bond (op). In case of calculating the value of the bond alone, the value of op is 0.

```
coeffb0 <- function(si, k, b){
bp1 <- 1 - ((1/6) * b * 3 * k * si) - ((1/6) * b * (k^2) * (si^2)) - ((1/20) * b * (si^2))
bp2 <- ((1/8) * (b^2) * (k^2) * (si^2)) - ((1/24) * b * (k^3) * (si^3)) -((7/240) * b * k * (si^3))
bp3 <- ((1/12) * (b^2) * (k^3) * (si^3)) + ((1/40) * (b^2) * k * (si^3))
bp4 <- ((-1/48)*(b^3)*(k^3)*(si^3))-((1/672)*b*(si^4))-((1/160)*(b^3)*(k^2)*(si^4))
bp5 <- ((1/384) * (b^4) * (k^4) * (si^4)) - ((1/120) * b * (k^4) * (si^4))
bp6 <- ((5/144) * (b^2) * (k^4) * (si^4)) - ((1/48) * (b^3) * (k^4) * (si^4))
bp7 <- ((1/800)*(b^2)*(si^4))-((3/280)*b*(k^2)*(si^4))+((11/480)*(b^2)*(k^2)*(si^4))
bp <- exp( - b) * (bp1 + bp2 + bp3 + bp4 + bp5 + bp6 + bp7)
return(bp)
}

coeffb1 <- function(si, k, b){
bq1 <- ((-1/6) * b * 2 * si) - ((5/24) * b * k * (si^2)) + ((1/6) * (b^2) * k * (si^2))
bq2 <- ((-1/24) * (b^3) * (k^2) * (si^3)) + ((23/144) * (b^2) * (k^2) * (si^3))
bq3 <- ((-31/1680) * b * (si^3)) - ((3/40) * b * (k^2) * (si^3)) + ((1/60) * (b^2) * (si^3))
bq4 <- ((-57/4480) * b * k * (si^4)) - ((31/576) * (b^3) * (k^3) * (si^4))
bq5 <- ((-1/120) * (b^3) * k * (si^4)) - ((7/360) * b * (k^3) * (si^4))
bq6 <- ((31/360) * (b^2) * (k^3) * (si^4)) + ((1/144) * (b^4) * (k^3) * (si^4))
bq7 <- ((37/1260) * (b^2) * k * (si^4))
bq <- exp( - b) * (3^0.5) * (bq1 + bq2 + bq3 + bq4 + bq5 + bq6 + bq7)
return(bq)
}

coeffb2 <- function(si, k, b){
br1 <- ((-1/5) * b * (si^2)) + ((1/6) * (b^2) * (si^2)) - ((1/12) * (b^3) * k * (si^3))
br2 <- ((37/120) * (b^2) * k * (si^3)) - ((11/80) * b * k * (si^3)) - ((113/720) * (b^3) * (k^2) * (si^4))
br3 <- ((1/48) * (b^4) * (k^2) * (si^4)) + ((31/128) * (b^2) * (k^2) * (si^4))
br4 <- ((-1/120) * (b^3) * (si^4)) - ((29/560) * b * (k^2) * (si^4))
br5 <- ((-13/1120) * b * (si^4)) + ((239/8400) * (b^2) * (si^4))
br <- exp( - b) * (br1 + br2 + br3 + br4 + br5)
return(br)
}
```

```
coeffb3 <- function(si, k, b){
bs1 <- ((-1/35) * b * (3^0.5) * (si^3)) + ((1/15) * (b^2) * (3^0.5) * (si^3))
bs2 <- ((-1/54) * (b^3) * (3^0.5) * (si^3)) - ((93/4480) * b * k * (3^0.5) * (si^4))
bs3 <- ((57/560) * (b^2) * k * (3^0.5) * (si^4)) + ((1/108) * (b^4) * k * (3^0.5) * (si^4))
bs4 <- ((-49/720) * (b^3) * k * (3^0.5) * (si^4))
bs <- exp( - b) * (bs1 + bs2 + bs3 + bs4)
return(bs)
}

coeffb4 <- function(si, k, b){
bt1 <- ((1/216) * (si^4) * (b^4)) - ((1/105) * b * (si^4))
bt2 <- ((17/350) * (b^2) * (si^4)) - ((1/30) * (b^3) * (si^4))
bt <- exp( - b) * (bt1 + bt2)
return(bt)
}

rootb <- function(si, k, b, op){
st1 <- polyroot(c((coeffb0(si, k, b) - op), coeffb1(si, k, b), coeffb2(si, k, b), coeffb3(si, k, b), coeffb4(si,
k, b)))
st2 <- st1
for(i in 1:4) {
if(abs(Im(st1[i])) > 1e-06)
st2[i] <- st1[i] - 1000000000
}
stt <- max(Re(st2))
return(stt)
}

intb0 <- function(si, k, b, op){
re <- rootb(si, k, b, op)
bb1 <- (coeffb0(si, k, b) - op) * pnorm(re)
return(bb1)
}

intb1 <- function(si, k, b, op){
re <- rootb(si, k, b, op)
bc1 <- coeffb1(si, k, b) * exp((re^2)/(-2)) * (-1/((2 * pi)^0.5))
return(bc1)
}

intb2 <- function(si, k, b, op){
re <- rootb(si, k, b, op)
bd1 <- re * exp((re^2)/(-2))
bd2 <- ((2 * pi)^0.5) * (1 - pnorm(re))
bd3 <- (bd1 + bd2) * (1/((2 * pi)^0.5))
bd <- coeffb2(si, k, b) * (1 - bd3)
return(bd)
}

intb3 <- function(si, k, b, op){
re <- rootb(si, k, b, op)
be1 <- (2 + (re^2)) * exp((re^2)/(-2))
be <- coeffb3(si, k, b) * be1 * (-1/((2 * pi)^0.5))
return(be)
}

intb4 <- function(si, k, b, op){
re <- rootb(si, k, b, op)
bf1 <- (re^3) * exp((re^2)/(-2))
```

```
bf2 <- 3 * re * exp((re^2)/(-2))
bf3 <- 3 * ((2 * pi)^0.5) * (1 - pnorm(re))
bf4 <- (bf1 + bf2 + bf3) * (1/((2 * pi)^0.5))
bf <- coeffb4(si, k, b) * (3 - bf4)
return(bf)
}


valueb <- function(si, a, b, op){
k <- a/si
bval <- 100 * (intb0(si, k, b, op) + intb1(si, k, b, op) + intb2(si, k, b, op) + intb3(si, k, b, op) + intb4(si,
k, b, op))
return(bval)
}
```

