Essays in Applied Economic Theory

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

Abstract

This thesis consists of three essays, all of which use the tools of economic theory to analyze specific situations in which multiple strategic agents interact with each other.

The first chapter studies the strategic transmission of information between an informed expert and a decision maker when the latter has access to imperfect private information relevant to the decision. The main insight of the paper is that the access to private information of the decision maker hampers the incentives of the expert to communicate. Surprisingly, in a wide range of environments, the decision maker's information cannot make up for the loss of communication and the welfare of both agents diminishes.

The second chapter presents a model of electoral competition between an incumbent and a challenger in which the voters receive more information about the quality of the incumbent. If the incumbent can manipulate the information received by the voters through costly effort, the model predicts an incumbency advantage, even though the two candidates are drawn from identical symmetric distributions, and the voters have rational expectations. It is also shown that a supermajority re-election rule improves welfare, mainly through discouraging low-quality politicians from manipulating the information.

Finally the third chapter uses a mechanism design approach to characterize the class of social choice functions which cannot be profitably manipulated, when the individuals have symmetric single-peaked preferences. Our result allows for the design of social choice functions to deal with feasibility constraints.

To Fernando and our children Fernando, Inés and Marta

Contents

Preface

This thesis consists of three independent and unrelated chapters. Each of them represents an area of my research interests. The first chapter of this thesis contributes to the fascinating and growing literature of cheap talk. This literature studies the transmission of soft information; information not verifiable and which cannot be contracted upon. Crawford and Sobel (1982) provided a tractable working framework to analyze this issue. In their seminal paper a privately informed expert sends a message to an uninformed decision maker who then has to make a decision. They show that, if the interests of the two agents are not perfectly aligned, only coarse information can be transmitted in equilibrium. A natural reaction from the decision maker to this poor information transmission, would be to acquire some information by herself. Chapter 1 presents a model in which the decision maker has access to an imperfect and private source of information. The presence of this extra source of information has a twofold effect on the expert's incentives to transmit information. On the one hand, more information allows the decision maker to choose better actions on average, reducing the implicit cost of being imprecise. This effect hampers communication. On the other hand, the extra information introduces uncertainty to the expert, since he is no longer certain of the decision maker's reaction to his messages. A risk averse expert has an incentive to report more precise messages to reduce the variance of the decision maker's actions. This effect favours communication. Chapter 1 provides some environments in which the first effect dominates the second and the access to information reduces (and sometimes prevents) the communication in equilibrium. More strikingly, the loss of communication can be such that, even if the decision maker has access to valuable information, the net welfare effect is negative; the decision maker should commit to acquiring no extra information.

The second chapter of this thesis contributes to the literature of political economy. More specifically, Chapter 2 presents a model of electoral competition between an incumbent and a challenger. In developed countries powerful causal incumbency advantages seem to be present in offices which have little direct control over the electoral process, and where there is no seniority rationale for re-election. The principal asymmetry between incumbent and challengers seems to be just in information: that voters are much more informed about incumbents. However, with rational expectations, extra information about the incumbent should not systematically bias voters' beliefs. In the second chapter of this thesis it is shown that if incumbents were able to manipulate the information that reaches the voters, then, even under rational expectations, incumbent power over information can lead to a systematic bias in *election*, such that incumbents are re-elected with a significantly higher probability than in the case without manipulation. More interestingly, Chapter 2 shows that voters can improve the efficiency of the electoral system by handicapping the incumbent, that is, by requiring the incumbent to be above average quality to win re-election. The handicap suggested is not time consistent, i.e. voters do not want to enforce it *ex post*. Chapter 2 proposes a simple constitutional mechanism for implementation: a supermajority rule, where incumbent politicians require a share of the vote strictly greater than one half in order to win re-election.

Finally, the third chapter of this thesis contributes to the literature of social choice. This literature studies mechanisms to aggregate individual preferences in order to reach a collective decision and such that certain desirable properties are satisfied. In particular, Chapter 3 is concerned with the property of strategy-proofness. An aggregation mechanism (or social choice function) is *strategy-proof* if no agent has ever incentives to strategically misrepresent his preference; in other words, the mechanism cannot be manipulated by the individuals. Chapter 3 provides a characterization of the class of strategy-proof social choice functions when the individuals have symmetric single-peaked preferences, which means that the preferences respond to the notion of distance.

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Lastly, I dedicate this work to my husband Fernando and my children Fernando, Inés and Marta in thanks for their love, patience and trust in me.

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Chapter 1

Cheap Talk With Two-sided Information

1.1 Introduction

Decision makers often seek advice from better informed experts before making a decision. Examples range from management consulting to political, financial and medical advice[1]. Frequently, the interests of the expert are not perfectly aligned with those of the decision maker and this creates an incentive for the expert to manipulate his information. Crawford and Sobel (1982)[2] (CS henceforth) studied the strategic information transmission between a biased expert (he) and an uninformed decision maker (she) when contracts or other commitment devices are not available[3]. They show that only coarse information can be transmitted in equilibrium, even though, when the divergence of preferences is small, the expert might like better to truthfully reveal his information than to provide coarse information. The problem is that the expert cannot credibly submit more precise information, because if he were trusted, he would have an incentive to lie.

[1]Cheap talk games have been applied to study communication in a wide variety of areas. See Morgan and Stocken (2003) for an application to finance, Gilligan and Krehbiel (1989); Stein (1989); Austen-Smith (1993); Krishna and Morgan (2001b) and Morgan and Stocken (2008) for applications to political science and Galeotti et al. (2009) for an application to organization design and sociology.

[2]Green and Stokey (2007), which circulated in 1981, also study the information transmission between two agents. They analyze the welfare implications of improving the information available to the expert. Along the same lines, Fischer and Stocken (2001) and Ivanov (2010) study the relation between the precision of the expert's information and the communication between the two players.

[3]Dessein (2002) and Alonso and Matouschek (2008) analyze the case in which the decision maker can commit to delegate her decision to the expert. Goltsman et al. (2009) and Kovac and Mylovanov (2009) study cases in which the decision maker can commit to a mechanism. Krishna and Morgan (2008) analyze optimal contracts with full and imperfect commitment.

A natural reaction from the decision maker to this poor information transmission, would be to acquire some information by herself, in addition to consulting the expert. I argue that the decision maker should be cautious before making such a move. In fact, I show that the presence of an informative signal may hamper the communication between the agents and as a result, in a wide range of environments, the decision maker would be better off by committing not to acquire extra information.

To gain some intuition of the results, consider a decision maker who wants to choose an action $y \in \mathbb{R}$ to minimize the distance to an unknown state of the world. For simplicity, suppose that the state of the world, θ, takes one of the values $\{0, \frac{1}{2}, 1\}$ with equal probability. The decision maker consults an expert who perfectly knows the true state of the world, but who would like a higher action to be implemented. For instance, suppose that the expert wants the decision maker to choose the action $y = \theta + \frac{1}{3}$, where $\frac{1}{3}$ represents the bias of the expert[4]. Along the lines of CS, full revelation is not possible in equilibrium because the expert observing the lowest state of the world has an incentive to deviate and pretend that he observed $\theta = \frac{1}{2}$. In the most informative equilibrium[5], the expert reveals the lowest state of the world and pools the two higher states. Notice that, in this case, when the expert observes the lowest state of the world, he does not have an incentive to behave as if he had observed a higher state, because doing so would lead to an action $y = \frac{3}{4}$, further away from his optimal action. Given this equilibrium, the ex-ante expected utility of an uninformed decision maker is $EU^D = -\frac{1}{24}$.

Suppose now that the decision maker has access to an informative signal, s, which takes values in $\{0, \frac{1}{2}, 1\}$ with the following conditional probability matrix:

$$P = \begin{array}{c} s \setminus \theta \\ 0 \\ \frac{1}{2} \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} \;0\; & \;\frac{1}{2}\; & \;1\; \end{array} \\ \begin{pmatrix} 0.7 & 0.15 & 0 \\ 0.3 & 0.7 & 0.3 \\ 0 & 0.15 & 0.7 \end{pmatrix} \end{array}$$

where $P_{s\theta} = Prob(s|\theta)$ is the conditional probability of observing signal s given that

[4]To be more precise I am assuming in this example that both agents have quadratic loss utilities given by $u(y, p) = -(y - p)^2$ where p represents the peak of the preferences which is θ for the decision maker and $\theta + \frac{1}{3}$ for the expert.

[5]There is an issue of multiplicity of equilibria in cheap talk games. In particular there is always a babbling equilibrium. CS show that for the case of quadratic-loss utilities, the most informative equilibrium is preferred by both agents to any other equilibrium and hence I focus on this one. For refinements of equilibria in cheap talk games, see Matthews et al. (1991); Farrell (1993); Rabin (1990) and Chen et al. (2008) among others.

the state of the world is θ.[6] Given this signal structure, the expert can no longer credibly separate the lowest state from the other two. The reason is that when he observes that $\theta = 0$, he knows that the decision maker will receive the signal $s = 0$ with high probability. If he lies and reports that $\theta \in \{\frac{1}{2}, 1\}$, with probability 0.7 the decision maker will choose $y = \frac{1}{2}$ and with probability 0.3 she will choose $y = \frac{13}{20}$, leading to an expected utility to the expert of $-\frac{1783}{36000} \simeq -0.0495$, which is higher than the expected utility he would have if he truthfully revealed that $\theta = 0$ (in that case the utility for the expert would be $-\frac{1}{9} \simeq -0.1111$). Therefore, the introduction of the private information prevents the expert from revealing any information at all.[7] Moreover, the ex-ante utility of the decision maker when she has access to the signal (and hence does not receive informative messages from the expert) is $EU^D = -\frac{6}{85}$, which is lower than what she had in the uninformed case.

This example shows that allowing the decision maker to have access to a private signal lowers the incentives of the expert to reveal information because he knows that the signal will shift the decision maker's action towards the true state of the world, making exaggeration more attractive. To generalize this intuition, I consider the CS model with a continuum of states and allow the decision maker to access a continuous signal distributed symmetrically around the state of the world before making her decision. The main contributions of the paper are as follows.

First, for general symmetric preferences, I show the existence of partition equilibria similar to those characterized by CS and extend the properties of the CS equilibria to this setup.

Second, for the quadratic-loss preferences case, I decompose into two opposing effects the impact of private information on the expert's incentives to communicate. On the one hand, there is an *information effect* which arises because more information allows the decision maker to choose better actions on average. The information effect reduces the incentives of the expert to report precise information because the signal in expectation pulls the decision maker's action towards the real state of the world. This makes exaggeration more attractive and leads to less communication between the agents. On the other hand, there is a *risk effect* which occurs because the expert is no longer certain of the decision maker's reactions to his messages. Since the expert is risk averse, he has an incentive to report more precise messages and thus reduce the variance of the decision maker's actions. This effect favours communication.

[6] Observe that the signal is not only informative, but it is *affiliated* with the state of the world, meaning that higher realizations of the signal lead to higher posterior beliefs about θ in the first order stochastic dominance sense.

[7] These results are not driven by the presence of zeros in the conditional probability matrix.

Third, I show that in some environments, the information effect dominates the risk effect, reducing (and sometimes preventing) the communication in equilibrium. I illustrate this result for two different models, the normal private information model and the uniform private information model, where I derive some comparative statics with respect to the accuracy of the signal: communication decreases with the accuracy of the signal.

Finally, I show through the normal and uniform models, that the acquisition of private information may lead to a decline in the welfare of both agents and hence in those cases the decision maker should commit to acquiring no information.

The rest of the paper is organized as follows. In Section 1.2, I discuss the related literature. In Section 1.3, I state the model and show the existence of the partition equilibria. In Section 1.4, I analyze the communication incentives and illustrate the welfare implications for the uniform and normal models. In Section 1.5, I relax some of the assumptions of the model and discuss the implications of the results; and finally, in Section 1.6, I conclude.

1.2 Related Literature

Only a few papers have studied information transmission when the decision maker is privately informed. Two early references are Seidmann (1990) and Watson (1996)[8]. They show different ways in which private information might facilitate communication. In Seidmann (1990) different types of expert share the same preferences over actions but differ in their preferences over lotteries. By introducing private information to the decision maker, experts can be partially ranked, whereas no information can ever be revealed in the uninformed case. In Watson (1996) the information of the two parties is complementary. The preferences of the two players depend on a two dimensional state of the world, and each player receives a signal about a different dimension. He finds conditions such that a fully revealing equilibrium exists. By contrast, this paper suggests that when the decision maker's information acts as a substitute for the expert's information, less communication arises in equilibrium.

My paper is most closely related to Chen (2010); Lai (2010) and Ishida and Shimizu (2011). These papers introduce information to the decision maker within the standard framework of CS. Chen (2010) studies the optimal timing of the sender's

[8] Olszewski (2004) also introduces private information to the decision maker alongside two kinds of expert; sincere non-strategic experts and experts who are exclusively concerned with being perceived as honest. He shows that full revelation is the unique equilibrium, because the decision maker can use her private information to cross-check the expert's statements.

report when the players have access to a public signal. The present paper differs from hers on the question. She is concerned with the timing of the report and hence she compares the equilibria which ensue if the public information was available before and after the communication stage, whereas I compare the equilibrium of the model with private information[9] to the equilibrium of the uninformed case. In other words, the question of this paper is whether it is worth acquiring information at all. Lai (2010) studies communication between an expert and an *amateur* who knows whether the state of the world is below or above a cutoff point which is her private information. As in the present paper, Lai finds that the expert in the *amateur* model is less willing to provide information. However, the decision maker always ex-ante benefits from having access to the extra information. The setup of this paper allows for more flexible signal structures. In particular, I am able to explore the communication as the signals become smoothly more precise and I find that in some cases having access to information reduces the ex-ante welfare of the decision maker. Finally, Ishida and Shimizu (2011), analyze the case when both the expert and the decision maker have discrete imperfect signals about a binary state of the world. They show that when the two agents are equally informed, no information can be revealed in equilibrium for arbitrarily small biases.

Also related to this paper are models which introduce multiple experts because each expert represents a different source of information to the decision maker. Austen-Smith (1993) analyzes the case of an uninformed House that refers legislation to two expert committees (which are imperfectly informed) under open rule. He finds that any single committee is willing to provide more information under single referral than multiple referral. However, the information content of multiple referral is superior to that of single referral. In Krishna and Morgan (2001a) a decision maker can sequentially consult experts with different biases. They find that if the experts have similar biases the decision maker cannot do better than to ignore the messages of the most biased expert. Galeotti et al. (2009) study communication across a network where all the agents are at the same time senders and receivers. They find that the willingness of a player to communicate with a neighbour declines with the number of opponents who communicates to this neighbour. In all these papers there is an equilibrium in which the decision maker ignores the report of all except one expert, and, as a result, consulting multiple experts cannot be detrimental. By contrast, in the setup of the present paper, it is never rational for the decision maker to ex-post ignore her signal and hence the welfare implications can be negative.

[9] Analogous to the case where the public signal arrives after the sender's report.

Finally there are three papers which study the effect of uncertainty (to both agents[10]) on the incentives to communicate. Krishna and Morgan (2004) introduce a jointly controlled lottery together with multiple rounds of communication in the CS framework and show that the resulting equilibria Pareto dominate those of the original model. Blume et al. (2007) introduce error in the message transmission. They show that adding noise to the model almost always leads to a Pareto improvement. Goltsman et al. (2009) study optimal mediation in communication games. They find that mediators should optimally introduce noise in their reports because this eases the incentive compatibility constraints of the expert. In all these papers the uncertainty is independent of the state of the world. By contrast, I show that if instead of pure noise the decision maker receives an informative noisy signal, the results can be reversed.

1.3 The Model

1.3.1 Setup

There are two players, an expert (E or he) and a decision maker (D or she). The expert privately and perfectly observes the value of the state of the world θ, while the decision maker only receives a noisy signal $s \in \mathbb{R}$. The conditional distribution of the signal is common knowledge, but the realization s is privately observed by the decision maker. I will refer to θ and s as the *type* of the expert and the decision maker respectively. After learning θ, the expert sends to the decision maker a costless message m from a message set $\mathcal{M}$. The decision maker, taking into account her private signal and the expert's message, chooses an action $y \in \mathbb{R}$ which affects both agents' payoffs.

The payoff functions of the players are defined by the following utility functions:

$u^D(y, \theta) = \tilde{u}^D(y - \theta)$

$u^E(y, \theta, b) = \tilde{u}^E(y - (\theta + b))$

where $\tilde{u}^D$ and $\tilde{u}^E$ are strictly concave, twice differentiable and symmetric functions around 0. The parameter b represents the bias of the expert; given a realisation of the state of the world θ, the expert would like the decision maker to choose action $\theta + b$, whereas the optimal action for the decision maker is to match the state of the world. I will consider the case $b > 0$ although all the results can be replicated for

[10] Another branch of the literature introduces uncertainty on the preferences of the expert. See for example Li and Madarasz (2007); Morgan and Stocken (2003); Wolinsky (2003) and Dimitrakas and Sarafidis (2005). They find that more information can be transmitted because the decision maker is less sensitive to the message of the expert.

negative biases $b < 0$.

The state of the world is a random variable uniformly distributed on $[0,1]$. I assume that the signal and the state of the world are affiliated[11], meaning that higher realizations of s lead to higher posterior beliefs about θ in the first-order stochastic dominance. I will further assume that, given θ, the signal which the decision maker receives is distributed symmetrically around θ, with conditional density distribution $f(s-\theta)$, where $f(\cdot)$ is symmetric around 0, single-peaked and positive everywhere.

The symmetry assumption of the payoff functions and the conditional distribution of the signal, together with the uniform prior, simplify the analysis of the problem, because, as will become clearer in Section 1.4, the preferences of the expert over messages in equilibrium, depend only on the distance between the state of the world and the states induced by the message and not on the particular values of these variables.

I will refer to this model as the *private information* model and denote it by $F-PI$ where F refers to its signal structure.

1.3.2 Equilibrium

The equilibrium concept I consider is the *Bayesian Nash Equilibrium* (BNE). Given θ, a message strategy for the expert is a probability distribution over $\mathcal{M}$ denoted by $q(m|\theta)$. Due to the concavity of $\tilde{u}^D$, the decision maker has a unique preferred action in any information set and we can restrict attention to pure strategies. Given s and a message m, an action strategy for the decision maker is denoted by $y(m,s)$. The strategies $(q(\cdot), y(\cdot))$ constitute a BNE if:

1. for each θ, $\int_{\mathcal{M}} q(m|\theta)dm = 1$, and if $q(m^*|\theta) > 0$ then $m^* \in \arg\max_m \int_S \tilde{u}^E(y(m,s) - (\theta + b))f(s-\theta)ds$;

2. for each m and s, such that $\int_0^1 q(m|t)f(s-t)dt > 0$, $y(m,s) \in \arg\max_y \int_0^1 \tilde{u}^D(y-\theta)g(\theta|m,s)d\theta$, where $g(\theta|m,s) = q(m|\theta)f(s-\theta)/\int_0^1 q(m|t)f(s-t)dt$

Since $f(\cdot|\theta)$ has full support for all $\theta \in [0,1]$, after any possible equilibrium message, all the receiver's types agree about which states of the world have positive probability. If an expert deviates pretending to be another type, the deviation will

[11] For unidimensional random variables, being affiliated is equivalent to saying that the joint density distribution is log-supermodular in (s,θ) or that the conditional density distribution f satisfies the *Monotone Likelihood Ratio Property* (MLRP): $f(s|\theta)f(s'|\theta') > f(s|\theta')f(s'|\theta)$ for all $s > s'$, $\theta > \theta'$.

not be discovered and the beliefs of the decision maker will be determined by Bayes's rule, as indicated in point (2). In particular, this implies that the Bayesian equilibria of this game can be made perfect by specifying some beliefs for unsent messages which replicate the beliefs after an equilibrium message; hence, perfection does not refine the set of equilibria.[12]

If the signal s were independent of θ, the setup would correspond to the canonical model of CS. However, when the signal is informative two main differences arise. First, the expert is no longer able to perfectly forecast the reaction of the decision maker to his message. Each message induces a lottery over actions and when the expert decides which message to send, he is in fact comparing lotteries and not actions. Second, since the signal depends on the state of the world, the distribution of the lotteries depends on the expert's type, and therefore two experts sending the same message face different lotteries. This implies that the set of experts who prefer one message to another does not need to form an interval as in CS[13]. This latter fact makes it difficult to provide a complete characterization of the equilibria. However, as shown below, equilibria of a special kind exist. These special equilibria share the structure of the partition equilibria characterized by CS and have the property that as the signal becomes less informative, they converge to the equilibria in CS. In the remainder of the paper I focus exclusively on these equilibria.

1.3.3 Monotone Partition Equilibria

In this section I show the existence of monotone partition equilibria similar to those characterized in CS. An equilibrium is said to be a monotone partition equilibrium if the state space, $[0,1]$, can be partitioned into intervals such that all the experts with types in a given interval use the same message strategy, which has disjoint support from the message strategies used in other intervals. Formally:

Definition 1.3.1. An BNE equilibrium $(q(\cdot), y(\cdot))$ is a *monotone partition equilibrium of size N*, if there exists a partition $0 = a_0 < a_1 < ... < a_N = 1$ such that

[12] If the support of the signal varies with θ, it might be the case that an expert deviating by sending a message which corresponded to another type would be discovered. In this case, perfection might impose an extra restriction since some out of equilibrium beliefs need to be specified and the beliefs should be consistent with the information revealed by the signal, i.e, it has to satisfy that if $g(\theta|m,s) > 0$, then $f(s-\theta) > 0$.

[13] Chen (2009) in a similar setup provides an example of an equilibrium in which low and high types pool together whereas middle types send a different message. Krishna and Morgan (2004) also show the existence of non-monotone equilibria (Example 2) when the expert faces the uncertainty of a jointly controlled lottery.

$q(m|\theta) = q(m|\theta')$ if $\theta, \theta' \in (a_i, a_{i+1})$[14], and if $q(m|\theta) > 0$ for $\theta \in (a_i, a_{i+1})$ then $q(m|\theta') = 0$ for all $\theta' \in (a_j, a_{j+1})$ with $j \neq i$.

Given a monotone partition equilibrium, the only information that the decision maker learns upon receiving a message is the interval in which the actual state of the world lies. As a result, I consider all the equilibria with the same partition as equivalent and with some abuse of notation I say that $m \equiv [\underline{a}, \overline{a}]$ if $[\underline{a}, \overline{a}] = cl(\{\theta \in [0,1] \mid q(m|\theta) > 0\})$, where $cl(A)$ denotes the closure of the set A.

Before turning to the characterization of monotone partition equilibria, I introduce two further pieces of notation which simplify the exposition of the argument. First I denote by $y(\underline{a}, \overline{a}, s; F)$ the best response of a decision maker with signal s upon receiving $[\underline{a}, \overline{a}]$:

$$y(\underline{a}, \overline{a}, s; F) = arg\max_y \int_{\underline{a}}^{\overline{a}} \tilde{u}^D(y-\theta) f(s-\theta) d\theta \tag{1.1}$$

Second, I denote by $U^E(\underline{a}, \overline{a}, \theta, b; F)$ the expected utility of an expert with type θ and bias b who sends message $m = [\underline{a}, \overline{a}]$:

$$U^E(\underline{a}, \overline{a}, \theta, b; F) = \int_{\mathbb{R}} \tilde{u}^E(y(\underline{a}, \overline{a}, s; F) - (\theta + b)) f(s-\theta) ds$$

The following proposition establishes that only a finite number of messages can be sent in a monotone partition equilibrium. The intuition behind this result is that the intervals sent in equilibrium cannot be too small (except for the first one). If the size of an interval were smaller than $2b$, the expert on the lower bound of the interval would strictly prefer all the actions induced by this message to any possible action induced by a lower interval. By continuity, an expert with type slightly below the lower bound of the interval would like to deviate and report that he belongs to the interval, violating the equilibrium conditions. Since a separating equilibrium is a partition equilibrium with an infinite number of messages, no separating equilibrium exists under the setup of this model.[15]

[14]The definition above does not determine the strategy of boundary types $\theta = a_i$. As we will see in the construction of the equilibria, those types are indifferent between the message strategies of adjacent intervals, and therefore there are many strategy specifications which lead to payoff equivalent equilibria.

[15]This result contrasts with the finding of Blume et al. (2007). In their setup there exist monotone partition equilibria with an infinite (even uncountable) number of intervals. In their model the message of the expert is lost with a fixed probability and replaced by a random message. As a result the decision maker can rationally choose an action outside the interval induced by the expert's message and the argument used in Proposition 1.3.2 does not apply.

Proposition 1.3.2. *The number of intervals sent in a partition equilibrium is finite. In particular, there is no separating equilibrium in the private information model.*[16]

Note that Proposition 1.3.2 is true for arbitrary supermodular payoff functions (not necessarily symmetric) and arbitrary affiliated signal structures.

As in CS, a monotone partition equilibrium is determined by a partition $0 = a_0 < a_1 < ... < a_N = 1$ which satisfies the following arbitrage condition:

$$U^E(a_{i-1}, a_i, a_i, b; F) = U^E(a_i, a_{i+1}, a_i, b; F) \qquad (A_F)$$

Condition (A_F) means that the boundary type a_i is indifferent between sending message $m_i \equiv [a_{i-1}, a_i]$ and message $m_{i+1} \equiv [a_i, a_{i+1}]$. In CS this condition was necessary and sufficient to determine an equilibrium. When the decision maker has private information correlated with the state of the world, condition (A_F) alone may not be sufficient. The reason is that when an expert chooses between two messages, he is not choosing between two different actions but between two different lotteries over actions. If an expert with type a_i is indifferent between m_i and m_{i+1}, he must prefer the actions induced by m_i when the realization of the signal is high, and the actions induced by m_{i+1} when the realization of the signal is low. Since θ and s are affiliated, an expert with type $\theta > a_i$ allocates higher probability to high signals and as a result he may prefer m_i over m_{i+1}.

To prevent such reversals of preferences, I impose the following condition on the signal structure:

Assumption A1: *The signal structure F satisfies $\frac{\int |f'(s)| ds}{1-F(\frac{1}{2})} < \frac{bK_{11}}{\tilde{u}^E(0)-\tilde{u}^E(1+b)}$, where $K_{11} > 0$ is a constant related to the concavity of $\tilde{u}^E$.*[17]

The ratio $\frac{\int |f'(s)| ds}{1-F(\frac{1}{2})}$ in Assumption A1 is a measure of the precision of the signal. Intuitively, as the signal becomes uninformative, the ratio tends to zero and as the signal becomes perfectly informative, the ratio tends to infinity. For the particular case of a signal distributed normally with standard deviation σ, $\int |f'(s)| ds = \frac{2}{\sigma^3\sqrt{2\pi}}$ and hence the ratio decreases in σ.

Assumption A1 bounds the precision of the signal so that a change in θ does not result in a strong shift of the probabilities of the signal, ensuring that $U^E(a_i, a_{i+1}, \theta, b; F) - U^E(a_{i-1}, a_i, \theta, b; F)$ is increasing in θ (see Lemma A.1.4 in Appendix A.1.2), and hence that (A_F) is also sufficient for equilibrium.

[16] All the proofs are relegated to Appendix A.1

[17] More specifically, K_{11} is a strictly positive constant satisfying $-(\tilde{u}_1^E(y) - \tilde{u}_1^E(y')) \geq K_{11}(y - y')$ for all $y, y' \in [0, 1 + b]$. This constant exists due to the concavity of $\tilde{u}^E$.

Finally, to be able to make welfare comparisons between equilibria, I henceforth assume:

Assumption A2: $U^E(\underline{a}, a, \underline{a}, b; F)$ *is single-peaked in* a *for* $a \geq \underline{a}$.

Assumption A2 guarantees that given $a_{i-1} \leq a_i$ there is at most a unique a_{i+1} which satisfies the arbitrage equation (A_F). It allows a stronger version of condition (M) in CS[18] to be proved, which in particular ensures that there is at most one partition of size N satisfying (A_F) and this allows comparative statics of the equilibria.[19]

The following theorem characterizes the monotone partition equilibria:

Theorem 1.3.3. *Under Assumptions A1 and A2, if* $b > 0$*, there exists an integer* $N(b, F)$ *such that, for every* $1 \leq N \leq N(b, F)$*:*

1. *there exists a unique monotone partition equilibrium characterized by the partition* $0 = a_0 < a_1 < ... < a_N = 1$ *satisfying* (A_F)*,*
2. $a_{i+1} - a_i > a_i - a_{i-1}$ *for all* $i = 1, ..., N - 1$.

Moreover, both the decision maker and the expert ex-ante prefer equilibrium partitions with more intervals.[20]

Theorem 1.3.3 establishes that for each positive integer up to a finite number $N(b, F)$, there exists a unique monotone partition equilibrium of that size, for which the intervals are increasing in length. In particular, this implies that the messages sent by the expert in equilibrium are less precise as the state of the world increases. Finally, Theorem 1.3.3 also states that the equilibria can be Pareto ranked, and that the equilibrium with size $N(b, F)$ ex-ante Pareto dominates all the others. On the basis of this last statement, for the welfare analysis in Section 1.4, I will focus on the equilibrium partition with the highest number of intervals.

1.4 Communication and Welfare

In this section I analyze how the access to private information affects the incentives of the expert to disclose information.

[18] See Proposition A.1.6 in Appendix A.1.2.

[19] Observe that Assumption A2 does not depend on the equilibrium partition and hence it is an assumption on the primitives of the model which can be checked for every particular signal structure.

[20] All the comparative statics with respect to the divergence of preferences b established in (CS) can also be transferred to the private information model. Since this is not the focus of the paper I do not state them here.

Observe that, given the symmetric setup of the model, the decision maker in the private information model has ex-ante the same preferences over partitions as the uninformed decision maker has. Hence, it is meaningful to say that one partition is more communicative than another if ex-ante the uninformed decision maker prefers the former over the latter.

For $0 \leq a_{i-1} \leq a_i \leq a_{i+1} \leq 1$, denote by $V(a_{i-1}, a_i, a_{i+1}, b; F)$ the difference in expected utility to the expert with type a_i between sending $m_{i+1} = [a_i, a_{i+1}]$ and $m_i = [a_{i-1}, a_i]$:

$$V(a_{i-1}, a_i, a_{i+1}, b; F) = U^E(a_i, a_{i+1}, a_i, b; F) - U^E(a_{i-1}, a_i, a_i, b; F)$$

In particular, the arbitrage condition (A_F) can be written as $V(a_{i-1}, a_i, a_{i+1}, b; F) = 0$.

Proposition 1.4.1 provides a sufficient condition to order different signal structures in terms of the communication transmitted in equilibrium. More precisely, it states that, to determine whether one signal structure leads to more communication than another, it is sufficient to study how the indifferent expert changes when the signal structure changes.

Proposition 1.4.1. *Suppose that F and F' are two signal structures satisfying the following condition:*

$$\text{(C):} \quad \textit{If } V(a_{i-1}, a_i, a_{i+1}, b; F) = 0, \textit{ then } V(a_{i-1}, a_i, a_{i+1}, b; F') > 0.$$

Then there is less communication transmitted in the $F' - PI$ model than in the $F - PI$ model.
Namely, if a and a' are two equilibrium partitions of size N of the $F - PI$ and the $F' - PI$ models respectively, then $a_i > a'_i$ for all $1 \leq i \leq N - 1$. Moreover, $N(b, F) \geq N(b, F')$.

Intuitively, if $V(a_{i-1}, a_i, a_{i+1}, b; F) > 0$, the expert with type a_i strictly prefers message m_{i+1} to message m_i. As a result, the new indifferent type a, such that $V(a_{i-1}, a, a_{i+1}, b; F) = 0$, would be to the left of a_i, but then the new partial partition $\{a_{i-1}, a, a_{i+1}\}$ provides less useful information to the decision maker. The reason is that m_{i+1} was larger than m_i, and hence a shift of a_i to the left makes the size of the intervals more uneven. Given the concavity of the decision maker's preferences, partition $\{a_{i-1}, a_i, a_{i+1}\}$ is preferred to partition $\{a_{i-1}, a, a_{i+1}\}$, because her ex-ante expected utility is higher under the former than under the latter.

Since the CS setup can be seen as a limiting case of the private information model (one in which the information structure is completely uninformative), to analyze how the communication is affected by the acquisition of private information, Proposition 1.4.1 says that it is enough to study how the preferences over messages change for the experts who were indifferent in the CS setup.

In order to proceed, I restrict attention to the case of quadratic-loss utilities.[21] The quadratic-loss utility functions are given by:

$\tilde{u}^D(y-\theta) = -(y-\theta)^2$

$\tilde{u}^E(y-(\theta+b)) = -(y-(\theta+b))^2$.

Given these utilities, the decision maker's optimal action when she receives message m and signal s is to match her expectation about the state of the world: $y(m,s) = E[\theta|m,s]$. Moreover, the expected utility of an expert with type θ who sends message m can be written as:

$$U^E(m,\theta) = -\hat{\sigma}^2(m,\theta) - (\hat{y}(m,\theta)-(\theta+b))^2 \tag{1.2}$$

where $\hat{y}(m,\theta)$ and $\hat{\sigma}^2(m,\theta)$ are the expectation and the variance of the actions chosen by the decision maker when the expert sends message m and has type θ.[22] Equation (1.2) states that the expert's expected utility depends only on the variance of the actions and the distance between the expert's peak and the expected action of the decision maker.

Denote by $y_{CS}(m)$ the action chosen by an uninformed decision maker upon receiving message m. The change in the expert's expected utility due to the introduction of private information is:

$$U^E(m,\theta) - U^E_{CS}(m,\theta) = \underbrace{-\hat{\sigma}^2(m,\theta)}_{Risk\ \ Effect} + \underbrace{(y_{CS}(m)-(\theta+b))^2 - (\hat{y}(m,\theta)-(\theta+b))^2}_{Information\ \ Effect} \tag{1.3}$$

The introduction of private information has two effects on the expert's expected utility: an *information effect* and a *risk effect*. The information effect arises because the signal allows the decision maker to choose better actions on average. In expectation, her actions will be closer to θ than they were before. For a boundary expert, an action closer to the actual state of the world is also an action closer to his peak. Hence, fixing a message, the information effect has a positive impact on the expected utility of a boundary expert. The risk effect occurs because the expert is no longer certain of the response of the decision maker to his message. Since the expert is risk averse, he dislikes this uncertainty and, if the message is fixed, the risk effect always

[21] Section 1.5 provides a discussion of the results for other symmetric preferences.

