Estimation of fractional co-integration with unknown integration orders

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Abstract

This thesis presents different methods of estimating the co-integrating parameter in a bivariate fractionally co-integrated model. The proposed estimates enjoy optimal convergence rates and standard asymptotic distributions, yielding Wald test statistics with χ^2 null limit distribution. In the last few years increasing interest has developed in the issue of fractional co-integration, where both the observable series and the co-integrating error can be fractional processes, nesting the familiar situation where their respective orders are 1 and 0. These values have typically been assumed known. Chapter 1 is mainly devoted to reviewing this traditional prescription and motivate the relevance of fractional co-integration. In Chapter 2, we analyse a fully parametric model where the co-integrating gap, that is the difference between the integration order of the observables and that of the co-integrating error, is larger than 0.5. There, we show that our estimates share with the Gaussian maximum likelihood estimate the same limiting distribution, irrespective of whether the orders of integration are known or unknown, subject in the latter case to their estimation with adequate rates of convergence. Chapter 3, still in a parametric framework, proposes estimates of the parameter of co-integration in case the co-integrating gap is less than 0.5. Again, we cover both situations where the orders of integration are known and unknown. Our estimates are inefficient relative to the Gaussian maximum likelihood, but share with this estimate optimal rate of convergence and asymptotic normality, being computationally much more convenient. Chapter 4 concentrates on both situations described in the previous two chapters from a semiparametric perspective, that is without assuming knowledge of the parametric structure of the input series generating the fractional processes in the model. Finally, Chapter 5 describes a simple procedure of testing for the equality of orders of integration of different series. This is as essential step in any empirical work in order to asses for the presence of co-integration in a certain estimated model.

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Chapter 1

Introduction

Traditionally, co-integration analysis has developed almost exclusively in the context of processes with non-fractional integration orders. Most popularly, observed series are assumed to have a single unit root, such that first differencing produces a weakly dependent, invertible stationary process, while co-integrating errors also satisfy the latter description. This basic setting has been greatly extended, to observed series in which twice differencing is required to produce stationary weak dependence, and to polynomial co-integration; polynomial time trends have also been introduced, and co-integration with respect to cyclic and seasonal frequencies has been examined. However, co-integration can exist among much more general non-stationary (and indeed stationary) observations, with stationary or non-stationary co-integrating errors, and it seems desirable to develop the topic in a broader context, nesting the integer-order cases in a more general class, allowing integration orders to be real-valued. Undoubtedly, dealing with fractional processes could entail some difficulties, but in recent times, knowledge of their statistical properties has advanced considerably, so that issues like their role in co-integration analysis can be explored. In fact, fractional co-integration has become a relatively popular issue in the last decade among both theoretical and empirical econometricians, and this thesis mainly concentrates on one of the most relevant issues in this field, that is the estimation of a relation of fractional co-integration.

Before describing our aim in detail, we need to place this work in the right perspective. This Introduction has been written with this idea in mind, stressing the connection between the wider framework that fractional co-integration allows and the traditional prescription of unit roots and standard co-integration.

Section 1.1 is devoted to describing in some detail the concept of integrated series, which is essential in order to define the concept of co-integration, analysed in Section 1.2. Section 1.3 relates directly to the bulk of the thesis, as it presents different methods of estimating, in a given co-integrated model, the co-integrating parameter. As will become clear, our estimates, presented in Chapters 2, 3 and 4, were inspired by some of these methods, but apply more generally than many of them, mainly under situations of less knowledge about the structure of the estimated model. Section 1.4 presents some empirical evidence of fractional co-integration, and finally, Section 1.5 describes briefly our main proposals in the thesis.

1.1 The concept of integration

Following Engle and Granger's (1987) seminal work, a scalar series ζ_t, $t \in Z$, $Z = \{t : t = 0, \pm 1, ...\}$, is integrated of order d, denoted traditionally $\zeta_t \sim I(d)$ (see Definitions 1.2 and 1.3 below), if it has no deterministic component and could be represented as a stationary, invertible autoregressive-moving average (ARMA) after differencing it d times. Usually, the parameter d has been assumed to be 0, 1 or 2, the original series being modelled as $I(0)$ processes without, or under first or twice differencing respectively. Undoubtedly, the key aspect of that definition is the concept of $I(0)$ process, which in popular terms has been referred to as "short memory", "weakly dependent", "short-range dependent" or, in our view the most appropriate description, "weakly autocorrelated" process. The $I(0)$ concept has taken different, although relatively closely related, shapes in the literature. Without the aim of being very exhaustive in an otherwise quite extensive field, we will comment on different ideas related to this issue.

Engle and Granger (1987) completed their above definition with some characteristics that they attributed to $I(0)$ processes. In particular, among other features, they stated that the spectral density of a covariance stationary $I(0)$ process ζ_t, $f_\zeta(\lambda)$, given by

$$f_\zeta(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_\zeta(j) e^{ij\lambda}, \tag{1.1}$$

where $\gamma_\zeta(j)$ represents the lag j autocovariance of the process ζ_t, should have the property

$$0 < f_\zeta(0) < \infty, \tag{1.2}$$

which clearly implies that its autocovariances decrease steadily in magnitude for large enough j, so that their sum is finite. This relates directly to the concept of $I(0)$ process implied by Robinson's (1993) definition of a covariance stationary $I(d)$ scalar process, which he defined as one with spectral density

$$g(\lambda) = \left|1 - e^{i\lambda}\right|^{-2d} \overline{g}(\lambda), \tag{1.3}$$

where $0 < \overline{g}(0) < \infty$. This implied definition of an $I(0)$ process also appears in Robinson (1994a), Marinucci and Robinson (2001) and Robinson and Yajima (2002). Robinson (1994a) stressed the appropriateness of the term "weakly autocorrelated" to design this class of processes, as only second moments are involved, but he admitted that other terminology in popular use was "short-range dependent" or "short memory". In his view, these are more global concepts referring not only to second moments, although, of course in the Gaussian case all these concepts are synonymous.

Other authors considered as $I(0)$ a very wide class of processes which are weakly dependent (in certain sense to be described subsequently) and possibly heterogeneously distributed. The main feature of these processes is that they satisfy the following invariance principle: let ζ_t be one of these processes, then with $[\cdot]$ denoting integer part and n the sample size, for $r \in [0,1]$,

$$n^{-\frac{1}{2}} \sum_{t=1}^{[nr]} \zeta_t \Rightarrow W\left(\sigma^2; r\right), \text{ as } n \to \infty, \tag{1.4}$$

where in case $W(A;r)$ is a scalar, it denotes a Brownian motion with variance A, whereas if it is a $k \times 1$ vector, it represents a k-dimensional Brownian motion with variance-covariance matrix A, for which the following notation will be used

$$W(A;r) = (W_1(A;r), ..., W_k(A;r))', \tag{1.5}$$

with the prime denoting transposition; σ^2 is a finite scalar given by

$$\sigma^2 = \lim_{n\to\infty} n^{-1} E\left(\left(\sum\nolimits_{t=1}^{n} \zeta_t\right)^2\right) > 0; \tag{1.6}$$

"$\Rightarrow$" denotes weak convergence of the associated probability measures. This approach was followed for example by Phillips (1986), Phillips and Durlauf (1986), Phillips (1987), Park and Phillips (1988, 1989), Lo (1991), Phillips (1991b). (1.4) has been established in the literature under various conditions on the process ζ_t. Billingsley (1968) proves it for a strictly stationary process under certain conditions on its dependence, but his results have been extended by several authors. Among them, Herrndorf (1984) presented a set of sufficient conditions, allowing for temporal dependence and a degree of non-trending heteroskedasticity in the process ζ_t, a strong mixing condition satisfied by ζ_t characterizing the typical "weak dependence" of the process.

Also, note that if we further assume that ζ_t is covariance stationary with spectral density $f_\zeta(\lambda)$, $\sigma^2 = 2\pi f_\zeta(0)$, so that (1.6) implies the familiar condition that ζ_t has finite and strictly positive spectral density at frequency 0. In any case, on theoretical grounds, the distinction between both versions is not that relevant, because while an $I(0)$ process is usually considered as stationary, proper extra conditions are usually set so that certain invariance principle holds. See for example our Assumptions 2.1 and 2.2 in Chapter 2. Nevertheless, we could adopt Robinson's (1993) implication as our benchmark for a definition of $I(0)$ process.

Definition 1.1. Integrated of order zero process
A zero-mean scalar covariance stationary process ζ_t, $t \in Z$, with spectral density $f_\zeta(\lambda)$ is integrated of order zero, denoted $\zeta_t \sim I(0)$, if

$$0 < f_\zeta(0) < \infty. \tag{1.7}$$

As mentioned before, in the last few years, increasing interest has developed in a wider framework which takes into account that $I(0)$, and also $I(1)$, $I(2)$,..., are very specific types of stationary and nonstationary processes respectively. In this vein, and in fact as a direct consequence of the definition of integrated process given in Engle and Granger (1987), one could think about a process which is $I(0)$ after d-differencing, where d needs not be an integer. Note first that by the binomial expansion, for any real α, $\alpha \neq -1, -2, ...$,

$$(1-z)^{-\alpha} = \sum_{j=0}^{\infty} a_j(\alpha) z^j, \; a_j(\alpha) = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}, \tag{1.8}$$

with Γ denoting the gamma function, so that defining $\Delta = 1 - L$, where L represents the lag operator and $1(\cdot)$ the indicator function, we could establish the following definition.

Definition 1.2. Type I fractionally integrated process
For any real number d, given a scalar $I(0)$ process ζ_t, $t \in Z$, $\widetilde{x}_t$, $t \in Z$, is a Type I fractionally integrated process of order d, denoted $\widetilde{x}_t \sim I_1(d)$, if defining

$$\psi_t = \Delta^{-(d-k)} \zeta_t, \tag{1.9}$$

for an integer k such that $d - 1/2 < k \leq d + 1/2$,

$$\begin{aligned} \widetilde{x}_t &= \psi_t, & k \leq 0, \quad (1.10)\\ \widetilde{x}_t &= \Delta^{-k}\{\psi_t 1(t > 0)\}, & k > 0. \quad (1.11) \end{aligned}$$

In case $0 < d < 1/2$, $k = 0$ and $\widetilde{x}_t$ is a covariance stationary process given by

$$\widetilde{x}_t = \sum_{j=0}^{\infty} a_j(d) \zeta_{t-j}, \tag{1.12}$$

with spectral density

$$f_{\widetilde{x}}(\lambda) = \left|1 - e^{i\lambda}\right|^{-2d} f_\zeta(\lambda). \tag{1.13}$$

In this case, Granger and Joyeux (1980) showed that, under certain additional conditions on the $I(0)$ process ζ_t, the lag j autocovariance of the process $\widetilde{x}_t$, $\gamma_{\widetilde{x}}(j)$, behaves like

$$\gamma_{\widetilde{x}}(j) \sim K(d)\, j^{2d-1},\ j \to \infty, \tag{1.14}$$

where $K(d)$ is a constant depending only on d, "$\sim$ " representing that the ratio of both sides of the relation tends to 1 as a certain specified condition holds (in this particular case $j \to \infty$). Note that, if for example ζ_t is a stationary and invertible finite ARMA process, its lag j autocovariance exhibits an exponential decay that contrasts heavily with the much slower hyperbolic decay of (1.14). This illustrates the "long-memory" aspect of the fractionally integrated process when $d > 0$. (1.14) also implies that the autocovariances are not summable, hence the spectral density of $\widetilde{x}_t$ at the origin is unbounded. More precisely, from (1.13),

$$f_{\widetilde{x}}(\lambda) \sim f_\zeta(0)\, \lambda^{-2d},\ \lambda \to 0. \tag{1.15}$$

As Robinson (1994a) indicates, the non-summability of the autocovariances and unbounded spectrum at the origin characterize a stationary but "strongly autocorrelated" sequence. On the contrary, when $d < 0$, the process $\widetilde{x}_t$ is covariance stationary, but with zero spectrum at the origin.

For larger d's, Definition 1.2 has to be taken with caution. For example, when $1/2 \leq d < 3/2$, the process $\Delta\widetilde{x}_t$, that is first differences of $\widetilde{x}_t$, is an $I_1(d-1)$ covariance stationary process, but $\widetilde{x}_t$ itself is nonstationary and $\widetilde{x}_t = 0$ for $t \leq 0$.

On this range of values of d, the most widely used in the literature is $d = 1$, for which Definition 1.2 states that

$$\widetilde{x}_t = \sum_{j=1}^{t} \zeta_j, \; t > 0, \tag{1.16}$$

$$\widetilde{x}_t = 0, \qquad t \leq 0, \tag{1.17}$$

noting that $a_j(1) = 1$, $j \geq 0$, so that when dealing with $d = 1$, Definition 1.2 represents the standard Engle and Granger's definition of an $I(1)$ process (given at the beginning of this section), with specific initial conditions given by (1.17). This is also the case for larger integer orders. The reason why $\widetilde{x}_t$ is defined as

$$\widetilde{x}_t = \Delta^{-d} \zeta_t, \tag{1.18}$$

only for $d < 1/2$ is that when $d \geq 1/2$, $\widetilde{x}_t$ in (1.18) is not well defined in mean square sense, as it does not have finite variance. On the contrary, when $d \geq 1/2$, Definition 1.2 implies that the variance of $\widetilde{x}_t$ is finite (albeit evolving at rate t^{2d-1}). Definition 1.2 is not the unique way of defining fractionally integrated processes, and next, we propose an alternative definition.

Definition 1.3. Type II fractionally integrated process
For any real number d, given a scalar $I(0)$ process ζ_t, $t \in Z$, x_t, $t \in Z$, is a Type II fractionally integrated process of order d, denoted $x_t \sim I_2(d)$, if

$$x_t = \zeta_t, \qquad d = 0, \tag{1.19}$$

$$x_t = \Delta^{-d} \{\zeta_t 1(t > 0)\}, \; d \neq 0. \tag{1.20}$$

This definition has different implications from those of Definition 1.2. For example, in case $d < 1/2$, $d \neq 0$, on the contrary of $\widetilde{x}_t$, x_t is nonstationary, although as showed in Lemma 3.4 of Robinson and Marinucci (2001), under relatively mild conditions, for all $j \geq 0$,

$$\lim_{t \to \infty} \{Cov(x_t, x_{t+j}) - Cov(\widetilde{x}_t, \widetilde{x}_{t+j})\} = 0. \tag{1.21}$$

Hence, x_t could be considered in this case as "asymptotically stationary", the nonstationarity being only due to the truncation on the right hand side of (1.20). For $d \geq 1/2$, x_t is purely nonstationarity, the truncation in (1.20) ensuring x_t is well defined in mean square sense. Note that both definitions are equivalent for $d = 0$ and positive integers.

Definitions 1.2 and 1.3 were proposed by Marinucci and Robinson (1999). Both concepts mirror different definitions of fractional Brownian motions (denoted also as Type I and II) to which the suitably normalised different fractionally integrated processes converge. Robinson (2002) provides bounds for differences between the two fractionally integrated processes. Throughout the thesis, due to notational convenience, we will mostly consider Type II fractionally integrated processes, and we will employ the simplifying notation $I(d)$ instead of $I_2(d)$ to denote this kind of processes. Undoubtedly, all the results in the thesis could be slightly modified to accommodate for Type I processes, the main implication of this change being the presence of Type I Brownian motions instead of Type II in some limiting distributions derived below. Marinucci and Robinson (1999) presented a very detailed analysis of the different types of convergence and the probabilistic properties of the two different classes of Brownian motions.

1.2 The concept of co-integration

Engle and Granger (1987) suggested that in case two processes x_t and y_t are both $I(d)$, then it is generally true that for a certain scalar $a \neq 0$, a linear combination $w_t = y_t - ax_t$ will also be $I(d)$, although it is possible that $w_t \sim I(d-b)$ with $b > 0$. This idea characterized the concept of co-integration, which they adapted from Granger (1981) and Granger and Weiss (1983). They provided the following definition for multivariate series.

Definition 1.4. $CI(d,b)$ **co-integration**
Given two real numbers d, b, the components of the vector z_t are said to be co-integrated of order d, b, denoted $z_t \sim CI(d,b)$, if

(i) all the components of z_t are $I(d)$,

(ii) there exists a vector α $(\neq 0)$ so that $w_t = \alpha' z_t \sim I(d-b)$, $b > 0$.

Here, α and w_t are called co-integrating vector and error respectively. This definition applies to both classes of fractionally integrated processes (see Definitions 1.1 and 1.2), but, as mentioned before, in the thesis we will mainly consider co-integration among Type II processes. These authors offered some intuition behind this crucial concept in modern time series econometrics, suggesting the existence of forces in economics which tend to keep series not too far apart. Given a vector of economic variables z_t, and a certain vector $\alpha \neq 0$, economic theory would say that the variables are in equilibrium if $\alpha' z_t = 0$, that is a specified linear constraint holds among those variables. This is a very tight notion of equilibrium, and it is a very narrow view that this equality could hold for every time period t. Alternatively, we might think of an equilibrium error, as $w_t = \alpha' z_t$, which accommodates deviations from equilibrium. If, for example, in Engle and Granger's (1987) definition $d = b = 1$, the variables in z_t are not stationary, with variances that go to infinity as t goes to infinity and non mean-reverting behaviour, that is the expected time between crossings of their mean is infinite. What characterizes in this case co-integration as a "long-run equilibrium" relationship is that a linear combination of $I(1)$ processes is $I(0)$, so that the series in z_t cannot drift too far apart.

To be fair, the idea of equilibrium between $I(1)$ processes was hinted long before in the statistics literature. In the autoregressive (AR) model

$$y_t = \rho y_{t-1} + \varepsilon_t, \quad t > 0, \tag{1.22}$$
$$y_t = 0, \quad t \leq 0, \tag{1.23}$$

ε_t being a sequence of independent normally distributed random variables with mean 0 and finite variance, Dickey and Fuller (1979) studied the properties of the regression estimate of ρ, $\widehat{\rho}$, under the assumption that $\rho = 1$. In fact, this represented a situation of co-integration between the $I(1)$ processes y_t and y_{t-1}, as the linear combination $y_t - y_{t-1}$ is $I(0)$. This is a particular case of what Park (1992) denoted as "singular co-integration", which was characterized by co-integrating errors being linear combinations of innovations driving also regressors. Dickey and Fuller's work was a direct consequence of a fertile line of research starting on the fifties. It is worth mentioning two works here which represented very important advances in this literature. Rubin (1950) showed the consistency of $\widehat{\rho}$ for any value of ρ. White (1958) obtained the limiting distribution of $\widehat{\rho} - \rho$ for $|\rho| \neq 1$, and for $\rho = 1$ was able to represent the limiting distribution of $n(\widehat{\rho} - 1)$ as that of the ratio of two integrals defined on a Brownian motion.

Engle and Granger (1987) introduced another important concept. If the multivariate $I(d)$ process z_t has $p \geq 2$ components, there may be more than simply one co-integrating vector α, representing this the case where several equilibrium relations drive the joint movement of the variables in z_t. It is easy to realize that the maximum number of linearly independent co-integrating vectors is $r \leq p - 1$, and the value r was defined as the "co-integrating rank" of z_t. Note that it does not make sense to possibly consider $r = p$, as in this case, any vector in p-dimensional Euclidean space would be a co-integrating vector, including for example vectors like $(1, 0, ..., 0)'$, $(0, 1, 0, ..., 0)'$ and so on, which would indicate that the first, second,..., components of z_t are $I(d - b)$, which is contradictory.

Also, even considering only integer orders of integration, a more general definition of co-integration than the one given by Engle and Granger (1987) is possible, allowing for a multivariate process with components having different orders of integration, noting that long-run economic relationships are possible among variables with different behaviours. Here, denoting d_1 and d_p the largest and smallest of these orders respectively, Johansen (1996) proposed that any vector $\alpha \neq 0$ such that $\alpha' z_t \sim I(d_w)$ with $d_w < d_1$ was a co-integrating vector. Flôres and Szafarz (1996) narrowed Johansen's definition, proposing instead that the vector series is co-integrated if there is a non-trivial linear combination of its components (with at least a non-zero scalar multiplying on d_1) which is integrated of order $d_w < d_1$. Alternatively, Robinson and Marinucci (1998) defined z_t to be co-integrated if there exists a vector $\alpha \neq 0$ such that $\alpha' z_t \sim I(d_w)$ with $d_w < d_p$, which is a much stronger requirement. Robinson and Yajima (2002) offered an alternative (rather more involved) definition and good comparisons among the different definitions appeared in the literature. Fortunately, we will avoid the problem of choosing among these definitions of co-integration in a multivariate framework, as throughout the thesis we only consider bivariate models, for which all the previous definitions are equivalent. This is an important limitation of our analysis, but we considered that at this point is more adequate to present results in a relatively simple framework, multivariate extension being mostly straightforward, but notationally much more involved, extensions of our work.

Thus, once fractionally integrated processes are defined, the concept of fractional co-integration appears as a natural extension of the traditional co-integration, where the observables were treated as $I(1)$ processes, and certain linear combinations of them as $I(0)$ processes. In fact, the standard definition of co-integration by Engle and Granger (1987) does not necessarily refer to integer orders of integration. Thus, by Definition 1.4, in the simple bivariate case, two series y_t, x_t sharing the same order of integration, say δ, are co-integrated $CI\,(\delta, \beta)$, if there exists a vector $\alpha \neq 0$ such that $\alpha' z_t \sim I\,(\gamma)$, $\delta > \gamma$, with $z_t = (y_t, x_t)'$ and

$$\beta = \delta - \gamma. \tag{1.24}$$

Throughout the thesis we will consider an extension of Phillips' (1991a) triangular system for this simple bivariate case, given by

$$\begin{aligned} y_t &= \nu x_t + \Delta^{-\gamma} u_{1t}^{\#}, \qquad (1.25) \\ x_t &= \Delta^{-\delta} u_{2t}^{\#}, \qquad (1.26) \end{aligned}$$

for $t = 0, \pm 1, \ldots$, where the $\#$ superscript attached to a scalar or vector sequence v_t has the meaning

$$v_t^{\#} = v_t 1\,(t > 0)\,. \tag{1.27}$$

Also, $u_t = (u_{1t}, u_{2t})'$ is a bivariate covariance stationary unobservable process with zero mean and spectral density matrix, $f\,(\lambda)$, satisfying

$$E(u_0 u_j') = \int_{-\pi}^{\pi} e^{ij\lambda} f\,(\lambda)\, d\lambda, \tag{1.28}$$

that is at least nonsingular and continuous at all frequencies; and finally

$$\begin{aligned} \nu &\neq 0, \qquad (1.29) \\ \delta &\geq \beta > 0, \qquad (1.30) \end{aligned}$$

noting that (1.30) implies $\gamma \geq 0$. As mentioned before, the truncation in (1.26) ensures that x_t has finite variance, and implies that $x_t = 0$, $t \leq 0$. The truncation in (1.25) is unnecessary if $\gamma < 1/2$ ($y_t - \nu x_t$ is covariance stationary without it and "asymptotically covariance stationary" with it), but is imposed there also for the sake of a uniform treatment, implying that $y_t = 0$, $t \leq 0$. In common parlance, u_t is an $I(0)$ vector process, x_t is an $I(\delta)$ process, as is (due to (1.25), (1.26), (1.29), (1.30)) y_t, while the co-integrating error $y_t - \nu x_t$ is an $I(\gamma)$ process, and we say that (x_t, y_t) is co-integrated of order (δ, β) ($CI(\delta, \beta)$), noting Definitions 1.3 and 1.4. If $\beta = 0$, there is no co-integration and ν is not identified. (1.25), (1.26) reduces to the bivariate version of Phillips' triangular form when $\gamma = 0$, $\delta = 1$, which is one of the most popular models displaying $CI\,(1,1)$ co-integration considered both in empirical and theoretical literature. (1.25), (1.26) allows greater flexibility in representing equilibrium relationship between economic variables than the traditional $CI\,(1,1)$ prescription. On the one hand, it is plausible the existence of long-run co-movements between nonstationary series which are not precisely $I\,(1)$. On the other, usually there is not any *a priori* reason for which to restrict to simply $I\,(0)$ co-integrating errors, as perhaps the convergence to equilibrium that any co-integrating relation ensures could be much slower than the adjustment imposed by for example a finite ARMA co-integrating error. Furthermore, we could also consider co-integration among (asymptotically) stationary variables, with some linear combinations producing co-integrating errors characterized by having weaker memory than that of the observed series. Also, it could be that the co-integrating error is purely nonstationary but mean reverting, so that a certain long-run equilibrium among perhaps non-mean reverting observables holds. Note that a normalisation has been carried out in (1.25), the co-integration vector corresponding to Engle and Granger's (1987) definition being now $(1, -\nu)'$. Note that a co-integrating vector is only identifiable up to a scale parameter, so that if α is a co-integrating vector, that is $\alpha' z_t \sim I(\gamma)$, $c\alpha' z_t \sim I(\gamma)$ for any scalar constant c, hence $c\alpha$ could also be considered a co-integrating vector.

As denoted by Phillips and Loretan (1991), (1.25), (1.26) with $\gamma = 0$, $\delta = 1$, represents "a typical co-integrated system" in structural form. (1.25) could be regarded as a stochastic version of the partial equilibrium relationship $y_t = \nu x_t$, with $\Delta^{-\gamma} u_{1t}^{\#}$ representing deviations from this equilibrium. (1.26) is a reduced form equation. (1.25), (1.26) is the key structural model in this thesis and Chapters 2, 3, 4 are devoted exclusively to investigate methods of estimating in this framework the parameter ν. Some other work on fractional co-integration has employed the alternative Type I definition of fractional integrated process, replacing (1.25), (1.26) by

$$\tilde{y}_t = \nu \tilde{x}_t + v_{1t}^{(\gamma)}, \qquad t \geq 1, \tag{1.31}$$

$$\tilde{x}_t = v_{21}^{(\delta)} + \ldots + v_{2t}^{(\delta)}, \qquad t \geq 1, \tag{1.32}$$

where $v_{1t}^{(\gamma)}$ and $v_{2t}^{(\delta)}$ are jointly stationary $I_1(\gamma)$ and $I_1(\delta - 1)$ processes, respectively, with $|\gamma| < 1/2$, $1/2 < \delta < 3/2$. When $\gamma = 0$, $\delta = 1$, $v_t(\gamma, \delta) = (v_{1t}^{(\gamma)}, v_{2t}^{(\delta)})' \equiv (u_{1t}, u_{2t})'$ implies $(\tilde{x}_t, \tilde{y}_t) \equiv (x_t, y_t)$, but more generally, with $v_t(\gamma, \delta)$ having spectral density matrix $\Lambda(\lambda; \gamma, \delta) f(\lambda) \Lambda(-\lambda; \gamma, \delta)$, for $\Lambda(\lambda; \gamma, \delta) = diag\left\{(1 - e^{i\lambda})^{-\gamma}, (1 - e^{i\lambda})^{1-\delta}\right\}$, this is not the case. In particular, note that (1.32) represents a Type I fractionally integrated process $I_1(\delta)$. Model (1.31), (1.32) covers a different range of γ, δ values from (1.25), (1.26), but higher δ can be involved by extending (1.32) to include two or more unit roots, while $\gamma \in (-1/2, 0)$ could be allowed in (1.25).

1.3 Estimation of co-integrating relations

During the last two decades, plenty of effort has been devoted to developing different estimates of the co-integrating parameter ν in (1.25), mainly assuming $\gamma = 0$, $\delta = 1$. Here, there is a clear distinction between what Jeganathan (1997) denotes as first and second stage procedures. Typically, limiting distributions of procedures in the first stage are nonstandard and unsuitable for use in statistical inference, whereas procedures in the second stage imply estimates of ν belonging to the locally asymptotic mixed normal family. This class of estimates enjoy several attractive features. They are symmetrically distributed, median unbiased and optimal theory of inference applies under Gaussian assumptions (see Saikkonen, 1991). Also, they lead to Wald test statistics with standard χ^2 null limit distribution. Jeganathan (1997) suggested that first stage procedures could be used to test for the presence of unit roots in a given model, and then, by second stage methods, one could estimate co-integrating relationships on the model where the unit roots tested in the first stage are imposed. Thus, as a practical consequence, the main difference between the two types of procedures is that first stage methods do not require knowledge of γ and/or δ, whereas second stage do. For example, in the standard $CI(1,1)$ case, first stage procedures implicitly estimate the unit roots present in (1.25), (1.26), hence nonstandard asymptotics appear. On the contrary, second stage methods incorporate the information about the values of γ and δ into the estimation procedure, achieving desirable asymptotic properties (see Phillips, 1991a). However, there are exceptions to this setting. For example, Hendry's methodology described below (see Hendry and Richard, 1982, 1983), makes use of the information $\gamma = 0$, $\delta = 1$ without achieving estimates of ν with optimal asymptotic properties. More importantly, in fractional circumstances, there could be situations where assuming γ and/or δ are known is highly unrealistic, even after pretesting. As it will become clear in Section 1.5, our purpose in this thesis is to provide estimation methods for ν in (1.25), (1.26), under different situations, which share in many cases the optimal asymptotic properties of the second stage procedures without assuming knowledge of γ and/or δ.

We present below the main approaches proposed in the literature for both classes of procedures, focussing mainly on the $CI(1,1)$ framework, where most theoretical and empirical contributions concentrate. Among different first stage methods, we will focus on two procedures that we also use throughout the thesis as preliminary estimates necessary to obtain our proposed second stage estimates. For the second stage ones, we will focus on two classes of estimates which are closely related to the ones we propose in Chapter 2, 3 and 4, and also one that has enjoyed great popularity in the $CI(1,1)$ situation, and has also been extended to fractional frameworks.

1.3.1 First stage procedures

Ordinary least squares (OLS)

Phillips and Durlauf (1986) analysed the asymptotic properties of the OLS estimate of ν in a multivariate version of (1.25) with $\gamma = 0$, $\delta = 1$, which for our particular bivariate situation is given by

$$\overline{\nu}_O = \frac{\sum_{t=1}^n y_t x_t}{\sum_{t=1}^n x_t^2}. \tag{1.33}$$

In case, we assume that the process u_t is independent and identically distributed with mean 0 and variance-covariance matrix Ω ($iid(0,\Omega)$), their results imply

$$n(\overline{\nu}_O - \nu) \Rightarrow \frac{\int_0^1 W_2(\Omega;r)\, dW_1(\Omega;r) + \omega_{12}}{\int_0^1 W_2^2(\Omega;r)\, dr}, \tag{1.34}$$

where $W(\Omega;r) = (W_1(\Omega;r), W_2(\Omega;r))'$ and ω_{ij} is the (i,j)th element of Ω. Note that the limit distribution on the right of (1.34) could be rewritten as

$$\frac{\int_0^1 W_2(\Omega;r)\, dW_{1.2}(\Omega;r)}{\int_0^1 W_2^2(\Omega;r)\, dr} + \frac{\omega_{12}\int_0^1 W_2(\Omega;r)\, dW_2(\Omega;r)}{\omega_{22}\int_0^1 W_2^2(\Omega;r)\, dr} + \frac{\omega_{12}}{\int_0^1 W_2^2(\Omega;r)\, dr}, \tag{1.35}$$

where

$$W_{1.2}(\Omega; r) = W_1(\Omega; r) - \frac{\omega_{12}}{\omega_{22}} W_2(\Omega; r), \tag{1.36}$$

which is uncorrelated with $W_2(\Omega; r)$, and thus by Gaussianity independent, so that the first term in (1.35) represents a mixed normal distribution. The second and third terms are the "unit root distribution" arising from the implicit estimation of the unit roots present in the model and the "second-order bias" originated by the endogeneity of the regressor x_t (due to the correlation between u_{1t} and u_{2t}) respectively. Stock (1987) had earlier suggested for a co-integrated model with *iid* errors that a result like (1.34) could be obtainable. In fact, Phillips and Durlauf (1986) showed that a multivariate version of this result holds under more general conditions on the error input series u_t. Denoting

$$\overline{\Sigma}_0 = \lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} E(u_t u_t'), \tag{1.37}$$

$$\overline{\Sigma}_1 = \lim_{n\to\infty} n^{-1} \sum_{t=2}^{n} \sum_{j=1}^{t-1} E(u_j u_t'), \tag{1.38}$$

$$\overline{\Sigma} = \overline{\Sigma}_0 + \overline{\Sigma}_1 + \overline{\Sigma}_1', \tag{1.39}$$

under some regularity conditions on the autocorrelated (and possibly heteroskedastic) process u_t

$$n(\overline{\nu}_O - \nu) \Rightarrow \frac{\int_0^1 W_2(\overline{\Sigma}; r)\, dW_1(\overline{\Sigma}; r) + \overline{\sigma}_{12}}{\int_0^1 W_2^2(\overline{\Sigma}; r)\, dr}, \tag{1.40}$$

where $\overline{\sigma}_{ij}$ is the (i,j)th element of $\overline{\Sigma}$.

In fractional circumstances, the properties of the OLS estimate (1.33) could be very distant from those in the traditional $CI(1,1)$ situation. Robinson (1994c) showed the inconsistency of the OLS in a similar model to (1.25), where the observable y_t, x_t were covariance stationary long-memory processes, sharing the same memory parameter, whereas the co-integrating error was also a covariance stationary long-memory process with memory strictly smaller than the memory of the observables. In this framework, the inconsistency of the OLS estimate is due to correlation between stationary regressor and co-integrating error. It can be easily shown that Robinson's conclusions would also hold for our model (1.25), (1.26), where $\gamma < \delta < 1/2$ implies that both observables and co-integrating error are asymptotically stationary.

Robinson and Marinucci (1998, 2001), for a model similar to (1.25), (1.26), but where the different processes considered belonged to a class closely related but wider than the Type II fractionally integrated, provided the asymptotic distribution of the OLS (with or without intercept) for the case $\delta \geq 1/2$, $\gamma \geq 0$. They showed that the rate of convergence of the OLS is $n^{\min(2\delta-1,\beta)}$, except for the case where $\delta > \beta$ and $2\delta - \beta = 1$, where the OLS is $n^{\beta}/\log n$-consistent. In all cases, the OLS have a nonstandard limiting distribution which, as mentioned before, complicates statistical inference. Finally, Chan and Terrin (1995) developed asymptotic theory for the OLS estimate in a general AR process with fractional innovations.

Narrow band least squares estimate (NBLS)

For $l = 0, 1$ and integer m, with $l \leq m \leq n/2$, we could estimate of ν in (1.25) by

$$\overline{\nu}_l(m) = \frac{\widehat{F}_{xy}(l,m)}{\widehat{F}_{xx}(l,m)}, \tag{1.41}$$

where given (perhaps identical) scalar or vector sequences a_t, b_t, $t = 1, ..., n$,

$$\widehat{F}_{ab}(l,m) = 2\,\mathrm{Re}\left\{\frac{2\pi}{n}\sum_{j=l}^{m} I_{ab}(\lambda_j)\right\} - \frac{2\pi}{n} I_{ab}(\pi)\, 1\,(m = n/2) \tag{1.42}$$

is the averaged (cross-) periodogram, where for integer j, $\lambda_j = 2\pi j/n$ are the Fourier frequencies,

$$I_{ab}(\lambda) = w_a(\lambda)\, w_b'(-\lambda) \tag{1.43}$$

being the (cross-) periodogram and

$$w_a(\lambda) = \frac{1}{(2\pi n)^{1/2}} \sum_{t=1}^{n} a_t e^{it\lambda}, \tag{1.44}$$

the discrete Fourier transform. Note that

$$\widehat{F}_{ab}(1,m) = \widehat{F}_{ab}(0,m) - \overline{a}\overline{b}, \tag{1.45}$$

with $\overline{a} = n^{-1}\sum_{t=1}^{n} a_t$, so omission of zero frequency implies sample-mean correction. Under the assumption

$$\frac{1}{m} + \frac{m}{n} \to 0 \text{ as } n \to \infty, \tag{1.46}$$

the averaged (cross-) periodograms are based on a degenerating band of frequencies around 0, so that (1.41) only considers low-frequency components of the series in the relation of co-integration. In this situation, $\overline{\nu}_l(m)$ is the narrow band estimate of ν. This is certainly a sensible approach, as co-integration defines a long-run relationship, and in order to estimate the co-integrating parameter, we could hope that extracting from the observable series the relevant elements, we avoid high-frequency components that could be distortive and uninformative in order to asses for a low-frequency phenomenon. Note also that from the orthogonality properties of the complex exponential (see (2.95) below), $\overline{\nu}_0([n/2]) = \overline{\nu}_O$ in (1.33), and similarly

$$\overline{\nu}_1([n/2]) = \frac{\sum_{t=1}^{n}(y_t - \overline{y})(x_t - \overline{x})}{\sum_{t=1}^{n}(x_t - \overline{x})^2}, \tag{1.47}$$

which are the OLS estimates without and with intercept respectively. The NBLS estimate was proposed by Robinson (1994c). It is related to the band estimate proposed by Hannan (1963), developed later by Engle (1974), with the fundamental difference that the band estimate focuses on a nondegenerate band of frequencies, so (1.46) does not hold. Due to (1.46), NBLS resembles nonparametric spectral estimation, where now the focus is the parameter ν instead of a spectral density at a given fixed frequency. Robinson (1994c) showed the consistency of the NBLS in case of stationary co-integration (with stationary or asymptotically stationary observables), where, as mentioned before, OLS is inconsistent. The reason for this is that focussing on a slowly degenerating band of low frequencies reduces the bias due to contemporaneous correlation between u_{1t} and u_{2t}. Robinson and Marinucci (1998) gave a rate of convergence (which they conjectured as sharp) for the NBLS estimate of ν, when the memory parameters of the observables and co-integrating error are $\delta < 1/2$ and $\gamma \geq 0$ respectively. In a similar framework, Christensen and Nielsen (2001), provided a better rate than that of Robinson and Marinucci (1998), and showed that under their assumptions, the NBLS has a normal asymptotic distribution. This was at cost of introducing a very strong condition, which in our model (1.25), (1.26) would imply that the coherency between the weak dependent processes u_{1t}, u_{2t}, at frequency 0 is 0, condition that is not satisfied if for example u_t is a bivariate finite ARMA. They only considered the case $0 \leq \gamma < \delta < 1/2$, $\delta + \gamma < 1/2$.

For the nonstationary case, Robinson and Marinucci (1998, 2001) also exploited the bias reduction achieved by focussing on a degenerating band of frequencies around 0, and showed that in case $2\delta - 1 < \beta$ or $2\delta - 1 = \beta$ with $\delta > \beta$, the rates of convergence previously given for the OLS can be improved upon. These are now $n^{\beta} m^{2\delta - \beta - 1}$ if $2\delta - 1 < \beta$, $n^{\beta} / \log m$ if $2\delta - 1 = \beta$ with $\delta > \beta$, and n^{β} otherwise, noting (1.46). As OLS, NBLS has nonstandard limiting distributions in all situations. For $CI\,(1,1)$ co-integration, convergence rates of $\overline{\nu}_l\,(m)$ and $\overline{\nu}_l\,([n/2])$ are identical, but $\overline{\nu}_1\,(m)$ eliminates the "second-order bias" present in the asymptotic distribution of $\overline{\nu}_1\,([n/2])$, which is similar to (1.34) with demeaned Brownian motions instead the undemeaned ones. The superiority of the NBLS over the OLS does not appear when comparing $\overline{\nu}_0\,(m)$ and $\overline{\nu}_0\,([n/2])$ however, for this standard $CI\,(1,1)$ case.

Other first stage estimation methods

The traditional $CI\,(1,1)$ literature has proposed other methods to estimate either ν in (1.25), or alternatively a basis for the co-integrating space. In general, these methods enjoy less popularity than the previous two (especially than OLS), and we also considered them as first stage procedures, as they do not require to incorporating information about (γ, δ). Stock and Watson (1988) proposed two tests for the number of stochastic trends driving the behaviour of a multivariate unit root process. Equivalently, these tests could be viewed as tests for co-integrating rank. As an intermediate step for the feasibility of their test statistics, these authors suggested a consistent estimate of a basis of the co-integrating space consisting of orthonormal co-integrating vectors. As these co-integrating vectors are linear combinations of the vector of observable $I\,(1)$ variables, say z_t, with bounded variance, they proposed the following approach. The first co-integrating vector forms the linear combination of z_t having the smallest variance, the second co-integrating vector having the next smallest variance and so on. Thus, in case the co-integrating rank is r, the co-integrating vectors are estimated as those linear combinations corresponding to the smallest r principal components, leading this method to estimates of the co-integrating vectors up to an arbitrary linear transformation.

Bossaerts (1988) proposed a different estimate of a basis of the co-integrating space. Given certain vector of $I(1)$ variables z_t with co-integrating rank r, his idea was to use canonical correlation analysis, which searches for linear combinations of elements of z_t and linear combinations of z_{t-1} which are maximally correlated subject to certain normalization constraint. He concluded that the last r canonical variables, which are the r canonical variables of z_t and z_{t-1} with smallest squared correlation coefficient between them, are defined by vectors in the co-integrating space, hence they are co-integrating vectors.

Finally, Phillips (1995) motivated by the well reported non-Gaussianity of financial data (mainly in terms of leptokurtosis and heavy tails), analysed asymptotic properties of the least absolute deviations (LAD) and M-estimates of ν in model (1.25), (1.26) with $\gamma = 0$, $\delta = 1$. Defining

$$\overline{\nu}_{LAD} = \arg\min_{\alpha} \sum_{t=1}^{n} |y_t - \alpha x_t|, \tag{1.48}$$

Phillips showed that like OLS, the LAD estimate although n-consistent, suffers from nonstandard asymptotics. Also, the limiting distribution of $\overline{\nu}_{LAD}$ depends on the value at the origin of the probability density function of u_{1t}, noting that due to the particular shape of this limiting distribution (similar to (1.40)), the scale effect due to this factor has a more distortive effect than just inflating the asymptotic variance of the estimate of ν. Phillips also proposed a general M-estimate given by

$$\overline{\nu}_{M} = \arg\min_{\alpha} \sum_{t=1}^{n} \Upsilon(y_t - \alpha x_t), \tag{1.49}$$

where Υ is a chosen function. Potentially, this general framework could include the LAD estimate (and indeed also the OLS), but Phillips set some restrictive conditions on Υ, as twice differentiability, which ruled out this possibility. Nevertheless, he gave some hints on how to treat the case where Υ is non-differentiable. As expected, the general M-estimate of ν has also a nonstandard limiting distribution, depending on a scale factor given by $E(\Upsilon''(u_{1t}))$, where Υ'' represents the second derivative of Υ.

As mentioned before, the nonstandard limiting distributions of first stage methods make statistical inference problematic, and our work in the thesis is devoted to providing estimates, that although computationally slightly more involved than simple first stage procedures, enjoy standard asymptotic theory without assuming knowledge of the $I(\gamma)/I(\delta)$ structure of the model. Also, although convergence rates of OLS and NBLS are optimal in some circumstances, in others, their rates seem capable of further improvements over some regions of the (γ, δ)-space. In Chapters 2, 3, 4, we will provide estimates which apart from enjoying asymptotic distributions leading to standard statistical inference, are, in some cases, faster than OLS or NBLS.

1.3.2 Second stage procedures

Full system parametric estimation

Phillips (1991a) proposed full system estimation of a multivariate triangular system error correction mechanism representation which, corresponding to (1.25), (1.26), with $\gamma = 0$, $\delta = 1$, is given by

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \nu \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + v_t, \tag{1.50}$$

where

$$v_t = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} u_t, \tag{1.51}$$

noting that linearity in the co-integrating parameter ν is kept, all the transient dynamics being absorbed by the error process v_t or equivalently u_t. The linearity imposed in the system produces equivalence between full system Gaussian maximum likelihood (ML) estimation and simple OLS in a suitably augmented model. In case u_t is assumed to be $iid\,(0, \Omega)$, the full system Gaussian ML estimate of ν is equivalent to the OLS estimate of ν in the augmented linear regression equation

$$y_t = \nu x_t + \varphi \Delta x_t + u_{1.2t}, \tag{1.52}$$

where

$$u_{1.2t} = u_{1t} - \varphi u_{2t}, \quad \varphi = \omega_{12}/\omega_{22}. \tag{1.53}$$

Prior information about the unit root present in the system is crucial, and in fact Phillips (1991a) admits that in our bivariate structural model rewriting (1.26) with $\delta = 1$ as

$$x_t = \eta x_{t-1} + u_{2t}, \tag{1.54}$$

with $\eta = 1$, the key to obtain optimal asymptotic theory is to incorporate in the estimation the valid information that $\eta = 1$, which is equivalent to knowledge that $\gamma = 0$, $\delta = 1$ in (1.25), (1.26). Full system estimation involving unrestricted parameters ν, η, would produce estimates of ν with non-optimal properties due to the, in this particular case, explicit estimation of the unit root parameter η. In fact, due to the triangularity of (1.25) with $\gamma = 0$, (1.54), with the second equation already in reduced form, two stages least squares (2SLS) is equivalent to the full information ML estimate of ν. Thus, taking x_{t-1} as instrument for x_t in (1.25), maintaining $u_t \sim iid\,(0, \Omega)$, the asymptotic distribution of the 2SLS estimate of ν, $\overline{\nu}_{2SLS}$ is

$$n\left(\overline{\nu}_{2SLS} - \nu\right) \Rightarrow \frac{\int_0^1 W_2(\Omega; r)\, dW_{1.2}(\Omega; r)}{\int_0^1 W_2^2(\Omega; r)\, dr} + \frac{\omega_{12} \int_0^1 W_2(\Omega; r)\, dW_2(\Omega; r)}{\omega_{22} \int_0^1 W_2^2(\Omega; r)\, dr}, \tag{1.55}$$

where the "second-order bias" term present in the asymptotic distribution of the OLS estimate is eliminated (see (1.35)), but not the unit root distribution. The white noise case is heavily stressed in Phillips (1991a), although a similar "augmentation" of the OLS could be done in case u_t has an AR representation of finite order. For example, in case

$$u_t = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} u_{t-1} + \varepsilon_t, \tag{1.56}$$

where ε_t is $iid\,(0, \Omega)$, the optimal estimate of ν would come from unrestricted OLS in the augmented regression

$$y_t = \nu x_t + \varphi \Delta x_t + b y_{t-1} - \nu b x_{t-1} + \varepsilon_{1.2t}, \tag{1.57}$$

where

$$\varepsilon_{1.2t} = \varepsilon_{1t} - \varphi \varepsilon_{2t}, \;\; \varphi = \omega_{12}/\omega_{22}. \tag{1.58}$$

The treatment of an arbitrary $I\,(0)$ linear process u_t is more delicate, however. Next, we propose asymptotically equivalent methods to full system Gaussian ML estimation. Given the AR representation for u_t

$$B\,(L)\,u_t = \varepsilon_t, \tag{1.59}$$

with $\varepsilon_t \sim iid\,(0, \Omega)$,

$$B\,(s) = I_2 - \sum\nolimits_{j=1}^{\infty} B_j s^j, \tag{1.60}$$

where I_r is the $r \times r$ identity matrix, the first method is a time-domain approximation to the infeasible generalised least squares estimate of ν, given by

$$\widehat{\nu} = \frac{\sum_{t=1}^{n} (B_1\,(L)\,x_t^{\#})'\Omega^{-1} B\,(L)\,(y_t^{\#}, \Delta x_t^{\#})'}{\sum_{t=1}^{n} (B_1\,(L)\,x_t^{\#})'\Omega^{-1} B_1\,(L)\,x_t^{\#}}, \tag{1.61}$$

where $B_1\,(L)$ denotes the first column of $B\,(L)$. Of course this estimate is infeasible, but replacing Ω, $B\,(L)$ by suitable consistent parametric estimates $\widehat{\Omega}$, $\widehat{B}\,(L)$ respectively, the feasible version of $\widehat{\nu}$ would have under relatively mild conditions the same asymptotic properties of $\widehat{\nu}$ to first order.

More elegant seems the proposal of Phillips (1991a) of a fully parametric frequency-domain approximation to the Gaussian likelihood, known as the Whittle approximation. Here, noting (1.28), we define

$$p\,(\lambda) = (1, 0)\, f^{-1}\,(\lambda)\,, \;\; q\,(\lambda) = (1, 0)\, f^{-1}\,(\lambda)\,(1, 0)'\,, \tag{1.62}$$

and the infeasible Whittle estimate of ν is given by

$$\widetilde{\nu} = \frac{\sum_{j=1}^{n} p\,(\lambda_j)\, w_x\,(-\lambda_j)\,(w_y\,(\lambda_j)\,, w_{\Delta x}\,(\lambda_j))'}{\sum_{j=1}^{n} q\,(\lambda_j)\, I_x\,(\lambda_j)}, \tag{1.63}$$

noting (1.43), (1.44). A feasible version could be obtained by replacing $p\,(\lambda)$, $q\,(\lambda)$ by consistent parametric estimates. All these approaches would produce optimal estimates under Gaussianity with mixed normal asymptotic distributions, but all of them require knowledge of the $I\,(1)\,/I\,(0)$ structure of the model, which is the reason for the presence of first differences of x_t throughout.

Jeganathan (1997) considered model (1.25) with $\gamma = 0$, (1.54), where $|\eta| \leq 1$ and u_t with known density function. His approach was based on a one-step iterative procedure from a suitable preliminary estimate. He showed that in order to obtain analogous optimality properties to previous methods (with mixed-normal asymptotic distribution and corresponding Wald tests with χ^2 null limit distribution) in case $\eta = \pm 1$, these unit roots needed to be imposed in the estimation procedure.

In fractional circumstances, Jeganathan (1999, 2001) considered ML estimation in (1.31), (1.32), stressing pure fractional $v_t(\gamma, \delta)$ (corresponding to white noise u_t in (1.25), (1.26)), having innovations with completely known, but not necessarily Gaussian, distribution. He obtained mixed normal asymptotics for his estimate of ν, in case γ and δ are known, though including some discussion of their estimation. In fact, he did not consider (1.32) explicitly, but

$$\tilde{x}_t = \eta \tilde{x}_{t-1} + v_{2t}^{(\delta)}, \tag{1.64}$$

with $|\eta| \leq 1$, but apart from also considering the case $\eta = -1$, Jeganathan's model allowing for a free extra parameter η, is not more general that (1.31), (1.32). Denoting for example $v_{2t}^{(\delta)} = \Delta^{-(\delta-1)} u_{2t}$, if $|\eta| < 1$, (1.64) implies that

$$\tilde{x}_t = \Delta^{-(\delta-1)} e_{2t}, \tag{1.65}$$

with

$$e_{2t} = \eta e_{2t-1} + u_{2t}, \tag{1.66}$$

so that $\tilde{x}_t$ is a completely standard Type I fractionally integrated process of order $\delta - 1$. If on the contrary $\eta = 1$, with the extra assumption $\tilde{x}_0 = 0$, (1.32) is the right representation of $\tilde{x}_t$. Thus, it seems that (1.32) for certain general $I(0)$ process u_{2t}, with $\delta \in (-1/2, 1/2) \cup (1/2, 3/2)$ captures both situations for a different definition of fractionally integrated process. It is true that in Jeganathan's framework the input $I(0)$ series generating the fractionally integrated process is different depending on whether $|\eta| < 1$ (e_{2t} is the input series generating $\tilde{x}_t$) or $\eta = 1$ (u_{2t} is the input series generating $\tilde{x}_t$), but this does not seem a very relevant difference.

We devote Chapters 2 and 3 in this thesis to investigate model (1.25), (1.26), from a fully parametric perspective, including cases where γ and δ are unknown.

Full system nonparametric frequency domain approach

Inspired by Hannan (1963), Phillips (1991b) proposed a narrow band frequency domain estimate with optimal asymptotic properties under Gaussianity which relies on a nonparametric estimate of the spectral density matrix of the error u_t in (1.25), (1.26) with $\gamma = 0$, $\delta = 1$ (or equivalently the one of v_t in (1.50)). The idea of his approach is that taking Fourier transforms in (1.25), (1.26), we obtain a triangular system in the frequency domain given by

$$\begin{pmatrix} w_y(\lambda) \\ w_{\Delta x}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \nu \end{pmatrix} w_x(\lambda) + w_u(\lambda). \tag{1.67}$$

Given the spectral density of u_t, $f(\lambda)$, we could estimate ν efficiently applying full band weighted least squares to (1.67), obtaining $\widetilde{\nu}$ in (1.63). Of course, this estimate is not feasible, as in practice $f(\lambda)$ is unknown. Focussing on a nonparametric approach, one could replace $f(\lambda)$ by some nonparametric estimate and show that the feasible estimate share the asymptotic properties, to first order, of the infeasible one. Phillips introduced two modifications on this idea. First, as co-integration is basically a long-run phenomenon, we could concentrate on a degenerating band of frequencies concentrated around frequency 0. Also, he replaced the (cross-) periodograms $w_x(-\lambda_j)(w_y(\lambda_j), w_{\Delta x}(\lambda_j))'$ and $I_x(\lambda_j)$ by consistent estimates of the corresponding (cross-) spectrums (more precisely these were averaged periodograms), although it is possible to show that this change does not matter asymptotically, at least to first order asymptotic properties. Furthermore, he also presented an estimate that is the narrow band equivalent of (1.63), where $p(\lambda)$, $q(\lambda)$, are replaced by nonparametric estimates of $p(0)$, $q(0)$ respectively. As Phillips (1991b) showed, this estimate also enjoys optimal asymptotic distribution under Gaussianity, the reason being that although $p(0)$, $q(0)$ are "imperfect" weights compared to $p(\lambda_j)$, $q(\lambda_j)$, as the estimate only concentrates on a degenerating band of frequencies around 0, the weights are approximately correct.

As an alternative to previous procedures, a different but asymptotically equivalent nonparametric approach would be to employ a similar AR orthogonalization to the one given in the parametric estimation, assuming u_t is an AR process of order p (AR(p)), with p tending suitably slow to infinity.

For fractional models, in a multivariate semiparametric version of (1.31), (1.32), and allowing also for the possibility of nonstationary $v_{1t}^{(\gamma)}$, Velasco (2000) considered a tapered version of local Whittle estimation of ν, γ and δ, for the case $1/2 < \delta < 3/2$, $0 \leq \gamma < \delta$ with $\beta > 1/2$, more particularly taking one Newton step from preliminary estimates with suitable convergence rates. This produces an estimate of ν which does not have optimal convergence rate but, unlike ours described in Chapter 4 and those in the other references, is asymptotically normal. In a similar setting, Hassler, Marmol and Velasco (2002) focused on log periodogram estimation of γ and δ given preliminary estimation of ν, developing rules of asymptotic inference. As explained in Section 1.5, our approach in Chapter 4 deals also with a nonparametric situation, being close in spirit to Phillips (1991b), but including cases where knowledge of the orders γ, δ is not assumed.

Fully modified OLS (FM-OLS)

Several authors have proposed modifications of the OLS in (1.25), with the aim of obtaining estimates of ν sharing the asymptotic properties of the fully parametric Gaussian ML estimate of ν. This was originated by the work of Phillips and Hansen (1990) for the case $\gamma = 0$, $\delta = 1$. These authors proposed an optimal single equation procedure based on appropriate treatment of the autocorrelation structure of the process u_t in a multivariate extension of our basic model (1.25), (1.26) with $\gamma = 0$, $\delta = 1$. The aim of the method is to remove bias and endogeneity effects that this autocorrelation in general produces. Their FM-OLS estimate of ν is given by

$$\overline{\nu}_{FM} = \frac{\sum_{t=1}^{n} x_t y_t^+ - n\widetilde{\Lambda}'\left(1, -\widetilde{\overline{\sigma}}_{22}^{-1}\widetilde{\overline{\sigma}}_{12}\right)}{\sum_{t=1}^{n} x_t^2}, \tag{1.68}$$

where

$$y_t^+ = y_t - \widetilde{\widetilde{\sigma}}_{22}^{-1}\widetilde{\widetilde{\sigma}}_{12}\Delta x_t, \tag{1.69}$$

$\widetilde{\Lambda}$ and $\widetilde{\widetilde{\sigma}}_{ij}$ being nonparametric estimates of $\Lambda = \sum_{k=0}^{\infty} E\left(u_{20}u_k\right)$ and of the (i,j)th element of $2\pi f\left(0\right)$ (the so-called long-run variance-covariance matrix of u_t) respectively. Again, note that in this approach, knowledge of the $I\left(1\right)/I\left(0\right)$ nature of the observables/co-integrating error is crucial, as it is precisely the use of this information which motivates the use of first differences of x_t in the modification of y_t and in the estimation of Λ and $2\pi f\left(0\right)$. The relevance of this work is that they achieved optimal estimation of ν under Gaussianity assumptions without the need of assuming a fully parametric structure for u_t, and also avoiding system estimation.

Park (1992), extending Park and Phillips (1988, 1989), proposed a similar modification to the OLS. Noting that a co-integrating relationship is not altered by certain modifications of the observables, in (1.25), (1.26) with $\gamma = 0$, $\delta = 1$, he proposed to transform the observables as

$$x_t^* = x_t - \left(\Sigma^{-1}\Gamma_2\right)' u_t, \tag{1.70}$$

$$y_t^* = y_t - \left(\Sigma^{-1}\Gamma_2\nu + \left(\begin{array}{cc} 0 & \overline{\sigma}_{12}\overline{\sigma}_{22}^{-1} \end{array}\right)\right)' u_t, \tag{1.71}$$

with $\Sigma = E\left(u_t u_t'\right)$, $\Gamma_2 = \left(\gamma_{12}, \gamma_{22}\right)'$, with

$$\gamma_{ij} = \sum_{k=0}^{\infty} E\left(u_{it}u_{jt-k}\right),\ i,j = 1,2. \tag{1.72}$$

Park showed that these transformations had nonnegligible effects on the limiting distribution of the least squares estimates based on the transformed variables and, in fact, this estimate enjoyed the mixed normal asymptotic distribution also achieved by Phillips and Hansen (1990). It is clear that modifications (1.70), (1.71) are close in spirit to those of these latter authors. Of course, these transformations are infeasible, but the unknown parameters related to the covariance structure of u_t could be replaced by appropriate nonparametric estimates, ν by its OLS estimate, and u_t by the residuals $\left(y_t - \overline{\nu}_O x_t, \Delta x_t\right)'$ (see (1.33)). Park showed the validity of a feasible estimate constructed following these lines. The main advantage of his procedure over Phillips and Hansen's one is that it requires only a once-and-for-all transformation of the data. Once the data are transformed, standard regression software will be enough to carry on any statistical analysis.

In a fractional framework, Dolado and Marmol (1996) considered a fractional extension of the FM-OLS estimate of ν, with nonparametric autocorrelation in u_t, and assuming knowledge of γ and δ. In relation to (1.31), (1.32), with ν a matrix and both equations vectors but depending still on only two integration orders γ and δ, Kim and Phillips (2000) consider an alternative extension of FM-OLS to that of Dolado and Marmol (1996), and its relation to Gaussian ML estimation. They assume parametric autocorrelation in $v_t(\gamma,\delta)$, obtaining limit distribution theory that differs from that of Jeganathan (1999, 2001), and from ours in Chapters 2 and 4 below (see (2.32)), even after replacing their version of fractional Brownian motion by ours. They also consider estimation of nuisance parameters, but only treated the cases $1 \leq \delta < 3/2$, $\gamma + \delta > 1$, $-1/2 < \gamma < 1/2$, which imply $\beta > 1/2$ and $1 \leq \delta < 3/2$, $1/2 < \gamma < 1$ for the case $\beta > 1$.

Other second stage estimation methods

A different research strategy was based on single equation error correction mechanism. Noting that from (1.25), (1.26) with $\gamma = 0$, $\delta = 1$,

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \begin{pmatrix} \Delta & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \tag{1.73}$$

so that in general

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = C(L)\,\varepsilon_t, \tag{1.74}$$

where ε_t is a bivariate *iid* process, for a certain moving average (MA) polynomial $C(L) = \sum_{j=0}^{\infty} C_j L^j$. For our particular situation, the Granger Representation Theorem implies that $C(1)$ is of rank 1, so there exists a 2×1 vector a, such that $C(1)\,a = 0$. This theorem also implies that there exists a vector ARMA representation

$$A(L) \begin{pmatrix} y_t \\ x_t \end{pmatrix} = d(L)\,\varepsilon_t, \tag{1.75}$$

for certain lag polynomials $A(L)$, $d(L)$, and also an error correction representation

$$A^*(L)(1-L) \begin{pmatrix} y_t \\ x_t \end{pmatrix} = -a\,(y_{t-1} - \nu x_{t-1}) + d(L)\,\varepsilon_t, \tag{1.76}$$

where

$$A(L) = A(1) + (1-L)\,A^*(L), \tag{1.77}$$

and $A^*(0) = I_2$. In general, $A(L)$, $A^*(L)$, $d(L)$ are infinite AR lag polynomials, but in practice finite-order approximations are used, the purely AR representation where $d(L) = 1$ having been stressed in the literature. Note that ν appears nonlinearly in (1.76), as a is unknown and must be estimated.

Stock (1987) analysed through a Monte Carlo experiment the case where $A^*(L) = (1 - \rho L)\,I_2$ and $d(L) = 1$ in (1.76), and estimated ν by means of nonlinear least squares in (1.76). His main finding was large Monte Carlo bias for this estimate. From Phillips and Loretan's (1991) arguments, it is clear that the asymptotic distribution of Stock's estimate is non-standard, with bias, asymmetry and non-scale nuisance parameters. Stock's approach is very related to Hendry's methodology, explained precisely in Hendry and Richard (1982, 1983). This approach suggests working back from a very general unrestricted dynamic specification towards certain more parsimonious model satisfying certain prescriptions, including that the model should fit the data up to a white noise innovation which is a martingale difference sequence relative to the selected data base. The starting point of this methodology is a general unrestricted regression, which in case $u_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}$ with $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ being $iid(0, \Omega)$, is equivalent to running least squares on the equation

$$y_t = \nu x_t + a(L)\,(y_t - \nu x_t) + b(L)\,\Delta x_t + \varepsilon_{1.2t}, \tag{1.78}$$

noting (1.58), where $a(L)$, $b(L)$ are lag polynomials of infinite order. Phillips (1988) and Phillips and Loretan (1991) showed that, in general, this single equation procedure does not lead to optimal inference, due to the improper account for autocorrelation given in (1.58). In fact, mixed normal asymptotics would be attained in case $\varepsilon_{1.2t}$ and u_{2t} were incoherent at frequency 0, but this is not usually the case, as in general, u_{2t} is not necessarily orthogonal to the past history of $\varepsilon_{1.2t}$. In any case, as Phillips (1988) admits, Hendry approach comes remarkably close to achieving optimal asymptotic properties.

Saikkonen (1991) presented asymptotically efficient estimates inspired by Hendry's error correction model methodology. In a multivariate version of (1.25), (1.26),with $\gamma = 0$, $\delta = 1$, based on the validity under certain regularity conditions of the projection

$$u_{1t} = \sum_{j=-\infty}^{\infty} \Pi_j u_{2t-j} + \eta_t, \tag{1.79}$$

where η_t is an $I(0)$ process such that

$$E(u_{2t}\eta_{t+j}) = 0,\ j = 0, \pm1, \pm2, ..., \tag{1.80}$$

this author proposed to estimate by OLS the linear regression equation

$$y_t = \nu x_t + \sum_{j=-p}^{p} \Pi_j \Delta x_{t-j} + \dot{\eta}_t, \tag{1.81}$$

where

$$\dot{\eta}_t = \eta_t + \sum_{|j|>p} \Pi_j \Delta u_{2t-j}, \tag{1.82}$$

so that proper orthogonalization is "almost" achieved, as heuristically Π_j is close to 0 for $|j| > p$ and p large enough. As, in general, one cannot assume that $\Pi_j = 0$ for $|j| > p$, for the asymptotic argument to go through, it is necessary to require that p tends to infinity with n at a suitable rate. Clearly, the choice of p is a delicate issue here, and the author suggests experimenting with a few values of p in empirical analysis. In any case, this difficulty is at the same level of the choice of bandwidth for consistent estimates of the long-run variance-covariance matrix of u_t in Phillips and Hansen's (1990) method, or even the choice of a parametric model for fully parametric methods like Phillips (1991a). An unpleasant issue related to Saikkonen's method is that his estimate is infeasible unless the future values $x_{n+1}, ...x_{n+p}$, are known. Thus, removal of the p most recent observations of y_t seems necessary in general. As p grows with n but at a slower rate, this removal could be negligible asymptotically, but the finite sample performance of the estimate will surely be affected.

Phillips and Loretan (1991) proposed a very similar method. The problem with estimation of the equation (1.78) is that u_{2t} is not necessarily orthogonal to the past history of $\varepsilon_{1.2t}$, hence these two processes are not incoherent at frequency 0. By means of the linear least squares projection

$$\widetilde{E}\left(\varepsilon_{1.2t} \middle| \{u_{2s}\}_{s=t+1}^{\infty}\right) = \sum_{k=1}^{\infty} c_k u_{2t+k}, \tag{1.83}$$

denoting $c(L) = \sum_{k=1}^{\infty} c_k L^k$, the error

$$\varepsilon_{1.2t}^{*} = \varepsilon_{1.2t} - c\left(L^{-1}\right) u_{2t}, \tag{1.84}$$

is a martingale difference sequence with respect to the filtration

$$M_{t-1} = \sigma\left(\{u_{1s}\}_{s=-\infty}^{t-1}, \{u_{2s}\}_{s=-\infty}^{\infty}\right). \tag{1.85}$$

Thus, nonlinear least squares in

$$y_t = \nu x_t + a(L)(y_t - \nu x_t) + b(L)\Delta x_t + c\left(L^{-1}\right)\Delta x_t + \varepsilon_{1.2t}^{*}, \tag{1.86}$$

would produce asymptotically efficient estimates under Gaussian assumptions. A similar problem as in Saikkonen (1991) also appears in order to deal with the possibly infinite lag polynomials in (1.86). In fact, Phillips and Loretan reported results for four different combinations of number of leads and lags in their Monte Carlo experiment. As Saikkonen suggested, Phillips and Loretan's procedure has the computational disadvantage of facing a nonlinear estimation problem, whereas Saikkonen's method was linear. On the contrary, the residual from the nonlinear regression in Phillips and Loretan approximates a white noise process, this not being the case in Saikkonen's approach. Thus, hypothesis testing on ν could be constructed in a very simple way, as normalisation only implies the estimation of a matrix that could be straightforwardly approximated by sample second moments of the residuals from the nonlinear regression. Stock and Watson (1993) extended this approach to situations of co-integration with general $I(d)$ variables and deterministic components, where d is integer but not necessarily 1.

Apart from these methods, a couple of procedures have been proposed which are useful in case the Gaussian assumption is unrealistic. First, we present the so-called adaptive estimates. Jeganathan (1995), in a multivariate version of model (1.25), (1.26), with $\gamma = 0$, $\delta = 1$, a single co-integrating relation and $u_t \sim iid(0, \Omega)$, proposed an adaptive estimate of the equivalent to ν in his multivariate model. The previously discussed second stage methods were Gaussian, in the sense that they were optimal in case the data were Gaussian. Sometimes, the Gaussian assumption is highly unrealistic, and proper ML estimation exploiting knowledge of the non-Gaussian joint density of the process u_t would achieve higher efficiency than Gaussian methods. Adaptive estimation produces estimates which share the asymptotic optimality properties of the ML estimate in case the density function of the error input series is unknown. Should the density be known, one could always obtain an asymptotically efficient estimate by a one-step procedure from certain adequate preliminary estimate of the parameter of interest. However, this requires knowledge of the score and information of the density of u_t. In case the density is unknown, one could compute nonparametric estimates of these quantities and substitute them by the true quantities in the iterative procedure. Jeganathan (1995) showed that this is an asymptotically valid method in the sense that the same asymptotic distribution as the ML estimate was achieved by his adaptive estimate. This limiting distribution is mixed normal with smaller conditional variance than the one offered by Gaussian methods in case data are not Gaussian. Jeganathan showed this under the relatively strong condition that the (in our case) bivariate joint density $p(a,b)$ of u_t should be elliptical symmetric, that is

$$p(a,b) = |\det \Omega|^{-\frac{1}{2}} f^* \left(\left\| \Omega^{-\frac{1}{2}} \begin{pmatrix} a & b \end{pmatrix}' \right\| \right), \tag{1.87}$$

for some function f^*, where here $\|\cdot\|$ denotes euclidean norm.

Hodgson (1998a) extended Jeganathan's work to allow for ARMA process of order r, q (ARMA(r,q)) with finite r, q, structure for the co-integrating error in (1.25), that is

$$u_{1t} = \sum_{j=1}^{r} a_j u_{1,t-j} + \sum_{j=1}^{q} b_j \varepsilon_{1,t-j} + \varepsilon_{1t}, \tag{1.88}$$

where $(\varepsilon_{1t}, u_{2t})'$ is an *iid* vector sequence. There could be controversy on whether to describe Hodgson's approach as adaptive, because the ARMA structure for u_{1t} was assumed to be known (of course without knowledge of the ARMA parameters), although the joint density of $(\varepsilon_{1t}, u_{2t})'$ was assumed unknown. "Adaptive" estimation of ν was proposed, assuming also joint density of $(\varepsilon_{1t}, u_{2t})'$ with the elliptic symmetry property.

As an alternative non-Gaussian robust method, corresponding to the first stage LAD and M-estimates, Phillips (1995) also proposed fully modified versions of these estimates. As opposite to most of the previously discussed second stage procedures, these fully modified estimates are non-Gaussian, as they do not share the asymptotic properties of the maximum likelihood estimates when the data are Gaussian. The fully modified LAD (FM-LAD) estimate requires a very similar correction to the one in Phillips and Hansen (1990), achieving also mixed normal asymptotics, with the extra requirement that certain nonparametric consistent estimate of the density of u_{1t} at 0 is necessary. Phillips also showed that provided u_{1t} is leptokurtic enough, the FM-LAD is more efficient than the FM-OLS, meaning this in the present framework smaller conditional variance in the limiting distribution. Using a very similar type of correction, Phillips also presented a fully modified M-estimate, which achieved mixed normal asymptotic distribution via a nonparametric correction similar to the one for the FM-LAD, the most distinctive feature being that the sample mean of $\Upsilon''(\widehat{u}_{1t})$ is involved, where $\widehat{u}_{1t}$ are residuals originated by certain preliminary consistent estimate of ν.

Johansen (1988) derived ML estimates of the co-integrating vectors for a co-integrated vector autoregressive (VAR) process with independent Gaussian errors. He assumed that a p-dimensional vector of random variables z_t had a VAR representation

$$z_t = \Pi_1 z_{t-1} + \Pi_2 z_{t-2} + \dots + \Pi_k z_{t-k} + \varepsilon_t, \tag{1.89}$$

where $\varepsilon_t \sim iid(0, \Omega)$. Johansen considered the case where the determinant of the polynomial

$$B(s) = I_p - \Pi_1 s - \Pi_2 s^2 - \dots - \Pi_k s^k, \tag{1.90}$$

has roots at $s = 1$. More specifically, he assumed that Δz_t was $I(0)$ and that

$$\Pi = I_p - \Pi_1 - \Pi_2 - \dots - \Pi_k, \tag{1.91}$$

had rank $r < p$, indicating this in terms of the Granger representation theorem that there are r co-integrating relations among the elements of z_t. Expressing Π as $\Pi = a\alpha'$ for suitable $p \times r$ matrices a, α, the linear combinations $\alpha' z_t$ are $I(0)$, and the space spanned by α is denoted as the co-integrating space. Johansen employed a method based on canonical correlations, and showed that a suitably normalised ML estimate of α was mixed normal asymptotically distributed, although, as he admits, this result is not very useful in practice as his normalization depends on the unknown matrix α. Knowledge of the co-integrating rank (the number of linearly independent co-integrating vectors) was essential in order to derive his result. Apart from this important result, which could be taken as an intermediate step in the whole of his analysis, one of the strongest contributions of his work is to propose a test of linear restrictions about the co-integrating vectors with standard χ^2 null limit distribution. He also proposed a likelihood ratio test for the dimension of the co-integration space.

In a similar setting, Ahn and Reinsel (1990) suggested a partial reduced rank estimating procedure that explicitly incorporated the unit roots present in the model, obtaining estimates of the co-integrating vectors with mixed Gaussian limiting distributions. Their key idea was to write (1.89) in error correction form as

$$\Delta z_t = -\Pi z_{t-1} + \sum_{j=1}^{k-1} \Pi_j^* \Delta z_{t-j} + \varepsilon_t, \tag{1.92}$$

where $\Pi_j^* = -\sum_{l=j+1}^{k} \Pi_l$, $j \leq k-1$, and they estimated Π with the reduced rank structure imposed, which they found to be equivalent to imposing in the estimation of the VAR model (1.89), $p-r$ unit roots. Through this procedure, optimal inference was achieved in contrast to the full rank least squares estimation of (1.89), which suffer from the typical problems originated by implicit estimation of the unit roots present in the model. The authors related their work with Johansen's (1988), claiming more flexibility for their approach, in terms of allowance of very straightforward incorporation of zero constraints on the stationary parameters.

Johansen (1991) showed that his proposed ML estimate of the co-integrating relations had a mixed normal asymptotic distribution in more general framework than in Johansen (1988), allowing for a constant term and seasonal dummies in his specified VAR model. As in his previous work, the co-integrating rank was assumed to be known, but Johansen also proposed a likelihood ratio test for the null of r linearly independent co-integrating vectors against diverse alternatives, including co-integration spaces of dimensions $r+1$ and p. The test statistics related to the previous tests have nonstandard null limit distributions, but depending only on the dimension of the problem $(p-r)$ and certain behaviour of the constant term. Furthermore, he also presented a test for the validity of linear restrictions of the co-integrating space with χ^2 null limit distribution in this wider framework.

Finally, Hodgson (1998b) in a multivariate co-integrated finite order VAR proposed an adaptive estimate of the vector of the different parameters including both long and short-run coefficients, obtaining corresponding results to Johansen (1988) and Ahn and Reinsel (1990) (derived under the Gaussianity assumption) in case the density of the input error series is unknown.

1.4 Empirical evidence of fractional co-integration

There are numerous empirical applications based on the notion of fractional co-integration. For example, Diebold, Husted and Rush (1991) examined the purchasing power parity (PPP), that is the tendency for nominal exchange rates and prices to adjust in such a way that the real exchange rate reverts (perhaps slowly) to its parity value. Thus, the (log) real exchange rate could be viewed as the co-integrating error in a linear combination of (log) nominal exchange rate and (log) prices with co-integrating vector $(1,-1)'$. Although, the authors approximated the log of the real exchange rate in a particular way, and not as the difference of the logs of nominal exchange rate and prices, what seems clear from their empirical analysis is that taking into account that is well assumed in the literature that the (log) nominal exchange rate is a unit root process (see e.g. Baillie and Bollerslev, 1994a,b), they reported situations where the estimated memory of the real exchange rates (e.g. France/Germany, Germany/UK) suggested non-stationary mean-reverting behaviour, whereas for other real exchange rates the authors provided evidence of stationary long memory. This evidence, in view of the lack of power of the traditional unit root tests against fractional alternatives was suggested by the authors as the main reason why lack of PPP, that is unit root behaviour in real exchange rates, was argued in many studies.

In a similar framework, Cheung and Lai (1993) proposed to check the PPP hypothesis via a regression of a foreign price index converted to domestic (US) currency units on a domestic price index, the errors of this relation capturing deviations from the PPP. While they provided evidence of the unit root character of the observables, they stated that the PPP will be characterized by certain stationary, or at least mean reverting behaviour of the co-integrating error. They computed semiparametric estimates of the degree of memory of the co-integrating error for different countries and bandwidths, and provided evidence of co-integrating errors with positive memory.

Similarly, Baillie and Bollerslev (1994a) argued whether seven spot exchange rates appear to be tied together in the long run or not, taking into account that there seems not to be discussion in the literature about the unit root character of those series, being much more fragile the idea that those series are co-integrated, see e.g. Sephton and Larsen (1991), Diebold, Gardeazabal and Yilmaz (1994), who concluded that "there exists substantial uncertainty regarding the existence of co-integrating relationships among nominal dollar exchange rates". Baillie and Bollerslev's (1994a) explanation for this finding was that unit root tests, which served traditionally to detect the presence of unit roots, had very low power against fractional alternatives, hence a situation of fractional co-integration with long memory co-integrating error could be hidden. In fact, their estimate of the memory of the co-integrating error was $\widehat{d} = 0.89$, over five standard errors away from 1.

Baillie and Bollerslev (1994b), analysed the so-called forward premium, $f_t - s_t$, where s_t and f_t are logs of the spot exchange rate and of the one month maturity forward rate respectively, having in mind the "overwhelming" evidence of presence of unit roots in spot exchange rates. Again, the difference $f_t - s_t$ could be considered as a co-integrating error with co-integrating vector $(1,-1)'$. They claimed that standard unit root tests, like augmented Dickey-Fuller (ADF), (see Engle and

Granger, 1987) and KPSS (see Kwiatkowski, Phillips, Schmidt and Shin, 1992) generally reject that the forward premium is $I(0)$, which is paradoxical as given that forward premium is associated with risk, it seems hard to see any theoretical reason for a $I(1)$ risk premium. The purpose of their paper was to show that the forward premium is indeed mean-reverting, the estimates of the memory of the forward premium for Canada, Germany and UK (with respect to US) being 0.45, 0.77 and 0.55 respectively.

In a similar setting as Diebold, Husted and Rush (1991), Crato and Rothman (1994) provided estimates of the (log) real bilateral sterling exchange rates with different countries, and for several of them it was reported evidence of fractional co-integration.

Dueker and Startz (1998), analysing a fractional co-integration relation between US and Canadian bond rates, suggest that it is desirable not to rely on an assumed value for the order of integration of the observables as was done in previous empirical analyses related to fractional co-integration, which most commonly considered this integration order to be one. Their estimates of the memory of the observables and co-integrating error were 0.674 and 0.200 respectively.

Kim and Phillips (2000) provided a similar analysis to the one by Baillie and Bollerslev (1994a), assuming also the memory of certain series of exchange rates to be one. Evidence of fractional co-integration was reported.

Marinucci and Robinson (2001) analysed two macroeconomic data sets used in earlier papers by Engle and Granger (1987) and Campbell and Shiller (1987). For consumption and income, Engle and Granger found evidence of $CI(1,1)$ co-integration. Marinucci and Robinson estimated the memory parameter of both observables and showed that for different semiparametric methods and bandwidths they were very close to one for both variables. They also estimated the memory of the co-integrating error, and the estimates ranged from 0.19 to 0.87, suggesting this that the $CI(1,1)$ framework could produce a good approximation for the behaviour of the observable series but not for the co-integrating errors. For stock prices and dividends data in Campell and Shiller, Marinucci and Robinson concluded that the evidence of co-integration was weak, as it seemed clear some evidence of mean reverting behaviour of the dividends, and the estimated memory of the co-integrating error ranged from 0.57 to 0.77 for different methods and bandwidths. This could provide an explanation to Campbell and Shiller's findings that, in their own words, were inconclusive about the existence of co-integration.

Andersen, Bollerslev, Diebold and Ebens (2001) examined "realized" daily equity return volatilities and correlations obtained from high-frequency transaction prices on individual stocks in the Dow Jones Industrial Average. They provided evidence of long memory for certain time series of logarithmic standard deviations and correlations, and stressed the evidence of comovements in volatility across assets. Christensen and Nielsen (2001) took a similar point, and claimed the existence of stationary co-integration between the volatility implied in option prices and the subsequent realized return volatility of the underlying asset, as in their view, the observables (log-volatilities) were fractionally integrated processes with estimated order ranging from 0.35 to 0.4, whereas the co-integrating error seemed weak dependent. By using a narrow band estimate, they obtained a much higher value for the estimate of the slope of their co-integration relation than the one provided in a similar work by Christensen an Prabhala (1998), who used an OLS estimator, as showed by Robinson (1994c) inconsistent in the case of stationary co-integration. Finally, stationary co-integration has been also considered by Robinson and Yajima (2002), who provided empirical work on testing for the rank of co-integration among spot closing prices of crude oil.

1.5 Description of the thesis

Throughout the thesis, we establish a very clear distinction between the cases where $\beta > 1/2$ or $\beta < 1/2$. We denote the former situation as "strong fractional co-integration", as the order of integration of the observables is reduced by the linear combination in more than $1/2$, nesting this the traditional $CI(1,1)$ co-integration framework where $\beta = 1$. As mentioned before, this situation was also theoretically analysed by Kim and Phillips (2000) and Velasco (2000). The latter case is denoted as "weak fractional co-integration", because the memory of the observables could be reduced by just a very small amount in the linear combination. This appears to be a framework that theoretical researchers have been not paying much attention to, but it seems to be supported by some data and also covers relevant situations. For example in financial data, it could be that the observables are "less" stationary than the co-integrating error, and in macroeconomic data, it could well be that the observables have a close to unit root behaviour, whereas the co-integrating error is nonstationary but mean-reverting. In fact, most of the empirical evidence provided in the previous section supports these two possibilities. Note that we have omitted from our analysis the $\beta = 1/2$ case, which, as it can be inferred from our results in Chapters 2 and 3, would require a separate treatment. Thus, although from a theoretical viewpoint it could be interesting to fill this gap, it is important to note that the treatment of this very particular case would undoubtedly entail some difficulties, while its interest from a practical perspective is limited, and also we felt that omission of this specific situation was of less relative importance in view of the great generality that our treatment of all except one $\beta > 0$ allows.

Regarding the strong fractional co-integration case, we propose estimates with analogous optimal properties to the Gaussian second stage procedures applied to the triangular system (1.25), (1.26) for $CI(1,1)$, which were discussed in Section 1.3. Our main contribution here is that those optimal properties hold irrespectively of whether δ and/or γ are known or unknown, subject to adequate estimation of these orders in this latter case. Thus, we provide theoretical evidence that, on the contrary to what the literature suggests, the incorporation of information about the true orders in the estimation is not necessary in order to obtain optimal Gaussian estimates. The reason for this outcome can be easily understood by comparison of the equivalent models

$$x_t = \eta x_{t-1} + u_{2t}, \quad t \geq 1, \quad x_t = 0, \quad t < 1, \tag{1.93}$$

$$x_t = \Delta^{-\delta} u_{2t}^{\#}, \tag{1.94}$$

when $\delta = \eta = 1$. In the traditional approach, the incorporation of the information that $\eta = 1$, hence avoiding the estimation of this parameter, was crucial, because estimation of the unit root η along with ν in a full system estimation procedure, produced non-standard asymptotics for the estimate of ν due to the discontinuity on the behaviour of x_t at $\eta = 1$ in (1.93). In our approach, we estimate the equivalent to the unit root in (1.93), which is the parameter $\delta = 1$ in model (1.94), where the discontinuity does not appear, and therefore the estimation of ν is not affected. We consider our approach to be a step ahead in the direction that, for example, Jeganathan (1997) indicates. This author states that there is a generalised opinion in the profession that "procedures whose limiting distribution involves unit root component and nuisance parameters are not to be highly recommended and that Wald-type procedures having the limit central chi-squared are the ones to have sound statistical basis". In Jeganathan's opinion these optimality results by themselves have little meaning, as what he considers as the crucial issue is whether the underlying structure of the model is reasonably supported by the data. For example, the optimal Gaussian procedures are reasonable in case the $I(1)/I(0)$ structure is assumed or imposed on the underlying structure of the model, and he states that how well this fits the data is the relevant issue here. Noting that the acceptance of a null hypothesis of unit root means basically that the unit root structure has support in view of the available data, it could well be that the model could correspond to one in which the root is only close to unity, with the nature of this closeness being unknown. We consider that our approach fits naturally in the essence of Jeganathan's interesting thoughts, as we try to accommodate in a more realistic way the underlying structure of the model to the actual data, avoiding complications due to pre-testing. Related to this $\beta > 1/2$ case, Chapter 2 is devoted to analysing this strong fractional co-integration framework from a fully parametric perspective, where the short memory model driving the error input series in (1.25), (1.26) is known up to some finite vector of unknown parameters. We propose different time and frequency domain estimates, which are relatively straightforward generalisations of (1.61), (1.63) respectively. We show that those estimates (and also a competitive but computationally simpler one when $\beta > 1$), have analogous optimal properties to the Gaussian second stage procedures, with mixed normal asymptotics leading to Wald tests with χ^2 null limiting distributions, implying straightforward inference on ν. Our results nest the traditional $CI(1,1)$ framework, where in order to obtain the same asymptotic properties as ours, the values $\gamma = 0$, $\delta = 1$ were assumed to be known. In Chapter 4, we show that parametric assumptions about the $I(0)$ structure of the error input series u_t are not necessary in order to obtain same results as given in Chapter 2. In this chapter, we propose several different frequency domain estimates, including both full band and narrow band approaches, whose feasibility is achieved through nonparametric estimates of the spectral density matrix of u_t and semiparametric estimates of the orders γ, δ.