Stationary Ornstein Uhlenbeck Process

The following subroutines calculate the value of the bond as well as the value of a European option priced on the bond. The subroutines coeffous0, coeffous1, coeffous2, coeffous3 and coeffous4 calculate the values of the co-efficients of the polynomial in "z". The subroutine rootous evaluates the roots of the polynomial and returns that largest real root as its output. The subroutines intous0, intous1, intous2, intous3 and intous4 calculate the value of the integrals. In fact, here the value of the integrals can be calculated exactly. Finally, the subroutine valueous combines the outputs obtained from the subroutines coeffous0, coeffous1, coeffous2, coeffous3, coeffous4 and intous0, intous1, intous2, intous3, intous4 to return the value of the bond or the value of the option priced on the bond. The user of this program only sees the subroutine valueous and all the user has to do is input the values of variance (si), the drift of the Brownian motion (a), the discount factor of the bond (b) and the strike price of the option of the bond (op). In case of calculating the value of the bond alone, the value of op is 0.

```
coeffous0 <- function(si, a, b){
cp1 <- (1/8) * b * ((2 * (a^3)) + (7 * a) - (6 * (a^2)) + (a * exp(-2 * a)) - (8 * a * exp( - a)))
cp2 <- (a^3) * (a + exp( - a) - 1)
cp3 <- 1 - ((cp1 * (si^2))/cp2)
cp4 <- (-108 * exp(-2 * a) * (a^4)) + (432 * (a^5)) - (432 * b * (a^5)) +(1152 * b * (a^4))
cp5 <- (-1440 * exp( - a) * (a^3)) + (882 * b * (a^2)) - (288 * b * (a^2) * exp(-3 * a))
cp6 <- (-576 * b * (a^4) * exp( - a)) - (1512 * b * (a^3)) - (96 * (a^3) * exp(-3 * a))
cp7 <- (72 * b * (a^4) * exp(-2 * a)) - (72 * (a^6)) - (216 * b * (a^3) *exp(-2 * a))
cp8 <- (1404*b*(a^2)*exp(-2*a))+(1728*b*(a^3)*exp(- a))-(2016*b*(a^2)*exp(- a))
cp9 <- (9 * exp(-4 * a) * (a^3)) - (1152 * (a^4)) + (18 * b * (a^2) * exp(-4 * a))
cp10 <- (216 * exp(-2 * a) * (a^3)) + (1311 * (a^3)) + (72 * b * (a^6))
cp11 <- (1/2304) * b * (cp4 + cp5 + cp6 + cp7 + cp8 + cp9 + cp10) * (1/(cp2^2))
cp <- exp( - b) * (cp3 + cp11)
return(cp)
}

coeffous1 <- function(si, a, b){
cq1 <- si * (1/((a^0.5)^3)) * ((a + exp( - a) - 1)^0.5)
cq2 <- (378*b*(a^2)*exp(- a))-(216*b*(a^3)*exp(- a))-(216*b*(a^2)*exp(-2 * a))
cq3 <- (3 * (a^2) * exp(-4 * a)) + (18 * b * (a^2) * exp(-3 * a)) + (189 * b * (a^3))
cq4 <- (36 * b * (a^4) * exp( - a)) + (27 * b * (a^3) * exp(-2 * a)) + (63 * b * a)
cq5 <- (216 * b * a * exp(-2 * a)) - (18 * b * a * exp(-3 * a)) + (171 * (a^3) * exp( - a))
cq6 <- (9 * b * a * exp(-4 * a)) + (102 * (a^2)) - (219 * (a^2) * exp( - a))
cq7 <- (135 * (a^2) * exp(-2 * a)) - (174 * (a^3)) + (3 * (a^3) * exp(3 * a))
cq8 <- (-21 * (a^2) * exp(-3 * a)) - (72 * b * a * exp(-3 * a)) - (18 * (a^5))
cq9 <- (90 * (a^4)) - (198 * b * a * exp( - a)) - (180 * b * (a^2))
cq10 <- (-90 * b * (a^4)) + (18 * b * (a^5))
cq11 <- ((a^0.5)^9) * (((a + exp( - a) - 1)^0.5)^5) * (1/72) * (si^3)
cq <- exp( - b) * b * ( - cq1 + cq11)
```

```
return(cq)
}

coeffous2 <- function(si, a, b){
cr1 <- (4 * (a^2)) - (7 * a) - (a * exp(-2 * a)) + (8 * a * exp( - a))
cr2 <- (2 * (a^2) * exp( - a)) - (4 * b * (a^2)) - (8 * b * a * exp( - a))
cr3 <- (8 * b * a) - (4 * b * exp(-2 * a)) + (8 * b * exp( - a)) - (4 * b)
cr4 <- (a^3) * (a + exp( - a) - 1)
cr5 <- (-1/8) * (cr1 + cr2 + cr3) * (1/cr4) * (si^2)
cr6 <- (1368 * (a^4)) + (4032 * b * (a^2) * exp( - a)) - (18 * (a^3) * exp(-4 * a))
cr7 <- (-2376*b*(a^3)*exp(- a))-(2808*b*(a^2)*exp(-2 * a))+(576*b*(a^2)*exp(-3 * a))
cr8 <- (2520 * b * (a^3)) + (120 * (a^3) * exp(-3 * a)) - (288 * (a^5))
cr9 <- (1800 * (a^3) * exp( - a)) + (144 * b * (a^4) * exp( - a)) - (72 *b * (a^4) * exp(-2 * a))
cr10 <- (-216 * b * (a^3) * exp(-2 * a)) + (216 * (a^3) * exp(-2 * a)) - (2118 * (a^3))
cr11 <- (2880 * (b^2) * (a^2)) - (1008 * (b^2) * a) + (1008 * (a^4) * exp( - a))
cr12 <- (-144 * (a^5) * exp( - a)) + (72 * b * (a^3) * exp(-3 * a)) - (2880 * b * (a^4))
cr13 <- (672*b*(a^2)*exp(-3*a)) (3168*(b^2)*a*exp(- a))-(576*(b^2)*(a^4)*exp(- a))
cr14 <- (-432 * (b^2) * (a^3) * exp(-2*a)) - (6048 * (b^2) * (a^2) * exp(-a)) - (4320 * b * (a^2) * exp(-
2 * a))
cr15 <- (1440 * (b^2) * (a^4)) - (3264 * b * (a^2)) + (1152 * (b^2) * a *exp(-3 * a))
cr16 <- (-3456*(b^2)*a*exp(-2*a)) + (3456 * (b^2) * (a^2) * exp(2 * a)) + (5568 * b * (a^3))
cr17 <- (576 * b * (a^5)) - (96 * b * (a^3) * exp(-3 * a)) - (288 * (b^2) * (a^5))
cr18 <- (7008 * b * (a^2) * exp( - a)) - (288 * (b^2) * (a^2) * exp( - a)) + (288 * (b^2) * a * exp(-3 *
a))
cr19 <- (-5472 * b * (a^3) * exp( - a)) - (144 * (b^2) * a * exp(-4 * a)) + (144 * b * (a^5) * exp( - a))
cr20 <- (3456 * (b^2) * (a^3) * exp( - a)) - (3024 * (b^2) * (a^3)) - (132 * b * (a^2) * exp(-4 * a))
cr21 <- (-1764 * b * (a^2)) - (1368 * b * (a^4)) + (144 * (a^4) * exp(2 * a)) + (288 * b * (a^5))
cr22 <- (cr6 + cr7 + cr8 + cr9 + cr10 + cr11 + cr12 + cr13 + cr14 + cr15 + cr16 + cr17 + cr18 + cr19 +
cr20 + cr21)
cr23 <- (1/2304) * (cr22/(cr4^2)) * (si^4)
cr <- exp( - b) * b * (cr5 + cr23)
return(cr)
}

coeffous3 <- function(si, a, b){