[22] Namely, $\hat{y}(m,\theta) = \int_{\mathbb{R}} y(m,s)f(s-\theta)ds$ and $\hat{\sigma}^2(m,\theta) = \int_{\mathbb{R}}(y(m,s)-\hat{y}(m,\theta))^2 f(s-\theta)ds$.

has a negative impact in the expert's expected utility.

I now compare the information and risk effect across messages for an expert with type $\theta = a_i$ who is indifferent between sending messages $m_i = [a_{i-1}, a_i]$ and $m_{i+1} = [a_i, a_{i+1}]$ in the CS model. If there were no divergence of preferences between the agents ($b = 0$), the length of the two intervals would be the same and due to the symmetric setup, the signal would influence the decision maker in a symmetric way and the expert would still be indifferent between the two messages. However, the presence of a bias $b > 0$ implies that m_{i+1} is larger than m_i, and therefore the lotteries over actions induced by these two messages are qualitatively different.

Consider first the information effect. Observe that the message sent by the expert determines the prior of the decision maker before hearing her signal. Since message m_{i+1} is larger than message m_i, sending m_{i+1} instead of m_i implies that the decision maker will have a less precise prior about the state of the world. But a less precise prior implies that the decision maker will rely more on her signal when updating her posterior. In other words, the actions of the decision maker are more sensitive to her private information the larger the sent message is. From the point of view of the expert with type a_i, it means that the expected action of the decision maker will shift towards him by more when he sends m_{i+1} than when he sends m_i. Hence the expert with type a_i, strictly prefers $\hat{y}(m_{i+1}, a_i)$ to $\hat{y}(m_i, a_i)$. Abstracting from risk aversion, this result implies that the message m_{i+1} becomes more attractive to the expert than the message m_i. Hence, the information effect of the signal lowers the incentives of the expert to communicate.

Proposition 1.4.2. *The information effect hampers communication. Namely, if* $0 \leq a_{i-1} \leq a_i < a_{i+1}$ *are such that the expert with type* a_i *is indifferent between* $y_{CS}(m_i)$ *and* $y_{CS}(m_{i+1})$*, where* $m_i = [a_{i-1}, a_i]$ *and* $m_{i+1} = [a_i, a_{i+1}]$*, then the expert strictly prefers* $\hat{y}(m_{i+1}, a_i)$ *to* $\hat{y}(m_i, a_i)$*.*

Consider now the risk effect. Intuitively, sending a larger message spreads the decision maker's actions across the interval, thereby increasing the variance of the lottery. Hence the risk effect is stronger in m_{i+1} than in m_i. Abstracting from the information effect, a risk averse expert prefers the lower and narrower interval, easing the communication between the agents.

Proposition 1.4.3. *The risk effect eases communication. Namely, if* $0 \leq a_{i-1} \leq a_i < a_{i+1}$ *are such that the expert with type* a_i *is indifferent between* $y_{CS}(m_i)$ *and* $y_{CS}(m_{i+1})$*, where* $m_i = [a_{i-1}, a_i]$ *and* $m_{i+1} = [a_i, a_{i+1}]$*, then* $\hat{\sigma}^2(m_{i+1}, a_i) > \hat{\sigma}^2(m_i, a_i)$*.*

Proposition 1.4.2 and 1.4.3 highlight two surprising effects of the acquisition of private information. More accurate actions deter communication whereas adding risk favours it. To understand these results, observe that in CS, low type experts were deterred from sending high messages because they led to actions too far away from their preferred actions. When private information is introduced, low type experts know that, even if they send a high message, the action chosen by the decision maker will be pulled towards their preferred action, because the signal received is affiliated with the state of the world. This pulling force makes high messages more attractive and hence, to avoid deviations, high intervals have to be even less precise in equilibrium, leading to a loss in communication. The opposite effect arises with the riskiness of the signal. A risk averse expert will try to balance the risk associated with the actions of the decision maker by making the intervals more even, hence favouring communication.

1.4.1 No Communication Results

The following results present some environments for which the introduction of information prevents all sort of communication.

Proposition 1.4.4. *For any information structure F, there exist $\overline{b} < \frac{1}{4}$ such that if $b > \overline{b}$, there is no communication in the $F - PI$ model. Moreover, for any $b > 0$, there exists a sufficiently precise signal structure such that there is no communication in the private information model.*

Observe that for $b < \frac{1}{4}$ there exists an informative equilibrium in the CS model, and hence Proposition 1.4.4 refers to situations where those equilibria are lost and only the babbling equilibrium subsists.

The intuition behind the proof of Proposition 1.4.4 is as follows. Consider an expert with bias $b = \frac{1}{4}$ in the CS model. In this case the expert with type $\theta = 0$ is indifferent between perfectly revealing his type or pooling with the rest of the interval. If the expert perfectly reveals his type, the introduction of private information does not involve any risk or information effect, since the decision maker's action is independent of the signal. However, if the expert pools with the rest of the interval, the addition of information has a positive impact on the expert's expected utility, because a better informed decision maker would tend to choose lower actions. In particular the information effect will dominate the risk effect and $V(0, 0, 1, \frac{1}{4}; F) > 0$[23].

[23]Observe that this is the case because the expert's preferences have the same shape as those of the decision maker. If the risk aversion of the expert were higher than that of the decision maker

By continuity this implies that for b sufficiently close to $\frac{1}{4}$, $V(0,0,1,b;F) > 0$ and no information can be transmitted in equilibrium. For the second statement, observe that for any b, there is a precision of the signal structure such that the lottery over actions induced by message $[0,1]$ is preferred by an expert with type $\theta = 0$ to the constant action $y = 0$. This being the case, no information can ever be transmitted in equilibrium.

1.4.2 Welfare

We have seen that in some environments the information effect dominates the risk effect and as a result there is less communication in equilibrium. Nevertheless, the signal itself may provide enough information to make up for the loss of communication. Clearly, if the divergence of preferences is such that there is no communication in the CS model ($b \geq \frac{1}{4}$), private information is always welfare improving[24]. Similarly, if the information is very precise, the decision maker is better off even if no information is ever transmitted from the expert.

However, acquiring information is not always welfare improving. In what follows I analyze two different families of signal structures: the normal family and the uniform family. In both cases, the communication from the expert declines as the accuracy of the signal increases. Moreover, for each family there is a range of parameters for which increasing the accuracy of the signal reduces welfare and, strikingly, the welfare falls below the welfare level of the uninformed decision maker.

Normal Private Information Model

Consider the case in which the signal is distributed normally around θ with variance σ^2. The parameter σ^2 is a measure of the dispersion of the signal.[25]

To be more specific, suppose that the bias of the expert is $b = \frac{1}{20}$.[26] For this bias, the most informative equilibrium in the standard CS model is determined by the following partition: $\{0, \frac{2}{15}, \frac{7}{15}, 1\}$. The expert reveals whether the state of the world lies in $[0, \frac{2}{15}]$, in $[\frac{2}{15}, \frac{7}{15}]$ or in $[\frac{7}{15}, 1]$, and the decision maker reacts by choosing

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this result could be reversed. In Section 1.5 I provide a discussion of the role of risk aversion and I give an example where the risk effect dominates the information effect.

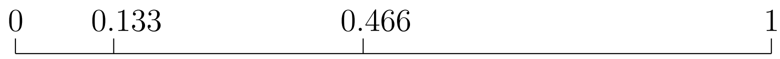
[24] See Persico (2000) and Athey and Levin (2001). They show that for decision problems where the signals are affiliated to the state of the world and the payoff of the decision maker satisfies the single crossing condition in (θ, y) (see Milgrom and Shannon (1994)), the ex-ante utility of the decision maker increases with the accuracy of the signals.

[25] Equivalently, $\frac{1}{\sigma^2}$ is a measure of the precision of the signal.

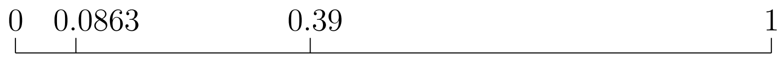
[26] This bias corresponds to the example illustrated in Crawford and Sobel (1982).

the midpoint in each interval[27].

In the private information model with $\sigma = 0.3$, the most informative equilibrium is determined by the partition $\{0, 0.0863, 0.59, 1\}$. Figure 1.1 provides a graphical illustration of the two equilibria.



(a) CS Equilibrium ($b = \frac{1}{20}$)



(b) PI Equilibrium ($b = \frac{1}{20}$, $\sigma = 0.3$)

Figure 1.1: Monotone Partition Equilibrium in CS model and in the Normal PI model, with $b = \frac{1}{20}$ and $\sigma = 0.3$

To compute the loss of communication due to the introduction of the signal, I compute the ex-ante utility of an uninformed decision maker under both partitions. The loss of communication is $EU^D_{CS,\{0,\frac{2}{15},\frac{7}{15},1\}} - EU^D_{CS,\{0,0.0863,0.39,1\}} = (-0.0159) - (-0.0213) = 0.0054$.

Figure 1.2 shows the partition equilibria as a function of the variance of the signal. For every σ the partition can be read by tracing the horizontal line at this level. The points of the partition correspond to the intersections with the solid lines. The case of $\sigma = 0.3$ is depicted as an example. The horizontal line cuts the solid lines at $a_1 = 0.0863$ and $a_2 = 0.39$, indicating that the partition equilibrium is $\{0, 0.0863, 0.39, 1\}$.

From Figure 1.2 we can see that for $\sigma < \sigma_0 \simeq 0.09$ no information is revealed in equilibrium. For $\sigma_0 < \sigma < \sigma_1 \simeq 0.193$ the partition equilibrium contains only two intervals and for $\sigma > \sigma_1$ the partition equilibrium is formed by three intervals. Finally, as σ increases, the equilibrium partition converges to the CS equilibrium.

Figure 1.2 suggests that the communication declines with the precision of the signal. This comparative statics is proven in Theorem 1.4.5 for the case of the family of uniform signals.

[27] It is easy to check that this in fact constitutes an equilibrium. For instance, when the expert observes $\theta = \frac{2}{15}$ he is indifferent between reporting the first interval and the second, because they lead respectively to actions $a_1 = \frac{1}{15}$ and $a_2 = \frac{9}{30}$, which are equidistant to his preferred action $\frac{2}{15} + \frac{1}{20}$.

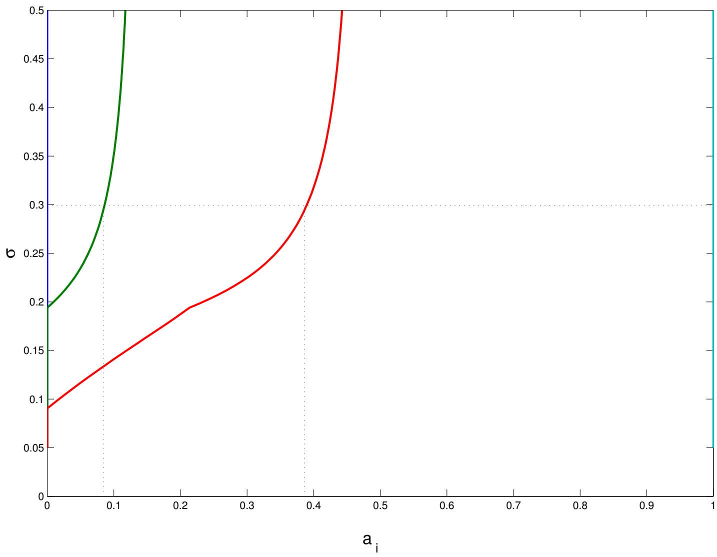


Figure 1.2: Monotone partition equilibria in the Normal PI model for different variances of the signal. ($b = \frac{1}{20}$)

Even if some communication is lost due to the introduction of the signal, the decision maker is able to choose better actions as a result of it. The welfare effect of making more accurate decisions can be computed as the difference in the ex-ante utility of the informed and the uninformed decision maker under the new partition. Coming back to the example depicted in Figure 1.1, the welfare gain is $EU^D_{PI,\{0,0.0863,0.39,1\}} - EU^D_{CS,\{0,0.0863,0.39,1\}} = (-0.0162) - (-0.0213) = 0.0051$.

The overall welfare effect of the addition of information can be computed by combining the loss in communication with the gain in accuracy. In this particular case the overall effect is negative: $EU^D_{PI,\{0,0.0863,0.39,1\}} - EU^D_{CS,\{0,\frac{2}{15},\frac{7}{15},1\}} = (-0.0162) - (-0.0159) = -0.0003$; the decision maker would be better off if she could commit to have no access to this additional information.

The next figure shows the ex-ante expected utility of the decision maker for different variances of the signal. The horizontal dashed line corresponds to the ex-ante utility of the CS model.

As can be seen from Figure 1.3 unless the precision of the signal is sufficiently high ($\sigma < 0.1735$), the decision maker is better off not seeking external information. The minimum ex-ante utility is reached at $\sigma = 0.1930$ which corresponds to the case where the partition equilibrium of the model passes from being of size 3 to size 2.

To understand why the loss in communication might outweigh the gain in infor-

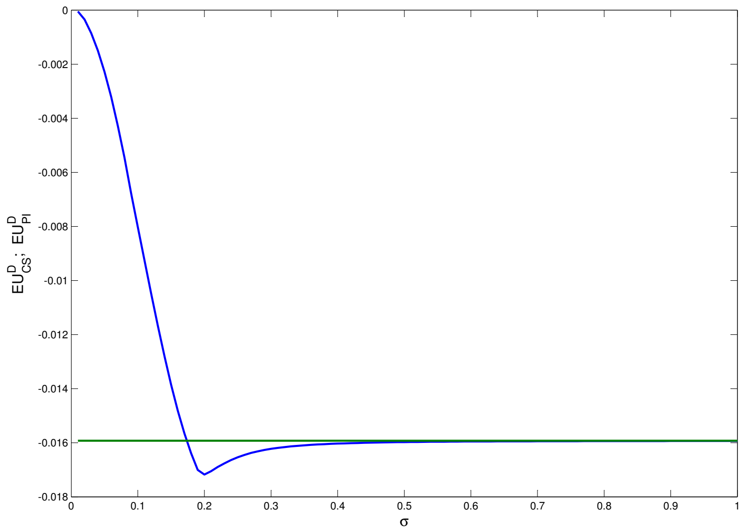


Figure 1.3: Ex-ante expected utility of the decision maker in the Normal PI model, for different variances of the signal. ($b = \frac{1}{20}$)

mation, it is useful to highlight two facts. First, as the signal is introduced, the largest interval becomes even larger, whereas the smallest ones are reduced (see Figure 1.2). The addition of information is most useful for those types lying in the big interval, but it is precisely this interval which becomes even larger, losing informativeness. Second, the introduction of information affects more the types determining the partition equilibrium than the rest of the types, so the change in communication is relatively high with respect to the overall change in welfare. To understand why this is the case, consider an interval in equilibrium: the introduction of an informative signal has a higher impact on welfare for those types close to the boundaries of the interval because they generate signals further away from the midpoint of the interval, which are the signals that lead to higher variation in the best response of the decision maker. So the boundary experts, those who determine the change in the communication, are the ones who are more affected by the introduction of the signal. If the signal is not precise enough, the welfare improvement due to the addition of the signal will not be able to outweigh the addition of noise due to the enlargement of the interval.

The Uniform Private Information Model

Suppose now that the signals are distributed uniformly on $[\theta - \delta, \theta + \delta]$.[28] The parameter δ plays the same role as σ in the normal example. Upon receiving a signal s and a message $m = [\underline{a}, \overline{a}]$ the decision maker's posterior distribution of θ is uniform on the interval $[\max\{\underline{a}, s-\delta\}, \min\{\overline{a}, s+\delta\}]$[29]. Given these beliefs, the optimal action for the decision maker is[30]:

$$y(\underline{a}, \overline{a}, s, \delta) = \frac{\max\{\underline{a}, s - \delta\} + \min\{\overline{a}, s + \delta\}}{2}$$

In this more tractable case the comparative static results which we observed for the family of normal signals can be proven.

Theorem 1.4.5. *In the Uniform Private Information model, an increase in the precision of the signal (a decrease in δ) leads to less communication in equilibrium. Namely, if a^{δ} and $a^{\delta'}$ are two monotone partition equilibria of size N of the $F_{\delta} - PI$ and $F_{\delta'} - PI$ models respectively, with $\delta' < \delta$, then $a_i^{\delta'} < a_i^{\delta}$ for all $i = 1, ..., N-1$. Moreover $N(b, \delta') \leq N(b, \delta)$.*

Theorem 1.4.5 states that the communication from the expert declines with the accuracy of the decision maker's information. Figure 1.4 illustrates the comparative static results for the family of uniform signals and Figure 1.5 shows the ex-ante expected utility of the decision maker for different precisions of the signal.

When $\delta \in [0.302, 0.343]$ the ex-ante welfare of the informed decision maker is lower than the welfare when the decision maker was uninformed. These levels of δ

[28] All the derivations related to this section, including the proof of Theorem 1.4.5, can be found in Appendix A.2

[29] Note that this signal structure does not satisfy the full support assumption. As is clear in the example, this assumption is not necessary for the existence of the monotone partition equilibria or for establishing its properties. However, the fact that the support of the signal varies with θ implies that an expert pretending to have another type might be discovered and this gives rise to more equilibria constructed by using out of equilibrium threats. For instance full revelation could be supported in an equilibrium using the following strategies:

$$q(m|\theta) = \begin{cases} 1 & if \quad m = \theta \\ 0 & otherwise \end{cases} \qquad y(m, s) = \begin{cases} m & if \quad m \in [s - \delta, s + \delta] \cap [0, 1] \\ -b^2 - 4\delta b & otherwise \end{cases}$$

Here I take a mild approach because I am interested in understanding how similar signals with full support (in which threatening with off equilibrium actions is not possible) affect the incentives to communicate. Note as well that full revelation cannot be supported if we impose perfection and require that out of equilibrium beliefs should be consistent with the information provided by the signal, that is, the support of the beliefs after signal s should be a subset of $[s - \delta, s + \delta] \cap [0, 1]$.

[30] All the functions previously defined will be indexed by δ to indicate the signal structure in consideration.

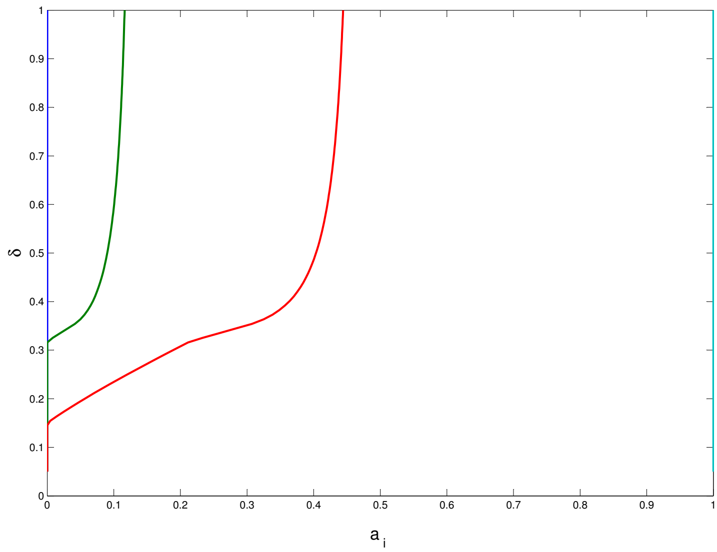


Figure 1.4: Monotone Partition equilibria in the Uniform PI model for different values of δ. ($b = \frac{1}{20}$)

correspond to the cases where the loss of communication is greater (see Figure 1.4). For these values of δ, a decision maker receiving a low signal is able to reject some high states of the world and hence the information effect in the upper interval is substantially stronger than in the lower interval, leading to a strong decrease in the communication, whereas the increase of welfare given the improvement of the signal is smoother. The minimum ex-ante utility is reached at $\delta = 0.3182$, which corresponds to the case where there are two payoff-equivalent equilibria, one with three intervals and one with two.

Although these results are not general, they do show that a better prepared decision maker does not necessarily lead to a better outcome in equilibrium. Several papers in the literature have suggested that more information might be worse (see, for example, Prendergast (1993) or Aghion and Tirole (1997)). However, the forces driving their results are completely different. Either the expert cares about his reputation and hence wants to pander to what he thinks the decision maker believes, or the decision maker's private information reduces the incentives of the expert to acquire information because she might overrule his proposal. In this setup, however, there are no reputation concerns and the expert already has the information relevant to the decision. The loss of welfare comes from the fact that private information makes it more difficult for the expert to credibly separate low types from high types,

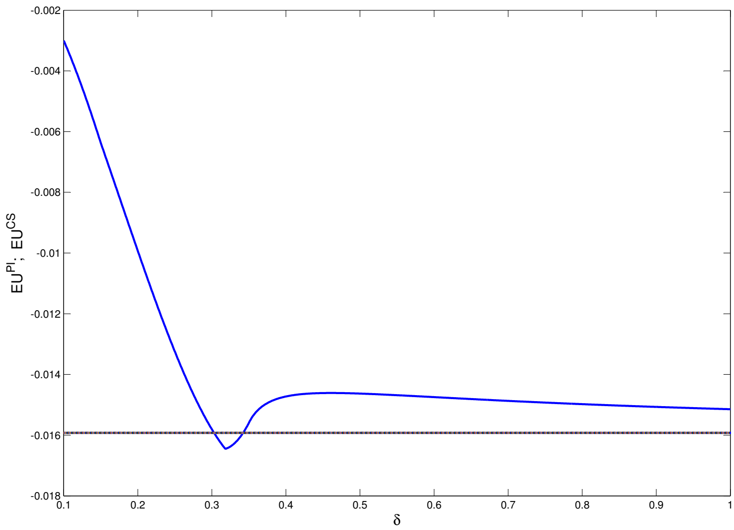


Figure 1.5: Ex-ante expected utility of the decision maker in the Uniform PI model for different dispersions of the signal. ($b = \frac{1}{20}$)

because the implicit cost of exaggerating (i.e. the risk that the decision maker will choose too high an action) is reduced by the private information.

1.5 Discussion

1.5.1 Information Acquisition

Up to now I have assumed that the decision maker had access to a free signal. Suppose instead that the acquisition of information is costly and that the decision maker has a quasi-linear utility in money.[31] In this case the timing of the acquisition of information matters. If the decision maker has to choose whether to acquire information or not before she meets the expert, then the results in Section 1.4.2 suggest that the decision maker would abstain from acquiring information unless the signal structure is precise enough and the information is not too costly.

Suppose now that the decision maker consults the expert and after hearing his message she decides whether or not to pay for the extra signal. In this situation it is worth acquiring information only if the message of the expert is coarse enough. To understand how the incentives to communicate are affected in this new framework,

[31] The decision maker's utility function can be written as $u^D(y, \theta, m) = m + \tilde{u}^D(y - \theta)$.

consider a partition equilibria with only two intervals in the CS setup. We know that the upper interval is larger than the lower interval. If the cost of acquiring information is sufficiently low for the decision maker to be willing to acquire information upon receiving any of the messages, then we are back to the private information model studied above. If contrariwise, the information is so costly that the decision maker never finds it profitable to acquire information, then we are in the CS framework. In the more interesting case in which the cost of information is such that the decision maker is willing to acquire information upon receiving the higher message but not upon receiving the lower one, the incentives to communicate become even weaker than with free private information. To understand why, notice that in this case the information effects on the incentives to communicate are present only on the upper interval. The boundary expert (in the uninformed CS model) knows that if he sends the upper message the decision maker will acquire information and therefore make a more accurate decision, closer to his preferences. If he sends the lower message, there is no acquisition of information and the middle action is chosen. Clearly, the expert prefers in this case to send the upper interval, shifting the indifferent type even further to the left than in the private information case studied before.

1.5.2 Other Information Structures

My analysis throughout Section 1.4 used the decomposition of the incentives of the expert to understand how the equilibria changed with the introduction of private information. The same decomposition of the incentives could be used to understand other communication results in the literature where the information structure of the game is modified. Consider for example the model proposed by Blume et al. (2007), in which the introduction of uncertainty leads to more communication in equilibrium and higher welfare. In their model the message sent by the expert is lost with some exogenous probability $\epsilon > 0$ and in such cases the decision maker receives a random message. The decision maker cannot distinguish between a message which has arrived by mistake and one sent by the sender, and hence upon receiving a message the decision maker chooses an action which is a weighted average between the ex-ante mean and the average expert who could have sent that message; $y_m^D = (1-\epsilon)d_m + \epsilon\frac{1}{2}$ where d_m is the midpoint of the interval (message) sent. [32].

[32] In this setup, the actual message strategy has a role in determining the equilibrium. The reason is that the message strategy affects the posterior beliefs of the decision maker on whether a certain message has arrived by mistake. This in turn determines the weights of the weighted average that represents her best response.

Consider the CS equilibrium with two intervals. As in the main text we can analyze how the incentives of the boundary expert are affected when the messages are random with a fixed probability. Since in the CS equilibria the intervals are increasing in length, the midpoint of the lower interval is further away from $\frac{1}{2}$ than the midpoint of the higher interval. Therefore, when noise is introduced, the shift in the best response of the decision maker is higher for the lower interval than for the higher one and the boundary expert now strictly prefers the lower interval.[33] In other words, the information effect in the lower interval is stronger than in the higher interval and this shifts the indifferent point to the right, leading to more communication in equilibrium.[34].

1.5.3 Other Preferences

In Section 1.4, I focused the analysis on the case of quadratic-loss utilities. Here I provide some intuition on the way in which the results might change if we consider other functional forms for the preferences of the agents. In particular, I consider the following families of utility functions:

$u^D(y,\theta) = -|y-\theta|^\rho$

$u^E(y,\theta,b) = -|y-(\theta+b)|^\varrho.$

where $\rho, \varrho \geq 1$. These families of utility functions were first introduced under this context by Krishna and Morgan (2004). One can interpret ρ and ϱ as a measure of risk-aversion since they measure the degree of concavity of $u^D(\cdot,\theta)$ and $u^E(\cdot,\theta,b)$ respectively. The case $\rho = \varrho = 2$ is equivalent to the quadratic-loss utilities studied before. The higher the ρ (ϱ) the more risk averse is the decision maker (expert).

In general, when $\varrho \neq 2$, the expected utility of the expert can no longer be written simply in terms of the expectation and the variance of the decision maker's action. However, it is useful to think of the information and risk effect in developing an intuition on these cases. Observe that the actions of the decision maker are completely independent of the preferences of the expert. Hence, if we fix the preferences of the decision maker and change the risk aversion of the expert, we are in fact comparing two fixed lotteries from the point of view of a risk averse agent. Intuitively, as ϱ decreases, the expert is more tolerant of risk and the risk effect diminishes. As a result, larger intervals become more attractive, leading to even less communication in

[33]Notice that there is no risk effect in this setup because the expert can perfectly forecast the reaction of the decision maker to his message if she gets it.

[34]Observe that when the indifferent expert has type $\theta > \frac{1}{2}$, the addition of noise has the opposite effect. Hence, when for small biases, i.e., when the most informative equilibrium has several steps, the communication implications are not so clear.

equilibrium. In contrast, as the expert's risk aversion increases, the risk effect becomes larger, reducing the impact of the information effect. For high enough risk aversion, it can even be the case that the risk effect outweighs the information effect leading to more communication in equilibrium. Consider for example an expert with preference parameter $\varrho = 6$ and bias $b = \frac{1}{4}$ who faces a decision maker with quadratic preferences ($\rho = 2$). In the uninformed case, no information can be transmitted in equilibrium. However, if the expert learns that the decision maker has access to a signal normally distributed around the state of the world and with standard deviation $\sigma = 0.5$, then his risk aversion allows him to reveal the following partition: $\{[0, 0.0119], [0.0119, 1]\}$. In this case obviously both agents are better off by the presence of the signal.

Alternatively, we could fix the preferences of the expert and change the preferences of the decision maker. In this case, however, a change in the preferences of the decision maker changes the lotteries over actions and hence changes indirectly the preferences of the expert. Intuitively, a more risk averse decision maker is less sensitive to her private information because she dislikes the risk associated with the signal; in order for her to choose an action below (above) the middle of the interval, she needs to receive a lower (higher) signal than when she was less risk averse, so that she is more certain that the true state of the world is actually low (high). In fact, an increase in the risk aversion of the decision maker has a similar effect on her actions to a decline in the accuracy of the signal structure. Hence, using the intuition of the comparative statics in Section 1.4, since the decision maker reacts less to her signal, the incentives to exaggerate are reduced and more communication arises in equilibrium.

For the case where $\rho = \varrho$, namely $u^E(y, \theta, 0) = u^D(y, \theta)$[35] the intuition is that although an increase in risk aversion smoothes the communication between the agents, the communication will still be worse compared to the canonical CS[36]. The reason is that, as discussed in Section 1.4, the value of information for the decision maker is bigger when her prior is less precise. A boundary expert with the same shape of preferences as the decision maker has nearly the same preferences as the decision maker when the state of the world is an extreme of the interval. Therefore the signal will make it more attractive for the expert to report the higher interval leading to less communication in equilibrium than in the CS case.

To sum up, as we increase the risk aversion of both agents the communication between them improves, and as a result the welfare of the agents increases (as a

[35] This assumption was made in CS to derive the comparative statics.

[36] Observe that in CS, the risk aversion of the agents does not play any role when the prior distribution is uniform in [0,1] and the preferences are symmetric, as in this model. In fact, the equilibria in the CS are the same with any set of symmetric preferences (i.e. for any $\rho, \varrho \geq 1$).

function of their risk aversion). This counterintuitive result was first highlighted by Krishna and Morgan (2004), although in their case only the risk aversion of the expert mattered because, in the absence of private information, the induced actions were independent of the risk aversion of the decision maker. This discussion extends their surprising conclusions to the risk aversion of the decision maker.

1.6 Conclusion

In this chapter, I analyze the strategic information transmission from an expert to a decision maker who has access to private information. The decision maker's information has two opposing effects. On the one hand, it allows the decision maker to choose better actions. On the other hand, it hampers the incentives of the expert to communicate because it makes exaggeration more attractive. As a consequence, the welfare of both agents might decline and hence the decision maker could benefit from committing to acquire no extra information. I provide two different environments in which this is the case.

Chapter 2

Handicapping Politicians: The Optimal Majority Rule in Incumbency Elections[1]

2.1 Introduction

In developing countries sitting politicians often are able to exercise considerable influence over the electoral process, and so engineer re-election. This power may explain observations of high re-election rates and voters' desire for term limits. However in developed countries powerful causal incumbency advantages seem to be present in offices which have little direct control over the electoral process, and where there is no seniority rationale for re-election (Ansolabehere and Snyder, 2002). The principal asymmetry between incumbent and challengers seems to be just in information: that voters are much more informed about incumbents. Cain et al. (1987) state this as follows:

Incumbents win because they are better known and more favorably evaluated by any wide variety of measures. And they are better known and more favorably evaluated because, among other factors, they bombard constituents with missives containing a predominance of favorable material, maintain extensive district office operations to service their constituencies, use modern technology to target groups of constituents with particular policy interests, and vastly outspend their opponents.[2]

This observation is difficult to reconcile with rational expectations, where extra

[1]This chapter draws on a joint work with Francecso Caselli, Tom Cunningham and Massimo Morelli. The idea of the paper was motivated by Francesco Caselli and Massimo Morelli willingness to provide a normative solution to deal with the well-documented problem of casual incumbency advantage. Tom Cunningham and myself carried out all the analytical results and writing of this paper with equal share; general decisions about the direction of the paper were made equally between the four authors.

[2]See Cain et al. (1987) p10.

information about the incumbent should not systematically bias voters' beliefs.

We show in this paper that even under rational expectations incumbent power over information can lead to a systematic bias in *election*, such that incumbents are re-elected with a significantly higher probability than in the case without the ability to manipulate the signal. Roughly, the reason is that medium-quality politicians exert a lot of effort to send signals similar to those sent by high-quality politicians, thus generating a skew distribution of signals. The median signal is above the mean signal, meaning that more than 50 percent of signals lead to posterior expected quality that is greater than the average quality, so more than 50 percent of politicians will be re-elected.

More interestingly we show that voters can improve the efficiency of the electoral system by handicapping the incumbent, that is by raising the threshold on expected quality needed to win re-election. A handicap will weaken the incentive of low-quality incumbents to exert effort, while strengthening the incentive of medium and high-quality incumbents to exert effort. The net effect is to raise the average quality of elected politicians.

The handicap we suggest is not time consistent, i.e. voters do not want to enforce it *ex post*. We thus suggest a simple constitutional mechanism for implementation: a supermajority rule, where incumbent politicians require a share of the vote strictly greater than one half in order to win re-election.

In the remainder of this Introduction we discuss related empirical and theoretical literature. Section sets up the model, and characterizes the equilibrium. For the rest of the paper we consider a simple symmetric 3-types distribution of politician types. Section analyzes the case of the simple majority rule. Section shows that under the simple majority rule an incumbency advantage exists in equilibrium. Section shows that the optimal re-election rule is a supermajority rule. Section gives an illustrative calibration, and Section discusses related issues and implementation of the supermajority rule.

2.1.1 Empirical Literature

The observed incumbency advantage is usually thought of as composed of a selection effect and a causal effect. The selection effect is due to incumbents being typically higher quality than challengers, and may be largely benign. A causal effect of incumbency could arise for a number of reasons, good and bad. For example, experience in office may improve the ability of a politician, or make them more effective through earning seniority. On the other hand, the privileges of office may allow them to unfairly influence the next election. The causal and selection effects are difficult to separate, though there are some notable recent attempts.

Levitt and Wolfram (1997) compare repeated pairings of candidates for election to the US Congress, in an attempt to control for the quality of incumbent and challenger. They find that the winner of the previous race has on average a 4% higher vote share in the second pairing.

Ansolabehere et al. (2000) compare county-level vote shares after redistricting in US Congressional elections. They find that incumbents receive 4% fewer votes in counties which have been redistricted into their constituencies, than in counties which remained in their constituency for both elections.

Lee (2008) compares bare winners and bare losers of elections. He finds that a party which barely wins a Congressional election has on average an 8% higher vote share and a 35% higher probability of winning the next election.[3]

Supermajority rules (also called "special majority") are common in constitutions, for example the US Congress can bypass the US President's veto only with a two thirds majority (Goodin and List, 2006).[4] More recently, the Turkish ruling party narrowly missed the supermajority threshold of two thirds of the seats that would have allowed them to change the Constitution unilaterally. However we are not aware of any supermajority rule being used to handicap the election of incumbents in the way that we suggest.