We also consider in the thesis the weak fractional co-integration situation. Here, our main contribution is to propose in the adverse situation where the co-integrating gap β is small, relatively simple estimates, which in all cases are asymptotically normal, and, at least in a fully parametric framework, enjoy optimal convergence rates, by which we mean that they match the rates achieved by the Gaussian ML estimate under suitable regularity conditions. The weak fractional co-integration case is more complex, and one could argue that has less interest than the $\beta > 1/2$ situation, which embodies the traditional $CI(1,1)$ framework. The $\beta < 1/2$ case contrasts heavily with this econometric prescription, but as noted in Section 1.4 empirical evidence has emerged of this possibility, and we will further motivate this issue in Chapter 3. We found that asymptotic inferential theory is very different in this case from members of the class $\beta > 1/2$. Chapter 3 is devoted to studying the $\beta < 1/2$ case assuming that the error input series u_t in (1.25), (1.26) is a VAR process of known finite order. Here, we propose time domain estimates of ν with $\sqrt{n}$ rate of convergence, asymptotically normal, less efficient that the Gaussian ML, but computationally more convenient, as only univariate optimization is involved. The estimate of ν depends on the estimates of γ, δ, and these estimates need to be $\sqrt{n}$-consistent, being the asymptotic variance of the estimate of ν sensitive to their precise form. In Chapter 4, we provide a nonparametric extension of Chapter 3, where u_t is an arbitrary $I(0)$ process of unknown form. Considering only different narrow band estimates, we showed that our estimates are asymptotically normal, slower than in the fully parametric case, and affected by the estimation of the orders γ, δ or $f(\lambda)$ in a subtle manner.

Finally, the thesis is completed with Chapter 5, which considers a test procedure for the equality of the orders of integration of two fractionally integrated processes. This topic does not directly refer to estimation of the co-integrating parameter, but, nevertheless, we found that this is a very relevant issue in any empirical analysis related to fractional co-integration, and of particular importance in order to justify the use of the techniques derived in the previous three chapters. Note that a necessary condition for two time series to be co-integrated is that their orders of integration be equal. Different test have been proposed in the literature from both parametric and semiparametric perspectives, but this latter approach has been showed to be invalid in case the series are actually co-integrated. In this chapter, we propose a simple testing procedure which does not suffer from this serious limitation.

Chapter 2

Parametric estimation of strong fractional co-integration

2.1 Introduction

As presented in Chapter 1, methods of estimating co-integrating vectors have been developed which have optimal asymptotic properties, with a limiting mixed normal distribution, thereby generating Wald test statistics with a standard, χ^2, null limit distribution (see our description of methods by Phillips and Hansen, 1990, Phillips, 1991a,b, Johansen, 1988). These methods have been justified under the assumption that integration orders of observed series and co-integrating errors are correctly specified integers, though it is standard practice to test these integration orders, particularly by unit root tests against stationary AR alternatives. Under fractional co-integration, the different orders of integrations involved in the estimated model are taken to be real numbers, and certainly, this consideration poses additional difficulties. For example, the "optimal" methods referred to above lose their most desirable properties (such as the χ^2 hypothesis tests, for example) when integration orders on which they are based are misspecified, a fair possibility under fractional circumstances. Also, the methodology developed by Engle and Granger (1987) and subsequent authors is not designed to detect such co-integrating relationships. Thus, our aim is to propose a general estimation method, nesting the traditional co-integration cases as $CI\,(1,1)$ (see Definition 1.4), and allowing integration orders to be unknown and real-valued.

In this chapter, we consider the bivariate model (1.25), (1.26), for the case of strong fractional co-integration, with

$$\delta \geq \beta > 1/2, \tag{2.1}$$

noting (1.24). In (1.25), (1.26) the possibility that γ and/or δ are known, but not necessarily integers, does not lack interest (in particular when $\delta = 1$ is fixed after pre-testing) but allowing both γ and δ to be unknown, thereby avoiding complications and ambiguities due to pre-testing, may be attractive. Fractional values may be difficult to interpret economically, though aggregation explanations have been developed, mean-reversion is nicely described, in the present context γ and δ are just nuisance parameters, while fractional, like non-fractional, co-integration is a kind of dimensionality-reducing structure.

As shown in Chapter 1, simple estimates of ν not requiring knowledge of γ and/or δ are readily available. For example OLS, with or without intercept, is $n^{\min(2\delta-1,\beta)}$-consistent (except in the case where $\delta > \beta$ and $2\delta - \beta = 1$, in which case it is $(n^\beta/\log n)$–consistent), as shown under mild conditions by Robinson and Marinucci (2001). Also, we saw in the previous chapter that in case $2\delta - 1 < \beta$, the rate of convergence can be improved upon by using a version of OLS in the frequency domain that focuses on a slowly degenerating band of low frequencies and thereby reduces the bias that is due to contemporaneous correlation between u_{1t}, u_{2t} (Robinson and Marinucci, 1998); these estimates were applied empirically by Marinucci and Robinson (2001). Both least squares and its narrow-band counterpart have nonstandard limit distributions, which are unsuitable for use in statistical inference, while their rate of convergence seems capable of still further improvement over some regions of (γ, δ)-space. In the present chapter, we develop and justify estimates of ν which have analogously optimal properties, in the presence of possibly unknown γ, δ, to those previously established by, for example, Phillips and Hansen (1990), Phillips (1991a,b), in case $\delta = \beta = 1$ is known. The estimates of ν are of generalised least squares (GLS) type, based on a constrained transformed bivariate regression model derived from (1.25), (1.26) and having the property that regressors are orthogonal to disturbances.

We allow for very general forms of parametric autocorrelation in u_t, in which circumstances a frequency-domain form of estimate of ν is convenient and flexible, though we also consider a time-domain form based on AR transformation. The model (1.25), (1.26) is perhaps the simplest interesting one possible. Our treatment of (1.25), (1.26), with parametric autocorrelation, itself requires lengthy proofs, whose ideas are relevant to more general models but best conveyed in a relatively simple setting. Admittedly, assuming knowledge of the structure of u_t could be a strong requirement, but the parametric approach has enjoyed great popularity among time series researchers, and, in any case, our work in this chapter could also be considered as a first step in order to investigate estimation issues in more general frameworks, where perhaps the spectral density of u_t is a nonparametric function. In fact, Chapter 4 will be devoted to analysing this situation, so focusing initially on a parametric setting could both fit naturally in the literature and also provide many useful results which undoubtedly will simplify subsequent analyses.

Our model presumes the existence of co-integration. The question of establishing such existence, or non-existence, is itself especially difficult in our fractional context, with unknown integration orders. Recently, Robinson and Yajima (2002) have developed methods for determining fractional co-integrating rank in a multivariate extension of (1.25), (1.26) based on sequential testing, principal components analysis, and a model choice procedure, while Marinucci and Robinson (2001) proposed and empirically applied a Hausman-type test for determining the existence of co-integration in (1.25), (1.26). In this chapter we do not consider this issue, but this is briefly explored in Chapter 5, where we present an alternative methodology for testing for the equality of orders of integration, which is a necessary condition for the existence of co-integration.

Our estimates of ν are described in the following section. Section 2.3 presents regularity conditions and the main results, also introducing simpler estimates that are asymptotically competitive when $\beta > 1$. In Appendix 2.A we outline the proofs, which rest heavily on a series of propositions which are proved in Appendix 2.B. Appendices 2.C and 2.D collect respectively some results used in the proofs of several propositions, and technical lemmas pertaining to properties of the $a_j(\alpha)$ in (1.8). Section 2.4 consists of a Monte Carlo study of finite-sample behaviour, Section 2.5 reports an empirical investigation of the purchasing power parity (PPP) hypothesis, and Section 2.6 discusses related topics.

2.2 Estimates of co-integrating parameters

For any sequence $\{w_t\}$, and any $c \geq 0$, introduce the notation

$$w_t(c) = \Delta^c w_t^{\#}, \tag{2.2}$$

noting (1.8), (1.27). Also define, for $c \geq 0$, $d \geq 0$,

$$z_t(c,d) = (y_t(c), x_t(d))'. \tag{2.3}$$

Thus (1.25), (1.26) can be written

$$z_t(\gamma, \delta) = \zeta x_t(\gamma)\nu + u_t^{\#}, \tag{2.4}$$

where

$$\zeta = (1,0)'. \tag{2.5}$$

In case u_t is white noise, with known, nonsingular covariance matrix Ω, and γ and δ are also known, GLS based on (2.4) and observations (x_t, y_t), $t = 1, ..., n$, is motivated by the orthogonality property $E\left(u_t'\Omega^{-1}\zeta x_t(\gamma)\right) = E\left(u_{2t}u_t'\right)\Omega^{-1}\zeta = 0$. More generally, GLS estimates can also be constructed in the presence of serial correlation in u_t, given known $2n \times 2n$ covariance matrix Ψ of $u = (u_1', ..., u_n')'$. If Ψ is a known function of an unknown finite-dimensional parameter vector θ, we might hope that insertion of sufficiently good estimates of γ, δ and θ, producing a feasible GLS estimate of ν, will not affect limiting distributional properties. However, Ψ and its estimate can be difficult to handle, both numerically and theoretically, so more convenient alternatives to such GLS or feasible GLS might be considered.

One such is based on AR transformation. Suppose u_t has an AR representation

$$B(L)u_t = \varepsilon_t, \tag{2.6}$$

where ε_t is a bivariate sequence that is at least (see Section 2.3 below) uncorrelated across t with nonsingular covariance matrix Ω, and

$$B(s) = I_2 - \sum_{j=1}^{\infty} B_j s^j, \tag{2.7}$$

where the B_j are 2×2 matrices satisfying conditions prescribed below. Suppose further that we know functions $\Omega(h)$, $B_j(h)$, where $h \in \mathbb{R}^p$, $p \geq 1$, such that for some $\theta \in \mathbb{R}^p$, we have $\Omega = \Omega(\theta)$, $B_j \equiv B_j(\theta)$. Define

$$B(s;h) = I_2 - \sum_{j=1}^{\infty} B_j(h)s^j, \tag{2.8}$$

and then

$$\widetilde{a}(c,d,h) = \sum_{t=1}^{n} \{B(L;h)\zeta x_t(c)\}' \Omega(h)^{-1} \{B(L;h)z_t(c,d)\}, \tag{2.9}$$

$$\widetilde{b}(c,h) = \sum_{t=1}^{n} \{B(L;h)\zeta x_t(c)\}' \Omega(h)^{-1} \{B(L;h)\zeta x_t(c)\}. \tag{2.10}$$

Note that each of the AR transformations automatically entails a truncation since $x_t(c) = 0$, $z_t(c,d) = 0$, $t \leq 0$. Now write

$$\widetilde{\nu}(c,d,h) = \frac{\widetilde{a}(c,d,h)}{\widetilde{b}(c,h)}, \tag{2.11}$$

and consider as estimates of ν

$$\widetilde{\nu}(\gamma,\delta,\theta),\ \widetilde{\nu}(\gamma,\delta,\widehat{\theta}),\ \widetilde{\nu}(\widehat{\gamma},\delta,\widehat{\theta}),\ \widetilde{\nu}(\gamma,\widehat{\delta},\widehat{\theta}),\ \widetilde{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta}), \tag{2.12}$$

given estimates $\widehat{\gamma}$, $\widehat{\delta}$, $\widehat{\theta}$. The estimates (2.12) respectively consider the cases in which γ, δ and θ are all known, the integration orders γ and δ are known but θ is not, followed by the cases in which one or other and then both of γ, δ are unknown and θ is also unknown: $\widetilde{\nu}(\gamma,\delta,\widehat{\theta})$ covers situations familiar from the integer integration order co-integration literature, where for example $\gamma = 0$, $\delta = 1$ is known; $\widetilde{\nu}(\widehat{\gamma},\delta,\widehat{\theta})$ extends this by assuming knowledge of the integration order of the observable x_t (say $\delta = 1$), but the order of the co-integrating error is not known to be 0; $\widetilde{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta})$ expresses the situation of least knowledge.

The estimates (2.12) are computationally convenient when u_t is a finite-degree AR process, but less so otherwise, for example when u_t is a finite-degree moving average (MA) or ARMA sequence, when the $B_j(h)$, though recursively calculable, do not have a very neat closed form. On the other hand, the spectral density matrix $f(\lambda)$, defined in (1.28) has a neat form in such cases, so a frequency-domain approach might be preferred, as was considered by Phillips (1991a) in the case $\gamma = 0$, $\delta = 1$ is known, and one can construct parametric models for which the gap between tractability of the spectral density on the one hand, and AR coefficients (or indeed autocovariances) on the other, is even greater (see e.g. Bloomfield, 1973, Robinson, 1978). A frequency-domain approach also has the advantage of approaching a well-established form of semiparametric estimate in which $f(\lambda)$ is a nonparametric function (see, e.g. Hannan, 1963, in case of regression models, and Phillips, 1991b, in case of $CI(1,1)$ co-integration).

To define the frequency-domain estimates, first introduce $f(\lambda;h)$, a known function of $\lambda \in (-\pi,\pi]$ and $h \in \mathbb{R}^p$, such that $f(\lambda;\theta) = f(\lambda)$ (see (1.28)). In terms of the AR representation (2.6), we have

$$f(\lambda;h) = (2\pi)^{-1}B(e^{i\lambda};h)^{-1}\Omega(h)B(e^{-i\lambda};h)^{-1'}, \tag{2.13}$$

so $f(\lambda; h)$ is of simple form in the finite ARMA models, replacing $B(e^{i\lambda}; h)^{-1}$ by $B(e^{i\lambda}; h)^{-1}A(e^{i\lambda}; h)$, A and this B both being finite-degree matrix polynomials. (Our assumptions below guarantee the existence where necessary of matrix inverses). Denoting

$$p(\lambda; h) = \zeta' f(\lambda; h)^{-1}, \quad q(\lambda; h) = \zeta' f(\lambda; h)^{-1}\zeta, \tag{2.14}$$

put

$$a(c,d,h) = \sum_{j=1}^{n} p(\lambda_j; h) w_{x(c)}(-\lambda_j) w_{z(c,d)}(\lambda_j), \tag{2.15}$$

$$b(c,h) = \sum_{j=1}^{n} q(\lambda_j; h) I_{x(c)}(\lambda_j), \tag{2.16}$$

noting (1.43), (1.44). Define

$$\widehat{\nu}(c,d,h) = \frac{a(c,d,h)}{b(c,h)}. \tag{2.17}$$

Corresponding to the five estimates (2.12) we may consider also

$$\widehat{\nu}(\gamma,\delta,\theta),\ \widehat{\nu}(\gamma,\delta,\widehat{\theta}),\ \widehat{\nu}(\widehat{\gamma},\delta,\widehat{\theta}),\ \widehat{\nu}(\gamma,\widehat{\delta},\widehat{\theta}),\ \widehat{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta}). \tag{2.18}$$

From the orthogonality properties of the complex exponential function (see (2.95) below), it readily follows that when u_t is *a priori* white noise, so that $B_j(h) \equiv 0$, $j \geq 1$, $f(\lambda; h) = (2\pi)^{-1}\Omega(h)$, we have $\widetilde{\nu}(c,d,h) \equiv \widehat{\nu}(c,d,h)$, so corresponding members of (2.12) and (2.18) are identical. Otherwise, when u_t is believed to be autocorrelated, they differ, but under regularity conditions all members of (2.12) and (2.18) have the same first-order asymptotic properties, as shown in Theorem 2.1 of the following section.

The $CI(1,1)$ literature has stressed error-correction model (ECM) formulations, on which parameter estimation can be based. We can rewrite (2.4) as

$$\Delta^{\delta} z_t = -\zeta(1-\Delta^{\beta})\left\{\Delta^{\delta-\beta}(1,-\nu)z_t\right\} + v_t^{\#}, \tag{2.19}$$

with $z_t = z_t(0,0) = (y_t, x_t)'$ and $v_t^{\#} = \left(u_{1t}^{\#} + \nu u_{2t}^{\#}, u_{2t}^{\#}\right)'$. When $\delta = \beta = 1$, (2.19) reduces to the triangular ECM representation of Phillips (1991a) for the $CI(1,1)$ case, on which he based a frequency-domain approximate Gaussian pseudo-ML estimate of ν. It is readily shown that this is equivalent to a corresponding Gaussian pseudo-ML estimate based on (2.4). In case u_t is known to be white noise, this is equivalent to the OLS estimate of ν in the extended regression $y_t(\gamma) = \nu x_t(\gamma) + \varphi x_t(\delta) + w_t^{\#}$, where $\varphi = E(u_{1t}u_{2t})/E(u_{2t}^2)$ and $w_t^{\#} = u_{1t}^{\#} - \varphi u_{2t}^{\#}$, namely $\overline{\nu}(\gamma,\delta)$, where

$$\overline{\nu}(c,d) = \frac{\sum_{t=1}^{n} x_t^2(d)\sum_{t=1}^{n} x_t(c)y_t(c) - \sum_{t=1}^{n} x_t(c)x_t(d)\sum_{t=1}^{n} x_t(d)y_t(c)}{\sum_{t=1}^{n} x_t^2(c)\sum_{t=1}^{n} x_t^2(d) - \left\{\sum_{t=1}^{n} x_t(c)x_t(d)\right\}^2}, \tag{2.20}$$

to extend Phillips' (1991a) observation in the $CI(1,1)$ case (though he derived from his ECM representation the OLS estimate of ν in $y_t(\gamma) = \nu\left\{x_t(\gamma) - x_t(\delta)\right\} + \chi x_t(\delta) +$

$w_t^{\#}$, with $\chi = \nu + \varphi$, which is identical to $\overline{\nu}(\gamma,\delta)$). Further, $\overline{\nu}(\gamma,\delta)$ can be shown to be equivalent to the GLS estimate $\widetilde{\nu}(\gamma,\delta,\overline{\theta}_I) = \widehat{\nu}(\gamma,\delta,\overline{\theta}_I)$, with $\overline{\theta}_I$ consisting of the three distinct elements of $\overline{\Omega}(\gamma,\delta)$, where

$$\overline{\Omega}(c,d) = n^{-1}\sum_{t=1}^{n}\left[y_t(c) - \overline{\nu}(c,d)x_t(c), x_t(d)\right]'\left[y_t(c) - \overline{\nu}(c,d)x_t(c), x_t(d)\right]. \quad (2.21)$$

Thus, our GLS approach can be seen to include Gaussian pseudo-ML estimation as a special case, where particular estimates of Ω are used, this interpretation continuing to apply when autocorrelation in u_t is incorporated. Based on (2.19) in the $CI(1,1)$ case, Phillips (1991b), employed a semiparametric version of GLS, involving smoothed nonparametric estimation of $f(\lambda)$ across a coarser grid than the Fourier frequencies, following Hannan (1963).

2.3 Conditions and main results

We present first a series of regularity conditions.

Assumption 2.1. *The process* u_t, $t = 0, \pm 1, ...,$ *has representation*

$$u_t = A(L)\,\varepsilon_t, \quad (2.22)$$

where

$$A(s) = I_2 + \sum_{j=1}^{\infty} A_j s^j, \quad (2.23)$$

and the A_j *are* 2×2 *matrices such that :*

(i)

$$\det\{A(s)\} \neq 0, \quad |s| = 1; \quad (2.24)$$

(ii) $A(e^{i\lambda})$ *is differentiable in* λ *with derivative in* $Lip(\eta)$, $\eta > 1/2$;

(iii) the ε_t *are independent and identically distributed vectors with mean zero, positive definite covariance matrix* Ω, *and* $E\|\varepsilon_t\|^q < \infty$, $q \geq 4$, $q > 2/(2\beta - 1)$.

Notice that (ii) implies $\sum_{j=1}^{\infty} j\|A_j\| < \infty$, because the derivative of $A(e^{i\lambda})$ has Fourier coefficients jA_j, whence Zygmund (1977, p.240) can be applied. Further, this also implies $\sum_{j=1}^{\infty} j\|A_j\|^2 < \infty$, which, along with the condition in (iii), enables us to apply the functional limit theorem of Marinucci and Robinson (2000) (developing earlier work of Akonom and Gourieroux, 1987, Silveira, 1991) to the nonstationary process $x_t(\gamma)$, as is required to characterize the limit distribution of our estimates of ν. Further, due to (i), $B(e^{i\lambda})$ (see (2.7)) satisfies the same smoothness condition as $A(e^{i\lambda})$ in (ii), and thus

$$\sum_{j=1}^{\infty} j\|B_j\| < \infty, \quad (2.25)$$

which implies the required conditions on the B_j in our other proofs, in particular of Propositions 2.1 and 2.2. It is Proposition 2.1 that requires the strongest conditions on the B_j, and this is possible by a lengthier proof under the milder requirement that $A(e^{i\lambda})$, and thus $B(e^{i\lambda})$, is boundedly differentiable, which itself implies (see Zygmund, 1977, p.251) $\sum_{j=1}^{\infty} j^{1/2} \|A_j\| < \infty$, and, from (i), $\sum_{j=1}^{\infty} j^{1/2} \|B_j\| < \infty$. However our present conditions seem satisfactorily mild, easily covering stationary and invertible ARMA systems. The moment assumption on ε_t is satisfied, for any $\beta > 1/2$, by Gaussianity.

The above assumption, with (1.25), (1.26), (1.29), (2.1), suffices in order to establish Theorem 2.1 below for the infeasible estimates $\widetilde{\nu}(\gamma, \delta, \theta)$ and $\widehat{\nu}(\gamma, \delta, \theta)$, but in order to insert estimated parameters further conditions are required. It is convenient to denote by Θ the set of all admissible values of $\widehat{\theta}$; often we may take Θ to be a bounded set, in part to satisfy stationarity conditions, while compactness of Θ would help to ensure existence of $\widehat{\theta}$.

Assumption 2.2

(i) $f(\lambda; \theta) = f(\lambda)$;

(ii) $f(\lambda; h)$ *has determinant bounded away from zero on* $([-\pi, \pi] \times \Theta)$;

(iii) $f(\lambda; h)$ *is boundedly differentiable in* h *on* $([-\pi, \pi] \times \Theta)$, *with derivative that is continuous in* h *at* $h = \theta$ *for all* λ;

(iv) $f(\lambda; \theta)$ *is differentiable in* λ, *with derivative satisfying a Lipschitz condition of order greater than* $1/2$ *in* λ;

(v) $(\partial/\partial h) f(\lambda; h)$ *is differentiable in* λ *at* $h = \theta$, *with derivative satisfying a Lipschitz condition of order greater than* $1/2$ *in* λ.

Given correct specification (i), these assumptions seem innocuous, again being easily satisfied by standard stationary and invertible ARMA parameterizations, for example, and could be slightly relaxed at cost of greater proof detail.

Assumption 2.3

(i) There exists $K < \infty$ *such that*

$$|\widehat{\gamma}| + \left|\widehat{\delta}\right| \le K, \tag{2.26}$$

and $\kappa > \max(0, 1 - \beta)$ *such that*

$$\widehat{\gamma} = \gamma + O_p\left(n^{-\kappa}\right), \ \widehat{\delta} = \delta + O_p\left(n^{-\kappa}\right); \tag{2.27}$$

(ii)

$$\widehat{\theta} = \theta + O_p(n^{-\frac{1}{2}}), \text{ where } \theta \in \Theta. \tag{2.28}$$

Condition (2.26) is innocuous if $\widehat{\gamma}$ and $\widehat{\delta}$ optimize over compact sets, as is standard for implicitly defined estimates. The convergence rates required in Assumption 2.3 are all less than those achieved of estimates (2.12) and (2.18) of ν in Theorem 2.1 below. In fact (ii) could be relaxed to the rate on $\widehat{\gamma}$ and $\widehat{\delta}$ of (i) if $f(\lambda; h)$ is smoother in h than required in Assumption 2.2, in particular if it is analytic in h (as in the ARMA case). We prefer our milder Assumption 2.2, and the relatively brief proof that (ii) affords, because $n^{1/2}-$consistency of parameter estimates in short memory time series models is familiar, for example in case of Whittle estimates, see eg. Hannan (1973). On the other hand, we might be content to assume $\kappa = 1/2$ in (2.27). The $n^{1/2}-$consistency and asymptotic normality of estimates of nonstationary integration orders (and indeed of parameters corresponding to θ in nonstationary fractional models), based on scalar series was established by Velasco and Robinson (2000), for Type I processes (see Definition 1.2). By bounding a measure of distance between Type I and Type II processes, Robinson (2002) showed that the same results hold for Type II processes, thereby checking (2.27) and (2.28) for estimates of δ and elements of θ identified by the u_{2t} process. Robinson (2002) likewise checked (2.27) and (2.28) for estimates (computed from residuals) of γ and elements of θ identified by $\{u_{1t}\}$, employing a preliminary estimate of ν, which satisfies a rate of convergence condition. This is satisfied by OLS when $\gamma + \delta \geq 1$, but not when $\gamma + \delta < 1$, where it is, however, satisfied by the NBLS estimate of Robinson and Marinucci (1998, 2001), using a bandwidth that increases sufficiently slowly; the strength of this rate condition is due in part to allowing the compact set of admissible values of γ to be arbitrarily large - if this is suitably reduced the condition can be relaxed so as to be satisfied by OLS even when $\gamma + \delta < 1$, so long as $\delta < 1/2$. The only gap left in demonstrating that Assumption 2.3 can be fully checked is that in general methods based on the bivariate series z_t are appropriate in order to estimate part of θ. However the extension of Velasco and Robinson's (2000) theory to cover bivariate series, and the subsequent adaptation to our setting, seems straightforward, while if $B(s;\theta)$ is *a priori* diagonal the only parameter not estimated by two univariate procedures is the off-diagonal element of Ω, which is estimated by an obvious side calculation, to satisfy (ii). Unless β is close to 1/2, (2.27) is capable of being satisfied also by "semiparametric" estimates of γ and δ, which might in any case be employed at an initial stage in determining the parametric model for f. On the other hand, from the viewpoint of a full co-integration analysis, efficient estimates of γ, δ and θ are desirable, suggesting construction of a Gaussian pseudo-ML approach, estimating all parameters jointly, which is computationally more onerous than the kind of step-by-step approach we have envisaged, but undoubtedly possible; asymptotic properties have yet to be explicitly derived, but the problem of differing convergence rates encountered by Saikkonen (1995) in a different setting can be avoided by concentrating out ν first.

We introduce notation to describe the limit distribution of our estimates. Denote by $W(r)$ the 2×1 vector Brownian motion with covariance matrix Ω (noting the simplifying notation $W(r) = W(\Omega; r)$ with respect to that of Chapter 1), and define

(Type II-see Marinucci and Robinson, 1999) fractional Brownian motion

$$W(r;\beta) = \int_0^r \frac{(r-s)^{\beta-1}}{\Gamma(\beta)} dW(s), \tag{2.29}$$

and then define

$$\widetilde{W}(r;\beta) = \xi' B(1)^{-1} W(r;\beta), \tag{2.30}$$

where

$$\xi = (0,1)'. \tag{2.31}$$

By "$\Rightarrow$" we will mean convergence in the Skorohod J_1 topology of $D[0,1]$.

Theorem 2.1. *Let (1.25), (1.26), (1.29), (2.1) and Assumptions 2.1-2.3 hold. Then, denoting by ν^* any of the estimates in (2.12) or (2.18), we have as $n \to \infty$,*

$$n^\beta(\nu^* - \nu) \Rightarrow \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 dr \right\}^{-1} 2\pi\zeta' B(1)' \Omega^{-1} \int_0^1 \widetilde{W}(r;\beta) dW(r). \tag{2.32}$$

The proof is outlined in Appendix 2.A, by a series of propositions whose proofs appear in Appendix 2.B. The rate of convergence in (2.32) is optimal for any regular parametric estimate in this model. Theorem 2.1 desirably implies that we can estimate ν as well, asymptotically, not knowing γ and/or δ and/or θ as knowing them, subject to the rate conditions of Assumption 2.3, with the implication that efficiency of estimation of γ, δ and θ does not matter.

The variates $\zeta' B(1)' \Omega^{-1} W(r)$ and $\widetilde{W}(r;\beta)$ are uncorrelated and thus, by Gaussianity, independent, so (2.32) indicates mixed normal asymptotics. As a consequence of this, and of the propositions in Appendix 2.A, we have

Corollary 2.1. *Denoting by b^* any of the quantities $\widetilde{b}(\gamma,\theta)$, $\widetilde{b}(\widehat{\gamma},\theta)$, $\widetilde{b}(\gamma,\widehat{\theta})$, $\widetilde{b}(\widehat{\gamma},\widehat{\theta})$, $b(\gamma,\theta)$, $b(\widehat{\gamma},\theta)$, $b(\gamma,\widehat{\theta})$, $b(\widehat{\gamma},\widehat{\theta})$, as $n \to \infty$, the Wald statistics*

$$b^*(\nu^* - \nu)^2 \to_d \chi_1^2. \tag{2.33}$$

The form of the limit distribution in (2.32), where spectral properties of u_t at only zero frequency are involved, and the nonstationarity of $x_t(\gamma)$, suggests simpler forms of estimate than (2.12), (2.18). We replace $p(\lambda_j;h)$, $q(\lambda_j;h)$ by $p(0;h)$, $q(0;h)$, and thence consider

$$\overline{\nu}(\gamma,\delta,\theta),\ \overline{\nu}(\gamma,\delta,\widehat{\theta}),\ \overline{\nu}(\widehat{\gamma},\delta,\widehat{\theta}),\ \overline{\nu}(\gamma,\widehat{\delta},\widehat{\theta}),\ \overline{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta}), \tag{2.34}$$

where

$$\overline{\nu}(c,d,h) = \frac{\overline{a}(c,d,h)}{\overline{b}(c,h)}, \tag{2.35}$$

in which

$$\overline{a}(c,d,h) = p(0;h) \sum_{t=1}^{n} z_t(c,d)x_t(c), \quad \overline{b}(c,h) = q(0;h) \sum_{t=1}^{n} x_t^2(c), \tag{2.36}$$

after applying (2.95) below. If we act on the belief that u_t is white noise, (2.34) is identical to (2.12), (2.18), but to cover other circumstances we have:

Theorem 2.2. *Let (1.25), (1.26), (1.29), (2.1) and Assumptions 2.1-2.3 hold. Then, denoting by ν° any of the estimates in (2.34), we have as $n \to \infty$:*
(i) for $1/2 < \beta < 1$,

$$n^{2\beta-1}(\nu^\circ - \nu) \Rightarrow \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 dr \right\}^{-1} p(0) \int_{-\pi}^{\pi} f(\lambda)\xi(1-e^{-i\lambda})^{-\beta} d\lambda; \tag{2.37}$$

(ii) for $\beta = 1$,

$$\begin{aligned} n(\nu^\circ - \nu) \Rightarrow & \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 dr \right\}^{-1} \\ & \times \left\{ p(0) \sum_{s=0}^{\infty} \psi_{-s} + 2\pi\zeta' B(1)' \Omega^{-1} \int_0^1 \widetilde{W}(r;1)\, dW(r) \right\}, \end{aligned} \tag{2.38}$$

where

$$\psi_s = E(u_0 u_s')\xi; \tag{2.39}$$

(iii) for $\beta > 1$,

$$n^{\beta}(\nu^\circ - \nu) \Rightarrow \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 dr \right\}^{-1} 2\pi\zeta' B(1)' \Omega^{-1} \int_0^1 \widetilde{W}(r;\beta)\, dW(r). \tag{2.40}$$

If u_t is white noise, so $f(\lambda) \equiv f(0)$, we have $p(0)f(\lambda)\xi \equiv 0$ and (2.37) becomes $\nu^\circ = \nu + o_p(n^{1-2\beta})$, but Theorem 2.1 applies here, with the sharp result (2.32); also, $p(0)\sum_{s=0}^{\infty} \psi_{-s} = p(0)\psi_0 = 2\pi p(0) f(0)\xi = 0$, so (2.38) reduces to (2.32). For autocorrelated u_t, when $\beta > 1$, (2.40) indicates that (2.34) still does as well as (2.12), (2.18), but when $\beta = 1$ the convergence rate in (2.38) is as good but the desirable mixed-normal asymptotics are lacking, due to "second-order bias" (see Chapter 1) appearing as the first term in the second factor on the right of (2.38), and when $\beta < 1$, in (2.37), not only are mixed-normal asymptotics lacking but convergence is slower. Indeed, for $1/2 < \beta < 1$ (2.34) never converges faster, and nearly always converges slower, than OLS of y_t on x_t. From Propositions 6.1, 6.2 and 6.5 of Robinson and Marinucci (2001), OLS is $n^{2\delta-1}$-consistent when $\gamma+\delta = 2\delta-\beta < 1$, $n^{2\delta-1}/\log n$-consistent when $\gamma + \delta = 2\delta - \beta = 1$ and $\gamma > 0$, n-consistent when

$\delta = 1$, $\gamma = 0$, and n^{β}-consistent when $\gamma + \delta = 2\delta - \beta > 1$, so over the intersection of these regions with $1/2 < \beta = \delta - \gamma < 1$ the rate in (2.37) is equalled when $\gamma = 0$ and exceeded when $\gamma > 0$, indicating that proper fractional differencing without proper accounting for $I(0)$ autocorrelation can do worse than simple methods based on unfiltered data.

Focusing more closely on $\gamma = 0$, where the central case (ii) is that of $I(1)$ x_t, while the widespread evidence of unit root behaviour based on tests against AR alternatives cannot be taken very seriously from a fractional viewpoint (see Diebold and Rudebusch, 1991, Robinson, 1994b), it might be reasonable to interpret this as suggesting that integration orders may often be close to 1, but either greater or less than 1, when the discontinuity in Theorem 2.2 at $\beta = 1$ makes use of (2.34) questionable. Even when $\beta > 1$, the detailed corrections for autocorrelation in (2.12) and (2.18) might be expected to produce better finite-sample properties than (2.34), which is based on an appeal to asymptotic theory due to a high degree of nonstationarity in $x_t(\gamma)$, while the extra computational burden of (2.12) and (2.18) does not seem prohibitive. Because this discussion indicates that it is less important than Theorem 2.1, and because its proof is in part embodied in that of Theorem 2.1 and in part straightforwardly uses Theorems 4.1, 4.3 and 4.4 of Robinson and Marinucci (2001), we have omitted the proof of Theorem 2.2. Theorem 4.3 of Robinson and Marinucci (2001) can also be applied to justify narrow-band frequency-domain versions of (2.34) which, at cost of introducing a user-chosen bandwidth, eliminate the second-order bias term in (2.38) and thereby achieve the asymptotics in (2.32), corresponding to an idea due to Phillips (1991b) in a semiparametric setting for the $CI(1,1)$ case. We will also pursue this idea in Chapter 4 below.

2.4 Monte Carlo evidence

With the main aim of studying the effect of estimating integration orders γ, δ on our estimates of ν and their limiting distributional properties, a Monte Carlo study was carried out on the case where in (2.23)

$$A(s) = diag\left\{\frac{1+\psi_1 s}{1-\phi_1 s}, \frac{1+\psi_2 s}{1-\phi_2 s}\right\}, \tag{2.41}$$

where ψ_i, ϕ_i, $i = 1, 2$, are allowed to take values which represent different situations where u_t is a bivariate:

1. white noise process, with $\phi_i = \psi_i = 0$, $i = 1, 2$;
2. purely AR(1) process, with $\phi_1 = \phi_2 = 0.5, 0.9$, $\psi_i = 0$, $i = 1, 2$;
3. purely MA(1) process, with $\psi_1 = \psi_2 = 0.5, 0.9$, $\phi_i = 0$, $i = 1, 2$;
4. ARMA(1,1) process, with $\phi_1 = \phi_2 = 0.4$, $\psi_1 = \psi_2 = 0.2$.

We generated Gaussian ε_t with covariance matrix Ω having ijth element ω_{ij}, varying the correlation $\rho = \omega_{12} / (\omega_{11}\, \omega_{22})^{1/2}$ (taking values 0, 0.5, -0.5, 0.75) and variance ratio $\tau = \omega_{22}/\omega_{11}$ (taking values 0.5, 1, 2 when u_t is a white noise and just 1 for the autocorrelated cases). The parameter ρ heavily influences the "simultaneous equation bias" in (1.25), regressors and disturbances being orthogonal only when $\rho = 0$, while τ affects the signal-to-noise ratio in (1.25), with increase in τ generally being associated with an increase in precision in estimation of ν. Our estimates are invariant to $\nu \neq 0$ and also to a scale factor of Ω, and so we fixed $\nu = \omega_{11} = 1$ with no loss of generality.

We generated 1000 series of lengths $n = 64, 128, 256$. For the white noise case, we computed the Infeasible estimate $\overline{\nu}_I$ and Feasible estimate $\overline{\nu}_F$, given by

$$\overline{\nu}_I = \widehat{\nu}(\gamma, \delta, \overline{\theta}_I) = \widetilde{\nu}(\gamma, \delta, \overline{\theta}_I) = \overline{\nu}(\gamma, \delta, \overline{\theta}_I), \tag{2.42}$$

$$\overline{\nu}_F = \widehat{\nu}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_F) = \widetilde{\nu}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_F) = \overline{\nu}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_F), \tag{2.43}$$

for $\widehat{\gamma}, \widehat{\delta}$ to be described subsequently. $\overline{\theta}_I, \overline{\theta}_F$ represent in this case 3×1 vectors of estimates of $\theta = (\omega_{11}, \omega_{12}, \omega_{22})'$, such that $\overline{\nu}_I = \overline{\nu}(\gamma, \delta), \overline{\nu}_F = \overline{\nu}(\widehat{\gamma}, \widehat{\delta})$, noting (2.20), $\overline{\theta}_I, \overline{\theta}_F$ consisting of the appropriate elements of $\overline{\Omega}_I = \widehat{\Omega}(\gamma, \delta), \overline{\Omega}_F = \widehat{\Omega}(\widehat{\gamma}, \widehat{\delta})$ (see (2.21)). Thus, we compare an optimal estimate ($\overline{\nu}_I$) in case γ, δ are known (one that is familiar from the unit root co-integration literature in case $(\gamma, \delta) = (0, 1)$) with one ($\overline{\nu}_F$) where γ, δ are unknown, and replaced by estimates.

Unlike in the white noise case, for the different autocorrelated situations, estimates in (2.12) and (2.18), although asymptotically equivalent, are not identical. Noting that time-domain estimates (2.12) are only computational convenient when u_t is a finite-degree AR process, as we deal in our experiment with MA and ARMA situations, for the sake of a uniform treatment, we present only results for the frequency domain estimates (2.18), and compare them with those in (2.34). Thus, in the different autocorrelated situations, we examined the performance of

$$\overline{\nu}_I = \widehat{\nu}(\gamma, \delta, \overline{\theta}_I),\ \overline{\nu}_F = \widehat{\nu}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_F), \tag{2.44}$$

$$\overline{\nu}_I^o = \overline{\nu}(\gamma, \delta, \overline{\theta}_I),\ \overline{\nu}_F^o = \overline{\nu}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_F), \tag{2.45}$$

for certain estimates $\overline{\theta}_I$, $\overline{\theta}_F$, $\widehat{\gamma}$, $\widehat{\delta}$ to be described subsequently.

To describe the procedure of estimation of the various short-memory parameters in the autocorrelated situations, let $\theta = (\theta_A', \theta_\Omega')'$, where θ_A, θ_Ω collect the ARMA parameters and the distinct elements of Ω respectively, i.e. $\theta_\Omega = (\omega_{11}, \omega_{12}, \omega_{22})'$. Now, given estimates $\widehat{\gamma}$, $\widehat{\delta}$ of the orders of integration, and estimates $\overline{\theta}_{AI}$, $\overline{\theta}_{AF}$ of θ_A, where in the computation of $\overline{\theta}_{AI}$ and $\overline{\theta}_{AF}$, we assumed that γ, δ, were known and unknown respectively, the corresponding estimates $\overline{\theta}_{\Omega I}$, $\overline{\theta}_{\Omega F}$ are the appropriate distinct elements of $\overline{\Omega}(\gamma, \delta, \overline{\theta}_{AI}; e)$, $\overline{\Omega}(\widehat{\gamma}, \widehat{\delta}, \overline{\theta}_{AF}; e)$, where

$$\overline{\Omega}(c, d, h; e) = \frac{2\pi}{n} \sum_{j=1}^{n} D(c, d, h; \lambda_j)^{-1} w_e(\lambda_j)\, w_e'(-\lambda_j)\, D(c, d, h; -\lambda_j)^{-1'}, \tag{2.46}$$

with

$$e_t = (y_t - \overline{\nu}_O x_t, x_t - x_{t-1} 1(\delta \geq 1))', \tag{2.47}$$

where $\overline{\nu}_O$ is the OLS estimate (see (1.33)), and

$$D(c,d,h;\lambda)=\begin{pmatrix}\left(1-e^{i\lambda}\right)^{-c} & 0\\ 0 & \left(1-e^{i\lambda}\right)^{-d+1(\delta\geq 1)}\end{pmatrix}A\left(h;e^{i\lambda}\right),\ A\left(\theta_A;s\right)=A\left(s\right). \tag{2.48}$$

Note that $\overline{\Omega}(\gamma,\delta,\theta_A;\widetilde{e})$, with $\widetilde{e}=(y_t-\nu x_t,x_t-x_{t-1}1\left(\delta\geq 1\right))'$ would be the standard parametric Whittle estimate of Ω based on the bivariate process $(y_t-\nu x_t,x_t)'$ or $(y_t-\nu x_t,x_t-x_{t-1})'$ depending on whether $\delta<1$ or $\delta\geq 1$ respectively. Of course, this estimate is infeasible, as it requires knowledge of the unknown parameters γ, δ, θ_A, ν. Also, $\overline{\nu}_O$ does not represent the current state of the art in estimating cointegrating vectors in the presence of unknown integration orders without employing estimates of these, Robinson and Marinucci (1998) having demonstrated how it can be improved upon by a narrow-band frequency domain least squares procedure. The use of such an estimate would presumably lead to an improvement in $\overline{\theta}_{\Omega I}$, $\overline{\theta}_{\Omega F}$ (and also, as it will be seen later, in $\overline{\theta}_{AI}$, $\overline{\theta}_{AF}$, $\widehat{\gamma}$), and thence in the different estimates in (2.18), but it involves choice of a bandwidth number, and in the purely "parametric" context of the current chapter we prefer to keep to the more familiar and simpler $\overline{\nu}_O$, whose performance as an estimate of ν we also compare with $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_I^{\circ}$ and $\overline{\nu}_F^{\circ}$.

In the situation when we consider known orders of integration, the short memory parameters in θ_A related to the process u_{2t} are estimated by the method described in Hannan (1973) after taking δ differences on the process x_t. In the case of unknown orders, we computed $\widehat{\delta}$ and the estimates of the short memory parameters in θ_A related to u_{2t} by variants of the univariate Whittle procedure of Velasco and Robinson (2000), using untapered x_t for $\delta<1$, but for $\delta\geq 1$ using untapered Δx_t and adding back one to the estimate of δ. The estimation of memory parameters of nonstationary series by means of integer-differenced stationary and invertible observations incurs no loss of efficiency (cf. Robinson, 1994b), but of course our use of knowledge of the actual δ in doing this may favour $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$ and the estimate of Ω (see (2.48)). On the other hand, it appears that Velasco and Robinson's (2000) estimates based on untapered data are only $n^{1/2}$-consistent (and asymptotically normal) when the memory parameter is less than $3/4$, so that our application of their procedure to first-differenced untapered data when $\delta=2$ (see (2.50) below) is not supported by their results, and may in itself lead to inferior $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$, compared to ones using memory parameter estimates which incorporate suitable tapering. The estimation of the parameters related to u_{1t} is more problematic, as even in the case of known orders of integration the process u_{1t} is not observable. If the orders are considered to be known, we estimated the short memory parameters in θ_A related to the process u_{1t} applying the Hannan's (1973) procedure to γ differences of the OLS residuals $y_t-\overline{\nu}_O x_t$. In case the orders are considered to be unknown, we computed $\widehat{\gamma}$ and the corresponding estimates in θ_A by the Velasco and Robinson's (2000) procedure applied to the residuals $y_t-\overline{\nu}_O x_t$.

The previous estimation procedure was applied to all autocorrelated situations except the one where u_{1t} and u_{2t} are purely AR processes with known γ, δ, where we preferred a more natural and computationally simpler way of estimating the short-memory parameters. In this situation, the estimates of ϕ_1 and ϕ_2 (see (2.41)) were obtained as the OLS coefficients in the regressions of $y_t\left(\gamma\right)-\overline{\nu}_O x_t\left(\gamma\right)$ on $y_{t-1}\left(\gamma\right)-$

$\overline{\nu}_O x_{t-1}(\gamma)$, and $x_t(\delta)$ on $x_{t-1}(\delta)$ respectively. Here, we computed our estimate of Ω as

$$\overline{\Omega}(\gamma,\delta,\overline{\theta}_{AI}) = \frac{1}{n}\begin{pmatrix} \sum_{t=1}^{n} r_{1t}^2 & \sum_{t=1}^{n} r_{1t}r_{2t} \\ \sum_{t=1}^{n} r_{1t}r_{2t} & \sum_{t=1}^{n} r_{2t}^2 \end{pmatrix}, \tag{2.49}$$

being r_{1t} and r_{2t} the OLS residuals obtained from those regressions respectively.

There are two parts to our Monte Carlo investigation, the first comparing performance in fractional circumstances of estimates assuming both γ and δ are known with ones where both are estimated, and the second focussing on the standard case $(\gamma,\delta) = (0,1)$ just for the white noise case, and considering also estimates in which one of γ or δ is estimated.

2.4.1 Performance for different combinations of orders

In the first part of the study, we employed all five (γ,δ) combinations of $\gamma = 0, 0.4$ with $\delta = 0.6, 1.2, 2$ where $\beta > 1/2$:

$$(\gamma,\delta) = (0,0.6), (0,1.2), (0,2), (0.4,1.2), (0.4,2). \tag{2.50}$$

The first case, $(\gamma,\delta) = (0,0.6)$, is one in which the bias of OLS is so strong as to dominate rate of convergence when $\rho \neq 0$ (see Robinson and Marinucci, 1998), while the remaining four cases are all ones in which OLS achieves the optimal rate. Table 2.1 records the convergence rates of OLS when $\rho \neq 0$, OLS when $\rho = 0$, and the optimal rates (achieved in Theorem 2.1).

TABLE 2.1
CONVERGENCE RATES:
OLS WITH $\rho \neq 0$, $\rho = 0$ AND OPTIMAL RATES

(γ,δ)	$(0,0.6)$	$(0,1.2)$	$(0,2)$	$(0.4,1.2)$	$(0.4,2)$
OLS, $\rho \neq 0$	$n^{.2}$	$n^{1.2}$	n^2	$n^{.8}$	$n^{1.6}$
OLS, $\rho = 0$	$n^{.6}$	$n^{1.2}$	n^2	$n^{.8}$	$n^{1.6}$
Optimal	$n^{.6}$	$n^{1.2}$	n^2	$n^{.8}$	$n^{1.6}$

Behaviour of the bias

We show the Monte Carlo bias (defined as the estimate minus ν) of the estimates corresponding to the white noise case in Tables 2.3-2.6. Overall, $\overline{\nu}_I$, $\overline{\nu}_F$ and $\overline{\nu}_O$ are individually no worse than any of the other estimates in 172, 108, and 75 out of 180 cases (considering all τ, ρ, n and order combinations) respectively, so that $\overline{\nu}_I$ is clearly best and our feasible estimate $\overline{\nu}_F$, although more complicated to calculate, seems worth relative to the computationally simpler $\overline{\nu}_O$. In fact, out of those 180 different cases, $\overline{\nu}_F$ behaves strictly better than $\overline{\nu}_O$ (with absolute value of the bias strictly smaller) 98 times, whereas that $\overline{\nu}_O$ is better than $\overline{\nu}_F$ just four times (we will say that they perform in relation or proportion, 98/4). The overall predominance of $\overline{\nu}_F$ over $\overline{\nu}_O$ in terms of bias, is clear for all values of ρ, although slightly less noticeable for $\rho = 0$. It is reassuring that while $\overline{\nu}_F$ is damaged by nuisance parameter estimation, it nevertheless emerges as worthwhile relative to OLS, whose bias is indeed unacceptably large in the case $(\gamma, \delta) = (0, 0.6)$, even for $n = 256$, except, of course, when $\rho = 0$. While the bias of $\overline{\nu}_I$ is virtually unaffected by varying ρ, there is evidence that the bias of $\overline{\nu}_F$ somewhat increases in absolute value with $|\rho|$, with sign opposite to that of ρ. As expected, biases tend to decrease with n. For all ρ, τ, bias tends to decrease in absolute value as β increases, as rates of convergence predict. There is also a tendency for bias to vary inversely with τ, but this is very noticeable only in the case $(\gamma, \delta) = (0, 0.6)$.

In Tables 2.7-2.10, we show the Monte Carlo bias for the purely AR(1) case. Overall, $\overline{\nu}_I^o$ outperforms the other four estimates as, out of 120 possible cases (for all combinations of ρ, n and (γ, δ)), is no worse than any of the others 96 times. It is followed by $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_O$ and $\overline{\nu}_F^o$, which are no worse than any of the other estimates in 84, 56, 54 and 41 cases respectively. Apart from the clear superiority of the infeasible estimates over the rest, the overall classification hides some very important features. For example, comparing $\overline{\nu}_F$ with $\overline{\nu}_O$, when $\phi_i = 0.5$, $i = 1, 2$, $\overline{\nu}_F$ is strictly better than $\overline{\nu}_O$ with relation 22/8. Similarly, when $\phi_i = 0.9$, $\overline{\nu}_O$ shows certain improvement (most noticeable for $(\gamma, \delta) = (0, 0.6)$ with $\rho \neq 0$), but the relation is still favourable to $\overline{\nu}_F$ (22/16). When u_t is close to the nonstationarity situation, the joint estimation of orders of integration and AR parameters gives for some replications estimates of these latter parameters very close to one. In this case, $\overline{\nu}_F$ behaves very poorly, although this effect seems to be noticeable only when β is small. $\overline{\nu}_I^o$ improves over $\overline{\nu}_I$ for all different combinations of orders of integration when $\rho \neq 0$. This is completely surprising for the cases where $\beta < 1$, as here the rate of convergence of $\overline{\nu}_I^o$ is smaller. This better behaviour of $\overline{\nu}_I^o$ is maintained when $\phi_i = 0.9$. The performance of $\overline{\nu}_F^o$ is somehow strange. It performs very badly in some cases, especially when $\phi_i = 0.9$. This estimate seems more affected than $\overline{\nu}_F$ when the estimates of ϕ_i are very close to one, in which case, bias for some replications is extremely large, affecting the overall behaviour of the estimate across all Monte Carlo replications. Nevertheless, this effect tends to disappear as the sample size gets larger, so for $n = 256$ and $\phi_i = 0.5$, $\overline{\nu}_F^o$ outperforms $\overline{\nu}_F$ in almost all cases, whereas for $\phi_i = 0.9$ just does it for the case $(\gamma, \delta) = (0, 0.6)$. $\overline{\nu}_F^o$ emerges as worth against $\overline{\nu}_O$ when $\phi_i = 0.5$ (in relation 20/13), but this changes completely when $\phi_i = 0.9$, where the relation is now in favour of $\overline{\nu}_O$ (43/2). Thus, it does not seem advisable to use $\overline{\nu}_F^o$ in a situation where we suspect there exists short-memory AR structure, especially if there is certain evidence of closeness to the nonstationary situation. As in the white noise case, for all ρ, bias tends to decrease in absolute value as β increases, as rates of convergence predict. In general bias of all the estimates increases in absolute value with $|\rho|$, with sign opposite to that of ρ and tend to decrease in absolute value as n increases.

For the purely MA(1) case (Tables 2.11-2.14), as in the AR case, bias improves as $|\rho|$ decreases and β increases. Overall, $\overline{\nu}_I^o$, $\overline{\nu}_F^o$, $\overline{\nu}_I$, $\overline{\nu}_F$ and $\overline{\nu}_O$ are individually no worse than any of the other estimates in 101, 81, 78, 71 and 55 out of 120 cases. The most surprising result here is that $\overline{\nu}_F^o$ performs better than the infeasible estimate $\overline{\nu}_I$ (with relation 18/7 out of 60 cases in both situations where $\psi_i = 0.5$ or $\psi_i = 0.9$), being this more noticeable for the cases where $\beta < 1$, against what convergence rates predict. It is completely clear that both feasible estimates improve over $\overline{\nu}_O$

(27/2 out of 60 cases for both $\psi_i = 0.5$ and $\psi_i = 0.9$ for the relation between $\overline{\nu}_F$ and $\overline{\nu}_O$, and 27/2 and 27/3 out of 60 cases for $\psi_i = 0.5$ and $\psi_i = 0.9$ respectively for the relation between $\overline{\nu}_F^{\text{o}}$ and $\overline{\nu}_O$). Estimates are in general not very affected by the increase in the parameter ψ_i. When this happens, $\overline{\nu}_I$ performs slightly worse in small sample sizes, but a bit better when $n = 256$, while $\overline{\nu}_F$ does not have a very clear reaction, except that it becomes worse for the case $(\gamma, \delta) = (0, 0.6)$. Also, $\overline{\nu}_I^{\text{o}}$ improves slightly in most of the cases, whereas $\overline{\nu}_F^{\text{o}}$ performs a bit worse. It has to be stressed that as opposite to the results for the AR situation, $\overline{\nu}_F^{\text{o}}$ does not seem to be very sensitive to values of short memory parameters close to noninvertibility, as it was the case for values of AR parameters close to nonstationarity. In general, our employed estimation method in AR circumstances, tends relatively often to give results where the orders are underestimated, whereas the AR parameters are overestimated, obtaining values very close to one. This seems not to occur when we approach the noninvertibility region.

Results for the ARMA(1,1) case are given in Tables 2.15, 2.16. Overall, out of 60 cases, $\overline{\nu}_I^{\text{o}}$, $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_O$ and $\overline{\nu}_F^{\text{o}}$ are individually no worse than any of the other estimates in 53, 43, 30, 29 and 21 cases respectively. Unlike the previous figures suggest, there is still clear evidence that $\overline{\nu}_F$ improves over $\overline{\nu}_O$ (with a relation of 23/11 out of those 60 cases). Now, $\overline{\nu}_F^{\text{o}}$ does not improve over $\overline{\nu}_O$ (22/15 in favour of $\overline{\nu}_O$), being the feasible estimate $\overline{\nu}_F^{\text{o}}$ better than $\overline{\nu}_O$ just for the case $(\gamma, \delta) = (0, 0.6)$ when $\rho \neq 0$. As before, biases react as theory predicts when n increases and also decrease in absolute value with increases in β and decreases in $|\rho|$.

Behaviour of the standard deviation

For the white noise case, Monte Carlo standard deviations are reported in Tables 2.17-2.20. Overall, out of 180 cases, $\overline{\nu}_I$, $\overline{\nu}_F$ and $\overline{\nu}_O$ are not worse than any of the other estimates 166, 78 and 75 times respectively, showing this clearly that $\overline{\nu}_I$ performs best. $\overline{\nu}_F$ performs better than $\overline{\nu}_O$ (with relation 63/50), standard deviations of the former estimate improving relatively faster when n and $|\rho|$ increase, and being also more favoured by negative ρ. As anticipated, standard deviations tend to decrease as τ and n increase. The standard deviations of both $\overline{\nu}_I$ and $\overline{\nu}_F$ show some tendency to decrease as $|\rho|$ increases, though it frequently increase for $\overline{\nu}_F$ when $(\gamma, \delta) = (0, 0.6)$. Otherwise, the close similarity in variability of $\overline{\nu}_I$ and $\overline{\nu}_F$ for $n = 256$ is encouraging. For $n = 64$, the change in sign of ρ is associated with some small improvement. Note that $\overline{\nu}_O$ is often more precise than $\overline{\nu}_F$, and even $\overline{\nu}_I$, when either n is small or $(\gamma, \delta) = (0, 0.6)$.

Results for the purely AR(1) case are presented in Tables 2.21-2.24. Out of the 120 cases reported, $\overline{\nu}_I^{\text{o}}$, $\overline{\nu}_I$, $\overline{\nu}_O$, $\overline{\nu}_F$ and $\overline{\nu}_F^{\text{o}}$ perform individually not worse than any of the remaining four estimates in 87, 80, 60, 39 and 23 cases respectively. Now, $\overline{\nu}_O$ beats $\overline{\nu}_F$ and $\overline{\nu}_F^{\text{o}}$, with relations 43/31 and 85/10 respectively. Although this is not an desirable result, it is certainly supportive that for $n = 256$ and $\phi_i = 0.5$, $\overline{\nu}_F$ and $\overline{\nu}_F^{\text{o}}$ perform slightly better than $\overline{\nu}_O$. $\overline{\nu}_I$ performs better than $\overline{\nu}_F$ with bigger differences for the cases $(\gamma, \delta) = (0, 0.6)$, $(0.4, 1.2)$. For $n = 64$, $\overline{\nu}_O$ is best for the case $(\gamma, \delta) = (0, 0.6)$, but this better performance vanishes as n increases. $\overline{\nu}_O$ is slightly better than $\overline{\nu}_F$ for all sample sizes in the case $(\gamma, \delta) = (0, 1.2)$. $\overline{\nu}_I^{\text{o}}$ is generally worse than $\overline{\nu}_I$, although differences tend to shrink, and for $n = 256$, both estimates behave quite similarly, $\overline{\nu}_I^o$ improving a bit over $\overline{\nu}_I$ when $\phi_i = 0.9$ in case $(\gamma, \delta) = (0, 1.2)$. On the other hand, $\overline{\nu}_F^o$ behaves much worse than $\overline{\nu}_F$ in almost all cases, with unacceptably large values of standard deviations when the sample size is small, results being even worse when $\phi_i = 0.9$. The feasible estimate in Theorem 2.2 appears to be very sensitive to estimates of the AR parameters very close to one. Finally, standard deviations tend to decrease as β, n and $|\rho|$ increase.

Results for the purely MA(1) case are given in Tables 2.25-2.28. As opposite to the AR case, in the MA(1) situation, all different estimates behave quite similarly, feasible estimates being much less damaged by the estimation of the short-memory parameters than in the AR(1) situation. The overall classification is $\overline{\nu}_I^o$, $\overline{\nu}_I$, $\overline{\nu}_O$, $\overline{\nu}_F^o$ and $\overline{\nu}_F$, being no worse than any of the others 107, 77, 56, 50 and 39 times out of 120 cases respectively. $\overline{\nu}_F^o$ clearly beats $\overline{\nu}_O$, with relation 49/26, this better behaviour being present in almost all cases when $n = 128, 256$ and $\rho \neq 0$. In fact, the only two cases where $\overline{\nu}_O$ performs consistently better than $\overline{\nu}_F^o$ are just $(\gamma, \delta) = (0, 0.6)$, $(0, 1.2)$ when $\rho = 0$. The other feasible estimate, $\overline{\nu}_F$, behaves worse than $\overline{\nu}_O$ with relation 42/29 in favour of $\overline{\nu}_O$, but this is completely driven by the bad behaviour of $\overline{\nu}_F$ when $n = 64$ (especially for $\psi_i = 0.9$). Fortunately, when $n = 256$ (and even when $n = 128$ for $\psi_i = 0.5$), apart from the cases $(\gamma, \delta) = (0, 0.6)$, $(0, 1.2)$ with $\rho = 0$, $\overline{\nu}_F$ beats $\overline{\nu}_O$. Clearly, $\overline{\nu}_I^o$ and $\overline{\nu}_I$ are best, and as n increases, $\overline{\nu}_O$ is normally worst, being this very noticeable for the case $(\gamma, \delta) = (0, 0.6)$ when $\rho \neq 0$. When the short memory parameters change to $\psi_i = 0.9$, $\overline{\nu}_I$ tends to behave slightly worse, while $\overline{\nu}_F$ gets damaged specially in case $(\gamma, \delta) = (0.4, 1.2)$ when $n = 64$, although it gets closer to the $\psi_i = 0.5$ situation as n increases. $\overline{\nu}_I^o$ and $\overline{\nu}_F^o$ remain similar to the $\psi_i = 0.5$ case, the main change being now that $\overline{\nu}_F^o$ is now preferable to $\overline{\nu}_F$ for most of the cases.

Results for the ARMA(1,1) are given in Tables 2.29, 2.30. Overall, out of 60 cases $\overline{\nu}_I$, $\overline{\nu}_I^o$, $\overline{\nu}_O$, $\overline{\nu}_F$ and $\overline{\nu}_F^o$ perform individually no worse than any of the other estimates in 48, 48, 27, 13 and 12 cases respectively. Thus, apart from the $(\gamma, \delta) = (0, 0.6)$ case with $\rho = 0$, where $\overline{\nu}_O$ is best, $\overline{\nu}_I$ and $\overline{\nu}_I^o$ are the dominant estimates. $\overline{\nu}_F$, and especially $\overline{\nu}_F^o$, behave clearly worse than $\overline{\nu}_O$, with relations 32/6 and 41/2 in favour of $\overline{\nu}_O$, although the behaviour of these three estimates becomes closer as sample size increases.

Behaviour of empirical sizes

We next examine the accuracy of the large sample χ^2 approximation of Corollary 2.1, looking at the size of Wald tests. For the white noise case, we define the Wald statistics $W_I = \overline{b}_I (\overline{\nu}_I - 1)^2$ and $W_F = \overline{b}_F (\overline{\nu}_F - 1)^2$, where

$$\begin{aligned} \overline{b}_I &= b\left(\gamma, \overline{\theta}_I\right) = \widetilde{b}\left(\gamma, \overline{\theta}_I\right) = \overline{v}\left(\gamma, \delta\right), \qquad (2.51) \\ \overline{b}_F &= b\left(\widehat{\gamma}, \overline{\theta}_F\right) = \widetilde{b}\left(\widehat{\gamma}, \overline{\theta}_F\right) = \overline{v}(\widehat{\gamma}, \widehat{\delta}), \qquad (2.52) \end{aligned}$$

with

$$\overline{v}\left(c, d\right) = \frac{n\left\{\Sigma_t x_t^2(c)\Sigma_t x_t^2(d) - \left\{\Sigma_t x_t(c) x_t(d)\right\}^2\right\}}{\Sigma_t x_t^2(d)\Sigma_t \widehat{\varepsilon}_t^2\left(c, d\right)}, \qquad (2.53)$$

where $\widehat{\varepsilon}_t\,(c,d)$ are residuals from the OLS regression of $y_t(c)$ on $x_t(c)$ and $x_t(d)$. Note that $\overline{v}\,(c,d)$ is the usual OLS estimate of variance of the coefficient of $x_t(c)$ in the OLS regression of $y_t(c)$ on $x_t(c)$ and $x_t(d)$. Table 2.31 contain empirical sizes (meaning percentage of rejections) for the white noise case, corresponding to nominal sizes $\alpha = 0.05,\, 0.10$, for the four values of ρ but $\tau = 1$ only, the results for $\tau = 0.5$ and 2 being very similar. The results for W_I are on average too large, but only slightly, and performance here seems very satisfactory over all (γ, δ) and ρ. The empirical sizes of W_F are clearly too large, but though the asymptotic theory would here only provide a good approximation in a larger sample size than any we have employed, nevertheless the sizes also decrease significantly over the range of n considered. The sizes of W_F tend to decrease in β for $|\rho| \geq 0.5$. The results for W_F are again worst when $(\gamma, \delta) = (0, 0.6)$, but are not so conspicuous as in the tables of biases.

For the different autocorrelated situations, we define

$$W_I = b\left(\gamma, \overline{\theta}_I\right)\left(\overline{\nu}_I - 1\right)^2,\ W_F = b\left(\widehat{\gamma}, \overline{\theta}_F\right)\left(\overline{\nu}_F - 1\right)^2, \tag{2.54}$$

$$W_I^{\circ} = \overline{b}\left(\gamma, \overline{\theta}_I\right)\left(\overline{\nu}_I^{\circ} - 1\right)^2,\ W_F^{\circ} = \overline{b}\left(\widehat{\gamma}, \overline{\theta}_F\right)\left(\overline{\nu}_F^{\circ} - 1\right)^2. \tag{2.55}$$

For the purely AR case (Tables 2.32-2.35), in general, the approximation gets worse as ρ increases in absolute value. For $\phi_i = 0.5$, results for W_I are on average too large, but performance is reasonably satisfactory, $(\gamma, \delta) = (0, 0.6)$ being the worst case, especially when $\rho = 0.75$. Empirical sizes of W_F are clearly too large, but they seem to react well to the increase in the sample size. The convergence is again very slow for the case $(\gamma, \delta) = (0, 0.6)$ (and even for $(\gamma, \delta) = (0.4, 1.2)$ when $\rho = 0.75$). Empirical sizes corresponding to W_I° and W_F° follow a similar pattern to the one described above, but surprisingly they are much smaller than those for W_I and W_F. This could make sense for the cases where $\beta > 1$, but is most surprising otherwise, as Theorem 2.2 implies that the estimates $\overline{\nu}_I^{\circ}$ and $\overline{\nu}_F^{\circ}$ do not enjoy mixed-normal asymptotics. In fact, in those cases, the theory says that the corresponding Wald statistics should explode as n increases, but this is not reflected in our experiment. For $\phi_i = 0.9$, as expected, the approximation gets worse. For $(\gamma, \delta) = (0, 0.6)$, sizes corresponding to W_F do not behave as theory predicts (getting the situation worse as $|\rho|$ increases), but we believe, this is due to the somehow extreme situation of an error process very close to non-stationarity with a value of β very close to the lower allowed limit of $1/2$. Sizes of W_F° for this case, also increase as n increases, and taking into account our previous results, is somehow noticeable that sizes also increase with the sample size for the case $(0.4, 1.2)$, as the theory predicts, whereas for this case, sizes of W_F react more appropriately to increases in n.

For the purely MA case, results for the case $\psi_i = 0.5$ are given in Tables 2.36, 2.37. Basically, similar conclusion as for the AR case apply. Sizes for W_I are on average too large, but performance is satisfactory, again results being worst when $(\gamma, \delta) = (0, 0.6)$. W_I° improves on these sizes, result not supported by the theory for the cases $(\gamma, \delta) = (0, 0.6)$, $(0.4, 1.2)$. Sizes for W_F are too large, but they seem to react well to the increase in sample size. Now, sizes for W_F° are similar to the ones for W_F, which represents a different behaviour compared to the results for the AR case, where in fact the improvement of W_F° over W_F was more important. In our present situation, this improvement is only very noticeable for the case $(\gamma,\delta) = (0,0.6)$, which in fact is a case for which sizes for W_F should behave better according to the asymptotic theory. Results for $\psi_i = 0.9$ are given in Tables 2.38, 2.39, and are somehow surprising. Sizes corresponding to W_I and W_F get in general worse with the increase in the value of the short memory parameters, but they again react well to the increase in the sample size. In contrast with this, sizes for W_I^o and W_F^o improve slightly when ψ_i increases, mainly for $n = 64$, although for the case $(\gamma,\delta) = (0,0.6)$ with $\rho = 0.75$ sizes corresponding to W_I^o increase when n increases, which supports the theory.

Results for the ARMA(1,1) case are reported in Tables 2.40, 2.41. Basically, the arguments presented for the previous autocorrelated cases also apply here. Improvements of W_I^o and W_F^o over W_I and W_F respectively, are most noticeable for the case $(\gamma,\delta) = (0,0.6)$, being not very important for W_I^o when β is large, W_F^o beating clearly W_F even in this situation.

2.4.2 Standard situation: $\gamma = 0,\ \delta = 1$

For the second part of the Monte Carlo study, we focus on the familiar case $(\gamma,\delta) = (0,1)$, presenting only results for the white noise situation. As discussed in Section 2.2, we include now also the "intermediate" estimates, employing prior knowledge of either γ or δ,

$$\overline{\nu}_\gamma = \overline{\nu}(0,\widehat{\delta}) = \widehat{\nu}(0,\widehat{\delta},\overline{\theta}_\gamma) = \widetilde{\nu}(0,\widehat{\delta},\overline{\theta}_\gamma), \tag{2.56}$$

$$\overline{\nu}_\delta = \overline{\nu}(\widehat{\gamma},1) = \widehat{\nu}(\widehat{\gamma},1,\overline{\theta}_\delta) = \widetilde{\nu}(\widehat{\gamma},1,\overline{\theta}_\delta), \tag{2.57}$$

where $\overline{\theta}_\gamma$, $\overline{\theta}_\delta$ consist of the appropriate elements of $\bar{\Omega}(0,\widehat{\delta})$, $\bar{\Omega}(\widehat{\gamma},1)$, respectively. Note that in this case $\overline{\nu}_O$ has the same rate of convergence as $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_\gamma$, $\overline{\nu}_\delta$, being n-consistent, but lacking the mixed normal asymptotics. We employed the same values of ρ and τ as before. Table 2.42 reports Monte Carlo biases. The best and worst estimates, when $\rho \neq 0$, are again $\overline{\nu}_I$ and $\overline{\nu}_O$ respectively. However, though $\overline{\nu}_\delta$ (which correctly assumes $\delta = 1$) is second-best, $\overline{\nu}_\gamma$ (which correctly assumes $\gamma = 0$) is inferior to $\overline{\nu}_F$; this is all the more surprising because γ is more problematic to estimate than δ as it uses residuals. It might appear that in $\overline{\nu}_F$ the contributions to bias from estimation of γ and δ to some degree cancel each other out. However, we must stress that this is in any case a small-sample phenomenon, being barely noticeable for $n = 128$ and absent when $n = 256$, while even for $n = 64$ the bias of $\overline{\nu}_\gamma$ is never so large as to cause serious concern.

As before, the standard deviations, reported in Table 2.43, are much less variable. For $|\rho| \geq 0.5$, $\overline{\nu}_O$ clearly performs worst, but there is little to be said about the optimal estimates, though for small n, $\overline{\nu}_I$ seems best, followed closely by $\overline{\nu}_\delta$, with almost identical values for $\overline{\nu}_\gamma$ and $\overline{\nu}_F$.

Table 2.44 reports empirical sizes, including now results for $W_\gamma = \overline{b}_\gamma\left(\overline{\nu}_\gamma - 1\right)^2$, $W_\delta = \overline{b}_\delta\left(\overline{\nu}_\delta - 1\right)^2$, where

$$\overline{b}_\gamma = b\left(0,\overline{\theta}_\gamma\right) = \widetilde{b}\left(0,\overline{\theta}_\gamma\right) = \overline{v}(0,\widehat{\delta}), \tag{2.58}$$

$$\overline{b}_\delta = b\left(\widehat{\gamma},\overline{\theta}_\delta\right) = \widetilde{b}\left(\widehat{\gamma},\overline{\theta}_\delta\right) = \overline{v}\left(\widehat{\gamma},1\right), \tag{2.59}$$

and

$$W_O = \frac{n(\overline{\nu}_O - 1)^2 \Sigma_t x_t^2}{\Sigma_t (y_t - \overline{\nu}_O x_t)^2}, \tag{2.60}$$

though W_O does not have a limiting null χ_1^2 distribution. We find that the empirical size of W_I is the most accurate, followed by W_γ, the discrepancy increasing with $|\rho|$. Even for $\rho \neq 0$, W_O often does better than W_δ and W_F, which perform quite similarly; clearly the effect of estimating γ is playing a dominant role here, and use of an improved preliminary estimate of ν, such as that proposed by Robinson and Marinucci (1998, 2001), or iteration, may be warranted.

2.5 Empirical investigation: the purchasing power parity hypotheses

Numerous empirical studies have cast significant doubt on the purchasing power parity (PPP) hypothesis with respect to the short run, but have yielded mixed evidence with respect to the long run (see e.g. Corbae and Ouliaris, 1988, Enders, 1988, Kim, 1990, Taylor, 1988). Cheung and Lai (1993) proposed a fractional version of the PPP specification, essentially (1.25), (1.26) with x_t representing the domestic price index and y_t the foreign price index, converted to domestic currency units. The coefficient ν in (1.25) is unity according to the absolute or homogeneous version of PPP, so this is testable by our Wald statistic of Corollary 2.1. Using unit root tests, Cheung and Lai (1993) failed to reject the hypothesis $\delta = 1$ and then, using differenced OLS residuals, they computed semiparametric log periodogram estimates of $\delta - \beta - 1$ and then tested the non-co-integration null hypothesis of $\beta = 0$ against the alternative $\beta > 0$, using critical values computed by simulation in view of the inapplicability of standard asymptotic theory in this case. They found evidence of co-integration in a number of bivariate series, but did not test $\nu = 1$. We employ a step-by-step approach, first testing whether the integration orders δ_x and δ_y of x_t and y_t are the same, then for the presence of co-integration, then for $\beta > 1/2$ and finally, given all these hurdles have been crossed, $\nu = 1$. In the first three steps we used semiparametric procedures (as did Cheung and Lai, 1993, Marinucci and Robinson, 2001), while in the final step, which is most relevant to the material of the current chapter, we identified parametric models for the autocorrelation in u_t and thence computed estimates of ν and Wald statistics.

The semiparametric estimates of integration orders were all Robinson's (1995a) versions of log periodogram estimates, but without trimming, using first differences and then adding back 1. We estimated δ_x and δ_y separately, and then tested $\delta_x = \delta_y (= \delta)$ by an adaptation of Robinson and Yajima's (2002) statistic $\widehat{T}_{ab}$ to log periodogram estimation, with their trimming sequence $h(n)$ chosen as b^{-5-2i} for $i = 1, ..., 4$, with b the bandwidth used in the estimation. Given $\delta_x = \delta_y$ is not rejected, we performed the Hausman test for no-co-integration of Marinucci and Robinson (2001), comparing the estimate $\widetilde{\delta}_x$ of δ_x with the more efficient bivariate one of Robinson (1995a), that uses the information $\delta_x = \delta_y$. Given co-integration is not rejected, the null $\beta = 1/2$ was rejected in favour of $\beta > 1/2$ if and only if a studentized $\widetilde{\delta}_x - \widetilde{\gamma} - 1/2$, was significantly large relative to the standard normal distribution, where $\widetilde{\gamma}$ is the estimate of γ using OLS residuals.

Using annual data (as is relevant to the long-run version of PPP) of Obstfeld and Taylor (2002) for the period 1870-1992 (with $n = 123$), we applied the above methodology to four bivariate series, the US ("domestic") versus the "foreign" countries Australia, Canada, Italy, UK. Strong evidence against equality of integration orders was found in case of Australia and Italy, and against co-integration in case of Canada. However, the UK "passed" all three initial tests. Across the range $b = 10, ..., 29$, $(\widetilde{\delta}_x, \widetilde{\delta}_y)$ varied between the extremes (1.341, 1.095) and (1.572, 1.376), and across $b = 16, ..., 25$ and the four $h(n)$ choices, $\delta_x = \delta_y$ was rejected in only 9 out of 40 cases, and these all at the 10% level. For the same b, no-co-integration was rejected at 10% in all cases, at 5% in 4 cases, and at 1% in 3 cases, while $\beta = 1/2$ was rejected against $\beta > 1/2$ at the 1% level in all cases.

For the US-UK data, we identified parametric models for $f(\lambda)$ as follows. Throughout, $A(L)$ in (2.22) was diagonal, and u_{1t}, u_{2t} treated separately. They were proxied by $\Delta^{\widetilde{\gamma}}(y_t - \bar{\nu}_O x_t)$, $\Delta^{\widetilde{\delta}_x} x_t$, for each of the extreme $\widetilde{\gamma}, \widetilde{\delta}_x$, namely $\widetilde{\gamma} = .374, .698$ and $\widetilde{\delta}_x = 1.572,\ 1.341$, and then Box-Jenkins-type procedures identified models within the ARMA class. This resulted in AR(1) and ARMA$(1,1)$ u_{1t} and white noise and ARMA$(1,1)$ u_{2t}, and we fitted all four combinations. We also fitted bivariate versions of Bloomfield's (1973) model, where

$$A(s) = diag\left\{\exp\left(\textstyle\sum_{j=1}^{p} \theta_{1j} s^j\right), \exp\left(\textstyle\sum_{j=1}^{p} \theta_{2j} s^j\right)\right\}, \tag{2.61}$$

for $p = 1, 2, 3$. For each model we applied the univariate Whittle procedure in Velasco and Robinson (2000), using untapered, differenced data and adding back 1. We summarize the seven models and the resulting $(\widehat{\delta}, \widehat{\gamma})$ as follows:

Model 1:	u_{1t} is $AR(1)$ and u_{2t} is white noise.	$(\widehat{\delta}, \widehat{\gamma}) = (1.612, .669)$.
Model 2:	u_{1t} is $AR(1)$ and u_{2t} is $ARMA(1,1)$.	$(\widehat{\delta}, \widehat{\gamma}) = (1.408, .669)$.
Model 3:	u_{1t} is $ARMA(1,1)$ and u_{2t} is white noise.	$(\widehat{\delta}, \widehat{\gamma}) = (1.612, .660)$.
Model 4:	u_{1t} is $ARMA(1,1)$ and u_{2t} is $ARMA(1,1)$.	$(\widehat{\delta}, \widehat{\gamma}) = (1.408, .660)$.
Model 5:	u_t is bivariate Bloomfield with $p = 1$.	$(\widehat{\delta}, \widehat{\gamma}) = (1.214, .710)$.
Model 6:	u_t is bivariate Bloomfield with $p = 2$.	$(\widehat{\delta}, \widehat{\gamma}) = (1.434, .701)$.
Model 7:	u_t is bivariate Bloomfield with $p = 3$.	$(\widehat{\delta}, \widehat{\gamma}) = (1.323, .547)$.

The $\widehat{\gamma}$ seem very robust to the short memory specification, the $\widehat{\delta}$ rather less so. We also took this opportunity to examine another question which in one form or another always arises with application of fractional models, and perhaps most acutely when nonstationary data are involved. This is the matter of truncation. When estimated innovations from a stationary fractional model are computed, the (infinite) AR representation has to be truncated because the data begins at time "1", not at time "$-\infty$". Now in our model (1.25), (1.26) for nonstationary data, the truncation is actually inherent in the model, so strictly speaking there is no "error" associated with it. However, the model reflects the time when the data begins, and if we were to drop the first observation, say, and start the model off at the next one, the degree of filtering applied to all subsequent observations would change, and it is possible that this could have a marked effect, especially with nonstationary data. Thus, in Table 2.2 we report computations of our estimates $\widehat{\nu}(\widehat{\gamma}, \widehat{\delta}, \widehat{\theta}) = \bar{\nu}_i$ and Wald statistics

$$b(\widehat{\gamma}, \hat{\theta}) \left\{ \widehat{\nu}\left(\widehat{\gamma}, \widehat{\delta}, \widehat{\theta}\right) - 1 \right\}^2 = W_i, \tag{2.62}$$

for models $i = 1, ..., 7$, based on the last $n' = n - j$ observations, for $j = 0, 1, .., 10$, in order to explore sensitivity to starting value.

TABLE 2.2
PPP EMPIRICAL EXAMPLE: ESTIMATES OF ν AND WALD TESTS OF $\nu = 1$ FOR MODELS 1-7 COMPUTED FROM THE LAST $n' = 113, ..., 123$ OBSERVATIONS OF US/UK DATA

n'	123	122	121	120	119	118	117	116	115	114	113
$\bar{\nu}_1$	1.139	1.050	1.014	.952	.889	.875	.871	.867	.864	.875	.875
W_1	26.23	.352	.017	.163	.759	.940	.986	1.035	1.082	.903	.890
$\bar{\nu}_2$	1.294	.959	1.030	.995	.949	.941	.941	.938	.936	.944	.943
W_2	117.3	.231	.078	.002	.159	.208	.206	.226	.243	.181	.182
$\bar{\nu}_3$	1.113	1.084	1.017	.955	.889	.871	.866	.863	.859	.871	.868
W_3	18.64	1.070	.027	.161	.823	1.079	1.138	1.196	1.251	1.051	1.059
$\bar{\nu}_4$	1.290	.966	1.028	.997	.950	.939	.939	.936	.934	.942	.939
W_4	122.6	.178	.078	.001	.170	.241	.240	.263	.281	.212	.227
$\bar{\nu}_5$	1.274	1.042	1.025	.986	.940	.933	.932	.931	.929	.939	.936
W_5	112.2	.225	.055	.014	.230	.283	.283	.296	.306	.223	.239
$\bar{\nu}_6$	1.278	.960	1.015	.983	.939	.932	.931	.930	.927	.937	.935
W_6	114.9	.211	.019	.020	.241	.292	.292	.306	.325	.246	.255
$\bar{\nu}_7$	1.298	.999	1.048	1.024	.975	.961	.962	.956	.956	.963	.958
W_7	116.9	.000	.279	.052	.047	.109	.105	.138	.136	.096	.122

Substantial variation is evident across the larger n', with all $\bar{\nu}_i$ exceeding 1 and the homogeneity hypothesis being strongly rejected when $n' = 123$, across all seven models, but as n' decreases, things stabilize. For $n' \leq 119$ some sensitivity to the u_{2t} specification was found, the white noise cases (Models 1 and 3) providing estimates of ν less than .9, whereas for the other models they all exceed .9, with the largest values for Model 7. For $n' \leq 122$ the homogeneity hypothesis $\nu = 1$ is never rejected even at the 10% level.

From certain perspectives, practitioners could considered our empirical analysis simplistic, as we do not take into account possible alternative features of our data. In particular, we did not check for the possibility of structural breaks or nonlinearities in our long time series. Admittedly, these are relevant issues, whose linkages with fractional processes are mainly undiscovered, but which already attracted the attention of some researchers. For example, Granger (1999) showed that structural break processes could produce "long-memory" properties of the data, while he suggested that, among nonlinear time series, there could be other plausible alternatives to $I(d)$ processes. Undoubtedly, a very rigorous and exhaustive analysis of the PPP hypothesis should contemplate these issues, but, at this stage, our intention was simply to propose a sensible methodology incorporating the techniques developed in this chapter, which, at the same time, motivated our testing problem appropriately.

2.6 Final comments

Our treatment of a bivariate system in a parametric setting is quite general, in that a very wide range of models for the $I(0)$ input series u_t is covered, while our regularity conditions seem to afford little scope for relaxation. Nevertheless, there are significant aspects we do not consider here, some of which are studied in subsequent chapters.

1. Our case $\beta > 1/2$ includes the familiar $CI(1,1)$ setting, but $0 < \beta < 1/2$ is also of interest. As discussed in Chapters 1 and 3 of this thesis, $x_t(\gamma)$ is then "asymptotically stationary" and our estimates are $n^{1/2}$-consistent and asymptotically normal, with limiting variance that is affected by the estimation (and the efficiency of estimation) of one or more of γ, δ and θ, because the requirement $\kappa > 1-\beta$ on κ in (2.27) still appears to be relevant when $\beta < 1/2$, but (2.27) is unachievable then because $\widehat{\gamma}$, $\widehat{\delta}$ are at most $n^{1/2}$-consistent, no matter the values of γ and δ, see eg. Velasco and Robinson (2000).

2. In view of the literature on non-fractional co-integration, there would be empirical interest in incorporating also in (1.25) and/or (1.26) deterministic components. Modification of the theory to cover polynomial time trends seems relatively straightforward, though our fractional focus suggests allowing for possibly non-integral powers of t in studying the relative importance of stochastic and deterministic trends, as Robinson and Marinucci (2000) did in connection with OLS and its narrow-band modification, while if such powers are unknown the extension is decidedly non-trivial.

3. Extension of our methods and theory to vector y_t and x_t, and matrix ν, seems straightforward when there is no variation in integration orders across elements of x_t and $y_t - \nu x_t$. However, multivariate data invite consideration not only of multiple co-integrating relationships but also of observables and/or co-integrating errors with differing integration orders, which would raise particular questions of identifiability and complicate estimation.