cs1 <- (-324 * b * (a^2) * exp( - a)) + (108 * b * (a^3) * exp( - a)) + (162 * b * (a^2) * exp(-2 * a))
cs2 <- ( - exp(-4 * a) * (a^2)) - (18 * b * (a^2) * exp(-3 * a)) - (135 *b * (a^3))
cs3 <- (18 * b * (a^4) * exp( - a)) + (27 * b * (a^3) * exp(-2 * a)) - (12 * (b^2))
cs4 <- (-63 * b * a) - (216 * b * a * exp(-2 * a)) - (72 * (b^2) * (a^2))
cs5 <- (48 * (b^2) * a) - (144 * (b^2) * a * exp( - a)) + (144 * (b^2) * (a^2) * exp( - a))
cs6 <- (-12 * (b^2) * (a^4)) - (48 * (b^2) * a * exp(-3 * a)) + (144 * (b^2) * a * exp(-2 * a))
cs7 <- (-72 * (b^2) * (a^2) * exp(-2 * a)) - (48 * (b^2) * a * exp(-3 * a)) + (18 * b * a * exp(-2 * a))
cs8 <- (48 * (b^2) * (a^3)) - (21 * (a^3) * exp( - a)) - (9 * b * a * exp(-4 * a))
cs9 <- (-28 * (a^2)) + (55 * (a^2) * exp( - a))
cs10 <- (-27 * (a^2) * exp(-2 * a)) + (40 * (a^3)) - ((a^3) * exp(-3 * a))
cs11 <- ((a^2) * exp(-3 * a)) + (72 * b * a * exp(-3 * a)) - (12 * (a^4))
cs12 <- (198 * b * a * exp( - a)) + (48 * (b^2) * exp(-3 * a)) - (72 * (b^2) * exp(-2 * a))
cs13 <- (48 * (b^2) * exp( - a)) + (18 * (b^2) * a * exp(-3 * a)) - (18 *(a^3) * exp(-2 * a))
cs14 <- (-18 * (a^4) * exp( - a)) - (12 * (b^2) * exp(-4 * a)) + (162 * b * (a^2)) + (36 * b * (a^4))
cs15 <- ((a^0.5)^9) * (((a + exp( - a) - 1)^0.5)^5)
cs16 <- (cs1 + cs2 + cs3 + cs4 + cs5 + cs6 + cs7 + cs8 + cs9 + cs10 + cs11 + cs12 + cs13 + cs14)
cs <- exp( - b) * b * (1/72) * (cs16/cs15) * (si^3)
return(cs)
}

coeffous4 <- function(si, a, b){
ct1 <- (-96 * (a^4)) - (2016 * b * (a^2) * exp( - a)) + (3 * (a^3) * exp(-4 * a))
ct2 <- (648 * b * (a^3) * exp( - a)) + (1404 * b * (a^2) * exp(-2 * a)) - (288 * b * (a^2) * exp(-3 * a))
ct3 <- (-1008 * b * (a^3)) - (8 * (a^3) * exp(-3 * a)) + (96 * (b^3))
ct4 <- (-120 * (a^3) * exp( - a)) + (288 * b * (a^4) * exp( - a)) + (72 * b * (a^4) * exp(-2 * a))
```

```
ct5 <- (432 * b * (a^3) * exp(-2 * a)) - (144 * (a^3) * exp(-2 * a)) + (269 * (a^3))
ct6 <- (-2592 * (b^2) * (a^2)) + (1008 * (b^2) * a) - (288 * (a^4) * exp( - a))
ct7 <- (-72 * b * (a^3) * exp(-3 * a)) + (384 * b * (a^4)) - (32 * b * (a^2) * exp(-3 * a))
ct8 <- (-3168 * (b^2) * a * exp( - a)) - (288 * (b^2) * (a^4) * exp( - a)) - (432 * (b^2) * (a^3) * exp(-2 *
a))
ct9 <- (5184 * (b^2) * (a^2) * exp( - a)) + (864 * b * (a^2) * exp(-2 * a)) - (576 * (b^2) * (a^4))
ct10 <- (896 * b * (a^2)) - (1152 * (b^2) * a * exp(-3 * a)) + (3456 * (b^2) * a * exp(-2 * a))
ct11 <- (-2592 * (b^2) * (a^2) * exp(-2 * a)) - (1280 * b * (a^3)) + (32 * b * (a^3) * exp(-3 * a))
ct12 <- (-1760 * b * (a^2) * exp( - a)) + (288 * (b^2) * (a^2) * exp(-3 *a)) - (288 * (b^2) * a * exp(-3 *
a))
ct13 <- (672 * b * (a^3) * exp( - a)) + (144 * (b^2) * a * exp(-4 * a)) - (1728 * (b^2) * (a^3) * exp( -
a))
ct14 <- (2160 * (b^2) * (a^3)) + (576 * b * (a^4) * exp( - a)) + (576 * b * (a^3) * exp(-2 * a))
ct15 <- (576 * (b^3) * (a^2)) - (288 * (b^2) * (a^2) * exp(-3 * a)) + (1152 * (b^3) * a * exp( - a))
ct16 <- (-1152 * (b^3) * (a^2) * exp( - a)) + (384 * (b^3) * (a^3) * exp( - a)) - (1152 * (b^3) * a * exp(-
2 * a))
ct17 <- (384 * (b^3) * a * exp(-3 * a)) + (576 * (b^3) * (a^2) * exp(-2 * a)) - (384 * (b^3) * a)
ct18 <- (96 * (b^3) * exp(-4 * a)) + (576 * (b^3) * exp(-2 * a)) - (384 * (b^3) * exp(-3 * a))
ct19 <- (96 * (b^3) * (a^4)) - (384 * (b^3) * (a^3)) - (384 * (b^3) * exp( - a)) + (50 * b * (a^2) * exp(-4
* a))
ct20 <- (882 * b * (a^2)) + (288 * b * (a^4)) - (36 * (a^4) * exp(-2 * a))
ct21 <- (a^6) * ((a + exp( - a) - 1)^2)
ct22 <- (ct1 + ct2 + ct3 + ct4 + ct5 + ct6 + ct7 + ct8 + ct9 + ct10 + ct11 + ct12 + ct13 + ct14 + ct15 +
ct16 + ct17 + ct18 + ct19 + ct20)
ct <- exp( - b) * b * (1/2304) * (ct22/ct21) * (si^4)
return(ct)
}


rootous <- function(si, a, b, op){
pt1 <- polyroot(c((coeffous0(si, a, b) - op), coeffous1(si, a, b), coeffous2(si, a, b), coeffous3(si, a, b),
coeffous4(si, a, b)))
pt2 <- pt1
for(i in 1:4) {
if(abs(Im(pt1[i])) > 1e-06)
pt2[i] <- pt1[i] - 1000000000
}
ptt <- max(Re(pt2))
return(ptt)
}


intous0 <- function(si, a, b, op){
re <- rootous(si, a, b, op)
cb1 <- (coeffous0(si, a, b) - op) * pnorm(re)
return(cb1)
}

intous1 <- function(si, a, b, op){
re <- rootous(si, a, b, op)
cc1 <- coeffous1(si, a, b) * exp((re^2)/(-2)) * (-1/((2 * pi)^0.5))
return(cc1)
}

intous2 <- function(si, a, b, op){
re <- rootous(si, a, b, op)
cd1 <- re * exp((re^2)/(-2))
cd2 <- ((2 * pi)^0.5) * (1 - pnorm(re))
cd3 <- (cd1 + cd2) * (1/((2 * pi)^0.5))
cd <- coeffous2(si, a, b) * (1 - cd3)
```

```
return(cd)
}

intous3 <- function(si, a, b, op){
re <- rootous(si, a, b, op)
ce1 <- (2 + (re^2)) * exp((re^2)/(-2))
ce <- coeffous3(si, a, b) * ce1 * (-1/((2 * pi)^0.5))
return(ce)
}

intous4 <- function(si, a, b, op){
re <- rootous(si, a, b, op)
cf1 <- (re^3) * exp((re^2)/(-2))
cf2 <- 3 * re * exp((re^2)/(-2))
cf3 <- 3 * ((2 * pi)^0.5) * (1 - pnorm(re))
cf4 <- (cf1 + cf2 + cf3) * (1/((2 * pi)^0.5))
cf <- coeffous4(si, a, b) * (3 - cf4)
return(cf)
}

valueous <- function(si, a, b, op){
cval <- 100 * (intous0(si, a, b, op) + intous1(si, a, b, op) + intous2(si, a, b, op) + intous3(si, a, b, op) +
intous4(si, a, b, op))
return(cval)
}
```