Finally note that although we know of no explicit incumbent supermajority rules, a similar effect is produced by existing institutions. Constitutional term limits can generally be overturned by an amendment, and constitutional amendments often require a supermajority among legislators. The net effect is then something similar to a supermajority rule on re-election. This is not uncommon in states with term limits, see for example in Colombia in 2004 and in Algeria in 2008.

2.1.2 Theoretical Literature

The theoretical literature discusses three mechanisms related to incumbency that are relevant to our model. The three mechanisms are (i) that re-election can function as a reward for good behavior, (ii) that extra information about incumbents allows signaling, and (iii) that there can be complementarities between politicians' terms.

An early literature on re-election incentives proposes that voters motivate politi-

[3] Note that Lee estimates the incumbency advantage that accrues to the party, not the candidate.

[4] Caplin and Nalebuff (1991) show that under certain conditions, a 64% supermajority voting rule can eliminate intransitivities in aggregation of preferences.

cian effort by using re-election as a reward for good behavior (e.g. Barro (1973)). In this kind of model a term limit would have an unambiguously negative effect on welfare because it would disable one of the principal mechanisms by which politicians are motivated.

A problem with models of this type is that the threat of punishment is only barely credible. When politicians all have the same quality voters are always indifferent between the incumbent and the challenger. This indifference means that an equilibrium in which voters punish badly behaved politicians can be subgame perfect.[5] However this indifference also means that if there exists any heterogeneity in politician talent, then the equilibrium disappears. For forward-looking voters any difference in perceived talent will dominate incentives to punish or reward sitting politicians. Voters may in fact be partly backward looking (see Smith et al. (1994)), which would of course complicate incentives for incumbents. We abstract away from these considerations in this paper.

A second strategic aspect of incumbency exists when politicians differ in ability, and voters observe signals related to ability when a politician is in power.[6] Early papers in this literature include Rogoff (1990), in which signaling produces a political budget cycle.

Smart and Sturm (2006) use this type of model, a signaling model, to analyze term limits. Their model has two types of politicians and information about the economy which is private to the politician. Re-election incentives cause both types to ignore their private information, for fear of being perceived as a low type. A term limit, which removes re-election incentives, can eliminate the distortion in policy choice and so raise welfare. Under certain parameters their model also predicts that a two-period term limit is superior to a one-period limit, because there will be some sorting of politicians in the first term.

Our model extends this signaling literature, and shows that it has a natural prediction for a rule on vote shares.

Finally, Gerbach (2008; 2009), like us propose a supermajority rule. In Gerbach (2008), the politician's type is fully revealed to voters once they are elected. However low quality incumbents are sometimes re-elected because while in power they implement policies with benefits that are contingent on re-election, thus generating complementarities between terms in office. A supermajority rule can deter such hostage-taking policies. In an independent work, Gerbach (2009) proposes a model

[5] Along with many other equilibria.

[6] This can be seen as a reinterpretation of the intuition in Barro (1973).

in which incumbents signal their ability with costly effort, with similar predictions to ours. Their different model however displays a continuum of pooling and semi-separating equilibria, and hence welfare judgments are derived under assumptions about the likelihood distribution over equilibria. Our modeling assumptions allow us to avoid such equilibrium selection problems.

2.2 The Model

The game is between two politicians - incumbent and challenger - and a continuum of voters. Both incumbent and challenger are defined by their talent $\theta \in \Theta$. Talent may be understood as the quality of the politician, a characteristic orthogonal to the political space, valued by every voter in the same way[7]. A few examples of what might be called talent are competency, honesty and charisma. The talents of both politicians are drawn from the same distribution with cumulative distribution function $F(\cdot)$ and mean $\hat{\theta}$.

The asymmetry between incumbent and challenger comes from the fact that during his period in office the incumbent can send a message about his talent to voters. More importantly, the incumbent can boost his message by exerting some costly effort. The message, denoted by $\tilde{\theta}$, is an additive combination of the politician's talent and his effort, $\tilde{\theta} = \theta + e$. The cost of effort is denoted by $c(\cdot)$ and is incurred only by the incumbent.[8]

Voters receive the message with some noise, representing the many unobservables which contribute to political outcomes, and constrain voters' ability to infer a politician's quality. Both the incumbency advantage and the supermajority result can be derived without noise, but noise eliminates pooling equilibria and hence allows us to explore the comparative statics of the equilibria. Also, noise generates a realistically continuous distribution of vote shares, which we use in our calibration.

To differentiate between the information sent by the incumbent and the information received by voters, we will call *message* what the incumbent sends and *signal* what the voters receive. The signal is equal to the original message, plus noise $s = \tilde{\theta} + \epsilon$, where ϵ is drawn from a continuous distribution with mean zero, symmetric and single peaked density distribution function $g(\cdot)$ and cumulative distribution

[7]This concept is also called in the literature *quality* or *valence* (Ansolabehere and Snyder, 2000; Carrillo and Castanheira, 2002; Ashworth and Bueno de Mesquita, 2008a;b).

[8]One might also think of the politician signaling their talent through the choice of a public policy as in Smart and Sturm (2006). We have abstracted of this because we wanted to isolate the informational effect of signalling and its indirect welfare implications.

function $G(\cdot)$.

Note that all voters receive the same signal, i.e. the noise is common to all voters.[9] However, voters differ in their preferences for the incumbent. We assume that the utility of voter i given an incumbent with talent θ is given by:

$$u_i(\theta) = \theta + \eta_i \tag{2.1}$$

where η_i represents voter i's relative preference for the incumbent over the challenger. We assume that η_i is continuously distributed, with full support on the real line, with cumulative distribution function $H(\cdot)$, density distribution $h(\cdot)$ with both mean and median equal to 0.

This model can be seen as a reduced form of a model in which, after the incumbent sends his message to the population, both incumbent and challenger announce their political platforms. In any subgame perfect equilibrium of such a model, there would be convergence of platforms to the median voter's preferences, and hence the choice of effort is taken as if the voters had preferences given by (2.1).[10] Finally, we assume that voters support the incumbent when indifferent, though because the noise distribution is atomless, the probability of an indifference occurring is vanishingly small.

Politicians are only office-motivated. Being in office leads to a reward of π. Their only cost is the cost of effort. Thus the incumbent chooses the level of effort to maximize

$$V(\theta, e) = \pi Pr(reelection|\theta, e) - c(e)$$

The game has two decision stages. In the first stage the incumbent sends a message that the voters receive with some noise. In the second stage the voters cast their vote. The outcome of the election depends on the votes cast and the re-election rule. We will denote a re-election rule by q when the incumbent needs at least the fraction q of the votes in order to be re-elected. In Section 2.3 we consider the particular case of simple majority rule for which $q = \frac{1}{2}$.

Given voters' preferences a simple majority rule is equivalent to giving all power to the median voter, which is in turn equivalent to maximizing a utilitarian social welfare

[9] We discuss in footnote 17 the general effects of heterogenous information.

[10] There is a recent literature that focuses on the interaction between the choice of effort and the choice of platform (see Ansolabehere and Snyder (2000); Aragones and Palfrey (2002); Carrillo and Castanheira (2002); Ashworth and Bueno de Mesquita (2008a); Meirowitz (2008)) when there is no asymmetry between the candidates. In some of these papers there is divergence of platforms in equilibrium. We abstract from the possibility of divergence and we choose instead to work with a model that corresponds to the more standard convergence outcome because the focus is not on the interaction between valence and political competition.

function. On the other hand, as we will discuss later in Section 2.5, a supermajority rule is equivalent to giving all the power to a voter who is opposed to or dislikes the incumbent. In order to be re-elected the incumbent's talent should be high enough to gain the support from this hostile voter.

Given a re-election rule q, an equilibrium is defined by an *effort rule*, $e_q : \Theta \rightarrow [0, +\infty)$ for the incumbent, and a *voting rule*, $v_q : \mathbb{R} \times \mathbb{R} \rightarrow \{0, 1\}$ for the voters such that:

$$\begin{array}{ll} (i) & e_q(\theta) \in argmax_e \quad \pi Pr_\epsilon(reelection|v_q(\cdot), \theta + e, q) - c(e) \\ (ii) & \quad v_q(s, \eta_i) = 1 \quad \textit{if and only if} \quad E[\theta|s, e_q(\cdot)] + \eta_i \geq \hat{\theta} \end{array} \tag{2.2}$$

where $Pr_\epsilon(reelection|v_q(\cdot), \theta + e, q)$ is the probability of re-election given the voting rule $v_q(\cdot)$, the message $\tilde{\theta} = \theta + e$ and the re-election rule q:

$$Pr_\epsilon(reelection|v_q(\cdot), \theta + e, q) = Pr_\epsilon \left(\int v_q(\theta + e + \epsilon, \eta_i) dH(\eta_i) \geq q \right)$$

and where $E[\theta|s, e_q(\cdot)]$ is the expected talent of the incumbent given that the public signal is s and using a posterior distribution of the incumbent's talent consistent with the equilibrium effort $e_q(\cdot)$.

Finally, we will say that the noise distribution $g(\cdot)$ satisfies the Monotone Likelihood Ratio Property (MLRP) if whenever $\tilde{\theta}_1 > \tilde{\theta}_2$, then $\frac{g(s-\tilde{\theta}_1)}{g(s-\tilde{\theta}_2)}$ increases in s.[11] The MLRP implies that higher signals lead to higher posterior distributions of the talent (higher here meaning first-order stochastic dominance).

The following proposition states than in equilibrium, the incumbent is re-elected whenever the public signal is equal to or above a certain threshold, and is not re-elected otherwise.

Proposition 2.2.1. *For any re-election rule q, if the cost of effort, $c(\cdot)$ is strictly convex and the distribution of noise satisfies the MLRP, then in any equilibrium $e^*_q(\cdot)$ and $v^*_q(\cdot)$, the incumbent is re-elected if and only if the public signal is above a threshold k_q, where k_q is given by:*

$$E[\theta|s = k_q, e^*_q(\cdot)] = \hat{\theta} - H^{-1}(1 - q) \tag{2.3}$$

Equation (2.3) states that the expected quality of an incumbent who sends a signal $s = k_q$ should equal the expected quality of a challenger, minus the partisan preference of the q^{th} percentile voter towards the incumbent.

[11] This definition corresponds to the special case of the MLRP defined by Milgrom (1981) when the signal structure is additive.

The proof of Proposition 2.2.1 is similar to that of Theorem 1 in Matthews and Mirman (1983) regarding limit pricing. Their setup is close to ours: a monopoly wants to deter the entrant of a possible challenger, and they do so by lowering their price, to signal lower profitability in the market. Analogously, a politician exerts effort to signal their type.

We begin with two preliminary results. In Lemma 2.2.2 we show that if the cost function is convex, the message sent by the incumbent is nondecreasing in his type.

Lemma 2.2.2. *Given a re-election rule q, if $c(\cdot)$ is strictly convex, and $e_q(\cdot)$ is a best response to $v_q(\cdot)$, then the corresponding message $\tilde{\theta}_q(\cdot)$ is nondecreasing in θ.*

Proof Let $\theta_1 < \theta_2$, and denote $\tilde{\theta}_q(\theta_i)$ by $\tilde{\theta}_i$ and $Pr_\epsilon(reelection|v_q(\cdot), \tilde{\theta}_i, q)$ by $P(\tilde{\theta}_i)$. Since $e_q(\cdot)$ (and therefore $\tilde{\theta}_q(\cdot)$) is a best response to $v_q(\cdot)$,

$$\begin{aligned} \pi P(\tilde{\theta}_1) - c(\tilde{\theta}_1 - \theta_1) &\geq \pi P(\tilde{\theta}_2) - c(\tilde{\theta}_2 - \theta_1) \\ \pi P(\tilde{\theta}_2) - c(\tilde{\theta}_2 - \theta_2) &\geq \pi P(\tilde{\theta}_1) - c(\tilde{\theta}_1 - \theta_2) \end{aligned}$$

Rearranging:

$$c(\tilde{\theta}_2 - \theta_1) - c(\tilde{\theta}_1 - \theta_1) \geq \pi(P(\tilde{\theta}_2) - P(\tilde{\theta}_1)) \geq c(\tilde{\theta}_2 - \theta_2) - c(\tilde{\theta}_1 - \theta_2)$$

Since the distance between the two sets of points is the same: $|(\tilde{\theta}_2 - \theta_1) - (\tilde{\theta}_1 - \theta_1)| = |(\tilde{\theta}_2 - \theta_2) - (\tilde{\theta}_1 - \theta_2)|$, the convexity of $c(\cdot)$ implies that $\tilde{\theta}_1 \leq \tilde{\theta}_2$. □

In Lemma 2.2.3 we find sufficient conditions so that each voter's best response is a threshold rule.

Lemma 2.2.3. *If $\tilde{\theta}(\cdot)$ is increasing and $g(\cdot)$ satisfies the MLRP, then voter i's best response is a threshold rule:*

$$v(s, \eta_i) = \begin{cases} 0 & if \quad s < k_i \\ 1 & if \quad s \geq k_i \end{cases}$$

where k_i is determined by $E[\theta|s = k_i, e_q(\cdot)] + \eta_i = \hat{\theta}$. Moreover, k_i is decreasing in the preference parameter η_i.

Proof If $\tilde{\theta}(\cdot)$ is increasing in θ and $g(\cdot)$ satisfies the MLRP, the conditional expectation of the talent is increasing in the signal received by the voter (Milgrom, 1981), i.e., if $s_1 < s_2$ then $E[\theta|s_1, e_q(\cdot)] < E[\theta|s_2, e_q(\cdot)]$.

Moreover, since no information is revealed from the challenger, the expected talent of the challenger coincides with the mean of the talent distribution. Therefore, a voter with partisan position η_i supports the incumbent if and only if:

$$E[\theta|s, e_q(\cdot)] + \eta_i \geq \hat{\theta} \tag{2.4}$$

Since the conditional expectation is increasing and continuous, there is a unique solution k_i to the equation $E[\theta|s = k_i, e_q(\cdot)] + \eta_i = \hat{\theta}$ and voter i follows a threshold rule in which $v(s, \eta_i) = 1$ if and only if $s \geq k_i$. Finally, by the monotonicity of the expectation, k_i is decreasing in η_i. $\square$

Now we prove Proposition 2.2.1.

Proof (Proposition 2.2.1) For any equilibrium $e_q^*(\cdot)$ and $v_q^*(\cdot)$, if $c(\cdot)$ is convex, Lemma 2.2.2 implies that $\tilde{\theta}_q^*(\cdot)$ is nondecreasing in theta. By the MLRP this implies that $E[\theta|s, e_q(\cdot)]$ is nondecreasing in s, and therefore $v_q^*(\cdot, \eta_i)$ is nondecreasing in s. If $v_q^*(\cdot)$ is constant, then $e_q^*(\cdot) \equiv 0$ since the effort is costly and does not change the behavior of voters. But then $\tilde{\theta}^*(\theta) = \theta$ is increasing in θ and Lemma 2.2.3 implies that $v_q^*(\cdot)$ is not constant. Therefore $v_q^*(\cdot, \eta_i)$ must be a threshold rule with some threshold $k_{i,q}$. By the monotonicity of the expectation, $k_{i,q}$ is decreasing in η_i. Denote by $\phi_q(\cdot)$ the decreasing function such that $\phi_q(\eta_i) = k_{i,q}$. The set of voters that support the incumbent given a signal s is $S_s = \{i : \eta_i \geq \eta_s\}$ where $\eta_s = \phi_q^{-1}(s)$. Define $\eta_q = H^{-1}(1-q)$ and $k_q = \phi_q(\eta_q)$. The signal $s = k_q$ is the minimal signal that guarantees reelection under rule q. In effect if $s \geq k_q$ then the share of votes for the incumbent is: $\Pr(\eta_i \geq \eta_s) \geq \Pr(\eta_i \geq \eta_q) = 1 - H(\eta_q) = q$. $\square$

Given a threshold k_q, the probability of re-election for an incumbent that sends message $\tilde{\theta}$ is $\Pr(\tilde{\theta} + \epsilon \geq k_q) = 1 - G(k_q - \tilde{\theta}) = G(\tilde{\theta} - k_q)$, where the last equality comes by the symmetry of the noise distribution.

We can now write the expected payoff of the incumbent as:

$$V(\theta, e, q) = \pi G(\theta + e - k_q) - c(e)$$

then the (local) first and second order conditions for the optimal effort level, $e_q^*(\cdot)$ are:

$$\begin{aligned} \pi g(\theta + e_q^*(\theta) - k_q) &= c'(e_q^*(\theta)) \\ \pi g'(\theta + e_q^*(\theta) - k_q) - c''(e_q^*(\theta)) &< 0 \end{aligned} \tag{2.5}$$

To guarantee that the local first and second order condition are sufficient for a global optimum we assume throughout the paper the following condition:

$$\inf_{e} c''(e) > \pi \sup_{\epsilon} g'(\epsilon) \tag{2.6}$$

Condition (2.6) requires the cost function to be sufficiently convex, so that the marginal cost cuts only once the marginal benefit.

Equation (2.5) together with the definition of the threshold (2.3), determine the equilibrium.

Up to now we have not made any assumption on the set Θ and the cumulative distribution of the talent $F(\cdot)$. For the rest of the paper we assume that the distribution of the politician talents is symmetric and has three types, i.e. $\Theta = \{\theta_L, \theta_M, \theta_H\}$, with $\theta_H - \theta_M = \theta_M - \theta_L \equiv \delta$ and $p = Pr(\theta_M) = Pr(\theta_L)$.

The symmetry assumption is here because, following the argument of Cain et al. (1987), we want to isolate the effect of the manipulation of the messages on the incumbency advantage. If the distribution of talent was skewed, for example if the median talent was above the mean, we would expect an incumbency advantage even without the manipulation of the messages. To see this, suppose that the voters could perfectly learn a politician's talent once he is in power. Then in more than 50% of elections the voters will discover that the incumbent has greater talent than the expected talent of the challenger, and hence they will strictly prefer to keep that politician. So more than 50% of the candidates will be re-elected. In a symmetric distribution, no incumbency advantage can arise without manipulation of the messages.

We consider three types because it is the simplest model that is rich enough to be able to explain all the mechanism we want to highlight. Simulations with continuous distributions make us believe that the results are true more generally, but we leave derivations for future work.

2.3 Simple Majority Rule

As a benchmark consider the simple majority rule $q = \frac{1}{2}$. Notice that given the assumptions on the voters' preferences for the incumbent, equation (2.3) becomes:[12]

$$E[\theta | s = k^*] = \hat{\theta} \tag{2.7}$$

[12] For clarity we suppress reference to the effort function $e^*_{1/2}(\cdot)$.

where k^* denotes the equilibrium threshold in the simple majority case. In other words, the simple majority rule is equivalent to giving all the power to the median voter, the voter that is ex-ante (before receiving the signal) indifferent between the incumbent and the challenger. The incumbent will be re-elected if and only if this voter believes him to have a higher than average talent.

The equilibrium for the simple majority rule has the following properties:

Proposition 2.3.1. *With a simple majority rule, the equilibrium is unique. The effort levels satisfy $e_M > e_L = e_H \equiv e^*$ with $e^* = c'^{-1}(\pi g(\theta_H - \theta_M))$ and the threshold signal is given by $k^* = \theta_M + e^*$.*

Proof For clarity we omit the reference to the electoral rule on the equilibrium variables. Given the talent distribution, upon receiving a signal $s = k^*$, equation (2.7) becomes:

$$\frac{\sum_j \theta_j g(k^* - \tilde{\theta}_j) Pr(\theta_j)}{\sum_j g(k^* - \tilde{\theta}_j) Pr(\theta_j)} = \theta_M$$

and given $Pr(\theta_H) = Pr(\theta_L)$ and $\theta_H - \theta_M = \theta_H - \theta_M$, it simplifies to:

$$g(k^* - \tilde{\theta}_H) = g(k^* - \tilde{\theta}_L) \tag{2.8}$$

In particular, given the symmetry of the noise distribution, equation (2.8) implies that the equilibrium threshold will be exactly half-way between the signals sent by the high and low types incumbents:

$$k^* = \frac{\tilde{\theta}_H + \tilde{\theta}_L}{2} = \theta_M + \frac{e_H + e_L}{2} \tag{2.9}$$

On the other hand, the first order conditions for the equilibrium effort (2.5) together with equation (2.8) imply that $e_H = e_L$. Denote by e^* this effort level. Then (2.9) implies $k^* = \theta_M + e^*$.

To see that $e_M > e^*$ notice that, from the single-peakedness and symmetry of g:

$$\pi g(k^* - \theta_M - e^*) > \pi g(k^* - \theta_L - e^*) = c'(e^*)$$

that is, the marginal benefit for an incumbent with type θ_M of exerting effort e^* outweighs the marginal cost of exerting this level of effort. Therefore, $e_M > e^*$.

Finally, replacing $k^* = \theta_M + e^*$ into the first order conditions for e^* given by

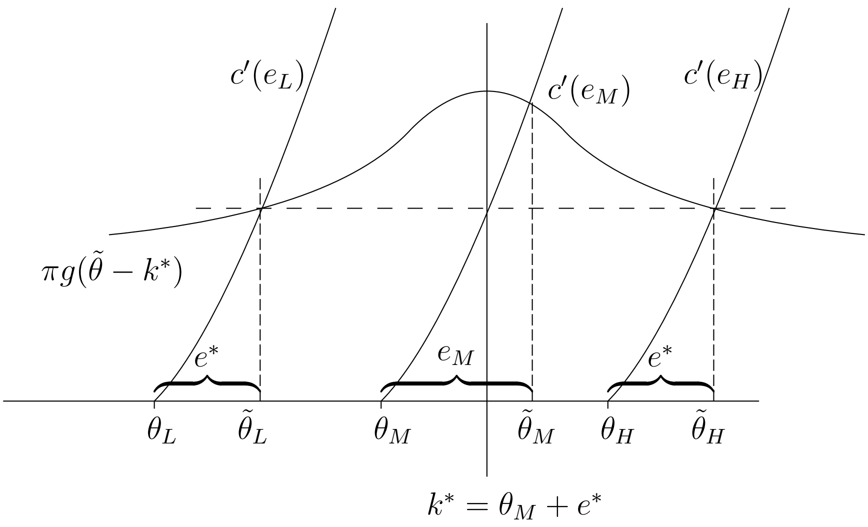


Figure 2.1: Equilibrium

equation (2.5), we obtain the equilibrium level e^*:

$$c'(e^*) = \pi g(\delta)$$

where $\delta \equiv \theta_H - \theta_M = \theta_M - \theta_L$ represents the dispersion of the talent distribution[13].
□

The equilibrium can be visualized in Figure 2.1. Both the talents and the messages can be read on the horizontal axis. The upward sloping lines represent the marginal costs of effort for each type. Equilibrium messages are determined by their intersections with the curve which represents the marginal benefit of exerting effort, $\pi g(\tilde{\theta} - k^*)$. The curve's peak is at $k^* = \theta_M + e^*$, the threshold above which incumbents are re-elected.

The effort level e^* is increasing in π, decreasing in the marginal cost, and decreasing in the dispersion of the incumbent's talent δ. These results are very intuitive, a direct change in the marginal benefit or cost changes the effort level accordingly. Moreover, if the distance between incumbents increases then it is more difficult to fool the voters by exerting effort and therefore the marginal benefits of effort goes down and they exert less effort.

Assuming that the noise is normally distributed with variance σ_ϵ^2 and mean zero, we can further study how the equilibrium effort level changes with the variance of

[13] The distance between the talents of the incumbents is a measure of the dispersion of the distribution. In fact the variance of the talents is given by: $Var(\theta) = 2p(\theta_H - \theta_M)^2 = 2p\delta^2$

the noise. The change in the equilibrium effort with respect to the variance of the noise depends on the relative size of the variance of the noise and the square of the dispersion of the incumbents:

$$\partial e^*/\partial \sigma_\epsilon^2 < 0 \quad \textit{if and only if} \quad \sigma_\epsilon^2 > \delta^2 \tag{2.10}$$

To understand this result consider the following two extreme scenarios. Suppose that the signal is *extremely noisy*, then voters do not infer much from the signal and incumbents exert very little effort. If the variance of the signal decreases making the signal more informative, then re-election will be more responsive to the signal received and incumbents will exert more effort. On the other hand, if the signal is *very precise*, incumbents are not going to be able to fool the voters and exert little effort. Condition (2.10) says that whether we consider the signal *extremely noisy* or *very precise* depends on the relative variances of the two distributions.

2.4 Incumbency Advantage

One interesting feature of the equilibrium is that the incumbents with middle talent are the ones that exert higher effort. The reason is that the equilibrium threshold is closer to their types and hence they have greater incentive to exert effort.

This extra effort from the incumbents with middle talent implies that the distribution of the messages, signals, and ultimately of expected types, will be negatively skewed (median is above the mean) leading to our result of an incumbency advantage.

We say that there is an incumbency advantage if the expected probability of being re-elected for an incumbent is greater than 50%.

Proposition 2.4.1. *The electoral competition model with the simple majority rule exhibits an incumbency advantage.*

Proof From Proposition 2.3.1, $e_M > e^*$. The probability of re-election for an incumbent with talent θ_j that sends message $\tilde{\theta}_j$ is then:

$$Pr(reelection \mid \tilde{\theta}_j) = Pr(\tilde{\theta}_j + \epsilon > k^*) = 1 - G(k^* - \theta_j - e_j)$$

The unconditional probability of re-election is therefore:

$$\begin{aligned}
Pr(s \geq k^*) &= p(1 - G(k^* - \tilde{\theta}_H)) + p(1 - G(k^* - \tilde{\theta}_L)) + (1 - 2p)(1 - G(k^* - \tilde{\theta}_M)) \\
&= p + (1 - 2p)(1 - G(k^* - \tilde{\theta}_M)) \\
&> \tfrac{1}{2}
\end{aligned} \tag{2.11}$$

Where the second equality follows because $G(k^* - \tilde{\theta}_H) = 1 - G(k^* - \tilde{\theta}_L)$ and the inequality because $e_M > e^*$ so $\tilde{\theta}_M > k^*$. □

Intuitively, when the median voter chooses whether to reappoint the incumbent or not, she compares her updated belief about the talent of the incumbent with the average talent of the challenger. In doing so, she can ignore middle type incumbents because they have just average talent, and hence taking into account the equilibrium messages of the incumbents, the threshold signal would be just the middle point between the messages sent by the low and the high signals. But given that the incumbents with middle talent exert more effort than the others, the message $\tilde{\theta}_M$ will exceed the threshold and therefore they will be re-elected more than half of the times. Because the average re-election probability for low and high types, when combined, is equal to exactly 50%, the total expected re-election probability will be greater than 50%.

2.5 Supermajority

In this section we consider the social planner's problem of maximizing the total welfare of the voters by choosing a re-election rule (we ignore the utility of the incumbent when computing the social welfare). We prove that the simple majority rule is suboptimal and that the welfare maximizing rule must be a supermajority rule ($q > \frac{1}{2}$).

We proceed in two steps. First, for the simple majority rule equilibrium we show that the voters would be better off if they could commit to a higher threshold to re-elect the incumbent. This commitment is not credible because ex post it is efficient to re-elect the incumbent if the updated beliefs indicate that he is above average (i.e., if the median voter would prefer him). We then propose a way to implement this commitment by setting a supermajority rule that takes decision power from the median voter and gives it to a voter with a partisan position somewhat against the incumbent.

Proposition 2.5.1. *In the electoral competition model with the simple majority rule, the welfare maximizing threshold is above the equilibrium threshold k^*.*

Proof Given a threshold k, the expected welfare can be expressed as the value of the outside option (the expected value of a challenger, θ_M), plus the expected change in value from retaining the incumbent:[14]

$$\begin{aligned} EW &= \theta_M + pPr(\tilde{\theta}_H + \epsilon \geq k)(\theta_H - \theta_M) + pPr(\tilde{\theta}_L + \epsilon \geq k)(\theta_L - \theta_M) \\ &= \theta_M + p\delta(G(\tilde{\theta}_H - k) - G(\tilde{\theta}_L - k)) \end{aligned} \tag{2.12}$$

The optimal threshold is then determined by the first order condition:

$$\frac{\partial EW}{\partial k} = p\delta\left(g(\tilde{\theta}_H - k)(\frac{\partial e_H}{\partial k} - 1) - g(\tilde{\theta}_L - k)(\frac{\partial e_L}{\partial k} - 1)\right) = 0 \tag{2.13}$$

At the equilibrium threshold, $g(\tilde{\theta}_H - k^*) = g(\tilde{\theta}_L - k^*)$, therefore if we evaluate the derivative (2.13) at k^*, the direct effect on welfare of a change in the threshold is zero. However, the change in the threshold also affects the choice of effort. Recall that the optimal level of effort given a threshold k satisfies the following first and second order conditions:

$$\begin{aligned} \pi g(\theta_j + e_j - k) &= c'(e_j) \\ \pi g'(\theta_j + e_j - k) - c''(e_j) &< 0 \end{aligned} \tag{2.14}$$

In particular, totally differentiating the first order condition with respect to k and rearranging:

$$\frac{\partial e_j}{\partial k} = \frac{\pi g'(\theta_j + e_j - k)}{\pi g'(\theta_j + e_j - k) - c''(e_j)} \tag{2.15}$$

and using the second order condition and the fact that $g'(\theta_L + e_L - k^*) > 0 > g'(\theta_H + e_H - k^*)$ we have that:

$$\left.\frac{\partial e_H}{\partial k}\right|_{k=k^*} > 0 \quad and \quad \left.\frac{\partial e_L}{\partial k}\right|_{k=k^*} < 0$$

Hence, plugging this into (2.13), the indirect effect on welfare of a raise in the threshold is positive. Increasing the threshold causes θ_H to exert more effort[15] while θ_L will reduce his effort, leading to more separation between the incumbents' signals and as a result an increase in welfare.

We have shown that welfare is improved by marginally increasing the threshold from its Nash equilibrium level. However this is not sufficient to show that a threshold higher than the Nash equilibrium threshold is optimal, because the welfare function

[14] The partisan preferences (η_i) disappear from this expression, because of their zero mean.

[15] A marginally higher threshold also leads the middle θ_M to exert more effort. To see this observe that $\theta_M + e_M - k^* = e_M - e^* > 0$ and hence $g'(\theta_M + e_M - k^*) < 0$ and $\frac{\partial e_M}{\partial k}|_{k=k^*} > 0$.

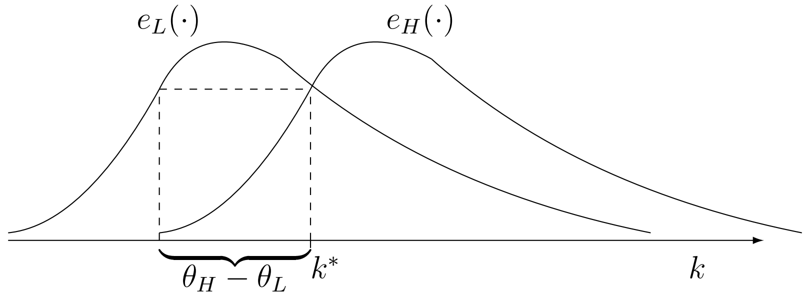


Figure 2.2: Effort functions

may not be single-peaked. We therefore demonstrate below that for any threshold $k < k^*$ the correspondent welfare is strictly lower than the welfare at the equilibrium threshold k^*. To see that, first notice that given θ_j, the optimal effort level e_j defined by equation (2.14) is a single-peaked function of the threshold k. For a given θ_j, the effort $e_j(\cdot)$ is increasing for $k < \theta_j + c'^{-1}(\pi g(0))$ and decreasing otherwise. Moreover, given equation (2.14), we have the following identity:

$$e_L(k - (\theta_H - \theta_L)) \equiv e_H(k)$$

so the optimal effort function of the low type is a horizontal shift to the left of the effort of the high type (see Figure 2.2).

At the equilibrium threshold, $e_L(k^*) = e_H(k^*) \equiv e^*$ so $e_L(k^*) = e_L(k^* - (\theta_H - \theta_L))$ which implies that k^* is on the downward-sloping part of curve $e_L(\cdot)$ and on the upward-sloping part of $e_H(\cdot)$. A representation of the effort functions can be seen in Figure 2.2.

Consider $k < k^*$, then $e_L(k) > e_H(k)$ and hence the distance between the high and low messages under threshold k is smaller than under threshold k^*:

$$\tilde{\theta}_H(k) - \tilde{\theta}_L(k) < \tilde{\theta}^*_H - \tilde{\theta}^*_L \tag{2.16}$$

Notice that by the symmetry of the noise distribution, the following two remarks are satisfied:

R1: Whenever two points are at a fixed distance h, $G(x) - G(x-h)$ is maximized at $x = \frac{h}{2}$, that is, when the two points are equidistant to the mean.[16]

R2: Given two points equidistant to the mean, the difference in the cumulative

[16]To see this consider the first order condition with respect to x: $g(x) - g(x-h) = 0$, and by the symmetry of $g(\cdot)$, this implies $x = -(x-h)$ or $x = \frac{h}{2}$.

distribution is increasing in the distance between the two points:

$$\frac{\partial}{\partial h}\left[G(\frac{h}{2}) - G(-\frac{h}{2})\right] = \frac{1}{2}(g(\frac{h}{2}) + g(-\frac{h}{2})) > 0$$

We can now conclude that for any threshold $k < k^*$ the welfare under threshold k is lower than under the equilibrium threshold k^*:

$$\begin{aligned} EW(k) &= \theta_M + p\delta(G(\tilde{\theta}_H(k) - k) - G(\tilde{\theta}_L(k) - k)) \\ &\leq \theta_M + p\delta(G(\tfrac{\tilde{\theta}_H(k)-\tilde{\theta}_L(k)}{2}) - G(-\tfrac{\tilde{\theta}_H(k)-\tilde{\theta}_L(k)}{2})) \\ &\leq \theta_M + p\delta(G(\tfrac{\tilde{\theta}^*_H-\tilde{\theta}^*_L}{2}) - G(-\tfrac{\tilde{\theta}^*_H-\tilde{\theta}^*_L}{2})) \\ &= EW(k^*) \end{aligned}$$

where the first inequality follows from R1 and the second from R2 and (2.16). □

Proposition 2.5.1 implies that the voters would be better off if they could commit to re-elect incumbents that have expected talent above a level which is strictly higher than the ex-ante average talent. An increase in the threshold will cause high types to exert more effort and low types to exert less effort. For both types their efforts will not offset the increase in the threshold, so both will be re-elected with a lower probability. But it is the larger fall in the probability of low-type re-election that increases welfare. In other words a supermajority rule makes voters better off entirely through discouraging low quality politicians from seeking re-election.