4. Our parametric treatment of autocorrelation in u_t follows a classical economic time series tradition and allows parsimony, but the unit root co-integration literature has stressed a nonparametric approach. Nonparametric estimation of $f(\lambda)$ should lead to the same outcomes as in Theorems 2.1 and 2.2, and corresponds in (2.12) to taking $B_j = 0$, $j > p$, but letting p go slowly to infinity in the asymptotic theory, while in (2.18) or (2.34) weighted autocovariance or periodogram estimation might be used. As shown in Chapter 4, the forms (2.34) are easiest to handle technically, while in (2.18), the variation in $f(\lambda_j)$ across the n Fourier frequencies might be dealt with by techniques like those used by Robinson (1991, pp.1354, 1355). Alternatively, one can employ estimates which are constant over slowly degenerating bands, as proposed in Hannan (1963) and employed by Phillips (1991b) in the $CI(1,1)$ case. Note that the slow convergence of nonparametric estimates of f is of concern because even

the refinement of Assumption 2.3 (ii) mentioned in the discussion of that assumption requires a convergence rate that approaches arbitrarily close to $n^{-1/2}$ as $\beta \to 1/2$. In principle $n^{\epsilon-1/2}$–consistent nonparametric spectral estimates can be found, for any $\epsilon > 0$ (where, for example, ϵ depends on kernel order, see eg. Cogburn and Davis, 1974), though, as β is unknown, one can never be sure that the ϵ achieved is sufficient. These issues are discussed in detail in Chapter 4, where we comment on the problem of relatively slow nonparametric estimates of the nuisance parameters.

2.7 Appendix 2

2.7.1 Appendix 2.A: Outline of proof of Theorem 2.1

Though the proof of (2.32) for the time-domain estimates (2.12) is not contained in that for the frequency-domain estimates (2.18), nevertheless the proof for the latter does involve approximation in the time domain so that many of the steps are similar. Thus, because it entails the greater technical challenge, computational elegance and generality, we give the proof only for (2.18).

Consider first the infeasible estimate $\widehat{\nu}(\gamma, \delta, \theta)$. We have

$$z_t(c,d) = \zeta x_t(c)\,\nu + v_t(c,d), \tag{2.63}$$

where

$$v_t(c,d) = (u_{1t}(c-\gamma), x_t(d))'. \tag{2.64}$$

Thus

$$\widehat{\nu}(c,d,h) - \nu = \frac{e(c,d,h)}{b(c,h)}, \tag{2.65}$$

where

$$e(c,d,h) = \sum_{j=1}^{n} p(\lambda_j; h)\, w_{x(c)}(-\lambda_j)\, w_{v(c,d)}(\lambda_j). \tag{2.66}$$

From (1.26), (2.64), $v_t(\gamma,\delta) = u_t^{\#}$, so that

$$\widehat{\nu}(\gamma,\delta,\theta) - \nu = \frac{e(\gamma)}{b(\gamma)}, \tag{2.67}$$

where

$$b(\gamma) = b(\gamma,\theta) = \sum_{j=1}^{n} q(\lambda_j) \left|w_{x(\gamma)}(\lambda_j)\right|^2, \tag{2.68}$$

$$e(\gamma) = e(\gamma,\delta,\theta) = \sum_{j=1}^{n} p(\lambda_j)\, w_{x(\gamma)}(-\lambda_j)\, w_u(\lambda_j), \tag{2.69}$$

with

$$p(\lambda) = p(\lambda;\theta),\ q(\lambda) = q(\lambda;\theta). \tag{2.70}$$

Also define

$$e^*(\gamma) = \sum_{m=1}^{n} \left\{ \zeta x_m(\gamma) - \sum_{s=1}^{m-1} B_s \zeta x_{m-s}(\gamma) \right\}' \Omega^{-1} \varepsilon_m, \tag{2.71}$$

$$e^{**}(\gamma) = \zeta' B(1)' \Omega^{-1} \sum_{m=1}^{n} x_{m-1}(\gamma) \varepsilon_m, \tag{2.72}$$

$$b^*(\gamma) = \sum_{m=1}^{n} \left\{ \zeta x_m(\gamma) - \sum_{s=1}^{m-1} B_s \zeta x_{m-s}(\gamma) \right\}' \Omega^{-1} \left\{ \zeta x_m(\gamma) - \sum_{s=1}^{m-1} B_s \zeta x_{m-s}(\gamma) \right\}, \tag{2.73}$$

$$b^{**}(\gamma) = \frac{q(0)}{2\pi} \sum_{m=1}^{n} x_m^2(\gamma). \tag{2.74}$$

Now (2.32) for $\widehat{\nu}(\gamma, \delta, \theta)$ follows on establishing the following six propositions.

Proposition 2.1. *As* $n \to \infty$,

$$e(\gamma) - e^*(\gamma) = o_p(n^\beta). \tag{2.75}$$

Proposition 2.2. *As* $n \to \infty$,

$$e^*(\gamma) - e^{**}(\gamma) = o_p(n^\beta). \tag{2.76}$$

Proposition 2.3. *As* $n \to \infty$,

$$n^{-\beta} e^{**}(\gamma) \Rightarrow \zeta' B(1)' \Omega^{-1} \int_0^1 \widetilde{W}(r;\beta)\, dW(r). \tag{2.77}$$

Proposition 2.4. *As* $n \to \infty$,

$$b(\gamma) - b^*(\gamma) = o_p(n^{2\beta}). \tag{2.78}$$

Proposition 2.5. *As* $n \to \infty$,

$$b^*(\gamma) - b^{**}(\gamma) = o_p(n^{2\beta}). \tag{2.79}$$

Proposition 2.6. *As* $n \to \infty$,

$$n^{-2\beta} b^{**}(\gamma) \Rightarrow \frac{q(0)}{2\pi} \int_0^1 \widetilde{W}(r;\beta)^2\, dr, \tag{2.80}$$

where the right side is almost surely positive.

To prove (2.32) for the remaining four estimates in (2.18), it suffices to consider only $\widehat{\nu}(\gamma, \delta, \widehat{\theta})$ and $\widehat{\nu}(\widehat{\gamma}, \widehat{\delta}, \widehat{\theta})$ as the proof for the other, intermediate cases, will essentially be implied. It thus remains to show that

$$\widehat{\nu}(\gamma, \delta, \widehat{\theta}) - \widehat{\nu}(\gamma, \delta, \theta) = o_p(n^\beta), \tag{2.81}$$

$$\widehat{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta}) - \widehat{\nu}(\gamma,\delta,\widehat{\theta}) = o_p\left(n^{\beta}\right). \tag{2.82}$$

We have first

$$\widehat{\nu}(\gamma,\delta,\widehat{\theta}) - \nu = \frac{e(\gamma,\delta,\widehat{\theta})}{b(\gamma,\widehat{\theta})}, \tag{2.83}$$

so that, from (2.65), the left side of (2.81) is

$$\frac{e(\gamma,\delta,\widehat{\theta}) - e(\gamma,\delta,\theta)}{b(\gamma,\widehat{\theta})} + e(\gamma,\delta,\theta)\left\{\frac{1}{b(\gamma,\widehat{\theta})} - \frac{1}{b(\gamma,\theta)}\right\}. \tag{2.84}$$

In view of Propositions 2.1-2.6, the proof of (2.81) follows on establishing the following two propositions.

Proposition 2.7. *As* $n \to \infty$,

$$e(\gamma,\delta,\widehat{\theta}) - e(\gamma,\delta,\theta) = o_p\left(n^{\beta}\right). \tag{2.85}$$

Proposition 2.8. *As* $n \to \infty$,

$$b(\gamma,\widehat{\theta}) - b(\gamma,\theta) = o_p\left(n^{2\beta}\right). \tag{2.86}$$

To prove (2.82), note that

$$\widehat{\nu}(\widehat{\gamma},\widehat{\delta},\widehat{\theta}) - \nu = \frac{e(\widehat{\gamma},\widehat{\delta},\widehat{\theta})}{b(\widehat{\gamma},\widehat{\theta})}, \tag{2.87}$$

so from (2.83) the left side of (2.82) is

$$\begin{aligned}
&\frac{e(\widehat{\gamma},\widehat{\delta},\widehat{\theta}) - e(\widehat{\gamma},\widehat{\delta},\theta) - e(\gamma,\delta,\widehat{\theta}) + e(\gamma,\delta,\theta)}{b(\widehat{\gamma},\widehat{\theta})} \\
&+\frac{e(\widehat{\gamma},\widehat{\delta},\theta) - e(\gamma,\delta,\theta)}{b(\widehat{\gamma},\widehat{\theta})} - \frac{e(\gamma,\delta,\widehat{\theta})}{b(\widehat{\gamma},\widehat{\theta})b(\gamma,\widehat{\theta})}\left\{b(\widehat{\gamma},\theta) - b(\gamma,\theta)\right\} \\
&-\frac{e(\gamma,\delta,\widehat{\theta})}{b(\widehat{\gamma},\widehat{\theta})b(\gamma,\widehat{\theta})}\left\{b(\widehat{\gamma},\widehat{\theta}) - b(\widehat{\gamma},\theta) - b(\gamma,\widehat{\theta}) + b(\gamma,\theta)\right\},
\end{aligned} \tag{2.88}$$

and (2.82) follows from Propositions 2.1-2.8 on establishing the following four propositions.

Proposition 2.9. *As* $n \to \infty$,

$$e(\widehat{\gamma},\widehat{\delta},\theta) - e(\gamma,\delta,\theta) = o_p\left(n^{\beta}\right). \tag{2.89}$$

Proposition 2.10. *As* $n \to \infty$,

$$e(\widehat{\gamma},\widehat{\delta},\widehat{\theta}) - e(\widehat{\gamma},\widehat{\delta},\theta) - e(\gamma,\delta,\widehat{\theta}) + e(\gamma,\delta,\theta) = o_p\left(n^{\beta}\right). \tag{2.90}$$

Proposition 2.11. *As* $n \to \infty$,

$$b(\widehat{\gamma},\theta) - b(\gamma,\theta) = o_p\left(n^{2\beta}\right). \tag{2.91}$$

Proposition 2.12. *As* $n \to \infty$,

$$b(\widehat{\gamma},\widehat{\theta}) - b(\widehat{\gamma},\theta) - b(\gamma,\widehat{\theta}) + b(\gamma,\theta) = o_p\left(n^{2\beta}\right). \tag{2.92}$$

2.7.2 Appendix 2.B: Proofs of propositions

Proof of Proposition 2.1. Write $e(\gamma)$ as

$$\frac{\zeta'}{n}\sum_{j=1}^{n}\sum_{l=-\infty}^{\infty}\overline{B}_l'e^{-il\lambda_j}\Omega^{-1}\sum_{m=-\infty}^{\infty}\overline{B}_m e^{im\lambda_j}\sum_{s=1}^{n}x_s(\gamma)\,e^{-is\lambda_j}\sum_{t=1}^{n}u_t e^{it\lambda_j}, \tag{2.93}$$

taking $\overline{B}_l = 0$, $l < 0$, $\overline{B}_0 = I_2$, $\overline{B}_l = -B_l$, $l > 0$. We can rewrite this as

$$\frac{\zeta'}{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{j=1}^{n}\sum_{l=-\infty}^{\infty}\overline{B}_{l-s}'e^{-i(l-s)\lambda_j}\Omega^{-1}\sum_{m=-\infty}^{\infty}\overline{B}_{m-t}e^{i(m-t)\lambda_j}x_s(\gamma)\,e^{-is\lambda_j}u_t e^{it\lambda_j}$$

$$=\sum_{m=1}^{\infty}\sum_{r=-\infty}^{\infty}\left\{\sum_{s=1}^{n}\overline{B}_{m-s+rn}\zeta x_s(\gamma)\right\}'\Omega^{-1}\sum_{t=1}^{n}\overline{B}_{m-t}u_t, \tag{2.94}$$

because

$$\sum_{j=1}^{n}e^{it\lambda_j} = n,\ t = 0, \mathrm{mod}\,(n)\,; = 0,\ \text{otherwise}. \tag{2.95}$$

The expectation of the absolute value of the difference between (2.94) and

$$\sum_{m=1}^{n}\sum_{r=-\infty}^{\infty}\left\{\sum_{s=1}^{n}\overline{B}_{m-s+rn}\zeta x_s(\gamma)\right\}'\Omega^{-1}\sum_{t=1}^{n}\overline{B}_{m-t}u_t \tag{2.96}$$

is bounded by

$$K\sum_{m=n+1}^{\infty}\left[E\left\|\sum_{r=-\infty}^{\infty}\sum_{s=1}^{n}\overline{B}_{m-s+rn}\sum_{v=1}^{s}a_{s-v}u_{2v}\right\|^2 E\left\|\sum_{t=1}^{n}\overline{B}_{m-t}u_t\right\|^2\right]^{\frac{1}{2}}, \tag{2.97}$$

with $a_t = a_t(\beta)$, K denoting throughout the thesis a generic positive constant. The second expectation is bounded by

$$tr\left\{\sum_{t=1}^{n}\sum_{s=1}^{n}\int_{-\pi}^{\pi}B_{m-t}f(\lambda)\,B_{m-s}'e^{i(s-t)\lambda}d\lambda\right\} \le K\int_{-\pi}^{\pi}\left\|\sum_{t=1}^{n}B_{m-t}e^{-it\lambda}\right\|^2 d\lambda$$

$$\le K\sum_{t=1}^{n}\|B_{m-t}\|^2 \le K\sum_{t=m-n}^{\infty}\|B_t\|^2, \tag{2.98}$$

for $m > n$. The first expectation in (2.97) is bounded by

$$tr\left\{\int_{-\pi}^{\pi}\sum_{r=-\infty}^{\infty}\sum_{s=1}^{n}\sum_{v=1}^{s}\overline{B}_{m-s+rn}a_{s-v}e^{-iv\lambda}f_{22}(\lambda)\sum_{q=-\infty}^{\infty}\sum_{t=1}^{n}\sum_{w=1}^{t}\overline{B}_{m-t+qn}'a_{t-w}e^{iw\lambda}d\lambda\right\}$$

$$\leq K \int_{-\pi}^{\pi} \left\| \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \sum_{v=1}^{s} \overline{B}_{m-s+rn} a_{s-v} e^{-iv\lambda} \right\|^2 d\lambda$$

$$\leq K \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \sum_{q=-\infty}^{\infty} \sum_{t=1}^{n} \left\| \overline{B}_{m-s+rn} \right\| \left\| \overline{B}_{m-t+qn} \right\| \sum_{v=1}^{\min(s,t)} a_{s-v} a_{t-v}, \tag{2.99}$$

where $f_{ii}(\lambda)$ is the $(i,i)th$ element of $f(\lambda)$, and thus is bounded. From Lemma 2.D.2, (2.99) is bounded by

$$K n^{2\beta-1} \left(\sum_{l=0}^{\infty} \|B_l\| \right)^2 = O\left(n^{2\beta-1}\right), \tag{2.100}$$

using (2.25). It follows that (2.97) is bounded by

$$\begin{aligned} K n^{\beta-\frac{1}{2}} \sum_{m=n+1}^{\infty} \left(\sum_{t=m-n}^{\infty} \|B_t\|^2 \right)^{\frac{1}{2}} &\leq K n^{\beta-\frac{1}{2}} \sum_{m=1}^{\infty} \left(\sum_{t=m}^{\infty} \|B_t\|^2 \right)^{\frac{1}{2}} \\ &\leq K n^{\beta-1/2} \sum_{m=1}^{\infty} \sum_{t=m}^{\infty} \|B_t\| \\ &\leq K n^{\beta-\frac{1}{2}} \sum_{j=1}^{\infty} j \|B_j\| = O(n^{\beta-\frac{1}{2}}), \end{aligned} \tag{2.101}$$

again using (2.25).

Next, the expectation of the absolute value of the difference between (2.96) and

$$\sum_{m=1}^{n} \sum_{r=-\infty}^{\infty} \left\{ \sum_{s=1}^{n} \overline{B}_{m-s+rn} \zeta x_s(\gamma) \right\}' \Omega^{-1} \varepsilon_m \tag{2.102}$$

is bounded by

$$K \sum_{m=1}^{n} \left[E \left\| \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \overline{B}_{m-s+rn} \sum_{v=1}^{s} a_{s-v} u_{2v} \right\|^2 E \left\| \sum_{t=-\infty}^{0} \overline{B}_{m-t} u_t \right\|^2 \right]^{\frac{1}{2}}. \tag{2.103}$$

Proceeding as in (2.98), the second expectation is bounded by $K \sum_{t=m}^{\infty} \|B_t\|^2$, so since the first expectation is bounded by (2.100), it follows that (2.103) is bounded by

$$K n^{\beta-\frac{1}{2}} \sum_{m=1}^{\infty} \left(\sum_{t=m+1}^{\infty} \|B_t\|^2 \right)^{\frac{1}{2}} = O(n^{\beta-\frac{1}{2}}), \tag{2.104}$$

as in (2.101). The expectation of the absolute value of the difference between (2.102) and $e^*(\gamma)$ is bounded by

$$K \sum_{m=1}^{n} \left[E \left\| \sum_{r>0} \sum_{s=1}^{n} \overline{B}_{m-s+rn} \sum_{v=1}^{s} a_{s-v} u_{2v} \right\|^2 \right]^{\frac{1}{2}}$$

$$\leq Kn^{\beta-\frac{1}{2}} \sum_{m=1}^{n} \left[\sum_{r>0} \sum_{s=1}^{n} \left\| \overline{B}_{m-s+rn} \right\|^2 \right]^{\frac{1}{2}}$$

$$\leq Kn^{\beta-\frac{1}{2}} \sum_{m=1}^{n} \left[\sum_{t=m}^{\infty} \left\| B_t \right\|^2 \right]^{\frac{1}{2}}, \tag{2.105}$$

which is $O(n^{\beta-1/2})$, to complete the proof.

Proof of Proposition 2.2. Consider first the difference

$$\zeta' \sum_{m=1}^{n} \sum_{s=1}^{m} \overline{B}'_{m-s} d_{m-1,s}(\gamma)\, \Omega^{-1} \varepsilon_m, \tag{2.106}$$

where $d_{m-1,s}(\gamma) = x_{m-1}(\gamma) - x_s(\gamma)$. Because there is a contribution to the mean only when $s = m$, (2.106) has expectation

$$-\sum_{m=1}^{n} \zeta' \Omega^{-1} E\left[\varepsilon_m \varepsilon'_m\right] \xi = -n\zeta'\xi = 0. \tag{2.107}$$

(2.106) has variance $c_1 + c_2 + c_3$, where

$$c_1 = \sum_{m=1}^{n} \sum_{q=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{q} \zeta' \overline{B}'_{m-s} \Omega^{-1} E\left[\varepsilon_m \varepsilon'_q\right] \Omega^{-1} \overline{B}_{q-t} \zeta E\left[d_{m-1,s}(\gamma)\, d_{q-1,t}(\gamma)\right], \tag{2.108}$$

$$c_2 = \sum_{m=1}^{n} \sum_{q=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{q} \zeta' \overline{B}'_{m-s} \Omega^{-1} E\left[\varepsilon_m d_{q-1,t}(\gamma)\right] E\left[\varepsilon'_q d_{m-1,s}(\gamma)\right] \Omega^{-1} \overline{B}_{q-t} \zeta, \tag{2.109}$$

and c_3 is a fourth cumulant term to be described subsequently. We have

$$\begin{aligned} d_{m-1,s}(\gamma) &= u_{2,m-1}(-\beta) - u_{2,s}(-\beta) \\ &= \sum_{v=1}^{s} (a_{m-1-v} - a_{s-v}) u_{2v} + \sum_{v=s+1}^{m-1} a_{m-1-v} u_{2v} 1\,(s \leq m-2), \end{aligned} \tag{2.110}$$

with $a_{-1} = 0$.

Considering first c_1, there is a contribution only when $q = m$, and then

$$\begin{aligned} \left| E\left[d_{m-1,s}(\gamma)\, d_{q-1,t}(\gamma)\right] \right| &= \left| \int_{-\pi}^{\pi} f_{22}(\lambda)\, r_{sm}(-\lambda) r_{tm}(\lambda) d\lambda \right| \\ &\leq K \left\{ \int_{-\pi}^{\pi} f_{22}(\lambda) \left| r_{sm}(\lambda) \right|^2 d\lambda \int_{-\pi}^{\pi} f_{22}(\lambda) \left| r_{tm}(\lambda) \right|^2 d\lambda \right\}^{1/2} \\ &\leq K (r_{sm} r_{tm})^{1/2}, \end{aligned} \tag{2.111}$$

writing

$$r_{sm}(\lambda) = \sum_{v=1}^{s}(a_{m-1-v} - a_{s-v})e^{iv\lambda} + \sum_{v=s+1}^{m-1} a_{m-1-v}e^{iv\lambda}1\,(s \le m-2)\,, \tag{2.112}$$

$$r_{sm} = \sum_{v=1}^{s}(a_{m-1-v} - a_{s-v})^2 + \sum_{v=s+1}^{m-1} a_{m-1-v}^2 1\,(s \le m-2)\,. \tag{2.113}$$

Then (2.111) is bounded by $K\{|m-s-1|\,|m-t-1|\}^{1/2} m^{\max(0,2\beta-2)}$, on taking $t = m-2$ in Lemma 2.D.3 for $s \le m-2$, then noting that $r_{m-1,m} = 0$, and that $r_{mm} = \sum_{v=1}^{m-1}(a_{m-v} - a_{m-1-v})^2 + 1 = O(m^{\max(0,2\beta-2)})$, on applying Lemma 2.D.3 with $s = m-1, t = m$. It follows that

$$\begin{aligned} |c_1| &\le K\sum_{m=1}^{n} m^{\max(0,2\beta-2)}\left\{\sum_{j=0}^{m} j^{\frac{1}{2}}\,\|B_j\|\right\}^2 \\ &= O\,(n)\,1(1/2 < \beta \le 1) + O\left(n^{2\beta-1}\right)1(\beta > 1). \end{aligned} \tag{2.114}$$

Next, note that c_2 is zero unless $m = q = s = t$, so $c_2 = O\,(n) = o(n^{2\beta})$. Finally, the fourth cumulant term, c_3, involves the fourth cumulant of ε_m, ε_q, $x_{m-1}(\gamma) - x_s(\gamma)\,,\ x_{q-1}(\gamma) - x_t(\gamma)\,,$ which is easily seen to be zero unless $m = q = s = t$, so that $c_3 = O\,(n)$ also.

It remains to show that

$$\zeta'\sum_{m=1}^{n}\left\{B\,(1) - \sum_{s=1}^{m}\overline{B}_{m-s}\right\}' x_{m-1}(\gamma)\,\Omega^{-1}\varepsilon_m = o_p\left(n^{\beta}\right). \tag{2.115}$$

Clearly the left side has mean zero. Its variance is, from arguments similar to those above, bounded by

$$K\sum_{m=1}^{n}\left\|B\,(1) - \sum_{s=0}^{m-1}\overline{B}_s\right\|^2 Ex_{m-1}^2(\gamma)\,E\,\|\varepsilon_m\|^2 \le K\sum_{m=1}^{n}\left(\sum_{s=m}^{\infty}\|B_s\|\right)^2 m^{2\beta-1}, \tag{2.116}$$

because $Ex_m^2(\gamma) = O(m^{2\beta-1})$ from Robinson and Marinucci (2001). Then, (2.116) is $o\left(n^{2\beta}\right)$ from the Toeplitz lemma, to complete the proof.

Proof of Proposition 2.3. Note that $\zeta' B\,(1)'\,\Omega^{-1}\varepsilon_m$ has mean zero and variance $q\,(0)\,/2\pi$; in view of Theorem 1 of Marinucci and Robinson (2000) and Assumption 2.1, the proof follows by Theorem 2.2 of Kurtz and Protter (1991).

Proof of Proposition 2.4. Omitted, as it is similar to the proof of Proposition 2.1 but significantly easier, especially in view of the norming $n^{-2\beta}$ rather than $n^{-\beta}$.

Proof of Proposition 2.5. This is likewise omitted due to its similarity to, and simplicity relative to, the proof of Proposition 2.2.

Proof of Proposition 2.6. Follows straightforwardly from Marinucci and Robinson (2000), the continuous mapping theorem and Assumption 2.1, and the fact that $\widetilde{W}(r;\beta)$ is almost surely nonzero, from (2.29), (2.30).

Proof of Proposition 2.7. By the mean-value theorem, $p(\lambda;\widehat{\theta}) - p(\lambda;\theta) = (\widehat{\theta} - \theta)'\overline{P}(\lambda)$, where $\overline{P}(\lambda)$ is the matrix $P(\lambda;h) = \partial p(\lambda;h)/\partial h$, with columns evaluated respectively at $\overline{\theta}^{(1)}$, $\overline{\theta}^{(2)}$, where $\left\|\overline{\theta}^{(i)} - \theta\right\| \leq \left\|\widehat{\theta} - \theta\right\|$, $i = 1,2$. Writing $P(\lambda) = P(\lambda;\theta)$,

$$\begin{aligned}\sup_{\lambda}\left\|\overline{P}(\lambda) - P(\lambda)\right\| &\leq 2\sup_{h\in N_\epsilon}\ \sup_{\lambda}\left\|P(\lambda;h) - P(\lambda)\right\| \\ &\quad +4\sup_{h\in\Theta}\ \sup_{\lambda}\left\|P(\lambda;h)\right\| 1\left(\left|\widehat{\theta}-\theta\right| \geq \epsilon\right), \end{aligned} \qquad (2.117)$$

where $\epsilon > 0$ and $N_\epsilon = \{h : \|h - \theta\| < \epsilon\}$. Noting Assumption 2.2 parts (ii) and (iii), since continuity in h for all λ implies uniform continuity on the compact set $[-\pi,\pi]$, the first term on the right of (2.117) tends to 0 as $\epsilon \to 0$. The second term is $o_p(1)$ as $n \to \infty$ for $\epsilon > 0$ from Assumption 2.2 (ii) and (iii) and Assumption 2.3 (ii). It follows that

$$\begin{aligned}\left\|\sum_{j=1}^{n}\left\{\overline{P}(\lambda_j) - P(\lambda_j)\right\} w_{x(\gamma)}(-\lambda_j)w_u(\lambda_j)\right\| &= o_p\left(\sum_{j=1}^{n}\left\|w_{x(\gamma)}(-\lambda_j)w_u(\lambda_j)\right\|\right) \\ &= o_p\left(\left\{\sum_{t=1}^{n}x_t^2(\gamma)\sum_{t=1}^{n}\|u_t\|^2\right\}^{\frac{1}{2}}\right),\end{aligned} \qquad (2.118)$$

which is $o_p(n^{\beta+1/2})$, where we use the Cauchy inequality, (2.95), $\sum_{t=1}^{n}\|u_t\|^2 = O_p(n)$ and

$$\sum_{t=1}^{n} x_t(\gamma)^2 = O_p\left(n^{2\beta}\right), \qquad (2.119)$$

from Robinson and Marinucci (2001). Thus, noting Assumption 2.3 (ii), it remains to show that

$$\sum_{j=1}^{n} P(\lambda_j)w_{x(\gamma)}(-\lambda_j)w_u(\lambda_j) = o_p\left(n^{\beta+\frac{1}{2}}\right). \qquad (2.120)$$

Denote by $P_L(\lambda)$ the partial sum, to L terms, of the Fourier series of $P(\lambda)$, so

$$P_L(\lambda) = \sum_{l=-L}^{L} P_l e^{-il\lambda}, \quad P_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\lambda)e^{il\lambda}d\lambda. \qquad (2.121)$$

From Assumption 2.2 (ii) and (v), and Zygmund (1977, p.64),

$$\sup_{\lambda}\left\|P(\lambda) - P_L(\lambda)\right\| = O\left(\frac{\log L}{L}\right), \qquad (2.122)$$

as $L \to \infty$. Thus

$$\left\| \sum_{j=1}^{n} \{P(\lambda_j) - P_L(\lambda_j)\} w_{x(\gamma)}(-\lambda_j) w_u(\lambda_j) \right\| \leq K \frac{\log L}{L} \left\{ \sum_{t=1}^{n} x_t(\gamma)^2 \sum_{t=1}^{n} \|u_t\|^2 \right\}^{\frac{1}{2}}, \tag{2.123}$$

proceeding as in (2.118). With $L = [n^{1/2}]$, (2.123) is $O_p\left((\log n)\, n^{\beta}\right) = o_p\left(n^{\beta+1/2}\right)$.

On the other hand, for $L < n$,

$$\sum_{j=1}^{n} P_L(\lambda_j) w_{x(\gamma)}(-\lambda_j) w_u(\lambda_j) = \frac{1}{2\pi} \sum_{l=-L}^{L} P_l \left\{ \sum_{t(l)}' x_t(\gamma) u_{t+l} \right.$$

$$\left. + \sum_{t(l)}'' x_t(\gamma) u_{t+l+n} + \sum_{t(l)}''' x_t(\gamma) u_{t+l-n} \right\}, \tag{2.124}$$

where

$$\sum_{t(l)}' = \sum_{1 \leq t, t+l \leq n}, \quad \sum_{t(l)}'' = \sum_{1 \leq t, t+l+n \leq n}, \quad \sum_{t(l)}''' = \sum_{1 \leq t, t+l-n \leq n}, \tag{2.125}$$

on applying (2.95). Looking first at the second and third terms in (2.124), we note that $1 \leq t, t+l+n \leq n$ and $1 \leq t, t+l-n \leq n$ are equivalent, respectively, to $1 \leq t \leq -l$, for $-L \leq l \leq -1$, and $1+n-l \leq t \leq n$, for $1 \leq l \leq L$. Then

$$E \left\| \sum_{t(l)}'' x_t(\gamma) u_{t+l+n} + \sum_{t(l)}''' x_t(\gamma) u_{t+l-n} \right\| \leq K\, |l| \sum_{s=0}^{n-1} |a_s(\beta)| \leq K\, |l|\, n^{\beta}, \tag{2.126}$$

from Lemma 2.D.1. Thus, because Assumption 2.2 (ii) and (v) implies

$$\sum_{l=-\infty}^{\infty} |l|\, \|P_l\| < \infty, \tag{2.127}$$

(Zygmund, 1977, p.240), the contribution from the final two terms of (2.124) is $O_p\left(n^{\beta}\right)$. Finally

$$\sum_{t(l)}' x_t(\gamma) u_{t+l} = O_p\left(n^{\max(\beta,1)}\right), \tag{2.128}$$

uniformly in l, from Lemmas 2.C.1 and 2.C.2, which, with (2.127) and Assumption 2.3 (ii), completes the proof of (2.120).

Proof of Proposition 2.8. Follows similarly to, but more easily than, the proof of Proposition 2.7.

Proof of Proposition 2.9. The left side of (2.89) is

$$\sum_{j=1}^{n} p(\lambda_j) \left\{ w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j) \right\} \left\{ w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j) \right\} \tag{2.129}$$

$$+\sum_{j=1}^{n} p(\lambda_j)\, w_{x(\gamma)}(-\lambda_j)\left\{w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right\} \tag{2.130}$$

$$+\sum_{j=1}^{n} p(\lambda_j)\left\{w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right\} w_u(\lambda_j). \tag{2.131}$$

Consider first (2.131)). Noting Assumption 2.2 (ii) and (iv) and proceeding as in the proof of Proposition 2.7, define

$$p_L(\lambda) = \sum_{l=-L}^{L} p_l e^{-il\lambda}, \quad p_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\lambda)\, e^{il\lambda} d\lambda, \tag{2.132}$$

where

$$\sup_{\lambda} \|p(\lambda) - p_L(\lambda)\| = O\left(\frac{\log L}{L}\right), \quad \sum_{l=-\infty}^{\infty} |l|\, \|p_l\| < \infty. \tag{2.133}$$

Thus

$$\sum_{j=1}^{n} \{p(\lambda_j) - p_L(\lambda_j)\}\left\{w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right\} w_u(\lambda_j) \tag{2.134}$$

is bounded in norm by

$$\frac{K \log L}{L}\left\{\sum_{t=1}^{n} \{x_t(\widehat{\gamma}) - x_t(\gamma)\}^2 \sum_{t=1}^{n} \|u_t\|^2\right\}^{\frac{1}{2}}, \tag{2.135}$$

using the Cauchy inequality and (2.95) again. Now choosing $L = [n^{1/2}]$ and taking $c = \delta - \gamma = \beta$, $\widehat{c} = \delta - \widehat{\gamma}$ in Lemma 2.C.5, (2.135) is $O_p\left((\log n)^2 n^{\beta-\kappa}\right) = o_p\left(n^{\beta}\right)$.

On the other hand, for $L < n$,

$$\sum_{j=1}^{n} p_L(\lambda_j)\left\{w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right\} w_u(\lambda_j) = \frac{1}{2\pi}\sum_{l=-L}^{L} p_l \left[\sum_{t(l)}{}' \{x_t(\widehat{\gamma}) - x_t(\gamma)\} u_{t+l}\right.$$

$$\left. + \sum_{t(l)}{}'' \{x_t(\widehat{\gamma}) - x_t(\gamma)\} u_{t+l+n} + \sum_{t(l)}{}''' \{x_t(\widehat{\gamma}) - x_t(\gamma)\} u_{t+l-n}\right]. \tag{2.136}$$

As in the proof of Lemma 2.C.5, we can write, for any $R \geq 2$,

$$\begin{aligned} x_t(\widehat{\gamma}) - x_t(\gamma) &= u_{2t}(\widehat{\gamma} - \delta) - u_{2t}(-\beta) \\ &= \sum_{r=1}^{R-1} \frac{(\gamma - \widehat{\gamma})^r}{r!} g^{(r)}(u_{2t}; \beta) + \frac{(\gamma - \widehat{\gamma})^R}{R!} g^{(R)}(u_{2t}; \delta - \overline{\gamma}), \end{aligned} \tag{2.137}$$

where, for a vector or scalar sequence φ_t, and real $b \geq 0$,

$$g^{(r)}(\varphi_t; b) = \sum_{s=1}^{t-1} a_s^{(r)}(b) \varphi_{t-s}, \tag{2.138}$$

with $a_s^{(r)}(b) = (d^r/db^r)a_s(b)$ and $|\overline{\gamma} - \gamma| \leq |\widehat{\gamma} - \gamma|$. Applying (2.159) of Lemma 2.C.4 with $r = R$, $c = \beta$, $\widehat{c} = \delta - \overline{\gamma}$, and Assumption 2.3 (i), indicates that the final term in (2.137) is uniformly $O_p\left(n^{-R\kappa}t^{\beta+\epsilon}\right)$, for any $\epsilon > 0$. Thus, the contribution of this term to (2.136) is, by the Cauchy inequality and (2.133), $O_p\left(n^{\beta+\epsilon+1-R\kappa}\right)$, which is $o_p\left(n^{\beta}\right)$ on choosing R large enough.

Next, as in (2.126), we have

$$E\left\|\sum_{t(l)}'' g^{(r)}\left(u_{2t};\beta\right)u_{t+l+n} + \sum_{t(l)}''' g^{(r)}\left(u_{2t};\beta\right)u_{t+l-n}\right\| \leq K|l|(\log n)^r n^{\beta}, \quad (2.139)$$

applying again Lemma 2.D.1, so on taking account of the $(\gamma - \widehat{\gamma})^r$ factors and invoking Assumption 2.3 (i) and (2.133), the contribution of the sums $\sum_{t(l)}''$ and $\sum_{t(l)}'''$ to (2.136) is $O_p\left((\log n)^r n^{\beta-\kappa}\right) + o_p\left(n^{\beta}\right) = o_p\left(n^{\beta}\right)$. It remains to consider the quantities

$$(\gamma - \widehat{\gamma})^r \sum_{l=-L}^{L} p_l \sum_{t(l)}' g^{(r)}\left(u_{2t};\beta\right)u_{t+l}, \quad 1 \leq r \leq R-1. \quad (2.140)$$

From (2.147) of Lemma 2.C.1 and (2.153) of Lemma 2.C.2 the sum over $\sum_{t(l)}'$ is $O_p\left((\log n)^r n^{\max(\beta,1)}\right)$, and thus, using (2.133) and Assumption 2.3 (i), (2.140) is $O_p\left(n^{\max(\beta,1)-\kappa}\log n\right)$ for $\kappa > \max(0, 1-\beta)$, that is, $o_p\left(n^{\beta}\right)$. This completes the proof that (2.131) is $o_p\left(n^{\beta}\right)$.

We next consider (2.130), and again wish to replace $p(\lambda)$ by $p_L(\lambda)$. First

$$\left\|\sum_{j=1}^{n}\left\{p\left(\lambda_j\right) - p_L\left(\lambda_j\right)\right\} w_{x(\gamma)}\left(-\lambda_j\right)\left\{w_{v(\widehat{\gamma},\widehat{\delta})}\left(\lambda_j\right) - w_u\left(\lambda_j\right)\right\}\right\| \quad (2.141)$$

is bounded by

$$\frac{K\log L}{L}\left\{\sum_{t=1}^{n} x_t(\gamma)^2 \sum_{t=1}^{n}\left\|v_t(\widehat{\gamma},\widehat{\delta}) - u_t\right\|^2\right\}^{\frac{1}{2}}. \quad (2.142)$$

Noting that $v_t(\widehat{\gamma},\widehat{\delta}) = \left(u_{1t}\left(\widehat{\gamma} - \gamma\right), u_{2t}(\widehat{\delta} - \delta)\right)'$ the second factor in braces is

$$\sum_{t=1}^{n}\left\|v_t(\widehat{\gamma},\widehat{\delta}) - u_t\right\|^2 = O_p\left(n^{1-2\kappa}\right), \quad (2.143)$$

from Lemma 2.C.5, so that, choosing $L = [n^{1/2}]$, and using (2.119), (2.141) is $O_p\left((\log n)\, n^{\beta-\kappa}\right) = o_p\left(n^{\beta}\right)$.

Next, proceeding as above, for $R \geq 2$,

$$\begin{aligned}
&\sum_{j=1}^{n} p_L\left(\lambda_j\right) w_{x(\gamma)}\left(-\lambda_j\right)\left\{w_{v(\widehat{\gamma},\widehat{\delta})}\left(\lambda_j\right) - w_u\left(\lambda_j\right)\right\} \\
&= \frac{1}{2\pi}\sum_{l=-L}^{L} p_l \sum_{r=1}^{R-1}\frac{1}{r!}\begin{pmatrix} (\widehat{\gamma} - \gamma)^r & 0 \\ 0 & (\widehat{\delta} - \delta)^r \end{pmatrix}\sum_{t(l)}' x_t(\gamma) g^{(r)}\left(u_{t+l};0\right) + o_p\left(n^{\beta}\right),
\end{aligned} \quad (2.144)$$

and the leading term is $o_p\left(n^{\beta}\right)$ from (2.148) of Lemma 2.C.1 and (2.154) of Lemma 2.C.2, (2.133) and Assumption 2.3 (i).

We are left with (2.129). It is clear from its structure, which involves both the differences appearing in (2.130) and (2.131), that application of similar arguments to those above will show it is $o_p\left(n^{\beta}\right)$, so we omit the details.

Proof of Proposition 2.10. The left side of (2.90) has norm bounded by

$$K \sup_{\lambda} \left\| f(\lambda;\widehat{\theta})^{-1} - f(\lambda;\theta)^{-1} \right\| \left[\left\{ \sum_{j=1}^{n} \left| w_{x(\widehat{\gamma})}(\lambda_j) \right|^2 \sum_{j=1}^{n} \left\| w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j) \right\|^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{j=1}^{n} \left| w_{x(\widehat{\gamma})}(\lambda_j) - w_{x(\gamma)}(\lambda_j) \right|^2 \sum_{j=1}^{n} \left\| w_u(\lambda_j) \right\|^2 \right\}^{\frac{1}{2}} \right], \tag{2.145}$$

and this is clearly $O_p\left(n^{\beta-\kappa+\epsilon}\right)$ for any $\epsilon > 0$, from earlier arguments.

Proof of Proposition 2.11. Omitted, being similar to but easier than the proof of Proposition 2.9.

Proof of Proposition 2.12. Omitted, in view of the remarks about the proofs of Propositions 2.10 and 2.11.

2.7.3 Appendix 2.C: Technical lemmas

Lemma 2.C.1. *Uniformly in $l \in [-L, L]$, $L < n$,*

$$E\left\{ \sum_{t(l)}' x_t(\gamma) u_{t+l} \right\} = O\left(n^{\max(\beta,1)}\right), \tag{2.146}$$

$$E\left\{ \sum_{t(l)}' g^{(r)}(u_{2t};\beta) u_{t+l} \right\} = O\left((\log n)^r n^{\max(\beta,1)}\right), \tag{2.147}$$

$$E\left\{ \sum_{t(l)}' x_t(\gamma) g^{(r)}(u_{t+l};0) \right\} = O\left((\log n)^r n^{\max(\beta,1)}\right). \tag{2.148}$$

Proof. The proofs are very similar, and in fact are possible under milder conditions following techniques of Robinson and Marinucci (2001), and we just discuss the proof of (2.148), which is slightly the most complicated. Writing $\Gamma_s = E\left(u_{2t} u_{t+s}\right)$, the left side is

$$\sum_{t(l)}' \sum_{s=1}^{t-1} a_s(\beta) \sum_{q=r}^{t+l-1} a_q^{(r)}(0) \Gamma_{s+l-q}, \tag{2.149}$$

which has norm bounded by

$$\sum_{t=1}^{n} \sum_{q=r}^{n} \left| a_q^{(r)}(0) \right| \sum_{s=1}^{n} \left\| \Gamma_s \right\| = O\left((\log n)^r n\right) \tag{2.150}$$

for $\beta < 1$, uniformly in l, and by

$$n^{\beta-1}\sum_{t=1}^{n}\sum_{q=r}^{n}\left|a_q^{(r)}(0)\right|\sum_{s=1}^{n}\|\Gamma_s\| = O\left((\log n)^r n^{\beta}\right) \tag{2.151}$$

for $\beta \geq 1$, by Lemma 2.D.4 and Assumption 2.2, to complete the proof.

Lemma 2.C.2. *Uniformly in $l \in [-L, L]$, $L < n$,*

$$Var\left\{\sum_{t(l)}{}' x_t(\gamma)u_{t+l}\right\} = O\left(n^{2\beta}\right), \tag{2.152}$$

$$Var\left\{\sum_{t(l)}{}' g^{(r)}(u_{2t};\beta)u_{t+l}\right\} = O\left((\log n)^{2r}n^{2\beta}\right), \tag{2.153}$$

$$Var\left\{\sum_{t(l)}{}' x_t(\gamma)g^{(r)}(u_{t+l};0)\right\} = O\left(n^{2\beta+\eta}\right), \tag{2.154}$$

for any $\eta > 0$.

Proof. The results follow from minor modifications of the proof of Theorem 5.1 of Robinson and Marinucci (2001). There are only two differences. The first is that the sums in the latter reference are over $t \in [1, n]$, whereas the lemma requires uniformity in l for sums over $t(l)$. But because the $t(l)$ are just a subset of $[1, n]$, this follows easily. The second difference is that in (2.153) and (2.154) (though not in (2.152)), the weights $a_s^{(r)}(\beta)$ and $a_s^{(r)}(0)$ that are involved are not covered by the weights of Robinson and Marinucci (2001), due to the presence of log factors. But allowance for such log factors is readily made, and they contribute the $(\log n)^{2r}$ and n^{η} factors in (2.153) and (2.154). We observe that the regularity conditions of Robinson and Marinucci (2001) are noticeably weaker than those on u_t in the present chapter.

Lemma 2.C.3. *For $i = 1, 2$, and uniformly in $r \geq 1$ and $t \geq 2$,*

$$E\left\{g^{(r)}(u_{it};0)^2\right\} = O(1), \tag{2.155}$$

and for $c > 1/2$

$$E\left\{g^{(r)}(u_{it};c)^2\right\} = O((\log t)^{2r}t^{2c-1}). \tag{2.156}$$

Proof. For any $c \geq 0$,

$$\begin{aligned} E\left\{g^{(r)}(u_{it};c)^2\right\} &= \sum_{s=1}^{t-1}\sum_{v=1}^{t-1} a_s^{(r)}(c)a_v^{(r)}(c)\int_{-\pi}^{\pi} f_{ii}(\lambda)e^{i(s-v)\lambda}d\lambda \\ &= \int_{-\pi}^{\pi} f_{ii}(\lambda)\left|\sum_{s=1}^{t-1} a_s^{(r)}(c)e^{is\lambda}\right|^2 d\lambda \leq K\int_{-\pi}^{\pi}\left|\sum_{s=1}^{t-1} a_s^{(r)}(c)e^{is\lambda}\right|^2 d\lambda \\ &\leq K\sum_{s=1}^{t-1} a_s^{(r)}(c)^2. \end{aligned} \tag{2.157}$$

From Lemmas 2.D.1 and 2.D.4, this is bounded by the right sides of (2.155) and (2.156), for $c = 0$ and $c > 1/2$ respectively.

Lemma 2.C.4. *For $i = 1, 2$, $\kappa > 0$, uniformly in $t \in [1, n]$, $r \geq 1$,*

$$g^{(r)}(u_{it}; \widehat{c}) = O_p(t^{\frac{1}{2}}), \tag{2.158}$$

if $\widehat{c} = O_p(n^{-\kappa})$, and

$$g^{(r)}(u_{it}; \widehat{c}) = O_p(t^{c+\epsilon}), \tag{2.159}$$

for any $\epsilon > 0$, if $\widehat{c} = c + O_p(n^{-\kappa})$, $c > 1/2$.

Proof. By the Cauchy inequality, for any $c \geq 0$,

$$\left|g^{(r)}(u_{it}; \widehat{c})\right| \leq \left\{ \sum_{s=1}^{t-1} a_s^{(r)}(\widehat{c})^2 \sum_{s=1}^{t-1} u_{is}^2 \right\}^{\frac{1}{2}}. \tag{2.160}$$

From Lemma 2.D.5, for $\epsilon > 0$,

$$\sum_{s=0}^{t-1} a_s^{(r)}(\widehat{c})^2 = O_p\left(\sum_{s=0}^{t-1} \{\log(s+1)\}^{2r} (s+1)^{2(c+\epsilon-1)} \right), \tag{2.161}$$

where $c = 0$ or $c > 1/2$. Thus, with $\sum_{s=1}^{t-1} u_{is}^2 = O_p(t)$, the bounds (2.158) and (2.159) follow.

Lemma 2.C.5. *For $i = 1, 2$, if $\widehat{c} = c + O_p(n^{-\kappa})$, $\kappa > 0$, uniformly in $t \in [1, n]$, as $n \to \infty$*

$$u_{it}(-\widehat{c}) - u_{it} = O_p\left(n^{-\kappa}\right), \quad c = 0, \tag{2.162}$$

$$u_{it}(-\widehat{c}) - u_{it}(-c) = O_p\left(n^{-\kappa} t^{c-\frac{1}{2}} \log t\right), \quad c > \frac{1}{2}. \tag{2.163}$$

Proof. We have, for $c \geq 0$,

$$u_{it}(-\widehat{c}) - u_{it}(-c) = \sum_{s=1}^{t-1} \{a_s(\widehat{c}) - a_s(c)\} u_{i,t-s}, \tag{2.164}$$

with $u_{it}(0) = u_{it}$. By Taylor's theorem, for any $R \geq 2$,

$$a_s(\widehat{c}) - a_s(c) = \sum_{r=1}^{R-1} a_s^{(r)}(c) \frac{(\widehat{c} - c)^r}{r!} + a_s^{(R)}(\overline{c}) \frac{(\widehat{c} - c)^R}{R!}, \tag{2.165}$$

where $|\overline{c} - c| \leq |\widehat{c} - c|$, so we can write (2.164) as

$$\sum_{r=1}^{R-1} \frac{(\widehat{c} - c)^r}{r!} g^{(r)}(u_{it}; c) + \frac{(\widehat{c} - c)^R}{R!} g^{(R)}(u_{it}; \overline{c}). \tag{2.166}$$

Taking $c = 0$, (2.155) and (2.158) indicate that (2.166) is $O_p\left(n^{-\kappa}\right) + O_p\left(n^{-R\kappa} t^{1/2}\right)$, whence (2.162) is proved by choosing R large enough and observing that $t \leq n$. In the same way, (2.163) is proved because (2.166) is $O_p\left(n^{-\kappa} t^{c-1/2} \log t\right) + O_p\left(n^{-R\kappa} t^{c+\eta}\right)$ for $\eta > 0$, due to (2.156) and (2.159).

2.7.4 Appendix 2.D: Lemmas concerning the a_s weights

Lemma 2.D.1. *For $c \in [c_0, C_0]$, $c_0 > 0$, $C_0 < \infty$, $s \geq 0$,*

$$\begin{aligned} |a_s(c)| &\leq K_0 (1+s)^{c-1}, & (2.167)\\ |a_s(c) - a_{s+1}(c)| &\leq K_0 (1+s)^{c-2}, & (2.168)\\ \left|a_s^{(r)}(c)\right| &\leq K_{0R} (\log(1+s))^r (1+s)^{c-1}, \ 1 \leq r \leq R, & (2.169)\end{aligned}$$

where $K_0 < \infty$ depends only on c_0 and C_0 and $K_{0R} < \infty$ depends only on c_0, C_0 and R.

Proof. First, (2.167) is familiar from Stirling's approximation, or derivable by induction, while (2.168) follows easily from the identity

$$a_{s+1}(c) = \{(s+c)/(s+1)\}\, a_s(c). \tag{2.170}$$

To prove (2.169), introduce the digamma function and its derivatives

$$\psi(x) = \frac{d}{dx} \log \Gamma(x), \quad \psi^{(r)}(x) = \frac{d^r \psi(x)}{dx^r}, \tag{2.171}$$

which exist for $r \geq 1$ and $x > 0$. We deduce from the chain rule that

$$a_s^{(r)}(c) = \sum_{i=0}^{r-1} \tau_i \left\{\psi^{(i)}(s+c) - \psi^{(i)}(c)\right\} a_s^{(r-1-i)}(c), \tag{2.172}$$

with the convention that $\psi^{(0)}(\cdot) = \psi(\cdot)$, $a^{(0)}(\cdot) = a(\cdot)$, and for finite constants τ_i, $0 \leq i \leq r-1$. Now from Gradshteyn and Ryzhik (1994, p.95), for $x > 0$

$$\psi(x) = \sum_{i=0}^{\infty} \frac{x-1}{(i+1)(x+i)} - \eta, \tag{2.173}$$

where η is Euler's constant. Thus for $x > 0$

$$\begin{aligned} |\psi(x)| &\leq \sum_{i=0}^{[x]} (i+1)^{-1} + |x-1| \sum_{i=[x]+1}^{\infty} i^{-2} + \eta \\ &\leq \log(x+1) + 1 + \eta \leq K \log(x+1), & (2.174)\end{aligned}$$

where K is independent of x. Also, for $l \geq 1$,

$$\psi^{(l)}(x) = (-1)^{l+1}\, l! \sum_{i=0}^{\infty} (x+i)^{-l-1}, \tag{2.175}$$

so that

$$\left|\psi^{(l)}(x)\right| \leq l!(x^{-l-1} + \frac{x^{-l}}{l}) \leq K_{0R} (1+x)^{-l}, \tag{2.176}$$

$1 \leq l \leq r \leq R$, for $x \geq c_0$. The proof is completed by applying (2.172) recursively, (2.176), and noting that $|\log(s+c+1)| \leq K_0 \log(s+1)$.

Lemma 2.D.2. *Uniformly in* $s, t \in [1, n]$, *for* $c > 1/2$

$$\sum_{v=1}^{\min(s,t)} a_{s-v}(c)\, a_{t-v}(c) = O\left(n^{2c-1}\right). \tag{2.177}$$

Proof. From (2.167), the left side of (2.177) is bounded in absolute value by

$$K \sum_{v=1}^{n} v^{c-1} \left(v + |s-t|\right)^{c-1}. \tag{2.178}$$

Since

$$\begin{aligned} (v+|s-t|)^{c-1} &\leq v^{c-1} \text{ for } c \leq 1, && (2.179)\\ &\leq Kn^{c-1} \text{ for } c > 1, && (2.180) \end{aligned}$$

(2.177) readily follows.

Lemma 2.D.3. *For* $1 \leq s \leq t-1$, $c > 1/2$,

$$\sum_{v=1}^{s} \{a_{t-v}(c) - a_{s-v}(c)\}^2 + \sum_{v=s+1}^{t} a_{t-v}^2(c) \leq K(t-s)t^{\max(0,2c-2)}. \tag{2.181}$$

Proof. Writing $a_s = a_s(c)$, for $1 \leq v \leq s, a_{t-v} - a_{s-v} = 0,\ c = 1$, while for $c \neq 1$ we have from (2.168)

$$|a_{t-v} - a_{s-v}| \leq \sum_{r=s+1}^{t} |a_{r-v} - a_{r-1-v}| \leq K \sum_{r=s+1}^{t} (r-v)^{c-2}. \tag{2.182}$$

Now (2.182) is bounded on the one hand by $K(s+1-v)^{c-1} 1(c<1) + Kt^{c-1}1(c>1)$, and on the other by $K(t-s)\left\{(s+1-v)^{c-2} 1(c<2) + t^{c-2} 1(c \geq 2)\right\}$. It follows that (2.182) is also bounded by

$$\begin{aligned} K(t-s)^{\frac{1}{2}}(s+1-v)^{c-\frac{3}{2}}, \quad \frac{1}{2} &< c < 1, && (2.183)\\ K(t-s)^{\frac{1}{2}} t^{\frac{c-1}{2}} (s+1-v)^{\frac{c}{2}-1}, \quad 1 &< c < 2, && (2.184)\\ K(t-s)^{\frac{1}{2}} t^{c-\frac{3}{2}}, \quad c &\geq 2. && (2.185) \end{aligned}$$

Thus $\sum_{v=1}^{s} \{a_{t-v}(c) - a_{s-v}(c)\}^2$ is bounded by

$$\begin{aligned} K(t-s)\sum_{v=1}^{s}(s+1-v)^{2c-3} &\leq K(t-s), \quad \frac{1}{2} < c < 1, && (2.186)\\ K(t-s)t^{c-1}\sum_{v=1}^{s}(s+1-v)^{c-2} &\leq K(t-s)t^{2(c-1)}, \quad 1 < c < 2, && (2.187)\\ K(t-s)t^{2c-3}s &\leq K(t-s)t^{2(c-1)}, \quad c \geq 2, && (2.188) \end{aligned}$$

that is by $K(t-s)\,t^{\max(0,2c-2)}$, $c>1/2$. On the other hand, for all $c>1/2$

$$\sum_{v=s+1}^{t} a_{t-v}^{2} \leq K(t-s)^{2c-1}, \tag{2.189}$$

whence the result immediately follows.

Lemma 2.D.4. *For $r \geq 1$,*

$$a_s^{(r)}(0) = 0, \quad s < r \tag{2.190}$$

and

$$\left|a_s^{(r)}(0)\right| \leq \frac{K_r (\log(s+1))^{r-1}}{(s-r+1)}, \quad s \geq r, \tag{2.191}$$

where $K_r < \infty$ depends only on r.

Proof. On taking logs in (1.8) and differentiating with respect to α we have

$$-\log(1-z)(1-z)^{-\alpha} = \sum_{s=0}^{\infty} a_s^{(1)}(\alpha)\, z^s. \tag{2.192}$$

Evaluating this expression at $\alpha = 0$ gives $a_0^{(1)}(0) = 0$ and $a_s^{(1)}(0) = s^{-1}$, $s \geq 1$. This proves the lemma for $r = 1$. For $r > 1$ we differentiate (2.192) $r-1$ times and evaluate at $\alpha = 0$ to get

$$\{-\log(1-z)\}^r = \sum_{s=0}^{\infty} a_s^{(r)}(0)\, z^s. \tag{2.193}$$

Clearly $a_s^{(r)}(0) = 0$, $s < r$. Also, we have the recursion

$$\sum_{s=0}^{\infty} a_s^{(r)}(0)\, z^s = -\log(1-z) \sum_{s=0}^{\infty} a_s^{(r-1)}(0)\, z^s, \ r \geq 2. \tag{2.194}$$

It follows that

$$a_s^{(r)}(0) = \frac{a_{r-1}^{(r-1)}(0)}{s-r+1} + \frac{a_r^{(r-1)}(0)}{s-r} + \dots + a_{s-1}^{(r-1)}(0), \quad s \geq r > 1. \tag{2.195}$$

If (2.191) is true with r replaced by $r-1$ we have

$$\begin{aligned} \left|a_s^{(r)}(0)\right| &\leq K_{r-1}(\log(s+1))^{r-2}\left\{\frac{1}{1.(s-r+1)} + \frac{1}{2(s-r)} + \dots + \frac{1}{1.(s-r+1)}\right\} \\ &\leq 2K_{r-1}\{\log(s+1)\}^{r-2}\frac{\log(s+1)}{s-r+1} \leq K_r \frac{(\log(s+1))^{r-1}}{s-r+1}, \end{aligned} \tag{2.196}$$

for $K_r \geq 2K_{r-1}$. The proof thus follows by induction.

Lemma 2.D.5. *Let $\widehat{c} = c + O_p(n^{-\kappa})$, $\kappa > 0$ such that $0 \leq c < K$ and $|\widehat{c}| \leq K$ for some $K < \infty$, and suppose $\bar{c}$ satisfies $|\bar{c} - c| \leq |\widehat{c} - c|$. Then uniformly in $s \in [0, n)$ as $n \to \infty$, and for any $\epsilon > 0$,*

$$a_s^{(r)}(\bar{c}) = O_p\left((\log(s+1))^r (s+1)^{c+\epsilon-1}\right), \tag{2.197}$$

as $n \longrightarrow \infty$.

Proof. From Lemma 2.D.1 and Lemma 2.D.4 we have, for any $\epsilon > 0$

$$\begin{aligned}
\left|a_s^{(r)}(\bar{c})\right| &\leq \left|a_s^{(r)}(\bar{c})\right| 1\left(|\widehat{c}-c| \leq \epsilon\right) + \left|a_s^{(r)}(\bar{c})\right| 1\left(|\widehat{c}-c| > \epsilon\right) \\
&\leq K\left(\log\left(s+1\right)\right)^r \left(\left(s+1\right)^{c+\epsilon-1} + \left(s+1\right)^{K-1} \frac{|\widehat{c}-c|^M}{\epsilon^M}\right) \\
&\leq K\left(\log\left(s+1\right)\right)^r \left(\left(s+1\right)^{c+\epsilon-1} + \left(s+1\right)^{K-1} n^{-M\kappa}\right), \qquad (2.198)
\end{aligned}$$

for any $M \geq 1$. We may choose $M \geq (K - c - \epsilon)/\kappa$ which, with $s \leq n$, completes the proof.

TABLE 2.3

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.006	-.003	-.007	-.002	-.002	-.003	-.001	-.001	.000
	0	1.2	-.002	-.001	-.002	-.001	-.001	-.001	.000	.000	.000
.5	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.003	-.009	-.001	-.002	-.005	-.001	-.001	-.002
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.002	-.005	-.001	-.002	-.002	.000	-.001	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
1	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.002	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.001	-.004	-.001	-.001	-.001	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
2	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.002	-.001	-.005	-.001	-.001	-.002	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 2.4

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = 0.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.002	-.056	.269	.004	-.019	.223	.002	-.013	.185
	0	1.2	.001	-.002	.010	.000	-.001	.003	.000	.000	.001
.5	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.002	-.013	.052	.002	-.003	.030	.001	-.002	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.001	-.040	.194	.003	-.014	.160	.001	-.010	.133
	0	1.2	.001	-.002	.007	.000	-.001	.002	.000	.000	.001
1	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.001	-.009	.038	.001	-.002	.022	.001	-.001	.012
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.001	-.029	.137	.002	-.009	.113	.001	-.007	.094
	0	1.2	.000	-.001	.005	.000	.000	.002	.000	.000	.000
2	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.001	-.007	.027	.001	-.002	.015	.000	-.001	.008
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 2.5

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = -0.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.000	.052	-.265	-.002	.020	-.218	-.002	.013	-.184
	0	1.2	.000	.002	-.010	.000	.001	-.003	.000	.000	-.001
.5	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.000	.010	-.053	-.001	.004	-.029	-.001	.001	-.016
	.4	2	.000	.001	-.001	.000	.000	.000	.000	.000	.000
	0	.6	.000	.038	-.191	-.001	.014	-.157	-.001	.009	-.132
	0	1.2	.000	.001	-.007	.000	.000	-.002	.000	.000	-.001
1	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.000	.007	-.038	-.001	.003	-.021	.000	.001	-.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.000	.026	-.135	-.001	.010	-.111	-.001	.007	-.094
	0	1.2	.000	.001	-.005	.000	.000	-.002	.000	.000	.000
2	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.000	.005	-.027	-.001	.002	-.015	.000	.001	-.008
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 2.6

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = 0.75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.002	-.166	.406	.003	-.062	.331	.001	-.044	.275
	0	1.2	.001	-.003	.015	.000	-.001	.005	.000	.000	.001
.5	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.002	-.014	.078	.001	-.003	.044	.000	-.001	.024
	.4	2	.000	-.001	.001	.000	.000	.000	.000	.000	.000
	0	.6	.002	-.119	.293	.002	-.045	.238	.001	-.033	.198
	0	1.2	.001	-.002	.011	.000	.000	.004	.000	.000	.001
1	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.001	-.010	.056	.001	-.002	.031	.000	-.001	.017
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.001	-.085	.207	.001	-.031	.169	.001	-.023	.140
	0	1.2	.000	-.002	.008	.000	.000	.003	.000	.000	.001
2	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.001	-.007	.040	.001	-.001	.022	.000	-.001	.012
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 2.7

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.005	-.006	-.006	-.001	-.003	-.002	.000	-.001	.000
	0	1.2	-.001	-.001	-.001	.000	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.003	-.007	-.001	-.002	-.003	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.052	.064	.121	.036	.044	.097	.023	.029	.078
	0	1.2	.000	-.002	.003	.000	-.001	.001	.000	.000	.000
.5	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.004	.033	.003	.001	.020	.001	.000	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.049	-.061	-.119	-.034	-.042	-.095	-.024	-.029	-.078
	0	1.2	.000	.002	-.003	.000	.001	-.001	.000	.001	.000
-.5	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.004	-.034	-.003	-.001	-.019	-.001	.001	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.078	.097	.183	.051	.063	.144	.034	.042	.116
	0	1.2	.000	-.002	.005	.000	-.001	.001	.000	.000	.000
.75	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.007	.049	.004	.002	.029	.001	.000	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 2.8

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.005	-.018	-.002	-.003	.000	-.001
	0	1.2	-.001	.000	.000	-.001	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.072	-.001	-.001	.000	-.001
	.4	2	.000	-.004	.000	.000	.000	.000
	0	.6	.031	.040	.021	.039	.014	.024
	0	1.2	.000	.000	.000	-.001	.000	.000
.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.003	-.114	.002	.004	.001	.000
	.4	2	.000	.007	.000	.000	.000	.000
	0	.6	-.031	-.099	-.020	-.037	-.014	-.024
	0	1.2	-.001	-.001	.000	.001	.000	.001
-.5	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.004	-.013	-.002	-.002	-.001	.000
	.4	2	.000	-.002	.000	.000	.000	.000
	0	.6	.047	.082	.030	-.989	.020	.035
	0	1.2	.001	-.003	.000	-.001	.000	.000
.75	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.006	-.046	.003	.019	.001	.001
	.4	2	.000	.018	.000	.000	.000	.000

TABLE 2.9

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.006	-.008	-.009	-.001	-.003	-.004	.001	.000	-.001
	0	1.2	-.001	.000	-.001	.000	.000	-.001	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.006	-.008	-.001	-.002	-.004	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.029	.092	.073	.019	.079	.056	.012	.068	.042
	0	1.2	.001	-.001	.003	.000	-.001	.001	.000	-.001	.000
.5	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.005	.014	.031	.001	.010	.018	.001	.007	.010
	.4	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	0	.6	-.027	-.097	-.075	-.018	-.078	-.054	-.011	-.066	-.041
	0	1.2	-.001	.001	-.004	.000	.001	-.001	.000	.002	.000
-.5	0	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.019	-.033	-.002	-.008	-.019	.000	-.006	-.010
	.4	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	0	.6	.042	.144	.111	.028	.117	.081	.018	.097	.061
	0	1.2	.000	-.002	.006	.000	-.001	.002	.000	-.001	.000
.75	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.020	.046	.003	.013	.026	.001	.010	.015
	.4	2	.000	-.001	.001	.000	-.001	.000	.000	.000	.000

TABLE 2.10

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.010	-3.97	-.003	-.001	-.001	-.001
	0	1.2	-.001	.051	-.001	.000	.000	.000
0	0	2	.000	-.006	.000	.000	.000	.000
	.4	1.2	.011	.438	-.002	.360	-.001	-.001
	.4	2	.000	.210	.000	-.015	.000	.000
	0	.6	.011	-.136	.008	.051	.005	.059
	0	1.2	.001	-.510	.000	.001	.000	-.002
.5	0	2	.000	.008	.000	.001	.000	.000
	.4	1.2	-.007	-.310	.000	.427	.000	.012
	.4	2	.002	8.50	.000	-.002	.000	.000
	0	.6	-.016	1.37	-.008	.007	-.005	-.056
	0	1.2	-.001	-1.28	.000	-.011	.000	.002
-.5	0	2	.000	-.016	.000	.000	.000	.000
	.4	1.2	-.004	-6.86	-.001	-.030	.000	-.012
	.4	2	.000	-.240	.000	-.018	.000	.000
	0	.6	.025	1.06	.017	.134	.010	.082
	0	1.2	.002	.389	.000	.004	.000	-.001
.75	0	2	.000	.004	.000	.000	.000	.000
	.4	1.2	.007	10.5	.002	-.286	.001	.018
	.4	2	.000	1.25	.000	.027	.000	.000

TABLE 2.11

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.005	-.005	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.004	-.007	-.001	-.002	-.003	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.065	.076	.147	.044	.052	.121	.030	.034	.099
	0	1.2	.001	-.002	.004	.000	-.001	.001	.000	.000	.000
.5	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.003	.035	.004	.001	.021	.001	.000	.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.062	-.073	-.145	-.042	-.049	-.118	-.030	-.034	-.099
	0	1.2	-.001	.001	-.004	.000	.000	-.001	.000	.000	.000
-.5	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.005	-.036	-.003	-.001	-.020	-.001	.000	-.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.097	.114	.223	.064	.075	.179	.043	.050	.147
	0	1.2	.001	-.002	.007	.000	.000	.002	.000	.000	.000
.75	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.012	.006	.052	.005	.002	.030	.002	.001	.017
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 2.12

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.004	-.006	-.001	-.002	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.002	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	.049	.059	.034	.040	.022	.025
	0	1.2	.001	-.002	.000	-.001	.000	.000
.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.001	.003	.000	.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.047	-.057	-.032	-.038	-.022	-.025
	0	1.2	-.001	.002	.000	.000	.000	.000
-.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.002	-.003	.000	-.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	.073	.087	.048	.056	.032	.035
	0	1.2	.001	-.002	.000	.000	.000	.000
.75	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.003	.004	.001	.001	.000
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 2.13

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.004	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.002	-.003	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.068	.085	.140	.045	.058	.115	.029	.038	.094
	0	1.2	.002	-.011	.004	.000	-.001	.001	.000	.000	.000
.5	0	2	.000	.001	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.012	.000	.035	.005	.004	.021	.002	.000	.011
	.4	2	.000	.007	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.065	-.081	-.139	-.043	-.056	-.113	-.029	-.038	-.094
	0	1.2	-.001	.001	-.004	.000	.000	-.001	.000	.000	.000
-.5	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.010	.001	-.036	-.004	-.006	-.020	-.001	-.001	-.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.102	.129	.212	.066	.085	.170	.042	.056	.140
	0	1.2	.002	-.001	.006	.000	-.001	.002	.000	.000	.000
.75	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.017	.012	.051	.006	.004	.030	.002	.001	.016
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 2.14

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.004	-.006	-.001	-.002	.000	-.001
	0	1.2	-.001	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.001	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	.046	.059	.032	.041	.021	.026
	0	1.2	.001	-.001	.000	-.001	.000	.000
.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.003	.003	.001	.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.044	-.058	-.029	-.038	-.021	-.027
	0	1.2	-.001	.001	.000	.000	.000	.000
-.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.004	-.003	-.001	-.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	.069	.089	.045	.058	.030	.037
	0	1.2	.001	-.002	.000	.000	.000	.000
.75	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.005	.004	.002	.001	.001
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 2.15

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	-.005	-.006	-.006	-.001	-.004	-.002	.000	-.001	.000
	0	1.2	-.001	-.001	-.001	.000	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.003	-.007	-.001	-.002	-.003	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	.056	.057	.123	.038	.039	.100	.025	.027	.081
	0	1.2	.001	-.003	.003	.000	-.002	.001	.000	-.001	.000
.5	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.001	.002	.033	.003	-.001	.020	.001	-.003	.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.054	-.056	-.122	-.036	-.037	-.098	-.025	-.027	-.081
	0	1.2	-.001	.003	-.003	.000	.002	-.001	.000	.001	.000
-.5	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.002	-.034	-.003	.002	-.020	-.001	.002	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.083	.083	.187	.054	.055	.148	.036	.039	.119
	0	1.2	.001	-.004	.005	.000	-.002	.001	.000	-.001	.000
.75	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.011	.004	.049	.005	.000	.029	.002	-.002	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 2.16

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.005	-.007	-.002	-.004	.000	-.001
	0	1.2	-.001	-.001	.000	.000	.000	.000
0	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.004	-.061	-.001	-.011	.000	-.001
	.4	2	.000	.005	.000	.014	.000	.000
	0	.6	.034	.058	.023	.039	.015	.024
	0	1.2	.000	-.003	.000	-.002	.000	-.001
.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.004	-.400	.002	-2.60	.001	-.001
	.4	2	.000	.021	.000	-.001	.000	.000
	0	.6	-.034	-.052	-.022	-.035	-.015	-.024
	0	1.2	-.001	.003	.000	.002	.000	.001
-.5	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.250	-.002	-.052	-.001	.001
	.4	2	.000	.000	.000	-.002	.000	.000
	0	.6	.051	.077	.032	.055	.021	.035
	0	1.2	.001	-.004	.000	-.003	.000	-.001
.75	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.007	1.64	.003	-.027	.001	-.002
	.4	2	.000	.024	.000	.012	.000	.000

TABLE 2.17

MONTE CARLO S.D. OF $\overline{v}_I, \overline{v}_F, \overline{v}_O$ FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.6	.164	.221	.120	.092	.098	.073	.057	.058	.046
	0	1.2	.035	.037	.035	.015	.015	.014	.006	.006	.006
.5	0	2	.004	.004	.004	.001	.001	.001	.000	.000	.000
	.4	1.2	.096	.103	.110	.050	.052	.064	.027	.028	.035
	.4	2	.012	.013	.015	.004	.004	.005	.001	.001	.002
	0	.6	.117	.158	.086	.066	.070	.052	.041	.041	.033
	0	1.2	.025	.026	.025	.010	.011	.010	.004	.004	.004
1	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.069	.074	.078	.035	.037	.046	.020	.020	.025
	.4	2	.009	.010	.010	.003	.003	.003	.001	.001	.001
	0	.6	.083	.113	.061	.047	.050	.037	.029	.030	.023
	0	1.2	.018	.019	.018	.007	.008	.007	.003	.003	.003
2	0	2	.002	.002	.002	.001	.001	.001	.000	.000	.000
	.4	1.2	.049	.053	.060	.025	.027	.033	.014	.014	.018
	.4	2	.006	.006	.007	.002	.002	.002	.001	.001	.001

TABLE 2.18

MONTE CARLO S.D. OF $\overline{v}_I, \overline{v}_F, \overline{v}_O$ FOR $\rho = 0.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.6	.142	.345	.140	.081	.119	.105	.048	.064	.079
	0	1.2	.030	.033	.031	.012	.013	.013	.005	.005	.005
.5	0	2	.004	.004	.004	.001	.001	.001	.000	.000	.000
	.4	1.2	.083	.110	.094	.043	.050	.056	.023	.026	.031
	.4	2	.011	.011	.013	.003	.003	.004	.001	.001	.001
	0	.6	.101	.247	.100	.058	.085	.075	.034	.046	.057
	0	1.2	.021	.024	.022	.009	.009	.009	.004	.004	.004
1	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.059	.079	.067	.031	.036	.040	.016	.018	.022
	.4	2	.008	.008	.009	.002	.002	.003	.001	.001	.001
	0	.6	.072	.176	.071	.041	.061	.053	.025	.033	.040
	0	1.2	.015	.017	.016	.006	.007	.006	.003	.003	.003
2	0	2	.002	.002	.002	.000	.000	.001	.000	.000	.000
	.4	1.2	.042	.056	.048	.022	.026	.029	.012	.013	.016
	.4	2	.005	.006	.007	.002	.002	.002	.001	.001	.001

TABLE 2.19

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = -0.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.136	.276	.135	.080	.113	.103	.051	.069	.081
	0	1.2	.028	.030	.031	.012	.013	.013	.005	.006	.005
.5	0	2	.003	.003	.004	.001	.001	.001	.000	.000	.000
	.4	1.2	.078	.100	.092	.042	.050	.056	.024	.027	.032
	.4	2	.010	.010	.013	.004	.004	.005	.001	.001	.001
	0	.6	.096	.205	.097	.057	.081	.074	.036	.049	.058
	0	1.2	.020	.021	.022	.009	.010	.010	.004	.004	.004
1	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.056	.071	.066	.030	.035	.040	.017	.020	.022
	.4	2	.007	.007	.009	.003	.003	.003	.001	.001	.001
	0	.6	.069	.141	.069	.041	.058	.053	.026	.035	.041
	0	1.2	.014	.015	.016	.006	.007	.007	.003	.003	.003
2	0	2	.002	.002	.002	.000	.001	.001	.000	.000	.000
	.4	1.2	.040	.051	.047	.022	.025	.029	.012	.014	.016
	.4	2	.005	.005	.006	.002	.002	.002	.001	.001	.001

TABLE 2.20

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\rho = 0.75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
τ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.108	1.80	.163	.062	.152	.128	.039	.084	.103
	0	1.2	.022	.030	.028	.010	.011	.011	.004	.004	.005
.5	0	2	.003	.003	.004	.001	.001	.001	.000	.000	.000
	.4	1.2	.063	.106	.081	.033	.044	.046	.019	.023	.026
	.4	2	.008	.009	.011	.003	.003	.004	.001	.001	.001
	0	.6	.076	1.27	.117	.043	.110	.092	.028	.060	.074
	0	1.2	.016	.021	.020	.007	.008	.008	.003	.003	.003
1	0	2	.002	.002	.003	.000	.001	.001	.000	.000	.000
	.4	1.2	.044	.076	.057	.023	.031	.033	.013	.016	.019
	.4	2	.006	.006	.008	.002	.002	.003	.001	.001	.001
	0	.6	.055	.916	.083	.031	.078	.065	.020	.043	.053
	0	1.2	.011	.015	.014	.005	.006	.006	.002	.002	.002
2	0	2	.001	.002	.002	.000	.000	.001	.000	.000	.000
	.4	1.2	.032	.054	.041	.017	.023	.024	.010	.011	.013
	.4	2	.004	.005	.005	.001	.001	.002	.000	.000	.001

TABLE 2.21

MONTE CARLO S.D. OF $\overline{v}_I, \overline{v}_F, \overline{v}_O$ FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.6	.106	.109	.096	.061	.064	.056	.038	.040	.035
	0	1.2	.025	.026	.026	.010	.011	.010	.004	.005	.004
0	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.070	.073	.083	.036	.040	.047	.020	.021	.025
	.4	2	.009	.009	.011	.003	.003	.003	.001	.001	.001
	0	.6	.096	.101	.092	.058	.062	.062	.036	.038	.042
	0	1.2	.022	.026	.022	.009	.011	.009	.004	.004	.004
.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.062	.071	.069	.032	.037	.040	.017	.020	.022
	.4	2	.008	.009	.010	.002	.003	.003	.001	.001	.001
	0	.6	.090	.096	.088	.057	.061	.061	.037	.039	.043
	0	1.2	.020	.024	.021	.009	.011	.009	.004	.005	.004
-.5	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.068	.067	.032	.038	.041	.018	.021	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
	0	.6	.082	.091	.089	.050	.056	.063	.033	.038	.048
	0	1.2	.017	.022	.018	.007	.010	.008	.003	.004	.003
.75	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.049	.061	.057	.025	.033	.032	.014	.019	.018
	.4	2	.006	.007	.008	.002	.003	.003	.001	.001	.001

TABLE 2.22

MONTE CARLO S.D. OF $\overline{v}_I^o, \overline{v}_F^o$ FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1,2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{v}_I^o$	$\overline{v}_F^o$	$\overline{v}_I^o$	$\overline{v}_F^o$	$\overline{v}_I^o$	$\overline{v}_F^o$
	0	.6	.115	.630	.065	.070	.040	.042
	0	1.2	.026	.037	.010	.011	.004	.005
0	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.072	1.42	.036	.040	.020	.021
	.4	2	.009	.087	.003	.003	.001	.001
	0	.6	.104	.414	.060	.068	.036	.041
	0	1.2	.021	.048	.010	.011	.004	.004
.5	0	2	.003	.005	.001	.001	.000	.000
	.4	1.2	.063	2.64	.032	.038	.017	.020
	.4	2	.008	.714	.002	.021	.001	.001
	0	.6	.097	1.54	.059	.068	.037	.042
	0	1.2	.020	.086	.009	.011	.004	.005
-.5	0	2	.002	.010	.001	.001	.000	.000
	.4	1.2	.059	2.84	.031	.038	.018	.021
	.4	2	.007	.165	.003	.004	.001	.001
	0	.6	.082	.118	.048	33.0	.031	.043
	0	1.2	.016	.044	.007	.010	.003	.004
.75	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.050	4.19	.025	.444	.014	.020
	.4	2	.006	.376	.002	.006	.001	.001

TABLE 2.23

MONTE CARLO S.D. OF $\overline{v}_I, \overline{v}_F, \overline{v}_O$ FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.6	.134	.123	.125	.078	.071	.071	.046	.045	.041
	0	1.2	.033	.028	.030	.014	.013	.011	.006	.006	.005
0	0	2	.004	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.086	.089	.110	.047	.050	.057	.024	.026	.028
	.4	2	.012	.012	.014	.004	.004	.004	.001	.001	.001
	0	.6	.119	.115	.109	.067	.077	.066	.040	.056	.039
	0	1.2	.027	.028	.025	.011	.012	.010	.004	.007	.004
.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.077	.082	.090	.039	.043	.047	.021	.023	.023
	.4	2	.010	.010	.012	.003	.003	.003	.001	.001	.001
	0	.6	.108	.119	.104	.067	.079	.064	.042	.057	.040
	0	1.2	.024	.026	.023	.011	.012	.011	.005	.007	.004
-.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.068	.072	.081	.041	.043	.049	.022	.024	.025
	.4	2	.008	.010	.011	.003	.003	.004	.001	.001	.001
	0	.6	.092	.117	.089	.053	.078	.053	.033	.066	.035
	0	1.2	.020	.024	.020	.008	.011	.009	.004	.007	.003
.75	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.059	.068	.071	.031	.036	.037	.017	.021	.019
	.4	2	.007	.009	.009	.002	.003	.003	.001	.001	.001

TABLE 2.24

MONTE CARLO S.D. OF $\overline{v}_I^o, \overline{v}_F^o$ FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{v}_I^o$	$\overline{v}_F^o$	$\overline{v}_I^o$	$\overline{v}_F^o$	$\overline{v}_I^o$	$\overline{v}_F^o$
	0	.6	.161	115	.081	.076	.046	.046
	0	1.2	.030	2.19	.011	.012	.005	.006
0	0	2	.003	.287	.001	.004	.000	.000
	.4	1.2	.423	33.5	.043	8.63	.021	.023
	.4	2	.011	4.26	.003	.845	.001	.001
	0	.6	.139	6.26	.075	.544	.041	.085
	0	1.2	.024	13.4	.009	.513	.004	.008
.5	0	2	.003	.259	.001	.043	.000	.000
	.4	1.2	.195	28.9	.039	6.04	.019	.024
	.4	2	.053	260	.003	.746	.001	.002
	0	.6	.134	22.0	.075	3.85	.044	.061
	0	1.2	.023	34.5	.010	.338	.004	.008
-.5	0	2	.003	.311	.001	.001	.000	.000
	.4	1.2	.101	156	.037	5.25	.020	.025
	.4	2	.012	6.46	.003	.977	.001	.004
	0	.6	.115	28.4	.059	.845	.035	.070
	0	1.2	.018	4.76	.008	.136	.003	.008
.75	0	2	.002	.244	.001	.001	.000	.000
	.4	1.2	.156	251	.030	13.8	.015	.027
	.4	2	.110	37.0	.002	.868	.001	.001

TABLE 2.25

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.102	.103	.091	.059	.059	.054	.038	.037	.034
	0	1.2	.025	.026	.025	.010	.011	.010	.004	.005	.004
0	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.068	.070	.080	.035	.036	.046	.020	.020	.025
	.4	2	.009	.009	.011	.003	.003	.003	.001	.001	.001
	0	.6	.094	.097	.093	.059	.060	.065	.038	.038	.047
	0	1.2	.021	.024	.022	.009	.009	.009	.004	.004	.004
.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.059	.067	.067	.031	.034	.040	.017	.018	.022
	.4	2	.008	.008	.009	.002	.003	.003	.001	.001	.001
	0	.6	.090	.095	.089	.058	.059	.064	.039	.039	.048
	0	1.2	.020	.022	.021	.009	.010	.009	.004	.004	.004
-.5	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.056	.063	.066	.031	.034	.040	.017	.019	.022
	.4	2	.007	.007	.009	.003	.003	.003	.001	.001	.001
	0	.6	.085	.097	.097	.055	.057	.073	.037	.038	.057
	0	1.2	.016	.021	.018	.007	.008	.008	.003	.003	.003
.75	0	2	.002	.003	.003	.000	.001	.001	.000	.000	.000
	.4	1.2	.046	.058	.056	.023	.029	.032	.013	.016	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001

TABLE 2.26

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.108	.110	.062	.062	.039	.039
	0	1.2	.025	.027	.010	.011	.004	.005
0	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.071	.035	.036	.019	.020
	.4	2	.009	.009	.003	.003	.001	.001
	0	.6	.096	.101	.059	.060	.036	.037
	0	1.2	.021	.025	.009	.009	.004	.004
.5	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.060	.069	.031	.035	.017	.018
	.4	2	.008	.008	.002	.002	.001	.001
	0	.6	.091	.097	.057	.059	.037	.038
	0	1.2	.020	.022	.009	.010	.004	.004
-.5	0	2	.002	.002	.001	.001	.000	.000
	.4	1.2	.057	.064	.030	.034	.017	.020
	.4	2	.007	.007	.003	.003	.001	.001
	0	.6	.080	.089	.050	.053	.033	.034
	0	1.2	.016	.022	.007	.008	.003	.003
.75	0	2	.002	.003	.000	.001	.000	.000
	.4	1.2	.045	.060	.023	.030	.013	.016
	.4	2	.006	.008	.002	.002	.001	.001

TABLE 2.27

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.103	.114	.092	.061	.060	.054	.038	.038	.034
	0	1.2	.025	.027	.025	.011	.011	.010	.005	.005	.004
0	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.069	.088	.080	.037	.046	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.098	.102	.093	.060	.065	.064	.038	.039	.046
	0	1.2	.023	.388	.022	.009	.010	.009	.004	.004	.004
.5	0	2	.003	.026	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.062	.400	.068	.032	.052	.040	.017	.021	.022
	.4	2	.008	.222	.009	.003	.003	.003	.001	.001	.001
	0	.6	.092	.099	.088	.060	.061	.063	.039	.039	.046
	0	1.2	.021	.052	.021	.009	.011	.009	.004	.004	.004
-.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.272	.066	.032	.051	.040	.018	.020	.023
	.4	2	.008	.012	.009	.003	.003	.003	.001	.001	.001
	0	.6	.092	.111	.094	.058	.063	.070	.038	.041	.055
	0	1.2	.018	.026	.018	.007	.019	.008	.003	.005	.003
.75	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.051	.084	.056	.024	.059	.032	.014	.021	.018
	.4	2	.006	.009	.008	.002	.003	.003	.001	.001	.001

TABLE 2.28

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.109	.111	.063	.064	.039	.039
	0	1.2	.025	.026	.010	.011	.004	.005
0	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.071	.036	.037	.020	.020
	.4	2	.009	.009	.003	.003	.001	.001
	0	.6	.097	.102	.059	.061	.036	.037
	0	1.2	.021	.024	.009	.009	.004	.004
.5	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.059	.069	.031	.035	.017	.018
	.4	2	.008	.008	.002	.002	.001	.001
	0	.6	.092	.100	.058	.060	.037	.038
	0	1.2	.020	.022	.009	.010	.004	.004
-.5	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.064	.031	.034	.017	.019
	.4	2	.007	.007	.003	.003	.001	.001
	0	.6	.080	.088	.050	.054	.032	.034
	0	1.2	.016	.021	.007	.008	.003	.003
.75	0	2	.002	.003	.000	.001	.000	.000
	.4	1.2	.045	.058	.023	.029	.013	.015
	.4	2	.006	.007	.002	.002	.001	.001