Subroutines to simulate the prices.

The three programs below are the three subroutines that can be used to generate a simulated set of data to obtain the simulated prices. The variables b, si and a0 represent the same thing as earlier.

Subroutine for generation of the data set in the Brownian Motion case.

```
for(i in 1:10000){
w1<-rnorm(1000,0,si/(1000^0.5))
w2<-cumsum(w1)
w3<-exp(w2)
w4<-mean(w3)
rx1[i]<-100*exp(-b*w4)
+}
```

Subroutine for generation of the data set in the Non-stationary Ornstein Uhlenbeck case.

```
j<-1:1000
j<-j/1000
for(i in 1:10000){
w1<-rnorm(1000,0,si/(1000^0.5))
w2<-w1*exp(a0*j)
w3<-cumsum(w2)
w4<-w3*exp(-a0*j)
w5<-exp(w4)
w6<-mean(w5)
rx1[i]<-100*exp(-b*w6)
+}
```

Subroutine for generation of the data set in the Stationary Ornstein Uhlenbeck case.

```
j<-1:1000
j<-j/1000
```

```
l<-1:1000
l<-l/l
for(i in 1:10000){
w1<-rnorm(1000,0,si/(1000^0.5))
w2<-w1*exp(a0*j)
w3<-cumsum(w2)
w4<-rnorm(1,0,si/((2*a)^0.5))
w5<-((w4*l)+w3)*exp(-a0*j)
w6<-exp(w5)
w7<-mean(w6)
rx1[i]<-100*exp(-b*w7)
+}
```

Valuation of Bonds based on two drivers.

These programs are the ones used to calculate the price of a bond where the interest rate is driven by two drivers where in general the two drivers could be correlated amongst themselves. The driving process is assumed to be a non-stationary Ornstein Uhlenbeck process. We present the programs for both the situations as outlined in chapter 4.

Interest Rate following a log-normal process.

The following set of programs calculates the value of the bond when the interest rate follows a log-normal process. This is the situation as described in Chapter 4, section 2. Here, we have r_t - the interest rate process defined as

$$r_t = \exp(Y_s^{(1)} + \gamma Y_s^{(2)})$$

where $Y_s^{(1)}$ and $Y_s^{(2)}$ are the two driving stochastic processes with a correlation between them to be ρ. The first subroutine mvz1 calculates the variance of the conditioning factor, mkug1 calculates the value of k_u, mdb1 calculates the value of the conditional expectation and finally mdbb1 calculates the value of the bond for different values of Z in the range $(-\infty, \infty)$. Here $t = 1$ and $b = 0.07$.

```
mvz1 <- function(si) {
q1 <- (2 * a1 * t) + (4 * exp( - a1 * t)) - exp(-2 * a1 * t) - 3
q2 <- (2 * a2 * t) + (4 * exp( - a2 * t)) - exp(-2 * a2 * t) - 3
qq1 <- (q1 * (si^2))/(2 * (a1^3))
qq2 <- (q2 * (si^2))/(2 * (a2^3))
q31 <- (a2-1+((a2+1)*(exp(-a1)-1)*(1/a1))-(2*(exp( -(a1+a2))-1)*(1/(a1 + a2))))
q32 <- ((1 - exp( - a2))/a2) + ((exp( - (a1 + a2)) - exp( - a2))/a1)
q3 <- q31 + q32 qq3 <- (((si^2) * rho * q3)/(a2 * (a1 + a2)))
qq <- qq1 + ((gamma^2) * qq2) + (2 * gamma * qq3)
return(qq)
}

mkug1 <- function(u) {
b1 <- 2 - (2 * exp( - a1 * u)) - exp( - a1 * (t - u)) + exp( - a1 * (t + u))
b2 <- 2 - (2 * exp( - a2 * u)) - exp( - a2 * (t - u)) + exp( - a2 * (t + u))
b3 <- a2-(exp(-a1*u)*(a2+1))+(2*exp((-a1*u)+(-a2*u)))+exp((u-1)*a2)exp((-a1*u)-a2)-1
b <- (b1/(2*(a1^2)))+(((gamma^2)*b2)/(2*(a2^2)))+((2*gamma*rho*b3)/(a2*(a1+a2)))
bb <- (si^2) * b * (1/(mvz1(si)^0.5))
return(bb)
}

mdb1 <- function(z, u) {
d1 <- mkug1(u) * z
d2 <- ((1 - exp(-2 * a1 * u))/(2 * a1))
d3 <- (gamma^2) * ((1 - exp(-2 * a2 * u))/(2 * a2))
```

```
d4 <- 2 * gamma * rho * (1 - exp(( - a1 * u) + ( - a2 * u))) * (1/(a1 + a2))
d5 <- (si^2) * (d2 + d3 + d4)
d6 <- mkugl(u)^2
d7 <- d1 + ((d5 - d6)/2)
return(exp(d7))
}


mdbb1 <- function(z) {
x <- 1:100
for(i in 1:100) {
x[i] <- mdb1(z, i/100)
        }
  return(exp( - b * mean(x)))
  }
```

Once we have the values of the function for different values of Z in the interval (-∞, ∞), all that we have to do then is to take the expectation over Z to find the value of the price of the bond. Remember that Z follows a standard normal distribution. In order to find the value of the contingent payment on the price of the bond, we need to restrict the integral while taking the expectation to the required region as described in chapter 4 section 2. For that we use the following subroutine.

```
rpb <- function(c) {
r1 <- 1:2000
r2 <- 1:2000
for(i in 1:2000) {
r1[i] <- max((rzc1[i] - c), 0)
r2[i] <- max((rzc2[i] - c), 0)
        }
rr <- r1 + r2
for(i in 1:200) {
ll[i] <- (rr[i] * exp(((y1[i]/tr)^2)/(-2)))/(tr * y[i])
        }
return(mean(ll)/((2 * pi)^0.5))
}
```

where y is a vector consisting of elements 2000 elements, the values varying from 0.005 to 1 and y1 is the logarithm of this vector. To calculate the lower bound to the price of the bond, we take *c =0*. To calculate the value of the contingent payment, we take *c* to be the value of the strike price at which the contingent payment is calculated.