This higher threshold is not optimal ex post, because it asks the voters not to re-elect some politicians with expected talent strictly greater than the expected talent of the challenger. As discussed in the introduction regarding Barro (1973), it is not clear that individual voters have access to credible commitment devices. However committing to a higher threshold has a natural interpretation with respect to the voters as a whole: a constitutional rule such that incumbents will only be allowed a second term if they exceed some threshold of the vote share strictly greater than one half, i.e. a supermajority rule.

If all voters are identical then this rule, of course, has no effect. However, if the voters differ in their preferences for the incumbent, in the way we have assumed, then a supermajority amendment transfers the decision power from the median voter to a voter that is ideologically opposed to the incumbent.[17] Therefore a supermajority

[17] Another source of voter heterogeneity may be differential information. However if agents are rational, and there is common knowledge of rationality, then it is difficult to argue that the heterogeneous information will not be efficiently aggregated. Information can be indirectly passed through, for example, opinion polls. If a voter compares her own private signal with the aggregated signals

rule acts in effect as a commitment device that sets a higher threshold of talent for re-election.

Proposition 2.5.2. *In the electoral competition model, the welfare maximizing re-election rule is a supermajority rule* ($q_W > \frac{1}{2}$)*.*

Proof Given a threshold k, there is a re-election rule that implements that threshold in equilibrium. Denote by $e_k(\cdot)$ the optimal effort the incumbent exerts if he faces threshold k,[18] as a function of his type. We define $q(k)$ as follows:

$$q(k) = 1 - H(\theta_M - E[\theta|s = k, e_k(\cdot)]) \tag{2.17}$$

Clearly, setting the re-election rule $q = q(k)$ leads to the equilibrium effort $e^*_{q(k)}(\cdot) \equiv e_k(\cdot)$ and to the equilibrium threshold $k_{q(k)} = k$. To prove Proposition 2.5.2 it would be sufficient to prove that $q(k)$ is increasing in k. However this need not be true everywhere. As the threshold gets past a certain point both high and low types will react to an increase in the threshold by lowering their levels of effort (see Figure 2.2), thus an increase in the threshold could correspond to a lower expected quality from a signal sent at the threshold.

To prove the result we proceed in two steps. First we show that the equation $q(k) = \frac{1}{2}$ has a unique solution at k^*. Then we show that $q(\cdot)$ is strictly increasing at k^*, the equilibrium threshold of the simple majority case. Since $q(\cdot)$ is continuous, this implies that for any $k > k^*$, $q(k) > q(k^*) = \frac{1}{2}$.

Formally, $q(k) = \frac{1}{2}$ if and only if $E[\theta|s = k, e_k(\cdot)] = \theta_M$. By equation (2.7), k^* satisfies $E[\theta|s = k^*, e_{k^*}(\cdot)] = \theta_M$. To see that k^* is the unique solution to this equation notice that if $E[\theta|s = k, e_k(\cdot)] = \theta_M$, it has to be the case that $\tilde{\theta}_L(k)$ and $\tilde{\theta}_H(k)$ are equidistant to the threshold k. This implies that $e_k(\theta_L) = e_k(\theta_H) = k - \theta_M$. Substituting this in the first order conditions leads to $e_k(\theta_L) = e^* = c'^{-1}(\pi(\theta_H - \theta_M))$ and $k = k^*$.

We now show that $q(k)$ is increasing at k^*, or equivalently, that $E[\theta|s = k, e_k(\cdot)]$ is increasing at k^*:

$$\left.\frac{\partial E[\theta|s = k, e_k(\cdot)]}{\partial k}\right|_{k=k^*} = \frac{\partial}{\partial k}\left[\frac{(\theta_H - \theta_M)p[g(\tilde{\theta}_H - k) - g(\tilde{\theta}_L - k)]}{p(g(\tilde{\theta}_H - k) + g(\tilde{\theta}_L - k)) + (1-2p)g(\tilde{\theta}_M - k)}\right]\Bigg|_{k=k^*}$$

of 1000 people in an opinion poll, then the latter would seem to swamp the former. Also voters should vote using the expectations conditional on being decisive; this force will generally make a supermajority rule less effective (see Feddersen and Pesendorfer (1998)).

[18] The effort function $e_k(\cdot)$ solves equation (2.14).

Denoting by D the denominator of this fraction:

$$\begin{aligned}\left.\frac{\partial E[\theta|s=k,e_k(\cdot)]}{\partial k}\right|_{k=k^*} &= \frac{(\theta_H-\theta_M)p}{D}\left[g'(\tilde{\theta}_H-k^*)(\frac{\partial e_H(k^*)}{\partial k}-1)-g'(\tilde{\theta}_L-k^*)(\frac{\partial e_L(k^*)}{\partial k}-1)\right]\\ &> 0\end{aligned}$$

where the inequality follows because $D > 0$, $g'(\tilde{\theta}_H - k^*) = -g'(\tilde{\theta}_L - k^*) < 0$ by the equilibrium condition (11) and $\frac{\partial e_j(k^*)}{\partial k} < 1$ for $j \in \{H, L\}$ by equation (2.15).

Therefore, denoting by k_W the welfare maximizing threshold defined by equation (2.13), $k_W > k^*$ by Proposition 2.5.1 and therefore the optimal re-election rule $q(k_W) > \frac{1}{2}$ is a supermajority rule. □

2.6 Calibration

In this section we do a simple calibration exercise to illustrate the magnitudes involved in our model. We assume that the noise and preference distributions are normal. We also assume a quadratic cost of effort function, with coefficient $\frac{c}{2}$: $c(e) = \frac{c}{2}e^2$, and without loss of generality we set $\theta_M = 0$.

The model has five free parameters (discussed below), and we do not estimate all those parameters. Instead we have two modest goals. First, to show that a set of parameters which seem intuitively reasonable (to the authors at least) can reproduce incumbency effects of the right magnitude. Second, to show that the implications for an optimal supermajority rule and its welfare effects are also of an intuitive magnitude.

We target the causal incumbency advantage numbers reported in Lee (2008). That paper uses a regression discontinuity analysis on U.S. Congressional elections, and finds that the difference in the probability of winning an election between a marginal winner and a marginal loser (i.e., a winner or loser of the previous election) is 35%, and that the difference in the average vote share is of 7-8%.[19]

The free parameters of the model are (1) the variance of the noise distribution σ_ϵ^2, (2) the variance of the voters' preferences σ_η^2, (3) the dispersion of the talent distribution $\delta = \theta^H - \theta^M$, (4) the probability of the high and low types p, and (5)

[19]These numbers correspond to the party rather than the candidate incumbency advantage and average vote share advantage. The problem with the establishment of a candidate incumbency advantage is that there is an endogenous attrition of candidates that distorts the results.

the relative cost of effort is c (we normalise $\pi = 1$).[20]

Given the quadratic cost and the normal distributions, the equilibrium of the model is the following:

$$\begin{aligned} e_H = e_L = e^* &= \tfrac{1}{c\sigma_\epsilon}\phi(\tfrac{\delta}{\sigma_\epsilon}) \\ e_M &= \tfrac{1}{c\sigma_\epsilon}\phi(\tfrac{e_M - e^*}{\sigma_\epsilon}) \\ k &= \tfrac{1}{c\sigma_\epsilon}\phi(\tfrac{\delta}{\sigma_\epsilon}) \end{aligned} \tag{2.18}$$

where $\phi(\cdot)$ is the standard Normal density distribution.

The probability of winning for an incumbent is given by equation (2.11) and hence the difference in the probability of winning between the incumbent and the challenger is:

$$x = 2Pr(reelection) - 1 = (1 - 2p)\left(1 - 2\Phi\left(\frac{k^* - e_M}{\sigma_\epsilon}\right)\right) \tag{2.19}$$

where $\Phi(\cdot)$ is the standard Normal cumulative distribution.

Note that Lee (2008) computes the difference in the probability of winning between a marginal winner and a marginal loser. This avoids the problem of unobserved heterogeneity between winners and losers, if there is sufficient unpredictable noise in votes. Posterior differences between a bare winner and loser thus must be caused by the fact of winning or losing. In our model, all the politicians come from the same distribution of talents and therefore they are ex-ante identical and the difference in the probability of winning comes entirely from having been incumbent.

To compute the average vote share, note that given a signal s the share of voters that support the incumbent is $H(E[\theta|s])$ (see equation (2.4) for the individual voting rule). Hence the average vote share is given by:

$$AVS = \sum_{j \in \{L,M,H\}} Pr(\theta_j) \int H(E[\theta|s])\, g(s|\tilde{\theta}_j)ds \tag{2.20}$$

and the difference in the average vote share between the incumbent and the challenger is

$$y = AVS - (1 - AVS)$$

We choose the parameters as follows: the difference in quality between θ_H and θ_M is $\delta = 1.4$ (which by symmetry is also the difference between θ_M and θ_L). The

[20] The sufficient condition (2.6) is translated in the following restriction for the parameters:

$$c \geq \frac{1}{\sigma_\epsilon^2\sqrt{2\pi}} e^{-\frac{1}{2}}$$

probability of the middle type is $(1-2p) = 0.7$. The noise has standard deviation $\sigma_\epsilon = 1$, and voters' preferences have standard deviation $\sigma_\eta = \frac{1}{2}$. Finally the cost of effort is $\frac{c}{\pi} = \frac{1}{4}$. These parameters deliver an incumbency advantage matching Lee's estimates, with a difference in the probability of winning of 35% and a difference in the average vote share of 7%.

We can now calculate, using equations (2.13) and (2.17), that the optimal supermajority rule is $q_W = 57\%$. This supermajority rule then leads to a welfare increase of 5%, by lowering the proportion of low-quality candidates who are re-elected.

2.7 Discussion

This paper suggests that if incumbents can use their term in office to influence the voters' perception of their ability, handicapping the incumbent by requiring a higher vote share to be re-elected can improve welfare.

Throughout the paper we have assumed the incumbent faces only a single challenger. To implement our supermajority rule in practice we suggest a two-part ballot: In the first part voters indicate whether they wish to retain the incumbent. In the second part they choose their preferred challenger. This has the advantage of not handicapping the incumbent's party for example, the Republican incumbent can run, and the Republicans can also field a challenger. This ballot structure has been used in some recall elections, e.g. that used for California Governor Gray Davis in 2003.

The type of model we use (screening with noise, in a continuous typespace, but with a discrete reward) is uncommon in the literature. As mentioned earlier, Matthews and Mirman (1983) has the most similar model, though they do not derive analogues of either our incumbency advantage or supermajority results. Besides limit pricing, our approach may be fruitful in a number of other contexts in which thresholds are observed, most naturally entry into jobs which require a minimum score on some test of skill.

A useful extension would be to build a model extending over more than two periods, allowing incumbents to fight multiple elections, and perhaps finding a stationary equilibrium. Unfortunately this is not a simple exercise because two new effects must be modeled: first, the posterior distribution of incumbent types will become asymmetric, through selection; second, voters must now take into account the option value of electing a challenger.

Finally we note that the incumbency advantage and supermajority result can also be shown to occur in a much simpler model with naïve voters. Suppose that there are two kinds of voters - sophisticated and naïve - and that the preferences of the median sophisticated voter coincide with social welfare. Suppose naïve voters always vote for the incumbent because, for example, they are irrationally influenced by advertising, and incumbents always advertise more than challengers. In this case we would expect an incumbency advantage equal to $x\%$ of the vote share, where x is the proportion of voters who are naïve. Further, a supermajority handicap on the incumbent of exactly $x\%$ would make the democratic outcome welfare maximizing. Much of the novelty of this paper is to show that similar results hold even when voters are rational.

Chapter 3

On Strategy-proofness and Symmetric Single-peakedness[1]

3.1 Introduction

Consider a society with n agents who have to collectively choose one alternative from a given set of social alternatives. Assume that this set is endowed with a natural strict order because alternatives have a common characteristic that makes the comparison between pairs of alternatives meaningful and objective. For instance, the set of alternatives may consist of physical locations (a public facility on a road or street), properties of a political project in terms of its left-right characteristics, the expenditure level on a public good, indexes reflecting the quality of a product, feasible temperatures in a room and so on.[2] In all these cases and in many others, this linear order structure permits to identify the set of alternatives with a subset of the real line. Agents have (potentially different) preferences on the set of alternatives. Black (1948) is the first to suggest that, given the linear order on the set of alternatives, agents' preferences ought to be single-peaked. The preference of an agent is *single-peaked* if there exists an alternative (called the top) which is strictly preferred to any other alternative and on each side of the top the preference is strictly monotonic, increasing on its left and decreasing on its right.[3]

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[1]The work in this chapter was carried out jointly with equal share by Jordi Massó and me. The paper is published in *Games and Economic Behavior* 2011 (72) 467-484.

[2]There is an extensive literature studying collective choice problems where the set of social alternatives is a linearly ordered set. See Moulin (1980), for instance. This class of problems also plays a fundamental role in Sprumont (1995) and Barberà (2001; 2010), three excellent surveys on strategy-proofness.

[3]The set of single-peaked preferences is extremely large and rich; for instance, for each alternative there are many single-peaked preferences that have as top this alternative. Moreover, no *a priori*

Society would like to select an alternative according to agents' preferences. But since they constitute private information, agents have to be asked about them. A social choice function on a domain of preferences requires each agent to report a preference and associates an alternative with the reported preference profile. Hence, a social choice function on a Cartesian product domain induces an (ordinal) direct revelation game where each agent's set of strategies is his set of possible preferences. A social choice function is *strategy-proof* if no agent has ever incentives to strategically misrepresent his preference; in other words, truth-telling is a (weakly) dominant strategy in the direct revelation game induced by the social choice function.

Moulin (1980) characterizes the class of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences as the set of generalized median voter schemes.[4] A generalized median voter scheme is, in general, a non-anonymous extension of the median voter. It can be interpreted as a particular way of distributing the power to influence the social outcome among all coalitions of agents. In addition, Moulin (1980) also identifies the two nested subclasses of strategy-proof, tops-only and anonymous social choice functions and strategy-proof, tops-only, anonymous and efficient social choice functions.[5] The ranges of all functions in Moulin (1980)'s characterizations are closed intervals. This implies that if some alternatives were banned or infeasible, either the social choice function would have to request from the agents more information than just their tops, or there would be a single-peaked preference profile and an agent with incentives to misreport his preferences.

In many applications however, the domain of preferences can be restricted even further because the linear order structure of the set of alternatives conveys to agents' preferences more than just an ordinal content. Often, an agent's preference on the set of alternatives is responsive also to the notion of distance, embedding to the preference its corresponding property of symmetry. A single-peaked preference is *symmetric* if the following additional condition holds: an alternative is strictly preferred to another one if and only if the former is strictly closer to the top. If an indifference class contains two alternatives then both are located in opposite sides of the top and are

restriction is imposed on how pairs of alternatives lying in different sides of the top are ordered. Ballester and Haeringer (forthcoming) identify two properties that are both necessary and sufficient to characterize the domain of single-peaked preference profiles.

[4]A social choice function is *tops-only* if the chosen alternative only depends on the profile of tops. Tops-only social choice functions are especially simple in terms of the amount of information they require about individual preferences. Ching (1997) gives an alternative description and several axiomatic characterizations of generalized median voter schemes; in particular, Ching (1997) shows that in Moulin (1980)'s characterization tops-onlyness can be replaced by continuity.

[5]A social choice function is *anonymous* if it is independent of the identities of the agents; it is *efficient* if it always selects a Pareto optimal alternative.

at the same distance of the top.[6]

To restrict further the domain of a social choice function is equivalent to shrink the set of agents' strategies in its induced direct revelation game. Thus, strategies that were dominant remain dominant while strategies that were not dominant in the larger domain may become dominant after the domain reduction. Therefore, two important facts hold. First, any strategy-proof social choice function on a domain remains strategy-proof on all of its subdomains. Second, a manipulable social choice function on a domain may become strategy-proof in a smaller subdomain.[7] Hence, we ask whether the set of strategy-proof and tops-only social choice functions on the domain of single-peaked preferences, identified by Moulin (1980) as the class of generalized median voter schemes, becomes larger when the domain of preferences where we want the social choice functions to operate is the subdomain of symmetric single-peaked preferences. We answer this question affirmatively by completely identifying the larger class of functions that emerge after restricting further the domain.

The new class of social choice functions can be described as generalized median voter schemes disturbed by discontinuity jumps. A social choice function f in the class coincides with a generalized median voter scheme except that at some (countable number of) discontinuity jumps (for instance, an interval (a, b) with midpoint d), instead of taking the value prescribed by the generalized median voter scheme, f takes the constant value a at $[a, d)$, either the value a or b at d and the constant value b at $(d, b]$. Our description of the class makes precise that the choice of either a or b at any of those profiles where the generalized median voter scheme would choose d must be monotonic in order to preserve strategy-proofness of the social choice function.

We want to stress the importance for applications of admitting discontinuous social choice functions that are non-onto because they have a disconnected range and

[6] The notion of symmetric single-peakedness has already been considered in the context of strategy-proofness; for example in Border and Jordan (1983); Peters et al. (1992); Klaus et al. (1998); Ehlers (2002); Nisan (2007); Kar and Kibris (2008); Klaus and Bochet (2010). It has also been considered in the context of Political Economy to model voters' preferences over policies identified with an interval; for example in McKelvey and Ordeshook (1993); Krehbiel (2006).

[7] Observe two things. First, this is just a possibility. For instance, for the case where the set of social alternatives is the family of all subsets of a given set of candidates Barberà et al. (1991) show that voting by committees is the class of strategy-proof and onto social choice functions on both, the domain of separable preferences as well as on the subdomain of additive preferences, although the set of additive preferences is strictly smaller than the set of separable preferences. No new strategy-proof social choice function appears after the domain reduction in this case. Second, given a tops-only social choice function on the domain of single-peaked preferences, the set of agents' strategies in its induced direct revelation game is smaller when single-peaked preferences are further restricted to be symmetric because the fact that the rule is constant (by tops-onlyness) in a large subset of profiles is unrelated with the fact that the set of preferences that agents may use to evaluate the outcomes of the social choice function is smaller.

this range can in fact be any closed subset of alternatives. Besides, this range can be chosen beforehand. Non-onto social choice functions are indispensable for the design of social choice functions that require that some subsets of alternatives are never chosen due to feasibility constraints. For instance when the range of the function has to be finite, or not all locations for a public facility are possible, or the set of indexes reflecting the quality of a product must be disconnected, or the thermostat controlling for the temperature in a room can not take all values and so on. In all these cases and in many others, discontinuities can not be regarded as pathological features of social choice functions but rather as indispensable requirements to deal with constraints on the set of feasible alternatives to be chosen.[8]

There is a large literature studying strategy-proofness on domains related to single-peakedness. Border and Jordan (1983) extend Moulin (1980)'s results to multi-dimensional environments. One of the domains they consider is the set of quadratic separable preferences that coincides with the domain of symmetric single-peaked preferences when the number of dimensions is equal to one. However, Border and Jordan (1983) only consider social choice functions that respect unanimity (*i.e.*, if all agents have the same top then the common top should be chosen). Hence, all their results apply only to social choice functions whose ranges coincide with the set of alternatives. In particular, they show that for the one-dimensional case strategy-proof social choice functions that respect unanimity on the domain of symmetric single-peaked preferences are uncompromising,[9] and the partial converse that all uncompromising social choice functions are strategy-proof; moreover, all uncompromising social choice functions on this domain are continuous. Nehring and Puppe (2007b;a) study strategy-proofness in rich domains satisfying a general notion of single-peakedness based on abstract betweenness relations. However, their richness condition explicitly excludes as an admitted domain the set of symmetric single-peaked preferences since it requires that for any triple of alternatives (y, x, z) with y not being between x and z there must exist a preference relation in the domain with top on x such that z is strictly preferred to y. Thus, their results and ours are logically unrelated.

Our result and its proof are closely related to the following papers. Theorem 3.4.2 partly retains the structure of Moulin (1980)'s characterization of strategy-proof and tops-only social choice functions under the single-peaked domain of preferences. Our

[8] Barberà et al. (1997; 1998; 2005) identify subclasses of strategy-proof social choice functions that are able to deal with constrained sets of alternatives in different environments.

[9] A social choice function is *uncompromising* if an agent's top lies to the right (respectively, left) of the chosen alternative x, then any change in the top which leaves it to the right (respectively, left) of x will not affect the choice.

result in Theorem 3.4.2 says that strategy-proof social choice functions on the symmetric single-peaked domain that are manipulable on the larger single-peaked domain consists of generalized median voter schemes that are perturbed by specific discontinuities. Our result is also related to Theorem 3 in Barberà and Jackson (1994) characterizing *all* strategy-proof social choice functions on the domain of single-peaked preferences. Their characterization includes social choice functions whose range is not an interval; however, the characterization is open because it relies on a family of tie-breaking rules (used to select between the two extremes of the discontinuity jumps) that are not fully described. Our characterization is closed because it explicitly describes the exact family of admissible tie-breaking rules needed to preserve strategy-proofness. Yet, we are able to provide this closed description because our domain contains only symmetric preferences. The proof of our result relays at some point on Berga and Serizawa (2000)'s characterization of all strategy-proof and onto social choice functions on a minimally rich domain as the class of generalized median voter schemes;[10] we use their result in the easier case when the given strategy-proof social choice function is continuous. In addition, our proof is substantially simpler than it would have been if we were not able to use Barberà et al. (2010) result identifying conditions of preference domains under which (individual) strategy-proofness is equivalent to group strategy-proofness. Their result allows us to avoid many steps of individual changes of preferences by instead moving simultaneously the preferences of all members of a given coalition.

The paper is organized as follows. In Section 3.2 we present preliminary notations and the most basic definitions. In Section 3.3 we state some previous results and give the main definitions and intuitions in order to understand why and how the class of generalized median voter schemes has to be enlarged in order to identify the full class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences. In Section 3.4 we state and prove our main result characterizing the complete class of strategy-proof social choice functions on the domain of symmetric single-peaked preferences (Theorem 3.4.2). After presenting some preliminaries of the proof in Subsection 3.4.2, we prove Theorem 3.4.2 in Section 3.4.3. In Section 3.5 we first state as corollaries of Theorem 3.4.2 the corresponding characterizations under strategy-proofness and anonymity (Corollary 3.5.2) and under strategy-proofness,

[10] A domain is *minimally rich* if (i) it is a subset of the single-peaked domain, (ii) for each alternative x there is a preference relation in the domain with top at x and (iii) for any pair of alternatives x and y ($x \neq y$) there is a preference in the domain that strictly orders x and y and whose top lies between x and y. Obviously, the set of symmetric single-peaked preferences is a minimally rich domain.

anonymity and efficiency (Corollary 3.5.3). We then argue about the importance for applications of allowing for non-onto social choice functions which were ruled out by the combination of strategy-proofness and tops-onlyness in Moulin (1980)'s characterization under single-peaked preferences and state Corollary 3.5.4 characterizing all strategy-proof social choice functions that are efficient relative to a given closed set of feasible alternatives. We finish with the remark that, as the consequence of the main result in Barberà et al. (2010), the four statements hold if we replace in them strategy-proofness by group strategy-proofness.

3.2 Preliminary notations and definitions

Let $N = \{1, ..., n\}$ be the set of *agents* of a society that has to choose an *alternative* x from the interval $[0, 1]$.[11] Subset of agents will be denoted by capital letters (like S) and their cardinalities by their corresponding small letters (like s). The *preference* of each agent $i \in N$ on the set of alternatives $[0, 1]$ is a complete, reflexive and transitive binary relation (a complete preorder) R_i on $[0, 1]$. Let $\mathcal{R}$ be the set of complete preorders on $[0, 1]$. A *preference profile* $R = (R_1, ..., R_n) \in \mathcal{R}^n$ is a n-tuple of preferences. To emphasize the role of agent i or subset of agents T, a preference profile R will be represented by (R_i, R_{-i}) or (R_T, R_{-T}), respectively. As usual, let P_i and I_i denote the strict and indifference preference relations induced by R_i, respectively. Given $R_i \in \mathcal{R}$, the *top of* R_i (if any) is the unique alternative $t(R_i)$ that is strictly preferred to any other alternative; *i.e.*, $t(R_i)P_i x$ for all $x \in [0, 1]\backslash\{t(R_i)\}$.

Given a subset of preferences $\mathcal{S} \subseteq \mathcal{R}$, a *social choice function* (SCF from now on) f on $\mathcal{S}$ is a function $f : \mathcal{S}^n \to [0, 1]$ selecting an alternative for each preference profile in $\mathcal{S}^n$. We will refer to this Cartesian product set $\mathcal{S}^n$ (or to the set $\mathcal{S}$ itself) as the domain of preferences. Given a SCF $f : \mathcal{S}^n \to [0, 1]$, denote its range by r_f; *i.e.*, $r_f = \{x \in [0, 1] \mid \text{there exists } R \in \mathcal{S}^n \text{ such that } f(R) = x\}$.

We will be interested in SCFs that induce truth-telling as a (weakly) dominant strategy in their associated (ordinal) direct revelation game.

Definition 3.2.1. A SCF $f : \mathcal{S}^n \to [0, 1]$ is *strategy-proof* if for all $R \in \mathcal{S}^n$, all $i \in N$ and all $R'_i \in \mathcal{S}$,

$$f(R_i, R_{-i})R_i f(R'_i, R_{-i}).$$

If $f(R'_i, R_{-i})P_i f(R)$ we say that i *manipulates* f at R via R'_i.

Definition 3.2.2. A SCF $f : \mathcal{S}^n \to [0, 1]$ is *group strategy-proof* if for all $R \in \mathcal{S}^n$, all

[11] Our results also hold for any linearly ordered metric space of alternatives. In particular, for any set of alternatives which is a closed interval of real numbers (as well as for the set $\mathbb{R} \cup \{-\infty, +\infty\}$).

$T \subseteq N$ and all $R'_T \in \mathcal{S}^t$ with $R'_i \neq R_i$ for all $i \in T$,

$$f(R_T, R_{-T}) R_i f(R'_T, R_{-T})$$

for some $i \in T$. If $f(R'_T, R_{-T}) P_i f(R)$ for all $i \in T$ we say that T *manipulates* f at R via R'_T.

We will also consider other properties of SCFs. A SCF $f : \mathcal{S}^n \to [0,1]$ is *anonymous* if it is invariant with respect to the agents' names; namely, for all one-to-one mappings $\sigma : N \to N$ and all $R \in \mathcal{S}^n$, $f(R_1, ..., R_n) = f(R_{\sigma(1)}, ..., R_{\sigma(n)})$. A SCF $f : \mathcal{S}^n \to [0,1]$ is *dictatorial* if there exists $i \in N$ such that for all $R \in \mathcal{S}^n$, $f(R) R_i x$ for all $x \in r_f$. A SCF $f : \mathcal{S}^n \to [0,1]$ is *efficient* if for all $R \in \mathcal{S}^n$, there is no $z \in [0,1]$ such that, for all $i \in N$, $z R_i f(R)$ and $z P_j f(R)$ for some $j \in N$.[12] A SCF $f : \mathcal{S}^n \to [0,1]$ is *unanimous* if for all $R \in \mathcal{S}^n$ such that $t(R_i) = x$ for all $i \in N$, $f(R) = x$. A SCF $f : \mathcal{S}^n \to [0,1]$ is *onto* if for all $x \in [0,1]$ there is $R \in \mathcal{S}^n$ such that $f(R) = x$ (*i.e.*, $r_f = [0,1]$). A SCF $f : \mathcal{S}^n \to [0,1]$ is *tops-only* if for all $R, R' \in \mathcal{S}^n$ such that $t(R_i) = t(R'_i)$ for all $i \in N$, $f(R) = f(R')$. Let $\mathcal{S} \subseteq \mathcal{R}$ be any subset of preferences with the property that for each $x \in [0,1]$ there exists at least one preference $R_i \in \mathcal{S}$ such that $t(R_i) = x$. Then, $\mathcal{S}^n$ is called a *rich domain* and with some abuse of notation, given a tops-only SCF $f : \mathcal{S}^n \to [0,1]$ we will refer to it by its corresponding voting scheme $f : [0,1]^n \to [0,1]$.

The version of the Gibbard-Satterthwaite Theorem when the set of alternatives is the interval $[0,1]$ states that a SCF $f : \mathcal{R}^n \to [0,1]$, with $\#r_f \neq 2$, is strategy-proof if and only if it is dictatorial (see Barberà and Peleg (1990)). An implicit assumption of the Gibbard-Satterthwaite Theorem is that the domain of the SCF is universal: the SCF operates on all preference profiles, because all of them are reasonable. However, for many applications, a linear order structure on the set of alternatives naturally induces a domain restriction in which there always exists a top and at each of the sides of the top the preference is strictly monotonic.

Definition 3.2.3. A preference $R_i \in \mathcal{R}$ is *single-peaked* if:
(1) there exists the top $t(R_i)$ of R_i and
(2) for all $x, y \in [0,1]$ such that $y < x \leq t(R_i)$ or $t(R_i) \leq x < y$, $x P_i y$.

Let $\mathcal{SP}$ be the set of single-peaked preferences on $[0,1]$. Observe that, given a single-peaked preference $R_i \in \mathcal{SP}$, $y P_i x$ may hold even if $|t(R_i) - x| < |t(R_i) - y|$; but then, x and y are necessarily located in different sides of the top $t(R_i)$. Often, the linear order structure of the set of alternatives and a distance conveys to the

[12] In Section 3.5.2 we will define the notion of efficiency relative to a subset of alternatives $A \subseteq r_f$ by replacing the above condition "there is no $z \in [0,1]$" by "there is no $z \in A$".

preference a symmetric property around the top (coming for instance, from a location interpretation of the set of alternatives) that naturally induces the restriction that preferences respond to the distance as follows.

Definition 3.2.4. A preference $R_i \in \mathcal{R}$ is *symmetric single-peaked* if:
(1) there exists the top $t(R_i)$ of R_i and
(2) for all $x, y \in [0,1]$, xP_iy if and only if $|t(R_i) - x| < |t(R_i) - y|$.

Obviously, a symmetric single-peaked preference is single-peaked. Let $\mathcal{SSP}$ be the set of symmetric single-peaked preferences on $[0,1]$. Given any alternative $x \in [0,1]$, there is a *unique* symmetric single-peaked preference R_i with its top $t(R_i) = x$ ($\mathcal{SSP}$ is a rich domain). Hence, there is a one-to-one mapping between the set of symmetric single-peaked preferences $\mathcal{SSP}$ and the set of alternatives $[0,1]$. Thus, we will use $t_i \in [0,1]$ to identify the (unique) $R_i \in \mathcal{SSP}$ such that $t(R_i) = t_i$ and $t = (t_1, ..., t_n)$ to denote the corresponding symmetric single-peaked preference profile $R = (R_1, ..., R_n)$ such that $t(R_i) = t_i$ for all $i \in N$. Note that, by this one-to-one identification, any SCF $f : \mathcal{SSP}^n \to [0,1]$ is tops-only. Thus, we will also denote a SCF $f : \mathcal{SSP}^n \to [0,1]$ by its corresponding voting scheme $f : [0,1]^n \to [0,1]$. Following Berga and Serizawa (2000) a subset $\mathcal{S} \subseteq \mathcal{SP}$ is a *minimally rich domain* if it is rich and for any pair of alternatives $x, y \in [0,1]$, $x \neq y$, there exists $R_i \in \mathcal{S}$ such that xP_iy and $t(R_i) \in (\min\{x,y\}, \max\{x,y\})$. Observe that $\mathcal{SSP}^n$ is a minimally rich domain.

3.3 Previous results and main intuition

3.3.1 Previous results

Moulin (1980) characterizes the family of strategy-proof and tops-only SCFs on the domain of single-peaked preferences as well as its anonymous subfamily.[13] The two characterizations are useful to develop helpful intuitions to understand our characterization of strategy-proof SCFs (and its anonymous subfamily) on the domain of symmetric single-peaked preferences. To state them, we need to define the median of an odd set of numbers and the notion of a monotonic family of fixed ballots. Given a set of odd real numbers $\{x_1, ..., x_K\}$, define its *median* as $med\{x_1, ..., x_K\} = y$, where y is such that $\#\{1 \leq k \leq K \mid x_k \leq y\} \geq \frac{K}{2}$ and $\#\{1 \leq k \leq K \mid x_k \geq y\} \geq \frac{K}{2}$; observe that since K is odd the median belongs to the set $\{x_1, ..., x_K\}$ and it is unique.

[13] Moulin (1980) also characterizes the subfamily of strategy-proof, tops-only, anonymous and efficient SCFs on the domain of single-peaked preferences. See Corollary 3.5.3 in Section 3.5 for the characterization of the same class of SCFs on the domain of symmetric single-peaked preferences.

A collection $\{p_S\}_{S\in 2^N}$ is a *monotonic family of fixed ballots* if $p_S \in [0,1]$ for all $S \in 2^N$ and $T \subset Q$ implies $p_Q \leq p_T$.

Proposition 3.3.1 (Moulin, 1980). *A SCF $f : \mathcal{SP}^n \to [0,1]$ is strategy-proof, tops-only and anonymous if and only if there exist $n+1$ fixed ballots $0 \leq p_n \leq ... \leq p_0 \leq 0$ such that for all $R \in \mathcal{SP}^n$,*

$$f(R) = med\{t(R_1), ..., t(R_n), p_n, ..., p_0\}.$$

Proposition 3.3.2 (Moulin, 1980). *A SCF $f : \mathcal{SP}^n \to [0,1]$ is strategy-proof and tops-only if and only if there exists a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$ such that for all $R \in \mathcal{SP}^n$,*

$$f(R) = \min_{S\in 2^N} \max_{i\in S}\{t(R_i), p_S\}.$$

The SCFs identified in Propositions 3.3.1 and 3.3.2 are called *median voter schemes* and *generalized median voter schemes*, respectively. A simple way of interpreting them is as follows. Each generalized median voting scheme (and its associated monotonic family of fixed ballots) can be understood as a particular way of distributing the power among coalitions to influence the social choice. To see that, take an arbitrary coalition S and its fixed ballot p_S. Then, coalition S can make sure that, by all of its members reporting a top alternative below p_S, the social choice will be at most p_S, independently of the reported top alternatives of the members of the complementary coalition.[14] An alternative way of describing this distribution of power among coalitions is as follows. Fix a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$ (*i.e.*, a generalized median voter scheme) and take a vector of tops $(t(R_1), ..., t(R_n))$. Start at the left extreme of the interval and push the outcome to the right until it reaches an alternative x for which the following two things happen simultaneously: (i) there exists a coalition of agents S such that all its members have reported a top alternative below or equal to x (*i.e.*, $t(R_i) \leq x$ for all $i \in S$) and (ii) the fixed ballot p_S associated to S is located also below x (*i.e.*, $p_S \leq x$). Median voter schemes are the anonymous subclass of generalized median voter schemes. Hence, the fixed ballots of any two coalitions with the same cardinality of any anonymous generalized median voter scheme are equal. From a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$ associated to an anonymous generalized median voter scheme f we can identify the $n+1$ ballots $p_n \leq p_{n-1} \leq ... \leq p_0$ needed to describe f as a median voter scheme as follows: for each $0 \leq s \leq n$, $p_s = p_S$ for all $S \in 2^N$ such that $\#S = s$. Moreover, if n is odd the (ordinary) median voter is obtained by choosing $p_n = ... = p_{\frac{n+1}{2}} = 0$ and

[14]See Barberà et al. (1997) for a similar interpretation for the case of a finite number of ordered alternatives.