TABLE 2.29

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_O$ FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.6	.105	.116	.095	.061	.069	.056	.038	.041	.035
	0	1.2	.025	.029	.025	.010	.012	.010	.004	.005	.004
0	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.070	.085	.082	.036	.045	.046	.020	.022	.025
	.4	2	.009	.010	.011	.003	.004	.003	.001	.001	.001
	0	.6	.095	.107	.092	.058	.067	.062	.036	.040	.043
	0	1.2	.022	.028	.022	.009	.012	.009	.004	.005	.004
.5	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.061	.084	.069	.032	.043	.040	.017	.023	.022
	.4	2	.008	.010	.010	.002	.003	.003	.001	.001	.001
	0	.6	.090	.101	.087	.058	.067	.061	.038	.042	.043
	0	1.2	.020	.025	.021	.009	.012	.009	.004	.005	.004
-.5	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.074	.067	.032	.045	.041	.018	.025	.023
	.4	2	.007	.009	.009	.003	.003	.003	.001	.001	.001
	0	.6	.082	.100	.089	.051	.064	.064	.034	.041	.049
	0	1.2	.017	.025	.018	.007	.011	.008	.003	.004	.003
.75	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.071	.056	.025	.041	.032	.014	.024	.018
	.4	2	.006	.008	.008	.002	.003	.003	.001	.001	.001

TABLE 2.30

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

		n	64	64	128	128	256	256
ρ	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.114	.126	.065	.072	.040	.043
	0	1.2	.026	.030	.010	.012	.004	.005
0	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.071	1.02	.036	.244	.020	.023
	.4	2	.009	.139	.003	.425	.001	.006
	0	.6	.102	.262	.060	.073	.036	.043
	0	1.2	.021	.028	.009	.012	.004	.005
.5	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.062	20.6	.032	67.0	.017	.022
	.4	2	.008	.648	.002	1.66	.001	.001
	0	.6	.096	.112	.059	.073	.037	.045
	0	1.2	.020	.025	.009	.012	.004	.005
-.5	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.059	7.77	.032	.798	.018	.023
	.4	2	.007	.999	.003	.066	.001	.001
	0	.6	.081	.112	.048	.073	.031	.048
	0	1.2	.016	2.68	.007	.012	.003	.005
.75	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.048	60.1	.025	3.31	.014	.061
	.4	2	.006	.737	.002	.246	.001	.001

TABLE 2.31

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.072	.194	.049	.125	.052	.090	.131	.261	.099	.166	.116	.137
	0	1.2	.059	.198	.062	.136	.048	.097	.113	.260	.110	.208	.122	.161
0	0	2	.054	.184	.057	.122	.058	.102	.109	.255	.108	.199	.120	.167
	.4	1.2	.060	.193	.050	.115	.051	.076	.125	.254	.109	.176	.099	.131
	.4	2	.051	.177	.071	.133	.059	.104	.108	.238	.123	.201	.121	.157
	0	.6	.064	.238	.054	.152	.052	.116	.128	.322	.113	.224	.105	.178
	0	1.2	.067	.203	.057	.132	.053	.097	.122	.289	.108	.202	.104	.157
.5	0	2	.065	.201	.055	.133	.059	.108	.116	.272	.112	.193	.111	.160
	.4	1.2	.067	.231	.051	.153	.049	.110	.127	.312	.102	.207	.092	.168
	.4	2	.065	.184	.055	.114	.058	.095	.122	.254	.114	.187	.111	.149
	0	.6	.062	.227	.059	.166	.059	.129	.128	.311	.120	.231	.109	.203
	0	1.2	.047	.209	.074	.161	.052	.095	.105	.292	.129	.225	.100	.149
-.5	0	2	.049	.199	.073	.163	.063	.112	.110	.264	.129	.222	.109	.157
	.4	1.2	.056	.238	.061	.167	.050	.109	.120	.318	.117	.222	.103	.174
	.4	2	.049	.186	.074	.146	.066	.094	.097	.248	.134	.214	.105	.152
	0	.6	.069	.332	.050	.259	.052	.247	.120	.416	.107	.337	.104	.327
	0	1.2	.066	.231	.054	.144	.053	.100	.127	.311	.099	.217	.112	.158
.75	0	2	.054	.221	.042	.144	.064	.104	.122	.293	.104	.208	.112	.150
	.4	1.2	.065	.292	.048	.195	.057	.141	.130	.383	.110	.278	.111	.199
	.4	2	.064	.210	.054	.130	.060	.097	.123	.267	.110	.193	.112	.148

TABLE 2.32

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.147	.225	.101	.186	.090	.163	.219	.286	.166	.259	.158	.229
	0	1.2	.103	.194	.075	.190	.062	.160	.166	.268	.137	.246	.121	.206
0	0	2	.092	.187	.073	.191	.068	.179	.162	.249	.133	.240	.130	.247
	.4	1.2	.119	.221	.080	.188	.062	.145	.211	.285	.150	.244	.120	.217
	.4	2	.095	.205	.072	.194	.075	.167	.156	.260	.137	.260	.117	.228
	0	.6	.211	.330	.173	.309	.160	.286	.301	.407	.264	.394	.258	.373
	0	1.2	.110	.215	.071	.214	.057	.176	.164	.293	.123	.276	.111	.240
.5	0	2	.111	.202	.082	.179	.067	.166	.176	.255	.141	.241	.114	.225
	.4	1.2	.140	.274	.079	.234	.075	.209	.208	.333	.149	.304	.129	.276
	.4	2	.105	.209	.075	.181	.062	.153	.165	.261	.124	.244	.105	.202
	0	.6	.210	.312	.188	.302	.180	.298	.288	.389	.268	.383	.251	.393
	0	1.2	.101	.233	.089	.211	.056	.176	.169	.309	.150	.281	.105	.255
-.5	0	2	.099	.198	.093	.192	.071	.168	.154	.258	.156	.252	.121	.236
	.4	1.2	.125	.263	.102	.242	.074	.190	.212	.339	.162	.309	.138	.263
	.4	2	.090	.214	.087	.200	.061	.157	.146	.262	.159	.246	.112	.218
	0	.6	.346	.520	.346	.519	.342	.509	.445	.591	.432	.605	.441	.597
	0	1.2	.116	.267	.072	.279	.062	.227	.192	.335	.127	.343	.109	.293
.75	0	2	.118	.185	.080	.197	.060	.160	.184	.250	.131	.252	.116	.226
	.4	1.2	.158	.312	.095	.306	.085	.281	.242	.390	.163	.380	.142	.346
	.4	2	.106	.180	.085	.186	.062	.139	.167	.243	.123	.246	.118	.191

TABLE 2.33

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\tau = 1$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.119	.164	.085	.163	.076	.148	.190	.230	.145	.217	.136	.209
	0	1.2	.093	.153	.070	.160	.055	.138	.153	.200	.121	.214	.122	.195
0	0	2	.077	.137	.063	.156	.064	.153	.138	.184	.124	.214	.125	.217
	.4	1.2	.098	.151	.073	.148	.053	.130	.173	.209	.132	.202	.102	.197
	.4	2	.090	.149	.073	.155	.071	.145	.139	.207	.123	.219	.123	.201
	0	.6	.150	.249	.112	.260	.097	.224	.221	.307	.187	.335	.169	.315
	0	1.2	.099	.178	.063	.181	.054	.170	.148	.232	.116	.251	.106	.233
.5	0	2	.099	.154	.063	.145	.064	.142	.156	.199	.121	.210	.110	.199
	.4	1.2	.111	.196	.073	.195	.069	.199	.184	.254	.131	.257	.112	.266
	.4	2	.093	.142	.062	.147	.059	.141	.143	.195	.117	.211	.105	.183
	0	.6	.144	.239	.127	.261	.110	.250	.229	.314	.203	.340	.186	.331
	0	1.2	.084	.180	.079	.191	.055	.167	.155	.244	.141	.248	.099	.245
-.5	0	2	.085	.144	.085	.158	.071	.155	.140	.195	.146	.214	.117	.212
	.4	1.2	.106	.181	.085	.199	.058	.180	.179	.247	.137	.272	.123	.252
	.4	2	.079	.148	.082	.159	.062	.143	.136	.196	.143	.214	.112	.210
	0	.6	.209	.378	.156	.399	.164	.389	.282	.451	.252	.484	.250	.474
	0	1.2	.092	.214	.063	.257	.060	.234	.156	.277	.117	.329	.109	.305
.75	0	2	.095	.157	.068	.162	.060	.143	.164	.211	.122	.219	.111	.202
	.4	1.2	.134	.248	.078	.252	.073	.270	.189	.329	.142	.321	.123	.342
	.4	2	.087	.140	.071	.146	.060	.117	.136	.184	.111	.206	.113	.178

TABLE 2.34

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.382	.265	.308	.181	.224	.137	.470	.341	.388	.261	.311	.213
	0	1.2	.407	.234	.357	.187	.284	.120	.479	.287	.433	.247	.353	.178
0	0	2	.430	.237	.376	.215	.326	.158	.517	.307	.458	.261	.419	.201
	.4	1.2	.398	.320	.307	.270	.235	.216	.467	.390	.384	.355	.325	.279
	.4	2	.406	.393	.360	.359	.309	.296	.509	.472	.442	.424	.405	.361
	0	.6	.425	.448	.337	.504	.264	.565	.504	.532	.430	.598	.345	.638
	0	1.2	.423	.253	.341	.210	.252	.160	.504	.315	.430	.285	.349	.233
.5	0	2	.445	.252	.385	.217	.325	.139	.523	.326	.475	.271	.390	.199
	.4	1.2	.405	.376	.309	.322	.234	.275	.494	.440	.400	.392	.315	.348
	.4	2	.425	.410	.367	.351	.286	.272	.506	.482	.450	.409	.365	.334
	0	.6	.418	.472	.343	.502	.288	.561	.506	.565	.419	.584	.359	.635
	0	1.2	.423	.256	.357	.215	.272	.154	.495	.318	.442	.290	.364	.239
-.5	0	2	.443	.252	.394	.219	.340	.134	.510	.307	.469	.270	.407	.190
	.4	1.2	.400	.380	.344	.325	.268	.271	.488	.450	.408	.392	.352	.367
	.4	2	.420	.417	.387	.346	.307	.273	.486	.471	.458	.420	.387	.334
	0	.6	.486	.679	.398	.751	.354	.790	.562	.733	.492	.810	.446	.841
	0	1.2	.428	.283	.342	.244	.272	.237	.499	.342	.423	.313	.350	.309
.75	0	2	.455	.247	.389	.217	.311	.142	.533	.312	.459	.283	.402	.196
	.4	1.2	.433	.393	.350	.360	.265	.360	.503	.484	.418	.431	.336	.444
	.4	2	.436	.427	.364	.347	.298	.275	.509	.488	.430	.411	.375	.335

TABLE 2.35

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\tau = 1$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.189	.153	.124	.117	.080	.078	.244	.213	.180	.185	.133	.148
	0	1.2	.177	.118	.124	.075	.077	.053	.231	.165	.180	.125	.144	.092
0	0	2	.169	.118	.117	.085	.095	.060	.230	.178	.167	.129	.165	.092
	.4	1.2	.157	.149	.105	.107	.073	.076	.218	.192	.164	.147	.124	.126
	.4	2	.155	.190	.121	.158	.090	.099	.216	.242	.177	.216	.153	.151
	0	.6	.211	.270	.159	.306	.114	.368	.263	.346	.221	.397	.169	.456
	0	1.2	.178	.139	.121	.120	.090	.119	.231	.191	.168	.178	.144	.202
.5	0	2	.193	.116	.130	.086	.092	.054	.256	.180	.180	.129	.153	.096
	.4	1.2	.171	.178	.129	.170	.096	.191	.231	.241	.183	.233	.149	.275
	.4	2	.163	.184	.120	.140	.090	.091	.218	.247	.167	.201	.144	.149
	0	.6	.220	.279	.167	.310	.137	.380	.278	.361	.233	.387	.189	.467
	0	1.2	.173	.145	.139	.115	.090	.112	.231	.196	.190	.191	.151	.192
-.5	0	2	.177	.126	.140	.090	.089	.056	.229	.184	.190	.133	.152	.088
	.4	1.2	.185	.172	.140	.171	.103	.177	.227	.217	.204	.232	.150	.271
	.4	2	.161	.184	.123	.148	.094	.098	.210	.230	.176	.209	.133	.148
	0	.6	.276	.423	.227	.507	.181	.611	.340	.505	.307	.587	.243	.682
	0	1.2	.186	.171	.123	.175	.095	.220	.237	.230	.185	.255	.154	.293
.75	0	2	.194	.126	.136	.096	.100	.052	.257	.177	.193	.152	.159	.089
	.4	1.2	.206	.227	.158	.270	.118	.369	.265	.290	.213	.357	.182	.460
	.4	2	.148	.164	.112	.142	.090	.090	.200	.228	.180	.193	.141	.140

TABLE 2.36

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.125	.195	.082	.122	.089	.105	.187	.249	.146	.189	.148	.159
	0	1.2	.076	.159	.065	.114	.050	.084	.127	.225	.124	.170	.122	.138
0	0	2	.061	.148	.068	.112	.065	.096	.126	.222	.114	.176	.120	.153
	.4	1.2	.096	.170	.069	.111	.065	.067	.167	.240	.140	.166	.115	.130
	.4	2	.060	.147	.070	.105	.064	.094	.117	.201	.128	.176	.120	.147
	0	.6	.206	.321	.194	.284	.199	.265	.302	.380	.281	.376	.299	.365
	0	1.2	.074	.191	.061	.116	.057	.092	.135	.255	.116	.185	.109	.144
.5	0	2	.076	.161	.066	.108	.067	.088	.135	.225	.115	.164	.110	.139
	.4	1.2	.107	.202	.075	.137	.061	.113	.179	.280	.123	.199	.107	.163
	.4	2	.068	.148	.062	.092	.061	.075	.129	.196	.114	.147	.110	.120
	0	.6	.217	.326	.203	.273	.212	.260	.288	.408	.285	.379	.291	.358
	0	1.2	.059	.182	.082	.149	.049	.081	.126	.252	.137	.192	.105	.140
-.5	0	2	.059	.158	.082	.123	.061	.090	.126	.209	.141	.189	.110	.144
	.4	1.2	.097	.208	.079	.159	.063	.113	.169	.292	.142	.218	.129	.178
	.4	2	.061	.142	.076	.105	.063	.074	.112	.185	.139	.158	.112	.129
	0	.6	.391	.525	.406	.531	.408	.513	.491	.627	.492	.616	.521	.612
	0	1.2	.074	.217	.055	.136	.055	.099	.135	.273	.104	.200	.110	.149
.75	0	2	.071	.156	.051	.114	.059	.078	.140	.211	.104	.162	.103	.123
	.4	1.2	.114	.262	.071	.180	.071	.137	.192	.325	.131	.258	.122	.198
	.4	2	.068	.119	.058	.092	.061	.066	.133	.177	.108	.133	.110	.102

TABLE 2.37

EMPIRICAL SIZES OF W_I^o AND W_F^o FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o
	0	.6	.115	.186	.083	.118	.074	.098	.177	.246	.140	.187	.144	.158
	0	1.2	.073	.161	.061	.105	.048	.084	.116	.210	.115	.161	.123	.138
0	0	2	.056	.139	.060	.114	.065	.101	.113	.213	.114	.159	.118	.153
	.4	1.2	.086	.162	.060	.105	.058	.062	.150	.231	.128	.154	.108	.123
	.4	2	.055	.137	.066	.109	.062	.087	.113	.196	.118	.160	.112	.150
	0	.6	.142	.252	.133	.209	.132	.192	.230	.317	.214	.296	.224	.266
	0	1.2	.067	.183	.060	.112	.056	.092	.138	.243	.112	.177	.106	.141
.5	0	2	.067	.166	.060	.104	.067	.085	.119	.216	.117	.155	.112	.137
	.4	1.2	.100	.198	.073	.130	.059	.110	.160	.268	.118	.206	.108	.161
	.4	2	.066	.144	.059	.086	.061	.075	.121	.187	.114	.136	.114	.122
	0	.6	.157	.262	.140	.209	.146	.198	.223	.350	.224	.312	.220	.282
	0	1.2	.060	.183	.074	.140	.050	.082	.115	.244	.134	.189	.102	.147
-.5	0	2	.054	.151	.074	.122	.064	.094	.122	.206	.135	.176	.110	.142
	.4	1.2	.085	.198	.072	.156	.061	.107	.163	.275	.132	.212	.116	.173
	.4	2	.060	.135	.073	.097	.066	.070	.113	.180	.124	.158	.106	.131
	0	.6	.240	.369	.248	.374	.263	.339	.344	.461	.352	.479	.358	.436
	0	1.2	.065	.213	.059	.132	.050	.098	.130	.261	.105	.201	.113	.157
.75	0	2	.061	.153	.044	.114	.063	.077	.123	.206	.102	.155	.110	.122
	.4	1.2	.091	.259	.065	.183	.058	.135	.156	.316	.124	.257	.122	.195
	.4	2	.061	.120	.058	.087	.059	.065	.119	.165	.107	.128	.114	.105

TABLE 2.38

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.151	.224	.105	.142	.088	.102	.221	.289	.176	.208	.152	.165
	0	1.2	.138	.222	.114	.147	.073	.093	.199	.276	.179	.227	.134	.147
0	0	2	.156	.229	.129	.173	.086	.110	.227	.293	.206	.242	.152	.172
	.4	1.2	.133	.209	.102	.136	.075	.088	.213	.280	.177	.201	.147	.149
	.4	2	.130	.210	.123	.158	.076	.097	.203	.268	.199	.234	.145	.168
	0	.6	.258	.378	.221	.339	.204	.309	.357	.459	.310	.439	.301	.413
	0	1.2	.140	.252	.105	.155	.080	.094	.201	.318	.167	.228	.134	.166
.5	0	2	.168	.245	.121	.171	.087	.106	.241	.314	.203	.253	.152	.159
	.4	1.2	.156	.232	.108	.155	.073	.119	.231	.307	.179	.220	.127	.170
	.4	2	.148	.210	.114	.141	.077	.083	.213	.263	.186	.208	.142	.136
	0	.6	.274	.393	.239	.336	.215	.307	.337	.477	.310	.425	.291	.420
	0	1.2	.144	.233	.126	.184	.067	.094	.209	.319	.192	.249	.124	.145
-.5	0	2	.174	.236	.142	.173	.082	.105	.255	.311	.230	.242	.137	.161
	.4	1.2	.139	.251	.115	.165	.072	.107	.210	.318	.191	.230	.133	.181
	.4	2	.148	.201	.138	.153	.079	.087	.212	.257	.216	.214	.133	.139
	0	.6	.434	.593	.430	.610	.405	.575	.532	.676	.516	.692	.501	.671
	0	1.2	.161	.277	.105	.168	.080	.102	.223	.340	.158	.241	.138	.163
.75	0	2	.191	.238	.131	.178	.098	.094	.251	.313	.199	.234	.143	.143
	.4	1.2	.174	.302	.115	.199	.081	.136	.256	.373	.177	.273	.141	.190
	.4	2	.166	.188	.127	.138	.094	.081	.229	.239	.178	.201	.142	.118

TABLE 2.39

EMPIRICAL SIZES OF W_I^o AND W_F^o FOR $\tau = 1$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o
	0	.6	.111	.170	.079	.125	.075	.092	.179	.239	.146	.183	.145	.163
	0	1.2	.063	.150	.061	.106	.048	.076	.116	.204	.113	.150	.121	.136
0	0	2	.049	.131	.062	.100	.063	.091	.112	.188	.117	.150	.120	.149
	.4	1.2	.083	.148	.059	.102	.056	.064	.139	.221	.126	.146	.108	.117
	.4	2	.053	.129	.068	.100	.062	.087	.111	.175	.127	.150	.117	.143
	0	.6	.136	.248	.128	.211	.126	.192	.206	.313	.201	.302	.206	.282
	0	1.2	.069	.162	.057	.099	.053	.079	.127	.220	.108	.167	.107	.135
.5	0	2	.065	.139	.057	.097	.065	.078	.114	.187	.116	.146	.116	.133
	.4	1.2	.089	.175	.070	.122	.054	.104	.151	.247	.124	.186	.107	.152
	.4	2	.070	.129	.055	.081	.061	.073	.115	.168	.113	.130	.112	.119
	0	.6	.148	.254	.126	.218	.140	.203	.215	.340	.207	.311	.206	.294
	0	1.2	.052	.143	.075	.134	.052	.079	.113	.226	.127	.182	.100	.132
-.5	0	2	.056	.130	.072	.106	.062	.082	.108	.183	.126	.164	.111	.131
	.4	1.2	.081	.183	.074	.137	.059	.097	.151	.246	.124	.195	.115	.158
	.4	2	.050	.126	.072	.096	.067	.069	.106	.167	.125	.140	.106	.125
	0	.6	.136	.359	.221	.382	.242	.355	.206	.457	.326	.474	.327	.463
	0	1.2	.069	.172	.053	.109	.050	.083	.127	.225	.095	.179	.112	.141
.75	0	2	.065	.131	.041	.096	.064	.072	.114	.178	.099	.139	.111	.109
	.4	1.2	.089	.221	.063	.161	.057	.121	.151	.293	.125	.232	.119	.170
	.4	2	.070	.103	.058	.078	.058	.062	.115	.144	.110	.117	.108	.095

TABLE 2.40

EMPIRICAL SIZES OF W_I AND W_F FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.148	.305	.098	.250	.090	.222	.219	.373	.169	.326	.158	.287
	0	1.2	.109	.308	.076	.261	.063	.208	.168	.371	.137	.316	.120	.271
0	0	2	.090	.294	.079	.255	.072	.226	.167	.360	.141	.319	.130	.286
	.4	1.2	.114	.373	.083	.307	.063	.239	.204	.435	.147	.371	.116	.310
	.4	2	.094	.371	.074	.315	.077	.239	.164	.420	.140	.372	.120	.290
	0	.6	.223	.390	.190	.361	.171	.306	.307	.473	.268	.445	.273	.404
	0	1.2	.101	.333	.072	.310	.062	.258	.169	.405	.129	.385	.111	.329
.5	0	2	.113	.310	.082	.263	.066	.211	.183	.377	.135	.327	.115	.278
	.4	1.2	.137	.413	.083	.374	.074	.298	.210	.466	.147	.433	.129	.355
	.4	2	.104	.364	.072	.315	.060	.212	.170	.432	.129	.364	.107	.277
	0	.6	.221	.373	.210	.352	.187	.327	.302	.456	.280	.437	.264	.410
	0	1.2	.098	.332	.093	.312	.059	.272	.177	.407	.153	.397	.098	.328
-.5	0	2	.099	.292	.092	.257	.074	.222	.167	.347	.154	.335	.124	.277
	.4	1.2	.134	.402	.107	.370	.081	.307	.202	.471	.168	.433	.138	.373
	.4	2	.088	.361	.089	.303	.063	.240	.152	.418	.156	.362	.111	.299
	0	.6	.365	.516	.368	.530	.358	.513	.465	.597	.445	.611	.450	.594
	0	1.2	.109	.368	.071	.392	.061	.292	.179	.438	.124	.454	.115	.353
.75	0	2	.119	.299	.075	.275	.062	.200	.186	.361	.134	.339	.118	.274
	.4	1.2	.164	.447	.096	.446	.084	.358	.238	.518	.165	.516	.140	.426
	.4	2	.104	.350	.081	.307	.061	.219	.163	.403	.122	.360	.115	.263

TABLE 2.41

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\tau = 1$, $\phi_i = .4$, $\psi_i = .2$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.125	.601	.085	.564	.075	.556	.198	.659	.152	.627	.141	.618
	0	1.2	.095	.580	.071	.583	.058	.543	.153	.640	.124	.645	.124	.597
0	0	2	.080	.572	.072	.584	.065	.535	.139	.630	.125	.647	.126	.607
	.4	1.2	.105	.589	.072	.576	.057	.578	.175	.642	.130	.638	.106	.622
	.4	2	.084	.573	.073	.596	.072	.540	.134	.622	.130	.639	.124	.607
	0	.6	.146	.680	.122	.685	.106	.654	.218	.735	.189	.733	.174	.707
	0	1.2	.097	.635	.062	.659	.056	.624	.145	.687	.117	.715	.110	.658
.5	0	2	.094	.584	.068	.585	.066	.559	.152	.639	.121	.648	.113	.628
	.4	1.2	.115	.651	.072	.630	.067	.604	.174	.694	.127	.689	.110	.661
	.4	2	.094	.587	.061	.578	.061	.556	.149	.637	.119	.635	.105	.617
	0	.6	.158	.670	.130	.680	.128	.659	.231	.716	.215	.743	.187	.714
	0	1.2	.083	.628	.081	.634	.057	.585	.152	.697	.143	.677	.097	.646
-.5	0	2	.085	.570	.088	.584	.070	.557	.134	.628	.145	.649	.115	.631
	.4	1.2	.111	.647	.089	.625	.064	.605	.181	.698	.143	.681	.124	.667
	.4	2	.082	.547	.083	.591	.062	.555	.129	.608	.152	.660	.107	.621
	0	.6	.216	.747	.191	.763	.165	.764	.296	.792	.277	.801	.253	.813
	0	1.2	.088	.671	.066	.697	.063	.626	.145	.714	.112	.752	.111	.672
.75	0	2	.096	.585	.063	.600	.063	.580	.153	.639	.117	.659	.116	.637
	.4	1.2	.128	.691	.081	.695	.076	.669	.189	.739	.139	.744	.132	.724
	.4	2	.090	.585	.073	.583	.064	.563	.141	.636	.112	.629	.116	.625

TABLE 2.42

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_\gamma, \overline{\nu}_\delta, \overline{\nu}_F, \overline{\nu}_O$, FOR $\delta = 1$, $\gamma = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

	τ	1			2			.5		
ρ	n	64	128	256	64	128	256	64	128	256
	$\overline{\nu}_I$	-.002	-.001	.000	-.001	.000	.000	-.003	-.001	.000
	$\overline{\nu}_\gamma$	-.002	-.001	.000	-.001	.000	.000	-.002	-.001	-.001
0	$\overline{\nu}_\delta$	-.002	-.001	.000	-.001	-.001	.000	-.002	-.001	.000
	$\overline{\nu}_F$	-.001	-.001	.000	-.001	-.001	.000	-.002	-.001	.000
	$\overline{\nu}_O$	-.002	-.001	.000	-.002	-.001	.000	-.003	-.001	.000
	$\overline{\nu}_I$	.001	.000	.000	.001	.000	.000	.001	.001	.000
	$\overline{\nu}_\gamma$	-.010	-.002	-.001	-.007	-.002	.000	-.014	-.003	-.001
.5	$\overline{\nu}_\delta$	.004	.001	.000	.003	.001	.000	.005	.001	.000
	$\overline{\nu}_F$	-.005	-.001	.000	-.003	-.001	.000	-.006	-.002	-.001
	$\overline{\nu}_O$	.030	.015	.007	.021	.011	.005	.041	.021	.010
	$\overline{\nu}_I$	.000	.000	.000	.000	.000	.000	.000	.000	.000
	$\overline{\nu}_\gamma$	.009	.003	.001	.007	.002	.001	.013	.004	.001
-.5	$\overline{\nu}_\delta$	-.003	-.001	.000	-.002	-.001	.000	-.004	-.002	.000
	$\overline{\nu}_F$	.004	.001	.001	.003	.001	.000	.005	.002	.001
	$\overline{\nu}_O$	-.028	-.014	-.007	-.020	-.010	-.005	-.039	-.020	-.010
	$\overline{\nu}_I$	.001	.001	.000	.000	.000	.000	.001	.001	.000
	$\overline{\nu}_\gamma$	-.016	-.004	-.001	-.012	-.003	-.001	-.023	-.005	-.001
.75	$\overline{\nu}_\delta$	.004	.001	.000	.003	.001	.000	.005	.001	.000
	$\overline{\nu}_F$	-.008	-.002	-.001	-.005	-.001	-.001	-.010	-.003	-.001
	$\overline{\nu}_O$	.044	.022	.011	.031	.016	.008	.061	.030	.015

TABLE 2.43

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_\gamma, \overline{\nu}_\delta, \overline{\nu}_F, \overline{\nu}_O$, FOR $\delta = 1$, $\gamma = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

ρ	τ / n	1: 64	1: 128	1: 256	2: 64	2: 128	2: 256	.5: 64	.5: 128	.5: 256
0	$\overline{\nu}_I$	.041	.019	.009	.029	.014	.007	.058	.027	.013
	$\overline{\nu}_\gamma$	.043	.020	.009	.031	.014	.007	.060	.028	.013
	$\overline{\nu}_\delta$	.043	.020	.009	.030	.014	.007	.060	.028	.013
	$\overline{\nu}_F$	.044	.020	.009	.031	.014	.007	.061	.028	.013
	$\overline{\nu}_O$	.040	.019	.009	.029	.014	.007	.056	.027	.013
.5	$\overline{\nu}_I$	.035	.016	.008	.025	.012	.006	.049	.023	.011
	$\overline{\nu}_\gamma$	.043	.017	.008	.030	.012	.006	.060	.024	.011
	$\overline{\nu}_\delta$	.037	.017	.008	.026	.012	.006	.052	.024	.012
	$\overline{\nu}_F$	.043	.018	.008	.031	.013	.006	.060	.025	.012
	$\overline{\nu}_O$	.040	.020	.010	.028	.014	.007	.056	.028	.014
-.5	$\overline{\nu}_I$	.033	.016	.008	.024	.012	.006	.046	.023	.011
	$\overline{\nu}_\gamma$	.040	.018	.009	.029	.013	.006	.056	.025	.012
	$\overline{\nu}_\delta$	.035	017.	.008	.025	.012	.006	.049	.024	.012
	$\overline{\nu}_F$	.039	.018	.009	.028	.013	.006	.055	.025	.012
	$\overline{\nu}_O$	.039	.020	.010	.028	.014	.007	.054	.028	.014
.75	$\overline{\nu}_I$	.026	.012	.006	.019	.009	.004	.037	.018	.009
	$\overline{\nu}_\gamma$	.042	.015	.007	.030	.011	.005	.059	.022	.010
	$\overline{\nu}_\delta$	.031	.015	.007	.022	.010	.005	.043	.021	.010
	$\overline{\nu}_F$	.042	.016	.007	.030	.011	.005	.059	.022	.010
	$\overline{\nu}_O$	.043	.021	.011	.030	.015	.007	.060	.029	.015

TABLE 2.44

EMPIRICAL SIZES OF $W_I, W_\gamma, W_\delta, W_F, W_O$ FOR $\delta = 1$, $\gamma = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

ρ	α / n	.05: W_I	.05: W_γ	.05: W_δ	.05: W_F	.05: W_O	.10: W_I	.10: W_γ	.10: W_δ	.10: W_F	.10: W_O
0	64	.061	.055	.199	.200	.058	.122	.125	.267	.264	.122
	128	.053	.053	.126	.126	.052	.107	.107	.191	.191	.113
	256	.048	.048	.090	.090	.046	.118	.115	.154	.153	.109
.5	64	.066	.106	.199	.218	.126	.127	.175	.266	.297	.213
	128	.056	.067	.137	.140	.126	.113	.129	.209	.202	.201
	256	.053	.064	.085	.095	.107	.095	.117	.146	.159	.180
-.5	64	.047	.104	.196	.223	.131	.110	.174	.274	.309	.210
	128	.068	.086	.145	.159	.121	.114	.148	.221	.218	.205
	256	.045	.061	.093	.100	.119	.101	.123	.148	.156	.199
.75	64	.066	.185	.211	.254	.212	.122	.262	.280	.333	.331
	128	.052	.116	.153	.156	.204	.099	.190	.217	.224	.330
	256	.056	.094	.102	.115	.197	.109	.159	.170	.170	.306

Chapter 3

Parametric estimation of weak fractional co-integration

3.1 Introduction

In this chapter, as opposite to the situation considered in Chapter 2, we will focus on the case where in model (1.25), (1.26), the co-integrating gap β is small, more precisely

$$\beta < 1/2, \tag{3.1}$$

and where the real numbers γ and δ satisfy

$$0 \leq \gamma < \delta. \tag{3.2}$$

As anticipated, we describe this situation as weak fractional co-integration, since the memory reduction achievable is small relative to the $CI\,(1,1)$ case, or other cases in which $\beta \geq 1/2$. We anticipate that in (1.25), (1.26),

$$Cov(u_{1t}, u_{2t}) \neq 0, \tag{3.3}$$

so that, viewing (1.25) as a regression model, the regressor x_t is contemporaneously correlated with the co-integrating error $\Delta^{-\gamma}u_{1t}^{\#}$. The most dramatic contrast with this familiar $CI(1,1)$ situation arises when

$$\delta < 1/2, \tag{3.4}$$

because the "simultaneous equation bias" inherent in (3.3) leads to inconsistency of the OLS due to the fact that x_t is asymptotically stationary and so its sum of squares does not asymptotically dominate that of $\Delta^{-\gamma}u_{1t}^{\#}$. To overcome this problem, Robinson (1994c) showed that the NBLS is consistent, due to the dominance near zero frequency of an $I(\gamma)$ spectral density by an $I(\delta)$ one. (He considered the purely stationary situation, where there is no truncation in (1.25), but our modification does not affect such basic asymptotic properties). The same method was subsequently studied by Robinson and Marinucci (1998, 2001) in case

$$\delta > 1/2, \tag{3.5}$$

where there is trending nonstationarity. Here, the OLS is consistent, with convergence rate depending on the location of γ and δ in the non-negative quadrant, but the NBLS still sometimes converges faster, and never converges slower, despite dropping high frequency information, as we showed in Chapter 1. In any case, as discussed in Chapter 2, the question which then arises is whether the rates of convergence of OLS and NBLS are optimal, by which we mean whether they match the rates achieved by the Gaussian ML estimate under suitable regularity conditions. They are optimal for the combination $\gamma+\delta > 1$, $\delta-\gamma > 1/2$, but otherwise not. In particular, the $n^{\delta-\gamma}$ rate is optimal for $\delta-\gamma > 1/2$ without the restriction $\gamma+\delta > 1$, and we have established it in Chapter 2 for estimates asymptotically equivalent to the ML, allowing for consistent estimation of unknown γ and δ and a vector θ of unknown parameters describing the autocovariance structure of u_t; these estimates of ν have mixed normal asymptotics, and a Wald test statistic with an asymptotic null χ^2 distribution, as established earlier in the $CI(1,1)$ case by Phillips (1991a). Indeed, we found the limit distribution unaffected by the question of whether θ, γ and δ are known or unknown.

In case of weak fractional co-integration with $\beta < 1/2$, a substantially different asymptotic inferential theory prevails, impacting also on the question of how δ and γ should be estimated. Under (3.1), since $y_t(\gamma)$ and $x_t(\gamma)$ are $I(\beta)$, they are asymptotically stationary, and so, intuitively, one anticipates the existence of $n^{1/2}$-consistent and asymptotically normal estimates of ν; the OLS and NBLS converge slower than this owing to the dominance of bias due to (3.3). Note that (3.1) excludes the traditional $CI(1,1)$ case and so might be thought of as less plausible than $\beta \geq 1/2$. However, the vast bulk of the co-integration literature has focused only on the $CI(1,1)$ possibility and there has been little study of fractional possibilities, or even the testing of the unit root hypothesis on y_t, x_t against fractional alternatives, as distinct from stationary AR ones. In fact, the fractional co-integration analysis by Robinson and Marinucci (1998) of two of the bivariate series originally analysed by Engle and Granger (1987) (namely M1/nominal GNP and M3/nominal GNP) and one analysed by Campbell and Shiller (1987) (stock prices/dividends) in the $CI(1,1)$ context was suggestive of (3.1). Moreover, we cover not only $\delta \geq 1/2$, but also the asymptotically stationary case $\delta < 1/2$, which may be relevant for many financial time series. In fact, some of the empirical evidence presented in Chapter 1 is also suggestive of this type of co-integration. Note that here, the NBLS of Robinson (1994c) is only $m^{1/2}$-consistent for m increasing slower than n (indeed the optimal minimum-mean-squared error rate is $n^{2/5}$), so that we again achieve an improvement.

We are principally concerned with estimation of ν. If γ and δ are known, while u_t is known to be white noise with unknown variance-covariance matrix Ω, then the ML estimate of ν is given in closed form, and may be computed by means of an added-variable least squares regression, as pursued in the following section, which also extends to VAR u_t, of known degree, but with unknown AR coefficients, when our estimate of ν is no longer as efficient as the ML but has the same, $\sqrt{n}$, rate of convergence, under (3.1). When γ and/or δ are unknown, and u_t has parametric autocorrelation (such as following a VAR), then it seems that the Gaussian ML of all the unknowns is again $\sqrt{n}$-consistent and asymptotically normal, but with limit covariance matrix that is not block-diagonal, so that in particular the asymptotic variance of the estimate of ν differs from that when γ and δ are known. If $\delta < 1/2$, *a priori*, conveying the implication that δ and γ are both estimated by optimizing over subsets of the intersection of (3.2) and (3.4), then the consistency and asymptotic distribution theory would largely follow the lines of authors such as Fox and Taqqu (1986) and Hosoya (1997), who were the first to develop such theory for standard scalar and vector long memory time series models respectively, the most notable difference perhaps being the fact that in our setting x_t and y_t would be only asymptotically stationary. If the possibility that $\delta \geq 1/2$ is admitted, and possibly $\gamma \geq 1/2$ also, then the situation is more delicate, as discussed in Section 3.4.

The preceding discussion makes it apparent that when γ and δ are unknown the issue of how they are estimated is of greater significance when $\beta < 1/2$ than when $\beta > 1/2$. It is indeed essential here (due to the correlation between x_t and u_{1t}) that they be estimated $\sqrt{n}$-consistently in order for ν to then be estimated $\sqrt{n}$ -consistently, so that simple closed-form semiparametric methods such as log periodogram regression will not suffice. Closed-form $\sqrt{n}$-consistent estimates of integration orders are available (see Kashyap and Eom, 1988, Moulines and Soulier, 1999), but these do not cover our bivariate situation and VAR u_t, and also entail logging the periodogram, which raises technical difficulties not present in estimates based on quadratic forms, such as the ML. In our setting some degree of numerical optimization seems inevitable. Since this is likely to entail an initial search of the parameter space to locate the vicinity of a global optimum, it is desirable if the computations can be arranged so that only univariate optimizations are involved. Even after concentrating out parameters, when both γ and δ are unknown the Gaussian ML estimation requires a bivariate optimization under white noise u_t, and at least a trivariate optimization when u_t is VAR. We propose $\sqrt{n}$-consistent and asymptotically normal estimates that require only univariate optimizations.

The basic structure of the estimates of ν is described in the following section. Section 3.3 provides asymptotic theory in case γ and δ are known. Section 3.4 considers estimation of γ and δ and the effect on estimating ν. Section 3.5 contains Monte Carlo evidence of finite sample behaviour, and Section 3.6 several empirical applications.

3.2 Estimation of ν

Noting (2.4) in Chapter 2, we take u_t to be generated by the VAR

$$u_t = \sum_{j=1}^{p} B_j u_{t-j} + \varepsilon_t, \tag{3.6}$$

where all zeros of $\det\{I_2 - \sum_{j=1}^{p} B_j z^j\}$ lie outside the unit circle, the B_j being 2×2 matrices, while ε_t is a bivariate sequence, uncorrelated and homoskedastic over t, with mean zero and covariance matrix Ω. We take (3.6) to mean white noise u_t when $p = 0$.

From (2.4) and (3.6) we have

$$z_t(\gamma,\delta) - \sum_{j=1}^{p} B_j z_{t-j}(\gamma,\delta) = \nu\left\{\zeta x_t(\gamma) - \sum_{j=1}^{p} B_j \zeta x_{t-j}(\gamma)\right\} + \varepsilon_t^+, \quad t \geq 1, \tag{3.7}$$

where

$$\begin{aligned} \varepsilon_1^+ &= u_1, \\ \varepsilon_t^+ &= u_t - \sum_{j=1}^{t-1} B_j u_{t-j}, \quad t = 2, ..., p, \\ \varepsilon_t^+ &= \varepsilon_t, \quad t > p. \end{aligned} \tag{3.8}$$

Denote by B_{ij} the ith row of B_j. Writing ε_{it} for the ith element of ε_t, for $t > p$, the second equation of (3.7) can be written as

$$x_t(\delta) - \sum_{j=1}^{p} B_{2j} z_{t-j}(\gamma,\delta) = -\nu \sum_{j=1}^{p} B_{2j}\zeta x_{t-j}(\gamma) + \varepsilon_{2t}, \tag{3.9}$$

whence the first equation can be written as

$$y_t(\gamma) = \nu x_t(\gamma) + \varphi x_t(\delta) + \sum_{j=1}^{p} (B_{1j} - \varphi B_{2j})\, z_{t-j}(\gamma,\delta) - \nu \sum_{j=1}^{p} (B_{1j} - \varphi B_{2j})\, \zeta x_{t-j}(\gamma) + \varepsilon_{1.2,t}, \tag{3.10}$$

where $\varepsilon_{1.2,t} = \varepsilon_{1t} - \varphi\varepsilon_{2t}$, $\varphi = E(\varepsilon_{1t}\varepsilon_{2t})/E(\varepsilon_{2t}^2)$; (3.10) is a form of error-correction representation.

We wish to cater for the possibility of prior zero restrictions on the B_j which serve to eliminate some $y_{t-j}(\gamma)$, $x_{t-j}(\gamma)$, $x_{t-j}(\delta)$, as this will improve efficiency. Thus we introduce a $q \times (3p+2)$ matrix, which is I_{3p+2} when there are no such restrictions, but for $q < 3p+2$, Q is formed by dropping rows corresponding to the restrictions. Thus we can write (3.10) as

$$y_t(\gamma) = \vartheta' Q Z_t(\gamma,\delta) + \varepsilon_{1.2,t}, \tag{3.11}$$

where

$$Z_t(c,d) = \left(x_t(c), x_t(d), w'_{t-1}(c,d), ..., w'_{t-p}(c,d)\right)', \tag{3.12}$$

$$w_t(c,d) = (x_t(c), x_t(d), y_t(c))'. \tag{3.13}$$

Since $E(\varepsilon_{1.2,t} Z_t(\gamma,\delta)) = 0$, we consider the (possibly constrained) least squares estimate

$$\widehat{\vartheta}(c,d) = G(c,d)^{-1} g(c,d), \tag{3.14}$$

taking $(c,d) = (\gamma,\delta)$, $(\gamma,\widetilde{\delta})$, $(\widetilde{\gamma},\delta)$ or $(\widetilde{\gamma},\widetilde{\delta})$, depending on whether γ and/or δ are known or estimated by $\widetilde{\gamma},\widetilde{\delta}$, and

$$G(c,d) = Q\frac{1}{n}\sum_{t=p+1}^{n} Z_t(c,d)Z_t'(c,d)Q', \quad g(c,d) = Q\frac{1}{n}\sum_{t=p+1}^{n} Z_t(c,d)y_t(c). \tag{3.15}$$

For example, in case $p = 1$, if u_{1t} is white noise while u_{2t} is AR(1), then $q = 3$ and (3.10) becomes

$$y_t(\gamma) = \nu x_t(\gamma) + \varphi x_t(\delta) - \varphi B_{221} x_{t-1}(\delta) + \varepsilon_{1.2,t}, \tag{3.16}$$

where B_{22j} is the second element of B_{2j}. Notice that ν, φ and B_{221} are all identified in (3.16), but it is apparent from comparison of (3.10) with (3.11) that in general, while ν and φ are expected to be identified, only some elements of the B_j are. However, we are treating the B_j as nuisance parameters, indeed it is principally ν that is of interest, so we stress

$$\widehat{\nu}(c,d) = 1' G(c,d)^{-1} g(c,d), \tag{3.17}$$

where $1 = (1, 0, ..., 0)'$.

The representation (3.10) is of error-correction type and in case $p = 0$, $\widehat{\nu}(\gamma, \delta)$ actually provides the Gaussian ML estimate of ν, given knowledge of γ, δ but lack of knowledge of Ω. For $p \geq 1$, it is less efficient than the ML for this case, but still $n^{1/2}$-consistent and computationally considerably simpler. Notice that over-specification of p results in a further efficiency loss, but under-specification of p produces inconsistency. In moderate sample sizes, a modest choice of p, even $p = 1$, might thus be a wise precaution. On the other hand, one could also regard (3.6) as approximating a more general infinite AR process with nonparametric $I(0)$ autocorrelation.

3.3 Asymptotic theory with known γ, δ

The present section establishes the $n^{1/2}$-consistency and asymptotic normality of $\widehat{\vartheta}(\gamma, \delta)$, and hence of $\widehat{\nu}(\gamma, \delta)$. We assume in addition to the description of (3.6) that the ε_t are stationary and ergodic with finite fourth moment, satisfying also

$$E\left(\varepsilon_t \middle| \mathcal{F}_{t-1}\right) = 0, \quad E\left(\varepsilon_t \varepsilon_t' \middle| \mathcal{F}_{t-1}\right) = \Omega \tag{3.18}$$

almost surely, where $\mathcal{F}_t$ is the σ-field of events generated by ε_s, $s \leq t$, and also assume that conditional (on $\mathcal{F}_{t-1}$) third and fourth moments and cross-moments of elements of ε_t equal the corresponding unconditional moments. Thus, the ε_t essentially behave like an *iid* sequence up to 4th moments. Now, noting from (1.26) that

$$x_t(\gamma) = \sum_{j=0}^{t-1} a_j(\beta) u_{2,t-j}, \quad t > 0; \quad = 0, \quad t \leq 0, \tag{3.19}$$

define

$$\overline{x}_t(\gamma) = \sum_{j=\max(t,0)}^{\infty} a_j(\beta) u_{2,t-j}, \quad \widetilde{x}_t(\gamma) = x_t(\gamma) + \overline{x}_t(\gamma), \tag{3.20}$$

so that because of (3.1), $\widetilde{x}_t(\gamma)$, $t = 0, \pm 1, ...$, is a covariance stationary sequence. Likewise, so is

$$\widetilde{y}_t(\gamma) = \nu \widetilde{x}_t(\gamma) + u_{1t}, \tag{3.21}$$

as is u_{2t}. Now define

$$\widetilde{w}_t = (\widetilde{x}_t(\gamma), u_{2t}, \widetilde{y}_t(\gamma))', \quad \widetilde{Z}_t = (\widetilde{x}_t(\gamma), u_{2t}, \widetilde{w}_{t-1}', ..., \widetilde{w}_{t-p}')', \tag{3.22}$$

$$\Phi = E(\widetilde{Z}_t\widetilde{Z}_t'), \quad \Psi = E\left(\varepsilon_{1.2,t}^2\widetilde{Z}_t\widetilde{Z}_t'\right). \tag{3.23}$$

The proof of the following theorem is left to Appendix 3.A.

Theorem 3.1. *As $n \to \infty$*

$$n^{1/2}\left\{\widehat{\vartheta}(\gamma,\delta) - \vartheta\right\} \to_d N\left(0, (Q\Phi Q')^{-1}Q\Psi Q'(Q\Phi Q')^{-1}\right), \tag{3.24}$$

and the covariance matrix on the right hand side is consistently estimated by

$$G(\gamma,\delta)^{-1}K(\gamma,\delta)G(\gamma,\delta)^{-1}, \tag{3.25}$$

where

$$K(c,d) = Q\frac{1}{n}\sum_{t=p+1}^{n}\widehat{\varepsilon}_{1.2,t}^2(c,d)Z_t(c,d)Z_t'(c,d)Q', \tag{3.26}$$

in which

$$\widehat{\varepsilon}_{1.2,t}(c,d) = y_t(c) - \widehat{\vartheta}(c,d)'QZ_t(c,d). \tag{3.27}$$

Remark 3.1.1. For $p \geq 1$, $\widehat{\nu}(\gamma,\delta)$ is inefficient relative to the Gaussian ML. Overparameterization in the B_j results in further loss of efficiency in estimation of ν. Consider the case where, in the estimation, the B_j are taken to be diagonal, with also u_{1t} white noise and u_{2t} $AR(p)$, to extend (3.16). Then, if in fact u_{2t} is also white noise the limiting variance of $n^{1/2}\{\widehat{\nu}(\gamma,\delta) - \nu\}$ is

$$\omega_{1.2}^2/\left(\omega_2^2\sum_{j=p+1}^{\infty}a_j^2(\beta)\right), \tag{3.28}$$

where $\omega_{1.2}^2 = E(\varepsilon_{1.2,t}^2)$, $\omega_2^2 = E(\varepsilon_{2t}^2)$; (3.28) is increasing in p. As a simpler alternative to (3.26), (3.27), we can consistently estimate (3.28) by

$$\widehat{\omega}_{1.2}^2(\gamma,\delta)\left(1'G(\gamma,\delta)1\right)^{-1}, \tag{3.29}$$

where

$$\widehat{\omega}_{1.2}^2(\gamma,\delta) = \frac{1}{n}\sum_{t=p+1}^{n}\widehat{\varepsilon}_{1.2,t}^2(\gamma,\delta). \tag{3.30}$$

Note that (3.28) and (3.29) also apply in case $p = 0$ is correctly taken in the estimation, when $\widehat{\nu}(\gamma,\delta)$ is equivalent to the Gaussian ML, and (3.28) becomes

$$\omega_{1.2}^2/\left(\omega_2^2\left\{\frac{2^{-4\beta}}{\pi}B\left(1/2-\beta, 1/2-\beta\right) - 1\right\}\right). \tag{3.31}$$

Note also that (3.28) and (3.31) do not depend on fourth cumulants of ε_t. However, if in fact u_t is not white noise, the limiting variance of $n^{1/2}\{\widehat{\nu}(\gamma,\delta) - \nu\}$, namely

$$1'(Q\Phi Q')^{-1}Q\Psi Q'(Q\Phi Q')^{-1}1, \tag{3.32}$$

(see (3.24)), in general depends on the fourth cumulant of $\varepsilon_{1.2,t}$, $\varepsilon_{1.2,t}$, ε_{2t} and ε_{2t}, though of course this is zero under Gaussianity.

Remark 3.1.2. On the other hand, under-parameterization of the B_j produces inconsistency of $\widehat{\nu}(\gamma, \delta)$, as when u_t is actually $AR(p+1)$. In this connection, note that in Chapter 2 we considered the Gaussian ML for $\beta > 1/2$ in case of a far more general parametric class than (3.6). We can view (3.6) more informally, as approximating an actual, unknown, time series model in the hope that bias is decreasing in p, a statement which can likely be justified in a rigorous way by allowing p to increase slowly with n. Our AR approach is computationally convenient, and is in a long tradition of macroeconometric estimation of linear simultaneous equations systems, as well as relating to Johansen's (1991) approach to $CI(1,1)$ co-integration. In case of ARMA models, over-parameterization of both AR and MA orders can have more serious consequences than those discussed in Remark 3.1.1.

Remark 3.1.3. So long as $p \geq 1$ and some B_j are non-diagonal, the endogeneity property (3.3) holds even when Ω is diagonal, i.e. $\varphi = 0$.

3.4 The case of unknown γ, δ

The main practical interest in fractional co-integration centres on the realistic situation in which γ and/or δ are unknown. We shall focus on the case where both γ and δ are unknown, as being the most difficult both computationally and theoretically.

First, suppose that u_t is correctly taken to be white noise, with unknown covariance matrix Ω satisfying (3.3). Considering the Gaussian log-likelihood, both Ω and ν can be concentrated out to leave an objective function of γ and δ. The resulting estimates of γ and δ can then be plugged into (3.17). As mentioned in Section 3.1, asymptotic theory under $\delta < 1/2$ is a relatively standard extension of that for Gaussian estimates in such models as stationary fractional ARIMAs. For fractional ARIMAs whose integration order is allowed to take nonstationary values, there has been difficulty with the consistency proof (an essential preliminary to limit distribution theory, because estimates are only implicitly defined). This is especially due to lack of uniformity of convergence of the objective function around admissible values 0.5 less than the true value of the integration order, as discussed by Velasco and Robinson (2000), who by means of tapering, and a different definition of fractional nonstationarity from ours, established $\sqrt{n}$-consistent and asymptotically normal frequency-domain estimation of integration orders and other parameters in quite general univariate models, while allowing the admissible set to be arbitrarily large. Tapering, however, inflates the variance, while time domain estimates conveniently exploit the simple white noise or VAR structure of u_t, and seem natural for our definition of nonstationarity, and are certainly justifiable if δ and γ are known to lie in intervals of length no greater than $1/2$, for example $(0, 1/2)$ or $(1/2, 1]$.

We propose estimates of γ, δ and ν that are $\sqrt{n}$-consistent and asymptotically normal and require two univariate nonlinear optimizations, in place of one bivariate one. Our procedure extends nicely to the VAR u_t case, where after cancelling out

Ω and the B_j, the Gaussian ML is a trivariate function; note that ν and the B_j are involved bilinearly as well as linearly in (3.7).

Pursuing the case of white noise u_t, i.e. $p = 0$ in (3.6), we can write (1.26) as

$$x_t(\delta) = \varepsilon_{2t}, \quad t \geq 1. \tag{3.33}$$

It is proposed to estimate δ by

$$\widetilde{\delta}_0 = \arg\min_{d \in \mathcal{D}} S_0(d), \tag{3.34}$$

for a compact set $\mathcal{D}$ and

$$S_0(d) = \sum_{t=1}^{n} x_t^2(d). \tag{3.35}$$

Then, we estimate γ by

$$\widetilde{\gamma}_0 = \arg\min_{c \in \mathcal{C}} T_0(c), \tag{3.36}$$

for a compact set $\mathcal{C}$ (presumably a subset of $[0, \widetilde{\delta}]$) and

$$T_0(c) = \sum_{t=1}^{n} \left\{ y_t(c) - \widehat{\nu}(c, \widetilde{\delta}_0) x_t(c) - \widehat{\varphi}(c, \widetilde{\delta}_0) x_t(\widetilde{\delta}_0) \right\}^2, \tag{3.37}$$

where $\widehat{\nu}(c, d)$ is given by (3.17), taking $p = 0$, and $\widehat{\varphi}(c, d)$ is the second element of $\widehat{\vartheta}(c, d)$ in this case. Notice that the presence of c as argument in $y_t(c)$, and indeed of d in $x_t(d)$ of (3.35), presents no barrier to consistent estimation because, for example, $y_t(c)$ involves c only in the coefficients of lagged values $y_{t-1}, y_{t-2}, ...$, not y_t.

In case of VAR u_t, we develop further the triangular structure of (1.25), (1.26) by assuming

$$B_j \text{ is upper-triangular}, \ j = 1, ..., p. \tag{3.38}$$

This corresponds to a kind of causal structure, with y_t formed from $y_{t-1}, y_{t-2}, ...$ and $x_t, x_{t-1}, ...$, but x_t being determined by

$$x_t(\delta) - \phi' R X_t(\delta) = \varepsilon_{2t}, \tag{3.39}$$

with

$$X_t(d) = (x_{t-1}(d), ..., x_{t-p}(d))', \tag{3.40}$$

and R an $r \times p$ matrix with $R = I_p$ in case $r = p$ but for $r < p$, R is formed by dropping specified rows from I_p, in case $B_{22j} = 0$ for some j. The prescription (3.39) includes the case of diagonal B_j, does not seem an excessive requirement given the allowance for non-diagonal Ω, and introduces an element of parsimony.

Define

$$\widehat{\phi}(d) = H(d)^{-1} h(d), \tag{3.41}$$

where

$$H(d) = R \frac{1}{n} \sum_{t=p+1}^{n} X_t(d) X_t'(d) R', \quad h(d) = R \frac{1}{n} \sum_{t=p+1}^{n} X_t(d) x_t(d). \tag{3.42}$$

First, estimate δ by

$$\widetilde{\delta}_p = \arg\min_{d \in \mathcal{D}} S_p(d), \tag{3.43}$$

where

$$S_p(d) = \sum_{t=p+1}^{n} \left\{ x_t(d) - \widehat{\phi}(d)' R X_t(d) \right\}^2. \tag{3.44}$$

Then, estimate γ by

$$\widetilde{\gamma}_p = \arg\min_{c \in \mathcal{C}} T_p(c), \tag{3.45}$$

where

$$T_p(c) = \sum_{t=p+1}^{n} \left\{ y_t(c) - \widehat{\vartheta}(c, \widetilde{\delta}_p)' Q Z_t(c, \widetilde{\delta}_p) \right\}^2. \tag{3.46}$$

As abbreviating notation, we denote throughout, for any $p \geq 0$, $\widetilde{\delta} = \widetilde{\delta}_p$, $\widetilde{\gamma} = \widetilde{\gamma}_p$. In the following theorem, we assume $\gamma \in \mathcal{C}$, $\delta \in \mathcal{D}$ and take the supports of $\mathcal{C}$ and $\mathcal{D}$ to be of width less than 0.5 to avoid a difficulty described earlier in this section. The proof is omitted as it is extremely complicated and lengthy, while not entailing any novel difficulty.

Theorem 3.2. *As $n \to \infty$*

$$n^{1/2} \begin{bmatrix} \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta}) - \nu \\ \widetilde{\gamma} - \gamma \\ \widetilde{\delta} - \delta \end{bmatrix} \to_d N(0, ABA'), \tag{3.47}$$

where A is a $3 \times (q+2)$ matrix and B is a $(q+2) \times (q+2)$ matrix, for which consistent estimates $\widehat{A}$ and $\widehat{B}$ are presented in Appendix 3.B.

Remark 3.2.1. Analytic formulae, in either the time or frequency domain, for A and B are excessively complicated, and thus omitted. Note that the estimate $\widehat{A}\widehat{B}\widehat{A}'$ provided by Appendix 3.B is guaranteed non-negative definite.

Remark 3.2.2. As well as being useful in inference on ν, the theorem could also be applied in inference on γ and δ, for example to set a confidence interval for β which could be useful in judging the suitability of the weak co-integration specification (3.1).

Remark 3.2.3. On the other hand, our estimation procedure, though not our asymptotic theory, can also be used when $\beta > 1/2$, though alternative, possibly computationally more convenient, methods, are available here.

Remark 3.2.4. One approach, suggested in Chapter 2 when $\beta > 1/2$, is the use of residuals from OLS or NBLS estimates of ν in the estimation of γ. However, these are always less-than-$n^{1/2}$-consistent under (3.1), and so it appears that the resulting estimates of γ will not achieve the essential $n^{1/2}$-consistency needed to provide an $n^{1/2}$-consistent estimate of ν.

Remark 3.2.5. Even when u_t is white noise, $\widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta})$, $\widetilde{\delta}$ and $\widetilde{\gamma}$ are inefficient relative to the Gaussian ML; intuitively, this is due to the estimation of δ from only the second equation of system (1.25), (1.26) (i.e. (3.34)), whereas the first equation also contains relative information. However, the estimates can be updated to efficiency by a single Newton step.

3.5 Monte Carlo evidence

With the main aim of investigating the performance in finite samples of the estimates of ν proposed in this chapter and associated rules of inference, and making comparisons with the simplest estimate, the OLS, a Monte Carlo experiment was carried out. In data generation from (1.25), (1.26), (3.6), we took $p = 1$ throughout, with

$$B_1 = diag\left\{b_1, b_2\right\}, \tag{3.48}$$

where each of the b_i was allowed to take each of the values 0, 0.5, 0.9. The case $b_1 = b_2 = 0$ actually corresponds to $p = 0$ in (3.6), where u_t is a white noise vector. Likewise, $b_1 = 0$, $b_2 \neq 0$ corresponds to (3.16). We have employed in (3.48) abbreviating notation compared to (3.16), so $b_2 = B_{221}$. The ε_t in (3.6) were generated as Gaussian with $E(\varepsilon_{1t}^2) = E(\varepsilon_{2t}^2) = 1$ and $E(\varepsilon_{1t}\varepsilon_{2t}) = \rho$, taking values -0.5, 0, 0.5, 0.75, via the g05ezf routine of the Fortran NAG library. We varied ρ in order to assess possible "simultaneous equation bias", x_t and u_{1t} being orthogonal only when $\rho = 0$. We employed four (γ, δ) combinations:

$$(\gamma, \delta) = (0, 0.4), \ (0.2, 0.4), \ (0.4, 0.8), \ (0.7, 1), \tag{3.49}$$

for all of which $\beta < 1/2$. Notice that variances of all estimates, both in finite samples and asymptotically, will inevitably vary across parameter values. For example, because the $E(\varepsilon_{it}^2)$ are fixed throughout, $E(\varepsilon_{1.2,t}^2)$ will decrease in $|\rho|$, while $E(u_{it}^2)$ will increase in b_i. Finite sample biases of our estimates will doubtless also be affected by such variation, though in a more subtle manner. We took $\nu = 1$.

For each combination of parameter values, 1000 series of $\{y_t, x_t\}$ of lengths $n = 64, 128, 256$ were generated. Fractional series were generated as in (3.19), using $a_0(\alpha) = 1$, $a_{j+1}(\alpha) = ((j+\alpha)/(j+1))a_j(\alpha)$, $j \geq 1$, for $\alpha > 0$. For each series, we computed estimates of the following three types:

(i) The OLS, given in (1.33).

(ii) The Infeasible estimate $\overline{\nu}_I = \widehat{\nu}(\gamma, \delta)$ based on correct specification and misspecification and/or over-specification.

(iii) The Feasible estimate $\overline{\nu}_F = \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta})$ based on correct specification and misspecification and/or over-specification.

By "correct specification" we mean that all prior zero restrictions on B_1 in (3.48), including the non-diagonal ones and any diagonal ones, are incorporated in the estimation, but not equality restrictions. By "mis-specification" we mean that for $b_1 \neq 0$ and $b_2 \neq 0$ we took $Z_t(c, d) = (x_t(c), x_t(d))'$. By "over-specification" we mean that for $b_1 = b_2 = 0$ we took $Z_t(c, d) = \left(x_t(c), x_t(d), w_{t-1}'(c, d)\right)'$. Of course, knowledge of $\rho = 0$ was never used. Table 3.1 records the convergence rates of the OLS and, under the heading "optimal", of $\overline{\nu}_I$, $\overline{\nu}_F$.

TABLE 3.1
CONVERGENCE RATES:
OLS WITH $\rho \neq 0$, $\rho = 0$ AND OPTIMAL RATES

(γ,δ)	$(0,0.4)$	$(0.2,0.4)$	$(0.4,0.8)$	$(0.7,1)$
Optimal	$n^{.5}$	$n^{.5}$	$n^{.5}$	$n^{.5}$
OLS, $\rho \neq 0$	inconsistent	inconsistent	$n^{.4}$	$n^{.3}$
OLS, $\rho = 0$	$n^{.5}$	$n^{.5}$	$n^{.4}$	$n^{.3}$

We describe how $\widetilde{\delta}$ and $\widetilde{\gamma}$ in $\overline{\nu}_F$ were computed. In estimating δ, we fixed $\mathcal{D}=[\widehat{\delta}-0.15,\widehat{\delta}+0.15]$ in (3.43), where $\widehat{\delta}$ is the version of the log periodogram estimate of Geweke and Porter-Hudak (1983) proposed by Robinson (1995a), applied to the series x_t without pooling or trimming, based on bandwidths $m = 20, 30, 60$, corresponding to $n = 64, 128, 256$, respectively in case u_{2t} is assumed in the estimation to be white noise, and on $m = 10, 15, 30$, corresponding to $n = 64, 128, 256$, in case u_{2t} is assumed in the estimation to be AR(1). In all cases, $\mathcal{D}$ contains the asymptotic 95% confidence interval $[\widehat{\delta}-1.96s.e.(\widehat{\delta}),\widehat{\delta}+1.96s.e.(\widehat{\delta})]$, where $s.e.(\widehat{\delta}) = \pi/\sqrt{24m}$ is the asymptotic standard error of $\widehat{\delta}$ (Robinson, 1995a). In estimating γ, we fixed $\mathcal{C}=[\widetilde{\delta}-0.50,\widetilde{\delta}-0.05]$ in (3.45). The lower bound corresponds to the assumption $\beta < 1/2$. The upper bound seems reasonable since a very small (less than 0.05) β is unlikely to be detectable, indeed there is then near loss of identificability and very poor behaviour of estimates of ν.

Tables 3.6-3.15 report Monte Carlo bias (defined as the estimate minus the true value) of $\overline{\nu}_O$, $\overline{\nu}_I$ and $\overline{\nu}_F$, each table referring to a particular (b_1, b_2) combination with either correct specification, mis-specification or over-specification. Generally, $\overline{\nu}_I$ performs best, followed by $\overline{\nu}_F$, with $\overline{\nu}_O$ worst, being these estimates no worse than any of the others in 387, 65 and 51 out of 480 cases (considering all ρ, n, (γ,δ), b_1, b_2 combinations) respectively.

We discuss first the cases of correct specification (Tables 3.6-3.12). The overall ordering is found in the full white noise case $b_1 = b_2 = 0$ (Table 3.6), and in the AR case (Tables 3.7-3.12) when $\rho \neq 0$, but not when $\rho = 0$ with $b_1 = b_2 \neq 0$, where $\overline{\nu}_O$ is best. For $b_1 = b_2 = 0.9$, $(\gamma,\delta) = (0.7,1)$ and small n, $\overline{\nu}_O$ usually beats $\overline{\nu}_F$ even when $\rho \neq 0$ (Table 3.8), but this effect does not occur for the same case when $b_1 = b_2 = 0.5$ (Table 3.7). For $b_1 = 0$, $b_2 \neq 0$ (Tables 3.9, 3.10), we are close to the white noise outcome when $b_2 = 0.5$, but for $b_2 = 0.9$ (Table 3.10), $\overline{\nu}_O$ improves relatively to the other estimates, and although it is still generally worst, its performance is relatively close to the one of $\overline{\nu}_F$. When $b_1 \neq 0$, $b_2 = 0$ the bias of $\overline{\nu}_O$ decays very slowly, and is unacceptably large when $b_1 = 0.9$ (Table 3.12). In any case, out of the 336 cases reported with correct specification, $\overline{\nu}_F$ beats $\overline{\nu}_O$ with relation 275/54 (see Chapter 2 for description of this concept). Focusing now more on variation across (γ,δ), the bias of $\overline{\nu}_I$ decreases in β, as is the case for $\overline{\nu}_F$ when $b_1 = b_2 = 0$. With AR structure, the worst performance of $\overline{\nu}_F$ is generally found for $(\gamma,\delta) = (0.2,0.4)$ or $(0.7,1)$. As for $\overline{\nu}_O$, bias varies with collective memory $\gamma+\delta$ when $\rho = 0$, but when $\rho \neq 0$, $(0,0.4)$ and $(0.2,0.4)$ are the worst cases, unsurprisingly in view of the OLS's inconsistency here. Generally, $\overline{\nu}_F$ works best under $(0.4,0.8)$. With respect to variation in ρ, overall, the bias shares the sign of ρ in case of $\overline{\nu}_O$, $\overline{\nu}_I$, but is opposite in case of

$\overline{\nu}_F$. $\overline{\nu}_I$ is relatively insensitive to ρ, though for $b_1 = 0.9$, $b_2 = 0$ (Table 3.12), bias increases in $|\rho|$, as is the case for $\overline{\nu}_O$, but no clear pattern can be found in the results for $\overline{\nu}_F$, though there is evidence of increase in bias with $|\rho|$. Looking at variation across (b_1, b_2), AR structure tends to reduce bias in $\overline{\nu}_O$ but increase it, and possibly change its sign, in $\overline{\nu}_I$. For $\overline{\nu}_F$, the worst performances occur when $b_1 \neq 0$, but even here bias decays rapidly as n increases, as it does also for $\overline{\nu}_I$.

Mis-specification (Tables 3.13, 3.14) has surprisingly little effect on $\overline{\nu}_I$, but seriously damages $\overline{\nu}_F$, especially when β is small, $(0.7, 1)$ being clearly the worst case, though when $\beta = 0.4$ and $b_1 = b_2 = 0.5$, bias decreases substantially with n. As expected, now $\overline{\nu}_O$ clearly dominates $\overline{\nu}_F$ with relations 27/18, 37/9 for $b_1 = b_2 = 0.5, 0.9$ respectively, out of 48 cases for each of the relations. As anticipated, over-specification (Table 3.15) makes little difference to $\overline{\nu}_I$, $\overline{\nu}_F$, which do much better than $\overline{\nu}_O$ (out of 48 cases, $\overline{\nu}_F$ beats $\overline{\nu}_O$ with relation 40/4).

Tables 3.16-3.25 contain Monte Carlo standard deviations. As noted before, variability is considerably affected by parameter values. In fact, $\overline{\nu}_O$ was superior to $\overline{\nu}_I$ for most of the combinations, with $\overline{\nu}_F$ a poor third, being these estimates no worse than any of the others 377, 115 and 0 times, out of 480 cases, respectively. With correct specification, this was most notably the case for small n and $b_1 = b_2 \neq 0$ (Tables 3.17, 3.18), in part due to the proliferation in regressors, five in $\overline{\nu}_I$ and $\overline{\nu}_F$ versus one in $\overline{\nu}_O$, with variability in $\widetilde{\delta}$ and $\widetilde{\gamma}$ considerably inflating standard deviations of $\overline{\nu}_F$ relatively to those of $\overline{\nu}_I$. Precision also increases with increasing n, and when one or both of the b_i is zero (see Tables 3.16, 3.19-3.22), the performance of $\overline{\nu}_I$ and $\overline{\nu}_F$ improves relative to that of $\overline{\nu}_O$. In fact, for the $b_1 = 0.9$, $b_2 = 0$ situation, $\overline{\nu}_I$ clearly beats $\overline{\nu}_O$ (with relation 37/11 out of 48 cases), while under the same AR structure, $\overline{\nu}_F$ also dominates $\overline{\nu}_O$ for cases $(\gamma, \delta) = (0.4, 0.8)$, $(0.7, 1)$ when $n = 256$, and $(\gamma, \delta) = (0.7, 1)$ when $n = 128$.

Mis-specification (Tables 3.23, 3.24) improves matters with respect to correct specification, especially when n is small, but the decrease in value of the standard deviation is quite slow, mainly for the case $(\gamma, \delta) = (0.7, 1)$. On the other hand, with over-specification (Table 3.25), $\overline{\nu}_I$ and $\overline{\nu}_F$ unsurprisingly deteriorate further, and generally larger sample sizes will be required in order for their faster convergence rate to consistently deliver smaller standard deviations than $\overline{\nu}_O$. Nevertheless, it must be borne in mind that this chapter's motivation is not to minimise variance but rather to achieve $n^{1/2}$-consistency and asymptotic normality in a fairly general context, which the OLS $\overline{\nu}_O$ does not provide.

We now go in to examine the usefulness of these limit distributional properties of $\overline{\nu}_I$ and $\overline{\nu}_F$ in finite-sample statistical inference, by examining the size of Wald tests. We computed

$$W_I = \frac{(\overline{\nu}_I - \nu)^2 n}{[G(\gamma, \delta)^{-1} K(\gamma, \delta) G(\gamma, \delta)^{-1}]_{(1)}}, \; W_F = \frac{(\overline{\nu}_F - \nu)^2 n}{\left[\widehat{A}\widehat{B}\widehat{A}'\right]_{(1)}}, \tag{3.50}$$

where $[\cdot]_{(i)}$ denotes ith diagonal element. Empirical sizes, with respect to nominal sizes $\alpha = 0.05$ and 0.1, again across 1000 replications, are reported in Tables 3.26-3.35, for each of the (b_1, b_2) for which biases and standard deviations were given.

With correct specification, even for $b_1 = b_2 = 0$ (Table 3.26), sizes of the infeasible statistic W_I are somewhat too large, and autocorrelation in u_t exarcebates this, with the case $b_1 \neq 0$, $b_2 = 0$ again worse than $b_1 = 0$, $b_2 \neq 0$, but not necessarily worse than $b_1 = b_2 \neq 0$ (Tables 3.27-3.32). Results for $\alpha = 0.1$ are clearly better than for $\alpha = 0.05$. Overall, there is improvement as n increases, and even for small n, the performance of W_I seems quite satisfactory. Predictably, mis-specification (Tables 3.33, 3.34) plays havoc, producing sizes that are unacceptably high, especially for $\alpha = 0.05$ and $b_1 = b_2 = 0.9$. With over-specification, performance is again good, though we would not expect high power.

For the feasible statistic W_F, with correct specification and no autocorrelation in u_t (Table 3.26), sizes are worse than for W_I, with less evidence of settling down as n increases and varying more across parameter values, sometimes actually being less than the nominal values. Indeed, with autocorrelation (Tables 3.27-3.32), sizes are emphatically too small and mostly further from the nominal values than the corresponding W_I are in the opposite direction, though this is by no means always the case, and for $n = 64$ and $\alpha = 0.05$ the results are extraordinarily good. However, we would not wish to draw over-optimistic general conclusions here, and certainly not from Tables 3.33, 3.34, where the mis-specification so evident in the results for W_I can barely be seen in those for W_F, superiority of W_F being even more dramatic when $b_1 = b_2 = 0.9$. With over-specification (Table 3.35), W_F mostly beats W_I, especially when $\alpha = 0.05$. It is possible that the performance of W_F relative to W_I is not accidental because W_I has an asymptotic formula in the denominator. Certainly, our overall experience with W_F is quite encouraging.

While we have stressed estimation of ν, estimates of δ and γ would also be of interest in any empirical analysis of fractional co-integration, and so we also give some space to the performance of $\widetilde{\delta}$ and $\widetilde{\gamma}$, and of Wald tests for δ and γ based on Theorem 3.2.

Tables 3.36 and 3.37 report Monte Carlo bias and standard deviation for $\widetilde{\delta}$ for the same values of δ (0.4, 0.8, 1), b_2 (0, 0.5, 0.9) and n (64, 128, 256) as before, again based on 1000 replications. However, we fix $\rho = 0.5$ here, using the same estimates of $\widetilde{\delta}$ computed in this case for the feasible estimates $\overline{\nu}_F$ and Wald statistics W_F discussed previously. We report results for minimization of both $S_0(d)$ and $S_1(d)$ (see (3.35), (3.44)), so that $S_0(d)$ with $b_2 = 0$ and $S_1(d)$ with $b_2 \neq 0$ both correspond to correct specification, $S_1(d)$ with $b_2 = 0$ to over-specification, and $S_0(d)$ with $b_2 \neq 0$ to mis-specification.

Biases based on $S_0(d)$ and $S_1(d)$ with $b_2 = 0$ increase somewhat with δ, but look satisfactory even for $n = 64$, and are decreasing in n. For $S_1(d)$ with $b_2 = 0.5$, there is some deterioration, but nevertheless performance is still acceptable, but for $b_2 = 0.9$, the results are very poor, even for $n = 256$, though this is not too surprising in view of the difficulties often caused by a near-unit root. Unsurprisingly, there is severe bias, increasing with b_2, when $S_0(d)$ is used with $b_2 \neq 0$. Standard deviations in the correctly specified and over-specified cases are pretty stable over δ, but, as expected, worse in the latter case.

Tables 3.38 and 3.39 report Monte Carlo sizes of Wald statistics for δ

$$W_\delta = \frac{(\widetilde{\delta} - \delta)^2 n}{\left[\widehat{A}\widehat{B}\widehat{A}'\right]_{(3)}}, \tag{3.51}$$

based on Theorem 3.2, with respect to the nominal sizes $\alpha = 0.05$, 0.1 respectively. As expected, under mis-specification they are far too large, and this is also the case using $S_1(d)$ with $b_2 = 0.9$. Otherwise, while still too large, they are not bad, and decrease in n, the ones for $\alpha = 0.1$ being best.

Tables 3.40-3.43 give corresponding results for $\widetilde{\gamma}$, with $b_1 = b_2 = b$ taking values 0, 0.5, 0.9, and for the four (γ, δ) combinations considered previously, the results (not reported) for the cases corresponding to ρ = 0, -0.5, 0.75 being very similar to the ones for $\rho = 0.5$. Our estimation procedure being sequential, we consider two categories, $S_0(d)$ followed by $T_0(c)$ (see (3.37)), and $S_1(d)$ followed by $T_1(c)$ (see (3.46)), so that in the former case there is correct specification for $b = 0$ and mis-specification for $b \neq 0$, and in the latter, over-specification for $b = 0$ and correct specification for $b \neq 0$. The bias and standard deviation results of Tables 3.40 and 3.41 exhibit somewhat some variation across (γ, δ), but otherwise the qualitative conclusions for $\widetilde{\delta}$ still apply. With the Wald statistic

$$W_\gamma = \frac{(\widetilde{\gamma} - \gamma)^2 n}{\left[\widehat{A}\widehat{B}\widehat{A}'\right]_{(2)}}, \tag{3.52}$$

more variation in sizes is also found, in Tables 3.42 and 3.43, than for W_δ, some of the sizes being smaller than the nominal ones.

3.6 Empirical examples

Using a methodology involving the OLS and NBLS of ν, and semiparametric estimates of ν, Robinson and Marinucci (1998) found evidence that $\beta < 1/2$ in some of the bivariate macroeconomic series originally examined by Engle and Granger (1987), Campbell and Shiller (1987), who were investigating only the possibility of $CI(1,1)$ co-integration. This experience motivates application of our present approach to the same data. The main departure from the methodology of the previous section was an attempt at greater realism by determining p in (3.6) from the data, rather than assuming its value *a priori*. For this purpose, we need proxies for the u_{it}, which can only be obtained by operating on the observed y_t, x_t, series with preliminary estimates of ν, γ and δ. To estimate ν here we used the OLS $\overline{\nu}_O$, given by (1.33) (and computed by Robinson and Marinucci, 1998). To estimate γ and δ, we used semiparametric estimates (already computed by Robinson and Marinucci, 1998, Marinucci and Robinson, 2001) in order to provide robustness against a range of short-memory specifications for u_t. Specifically, the estimates of γ and δ computed by these authors were of log periodogram (LP) and semiparametric Gaussian (SG) type (of the precise form considered by Robinson 1995a,b), using various bandwidths and based either on raw data/residuals or on first differenced ones followed by adding back 1. For asymptotic theory under stationarity we appeal to Robinson (1995a,b), and under nonstationarity, to Velasco (1999a,b). For preliminary estimates of γ, δ, ν, sample correlograms and partial correlograms were computed (to lag length 36) in order to identify, in the spirit of Box and Jenkins (1971), the AR orders of the u_{it}. For each data set, this was done for both the smallest and largest of the various univariate estimates based on the series x_t/residuals provided by Robinson and Marinucci (1998), Marinucci and Robinson (2001), and implications of both provided when the results could not be reconciled, recognizing the imprecision in semiparametric estimation. As in Chapter 2, to check for stability with respect to the truncation phenomenon, we report computations based on the last $n' = n - j$ observations, for $j = 0, 1, ..., 10$.

We look first at Engle and Granger's (1987) quarterly consumption and income data, 1947Q1-1981Q2 ($n = 138$). They found evidence of $CI\,(1,1)$ co-integration, but did not investigate fractional possibilities. Marinucci and Robinson's (2001) analysis tends to support the notion of $\delta = 1$, but not of $\gamma = 0$, with positive estimates of γ that sometimes fall in the nonstationary region, thereby hinting that $\beta < 1/2$ is possible.

Taking y=consumption, x=income, the OLS of ν, from Robinson and Marinucci (1998), is 0.229. The two preliminary estimates of δ taken from Marinucci and Robinson (2001) were 0.89 (LP applied to first differences of x and adding back 1, with bandwidth 22) and 1.08 (SG applied to first differences of x and adding back 1, with bandwidth 40). In each case, the corresponding correlograms and partial correlograms suggested modelling u_{2t} as white noise. The preliminary estimates of γ were 0.19 (LP applied to raw residuals with bandwidth 22) and 0.87 (SG applied to first differenced residuals and adding back 1, with bandwidth 40). This large gap results in identification of an AR(1) u_{1t} in the first case, and white noise u_{1t} in the second. In view of these investigations, we carried out two distinct co-integration analyses, one with $p = 0$ in (3.6), the other with $p = 1$ in (3.6) with $B_1 = diag\,(b_1, 0)$.

In case u_{1t} and u_{2t} are both white noise, Table 3.2 reports values of the following statistics with n replaced by $n' = n - j$, $j = 0, ..., 10$: $\widehat{\nu} = \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta})$, $\widetilde{\delta}$, $\widetilde{\gamma}$, and their estimated standard errors $SE(\widehat{\nu})$, $SE(\widetilde{\delta})$, $SE(\widetilde{\gamma})$ from Theorem 3.2, $\widehat{\varphi} = \widehat{\varphi}(\widetilde{\gamma}, \widetilde{\delta})$, which is the estimated coefficient of $x_t(\widetilde{\delta})$ in (3.10) for $p = 0$ with $\widetilde{\gamma}$, $\widetilde{\delta}$, replacing γ, δ, and the correlation $Corr\,(\varepsilon_{1t}, \varepsilon_{2t})$ is estimated by

$$r = \widehat{\varphi}(\widetilde{\gamma}, \widetilde{\delta})(\widehat{\omega}_{22}/\widehat{\omega}_{11})^{\frac{1}{2}}, \tag{3.53}$$

where

$$\widehat{\omega}_{11} = n^{-1} \sum\nolimits_t' \left(y_t(\widetilde{\gamma}) - \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta}) x_t(\widetilde{\gamma}) \right)^2, \; \widehat{\omega}_{22} = n^{-1} \sum\nolimits_t' x_t^2(\widetilde{\delta}), \tag{3.54}$$

with $\sum_t'$ meaning summation over the last n' observations.

As n' falls, $\widehat{\nu}$ and $\widetilde{\delta}$ tend to increase, and $\widetilde{\gamma}$ to decrease, but there is high stability for $n' \leq 133$, and generally the changes are insignificant relative to standard errors, $\widehat{\nu}$ for $n' = 128$ being one standard error larger than $\widehat{\nu}$ for $n' = 138$ (and also somewhat larger than the OLS). The estimates of δ and γ are certainly consistent with $\beta < 1/2$. More especially, exploiting the standard errors provided by our approach, the hypothesis that $\delta = 1$ seems rejectable against $\delta > 1$, but (though we do not report standard errors of $\widetilde{\beta} = \widetilde{\delta} - \widetilde{\gamma}$, which could be computed using Theorem 3.2) there is no evidence against $\beta < 1/2$. Substantial negative contemporaneous correlation between u_{1t} and u_{2t} is suggested. Note that dropping the first observation does not affect $\widetilde{\delta}$, as $x_1(d) = x_1$ for any d.

TABLE 3.2

Consumption and Income: u_t white noise

n'	138	137	136	135	134	133	132	131	130	129	128
$\widehat{\nu}$	.223	.222	.251	.252	.251	.248	.247	.242	.243	.245	.246
$SE(\widehat{\nu})$	.027	.031	.024	.022	.023	.022	.023	.021	.022	.023	.023
$\widetilde{\delta}$	1.07	1.07	1.09	1.15	1.15	1.17	1.18	1.18	1.18	1.18	1.18
$SE(\widetilde{\delta})$	.028	.028	.059	.068	.073	.080	.083	.082	.083	.082	.084
$\widetilde{\gamma}$	.714	.745	.715	.692	.694	.696	.696	.685	.692	.694	.693
$SE(\widetilde{\gamma})$	.084	.092	.087	.087	.089	.090	.090	.089	.093	.093	.093
$\widehat{\varphi}$	-.024	-.055	-.085	-.090	-.090	-.086	-.085	-.072	-.073	-.073	-.074
r	-.195	-.189	-.297	-.311	-.310	-.294	-.285	-.247	-.251	-.250	-.253

The analysis with u_{1t} AR(1) in Table 3.3 presents a very different picture. Here, we also report $\widehat{b}_1$ and $\widehat{\nu b_1}$, which are the estimated coefficients of $y_{t-1}(\widetilde{\gamma})$ and $-x_{t-1}(\widetilde{\gamma})$ in the regression (cf. (3.10)) used to compute $\widehat{\nu}$ and $\widehat{\varphi}$, and $\widehat{\omega}_{11}$ in r is now the sample average of the squared residuals from the regression of $y_t(\widetilde{\gamma}) - \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta})x_t(\widetilde{\gamma})$ on $y_{t-1}(\widetilde{\gamma}) - \widehat{\nu}(\widetilde{\gamma}, \widetilde{\delta})x_{t-1}(\widetilde{\gamma})$.