Interest Rate following a sum of two log-normal processes.

The following set of programs calculates the value of the bond when the interest rate follows a log-normal process. This is the situation as described in Chapter 4, section 3. Here, we have r_t - the interest rate process defined as

$$r_t = \exp(Y_s^{(1)}) + \gamma \exp(\beta Y_s^{(2)})$$

where $Y_s^{(1)}$ and $Y_s^{(2)}$ are the two driving stochastic processes with a correlation between them to be ρ. The first subroutine tdr1 calculates the variance of the conditioning factor, tdr11c and tdr12c calculates the value of $k_u^{(1)}$ and $k_u^{(2)}$ respectively, tdr21c and tdr22c calculates the value of the conditional expectation. Finally tdr3 calculates the value of the bond for different values of Z in the range (-∞, ∞). Here *t =1* and *b = 0.07.*

```
tdr1 <- function(si) {
q1 <- (2 * a1 * t) + (4 * exp( - a1 * t)) - exp(-2 * a1 * t) - 3
q2 <- (2 * a2 * t) + (4 * exp( - a2 * t)) - exp(-2 * a2 * t) - 3
qq1 <- (q1 * (si^2))/(2 * (a1^3))
qq2 <- (q2 * (si^2))/(2 * (a2^3))
```

```
q31 <- (a2-1+((a2+1)*(exp(-a1)-1)*(1/a1))-(2*(exp(-(a1+a2))-1)*(1/(a1+a2))))
q32 <- ((1 - exp( - a2))/a2) + ((exp( - (a1 + a2)) - exp( - a2))/a1)
q3 <- q31 + q32 qq3 <- (((si^2) * rho * q3)/(a2 * (a1 + a2)))
qq <- qq1 + ((beta^2) * (gamma^2) * qq2) + (2 * beta * gamma * qq3)
return(qq)
}

tdr11c <- function(u) {
b1 <- 2 - (2 * exp( - a1 * u)) - exp( - a1 * (t - u)) + exp( - a1 * (t + u))
b2 <- (((si^2) * b1)/(2 * (a1^2)))
b3 <- (a2-((a2+1)*exp(-a1*u))+(2*exp(-(a1+a2)*u))+exp((u - 1)*a2)-exp(-((a1*u)+a2))-1)
b4 <- ((beta * gamma * b3 * (si^2) * rho)/(a2 * (a1 + a2)))
bb <- (b2 + b4) * (1/(tdr1(si)^0.5))
return(bb)
}


tdr12c <- function(u) {
c1 <- 2 - (2 * exp( - a2 * u)) - exp( - a2 * (t - u)) + exp( - a2 * (t + u))
c2 <- (((beta^2) * gamma * (si^2) * c1)/(2 * (a2^2)))
c3 <- (a2-((a2+1)*exp(-a1*u))+(2 *exp(-(a1+a2)*u))+exp((u -1)*a2)-exp(-((a1*u)+a2))-1)
c4 <- ((beta * c3 * (si^2) * rho)/(a2 + (a1 + a2)))
cc <- (c2 + c4) * (1/(tdr1(si)^0.5)) return(cc)
}


tdr21c <- function(z, u) {
d1 <- tdr11c(u) * z
d2 <- ((1 - exp(-2 * a1 * u))/(2 * a1))
d3 <- (si^2) * d2
d4 <- tdr11c(u)^2
d5 <- d1 + ((d3 - d4)/2)
return(exp(d5))
}


tdr22c <- function(z, u) {
e1 <- tdr12c(u) * z
e2 <- ((1 - exp(-2 * a2 * u))/(2 * a2))
e3 <- (si^2) * e2
e4 <- tdr12c(u)^2
e5 <- e1 + ((e3 - e4)/2)
return(exp(e5))
}


tdr3 <- function(z) {
x <- 1:100
for(i in 1:100) {
x[i] <- tdr21c(z, i/100) + (gamma * tdr22c(z, i/100))
                }
return(exp( - b * mean(x)))
}
```

Once we have the values of the function for different values of Z in the interval $(-\infty, \infty)$, all that we have to do then is to take the expectation over Z to find the value of the price of the bond. Remember that Z follows a standard normal distribution. In order to find the value of the contingent payment on the price of the bond, we need to restrict the integral while taking the expectation to the required region as described in chapter 4 section 3. For that we use the following subroutine rpb as defined above.

Subroutines used for simulation

This subroutine is used for generating the prices of the bond when r_t - the interest rate process defined as

$$r_t = \exp(Y_s^{(1)} + \gamma Y_s^{(2)})$$

where $Y_s^{(1)}$ and $Y_s^{(2)}$ are the two driving stochastic processes with a correlation between them to be ρ.

```
gentdrll <- function(op) {
w1 <- rnorm(1000, 0, (si/(1000^0.5)))
w2 <- rnorm(1000, 0, (si/(1000^0.5)))
w3 <- w1 * exp(a1 * j)
w4 <- cumsum(w3)
w5 <- w4 * exp( - a1 * j)
w6 <- ((rho * w1) + (((1 - (rho^2))^0.5) * w2)) * exp(a2 * j)
w7 <- cumsum(w6)
w8 <- w7 * exp( - a2 * j)
w9 <- mean(exp(w5 + (gamma * w8)))
return(exp( - b * w9))
```
where si is the instantaneous variance.

This subroutine is used for generating the prices of the bond when r_t - the interest rate process defined as

$$r_t = \exp(Y_s^{(1)}) + \gamma \exp(\beta Y_s^{(2)}))$$

where $Y_s^{(1)}$ and $Y_s^{(2)}$ are the two driving stochastic processes with a correlation between them to be ρ and si is the instantaneous variance.

```
gent1 <- function(op) {
w1 <- rnorm(1000, 0, (si/(1000^0.5)))
w2 <- rnorm(1000, 0, (si/(1000^0.5)))
w3 <- w1 * exp(a1 * j)
w4 <- cumsum(w3)
w5 <- w4 * exp( - a1 * j)
w6 <- ((rho * w1) + (((1 - (rho^2))^0.5) * w2)) * exp(a2 * j)
w7 <- cumsum(w6)
w8 <- w7 * exp( - a2 * j)
w9 <- exp(w5) + (gam * exp(beta * w8))
return(exp( - b * mean(w9)))
}
```

Pricing of European options using conditioning factor.

These programs are the ones used to calculate the price of a European option and its associated correction factor. In the first case, it is assumed that the volatility process follows a simple one-dimensional Brownian Motion, while in the second case, the volatility is assumed to follow an Ornstein Uhlenbeck process.