$p_{\frac{n+1}{2}-1} = ... = p_0 = 1$ since for any $R \in \mathcal{SP}^n$,

$$\begin{aligned} med\{t(R_1), ..., t(R_n), p_n, ..., p_0\} &= med\{t(R_1), ..., t(R_n), \underbrace{0, ..., 0}_{\frac{n+1}{2}-\text{times}}, \underbrace{1, ..., 1}_{\frac{n+1}{2}-\text{times}}\} \\ &= med\{t(R_1), ..., t(R_n)\}. \end{aligned}$$

Finally, the SCF f where agent $j \in N$ is the dictator (*i.e.*, for all $R \in \mathcal{SP}^n$, $f(R) = t(R_j)$) can be described as a generalized median voter scheme by setting $p_T = 0$ for all $T \subset N$ such that $j \in T$ and $p_S = 1$ for all $S \subset N$ such that $j \notin S$. Then, for any $R \in \mathcal{SP}^n$, $\max\{t(R_j), p_{\{j\}}\} = t(R_j)$, for any $T \subset N$ such that $j \in T$, $t(R_j) \leq \max_{i \in T}\{t(R_i), p_T\}$ and for any $S \subset N$ such that $j \notin S$, $\max_{i \in S}\{t(R_i), p_S\} = 1$. Thus, $\min_{S' \in 2^N} \max_{i' \in S'}\{t(R_{i'}), p_{S'}\} = t(R_j)$.

Moulin (1980) also shows that the class of group strategy-proof and tops-only SCFs on the domain of single-peaked preferences coincides with the set of generalized median voter schemes. From the main result in Barberà et al. (2010) we can conclude that any strategy-proof SCF on the domain of symmetric single-peaked preferences is group strategy-proof as well. Since we will later use this fact we state it here as a remark. [15]

Remark 3.3.3 (Barberà, Berga and Moreno, 2010). *Let $f : \mathcal{SSP}^n \to [0,1]$ be a strategy-proof SCF. Then, $f : \mathcal{SSP}^n \to [0,1]$ is group strategy-proof.*

To see that in the statements of Propositions 3.3.1 and 3.3.2 tops-onlyness does not follow from strategy-proofness consider the SCF $f : \mathcal{SP}^n \to [0,1]$ where for all $R \in \mathcal{SP}^n$,

$$f(R) = \begin{cases} 0 & \text{if } \#\{i \in N \mid 0R_i1\} \geq \#\{i \in N \mid 1P_i0\} \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

Notice that f is strategy-proof and anonymous but it is not tops-only. It also violates efficiency, unanimity and ontoness. In the last section of the paper we will describe how our characterization includes this class of SCFs on the domain of symmetric single-peaked preferences.

3.3.2 Main intuition and definitions

[15] Barberà et al. (2010) give sufficient conditions defining domains of preferences under which strategy-proofness is equivalent to group strategy-proofness. The domain of symmetric single-peaked preferences satisfies these sufficient conditions.

Consider Propositions 3.3.1 and 3.3.2 for the simplest case where $n = 1$.[16] Figure 3.1 depicts the voting scheme $f : [0,1] \to [0,1]$ of a strategy-proof and tops-only SCF $f : \mathcal{SP} \to [0,1]$ with the two associated fixed ballots $0 < p_1 < p_0 < 1$. Observe that for any pair of fixed ballots $0 \leq p_1 \leq p_0 \leq 1$ the corresponding voting scheme $f : [0,1] \longrightarrow [0,1]$ is always increasing (*i.e.*, $0 \leq x < y \leq 1$ implies $f(x) \leq f(y)$), continuous and $r_f = [p_1, p_0]$. For any $n \geq 1$, a voting scheme $f : [0,1]^n \longrightarrow [0,1]$ is *increasing* if $f(t) \leq f(t')$ for all $t, t' \in [0,1]^n$ such that $t_i \leq t'_i$ for all $i \in N$.

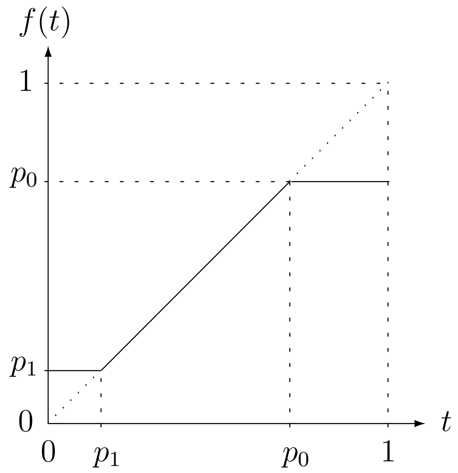

Figure 3.1: Voting scheme of a strategy-proof and tops-only SCF on $\mathcal{SP}$, for $n = 1$ and two fixed ballots $0 < p_1 < p_0 < 1$.

More generally, let $\mathcal{S}$ be a subset of $\mathcal{SP}$. A SCF $f : \mathcal{S}^n \to [0,1]$ is *increasing* if $f(R) \leq f(R')$ for all $R, R' \in \mathcal{S}^n$ such that $t(R_i) \leq t(R'_i)$ for all $i \in N$. By Proposition 3.3.2 the following remark holds.

Remark 3.3.4. *Let $f : \mathcal{SP}^n \longrightarrow [0,1]$ be a strategy-proof and tops-only SCF. Then, its corresponding voting scheme $f : [0,1]^n \longrightarrow [0,1]$ is increasing and continuous.*

Lemma 3.3.5 below states that, for any $n \geq 1$, any strategy-proof SCF is increasing on the domain of symmetric single-peaked preferences (observe that tops-only is not required explicitly since for each $x \in [0,1]$ there exists a unique $R_i \in \mathcal{SSP}$ such that $t(R_i) = x$).

Lemma 3.3.5. *Let $f : \mathcal{SSP}^n \to [0,1]$ be a strategy-proof SCF. Then, f is increasing.*

Proof The statement follows from the iterated application of Claim A.

[16] When $n = 1$ anonymity is vacuous. Indeed, we can uniquely identify the two fixed ballots of the propositions as $p_1 = p_{\{1\}}$ and $p_0 = p_\emptyset$.

Claim A *Let $f : \mathcal{SSP}^n \to [0,1]$ be a strategy-proof SCF. Let $t, t' \in \mathcal{SSP}^n$ be such that for some $i \in N$, $t_i < t'_i$ and $t_{-i} = t'_{-i}$. Then, $f(t) \leq f(t')$.*

Proof of Claim A Assume otherwise; that is, there exist $t, t' \in \mathcal{SSP}^n$ and $i \in N$ such that

$$t_i < t'_i, \tag{3.2}$$

$t_{-i} = t'_{-i}$ and $f(t') < f(t)$. We distinguish among six possible cases. The first three cases (i) $f(t') < f(t) \leq t_i < t'_i$, (ii) $t_i \leq f(t') < f(t) \leq t'_i$ and (iii) $f(t') < t_i \leq f(t) \leq t'_i$ contradict strategy-proofness of f since in all three cases i manipulates f at t' via t_i. The two cases (iv) $t_i < t'_i \leq f(t') < f(t)$ and (v) $t_i \leq f(t') \leq t'_i \leq f(t)$ contradict strategy-proofness of f since in both cases i manipulates f at t via t'_i. The remaining case is (vi) $f(t') \leq t_i < t'_i \leq f(t)$. Since $t_i, t'_i \in \mathcal{SSP}$ and f is strategy-proof,

$$\begin{aligned} f(t) - t_i &\leq t_i - f(t') \\ t'_i - f(t') &\leq f(t) - t'_i. \end{aligned}$$

Adding up,

$$\begin{aligned} f(t) - t_i + t'_i - f(t') &\leq t_i - f(t') + f(t) - t'_i \\ t'_i - t_i &\leq t_i - t'_i \\ t'_i &\leq t_i, \end{aligned}$$

a contradiction with (3.2). □

We have shown that the monotonicity of strategy-proof SCFs is preserved when we restrict the domain of single-peaked preferences to be symmetric. However, continuity (of its corresponding voting scheme) does not follow from strategy-proofness and tops-onlyness in this smaller domain. Indeed, a special class of discontinuities may arise. It is very easy to understand why when $n = 1$. First, take any $\tau, \delta \in (0,1)$ such that $\delta \leq \min\{\tau, 1-\tau\}$ and define the SCFs $f^- : \mathcal{SSP} \longrightarrow [0,1]$ and $f^+ : \mathcal{SSP} \longrightarrow [0,1]$ where for each $t_i \in \mathcal{SSP}$,

$$f^-(t_i) = \begin{cases} \tau - \delta & \text{if } t_i \leq \tau \\ \tau + \delta & \text{if } \tau < t_i \end{cases}$$

and

$$f^+(t_i) = \begin{cases} \tau - \delta & \text{if } t_i < \tau \\ \tau + \delta & \text{if } \tau \leq t_i. \end{cases}$$

In Figure 3.2 we depict f^-. Both f^- and f^+ are strategy-proof on the domain of symmetric single-peaked preferences. At any $t_i \in \mathcal{SSP}$ such that either $t_i > \tau$ or $t_i < \tau$ agent i can not manipulate them. Let $t_i \in \mathcal{SSP}$ be such that $t_i = \tau$. Then, $(\tau - \delta)I_i(\tau + \delta)$ since $(\tau - \delta)$ and $(\tau + \delta)$ are at the same distance δ to τ. The function $f^- : [0,1] \longrightarrow [0,1]$ is left-continuous while the function $f^+ : [0,1] \longrightarrow [0,1]$ is right-continuous.[17] Observe that neither f^- nor f^+ are strategy-proof on the domain of single-peaked preferences since, for instance, for $\tau = 1/2$, $\delta = 1/4$ and any $R_i \in \mathcal{SP}$ such that $t(R_i) = 3/8$ and $3/4P_i1/4$ agent i manipulates f^- and f^+ at R_i via any R_i' such that $t(R_i') = 7/8$ since $f^-(R_i') = f^+(R_i') = 3/4P_i1/4 = f^+(R_i) = f^-(R_i)$.

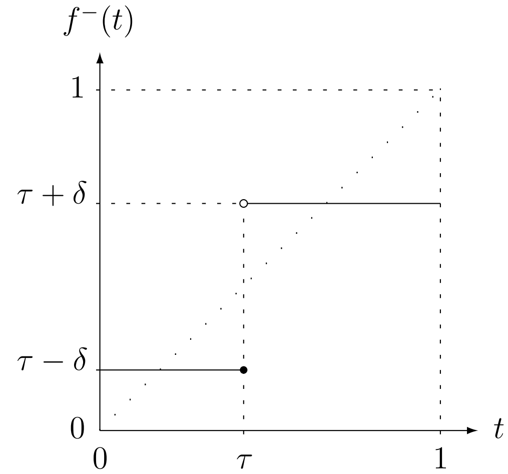

Figure 3.2: A discontinuity that preserves strategy-proofness on $\mathcal{SSP}$: f^-.

More generally, a strategy-proof SCF $f : \mathcal{SSP} \longrightarrow [0,1]$ could have a countable number of discontinuities as long as the midpoint of each discontinuity jump is the discontinuity point itself; namely, for the point $d \in [0,1]$ where f is discontinuous at d,

$$d = \frac{\lim_{x \to d^-} f(x) + \lim_{x \to d^+} f(x)}{2}$$

must hold, otherwise, f is not strategy-proof. Thus, discontinuity jumps have to be symmetric around the discontinuity point.

As we will show in Section 3.4, the class of strategy-proof SCFs on the domain of symmetric single-peaked preferences is the class of generalized median voter schemes identified by Moulin (1980) plus the SCFs obtained after perturbing each generalized

[17]A function $g : [0,1] \longrightarrow [0,1]$ is *left-continuous* (respectively, *right-continuous*) if for all $x \in [0,1]$ and for any sequence $\{x_m\}_{m\in\mathbb{N}}$ such that $x_m \leq x$ (respectively, $x_m \geq x$) for all $m \in \mathbb{N}$, $\{x_m\}_{m\in\mathbb{N}} \longrightarrow x$ implies $\{g(x_m)\}_{m\in\mathbb{N}} \longrightarrow g(x)$.

median voter scheme by admitting these very particular kind of discontinuities. We will call them *disturbed minmax*. Formally,

Definition 3.3.6. Let $\{p_S\}_{S\in 2^N}$ be a monotonic family of fixed ballots. A collection of open intervals $I = \{I_m\}_{m\in M}$, where M is an indexation set, is a *family of discontinuity jumps compatible with* $\{p_S\}_{S\in 2^N}$ if:

(1) M is countable,

(2) for all $m \in M$, $I_m = (a_m, b_m) \subset [p_N, p_\emptyset]$,

(3) for all $m, m' \in M$ such that $m \neq m'$, $I_m \cap I_{m'} = \emptyset$,

(4) for all $S \in 2^N$, $p_S \notin \bigcup\limits_{m\in M} I_m$.

Given a family of discontinuity jumps $I = \{I_m\}_{m\in M}$ we denote the *midpoint* of each open interval $I_m = (a_m, b_m)$ by $d_m = \frac{a_m+b_m}{2}$ and we preliminary perturb the identity function as follows.

Definition 3.3.7. Given a family of discontinuity jumps $I = \{I_m\}_{m\in M}$, the corresponding *perturbation* function $\Pi^I : [0,1] \to [0,1]$ is defined as follows: for each $x \in [0,1]$,

$$\Pi^I(x) = \begin{cases} x & \text{if } x \notin \bigcup\limits_{m\in M} I_m \\ a_m & \text{if } x \in (a_m, d_m] \\ b_m & \text{if } x \in (d_m, b_m). \end{cases} \tag{3.3}$$

Let I be a family of discontinuity jumps compatible with the monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$. A possible perturbation of the generalized median voter scheme associated to $\{p_S\}_{S\in 2^N}$ that preserves its strategy-proofness in the symmetric single-peaked domain is as follows: for each $t = (t_1, ..., t_n) \in \mathcal{SSP}^n$,

$$f(t_1, ..., t_n) = \Pi^I(\min_{S\in 2^N} \max_{i\in S}\{t_i, p_S\}).$$

We will show that these perturbed functions (of generalized median voter schemes) are the basis to characterize the class of all strategy-proof SCFs on the domain of symmetric single-peaked preferences.

Figure 3.3 illustrates the perturbation for the case $n = 1$, $M = \{m\}$ and $I = \{I_m = (a_m, b_m)\}$; *i.e.*, $f(t) = \Pi^I(med\{t, p_1, p_0\})$.

Notice that Π^I arbitrarily assigns the value a_m to the point d_m. If instead $\Pi^I(d_m) = b_m$, the perturbed median voter scheme would still be strategy-proof. When $n = 1$, there are just two ways of perturbing the generalized median voter scheme at each discontinuity jump while preserving its strategy-proofness. When $n > 1$ the process of assigning values to the discontinuity points in a way that maintains strategy-proofness is more complex.

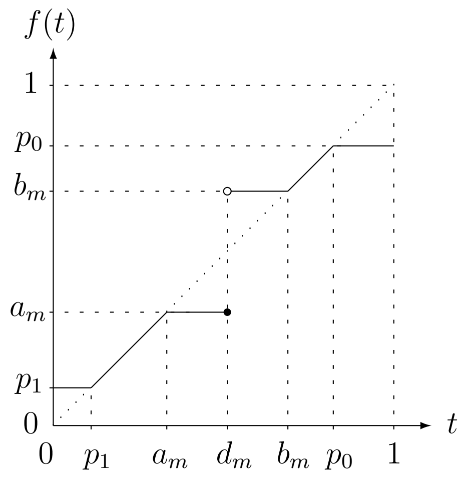

Figure 3.3: A strategy-proof SCF for $n = 1$ and $M = \{m\}$ and $I_m = (a_m, b_m)$.

Figure 3.4 illustrates the perturbation of an anonymous SCF for the case $n = 2$, $M = \{m\}$, $I = \{I_m\}$ and $0 < p_2 < a_m < d_m < b_m < p_1 < p_0 < 1$; *i.e.*, $f(t_1, t_2) = \Pi^I(med\{t_1, t_2, p_2, p_1, p_0\})$. The tops of the two agents are measured on the axes and in bold-italic is represented the value of the SCF in each region. The bold line indicates the discontinuity points of the SCF.

It is easy to see that if Π^I had assigned the value b_m, instead of a_m, to d_m the perturbation of the generalized median voter scheme would still have remained strategy-proof on the domain of symmetric single-peaked preferences. But now there are more ways of assigning values to the discontinuity points that preserve the strategy-proofness of f. For the particular case depicted in Figure 3.4, the SCF would have remained strategy-proof and anonymous if it had assigned the value a_m to the points in the set $B_1 = \{(t_1, t_2) \in [0, 1]^2 \mid 0 \leq t_1 < d_m \text{ and } t_2 = d_m\}$, as well as to the points in the set $B_2 = \{(t_1, t_2) \in [0, 1]^2 \mid t_1 = d_m \text{ and } 0 \leq t_2 < d_m\}$, whereas it had assigned b_m to the point (d_m, d_m) . Actually, if anonymity was not required then it could also have assigned the value a_m to the points in B_1 and b_m to the rest of points in $B_2 \cup \{(d_m, d_m)\}$. However assigning the value a_m to the point (d_m, d_m) and b_m to the rest of points in $B_1 \cup B_2$ would violate strategy-proofness because at any profile (t_1, d_m) with $0 < t_1 < d_m$ agent 1 could manipulate the SCF via $t_1' = d_m$.

Intuitively, the perturbation of the generalized median voter scheme should preserve the increasing monotonicity of the SCF; otherwise, some agent could manipulate it at some profile. We next formalize all these possibilities.

Consider a generalized median voter scheme with its associated monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$. Let $I = \{I_m\}_{m \in M}$ be a family of discontinuity jumps

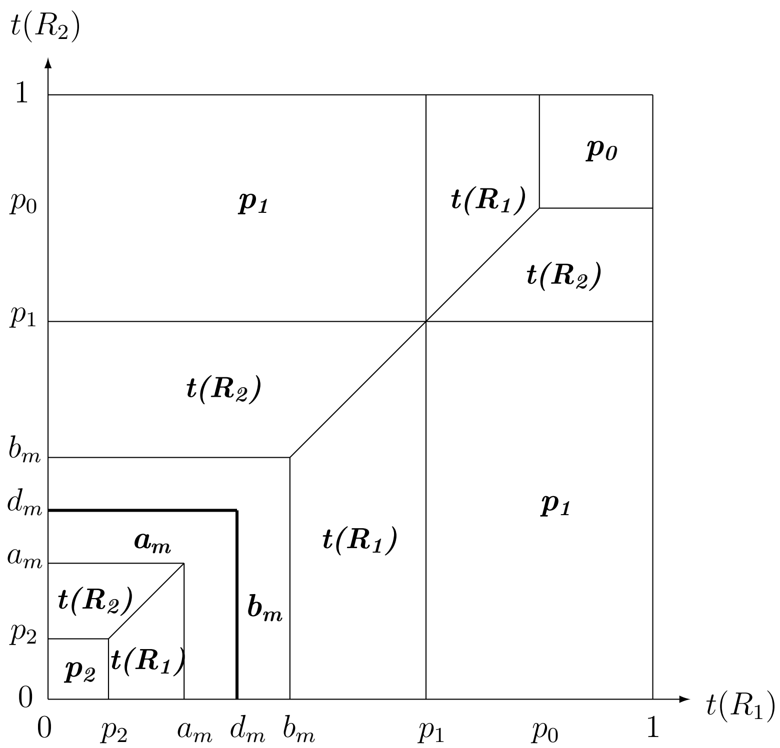

Figure 3.4: An anonymous and strategy-proof SCF for $n = 2$, $M = \{m\}$, $I = \{I_m\}$ and $0 < p_2 < a_m < d_m < b_m < p_1 < p_0 < 1$.

compatible with $\{p_S\}_{S\in 2^N}$ and assume $M \neq \varnothing$. Fix $m \in M$ and define

$$D_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^n \mid \min_{S\in 2^N} \max_{i\in S}\{t_i, p_S\} = d_m\};$$

namely, D_m is the set of symmetric single-peaked preference profiles at which the generalized median voter scheme will select d_m and thus the corresponding perturbation function Π^I will generate a discontinuity point. We refer to any set D_m as a *discontinuity set*. We want to determine the shape of the discontinuity sets because, in order to maintain strategy-proofness, we must preserve the increasing monotonicity of the function. To do that we need to track the agents with tops strictly below, equal and strictly above d_m.

Note that, since no fixed ballot belongs to any discontinuity jump, if $t \in D_m$ then there is at least one agent $i \in N$ such that $t_i = d_m$.

For each $t \in D_m$ define the vector of *extreme votes* $ev^m(t) = (ev_1^m(t), ..., ev_n^m(t)) \in$

$\{0, d_m, 1\}^n$, where for each $i \in N$,

$$ev_i^m(t) = \begin{cases} 0 & \text{if } 0 \leq t_i < d_m \\ d_m & \text{if } t_i = d_m \\ 1 & \text{if } d_m < t_i \leq 1. \end{cases}$$

The vector $ev^m(t)$ describes at the profile t the location of the top of each agent relative to d_m (0 if it is strictly below, 1 if it is strictly above and d_m if it is exactly located at d_m). Let $EV(D_m)$ denote the set $\{ev^m(t) \mid t \in D_m\}$. Namely, the set $EV(D_m)$ describes all the extreme votes at which d_m is chosen by the generalized median voter scheme associated to the monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$. Notice that since $\min_{S \in 2^N} \max_{i \in S}\{t_i, p_S\} = d_m$, if we reallocate the tops below d_m to 0 and the tops above d_m to 1, the minmax is not affected. Therefore, $\min_{S \in 2^N} \max_{i \in S}\{t_i, p_S\} = d_m = \min_{S \in 2^N} \max_{i \in S}\{ev_i^m(t), p_S\}$.

We now turn to describe how strategy-proof SCFs on the symmetric single-peaked domain may choose between a_m and b_m at those profiles that induce a discontinuity at $d_m = \frac{a_m + b_m}{2}$. Define the preorder $\preceq$ on $\mathbb{R}^n$ as follows: for all $x, x' \in \mathbb{R}^n$,

$$x \preceq x' \Leftrightarrow x_i \leq x'_i \text{ for all } i \in \{1, ..., N\}$$

and, given $m \in M$, denote the restriction of $\preceq$ on the set $EV(D_m)$ by $\preceq_m$. Observe that the natural preorder $\preceq$ on $\mathbb{R}^n$ induces an incomplete, reflexive and transitive binary relation $\preceq_m$ on $EV(D_m)$ with the property that $\widehat{ev}^m \preceq_m ev^m$ if and only if ev^m represents a shift to the right of some of the extreme votes of $\widehat{ev}^m$. Thus, $\succeq_m$ can be read as the relation "to be more rightist than".

Let Y_m be a non-empty subset of $EV(D_m)$. Denote by $X_m = U(Y_m)$ the upper contour set of Y_m (according to $\preceq_m$) as

$$X_m = U(Y_m) = \{ev^m \in EV(D_m) \mid \widehat{ev}^m \preceq_m ev^m \text{ for some } \widehat{ev}^m \in Y_m\}.$$

By convention, set $U(\varnothing) = \varnothing$. Now, given $X_m \subseteq EV(D_m)$ with the property that $X_m = U(X_m)$, define $g^{X_m} : D_m \longrightarrow \{a_m, b_m\}$ as follows: for every $t \in D_m$,

$$g^{X_m}(t) = \begin{cases} b_m & \text{if } ev^m(t) \in X_m \\ a_m & \text{otherwise.} \end{cases}$$

The functions g^{X_m} cover all different ways of assigning values a_m and b_m to the preference profiles that generate a discontinuity point at d_m preserving the monotonicity of the perturbation. For each particular $m \in M$ there are many such functions because there are many subsets $X_m \subseteq EV(D_m)$ with the property that $X_m = U(X_m)$. Given a family of discontinuity jumps $I = \{I_m\}_{m\in M}$ we say that $\{X_m\}_{m\in M}$ is a *family of tie-breaking sets of* M if for all $m \in M$, $X_m \subseteq EV(D_m)$ and $X_m = U(X_m)$.

3.4 Characterization

We are now ready to define disturbed minimax SCFs and state and prove that they constitute the class of all strategy-proof SCFs on the domain of symmetric single-peaked preferences.

3.4.1 Definition and statement

Definition 3.4.1. A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is a *disturbed minmax* if there exist:
(1) a monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$;
(2) a family of discontinuity jumps $I = \{I_m\}_{m\in M}$ compatible with $\{p_S\}_{S\in 2^N}$; and
(3) a family of tie-breaking sets $\{X_m\}_{m\in M}$ of M
such that, for all $t = (t_1, ..., t_n) \in \mathcal{SSP}^n$,

$$f(t) = \begin{cases} \Pi^I(\min\limits_{S\in 2^N} \max\limits_{i\in S}\{t_i, p_S\}) & \text{if } \min\limits_{S\in 2^N} \max\limits_{i\in S}\{t_i, p_S\} \neq d_m \text{ for all } m \in M \\ g^{X_m}(t_1, ..., t_n) & \text{if } \min\limits_{S\in 2^N} \max\limits_{i\in S}\{t_i, p_S\} = d_m \text{ for some } m \in M. \end{cases} \tag{3.4}$$

Theorem 3.4.2. *A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is strategy-proof if and only if it is a disturbed minmax.*

Before moving to the proof of Theorem 3.4.2 consider again the SCF f defined in (3.1) but restricted to the domain of symmetric single-peaked preferences, where for all $R \in \mathcal{SSP}^n$,

$$f(R) = \begin{cases} 0 & \text{if } \#\{i \in N \mid 0R_i1\} \geq \#\{i \in N \mid 1P_i0\} \\ 1 & \text{otherwise.} \end{cases}$$

Observe that for any $R_i \in \mathcal{SSP}$, $0R_i1$ if and only if $t(R_i) \leq \frac{1}{2}$. It is easy to see that in the domain of single-peaked preferences f is strategy-proof and anonymous but it is not tops-only. Hence, while it is excluded in Moulin (1980)'s characterization under the domain of single-peaked preferences stated above as Proposition 3.3.2, it has the following representation as a disturbed minmax under the domain of symmetric single-peaked preferences. Its family of monotonic fixed ballots is

$$p_S = \begin{cases} 0 & \text{if } \#S \geq \lceil \frac{n}{2} \rceil \\ 1 & \text{if } \#S < \lceil \frac{n}{2} \rceil, \end{cases}$$

where $\lceil \frac{n}{2} \rceil$ is the smallest integer larger or equal to $\frac{n}{2}$. The family I of discontinuity jumps compatible with the monotonic family of fixed ballots contains only one discontinuity interval $I_m = (a_m, b_m) = (0,1)$ with $d_m = \frac{1}{2}$ and the tie-breaking set of $M = \{m\}$ is $X_m = \{ev \in \{0, \frac{1}{2}, 1\}^n \mid \#\{i \in N \mid ev_i \in \{0, \frac{1}{2}\}\} < \lceil \frac{n}{2} \rceil\}$.

3.4.2 Preliminaries of the proof of Theorem 3.4.2

We start with some additional notation. Given $x \in [0,1]$, $S \subseteq N$ with $s = \#S$ and $t \in \mathcal{SSP}^n$, define $x^S \equiv (\underbrace{x, ..., x}_{s-\text{times}})$ and $t_S \equiv (t_j)_{j \in S}$. Thus, $(x^S, t_{-S}) \equiv (y_1, ..., y_n)$, where $y_j = x$ if $j \in S$ and $y_j = t_j$ if $j \notin S$. Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a SCF and $S \subseteq N$. Define the SCF $\Delta_f^S : [0,1] \times \mathcal{SSP}^{n-s} \longrightarrow [0,1]$ as follows. For all $(x, t_{-S}) \in [0,1] \times \mathcal{SSP}^{n-s}$,

$$\Delta_f^S(x, t_{-S}) = f(x^S, t_{-S}).$$

We will denote the diagonal function associated to f by $\Delta_f \equiv \Delta_f^N$.

Given $t \in [0,1]^n$ and $x \in [0,1]$, define the subset of profiles of tops $\mathcal{C}_{t,x}$ as:

$$\begin{aligned} \mathcal{C}_{t,x} = \quad & \{t' \in \mathcal{SSP}^n \mid x \leq t'_i \leq t_i \text{ for all } i \text{ such that } x \leq t_i \text{ and} \\ & \qquad t_i \leq t'_i \leq x \text{ for all } i \text{ such that } t_i \leq x\}; \end{aligned}$$

namely, $\mathcal{C}_{t,x}$ is the set of profiles t' with the property that the top t'_i of each agent i lies between t_i and x. Given a SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$, a subset $\mathcal{T} \subseteq \mathcal{SSP}^n$ and $x \in [0,1]$ the notation $f\mid_{\mathcal{T}} \equiv x$ means that for all $t \in \mathcal{T}$, $f(t) = x$.

As a consequence of Remark 3.3.3 and Lemma 3.3.5 the following statements hold.

Remark 3.4.3. *Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a strategy-proof SCF. Then,*
(R1) f is unanimous on its range r_f; namely, $x \in r_f$ implies $f(x^N) = x$;
(R2) for all $S \subseteq N$, $\Delta_f^S : [0,1] \times \mathcal{SSP}^{n-s} \longrightarrow [0,1]$ is strategy-proof; and
(R3) if $t \in \mathcal{SSP}^n$ is such that $f(t) = x$ then, $f|_{\mathcal{C}_{t,x}} \equiv x$.

The two first statements follow from group strategy-proofness (Remark 3.3.3) and the last one from monotonicity (Lemma 3.3.5) and (R1).

We now state and prove the following three lemmata that will be useful in the proof of Theorem 3.4.2. Lemma 3.4.4 says that the range of a strategy-proof SCF and the range of its associated diagonal function coincide and it is a closed subset of $[0,1]$ (see also Zhou (1991)).

Lemma 3.4.4. *Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a strategy-proof SCF. Then, $r_f = r_{\Delta_f}$. Moreover, r_f is closed.*

Proof By definition of Δ_f, $r_{\Delta_f} \subseteq r_f$. Take $x \in r_f$. Then, by (R1), $f(x^N) = x$. Thus, $x \in r_{\Delta_f}$. Let $\{x_k\} \to x$ be such that $x_k \in r_f$ for all $k \geq 1$ and assume $x \notin r_f$. Define $y = f(x^N) \neq x$ and let x_k be such that $|x_k - x| < |y - x|$. By (R1), $f(x_k^N) = x_k$. Thus, N manipulates f at x via x_k. □

Lemmata 3.4.5 and 3.4.6 roughly say that if a strategy-proof SCF is constant and equal to x on one variable over some interval containing this constant x, but it is not constant over the whole interval $[0,1]$, then there is a discontinuity at some point z and the discontinuity leaves indifferent the agent with top at z (see Figures 3.2 and 3.3). In the proof of Theorem 3.4.2, z will correspond to the midpoint d_m of a discontinuity jump $I_m = (a_m, b_m)$, where $a_m = x$ and $b_m = 2z - x$.

Lemma 3.4.5. *Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a strategy-proof SCF with the property that there are $i \in N$, $x \in [a,b) \subset [0,1]$ and $t_{-i} \in \mathcal{SSP}^{n-1}$ such that*
(1) $f(t_i, t_{-i}) = x$ for all $t_i \in [a,b)$ and
(2) $f(1, t_{-i}) = y > x$.
Then, there exists $z \in [b, \frac{x+y}{2}]$ such that $f(\cdot, t_{-i})$ is discontinuous at z and

$$f|_{[a,z)\times\{t_{-i}\}} \equiv x$$
$$f|_{(z,2z-x]\times\{t_{-i}\}} \equiv 2z - x.$$

Proof Let $i \in N$, $x \in [a,b)$ and $t_{-i} \in \mathcal{SSP}^{n-1}$ be such that conditions *(1)* and *(2)* hold for f. First note that the interval $[b, \frac{x+y}{2}]$ is not empty since $b \leq \frac{x+y}{2}$: If $b > \frac{x+y}{2}$ then b would be closer to y than to x and for a small enough $\epsilon > 0$, i would manipulate f at $(b - \epsilon, t_{-i})$ via $t_i' = 1$.