TABLE 3.3

Consumption and Income: u_{1t} AR(1), u_{2t} white noise

n'	137	136	135	134	133	132	131	130	129	128	127
$\widehat{\nu}$	.163	.257	.264	.267	.263	.265	.258	.261	.262	.263	.262
$SE(\widehat{\nu})$	.179	.055	.054	.057	.053	.056	.051	.056	.055	.055	.054
$\widetilde{\delta}$	1.07	1.09	1.15	1.15	1.17	1.18	1.18	1.18	1.18	1.18	1.18
$SE(\widetilde{\delta})$	.028	.059	.068	.073	.080	.083	.082	.083	.082	.084	.084
$\widetilde{\gamma}$	-.101	-.167	-.183	-.184	-.184	-.179	-.193	-.180	-.184	-.189	-.186
$SE(\widetilde{\gamma})$	.234	.187	.181	.183	.185	.193	.180	.193	.192	.191	.192
$\widehat{b}_1$	.798	.843	.842	.839	.837	.832	.845	.842	.842	.842	.843
$\widehat{\nu b_1}$	.116	.221	.228	.230	.226	.226	.223	.225	.226	.227	.226
$\widehat{\varphi}$	.009	-.088	-.102	-.104	-.102	-.105	-.093	-.096	-.094	-.095	-.094
r	.009	-.128	-.122	-.119	-.126	-.127	-.128	-.128	-.119	-.117	-.121

In view of the AR(1) component, we effectively lose one observation, so n' goes from 127 to 137, the effect of then dropping the first observation being very striking, but the estimates subsequently exhibiting little variation across n'. As u_{2t} is still considered a white noise, the estimates of δ are identical to those of Table 3.2, but estimates of γ are all now less than zero, although not significantly, Engle and Granger's (1987) $CI(1,1)$ conclusion now being supported. The AR component in u_{1t} clearly accounts for the bulk of the autocorrelation in co-integrating errors, resulting in the small estimates of γ, which are based on AR-transformed data. The ML, which estimates γ simultaneously with b_1 and the other parameters, would allow AR and fractional features to compete more favourably, though, as discussed in the Introduction, it would require much heavier computation. Notice that $\widehat{\nu b_1}$ looks quite consistent with the values of $\widehat{\nu}$ and $\widehat{b}_1$, possibly providing some support for the present specification. Note also that the various $\widehat{\nu}$ are larger than before, but that, if indeed $\beta > 1/2$, their standard errors have to be interpreted with caution, as $\widehat{\nu}$ is then no longer asymptotically normal.

Engle and Granger (1987) found no evidence of $CI\,(1,1)$ co-integration between $\log M_1$ (y) and $\log GNP$ (x), on the basis of 90 quarterly observations, 1959Q1-1981Q2. Robinson and Marinucci's (1998) fractional analysis admitted the possibility of co-integration, with $\beta < 1/2$. In our preliminary analysis of autocorrelation in u_t, we took from their estimates of δ the values 1.22 (SG applied to first differences of x and adding back 1, using bandwidth 30) and 1.36 (LP applied to first differences of x and adding back 1, using bandwidth 22), and from their estimates of γ the values 0.76, 1.2, both LP estimates but applied respectively to raw residuals using bandwidth 22, and first differences of residuals and adding back 1, using bandwidth 16. Employing also the OLS of ν, 0.643, we found no evidence of autocorrelation in u_t, so proceeded to a co-integration analysis on the basis of $p = 0$ in (3.6). The results are reported in Table 3.4. We found large variation across the largest n', but a good degree of stability is then achieved, with substantially larger values of $\widetilde{\delta}$ and $\widetilde{\gamma}$ (and of their standard errors). Clearly, $\widetilde{\delta}$ significantly exceeds 1, while $\widetilde{\gamma}$ does not, and the resulting $\widetilde{\beta} = \widetilde{\delta} - \widetilde{\gamma}$ are extremely close to the threshold value of 1/2. There is considerable negative correlation between u_{1t} and u_{2t}, and for the smaller n', $\widehat{\nu}$ is close to the OLS.

TABLE 3.4

LogM1 and LogGNP: u_t white noise

n'	90	89	88	87	86	85	84	83	82	81	80
$\widehat{\nu}$	.704	.740	.578	.564	.608	.640	.638	.644	.643	.649	.658
$SE(\widehat{\nu})$	.077	.145	.040	.058	.058	.054	.054	.061	.061	.061	.061
$\widetilde{\delta}$	1.06	1.06	1.91	1.88	1.74	1.63	1.64	1.63	1.63	1.61	1.59
$SE(\widetilde{\delta})$	.057	.057	.025	.121	.117	.068	.083	.082	.086	.084	.076
$\widetilde{\gamma}$	.884	.928	1.12	1.16	1.11	1.09	1.09	1.11	1.10	1.10	1.09
$SE(\widetilde{\gamma})$	.108	.122	.121	.121	.131	.136	.138	.140	.140	.139	.139
$\widehat{\varphi}$	-.134	-.222	-.261	-.268	-.315	-.352	-.350	-.379	-.376	-.391	-.408
r	-.839	-.543	-.402	-.413	-.455	-.475	-.473	-.507	-.504	-.515	-.522

Finally, we looked at the $n = 116$ annual observations, 1871-1986, on stock prices (y) and dividends (x), analysed by Campbell and Shiller (1987). Their findings with respect to $CI\,(1,1)$ co-integration were inconclusive, but Robinson and Marinucci's (1998) and Marinucci and Robinson's (2001) analyses again offered the possibility of co-integration with $\beta < 1/2$. The preliminary estimates of δ taken from Marinucci and Robinson (2001) were 0.86 and 0.95, being SG based on first differences of x and adding back 1, with bandwidths respectively 30 and 40. The preliminary estimates of γ were 0.57, 0.77, being LP on first differences of residuals and adding back one, with bandwidth 30, and SG on raw residuals with bandwidth 22, respectively. We also used the OLS of ν, 31. In this case, both γ estimates suggested white noise u_{1t}, while the δ estimates variously suggested white noise and AR(1) u_{2t}, but our subsequent fractional co-integration analysis produced $\widetilde{\gamma}$ and $\widetilde{\delta}$ that were too close to admit the likelihood of any co-integration. Thus, we report, in Table 3.5, only the results with both u_{1t} and u_{2t} white noise. There is little variation with n', and strong support for the unit root hypothesis on δ, and, since $\widetilde{\gamma}$ is significantly larger than $1/2$ at the 5% level, co-integration with $\beta < 1/2$ is certainly a possibility. We find that $\widehat{\nu}$ is somewhat larger than the OLS value, though not significantly so.

TABLE 3.5
Stock Prices and Dividends: u_t white noise

n'	116	115	114	113	112	111	110	109	108	107	106
$\widehat{\nu}$	32.7	32.7	32.2	31.9	31.7	31.8	31.7	32.0	32.1	32.1	32.1
$SE(\widehat{\nu})$	7.56	7.64	7.80	7.83	7.81	7.93	7.91	7.99	8.02	7.99	8.01
$\widetilde{\delta}$	1.04	1.04	1.08	1.09	1.09	1.09	1.09	1.09	1.10	1.10	1.10
$SE(\widetilde{\delta})$	.077	.077	.090	.092	.092	.092	.093	.093	.095	.095	.095
$\widetilde{\gamma}$	.749	.751	.751	.752	.751	.752	.752	.751	.749	.749	.749
$SE(\widetilde{\gamma})$	.114	.116	.116	.117	.116	.117	.117	.116	.116	.116	.116
$\widehat{\varphi}$	-8.97	-9.52	-9.13	-8.82	-8.56	-8.67	-8.54	-8.52	-.8.64	-8.59	-8.69
r	-.299	-.283	-.272	-.263	-.256	-.259	-.255	-.252	-.255	-.253	-.256

3.7 Appendix 3

3.7.1 Appendix 3.A: Proof of Theorem 3.1

We prove first that Φ is nonsingular, which ensures existence of the inverses in (3.24). Define

$$\Phi^{+} = E\left(\widetilde{Z}_t^{+}\widetilde{Z}_t^{+\prime}\right), \quad \widetilde{Z}_t^{+} = \left(\widetilde{w}_t', \widetilde{w}_{t-1}', ..., \widetilde{w}_{t-p}'\right)'. \tag{3.55}$$

It clearly suffices to show that Φ^{+} is positive definite. Defining

$$\overline{\Phi}^{+} = E\left(\overline{Z}_t\overline{Z}_t'\right), \quad \overline{Z}_t = \left(\overline{w}_t', \overline{w}_{t-1}', ..., \overline{w}_{t-p}'\right)', \tag{3.56}$$

for $\overline{w}_t = (\widetilde{x}_t(\gamma), u_{2t}, u_{1t})'$, from (3.21) it suffices to show that $\overline{\Phi}^{+}$ is positive definite, and similarly, defining

$$\overline{\Phi}^{++} = E\left(R\overline{Z}_t\overline{Z}_t'R'\right), \tag{3.57}$$

where R is a full rank $3(p+1) \times 3(p+1)$ matrix whose columns are orthonormal vectors such that

$$R\overline{Z}_t = [\overline{x}(\gamma)', \overline{u}_2', \overline{u}_1']', \tag{3.58}$$

where $\overline{x}(\gamma) = (\widetilde{x}_t(\gamma), ..., \widetilde{x}_{t-p}(\gamma))'$, $\overline{u}_2 = (u_{2t}, ..., u_{2,t-p})'$, $\overline{u}_1 = (u_{1t}, ..., u_{1,t-p})'$, it suffices to show that $\overline{\Phi}^{++}$ is positive definite. Define the vectors

$$e(\lambda) = \left(1, e^{i\lambda}, ..., e^{ip\lambda}\right)', \quad d(\lambda) = (1 - e^{i\lambda})^{-\beta}e(\lambda), \tag{3.59}$$

and the $3(p+1) \times 2$ matrix

$$E(\lambda) = \begin{bmatrix} 0' & 0' & e(\lambda)' \\ d(\lambda)' & e(\lambda)' & 0' \end{bmatrix}', \tag{3.60}$$

where $0'$ is here a $1 \times (p+1)$ vector of zeros. As in our previous chapter, $f(\lambda)$ is the spectral density matrix of u_t, and note from positive finiteness of Ω and finiteness of the B_j that the smallest eigenvalue of the Hermitian matrix $f(\lambda)$ is bounded from below by a positive constant c, uniformly in λ. Then we can write

$$\overline{\Phi}^{++} = \int_{-\pi}^{\pi} E(\lambda) f(\lambda) E(-\lambda)' d\lambda, \tag{3.61}$$

which for some $c > 0$ exceeds

$$c \int_{-\pi}^{\pi} E(\lambda) E(-\lambda)' d\lambda = c \begin{bmatrix} A & B & 0 \\ B' & I_{p+1} & 0 \\ 0 & 0 & I_{p+1} \end{bmatrix} \tag{3.62}$$

by a non-negative definite matrix, where 0, A and B are $(p+1) \times (p+1)$ matrices, having (i,j)th elements 0, $\sum_{\ell=0}^{\infty} a_\ell a_{\ell+|i-j|}$ and $a_{j-i} 1(j \geq i)$ respectively, with $a_j = a_j(\beta)$. It thus suffices to show that $A - BB'$ is positive definite. But for a $(p+1) \times 1$ vector $\zeta = (\zeta_i)$,

$$\zeta'(A - BB')\zeta = \sum_{\ell=1}^{\infty} \left(a_\ell \zeta_{p+1} + \ldots + a_{\ell+p}\zeta_1\right)^2, \tag{3.63}$$

which is positive unless $\zeta = 0$ because $a_\ell / a_{\ell-1} = (\ell + \beta - 1)/\ell$ is strictly increasing in $\ell \geq 1$ for $\beta < 1$.

We now have to show that

$$\frac{1}{n} \sum{}' Z_t(\gamma,\delta) Z_t'(\gamma,\delta) \;\to_p\; \Phi, \tag{3.64}$$

$$n^{-1/2} \sum{}' Z_t(\gamma,\delta) \varepsilon_{1.2,t} \;\to_d\; N(0,\Psi), \tag{3.65}$$

writing $\sum' = \sum_{t=p+1}^{n}$. To prove (3.65), note first that it suffices to show

$$n^{-1/2} \sum{}' \widetilde{Z}_t \varepsilon_{1.2,t} \to_d N(0,\Psi), \tag{3.66}$$

because

$$\begin{aligned}
E \left\| n^{-1/2} \sum{}' \left\{ Z_t(\gamma,\delta) - \widetilde{Z}_t \right\} \varepsilon_{1.2,t} \right\|^2 &\leq \frac{K}{n} \sum{}' E \left\| Z_t(\gamma,\delta) - \widetilde{Z}_t \right\|^2 \\
&\leq \frac{K}{n} \sum{}' \sum_{j=1}^{p} E\overline{x}_{t-j}^2(\gamma) \\
&\leq \frac{K}{n} \sum{}' \sum_{j=1}^{p} \int_{-\pi}^{\pi} \left| \sum_{s=t-j}^{\infty} a_s e^{-is\lambda} \right|^2 \|f(\lambda)\| \, d\lambda \\
&\leq \frac{K}{n} \sum_{t=1}^{n} \sum_{s=t}^{\infty} a_s^2 \to 0,
\end{aligned} \tag{3.67}$$

as $n \to \infty$, by the Toeplitz lemma, the last inequality following because $f(\lambda)$ is bounded due to the assumption on the B_ℓ. Write $\widetilde{Z}_t = Z_{at} + Z_{bt}$, where the first two elements of Z_{at}, and the last $3p$ elements of Z_{bt}, equal corresponding ones of $\widetilde{Z}_t$. Thus Z_{bt} is $\mathcal{F}_{t-1}$-measurable and

$$E\left(\varepsilon_{1.2,t} \widetilde{Z}_t) | \mathcal{F}_{t-1}\right) = E\left(\varepsilon_{1.2,t} Z_{at}\right) + Z_{bt} E\left(\varepsilon_{1.2,t} | \mathcal{F}_{t-1}\right) = 0, \; a.s. \tag{3.68}$$

Further,

$$E\left(\varepsilon_{1.2,t}^2 \widetilde{Z}_t \widetilde{Z}_t' \,|\mathcal{F}_{t-1}\right) = E\left(\varepsilon_{1.2,t}^2 Z_{at} Z_{at}'\right) + E\left(\varepsilon_{1.2,t}^2 Z_{at}\right) Z_{bt}'$$
$$+ Z_{bt} E\left(\varepsilon_{1.2,t}^2 Z_{at}'\right) + E(\varepsilon_{1.2,t}^2) Z_{bt} Z_{bt}', \; a.s., \tag{3.69}$$

and so

$$\frac{1}{n}\sum{}' \left[E\left\{\varepsilon_{1.2,t}^2 \widetilde{Z}_t \widetilde{Z}_t' \,|\mathcal{F}_{t-1}\right\} - E\left\{\varepsilon_{1.2,t}^2 \widetilde{Z}_t \widetilde{Z}_t'\right\}\right] \to_p 0, \tag{3.70}$$

because Z_{bt} and $Z_{bt} Z_{bt}' - E(Z_{bt} Z_{bt}')$ are stationary and ergodic with zero means. Since (3.69) has expectation Ψ, (3.66) then follows from the Cramer-Wold device and Theorem 1 of Brown (1971), noting that the Lindeberg condition in the latter reference is trivially satisfied because $\varepsilon_{1.2,t}\widetilde{Z}_t$ is stationary with finite variance. Thus (3.65) is proved. The proof of (3.64) follows from (3.67) and elementary inequalities. This concludes the proof of (3.24). The proof of the final statement of the theorem is omitted as it is standard given (3.24) and its proof.

3.7.2 Appendix 3.B: Definitions of $\widehat{A}$ and $\widehat{B}$

For brevity we write $\tilde{G} = G(\tilde{\gamma}, \tilde{\delta})$, $\widetilde{\vartheta} = \widetilde{\vartheta}(\tilde{\gamma}, \tilde{\delta})$, $\tilde{H} = H(\tilde{\delta})$, $\tilde{\phi} = \hat{\phi}(\tilde{\delta})$. We have

$$\widehat{A} = \begin{bmatrix} \hat{a}_1' & \hat{a}_2 & \hat{a}_3 \\ 0' & \hat{a}_4 & \hat{a}_5 \\ 0' & 0 & \hat{a}_6 \end{bmatrix}, \tag{3.71}$$

where

$$\hat{a}_1' = 1'\tilde{G}^{-1}, \quad \hat{a}_2 = -1'\widetilde{\vartheta}_c \tilde{s}_{cc}^{-1}, \tag{3.72}$$
$$\hat{a}_3 = 1'\widetilde{\vartheta}_c \tilde{s}_{cc}^{-1} \tilde{s}_{cd} \tilde{s}_{dd}^{-1} - 1'\widetilde{\vartheta}_d \tilde{s}_{dd}^{-1}, \quad \hat{a}_4 = -\tilde{s}_{cc}^{-1}, \tag{3.73}$$
$$\hat{a}_5 = \tilde{s}_{cc}^{-1} \tilde{s}_{cd} \tilde{s}_{dd}^{-1}, \quad \hat{a}_6 = -\tilde{s}_{dd}^{-1}, \tag{3.74}$$

in which

$$\widetilde{\vartheta}_c = \tilde{G}^{-1}\left(\tilde{g}_c - \tilde{G}_c \widetilde{\vartheta}\right), \quad \widetilde{\vartheta}_d = \tilde{G}^{-1}\left(\tilde{g}_d - \tilde{G}_d \widetilde{\vartheta}\right), \tag{3.75}$$
$$\tilde{g}_c = Q\frac{1}{n}\sum{}'\left\{Z_{tc}(\tilde{\gamma}) y_t(\tilde{\gamma}) + Z_t(\tilde{\gamma}, \tilde{\delta}) y_{tc}(\tilde{\gamma})\right\}, \tag{3.76}$$
$$\tilde{G}_c = Q\frac{1}{n}\sum{}'\left\{Z_{tc}(\tilde{\gamma}) Z_t'(\tilde{\gamma}, \tilde{\delta}) + Z_t(\tilde{\gamma}, \tilde{\delta}) Z_{tc}'(\tilde{\gamma})\right\} Q', \tag{3.77}$$
$$\tilde{g}_d = Q\frac{1}{n}\sum{}' Z_{td}(\tilde{\delta}) y_t(\tilde{\gamma}), \tag{3.78}$$
$$\tilde{G}_d = Q\frac{1}{n}\sum{}'\left\{Z_{td}(\tilde{\delta}) Z_t'(\tilde{\gamma}, \tilde{\delta}) + Z_t(\tilde{\gamma}, \tilde{\delta}) Z_{td}'(\tilde{\delta})\right\} Q', \tag{3.79}$$

with

$$y_{tc}(\tilde{\gamma}) = \log(1-L) y_t(\tilde{\gamma}), \tag{3.80}$$
$$Z_{tc}(\tilde{\gamma}) = \log(1-L)\left\{x_t(\tilde{\gamma}), 0, x_{t-1}(\tilde{\gamma}), 0, y_{t-1}(\tilde{\gamma}), ..., x_{t-p}(\tilde{\gamma}), 0, y_{t-p}(\tilde{\gamma})\right\}', \tag{3.81}$$
$$Z_{td}(\widetilde{\delta}) = \log(1-L)\left\{0, x_t(\widetilde{\delta}), 0, x_{t-1}(\widetilde{\delta}), 0, ..., 0, x_{t-p}(\widetilde{\delta}), 0\right\}', \tag{3.82}$$

and where

$$\widetilde{s}_{cc} = \frac{1}{n}\sum{}'\tilde{v}_{tc}^2, \quad \widetilde{s}_{cd} = \frac{1}{n}\sum{}'\tilde{v}_{tc}\tilde{v}_{td}, \quad \widetilde{s}_{dd} = \frac{1}{n}\sum{}'\tilde{w}_{td}^2, \tag{3.83}$$

with

$$\begin{aligned}
\tilde{v}_{tc} &= y_{tc}(\tilde{\gamma}) - \widetilde{\vartheta}_c' Q Z_t(\tilde{\gamma},\tilde{\delta}) - \widetilde{\vartheta}' Q Z_{tc}(\tilde{\gamma}), && (3.84)\\
\tilde{v}_{td} &= -\widetilde{\vartheta}_d' Q Z_t(\tilde{\gamma},\tilde{\delta}) - \widetilde{\vartheta}' Q Z_{td}(\tilde{\delta}), && (3.85)\\
\tilde{w}_{td} &= x_{td}(\tilde{\delta}) - \tilde{\phi}_d' R X_t(\tilde{\delta}) - \tilde{\phi}' R X_{td}(\tilde{\delta}), && (3.86)\\
x_{td}(\tilde{\delta}) &= \log(1-L)x_t(\tilde{\delta}), && (3.87)\\
X_{td}(\tilde{\delta}) &= \log(1-L)X_t(\tilde{d}), && (3.88)\\
\tilde{\phi}_d &= \tilde{H}^{-1}(\tilde{h}_d - \tilde{H}_d\tilde{\phi}), && (3.89)\\
\tilde{h}_d &= R\frac{1}{n}\sum{}'\left\{X_{td}(\tilde{\delta})x_t(\tilde{\delta}) + X_t(\tilde{\delta})x_{td}(\tilde{\delta})\right\}, && (3.90)\\
\tilde{H}_d &= R\frac{1}{n}\sum{}'\left\{X_{td}(\tilde{\delta})X_t'(\tilde{\delta}) + X_t(\tilde{\delta})X_{td}'(\tilde{\delta})\right\}R'. && (3.91)
\end{aligned}$$

We also have

$$\widehat{B} = \frac{1}{n}\sum{}'\begin{bmatrix} \hat{\varepsilon}_{1.2,t}(\tilde{\gamma},\tilde{\delta})QZ_t(\tilde{\gamma},\tilde{\delta}) \\ \hat{\varepsilon}_{1.2,t}(\tilde{\gamma},\tilde{\delta})\tilde{v}_{tc} \\ \hat{\varepsilon}_{2t}(\tilde{\delta})\tilde{w}_{td} \end{bmatrix}\begin{bmatrix} \hat{\varepsilon}_{1.2,t}(\tilde{\gamma},\tilde{\delta})QZ_t(\tilde{\gamma},\tilde{\delta}) \\ \hat{\varepsilon}_{1.2,t}(\tilde{\gamma},\tilde{\delta})\tilde{v}_{tc} \\ \hat{\varepsilon}_{2t}(\tilde{\delta})\tilde{w}_{td} \end{bmatrix}', \tag{3.92}$$

where

$$\hat{\varepsilon}_{2t}(d) = x_t(d) - \tilde{\phi}' R X_t(d). \tag{3.93}$$

TABLE 3.6

MONTE CARLO BIAS, $b_1 = b_2 = 0$, correct specification

		n	64			128			256		
ρ	γ	δ	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$
	0	.4	-.006	-.005	-.007	-.001	-.001	-.003	-.001	-.002	.000
	.2	.4	-.014	-.036	-.011	.000	-.004	-.005	-.003	-.009	.000
0	.4	.8	-.006	-.002	-.015	-.001	-.002	-.009	-.001	-.001	-.002
	.7	1	-.009	-.024	-.031	.000	-.002	-.023	-.002	-.003	-.005
	0	.4	.001	-.117	.337	.005	-.032	.320	.003	-.009	.308
	.2	.4	-.001	-.268	.394	.009	-.143	.384	.006	-.071	.376
.5	.4	.8	.001	-.124	.192	.005	-.029	.155	.003	-.009	.120
	.7	1	.000	-.246	.214	.006	-.074	.182	.004	-.024	.143
	0	.4	.000	.104	-.338	-.002	.031	-.320	-.003	.007	-.307
	.2	.4	.000	.212	-.401	-.005	.137	-.387	-.010	.061	-.377
-.5	.4	.8	.000	.091	-.193	-.002	.027	-.151	-.003	.007	-.120
	.7	1	.000	.181	-.220	-.003	.065	-.176	-.006	.019	-.142
	0	.4	.002	-.178	.511	.003	-.042	.481	.002	-.011	.460
	.2	.4	.003	-.353	.599	.007	-.209	.578	.006	-.097	.562
.75	.4	.8	.002	-.177	.287	.003	-.043	.226	.002	-.010	.176
	.7	1	.003	-.308	.315	.005	-.120	.258	.004	-.031	.206

TABLE 3.7

MONTE CARLO BIAS, $b_1 = b_2 = 0.5$, correct specification

		n	64			128			256		
ρ	γ	δ	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_O$
	0	.4	-.042	-.053	-.008	-.032	.002	-.003	-.006	.012	.000
	.2	.4	-.069	-.131	-.015	-.044	-.060	-.006	-.003	-.013	-.001
0	.4	.8	-.042	-.139	-.017	-.032	.051	-.010	-.006	.009	-.002
	.7	1	-.052	-.111	-.033	-.036	.023	-.024	-.005	.006	-.005
	0	.4	.004	-.072	.240	-.004	-.041	.222	-.006	-.004	.208
	.2	.4	.016	-.012	.337	.007	.035	.326	-.009	.008	.314
.5	.4	.8	.004	-.065	.164	-.004	-.044	.135	-.006	-.005	.105
	.7	1	.009	-.073	.204	.000	-.070	.177	-.007	-.032	.140
	0	.4	-.017	.095	-.242	-.014	-.003	-.221	.005	.034	-.208
	.2	.4	-.012	-.026	-.346	-.019	.011	-.328	.009	.034	-.316
-.5	.4	.8	-.017	.081	-.167	-.014	.011	-.131	.005	.033	-.105
	.7	1	-.015	.058	-.212	-.016	-.015	-.170	.007	.062	-.138
	0	.4	-.001	-.160	.365	-.006	-.079	.332	-.009	-.015	.310
	.2	.4	-.010	-.122	.513	-.007	-.036	.487	-.018	-.033	.469
.75	.4	.8	-.001	-.128	.244	-.006	-.092	.196	-.009	-.017	.154
	.7	1	-.004	-.258	.300	-.006	-.168	.250	-.012	-.043	.201

TABLE 3.8

MONTE CARLO BIAS, $b_1 = b_2 = 0.9$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.4	-.026	-.150	-.014	-.016	.110	-.005	-.008	.025	.000
	.2	.4	-.057	.028	-.027	-.033	-.038	-.012	-.009	.013	-.001
0	.4	.8	-.026	.019	-.025	-.016	.052	-.014	-.008	-.011	-.003
	.7	1	-.036	-.012	-.043	-.022	-.153	-.030	-.008	.001	-.006
	0	.4	.016	.050	.158	.004	-.023	.137	.005	-.003	.120
	.2	.4	.028	-.094	.281	.010	.135	.267	.008	.086	.247
.5	.4	.8	.016	-.109	.140	.004	-.052	.116	.005	-.020	.090
	.7	1	.019	-.287	.195	.006	-.191	.170	.006	-.034	.134
	0	.4	-.015	-.001	-.161	-.003	-.025	-.136	-.005	.010	-.120
	.2	.4	-.041	.130	-.293	-.008	-.023	-.266	-.006	-.140	-.248
-.5	.4	.8	-.015	.065	-.147	-.003	.024	-.113	-.005	.040	-.088
	.7	1	-.024	.299	-.207	-.005	.121	-.166	-.006	.136	-.131
	0	.4	.027	.037	.237	.010	-.025	.202	.007	.018	.176
	.2	.4	.047	-.025	.421	.020	.093	.390	.010	.134	.364
.75	.4	.8	.027	-.194	.206	.010	-.038	.165	.007	.005	.129
	.7	1	.034	-.483	.283	.013	-.270	.236	.008	-.116	.192

TABLE 3.9

MONTE CARLO BIAS, $b_1 = 0$, $b_2 = 0.5$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.4	-.001	-.003	-.004	.001	.004	-.001	.001	.001	.000
	.2	.4	.001	-.016	-.008	.004	-.001	-.003	.003	.009	.000
0	.4	.8	-.001	-.022	-.008	.001	.005	-.005	.001	.001	-.001
	.7	1	.000	-.044	-.017	.002	.012	-.012	.002	-.001	-.002
	0	.4	.006	.009	.142	.004	-.003	.129	.001	.001	.119
	.2	.4	.016	.028	.201	.010	-.013	.189	.004	.000	.180
.5	.4	.8	.006	.010	.082	.004	.001	.067	.001	.001	.052
	.7	1	.009	.002	.102	.006	-.006	.088	.002	-.004	.069
	0	.4	-.001	.001	-.142	.000	.005	-.128	.000	.004	-.119
	.2	.4	-.002	-.031	-.203	.001	.011	-.189	-.001	.021	-.181
-.5	.4	.8	-.001	-.003	-.083	.000	.008	-.065	.000	.004	-.052
	.7	1	-.001	-.009	-.106	.000	.015	-.085	.000	.017	-.069
	0	.4	.004	.005	.216	.002	.002	.192	.000	.000	.178
	.2	.4	.011	.042	.305	.006	-.004	.283	.001	-.017	.269
.75	.4	.8	.004	.002	.123	.002	.000	.097	.000	.001	.076
	.7	1	.006	-.012	.151	.003	-.018	.124	.001	-.010	.100

TABLE 3.10

MONTE CARLO BIAS, $b_1 = 0$, $b_2 = 0.9$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.4	-.001	.003	-.001	-.001	.002	.000	.000	.001	.000
	.2	.4	-.002	.002	-.003	-.002	.002	-.001	.001	.001	.000
0	.4	.8	-.001	.002	-.003	-.001	.001	-.001	.000	.001	.000
	.7	1	-.001	.003	-.049	-.001	-.001	-.003	.000	.001	-.001
	0	.4	.002	-.007	.015	.001	-.008	.011	.000	-.009	.009
	.2	.4	.006	-.019	.034	.002	-.017	.028	.000	-.016	.025
.5	.4	.8	.002	-.005	.012	.001	-.005	.010	.000	-.005	.008
	.7	1	.003	.002	.020	.001	.000	.016	.000	-.002	.013
	0	.4	-.001	.012	-.014	.000	.010	-.011	.001	.010	-.009
	.2	.4	-.004	.026	-.033	.000	.020	-.028	.002	.017	-.025
-.5	.4	.8	-.001	.005	-.013	.000	.006	-.009	.001	.006	-.007
	.7	1	-.002	.001	-.021	.000	.002	-.016	.001	.003	-.012
	0	.4	.002	-.016	.022	.000	-.014	.016	.000	-.013	.014
	.2	.4	.005	-.035	.050	.001	-.028	.041	-.001	-.025	.037
.75	.4	.8	.002	-.009	.018	.000	-.008	.014	.000	-.007	.011
	.7	1	.003	.003	.029	.001	-.004	.023	.000	-.004	.018

TABLE 3.11

MONTE CARLO BIAS, $b_1 = 0.5$, $b_2 = 0$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_O$
	0	.4	.008	.029	-.012	.004	-.019	-.005	.004	.003	.000
	.2	.4	.012	.067	-.021	.010	-.016	-.009	.005	-.022	-.002
0	.4	.8	.008	.033	-.029	.004	-.023	-.018	.004	.003	-.003
	.7	1	.010	.060	-.061	.006	-.022	-.046	.005	-.002	-.009
	0	.4	.019	-.162	.429	.007	-.101	.416	.000	-.037	.403
	.2	.4	.076	-.144	.525	.036	-.128	.526	.013	-.097	.522
.5	.4	.8	.019	-.162	.332	.007	-.103	.280	.000	-.037	.221
	.7	1	.040	-.194	.403	.018	-.140	.354	.005	-.065	.282
	0	.4	.014	.229	-.437	.011	.109	-.418	.007	.049	-.404
	.2	.4	-.037	.191	-.544	-.009	.140	-.534	-.008	.098	-.525
-.5	.4	.8	.014	.216	-.339	.011	.110	-.274	.007	.049	-.222
	.7	1	-.006	.256	-.419	.003	.138	-.342	.001	.082	-.280
	0	.4	.002	-.301	.654	-.003	-.129	.625	-.004	-.049	.603
	.2	.4	.071	-.260	.800	.028	-.201	.792	.011	-.147	.781
.75	.4	.8	.002	-.300	.496	-.003	-.124	.408	-.004	-.048	.325
	.7	1	.029	-.325	.594	.009	-.172	.501	.002	-.098	.407

TABLE 3.12

MONTE CARLO BIAS, $b_1 = 0.9$, $b_2 = 0$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.4	.006	-.056	-.039	.005	.045	-.015	.005	.013	-.002
	.2	.4	-.002	-.056	-.065	.009	.020	-.030	.005	-.001	-.005
0	.4	.8	.006	-.053	-.119	.005	.053	-.082	.005	.014	-.013
	.7	1	.003	-.063	-.251	.006	.039	-.210	.005	.009	-.038
	0	.4	.129	.325	.714	.052	.165	.740	.018	.050	.741
	.2	.4	.258	.223	.970	.126	.141	1.07	.056	.061	1.12
.5	.4	.8	.129	.333	.994	.052	.167	.981	.018	.055	.854
	.7	1	.177	.240	1.42	.079	.148	1.46	.032	.053	1.27
	0	.4	-.118	-.457	-.758	-.040	-.144	-.755	-.014	-.043	-.746
	.2	.4	-.264	-.403	-1.05	-.110	-.153	-1.11	-.054	-.094	-1.14
-.5	.4	.8	-.118	-.475	-1.05	-.040	-.143	-.965	-.014	-.045	-.852
	.7	1	-.172	-.397	-1.51	-.066	-.159	-1.41	-.029	-.068	-1.26
	0	.4	.167	.419	1.09	.065	.192	1.11	.022	.036	1.11
	.2	.4	.363	.379	1.48	.172	.213	1.61	.079	.064	1.68
.75	.4	.8	.167	.423	1.48	.065	.191	1.42	.022	.036	1.25
	.7	1	.242	.376	2.08	.106	.166	2.05	.043	.049	1.83

TABLE 3.13

MONTE CARLO BIAS, $b_1 = b_2 = 0.5$, mis-specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.4	-.005	.003	-.008	.000	.000	-.003	.000	-.004	.000
	.2	.4	-.010	-.027	-.016	.002	-.002	-.006	-.001	-.015	-.001
0	.4	.8	-.005	.009	-.017	.000	.001	-.010	.000	-.002	-.002
	.7	1	-.007	-.004	-.033	.000	.005	-.024	.000	-.015	-.005
	0	.4	.004	-.240	.240	.006	-.188	.222	.003	-.096	.208
	.2	.4	.008	-.361	.337	.013	-.365	.326	.007	-.343	.314
.5	.4	.8	.004	-.352	.164	.006	-.230	135	.003	-.140	.105
	.7	1	.005	-.808	.204	.008	-.842	177	.004	-.866	.140
	0	.4	.000	.176	-.242	-.001	.174	-.221	-.003	.101	-.208
	.2	.4	.000	.287	-.346	-.003	.356	-.328	-.009	.304	-.316
-.5	.4	.8	.000	.299	-.167	-.001	.244	-.132	-.003	.146	-.105
	.7	1	.000	.790	-.212	-.002	.818	-.170	-.005	.883	-.138
	0	.4	.004	-.318	.365	.003	-.217	.332	.002	-.117	.310
	.2	.4	.009	-.500	.513	.008	-.564	.487	.006	-.493	.469
.75	.4	.8	.004	-.457	.244	.003	-.280	.196	.002	-.154	.154
	.7	1	.006	-1.20	.300	.005	-1.18	.250	.003	-1.18	.201

TABLE 3.14

MONTE CARLO BIAS, $b_1 = b_2 = 0.9$, mis-specification

ρ	γ	n / δ	64 $\bar{v}_I$	64 $\bar{v}_F$	64 $\bar{v}_O$	128 $\bar{v}_I$	128 $\bar{v}_F$	128 $\bar{v}_O$	256 $\bar{v}_I$	256 $\bar{v}_F$	256 $\bar{v}_O$
0	0	.4	-.013	-.003	-.014	-.007	.007	-.005	-.001	-.006	.000
	.2	.4	-.030	-.033	-.027	-.015	-.002	-.012	-.002	-.014	-.001
	.4	.8	-.013	.025	-.025	-.007	.010	-.014	-.001	-.009	-.003
	.7	1	-.020	.010	-.043	-.010	.006	-.030	-.001	-.005	-.006
.5	0	.4	.004	-.284	.158	.005	-.144	.137	.002	-.073	.120
	.2	.4	.013	-.463	.281	.016	-.347	.267	.006	-.278	.247
	.4	.8	.004	-1.30	.140	.005	-1.27	.116	.002	-1.26	.090
	.7	1	.009	-1.39	.195	.009	-1.34	.170	.004	-1.38	.134
-.5	0	.4	-.005	.236	-.161	-.003	.142	-.136	-.001	.068	-.120
	.2	.4	-.013	.393	-.293	-.007	.316	-.266	-.004	.250	-.248
	.4	.8	-.005	1.26	-.147	-.003	1.25	-.113	-.001	1.24	-.088
	.7	1	-.008	1.38	-.207	-.004	1.35	-.166	-.002	1.36	-.131
.75	0	.4	.002	-.385	.237	.003	-.167	.202	.000	-.095	.176
	.2	.4	.006	-.654	.421	.008	-.534	.390	.000	-.402	.364
	.4	.8	.002	-1.90	.206	.003	-1.76	.165	.000	-1.73	.129
	.7	1	.003	-2.06	.282	.005	-2.01	.236	.000	-2.02	.192

TABLE 3.15

MONTE CARLO BIAS, $b_1 = b_2 = 0$, over-specification

ρ	γ	n / δ	64 $\bar{v}_I$	64 $\bar{v}_F$	64 $\bar{v}_O$	128 $\bar{v}_I$	128 $\bar{v}_F$	128 $\bar{v}_O$	256 $\bar{v}_I$	256 $\bar{v}_F$	256 $\bar{v}_O$
0	0	.4	-.032	-.138	-.007	-.006	.013	-.003	.007	.006	.000
	.2	.4	-.036	.007	-.011	.023	.021	-.005	.027	.024	.000
	.4	.8	-.032	-.091	-.015	-.006	.000	-.009	.007	-.001	-.002
	.7	1	-.034	-.036	-.031	.003	.000	-.023	.014	-.009	-.005
.5	0	.4	.006	.040	.337	.017	.044	.320	-.005	.014	.308
	.2	.4	.021	-.122	.394	.061	.020	.384	.004	-.036	.376
	.4	.8	.006	.019	.192	.017	.021	.155	-.005	.007	.120
	.7	1	.012	-.133	.214	.032	.043	.182	-.001	.008	.143
-.5	0	.4	.020	-.047	-.338	.013	.032	-.320	.021	.018	-.307
	.2	.4	.065	-.129	-.401	.042	.137	-.387	.035	.086	-.377
	.4	.8	.020	-.053	-.193	.013	.045	-.151	.021	.033	-.120
	.7	1	.035	-.044	-.220	.022	.028	-.176	.026	.062	-.142
.75	0	.4	-.018	.002	.511	.002	.086	.481	-.016	.001	.460
	.2	.4	-.034	-.083	.599	.016	-.127	.578	-.021	-.124	.562
	.4	.8	-.018	-.016	.287	.002	.058	.226	-.016	-.013	.176
	.7	1	-.023	-.118	.315	.007	-.051	.258	-.017	-.037	.206

TABLE 3.16

MONTE CARLO S.D., $b_1 = b_2 = 0$, correct specification

ρ	γ	n / δ	64 $\overline{v}_I$	64 $\overline{v}_F$	64 $\overline{v}_O$	128 $\overline{v}_I$	128 $\overline{v}_F$	128 $\overline{v}_O$	256 $\overline{v}_I$	256 $\overline{v}_F$	256 $\overline{v}_O$
0	0	.4	.212	.383	.107	.128	.160	.073	.086	.092	.049
	.2	.4	.489	1.03	.141	.310	.559	.105	.217	.318	.076
	.4	.8	.212	.387	.171	.128	.159	.128	.086	.092	.093
	.7	1	.305	.679	.322	.189	.323	.278	.130	.153	.214
.5	0	.4	.184	.566	.112	.113	.218	.084	.073	.098	.063
	.2	.4	.426	1.12	.136	.276	.650	.104	.187	.366	.078
	.4	.8	.184	.569	.160	.113	.194	.127	.073	.098	.092
	.7	1	.266	.913	.283	.168	.376	.247	.112	.176	.192
-.5	0	.4	.178	.528	.109	.112	.227	.084	.076	.101	.065
	.2	.4	.419	1.01	.131	.274	.614	.102	.193	.359	.077
	.4	.8	.178	.485	.154	.112	.221	.122	.076	.103	.092
	.7	1	.259	.758	.270	.167	.361	.237	.116	.185	.188
.75	0	.4	.140	.711	.114	.087	.237	.091	.058	.102	.075
	.2	.4	.328	1.08	.116	.213	.706	.092	.146	.426	.073
	.4	.8	.140	.734	.140	.087	.260	.111	.058	.101	.086
	.7	1	.203	.973	.226	.129	.537	.188	.088	.197	.152

TABLE 3.17

MONTE CARLO S.D., $b_1 = b_2 = 0.5$, correct specification

ρ	γ	n / δ	64 $\overline{v}_I$	64 $\overline{v}_F$	64 $\overline{v}_O$	128 $\overline{v}_I$	128 $\overline{v}_F$	128 $\overline{v}_O$	256 $\overline{v}_I$	256 $\overline{v}_F$	256 $\overline{v}_O$
0	0	.4	1.50	3.09	.127	.873	1.36	.084	.526	.768	.057
	.2	.4	2.96	4.35	.197	1.71	2.80	.145	1.06	1.69	.106
	.4	.8	1.50	3.10	.195	.873	1.48	.141	.526	.746	.099
	.7	1	1.96	3.87	.348	1.13	2.17	.292	.693	1.14	.219
.5	0	.4	1.34	2.66	.123	.779	1.51	.089	.472	.673	.064
	.2	.4	2.61	4.02	.183	1.55	2.59	.140	.950	1.42	.102
	.4	.8	1.34	2.71	.176	.779	1.55	.133	.472	.666	.094
	.7	1	1.74	3.53	.304	1.02	1.98	.257	.623	1.04	.196
-.5	0	.4	1.35	2.84	.118	.766	1.32	.086	.468	.713	.064
	.2	.4	2.65	4.27	.176	1.52	2.20	.135	.952	1.56	.100
	.4	.8	1.35	2.98	.168	.766	1.33	.128	.468	.752	.093
	.7	1	1.76	3.66	.287	1.00	1.75	.245	.620	1.10	.192
.75	0	.4	1.06	2.17	.114	.595	1.04	.086	.365	.469	.068
	.2	.4	2.08	3.06	.150	1.19	2.07	.115	.738	1.23	.088
	.4	.8	1.06	2.20	.143	.595	1.10	.108	.365	.479	.082
	.7	1	1.38	2.74	.242	.783	1.63	.194	.483	.784	.154

TABLE 3.18

MONTE CARLO S.D., $b_1 = b_2 = 0.9$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
0	0	.4	1.06	4.16	.192	.553	2.10	.122	.306	1.43	.079
	.2	.4	2.04	5.28	.354	1.10	3.06	.253	.634	1.93	.177
	.4	.8	1.06	4.14	.282	.553	2.31	.191	.306	1.24	.120
	.7	1	1.37	4.22	.483	.729	2.10	.370	.411	1.01	.249
.5	0	.4	.901	3.38	.172	.472	2.18	.115	.266	1.23	.075
	.2	.4	1.76	4.46	.319	.953	3.01	.233	.553	1.65	.161
	.4	.8	.901	3.51	.241	.472	2.15	.170	.266	1.19	.109
	.7	1	1.17	4.10	.405	.625	2.39	.313	.358	1.02	.219
-.5	0	.4	.918	3.47	.164	.480	1.93	.112	.271	1.16	.075
	.2	.4	1.78	5.12	.300	.961	2.85	.225	.557	1.67	.159
	.4	.8	.918	3.73	.225	.480	1.90	.161	.271	1.09	.108
	.7	1	1.19	3.82	.374	.633	2.15	.296	.363	1.26	.216
.75	0	.4	.717	2.75	.138	.372	1.67	.093	.212	.946	.066
	.2	.4	1.39	3.93	.248	.747	2.26	.179	.441	1.37	.131
	.4	.8	.717	3.22	.195	.372	1.52	.128	.212	.823	.088
	.7	1	.930	3.38	.331	.491	1.83	.232	.286	.938	.169

TABLE 3.19

MONTE CARLO S.D., $b_1 = 0$, $b_2 = 0.5$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
0	0	.4	.156	.529	.074	.088	.219	.047	.054	.075	.031
	.2	.4	.428	1.01	.107	.262	.611	.076	.168	.284	.055
	.4	.8	.156	.508	.100	.088	.185	.071	.054	.071	.050
	.7	1	.244	.743	.177	.144	.359	.147	.090	.146	.110
.5	0	.4	.128	.307	.074	.077	.156	.052	.046	.062	.037
	.2	.4	.359	.759	.101	.225	.491	.076	.143	.271	.055
	.4	.8	.128	.322	.090	.077	.146	.067	.046	.066	.047
	.7	1	.202	.560	.154	.124	.297	.129	.077	.152	.098
-.5	0	.4	.124	.296	.071	.077	.152	.051	.048	.071	.038
	.2	.4	.348	.691	.098	.225	.463	.073	.145	.277	.055
	.4	.8	.124	.290	.086	.077	.179	.065	.048	.071	.047
	.7	1	.196	.470	.146	.124	.296	.123	.079	.162	.096
.75	0	.4	.101	.245	.074	.058	.116	.052	.036	.056	.041
	.2	.4	.279	.596	.089	.171	.382	.066	.111	.256	.051
	.4	.8	.101	.243	.074	.058	.137	.054	.036	.057	.041
	.7	1	.158	.442	.123	.094	.267	.097	.060	.126	.077

TABLE 3.20

MONTE CARLO S.D., $b_1 = 0$, $b_2 = 0.9$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.4	.058	.088	.030	.029	.046	.016	.015	.026	.009
	.2	.4	.174	.112	.049	.092	.057	.029	.053	.032	.019
0	.4	.8	.058	.072	.035	.029	.036	.021	.015	.019	.012
	.7	1	.094	.142	.058	.048	.056	.040	.027	.027	.025
	0	.4	.049	.077	.025	.026	.040	.014	.014	.022	.008
	.2	.4	.149	.099	.042	.084	.052	.027	.048	.029	.017
.5	.4	.8	.049	.058	.029	.026	.031	.018	.014	.017	.011
	.7	1	.080	.104	.048	.044	.050	.034	.024	.024	.022
	0	.4	.047	.083	.025	.024	.039	.014	.013	.022	.009
	.2	.4	.143	.098	.040	.080	.049	.026	.046	.029	.017
-.5	.4	.8	.047	.068	.028	.024	.031	.018	.013	.017	.011
	.7	1	.077	.112	.045	.042	.049	.032	.023	.024	.022
	0	.4	.040	.064	.021	.019	.031	.011	.011	.019	.007
	.2	.4	.122	.087	.034	.063	.043	.021	.036	.026	.014
.75	.4	.8	.040	.049	.024	.019	.028	.014	.011	.014	.009
	.7	1	.066	.126	.039	.033	.045	.025	.019	.020	.017

TABLE 3.21

MONTE CARLO S.D., $b_1 = 0.5$, $b_2 = 0$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
	0	.4	.593	1.25	.164	.367	.571	.115	.229	.279	.079
	.2	.4	1.03	1.71	.246	.660	1.06	.191	.425	.721	.143
0	.4	.8	.593	1.25	.331	.367	.580	.255	.229	.279	.186
	.7	1	.726	1.48	.633	.457	.831	.553	.289	.428	.427
	0	.4	.507	1.12	.157	.316	.614	.117	.196	.281	.086
	.2	.4	.908	1.66	.229	.559	1.01	.179	.357	.628	.134
.5	.4	.8	.507	1.11	.305	.316	.612	.244	.196	.279	.179
	.7	1	.628	1.39	.557	.387	.797	.490	.242	.439	.383
	0	.4	.500	1.07	.152	.315	.613	.115	.200	.312	.087
	.2	.4	.886	1.58	.221	.565	1.06	.173	.364	.652	.129
-.5	.4	.8	.500	1.03	.292	.315	.615	.235	.200	.311	.178
	.7	1	.614	1.38	.528	.390	.800	.469	.248	.479	.375
	0	.4	.398	1.12	.137	.243	.617	.110	.156	.262	.089
	.2	.4	.708	1.43	.182	.424	.862	.143	.277	.612	.107
.75	.4	.8	.398	1.13	.249	.243	.583	.202	.156	.256	.160
	.7	1	.485	1.27	.441	.290	.716	.372	.187	.450	.302

TABLE 3.22

MONTE CARLO S.D., $b_1 = 0.9$, $b_2 = 0$, correct specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
0	0	.4	.666	2.08	.466	.399	1.22	.373	.239	.560	.280
	.2	.4	1.14	2.65	.864	.711	1.66	.764	.443	.882	.615
	.4	.8	.666	2.05	1.36	.399	1.20	1.18	.239	.567	.907
	.7	1	.813	2.24	2.70	.496	1.33	2.60	.301	.655	2.09
.5	0	.4	.615	2.06	.450	.358	1.09	.353	.205	.468	.262
	.2	.4	1.09	2.55	.849	.657	1.62	.729	.408	.816	.585
	.4	.8	.615	2.15	1.24	.358	1.10	1.08	.205	.486	.816
	.7	1	.768	2.25	2.38	.451	1.28	2.27	.268	.586	1.85
-.5	0	.4	.608	2.16	.434	.346	1.13	.338	.210	.510	.249
	.2	.4	1.09	2.72	.831	.642	1.51	.714	.403	.903	.555
	.4	.8	.608	2.21	1.18	.346	1.14	1.04	.210	.515	.818
	.7	1	.761	2.34	2.23	.438	1.34	2.16	.270	.678	1.81
.75	0	.4	.529	2.01	.383	.295	1.02	.297	.166	.349	.217
	.2	.4	.986	2.67	.769	.590	1.55	.652	.362	.787	.508
	.4	.8	.529	2.05	.974	.295	.941	.835	.166	.359	.678
	.7	1	.681	2.24	1.89	.391	1.13	1.72	.228	.521	1.44

TABLE 3.23

MONTE CARLO S.D., $b_1 = b_2 = 0.5$, mis-specification

		n	64			128			256		
ρ	γ	δ	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$	$\overline{v}_I$	$\overline{v}_F$	$\overline{v}_O$
0	0	.4	.228	.542	.127	.137	.338	.084	.090	.150	.057
	.2	.4	.541	1.25	.197	.346	.834	.145	.238	.544	.106
	.4	.8	.228	.620	.195	.137	.358	.141	.090	.171	.099
	.7	1	.333	1.13	.348	.207	.778	.292	.139	.525	.219
.5	0	.4	.195	.840	.123	.121	.556	.089	.077	.245	.064
	.2	.4	.468	1.28	.183	.302	.881	.140	.204	.685	.102
	.4	.8	.195	.919	.176	.121	.578	.133	.077	.316	.094
	.7	1	.287	1.30	.304	.181	1.03	.257	.119	.885	.196
-.5	0	.4	.188	.643	.118	.120	.498	.086	.080	.313	.064
	.2	.4	.457	1.14	.176	.304	.868	.135	.209	.668	.100
	.4	.8	.188	.799	.168	.120	.614	.128	.080	.375	.093
	.7	1	.278	1.27	.287	.181	1.01	.245	.123	.907	.192
.75	0	.4	.148	.956	.114	.091	.632	.086	.061	.259	.068
	.2	.4	.356	1.24	.150	.230	1.00	.115	.159	.803	.088
	.4	.8	.148	1.10	.143	.091	.691	.108	.061	.290	.082
	.7	1	.217	1.45	.242	.137	1.28	.194	.094	1.17	.154

TABLE 3.24

MONTE CARLO S.D., $b_1 = b_2 = 0.9$, mis-specification

ρ	γ	n / δ	64 $\overline{v}_I$	64 $\overline{v}_F$	64 $\overline{v}_O$	128 $\overline{v}_I$	128 $\overline{v}_F$	128 $\overline{v}_O$	256 $\overline{v}_I$	256 $\overline{v}_F$	256 $\overline{v}_O$
0	0	.4	.318	.571	.192	.189	.290	.122	.116	.150	.079
	.2	.4	.844	1.31	.354	.540	.827	.253	.357	.489	.177
	.4	.8	.318	1.11	.282	.189	.734	.191	.116	.520	.120
	.7	1	.490	1.71	.483	.303	1.09	.370	.194	.743	.249
.5	0	.4	.271	.807	.172	.171	.438	.115	.103	.214	.075
	.2	.4	.732	1.33	.319	.489	.876	.233	.313	.628	.161
	.4	.8	.271	1.37	.241	.171	1.11	.170	.103	1.03	.109
	.7	1	.422	1.57	.405	.275	1.03	.313	.170	.766	.219
-.5	0	.4	.259	.729	.164	.166	.451	.112	.105	.221	.075
	.2	.4	.701	1.22	.300	.479	.841	.225	.321	.640	.159
	.4	.8	.259	1.30	.225	.166	1.10	.161	.105	1.01	.108
	.7	1	.403	1.53	.374	.267	1.05	.296	.175	.792	.216
.75	0	.4	.213	1.00	.138	.125	.454	.093	.082	.263	.066
	.2	.4	.572	1.28	.248	.360	.983	.179	.248	.773	.131
	.4	.8	.213	1.58	.195	.125	1.44	.128	.082	1.42	.088
	.7	1	.330	1.36	.331	.201	1.02	.232	.135	.831	.169

TABLE 3.25

MONTE CARLO S.D., $b_1 = b_2 = 0$, over-specification

ρ	γ	n / δ	64 $\overline{v}_I$	64 $\overline{v}_F$	64 $\overline{v}_O$	128 $\overline{v}_I$	128 $\overline{v}_F$	128 $\overline{v}_O$	256 $\overline{v}_I$	256 $\overline{v}_F$	256 $\overline{v}_O$
0	0	.4	2.04	4.46	.107	1.19	2.23	.073	.748	.929	.049
	.2	.4	4.03	6.98	.141	2.37	4.40	.105	1.52	2.70	.076
	.4	.8	2.04	4.40	.171	1.19	2.24	.128	.748	.914	.093
	.7	1	2.66	5.39	.322	1.56	3.38	.278	.988	1.59	.214
.5	0	.4	1.74	3.22	.112	1.06	1.79	.084	.668	.907	.063
	.2	.4	3.39	6.04	.136	2.12	3.90	.104	1.35	2.39	.078
	.4	.8	1.74	3.47	.160	1.06	1.72	.127	.668	.899	.092
	.7	1	2.26	4.53	.283	1.40	2.85	.247	.881	1.44	.192
-.5	0	.4	1.78	3.55	.109	1.07	1.91	.084	.670	.925	.065
	.2	.4	3.46	5.48	.131	2.14	3.92	.102	1.36	2.33	.077
	.4	.8	1.78	3.42	.154	1.07	1.92	.122	.670	.971	.092
	.7	1	2.30	4.52	.270	1.41	2.85	.237	.887	1.53	.188
.75	0	.4	1.42	2.73	.114	.831	1.63	.091	.519	.651	.075
	.2	.4	2.74	4.51	.116	1.67	3.24	.092	1.05	1.85	.073
	.4	.8	1.42	2.76	.140	.831	1.57	.111	.519	.636	.086
	.7	1	1.83	3.48	.226	1.09	2.31	.188	.686	1.06	.152

TABLE 3.26

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0$, correct specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.078	.061	.053	.056	.057	.059	.136	.122	.112	.094	.125	.114
	.2	.4	.077	.045	.054	.032	.062	.034	.133	.083	.104	.072	.114	.069
0	.4	.8	.078	.059	.053	.055	.057	.059	.136	.125	.112	.087	.125	.114
	.7	1	.076	.058	.058	.057	.053	.053	.134	.107	.105	.103	.120	.098
	0	.4	.074	.057	.055	.061	.055	.065	.136	.089	.119	.092	.117	.111
	.2	.4	.073	.105	.055	.082	.054	.079	.141	.153	.120	.128	.111	.112
.5	.4	.8	.074	.059	.055	.057	.055	.066	.136	.089	.119	.094	.117	.111
	.7	1	.068	.088	.055	.076	.050	.069	.140	.125	.121	.117	.116	.109
	0	.4	.076	.063	.072	.061	.068	.068	.124	.103	.124	.107	.122	.118
	.2	.4	.076	.123	.059	.106	.058	.084	.134	.168	.117	.145	.130	.119
-.5	.4	.8	.076	.071	.072	.059	.068	.069	.124	.101	.124	.105	.122	.118
	.7	1	.073	.102	.066	.086	.060	.078	.129	.144	.118	.142	.128	.117
	0	.4	.075	.052	.059	.054	.063	.070	.136	.083	.112	.097	.116	.111
	.2	.4	.073	.168	.058	.136	.069	.094	.143	.207	.113	.166	.116	.132
.75	.4	.8	.075	.049	.059	.054	.063	.073	.136	.083	.112	.097	.116	.110
	.7	1	.076	.120	.060	.105	.064	.078	.143	.155	.113	.138	.110	.117

TABLE 3.27

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0.5$, correct specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.100	.036	.078	.046	.060	.034	.151	.070	.127	.079	.103	.066
	.2	.4	.097	.035	.065	.034	.064	.024	.144	.070	.126	.062	.098	.051
0	.4	.8	.100	.040	.078	.041	.060	.032	.151	.073	.127	.080	.103	.067
	.7	1	.103	.039	.071	.041	.057	.028	.154	.077	.130	.073	.101	.055
	0	.4	.092	.030	.077	.033	.068	.037	.156	.062	.129	.066	.121	.081
	.2	.4	.085	.043	.076	.048	.060	.042	.157	.084	.129	.080	.105	.077
.5	.4	.8	.092	.027	.077	.030	.068	.041	.156	.058	.129	.071	.121	.074
	.7	1	.094	.042	.082	.042	.060	.038	.159	.073	.128	.062	.122	.073
	0	.4	.093	.046	.075	.045	.058	.041	.151	.072	.133	.080	.109	.073
	.2	.4	.089	.046	.076	.051	.053	.039	.140	.079	.138	.095	.107	.079
-.5	.4	.8	.093	.049	.075	.052	.058	.038	.151	.077	.133	.089	.109	.067
	.7	1	.091	.054	.077	.048	.052	.041	.147	.082	.131	.087	.105	.069
	0	.4	.099	.038	.068	.036	.068	.038	.165	.067	.124	.065	.114	.073
	.2	.4	.101	.056	.069	.067	.062	.075	.164	.093	.124	.103	.112	.114
.75	.4	.8	.099	.038	.068	.032	.068	.039	.165	.074	.124	.063	.114	.074
	.7	1	.094	.050	.073	.044	.060	.049	.165	.093	.126	.077	.114	.089

TABLE 3.28

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0.9$, correct specification

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.122	.038	.080	.035	.077	.025	.187	.066	.150	.064	.129	.053
	.2	.4	.125	.033	.092	.023	.063	.024	.191	.069	.146	.051	.130	.050
0	.4	.8	.122	.032	.080	.033	.077	.033	.187	.068	.150	.068	.129	.064
	.7	1	.125	.043	.079	.035	.075	.018	.192	.073	.146	.055	.122	.055
	0	.4	.112	.027	.097	.031	.067	.030	.177	.054	.160	.064	.145	.063
	.2	.4	.118	.035	.094	.042	.071	.055	.182	.069	.161	.084	.139	.096
.5	.4	.8	.112	.038	.097	.035	.067	.036	.177	.080	.160	.069	.145	.064
	.7	1	.121	.048	.090	.039	.073	.055	.179	.075	.165	.070	.133	.081
	0	.4	.114	.037	.092	.034	.084	.028	.184	.080	.161	.071	.132	.063
	.2	.4	.109	.048	.098	.046	.074	.054	.180	.088	.158	.088	.138	.101
-.5	.4	.8	.114	.054	.092	.039	.084	.036	.184	.079	.161	.070	.132	.068
	.7	1	.112	.060	.097	.044	.082	.053	.182	.089	.161	.072	.136	.093
	0	.4	.115	.035	.100	.026	.079	.035	.185	.069	.161	.069	.151	.059
	.2	.4	.107	.057	.096	.063	.081	.105	.188	.108	.162	.104	.146	.156
.75	.4	.8	.115	.047	.100	.033	.079	.033	.185	.073	.161	.062	.151	.061
	.7	1	.112	.046	.101	.059	.079	.061	.181	.090	.159	.087	.141	.106

TABLE 3.29

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = 0$, $b_2 = 0.5$, correct specification

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.069	.010	.067	.022	.059	.028	.113	.018	.122	.048	.106	.072
	.2	.4	.066	.020	.064	.023	.065	.018	.114	.035	.120	.041	.112	.038
0	.4	.8	.069	.010	.067	.017	.059	.029	.113	.018	.122	.050	.106	.068
	.7	1	.070	.015	.067	.027	.065	.023	.114	.034	.125	.054	.107	.062
	0	.4	.062	.020	.054	.024	.049	.034	.124	.042	.115	.053	.105	.054
	.2	.4	.061	.044	.053	.064	.049	.059	.127	.078	.110	.091	.103	.096
.5	.4	.8	.062	.019	.054	.022	.049	.037	.124	.039	.115	.051	.105	.057
	.7	1	.066	.040	.051	.045	.047	.054	.127	.076	.118	.069	.102	.076
	0	.4	.067	.017	.067	.018	.055	.033	.125	.033	.117	.045	.100	.059
	.2	.4	.067	.053	.063	.063	.055	.059	.119	.082	.119	.095	.094	.088
-.5	.4	.8	.067	.013	.067	.019	.055	.031	.125	.035	.117	.046	.100	.054
	.7	1	.067	.045	.066	.038	.058	.047	.122	.071	.120	.074	.103	.073
	0	.4	.073	.024	.055	.025	.054	.022	.145	.037	.107	.053	.096	.043
	.2	.4	.069	.108	.054	.126	.057	.113	.131	.158	.104	.164	.099	.151
.75	.4	.8	.073	.031	.055	.024	.054	.023	.145	.051	.107	.056	.096	.051
	.7	1	.067	.082	.058	.055	.051	.065	.137	.117	.106	.096	.103	.106

TABLE 3.30

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = 0$, $b_2 = 0.9$, correct specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.074	.002	.064	.001	.046	.000	.122	.002	.114	.001	.099	.001
	.2	.4	.074	.000	.064	.005	.053	.002	.125	.004	.113	.008	.097	.010
0	.4	.8	.074	.004	.064	.005	.046	.014	.122	.012	.114	.010	.099	.028
	.7	1	.074	.017	.063	.032	.052	.028	.123	.036	.113	.052	.096	.064
	0	.4	.067	.000	.074	.000	.053	.000	.122	.000	.127	.004	.122	.003
	.2	.4	.066	.000	.079	.001	.054	.004	.118	.002	.124	.009	.110	.010
.5	.4	.8	.067	.005	.074	.005	.053	.008	.122	.008	.127	.019	.122	.022
	.7	1	.069	.016	.079	.023	.054	.028	.118	.037	.125	.057	.115	.051
	0	.4	.073	.001	.066	.000	.045	.000	.130	.004	.128	.003	.097	.000
	.2	.4	.070	.004	.058	.004	.038	.005	.129	.008	.122	.011	.102	.012
-.5	.4	.8	.073	.002	.066	.008	.045	.009	.130	.006	.128	.016	.097	.021
	.7	1	.070	.019	.065	.027	.045	.027	.128	.045	.124	.050	.099	.071
	0	.4	.080	.001	.076	.002	.059	.003	.153	.003	.123	.006	.112	.007
	.2	.4	.086	.004	.073	.001	.053	.007	.151	.007	.124	.013	.100	.025
.75	.4	.8	.080	.005	.076	.009	.059	.010	.153	.013	.123	.021	.112	.025
	.7	1	.081	.016	.077	.020	.055	.030	.155	.032	.117	.050	.105	.065

TABLE 3.31

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = 0.5$, $b_2 = 0$, correct specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.071	.043	.068	.048	.069	.054	.137	.075	.136	.093	.121	.087
	.2	.4	.070	.051	.078	.048	.065	.043	.135	.083	.129	.091	.124	.080
0	.4	.8	.071	.045	.068	.051	.069	.053	.137	.076	.136	.089	.121	.086
	.7	1	.075	.049	.072	.055	.065	.050	.133	.081	.133	.091	.118	.084
	0	.4	.067	.046	.068	.045	.059	.041	.125	.079	.130	.095	.095	.074
	.2	.4	.078	.045	.055	.052	.053	.042	.139	.077	.108	.081	.112	.076
.5	.4	.8	.067	.044	.068	.044	.059	.039	.125	.087	.130	.089	.095	.071
	.7	1	.072	.051	.058	.046	.053	.044	.135	.083	.124	.087	.100	.066
	0	.4	.077	.032	.071	.049	.062	.047	.145	.083	.122	.082	.112	.091
	.2	.4	.078	.049	.076	.049	.057	.045	.136	.078	.134	.082	.098	.086
-.5	.4	.8	.077	.033	.071	.050	.062	.047	.145	.085	.122	.082	.112	.089
	.7	1	.081	.040	.077	.046	.057	.058	.134	.082	.126	.089	.106	.111
	0	.4	.066	.048	.058	.042	.053	.036	.129	.075	.111	.078	.089	.070
	.2	.4	.080	.050	.053	.054	.048	.057	.151	.086	.113	.092	.111	.084
.75	.4	.8	.066	.049	.058	.042	.053	.037	.129	.071	.111	.080	.089	.069
	.7	1	.072	.053	.056	.050	.052	.050	.140	.083	.115	.090	.091	.076

TABLE 3.32

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = 0.9$, $b_2 = 0$, correct specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.097	.053	.086	.042	.071	.042	.162	.087	.157	.077	.125	.072
	.2	.4	.090	.038	.091	.026	.077	.021	.166	.065	.150	.045	.127	.042
0	.4	.8	.097	.044	.086	.042	.071	.045	.162	.080	.157	.075	.125	.073
	.7	1	.092	.039	.089	.035	.070	.030	.155	.068	.150	.068	.124	.056
	0	.4	.112	.041	.073	.031	.053	.031	.165	.063	.141	.059	.101	.066
	.2	.4	.097	.023	.078	.027	.064	.019	.161	.045	.139	.049	.120	.045
.5	.4	.8	.112	.043	.073	.030	.053	.031	.165	.063	.141	.058	.101	.062
	.7	1	.109	.027	.082	.031	.060	.032	.164	.054	.147	.064	.110	.064
	0	.4	.101	.051	.081	.033	.068	.030	.171	.082	.140	.062	.115	.062
	.2	.4	.105	.031	.087	.023	.060	.021	.178	.059	.139	.046	.123	.031
-.5	.4	.8	.101	.051	.081	.034	.068	.030	.171	.081	.140	.060	.115	.061
	.7	1	.101	.036	.086	.031	.061	.031	.175	.068	.140	.052	.119	.055
	0	.4	.117	.032	.082	.024	.051	.021	.185	.053	.133	.052	.104	.051
	.2	.4	.107	.028	.078	.026	.065	.024	.173	.044	.133	.042	.114	.043
.75	.4	.8	.117	.033	.082	.022	.051	.021	.185	.053	.133	.053	.104	.051
	.7	1	.111	.030	.081	.028	.058	.029	.184	.059	.143	.053	.106	.054

TABLE 3.33

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0.5$, mis-specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.258	.026	.245	.027	.248	.037	.344	.063	.319	.060	.325	.079
	.2	.4	.242	.013	.214	.012	.229	.013	.327	.043	.296	.024	.310	.035
0	.4	.8	.258	.022	.245	.024	.248	.035	.344	.061	.319	.052	.325	.073
	.7	1	.255	.019	.229	.006	.241	.017	.339	.042	.308	.029	.322	.031
	0	.4	.264	.040	.246	.030	.248	.032	.356	.070	.324	.052	.324	.069
	.2	.4	.245	.072	.230	.051	.224	.052	.341	.105	.303	.079	.317	.064
.5	.4	.8	.264	.033	.246	.028	.248	.029	.356	.054	.324	.046	.324	.070
	.7	1	.253	.031	.239	.028	.239	.014	.347	.047	.306	.043	.325	.025
	0	.4	.274	.033	.250	.026	.255	.030	.349	.067	.333	.053	.341	.068
	.2	.4	.258	.077	.228	.053	.228	.047	.331	.117	.317	.080	.317	.073
-.5	.4	.8	.274	.031	.250	.024	.255	.024	.349	.058	.333	.046	.341	.070
	.7	1	.270	.036	.243	.019	.233	.011	.343	.050	.331	.033	.334	.022
	0	.4	.274	.035	.244	.024	.251	.025	.360	.057	.329	.043	.333	.064
	.2	.4	.249	.119	.221	.079	.218	.054	.336	.155	.310	.099	.313	.071
.75	.4	.8	.274	.028	.244	.022	.251	.025	.360	.044	.329	.034	.333	.063
	.7	1	.262	.041	.240	.032	.238	.010	.350	.051	.318	.040	.318	.013

TABLE 3.34

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0.9$, mis-specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.622	.040	.647	.037	.651	.058	.681	.078	.694	.076	.709	.114
	.2	.4	.581	.020	.605	.019	.620	.018	.651	.053	.662	.043	.677	.056
0	.4	.8	.622	.027	.647	.015	.651	.019	.681	.043	.694	.031	.709	.042
	.7	1	.603	.006	.627	.003	.644	.002	.667	.011	.682	.006	.692	.008
	0	.4	.617	.040	.644	.039	.637	.041	.670	.064	.700	.067	.695	.086
	.2	.4	.614	.068	.612	.065	.594	.056	.658	.098	.668	.087	.661	.081
.5	.4	.8	.617	.020	.644	.009	.637	.013	.670	.027	.700	.020	.695	.033
	.7	1	.615	.010	.631	.003	.619	.006	.663	.011	.688	.006	.679	.008
	0	.4	.643	.038	.643	.041	.648	.047	.700	.075	.705	.064	.705	.083
	.2	.4	.616	.081	.623	.066	.620	.067	.677	.109	.673	.106	.670	.090
-.5	.4	.8	.643	.019	.643	.012	.648	.006	.700	.031	.705	.024	.705	.029
	.7	1	.635	.011	.639	.007	.632	.002	.687	.015	.696	.010	.692	.003
	0	.4	.637	.034	.645	.022	.623	.033	.684	.045	.701	.042	.690	.057
	.2	.4	.618	.110	.606	.091	.594	.072	.676	.137	.687	.116	.681	.092
.75	.4	.8	.637	.012	.645	.012	.623	.005	.684	.016	.701	.016	.690	.016
	.7	1	.628	.013	.631	.009	.615	.004	.682	.018	.696	.011	.683	.004

TABLE 3.35

EMPIRICAL SIZES OF W_I AND W_F, $b_1 = b_2 = 0$, over-specification

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
ρ	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.078	.047	.061	.047	.050	.042	.127	.085	.115	.088	.100	.082
	.2	.4	.072	.040	.054	.042	.047	.027	.135	.075	.107	.070	.086	.048
0	.4	.8	.078	.049	.061	.041	.050	.042	.127	.091	.115	.083	.100	.080
	.7	1	.075	.049	.052	.037	.049	.033	.132	.083	.104	.074	.094	.074
	0	.4	.068	.037	.063	.052	.056	.048	.124	.071	.118	.093	.105	.082
	.2	.4	.071	.052	.064	.045	.061	.026	.113	.079	.116	.071	.110	.046
.5	.4	.8	.068	.039	.063	.050	.056	.047	.124	.071	.118	.088	.105	.083
	.7	1	.065	.043	.056	.047	.060	.046	.120	.076	.110	.087	.110	.074
	0	.4	.091	.057	.072	.048	.066	.048	.143	.087	.109	.093	.112	.095
	.2	.4	.084	.051	.065	.049	.053	.021	.139	.088	.115	.080	.099	.056
-.5	.4	.8	.091	.054	.072	.051	.066	.051	.143	.092	.109	.090	.112	.100
	.7	1	.088	.062	.067	.051	.058	.040	.137	.103	.112	.094	.105	.084
	0	.4	.085	.052	.072	.047	.060	.047	.144	.087	.129	.081	.113	.085
	.2	.4	.074	.051	.073	.057	.057	.026	.138	.099	.126	.084	.114	.045
.75	.4	.8	.085	.049	.072	.042	.060	.047	.144	.084	.129	.076	.113	.088
	.7	1	.080	.056	.080	.051	.058	.044	.143	.093	.125	.093	.112	.090

TABLE 3.36

MONTE CARLO BIAS OF $\widetilde{\delta}$, $\rho = 0.5$

	n	64			128			256		
estimation	$\delta \backslash b_2$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d)$	.4	-.023	.377	.795	-.011	.358	.818	-.005	.363	.833
	.8	-.025	.328	.493	-.008	.332	.524	-.004	.343	.545
	1	-.036	.227	.267	-.014	.232	.292	-.006	.236	.290
$S_1(d)$	.4	-.045	.127	.662	-.029	.047	.595	-.015	.025	.570
	.8	-.040	.105	.405	-.017	.048	.379	-.011	.029	.356
	1	-.051	.047	.196	-.033	.015	.166	-.016	.007	.150

TABLE 3.37

MONTE CARLO S.D. OF $\widetilde{\delta}$, $\rho = 0.5$

	n	64			128			256		
estimation	$\delta \backslash b_2$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d)$	.4	.125	.135	.139	.082	.105	.104	.052	.073	.071
	.8	.125	.145	.206	.082	.110	.203	.051	.079	.193
	1	.122	.164	.217	.079	.139	.222	.050	.113	.211
$S_1(d)$	.4	.253	.240	.259	.161	.171	.222	.093	.116	.172
	.8	.257	.245	.275	.170	.174	.254	.095	.119	.232
	1	.240	.224	.278	.163	.168	.249	.092	.116	.221

TABLE 3.38

EMPIRICAL SIZES ($\alpha = 0.05$) OF W_δ, $\rho = 0.5$

	n	64			128			256		
estimation	$\delta \backslash b_2$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d)$	.4	.134	.902	1.00	.099	.968	1.00	.073	1.00	1.00
	.8	.126	.839	.984	.095	.952	.993	.068	.997	1.00
	1	.121	.611	.786	.082	.800	.923	.064	.918	.981
$S_1(d)$	.4	.129	.140	.685	.103	.084	.741	.074	.063	.877
	.8	.123	.125	.337	.115	.090	.424	.080	.058	.473
	1	.088	.083	.141	.090	.048	.146	.069	.035	.177

TABLE 3.39

EMPIRICAL SIZES ($\alpha = 0.10$) OF W_δ, $\rho = 0.5$

	n	64			128			256		
estimation	$\delta \backslash b_2$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d)$	.4	.188	.935	1.00	.147	.975	1.00	.122	1.00	1.00
	.8	.191	.889	.989	.151	.970	.996	.123	.997	1.00
	1	.177	.705	.851	.136	.856	.939	.111	.941	.983
$S_1(d)$	.4	.190	.190	.752	.175	.127	.792	.129	.091	.930
	.8	.186	.168	.397	.187	.129	.479	.137	.099	.529
	1	.150	.116	.158	.150	.088	.182	.130	.066	.210

TABLE 3.40

MONTE CARLO BIAS OF $\tilde{\gamma}$, $\rho = 0.5$, $b_1 = b_2 = b$

		n	64			128			256		
estimation	γ	$\delta \backslash b$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d), T_0(c)$	0	.4	-.008	.420	.857	-.006	.413	.866	-.004	.414	.871
	.2	.4	-.046	.372	.809	-.020	.388	.844	-.006	.407	.865
	.4	.8	-.008	.405	.743	-.005	.409	.775	-.004	.413	.794
	.7	1	-.034	.347	.494	-.015	.371	.520	-.006	.389	.523
$S_1(d), T_1(c)$	0	.4	-.047	.094	.642	-.027	.024	.582	-.015	.001	.561
	.2	.4	-.176	-.051	.481	-.103	-.098	.412	-.040	-.089	.376
	.4	.8	-.043	.079	.414	-.020	.026	.387	-.013	.003	.343
	.7	1	-.116	-.042	.173	-.070	-.058	.149	-.032	-.056	.115

TABLE 3.41

MONTE CARLO S.D. OF $\tilde{\gamma}$, $\rho = 0.5$, $b_1 = b_2 = b$

		n	64			128			256		
estimation	γ	$\delta \backslash b$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d), T_0(c)$	0	.4	.096	.106	.107	.067	.078	.079	.048	.057	.056
	.2	.4	.106	.111	.111	.075	.085	.085	.051	.058	.058
	.4	.8	.095	.110	.124	.066	.081	.107	.048	.058	.090
	.7	1	.103	.113	.173	.074	.087	.177	.051	.061	.172
$S_1(d), T_1(c)$	0	.4	.233	.220	.254	.133	.159	.224	.077	.115	.192
	.2	.4	.266	.224	.270	.180	.179	.253	.094	.151	.233
	.4	.8	.232	.220	.288	.142	.155	.265	.077	.114	.267
	.7	1	.237	.216	.277	.159	.162	.241	.089	.134	.216

TABLE 3.42

EMPIRICAL SIZES ($\alpha = 0.05$) OF W_γ, $\rho = 0.5$, $b_1 = b_2 = b$

		n	64			128			256		
estimation	γ	$\delta \backslash b$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d), T_0(c)$	0	.4	.038	.979	1.00	.041	1.00	1.00	.051	1.00	1.00
	.2	.4	.087	.939	1.00	.085	.995	1.00	.059	1.00	1.00
	.4	.8	.039	.969	1.00	.040	1.00	1.00	.045	1.00	1.00
	.7	1	.064	.917	.993	.076	.992	.998	.059	1.00	1.00
$S_1(d), T_1(c)$	0	.4	.072	.098	.683	.039	.058	.722	.034	.033	.826
	.2	.4	.096	.053	.455	.106	.077	.464	.090	.092	.505
	.4	.8	.070	.084	.383	.047	.055	.412	.034	.032	.465
	.7	1	.070	.061	.163	.060	.045	.152	.074	.058	.161

TABLE 3.43

EMPIRICAL SIZES ($\alpha = 0.10$) OF W_γ, $\rho = 0.5$, $b_1 = b_2 = b$

		n	64			128			256		
estimation	γ	$\delta \backslash b$	0	.5	.9	0	.5	.9	0	.5	.9
$S_0(d), T_0(c)$	0	.4	.075	.993	1.00	.081	1.00	1.00	.107	1.00	1.00
	.2	.4	.151	.964	1.00	.128	.997	1.00	.111	1.00	1.00
	.4	.8	.082	.984	1.00	.080	1.00	1.00	.108	1.00	1.00
	.7	1	.122	.946	.997	.123	.995	.998	.123	1.00	1.00
$S_1(d), T_1(c)$	0	.4	.116	.135	.733	.080	.088	.778	.064	.068	.890
	.2	.4	.157	.094	.515	.179	.113	.520	.151	.143	.569
	.4	.8	.120	.127	.427	.094	.087	.461	.062	.068	.512
	.7	1	.112	.083	.191	.121	.075	.189	.129	.092	.195

Chapter 4

Semiparametric estimation of strong and weak co-integration

4.1 Introduction

As presented in Chapters 2 and 3, fully parametric estimation of parameter ν in (1.25) enjoys several attractive properties but, undoubtedly, is not free from the usual concern associated to any parametric prescription: the possible misspecification of the model driving the process u_t. Thus, although any criticism to our approach in those chapters would be shared by any parametric methodology, we felt that extending our analysis to a situation where knowledge of a parametric model for u_t is difficult to justify, would nicely complete our discussion about the estimation of ν in (1.25). We denote our approach as semiparametric, because while we consider the spectral density of u_t to be an unknown nonparametric function, we still deal with Type II fractional integrated processes (see Definition 1.3), which are less general than those of Robinson and Marinucci (1998, 2001) or other stationary and non-stationary long memory processes which commonly appear in the literature.

We analyse the cases of strong and weak co-integration in model (1.25), (1.26) simultaneously. As mentioned before, we do not treat the borderline case $\beta = 1/2$, which is indeed very specific. Although this could be a limitation of our analysis, we believe that, especially from an empirical perspective, is a minor one, as in practice it is not possible to asses whether the co-integrating gap β is exactly $1/2$ or arbitrarily close to it, situation covered by our allowed range of values for β. We propose two different classes of frequency domain estimates of ν, which are directly related to those in (2.18), (2.34) of Chapter 2. As anticipated, it could have been equally possible to consider time domain estimates in the spirit of those in Chapter 3, based on an $AR(p)$ representation of u_t, with p tending suitably to infinity with n. We preferred instead the more aesthetic and computationally appealing frequency domain ones, for which we allow for simultaneous consideration of both full and narrow band approaches.

In case of strong co-integration, properties of our estimates mimic those achieved in the parametric setting: n^{β}-consistency, mixed-normal asymptotics, and first order asymptotics unaffected by insertion of estimates of the nuisance parameters which in our present framework are γ, δ and the nonparametric function $f(\lambda)$. In fact, this result extends to the situation of strong fractional co-integration with possibly unknown orders, the well established fact, developed in the $CI(1,1)$ co-integration literature, that parametric assumptions about the model generating the observables are not necessary in order to obtain optimal asymptotic theory under Gaussianity. For example, as described in detail in Chapter 1, Phillips and Hansen (1990) and Phillips (1991b) showed that the same result as in Phillips (1991a) is obtainable without assuming knowledge of the parametric structure generating the observables.

We also consider the case of weak fractional co-integration, and proposed narrow band estimates of the co-integrating parameter ν, which although do not share the parametric optimal rate, $\sqrt{n}$ (achieved by our estimates in Chapter 3), are also asymptotically normal. Our estimates are comparable to those of Christensen and Nielsen (2001), who achieved similar results to ours (only for the case of stationary co-integration) under much stronger conditions on the structure of the underlying error input process u_t.

In the present semiparametric situation, the issue of estimating nuisance parameters with certain required properties is more delicate than in our previous parametric setting, and in this chapter we also comment on sensible estimation procedures for those parameters. We do not provide a proper theoretical justification of these methods, but this would mainly require extending previously derived results, as those developed by Robinson (1995a,b), Velasco (1999,a,b) and more recently by Robinson and Henry (2003). Thus, we are content with just proposing some methods which surely would offer, under certain regularity conditions, the desired properties. As presented in Assumptions 4.2, 4.3, 4.2°, below, conditions on the estimates of the nuisance parameters involve both the rate of convergence of these estimates and also the rate at which the bandwidth, m, which defines the "band" structure of the proposed estimate of ν (see (4.2), (4.3), (4.35), (4.36) below), evolves. As it will become apparent later, even if the estimates of the nuisance parameters are relatively slow, those conditions could be still satisfied by simply constraining the rate of growth of m. In case of strong co-integration, we could nicely take advantage of this result, as in this case, the rate of convergence of our estimates of ν is not affected by m. However, under weak co-integration the picture changes, as the convergence rate of our estimates of ν is positively related to the rate at which m increases, and in particular, slower estimates of the nuisance parameters imply slower feasible estimates of ν (although still asymptotically normal).

Next section is devoted to describing the first class of estimates, denoted as "optimally" weighted estimates. Section 4.3 presents and analyses the properties of the second class, the "zero-frequency" weighted estimates. Proofs for the main results in these two sections are collected in the Appendix 4. Finally, Section 4.4 contains Monte Carlo evidence of finite sample performance.