Volatility following a Brownian Motion

The first set of programs are to calculate the expected value of P conditionally on Z, where both P and Z have been defined earlier (see 4.2.1). Here t2 calculates the value of the conditioning factor for different values of u. The subroutine thw calculates the value of the conditional expectation for given values of z and u and the subroutine tthw performs the function of the integration over u, the range of u being between [0,1].

```
t2 <- function(u){
a <- (3^0.5) * (u - ((u^2)/2))
return(a)
}

thw <- function(z, u){
p1 <- k * t2(u) * z
p2 <- ((k^2) * (u - (t2(u)^2)))/2
p <- p1 + p2
return(sum(exp(p)))
}

tthw <- function(z){
x <-1:1000
for( i in 1:1000 ){
x[i]<-thw(z, i/1000)
}
return(mean(x))
}
```

Finally, we perform the integration over z, by allowing z to take vales in the range of (-∞,∞); for all practical purposes, we let z take values in the range [-3, 3] in very small steps - the entire range is sub-divided into 1000. This gives a vector of values corresponding to the values of z in that range, say hs1.

We repeat a similar exercise to calculate the expected value of Q conditionally on Z, where both Q and Z have been defined earlier (see 5.2.1). Here t2 calculates the value of the conditioning factor for different values of u. The subroutine thw1 calculates the value of the conditional expectation for given values of z and u and the subroutine tthw1 performs the function of the integration over u, the range of u being between [0,1].

```
t2 <- function(u){
a <- (3^0.5) * (u - ((u^2)/2))
return(a)
}

thw1 <- function(z, u){
q1 <- (k/2) * t2(u) * z
q2 <- (((k/2)^2) * (u - (t2(u)^2)))/2
q<-q1+q2
return(sum(exp(q)))
}

tthw1 <- function(z){
x<-1:1000
for( i in 1:1000 ){
x[i]<-thw1(z, i/1000)
}
return(mean(x))
}
```

Finally, we perform the integration over z, by allowing z to take vales in the range of (-∞,∞); for all practical purposes, we let z take values in the range [-3, 3] in very small steps - the entire range is sub-divided into 1000. This gives a vector of values corresponding to the values of z in that range, say hs2.

Once we have the vectors hs1 and hs2, we can then calculate the vectors *a* and *sigma* so that we can calculate the price of the option as well as the associated correction factor. To calculate *a* , we first calculate r1, where r1 is given by

$$r1 = \left[\frac{e^{\frac{kz\sqrt{3}}{2} + \frac{k^2}{32}} - 1}{\frac{k}{2}} \right].$$

We calculate the value of r1 using the same value of z as used in the calculation of the two integrals. Once we have r1, the we easily calculate the value of *a* and *sigma* as defined below and use these values to calculate the value of the option along with its correction factor.

$$a = r - \tfrac{1}{2}\sigma^2 hs1 + \rho\sigma\left[r1 - \left(\tfrac{k}{4} hs\,2\right)\right]$$

$$sigma \;= \tfrac{1}{2}\sigma^2(1-\rho^2)hs1$$

Having obtained these values, the corresponding values of the price of the option and the associated correction factor is given by

```
ppb <- function(b){
q1 <- pnorm((a + sigma - (l * log(b/100)))/(sigma^0.5))
q1 <- q1 * exp(a+(sigma/2))
q2 <- pnorm((a - (l * log(b/100)))/(sigma^0.5))
q2 <- q2 * (b/100)
q <- q1 - q2
q3 <- exp(a + (sigma/2))*exp(((a + sigma - (l * log(b/100)))^2)/(-2 * sigma))
q3 <- q3/((2 * pi *sigma)^0.5)
qq <- (q1 + q3) * oo
for(I in 1:1000){
chk <- (-3 + ((6 * i)/1000))
ll[i] <- (q[i] * exp((chk^2)/(-2)))/((2 * pi)^0.5)
ll1[i] <- (qq[i] * exp((3^0.5) * chk * k * 0.5) * exp((chk^2)/(2)))/((2 * pi)^0.5)
}
y1 <-sum(ll) * 0.6
y2 <-sum(ll1) * 0.6
return(y1, y2)
}
```

The value returned as y1 is the value of the option without the correction factor, while the value returned as y2 is the value of the correction factor. Thus the corrected calculated value of the option is the sum of the two values, namely y1 + y2. Here l is a vector of 1 of suitable length. Also, oo is a constant and is given by

$$oo = \frac{1}{2}\left(\frac{\rho^2\sigma^2}{\frac{k^2}{4}}\right)e^{\frac{zk\sqrt{3}}{2}}\left\{e^{\frac{k^2}{8}} - e^{\frac{k^2}{16}}\right\}.$$

Throughout the examples, a0, the initial value has been taken as 0.

Volatility following an Ornstein Uhlenbeck process

The first set of programs are to calculate the expected value of P conditionally on Z, where both P and Z have been defined earlier (see 4.2.2). Here o1 calculates the value of the conditioning factor for different values of u. The subroutine ow calculates the value of the conditional expectation for given values of z and u and the subroutine oow performs the function of the integration over u, the range of u being between [0,1]. Here, a1 is the mean reversion force of the Ornstein Uhlenbeck process and v0 is the initial value.

```
o1 <- function(u, a1){
p1 <- (2 * a1) - ((1 - exp(-a1)) * (3 - exp(-a1)))
p2 <- ((2 * a1)/p1)^0.5
p3 <- exp(-a1 * u) * (Cosh(a1 * u) -1)
p4 <- Sinh(a1 * u) * (exp(-a1 * u) - exp(-a1))
```

```
p5 <- (p2 * (p3 + p4))/a1
return(p5)
}

ow <- function(z, u){
p1 <- ((1 - exp(-2 * a1 *u))/(2 * a1)) - (o1(u, a1)^2)
p2 <- k * o1(u, a1) * z
p3 <- p2 + ((k^2) * p1 * 0.5) + (k * v0 * exp(-a1 *u))
return(sum(exp(p3)))
}

oow <- function(z){
x <- 1:1000
for( i in 1:1000){
x[i] <- ow(z, i/1000)
}
return(mean(x))
}
```

Finally, we perform the integration over z, by allowing z to take vales in the range of $(-\infty,\infty)$; for all practical purposes, we let z take values in the range [-3, 3] in very small steps - the entire range is sub-divided into 1000. This gives a vector of values corresponding to the values of z in that range, say hs3.

We repeat a similar exercise to calculate the expected value of Q conditionally on Z, where both Q and Z have been defined earlier (see 4.2.1). However, in this case, we need to calculate two integrals to obtain the expected value of Q conditionally on Z, as shown in 4.2.2. In both these cases, o1 calculates the value of the conditioning factor for different values of u. The differences are in the subroutines ow1 and owc. The subroutine ow1 calculates the value of the conditional expectation for given values of z and u for the exponent with a factor of (k/2). The sub-routine owc calculates the conditional expectation of the integration within the range [0,1] of V_t $\exp((k\ V_t)/2)$ with respect to dt. The subroutine oow1 performs the function of the integration over u, the range of u being between [0,1] corresponding to the conditional expectation obtained through ow1, while owc1 does the same for the function corresponding to owc1.

```
o1 <- function(u, a1){
p1 <- (2 * a1) - ((1 - exp(-a1)) * (3 - exp(-a1)))
p2 <- ((2 * a1)/p1)^0.5
p3 <- exp(-a1 * u) * (Cosh(a1 * u) -1)
p4 <- Sinh(a1 * u) * (exp(-a1 * u) - exp(-a1))
p5 <- (p2 * (p3 + p4))/a1
return(p5)
}

ow1 <- function(z, u){
p1 <- ((1 - exp(-2 * a1 * u))/(2 * a1)) - (o1(u, a1)^2)
p2 <- (k/2) * o1(u, a1) * z
p3 <- p2 + (((k/2)^2) * p1 * 0.5) + ((k/2)* v0 *exp(-a1 * u))
return(sum(exp(p3)))
}

oow1 <- function(z){
x <- 1:1000
for(i in 1:1000){
x[i] <- ow1(z, i/1000)
}
return(mean(x))
}
```

Finally, we perform the integration over z, by allowing z to take vales in the range of $(-\infty,\infty)$; for all practical purposes, we let z take values in the range [-3, 3] in very small steps - the entire range is sub-divided into 1000. This gives a vector of values corresponding to the values of z in that range, say hs4.