Define $z = \sup\{t_i \in [0,1] \mid f(t_i, t_{-i}) = x\}$. Obviously $z \geq b > x$ and, by the monotonicity of f, $\lim_{t_i \to z^-} f(t_i, t_{-i}) = x$ and $f|_{[a,z)\times\{t_{-i}\}} \equiv x$. We now prove that $\lim_{t_i \to z^+} f(t_i, t_{-i}) = 2z - x$. Suppose that $\lim_{t_i \to z^+} f(t_i, t_{-i}) < 2z - x$. Then, there exists $\epsilon > 0$ such that $f(z + \epsilon, t_{-i}) + 2\epsilon < 2z - x$ and $f(z - \epsilon, t_{-i}) = x$. Either $f(z + \epsilon, t_{-i}) > z - \epsilon$, in which case $0 < f(z + \epsilon, t_{-i}) - (z - \epsilon) < (z - \epsilon) - x = (z - \epsilon) - f(z - \epsilon, t_{-i})$ and hence, i would manipulate f at $(z - \epsilon, t_{-i})$ via $t_i' = z + \epsilon$. Or $f(z + \epsilon, t_{-i}) < z - \epsilon$ and therefore $f(z - \epsilon, t_{-i}) = x < f(z + \epsilon, t_{-i}) < z - \epsilon$ and i would manipulate f at $(z - \epsilon, t_{-i})$ via $t_i' = z + \epsilon$. Similarly, if $\lim_{t_i \to z^+} f(t_i, t_{-i}) > 2z - x$, there exists $\epsilon > 0$ such that $f(z + \epsilon, t_{-i}) - 2\epsilon > 2z - x$ and $f(z - \epsilon, t_{-i}) = x$. But then $f(z + \epsilon, t_{-i}) - (z + \epsilon) > (z + \epsilon) - x = (z + \epsilon) - f(z - \epsilon, t_{-i}) > 0$ and hence, i would manipulate f at $(z + \epsilon, t_{-i})$ via $z - \epsilon$. Thus, $\lim_{t_i \to z^+} f(t_i, t_{-i}) = 2z - x$ and $f(\cdot, t_{-i})$ is discontinuous at z. Now by (R3), $f|_{(z,2z-x]\times\{t_{-i}\}} \equiv 2z - x$. Finally, by monotonicity of f, $2z - x \leq y$ and hence, $z \in [b, \frac{x+y}{2}]$. □

Lemma 3.4.6. *Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a strategy-proof SCF with the property that there are $i \in N$, $x \in (a,b] \subset [0,1]$ and $t_{-i} \in \mathcal{SSP}^{n-1}$ such that*
(1) $f(t_i, t_{-i}) = x$ for all $t_i \in (a,b]$ and
(2) $f(0, t_{-i}) = y < x$.
Then, there exists $z \in [\frac{x+y}{2}, a]$ such that $f(\cdot, t_{-i})$ is discontinuous at z and

$$f|_{(z,b]\times\{t_{-i}\}} \equiv x$$
$$f|_{[2z-x,z)\times\{t_{-i}\}} \equiv 2z - x.$$

Proof Omitted since it is symmetric to the proof of Lemma 3.4.5. □

3.4.3 Proof of Theorem 3.4.2

It is easy to check that any disturbed minmax SCF is strategy-proof on the symmetric single-peaked domain. To see this notice that if f is a disturbed minmax SCF, for all $t \in \mathcal{SSP}^n$,

$$|f(t) - \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}| = \min\{|x - \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}| \mid x \in r_f\}. \tag{3.5}$$

Fix a profile $t \in \mathcal{SSP}^n$ and an agent $i \in N$. If $t_i = \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}$, then by (3.5) i cannot benefit from reporting a different preference. Suppose that $t_i < \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}$ (the case $t_i > \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}$ is symmetric). The only way i can affect the value of the SCF is by reporting a preference $t_i' > \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}$. Since disturbed minmax SCFs are increasing, $f(t_i', t_{-i}) \geq f(t)$. We distinguish between two cases:
Case 1: $f(t) \geq t_i$. Then $|f(t_i', t_{-i}) - t_i| = f(t_i', t_{-i}) - t_i \geq f(t) - t_i = |f(t) - t_i|$ and the deviation is not profitable.
Case 2: $f(t) < t_i < \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\}$. By the definition of the disturbed minmax, it must be that $f(t) = a_m$ for some $m \in M$ and $a_m < t_i < \min_{S\in 2^N} \max_{j\in S}\{t_j, p_S\} \leq d_m$. Hence, either $f(t_i', t_{-i}) = a_m = f(t)$, in which case the deviation is not profitable, or $f(t_i', t_{-i}) \geq b_m$ and $|f(t_i', t_{-i}) - t_i| = f(t_i', t_{-i}) - t_i \geq b_m - t_i > \frac{b_m - a_m}{2} \geq |f(t) - t_i|$ and again the deviation is not profitable.

Thus, any disturbed minmax SCF is strategy-proof.

Let $f : \mathcal{SSP}^n \longrightarrow [0,1]$ be a strategy-proof SCF. To show that f is a disturbed minmax we first have to identify its associated monotonic family of fixed ballots $\{p_S\}_{S\in 2^N}$, family $I = \{I_m\}_{m\in M}$ of discontinuity jumps compatible with $\{p_S\}_{S\in 2^N}$ and family of tie-breaking sets $\{X_m\}_{m\in M}$ of M. Then, we will show that f coincides with the disturbed minmax SCF obtained by (3.4) in Definition 3.4.1, applied to all of them.

For each $S \in 2^N$, define its associated fixed ballot by setting

$$p_S \equiv f(0^S, 1^{N \backslash S}); \tag{3.6}$$

i.e., p_S is the image of f at the profile where all agents in S have their top at 0 and all agents not in S have their top at 1.

Consider the diagonal function $\Delta_f : \mathcal{SSP} \longrightarrow [0,1]$ associated to f. By (R2) Δ_f is strategy-proof. Thus, by Lemma 3.3.5, Δ_f is increasing and hence it has at most a countable number of discontinuities.[18] Denote by $\{d_m\}_{m \in M}$ the discontinuity points of Δ_f, where M is a countable set. For each $m \in M$, define $a_m = \lim_{x \to d_m^-} \Delta_f(x)$ and $b_m = \lim_{x \to d_m^+} \Delta_f(x)$. Since Δ_f is discontinuous at d_m and increasing on $[0,1]$, a_m and b_m exist and $a_m < b_m$. By Lemma 3.4.4, r_{Δ_f} is closed and therefore $a_m, b_m \in r_{\Delta_f}$ and by (R1), $\Delta_f(a_m) = a_m$ and $\Delta_f(b_m) = b_m$. Moreover, since Δ_f is strategy-proof, d_m must be the midpoint of $I_m \equiv (a_m, b_m)$. Otherwise, if $d_m < \frac{a_m+b_m}{2}$, there would exist an $\epsilon > 0$ such that $d_m < \frac{a_m+b_m}{2} - \epsilon$ and $\Delta_f(\frac{a_m+b_m}{2} - \epsilon) \geq b_m$, which would imply that Δ_f is manipulable at $\frac{a_m+b_m}{2} - \epsilon$ via $t' = a_m$. Similarly, if $d_m > \frac{a_m+b_m}{2}$, there would exist an $\epsilon > 0$ such that $d_m > \frac{a_m+b_m}{2} + \epsilon$ and $\Delta_f(\frac{a_m+b_m}{2} + \epsilon) \leq a_m$, which would imply that Δ_f is manipulable at $\frac{a_m+b_m}{2} + \epsilon$ via $t' = b_m$.

Notice that the family of discontinuity jumps $I = \{I_m\}_{m \in M}$ is compatible with $\{p_S\}_{S \in 2^N}$ since:

(1) M is countable.

(2) By the monotonicity of Δ_f, $a_m = \Delta_f(a_m) \geq \Delta_f(0) = p_N$ and $b_m = \Delta_f(b_m) \leq \Delta_f(1) = p_\emptyset$ and therefore $I_m = (a_m, b_m) \subset [p_N, p_\emptyset]$.

(3) By the monotonicity of Δ_f and the definition of a_m and b_m, $I_m \cap I_{m'} = \emptyset$ for any $m, m' \in M$, $m' \neq m$.

(4) Finally, by (3.6) and Lemma 3.4.4, for each $S \in 2^N$, $p_S \in r_f = r_{\Delta_f}$, $r_f \cap (a_m, b_m) = r_{\Delta_f} \cap (a_m, b_m) = \emptyset$. Thus, for all $S \in 2^N$, $p_S \notin \bigcup\limits_{m \in M} I_m$.

In fact,

$$r_f = r_{\Delta_f} = [p_N, p_\varnothing] \backslash \{ \bigcup_{m \in M} I_m \}. \tag{3.7}$$

[18] Any real-valued monotone function of a real variable has at most a countable number of discontinuities. This result is due to Froda (1929) although in the literature it is widely used without Froda's name being mentioned.

If M is empty (*i.e.*, Δ_f is continuos and its range is equal to $[p_N, p_\varnothing]$), the statement of Theorem 3.4.2 follows because f is a generalized median voter scheme defined on the minimally rich domain $\mathcal{SSP}^n$ (see Theorem 1 in Berga and Serizawa (2000)).[19]

Assume M is non-empty and fix $m \in M$. To identify the element X_m in the family of tie-breaking sets of M, consider the previously defined discontinuity set

$$D_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^n \mid \min_{S \in 2^N} \max_{i \in S}\{t_i\} = d_m\},$$

the set of profiles of extreme votes that induce d_m through the minmax

$$EV(D_m) = \{ev^m(t) \mid t \in D_m\},$$

and its associated preorder $\preceq_m$. Then, define

$$X_m = \{ev^m \in EV(D_m) \mid f(ev^m) > d_m\}. \tag{3.8}$$

By Lemma 3.3.5, f is increasing and therefore X_m coincides with its upper contour set relative to $\preceq_m$; *i.e.*, $X_m = U(X_m)$.

So far we have identified from f the monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$, the family $I = \{I_m\}_{m \in M}$ of discontinuity jumps compatible with $\{p_S\}_{S \in 2^N}$ (we are now assuming that $M \neq \varnothing$) and the family $\{X_m\}_{m \in M}$ of tie-breaking sets of M (and hence, its corresponding family of tie-breaking functions $\{g^{X_m} : D_m \longrightarrow \{a_m, b_m\}\}_{m \in M}$). Given all of them, let F be the SCF defined by condition (3.4) in Definition 3.4.1. We want to show that $f = F$.

Let $t = (t_1, ..., t_n) \in \mathcal{SSP}^n$ be arbitrary. To show that $f(t) = F(t)$ define $q = \min_{T \in 2^N} \max_{i \in T}\{t_i, p_T\}$. We distinguish among four different cases relating q, t and $f(t)$.

Case 1: $q \notin \{t_1, ..., t_n\}$.

Consider $S = \{i \in N \mid t_i < q\}$. Then $p_S = q$. To see that observe that if $p_S < q$ then $\max_{i \in S}\{t_i, p_S\} < q$ contradicting the definition of q. Further, since $q = \min_{T \in 2^N} \max_{i \in T}\{t_i, p_T\} \notin \{t_1, ..., t_n\}$, there exists $\bar{T} \in 2^N$, such that $p_{\bar{T}} = q$ and $t_j < p_{\bar{T}}$ for all $j \in \bar{T}$. But then, $\bar{T} \subseteq S$ and, by the monotonicity of $p = \{p_T\}_{T \in 2^N}$, $p_S \leq p_{\bar{T}}$. Therefore, by the definition of q, $p_S = p_{\bar{T}} = q$.

[19]Observe that all results in Berga and Serizawa (2000) refer only to onto SCFs. Hence, to be more precise with the application of their result, notice that the restriction of $\mathcal{SSP}$ on the interval $[p_N, p_\varnothing]$ is a symmetric single-peaked domain (on $[p_N, p_\varnothing]$) and it is a minimally rich domain (on $[p_N, p_\varnothing]$). Denote it by $\mathcal{SSP} \mid_{[p_N, p_\varnothing]}$. Thus, we can identify the notation of Berga and Serizawa (2000) for the image set $Z = [\alpha, \beta]$ with our identified interval $[p_N, p_\varnothing]$ and apply their Theorem 1 to the SCF $f^* : (\mathcal{SSP} \mid_{[p_N, p_\varnothing]})^n \longrightarrow [p_N, p_\varnothing]$. Finally, observe that their generalized median voter schemes (defined through a left-coalition system) satisfy voter sovereignty and hence, $r_{f^*} = [p_N, p_\varnothing]$.

By the definition of S and the assumption that $q \notin \{t_1, ..., t_n\}$, $t_j > p_S$ for all $j \notin S$. Then, $t \in \mathcal{C}_{(0^S,1^{N\setminus S}),p_S}$ and, by (R3) and the definition of p_S, $f|_{\mathcal{C}_{(0^S,1^{N\setminus S}),p_S}} \equiv p_S$. Therefore, $f(t) = p_S$.

Moreover, by (3.7), $p_S \notin \cup_{m\in M} I_m$. Hence, by (3.4) in Definition 3.4.1 and the definition of Π^I in (3.3), $F(t) = \Pi^I(\min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\}) = \min_{T\in 2^N} \max_{i\in T}\{t_i, p_T\} = p_S$. Thus, $f(t) = F(t)$.

CASE 2: $q = t_i$ for some $i \in N$ and $f(t) = t_i$.

If $t_i = f(t)$, then $t_i \in r_f$ and therefore, by (3.7), $t_i \notin \cup_{m\in M} I_m$. By (3.4) in Definition 3.4.1 and the definition of Π^I in (3.3), $F(t) = \Pi^I(\min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}) = \Pi^I(t_i) = t_i$. Thus, $f(t) = F(t)$.

CASE 3: $q = t_i$ for some $i \in N$, $f(t) \equiv x \neq t_i$ and $t_i \notin \cup_{m\in M}\{d_m\}$.

To show that in this case $f(t) = F(t)$ we proceed in two steps. First we prove that $f(t) = f(t_i^N)$ and then we prove that $f(t_i^N) = F(t)$.

Step 1: $f(t) = f(t_i^N)$.

Define $S_i^< = \{j \in N \mid t_j < t_i\}$, $S_i^= = \{j \in N \mid t_j = t_i\}$ and $S_i^> = \{j \in N \mid t_j > t_i\}$. We will denote $S_i^\leq = S_i^< \cup S_i^=$ and $S_i^\geq = S_i^> \cup S_i^=$.

Because $t_i = \min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}$, it must be that $t_i \in [p_{S_i^\leq}, p_{S_i^<}]$. Otherwise, suppose first that $t_i < p_{S_i^\leq}$ and consider $T \in 2^N$. If $T \subset S_i^\leq$, we have that $t_i < p_{S_i^\leq} \leq p_T$ and therefore $t_i < \max_{j\in T}\{t_j, p_T\}$. If $T \cap S_i^> \neq \emptyset$, then by the definition of $S_i^>$, $\max_{j\in T}\{t_j, p_T\} > t_i$. Hence, we have a contradiction with $t_i = \min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}$. Similarly, if $p_{S_i^<} < t_i$, then $\max_{j\in S_i^<}\{t_j, p_{S_i^<}\} < t_i$ again contradicting $t_i = \min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}$.

We now show that $f(t_i^N) \in [p_{S_i^\leq}, p_{S_i^<}]$. If $f(t_i^N) < p_{S_i^\leq} \leq t_i$, then N manipulates f at t_i^N via $(0^{S_i^\leq}, 1^{S_i^>})$ since $f(0^{S_i^\leq}, 1^{S_i^>}) = p_{S_i^\leq}$ and if $t_i \leq p_{S_i^<} < f(t_i^N)$, then N manipulates f at t_i^N via $(0^{S_i^<}, 1^{S_i^\geq})$ since $f(0^{S_i^<}, 1^{S_i^\geq}) = p_{S_i^<}$.

We prove that $f(t) = f(t_i^N)$ by contradiction. Suppose $f(t_i^N) \neq f(t) = x$. Then, either $x < f(t_i^N) \leq p_{S_i^<}$ or $p_{S_i^\leq} \leq f(t_i^N) < x$. The two cases are symmetric and therefore we omit the proof for the second case (which uses Lemma 3.4.6 instead of Lemma 3.4.5).

Suppose $x < f(t_i^N) \leq p_{S_i^<}$. The condition $x < f(t_i^N)$ implies $x < t_i$ since we are assuming that $x \neq t_i$ holds and if $x > t_i$, then N would manipulate f at t_i^N via t . By (R3), the definition of $S_i^\geq$ and $f(t) = x$,

$$\Delta_f^{S_i^\geq}(\tau, t_{S_i^<}) = x \text{ for all } \tau \in [x, t_i].$$

On the other hand, since $t_j < t_i \leq p_{S_i^<}$ for all $j \in S_i^<$, $(\tau^{S_i^{\geq}}, t_{S_i^<}) \in \mathcal{C}_{(0^{S_i^<}, 1^{S_i^{\geq}}), p_{S_i^<}}$ for all $\tau \in [p_{S_i^<}, 1]$ and therefore by (3.6) and (R3),

$$\Delta_f^{S_i^{\geq}}(\tau, t_{S_i^<}) = p_{S_i^<} \text{ for all } \tau \in [p_{S_i^<}, 1].$$

By Lemma 3.4.5, applied to the strategy-proof SCF $\Delta_f^{S_i^{\geq}} : [0,1] \times [0,1]^{S_i^<} \longrightarrow [0,1]$, where $[a,b) = [x, t_i)$ and $y = p_{S_i^<}$, there exists $z \in [t_i, \frac{x + p_{S_i^<}}{2}]$ such that $\Delta_f^{S_i^{\geq}}(\cdot, t_{S_i^<})$ is discontinuous at z and

$$\Delta_f^{S_i^{\geq}}|_{[x,z) \times \{t_{S_i^<}\}} \equiv x \text{ and } \Delta_f^{S_i^{\geq}}|_{(z, 2z-x] \times \{t_{S_i^<}\}} \equiv 2z - x.$$

Applying (R3) again, if $\tau \in (z, 2z - x]$ and $t'_j \in [t_j, 2z - x]$[20] for all $j \in S_i^<$, then

$$\Delta_f^{S_i^{\geq}}(\tau, t'_{S_i^<}) = 2z - x. \tag{3.9}$$

Note that z is a discontinuity point of Δ_f as well. To see that observe that by (3.9), $f(w^N) = 2z - x$ for all $w \in (z, 2z - x]$. On the other hand, $f(t) = x$ and hence, $x \in r_f$ and by (R1), $f(x^N) = x$. Assume that there exists $\hat{w} \in (x, z)$ such that $f(\hat{w}^N) \neq x$. By monotonicity of f, $x < f(\hat{w}^N) \leq 2z - x$. Then, either $f(\hat{w}^N) = 2z - x$ and N manipulates f at $\hat{w}^N$ via x^N, or $f(\hat{w}^N) < 2z - x$ and for any $0 < \epsilon < z - \hat{w}$, N manipulates f at $(z + \epsilon)^N$ via $\hat{w}^N$. Thus, $f(\hat{w}^N) = x$. Therefore, Δ_f has the property that

$$\Delta_f(w) = \begin{cases} x & \text{if } w \in [x, z) \\ 2z - x & \text{if } w \in (z, 2z - x]. \end{cases}$$

This means that Δ_f is discontinuous at z and hence there exists $m \in M$ such that $d_m = z$. Since under Case 3, t_i is not a discontinuity point of Δ_f, $t_i \neq z$ and therefore, by the definition of z, $t_i < z$.

By monotonicity of f and (3.9), $f(t_i^N) \leq \Delta_f^{S_i^{\geq}}(z + \epsilon, t_i^{S_i^<}) = 2z - x$ for all sufficiently small $\epsilon > 0$ (in the next paragraph we will find an upper bound for such ϵ's). We want to show that the inequality is strict; *i.e.*, $f(t_i^N) < 2z - x$ holds. Suppose $f(t_i^N) = 2z - x$; then, since $t_i < z$ can be re-written as $t_i - x < 2z - x - t_i$, this means that N would manipulate f at t_i^N via t which contradicts strategy-proofness of f.

To sum up, we have shown that if $x < f(t_i^N) \leq p_{S_i^<}$, then $f(t_i^N) < 2z - x$ and $\lim_{\tau \to z^+} \Delta_f^{S_i^{\geq}}(\tau, t_i^{S_i^<}) = 2z - x$. But then it is easy to see that for a small $\epsilon > 0$, $S_i^{\geq}$ manipulates f at $((z + \epsilon)^{S_i^{\geq}}, t_i^{S_i^<})$ via $t_i^{S_i^{\geq}}$. Namely, if $0 < \epsilon < \frac{f(t_i^N) - x}{2}$, then

[20] Notice that if $j \in S_i^<$ then $t_j < t_i \leq 2z - x$ and therefore the interval is well defined.

$-(2z - x - (z+\epsilon)) < f(t_i^N) - (z+\epsilon) < 2z - x - (z+\epsilon)$ where the first inequality is equivalent to the assumption $\epsilon < \frac{f(t_i^N)-x}{2}$ and the second inequality follows from $f(t_i^N) < 2z - x$. Therefore,

$$\left|f(t_i^N) - (z+\epsilon)\right| < 2z - x - (z+\epsilon),$$

which means that $S_i^{\geq}$ manipulates f at $((z+\epsilon)^{S_i^{\geq}}, t_i^{S_i^{<}})$ via $t_i^{S_i^{\geq}}$; a contradiction. This concludes the proof of *Step 1.*

Step 2: $f(t_i^N) = F(t)$.

By strategy-proofness of f, Δ_f is strategy-proof and since $\Delta_f(t_i) \equiv f(t_i^N) \neq t_i$, by (R1), $t_i \notin r_{\Delta_f}$. By (3.7), there exists $m \in M$ such that $t_i \in (a_m, b_m)$. By (R1), $\Delta_f(a_m) = a_m$ and $\Delta_f(b_m) = b_m$. Since Δ_f is strategy-proof,

$$x = \Delta_f(t_i) = \begin{cases} a_m & \text{if } a_m < t_i < d_m \\ b_m & \text{if } d_m < t_i < b_m, \end{cases} \tag{3.10}$$

which coincides with the value of $F(t) = \Pi^I(\min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\} = \Pi^I(t_i) = x$. Thus, $f(t_i^N) \equiv \Delta_f(t_i) = F(t)$. This concludes the proof of *Step 2.*

Putting together *Step 1* and *Step 2*, we have shown that $f(t) = F(t)$.

CASE 4: $q = t_i$ for some $i \in N$, $f(t) \equiv x \neq t_i$ and $t_i = d_m$ for some $m \in M$.

Denote by $I_m = (a_m, b_m)$ the discontinuity jump corresponding to d_m. Denote $S_m^{=} = \{j \in N \mid t_j = d_m\}$, $S_m^{<} = \{j \in N \mid t_j < d_m\}$ and $S_m^{>} = \{j \in N \mid t_j > d_m\}$ and let ϵ be such that $0 < \epsilon < \min_{j\in S_m^{<},\, k\in S_m^{>}}\{d_m - a_m, d_m - t_j, t_k - d_m\}$. Given this $\epsilon > 0$, consider the two profiles of tops $t^{\epsilon-} = (t_{S_m^{<}}, (d_m - \epsilon)^{S_m^{=}}, t_{S_m^{>}})$ and $t^{\epsilon+} = (t_{S_m^{<}}, (d_m + \epsilon)^{S_m^{=}}, t_{S_m^{>}})$. By construction of $t^{\epsilon-}$ and $t^{\epsilon+}$, the fact that $t_i = \min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\}$ and since $p_T \notin I_m$ for all $T \in 2^N$, $\min_{T\in 2^N} \max_{j\in T}\{t_j^{\epsilon-}, p_T\} = d_m - \epsilon$ and $\min_{T\in 2^N} \max_{j\in T}\{t_j^{\epsilon+}, p_T\} = d_m + \epsilon$. Both $d_m - \epsilon$ and $d_m + \epsilon$ belong to I_m and therefore they do not belong to r_f. Moreover, since $I_m \cap I_{m'} = \emptyset$, neither $d_m - \epsilon$ nor $d_m + \epsilon$ are discontinuity points of Δ_f. We are therefore under the assumptions of CASE 3 and by *Step1*:

$$f(t^{\epsilon-}) = \Delta_f(d_m - \epsilon) = a_m$$
$$f(t^{\epsilon+}) = \Delta_f(d_m + \epsilon) = b_m,$$

where the second equality in both statements follow from the strategy-proofness of Δ_f. By monotonicity, $f(t^{\epsilon-}) \leq f(t) \leq f(t^{\epsilon+})$, which together with (3.7) implies that $f(t) \in \{a_m, b_m\}$. Thus, we have shown that if t is such that $\min_{T\in 2^N} \max_{j\in T}\{t_j, p_T\} =$

$t_i = d_m$ for some $m \in M$ then,

$$f(t) \in \{a_m, b_m\}. \tag{3.11}$$

To show that $f(t) = F(t)$, assume first that t is such that $ev^m(t) \notin X_m$. By definition of F, $F(t) = a_m$. Since $ev^m(t) \notin X_m$, by (3.8), $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) \le d_m$ which means, by (3.7), that $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) \le a_m$. Moreover, $t' = (0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>})$ is such that $\min_{T \in 2^N} \max_{j \in T}\{t'_j, p_T\} = d_m$ and, by (3.11), $f(0^{S_m^<}, d_m^{S_m^=}, 1^{S_m^>}) = a_m$. By (R3),

$$f(0^{S_m^<}, d_m^{S_m^=}, t_{S_m^>}) = a_m. \tag{3.12}$$

If $S_m^< = \varnothing$, then $(0^{S_m^<}, d_m^{S_m^=}, t_{S_m^>}) = t$ and $f(t) = a_m$. If $S_m^< \neq \varnothing$ then $f(t) = a_m$ or otherwise $S_m^<$ manipulates f at t via $0^{S_m^<}$. Thus, we have shown that $f(t) = a_m = F(t)$. Symmetrically, we can show that if t is such that $ev^m(t) \in X_m$ then $f(t) = F(t) = b_m$.

This finishes the proof of Theorem 3.4.2.

3.5 Final remarks

As direct consequences of Theorem 3.4.2, Corollaries 1, 2 and 3 below characterize three relevant subclasses of strategy-proof SCFs on the domain of symmetric single-peaked preferences.

3.5.1 Anonymity and efficiency

Corollaries 3.5.2 and 3.5.3 characterize two nested subclasses: the class of strategy-proof and anonymous SCFs (Corollary 3.5.2) and the class of strategy-proof, anonymous and efficient SCFs (Corollary 3.5.3).

To state Corollary 3.5.2 we first need to translate the definitions of extreme votes and tie-breaking sets of M to the anonymous case. Consider the family of $n+1$ fixed ballots $0 \le p_n \le ... \le p_1 \le p_0 \le 1$ associated to a median voter scheme and let $m \in M$. The set of profiles at which the median voter scheme will select d_m is

$$\widetilde{D}_m = \{t = (t_1, ..., t_n) \in \mathcal{SSP}^n \mid med\{t_1, ..., t_n, p_n, ..., p_0\} = d_m\}.$$

By anonymity, we only need to track the number of agents with tops strictly below, equal and strictly above d_m. Hence, for each $t = (t_1, ..., t_n) \in \mathcal{SSP}^n$, define the triple $l^m(t) = (l^m_<(t), l^m_=(t), l^m_>(t))$ where:
(1) $l^m_<(t) = \#\{i \in N \mid t_i < d_m\}$,
(2) $l^m_=(t) = \#\{i \in N \mid t_i = d_m\}$ and

(3) $l^m_>(t) = \#\{i \in N \mid t_i > d_m\}$.
Observe that $l^m_<(t) + l^m_=(t) + l^m_>(t) = n$ and since fixed ballots do not belong to any discontinuity jump, if $t \in \widetilde{D}_m$ then, there is $i \in N$ such that $t_i = d_m$ (*i.e.*, $l^m_=(t) \geq 1$). Let $\nabla^n = \{(x, y, z) \in \{0, 1, ..., n\}^3 \mid x + y + z = n \text{ and } y \geq 1\}$ be the set of triples with positive integer components adding up to n and whose middle component is equal or larger than 1 and define $L(\widetilde{D}_m) = \{l^m(t) \in \nabla^n \mid t = (t_1, ..., t_n) \in \widetilde{D}_m\}$; namely, $L(\widetilde{D}_m)$ describes all anonymous distributions of tops (number of tops strictly below d_m, number of tops at d_m, number of tops strictly above d_m) at which the median voter selects d_m. Define the preorder $\widetilde{\preceq}$ on $\{0, 1, ..., n\}^3$ as follows: for all $(x, y, z), (x', y', z') \in \{0, 1, ..., n\}^3$,

$$(x', y', z')\widetilde{\preceq}(x, y, z) \Leftrightarrow z' \leq z \text{ and } x' \geq x.$$

Denote the restriction of the preorder $\widetilde{\preceq}$ on the set $L(\widetilde{D}_m)$ by $\widetilde{\preceq}_m$ and let $\widetilde{Y}_m$ be a non-empty subset of $L(\widetilde{D}_m)$. Denote by $\widetilde{X}_m = U(\widetilde{Y}_m)$ the upper contour set of $\widetilde{Y}_m$ (according to $\widetilde{\preceq}_m$) as the set of triples in $L(\widetilde{D}_m)$ such that they are more rightist than some triple in $\widetilde{Y}_m$; namely,

$$\widetilde{X}_m = U(\widetilde{Y}_m) = \{(l_<, l_=, l_>) \in L(\widetilde{D}_m) \mid (x, y, z)\widetilde{\preceq}_m(l_<, l_=, l_>) \text{ for some } (x, y, z) \in \widetilde{Y}_m\}.$$

By convention, set $U(\varnothing) = \varnothing$. Given $\widetilde{X}_m \subseteq L(\widetilde{D}_m)$ with the property that $\widetilde{X}_m = U(\widetilde{X}_m)$, define $g^{\widetilde{X}_m} : \widetilde{D}_m \longrightarrow \{a_m, b_m\}$ as follows: for every $t \in \widetilde{D}_m$,

$$g^{\widetilde{X}_m}(t) = \begin{cases} b_m & \text{if } l^m(t) \in \widetilde{X}_m \\ a_m & \text{otherwise.} \end{cases}$$

Given a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ we say that $\{\widetilde{X}_m\}_{m \in M}$ is an *anonymous family of tie-breaking sets of* M if for all $m \in M$, $\widetilde{X}_m \subseteq L(\widetilde{D}_m)$ and $\widetilde{X}_m = U(\widetilde{X}_m)$.

Definition 3.5.1. A SCF $f : \mathcal{SSP}^n \longrightarrow [0, 1]$ is a *disturbed median* if there exist:
(1) a family of $n + 1$ fixed ballots $0 \leq p_n \leq ... \leq p_1 \leq p_0 \leq 1$;
(2) a family of discontinuity jumps $I = \{I_m\}_{m \in M}$ compatible with $p_n, ..., p_1, p_0$; and
(3) an anonymous family of tie-breaking sets $\{\widetilde{X}_m\}_{m \in M}$ of M
such that, for all $t = (t_1, ..., t_n) \in \mathcal{SSP}^n$,

$$f(t) = \begin{cases} \Pi^I(med\{t_1, ..., t_n, p_n, ..., p_0\}) & \text{if } med\{t_1, ..., t_n, p_n, ..., p_0\} \neq d_m \text{ for all } m \in M \\ g^{\widetilde{X}_m}(t_1, ..., t_n) & \text{if } med\{t_1, ..., t_n, p_n, ..., p_0\} = d_m \text{ for an } m \in M. \end{cases}$$

Corollary 3.5.2. *A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is strategy-proof and anonymous if and only if it is a disturbed median.*

Corollary 3.5.3. *A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is strategy-proof, anonymous and efficient if and only if it is a median voter scheme with the property that $p_n = 0$ and $p_0 = 1$.*

Efficiency requires that f respects unanimity and hence, $r_f = [0,1]$. Thus, (i) its associated family of $n+1$ fixed ballots has the property that $0 = p_n \leq p_{n-1} \leq ... \leq p_0 = 1$ and (ii) the family of discontinuity sets M is empty. Observe that since $p_n = 0$ and $p_0 = 1$ they cancel each other out in the computation of the median at any profile t and therefore, the generalized median voter scheme can also be described as the median of the n tops and the $n-1$ fixed ballots $p_{n-1} \leq ... \leq p_1$. This corresponds to Moulin (1980)'s characterization of the class of strategy-proof, anonymous and efficient SCFs on the domain of single-peaked preferences. Thus, the reduction of the domain does not generate in this case new strategy-proof, anonymous and efficient SCFs.

3.5.2 Feasibility constraints

Our result has important implications for the design of strategy-proof SCFs on the domain of symmetric single-peaked preferences under feasibility constraints. Often, some subsets of alternatives (although conceivable) can not be chosen due to feasibility constraints. Then, discontinuities are compulsory rather than pathological because discontinuity jumps on the range of strategy-proof SCFs are necessary. Our result precisely describes their nature and how the strategy-proof SCF may select its value at these discontinuity points. However, if f is a strategy-proof and discontinuous SCF then, $r_f \subsetneq [0,1]$ and hence, f will not be efficient; in particular, f will not respect unanimity. SCFs that are not efficient but they are efficient relative to the feasible set of alternatives are specially interesting. Thus, let $A \subsetneq [0,1]$ be a closed set of feasible alternatives.[21] A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is *efficient relative to* A if $r_f \subseteq A$ and for all $R \in \mathcal{SSP}^n$ there is no $z \in A$ such that, for all $i \in N$, $zR_i f(R)$ and $zP_j f(R)$ for some $j \in N$. The following result follows from Theorem 3.4.2.

Corollary 3.5.4. *Let A be a closed subset of $[0,1]$. A SCF $f : \mathcal{SSP}^n \longrightarrow [0,1]$ is strategy-proof and efficient relative to A if and only if it is a disturbed minmax with $r_f = A$.*

Note that the requirement $r_f = A$ imposes certain conditions on the monotonic family of fixed ballots $\{p_S\}_{S \in 2^N}$ and on the discontinuity jumps. For instance $p_N =$

[21] Remember that, by Lemma 3.4.4, strategy-proof SCFs have a closed range.

$\min\{x \in A\}$, $p_\varnothing = \max\{x \in A\}$ and $p_S \in A$ for all $S \in 2^N$. Moreover since A is closed the set $[p_N, p_\varnothing] \backslash A$ is open and therefore it can be written as a countable and disjoint union of open intervals: $[p_N, p_\varnothing] \backslash A = \cup_{m \in M} I_m$ where I_m is an open interval for all $m \in M$ and $I_m \cap I_{m'} = \varnothing$ for all $m, m' \in M$. This representation is unique up to permutations in M and in fact the requirement $r_f = A$ implies that the family of discontinuity jumps compatible with $\{p_S\}_{S \in 2^N}$ is exactly $I = \{I_m\}_{m \in M}$.