4.2 The "optimally" weighted class of estimates

As in the previous two chapters, we also consider now model (1.25), (1.26) with (1.30), $\beta \neq 1/2$. Noting (2.2), (2.3), (2.4), (2.5), considering a certain nonparametric estimate of $f(\lambda)$ (see (1.28)), $\widetilde{f}(\lambda)$, we could define

$$\widetilde{p}(\lambda) = \zeta' \widetilde{f}(\lambda)^{-1}, \ \widetilde{q}(\lambda) = \zeta' \widetilde{f}(\lambda)^{-1} \zeta. \tag{4.1}$$

Denoting by λ_j the Fourier frequencies (see Chapter 1), we could set in a similar way as in Robinson and Marinucci (1998, 2001),

$$\widetilde{a}_m(c,d) = \sum_{j=0}^{m} \operatorname{Re}\left\{c_j \widetilde{p}(\lambda_j) I_{z(c,d)x(c)}(\lambda_j)\right\}, \tag{4.2}$$

$$\widetilde{b}_m(c) = \sum_{j=0}^{m} \operatorname{Re}\left\{c_j \widetilde{q}(\lambda_j) I_{x(c)}(\lambda_j)\right\}, \tag{4.3}$$

noting (1.43), (1.44), for an integer m such that

$$m \to \infty \text{ as } n \to \infty, \quad 1 \leq m \leq n/2, \tag{4.4}$$

where $c_j = 1$, $j = 0, n/2$, $c_j = 2$, otherwise. Similarly, we could also define $a_m(c,d)$, $b_m(c)$ as (4.2), (4.3), but replacing $p(\lambda)$, $q(\lambda)$ (see (1.62)) by $\widetilde{p}(\lambda)$, $\widetilde{q}(\lambda)$ respectively. Thus, defining

$$\overline{\nu}_m(c,d) = \frac{a_m(c,d)}{b_m(c)}, \ \widetilde{\nu}_m(c,d) = \frac{\widetilde{a}_m(c,d)}{\widetilde{b}_m(c)}, \tag{4.5}$$

we could consider, as in Chapter 2, the following set of estimates

$$\overline{\nu}_m(\gamma,\delta), \ \widetilde{\nu}_m(\gamma,\delta), \ \widetilde{\nu}_m(\widehat{\gamma},\delta), \ \widetilde{\nu}_m(\gamma,\widehat{\delta}), \ \widetilde{\nu}_m(\widehat{\gamma},\widehat{\delta}), \tag{4.6}$$

for certain estimates $\widehat{\gamma}$, $\widehat{\delta}$ of γ, δ, to be described subsequently. Note that for the particular choice $m = [n/2]$, our set of estimates (4.6) is closely related to (2.18) in Chapter 2. This similarity comes from the fact that due to the symmetry of the real part of a periodogram about $\lambda = 0$ and $\lambda = \pi$,

$$a_{[n/2]}(c,d) = \sum_{j=1}^{n} p(\lambda_j) I_{z(c,d)x(c)}(\lambda_j), \ b_{[n/2]}(c) = \sum_{j=1}^{n} q(\lambda_j) I_{x(c)}(\lambda_j), \tag{4.7}$$

which are the same expressions as (2.15), (2.16) in Chapter 2 when evaluated at $h = \theta$, implying that $\overline{\nu}_{[n/2]}(\gamma,\delta) = \widehat{\nu}(\gamma,\delta,\theta)$ given in (2.18) in Chapter 2, whereas the rest of the estimates in (4.6) represent a natural extension of estimates (2.18) in Chapter 2, allowing for non-parametric estimates of the spectral density at different frequencies instead of parametric ones. For this reason, we refer to these estimates as full band estimates. Note also that for $m = [n/2]$, expressions inside braces in (4.2), (4.3) are real, so our notation is certainly redundant in this case.

When $m < [n/2]$, the most interesting case is when $m/n \to 0$ as $n \to \infty$, in which case estimates (4.6) are the narrow band versions of estimates with $m = [n/2]$, being this the only situation we consider in case of weak fractional co-integration with $\beta < 1/2$. The motivation for considering also a narrow band approach is basically that estimation of the parameter of the relation of co-integration, relates to estimation of a long run equilibrium relationship, so we could just focus on a band of frequencies near frequency 0, with the hope that the suppression of "high" frequencies does not affect the asymptotic properties (to first order) of our estimates, while perhaps dropping frequencies helps bias reduction in finite samples, as for the narrow band estimate of Robinson and Marinucci (1998, 2001).

As we might hint later, the rates of convergence of the estimates of the spectral density $f(\lambda)$ and orders of integration are very much dependent on the smoothness of $f(\lambda)$. These estimates, as it could be inferred from Chapter 2, need to reach rates very close to the parametric ones in certain circumstances, but this is only achievable in case $f(\lambda)$ is smooth enough, with a possibly very large number of existing derivatives. Thus, in case we suspect this kind of smoothness condition does not hold in our case, in principle, feasible full band estimates would not share the same asymptotic properties as the infeasible one when β is only slightly bigger than $1/2$. Then, as we will show later, the main point in favour of narrow band estimates is that even in the hypothetical case that relatively slow estimates of the nuisance parameters are available, feasible narrow band estimates could enjoy the same asymptotics as infeasible full band estimates given in Chapter 2, being this achievable by constraining the rate of growth of m accordingly. Thus, apart from its plausible "improved" finite sample behaviour over estimates with $m = [n/2]$, narrow band estimates are interesting from a theoretical point of view, as in certain cases, they will reflect the "sacrifice" made on the rate of growth of m under minimal conditions of smoothness of $f(\lambda)$. This narrow band approach was followed by Phillips (1991b) for the standard case in the unit root literature $\gamma = 0$, $\delta = 1$.

Our aim will be to find conditions which guaranteed a uniform behaviour for all the estimates in (4.6) under both situations of strong and weak fractional cointegration. First, related to the bivariate process u_t, we will work under Assumption 2.1 (see Chapter 2). As in Chapter 2, this assumption enables us to apply the functional limit theorem of Marinucci and Robinson (2000) to the purely nonstationary process $x_t(\gamma)$ when $\beta > 1/2$, as is required to characterize the limit distribution of our estimates of ν. The application of this functional central limit theorem is the reason for the need of a global smoothness assumption, even for the narrow band estimates. Assumption 2.1 is the only condition needed in order to calculate the asymptotic distribution of the infeasible estimate $\overline{\nu}_m(\gamma,\delta)$ (the one given in Theorem 2.1, for $\beta > 1/2$) as long as $m \to \infty$.

Noting (2.29), (2.30), (2.31), define the random variable

$$V = \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 \, dr \right\}^{-1} 2\pi\zeta' A(1)^{-1'} \Omega^{-1} \int_0^1 \widetilde{W}(r;\beta) \, dW(r), \tag{4.8}$$

and denote by $f_{ij}(\lambda)$, $f^{ij}(\lambda)$ the (i,j)th components of $f(\lambda)$, $f^{-1}(\lambda)$ respectively.

Theorem 4.1. *Let (1.25), (1.26), (1.29), (1.30), (4.4) and Assumption 2.1 hold. Then, as $n \to \infty$,*

(i) if $\beta > 1/2$
$$n^{\beta}(\overline{\nu}_m(\gamma,\delta) - \nu) \Rightarrow V; \tag{4.9}$$

(ii) if $\beta < 1/2$ *and*

$$m^{\beta-1/2}\log^{1/2} n + m^{3+2\eta}n^{-2-2\eta} \to 0, \tag{4.10}$$

$$m^{\frac{1}{2}}\lambda_m^{-\beta}(\overline{\nu}_m(\gamma,\delta) - \nu) \to_d N\left(0, \frac{1-2\beta}{2f^{11}(0)\, f_{22}(0)}\right). \tag{4.11}$$

As mentioned in Chapter 2, the variates $\zeta' A(1)^{-1\prime}\Omega^{-1}W(r)$ and $\widetilde{W}(r;\beta)$ are uncorrelated, and thus by Gaussianity independent, so (4.8) indicates mixed-normal asymptotics. As a consequence of this property and of some steps in the proof of Theorem 4.1, given in the Appendix 4,

$$b_m(\gamma)(\overline{\nu}_m(\gamma,\delta) - \nu)^2 \to_d \chi_1^2, \tag{4.12}$$

under the various conditions specified in Theorem 4.1. Also, note that the condition on the second term of the left side of (4.10) is similar to A4' and A4 in Robinson (1995b) and Lobato (1999) respectively, imposing an upper bound to the rate of increase of m with n. Under our Assumption 2.1, in their notation $\beta = 1+\eta$. (4.11) indicates that, in presence of weak fractional co-integration, our proposed estimate is in general faster than the narrow band estimate of Robinson and Marinucci (1998, 2001) (see Chapter 1). As shown in Christensen and Nielsen (2001), this latter estimate enjoys the same rate of convergence as our estimate under their very strong A', which in our framework would imply that the coherency at frequency 0 between the processes u_{1t} and u_{2t} is zero. This condition does not hold in general for u_t being an ARMA process, where the rate of convergence of the narrow band estimates would be given by Theorem 3.1 in Robinson and Marinucci (1998), where they conjectured that their derived rate, $n^{\beta}m^{-\beta}$, was sharp. Note also that Christensen and Nielsen gave results for a similar model to (1.25), (1.26), with covariance stationary observables and co-integrating error, with memories δ, γ respectively, satisfying $0 \leq \gamma < \delta < 1/2$, $\delta + \gamma < 1/2$.

As in the fully parametric cases described in Chapters 2 and 3, in order to insert estimated parameters further regularity conditions are needed.

Assumption 4.1. *There exists* $K < \infty$ *such that*

$$|\widehat{\gamma}| + \left|\widehat{\delta}\right| \leq K, \tag{4.13}$$

and $\kappa > 0$ *such that*

$$\widehat{\gamma} = \gamma + O_p\left(n^{-\kappa}\right), \quad \widehat{\delta} = \delta + O_p\left(n^{-\kappa}\right), \tag{4.14}$$

where, as $n \to \infty$

$$n^{-\kappa}m^{1-\max\{\min\{\beta,1\},1/2\}}\log m \to 0. \tag{4.15}$$

Assumption 4.1 is unprimitive and very similar to Assumption 2.3. Undoubtedly, the search for particular estimates of γ, δ to satisfy (4.14) and (4.15) in the present framework could entail some difficulties. In view of (4.15), semiparametric methods, like the log-periodogram due to Geweke and Porter-Hudak (1983), whose asymptotic properties were developed by Robinson (1995a), the Gaussian semiparametric Whittle estimator hinted by Künsch (1987) and studied by Robinson (1995b), or the ones proposed by Hurvich, Deo and Brodsky (1998), Velasco (1999a,b), might not be valid in certain cases (when β is close to $1/2$ and $m = [n/2]$ for example) due to their relatively slow rate of convergence. As shown by Giraitis, Robinson and Samarov (1997) and Hurvich, Deo and Brodsky (1998), the typical rate of convergence of these local semiparametric methods is limited to $n^{2/5}$. Alternative methods based on nonparametric assumptions about the short memory component of a fractionally integrated process have been proposed in order to improve this rate of convergence. Here, Moulines and Soulier (1999), Bhansali and Kokoszka (1999) and Hurvich and Brodsky (2001) have proposed broad-band approaches originally motivated by Janacek (1982), where a nonparametric estimate of the spectrum at all frequencies is used. Typically, the rate of convergence of the estimate of the order of integration depends on the smoothness of the short memory component of the process, and the rate (4.14) with

$$\kappa = \kappa_n = \frac{1}{2}\left(1 - \log^{-1} n \log\left(\log n\right)\right), \tag{4.16}$$

that is the rate $n^{1/2} \log^{-1/2} n$, is achievable in case the short memory component $f(\lambda)$ is analytic. This is a relevant result here, as when m/n does not tend to 0 as n tends to infinity, for the case of strong fractional co-integration, this rate would suffice for any value of $\beta > 1/2$. A different approach with the aim of obtaining bias-reduction and hence improvements in rates of convergence was considered by Robinson and Henry (2003). They proposed a very general narrow-band estimate which, depending on certain user-chosen parameter and function, could be viewed as Gaussian semiparametric, log-periodogram or a mixture of both, achieving bias-reduction by the use of higher order kernels. They obtained for the estimates of the orders rates as $n^{1/2-\epsilon}$, for possibly arbitrarily small $\epsilon > 0$, where ϵ basically depends on the number of existing derivatives of the spectrum of a long memory process near frequency 0. Undoubtedly, their approach could be accommodated to our framework of (possibly non-observable) processes with arbitrarily large memory, but, as in practice β is unknown, even in the situation where $f(\lambda)$ is analytic near frequency 0, this method does not allow us treat all $\beta > 1/2$. Andrews and Sun (2001) achieved similar improvements in convergence rates by extending the Gaussian semiparametric estimate in Robinson (1995b) through the use of local polynomials instead. In any case, all these estimation methods are given for covariance stationary long memory processes, including the so-called Type I fractionally integrated (see Definition 1.2). This differs substantially from our situation, where, as it is clear from (1.25), (1.26), we have to deal with Type II fractionally integrated processes of arbitrarily large memory. Undoubtedly, the use of tapering seems unavoidable in our case, but our guess is that similar results, just taking into account the inflation in the variance due to tapering, to the ones in Robinson and Henry (2003) or Andrews and Sun (2001) are going to apply to our type of processes following the results in Robinson (2002), at least when $\beta > 1/2$. Related to this, it is important to note that estimation of γ requires a preliminary estimate of ν, as the process $y_t - \nu x_t$ is unobservable. In view of results in Robinson (2002), it could be shown that this would not affect first order asymptotic properties of the chosen estimation procedure of γ when $\beta > 1/2$, but if $\beta < 1/2$, any estimation method produces relatively slow estimates of ν. Here, our guess is that κ in (4.14), could only be arbitrarily close to β, not to $1/2$, regardless of the smoothness conditions enjoyed by $f(\lambda)$, in case we have a preliminary estimate of ν whose rate of convergence is $n^{\beta-\epsilon}$ for a certain arbitrarily small $\epsilon > 0$. This is a strong assumption, and certainly the OLS does not satisfy this condition for every (γ, δ) combination if $\beta < 1$, but in this case the narrow band estimate proposed by Robinson and Marinucci (1998, 2001) suffices.

Similarly, we establish unprimitive conditions related to the nonparametric estimate of the spectral density $f(\lambda)$.

Assumption 4.2. *Uniformly in j, there exist $\varkappa > 0$, $\phi > 0$, such that*

$$\widetilde{f}(\lambda_j) - f(\lambda_j) = O_p\left(n^{-\varkappa}\right), \tag{4.17}$$

$$\widetilde{f}(\lambda_{j+1}) - f(\lambda_{j+1}) - \left(\widetilde{f}(\lambda_j) - f(\lambda_j)\right) = O_p\left(n^{-\phi}\right), \tag{4.18}$$

where, as $n \to \infty$

$$n^{-\varkappa} m^{1-\max\{\min\{\beta,1\},1/2\}} \to 0, \tag{4.19}$$

$$n^{-\phi} m^{2-\max\{\min\{\beta,1\},1/2\}} \to 0. \tag{4.20}$$

As for the estimates of the orders, when m/n does not tend to zero, estimates with $\varkappa$, ϕ arbitrarily close to $1/2$ and $3/2$ respectively could be needed in case of strong fractional co-integration with β just above $1/2$. As noted before, the full band case is not allowed for the weak fractional co-integration situation, so depending on $\varkappa$, ϕ, we could adjust m accordingly so that (4.19), (4.20) are satisfied, procedure also valid when $\beta > 1/2$. While we could guess that a value for $\varkappa$ similar to the one for κ in (4.16) is achievable under analycity of $f(\lambda)$, as it could be inferred from Moulines and Soulier (1999), it could be proven that for standard spectral density estimates similar results as for the estimates of the orders hold. These estimates could be based on residuals

$$\widetilde{u}_t = \left[y_t(\widehat{\gamma}) - \widehat{\nu} x_t(\widehat{\gamma}), x_t(\widehat{\delta})\right]', \tag{4.21}$$

for certain preliminary estimate of ν, $\widehat{\nu}$, and $\widetilde{f}(\lambda)$ could be for example the weighted periodogram

$$\widetilde{f}_1(\lambda) = \frac{2\pi b}{n} \sum_{j=-\infty}^{\infty} \overline{K}(b(\lambda - \lambda_j)) I_{\widetilde{u}}(\lambda_j) = \frac{2\pi}{n} \sum_{j=1}^{n} \overline{K}_b(\lambda - \lambda_j) I_{\widetilde{u}}(\lambda_j), \tag{4.22}$$

where

$$\overline{K}_b(\lambda) = b \sum_{j=-\infty}^{\infty} \overline{K}(b(\lambda + 2\pi j)), \tag{4.23}$$

and $\overline{K}$ is a real and even function, $b \to \infty$ as $n \to \infty$ but more slowly than n, or the weighted autocovariances estimate

$$\widetilde{f}_2(\lambda) = \int_{-\pi}^{\pi} \overline{K}_b(\lambda - \mu) I_{\widetilde{u}}(\mu) d\mu = \frac{1}{2\pi} \sum_{s=1-n}^{n-1} k\left(\frac{s}{b}\right) \widetilde{\gamma}_u(s) e^{-is\lambda}, \tag{4.24}$$

where

$$\begin{aligned} \widetilde{\gamma}_u(s) &= \frac{1}{n}\sum_{t=1}^{n-s} \widetilde{u}_t \widetilde{u}'_{t+s},\ s \geq 0, & (4.25)\\ &= \widetilde{\gamma}_u{}'(-s),\ s < 0, & (4.26) \end{aligned}$$

and

$$k(x) = \int_{-\infty}^{\infty} \overline{K}(\lambda) e^{ix\lambda} d\lambda,\ x \in \Re,\ k\left(\frac{r}{b}\right) = \int_{-\pi}^{\pi} \overline{K}_b(\lambda) e^{ir\lambda} d\lambda. \tag{4.27}$$

For both estimates, it could be proven that similar results as the ones for the estimates of the orders of integration could be achieved for appropriate choices of the kernel functions $\overline{K}$ and k. In fact, apart from other regularity conditions, assuming for example

$$|1 - k(x)| < K|x|^h \text{ for some } h > 0, \tag{4.28}$$

where $h \geq s$, s indicating that $A(e^{i\lambda})$ is s times differentiable in $\lambda \in [-\pi, \pi]$ with sth derivative in $Lip(\eta)$, $\eta > 1/2$, means that the function $k(x)$ is locally (in a neighbourhood of 0) $Lip(h)$. If $h > 1$, this implies that $d^c k(x)/dx^c = 0$ for any $c < h$, so bias reduction is possible provided the spectral density $f(\lambda)$ is smooth enough. Condition (4.28) relates to what Parzen (1957) describes as characteristic exponent of a kernel $k(x)$, that is the largest number h such that $k^{(h)}$ exists and is finite (nonzero), where

$$k^{(h)} = \lim_{x \to 0} \frac{1 - k(x)}{|x|^h}. \tag{4.29}$$

In fact, this condition relates closely to the idea of higher order kernels, as (4.29) implies that $d^c k(x)/dx^c = 0$ for any $c < h$. This, in view of (4.27), readily implies that

$$\int_{-\pi}^{\pi} \mu^c \overline{K}_b(\mu) d\mu = 0, \tag{4.30}$$

for any $c < h$, which is the basis for bias reduction in nonparametric estimation of the spectral density. Of course, the higher h is chosen, the higher the rate of convergence of our estimates will be. As Robinson (1991) mentions, condition (4.29) holds for $h = 1, 2$ for many of the usual kernels, but it seems that in case the h required is very large, a careful choice of the covariance averaging kernel is required. One could use, for example, a continuous impulse spline,

$$k(x) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \frac{\alpha^q}{r^q + \alpha^q} e^{-irx}, \tag{4.31}$$

as proposed by Cogburn and Davis (1974) with differential operator $L = d^{q/2}/dx^{q/2}$ (which in their view is the natural choice), with q being an even number. For an appropriate choice of α, (4.31) could lead to estimates of the spectral density with rates of convergence arbitrarily close to $n^{1/2}$, for certain smoothness conditions on $f(\lambda)$. Finally, note that under certain regularity conditions, estimating the spectral density from the true error process u_t would produce consistent estimates for any choice of b. As we estimate $f(\lambda)$ from residuals, the rate of growth of b in general cannot be chosen freely, and has to account for this "residuals" effect.

Now, we present a theorem collecting the results related to the infeasible estimate stressed in the traditional co-integration literature with $\gamma = 0$, $\delta = 1$, $\widetilde{\nu}_m(\gamma, \delta)$, the "more" feasible estimates $\widetilde{\nu}_m(\widehat{\gamma}, \delta)$, $\widetilde{\nu}_m(\gamma, \widehat{\delta})$ and the fully feasible estimate $\widetilde{\nu}_m(\widehat{\gamma}, \widehat{\delta})$. We denote by ν_m^* any of this four estimates.

Theorem 4.2. *Let (1.25), (1.26), (1.29), (1.30), (4.4) and Assumptions 2.1, 4.1,.4.2 hold. Then, as $n \to \infty$,*

(i) if $\beta > 1/2$

$$n^{\beta}(\nu_m^* - \nu) \Rightarrow V; \tag{4.32}$$

(ii) if $\beta < 1/2$ and (4.10) hold

$$m^{\frac{1}{2}}\lambda_m^{-\beta}(\nu_m^* - \nu) \to_d N\left(0, \frac{1-2\beta}{2f^{11}(0)\, f_{22}(0)}\right). \tag{4.33}$$

The proof of Theorem 4.2 is given in the Appendix 4. As in the fully infeasible case, following the steps given in the proof of Theorem 4.2, it can be easily shown that denoting by b_m^* either $\widetilde{b}_m(\gamma)$ or $\widetilde{b}_m(\widehat{\gamma})$,

$$b_m^*(\nu_m^* - \nu)^2 \to_d \chi_1^2, \tag{4.34}$$

under the various conditions of Theorem 4.2.

4.3 The "zero-frequency" weighted class of estimates

As in (4.2), (4.3), we could define

$$\widetilde{a}_m^{o}(c,d) = \mathrm{Re}\left\{\widetilde{p}(0)\sum_{j=0}^{m} c_j I_{z(c,d)x(c)}(\lambda_j)\right\}, \tag{4.35}$$

$$\widetilde{b}_m^{o}(c) = \mathrm{Re}\left\{\widetilde{q}(0)\sum_{j=0}^{m} c_j I_{x(c)}(\lambda_j)\right\}, \tag{4.36}$$

and $a_m^{o}(c,d)$, $b_m^{o}(c)$, where $p(0)$ and $q(0)$ replace $\widetilde{p}(0)$ and $\widetilde{q}(0)$ respectively in (4.35), (4.36). Thus, defining

$$\overline{\nu}_m^{o}(c,d) = \frac{a_m^{o}(c,d)}{b_m^{o}(c)},\ \widetilde{\nu}_m^{o}(c,d) = \frac{\widetilde{a}_m^{o}(c,d)}{\widetilde{b}_m^{o}(c)}, \tag{4.37}$$

we could consider, as before, the following set of estimates

$$\overline{\nu}_m^{\mathrm{o}}(\gamma,\delta),\ \widetilde{\nu}_m^{\mathrm{o}}(\gamma,\delta),\ \widetilde{\nu}_m^{\mathrm{o}}(\widehat{\gamma},\delta),\ \widetilde{\nu}_m^{\mathrm{o}}(\gamma,\widehat{\delta}),\ \widetilde{\nu}_m^{\mathrm{o}}(\widehat{\gamma},\widehat{\delta}). \tag{4.38}$$

Note that for the particular choice $m = [n/2]$,

$$\widetilde{a}_{[n/2]}^{\mathrm{o}}(c,d) = \frac{\widetilde{p}(0)}{2\pi}\sum_{t=1}^{n} z_t(c,d)\,x_t(c),\ \widetilde{b}_{[n/2]}^{\mathrm{o}}(c) = \frac{\widetilde{q}(0)}{2\pi}\sum_{t=1}^{n} x_t^2(c), \tag{4.39}$$

so in this case, our estimates could be naturally expressed in the time domain. These estimates are related to the ones of Theorem 2.2. As stated there, the lack of "optimal" weighting does not only produce dramatic change in the limiting distributional properties of the estimates, but also slower rate of convergence in some cases. Thus, we could exploit the bias reduction obtained by averaging over a degenerate band of frequencies (as opposite to full-band averaging), as shown in Robinson and Marinucci (1998, 2001), and thus compensate for the lack of optimal weighting. However, it is true that for frequencies arbitrarily close to 0, the weighting of these estimates is close to the optimal one, and this is precisely the reason why our approach works in this case. In fact, it was already mentioned in Chapter 2 that a narrow band approach could make the estimates in Theorem 2.2 have mixed normal asymptotics for the case $\beta = 1$, being this a straightforward implication of Theorem 4.3 of Robinson and Marinucci (2001). It could have been conjectured that this was also going to be the case for $1/2 < \beta < 1$, but this is not a completely straightforward implication of the results in Robinson and Marinucci (2001). Our purpose is to prove that under certain conditions the bias reduction due to narrow-band averaging is strong enough so to make the "zero-frequency" weighted estimates have mixed-normal asymptotics and optimal convergence rates even when $\beta \le 1$, in case of strong fractional co-integration.

The main advantage of these estimates is their simplicity, as no estimation of the spectral density at different Fourier frequencies is needed, so they could be preferable to the "optimally" weighted ones in terms of computational convenience.

We simplify slightly Assumptions 2.1 and 4.2 to accommodate for this kind of estimates.

Assumption 2.1°. *Assumption 2.1 holds with the condition*

$$\det\{A(1)\} \neq 0 \tag{4.40}$$

replacing (2.24).

Assumption 4.2°. *There exist $\varkappa > 0$ such that*

$$\widetilde{f}(0) - f(0) = O_p\left(n^{-\varkappa}\right), \tag{4.41}$$

where, as $n \to \infty$

$$n^{-\varkappa} m^{1-\max\{\beta,1/2\}} \to 0. \tag{4.42}$$

Denoting by $\nu_m^{\mathrm{o}*}$ any of the estimates in (4.38), we collect in one theorem the equivalent to Theorems 4.1 and 4.2 in the previous section.

Theorem 4.3. *Let (1.25), (1.26), (1.29), (1.30), (4.4) and Assumptions 2.1°, 4.1, 4.2° hold. Then, as $n \to \infty$,* *(i) if $\beta > 1/2$ and*

$$mn^{-\beta} \to 0, \tag{4.43}$$

$$n^{\beta}(\nu_m^{o*} - \nu) \Rightarrow V; \tag{4.44}$$

(ii) if $\beta < 1/2$ and (4.10) hold

$$m^{\frac{1}{2}}\lambda_m^{-\beta}(\nu_m^{o*}-\nu) \to_d N\left(0, \frac{1-2\beta}{2f^{11}(0)\, f_{22}(0)}\right). \tag{4.45}$$

We give the proof of this theorem in the Appendix 4, but just for the infeasible estimate $\overline{\nu}_m^{o}(\gamma, \delta)$, as the proof for the rest of the estimates in (4.38) follows immediately from the proof of Theorem 4.2. Note here that (4.43) is a new condition, which is not restrictive when $\beta > 1$, but for $\beta \leq 1$ has a very important implication, as in this case a choice of m such that m/n^{β} does not tend to 0 is not allowed for this type of estimates. As hinted before, the reason for this is that when $\beta \leq 1$ the nonstationarity of the process $x_t(\gamma)$ is not strong enough to compensate for the lack of optimal weighting, compensations which could be achieved by using a narrow instead of a full band approach. Again, under the various conditions of Theorem 4.3, denoting by b_m^{o*} any of $b_m^{o}(\gamma)$, $\widetilde{b}_m^{o}(\gamma)$ or $\widetilde{b}_m^{o}(\widehat{\gamma})$

$$b_m^{o*}(\nu_m^{o*} - \nu)^2 \to_d \chi_1^2. \tag{4.46}$$

4.4 Monte Carlo evidence

A Monte Carlo study was carried out with the aim of comparing the performance in terms of bias and standard deviation of our proposed estimates (in both situations where the orders of integration are assumed known and unknown), with an estimate which does not require any knowledge of either the orders of integration or the short memory structure of u_t in (1.25), (1.26), which is a band estimate given by $\overline{\nu}_0(m)$ (see (1.41)), for $m \leq [n/2]$, which, provided the chosen bandwidth m tends to infinity at a relatively slower rate than the sample size n, is the so-called narrow band estimate discussed in Chapter 1. Note that when $m = [n/2]$, this band estimate is identical to the OLS estimate given in (1.33). Apart from comparing these estimates, we also analyse the adjustment of the Wald statistics, corresponding to our different estimates of ν, to its limiting χ_1^2 distribution.

As in Chapter 2, we generated Gaussian ε_t with covariance matrix Ω having ijth element ω_{ij}, varying the correlation $\rho = \omega_{12}/(\omega_{11}\,\omega_{22})^{1/2}$, taking values 0, 0.5, -0.5, 0.75, fixing $\nu = \omega_{11} = \omega_{22} = 1$. We consider the combinations of integration orders corresponding to strong and weak fractional co-integration cases given in Chapters 2 and 3 respectively. Table 4.1 presents the different convergence rates of our proposed estimates and also the ones of the band estimate for both cases where $\rho \neq 0$ and $\rho = 0$. These rates are derived from our results in Theorem 4.1 and 4.2, and Robinson and Marinucci (1998, 2001). For the strong fractional co-integration case, the described rates for the band estimates apply for any $m \leq [n/2]$, $m \to \infty$, noting that the rates of our proposed estimates are optimal in this case. For the weak co-integration situation, we only consider narrow band estimates with $m/n \to 0$ as $n \to \infty$, noting that (4.10) needs to be satisfied.

TABLE 4.1
CONVERGENCE RATES:
BAND WITH $\rho \neq 0$, $\rho = 0$ AND OUR PROPOSED ESTIMATES

(γ, δ)	BAND, $\rho \neq 0$	BAND, $\rho = 0$	PROPOSED
$(0, 0.6)$	$n^{.6}m^{-.4}$	$n^{.6}$	$n^{.6}$
$(0, 1.2)$	$n^{1.2}$	$n^{1.2}$	$n^{1.2}$
$(0, 2)$	n^{2}	n^{2}	n^{2}
$(0.4, 1.2)$	$n^{.8}$	$n^{.8}$	$n^{.8}$
$(0.4, 2)$	$n^{1.6}$	$n^{1.6}$	$n^{1.6}$
$(0, 0.4)$	$n^{.4}m^{-.4}$	$n^{.4}m^{.1}$	$n^{.4}m^{.1}$
$(0.2, 0.4)$	$n^{.2}m^{-.2}$	$n^{.2}m^{.3}$	$n^{.2}m^{.3}$
$(0.4, 0.8)$	$n^{.4}$	$n^{.4}$	$n^{.4}m^{.1}$
$(0.7, 1)$	$n^{.3}$	$n^{.3}$	$n^{.3}m^{.2}$

We generated 1000 series of lengths $n = 64, 128, 256$, and choosing different bandwidths b (taking values 15,25,45, depending on whether n is 64,128,256 respectively), computed the nonparametric estimate of the spectral density as

$$\widetilde{f}(\lambda_j) = \frac{1}{2b+1} \sum_{k=j-b}^{j+b} I_{\widetilde{u}}(\lambda_k), \tag{4.47}$$

with

$$\widetilde{u}_t(c, d, a) = (y_t(c) - a x_t(c), x_t(d))', \tag{4.48}$$

where in all cases $a = \overline{\nu}_0(b)$ and $(c, d) = (\gamma, \delta)$ or $(\widehat{\gamma}, \widehat{\delta})$ depending on whether the orders of integration are considered to be known or unknown respectively. The estimates $\widehat{\gamma}$, $\widehat{\delta}$, are Robinson's (1995a) version of the log-periodogram estimates of Geweke and Porter-Hudak (1983) without trimming or pooling applied to the untapered series $y_t - \overline{\nu}_0(b) x_t$ and $\overline{x}_t$, where $\overline{x}_t = x_t$ for $\delta < 1$, $\overline{x}_t = \Delta x_t$ for $\delta \geq 1$, adding back one to the estimate of the order of $\overline{x}_t$ in this case to compute the final estimate of δ. b is also the chosen bandwidth for the semiparametric estimates of the orders of integration.

4.4.1 Strong fractional co-integration

In this case, we computed "optimally" weighted Infeasible estimates $\overline{\nu}_I$ and Feasible estimates $\overline{\nu}_F$, defined as

$$\overline{\nu}_I = \widetilde{\nu}_m(\gamma, \delta), \quad \overline{\nu}_F = \widetilde{\nu}_m(\widehat{\gamma}, \widehat{\delta}), \tag{4.49}$$

and also "zero-frequency" weighted ones, given by

$$\overline{\nu}_I^{o} = \widetilde{\nu}_m^{o}(\gamma, \delta), \quad \overline{\nu}_F^{o} = \widetilde{\nu}_m^{o}(\widehat{\gamma}, \widehat{\delta}). \tag{4.50}$$

We reported results for these four estimates and also for the Band estimate in (1.41), denoting $\overline{\nu}_B = \overline{\nu}_0(m)$, for three different sets of bandwidths m, given by (I,II,III)=(10,20,32), (20,40,64), (40,80,128), depending on whether $n = 64, 128, 256$ respectively. Note that the highest bandwidth (III) for each sample size corresponds to the full band case. In our experiment, related to the structure of u_t, we only considered cases 1.,2. and 3. described in the Monte Carlo section of Chapter 2.

Behaviour of the bias

Results concerning Monte Carlo bias (defined as the estimate minus ν) for the situation where u_t in (1.25), (1.26) is a purely white noise process are presented in Tables 4.2-4.9. Overall, $\overline{\nu}_I^o$, $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_F^o$, $\overline{\nu}_B$ are no worse than any of the other estimates in 167, 144, 104, 99 and 81 out of 180 cases respectively. These figures show the expected dominance of the infeasible estimates and also, the relatively better performance of the feasible estimates over the computationally simpler band estimate. In fact, when comparing *vis a vis* the behaviour of $\overline{\nu}_F$ and $\overline{\nu}_F^o$ relative to $\overline{\nu}_B$, the results are totally clear in favour of our two feasible estimates with relations (note the definition of this concept given in Chapter 2) 89/7 and 91/13 respectively, out of 180 cases. As expected, biases increase in absolute value with $|\rho|$, being this most noticeable for the cases where $\beta < 1$, for which there is also the biggest differences between our proposed feasible estimates and the band one. The "zero-frequency" weighted (ZW) infeasible estimate is slightly superior, especially for $\rho \neq 0$, to the "optimally" weighted (OW) infeasible one (with proportion 33/10), with the exception of the full band situation with $\beta = 0.6$, where as the theory predicts, $\overline{\nu}_I$ beats $\overline{\nu}_I^o$. On the contrary, the OW feasible estimate outperforms the ZW feasible with relation 53/12, differences being most noticeable for the full band situation. In general, biases decrease as n, β increase and m (with the exception of the case $\rho = 0$) decreases. The sign of the bias for the infeasible, band and feasible estimates (just when $\beta < 1$ for this latter class of estimates) is the one of ρ, being the opposite of ρ for the feasible estimates when $\beta > 1$.

Results for the AR situation are presented in Tables 4.10-4.25. Comparing these results with the ones for the white noise case, the only estimates which enjoy big improvements in the AR framework are $\overline{\nu}_B$ and the infeasible estimates, especially for the case $\beta = 0.6$, being this effect stronger the bigger are the AR parameters ϕ_i, $i = 1, 2$. $\overline{\nu}_F$ tends also to perform slightly better when the strongest autocorrelation structure is imposed for the case $\beta = 0.6$. When $\phi_i = 0.5$, overall, $\overline{\nu}_I^o$, $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_B$, $\overline{\nu}_F^o$ are no worse than any of the other estimates in 167, 126, 90, 89 and 87 out of 180 cases respectively. This general ordering shows the predominance of the infeasible estimates, but suggests an undervalued image of our proposed feasible estimates. In fact, both, $\overline{\nu}_F$ and $\overline{\nu}_F^o$ clearly beat $\overline{\nu}_B$, with relations 83/15 and 79/18 out of 180 cases respectively, being this predominance more noticeable as β decreases, with $\overline{\nu}_B$ showing a competitive behaviour only when n is small and β large. The ZW infeasible estimates clearly outperform the OW infeasible ones, with relation 42/1, being this superiority more evident when $\beta < 1$ and $\rho \neq 0$, even for the full band estimates, although in general the differences between these two classes of estimates are very small. Both feasible estimates behave in a rather similar way, as $\overline{\nu}_F$ beats

$\overline{\nu}_F^{\circ}$ just with relation 30/28. Here, $\overline{\nu}_F^{\circ}$ tends to behave slightly better than $\overline{\nu}_F$ when $\beta = 0.6$, especially for the two narrow band cases, whereas $\overline{\nu}_F$ improves over $\overline{\nu}_F^{\circ}$ when $(\gamma, \delta) = (0.4, 1.2)$, or for the full band situation when $(\gamma, \delta) = (0, 0.6)$, being this performance relatively better for smaller n. When ϕ_i moves to 0.9, the picture slightly changes. The overall ordering is pretty similar to the $\phi_i = 0.5$ situation, being $\overline{\nu}_I^{\circ}$, $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$, $\overline{\nu}_B$, no worse than any of the other estimates in 146, 136, 90, 81 and 76 out of 180 cases respectively. Again, $\overline{\nu}_F$ and $\overline{\nu}_F^{\circ}$ beat $\overline{\nu}_B$ (with relations 88/30 and 73/44 respectively), showing these figures the previously referred relative improvement of $\overline{\nu}_B$ when the AR structure is stronger. As before, $\overline{\nu}_B$ is competitive for the cases where β is large, while $\overline{\nu}_F^{\circ}$ does not show clear superiority over $\overline{\nu}_B$ when $|\rho| \neq 0$. While $\overline{\nu}_I^{\circ}$ still seems preferable to $\overline{\nu}_I$, this is less clear than in the $\phi_i = 0.5$ case, as now the relation is 13/4 in favour of the ZW estimate. On the contrary, the OW feasible superiority over the ZW one is now much clearer, with relation 58/4 in its favour. As in the white noise case, biases decrease as n and β increase and $|\rho|$ decreases, the sign of the bias following the same pattern as in the white noise situation.

Results for the MA case are given in Tables 4.26-4.41. Overall, the most remarkable feature here is that results are mainly unaffected by the value the MA parameter takes. The only relevant difference appears to be an small improvement when we move from $\psi_i = 0.5$ to $\psi_i = 0.9$ for the case $\beta = 0.6$, especially for the full band estimates. Also, results for both infeasible and feasible estimates are extremely similar to those in the white noise situation, with some small improvements over the white noise case for the band estimate when $\beta = 0.6$. The overall ordering of the different estimates is $\overline{\nu}_I^{\circ}$, $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$, $\overline{\nu}_B$, which are no worse than any of the others in 176, 130, 101, 94, 84 (or 176, 128, 103, 89, 83) respectively when $\psi_i = 0.5$ (or $\psi_i = 0.9$), out of 180 cases. Both feasible estimates present strong predominance over $\overline{\nu}_B$ in a more evident way than in the AR case, being this mainly based on a better behaviour when $\beta = 0.6, 0.8$. $\overline{\nu}_F$ dominates $\overline{\nu}_B$ with relation 87/10 (88/10) when $\psi_i = 0.5$ (0.9). Similarly, $\overline{\nu}_F^{\circ}$ beats $\overline{\nu}_B$ with proportion 84/14 (84/15) when $\psi_i = 0.5$ (0.9). In general, these values are very close to those obtained in the white noise case, and more favourable to our feasible estimates than in the AR situation (especially when $\phi_i = 0.9$). The main difference with respect to the white noise framework appears from the comparison of ZW and OW estimates. In general, the MA situation gives more support to the use of ZW estimates, as $\overline{\nu}_I^{\circ}$ is superior to $\overline{\nu}_I$ with relation 47/1 (48/0) for the case $\psi_i = 0.5$ (0.9), being these differences mainly based on the cases where $\beta < 1$. For the feasible estimates, the superiority of the ZW is less clear, but still evident, with relations 31/22 and 32/18 for $\psi_i = 0.5$ and 0,9 respectively, although differences are only noticeable for $\beta = 0.6$, with mixed evidence for $\beta = 0.8$. The general behaviour of biases (including their signs) when n, β, ρ and m change, described for the white noise and AR situations is maintained.

Behaviour of the standard deviation

Results corresponding to the white noise case are presented in Tables 4.42-4.49. Overall, the superiority of the infeasible estimates is clear, with a general ranking of $\overline{\nu}_I$, $\overline{\nu}_I^{\circ}$, $\overline{\nu}_B$, $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$, which are no worse than any of the other estimates in 145, 144, 82, 58 and 56 out of 180 cases respectively. $\overline{\nu}_B$ is only better than the infeasible estimates when $\beta = 0.6$ and $\rho = 0$, showing also certain less clear superiority when $\rho \neq 0$ for the same β, when n and m are small. The main difference with the results for the bias is that now, the band estimate emerges as competitive relative to the feasible estimates. For example, $\overline{\nu}_F$ and $\overline{\nu}_B$ show an extremely close behaviour, with relation 57/56 in favour of our proposed feasible estimate. Here, two features seem relevant. $\overline{\nu}_F$ is clearly superior to $\overline{\nu}_B$ for the two cases where $\beta < 1$, the relative dominance of the band estimate being most noticeable for the case $(\gamma, \delta) = (0, 1.2)$. Also, as sample size increases $\overline{\nu}_F$ performs relatively better than $\overline{\nu}_B$, being this a certainly supportive result. $\overline{\nu}_F^o$ clearly shows a worse behaviour than $\overline{\nu}_F$, which is superior to its zero-frequency counterpart with relation 68/15. Also, $\overline{\nu}_F^o$ is beaten by the band estimate with relation 75/32. Most of this effect is caused by the relative deterioration of $\overline{\nu}_F^o$ for the cases $\beta < 1$, especially for the full band situation. Both infeasible estimates behave in a similar way, with relation 29/27 in favour of $\overline{\nu}_I$, ZW being better than OW when $\beta = 0.6$ for the smallest bandwidth, the opposite happening for the full band estimates when $\beta < 1$. As expected, standard deviations decrease when n, β increase, but in general they are not very affected by variations in bandwidth. When $\rho = 0$, some decrease in standard deviations when m increases is apparent when $\beta = 0.6$, but for this β case this effect is reversed when $\rho = 0.75$.

Results for the AR cases are presented in Tables 4.50-4.65. The overall ranking of the estimates is relatively similar to the one in the white noise situation. When $\phi_i = 0.5$, $\overline{\nu}_I^o$, $\overline{\nu}_I$, $\overline{\nu}_B$, $\overline{\nu}_F$, $\overline{\nu}_F^o$, are no worse than any of the other estimates in 165, 148, 83, 46 and 45 out of 180 cases respectively. When $\phi_i = 0.9$, the main difference is that $\overline{\nu}_I$ performs better than $\overline{\nu}_I^o$, the rest of the estimates being ranked in the same way, with 147, 139, 92, 46 and 36 out of 180 cases being no worse than any of the other estimates respectively. In general, standard deviations tend to decrease as ϕ_i decreases (especially for both cases where $\beta < 1$). As the overall ranking shows, the infeasible estimates are superior to $\overline{\nu}_B$ except for the case $\beta = 0.6$ when $\rho = 0$ and $\phi_i = 0.5$, and also for some cases with $|\rho| = 0.5$ and n small. When $\phi_i = 0.9$, with $\rho \leq 0.5$ and $\beta = 0.6$, $\overline{\nu}_B$ is also better than the infeasible estimates, this fact being also apparent for the same β case when $\rho = 0.75$ and $n = 64$. One of the main differences with the white noise situation is that in general $\overline{\nu}_B$ beats the feasible estimates when $\phi_i = 0.5$ (with relations 77/29 and 97/14 respect to $\overline{\nu}_F$ and $\overline{\nu}_F^o$). When $\phi_i = 0.9$, this relative superiority of the band estimate is even more evident, as it dominates $\overline{\nu}_F$ and $\overline{\nu}_F^o$ with relations 92/30 and 120/9 respectively. This improved behaviour is mainly based on the results for $n = 64$ and $(\gamma, \delta) = (0, 1.2)$, the most favourable case for the feasible estimates being $(\gamma, \delta) = (0.4, 1.2)$. The reported similarity between the two infeasible estimates in the white noise case is less clear now. Although there are not big differences in values, when $\phi_i = 0.5$, $\overline{\nu}_I^o$ beats $\overline{\nu}_I$ with relation 26/6, whereas the opposite happens when $\phi_i = 0.9$, where the relation is 16/2 in favour of $\overline{\nu}_I$, the superiority in this case being almost exclusively based on the full band situation. On the contrary, the gap between the feasible estimates reported in the white noise situation is now augmented. $\overline{\nu}_F$ is now clearly better than $\overline{\nu}_F^o$ with relations 85/2 and 100/1 when $\phi_i = 0.5$ and 0.9 respectively.

Results for the MA case are given in Tables 4.66-4.81. For both feasible and infeasible estimates the values of standard deviations are very close to the ones in the white noise situation, without showing important differences with respect to the value of ψ_i. Certain general improvement in the behaviour of $\overline{\nu}_B$ over the white noise situation is noticeable. The infeasible estimates are superior to the rest, with overall ranking, out of 180 cases, of $\overline{\nu}_I^{\circ}$, $\overline{\nu}_I$, $\overline{\nu}_B$, $\overline{\nu}_F$, $\overline{\nu}_F^{\circ}$, being no worse than any of the other estimates in 164, 127, 76, 57, 51 and 165, 124, 86, 55, 49 cases when $\psi_i = 0.5$ and 0.9 respectively. The situation $\beta = 0.6$ with $n = 64$ is the only one where $\overline{\nu}_B$ performs better than the infeasible estimates. The MA framework is much more supportive than the AR one for our feasible estimate $\overline{\nu}_F$ in relation to $\overline{\nu}_B$. Thus, $\overline{\nu}_B$ is superior to $\overline{\nu}_F$, but only with relation 61/53 for both $\psi_i = 0.5$ and 0.9 cases. $\overline{\nu}_B$ clearly dominates $\overline{\nu}_F$ when $n = 64$, especially for the cases $(\gamma, \delta) = (0, 0.6)$ and $(0, 1.2)$, whereas $\overline{\nu}_F$ performs better than $\overline{\nu}_B$ for the case $(0.4, 1.2)$. As in the AR framework, $\overline{\nu}_B$ is superior to $\overline{\nu}_F^{\circ}$ with relations 76/36 and 79/35 for $\psi_i = 0.5$ and 0.9 cases respectively, being $(0.4, 1.2)$ the only case where in general $\overline{\nu}_F^{\circ}$ beats $\overline{\nu}_B$. In any case, the differences, especially when n is large, are not of serious concern. The ZW infeasible estimates clearly outperform $\overline{\nu}_I$, with relations 43/6 and 44/6 for $\psi_i = 0.5$ and 0.9 cases respectively, being $\rho = 0$ the case where both estimates differ the least. On the contrary, $\overline{\nu}_F$ beats $\overline{\nu}_F^{\circ}$, with relations 78/7 and 80/10 for $\psi_i = 0.5$ and 0.9 cases respectively, being the situation with $\beta < 1$ where the biggest differences appear. In any case, the values for both ZW and OW estimates, although showing the previous general patterns, are very similar.

Behaviour of empirical sizes

We next analyse the adjustment to their limiting χ_1^2 distribution of the Wald statistics W_I, W_F, W_I°, W_F°, defined as

$$W_I = \overline{b}_I \left(\overline{\nu}_I - 1\right)^2, \quad W_F = \overline{b}_F \left(\overline{\nu}_F - 1\right)^2, \tag{4.51}$$

$$W_I^{\circ} = \overline{b}_I^{\circ} \left(\overline{\nu}_I^{\circ} - 1\right)^2, \quad W_F^{\circ} = \overline{b}_F^{\circ} \left(\overline{\nu}_F^{\circ} - 1\right)^2, \tag{4.52}$$

where

$$\overline{b}_I = \widetilde{b}_m(\gamma), \quad \overline{b}_F = \widetilde{b}_m(\widehat{\gamma}), \tag{4.53}$$

$$\overline{b}_I^{\circ} = \widetilde{b}_m^{\circ}(\gamma), \quad \overline{b}_F^{\circ} = \widetilde{b}_m^{\circ}(\widehat{\gamma}). \tag{4.54}$$

Tables 4.82-4.89 contain empirical sizes corresponding to nominal $\alpha = 0.05, 0.10$, for the four values of ρ, when u_t is generated by a white noise process. Results corresponding to the infeasible Wald statistic W_I are on average too large, but certainly close to the nominal sizes, even for $n = 64$, for all values of ρ and m when $\beta > 1$, empirical sizes reacting as theory predicts when n increases. For the case $\beta = 0.8$, empirical sizes of W_I behave worse than in the previous situation when $n = 64$, but they react quickly in the appropriate direction, so that when $n = 256$, sizes for $\beta = 0.8$ are comparable to those corresponding to larger β. For this case, sizes are not very affected by changes in ρ, but the combination of simultaneous increases in $|\rho|$ and m seems to have certain deterioration effect. This is much more evident when $\beta = 0.6$, where in general empirical sizes are substantially higher than for all the previous β cases. For this situation, there is a clear worsening of the empirical sizes when $|\rho|$ and m increase, and also there is not evidence of the expected response to the increase in sample size, except for the case where $\rho = 0$, where values for W_I are not very far from those corresponding to cases with higher β. Empirical sizes for W_F are substantially larger than the ones for W_I, although in almost all cases react appropriately when n increases, being the worst case the one for $\beta = 0.6$ with $\rho = 0.75$. Clearly, results are closer to the nominal sizes for $\alpha = 0.10$ than for $\alpha = 0.05$. In general, W_F behaves better as $|\rho|$ decreases and β increases, the deterioration of the empirical sizes when $|\rho|$ increases being more evident when $\beta < 1$. The increase in m does not almost have any effect in W_F when $\rho = 0$, or $\rho \neq 0$ with $\beta > 1$. When $\rho \neq 0$, W_F suffers certain deterioration when $\beta < 1$, which is very important for the case $\beta = 0.6$, with empirical sizes corresponding to $\rho = 0.75$ and the largest bandwidth being unacceptably large in this latter case. When $\beta > 1$, empirical sizes of W_I^o and W_F^o are extremely similar to the ones of W_I, W_F, for all ρ, m, n and β. For $\beta = 0.8$, both W_I^o and especially W_F^o, behave worse than W_I and W_F respectively, this effect being more noticeable as m increases, which is a predicted result by the theory. Similar effect is evident when $\beta = 0.6$, the relative deterioration of W_I^o and W_F^o being more important now.

Results on empirical sizes for the AR situation are given in Tables 4.90-4.105. Clearly, W_I is heavily damaged with respect to the white noise framework, with values corresponding to $\phi_i = 0.9$ being unacceptably large. When $\beta > 1$, these values are relatively unaffected by ρ, m and β, decreasing in all cases when n increases, quite slowly for $\phi_i = 0.9$, however. For $\phi_i = 0.5$, $\beta = 0.8$, W_I is also not very affected by the value of ρ, but certain increase in sizes along with increases in m is noticeable, especially for large $|\rho|$. When $\phi_i = 0.9$, this effect is less important. For the case $\beta = 0.6$ and $\phi_i = 0.5$, sizes of W_I clearly increase with $|\rho|$ and m, being this latter effect stronger as $|\rho|$ increases. Again, this is less evident when $\phi_i = 0.9$, but sizes are very large here, although not far from ones corresponding to bigger β. The behaviour of W_F is one of the most striking results in our Monte Carlo experiment. For $\phi_i = 0.5$ and $\rho \leq 0.5$, empirical sizes are substantially smaller than those corresponding to the infeasible statistic W_I, especially when β is large. Again, when $\beta > 1$, sizes are relatively unaffected by m, with small increments as $|\rho|$ increases (especially for $\beta = 1.2$), and always decrease as n increases, with empirical sizes very often being smaller than the nominal ones when $n = 256$. In fact, when $\phi_i = 0.9$, empirical sizes when $\beta > 1$ behave qualitatively in a similar way to the $\phi_i = 0.5$ case, but they are importantly pushed down, so that when $n = 256$ empirical sizes are much smaller than the nominal ones. The behaviour of the empirical sizes when $\beta < 1$ is interesting. When $\phi_i = 0.5$ and $\rho = 0$, they are substantially smaller than those corresponding to W_I, being very close to the nominal ones when $n = 256$. As $|\rho|$ increases, this pattern is less clear, and while when $|\rho| = 0.5$ sizes are still better for W_F (only slightly when $\beta = 0.6$ though), they are clearly worse for $\rho = 0.75$, a very important deterioration as $|\rho|$ increases taking place, whose effect is more evident as m increases, especially for $\beta = 0.6$. This very strong worsening of the behaviour of W_F as $|\rho|$ increases is also observed when $\phi_i = 0.9$, but here, even for the most adverse situation where $\beta = 0.6$ and $\rho = 0.75$, empirical sizes of W_F are better than the ones of W_I for any m, as now sizes corresponding to W_F decrease when ϕ_i increases. Generally, W_I^o, W_F^o perform very similarly but slightly better than W_I, W_F, except for the cases where $\beta = 0.8$

or $\beta = 0.6$ and $\phi_i = 0.9$, for which W_F tends to behave better than W_F°.

Results for the MA framework are presented in Tables 4.106-4.121. W_I behaves in a very similar way (with sizes slightly larger) to the white noise situation. For $n = 256$, empirical sizes are quite close to the nominal ones, except for the $\beta = 0.6$ case. This holds for both values of ψ_i, the "close to noninvertibility" situation not showing any important difference with the one where $\psi_i = 0.5$. Sizes for W_F, although still worse than those of W_I, are closer to them now than in the situation with $\phi_i = \psi_i = 0$, $i = 1, 2$. Again, the effect of increasing the MA parameter does not have any important effect. Also, W_I° and W_F° perform relatively better than W_I and W_F respectively, the clearest improvement appearing when $\beta = 0.6$.

4.4.2 Weak fractional co-integration

For this case, we simplified and modified substantially the content of the Monte Carlo experiment. We just present results corresponding to the simplest case where $\phi_i = \psi_i = 0$, $i = 1, 2$, in (2.41) for estimates in (4.49) and also for the band estimate $\overline{\nu}_B$, for the different sets of bandwidths (I,II,III)=(2,8,15), (2,12,20), (3,15,25), for $n = 64, 128, 256$ respectively. These different choices for (I,II,III) represent in all cases narrow band situations. Instead of "zero-frequency" weighted estimates, we report results corresponding to infeasible and feasible two-steps estimates, given by $\overline{\nu}_2^I$ and $\overline{\nu}_2^F$ respectively. Now, $\overline{\nu}_2^I$ is calculated from an estimate of the spectral density based on (two-steps) residuals $\widetilde{u}_t\,(\gamma, \delta, \overline{\nu}_I)$, noting (4.48). Similarly, in order to compute $\overline{\nu}_2^F$, the estimate of γ is calculated from residuals $y_t - \overline{\nu}_F x_t$, and given this estimate, say $\widehat{\gamma}_2$, the estimate of the spectral density is based on residuals $\widetilde{u}_t(\widehat{\gamma}_2, \widehat{\delta}, \overline{\nu}_F)$. Results for these two-steps estimates are reported for the same set of bandwidths specified before.

Behaviour of the bias

Results for the bias are presented in Tables 4.122-4.129. The overall ranking presents an overwhelming dominance of the two-steps infeasible estimate. This ranking is $\overline{\nu}_2^I$, $\overline{\nu}_I$, $\overline{\nu}_B$, $\overline{\nu}_2^F$, $\overline{\nu}_F$, which are no worse than any of the other estimates in 134, 10, 9, 8 and 3 out of 144 cases respectively. As we will show later, this ranking damages strongly the image of the performance of the feasible estimates, which, specially for the two-steps estimate is excellent. The behaviour of the bias differs substantially depending on whether $\rho = 0$ or $\rho \neq 0$. In the former case, although $\overline{\nu}_2^I$ is clearly best, dominating for example $\overline{\nu}_I$ with relation 22/4 out of 36 cases, the same does not happen for the feasible two-steps estimate which is inferior to $\overline{\nu}_I$ and $\overline{\nu}_F$ with relations 21/10 and 13/11 respectively, out of those 36 cases, smaller bandwidths benefitting clearly one-step estimates. $\overline{\nu}_F$ and $\overline{\nu}_2^F$ perform better than $\overline{\nu}_B$, with relations 18/12 and 16/13 respectively, the band estimate being superior only when m and n are small. As theory predicts, biases decrease in absolute value when β and n increase, and, unexpectedly, tend to decrease as m increases.

This picture changes dramatically when $\rho \neq 0$. Here, in all cases, the biases share the sign of ρ, increase in absolute value when m increases and show the same pattern as when $\rho = 0$ with respect to β and n. There are two important features to note when $\rho \neq 0$, however. First, both feasible estimates are better than the band estimate in all cases. Second, $\overline{\nu}_2^F$, whose corresponding biases are in almost all cases slightly bigger than those of $\overline{\nu}_2^I$, performs much better not only than $\overline{\nu}_F$, but also, and more importantly, than $\overline{\nu}_I$. The relation with respect to the one-step infeasible estimate is 91/16 out of 108 cases in favour of $\overline{\nu}_2^F$, being this a certainly encouraging result, providing evidence of an important bias reduction achieved through our proposed iterative procedure. In fact, we suspect that more iterations could lead to further improvements. The only cases where $\overline{\nu}_I$ is competitive correspond to $(\gamma, \delta) = (0.4, 0.8)$, $(0.7, 1)$, for high bandwidths when n is small. Finally, as expected, biases increase with $|\rho|$.

Behaviour of the standard deviation

Results for standard deviations are presented in Tables 4.130-4.137. Over the four values of ρ, $\overline{\nu}_B$ is clearly superior to the other estimates, with a complete predominance for the two cases where $\gamma + \delta < 1$, i.e. $(\gamma, \delta) = (0, 0.4)$, $(0.2, 0.4)$ for all ρ, m and n. This fact is reflected in the overall ranking, which is $\overline{\nu}_B$, $\overline{\nu}_I$, $\overline{\nu}_2^I$, $\overline{\nu}_F$, $\overline{\nu}_2^F$, no worse than any of the rest in 98, 23, 22, 4 and 0 out of 144 cases respectively. For all estimates, standard deviations decrease as β, n, ρ and m increase. $\overline{\nu}_B$ is the least affected (although still very noticeably) by increments in m, hence the gap between this estimate and the rest tends to shrink as m increases. $\overline{\nu}_B$ beats $\overline{\nu}_F$ with relation 108/34, showing $\overline{\nu}_F$'s predominance over $\overline{\nu}_B$ only when $(\gamma, \delta)=(0.4, 0.8)$ for the highest m, and $(\gamma, \delta) = (0.7, 1)$ for the two highest bandwidths. Similarly, $\overline{\nu}_B$ beats $\overline{\nu}_2^F$ with relation 124/20. Also, $\overline{\nu}_2^F$ is superior to $\overline{\nu}_B$ when $(\gamma, \delta) = (0.7, 1)$ for the two highest bandwidths.

As opposite to the evidence related to the bias, the two-steps estimates perform in terms of standard deviations clearly worse than the one-step ones. $\overline{\nu}_I$ dominates $\overline{\nu}_2^I$ with relation 122/22, $\overline{\nu}_2^I$ being only superior to $\overline{\nu}_I$ (with small differences though) when $(\gamma, \delta) = (0.7, 1)$ for the two highest bandwidths. Even more striking is the difference between the feasible estimates, as $\overline{\nu}_F$ outperforms $\overline{\nu}_2^F$ with relation 137/6, $\overline{\nu}_2^F$ being only superior for some cases with $(\gamma, \delta) = (0.7, 1)$ for the highest bandwidth.

Behaviour of empirical sizes

We next analyse the adjustment to their limiting χ_1^2 distribution of the Wald statistics W_I, W_F, W_2^I, W_2^F, where the two-steps Wald statistics are defined as

$$W_2^I = \overline{b}_{2I}\left(\overline{\nu}_2^I - 1\right)^2, \quad W_2^F = \overline{b}_{2F}\left(\overline{\nu}_2^F - 1\right)^2, \tag{4.55}$$

where $\overline{b}_{2I}$ and $\overline{b}_{2F}$ differ from their respective one-step counterparts, $\overline{b}_I$ and $\overline{b}_F$ respectively, in the same way as $\overline{\nu}_2^I$ and $\overline{\nu}_2^F$ differed from $\overline{\nu}_I$ and $\overline{\nu}_F$.

Results for the empirical sizes corresponding to the different Wald statistics are given in Tables 4.138-4.145. Sizes for all cases are too large, in most of the situations being very far from the nominal ones, showing in some cases certain convergence as n increases, although this is usually very slow. Also, as expected, their values increase as β decreases. Overall, results are not at all encouraging here.

When $\rho = 0$, empirical sizes corresponding to W_I are too large, but somewhat acceptable. For the smallest bandwidth, they react in the appropriate direction as n increases, being this less clear for the other two bandwidths, except for the case $(\gamma, \delta) = (0.4, 0.8)$. For $(\gamma, \delta) = (0.2, 0.4)$ sizes tend to be smaller as m increases, the opposite clearly happening with $(\gamma, \delta) = (0.7, 1)$, and in a less evident way with $(\gamma, \delta) = (0.4, 0.8)$. Sizes corresponding to the two-steps infeasible estimate for this $\rho = 0$ situation are clearly larger than those of W_I, with the exception of some cases for $(\gamma, \delta) = (0.7, 1)$ for the two highest bandwidths. These sizes behave in a qualitatively similar way to those of W_I, including the very important deterioration of the sizes as n increases for $(\gamma, \delta) = (0.2, 0.4)$ associated with the highest bandwidth. As $|\rho|$ increases, sizes suffer from further increments, which are especially evident for cases $(\gamma, \delta) = (0.2, 0.4)$, $(0.7, 1)$. Also, there is now a substantial deterioration of the empirical sizes as m increases for all β, without evidence of the appropriate reaction as n increases for the case $(\gamma, \delta) = (0.2, 0.4)$ for the two highest bandwidths. For the smallest bandwidth and $|\rho| = 0.5$, sizes of W_2^I are still larger than those of W_I, but although they also suffer increments as m increases, W_2^I is less damaged than W_I under those increases. Also, the deterioration of W_2^I as ρ increases is less important that the one of W_I, so that when $\rho = 0.75$, in almost all cases, W_2^I presents smaller sizes than W_I (especially for $(\gamma, \delta) = (0, 0.4)$). This relative better performance of W_2^I is also evident for $|\rho| = 0.5$, but only for the two highest bandwidths. When $\rho \neq 0$, W_2^I also shows a better behaviour than W_I when n increases.

Sizes corresponding to W_F and W_2^F follow in general the same pattern as their corresponding infeasible counterparts, but they are in almost all cases larger, the gap between sizes of infeasible and corresponding feasible statistics increasing as $|\rho|$ increases.

4.5 Appendix 4

Proof of Theorem 4.1. We show first (4.9). Clearly

$$\overline{\nu}_m(\gamma, \delta) - \nu = \frac{e_m(\gamma)}{b_m(\gamma)}, \tag{4.56}$$

where

$$e_m(\gamma) = \mathrm{Re}\left\{ \sum_{j=0}^{m} c_j p(\lambda_j) I_{ux(\gamma)}(\lambda_j) \right\}. \tag{4.57}$$

First, we show that

$$E(e_m(\gamma)) = o(n^{\beta}). \tag{4.58}$$

We can write the left side of (4.58) as the real part of

$$\frac{1}{2\pi n} \sum_{j=0}^{m} c_j p(\lambda_j) \int_{-\pi}^{\pi} D_n(\lambda_j - \mu) \sum_{t=1}^{n} a_{n-t} e^{-i(n-t)\lambda_j} D_t(\mu - \lambda_j) f(\mu) \xi d\mu, \tag{4.59}$$

where $a_t = a_t(\beta)$, $D_t(\lambda) = \sum_{s=1}^{t} e^{is\lambda}$, the Dirichlet kernel, where for $0 < \lambda < \pi$,

$$|D_t(\lambda)| < K \min\left\{|\lambda|^{-1}, t\right\}. \tag{4.60}$$

Noting that for any λ,

$$p(\lambda) f(\lambda) \xi = 0, \tag{4.61}$$

by periodicity, we can write (4.59) as

$$\frac{1}{2\pi n} \sum_{j=0}^{m} c_j p(\lambda_j) \int_{-\pi}^{\pi} D_n(-\mu) \sum_{t=0}^{n-1} a_t e^{-it\lambda_j} D_{n-t}(\mu) \left[f(\mu + \lambda_j) - f(\lambda_j)\right] \xi d\mu. \tag{4.62}$$

Next, by summation by parts, (4.62) is

$$\begin{aligned} &\frac{1}{2\pi n} \sum_{j=0}^{m} c_j p(\lambda_j) \int_{-\pi}^{\pi} D_n(-\mu) a_{n-1} D_1(\mu) \left[f(\mu + \lambda_j) - f(\lambda_j)\right] \xi \sum_{t=0}^{n-1} e^{-it\lambda_j} d\mu \\ &- \frac{1}{2\pi n} \sum_{j=0}^{m} c_j p(\lambda_j) \int_{-\pi}^{\pi} D_n(-\mu) \left[f(\mu + \lambda_j) - f(\lambda_j)\right] \xi \\ &\times \sum_{t=0}^{n-2} \left(a_{t+1} D_{n-t-1}(\mu) - a_t D_{n-t}(\mu)\right) \sum_{h=0}^{t} e^{-ih\lambda_j} d\mu. \end{aligned} \tag{4.63}$$

Clearly, the first term in (4.63) is

$$\frac{1}{2\pi} p(0) \int_{-\pi}^{\pi} D_n(-\mu) a_{n-1} D_1(\mu) \left[f(\mu) - f(0)\right] \xi d\mu, \tag{4.64}$$

noting (2.95). Now, (4.64) is bounded in modulus by

$$K \left|a_{n-1}\right| \int_{-\pi}^{\pi} \left|D_n(\mu)\right| d\mu = O\left(n^{\beta-1} \log n\right), \tag{4.65}$$

as f is a differentiable function, for any finite $c > 0$, by the Stirling's approximation

$$\left|a_s(c)\right| \leq K (1+s)^{c-1}, \; s \geq 0, \tag{4.66}$$

and

$$\int_{-\pi}^{\pi} \left|D_n(\mu)\right| d\mu = O(\log n), \tag{4.67}$$

(see e.g. Zygmund, 1977). Regarding the second term in (4.63), note that

$$a_{t+1} D_{n-t-1}(\mu) - a_t D_{n-t}(\mu) = \left(a_{t+1} - a_t\right) D_{n-t-1}(\mu) - e^{i(n-t)\mu} a_t. \tag{4.68}$$

First, the contribution of the first term on the right of (4.68) to the second term of (4.63) is 0 for $\beta = 1$, as in this case $a_{t+1} = a_t$, $t = 0, ..., n-2$. For $\beta \neq 1$, this contribution is bounded in modulus by

$$Kn^{-1}\left\{\sum_{j=0}^{m}\int_{-\pi}^{\pi}|D_n(\mu)|^2\|f(\mu+\lambda_j)-f(\lambda_j)\|\,d\mu\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{j=0}^{m}\int_{-\pi}^{\pi}\left|\sum_{t=0}^{n-2}(a_{t+1}-a_t)\,D_{n-t-1}(\mu)\,(D_t(-\lambda_j)+1)\right|^2\|f(\mu+\lambda_j)-f(\lambda_j)\|\,d\mu\right\}^{\frac{1}{2}}. \tag{4.69}$$

Now, the term inside the first braces is bounded by

$$Km\int_{-\pi}^{\pi}|\mu|\,|D_n(\mu)|^2\,d\mu = O(m\log n), \tag{4.70}$$

by (4.60) and (4.67), noting that by Assumption 2.1 f is boundedly differentiable. Next, the inside of the second braces is bounded by

$$\begin{aligned} &K\sum_{j=0}^{m}\int_{-\pi}^{\pi}|\mu|\sum_{t=0}^{n-2}\sum_{s=0}^{n-2}(a_{t+1}-a_t)\,D_{n-t-1}(\mu)\,(D_t(-\lambda_j)+1)\\ &\times(a_{s+1}-a_s)\,D_{n-s-1}(-\mu)\,(D_s(\lambda_j)+1)\\ =\;& O\left(n^2\log n\sum_{j=1}^{m}j^{-2}\left(\sum_{t=1}^{n}t^{\beta-2}\right)^2\right), \end{aligned} \tag{4.71}$$

by Lemma 2.C.1 of Chapter 2 and (4.60), which is

$$\begin{aligned} O\left(n^2\log n\right),\ \beta &< 1,\\ O\left(n^{2\beta}\log n\right),\ \beta &> 1, \end{aligned} \tag{4.72}$$

implying that (4.69) is

$$\begin{aligned} O\left(m^{\frac{1}{2}}\log n\right),\ \beta &< 1,\\ O\left(n^{\beta-1}m^{\frac{1}{2}}\log n\right),\ \beta &> 1. \end{aligned} \tag{4.73}$$

Finally, the contribution of the second term on the right of (4.68) to the second term of (4.63) is bounded in modulus by

$$Kn^{-1}\sum_{j=0}^{m}\left\{\int_{-\pi}^{\pi}|\mu D_n(\mu)|^2\,d\mu\int_{-\pi}^{\pi}\left|\sum_{t=0}^{n-2}e^{i(n-t)\mu}a_t\,(D_t(-\lambda_j)+1)\right|^2d\mu\right\}^{\frac{1}{2}}. \tag{4.74}$$

Now, the first integral inside braces is $O(1)$ by (4.60), whereas the second one is bounded by $K\sum_{t=1}^{n} a_t^2 |D_t(\lambda_j)|^2$, so that (4.74) is bounded by

$$Kn^{-1}\sum_{j=1}^{m}\left\{n^{2\beta+1}j^{-2}\right\}^{\frac{1}{2}} = O\left(n^{\beta-\frac{1}{2}}\log m\right), \tag{4.75}$$

to conclude the proof of (4.58).

Next, we prove that as $n \to \infty$,

$$n^{-\beta}(e_m(\gamma) - E(e_m(\gamma))) \Rightarrow \zeta' A(1)^{-1\prime}\Omega^{-1}\int_0^1 \widetilde{W}(r;\beta)\,dW(r). \tag{4.76}$$

This proof will just consist on showing that as $n \to \infty$,

$$e_m(\gamma) - E\{e_m(\gamma)\} = \frac{p(0)}{2\pi}\sum_{t=1}^{n} x_{t-1}(\gamma)A(1)\varepsilon_t + o_p(n^\beta), \tag{4.77}$$

because

$$\frac{p(0)}{2\pi n^\beta}\sum_{t=1}^{n} x_{t-1}(\gamma)A(1)\varepsilon_t \Rightarrow \zeta' A(1)^{-1\prime}\Omega^{-1}\int_0^1 \widetilde{W}(r;\beta)dW(r), \tag{4.78}$$

by Proposition 2.3 of Chapter 2. Now, in view of Propositions 2.1, 2.2, (4.77) holds on showing

$$Var\left\{\mathrm{Re}\left\{\sum_{j=m+1}^{[n/2]} c_j p(\lambda_j) I_{ux(\gamma)}(\lambda_j)\right\}\right\} = o\left(n^{2\beta}\right), \tag{4.79}$$

but, as mentioned in Robinson and Marinucci (2001), (4.79) follows by a simple modification of their Theorem 5.1, as $p(\lambda)$ is a well-behaved function without poles.

Finally, to complete the proof of (4.9), we show that as $n \to \infty$,

$$n^{-2\beta}b_m(\gamma) \Rightarrow \frac{q(0)}{2\pi}\int_0^1 \widetilde{W}(r;\beta)^2\,dr, \tag{4.80}$$

where the right side is almost surely positive. This result follows in view of Propositions 2.4, 2.5, 2.6, as by Theorem 4.4 and simple modification of Theorem 5.1 of Robinson and Marinucci (2001) and Assumption 2.1,

$$\mathrm{Re}\left\{\sum_{j=m+1}^{[n/2]} c_j q(\lambda_j) I_{x(\gamma)}(\lambda_j)\right\} = o_p(n^{2\beta}). \tag{4.81}$$

Now, we prove (4.11). First, defining

$$\widetilde{x}_t(\gamma) = \sum_{j=0}^{\infty} a_j u_{2,t-j}, \tag{4.82}$$

(4.11) follows on showing

$$\begin{aligned}
\sum_{j=0}^{m} \operatorname{Re}\left\{c_j p\left(\lambda_j\right) I_{ux(\gamma)}(\lambda_j)\right\} &= 2\sum_{j=1}^{m} \operatorname{Re}\left\{p\left(\lambda_j\right) I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} \\
&\quad + o_p\left(n^{\beta} m^{\frac{1}{2}-\beta}\right), \qquad (4.83)\\
\sum_{j=0}^{m} \operatorname{Re}\left\{c_j q\left(\lambda_j\right) I_{x(\gamma)}(\lambda_j)\right\} &= 2\sum_{j=1}^{m} \operatorname{Re}\left\{q\left(\lambda_j\right) I_{\widetilde{x}(\gamma)}(\lambda_j)\right\} \\
&\quad + o_p\left(n^{2\beta} m^{1-2\beta}\right), \qquad (4.84)\\
m^{\frac{1}{2}} \lambda_m^{\beta-1} \frac{2\pi}{n} \sum_{j=1}^{m} \operatorname{Re}\left\{p\left(\lambda_j\right) I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} &\rightarrow {}_d N\left(0, \frac{f^{11}(0) f_{22}(0)}{2(1-2\beta)}\right), \qquad (4.85)\\
\lambda_m^{2\beta-1} \frac{2\pi}{n} \sum_{j=1}^{m} \operatorname{Re}\left\{q\left(\lambda_j\right) I_{\widetilde{x}(\gamma)}(\lambda_j)\right\} &\rightarrow {}_p \frac{f^{11}(0) f_{22}(0)}{1-2\beta}, \qquad (4.86)
\end{aligned}$$

by simple application of Cramer's Theorem. First, we show (4.83). Now, the left side of (4.83) is

$$\begin{aligned}
&2\sum_{j=1}^{m} \operatorname{Re}\left\{p\left(\lambda_j\right) I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} + p(0) I_{u\widetilde{x}(\gamma)}(0) + p(0)\left(I_{ux(\gamma)}(0) - I_{u\widetilde{x}(\gamma)}(0)\right) \\
&+2\sum_{j=1}^{m} \operatorname{Re}\left\{p\left(\lambda_j\right)\left(I_{ux(\gamma)}(\lambda_j) - I_{u\widetilde{x}(\gamma)}(\lambda_j)\right)\right\}. \qquad (4.87)
\end{aligned}$$

Now, the second term in (4.87) is

$$\frac{p(0)}{2\pi n} \sum_{t=1}^{n} u_t \sum_{s=1}^{n} \widetilde{x}_t(\gamma) = O_p\left(n^{\beta}\right) = o_p\left(n^{\beta} m^{\frac{1}{2}-\beta}\right), \qquad (4.88)$$

as under Assumption 2.1, $\sum_{t=1}^{n} u_t = O_p\left(n^{1/2}\right)$, $\sum_{t=1}^{n} \widetilde{x}_t(\gamma) = O_p\left(n^{1/2+\beta}\right)$ (see eg. Robinson, 1994a). Next, the third term in (4.87) is

$$\frac{p(0)}{2\pi n} \sum_{t=1}^{n} u_t \sum_{s=1}^{n} \left(x_s(\gamma) - \widetilde{x}_s(\gamma)\right), \qquad (4.89)$$

where the expectation of the second summation in (4.89) is 0, whereas its variance is bounded by

$$\begin{aligned}
& K \int_{-\pi}^{\pi} \left| \sum_{s=1}^{n} \sum_{l=0}^{\infty} a_{s+l} e^{il\mu} \right|^2 d\mu \\
\leq\ & K \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{l=0}^{\infty} (t+l)^{\beta-1} (s+l)^{\beta-1} \\
\leq\ & K \sum_{t=1}^{n} \sum_{l=0}^{\infty} (t+l)^{2\beta-2} + K \sum_{t=2}^{n} \sum_{s=1}^{t-1} \sum_{l=0}^{\infty} (t+l)^{\beta-1} (s+l)^{\beta-1}
\end{aligned}$$

$$\leq K\sum_{t=1}^{n}\sum_{l=t}^{\infty}l^{2\beta-2}+K\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{l=0}^{\infty}(s+l)^{2\beta-2}\leq Kn^{2\beta+1}, \tag{4.90}$$

implying that

$$\sum_{s=1}^{n}\left(x_s(\gamma)-\widetilde{x}_s(\gamma)\right)=O_p\left(n^{\beta+\frac{1}{2}}\right), \tag{4.91}$$

hence we conclude as in (4.88). Finally, regarding the fourth term in (4.87), we first calculate the order of magnitude of its expectation, which is the real part of

$$\frac{1}{\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m}p(\lambda_j)\,D_n(\lambda_j-\mu)\sum_{s=0}^{\infty}\sum_{k=1}^{n}a_{k+s}e^{-ik\lambda_j}f(\mu)\,\xi e^{-is\mu}d\mu, \tag{4.92}$$

which by (4.61) and periodicity is equal to

$$\frac{1}{\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m}p(\lambda_j)\,D_n(-\mu)\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}a_k e^{-ik\lambda_j}\left(f(\mu+\lambda_j)-f(\lambda_j)\right)\xi e^{-is(\mu-\lambda_j)}d\mu$$

$$\leq Kn^{-1}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}|D_n(-\mu)|\,\|f(\mu+\lambda_j)-f(\lambda_j)\|^2 d\mu\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}a_k e^{-ik\lambda_j}e^{-is(\mu-\lambda_j)}\right|^2 d\mu\right\}^{\frac{1}{2}}. \tag{4.93}$$

Now, by Assumption 2.1 and (4.60) the first element inside braces in (4.93) is $O(m)$. The second element is

$$2\pi\sum_{j=1}^{m}\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}\sum_{l=s+1}^{n+s}a_k a_l e^{i(l-k)\lambda_j}$$

$$\leq K\sum_{j=1}^{m}\sum_{s=0}^{\infty}\frac{(s+1)^{2\beta-2}}{|\lambda_j|^2}=O\left(n^2\right), \tag{4.94}$$

by Lemma 3.2 in Robinson and Marinucci (2001), to conclude that the expectation is $O(m^{1/2})$. Next, we calculate the order of the variance of the fourth term in (4.87), which is bounded by the real part of

$$\frac{1}{\pi^2 n^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{t=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}a_{s+l}a_{q+p}e^{i\lambda_j(t-s)-i\lambda_k(r-q)}$$

$$\times p(\lambda_j)\left\{E\left(u_t u_r'\right)E\left(u_{2,-l}u_{2,-p}\right)+E\left(u_t u_{2,-p}\right)E\left(u_r' u_{2,-l}\right)+k\right\}p'(-\lambda_k), \tag{4.95}$$

where k is a fourth cumulant term of the processes u_t, u_r, $u_{2,-l}$, $u_{2,-p}$. We just give detail of the contribution to the variance of the first term in braces in (4.95). It can be shown by simple application of the Cauchy inequality that the contribution of the second and third terms is of the same order as the one of the first term. Now, this contribution is bounded by

$$
\begin{aligned}
& Kn^{-2}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}e^{-i\lambda_j s}\sum_{q=1}^{n}a_{q+l}e^{i\lambda_k q}\sum_{t=1}^{n}e^{it(\lambda_j-\lambda_k)} \\
\leq\; & Kn^{-1}\sum_{j=1}^{m}\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}e^{-i\lambda_j s}\sum_{q=1}^{n}a_{q+l}e^{i\lambda_j q}, \qquad (4.96)
\end{aligned}
$$

by (2.95). Now, (4.96) is bounded by

$$
Kn^{-1}m\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}^2+Kn^{-1}\sum_{j=1}^{m}\sum_{l=0}^{\infty}\sum\sum_{s\neq q}a_{s+l}a_{q+l}e^{i\lambda_j(q-s)}. \qquad (4.97)
$$

Clearly, the first term in (4.97) is $O\left(mn^{2\beta-1}\right)$, and by (4.60) the second is bounded by

$$
\begin{aligned}
Kn^{-1}\sum_{l=0}^{\infty}\sum\sum_{s\neq q}^{n}a_{s+l}a_{q+l}\frac{1}{|\lambda_{q-s}|} &\leq K\sum_{l=0}^{\infty}\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{(s+l)^{\beta-1}(q+l)^{\beta-1}}{q-s} \\
&\leq K\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{1}{q-s}\sum_{l=s}^{\infty}l^{2\beta-2}\leq K\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{s^{2\beta-1}}{q-s} \\
&= K\sum_{q=1}^{n-1}q^{-1}\sum_{s=1}^{n-q}s^{2\beta-1}\leq Kn^{2\beta}\log n. \qquad (4.98)
\end{aligned}
$$

Thus, the fourth term in (4.87) is

$$
O_p\left(m^{\frac{1}{2}}+n^{\beta}\log^{\frac{1}{2}}n\right)=o_p\left(n^{\beta}m^{\frac{1}{2}-\beta}\right), \qquad (4.99)
$$

by (4.10), to conclude the proof of (4.83). Next, we show (4.84). First, noting that from previous arguments

$$
q(0)\frac{1}{2\pi n}\left(\sum_{t=1}^{n}x_t(\gamma)\right)^2=O_p\left(n^{2\beta}\right)=o_p\left(n^{2\beta}m^{1-2\beta}\right), \qquad (4.100)
$$

(4.84) follows on showing

$$
\sum_{j=1}^{m}\operatorname{Re}\left\{q(\lambda_j)\,w_{x(\gamma)}(\lambda_j)\left(w_{x(\gamma)}(-\lambda_j)-w_{\tilde{x}(\gamma)}(-\lambda_j)\right)\right\}=o_p\left(n^{2\beta}m^{1-2\beta}\right). \qquad (4.101)
$$

First the expectation of the left side of (4.101) is the real part of

$$
\frac{1}{2\pi n}\sum_{j=1}^{m}q(\lambda_j)\sum_{t=1}^{n}\sum_{q=0}^{n-t}\sum_{s=1}^{n}\sum_{l=0}^{\infty}a_q e^{iq\lambda_j}a_{s+l}e^{i\lambda_j(t-s)}\int_{-\pi}^{\pi}f_{22}(\mu)\,e^{-i(l+t)\mu}d\mu
$$

$$
\begin{aligned}
&= \frac{1}{2\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m} q(\lambda_j)\sum_{t=1}^{n} a_{n-t}e^{i(n-t)\lambda_j}D_t(\lambda_j-\mu) \\
&\quad\times\sum_{s=1}^{n}\sum_{l=0}^{\infty} a_{s+l}e^{-i\lambda_j s}e^{-il\mu}\left(f_{22}(\mu)-f_{22}(\lambda_j)\right)d\mu, \qquad (4.102)
\end{aligned}
$$

as

$$
\int_{-\pi}^{\pi} e^{-i(l+t)\mu}d\mu = 0, \qquad (4.103)
$$

for all $t \geq 1$, $l \geq 0$. Then, (4.102) is bounded by

$$
\begin{aligned}
&Kn^{-1}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{t=1}^{n} a_{n-t}e^{i(n-t)\lambda_j}D_t(\lambda_j-\mu)\left(f_{22}(\mu)-f_{22}(\lambda_j)\right)\right|^2 d\mu\right\}^{\frac{1}{2}} \\
&\times\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{s=1}^{n}\sum_{l=0}^{\infty} a_{s+l}e^{-i\lambda_j s}e^{-il\mu}\right|^2 d\mu\right\}^{\frac{1}{2}}. \qquad (4.104)
\end{aligned}
$$

Now, the first element in braces in (4.104) is bounded by

$$
\begin{aligned}
&K\sum_{j=1}^{m}\sum_{t=1}^{n}\sum_{q=1}^{n} a_{n-t}a_{n-q}\left|D_t(\lambda_j-\mu)\right|\left|D_q(\mu-\lambda_j)\right|\left(f_{22}(\mu)-f_{22}(\lambda_j)\right)^2 \\
&\leq Kmn^{2\beta}, \qquad (4.105)
\end{aligned}
$$

by (4.60). The second element in braces is

$$
\begin{aligned}
&2\pi\sum_{j=1}^{m}\sum_{s=1}^{n}\sum_{p=1}^{n}\sum_{l=0}^{\infty} a_{s+l}a_{p+l}e^{i(p-s)\lambda_j} \\
&\leq Km\sum_{s=1}^{n}\sum_{l=s}^{\infty} a_l^2 + Kn\sum\sum_{s\neq p}^{n}\sum_{l=0}^{\infty}\frac{a_{s+l}a_{p+l}}{|s-p|} \\
&= O\left(mn^{2\beta}+n^{2\beta+1}\log n\right), \qquad (4.106)
\end{aligned}
$$

where the order corresponding to the second term in the right of the inequality in (4.106) is calculated as in (4.98). Thus, the expectation of the left side of (4.101) is $O\left(n^{2\beta-1/2}m^{1/2}\log^{1/2} n\right)$. Next, we consider the variance of the left side of (4.101) which is bounded by the real part of

$$
\begin{aligned}
&\frac{1}{4\pi^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{t=1}^{n}\sum_{r=1}^{n}\sum_{q=0}^{n-t}\sum_{p=0}^{n-r}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty} q(\lambda_j)q(-\lambda_k) \\
&\times a_q e^{iq\lambda_j}a_p e^{-ip\lambda_k}a_{s+l}a_{u+v}e^{i\lambda_j(t-s)}e^{-i\lambda_k(r-u)} \\
&\times\left\{E\left(u_{2t}u_{2r}\right)E\left(u_{2,-l}u_{2,-v}\right)+E\left(u_{2t}u_{2,-v}\right)E\left(u_{2r}u_{2,-l}\right)+k\right\}, \qquad (4.107)
\end{aligned}
$$

where k is the fourth cumulant term of u_{2t}, u_{2r}, $u_{2,-l}$, $u_{2,-v}$. As before, we just consider the contribution of the first term in braces, the treatment of the remaining terms being very similar. Now, this contribution is bounded by

$$Kn^{-2}\sum_{t=1}^{n}\left\{\sum_{l=0}^{n}+\sum_{l=n+1}^{\infty}\right\}\left|\sum_{j=1}^{m}\sum_{q=0}^{n-t}a_q e^{iq\lambda_j}\sum_{s=l+1}^{n+l}a_s e^{-is\lambda_j}e^{i(t+l)\lambda_j}\right|^2. \quad (4.108)$$

Now, noting that by Lemma 3.2 in Robinson and Marinucci (2001)

$$\left|\sum_{s=l+1}^{n+l}a_s e^{-is\lambda_j}\right| \le K\frac{(l+1)^{\beta/2-1/2}}{|\lambda_j|^{\beta/2+1/2}}, \quad (4.109)$$

the contribution of the summation in l from 0 to n to (4.108) is bounded by

$$\begin{aligned} Kn^{-1}\sum_{l=1}^{n}\left|l^{\beta/2-1/2}\sum_{j=1}^{m}\frac{1}{|\lambda_j|^{3\beta/2+1/2}}\right|^2 &\le Kn^{4\beta},\ \beta>1/3,\\ &\le Kn^{4\beta}\log^2 m,\ \beta=1/3,\\ &\le Kn^{4\beta}m^{1-3\beta},\ \beta<1/3. \quad (4.110)\end{aligned}$$

Next, by Lemma 3.2 in Robinson and Marinucci (2001), the contribution of the second summation in l in (4.108) is bounded by

$$Kn^{-1}\sum_{l=n+1}^{\infty}\left|l^{\beta-1}n^{1+\beta}\sum_{j=1}^{m}j^{-1-\beta}\right|^2 \le Kn^{2\beta+1}\sum_{l=n+1}^{\infty}l^{2\beta-2}\le Kn^{4\beta}. \quad (4.111)$$

Thus, we conclude that the left of (4.101) is

$$\begin{aligned} &O_p\left(n^{2\beta-1/2}m^{1/2}\log^{1/2}n+n^{2\beta}\right),\ \beta>1/3,\\ &O_p\left(n^{2\beta-1/2}m^{1/2}\log^{1/2}n+n^{2\beta}\log m\right),\ \beta=1/3,\\ &O_p\left(n^{2\beta-1/2}m^{1/2}\log^{1/2}n+n^{2\beta}m^{1/2-3\beta/2}\right),\ \beta<1/3,\end{aligned} \quad (4.112)$$

in all cases $o_p\left(n^{2\beta}m^{1-2\beta}\right)$. Finally, (4.85), (4.86) follow as in the proof of Theorem 2 of Christensen and Nielsen (2001) who adapted the steps in Lobato (1999) to a somewhat different situation. Following these references, it can be easily shown that

$$m^{\frac{1}{2}}\lambda_m^{\beta-1}\frac{2\pi}{n}\sum_{j=1}^{m}\mathrm{Re}\left\{p(\lambda_j)\,I_{u\tilde{x}(\gamma)}(\lambda_j)\right\}=\sum_{t=2}^{n}\zeta_t\sum_{s=1}^{t-1}c_{t-s}\zeta_s+o_p\left(n^{\beta}m^{1/2-\beta}\right), \quad (4.113)$$

where $\zeta_t=\Omega^{-1/2}\varepsilon_t$,

$$c_t=\frac{1}{2\pi n m^{1/2}}\sum_{j=1}^{m}\varrho(\lambda_j)\cos(t\lambda_j), \quad (4.114)$$

and

$$\varrho(\lambda)=\lambda_m^{\beta}\left[B'(\lambda)\,p'(\lambda)\,\xi'\left(1-e^{-i\lambda}\right)^{-\beta}B(-\lambda)+\left(1-e^{i\lambda}\right)^{-\beta}B'(\lambda)\,\xi p(-\lambda)\,B(-\lambda)\right], \quad (4.115)$$

with $B(\lambda) = A\left(e^{i\lambda}\right)\Omega^{1/2}$. Now, probably the only point that is worth mentioning is that

$$\begin{aligned}
& \frac{1}{4\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{\varrho'(-\lambda_j)\varrho(\lambda_j)\right\}\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos^2((t-s)\lambda_j) \\
= & \frac{(n-1)^2}{16\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{\varrho'(-\lambda_j)\varrho(\lambda_j)\right\} \\
= & \frac{(n-1)^2\lambda_m^{2\beta}}{8\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{B'(\lambda_j)p'(\lambda_j)\xi'\left(1-e^{-i\lambda_j}\right)^{-\beta}B(-\lambda_j)\right. \\
& \left.\times\left(1-e^{i\lambda_j}\right)^{-\beta}B'(\lambda_j)\xi p(-\lambda_j)B(-\lambda_j)\right\} \qquad (4.116)
\end{aligned}$$

some cancellations taking place due to (4.61), so that (4.116) is equal to

$$\frac{(n-1)^2\lambda_m^{2\beta}}{2n^2 m}\sum_{j=1}^{m}\left|1-e^{i\lambda_j}\right|^{-2\beta} f_{22}(\lambda_j) f^{11}(-\lambda_j) \to \frac{f_{22}(0)f^{11}(0)}{2(1-2\beta)}, \qquad (4.117)$$

as $n \to \infty$, by (4.4).