To calculate the final integral, that is to obtain owc, we proceed as follows.

```
o1 <- function(u, a1){
p1 <- (2 * a1) - ((1 - exp(-a1)) * (3 - exp(-a1)))
p2 <- ((2 * a1)/p1)^0.5
p3 <- exp(-a1 * u) * (Cosh(a1 * u) -1)
p4 <- Sinh(a1 * u) * (exp(-a1 * u) - exp(-a1))
p5 <- (p2 * (p3 + p4))/a1
return(p5)
}

owc <- function(z, u){
p1 <- o1(u, a1) * z
p2 <- ((1 - exp(-2 * a1 * u))/(2 * a1)) - (o1(u,a1)^2)
p3 <- p1 + (p2/2)
p4 <- (p1/2) + (p2/8)
p <- p3 * exp(p4)
return(sum(p))
}

oowc<- function(z){
x <- 1:1000
for(i in 1:1000){
x[i] <- owc(z, i/1000)
}
return(mean(x))
}
```

Finally, we perform the integration over z, by allowing z to take vales in the range of $(-\infty,\infty)$; for all practical purposes, we let z take values in the range [-3, 3] in very small steps - the entire range is sub-divided into 1000. This gives a vector of values corresponding to the values of z in that range, say hs4.

Once we have the vectors hs3, hs4 and hs5, we can then calculate the vectors *a* and *sigma* so that we can calculate the price of the option as well as the associated correction factor. To calculate *a* , we first calculate r1, where r1 is given by

$$r1 = \left[\frac{e^{\frac{kzL}{2} + \frac{k^2}{4}\left\{\frac{1-e^{-2a1}}{2a1} - L^2\right\} + \frac{kv0e^{-a1}}{2}} - 1}{\frac{k}{2}} \right].$$

We calculate the value of r1 using the same value of z as used in the calculation of the two integrals. Once we have r1, the we easily calculate the value of *a* and *sigma* as defined below and use these values to calculate the value of the option along with its correction factor.

$$a = r - \tfrac{1}{2}\sigma^2 hs3 + \rho\sigma\left[r1 - \left(\tfrac{k}{4} hs4\right) + \left(\frac{a1}{k} hs5\right)\right]$$

$$sigma = \tfrac{1}{2}\sigma^2(1-\rho^2)hs3$$

Having obtained these values, the corresponding values of the price of the option and the associated correction factor is given by

```
ppou<- function(b){
q1 <- pnorm((a + sigma - (l * log(b/100)))/(sigma^0.5))
q1 <- q1 * exp(a + (sigma/2))
q2 <- pnorm((a - (l * log(b/100)))/(sima^0.5))
q2 <- q2 *(b/100)
q <- q1 - q2
q3 <- exp(a + (sigma/2)) * exp(((a + sigma - (l* log(b/100)))^2)/(-2 * sigma))
q3 <- q3/((2 * pi * sigma)^0.5)
qq <- (q1 + q3) * oo
for(i in 1:1000){
chk <- (-3 + ((6*i)/1000))
ll[i] <- (q[i] * exp((chk^2)/(-2)))/((2 * pi)^0.5)
ll1[i] <- (qq[i] * exp(2 * (k/2) * L * chk)*exp((chk^2)/(-2)))/((2 * pi)^0.5)
}
y1 <- sum(ll) * 0.6
y2 <- sum(ll1) * 0.6
return(y1, y2)
}
```

The value returned as y1 is the value of the option without the correction factor, while the value returned as y2 is the value of the correction factor. Thus the corrected calculated value of the option is the sum of the two values, namely y1 + y2. Here l is a vector of 1 of suitable length. Also, oo is a constant and is given by

$$oo = \frac{1}{2}\left(\frac{\rho^2\sigma^2}{\frac{k^2}{4}}\right).\left\{e^{\frac{k^2}{2}\left(\frac{1-e^{-2al}}{2al}-L^2\right)+kv0e^{-al}} - e^{\frac{k^2}{4}\left(\frac{1-e^{-2al}}{2al}-L^2\right)+kv0e^{-al}}\right\},$$

where,

$$L = \frac{\left(1-e^{-al}\right)^2}{2al^2}.\frac{1}{\sqrt{\frac{2al-\left(1-e^{-al}\right)\left(3-e^{-al}\right)}{2al^3}}}$$

Throughout the examples, v0, the initial value has been taken as 0.

Subroutines to simulate the prices

These two programs are used to generate a simulated set of data to obtain the simulated prices. The variables are the same as before. Also we start with Y0 = 100.

Subroutines for generation of the data set in the Brownian Motion case

```
z <- 1:10000
for(i in 1:10000){
 w1 <- rnorm(1000, 0, 1/(1000^0.5))
w2 <- rnorm(1000, 0, 1/(1000^0.5))
v <- k * cumsum(w1)
v1 <- c(0, v)
w11 <- c(w1, 0)
w12 <- c(w2, 0)
q <- ((r - (0.5 * (si^2) *exp(v1)))/1000)
v2 <- q + (si * * exp(v1/2) *((rho * w11) + (((1 - (rho^2))^0.5) * w12)))
z[i] <- Y0 * exp(sum(v2))
```

```
}
```

Subroutines for generation of the data set in the Ornstein Uhlenbeck case

```
z <- 1:10000
j <- 1:1000
j <- j/1000
for(i in 1:10000){
 w1 <- rnorm(1000, 0, 1/(1000^0.5))
w2 <- rnorm(1000, 0, 1/(1000^0.5))
w01 <- w1 * exp(a1 * j)
v01 <- k * cumsum(w01)
v <- (v01 + v0) * exp(-a1 * j)
v1 <- c(0, v)
w11 <- c(w1, 0)
w12 <- c(w2, 0)
q <- ((r - (0.5 * (si^2) *exp(v1)))/1000)
v2 <- q + (si * * exp(v1/2) *((rho * w11) + (((1 - (rho^2))^0.5) * w12)))
z[i] <- Y0 * exp(sum(v2))
}
```

Valuing stop-loss reinsurance contracts

Here, we give the Splus codes to calculate the value of the stop-loss reinsurance contracts when we have a doubly stochastic Poisson (Cox) process. The code out here is under the assumption that the stochastic process in question is a non-stationary Ornstein Uhlenbeck process with a mean reversion force *a*. Also, throughout the calculations, $t = 1$. The approach is the same as in the earlier situations of pricing of bonds. We first look at a conditioning factor and find the conditional value of the stop-loss contract. The conditioning factor is so chosen that it has a standard normal distribution. Finally, we take expectation over the distribution of the conditioning factor to get the unconditional value. We also present a simulation routine - this is to compare the values obtained by using the conditioning factor approach.