As an illustration of Corollary 3.5.4, suppose that the set of feasible alternatives is $A = \{0\} \cup \{0.1\} \cup [0.2, 0.8] \cup \{0.9\}$. In that case the only general requirements on the fixed ballots are that $p_N = 0$, $p_\varnothing = 0.9$ and p_S has to belong to A for all $S \in 2^N$. The family of discontinuity jumps is given by $I_1 = (0, 0.1)$, $I_2 = (0.1, 0.2)$ and $I_3 = (0.8, 0.9)$ and therefore the discontinuity points are $d_1 = 0.05$, $d_2 = 0.15$ and $d_3 = 0.85$. To proceed with the illustration and in order to design a particular strategy-proof and anonymous SCF f whose range r_f be equal to A let $N = \{1, 2, 3\}$ be the set of agents and let $p_3 = p_2 = 0$ and $p_1 = p_0 = 0.9$ be the family of four fixed ballots. In this particular case the ballots cancel each other and hence, for all $(t_1, t_2, t_3) \in \mathcal{SSP}^3$, $med\{t_1, t_2, t_3, 0, 0, 0.9, 0.9\} = med\{t_1, t_2, t_3\}$. For each discontinuity point d_m the set $L(\widetilde{D}_m)$ consists of four triplets: $L(\widetilde{D}_m) = \{(1, 2, 0), (0, 3, 0), (1, 1, 1), (0, 2, 1)\}$ where for example, the triplet $(1, 2, 0)$ means that one top is strictly below d_m and the remaining two tops are exactly equal to d_m. Note, that in all the four cases the median of the tops coincides with d_m and hence all the profiles of tops that are represented by $L(\widetilde{D}_m)$ result in discontinuity points. Moreover, and since $L(\widetilde{D}_1) = L(\widetilde{D}_2) = L(\widetilde{D}_3)$, $\widetilde{\preceq}_1 = \widetilde{\preceq}_2 = \widetilde{\preceq}_3$ as well. Denote it by $\widetilde{\preceq}'$ and observe that $(1, 2, 0) \widetilde{\preceq}' (1, 1, 1) \widetilde{\preceq}' (0, 2, 1)$, $(1, 2, 0) \widetilde{\preceq}' (0, 3, 0) \widetilde{\preceq}' (0, 2, 1)$ and that $(1, 1, 1)$ and $(0, 3, 0)$ are not comparable by $\widetilde{\preceq}'$. To assign a value to the SCF on these discontinuity points preserving the monotonicity of the SCF f we need to select for each d_m a tie-breaking set $\widetilde{X}_m$ such that $\widetilde{X}_m = U(\widetilde{X}_m)$. Given $L(\widetilde{D}_m)$, there are six different ways of doing so: $\widetilde{X}_m \in \{\emptyset, \{(0, 2, 1)\}, \{(1, 1, 1), (0, 2, 1)\}, \{(0, 3, 0), (0, 2, 1)\}, \{(1, 1, 1,), (0, 3, 0), (0, 2, 1)\}, L(\widetilde{D}_m)\}$. For instance, choose $\widetilde{X}_1 = \{(1, 1, 1), (0, 2, 1)\}$, $\widetilde{X}_2 = \{(0, 2, 1)\}$ and $\widetilde{X}_3 = L(\widetilde{D}_m)$. Thus, the disturbed median f that we may define applying Definition 3.5.1 to the family of four fixed ballots $0 = p_3 = p_2 < p_1 = p_0 = 0.9$, the family of discontinuity jumps $I_1 = (0, 0.1)$, $I_2 = (0.1, 0.2)$ and $I_3 = (0.8, 0.9)$ and the anonymous family of tie-breaking sets $\widetilde{X}_1 = \{(1, 1, 1), (0, 2, 1)\}$, $\widetilde{X}_2 = \{(0, 2, 1)\}$ and $\widetilde{X}_3 = L(\widetilde{D}_3)$ has range equal to A and it is efficient relative to A. The disturbed median f could also be defined as follows. For all $t = (t_1, t_2, t_3) \in \mathcal{SSP}^3$ and after setting $y \equiv med\{t_1, t_2, t_3\}$,

$$f(t) = \begin{cases} 0 & \text{if } y < 0.05 \text{ or } y = 0.05 \text{ and } \#\{i \mid t_i \leq 0.05\} = 3 \\ 0.1 & \text{if } y = 0.05 \text{ and } \#\{i \mid t_i \leq 0.05\} < 3 \text{ or } 0.05 < y < 0.15 \\ & \quad \text{or } y = 0.15 \text{ and either } \exists j \text{ s.t. } t_j < 0.15 \text{ or } t_1 = t_2 = t_3 = 0.15 \\ 0.2 & \text{if } y = 0.15 \text{ and } \#\{i \mid t_i \geq 0.15\} = 3 \text{ and } \exists j \text{ s.t. } t_j > 0.15 \\ & \quad \text{or } 0.15 < y < 0.2 \\ y & \text{if } 0.2 \leq y \leq 0.8 \\ 0.8 & \text{if } 0.8 < y < 0.85 \\ 0.9 & \text{if } y \geq 0.85. \end{cases}$$

The complexity of this description indicates the usefulness of Theorem 3.4.2's characterization.

Finally, by Remark 3.3.3, the four statements above (Theorem 3.4.2 and Corollaries 3.5.2, 3.5.3 and 3.5.4) also hold after replacing strategy-proofness by group strategy-proofness.

Appendix A

Appendix to Chapter 1

A.1 Proofs

A.1.1 Proof of Proposition 1.3.2

Proposition 1.3.2 *The number of intervals sent in a partition equilibrium is finite. In particular, there is no separating equilibrium in the private information model.*

Proof of Proposition 1.3.2 The proposition follows as an immediate corollary of Lemma A.1.1 and the fact that $[0, 1]$ is bounded. $\square$

Lemma A.1.1. *If $b > 0$ and $m = [\underline{a}, \overline{a}]$ is a message sent in a partition equilibrium with $\underline{a} > 0$, then $\overline{a} - \underline{a} \geq 2b$.*

Proof of Lemma A.1.1: Suppose by way of contradiction that we could find a partition equilibrium in which message $m = [\underline{a}, \overline{a}]$ with $\underline{a} > 0$ and $\overline{a} - \underline{a} < 2b$ was sent. Then in particular $|\overline{a} - (\underline{a} + b)| < b = (\underline{a} + b) - \underline{a}$ implying that an expert with type $\theta = \underline{a}$ strictly prefers action $y = \overline{a}$ to action $y' = \underline{a}$. By continuity of preferences, there exists $\epsilon > 0$ such that $\underline{a} - \epsilon > 0$ and an expert with type $\theta' = \underline{a} - \epsilon$ strictly prefers $y = \overline{a}$ to $y' = \underline{a}$. Hence, by the concavity of the expert's preferences, all the actions $y(\underline{a}, \overline{a}, s)$, $s \in \mathbb{R}$ are preferred to $y' = \underline{a}$, which implies that type θ' strictly prefers message m to any interval message $m' \subset [0, \underline{a}]$, contradicting the view that m belongs to a partition equilibrium. $\square$

A.1.2 Proof of Theorem 1.3.3

Theorem 1.3.3 *Under Assumptions A1 and A2, if $b > 0$, there exists an integer $N(b, F)$ such that, for every $1 \leq N \leq N(b, F)$:*

1. *there exists a unique monotone partition equilibrium characterized by the partition* $0 = a_0 < a_1 < ... < a_N = 1$ *satisfying*

$$U^E(a_{i-1}, a_i, a_i, b; F) = U^E(a_i, a_{i+1}, a_i, b; F) \qquad (A_F),$$

2. $a_{i+1} - a_i > a_i - a_{i-1}$ *for all* $i = 1, ..., N-1$.

Moreover, both the decision maker and the expert ex-ante prefer equilibrium partitions with more intervals

Before moving to the proof of the theorem, I derive some previous results which will be used in the proof. Lemma A.1.2 establishes some monotonicity properties of the decision maker's best action:

Lemma A.1.2. *Given a message* $m = [\underline{a}, \overline{a}]$, $y(\underline{a}, \overline{a}, s)$ *is increasing in all its arguments and* $\underline{a} \leq y(\underline{a}, \overline{a}, s) \leq \overline{a}$ *for all* $s \in \mathbb{R}$

Proof of Lemma A.1.2: $y(\underline{a}, \overline{a}, s)$ solves the first order condition[1]:

$$\int_{\underline{a}}^{\overline{a}} \tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \theta) f(s - \theta) d\theta = 0 \tag{A.1}$$

Since $\tilde{u}_{11}^D(\cdot) < 0$ and $f(\cdot) \geq 0$, there exists a $\bar{\theta} \in (\underline{a}, \overline{a})$ such that $\tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \bar{\theta}) = 0$ and therefore:

$$\tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \underline{a}) < 0 \quad \textit{and} \quad \tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \overline{a}) > 0 \tag{A.2}$$

Differentiating (A.1) with respect to its first argument and rearranging:

$$y_1(\underline{a}, \overline{a}, s) = \frac{\tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \underline{a}) f(s - \underline{a})}{\int_{\underline{a}}^{\overline{a}} \tilde{u}_{11}^D(y(\underline{a}, \overline{a}, s) - \theta) f(s - \theta) d\theta} > 0$$

where the inequality follows by (A.2) and $\tilde{u}_{11}^D(\cdot) < 0$. Analogously, differentiating (A.1) with respect to its second argument:

$$y_2(\underline{a}, \overline{a}, s) = -\frac{\tilde{u}_1^D(y(\underline{a}, \overline{a}, s) - \overline{a}) f(s - \overline{a})}{\int_{\underline{a}}^{\overline{a}} \tilde{u}_{11}^D(y(\underline{a}, \overline{a}, s) - \theta) f(s - \theta) d\theta} > 0$$

To show that $y(\underline{a}, \overline{a}, s)$ is increasing in s it is sufficient to prove that $U(y, s) = \int_{\underline{a}}^{\overline{a}} \tilde{u}^D(y - \theta) f(s - \theta) d\theta$ is supermodular in (y, s) (see Athey (2002)). Given $y' > y$, $U(y', s) - U(y, s) = \int_{\underline{a}}^{\overline{a}} (\tilde{u}^D(y' - \theta) - \tilde{u}^D(y - \theta)) f(s - \theta) d\theta$ which is increasing in s because $\tilde{u}^D(y' - \theta) - \tilde{u}^D(y - \theta)$ is increasing in θ by $\tilde{u}_{11}^D(\cdot) < 0$, and $f(s - \theta)$ is

[1] Partial derivatives are denoted with subscripts.

ordered in the FOSD (Milgrom, 1981). Therefore $U(y,s)$ is supermodular in (y,s) and $y(\underline{a},\overline{a},s)$ is increasing in s.
Finally, (A.2) and $u^D_{11}(\cdot)<0$ imply that $\underline{a}\le y(\underline{a},\overline{a},s)\le\overline{a}$ for all $s\in\mathbb{R}$. □

Lemma A.1.3 shows that, given the symmetry of the setup, the decision maker's best response is completely determined by the length and the initial point of the interval sent and it is symmetric with respect to the mid-point of the interval which the expert sends.

Lemma A.1.3. *If $\tilde{u}^D(\cdot)$ and $f(\cdot)$ are symmetric:*

1. $Pr(\theta\in[a,a+h)|s)=Pr(\theta\in[0,h)|s-a)$.
2. $Pr(\theta\in[0,h)|\frac{h}{2}-s)=Pr(\theta\in[0,h)|\frac{h}{2}+s))$.
3. $g(\frac{h}{2}-\theta|0,h,\frac{h}{2}-s)=g(\frac{h}{2}+\theta|0,h,\frac{h}{2}+s)$.
4. $y(a,a+h,s)=a+y(0,h,s-a)$.
5. $y(0,h,\frac{h}{2}+s)-\frac{h}{2}=\frac{h}{2}-y(0,h,\frac{h}{2}-s)$. *In particular* $y(0,h,\frac{h}{2})=\frac{h}{2}$.

Proof of Lemma A.1.3: All the results are immediate implications of the symmetry of the functions and a change in variable.
1. $Pr(\theta\in[a,a+h)|s)=\int_a^{a+h}f(s-\theta)d\theta=\int_0^h f(s-a-\theta)d\theta=Pr(\theta\in[0,h)|s-a)$.
2. $Pr(\theta\in[0,h)|\frac{h}{2}-s)=\int_0^h f(\frac{h}{2}-s-\theta)d\theta=\int_0^h f(h-\theta-(\frac{h}{2}+s))=\int_0^h f(\theta-(\frac{h}{2}+s))d\theta=$
$Pr(\theta\in[0,h)|\frac{h}{2}+s))$.
3. $g(\frac{h}{2}-\theta|0,h,\frac{h}{2}-s)=\frac{f(s-\theta)}{Pr(\theta\in[0,h)|\frac{h}{2}-s)}=\frac{f(\theta-s)}{Pr(\theta\in[0,h)|\frac{h}{2}+s)}=g(\frac{h}{2}+\theta|0,h,\frac{h}{2}+s)$.
4. $0=\int_a^{a+h}\tilde{u}^D_1(y(a,a+h,s)-\theta)f(\theta-s)d\theta=\int_0^h\tilde{u}^D_1(y(a,a+h,s)-a-\theta)f(\theta-(s-a))d\theta$ therefore $y(a,a+h,s)-a$ solves $\int_0^h\tilde{u}^D_1(y-\theta)f(\theta-(s-a))d\theta=0$ which implies that $y(0,h,s-a)=y(a,a+h,s)-a$.
5. $0=\int_0^h\tilde{u}^D_1(y(0,h,\frac{h}{2}+s)-\theta)f(\theta-(\frac{h}{2}+s))d\theta=\int_0^h-\tilde{u}^D_1(h-y(0,h,\frac{h}{2})-\theta)f(\frac{h}{2}-s-\theta)d\theta$ and therefore $y(0,h,\frac{h}{2}-s)=h-y(0,h,\frac{h}{2})$.
Finally, using this equation for $s=0$, $y(0,h,\frac{h}{2})=\frac{h}{2}$. □

Lemma A.1.4 shows that, given Assumption A1, $U^E(a_i,a_{i+1},\theta)-U^E(a_{i-1},a_i,\theta)$ is increasing in θ and hence the arbitrage condition (A_F) is sufficient to characterize an equilibrium.

Lemma A.1.4. *Under Assumption A1,* $\frac{\partial}{\partial\theta}\left(U^E(a_i,a_{i+1},\theta)-U^E(a_{i-1},a_i,\theta)\right)>0$

Proof of Lemma A.1.4: The derivative of $U^E(a_i, a_{i+1}, \theta) - U^E(a_{i-1}, a_i, \theta)$ is given by:

$$\begin{aligned}
&\tfrac{\partial}{\partial\theta}\left(U^E(a_i, a_{i+1}, \theta) - U^E(a_{i-1}, a_i, \theta)\right) = \\
&\quad \int_{\mathbb{R}} -\left[\tilde{u}_1^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}_1^E(y(a_i, a_{i-1}, s) - (\theta+b))\right] f(s-\theta)ds \\
&\quad + \int_{\mathbb{R}} -\left[\tilde{u}^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}^E(y(a_i, a_{i-1}, s) - (\theta+b))\right] f'(s-\theta)ds
\end{aligned} \tag{A.3}$$

Consider the first integral in (A.3). By Lemma A.1.2 and Lemma A.1.3.5, $y(a_i, a_{i+1}, s) > \frac{a_i + a_{i+1}}{2}$ for all $s > \frac{a_i+a_{i+1}}{2}$. By Lemma A.1.1 $a_{i+1} - a_i > 2b$ so

$$y(a_i, a_{i+1}, s) - y(a_{i-1}, a_i, s) > b \quad for\ \ all \quad s > \frac{a_i + a_{i+1}}{2} \tag{A.4}$$

By the concavity of $\tilde{u}^E(\cdot)$ there exists a constant $K_{11} > 0$ such that for all $y, y' \in [0, 1+b]$, $-(\tilde{u}_1^E(y) - \tilde{u}_1^E(y')) \geq K_{11}(y - y')$. Moreover,

$$\int_{s > \frac{a_i+a_{i+1}}{2}} f(s-\theta)ds = 1 - F(\frac{a_i + a_{i+1}}{2} - \theta) > 1 - F(\frac{a_{i+1} - a_i}{2}) > 1 - F(1/2) \tag{A.5}$$

Putting everything together:

$$\begin{aligned}
&\int_{\mathbb{R}} -\left[\tilde{u}_1^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}_1^E(y(a_i, a_{i-1}, s) - (\theta+b))\right] f(s-\theta)ds > \\
&\int_{s > \frac{a_i+a_{i+1}}{2}} -\left[\tilde{u}_1^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}_1^E(y(a_i, a_{i-1}, s) - (\theta+b))\right] f(s-\theta)ds > \\
&K_{11}b \int_{s > \frac{a_i+a_{i+1}}{2}} f(s-\theta)ds > \\
&K_{11}b(1 - F(1/2))
\end{aligned} \tag{A.6}$$

where the first inequality follows because the integrand is positive for all s given the concavity of $\tilde{u}^E$. The second inequality follows by (A.4) and the definition of K_{11} and the third inequality follows by (A.5).

Turning to the second integral in (A.3):

$$\tilde{u}^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}^E(y(a_i, a_{i-1}, s) - (\theta+b)) < \tilde{u}^E(0) - \tilde{u}^E(1+b) \quad for\ \ all \quad s \in \mathbb{R}$$

hence:

$$\begin{aligned}
&\int_{\mathbb{R}} -\left[\tilde{u}^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}^E(y(a_i, a_{i-1}, s) - (\theta+b))\right] f'(s-\theta)ds > \\
&-\int_{\mathbb{R}} \left|\tilde{u}^E(y(a_{i-1}, a_i, s) - (\theta+b)) - \tilde{u}^E(y(a_i, a_{i-1}, s) - (\theta+b))\right| |f'(s-\theta)|ds > \\
&-(\tilde{u}^E(0) - \tilde{u}^E(1+b)) \int_{\mathbb{R}} |f'(s-\theta)|ds
\end{aligned} \tag{A.7}$$

Combining (A.6) and (A.7) we have that if $\int |f'(s)|ds < \frac{K_{11}b(1-F(\frac{1}{2}))}{\tilde{u}^E(0) - \tilde{u}^E(1+b)}$, then $\frac{\partial}{\partial\theta}\left(U^E(a_i, a_{i+1}, \theta) - U^E(a_{i-1}, a_i, \theta)\right) > 0$ □

Note that Lemma A.1.4 is sufficient but not necessary to prove that (A_F) is also a sufficient condition for equilibrium. In fact, it would be enough to show that whenever $\{a_{i-1}, a_i, a_{i+1}\}$ satisfy (A_F), then

$$\begin{aligned} U^E(a_i, a_{i+1}, \theta) - U^E(a_{i-1}, a_i, \theta) > 0 \qquad & if \quad \theta \in [a_i, a_{i+1}] \\ U^E(a_i, a_{i+1}, \theta) - U^E(a_{i-1}, a_i, \theta) < 0 \qquad & if \quad \theta \in [a_{i-1}, a_i]. \end{aligned}$$

which is a weaker condition that the monotonicity shown in Lemma A.1.4. Moreover, from the proof of Lemma A.1.4 we can see that the bound in Assumption A1 does not bind. In fact, with the normal and uniform signal structure, (A^F) is always a sufficient condition for the equilibrium, even if Assumption A1 is not satisfied.

Lemma A.1.5 will be used in the proof of Proposition A.1.6 and Proposition 1.4.1. Lemma A.1.5 uses Assumption A2 to derive some properties of the function $V(\cdot)$. Recall that $V(a_{i-1}, a_i, a_{i+1}, b) = U^E(a_i, a_{i+1}, a_i, b) - U^E(a_{i-1}, a_i, a_i, b)$[2].

Lemma A.1.5. *If* $0 \le a_{i-1} < a_i < a_{i+1} \le 1$ *and* $V(a_{i-1}, a_i, a_{i+1}, b) = 0$*, then* $U_1^E(a, a_i, a_i, b) > 0$ *and* $V_1(a, a_i, a_{i+1}, b) < 0$ *for all* $a \in [0, a_i)$*,* $U_2^E(a_i, a, a_i, b) < 0$ *and* $V_3(a_{i-1}, a_i, a, b) < 0$ *for all* $a \in [a_{i+1}, 1]$*, and* $V(a_{i-1}, a_i, a, b) > 0$ *for all* $a \in [a_i, a_{i+1})$*.*

Proof of Lemma A.1.5: By the concavity of $\tilde{u}^E(\cdot)$ and lemma A.1.2,

$$\frac{\partial}{\partial a} U^E(a, a_i, a_i) = \int_{\mathbb{R}} \tilde{u}_1^E(y(a, a_i, s) - (a_i + b)) \frac{\partial y}{\partial a}(a, a_i, s) f(s - a_i) ds > 0 \tag{A.8}$$

so $U^E(a, a_i, a_i)$ is strictly increasing in a for all $a \le a_i$ and hence $V_1(a, a_i, a_{i+1}, b) < 0$ for all $a \in [0, a_i]$. Assumption A2 and the fact that $V(a_{i-1}, a_i, a_i) > 0$ (which follows by $U_1^E(a, a_i, a_i, b) < 0$ for all $a \in [0, a_i]$ and $V(a_{i-1}, a_i, a_{i+1}, b) = 0$) entails that $U_2^E(a_i, a, a_i, b) < 0$ and $V_3(a_{i-1}, a_i, a, b) < 0$ for all $a \in [a_{i+1}, 1]$, and $V(a_{i-1}, a_i, a, b) > 0$ for all $a \in [a_i, a_{i+1})$. □

The following proposition uses Assumption A2 to prove a stronger version of the monotonicity condition (M) in CS which in particular ensures that there is at most one partition of size N satisfying (A_F).

Proposition A.1.6. *If* $\hat{a}$ *and* $\tilde{a}$ *are two partial partitions satisfying* (A_F) *with* $\hat{a}_0 = \tilde{a}_0$ *and* $\hat{a}_1 > \tilde{a}_1$*, then* $\hat{a}_i - \hat{a}_{i-1} > \tilde{a}_i - \tilde{a}_{i-1}$ *for all* $i \ge 1$*.*

Proof of Proposition A.1.6: Denote by $h_{i+1} = a_{i+1} - a_i$ and $h_i = a_i - a_{i-1}$. By Lemma A.1.3 $U^E(a_i, a_{i+1}, a_i, b) = U^E(0, h_{i+1}, 0, b)$ and $U^E(a_{i-1}, a_i, a_i, b) = U^E(0, h_i, h_i, b)$. Hence $V(a_{i-1}, a_i, a_{i+1}, b)$ is a function only of the length of the intervals h_i and h_{i+1}

[2] For clarity of exposition I omit the reference to the signal structure F whenever it does not lead to confusion to do so.

and not of the location of the intervals. Denote this function as $\tilde{V}(h_i, h_{i+1})$. Namely, $\tilde{V}(h_i, h_{i+1}) = U^E(0, h_{i+1}, 0, b) - U^E(0, h_i, h_i, b)$. Given h, define $\phi(h)$ as the positive number, if one exists, which solves $\tilde{V}(h, \phi(h)) = 0$. (If this equation does not have a solution then I will consider that $\phi(h) = +\infty$). By Assumption A2, there is at most one solution to this equation and therefore $\phi(h)$ is a well defined function of h. Proposition A.1.6 is then reduced to prove that $\phi(h)$ is increasing in h, which is immediate from Lemma A.1.5. □

We are now ready to prove Theorem 1.3.3.

Proof of Theorem 1.3.3: The proof of the first statement of the theorem follows closely the proof of Theorem 1 of CS. I start by proving that there exists an integer $N(b, F)$, such that for every N, $1 \leq N \leq N(b, F)$, there exists a partition of size N satisfying the arbitrage condition (A_F).
First, note that, by equation A.8 $U^E(a, a_i, a_i)$ is strictly increasing in a. Denote by $\hat{a}^i$ the strictly decreasing partial partition $\hat{a}_0 > \hat{a}_1 > ... > \hat{a}_i$ which satisfies (A_F). By the monotonicity of $U^E(a, \hat{a}_i, \hat{a}_i)$, there can at most be one value $\hat{a}_{i+1} < \hat{a}_i$ satisfying (A_F).[3]
Define $K(\hat{a}) \equiv \max\{i :$ there exists $0 \leq \hat{a}_i < .... < \hat{a}_2 < \hat{a} < 1$ satisfying $(A_F)\}$. By Lemma A.1.1, $K(\hat{a})$ is finite, well defined and uniformly bounded. Define $N(b, F) = sup_{\hat{a} \in [0,1)} K(\hat{a}) < \infty$. Note that $N(b, F)$ is achieved for certain $\bar{a} \in [0, 1)$ because $K(\hat{a}) \in \mathbb{N}$ and bounded. It remains to be proven that for each $1 \leq N \leq N(b, F)$ there is a partition a satisfying (A_F). Denote $a^{K(a)}$ the decreasing partial partition of size $K(a)$ satisfying (A_F) and such that $a_1^{K(a)} = a$. The partition changes continuously with a and therefore $K(a)$ is locally constant and can at most change by one at a discontinuity. Finally $K(0) = 1$, so $K(a)$ takes on all integer values between one and $N(b, F)$.

Now, I argue that the arbitrage condition (A_F) is also sufficient for the equilibrium. By $u_{11}^E(\cdot) < 0$, $U^E(m_i, \theta)$ is single-peaked in i and condition (A_F) and Lemma A.1.4, that $U^E(m_i, \theta) = \max_j U^E(m_j, \theta)$ for $\theta \in [a_{i-1}, a_i]$.

For the second statement of Theorem 1.3.3, let a be a partition which supports an equilibrium, and let $h_i = a_i - a_{i-1}$ and $h_{i+1} = a_{i+1} - a_i$. Suppose that $h_{i+1} \leq h_i$, then for all $s \in \mathbb{R}$, $y(a_i, a_{i+1}, a_i + s) - a_i \leq y(a_i, a_i + h_i, a_i + s) - a_i = a_i - y(a_i - h_i, a_i, a_i - s) = a_i - y(a_{i-1}, a_i, a_i - s)$, where the inequality follows because $h_{i+1} \leq h_i$ and lemma

[3] In CS the authors use a symmetric argument with strictly increasing partial partitions. The reason that I use decreasing partitions is that, given that $b > 0$ the expected utility of an expert of type t_i when he sends message $m = [t, t_i]$ strictly decreases as t decreases. For increasing partitions, the monotonicity is harder to prove because there are actions on both sides of the expert's peak. (This is the role of Assumption A2, although it is not necessary for this stage.).

A.1.2 and the equality follows by lemma A.1.3. But then if $b > 0$, an expert with type a_i strictly prefers $y(a_i, a_{i+1}, a_i + s)$ to $y(a_{i-1}, a_i, a_i - s)$ for all s, and since $f((a_i+s)-a_i) = f((a_i-s)+a_i)$ by the symmetry of $f(\cdot)$, $U^E(m_{i+1}, a_i) > U^E(m_i, a_i)$, contradicting the equilibrium condition.[4]

Finally, the proof of the third statement of the theorem mimics the proofs of Theorem 3 and 5 of CS using Proposition A.1.6 in the place of condition (M). □

A.1.3 Proof of Proposition 1.4.1

Proposition 1.4.1 *Suppose that F and F' are two signal structures satisfying the following condition:*

(C): If $V(a_{i-1}, a_i, a_{i+1}, b, F) = 0$, then $V(a_{i-1}, a_i, a_{i+1}, b, F') > 0$.

Then there is less communication transmitted in the $F'-PI$ model than in the $F-PI$ model.
Namely, if a and a' are two equilibrium partitions of size N of the $F - PI$ and the $F' - PI$ models respectively, then $a_i > a'_i$ for all $1 \leq i \leq N - 1$. Moreover, $N(b, F) \geq N(b, F')$.

Proof of Proposition 1.4.1: Suppose that for any $0 \leq a_{i-1} \leq a_i \leq a_{i+1} \leq 1$ such that $V(a_{i-1}, a_i, a_{i+1}, b, F) = 0$, we have $V(a_{i-1}, a_i, a_{i+1}, b, F') > 0$[5]. First I prove that if $a(K)$ and $a'(K)$ are two partial partitions of size K satisfying (A_F) and $(A^{F'})$ respectively, with $a_0(K) = a'_0(K)$ and $a_K(K) = a'_K(K)$ then $a_i(K) > a'_i(K)$. The proof is made by induction on the size of the partition K. If $K = 1$ the statement is vacuous. Suppose $K > 1$ and the statement is true for all $K' = 1, .., K-1$. Suppose by way of contradiction that $a_j(K) \leq a'_j(K)$ for some $j = 1, ..., K-1$. Suppose further that j is the highest index for which this inequality is satisfied and hence $a_i(K) > a'_i(K)$ for all $j < i < K$. Define ${}^x a \equiv ({}^x a_0, {}^x a_1, ..., {}^x a_j)$ the partial partition which satisfies (A_F) such that ${}^x a_0 = 0$ and ${}^x a_1 = x$. By definition ${}^{a_1(K)} a_j = a_j(K) \leq a'_j(K)$. By the continuity of ${}^x a$ in x there exists an $\bar{x} \geq a_1(K)$ such that ${}^{\bar{x}} a_j = a'_j(K)$ and by Proposition A.1.6, ${}^{\bar{x}} a_i \geq a_i(K)$ for all $0 < i < j$. Denote by $\bar{a} \equiv {}^{\bar{x}} a$. By Lemma A.1.5, there exists a unique $\bar{a}_{j+1} > \bar{a}_j$ such that $V(\bar{a}_{j-1}, \bar{a}_j, \bar{a}_{j+1}, F) = 0$. By the condition of the Proposition, $V(\bar{a}_{j-1}, \bar{a}_j, \bar{a}_{j+1}, F') > 0$. By Proposition A.1.6 $\bar{a}_{j+1} \geq a_{j+1}(K) > a'_{j+1}(K)$, and hence using the fact that $\bar{a}_j = a'_j(K)$ and Lemma A.1.5:

$$V(\bar{a}_{j-1}, a'_j(K), a'_{j+1}(K), F') > 0 \tag{A.9}$$

[4] This second statement follows directly from Proposition A.1.6, but I prove it here because the proof of Proposition A.1.6 requires Assumption A2, whereas this proof does not.

[5] The opposite case is symmetric.

At the same time, applying the induction hypothesis to $(\bar{a}_0, ..., \bar{a}_j)$ and $(a'_0(K), ..., a'_j(K))$, $a'_i(K) < \bar{a}'_i$ for all $0 < i < j$. But then using Lemma A.1.5,

$$V(\bar{a}_{j-1}, a'_j(K), a'_{j+1}(K), F') < V(a'_{j-1}(K), a'_j(K), a'_{j+1}(K), F') = 0$$

which contradicts (A.9) and establishes the result.

Finally, let $a'(N(b, F'))$ be the partition equilibrium of $F' - PI$ of size $N(b, F')$. Let $\bar{a}$ be the partial partition satisfying (A_F) such that $\bar{a}_1 = a'_1(N(b, F'))$, then by Proposition A.1.6 and the previous result, $\bar{a}_i < a'_i(N(b, F'))$. In particular, $\bar{a}$ is at least of length $N(b, F')$. Hence $N(b, F) \geq N(b, F')$ □

A.1.4 Proof of Proposition 1.4.2

Proposition 1.4.2 *The information effect hampers communication. Namely, if* $0 \leq a_{i-1} \leq a_i < a_{i+1}$ *are such that the expert with type* a_i *is indifferent between* $y_{CS}(m_i)$ *and* $y_{CS}(m_{i+1})$*, where* $m_i = [a_{i-1}, a_i]$ *and* $m_{i+1} = [a_i, a_{i+1}]$*, then the expert strictly prefers* $\hat{y}(m_{i+1}, a_i)$ *to* $\hat{y}(m_i, a_i)$*.*

The following results are used for the proof of Proposition 1.4.2. Lemma A.1.7 transfers the symmetric properties of the best response established in Lemma A.1.3 to the expected best response.

Lemma A.1.7. *Given a message* $m = [\underline{a}, \overline{a})$*, and a type* θ*, the expected action of the decision maker* $\hat{y}(\underline{a}, \overline{a}, \theta)$ *satisfies the following properties:*

1. $\hat{y}(\underline{a}, \overline{a}, \theta)$ *is increasing in all its arguments and* $\underline{a} < \hat{y}(\underline{a}, \overline{a}, \theta) < \overline{a}$
2. $\hat{y}(a, a + h, \theta) = a + \hat{y}(0, h, \theta - a)$.
3. $\hat{y}(0, h, \frac{h}{2} + \theta) - \frac{h}{2} = \frac{h}{2} - \hat{y}(0, h, \frac{h}{2} - \theta)$. *In particular* $\hat{y}(0, h, \frac{h}{2}) = \frac{h}{2}$.

Proof of Lemma A.1.7: All the results are immediate implications of Lemma A.1.3, Lemma A.1.2 and a change in variable.

1. It is a direct implication of Lemma A.1.2 and the fact that s and θ are affiliated.

2. $\hat{y}(a, a + h, \theta) = \int_{\mathbb{R}} y(a, a + h, s) f(s - \theta) ds = \int_{\mathbb{R}} a + y(0, h, s - a) f(s - \theta) ds =$
$= a + \int_{\mathbb{R}} y(0, h, s) f(s - (\theta - a)) ds = a + \hat{y}(0, h, \theta - a)$.

3. $\hat{y}(0, h, \frac{h}{2} + \theta) - \frac{h}{2} = \int_{\mathbb{R}} y(0, h, s) f(s - (\frac{h}{2} + \theta)) ds - \frac{h}{2} =$
$= \int_{\mathbb{R}} h - y(0, h, h - s) f(s - (\frac{h}{2} + \theta)) ds - \frac{h}{2} =$
$= \frac{h}{2} - \int_{\mathbb{R}} y(0, h, s) f(\frac{h}{2} - \theta - s) ds = \frac{h}{2} - \hat{y}(0, h, \frac{h}{2} - \theta)$.

Finally using this equation for $\theta = 0$, $\hat{y}(0, h, \frac{h}{2}) = \frac{h}{2}$. □

Lemma A.1.8 is the key result for Proposition 1.4.2. It states that as the length of the interval increases, the distance between the (CS) action and the expected action from the point of view of the boundary type increases.