Proof of Theorem 4.2. The result follows on showing that as $n \to \infty$

$$\begin{aligned}
\widetilde{\nu}_m(\gamma,\delta) - \overline{\nu}_m(\gamma,\delta) &= o_p\left(n^{\beta}m^{1/2-\min\{\beta,1/2\}}\right), \qquad (4.118)\\
\widetilde{\nu}_m(\widehat{\gamma},\widehat{\delta}) - \widetilde{\nu}_m(\gamma,\delta) &= o_p\left(n^{\beta}m^{1/2-\min\{\beta,1/2\}}\right), \qquad (4.119)
\end{aligned}$$

noting that the proof for $\widetilde{\nu}_m(\widehat{\gamma},\delta)$ and $\widetilde{\nu}_m(\gamma,\widehat{\delta})$ is implied by the proof of (4.119). First, (4.118) follows on showing

$$\begin{aligned}
\widetilde{e}_m(\gamma) - e_m(\gamma) &= o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}}), \qquad (4.120)\\
\widetilde{b}_m(\gamma) - b_m(\gamma) &= o_p(n^{2\beta}m^{1-2\min\{1/2,\beta\}}). \qquad (4.121)
\end{aligned}$$

We just prove (4.120) as the proof for (4.121) is similar, but significantly simpler. Now the left side of (4.120) is

$$\mathrm{Re}\left\{\sum_{j=0}^{m} c_j(\widetilde{p}(\lambda_j) - p(\lambda_j))I_{ux(\gamma)}(\lambda_j)\right\}, \qquad (4.122)$$

and noting that

$$\widetilde{p}(\lambda) - p(\lambda) = \zeta' f(\lambda)^{-1}\left[f(\lambda) - \widetilde{f}(\lambda)\right]\widetilde{f}(\lambda)^{-1}, \qquad (4.123)$$

the two possible terms for which $c_j = 1$ are $O_p(n^{\beta-\varkappa}) = o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ by (4.19), as by Assumption 2.1, $\sum_{t=1}^{n} u_t = O_p(n^{1/2})$, and by results in Robinson and

Marinucci (2001) and previous arguments, $\sum_{t=1}^{n} x_t(\gamma) = O_p(n^{\beta+1/2})$. Next, by summation by parts, the remaining terms in (4.122) are equal to

$$2\operatorname{Re}\left\{(\widetilde{p}(\lambda_{m^*}) - p(\lambda_{m^*}))\sum_{j=1}^{m^*} I_{ux(\gamma)}(\lambda_j)\right\}$$
$$-2\operatorname{Re}\left\{\sum_{j=1}^{m^*-1}(\widetilde{p}(\lambda_{j+1}) - p(\lambda_{j+1}) - (\widetilde{p}(\lambda_j) - p(\lambda_j)))\sum_{h=1}^{j} I_{ux(\gamma)}(\lambda_h)\right\}, \tag{4.124}$$

where $m^* = m-1$ if $m = n/2$ or $m^* = m$, otherwise. Now, we consider the order of magnitude of the expectation of $\sum_{h=1}^{j} I_{u_i x(\gamma)}(\lambda_h)$, $i = 1,2$, uniformly in $j \in [1,m]$. First, for $1/2 < \beta < 1$, as in the proof of Proposition 4.1 of Robinson and Marinucci (1998)

$$\left|E\left(\sum_{h=1}^{j} I_{u_i x(\gamma)}(\lambda_h)\right)\right| \le K n^{-\frac{1}{2}} \sum_{h=1}^{j}\left\{\sum_{t=1}^{n}\left|\sum_{s=0}^{n-t} a_s e^{is\lambda_h}\right|^2\right\}^{\frac{1}{2}}, \tag{4.125}$$

where by Lemma 3.2 in Robinson and Marinucci (2001), for $0 < |\lambda| < \pi$,

$$\left|\sum_{s=0}^{t} a_s e^{is\lambda}\right| = O\left(\frac{1}{|\lambda|^{\beta}}\right), \tag{4.126}$$

implying that, uniformly in $j \in [1,m]$,

$$\left|E\left(\sum_{h=1}^{j} I_{ux(\gamma)}(\lambda_h)\right)\right| = O\left(n^{\beta} m^{1-\beta}\right). \tag{4.127}$$

Next, for $\beta = 1$, noting that Assumption 2.1 ensures that the conditions for Lemma 5.4 of Robinson and Marinucci (1998) hold, uniformly in $j \in [1,m]$,

$$E\left(\sum_{h=1}^{j} I_{u_i x(\gamma)}(\lambda_h)\right) = O(m), \tag{4.128}$$

noting that uniformity in j follows easily from the arguments in the proof in that lemma. Now, for $\beta > 1$, the left side of (4.128) is

$$\frac{1}{2\pi n}\sum_{h=1}^{j}\int_{-\pi}^{\pi} D_n(-\mu)\sum_{t=0}^{n-1} a_t e^{-it\lambda_h} D_{n-t}(\mu) f_{i2}(\mu+\lambda_h)\, d\mu \tag{4.129}$$

which by summation by parts is equal to

$$\frac{1}{2\pi n}\int_{-\pi}^{\pi} D_n(-\mu)\sum_{t=1}^{n-1} a_t D_{n-t}(\mu) f_{i2}(\mu+\lambda_j)\, D_j(-\lambda_t)\, d\mu$$

$$-\frac{1}{2\pi n}\int_{-\pi}^{\pi} D_n(-\mu)\sum_{t=1}^{n-1} a_t D_{n-t}(\mu)\sum_{h=1}^{j-1}\left(f_{i2}(\mu+\lambda_{h+1})-f_{i2}(\mu+\lambda_h)\right)$$

$$\times D_h(-\lambda_t)\,d\mu+\frac{j}{2\pi n}\int_{-\pi}^{\pi}|D_n(\mu)|^2 f_{i2}(\mu+\lambda_h)\,d\mu. \qquad (4.130)$$

Now, the third term in (4.130) is $O(j)$, noting that for any $r\geq 1$

$$\int_{-\pi}^{\pi}|D_r(\mu)|^2\,d\mu=2\pi r. \qquad (4.131)$$

Next, uniformly in j, the first term is bounded by

$$Kn^{-1}\sum_{t=1}^{n-1}|a_t D_j(-\lambda_t)|\left\{\int_{-\pi}^{\pi}|D_n(\mu)|^2\,d\mu\int_{-\pi}^{\pi}|D_{n-t}(\mu)|^2\,d\mu\right\}^{\frac{1}{2}}$$

$$\leq\ Kn\sum_{t=1}^{n}t^{\beta-2}=O\left(n^{\beta}\right), \qquad (4.132)$$

by (4.60). Also, noting that by Assumption 2.1, uniformly in μ and h,

$$f_{i2}(\mu+\lambda_{h+1})-f_{i2}(\mu+\lambda_h)=O\left(n^{-1}\right), \qquad (4.133)$$

using a similar analysis to the one of the first term, it is easy to show that the second term of (4.130) is $O\left(n^{\beta-1}j\right)$. Next, by the proof of Proposition 4.2 of Robinson and Marinucci (1998), for any $i=1,2$

$$Var\left(\sum_{h=1}^{j}I_{u_ix(\gamma)}(\lambda_h)\right)\leq Kn^{-1}\sum_{h=1}^{j}\sum_{t=1}^{n}\left|\sum_{s=0}^{n-t}a_s e^{is\lambda_h}\right|^2, \qquad (4.134)$$

which implies by (4.126) that, uniformly in $j\in[1,m]$,

$$Var\left(\sum_{h=1}^{j}I_{u_ix(\gamma)}(\lambda_h)\right)=O\left(n^{2\beta}\right). \qquad (4.135)$$

Finally, for $\beta<1/2$,

$$\sum_{h=1}^{j}I_{u_ix(\gamma)}(\lambda_h)\leq\left\{\sum_{h=1}^{j}I_{u_i}(\lambda_h)\sum_{h=1}^{j}I_{x(\gamma)}(\lambda_h)\right\}^{\frac{1}{2}}=O\left(n^{\beta}j^{1-\beta}\right), \qquad (4.136)$$

by the properties of the periodogram described in Robinson (1995a), to conclude finally that

$$\begin{aligned}\sum_{h=1}^{j}I_{ux(\gamma)}(\lambda_h)\ &=\ O_p\left(n^{\beta}m^{1-\beta}\right),\ \beta<1,\\ &=\ O_p\left(n^{\beta}\right),\ \beta\geq 1,\end{aligned} \qquad (4.137)$$

uniformly in $j \in [1,m]$. Thus, by Assumption 4.2, the first term of (4.124) is

$$\begin{aligned} O_p\left(n^{\beta-\varkappa}m^{1-\beta}\right), \beta &< 1, \\ O_p\left(n^{\beta-\varkappa}\right), \beta &\geq 1, \end{aligned} \tag{4.138}$$

so the first term of (4.124) is $o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ noting (4.19). Now, $\widetilde{p}_{j+1}-p_{j+1}-(\widetilde{p}_j-p_j)$ is

$$\begin{aligned} &\zeta'\widetilde{f}_j^{-1}\left[\widetilde{f}_j-\widetilde{f}_{j+1}-(f_j-f_{j+1})\right]\widetilde{f}_{j+1}^{-1} \\ &+\zeta'\left(\widetilde{f}_j^{-1}-f_j^{-1}\right)(f_j-f_{j+1})\widetilde{f}_j^{-1} \\ &+\zeta' f_j^{-1}(f_j-f_{j+1})\left(\widetilde{f}_{j+1}^{-1}-f_{j+1}^{-1}\right). \end{aligned} \tag{4.139}$$

First, by (4.137) and Assumptions 2.1, 4.2, the contribution of the second and third terms in (4.139) to the second term of (4.124) is

$$\begin{aligned} O_p\left(n^{\beta-1-\varkappa}m^{2-\beta}\right), \beta &< 1, \\ O_p\left(n^{\beta-1-\varkappa}m\right), \beta &\geq 1, \end{aligned} \tag{4.140}$$

which is $o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ by (4.19). Finally, by Assumption 4.2 and (4.137), the contribution of the first term in (4.139) to the second term of (4.124) is

$$\begin{aligned} O_p\left(n^{\beta-\phi}m^{2-\beta}\right), \beta &< 1, \\ O_p\left(n^{\beta-\phi}m\right), \beta &\geq 1, \end{aligned} \tag{4.141}$$

again $o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ by (4.20), to conclude the proof of (4.118).

Next, noting that

$$\widetilde{\nu}_m(\widehat{\gamma},\widehat{\delta})-\nu=\frac{\widetilde{e}_m(\widehat{\gamma},\widehat{\delta})}{\widetilde{b}_m(\widehat{\gamma})}, \tag{4.142}$$

where

$$\widetilde{e}_m(\widehat{\gamma},\widehat{\delta})=\mathrm{Re}\left\{\sum_{j=0}^{m}c_j\widetilde{p}(\lambda_j)\,I_{v(\widehat{\gamma},\widehat{\delta})x(\widehat{\gamma})}(\lambda_j)\right\}, \tag{4.143}$$

and

$$v(\widehat{\gamma},\widehat{\delta})=(u_{1t}(\widehat{\gamma}-\gamma)\,,x_t(\widehat{\delta}))', \tag{4.144}$$

(4.119) follows on establishing

$$\begin{aligned} e_m(\widehat{\gamma},\widehat{\delta})-e_m(\gamma) &= o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}}), && (4.145)\\ \widetilde{e}_m(\widehat{\gamma},\widehat{\delta})-e_m(\widehat{\gamma},\widehat{\delta})-\widetilde{e}_m(\gamma)+e_m(\gamma) &= o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}}), && (4.146)\\ b_m(\widehat{\gamma})-b_m(\gamma) &= o_p(n^{2\beta}m^{1-2\min\{1/2,\beta\}}), && (4.147)\\ \widetilde{b}_m(\widehat{\gamma})-b_m(\widehat{\gamma})-\widetilde{b}_m(\gamma)+b_m(\gamma) &= o_p(n^{2\beta}m^{1-2\min\{1/2,\beta\}}), && (4.148) \end{aligned}$$

where $e_m(\widehat{\gamma},\widehat{\delta})$ is like $\widetilde{e}_m(\widehat{\gamma},\widehat{\delta})$ but with $p(\lambda)$ replacing $\widetilde{p}(\lambda)$ in (4.143). We just prove (4.145), (4.146), the proofs for (4.147), (4.148) being similar but simpler.

Now, the left side of (4.145) is the real part of

$$\sum_{j=0}^{m} c_j p(\lambda_j) \left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right] \left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right] \quad (4.149)$$

$$+\sum_{j=0}^{m} c_j p(\lambda_j) w_{x(\gamma)}(-\lambda_j) \left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right] \quad (4.150)$$

$$+\sum_{j=0}^{m} c_j p(\lambda_j) \left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right] w_u(\lambda_j). \quad (4.151)$$

Considering first (4.151), by Taylor's theorem, this is the real part of

$$\sum_{r=1}^{R-1} \frac{(\gamma-\widehat{\gamma})^r}{r!} \sum_{j=0}^{m} c_j p(\lambda_j) w_{u_2}^{(r)}(-\lambda_j;\beta) w_u(\lambda_j)$$
$$+\frac{(\gamma-\widehat{\gamma})^R}{R!} \sum_{j=0}^{m} c_j p(\lambda_j) w_{u_2}^{(R)}(-\lambda_j;\delta-\overline{\gamma}) w_u(\lambda_j), \quad (4.152)$$

where for a vector or scalar sequence φ_t, and real $b \geq 0$,

$$w_{\varphi}^{(r)}(\lambda;b) = \frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_{t=2}^{n} \sum_{s=1}^{t-1} a_s^{(r)}(b) \varphi_{t-s} e^{it\lambda}, \quad (4.153)$$

with

$$a_s^{(r)}(b) = \frac{d^r a_s(b)}{db^r}, \quad (4.154)$$

and $|\overline{\gamma}-\gamma| \leq |\widehat{\gamma}-\gamma|$. Now, by a straightforward extension of results in Robinson and Marinucci (1998, 2001)

$$\begin{aligned}\sum_{j=0}^{m} c_j p(\lambda_j) w_{u_2}^{(r)}(-\lambda_j;\beta) w_u(\lambda_j) &= O_p\left(n^{\beta} m^{1-\beta} (\log m)^r\right), \ \beta < 1, \\ &= O_p\left(n^{\beta} (\log m)^r\right), \ \beta \geq 1, \end{aligned} \quad (4.155)$$

the only differences being that the weights $a_s^{(r)}(\beta)$ that are involved (see Lemma 2.C.1), are not covered by the weights of Robinson and Marinucci (2001) (but it can be easily shown that they just contribute the $(\log m)^r$ factors), and the smooth weighting factor $c_j p(\lambda_j)$, which, as mentioned before, can be handled by simple modification of the proofs of Robinson and Marinucci (1998, 2001). Next, the summation in the second term of (4.152) is bounded by

$$\begin{aligned} K\sum_{j=0}^{m} \left|w_{u_2}^{(R)}(-\lambda_j;\delta-\overline{\gamma})\right| \|w_u(\lambda_j)\| &\leq Kn^2 \sum_{j=1}^{m} \left|a_j^{(R)}(\delta-\overline{\gamma})\right| \\ &= O_p\left(n^{\beta+\epsilon+2}\right), \end{aligned} \quad (4.156)$$

for any $\epsilon > 0$ in view of Lemma 2.C.5. Thus, by Assumption 4.1, choosing $R > (\kappa + 2)/\kappa$, (4.151) is

$$\begin{aligned} O_p\left(n^{\beta-\kappa} m^{1-\beta} \log m\right), \ \beta &< 1, \\ O_p\left(n^{\beta-\kappa} \log m\right), \ \beta &\geq 1, \end{aligned} \tag{4.157}$$

orders which are $o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}})$ in view of (4.15). Now, again by Taylor's theorem, (4.150) is the real part of

$$\begin{aligned} &\sum_{r=1}^{R-1} \frac{1}{r!} \sum_{j=0}^{m} c_j p\left(\lambda_j\right) w_{x(\gamma)}(-\lambda_j) \begin{pmatrix} (\widehat{\gamma}-\gamma)^r & 0 \\ 0 & (\widehat{\delta}-\delta)^r \end{pmatrix} w_u^{(r)}(\lambda_j; 0) \\ &+\frac{1}{R!} \sum_{j=0}^{m} c_j p\left(\lambda_j\right) w_{x(\gamma)}(-\lambda_j) \begin{pmatrix} (\widehat{\gamma}-\gamma)^R & 0 \\ 0 & (\widehat{\delta}-\delta)^R \end{pmatrix} w_u^{(R)}(\lambda_j; \widetilde{\gamma}, \widetilde{\delta}), \end{aligned} \tag{4.158}$$

where

$$w_u^{(R)}(\lambda; \widetilde{\gamma}, \widetilde{\delta}) = \left(w_{u_1}^{(R)}(\lambda; \widetilde{\gamma}), w_{u_2}^{(R)}(\lambda_j; \widetilde{\delta})\right)', \tag{4.159}$$

and $|\widetilde{\gamma}| \leq |\widehat{\gamma} - \gamma|$, $\left|\widetilde{\delta}\right| \leq \left|\widehat{\delta} - \delta\right|$. Again, by straightforward modification of results in Robinson and Marinucci (1998, 2001), the orders in (4.155) apply to the summation over j in the first term of (4.158) as the weights $a_s^{(r)}$ (0) involved (see Lemma 2.C.4), just contribute the $(\log m)^r$ factors. Next, the summation in the second term of (4.158) is bounded by

$$\begin{aligned} &K \sum_{q=0}^{m} \left|w_{x(\gamma)}(-\lambda_q)\right| \left\|w_u^{(R)}(\lambda_q; \widetilde{\gamma}, \widetilde{\delta})\right\| \\ &\leq K n^{\beta+\frac{3}{2}} \sum_{q=0}^{m} \left\{\left|a_q^{(R)}(\widetilde{\gamma})\right| + \left|a_q^{(R)}(\widetilde{\delta})\right|\right\}, \end{aligned} \tag{4.160}$$

which is $O_p(n^{\beta+3/2+\epsilon})$ for any $\epsilon > 0$, in view of Lemma 2.C.5. Thus, by Assumption 4.1, choosing $R > (3/2+\kappa)/\kappa$, the orders (4.157) apply to (4.150). Finally, by same arguments as the ones above, we can easily show that (4.149) is $o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}})$, to complete the proof of (4.145).

Next, the left side of (4.146) is the real part of

$$\begin{aligned} &\sum_{j=0}^{m} c_j \left(\widetilde{p}_j - p_j\right) \left\{w_{x(\widehat{\gamma})}(-\lambda_j) \left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right]\right. \\ &\left.+ \left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right] w_u(\lambda_j)\right\}. \end{aligned} \tag{4.161}$$

First, by arguments discussed in the proof of (4.145), (4.161) is dominated by the real part of

$$\begin{aligned} &\sum_{j=0}^{m} \left(\widetilde{p}_j - p_j\right) \left\{\begin{pmatrix} (\widehat{\gamma}-\gamma) & 0 \\ 0 & (\widehat{\delta}-\delta) \end{pmatrix} w_{x(\gamma)}(-\lambda_j) w_u^{(1)}(\lambda_j; 0)\right. \\ &\left.+(\gamma - \widehat{\gamma}) w_{u_2}^{(1)}(-\lambda_j; \beta) w_u(\lambda_j)\right\}. \end{aligned} \tag{4.162}$$

As before, the terms where $c_j = 1$ are of smaller order. For the rest, by summation by parts, the second term of (4.162) is

$$2(\gamma-\widehat{\gamma})\left\{[\widetilde{p}_m - p_m]\sum_{j=1}^{m^*} w_{u_2}^{(1)}(-\lambda_j;\beta)w_u(\lambda_j)\right.$$
$$\left.-\sum_{j=1}^{m^*-1}[\widetilde{p}_{j+1}-p_{j+1}-(\widetilde{p}_j-p_j)]\sum_{h=1}^{j} w_{u_2}^{(1)}(-\lambda_h;\beta)w_u(\lambda_h)\right\}. \qquad (4.163)$$

Now, by a straightforward extension of (4.137), uniformly in $j \in [1,m]$,

$$\begin{aligned}\sum_{h=1}^{j} w_{u_2}^{(1)}(-\lambda_h;\beta)w_u(\lambda_h) &= O_p\left(n^{\beta}m^{1-\beta}\log m\right), \ \beta<1,\\ &= O_p\left(n^{\beta}\log m\right), \ \beta \geq 1, \end{aligned} \qquad (4.164)$$

to conclude by Assumptions 4.1, 4.2, that (4.163) is

$$\begin{aligned} O_p\left(n^{\beta-\kappa}m^{1-\beta}\log m(n^{-\varkappa}+n^{-\phi}m)\right), \ \beta &< 1,\\ O_p\left(n^{\beta-\kappa}\log m(n^{-\varkappa}+n^{-\phi}m)\right), \ \beta &\geq 1, \end{aligned} \qquad (4.165)$$

orders which are $o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ by (4.15), (4.19), (4.20).

Now, by same arguments as above and the ones described in the proof of (4.145), it can be easily shown that the orders in (4.165) also apply to the first term of (4.162), to complete the proof.

Proof of Theorem 4.3. Now, for $\beta > 1$, (4.44) follows in view of Theorem 2.2 when $m = [n/2]$. For $m < [n/2]$,

$$\mathrm{Re}\left\{\sum_{j=0}^{m} c_j I_{ux(\gamma)}(\lambda_j)\right\} = \sum_{j=1}^{n} I_{ux(\gamma)}(\lambda_j) + o_p(n^{\beta}), \qquad (4.166)$$

$$\mathrm{Re}\left\{\sum_{j=0}^{m} c_j I_{x(\gamma)}(\lambda_j)\right\} = \sum_{j=1}^{n} I_{x(\gamma)}(\lambda_j) + o_p(n^{2\beta}), \qquad (4.167)$$

by Propositions 4.1, 4.2 of Robinson and Marinucci (1998), and then we conclude as in the case $m = [n/2]$. For $\beta = 1$, as mentioned in Chapter 2, (4.44) follows by Theorem 4.3 of Robinson and Marinucci (2001) and (4.43). For $1/2 < \beta < 1$, noting that

$$\overline{\nu}_m^{\mathrm{o}}(\gamma,\delta) - \nu = \frac{e_m^{\mathrm{o}}(\gamma)}{b_m^{\mathrm{o}}(\gamma)}, \qquad (4.168)$$

where

$$e_m^{\mathrm{o}}(\gamma) = \mathrm{Re}\left\{p(0)\sum_{j=0}^{m} c_j I_{ux(\gamma)}(\lambda_j)\right\}, \qquad (4.169)$$

we first prove that

$$E(e_m^{\mathrm{o}}(\gamma)) = o(n^{\beta}). \qquad (4.170)$$

By the orthogonality condition (4.61), we can write the left side of (4.170) as the real part of

$$\frac{1}{2\pi n}\sum_{j=0}^{m}\int_{-\pi}^{\pi} D_n(\lambda_j-\mu)\sum_{t=1}^{n} a_{n-t}e^{-i(n-t)\lambda_j} D_t(\mu-\lambda_j)\left\{\Xi(\mu,\lambda_j)+\Xi(\lambda_j,0)\right\}d\mu, \quad (4.171)$$

where

$$\Xi(a,b)=p(0)\left\{f(a)-f(b)\right\}\xi. \quad (4.172)$$

The contribution of the second term in braces in (4.171) is

$$n^{-1}\sum_{j=1}^{m}\Xi(\lambda_j,0)\sum_{t=0}^{n-1} a_t\,(n-t)\,e^{-it\lambda_j}. \quad (4.173)$$

By summation by parts, (4.173) is bounded in modulus by

$$n^{-1}\sum_{t=0}^{n-1}|a_t|\,(n-t)\left|\sum_{j=1}^{m-1}\left[\Xi(\lambda_j,0)-\Xi(\lambda_{j+1},0)\right]D_j\left(-\lambda_t\right)+\Xi(\lambda_m,0)D_m\left(-\lambda_t\right)\right|$$

$$\leq Km\sum_{t=1}^{n} t^{\beta-2}\leq Km, \quad (4.174)$$

as we only consider $\beta<1$, to conclude by (4.43). Finally, the proof of (4.58) readily implies that the contribution of the first term in braces in (4.171) is $o(n^{\beta})$.

Next, we show that, as $n\to\infty$,

$$n^{-\beta}(e_m^{\circ}(\gamma)-E(e_m^{\circ}(\gamma)))\Rightarrow\zeta' A\,(1)^{-1\prime}\,\Omega^{-1}\int_0^1\widetilde{W}\,(r;\beta)\,dW\,(r)\,. \quad (4.175)$$

First, note that by Theorem 5.1 of Robinson and Marinucci (2001), as $n\to\infty$,

$$Var(e_m^{\circ}(\gamma))=Var\left(p\,(0)\sum_{j=1}^{n} I_{ux(\gamma)}(\lambda_j)\right)+o(n^{2\beta}), \quad (4.176)$$

implying that

$$e_m^{\circ}(\gamma)-E(e_m^{\circ}(\gamma))=\frac{p\,(0)}{2\pi}\sum_{t=1}^{n}\left\{x_t(\gamma)u_t-E\left[x_t(\gamma)u_t\right]\right\}+o_p(n^{\beta}). \quad (4.177)$$

Thus, in view of the proof of Theorem 4.1, it just remain to prove that

$$\frac{p\,(0)}{2\pi}\sum_{t=1}^{n}\left\{x_t(\gamma)u_t-E\left[x_t(\gamma)u_t\right]\right\}-\frac{p\,(0)}{2\pi}\sum_{t=2}^{n} x_{t-1}(\gamma)A\,(1)\,\varepsilon_t=o_p(n^{\beta}). \quad (4.178)$$

First, note that

$$\sum_{t=1}^{n}\left\{x_t(\gamma)u_t-E\left[x_t(\gamma)u_t\right]\right\}-\sum_{t=1}^{n}\left\{x_t(\gamma)A\,(1)\,\varepsilon_t-E\left[x_t(\gamma)A\,(1)\,\varepsilon_t\right]\right\}$$

$$= \sum_{t=1}^{n} \{x_t(\gamma)(v_{t-1} - v_t) - E[x_t(\gamma)(v_{t-1} - v_t)]\}, \tag{4.179}$$

where

$$v_t = \sum_{j=0}^{\infty} \widetilde{A}_j \varepsilon_{t-j}, \ \widetilde{A}_j = \sum\nolimits_{k=j+1}^{\infty} A_k, \tag{4.180}$$

and

$$\sum_{t=1}^{n} x_t(\gamma)(v_{t-1} - v_t) = \sum_{t=2}^{n} \{x_t(\gamma) - x_{t-1}(\gamma)\} v_{t-1} + x_1(\gamma)v_0 - x_n(\gamma)v_n. \tag{4.181}$$

Now, as in the proof of Theorem 5.1 of Robinson and Marinucci (2001), as Assumption 2.1 ensures boundedness of the spectrum of the process v_t and the crosspectrum of v_t with u_{2t}, it can be easily shown that

$$Var\left\{\sum_{t=2}^{n} \{x_t(\gamma) - x_{t-1}(\gamma)\} v_{t-1}\right\} = O(n). \tag{4.182}$$

Next,

$$E|x_1(\gamma)v_0| \leq \{Ex_1(\gamma)^2 Ev_0^2\}^{\frac{1}{2}} \leq \infty, \tag{4.183}$$

due to the truncation in (1.26) and Assumption 2.1. Similarly, by Robinson and Marinucci (1998, 2001),

$$E|x_n(\gamma)v_n| \leq \{Ex_n(\gamma)^2 Ev_n^2\}^{\frac{1}{2}} \leq Kn^{\beta-\frac{1}{2}}, \tag{4.184}$$

to conclude that (4.179) is $o_p(n^\beta)$. Finally, we have to prove that

$$\sum_{t=2}^{n} x_{t-1}(\gamma)A(1)\varepsilon_t - \sum_{t=2}^{n} \{x_t(\gamma)A(1)\varepsilon_t - E[x_t(\gamma)A(1)\varepsilon_t]\} = o_p(n^\beta), \tag{4.185}$$

but this immediately follows, as

$$Var\left\{\sum_{t=2}^{n} [x_{t-1}(\gamma) - x_t(\gamma)] A(1)\varepsilon_t\right\} = O(n), \tag{4.186}$$

by similar arguments to the ones in the proof of Theorem 5.1 of Robinson and Marinucci (2001), to complete the proof of (4.44).

Finally, (4.45) follows on showing that

$$e_m^{\circ}(\gamma) - e_m(\gamma) = o_p(n^\beta m^{1/2-\beta}), \tag{4.187}$$

$$b_m^{\circ}(\gamma) - b_m(\gamma) = o_p(n^{2\beta} m^{1-2\beta}). \tag{4.188}$$

Now, by the bounds for the periodograms given in Robinson (1995a), Robinson (2002) and Assumption 2.1, the left side of (4.187) is bounded in modulus by

$$K\left\{\sum_{j=1}^{m} \|p(\lambda_j) - p(0)\| \, \|I_u(\lambda_j)\| \sum_{k=1}^{m} \|p(\lambda_k) - p(0)\| \, I_{x(\gamma)}(\lambda_k)\right\}^{\frac{1}{2}}$$

$$\leq K\left\{n^{2\beta-2-2\eta} \sum_{j=1}^{m} j^{1+\eta} \sum_{k=1}^{m} k^{1+\eta-2\beta}\right\}^{\frac{1}{2}} \leq Kn^{\beta-1-\eta} m^{2+\eta-\beta}, \tag{4.189}$$

so that (4.187) holds as

$$\frac{m^{3/2+\eta}}{n^{1+\eta}} \to 0 \text{ as } n \to \infty, \tag{4.190}$$

by (4.10). Finally, by the same arguments, the left side of (4.188) is bounded by

$$K \sum_{j=1}^{m} \lambda_j^{1+\eta} I_{x(\gamma)}(\lambda_j) \leq K n^{2\beta-1-\eta} m^{2+\eta-2\beta}, \tag{4.191}$$

so that (4.188) holds as

$$\frac{m^{1+\eta}}{n^{1+\eta}} \to 0 \text{ as } n \to \infty, \tag{4.192}$$

again by (4.10), to conclude the proof.

TABLE 4.2

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.003	-.005	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.002	-.004	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.005	-.005	-001	-.002	-.002	.000	.000	.000
	0	1.2	-001	-.001	-.001	.000	.000	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.007	-.001	-.001	-.003	.000	-.001	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.005	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.007	-.001	-.002	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.3

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.003	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.007	.000	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.001	-.001	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.006	.000	-.001	.000	.001
	0	1.2	-.001	-.001	.000	-.001	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.006	-.001	-.002	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.4

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	.050	.063	.116	.034	.043	.095	.023	.026	.077
	0	1.2	.001	-.001	.003	.000	-.001	.001	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.008	.014	.033	.004	.007	.020	.002	.003	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
II	0	.6	.063	.078	.155	.043	.053	.128	.029	.033	.105
	0	1.2	.001	-.001	.005	.000	-.001	.001	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.016	.036	.004	.008	.021	.002	.003	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
III	0	.6	.071	.087	.194	.048	.058	.160	.032	.037	.133
	0	1.2	.001	-.001	.007	.000	-.001	.002	.000	.000	.001
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.019	.038	.005	.009	.022	.002	.003	.012
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.5

MONTE CARLO BIAS OF $\overline{\nu}_I^\circ, \overline{\nu}_F^\circ$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$
I	0	.6	.045	.063	.031	.042	.021	.025
	0	1.2	.001	-.001	.000	-.001	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.014	.003	.007	.001	.002
	.4	2	.000	-.001	.000	.000	.000	.000
II	0	.6	.059	.086	.041	.058	.028	.037
	0	1.2	.001	-.001	.000	-.001	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.008	.020	.004	.010	.002	.004
	.4	2	.000	-.001	.000	.000	.000	.000
III	0	.6	.076	.109	.052	.073	.035	.049
	0	1.2	.001	.001	.000	.000	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.029	.005	.013	.002	.005
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.6

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.046	-.063	-.115	-.031	-.041	-.093	-.022	-.026	-.077
	0	1.2	-.001	.000	-.003	.000	.001	-.001	.000	.000	.000
I	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.016	-.035	-.003	-.006	-.020	-.001	-.002	-.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.059	-.078	-.153	-.040	-.051	-.125	-.028	-.034	-.105
	0	1.2	-.001	.000	-.004	.000	.000	-.001	.000	.000	.000
II	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.019	-.036	-.003	-.008	-.020	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.066	-.085	-.191	-.045	-.056	-.157	-.031	-.037	-.133
	0	1.2	-.001	.000	-.007	.000	.000	-.002	.000	.000	-.001
III	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.009	-.021	-.038	-.004	-.009	-.021	-.001	-.003	-.011
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.7

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.043	-.064	-.028	-.040	-.020	-.025
	0	1.2	-.001	.001	.000	.001	.000	.001
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.017	-.002	-.007	-.001	-.002
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.056	-.086	-.038	-.056	-.027	-.037
	0	1.2	.000	.000	.000	.001	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.007	-.023	-.003	-.010	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.071	-.106	-.048	-.072	-.034	-.049
	0	1.2	-.001	-.002	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.009	-.030	-.004	-.013	-.001	-.005
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.8

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	.074	.091	.176	.049	.060	.140	.033	.038	.114
	0	1.2	.001	-.001	.004	.000	.000	.001	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.023	.050	.005	.012	.029	.002	.004	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
II	0	.6	.095	.117	.235	.063	.077	.189	.043	.049	.156
	0	1.2	.001	-.001	.007	.000	.000	.002	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.013	.026	.053	.006	.013	.030	.002	.005	.017
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
III	0	.6	.108	.132	.293	.071	.085	.238	.048	.055	.198
	0	1.2	.001	.000	.011	.000	.000	.004	.000	.000	.001
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.014	.029	.056	.006	.014	.031	.002	.006	.017
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.9

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
I	0	.6	.068	.091	.044	.058	.030	.035
	0	1.2	.001	-.001	.000	.000	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.024	.004	.012	.001	.005
	.4	2	.000	-.001	.000	.000	.000	.000
II	0	.6	.089	.127	.059	.082	.040	.053
	0	1.2	.001	-.001	.000	.000	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.011	.031	.005	.015	.002	.006
	.4	2	.000	-.001	.000	.000	.000	.000
III	0	.6	.112	.164	.075	.106	.051	.072
	0	1.2	.001	.002	.000	.000	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.014	.041	.006	.018	.002	.007
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.10

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0,\ \phi_i = .5,\ \psi_i = 0,\ i = 1,2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.003	-.003	-.005	-.001	-.001	-.002	.000	.000	.000
	0	1.2	-.001	.000	-.001	.000	.000	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.003	-.006	-.001	-.001	-.002	.000	.000	.000
	0	1.2	-.001	.000	-.001	.000	.000	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.004	-.006	-.001	-.001	-.002	.000	.000	.000
	0	1.2	-001	.000	-.001	.000	.000	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.11

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0,\ \phi_i = .5,\ \psi_i = 0,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.004	-.005	-.001	.000	.000	.000
	0	1.2	-.001	.000	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.001	.000	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.005	-.001	.000	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	.000	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.005	-.001	.000	.000	.001
	0	1.2	-.001	.000	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.12

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.041	.063	.101	.029	.045	.082	.019	.029	.066
	0	1.2	.001	-.002	.002	.000	-.001	.001	.000	.000	.000
I	0	2	.000	-.001	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.019	.032	.004	.010	.019	.001	.004	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.048	.076	.114	.034	.053	.092	.023	.035	.074
	0	1.2	.001	-.002	.003	.000	-.001	.001	.000	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.008	.021	.033	.004	.012	.020	.002	.005	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.052	.082	.121	.037	.057	.097	.025	.038	.078
	0	1.2	.001	-.001	.003	.000	-.001	.001	.000	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.024	.033	.004	.012	.020	.002	.005	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.13

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.039	.061	.027	.042	.018	.026
	0	1.2	.001	-.002	.000	-.001	.000	.000
I	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.007	.019	.003	.010	.001	.004
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.043	.076	.030	.051	.019	.032
	0	1.2	.001	-.002	.000	-.001	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.007	.023	.004	.012	.001	.005
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.046	.089	.032	.058	.021	.037
	0	1.2	.001	.000	.000	-.001	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.007	.029	.004	.013	.001	.005
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.14

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.038	-.062	-.101	-.026	-.042	-.080	-.019	-.028	-.066
	0	1.2	-.001	.002	-.003	.000	.001	-.001	.000	.001	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.020	-.034	-.003	-.009	-.019	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.045	-.074	-.112	-.031	-.051	-.090	-.022	-.035	-.074
	0	1.2	-.001	.001	-.003	.000	.001	-.001	.000	.001	.000
II	0	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.007	-.023	-.034	-.003	-.011	-.019	-.001	-.004	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.049	-.079	-.119	-.034	-.054	-.095	-.024	-.038	-.078
	0	1.2	-.001	.001	-.003	.000	.001	-.001	.000	.001	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.025	-.034	-.003	-.012	-.019	-.001	-.004	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.15

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.036	-.061	-.024	-.039	-.017	-.025
	0	1.2	-.001	.002	.000	.001	.000	.001
I	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.006	-.021	-.002	-.009	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.040	-.075	-.027	-.049	-.019	-.032
	0	1.2	-.001	.001	.000	.001	.000	.001
II	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.006	-.026	-.003	-.011	-.001	-.004
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.043	-.086	-.029	-.056	-.020	-.036
	0	1.2	-.001	.000	.000	.001	.000	.001
III	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.007	-.030	-.003	-.013	-.001	-.004
	.4	2	.000	.001	.000	.000	.000	.000

TABLE 4.16

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.061	.091	.154	.041	.063	.121	.028	.041	.097
	0	1.2	.001	-.001	.004	.000	.000	.001	.000	.000	.000
I	0	2	.000	-.001	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.031	.048	.005	.016	.028	.002	.007	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.072	.111	.172	.049	.076	.135	.033	.051	.109
	0	1.2	.001	-.001	.005	.000	.000	.001	.000	.000	.000
II	0	2	.000	-.001	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.011	.035	.049	.005	.018	.028	.002	.008	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.079	.121	.183	.053	.082	.144	.036	.056	.109
	0	1.2	.001	-.001	.005	.000	.000	.001	.000	.000	.000
III	0	2	.000	-.001	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.012	.037	.049	.006	.019	.029	.002	.008	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.17

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.057	.089	.038	.058	.025	.036
	0	1.2	.001	-.002	.000	-.001	.000	.000
I	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.009	.032	.004	.017	.001	.007
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.064	.110	.042	.071	.028	.045
	0	1.2	.001	-.001	.000	.000	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.038	.004	.019	.002	.008
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.068	.128	.045	.080	.030	.052
	0	1.2	.001	.000	.000	.000	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.043	.005	.020	.002	.008
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.18

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.008	-.005	-.009	-.004	.000	-.004	-.001	-.001	-.001
	0	1.2	-.001	.000	-.001	-.001	.000	-.001	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.002	-.008	-.002	.000	-.004	-.001	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.009	-.005	-.009	-.004	-.001	-.004	-.001	-.001	-.001
	0	1.2	-.001	.000	-.001	-.001	.000	-.001	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.002	-.008	-.002	.000	-.004	-.001	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.008	-.006	-.009	-.004	-.001	-.004	-.001	.000	-.001
	0	1.2	-.001	.000	-.001	-.001	.000	-.001	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.003	-.008	-.002	-.001	-.004	-.001	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.19

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.008	-.007	-.004	.000	-.001	-.001
	0	1.2	-.001	.000	-.001	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.003	-.002	.000	-.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.009	-.007	-.004	.001	-.001	.000
	0	1.2	-.001	.000	-.001	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.003	-.002	.000	-.001	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.009	-.007	-.004	.000	-.001	.000
	0	1.2	-.001	.000	-.001	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.005	-.004	-.002	-.001	-.001	.000
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.20

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.034	.053	.072	.022	.039	.055	.013	.024	.041
	0	1.2	.001	.000	.003	.000	.000	.001	.000	.000	.000
I	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.026	.031	.005	.016	.018	.002	.008	.010
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000
	0	.6	.035	.060	.073	.022	.044	.055	.013	.028	.041
	0	1.2	.001	.000	.003	.000	.000	.001	.000	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.028	.031	.005	.017	.018	.002	.008	.010
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000
	0	.6	.036	.065	.073	.023	.046	.056	.014	.030	.042
	0	1.2	.001	.000	.003	.000	.000	.001	.000	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.011	.030	.031	.005	.017	.018	.002	.009	.010
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000

TABLE 4.21

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.034	.055	.022	.040	.013	.024
	0	1.2	.001	.000	.000	.000	.000	.000
I	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.028	.005	.017	.002	.008
	.4	2	.000	-.002	.000	-.001	.000	.000
	0	.6	.034	.067	.022	.047	.013	.029
	0	1.2	.001	.000	.000	.000	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.033	.005	.019	.002	.009
	.4	2	.000	-.002	.000	-.001	.000	.000
	0	.6	.034	.080	.022	.053	.013	.032
	0	1.2	.001	.002	.000	.000	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.040	.005	.021	.002	.010
	.4	2	.000	-.002	.000	-.001	.000	.000

TABLE 4.22

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.035	-.055	-.074	-.021	-.035	-.053	-.012	-.024	-.040
	0	1.2	-.002	.001	-.004	.000	.001	-.001	.000	.001	.000
I	0	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.012	-.026	-.033	-.005	-.014	-.019	-.002	-.007	-.010
	.4	2	.000	.002	-.001	.000	.001	.000	.000	.000	.000
	0	.6	-.036	-.061	-.074	-.021	-.041	-.054	-.013	-.028	-.041
	0	1.2	-.002	.000	-.004	.000	.001	-.001	.000	.001	.000
II	0	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.012	-.028	-.033	-.005	-.016	-.019	-.001	-.008	-.010
	.4	2	.000	.002	-.001	.000	.001	.000	.000	.000	.000
	0	.6	-.037	-.065	-.075	-.021	-.043	-.054	-.013	-.029	-.041
	0	1.2	-.002	.000	-.004	.000	.001	-.001	.000	.001	.000
III	0	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.012	-.030	-.033	-.005	-.016	-.019	-.001	-.008	-.010
	.4	2	.000	.002	-.001	.000	.001	.000	.000	.000	.000

TABLE 4.23

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.035	-.058	-.021	-.036	-.012	-.023
	0	1.2	-.002	.000	.000	.001	.000	.001
I	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.012	-.030	-.005	-.016	-.002	-.007
	.4	2	.000	.002	.000	.001	.000	.000
	0	.6	-.036	-.069	-.021	-.044	-.012	-.028
	0	1.2	-.002	-.001	.000	.001	.000	.001
II	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.012	-.036	-.005	-.019	-.002	-.008
	.4	2	.000	.002	.000	.001	.000	.000
	0	.6	-.036	-.079	-.021	-.050	-.013	-.032
	0	1.2	-.002	-.002	.000	.000	.000	.001
III	0	2	.000	.001	.000	.000	.000	.000
	.4	1.2	-.012	-.041	-.005	-.021	-.002	-.009
	.4	2	.000	.002	.000	.001	.000	.000

TABLE 4.24

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75,\ \phi_i = .9,\ \psi_i = 0,\ i = 1,2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	.051	.081	.108	.031	.058	.080	.018	.036	.060
	0	1.2	.002	.001	.006	.001	.001	.002	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.016	.042	.046	.007	.025	.026	.002	.013	.015
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000
II	0	.6	.052	.090	.110	.032	.064	.081	.019	.041	.060
	0	1.2	.002	.001	.006	.001	.001	.002	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.016	.044	.046	.007	.026	.026	.002	.013	.015
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000
III	0	.6	.053	.096	.111	.032	.067	.081	.019	.043	.061
	0	1.2	.002	.002	.006	.001	.001	.002	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	.016	.046	.046	.006	.026	.026	.002	.014	.015
	.4	2	.000	-.002	.001	.000	-.001	.000	.000	.000	.000

TABLE 4.25

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75,\ \phi_i = .9,\ \psi_i = 0,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
I	0	.6	.050	.084	.031	.059	.018	.036
	0	1.2	.002	.002	.001	.001	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.016	.046	.007	.028	.002	.014
	.4	2	.000	-.002	.000	-.001	.000	.000
II	0	.6	.051	.099	.032	.067	.018	.041
	0	1.2	.002	.003	.001	.001	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.016	.052	.007	.030	.002	.015
	.4	2	.000	-.002	.000	-.001	.000	.000
III	0	.6	.051	.113	.032	.073	.018	.044
	0	1.2	.002	.005	.001	.001	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.016	.058	.007	.030	.002	.015
	.4	2	.000	-.002	.000	-.001	.000	.000

TABLE 4.26

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.003	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.007	-.001	-.002	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.27

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.003	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.003	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.28

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.048	.066	.113	.034	.045	.092	.023	.028	.074
	0	1.2	.001	-.001	.003	.000	-.001	.001	.000	.000	.000
I	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.008	.016	.033	.004	.008	.020	.002	.003	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.062	.085	.138	.043	.057	.113	.029	.036	.092
	0	1.2	.001	-.001	.004	.000	-.001	.001	.000	.000	.000
II	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.020	.034	.005	.010	.021	.002	.004	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.067	.090	.147	.046	.060	.121	.031	.039	.099
	0	1.2	.001	-.001	.004	.000	-.001	.001	.000	.000	.000
III	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.022	.035	.005	.010	.021	.002	.004	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.29

MONTE CARLO BIAS OF $\overline{\nu}_I^{\circ}, \overline{\nu}_F^{\circ}$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$
	0	.6	.043	.063	.030	.042	.020	.025
	0	1.2	.001	-.002	.000	-.001	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.015	.003	.008	.001	.003
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.052	.081	.037	.054	.024	.034
	0	1.2	.001	-.002	.000	-.001	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.020	.004	.010	.001	.004
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.056	.089	.039	.059	.026	.037
	0	1.2	.001	-.001	.000	-.001	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.008	.023	.004	.011	.002	.004
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.30

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	-.045	-.065	-.112	-.031	-.043	-.090	-.022	-.028	-.074
	0	1.2	-.001	.001	-.003	.000	.001	-.001	.000	.001	.000
	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.017	-.034	-.003	-.007	-.020	-.001	-.002	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
II	0	.6	-.058	-.082	-.136	-.040	-.054	-.111	-.028	-.036	-.092
	0	1.2	-.001	.000	-.004	.000	.001	-.001	.000	.000	.000
	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.021	-.036	-.004	-.009	-.020	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
III	0	.6	-.062	-.087	-.145	-.043	-.058	-.118	-.030	-.038	-.099
	0	1.2	-.001	.000	-.004	.000	.001	-.001	.000	.000	.000
	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.009	-.023	-.036	-.004	-.010	-.020	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.31

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
I	0	.6	-.041	-.063	-.027	-.040	-.020	-.025
	0	1.2	-.001	.001	.000	.001	.000	.001
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.018	-.002	-.007	-.001	-.002
	.4	2	.000	.001	.000	.000	.000	.000
II	0	.6	-.049	-.080	-.030	-.052	-.024	-.034
	0	1.2	-.001	.001	.000	.001	.000	.001
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.023	-.003	-.009	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000
III	0	.6	-.053	-.088	-.036	-.057	-.026	-.037
	0	1.2	-.001	.000	.000	.001	.000	.000
	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.007	-.025	-.003	-.011	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000

TABLE 4.32

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.072	.096	.171	.048	.063	.136	.033	.040	.110
	0	1.2	.001	-.001	.004	.000	.000	.001	.000	.000	.000
I	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.026	.049	.005	.013	.029	.002	.005	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.094	.126	.209	.063	.082	.167	.042	.054	.137
	0	1.2	.001	-.001	.006	.000	.000	.002	.000	.000	.000
II	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.013	.031	.051	.006	.015	.030	.002	.006	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.101	.135	.223	.068	.088	.179	.046	.058	.147
	0	1.2	.001	.000	.007	.000	.000	.002	.000	.000	.000
III	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.014	.033	.052	.006	.016	.030	.002	.006	.017
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.33

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.065	.091	.043	.058	.029	.035
	0	1.2	.001	-.002	.000	.000	.000	.000
I	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.009	.026	.004	.013	.001	.005
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.078	.118	.052	.075	.035	.048
	0	1.2	.001	-.001	.000	.000	.000	.000
II	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.032	.005	.015	.002	.006
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.083	.131	.056	.083	.038	.053
	0	1.2	.001	-.001	.000	.000	.000	.000
III	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.011	.035	.005	.016	.002	.006
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.34

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.003	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	.000	-.001	.000	.000	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.003	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.007	-.001	-.001	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.004	-.005	-.001	-.002	-.002	.000	.000	.000
	0	1.2	-.001	-.001	-.001	.000	.000	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.007	-.001	-.002	-.003	.000	.000	-.001
	.4	2	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 4.35

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.003	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.004	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000
	0	.6	-.004	-.006	-.001	-.001	.000	.000
	0	1.2	-.001	-.001	.000	.000	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.003	-.005	-.001	-.001	.000	.000
	.4	2	.000	.000	.000	.000	.000	.000

TABLE 4.36

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.048	.067	.113	.034	.046	.092	.023	.028	.074
	0	1.2	.001	-.001	.003	.000	-.001	.001	.000	.000	.000
I	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.016	.033	.004	.008	.020	.002	.003	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.062	.087	.136	.043	.058	.111	.029	.037	.090
	0	1.2	.001	-.001	.004	.000	-.001	.001	.000	.000	.000
II	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.009	.021	.034	.005	.010	.020	.002	.004	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	.065	.091	.140	.045	.060	.115	.030	.039	.094
	0	1.2	.001	-.001	.004	.000	-.001	.001	.000	.000	.000
III	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.022	.035	.005	.010	.021	.002	.004	.011
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.37

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.043	.063	.030	.042	.020	.025
	0	1.2	.001	-.002	.000	-.001	.000	.000
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.006	.016	.003	.008	.001	.003
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.051	.080	.036	.053	.024	.033
	0	1.2	.001	-.002	.000	-.001	.000	.000
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.020	.004	.010	.001	.004
	.4	2	.000	-.001	.000	.000	.000	.000
	0	.6	.053	.084	.037	.056	.025	.035
	0	1.2	.001	-.001	.000	-.001	.000	.000
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	.007	.022	.004	.010	.001	.004
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.38

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1,2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	-.045	-.066	-.111	-.031	-.043	-.090	-.022	-.028	-.074
	0	1.2	-.001	.001	-.003	.000	.001	-.001	.000	.001	.000
I	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.018	-.034	-.003	-.007	-.020	-.001	-.002	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.059	-.085	-.134	-.040	-.055	-.109	-.028	-.037	-.091
	0	1.2	-.001	.000	-.004	.000	.001	-.001	.000	.000	.000
II	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.008	-.022	-.035	-.004	-.009	-.020	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000
	0	.6	-.061	-.088	-.139	-.042	-.058	-.113	-.030	-.038	-.094
	0	1.2	-.001	.000	-.004	.000	.001	-.001	.000	.000	.000
III	0	2	.000	.000	.001	.000	.000	.000	.000	.000	.000
	.4	1.2	-.009	-.023	-.036	-.004	-.010	-.020	-.001	-.003	-.011
	.4	2	.000	.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.39

MONTE CARLO BIAS OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	-.041	-.063	-.027	-.040	-.019	-.025
	0	1.2	-.001	.001	.000	.001	.000	.001
I	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.018	-.002	-.007	-.001	-.002
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.048	-.079	-.033	-.051	-.024	-.033
	0	1.2	-.001	.001	.000	.001	.000	.001
II	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.006	-.023	-.003	-.009	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000
	0	.6	-.050	-.084	-.034	-.054	-.024	-.035
	0	1.2	-.001	.000	.000	.001	.000	.001
III	0	2	.000	.000	.000	.000	.000	.000
	.4	1.2	-.007	-.024	-.003	-.010	-.001	-.003
	.4	2	.000	.001	.000	.000	.000	.000

TABLE 4.40

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75,\ \phi_i = 0,\ \psi_i = .9,\ i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	.072	.097	.171	.048	.064	.135	.033	.040	.110
	0	1.2	.001	-.001	.004	.000	.000	.001	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.010	.026	.049	.005	.013	.029	.002	.005	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
II	0	.6	.094	.130	.205	.063	.084	.164	.043	.054	.134
	0	1.2	.001	-.001	.006	.000	.000	.002	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.013	.032	.051	.006	.015	.030	.002	.006	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000
III	0	.6	.099	.136	.212	.066	.088	.170	.045	.058	.140
	0	1.2	.001	.000	.006	.000	.000	.002	.000	.000	.000
	0	2	.000	.000	-.001	.000	.000	.000	.000	.000	.000
	.4	1.2	.014	.034	.051	.006	.016	.030	.002	.006	.016
	.4	2	.000	-.001	.000	.000	.000	.000	.000	.000	.000

TABLE 4.41

MONTE CARLO BIAS OF $\overline{\nu}^o_I, \overline{\nu}^o_F$ FOR $\rho = .75,\ \phi_i = 0,\ \psi_i = .9,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$
I	0	.6	.065	.091	.043	.058	.029	.035
	0	1.2	.001	-.002	.000	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.009	.026	.004	.013	.001	.005
	.4	2	.000	-.001	.000	.000	.000	.000
II	0	.6	.077	.117	.051	.074	.035	.047
	0	1.2	.001	-.002	.000	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.032	.005	.015	.002	.006
	.4	2	.000	-.001	.000	.000	.000	.000
III	0	.6	.079	.124	.053	.079	.036	.050
	0	1.2	.001	-.001	.000	.000	.000	.000
	0	2	.000	-.001	.000	.000	.000	.000
	.4	1.2	.010	.034	.005	.016	.002	.006
	.4	2	.000	-.001	.000	.000	.000	.000

TABLE 4.42

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1, 2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.111	.115	.098	.065	.068	.058	.040	.042	.036
	0	1.2	.026	.030	.025	.011	.012	.010	.004	.005	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.072	.077	.080	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.106	.109	.092	.062	.063	.054	.038	.040	.034
	0	1.2	.026	.029	.025	.010	.011	.010	.004	.005	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.075	.079	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.010	.003	.003	.003	.001	.001	.001
	0	.6	.103	.105	.086	.061	.062	.052	.038	.039	.033
	0	1.2	.026	.029	.025	.010	.011	.010	.004	.005	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.075	.078	.037	.041	.046	.020	.021	.025
	.4	2	.009	.009	.010	.003	.003	.003	.001	.001	.001

TABLE 4.43

MONTE CARLO S.D. OF $\overline{\nu}^{\circ}_I, \overline{\nu}^{\circ}_F$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}^{\circ}_I$	$\overline{\nu}^{\circ}_F$	$\overline{\nu}^{\circ}_I$	$\overline{\nu}^{\circ}_F$	$\overline{\nu}^{\circ}_I$	$\overline{\nu}^{\circ}_F$
	0	.6	.110	.114	.065	.068	.039	.041
	0	1.2	.025	.030	.010	.011	.004	.005
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.077	.037	.041	.020	.021
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.106	.111	.064	.065	.040	.041
	0	1.2	.025	.029	.010	.011	.004	.005
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.079	.037	.043	.020	.022
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.108	.114	.068	.069	.042	.044
	0	1.2	.025	.030	.010	.012	.004	.005
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.073	.087	.039	.048	.021	.024
	.4	2	.009	.009	.003	.003	.001	.001

TABLE 4.44

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.100	.106	.093	.060	.064	.061	.037	.040	.042
	0	1.2	.021	.027	.022	.009	.011	.009	.004	.004	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.061	.072	.068	.032	.039	.040	.017	.021	.022
	.4	2	.008	.009	.009	.002	.003	.003	.001	.001	.001
	0	.6	.098	.103	.096	.060	.063	.067	.038	.040	.048
	0	1.2	.021	.027	.022	.009	.011	.009	.004	.004	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.061	.069	.067	.032	.037	.040	.017	.020	.022
	.4	2	.008	.009	.009	.002	.003	.003	.001	.001	.001
	0	.6	.097	.100	.100	.060	.063	.075	.039	.040	.057
	0	1.2	.021	.026	.022	.009	.011	.009	.004	.004	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.067	.067	.032	.037	.040	.017	.020	.022
	.4	2	.008	.009	.009	.002	.003	.003	.001	.001	.001

TABLE 4.45

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.098	.105	.059	.064	.036	.039
	0	1.2	.021	.028	.009	.011	.004	.004
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.060	.073	.032	.040	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.096	.106	.060	.068	.038	.043
	0	1.2	.021	.028	.009	.011	.004	.004
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.060	.071	.032	.039	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.101	.123	.064	.079	.042	.055
	0	1.2	.021	.027	.009	.011	.004	.004
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.062	.077	.033	.040	.018	.022
	.4	2	.008	.009	.002	.003	.001	.001

TABLE 4.46

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.094	.102	.088	.059	.065	.060	.037	.040	.041
	0	1.2	.020	.024	.021	.008	.011	.009	.004	.005	.004
I	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.068	.066	.031	.038	.040	.017	.021	.022
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
	0	.6	.091	.098	.090	.059	.064	.066	.038	.040	.050
	0	1.2	.020	.024	.022	.009	.011	.009	.004	.005	.004
II	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.065	.066	.032	.037	.040	.017	.020	.022
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
	0	.6	.090	.096	.097	.060	.064	.074	.039	.041	.058
	0	1.2	.020	.023	.022	.009	.011	.010	.004	.005	.004
III	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.064	.066	.032	.037	.040	.018	.020	.022
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001

TABLE 4.47

MONTE CARLO S.D. OF $\overline{\nu}_I^\circ, \overline{\nu}_F^\circ$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$
	0	.6	.092	.102	.058	.064	.037	.039
	0	1.2	.020	.024	.009	.011	.004	.005
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.068	.031	.039	.017	.021
	.4	2	.007	.009	.003	.003	.001	.001
	0	.6	.091	.106	.060	.071	.040	.046
	0	1.2	.020	.024	.009	.011	.004	.005
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.069	.032	.039	.018	.021
	.4	2	.007	.009	.003	.003	.001	.001
	0	.6	.097	.122	.065	.086	.043	.058
	0	1.2	.020	.024	.009	.010	.004	.004
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.059	.076	.034	.044	.019	.023
	.4	2	.007	.008	.003	.003	.001	.001

TABLE 4.48

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.083	.095	.087	.051	.057	.061	.034	.036	.046
	0	1.2	.016	.024	.018	.007	.010	.008	.003	.004	.003
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.063	.056	.025	.034	.032	.014	.020	.018
	.4	2	.006	.007	.008	.002	.002	.003	.001	.001	.001
	0	.6	.087	.097	.100	.055	.061	.076	.037	.039	.060
	0	1.2	.016	.025	.018	.007	.010	.008	.003	.004	.003
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.059	.056	.024	.032	.032	.014	.019	.019
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001
	0	.6	.089	.098	.117	.057	.063	.092	.039	.040	.074
	0	1.2	.016	.023	.020	.007	.010	.008	.003	.004	.003
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.056	.057	.024	.031	.033	.014	.018	.019
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001

TABLE 4.49

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.080	.095	.048	.056	.032	.035
	0	1.2	.016	.026	.007	.011	.003	.004
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.063	.024	.036	.013	.021
	.4	2	.006	.008	.002	.002	.001	.001
	0	.6	.083	.105	.052	.068	.035	.043
	0	1.2	.016	.026	.007	.010	.003	.004
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.058	.024	.033	.014	.018
	.4	2	.006	.009	.002	.002	.001	.001
	0	.6	.092	.137	.060	.090	.041	.061
	0	1.2	.016	.023	.007	.009	.003	.004
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.049	.068	.026	.034	.014	.017
	.4	2	.006	.008	.002	.002	.001	.001

TABLE 4.50

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0,\ \phi_i = .5,\ \psi_i = 0,\ i = 1,2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.113	.117	.102	.065	.069	.059	.040	.043	.036
	0	1.2	.026	.033	.026	.010	.013	.010	.004	.005	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.072	.081	.083	.037	.043	.047	.020	.022	.025
	.4	2	.009	.011	.011	.003	.003	.003	.001	.001	.001
	0	.6	.111	.107	.098	.064	.063	.057	.039	.040	.035
	0	1.2	.026	.031	.026	.010	.012	.010	.004	.005	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.073	.077	.083	.037	.041	.047	.020	.021	.025
	.4	2	.009	.011	.011	.003	.003	.003	.001	.001	.001
	0	.6	.110	.105	.096	.064	.063	.056	.039	.039	.035
	0	1.2	.026	.031	.026	.010	.012	.010	.004	.005	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.073	.076	.083	.037	.041	.047	.020	.021	.025
	.4	2	.009	.011	.011	.003	.003	.003	.001	.001	.001

TABLE 4.51

MONTE CARLO S.D. OF $\overline{\nu}_I^{\circ}, \overline{\nu}_F^{\circ}$ FOR $\rho = 0,\ \phi_i = .5,\ \psi_i = 0,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$
	0	.6	.114	.117	.065	.069	.040	.042
	0	1.2	.026	.033	.010	.013	.004	.005
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.071	.080	.037	.043	.020	.022
	.4	2	.009	.011	.003	.003	.001	.001
	0	.6	.112	.110	.064	.065	.039	.041
	0	1.2	.026	.032	.010	.013	.004	.005
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.071	.079	.037	.043	.020	.023
	.4	2	.009	.011	.003	.003	.001	.001
	0	.6	.111	.112	.065	.067	.040	.043
	0	1.2	.026	.032	.010	.013	.004	.005
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.071	.084	.037	.045	.020	.023
	.4	2	.009	.011	.003	.003	.001	.001

TABLE 4.52

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.099	.108	.093	.060	.065	.060	.036	.041	.039
	0	1.2	.021	.030	.022	.009	.013	.009	.004	.005	.004
I	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.076	.070	.032	.042	.041	.017	.023	.022
	.4	2	.008	.010	.010	.002	.003	.003	.001	.001	.001
	0	.6	.098	.100	.092	.060	.062	.061	.037	.039	.041
	0	1.2	.021	.030	.022	.009	.012	.009	.004	.005	.004
II	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.070	.069	.032	.038	.041	.017	.021	.022
	.4	2	.008	.010	.010	.002	.003	.003	.001	.001	.001
	0	.6	.097	.097	.092	.060	.061	.062	.037	.039	.042
	0	1.2	.021	.029	.022	.009	.012	.009	.004	.005	.004
III	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.061	.068	.069	.033	.037	.040	.017	.021	.022
	.4	2	.008	.010	.010	.002	.003	.003	.001	.001	.001

TABLE 4.53

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.098	.110	.059	.065	.036	.041
	0	1.2	.021	.031	.009	.013	.004	.005
I	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.060	.078	.032	.043	.017	.023
	.4	2	.008	.011	.002	.003	.001	.001
	0	.6	.097	.104	.059	.064	.036	.041
	0	1.2	.021	.031	.009	.013	.004	.005
II	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.059	.074	.032	.041	.017	.022
	.4	2	.008	.011	.002	.003	.001	.001
	0	.6	.097	.110	.059	.066	.036	.043
	0	1.2	.021	.030	.009	.013	.004	.005
III	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.059	.075	.032	.040	.017	.022
	.4	2	.008	.010	.002	.003	.001	.001

TABLE 4.54

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n=64$			$n=128$			$n=256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.094	.104	.089	.058	.065	.058	.036	.040	.040
	0	1.2	.020	.028	.022	.009	.013	.009	.004	.005	.004
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.072	.068	.031	.041	.041	.017	.023	.023
	.4	2	.007	.009	.009	.003	.003	.003	.001	.001	.001
	0	.6	.093	.097	.088	.059	.062	.059	.037	.040	.042
	0	1.2	.020	.027	.022	.009	.012	.009	.004	.005	.004
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.067	.067	.032	.038	.041	.018	.021	.023
	.4	2	.007	.009	.009	.003	.003	.003	.001	.001	.001
	0	.6	.092	.095	.088	.059	.062	.061	.038	.040	.043
	0	1.2	.020	.026	.021	.009	.012	.009	.004	.005	.004
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.065	.067	.032	.038	.041	.018	.021	.023
	.4	2	.007	.009	.009	.003	.003	.003	.001	.001	.001

TABLE 4.55

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.094	.105	.058	.065	.036	.041
	0	1.2	.020	.029	.009	.013	.004	.006
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.058	.073	.031	.042	.017	.024
	.4	2	.007	.010	.003	.003	.001	.001
	0	.6	.093	.102	.058	.066	.037	.042
	0	1.2	.020	.028	.009	.013	.004	.005
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.058	.070	.031	.041	.018	.023
	.4	2	.007	.010	.003	.003	.001	.001
	0	.6	.093	.109	.059	.071	.037	.045
	0	1.2	.020	.026	.009	.012	.004	.005
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.058	.071	.032	.042	.018	.022
	.4	2	.007	.009	.003	.003	.001	.001

TABLE 4.56

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.079	.095	.082	.048	.056	.056	.032	.036	.042
	0	1.2	.016	.029	.018	.007	.012	.008	.003	.005	.003
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.067	.057	.024	.038	.032	.014	.022	.018
	.4	2	.006	.009	.008	.002	.003	.003	.001	.001	.001
	0	.6	.080	.090	.085	.050	.056	.060	.033	.036	.045
	0	1.2	.016	.028	.018	.007	.011	.008	.003	.005	.003
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.060	.057	.024	.033	.032	.014	.020	.018
	.4	2	.006	.009	.008	.002	.003	.003	.001	.001	.001
	0	.6	.082	.089	.089	.051	.056	.063	.034	.037	.048
	0	1.2	.016	.026	.018	.007	.011	.008	.003	.005	.003
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.057	.057	.024	.032	.032	.014	.019	.018
	.4	2	.006	.009	.008	.002	.003	.003	.001	.001	.001

TABLE 4.57

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.078	.096	.047	.057	.030	.037
	0	1.2	.016	.030	.007	.013	.003	.005
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.047	.070	.024	.041	.014	.024
	.4	2	.006	.010	.002	.003	.001	.001
	0	.6	.078	.094	.047	.057	.031	.036
	0	1.2	.016	.030	.007	.013	.003	.005
II	0	2	.002	.004	.001	.001	.000	.000
	.4	1.2	.046	.064	.024	.038	.014	.022
	.4	2	.006	.010	.002	.003	.001	.001
	0	.6	.079	.107	.048	.062	.032	.039
	0	1.2	.016	.027	.007	.012	.003	.005
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.047	.064	.024	.035	.014	.020
	.4	2	.006	.009	.002	.003	.001	.001

TABLE 4.58

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.142	.128	.126	.079	.076	.071	.045	.047	.041
	0	1.2	.030	.036	.030	.011	.017	.011	.005	.008	.005
I	0	2	.004	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.089	.093	.110	.043	.052	.057	.022	.029	.028
	.4	2	.011	.014	.014	.003	.005	.004	.001	.002	.001
	0	.6	.142	.121	.125	.079	.071	.071	.045	.043	.041
	0	1.2	.030	.034	.030	.011	.016	.011	.005	.007	.005
II	0	2	.004	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.090	.091	.110	.043	.050	.057	.022	.027	.028
	.4	2	.011	.013	.014	.003	.004	.004	.001	.002	.001
	0	.6	.140	.119	.125	.078	.070	.071	.045	.043	.041
	0	1.2	.030	.033	.030	.011	.016	.011	.005	.007	.005
III	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.089	.090	.110	.043	.049	.057	.021	.026	.028
	.4	2	.011	.013	.014	.003	.004	.004	.001	.002	.001

TABLE 4.59

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.142	.128	.079	.076	.045	.047
	0	1.2	.030	.036	.011	.017	.005	.008
I	0	2	.004	.004	.001	.001	.000	.000
	.4	1.2	.089	.092	.043	.052	.022	.029
	.4	2	.011	.014	.003	.005	.001	.002
	0	.6	.142	.125	.079	.074	.045	.047
	0	1.2	.030	.034	.011	.016	.005	.008
II	0	2	.004	.004	.001	.001	.000	.000
	.4	1.2	.089	.094	.043	.053	.022	.029
	.4	2	.011	.013	.003	.005	.001	.002
	0	.6	.142	.137	.079	.079	.045	.049
	0	1.2	.030	.035	.011	.017	.005	.008
III	0	2	.004	.004	.001	.001	.000	.000
	.4	1.2	.089	.105	.043	.057	.022	.031
	.4	2	.011	.013	.003	.005	.001	.002

TABLE 4.60

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.119	.119	.110	.071	.072	.066	.041	.047	.039
	0	1.2	.024	.033	.025	.010	.016	.010	.004	.007	.004
I	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.073	.088	.090	.038	.048	.047	.019	.029	.023
	.4	2	.009	.011	.012	.003	.004	.003	.001	.001	.001
	0	.6	.119	.110	.109	.071	.066	.066	.041	.042	.039
	0	1.2	.024	.032	.025	.009	.016	.010	.004	.007	.004
II	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.073	.084	.090	.038	.044	.047	.019	.027	.023
	.4	2	.009	.012	.012	.003	.004	.003	.001	.001	.001
	0	.6	.118	.105	.109	.070	.064	.066	.040	.041	.039
	0	1.2	.024	.031	.025	.009	.015	.010	.004	.007	.004
III	0	2	.003	.004	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.073	.080	.090	.037	.043	.047	.019	.026	.023
	.4	2	.009	.011	.012	.003	.004	.003	.001	.001	.001

TABLE 4.61

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.119	.120	.071	.073	.041	.048
	0	1.2	.024	.034	.010	.017	.004	.008
I	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.073	.090	.038	.050	.019	.031
	.4	2	.009	.012	.003	.004	.001	.001
	0	.6	.119	.114	.071	.069	.041	.045
	0	1.2	.024	.034	.010	.017	.004	.008
II	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.073	.088	.038	.048	.019	.029
	.4	2	.009	.012	.003	.004	.001	.001
	0	.6	.119	.121	.071	.073	.041	.047
	0	1.2	.024	.033	.010	.016	.004	.008
III	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.073	.092	.038	.048	.019	.029
	.4	2	.009	.012	.003	.004	.001	.001

TABLE 4.62

MONTE CARLO S.D. OF $\bar{\nu}_I, \bar{\nu}_F, \bar{\nu}_B$ FOR $\rho = -.5, \phi_i = .9,\ \psi_i = 0,\ i = 1,2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_B$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_B$	$\bar{\nu}_I$	$\bar{\nu}_F$	$\bar{\nu}_B$
	0	.6	.116	.115	.105	.069	.071	.064	.041	.045	.040
	0	1.2	.022	.031	.023	.010	.016	.011	.004	.008	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.070	.082	.081	.037	.051	.049	.019	.030	.025
	.4	2	.008	.011	.011	.003	.004	.004	.001	.001	.001
	0	.6	.115	.106	.105	.069	.065	.064	.041	.042	.040
	0	1.2	.022	.030	.023	.010	.015	.011	.004	.008	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.069	.076	.081	.037	.048	.049	.019	.027	.025
	.4	2	.008	.011	.011	.003	.004	.004	.001	.001	.001
	0	.6	.113	.103	.104	.069	.064	.064	.041	.041	.040
	0	1.2	.022	.029	.023	.010	.015	.011	.004	.008	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.068	.073	.081	.037	.047	.049	.019	.026	.025
	.4	2	.008	.010	.011	.003	.004	.004	.001	.001	.001

TABLE 4.63

MONTE CARLO S.D. OF $\bar{\nu}_I^o, \bar{\nu}_F^o$ FOR $\rho = -.5,\ \phi_i = .9,\ \psi_i = 0,\ i = 1,2$

		n	64	64	128	128	256	256
m	γ	δ	$\bar{\nu}_I^o$	$\bar{\nu}_F^o$	$\bar{\nu}_I^o$	$\bar{\nu}_F^o$	$\bar{\nu}_I^o$	$\bar{\nu}_F^o$
	0	.6	.116	.116	.069	.072	.041	.047
	0	1.2	.022	.032	.010	.017	.004	.008
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.082	.037	.052	.019	.031
	.4	2	.008	.011	.003	.004	.001	.001
	0	.6	.115	.112	.069	.071	.041	.046
	0	1.2	.022	.031	.010	.016	.004	.008
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.081	.037	.053	.019	.030
	.4	2	.008	.012	.003	.004	.001	.001
	0	.6	.115	.122	.069	.078	.041	.048
	0	1.2	.022	.030	.010	.015	.004	.008
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.085	.037	.056	.019	.029
	.4	2	.008	.010	.003	.004	.001	.001

TABLE 4.64

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.096	.103	.089	.054	.063	.053	.033	.043	.035
	0	1.2	.018	.031	.020	.008	.015	.009	.003	.007	.003
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.077	.072	.029	.046	.037	.015	.029	.019
	.4	2	.007	.010	.010	.002	.003	.003	.001	.001	.001
	0	.6	.095	.094	.089	.053	.057	.053	.033	.038	.035
	0	1.2	.018	.030	.020	.008	.014	.009	.003	.007	.003
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.071	.072	.029	.041	.037	.015	.026	.019
	.4	2	.007	.011	.010	.002	.003	.003	.001	.001	.001
	0	.6	.095	.090	.089	.053	.054	.053	.033	.036	.035
	0	1.2	.018	.029	.020	.007	.014	.009	.003	.007	.003
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.066	.071	.028	.038	.037	.015	.025	.019
	.4	2	.007	.010	.009	.002	.003	.003	.001	.001	.001

TABLE 4.65

MONTE CARLO S.D. OF $\overline{\nu}_I^{\circ}, \overline{\nu}_F^{\circ}$ FOR $\rho = .75$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$	$\overline{\nu}_I^{\circ}$	$\overline{\nu}_F^{\circ}$
	0	.6	.096	.105	.053	.066	.033	.046
	0	1.2	.018	.033	.008	.016	.003	.008
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.058	.080	.029	.050	.015	.032
	.4	2	.007	.012	.002	.004	.001	.001
	0	.6	.095	.100	.053	.061	.033	.040
	0	1.2	.018	.033	.008	.016	.003	.008
II	0	2	.002	.004	.001	.001	.000	.000
	.4	1.2	.058	.075	.029	.050	.015	.029
	.4	2	.007	.012	.002	.004	.001	.001
	0	.6	.095	.111	.054	.062	.033	.039
	0	1.2	.018	.029	.008	.014	.003	.007
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.058	.076	.029	.042	.015	.026
	.4	2	.007	.011	.002	.003	.001	.001

TABLE 4.66

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.110	.114	.099	.065	.068	.058	.040	.042	.036
	0	1.2	.026	.031	.025	.011	.012	.010	.004	.005	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.077	.081	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.105	.105	.093	.062	.062	.054	.038	.039	.034
	0	1.2	.026	.030	.025	.010	.011	.010	.004	.005	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.075	.080	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.104	.103	.091	.062	.062	.054	.039	.039	.034
	0	1.2	.026	.029	.025	.010	.011	.010	.004	.005	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.075	.080	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001

TABLE 4.67

MONTE CARLO S.D. OF $\overline{\nu}^o_I, \overline{\nu}^o_F$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$	$\overline{\nu}^o_I$	$\overline{\nu}^o_F$
	0	.6	.110	.114	.065	.068	.039	.041
	0	1.2	.026	.030	.010	.012	.004	.005
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.077	.036	.041	.020	.021
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.106	.108	.063	.064	.039	.040
	0	1.2	.025	.030	.010	.012	.004	.005
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.077	.036	.042	.020	.022
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.106	.108	.063	.064	.039	.041
	0	1.2	.025	.030	.010	.012	.004	.005
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.078	.037	.043	.020	.022
	.4	2	.009	.010	.003	.003	.001	.001

TABLE 4.68

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.098	.106	.093	.060	.064	.061	.037	.040	.041
	0	1.2	.021	.028	.022	.009	.011	.009	.004	.004	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.073	.068	.032	.040	.040	.017	.021	.022
	.4	2	.008	.010	.009	.002	.003	.003	.001	.001	.001
	0	.6	.096	.100	.093	.060	.063	.064	.038	.040	.045
	0	1.2	.021	.027	.022	.009	.011	.009	.004	.004	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.067	.068	.032	.037	.040	.017	.020	.022
	.4	2	.008	.010	.009	.002	.003	.003	.001	.001	.001
	0	.6	.096	.099	.093	.060	.063	.065	.038	.040	.047
	0	1.2	.021	.026	.022	.009	.011	.009	.004	.004	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.066	.067	.032	.036	.040	.017	.020	.022
	.4	2	.008	.009	.009	.002	.003	.003	.001	.001	.001

TABLE 4.69

MONTE CARLO S.D. OF $\overline{\nu}_I^\circ, \overline{\nu}_F^\circ$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$	$\overline{\nu}_I^\circ$	$\overline{\nu}_F^\circ$
	0	.6	.097	.106	.059	.064	.036	.039
	0	1.2	.021	.028	.009	.012	.004	.004
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.059	.074	.032	.040	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.095	.103	.058	.065	.037	.041
	0	1.2	.021	.028	.009	.011	.004	.004
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.059	.071	.032	.039	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.095	.106	.059	.067	.037	.043
	0	1.2	.021	.028	.009	.011	.004	.004
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.059	.070	.032	.038	.017	.020
	.4	2	.008	.010	.002	.003	.001	.001

TABLE 4.70

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.093	.102	.088	.058	.064	.059	.037	.039	.041
	0	1.2	.020	.025	.021	.009	.011	.009	.004	.005	.004
I	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.068	.066	.031	.038	.040	.017	.021	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
	0	.6	.091	.097	.088	.059	.064	.063	.038	.041	.046
	0	1.2	.020	.024	.021	.009	.011	.009	.004	.005	.004
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.064	.066	.032	.037	.040	.018	.020	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
	0	.6	.090	.096	.089	.060	.064	.064	.039	.041	.048
	0	1.2	.020	.023	.021	.009	.011	.009	.004	.005	.004
III	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.063	.066	.032	.037	.040	.018	.020	.022
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001

TABLE 4.71

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.092	.102	.058	.064	.036	.040
	0	1.2	.020	.026	.009	.012	.004	.005
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.069	.031	.039	.017	.022
	.4	2	.007	.009	.003	.003	.001	.001
	0	.6	.091	.101	.058	.067	.038	.043
	0	1.2	.020	.025	.009	.011	.004	.005
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.067	.031	.039	.018	.021
	.4	2	.007	.009	.003	.003	.001	.001
	0	.6	.091	.105	.059	.071	.039	.045
	0	1.2	.020	.024	.009	.011	.004	.005
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.068	.032	.039	.018	.021
	.4	2	.007	.008	.003	.003	.001	.001