Here, *si* is the value of the instantaneous variance and con is the scaling factor of the aggregated process as defined in chapter 5, section 3. The subroutines *vz*, *kug*, *db*, *pdbbcou* and *pdbb1ou* are used to calculate the conditional value of the stop-loss reinsurance contract and *rcal* is used to calculate the unconditional value once the conditional value has been obtained. We assume that the conditional values are stored in *rxss*.

```
vz <- function(si) {
q <- (2 * a * t) + (4 * exp( - a * t)) - exp(-2 * a * t) - 3
qq <- (si^2) * (q/(2 * (a^3)))
return(qq)
}

kug <- function(u) {
b1 <- 1 - exp( - a * u)
b2 <- 1 - exp( - a * (t - u))
b3 <- exp( - a * u) - exp( - a * (t + u))
b <- (b1 + b2 - b3)/a
bb <- (si^2) * b * (1/(2 * a)) * (1/(vz(si)^0.5))
return(bb)
}

db <- function(z, u) {
a1 <- kug(u) * z
a2 <- 1 - exp(-2 * a * u)
```

```
a2 <- (si^2) * a2 * (1/(2 * a))
a3 <- kug(u)^2
a4 <- a1 + ((a2 - a3)/2)
return(exp(a4))
}

pdbbcou <- function(z) {
x <- 1:1000
for(i in 1:1000) {
x[i] <- db(z, i/1000)
}
return(con * mean(x))
}

pdbb1ou <- function(z, k) {
pq <- (pdbbcou(z) * pgamma(pdbbcou(z), k)) - (k * pgamma(pdbbcou(z), (k + 1)))
return(pq)
}

rcal<- function(op) {
for(i in 1:200) {
qq1[i] <- (rxss[i] * exp(((y1[i])^2)/(-2)))/(y[i])
}
return((mean(qq1)/((2 * pi)^0.5)))
}
```

Simulation sub-routine

The subroutine, *genou*, used for simulation purposes and *rsim* is them used to calculate the value of the stop-loss reinsurance contract for different values of the strike price *b*.

```
genou <- function(con) {
w1 <- rnorm(1000, 0, 1/(1000^0.5))
w2 <- (exp(a * j)) * w1
w3 <- cumsum(w2)
w4 <- (exp( - a * j)) * w3
w5 <- exp(si * w4)
w6 <- con * mean(w5)
w7 <- rpois(1, w6)
return(w7)
}

rsim <- function(b) {
yck <- (abs(rxs - b) + (rxs - b))/2
yyck <- (var(yck)/50000)^0.5
return(mean(yck), yyck)
}
```

Subroutines to calculate the prices of an Asian option, using the alternative method.

The following set of subroutines calculates the value of an Asian option using the alternative method as described in chapter 6. This is an alternative method to calculate the prices as compared to the one proposed by Rogers and Shi (1995). The following subroutines calculate the value of the Asian option. The subroutines coeff0, coeff1, coeff2, coef3 and coeff4 calculate the values of the co-efficients of the polynomial in "z". The subroutine root evaluates the roots of the polynomial and returns that largest real root as its output. The subroutines int0, int1, int2, int3 and int4 calculate the value of the integrals. In fact, here the value of the integrals can be calculated exactly. Finally, the subroutine value combines the outputs obtained from the subroutines coeff0, coeff1, coeff2, coeff3, coeff4 and int0, int1, int2, int3, int4 to return the value of the bond or the value of the option priced on the bond. The user of this program only sees the subroutine value and all the user has to do is input the values of variance (si), the interest rate (r) and the strike price of the option (b). In the case of calculating the value of the underlying asset, the value of the strike price, b, is 0.

```
coeff0 <- function(si, k){
ap1 <- 1 - ((1/5) * (si^2)) + ((1/35) * (si^4)) - ((29/560) * (si^4) * (k^2))
ap2 <- ((1/24) * (si^3) * (k^3)) - ((11/80) * (si^3) * k) + ((1/6) * (si^2) * (k^2))
ap3 <- ((1/120) * (si^4) * (k^4)) + ((1/2) * si * k)
ap <- ap1 + ap2 + ap3
return(ap)
}

coeff1 <- function(si, k){
aq1 <- ((5/24)*(3^0.5)*(si^2)*k)+((1/3)*(3^0.5)*si)+((3/40)*(3^0.5)*(si^3)*(k^2))
aq2 <- ((-279/4480) * (3^0.5) * (si^4) * k) - ((3/35) * (3^0.5) * (si^3))
aq3 <- ((7/360) * (3^0.5) * (si^4) * (k^3))
aq <- aq1 + aq2 + aq3
return(aq)
}

coeff2 <- function(si, k){
ar1 <- ((-2/35) * (si^4)) + ((29/560) * (si^4) * (k^2))
ar2 <- ((11/80) * (si^3) * k) + ((1/5) * (si^2))
ar <- ar1 + ar2
return(ar)
}

coeff3 <- function(si, k){
as1 <- ((93/4480) * (3^0.5) * (si^4) * k) + ((1/35) * (3^0.5) * (si^3))
return(as1)
}

coeff4 <- function(si, k){
at1 <- (1/105) * (si^4)
return(at1)
}

root <- function(si, k, b){
rt1 <- polyroot(c((coeff0(si, k) - b), coeff1(si, k), coeff2(si, k), coeff3(si, k), coeff4(si, k)))
rt2 <- rt1
for(i in 1:4) {
if(abs(Im(rt1[i])) > 1e-06)
rt2[i] <- rt1[i] - 1000000000
}
rtt <- max(Re(rt2))
return(rtt)
}

int0 <- function(si, k, b){
re <- root(si, k, b)
ab1 <- (coeff0(si, k) - b) * (1 - pnorm(re))
return(ab1)
}

int1 <- function(si, k, b){
re <- root(si, k, b)
ae1 <- coeff1(si, k) * (1/((2 * pi)^0.5)) * exp((re^2)/(-2))
return(ae1)
}
```

```
int2 <- function(si, k, b){
re <- root(si, k, b)
ad1 <- re * exp((re^2)/(-2))
ad2 <- ((2 * pi)^0.5) * (1 - pnorm(re))
ad <- coeff2(si, k) * (ad1 + ad2) * (1/((2 * pi)^0.5))
return(ad)
}

.int3 <- function(si, k, b){
re <- root(si, k, b)
ae1 <- (2 + (re^2)) * exp((re^2)/(-2))
ae <- coeff3(si, k) * ae1 * (1/((2 * pi)^0.5))
return(ae)
}

int4 <- function(si, k, b){
re <- root(si, k, b)
af1 <- (re^3) * exp((re^2)/(-2))
af2 <- 3 * re * exp((re^2)/(-2))
af3 <- 3 * ((2 * pi)^0.5) * (1 - pnorm(re))
af <- coeff4(si, k) * (af1 + af2 + af3) * (1/((2 * pi)^0.5))
return(af)
}


value <- function(si, r, b){
k <- r/si
val <- int0(si, k, b) + int1(si, k, b) + int2(si, k, b) + int3(si, k, b) + int4(si, k, b)
val1 <- val * exp( - r) * 100
return(val1)
}
```