Lemma A.1.8. $\frac{\partial}{\partial h}(h/2 - \hat{y}(0,h,0)) > 0$

Proof of Lemma A.1.8: By Lemma A.1.7.3, $\hat{y}(0,h,\theta) + \hat{y}(0,h,h-\theta) = h$. Totally differentiating this equation with respect to h:

$$\hat{y}_2(0,h,\theta) + \hat{y}_2(0,h,h-\theta) + \hat{y}_3(0,h,h-\theta) = 1$$

where all the terms on the left hand side are positive by Lemma A.1.7.1. It is therefore enough to show that if $\theta < h/2$ then $\hat{y}_2(0,h,\theta) \le \hat{y}_2(0,h,h-\theta)$ since this would imply that $\hat{y}_2(0,h,\theta) < 1/2$ for all $\theta < h/2$, and in particular that $h/2 - \hat{y}(0,h,0)$ is increasing in h.
First note that given quadratic loss utilities, $y(0,h,s) = \int_0^h \theta \frac{f(\theta-s)}{\int_0^h f(t-s)dt} d\theta$, and therefore:

$$y_2(0,h,s) = \int_0^h (h-\theta)\frac{f(h-s)f(\theta-s)}{(\int_0^h f(t-s)dt)^2} d\theta = \int_0^h (h-\theta) g(h|0,h,s) g(\theta|0,h,s) d\theta$$

and therefore if $s > 0$:

$$\begin{aligned} y_2(0,h,\tfrac{h}{2}+s) - y_2(0,h,\tfrac{h}{2}-s) &= \textstyle\int_0^h (h-\theta)\left[g(h|0,h,\tfrac{h}{2}+s)g(\theta|0,h,\tfrac{h}{2}+s)\right. \\ &\quad \left. -g(h|0,h,\tfrac{h}{2}-s)g(\theta|0,h,\tfrac{h}{2}-s)\right] d\theta \\ &= \textstyle\int_0^h (h-\theta)\left[g(h|0,h,\tfrac{h}{2}+s)g(h-\theta|0,h,\tfrac{h}{2}-s)\right. \\ &\quad \left. -g(h|0,h,\tfrac{h}{2}-s)g(h-\theta|0,h,\tfrac{h}{2}+s)\right] d\theta \\ &> 0 \end{aligned} \tag{A.10}$$

where the equality follows by Lemma A.1.3-3 and the inequality follows because $g(\theta|\cdot,s)$ is log-supermodular in (θ,s) (recall that θ and s are affiliated).
Finally, if $\theta < \frac{h}{2}$,

$$\begin{aligned} &\hat{y}_2(0,h,\theta) - \hat{y}_2(0,h,h-\theta) = \textstyle\int_{\mathbb{R}} y_2(0,h,s)(f(s-\theta) - f(s-h+\theta))ds \\ &= \textstyle\int_{s>0} \left(y_2(0,h,\tfrac{h}{2}+s) - y_2(0,h,\tfrac{h}{2}-s)\right)\left(f(\tfrac{h}{2}+s-\theta) - f(\tfrac{h}{2}+s-(h-\theta))\right) ds \le 0 \end{aligned}$$

where the second equality follows by dividing the signal space at $h/2$, and the inequality follows because the first term is always positive by (A.10) and the second is negative whenever $\theta < \frac{h}{2}$. □

Proof of Proposition 1.4.2: If $\hat{y}(m_{i+1},a_i) \le a_i + b$, then by Lemma A.1.7.1 $a_i < \hat{y}(m_{i+1},a_i) \le a_i + b$ and $\hat{y}(m_i,a_i) < a_i$. So clearly $(\hat{y}(m_{i+1},a_i) - (a_i+b))^2 \le b^2 <$

$(\hat{y}(m_i, a_i) - (a_i + b))^2$. This together with the fact that $a_i + b$ is equidistant to $y_{CS}(a_{i-1}, a_i)$ and $y_{CS}(a_i, a_{i+1})$, implies that the information effect for message m_{i+1} is greater than for message m_i.

Suppose now that $\hat{y}(m_{i+1}, a_i) > a_i+b$. In this case comparing the distance between the expected actions and the expert's peak is equivalent to comparing the distance between the expected actions and the respective CS actions. The bigger the distance between the expected action and the CS action, the closer is the expected action to the expert's peak and hence the bigger is the information effect.

Using Lemma A.1.7.2 and A.1.7.3, the distance between the expected actions and the CS actions can be written as a function which depends only on the length of the intervals:

$$\begin{aligned} y_{CS}(m_{i+1}) - \hat{y}(a_i, a_{i+1}, a_i) &= \tfrac{a_i+a_{i+1}}{2} - \hat{y}(a_i, a_{i+1}, a_i) = \tfrac{h_{i+1}}{2} - \hat{y}(0, \tfrac{h_{i+1}}{2}, 0) \\ \hat{y}(a_{i-1}, a_i, a_i) - y_{CS}(m_i) &= \hat{y}(a_{i-1}, a_i, a_i) - \tfrac{a_{i-1}+a_i}{2} = \hat{y}(0, h_i, h_i) - \tfrac{h_i}{2} = \tfrac{h_i}{2} - \hat{y}(0, \tfrac{h_i}{2}, 0) \end{aligned} \tag{A.11}$$

where $h_{i+1} = a_{i+1} - a_1$ and $h_i = a_i - a_{i-1}$.

Since $h_{i+1} > h_i$, then to conclude that the information effect for message m_{i+1} is greater than for message m_i it is enough to show that $\frac{h}{2} - \hat{y}(0, h, 0)$ increases with h, which is proved in Lemma A.1.8. $\square$

A.1.5 Proof of Proposition 1.4.3

Proposition 1.4.3 *The risk effect eases communication. Namely, if* $0 \leq a_{i-1} \leq a_i < a_{i+1}$ *are such that the expert with type* a_i *is indifferent between* $y_{CS}(m_i)$ *and* $y_{CS}(m_{i+1})$, *where* $m_i = [a_{i-1}, a_i]$ *and* $m_{i+1} = [a_i, a_{i+1}]$, *then* $\hat{\sigma}^2(m_{i+1}, a_i) > \hat{\sigma}^2(m_i, a_i)$.

Lemma A.1.9 is used in the proof of Proposition 1.4.3. It establishes some useful symmetric properties to the variance of the decision maker's actions:

Lemma A.1.9. *Given a message* $m = [\underline{a}, \overline{a})$, *and a type* θ, *the variance of the actions of the decision maker* $\hat{\sigma}(\underline{a}, \overline{a}, \theta)$ *satisfies the following properties:*

1. $\hat{\sigma}(a, a+h, \theta) = \hat{\sigma}(0, h, \theta - a)$.
2. $\hat{\sigma}(0, h, \frac{h}{2} + \theta) = \hat{\sigma}(0, h, \frac{h}{2} - \theta)$.

Proof of Lemma A.1.9: All the results are immediate implications of Lemma A.1.3, Lemma A.1.7 and a change in variable.

1. $\hat{\sigma}(a, a+h, \theta) = \int_{\mathbb{R}}(y(a, a+h, s) - \hat{y}(a, a+h, \theta))^2 f(s-\theta)ds =$
$\int_{\mathbb{R}}(y(0, h, s-a) - \hat{y}(0, h, \theta - a))^2 f(s-\theta)ds =$
$\int_{\mathbb{R}}(y(0, h, s) - \hat{y}(0, h, \theta))^2 f(s-(\theta - a))ds = \hat{\sigma}(0, h, \theta - a)$.

2. $\hat{\sigma}(0,h,\frac{h}{2}+\theta) = \int_{\mathbb{R}}(y(0,h,s)-\hat{y}(0,h,\frac{h}{2}+\theta))^2 f(s-(\theta+\frac{h}{2}))ds =$
   $\int_{\mathbb{R}}(y(0,h,h-s)-\hat{y}(0,h,\frac{h}{2}-\theta))^2 f(s-(\theta+\frac{h}{2}))ds =$
   $\int_{\mathbb{R}}(y(0,h,s)-\hat{y}(0,h,\frac{h}{2}-\theta))^2 f(\frac{h}{2}-\theta-s)ds = \hat{\sigma}(0,h,\frac{h}{2}-\theta).$ □

Proof of Proposition 1.4.3: Using Remark A.1.9 the information effect for a boundary type can be written as a function of the size of the interval sent only: $\hat{\sigma}(a_i,a_{i+1},a_i) = \hat{\sigma}(0,h_{i+1},0)$ and $\hat{\sigma}(a_{i-1},a_i,a_i) = \hat{\sigma}(0,h_i,0)$. Hence to compare the risk effect of sending m_i versus m_{i+1} it is enough to show that $\frac{\partial}{\partial h}\hat{\sigma}^2(0,h,0) > 0$. But this follows because by (A.10) the distance between the decision maker's actions increases with h. □

A.1.6 Proof of Proposition 1.4.4

Proposition 1.4.4 *For any information structure F, there exist $\overline{b} < \frac{1}{4}$ such that if $b > \overline{b}$, there is no communication in the $F-PI$ model. Moreover, for any b, there exists a sufficiently precise signal structure such that there is no communication in the private information model.*

The following Lemma will be used in the proof of Proposition 1.4.4. It states that an expert with type $\theta = 0$ strictly prefers to send message $[0,4b]$ to perfectly reveal himself.

Lemma A.1.10. $V(0,0,4b,b) > 0$

Proof of Lemma A.1.10: Recall that $V(0,0,4b,b) = U^E(0,4b,0,b) - U^E(0,0,0,b)$.

$$\begin{aligned}
&V(0,0,4b,b) = -\int_{\mathbb{R}}(y(0,4b,s)-b)^2 f(s)ds + b^2 \\
&= \int_{\mathbb{R}}(2b-y(0,4b,s))y(0,4b,s)f(s)ds \\
&= \int_{s>0}\left[(2b-y(0,4b,2b+s))y(0,4b,2b+s)f(2b+s)+\right. \\
&\qquad \left.(2b-y(0,4b,2b-s))y(0,4b,2b-s)f(2b-s)\right]ds \\
&= \int_{s>0}(2b-y(0,4b,2b-s))(y(0,4b,2b-s)f(2b-s)-y(0,4b,2b+s)f(2b+s))ds
\end{aligned}$$

where the third equality follows by dividing the signal space at $2b$, and the last equality follows by the symmetric properties of the functions (see Lemma A.1.3). The first factor in the integral is always positive and using the fact that for quadratic-loss preferences $y(0,4b,s) = \int_0^{4b}\theta g(\theta|0,4b,s)d\theta$ the second factor can be written as:

$$\begin{aligned}
&y(0,4b,2b-s)f(2b-s) - y(0,4b,2b+s)f(2b+s) = \\
&= \int_0^{4b} f(t-2b+s)dt \cdot \\
&\qquad \left[\int_0^{4b}\theta\left(g(\theta|0,4b,2b-s)g(4b|0,4b,2b+s) - g(\theta|0,4b,2b+s)g(4b|0,4b,2b-s)\right)d\theta\right] \\
&> 0
\end{aligned}$$

where the equality follows from the fact that $g(4b|0,4b,2b+s) = \frac{f(2b-s)}{\int_0^{4b} f(t-(2b+s))}$, $g(4b|0,4b,2b-s) = \frac{f(2b+s)}{\int_0^{4b} f(t-(2b-s))}$ and $\int_0^{4b} f(t-(2b+s)) = \int_0^{4b} f(t-(2b-s))$, and the inequality uses the affiliation of s and θ. □

Proof of Proposition 1.4.4: To prove the first statement of the Proposition, observe that by Lemma A.1.10, $V(0,0,1,\frac{1}{4}) > 0$. By continuity of $V(\cdot)$ in b, there exists a $\bar{b} < \frac{1}{4}$ such that $V(0,0,1,b) > 0$ for all $b \in (\bar{b}, \frac{1}{4}]$. By Lemma A.1.5 $V(0,0,a,b) > 0$ for all $a \in [0,1]$, so there can be no information transmitted in equilibrium.

Finally, for the second statement suppose that the conditional distribution of the signal belongs to a parameterized family $\{F^\lambda(\cdot|\theta), \lambda \in (0,\infty)\}$, where λ represents the precision[6] of the signal, and such that in the limit, when $\lambda \to \infty$, it corresponds to the degenerate distribution in θ. Then the second statement follows by the fact that as $\lambda \to \infty$ the conditional distribution $G(\theta|s)$ converges to the degenerate distribution on s. And hence, there is a precision λ^b, such that the lottery induced by message $[0,1]$ is preferred by the expert with type $\theta = 0$ and bias b to the constant action $y = 0$. Namely, $V(0,0,1,b,F^\lambda) > 0$, and by Lemma A.1.5, $V(0,0,a,b,F^\lambda) > 0$ for all $a \in [0,1]$, so there can be no information transmitted in equilibrium. □

A.2 Uniform Private Information Model

Recall that the optimal action in this model is:

$$y(\underline{a}, \overline{a}, s, F_\delta) = \frac{\max\{\underline{a}, s-\delta\} + \min\{\overline{a}, s+\delta\}}{2}$$

If $\overline{a} - \underline{a} \leq 2\delta$ the expectation and the second moment of the decision maker's actions from the point of view of the expert are given by:

$$\begin{aligned}
\hat{y}(\underline{a}, \overline{a}, \theta, F_\delta) &= \tfrac{\underline{a}+\overline{a}}{2} + \tfrac{1}{8\delta}(\overline{a}-\underline{a})(2\theta - \overline{a} - \underline{a}) \\
E(y^2|\underline{a}, \overline{a}, \theta, F_\delta) &= \tfrac{(\underline{a}+\overline{a})^2}{4} + \tfrac{1}{24\delta}[(\theta+\overline{a})^3 - (\theta+\underline{a})^3 - 3(\underline{a}+\overline{a})^2(\overline{a}-\underline{a})]
\end{aligned}$$

If $\overline{a} - \underline{a} > 2\delta$, the expectation and second moment of the decision maker's actions

[6] One signal is more precise than another if the latter is a mean preserving spread of the former.

are:

$$\hat{y}(\underline{a},\overline{a},\theta,F_\delta) = \begin{cases} \frac{\delta+\underline{a}+\theta}{2} + \frac{1}{8\delta}(\underline{a}-\theta)^2 & if \quad \theta < \min\{\underline{a}+2\delta, \overline{a}-2\delta\} \\ \frac{\underline{a}+\overline{a}}{2} + \frac{1}{8\delta}(\overline{a}-\underline{a})(2\theta-\overline{a}-\underline{a}) & if \quad \overline{a}-2\delta < \theta < \underline{a}+2\delta \\ \theta & if \quad \underline{a}+2\delta < \theta < \overline{a}-2\delta \\ \frac{\theta+\overline{a}-\delta}{2} + \frac{1}{8\delta}(\theta-\overline{a})^2 & if \quad \theta \geq \max\{\underline{a}+2\delta, \overline{a}-2\delta\} \end{cases}$$

$$E(y^2|\underline{a},\overline{a},\theta,F_\delta) = \begin{cases} \frac{1}{24\delta}[4(\underline{a}+\delta)^3 + 4(\theta+\delta)^3 - (\underline{a}+\theta)^3] & if \quad \theta < \min\{\underline{a}+2\delta, \overline{a}-2\delta\} \\ \frac{1}{24\delta}[4(\underline{a}+\delta)^3 - 4(\overline{a}-\delta)^3 + (\theta+\overline{a})^3 - (\underline{a}+\theta)^3] & if \quad \overline{a}-2\delta < \theta < \underline{a}+2\delta \\ \theta^2 + \frac{\delta^2}{3} & if \quad \underline{a}+2\delta < \theta < \overline{a}-2\delta \\ \frac{1}{24\delta}[(\overline{a}+\theta)^3 - 4(\overline{a}-\delta)^3 - 4(\theta-\delta)^3] & if \quad \theta \geq \max\{\underline{a}+2\delta, \overline{a}-2\delta\} \end{cases}$$

Given quadratic-loss utilities, $U^E(\underline{a},\overline{a},\theta,b,F_\delta) = -E(y^2|\underline{a},\overline{a},\theta,b,F_\delta) + 2\hat{y}(\underline{a},\overline{a},\theta,F_\delta) - (\theta+b)^2$. In particular, denoting by $h_i = a_i - a_{i-1}$ and $h_{i+1} = a_{i+1} - a_i$, the expected utilities of an expert with type $\theta = a_i$ who sends message $[a_{i-1}, a_i]$ and $[a_i, a_{i+1}]$ are respectively:

$$\begin{aligned} U^E(a_{i-1},a_i,a_i,b,F_\delta) &= \begin{cases} -(\frac{h_i}{2}+b)^2 + \frac{1}{12\delta}h_i^3 + \frac{b}{4\delta}h_i^2 & if \quad h_i \leq 2\delta \\ -\delta b - \frac{\delta^2}{3} - b^2 & if \quad h_i > 2\delta \end{cases} \\ U^E(a_i,a_{i+1},a_i,b,F_\delta) &= \begin{cases} -(\frac{h_{i+1}}{2}-b)^2 + \frac{1}{12\delta}h_{i+1}^3 - \frac{b}{4\delta}h_{i+1}^2 & if \quad h_{i+1} \leq 2\delta \\ \delta b - \frac{\delta^2}{3} - b^2 & if \quad h_{i+1} > 2\delta \end{cases} \end{aligned} \tag{A.12}$$

Remark A.2.1. *Assumption A2 is satisfied in the Uniform private information model.*

Proof of Remark A.2.1: Taking the derivative of $U^E(0,h,0,b,F_\delta)$ in Equation (A.12) with respect to h: $\frac{\partial}{\partial h}U^E(0,h,0,b,F_\delta) = \frac{1}{4\delta}(h-2b)(h-4\delta)$ if $h \leq 2\delta$, 0 otherwise. If $b > 2\delta$ no information can be sent in equilibrium and there is nothing to check[7]. If $b < 2\delta$, $U^E(0,h,0,b,F_\delta)$ is increasing for $h < 2b$ and decreasing for $h > 2b$. □

A.2.1 Proof of Theorem 1.4.5

Theorem 1.4.5 *In the Uniform Private Information model, an increase in the precision of the signal (a decrease in δ) leads to less communication in equilibrium. Namely, if a^δ and $a^{\delta'}$ are two monotone partition equilibria of size N of the $F_\delta - PI$ and $F_{\delta'} - PI$ models respectively, with $\delta' < \delta$, then $a_i^{\delta'} < a_i^\delta$ for all $i = 1, ..., N-1$. Moreover $N(b, F_{\delta'}) \leq N(b, F_\delta)$.*

For the proof of Theorem 1.4.5 I use the following results:

[7]See Remark A.2.3

Lemma A.2.2. *Let a^δ be a monotone partition equilibrium of the $F_\delta - PI$ model. Suppose that $a^\delta_{i+1} - a^\delta_i < 2\delta$, then $V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_{\delta'}) > 0$ for all $\delta' < \delta$.*

Proof of Lemma A.2.2: By Theorem 1.3.3, $h_{i+1} \equiv a^\delta_{i+1} - a^\delta_i > a^\delta_i - a^\delta_{i-1} \equiv h_i$, and hence, $h_{i+1} < 2\delta$ implies $h_i < 2\delta$. Since $V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_\delta) = 0$, for $\delta' \in (\frac{h_{i+1}}{2}, \delta)$:

$$\begin{aligned} V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_{\delta'}) &= V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_{\delta'}) - V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_\delta) \\ &= (\tfrac{1}{12\delta'} - \tfrac{1}{12\delta})(h^2_{i+1}(h_{i+1} - 3b) - h^2_i(h_i - 3b)) \end{aligned} \tag{A.13}$$

which is positive for $\delta' < \delta$ as long as $h_{i+1} > 3b$. Note that as δ goes to infinity, the signal becomes uninformative resulting in the CS setup where $h^{CS}_{i+1} = h^{CS}_i + 4b \geq 4b$. Therefore, by (A.13), as the signal becomes more informative, the required h_{i+1} which makes $\theta = a^\delta_i$ indifferent between m_i and m_{i+1} becomes larger, implying that $h_{i+1} \geq 4b > 3b$ always holds, and thus $V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_{\delta'}) > 0$. If $\delta' < \frac{h_i}{2}$, then by (A.12), $V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_{\delta'}) > 0$. $\square$

Consider now the case $a^\delta_{i+1} - a^\delta_i > 2\delta$. Observe that it cannot be that $a^\delta_i - a^\delta_{i-1} > 2\delta$ as well, because in that case by (A.12) the expert with type a^δ_i strictly prefers m_{i+1}. Since by Theorem 1.3.3, intervals in equilibrium are increasing in size, the only interval which might be larger than 2δ is the last one. The following remark summarizes this argument.

Remark A.2.3. *For any equilibrium partition $a^\delta = \{0 = a^\delta_0 < a^\delta_1 < ... < a^\delta_{N-1} < a^\delta_N\}$ of the $F_\delta - PI$ model, $h_i = a^\delta_i - a^\delta_{i-1} < 2\delta$ for $1 \leq i \leq N-1$.*

The following lemma shows that whenever $a^\delta_i - a^\delta_{i-1} < 2\delta < 1 - a^\delta_i$ and $V(a^\delta_{i-1}, a^\delta_i, 1, F_\delta) = 0$ then $V(a^\delta_{i-1}, a^\delta_i, 1, F_{\delta'}) > 0$ for $\frac{a^\delta_i - a^\delta_{i-1}}{2} < \delta' < \delta$.

Lemma A.2.4. *Suppose that $a^\delta_i - a^\delta_{i-1} < 2\delta < 1 - a^\delta_i$ and $V(a^\delta_{i-1}, a^\delta_i, 1, F_\delta) = 0$ then $V(a^\delta_{i-1}, a^\delta_i, 1, F_{\delta'}) > 0$ for $\delta' < \delta$.*

Proof of Lemma A.2.4: By (A.12), $V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_\delta) = \delta b - \frac{\delta^2}{3} - b^2 + (\frac{h_i}{2} + b)^2 - \frac{1}{12\delta} h^3_i - \frac{b}{4\delta} h^2_i$. Taking the derivative with respect to δ:

$$\begin{aligned} \frac{\partial}{\partial\delta} V(a^\delta_{i-1}, a^\delta_i, a^\delta_{i+1}, F_\delta) &= b - \frac{2\delta}{3} + \frac{1}{12\delta^2} h^3_i + \frac{b}{4\delta^2} h^2_i \\ &< b - \frac{2\delta}{3} + \frac{1}{12\delta^2}(2\delta)^3 + \frac{b}{4\delta^2}(2\delta)^2 = 0 \end{aligned}$$

where the inequality follows because, by assumption $h_i < 2\delta$. $V(\cdot)$ decreasing in δ combined with $V(a^\delta_{i-1}, a^\delta_i, 1, F_\delta) = 0$ implies $V(a^\delta_{i-1}, a^\delta_i, 1, F_{\delta'}) > 0$ for $\frac{h_i}{2} < \delta' < \delta$. If $\delta' < \frac{h_i}{2} < \frac{h_{i+1}}{2}$, by (A.12), $V(a^\delta_{i-1}, a^\delta_i, 1, F_{\delta'}) > 0$. $\square$

Proof of Theorem 1.4.5: The theorem is a direct implication of Propositions A.2.2, A.2.4, Remark A.2.3 and Proposition 1.4.1. $\square$

Bibliography

Aghion, P. and Tirole, J. (1997). Formal and Real Authority in Organizations, *Journal of Political Economy* **105**.

Alonso, R. and Matouschek, N. (2008). Optimal Delegation, *Review of Economic Studies* **75**: 259–293.

Ansolabehere, S. and Snyder, J. M. (2000). Valence politics and equilibrium in spatial election models, *Public Choice* **103**: 327–336.

Ansolabehere, S. and Snyder, J. M. (2002). The Incumbency Advantage in U.S. Elections: An Analysis of State and Federal Offices, 1942-2000, *Election Law Journal 1* **3**: 315–338.

Ansolabehere, S., Snyder, J. M. and Stewart, C. (2000). Old Voters, New Voters, and the Personal Vote: Using Redistricting to Measure the Incumbency Advantage, *American Journal of Political Science* **44**: 17–34.

Aragones, E. and Palfrey, T. R. (2002). Mixed Equilibrium in a Downsian Model with a Favoured Candidate, *Journal of Economic Theory* **103**: 131–161.

Ashworth, S. and Bueno de Mesquita, E. (2008a). Elections with platform and valence competition., *Games and Economic Behavior* .

Ashworth, S. and Bueno de Mesquita, E. (2008b). Electoral Selection, Strategic Challenger Entry, and the Incumbency Advantage, *The Journal of Politics* **70**: 1006–1025.

Athey, S. (2002). Monotone Comparative Statics Under Uncertainty, *Quarterly Journal of Economics* **117**: 187–223.

Athey, S. and Levin, J. (2001). The Value of Information in Monotone decision Problems, *Econometrica* **71**: 1691–1660.

Austen-Smith, D. (1993). Interested Experts and Policy Advice: Multiple Referrals under Open Rule, *Games and Economic Behavior* **5**: 3–43.

Ballester, M. A. and Haeringer, G. (forthcoming). A characterization of the single-peaked domain, *Social Choice and Welfare* .

Barberà, S. (2001). An introduction to strategy-proof social choice functions, *Social Choice and Welfare* **18**: 619–653.

Barberà, S. (2010). Strategy-proof social choice, *Handbook of Social Choice and Welfare* **2**. edited by K. Arrow and K. Suzumura. Kluwer Academic Publishers.

Barberà, S., Berga, D. and Moreno, B. (2010). Individual versus group strategy-proofness: when do they coincide?, *Journal of Economic Theory* **145**: 1648–1674.

Barberà, S. and Jackson, M. O. (1994). A characterization of strategy-proof social choice functions for economies with pure public goods, *Social Choice and Welfare* **11**: 241–252.

Barberà, S., Massó, J. and Neme, A. (1997). Voting under constraints, *Journal of Economic Theory* **76**: 298–321.

Barberà, S., Massó, J. and Neme, A. (2005). Voting by committees under constraints, *Journal of Economic Theory* **122**: 185–205.

Barberà, S., Massó, J. and Serizawa, S. (1998). Strategy-proof voting on compact ranges, *Games and Economic Behavior* **25**: 272–291.

Barberà, S. and Peleg, B. (1990). Strategy-proof voting schemes with continuous preferences, *Social Choice and Welfare* **7**: 31–38.

Barberà, S., Sonnenschein, H. and Zhou., L. (1991). Voting by committees, *Econometrica* **59**: 595–609.

Barro, R. (1973). The control of politicians: an economic model, *Public Choice* **14**(1): 19–42.

Berga, D. and Serizawa, S. (2000). Maximal domain for strategy-proof rules with one public good, *Journal of Economic Theory* **90**: 39–61.

Black, D. (1948). On the rationale of group decision making, *Journal of Political Economy* **56**: 23–34.

Blume, A., Board, O. J. and Kawamura, K. (2007). Noisy Talk, *Theoretical Economics* **2**: 395–440.

Border, K. and Jordan, J. (1983). Straightforward elections, unanimity and phantom voters, *The Review of Economic Studies* **50**: 153–170.

Cain, B., Ferejohn, J. and Fiorina, M. (1987). *The Personal Vote: Constituency Service and Electoral Independence.*

Caplin, A. and Nalebuff, B. (1991). Aggregation and Social Choice, *Econometrica* **59 (1)**: 1–24.

Carrillo, J. D. and Castanheira, M. (2002). Platform Divergence, Political Efficiency and the Median Voter Theorem. CEPR Discussion Paper No. 3180.

Chen, Y. (2009). Communication with Two-sided Asymmetric Information. Working paper Arizona State University.

Chen, Y. (2010). Value of Public Information in Sender-Receiver Games. Working paper Arizona State University.

Chen, Y., Kartik, N. and Sobel, J. (2008). Selecting Cheap-Talk Equilibria, *Econometrica* **76**: 117–136.

Ching, S. (1997). Strategy-proofness and 'median voters', *International Journal of Game Theory* **26**: 473–490.

Crawford, V. P. and Sobel, J. (1982). Strategic Information Transmission, *Econometrica* **50**: 1431–1451.

Dessein, W. (2002). Authority and Communication in Organizations, *Review of Economics Studies* **69**: 811–838.

Dimitrakas, V. and Sarafidis, Y. (2005). Advice from an Expert with Unknown Motives. Mimeo INSEAD.

Ehlers, L. (2002). Resource-monotonic allocation when preferences are single-peaked, *Economic Theory* **20**: 113–131.

Farrell, J. (1993). Meaning and Credibility in Cheap-Talk Games, *Games and Economic Behavior* **5**: 514–531.

Feddersen, T. and Pesendorfer, W. (1998). Convicting the Inocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting, *American Political Science Review* **92 (1)**.

Fischer, P. E. and Stocken, P. C. (2001). Imperfect Information and Credible Communication, *Journal of Accounting Research* **39**: 119–134.

Galeotti, A., Ghiglino, C. and Squintani, F. (2009). Strategic Information Transmission in Networks. Mimeo.

Gerbach, H. (2008). Vote-Share Contracts and Democracy. Unpublished.

Gerbach, H. (2009). Higher Vote Thresholds for Incumbents, Effort and Selection. CEPR Discussion Paper No. DP7320.

Gilligan, T. W. and Krehbiel, K. (1989). Asymmetric Information and Legislative Rules with Heterogeneous Committee, *American Journal of Political Science* **33**: 459–490.

Goltsman, M., Horner, J., Pavlov, G. and Squintani, F. (2009). Arbitration, Mediation and Cheap Talk, *Journal of Economic Theory* .

Goodin, R. E. and List, C. (2006). Special majorities rationalized, *British journal of political science* **36 (2)**: 213–241.

Green, J. R. and Stokey, N. (2007). A Two-Person Game of Information Transmission, *Journal of Economic Theory* **135**: 90–104.

Ishida, J. and Shimizu, T. (2011). Cheap Talk with an Informed Receiver. Working Paper.

Ivanov, M. (2010). Informational control and organizational design, *Journal of Economic Theory* **145**: 721–751.

Kar, A. and Kibris, O. (2008). Allocating multiple states among agents with single-peaked preferences, *Social Choice and Welfare* **31**: 641–666.

Klaus, B. and Bochet, O. (2010). The relation between monotonicity and strategy-proofness. Cahier de recherches économiques de DEEP No 10.01.

Klaus, B., Peters, H. and Storken, T. (1998). Strategy-proof division with single-peaked preferences and individual endowments, *Social Choice and Welfare* **15**: 297–311.

Kovac, E. and Mylovanov, T. (2009). Stochastic mechanisms in settings without monetary transfers: The regular case, *Journal of Economic Theory* **144**: 1373–1395.

Krehbiel, K. (2006). *Pivots.* The Oxford Handbook of Political Economy. Edited by B.B. Weingast and D. Wittman. Oxford University Press.

Krishna, V. and Morgan, J. (2001a). A Model of Expertise, *The Quarterly Journal of Economics* **116**: 747–775.

Krishna, V. and Morgan, J. (2001b). Asymetric Information and Legislative Rules: Some Amendments, *American Political Science Review* **95**: 435–452.

Krishna, V. and Morgan, J. (2004). The art of conversation: eliciting information from experts through multi-stage communication, *Journal of Economic Theory* **117**: 147–179.

Krishna, V. and Morgan, J. (2008). Contracting for information under imperfect commitment, *RAND Journal of Economics* **39**: 905–925.

Lai, E. K. (2010). Expert Advice for Amateurs. Mimeo.

Lee, D. S. (2008). Randomized experiments from non-random selection in U.S. House elections, *Journal of Econometrics* **142**: 675–697.

Levitt, S. D. and Wolfram, C. D. (1997). Decomposing the Sources of Incumbency Advantage in the U.S. House, *Legislative Studies Quarterly* **22**: 45–60.

Li, M. and Madarasz, K. (2007). When Mandatory Disclosure Hurst: Expert Advice and Conflicting Interest, *Journal of Economic Theory* .

Matthews, S. A. and Mirman, L. J. (1983). Equilibrium Limit Pricing: The Effects of Private Information and Stochastic Demand, *Econometrica* **51 (4)**: 981–996.

Matthews, S. A., Okuno-Fujiwara, M. and Postlewaite, A. (1991). Refining Cheap-talk Equilibria, *Journal of Economic Theory* **55**: 247–273.

McKelvey, R. and Ordeshook, P. (1993). Information and elections: retrospective voting and rational expectations, *Experimental Foundations of Political Science* **31**: 641–666. edited by D.R. Kinder and T.R. Palfrey. Michigan Studies in Political Analysis.

Meirowitz, A. (2008). Electoral Contests, Incumbency Advantages, and Campaign Finance, *The Journal of Politics* **70**: 681–699.

Milgrom, P. R. (1981). Good news and bad news: Representation theorems and applications, *Bell Journal of Economics* **12**: 380–391.

Milgrom, P. and Shannon, C. (1994). Monotone Comparative Statics, *Econometrica* **62**: 157–180.

Morgan, J. and Stocken, P. C. (2003). An Analysis of Stock Recommendations, *RAND Journal of Economics* **34**: 183–203.

Morgan, J. and Stocken, P. C. (2008). Information Aggregation in Polls, *American Economic Review* **98**: 864–896.

Moulin, H. (1980). On strategy-proofness and single peakedness, *Public Choice* **35**: 437–455.

Nehring, K. and Puppe, C. (2007a). Efficient and strategy-proof voting rules: A characterization, *Games and Economic Behavior* **59**: 132–153.

Nehring, K. and Puppe, C. (2007b). The structure of strategy-proof social choice−Part I: General characterization and possibility results on median spaces, *Journal of Economic Theory* **135**: 269–305.

Nisan, N. (2007). *Introduction to mechanism design (for computer scientists)*. Algorithmic Game Theory. Edited by N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Cambridge University Press.

Olszewski, W. (2004). Informal communication, *Journal of Economic Theory* **117**: 180–200.

Persico, N. (2000). Information Acquisition in Auctions, *Econometrica* **68**: 135–149.

Peters, H., van der Stel, H. and Storken, T. (1992). Pareto optimality, anonymity, and strategy-proofness in location problems, *International Journal of Game Theory* **21**: 221–235.

Prendergast, C. (1993). A Theory of ”Yes Men”, *The American Economic Review* **83**: 757–770.

Rabin, M. (1990). Communication Between Rational Agents, *Journal of Economic Theory* **51**: 144–170.

Rogoff, K. (1990). Equilibrium political budget cycles, *The American Economic Review* pp. 21–36.

Seidmann, D. J. (1990). Effective Cheap Talk with Conflicting Interests, *Journal of Economic Theory* **50**: 445–458.

Smart, M. and Sturm, D. M. (2006). Term Limits and Electoral Accountability. CEPR Discussion Paper No. 4272.

Smith, D. J., Dua, P. and Taylor, S. W. (1994). Voters and macroeconomics: Are they forward looking or backward looking, *Public Choice* **78**: 283–293.

Sprumont, Y. (1995). Strategy-proof collective choice in economic and political environments, *Canadian Journal of Economics* **28**: 68–107.

Stein, J. C. (1989). Cheap Talk and the Fed: A Theory of Imprecise Policy Announcements, *The American Economic Review* **79**: 32–42.

Watson, J. (1996). Information Transmission when the Informed Party is Confused, *Games and Economic Behavior* **12**: 143–161.

Wolinsky, A. (2003). Information Transmission when the Sender's Preferences are Uncertain, *Games and Economic Behavior* **42**: 319–326.