TABLE 4.72

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.082	.095	.085	.050	.057	.060	.033	.036	.045
	0	1.2	.016	.026	.018	.007	.011	.008	.003	.004	.003
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.063	.056	.024	.035	.032	.014	.020	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001
	0	.6	.085	.097	.093	.054	.062	.069	.037	.040	.054
	0	1.2	.016	.025	.018	.007	.010	.008	.003	.004	.003
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.056	.056	.024	.031	.032	.014	.018	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001
	0	.6	.087	.099	.097	.056	.064	.073	.038	.041	.057
	0	1.2	.016	.023	.018	.007	.010	.008	.003	.004	.003
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.055	.056	.025	.030	.032	.014	.017	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001

TABLE 4.73

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.079	.094	.047	.056	.031	.035
	0	1.2	.016	.027	.007	.011	.003	.004
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.064	.024	.037	.013	.021
	.4	2	.006	.009	.002	.003	.001	.001
	0	.6	.079	.097	.050	.061	.033	.039
	0	1.2	.016	.027	.007	.011	.003	.004
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.059	.024	.034	.013	.019
	.4	2	.006	.009	.002	.003	.001	.001
	0	.6	.081	.106	.051	.066	.034	.042
	0	1.2	.016	.025	.007	.011	.003	.004
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.059	.024	.032	.014	.018
	.4	2	.006	.009	.002	.003	.001	.001

TABLE 4.74

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.110	.114	.099	.065	.068	.058	.040	.042	.036
	0	1.2	.026	.031	.026	.011	.012	.010	.004	.005	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.071	.078	.081	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.105	.104	.093	.062	.062	.055	.038	.039	.034
	0	1.2	.026	.030	.025	.010	.012	.010	.004	.005	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.072	.075	.080	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001
	0	.6	.104	.103	.092	.062	.062	.054	.039	.039	.034
	0	1.2	.026	.029	.025	.010	.012	.010	.004	.005	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.072	.075	.080	.037	.041	.046	.020	.021	.025
	.4	2	.009	.010	.011	.003	.003	.003	.001	.001	.001

TABLE 4.75

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.110	.115	.065	.068	.039	.041
	0	1.2	.026	.031	.010	.012	.004	.005
I	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.070	.078	.036	.041	.020	.021
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.106	.107	.063	.064	.039	.040
	0	1.2	.025	.030	.010	.012	.004	.005
II	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.076	.036	.041	.020	.022
	.4	2	.009	.010	.003	.003	.001	.001
	0	.6	.106	.107	.063	.064	.039	.041
	0	1.2	.025	.030	.010	.012	.004	.005
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.069	.077	.037	.042	.020	.022
	.4	2	.009	.010	.003	.003	.001	.001

TABLE 4.76

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.098	.106	.093	.060	.064	.061	.037	.040	.041
	0	1.2	.021	.028	.022	.009	.011	.009	.004	.004	.004
I	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.073	.068	.032	.040	.040	.017	.021	.022
	.4	2	.008	.010	.010	.002	.003	.003	.001	.001	.001
	0	.6	.096	.101	.092	.060	.063	.064	.038	.040	.045
	0	1.2	.021	.027	.022	.009	.011	.009	.004	.004	.004
II	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.067	.068	.032	.037	.040	.017	.020	.022
	.4	2	.008	.010	.009	.002	.003	.003	.001	.001	.001
	0	.6	.095	.100	.093	.060	.063	.064	.038	.041	.046
	0	1.2	.021	.026	.022	.009	.011	.009	.004	.004	.004
III	0	2	.003	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.060	.066	.068	.032	.036	.040	.017	.020	.022
	.4	2	.008	.009	.009	.002	.003	.003	.001	.001	.001

TABLE 4.77

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.097	.106	.059	.064	.036	.039
	0	1.2	.021	.029	.009	.012	.004	.004
I	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.059	.074	.032	.041	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.095	.102	.058	.064	.036	.041
	0	1.2	.021	.029	.009	.011	.004	.004
II	0	2	.003	.004	.001	.001	.000	.000
	.4	1.2	.059	.071	.032	.039	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001
	0	.6	.095	.104	.059	.065	.037	.042
	0	1.2	.021	.028	.009	.011	.004	.004
III	0	2	.003	.003	.001	.001	.000	.000
	.4	1.2	.059	.070	.032	.038	.017	.021
	.4	2	.008	.010	.002	.003	.001	.001

TABLE 4.78

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
I	0	.6	.093	.102	.088	.058	.064	.059	.037	.039	.041
	0	1.2	.020	.025	.021	.009	.011	.009	.004	.005	.004
	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.069	.066	.031	.038	.040	.017	.021	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
II	0	.6	.091	.097	.088	.059	.064	.062	.038	.041	.046
	0	1.2	.020	.024	.022	.009	.011	.009	.004	.005	.004
	0	2	.002	.002	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.058	.064	.066	.032	.037	.040	.018	.020	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001
III	0	.6	.090	.097	.088	.059	.065	.063	.039	.041	.046
	0	1.2	.020	.023	.021	.009	.011	.009	.004	.005	.004
	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.057	.063	.066	.032	.037	.040	.018	.020	.023
	.4	2	.007	.008	.009	.003	.003	.003	.001	.001	.001

TABLE 4.79

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
I	0	.6	.093	.103	.058	.064	.036	.040
	0	1.2	.020	.026	.009	.012	.004	.005
	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.069	.031	.039	.017	.022
	.4	2	.007	.009	.003	.003	.001	.001
II	0	.6	.091	.101	.058	.066	.038	.043
	0	1.2	.020	.025	.009	.011	.004	.005
	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.067	.031	.039	.018	.021
	.4	2	.007	.009	.003	.003	.001	.001
III	0	.6	.091	.103	.058	.068	.038	.044
	0	1.2	.020	.025	.009	.011	.004	.005
	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.057	.067	.031	.039	.018	.021
	.4	2	.007	.009	.003	.003	.001	.001

TABLE 4.80

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.6	.082	.095	.085	.050	.057	.060	.033	.036	.045
	0	1.2	.016	.026	.018	.007	.011	.008	.003	.004	.003
I	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.063	.056	.024	.035	.032	.014	.020	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001
	0	.6	.085	.098	.092	.055	.063	.068	.037	.040	.053
	0	1.2	.016	.024	.018	.007	.010	.008	.003	.004	.003
II	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.047	.055	.056	.025	.031	.032	.014	.018	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001
	0	.6	.086	.100	.094	.056	.064	.070	.038	.042	.055
	0	1.2	.016	.023	.018	.007	.010	.008	.003	.004	.003
III	0	2	.002	.003	.003	.001	.001	.001	.000	.000	.000
	.4	1.2	.048	.055	.056	.025	.030	.032	.014	.017	.018
	.4	2	.006	.008	.008	.002	.002	.003	.001	.001	.001

TABLE 4.81

MONTE CARLO S.D. OF $\overline{\nu}_I^o, \overline{\nu}_F^o$ FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$	$\overline{\nu}_I^o$	$\overline{\nu}_F^o$
	0	.6	.079	.094	.047	.056	.031	.035
	0	1.2	.016	.027	.007	.012	.003	.004
I	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.065	.024	.037	.014	.021
	.4	2	.006	.009	.002	.003	.001	.001
	0	.6	.079	.096	.049	.061	.033	.038
	0	1.2	.016	.027	.007	.011	.003	.004
II	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.060	.024	.034	.013	.020
	.4	2	.006	.009	.002	.003	.001	.001
	0	.6	.080	.101	.050	.063	.034	.040
	0	1.2	.016	.026	.007	.011	.003	.004
III	0	2	.002	.003	.001	.001	.000	.000
	.4	1.2	.046	.059	.024	.033	.013	.019
	.4	2	.006	.009	.002	.003	.001	.001

TABLE 4.82

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.106	.171	.096	.146	.083	.121	.175	.222	.146	.206	.140	.191
	0	1.2	.075	.190	.062	.147	.052	.125	.144	.247	.123	.195	.120	.169
I	0	2	.065	.165	.070	.145	.066	.120	.119	.214	.117	.203	.120	.175
	.4	1.2	.094	.179	.078	.139	.061	.122	.157	.244	.132	.198	.116	.165
	.4	2	.062	.172	.072	.154	.066	.118	.125	.222	.131	.198	.122	.176
	0	.6	.119	.182	.089	.142	.081	.116	.181	.231	.159	.196	.146	.184
	0	1.2	.071	.191	.063	.144	.048	.124	.136	.246	.120	.193	.124	.171
II	0	2	.068	.154	.066	.144	.065	.119	.125	.215	.119	.201	.122	.177
	.4	1.2	.104	.183	.082	.138	.059	.121	.170	.249	.130	.194	.116	.162
	.4	2	.070	.172	.076	.155	.064	.121	.118	.226	.130	.194	.122	.173
	0	.6	.127	.185	.100	.146	.090	.123	.198	.233	.159	.204	.155	.192
	0	1.2	.074	.192	.065	.142	.052	.126	.141	.246	.122	.199	.123	.170
III	0	2	.071	.163	.065	.147	.066	.118	.126	.219	.119	.198	.123	.176
	.4	1.2	.104	.187	.086	.142	.064	.121	.171	.251	.139	.200	.121	.168
	.4	2	.067	.174	.071	.152	.062	.117	.121	.219	.126	.202	.122	.172

TABLE 4.83

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.103	.177	.090	.143	.078	.117	.175	.224	.151	.203	.133	.180
	0	1.2	.073	.192	.065	.149	.050	.126	.137	.247	.122	.194	.122	.172
I	0	2	.066	.165	.071	.150	.065	.118	.118	.215	.117	.201	.121	.176
	.4	1.2	.091	.177	.077	.134	.053	.120	.163	.239	.126	.192	.111	.165
	.4	2	.064	.169	.075	.156	.065	.119	.123	.224	.128	.199	.117	.174
	0	.6	.120	.194	.096	.151	.081	.126	.204	.245	.161	.216	.148	.197
	0	1.2	.070	.188	.063	.145	.049	.127	.130	.249	.123	.193	.126	.173
II	0	2	.070	.159	.071	.149	.064	.118	.116	.218	.116	.202	.121	.180
	.4	1.2	.097	.192	.085	.147	.066	.132	.169	.251	.134	.208	.117	.178
	.4	2	.068	.171	.076	.157	.062	.121	.121	.223	.125	.194	.118	.171
	0	.6	.149	.219	.131	.186	.118	.163	.217	.275	.205	.252	.188	.243
	0	1.2	.066	.192	.066	.142	.051	.127	.143	.250	.128	.196	.123	.174
III	0	2	.071	.162	.070	.145	.064	.117	.119	.221	.122	.202	.122	.176
	.4	1.2	.117	.228	.103	.164	.079	.143	.192	.286	.153	.225	.139	.201
	.4	2	.066	.172	.074	.155	.063	.118	.125	.223	.124	.196	.118	.176

TABLE 4.84

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
I	0	.6	.143	.233	.128	.232	.132	.203	.229	.304	.221	.308	.221	.290
	0	1.2	.078	.171	.064	.149	.061	.128	.140	.225	.111	.195	.108	.184
	0	2	.075	.196	.068	.151	.069	.128	.131	.248	.126	.195	.119	.176
	.4	1.2	.105	.204	.073	.190	.057	.161	.165	.273	.133	.250	.111	.229
	.4	2	.072	.189	.062	.140	.061	.123	.130	.233	.116	.182	.110	.175
II	0	.6	.191	.279	.180	.275	.181	.248	.281	.352	.267	.362	.279	.341
	0	1.2	.083	.173	.063	.146	.060	.128	.144	.223	.114	.194	.112	.186
	0	2	.071	.187	.064	.154	.068	.129	.133	.247	.126	.194	.117	.175
	.4	1.2	.112	.216	.076	.185	.061	.163	.167	.287	.138	.246	.113	.226
	.4	2	.072	.182	.066	.141	.061	.121	.131	.234	.115	.180	.110	.170
III	0	.6	.228	.311	.229	.315	.217	.291	.343	.399	.311	.396	.321	.391
	0	1.2	.083	.170	.065	.149	.059	.125	.142	.224	.111	.191	.108	.181
	0	2	.077	.192	.066	.152	.068	.128	.129	.250	.123	.195	.117	.172
	.4	1.2	.123	.226	.079	.185	.063	.163	.177	.289	.139	.255	.114	.228
	.4	2	.075	.181	.063	.146	.062	.118	.132	.235	.112	.182	.109	.177

TABLE 4.85

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
I	0	.6	.137	.231	.118	.224	.117	.193	.227	.299	.198	.305	.197	.270
	0	1.2	.077	.179	.063	.149	.059	.129	.145	.228	.104	.198	.106	.190
	0	2	.079	.191	.067	.152	.068	.128	.133	.248	.124	.191	.117	.174
	.4	1.2	.102	.212	.073	.194	.055	.170	.162	.276	.122	.254	.105	.228
	.4	2	.071	.187	.060	.140	.059	.124	.133	.235	.117	.182	.110	.176
II	0	.6	.187	.306	.172	.307	.185	.281	.279	.389	.256	.388	.258	.380
	0	1.2	.083	.182	.060	.147	.061	.130	.143	.228	.111	.196	.109	.189
	0	2	.076	.186	.065	.152	.069	.129	.134	.251	.127	.192	.117	.175
	.4	1.2	.116	.227	.072	.200	.061	.167	.172	.294	.130	.261	.108	.220
	.4	2	.073	.186	.063	.143	.061	.119	.138	.240	.119	.181	.107	.171
III	0	.6	.281	.414	.269	.407	.253	.372	.356	.496	.353	.476	.341	.458
	0	1.2	.080	.177	.059	.149	.059	.126	.134	.229	.109	.196	.109	.187
	0	2	.078	.190	.065	.153	.067	.127	.128	.248	.123	.195	.119	.172
	.4	1.2	.126	.253	.080	.199	.076	.172	.201	.323	.150	.278	.129	.224
	.4	2	.074	.178	.062	.143	.061	.118	.136	.237	.113	.182	.107	.173

TABLE 4.86

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.143	.241	.134	.246	.145	.212	.215	.312	.221	.327	.211	.277
	0	1.2	.062	.180	.080	.177	.054	.120	.128	.246	.133	.229	.107	.176
I	0	2	.063	.161	.077	.166	.065	.129	.127	.215	.137	.227	.111	.177
	.4	1.2	.085	.226	.075	.196	.065	.167	.164	.298	.145	.272	.129	.222
	.4	2	.063	.156	.078	.164	.061	.118	.125	.208	.132	.211	.111	.170
	0	.6	.189	.291	.191	.287	.200	.256	.271	.379	.273	.360	.282	.330
	0	1.2	.065	.182	.078	.177	.052	.122	.130	.248	.133	.227	.104	.181
II	0	2	.061	.167	.076	.167	.064	.129	.122	.216	.134	.223	.113	.180
	.4	1.2	.095	.237	.088	.199	.070	.165	.171	.299	.147	.272	.131	.216
	.4	2	.061	.161	.076	.162	.063	.117	.123	.207	.134	.211	.112	.167
	0	.6	.220	.336	.224	.323	.231	.291	.309	.414	.311	.394	.309	.369
	0	1.2	.064	.186	.079	.177	.052	.119	.129	.246	.135	.228	.102	.179
III	0	2	.064	.166	.079	.168	.064	.128	.125	.219	.132	.221	.115	.178
	.4	1.2	.102	.237	.090	.198	.074	.170	.169	.306	.155	.270	.137	.223
	.4	2	.064	.160	.077	.165	.063	.115	.122	.211	.138	.211	.111	.168

TABLE 4.87

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.145	.240	.125	.237	.136	.207	.204	.316	.205	.311	.198	.268
	0	1.2	.060	.182	.078	.178	.052	.123	.130	.249	.135	.224	.106	.179
I	0	2	.059	.164	.078	.166	.065	.127	.128	.213	.135	.226	.113	.176
	.4	1.2	.085	.233	.084	.202	.061	.171	.160	.296	.138	.286	.128	.227
	.4	2	.068	.156	.073	.168	.063	.118	.120	.212	.133	.210	.113	.172
	0	.6	.190	.324	.181	.320	.200	.301	.257	.404	.260	.407	.271	.367
	0	1.2	.070	.189	.081	.180	.051	.125	.132	.241	.134	.227	.107	.184
II	0	2	.062	.167	.076	.169	.063	.128	.124	.215	.133	.224	.112	.178
	.4	1.2	.088	.247	.086	.211	.073	.173	.183	.330	.148	.290	.135	.241
	.4	2	.064	.158	.072	.166	.061	.116	.117	.205	.132	.215	.110	.167
	0	.6	.258	.411	.261	.413	.260	.395	.339	.491	.343	.499	.348	.477
	0	1.2	.070	.184	.080	.178	.050	.122	.130	.249	.137	.215	.107	.184
III	0	2	.060	.172	.078	.171	.064	.128	.128	.213	.132	.224	.114	.181
	.4	1.2	.124	.277	.108	.228	.089	.183	.190	.352	.174	.291	.145	.254
	.4	2	.062	.159	.073	.167	.059	.116	.112	.208	.134	.211	.112	.170

TABLE 4.88

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.227	.336	.237	.364	.254	.335	.313	.420	.331	.455	.347	.427
	0	1.2	.078	.205	.057	.184	.052	.156	.138	.262	.115	.247	.113	.216
I	0	2	.064	.183	.052	.154	.064	.148	.119	.242	.116	.198	.105	.208
	.4	1.2	.099	.285	.065	.278	.061	.246	.157	.352	.137	.356	.123	.314
	.4	2	.065	.170	.063	.144	.061	.142	.118	.231	.118	.202	.115	.186
	0	.6	.346	.474	.364	.478	.385	.474	.459	.552	.470	.560	.478	.568
	0	1.2	.074	.201	.057	.179	.055	.156	.141	.258	.114	.245	.115	.215
II	0	2	.067	.180	.053	.154	.065	.148	.119	.240	.110	.204	.109	.205
	.4	1.2	.113	.294	.071	.275	.061	.238	.176	.355	.146	.349	.128	.313
	.4	2	.066	.175	.065	.146	.059	.139	.119	.228	.116	.201	.115	.185
	0	.6	.462	.557	.445	.557	.455	.542	.552	.634	.546	.633	.544	.638
	0	1.2	.074	.199	.057	.178	.053	.157	.144	.262	.113	.245	.108	.211
III	0	2	.068	.184	.056	.155	.066	.146	.127	.241	.113	.203	.109	.205
	.4	1.2	.122	.304	.082	.280	.066	.233	.187	.360	.154	.348	.126	.306
	.4	2	.068	.174	.063	.150	.059	.143	.125	.225	.113	.199	.113	.185

TABLE 4.89

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.192	.338	.202	.355	.214	.309	.294	.415	.302	.438	.313	.402
	0	1.2	.071	.210	.058	.195	.051	.164	.135	.270	.112	.256	.116	.219
I	0	2	.066	.182	.051	.152	.063	.148	.123	.244	.117	.201	.105	.206
	.4	1.2	.098	.292	.060	.292	.057	.262	.150	.365	.122	.385	.113	.338
	.4	2	.066	.173	.064	.147	.062	.143	.117	.226	.116	.203	.114	.186
	0	.6	.312	.500	.332	.501	.356	.497	.422	.567	.424	.589	.450	.593
	0	1.2	.074	.209	.061	.187	.054	.163	.138	.279	.111	.254	.118	.219
II	0	2	.066	.180	.052	.154	.064	.147	.121	.242	.114	.202	.108	.206
	.4	1.2	.112	.309	.071	.291	.065	.248	.173	.378	.125	.373	.125	.321
	.4	2	.067	.178	.067	.147	.060	.141	.122	.228	.115	.201	.113	.190
	0	.6	.460	.637	.465	.625	.479	.634	.549	.703	.560	.707	.596	.723
	0	1.2	.078	.200	.057	.183	.052	.157	.142	.268	.113	.244	.114	.213
III	0	2	.067	.180	.054	.154	.066	.146	.130	.245	.114	.198	.108	.208
	.4	1.2	.135	.341	.089	.285	.078	.228	.212	.415	.152	.370	.125	.295
	.4	2	.067	.177	.065	.147	.058	.144	.125	.224	.113	.200	.111	.185

TABLE 4.90

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.227	.133	.166	.101	.140	.068	.303	.196	.237	.147	.231	.118
	0	1.2	.190	.095	.151	.067	.142	.045	.265	.147	.240	.102	.206	.070
I	0	2	.187	.078	.145	.073	.140	.050	.275	.114	.242	.103	.224	.070
	.4	1.2	.221	.118	.160	.078	.128	.046	.277	.167	.229	.115	.212	.088
	.4	2	.185	.079	.150	.070	.145	.039	.261	.126	.244	.100	.215	.062
	0	.6	.233	.136	.177	.091	.151	.072	.301	.190	.251	.157	.233	.119
	0	1.2	.191	.091	.151	.062	.141	.042	.262	.141	.239	.096	.204	.069
II	0	2	.181	.077	.144	.075	.142	.050	.273	.116	.236	.103	.219	.071
	.4	1.2	.228	.121	.164	.076	.131	.046	.289	.170	.236	.115	.216	.085
	.4	2	.179	.079	.148	.063	.146	.039	.262	.126	.242	.100	.214	.060
	0	.6	.229	.139	.183	.099	.159	.079	.317	.201	.260	.163	.249	.125
	0	1.2	.196	.095	.163	.063	.139	.044	.269	.142	.246	.101	.202	.071
III	0	2	.190	.080	.152	.075	.138	.049	.270	.120	.241	.102	.214	.073
	.4	1.2	.229	.121	.175	.084	.134	.050	.302	.173	.241	.116	.224	.089
	.4	2	.187	.080	.157	.067	.138	.040	.265	.122	.244	.103	.220	.061

TABLE 4.91

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = 0$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.224	.127	.169	.095	.138	.068	.298	.198	.239	.141	.229	.111
	0	1.2	.193	.095	.151	.064	.141	.044	.265	.141	.241	.098	.209	.070
I	0	2	.186	.075	.147	.074	.138	.050	.274	.118	.243	.098	.225	.070
	.4	1.2	.223	.112	.159	.075	.125	.043	.277	.162	.221	.110	.209	.089
	.4	2	.185	.077	.151	.072	.146	.039	.266	.124	.242	.097	.215	.062
	0	.6	.225	.138	.168	.097	.142	.070	.296	.195	.244	.153	.231	.124
	0	1.2	.191	.096	.149	.061	.143	.042	.263	.143	.236	.099	.205	.073
II	0	2	.187	.075	.145	.074	.140	.049	.275	.121	.238	.102	.221	.072
	.4	1.2	.219	.123	.158	.076	.126	.047	.289	.175	.234	.120	.211	.093
	.4	2	.178	.077	.151	.068	.148	.040	.264	.125	.245	.097	.213	.060
	0	.6	.220	.157	.173	.107	.145	.087	.292	.215	.250	.162	.233	.135
	0	1.2	.191	.091	.147	.064	.144	.043	.267	.143	.237	.099	.204	.073
III	0	2	.182	.076	.149	.075	.138	.048	.274	.118	.241	.103	.222	.072
	.4	1.2	.222	.129	.157	.084	.128	.049	.292	.192	.232	.132	.216	.102
	.4	2	.178	.078	.151	.069	.148	.040	.261	.124	.244	.097	.213	.060

TABLE 4.92

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.256	.202	.222	.206	.203	.173	.360	.274	.319	.286	.290	.259
	0	1.2	.190	.103	.142	.078	.128	.053	.267	.149	.225	.101	.204	.084
I	0	2	.198	.093	.150	.082	.132	.050	.269	.132	.216	.106	.193	.070
	.4	1.2	.200	.155	.154	.117	.121	.100	.285	.217	.241	.187	.203	.162
	.4	2	.189	.079	.142	.065	.126	.037	.273	.113	.217	.090	.193	.069
	0	.6	.280	.236	.247	.237	.233	.206	.369	.313	.338	.317	.323	.298
	0	1.2	.192	.101	.144	.077	.129	.055	.273	.145	.215	.103	.204	.082
II	0	2	.196	.092	.144	.079	.129	.048	.273	.136	.218	.108	.196	.071
	.4	1.2	.204	.154	.159	.112	.128	.096	.294	.229	.249	.182	.211	.155
	.4	2	.194	.078	.141	.065	.127	.039	.272	.111	.209	.093	.192	.072
	0	.6	.316	.259	.274	.266	.255	.240	.391	.353	.353	.350	.346	.332
	0	1.2	.191	.102	.147	.076	.127	.052	.278	.142	.216	.103	.204	.084
III	0	2	.197	.097	.147	.083	.131	.052	.278	.135	.218	.106	.193	.071
	.4	1.2	.216	.159	.166	.113	.132	.097	.297	.227	.252	.183	.207	.154
	.4	2	.195	.077	.139	.066	.126	.039	.268	.116	.223	.096	.197	.071

TABLE 4.93

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.248	.203	.212	.202	.177	.161	.345	.280	.303	.274	.269	.239
	0	1.2	.187	.106	.144	.078	.127	.055	.268	.149	.221	.108	.203	.088
I	0	2	.195	.094	.147	.082	.130	.049	.269	.138	.218	.105	.194	.070
	.4	1.2	.194	.159	.153	.132	.117	.111	.281	.224	.235	.199	.202	.172
	.4	2	.185	.078	.145	.067	.125	.037	.277	.117	.217	.094	.195	.071
	0	.6	.259	.250	.227	.229	.198	.190	.360	.326	.322	.308	.282	.281
	0	1.2	.188	.103	.145	.079	.128	.056	.275	.148	.221	.112	.204	.090
II	0	2	.192	.094	.145	.082	.128	.048	.269	.138	.216	.106	.194	.071
	.4	1.2	.202	.165	.150	.139	.115	.108	.284	.235	.235	.196	.203	.168
	.4	2	.189	.080	.142	.066	.124	.037	.273	.116	.219	.094	.196	.073
	0	.6	.279	.306	.231	.264	.208	.227	.367	.368	.328	.351	.294	.320
	0	1.2	.189	.101	.145	.077	.127	.055	.274	.147	.218	.108	.204	.089
III	0	2	.193	.092	.148	.082	.128	.049	.275	.135	.216	.104	.195	.070
	.4	1.2	.203	.173	.157	.131	.115	.111	.282	.255	.243	.198	.201	.165
	.4	2	.189	.079	.142	.065	.125	.038	.269	.115	.221	.093	.194	.073

TABLE 4.94

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.244	.203	.225	.192	.203	.187	.326	.262	.313	.276	.288	.248
	0	1.2	.201	.110	.159	.087	.117	.058	.284	.160	.228	.131	.193	.090
I	0	2	.183	.083	.165	.067	.132	.048	.268	.121	.242	.106	.206	.073
	.4	1.2	.221	.152	.171	.137	.146	.100	.300	.226	.247	.187	.211	.151
	.4	2	.172	.079	.168	.066	.128	.042	.262	.114	.246	.105	.198	.064
	0	.6	.272	.238	.259	.235	.239	.225	.356	.322	.333	.313	.316	.300
	0	1.2	.200	.116	.164	.081	.117	.057	.282	.159	.229	.132	.198	.088
II	0	2	.181	.083	.163	.072	.130	.048	.267	.126	.239	.104	.206	.072
	.4	1.2	.227	.154	.178	.126	.150	.094	.311	.224	.252	.184	.208	.147
	.4	2	.174	.084	.168	.066	.128	.043	.269	.111	.243	.107	.197	.064
	0	.6	.281	.267	.276	.260	.252	.250	.376	.339	.355	.342	.342	.327
	0	1.2	.203	.108	.170	.084	.120	.056	.289	.152	.236	.129	.199	.091
III	0	2	.185	.087	.170	.071	.135	.045	.274	.124	.244	.111	.211	.075
	.4	1.2	.235	.156	.179	.132	.154	.096	.322	.222	.254	.186	.214	.156
	.4	2	.177	.082	.174	.070	.127	.044	.272	.112	.246	.108	.193	.060

TABLE 4.95

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = -.5$, $\phi_i = .5$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.244	.201	.219	.182	.189	.171	.316	.266	.303	.270	.272	.245
	0	1.2	.202	.116	.160	.085	.119	.060	.283	.165	.232	.137	.195	.097
I	0	2	.182	.085	.163	.066	.131	.049	.263	.121	.241	.105	.203	.072
	.4	1.2	.216	.160	.165	.136	.143	.111	.298	.237	.244	.192	.209	.174
	.4	2	.172	.079	.165	.070	.127	.042	.263	.115	.242	.107	.196	.065
	0	.6	.250	.235	.236	.221	.210	.213	.335	.327	.307	.313	.284	.303
	0	1.2	.202	.120	.163	.091	.117	.058	.281	.166	.228	.131	.195	.095
II	0	2	.184	.086	.164	.071	.132	.049	.267	.123	.237	.104	.202	.070
	.4	1.2	.224	.179	.174	.138	.142	.119	.305	.245	.245	.194	.212	.182
	.4	2	.171	.081	.163	.065	.126	.043	.268	.114	.241	.106	.201	.063
	0	.6	.266	.292	.249	.269	.216	.242	.342	.378	.313	.365	.291	.329
	0	1.2	.203	.118	.165	.089	.119	.057	.276	.159	.231	.131	.196	.094
III	0	2	.183	.085	.167	.070	.130	.047	.268	.126	.240	.105	.200	.071
	.4	1.2	.223	.186	.178	.140	.144	.113	.308	.252	.244	.192	.214	.178
	.4	2	.172	.080	.164	.069	.129	.043	.267	.113	.246	.106	.196	.065

TABLE 4.96

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
I	0	.6	.319	.364	.303	.375	.307	.353	.412	.434	.395	.466	.393	.438
	0	1.2	.191	.121	.145	.086	.135	.079	.271	.180	.212	.143	.201	.121
	0	2	.176	.098	.145	.084	.115	.066	.258	.136	.221	.110	.195	.094
	.4	1.2	.205	.246	.150	.227	.131	.196	.296	.319	.228	.305	.211	.275
	.4	2	.158	.082	.144	.066	.132	.051	.248	.109	.217	.099	.198	.074
II	0	.6	.394	.431	.385	.467	.383	.474	.484	.524	.471	.561	.467	.558
	0	1.2	.184	.123	.142	.083	.132	.081	.276	.173	.213	.139	.201	.117
	0	2	.183	.101	.144	.084	.116	.063	.259	.138	.221	.111	.201	.094
	.4	1.2	.218	.247	.157	.213	.138	.174	.309	.325	.238	.295	.216	.242
	.4	2	.164	.084	.141	.071	.133	.048	.243	.112	.215	.100	.194	.076
III	0	.6	.432	.494	.423	.523	.419	.516	.518	.588	.502	.606	.512	.610
	0	1.2	.194	.125	.145	.083	.130	.083	.265	.172	.224	.135	.203	.116
	0	2	.182	.107	.145	.085	.120	.060	.261	.140	.227	.109	.192	.094
	.4	1.2	.230	.245	.165	.213	.138	.174	.319	.321	.242	.294	.216	.244
	.4	2	.171	.085	.145	.071	.129	.052	.250	.114	.217	.097	.199	.075

TABLE 4.97

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .75$, $\phi_i = .5$, $\psi_i = 0$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
I	0	.6	.302	.342	.280	.356	.273	.312	.399	.437	.371	.441	.361	.399
	0	1.2	.191	.133	.140	.098	.132	.086	.274	.189	.214	.157	.199	.124
	0	2	.174	.101	.148	.085	.115	.066	.260	.137	.219	.107	.195	.094
	.4	1.2	.196	.266	.143	.248	.129	.227	.290	.335	.226	.342	.210	.307
	.4	2	.156	.083	.144	.069	.133	.054	.245	.115	.216	.102	.196	.077
II	0	.6	.337	.429	.322	.420	.306	.386	.433	.506	.408	.512	.406	.485
	0	1.2	.188	.136	.142	.091	.132	.085	.276	.194	.215	.157	.199	.127
	0	2	.179	.102	.140	.085	.114	.067	.260	.139	.219	.107	.196	.093
	.4	1.2	.206	.284	.144	.246	.128	.214	.298	.354	.227	.349	.213	.295
	.4	2	.161	.087	.141	.069	.131	.054	.243	.116	.213	.100	.195	.078
III	0	.6	.371	.488	.348	.477	.334	.462	.453	.568	.430	.577	.426	.543
	0	1.2	.189	.129	.145	.089	.134	.083	.271	.181	.216	.147	.198	.123
	0	2	.179	.103	.144	.083	.116	.066	.260	.137	.218	.106	.193	.093
	.4	1.2	.214	.286	.144	.230	.132	.191	.311	.366	.233	.322	.210	.269
	.4	2	.164	.086	.143	.068	.131	.054	.243	.117	.214	.099	.195	.080

TABLE 4.98

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.553	.130	.529	.091	.515	.056	.611	.195	.599	.146	.587	.103
	0	1.2	.578	.073	.546	.040	.484	.014	.646	.113	.615	.061	.554	.034
I	0	2	.602	.052	.540	.029	.486	.006	.650	.077	.603	.051	.557	.009
	.4	1.2	.570	.093	.549	.059	.496	.034	.630	.148	.613	.103	.574	.084
	.4	2	.559	.048	.544	.023	.482	.005	.617	.067	.611	.044	.558	.013
	0	.6	.563	.143	.535	.083	.518	.051	.618	.198	.603	.143	.590	.110
	0	1.2	.585	.065	.547	.041	.485	.014	.640	.111	.610	.061	.555	.032
II	0	2	.604	.052	.548	.029	.484	.007	.649	.078	.599	.052	.556	.010
	.4	1.2	.571	.097	.554	.058	.503	.028	.630	.148	.617	.097	.569	.070
	.4	2	.551	.046	.544	.022	.482	.006	.624	.069	.616	.042	.555	.015
	0	.6	.567	.145	.545	.090	.529	.055	.633	.216	.607	.156	.593	.115
	0	1.2	.584	.068	.547	.040	.494	.015	.636	.112	.611	.064	.565	.031
III	0	2	.588	.054	.545	.029	.491	.007	.671	.078	.604	.053	.560	.010
	.4	1.2	.588	.099	.560	.058	.504	.029	.641	.160	.613	.103	.577	.076
	.4	2	.566	.050	.549	.022	.491	.007	.640	.070	.616	.045	.561	.015

TABLE 4.99

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = 0$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.555	.126	.531	.084	.515	.056	.612	.190	.599	.137	.588	.104
	0	1.2	.580	.070	.545	.041	.484	.015	.645	.107	.613	.064	.554	.033
I	0	2	.601	.052	.541	.029	.486	.006	.649	.078	.604	.053	.557	.009
	.4	1.2	.569	.091	.550	.058	.496	.036	.630	.152	.614	.102	.574	.086
	.4	2	.559	.049	.544	.022	.483	.005	.616	.069	.611	.044	.558	.014
	0	.6	.553	.151	.529	.086	.518	.063	.616	.211	.599	.146	.588	.119
	0	1.2	.582	.070	.545	.039	.485	.016	.641	.116	.611	.063	.551	.031
II	0	2	.599	.049	.541	.028	.486	.006	.647	.079	.603	.053	.556	.010
	.4	1.2	.570	.103	.545	.058	.497	.041	.628	.168	.615	.107	.574	.093
	.4	2	.555	.048	.543	.020	.482	.006	.621	.070	.610	.043	.557	.014
	0	.6	.555	.191	.531	.116	.517	.086	.616	.256	.601	.180	.589	.159
	0	1.2	.584	.072	.543	.040	.484	.017	.644	.119	.610	.066	.551	.032
III	0	2	.599	.051	.541	.028	.488	.006	.649	.078	.601	.052	.556	.009
	.4	1.2	.572	.135	.546	.075	.501	.057	.629	.208	.613	.126	.575	.096
	.4	2	.553	.049	.544	.020	.482	.006	.622	.072	.611	.043	.556	.015

TABLE 4.100

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.563	.193	.544	.197	.517	.168	.628	.268	.612	.279	.589	.232
	0	1.2	.570	.075	.531	.039	.476	.024	.615	.117	.614	.077	.551	.044
I	0	2	.589	.060	.527	.039	.457	.005	.645	.102	.600	.063	.551	.009
	.4	1.2	.560	.135	.532	.124	.496	.099	.629	.205	.600	.196	.566	.177
	.4	2	.571	.046	.541	.024	.451	.004	.614	.073	.604	.045	.542	.007
	0	.6	.572	.214	.551	.203	.523	.165	.634	.283	.618	.288	.591	.243
	0	1.2	.571	.073	.542	.035	.475	.022	.618	.113	.604	.074	.554	.043
II	0	2	.595	.067	.530	.040	.458	.005	.642	.098	.605	.061	.547	.009
	.4	1.2	.561	.142	.529	.111	.503	.084	.631	.202	.598	.185	.565	.157
	.4	2	.567	.047	.544	.022	.456	.003	.618	.072	.606	.042	.542	.007
	0	.6	.588	.231	.556	.225	.535	.179	.643	.304	.622	.305	.600	.255
	0	1.2	.572	.071	.545	.036	.481	.020	.628	.108	.612	.075	.561	.042
III	0	2	.598	.066	.531	.040	.470	.005	.650	.097	.608	.065	.559	.008
	.4	1.2	.573	.138	.528	.113	.499	.088	.628	.209	.603	.187	.570	.157
	.4	2	.568	.048	.546	.022	.477	.004	.634	.070	.621	.045	.554	.007

TABLE 4.101

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.562	.216	.545	.205	.517	.176	.628	.280	.612	.286	.589	.234
	0	1.2	.569	.078	.530	.044	.476	.028	.616	.121	.612	.083	.548	.054
I	0	2	.586	.062	.526	.041	.459	.006	.645	.098	.599	.063	.553	.008
	.4	1.2	.558	.147	.533	.143	.496	.128	.628	.210	.599	.217	.565	.203
	.4	2	.572	.046	.542	.024	.450	.004	.616	.073	.603	.046	.544	.009
	0	.6	.564	.243	.546	.218	.519	.174	.633	.314	.613	.299	.589	.258
	0	1.2	.572	.081	.531	.049	.474	.027	.620	.118	.612	.090	.550	.054
II	0	2	.588	.066	.528	.040	.458	.006	.643	.100	.601	.062	.554	.008
	.4	1.2	.556	.155	.533	.149	.497	.129	.629	.237	.602	.230	.566	.195
	.4	2	.573	.047	.540	.024	.453	.004	.614	.074	.605	.043	.545	.009
	0	.6	.565	.288	.547	.254	.518	.201	.629	.389	.615	.348	.589	.282
	0	1.2	.570	.081	.532	.047	.476	.030	.618	.117	.610	.088	.550	.049
III	0	2	.589	.068	.524	.041	.458	.006	.644	.099	.602	.064	.555	.008
	.4	1.2	.556	.188	.533	.142	.496	.112	.629	.261	.602	.221	.564	.195
	.4	2	.573	.048	.540	.024	.454	.004	.613	.071	.607	.042	.545	.009

TABLE 4.102

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.586	.199	.554	.182	.531	.154	.664	.266	.622	.244	.599	.226
	0	1.2	.575	.078	.533	.048	.507	.019	.633	.129	.600	.077	.582	.044
I	0	2	.581	.054	.530	.034	.493	.006	.629	.083	.594	.052	.564	.014
	.4	1.2	.583	.148	.564	.125	.515	.095	.640	.221	.626	.174	.581	.164
	.4	2	.555	.042	.533	.026	.491	.005	.617	.070	.600	.042	.574	.014
	0	.6	.591	.224	.562	.199	.529	.167	.660	.286	.623	.278	.599	.240
	0	1.2	.574	.076	.538	.050	.512	.017	.631	.125	.604	.071	.576	.039
II	0	2	.579	.055	.529	.033	.490	.006	.636	.089	.593	.054	.565	.016
	.4	1.2	.589	.151	.570	.120	.518	.091	.642	.222	.623	.162	.582	.145
	.4	2	.559	.046	.540	.024	.485	.004	.618	.064	.610	.044	.571	.013
	0	.6	.598	.230	.570	.214	.529	.179	.657	.303	.631	.289	.597	.250
	0	1.2	.582	.073	.554	.048	.524	.018	.641	.121	.597	.075	.587	.037
III	0	2	.586	.056	.532	.033	.505	.006	.654	.087	.588	.057	.556	.015
	.4	1.2	.589	.151	.578	.122	.522	.085	.647	.219	.632	.163	.582	.144
	.4	2	.575	.045	.552	.023	.499	.004	.629	.064	.613	.045	.569	.015

TABLE 4.103

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = -.5$, $\phi_i = .9$, $\psi_i = 0$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.585	.208	.554	.180	.531	.164	.665	.270	.623	.260	.600	.250
	0	1.2	.576	.081	.533	.049	.507	.023	.632	.132	.600	.077	.581	.052
I	0	2	.582	.054	.529	.033	.494	.006	.629	.082	.596	.053	.563	.014
	.4	1.2	.582	.157	.564	.134	.514	.116	.639	.238	.625	.201	.581	.197
	.4	2	.556	.043	.531	.026	.491	.005	.617	.070	.602	.043	.575	.014
	0	.6	.587	.231	.557	.208	.531	.190	.665	.321	.622	.298	.600	.280
	0	1.2	.576	.084	.532	.054	.507	.023	.631	.139	.600	.078	.583	.049
II	0	2	.579	.057	.526	.030	.493	.006	.633	.085	.595	.054	.565	.015
	.4	1.2	.581	.183	.564	.141	.517	.127	.641	.249	.624	.205	.581	.198
	.4	2	.551	.047	.531	.025	.491	.005	.617	.073	.606	.045	.574	.012
	0	.6	.588	.287	.558	.262	.530	.212	.663	.369	.623	.347	.600	.286
	0	1.2	.574	.079	.534	.048	.508	.021	.632	.133	.600	.074	.582	.044
III	0	2	.579	.056	.530	.032	.493	.007	.633	.086	.596	.052	.567	.015
	.4	1.2	.579	.194	.565	.142	.517	.119	.639	.268	.624	.205	.581	.187
	.4	2	.555	.046	.530	.024	.493	.004	.616	.069	.606	.046	.574	.013

TABLE 4.104

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.610	.352	.577	.357	.545	.317	.668	.412	.639	.439	.611	.394
	0	1.2	.576	.084	.531	.070	.480	.045	.623	.139	.601	.111	.556	.077
I	0	2	.565	.063	.530	.043	.463	.009	.621	.095	.598	.077	.541	.025
	.4	1.2	.576	.244	.543	.249	.489	.239	.618	.332	.597	.356	.556	.339
	.4	2	.540	.049	.523	.025	.451	.008	.596	.074	.599	.043	.533	.013
	0	.6	.621	.379	.585	.386	.558	.360	.676	.455	.644	.474	.611	.437
	0	1.2	.569	.086	.531	.064	.478	.045	.621	.141	.608	.105	.551	.073
II	0	2	.562	.061	.534	.042	.465	.007	.621	.099	.599	.076	.549	.026
	.4	1.2	.575	.245	.544	.238	.484	.222	.619	.335	.597	.335	.557	.305
	.4	2	.543	.048	.522	.022	.454	.009	.595	.078	.595	.040	.533	.013
	0	.6	.629	.404	.590	.416	.565	.391	.687	.486	.641	.507	.626	.477
	0	1.2	.586	.086	.543	.058	.482	.043	.636	.134	.619	.108	.556	.073
III	0	2	.577	.066	.533	.042	.469	.007	.634	.107	.606	.078	.557	.023
	.4	1.2	.579	.246	.546	.236	.478	.218	.635	.338	.601	.325	.553	.296
	.4	2	.558	.048	.520	.022	.470	.009	.612	.075	.594	.038	.544	.014

TABLE 4.105

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .75$, $\phi_i = .9$, $\psi_i = 0$, $i = 1,2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.613	.359	.573	.367	.544	.338	.667	.417	.639	.449	.611	.403
	0	1.2	.571	.094	.531	.083	.479	.060	.622	.150	.601	.132	.556	.094
I	0	2	.564	.063	.530	.043	.463	.009	.620	.097	.600	.076	.540	.023
	.4	1.2	.573	.274	.542	.296	.488	.293	.617	.365	.595	.399	.557	.379
	.4	2	.539	.050	.524	.024	.452	.009	.595	.086	.599	.044	.532	.015
	0	.6	.613	.416	.578	.403	.546	.356	.670	.487	.640	.486	.612	.437
	0	1.2	.574	.104	.528	.082	.478	.062	.623	.162	.602	.134	.553	.090
II	0	2	.561	.061	.530	.041	.463	.009	.625	.105	.600	.073	.543	.024
	.4	1.2	.574	.301	.542	.305	.490	.276	.619	.400	.595	.412	.557	.368
	.4	2	.542	.055	.524	.024	.454	.009	.596	.087	.602	.042	.531	.015
	0	.6	.618	.470	.577	.434	.549	.366	.672	.538	.641	.525	.613	.474
	0	1.2	.575	.090	.531	.076	.477	.056	.625	.150	.603	.120	.553	.084
III	0	2	.564	.065	.526	.043	.463	.009	.623	.103	.601	.075	.542	.024
	.4	1.2	.573	.305	.540	.259	.489	.235	.620	.398	.595	.360	.558	.319
	.4	2	.546	.051	.527	.022	.453	.010	.596	.082	.599	.039	.531	.014

TABLE 4.106

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.139	.164	.118	.134	.096	.105	.217	.214	.169	.182	.154	.167
	0	1.2	.106	.154	.079	.113	.066	.104	.178	.204	.142	.174	.138	.148
I	0	2	.091	.135	.082	.118	.078	.097	.165	.178	.136	.169	.140	.149
	.4	1.2	.132	.163	.097	.119	.070	.102	.198	.218	.155	.166	.132	.146
	.4	2	.094	.146	.092	.120	.075	.094	.164	.183	.144	.176	.142	.145
	0	.6	.146	.162	.113	.128	.088	.100	.220	.212	.176	.184	.158	.166
	0	1.2	.103	.159	.080	.113	.065	.106	.173	.204	.143	.169	.142	.149
II	0	2	.094	.131	.086	.119	.081	.097	.162	.176	.137	.165	.144	.152
	.4	1.2	.137	.164	.099	.123	.072	.102	.207	.219	.163	.168	.140	.147
	.4	2	.090	.147	.091	.123	.075	.095	.160	.189	.142	.175	.139	.144
	0	.6	.154	.170	.120	.133	.106	.110	.227	.219	.184	.188	.172	.177
	0	1.2	.110	.158	.083	.114	.069	.105	.174	.209	.151	.170	.139	.152
III	0	2	.094	.137	.085	.126	.080	.097	.163	.181	.142	.174	.139	.140
	.4	1.2	.139	.170	.107	.129	.075	.105	.213	.221	.167	.173	.143	.146
	.4	2	.094	.152	.092	.129	.073	.098	.163	.185	.148	.176	.140	.146

TABLE 4.107

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.138	.154	.109	.126	.089	.098	.215	.212	.176	.181	.149	.160
	0	1.2	.105	.158	.075	.113	.068	.103	.171	.200	.138	.171	.137	.150
I	0	2	.091	.133	.084	.117	.079	.098	.161	.179	.138	.167	.142	.149
	.4	1.2	.121	.162	.094	.114	.061	.100	.198	.212	.148	.160	.127	.140
	.4	2	.092	.147	.091	.120	.078	.093	.161	.185	.142	.176	.141	.142
	0	.6	.146	.164	.108	.128	.090	.106	.219	.214	.172	.182	.158	.164
	0	1.2	.101	.160	.077	.112	.069	.103	.169	.199	.143	.167	.141	.149
II	0	2	.092	.128	.084	.117	.079	.094	.164	.171	.137	.167	.141	.152
	.4	1.2	.128	.168	.098	.116	.066	.105	.202	.218	.154	.170	.132	.149
	.4	2	.092	.146	.092	.122	.076	.094	.158	.186	.144	.174	.140	.144
	0	.6	.148	.163	.115	.136	.102	.113	.225	.224	.179	.192	.160	.178
	0	1.2	.101	.159	.081	.113	.069	.102	.171	.196	.142	.169	.142	.150
III	0	2	.092	.131	.083	.120	.082	.095	.160	.174	.136	.171	.141	.149
	.4	1.2	.131	.174	.104	.125	.071	.111	.208	.228	.160	.176	.133	.155
	.4	2	.092	.146	.091	.123	.074	.095	.162	.183	.141	.176	.139	.145

TABLE 4.108

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.174	.221	.158	.226	.154	.203	.273	.293	.243	.305	.240	.284
	0	1.2	.108	.150	.078	.123	.069	.102	.167	.194	.140	.162	.122	.155
I	0	2	.099	.159	.084	.131	.077	.109	.170	.203	.146	.164	.129	.155
	.4	1.2	.131	.187	.088	.172	.068	.143	.189	.246	.158	.227	.128	.211
	.4	2	.096	.138	.081	.118	.073	.098	.165	.190	.136	.156	.129	.148
	0	.6	.211	.278	.202	.283	.205	.253	.324	.369	.298	.370	.306	.355
	0	1.2	.106	.150	.077	.124	.074	.099	.173	.191	.140	.162	.124	.155
II	0	2	.098	.157	.085	.130	.080	.107	.177	.201	.143	.167	.127	.153
	.4	1.2	.139	.202	.094	.167	.074	.147	.201	.262	.159	.222	.127	.204
	.4	2	.098	.138	.083	.117	.073	.099	.173	.187	.136	.158	.125	.146
	0	.6	.250	.308	.245	.313	.228	.287	.355	.398	.327	.402	.339	.386
	0	1.2	.109	.153	.077	.129	.074	.100	.172	.192	.141	.166	.125	.154
III	0	2	.105	.155	.088	.135	.080	.106	.174	.203	.143	.167	.132	.156
	.4	1.2	.141	.202	.103	.166	.073	.140	.216	.263	.157	.228	.133	.205
	.4	2	.104	.136	.082	.124	.075	.097	.172	.194	.134	.160	.127	.141

TABLE 4.109

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.167	.218	.133	.213	.128	.191	.256	.289	.215	.298	.209	.255
	0	1.2	.102	.152	.076	.123	.072	.102	.164	.197	.138	.168	.122	.156
I	0	2	.100	.158	.087	.129	.076	.109	.169	.204	.144	.162	.126	.155
	.4	1.2	.129	.194	.089	.175	.066	.150	.187	.245	.147	.238	.120	.206
	.4	2	.096	.141	.078	.115	.073	.098	.164	.193	.139	.156	.125	.150
	0	.6	.199	.276	.163	.263	.160	.240	.284	.357	.251	.354	.247	.324
	0	1.2	.101	.155	.074	.126	.070	.099	.169	.198	.135	.172	.123	.159
II	0	2	.097	.156	.081	.129	.081	.108	.173	.198	.144	.163	.127	.152
	.4	1.2	.127	.203	.074	.178	.071	.151	.191	.262	.135	.238	.123	.210
	.4	2	.096	.140	.080	.115	.074	.098	.170	.193	.137	.157	.125	.148
	0	.6	.213	.307	.183	.289	.186	.261	.310	.386	.273	.382	.266	.361
	0	1.2	.099	.154	.077	.126	.069	.100	.170	.199	.135	.171	.123	.157
III	0	2	.101	.155	.084	.131	.081	.108	.168	.199	.143	.162	.127	.153
	.4	1.2	.139	.211	.090	.178	.070	.150	.196	.272	.152	.230	.124	.208
	.4	2	.096	.137	.078	.117	.073	.098	.167	.192	.139	.156	.124	.145

TABLE 4.110

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.179	.220	.155	.238	.158	.213	.246	.306	.247	.317	.231	.271
	0	1.2	.093	.156	.098	.155	.069	.112	.167	.211	.155	.198	.124	.150
I	0	2	.089	.138	.098	.140	.076	.104	.163	.187	.156	.187	.134	.152
	.4	1.2	.126	.189	.095	.176	.078	.147	.195	.276	.166	.241	.148	.205
	.4	2	.096	.131	.091	.130	.080	.091	.151	.172	.160	.181	.128	.137
	0	.6	.220	.284	.211	.289	.215	.259	.298	.379	.298	.369	.292	.340
	0	1.2	.098	.155	.096	.149	.068	.105	.173	.205	.153	.201	.118	.149
II	0	2	.090	.138	.095	.136	.076	.106	.160	.181	.156	.189	.132	.154
	.4	1.2	.138	.200	.100	.172	.083	.153	.208	.277	.177	.237	.155	.200
	.4	2	.092	.137	.087	.132	.080	.089	.148	.172	.157	.179	.123	.136
	0	.6	.237	.308	.242	.318	.238	.281	.326	.410	.328	.395	.323	.373
	0	1.2	.099	.160	.099	.152	.066	.104	.172	.204	.160	.199	.121	.147
III	0	2	.091	.144	.096	.142	.076	.106	.167	.184	.165	.193	.135	.147
	.4	1.2	.134	.201	.108	.177	.087	.149	.216	.276	.184	.241	.156	.203
	.4	2	.090	.139	.089	.136	.078	.090	.151	.179	.165	.183	.129	.139

TABLE 4.111

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.168	.215	.145	.225	.143	.197	.236	.301	.234	.302	.209	.266
	0	1.2	.091	.154	.097	.158	.068	.108	.170	.208	.157	.204	.121	.156
I	0	2	.089	.137	.097	.138	.076	.105	.161	.186	.156	.187	.135	.152
	.4	1.2	.122	.196	.093	.177	.076	.157	.193	.275	.167	.261	.144	.215
	.4	2	.091	.132	.092	.128	.082	.091	.150	.171	.157	.180	.129	.137
	0	.6	.203	.270	.194	.285	.184	.261	.269	.368	.257	.361	.255	.330
	0	1.2	.096	.159	.092	.152	.065	.106	.175	.210	.154	.207	.118	.157
II	0	2	.091	.132	.096	.134	.073	.107	.158	.181	.152	.191	.130	.152
	.4	1.2	.124	.209	.100	.186	.080	.160	.208	.295	.170	.252	.152	.225
	.4	2	.091	.135	.089	.131	.080	.090	.149	.169	.153	.181	.125	.136
	0	.6	.211	.307	.202	.316	.201	.283	.283	.401	.271	.393	.268	.355
	0	1.2	.097	.157	.095	.149	.065	.105	.174	.210	.155	.206	.120	.156
III	0	2	.088	.136	.096	.135	.073	.106	.158	.182	.155	.193	.131	.149
	.4	1.2	.128	.216	.103	.189	.083	.157	.210	.300	.173	.251	.151	.222
	.4	2	.089	.134	.090	.131	.078	.090	.150	.170	.156	.179	.126	.135

TABLE 4.112

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.259	.358	.269	.387	.274	.346	.360	.433	.357	.467	.367	.436
	0	1.2	.104	.175	.075	.154	.070	.132	.166	.227	.134	.220	.135	.188
I	0	2	.086	.154	.072	.125	.076	.125	.160	.198	.139	.168	.119	.179
	.4	1.2	.130	.258	.090	.266	.076	.230	.191	.341	.154	.340	.131	.298
	.4	2	.083	.137	.080	.120	.071	.123	.149	.183	.141	.168	.133	.161
	0	.6	.393	.512	.394	.514	.404	.501	.503	.594	.489	.610	.502	.599
	0	1.2	.106	.182	.074	.150	.068	.133	.171	.231	.137	.217	.133	.186
II	0	2	.087	.153	.073	.126	.075	.128	.156	.202	.135	.170	.122	.178
	.4	1.2	.137	.258	.093	.264	.078	.206	.213	.338	.167	.332	.140	.286
	.4	2	.086	.143	.080	.121	.069	.121	.157	.178	.139	.169	.133	.163
	0	.6	.442	.568	.445	.577	.450	.558	.554	.651	.543	.652	.545	.653
	0	1.2	.110	.176	.074	.144	.069	.136	.176	.227	.137	.217	.131	.181
III	0	2	.094	.147	.070	.128	.076	.128	.160	.202	.137	.173	.123	.174
	.4	1.2	.156	.270	.103	.264	.084	.207	.225	.341	.175	.324	.144	.280
	.4	2	.092	.142	.078	.122	.073	.121	.165	.179	.138	.172	.129	.157

TABLE 4.113

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .5$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.221	.341	.219	.351	.225	.305	.324	.421	.320	.439	.326	.396
	0	1.2	.102	.181	.073	.162	.070	.138	.168	.234	.133	.232	.134	.196
I	0	2	.085	.157	.071	.127	.075	.126	.160	.202	.140	.169	.118	.180
	.4	1.2	.123	.264	.082	.290	.067	.246	.184	.352	.137	.368	.128	.328
	.4	2	.085	.139	.082	.120	.070	.123	.147	.186	.138	.170	.133	.162
	0	.6	.293	.450	.309	.463	.312	.437	.395	.536	.404	.545	.416	.552
	0	1.2	.105	.191	.072	.155	.068	.135	.163	.242	.136	.227	.133	.197
II	0	2	.086	.156	.075	.128	.075	.129	.158	.200	.135	.171	.119	.179
	.4	1.2	.132	.284	.079	.282	.075	.237	.194	.360	.145	.358	.139	.320
	.4	2	.083	.142	.079	.121	.069	.123	.154	.184	.142	.166	.131	.164
	0	.6	.327	.505	.333	.503	.349	.490	.437	.580	.433	.589	.439	.586
	0	1.2	.107	.187	.071	.155	.066	.137	.167	.236	.138	.223	.131	.194
III	0	2	.087	.152	.071	.127	.075	.128	.155	.200	.134	.170	.121	.176
	.4	1.2	.136	.287	.082	.277	.076	.227	.202	.368	.149	.348	.138	.304
	.4	2	.088	.143	.082	.123	.072	.124	.153	.182	.141	.166	.134	.162

TABLE 4.114

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.141	.161	.121	.132	.097	.103	.221	.211	.174	.183	.155	.165
	0	1.2	.113	.155	.080	.110	.068	.102	.180	.199	.147	.170	.141	.146
I	0	2	.095	.129	.085	.117	.080	.095	.171	.175	.137	.164	.144	.142
	.4	1.2	.135	.157	.102	.116	.071	.098	.203	.218	.160	.164	.135	.144
	.4	2	.103	.142	.093	.116	.076	.090	.167	.182	.146	.169	.144	.139
	0	.6	.151	.160	.118	.125	.091	.099	.227	.213	.182	.183	.161	.164
	0	1.2	.105	.154	.082	.113	.071	.103	.179	.200	.144	.166	.144	.145
II	0	2	.095	.130	.088	.116	.081	.094	.169	.168	.138	.162	.148	.147
	.4	1.2	.139	.162	.107	.116	.072	.100	.214	.221	.166	.168	.144	.145
	.4	2	.101	.142	.095	.117	.076	.090	.172	.187	.145	.167	.141	.140
	0	.6	.157	.162	.125	.134	.100	.105	.232	.218	.191	.188	.171	.176
	0	1.2	.115	.155	.087	.113	.072	.101	.180	.203	.155	.165	.139	.148
III	0	2	.099	.133	.087	.120	.084	.095	.172	.178	.143	.173	.149	.139
	.4	1.2	.142	.171	.105	.120	.077	.103	.218	.222	.177	.174	.142	.146
	.4	2	.101	.144	.093	.127	.077	.093	.166	.181	.153	.170	.143	.144

TABLE 4.115

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = 0$, $\phi_i = 0$, $\psi_i = .9$, $i = 1,2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.142	.150	.110	.126	.095	.094	.219	.208	.176	.178	.150	.159
	0	1.2	.111	.157	.082	.111	.069	.100	.175	.194	.146	.168	.141	.146
I	0	2	.092	.129	.086	.117	.083	.094	.170	.170	.139	.163	.146	.146
	.4	1.2	.127	.157	.097	.110	.062	.096	.203	.207	.151	.157	.130	.138
	.4	2	.099	.141	.091	.115	.078	.091	.165	.179	.146	.171	.146	.138
	0	.6	.150	.159	.111	.127	.092	.101	.223	.211	.177	.178	.158	.160
	0	1.2	.106	.156	.081	.108	.072	.099	.173	.193	.146	.166	.141	.148
II	0	2	.095	.127	.089	.116	.082	.095	.172	.166	.139	.160	.144	.145
	.4	1.2	.131	.161	.096	.113	.067	.102	.205	.215	.157	.165	.141	.146
	.4	2	.100	.143	.092	.117	.077	.093	.170	.183	.147	.171	.145	.140
	0	.6	.145	.158	.113	.131	.096	.107	.224	.216	.178	.184	.161	.168
	0	1.2	.107	.157	.082	.108	.070	.101	.173	.193	.144	.166	.143	.148
III	0	2	.095	.127	.086	.115	.083	.094	.172	.168	.140	.159	.142	.143
	.4	1.2	.132	.164	.102	.117	.069	.105	.212	.220	.158	.170	.137	.149
	.4	2	.095	.143	.092	.117	.078	.091	.161	.181	.147	.173	.143	.140

TABLE 4.116

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.178	.223	.160	.229	.155	.205	.278	.294	.244	.312	.246	.283
	0	1.2	.110	.147	.080	.120	.070	.100	.174	.191	.141	.157	.127	.151
I	0	2	.108	.154	.086	.125	.079	.107	.175	.196	.148	.158	.132	.151
	.4	1.2	.137	.185	.091	.170	.071	.141	.193	.241	.162	.222	.131	.209
	.4	2	.101	.134	.081	.116	.075	.096	.170	.174	.142	.155	.130	.144
	0	.6	.226	.283	.211	.284	.206	.258	.333	.370	.307	.370	.308	.357
	0	1.2	.111	.145	.079	.123	.073	.096	.176	.190	.143	.159	.127	.150
II	0	2	.104	.151	.088	.125	.083	.105	.181	.197	.150	.157	.129	.147
	.4	1.2	.142	.198	.098	.165	.075	.144	.210	.259	.165	.219	.137	.201
	.4	2	.102	.134	.084	.115	.078	.096	.181	.176	.138	.155	.128	.142
	0	.6	.244	.305	.236	.311	.228	.283	.357	.392	.328	.399	.333	.379
	0	1.2	.113	.149	.077	.126	.076	.099	.181	.190	.146	.167	.129	.150
III	0	2	.112	.154	.094	.132	.081	.102	.183	.194	.146	.163	.133	.147
	.4	1.2	.151	.203	.102	.160	.075	.138	.216	.262	.163	.225	.134	.203
	.4	2	.111	.136	.086	.120	.077	.096	.181	.184	.138	.158	.128	.138

TABLE 4.117

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.172	.219	.139	.216	.127	.189	.262	.283	.218	.297	.211	.255
	0	1.2	.104	.149	.079	.123	.074	.099	.169	.195	.139	.165	.127	.154
I	0	2	.110	.152	.086	.124	.079	.107	.175	.195	.146	.158	.129	.153
	.4	1.2	.134	.192	.091	.173	.068	.149	.189	.241	.150	.229	.122	.205
	.4	2	.101	.137	.080	.114	.075	.097	.172	.179	.143	.154	.128	.146
	0	.6	.202	.270	.166	.264	.160	.234	.287	.350	.248	.348	.247	.313
	0	1.2	.102	.149	.077	.124	.073	.097	.174	.198	.140	.169	.124	.155
II	0	2	.108	.155	.085	.124	.082	.107	.177	.194	.147	.159	.128	.149
	.4	1.2	.129	.200	.090	.176	.072	.148	.194	.258	.148	.231	.123	.206
	.4	2	.099	.136	.081	.113	.076	.097	.176	.181	.139	.155	.126	.144
	0	.6	.207	.289	.174	.275	.176	.247	.298	.359	.258	.366	.256	.346
	0	1.2	.104	.150	.076	.123	.072	.097	.174	.198	.138	.168	.124	.154
III	0	2	.106	.154	.087	.126	.083	.106	.180	.191	.147	.160	.127	.148
	.4	1.2	.137	.206	.091	.179	.071	.148	.198	.265	.153	.229	.126	.208
	.4	2	.101	.132	.080	.112	.077	.096	.174	.183	.142	.155	.127	.143

TABLE 4.118

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.182	.220	.160	.233	.162	.214	.253	.309	.252	.318	.232	.274
	0	1.2	.102	.156	.101	.150	.070	.109	.176	.207	.158	.197	.124	.149
I	0	2	.095	.135	.099	.137	.076	.104	.170	.182	.157	.182	.136	.150
	.4	1.2	.130	.180	.099	.173	.080	.144	.202	.275	.168	.238	.150	.204
	.4	2	.102	.127	.090	.125	.082	.088	.158	.170	.164	.177	.130	.130
	0	.6	.227	.286	.213	.293	.219	.262	.303	.380	.303	.372	.293	.348
	0	1.2	.105	.150	.097	.144	.069	.104	.177	.201	.156	.200	.121	.149
II	0	2	.098	.132	.101	.130	.077	.102	.169	.178	.156	.183	.135	.148
	.4	1.2	.139	.193	.103	.171	.085	.149	.218	.276	.183	.234	.158	.195
	.4	2	.097	.134	.089	.128	.082	.088	.156	.166	.159	.176	.129	.130
	0	.6	.235	.304	.239	.317	.234	.281	.318	.404	.323	.395	.318	.368
	0	1.2	.103	.155	.101	.148	.066	.103	.181	.200	.165	.198	.125	.144
III	0	2	.097	.140	.098	.139	.077	.102	.168	.186	.166	.185	.140	.150
	.4	1.2	.141	.189	.117	.177	.091	.149	.225	.273	.189	.238	.158	.198
	.4	2	.094	.132	.090	.135	.080	.088	.157	.169	.168	.177	.132	.130

TABLE 4.119

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = -.5$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.176	.217	.145	.225	.145	.197	.242	.295	.234	.303	.212	.263
	0	1.2	.097	.154	.100	.154	.070	.108	.179	.203	.159	.201	.123	.152
I	0	2	.092	.131	.099	.138	.076	.105	.167	.179	.156	.182	.136	.150
	.4	1.2	.129	.191	.093	.176	.079	.155	.202	.277	.170	.258	.145	.212
	.4	2	.095	.127	.093	.125	.083	.088	.158	.168	.162	.177	.129	.131
	0	.6	.204	.265	.192	.279	.182	.257	.267	.362	.257	.352	.255	.324
	0	1.2	.095	.156	.099	.151	.067	.107	.182	.208	.157	.203	.121	.154
II	0	2	.095	.131	.101	.132	.074	.101	.167	.177	.154	.185	.133	.148
	.4	1.2	.130	.210	.100	.182	.084	.156	.215	.289	.173	.253	.151	.220
	.4	2	.097	.132	.089	.128	.083	.090	.154	.168	.158	.179	.129	.131
	0	.6	.206	.286	.197	.289	.192	.268	.277	.379	.269	.375	.256	.338
	0	1.2	.099	.158	.097	.148	.068	.106	.180	.207	.159	.207	.120	.155
III	0	2	.093	.131	.099	.131	.075	.103	.163	.179	.154	.186	.133	.149
	.4	1.2	.133	.209	.103	.182	.083	.157	.214	.289	.173	.249	.150	.219
	.4	2	.096	.132	.092	.129	.083	.090	.153	.164	.157	.177	.129	.131

TABLE 4.120

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.261	.364	.276	.390	.275	.348	.365	.439	.361	.469	.372	.447
	0	1.2	.109	.174	.076	.153	.072	.129	.170	.220	.134	.215	.138	.184
I	0	2	.086	.150	.073	.120	.077	.121	.167	.198	.146	.167	.121	.174
	.4	1.2	.130	.253	.094	.267	.079	.223	.196	.337	.155	.337	.137	.297
	.4	2	.089	.134	.083	.118	.075	.121	.155	.178	.141	.165	.136	.160
	0	.6	.405	.528	.400	.528	.409	.510	.516	.603	.494	.614	.504	.607
	0	1.2	.110	.180	.077	.144	.069	.127	.176	.226	.140	.211	.136	.184
II	0	2	.093	.149	.075	.122	.078	.124	.165	.200	.141	.165	.122	.172
	.4	1.2	.142	.257	.096	.258	.079	.203	.224	.332	.167	.323	.142	.282
	.4	2	.090	.138	.083	.119	.072	.120	.162	.171	.142	.162	.134	.160
	0	.6	.436	.565	.440	.567	.438	.546	.543	.646	.521	.648	.531	.642
	0	1.2	.112	.175	.077	.143	.069	.133	.176	.221	.140	.210	.132	.178
III	0	2	.096	.146	.073	.125	.078	.129	.161	.201	.144	.169	.126	.177
	.4	1.2	.154	.262	.103	.260	.083	.204	.229	.336	.172	.321	.143	.274
	.4	2	.097	.143	.082	.119	.074	.120	.162	.173	.141	.166	.134	.156

TABLE 4.121

EMPIRICAL SIZES OF W_I° AND W_F° FOR $\rho = .75$, $\phi_i = 0$, $\psi_i = .9$, $i = 1, 2$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°	W_I°	W_F°
	0	.6	.225	.340	.219	.355	.227	.306	.329	.422	.325	.437	.325	.394
	0	1.2	.105	.182	.074	.158	.072	.134	.169	.231	.135	.227	.137	.192
I	0	2	.088	.149	.073	.120	.076	.122	.167	.198	.143	.166	.121	.173
	.4	1.2	.125	.265	.082	.289	.070	.246	.186	.353	.143	.363	.133	.328
	.4	2	.090	.138	.083	.117	.072	.123	.150	.180	.140	.165	.134	.161
	0	.6	.288	.448	.299	.451	.307	.429	.395	.528	.398	.539	.406	.538
	0	1.2	.108	.188	.075	.155	.068	.133	.167	.240	.142	.224	.135	.192
II	0	2	.092	.150	.077	.122	.077	.124	.164	.198	.140	.166	.120	.175
	.4	1.2	.134	.279	.082	.281	.075	.236	.199	.362	.148	.353	.142	.321
	.4	2	.088	.139	.086	.119	.072	.122	.158	.182	.139	.163	.134	.161
	0	.6	.311	.469	.321	.480	.325	.459	.408	.555	.415	.560	.426	.569
	0	1.2	.107	.187	.075	.153	.068	.133	.169	.239	.140	.221	.133	.192
III	0	2	.089	.149	.076	.121	.077	.124	.163	.197	.140	.166	.119	.176
	.4	1.2	.134	.281	.082	.277	.078	.228	.201	.365	.150	.347	.142	.312
	.4	2	.092	.139	.082	.121	.074	.120	.157	.179	.141	.164	.134	.161

TABLE 4.122

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	-.016	-.023	-.014	-.001	-.004	-.003	-.001	-.003	-.001
I	.2	.4	-.034	-.033	-.025	-.004	-.008	-.010	-.001	-.006	-.001
	.4	.8	-.019	-.025	-.020	-.005	-.004	-.013	-.002	-.003	-.003
	.7	1	-.033	-.041	-.036	-.013	-.015	-.030	-.003	-.005	-.007
	0	.4	-.008	-.012	-.009	-.001	-.002	-.002	.000	-.001	.000
II	.2	.4	-.020	-.024	-.017	-.005	-.006	-.007	.000	-.001	.000
	.4	.8	-.012	-.015	-.016	-.005	-.005	-.010	.000	-.002	-.002
	.7	1	-.027	-.030	-.032	-.016	-.017	-.024	-.003	-.004	-.005
	0	.4	-.006	-.007	-.008	-.002	-.003	-.003	.000	.000	.000
III	.2	.4	-.013	-.013	-.014	-.007	-.006	-.007	-.001	-.001	-.001
	.4	.8	-.011	-.012	-.016	-.006	-.006	-.009	.000	-.001	-.002
	.7	1	-.026	-.027	-.032	-.018	-.019	-.023	-.003	-.004	-.005

TABLE 4.123

MONTE CARLO BIAS OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = 0,\ \phi_i = \psi_i = 0,\ i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	-.017	-.028	-.001	-.004	-.001	-.005
I	.2	.4	-.041	-.034	-.001	-.008	-.001	-.014
	.4	.8	-.018	-.029	-.002	-.001	-.001	-.005
	.7	1	-.032	-.046	-.006	-.009	-.002	-.007
	0	.4	-.007	-.012	.000	-.002	.000	-.001
II	.2	.4	-.023	-.027	-.004	-.007	000	-.003
	.4	.8	-.010	-.015	-.002	-.004	.000	-.002
	.7	1	-.023	-.030	-.010	-.015	-.001	-.004
	0	.4	-.005	-.005	-.002	-.004	.001	.000
III	.2	.4	-.012	-.010	-.007	-.007	-.001	-.001
	.4	.8	-.009	-.010	-.004	-.005	.000	-.001
	.7	1	-.022	-.023	-.014	-.016	-.002	-.003

TABLE 4.124

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.060	.072	.149	.043	.054	.119	.028	.031	.096
I	.2	.4	.182	.204	.276	.164	.194	.254	.131	.140	.226
	.4	.8	.055	.069	.140	.040	.052	.109	.025	.031	.085
	.7	1	.100	.122	.194	.080	.107	.163	.055	.066	.132
	0	.4	.119	.141	.211	.093	.108	.181	.063	.070	.146
II	.2	.4	.263	.282	.321	.240	.256	.301	.203	.212	.273
	.4	.8	.092	.116	.161	.068	.088	.131	.044	.054	.099
	.7	1	.142	.173	.204	.116	.144	.176	.082	.098	.138
	0	.4	.177	.197	.259	.127	.143	.212	.085	.093	.170
III	.2	.4	.318	.329	.351	.279	.289	.323	.235	.242	.292
	.4	.8	.120	.143	.174	.084	.103	.137	.052	.064	.103
	.7	1	.169	.195	.208	.133	.157	.178	.092	.109	.140

TABLE 4.125

MONTE CARLO BIAS OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.025	.031	.019	.026	.010	.005
I	.2	.4	.120	.164	.110	.162	.079	.089
	.4	.8	.024	.031	.018	.026	.009	.009
	.7	1	.053	.077	.044	.075	.025	.033
	0	.4	.069	.105	.050	.075	.029	.038
II	.2	.4	.216	.259	.191	.225	.153	.172
	.4	.8	.054	.091	.037	.065	.021	.032
	.7	1	.101	.150	.078	.121	.050	.072
	0	.4	.123	.164	.078	.106	.044	.057
III	.2	.4	.288	.315	.242	.266	.189	.207
	.4	.8	.085	.123	.053	.081	.028	.042
	.7	1	.138	.182	.101	.140	.062	.086

TABLE 4.126

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	-.064	-.088	-.154	-.039	-.045	-.116	-.026	-.029	-.095
I	.2	.4	-.197	-.229	-.288	-.156	-.175	-.249	-.129	-.139	-.225
	.4	.8	-.062	-.086	-.149	-.035	-.045	-.106	-.023	-.028	-.083
	.7	1	-.113	-.144	-.210	-.074	-.095	-.160	-.051	-.061	-.128
	0	.4	-.122	-.148	-.216	-.090	-.108	-.180	-.064	-.072	-.147
II	.2	.4	-.277	-.295	-.333	-.242	-.258	-.304	-.207	-.218	-.276
	.4	.8	-.094	-.123	-.165	-.064	-.084	-.127	-.044	-.055	-.098
	.7	1	-.150	-.182	-.213	-.110	-.137	-.170	-.081	-.098	-.137
	0	.4	-.176	-.196	-.261	-.125	-.142	-.212	-.085	-.094	-.171
III	.2	.4	-.324	-.331	-.360	-.282	-.292	-.326	-.237	-.245	-.294
	.4	.8	-.119	-.141	-.176	-.080	-.098	-.134	-.052	-.064	-.103
	.7	1	-.173	-.195	-.216	-.127	-.150	-.172	-.091	-.108	-.138

TABLE 4.127

MONTE CARLO BIAS OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	-.030	-.056	-.015	-.008	-.008	-.002
I	.2	.4	-.138	-.192	-.100	-.127	-.075	-.088
	.4	.8	-.029	-.058	-.014	-.012	-.007	-.004
	.7	1	-.064	-.108	-.037	-.052	-.021	-.024
	0	.4	-.071	-.117	-.046	-.074	-.030	-.040
II	.2	.4	-.232	-.273	-.193	-.228	-.158	-.180
	.4	.8	-.055	-.099	-.033	-.061	-.021	-.033
	.7	1	-.107	-.162	-.072	-.114	-.051	-.074
	0	.4	-.119	-.163	-.075	-.107	-.043	-.058
III	.2	.4	-.292	-.313	-.245	-.268	-.192	-.211
	.4	.8	-.081	-.120	-.048	-.077	-.027	-.042
	.7	1	-.139	-.178	-.095	-.132	-.061	-.086

TABLE 4.128

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.092	.112	.227	.057	.073	.173	.036	.047	.140
I	.2	.4	.282	.318	.419	.220	.250	.366	.182	.200	.328
	.4	.8	.084	.114	.208	.052	.076	.159	.031	.046	.121
	.7	1	.149	.195	.285	.104	.142	.234	.072	.097	.188
	0	.4	.183	.212	.323	.136	.156	.270	.091	.102	.217
II	.2	.4	.405	.432	.492	.357	.376	.450	.298	.313	.406
	.4	.8	.138	.174	.241	.097	.127	.190	.061	.081	.143
	.7	1	.211	.258	.301	.162	.204	.249	.115	.144	.199
	0	.4	.269	.298	.394	.189	.209	.317	.125	.136	.254
III	.2	.4	.484	.496	.535	.418	.430	.483	.349	.359	.435
	.4	.8	.178	.208	.260	.120	.147	.199	.075	.095	.151
	.7	1	.248	.283	.307	.187	.223	.252	.131	.159	.201

TABLE 4.129

MONTE CARLO BIAS OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.040	.054	.020	.029	.009	.011
I	.2	.4	.192	.260	.133	.178	.100	.127
	.4	.8	.037	.069	.019	.037	.007	.016
	.7	1	.081	.145	.048	.088	.027	.050
	0	.4	.107	.160	.071	.106	.040	.058
II	.2	.4	.336	.396	.283	.329	.221	.254
	.4	.8	.082	.140	.051	.095	.027	.052
	.7	1	.150	.226	.107	.172	.067	.111
	0	.4	.186	.246	.114	.153	.063	.085
III	.2	.4	.438	.471	.363	.394	.280	.306
	.4	.8	.124	.176	.074	.118	.038	.066
	.7	1	.201	.261	.140	.200	.087	.131

TABLE 4.130

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.294	.341	.221	.216	.255	.169	.131	.146	.105
I	.2	.4	.569	.601	.371	.515	.555	.335	.341	.371	.234
	.4	.8	.295	.341	.250	.219	.257	.200	.133	.145	.126
	.7	1	.429	.495	.406	.353	.433	.369	.227	.264	.251
	0	.4	.192	.197	.147	.131	.135	.101	.092	.096	.072
II	.2	.4	.296	.293	.217	.232	.227	.170	.189	.191	.137
	.4	.8	.207	.209	.191	.143	.147	.142	.097	.100	.102
	.7	1	.316	.323	.339	.250	.265	.290	.176	.190	.220
	0	.4	.152	.161	.127	.112	.114	.091	.083	.086	.066
III	.2	.4	.213	.219	.178	.180	.180	.146	.154	.156	.120
	.4	.8	.185	.185	.181	.133	.137	.137	.093	.096	.099
	.7	1	.307	.306	.329	.249	.257	.285	.179	.190	.218

TABLE 4.131

MONTE CARLO S.D. OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.324	.448	.232	.320	.139	.175
I	.2	.4	.711	.825	.631	.752	.406	.503
	.4	.8	.324	.449	.232	.323	.139	.171
	.7	1	.478	.637	.380	.538	.237	.317
	0	.4	.222	.240	.149	.163	.102	.114
II	.2	.4	.371	.366	.289	.284	.235	.243
	.4	.8	.228	.240	.153	.165	.103	.113
	.7	1	.326	.346	.244	.274	.170	.194
	0	.4	.177	.190	.129	.135	.094	.101
III	.2	.4	.253	.261	.217	.216	.188	.192
	.4	.8	.197	.202	.139	.147	.097	.105
	.7	1	.300	.312	.234	.256	.165	.189

TABLE 4.132

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.262	.291	.201	.185	.207	.146	.115	.129	.094
I	.2	.4	.521	.530	.342	.429	.455	.289	.300	.325	.207
	.4	.8	.261	.286	.221	.188	.207	.174	.116	.128	.113
	.7	1	.382	.413	.354	.297	.336	.308	.197	.221	.219
	0	.4	.172	.179	.137	.117	.121	.097	.084	.088	.070
II	.2	.4	.268	.267	.201	.206	.205	.158	.171	.170	.127
	.4	.8	.185	.191	.174	.129	.136	.133	.089	.094	.095
	.7	1	.283	.285	.298	.222	.233	.256	.159	.171	.197
	0	.4	.144	.153	.123	.105	.110	.091	.076	.078	.066
III	.2	.4	.198	.204	.167	.167	.170	.140	.140	.139	.112
	.4	.8	.169	.171	.166	.123	.128	.130	.084	.087	.093
	.7	1	.273	.269	.289	.223	.228	.252	.159	.167	.195

TABLE 4.133

MONTE CARLO S.D. OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.288	.380	.199	.268	.121	.162
I	.2	.4	.648	.708	.520	.637	.357	.440
	.4	.8	.286	.363	.199	.248	.121	.157
	.7	1	.427	.514	.319	.408	.206	.266
	0	.4	.196	.214	.130	.142	.091	.103
II	.2	.4	.332	.328	.252	.246	.209	.212
	.4	.8	.202	.221	.135	.153	.092	.107
	.7	1	.292	.311	.214	.242	.153	.177
	0	.4	.164	.179	.116	.125	.082	.088
III	.2	.4	.233	.241	.196	.198	.167	.164
	.4	.8	.178	.187	.125	.137	.085	.095
	.7	1	.269	.275	.209	.228	.146	.163

TABLE 4.134

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.247	.277	.187	.182	.219	.144	.116	.135	.095
I	.2	.4	.483	.499	.313	.425	.480	.283	.293	.316	.201
	.4	.8	.249	.273	.213	.183	.206	.164	.116	.128	.112
	.7	1	.356	.386	.333	.289	.326	.287	.195	.218	.221
	0	.4	.161	.169	.128	.114	.123	.094	.085	.091	.070
II	.2	.4	.255	.254	.191	.201	.204	.152	.170	.173	.125
	.4	.8	.173	.177	.165	.124	.131	.128	.090	.095	.095
	.7	1	.263	.262	.280	.210	.217	.243	.159	.169	.194
	0	.4	.136	.147	.117	.102	.109	.088	.077	.081	.066
III	.2	.4	.188	.193	.159	.161	.164	.134	.138	.139	.110
	.4	.8	.160	.165	.159	.118	.122	.125	.085	.087	.093
	.7	1	.259	.254	.275	.211	.212	.240	.157	.161	.191

TABLE 4.135

MONTE CARLO S.D. OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.272	.381	.194	.315	.122	.167
I	.2	.4	.604	.693	.515	.708	.349	.421
	.4	.8	.272	.360	.194	.277	.122	.160
	.7	1	.400	.501	.313	.425	.205	.262
	0	.4	.184	.207	.128	.148	.092	.110
II	.2	.4	.318	.312	.248	.254	.209	.221
	.4	.8	.189	.206	.131	.148	.093	.111
	.7	1	.271	.283	.205	.227	.154	.180
	0	.4	.154	.173	.113	.127	.084	.093
III	.2	.4	.221	.227	.189	.195	.166	.169
	.4	.8	.168	.181	.121	.131	.086	.095
	.7	1	.254	.261	.198	.209	.145	.158

TABLE 4.136

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_F, \overline{\nu}_B$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

			$n = 64$			$n = 128$			$n = 256$		
m	γ	δ	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$	$\overline{\nu}_I$	$\overline{\nu}_F$	$\overline{\nu}_B$
	0	.4	.199	.246	.156	.136	.160	.113	.094	.109	.079
I	.2	.4	.398	.432	.265	.328	.348	.221	.242	.255	.169
	.4	.8	.200	.236	.176	.137	.154	.129	.095	.110	.093
	.7	1	.294	.325	.279	.219	.233	.227	.157	.174	.171
	0	.4	.140	.153	.117	.094	.099	.084	.069	.072	.062
II	.2	.4	.216	.219	.162	.162	.162	.127	.136	.136	.104
	.4	.8	.146	.153	.140	.098	.103	.106	.072	.079	.081
	.7	1	.223	.224	.236	.164	.164	.192	.126	.132	.155
	0	.4	.128	.145	.113	.091	.100	.084	.067	.068	.063
III	.2	.4	.162	.172	.138	.135	.141	.114	.116	.115	.094
	.4	.8	.139	.143	.138	.098	.101	.106	.071	.075	.081
	.7	1	.218	.215	.230	.168	.167	.190	.126	.131	.154

TABLE 4.137

MONTE CARLO S.D. OF $\overline{\nu}_2^I, \overline{\nu}_2^F$ FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		n	64	64	128	128	256	256
m	γ	δ	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$	$\overline{\nu}_2^I$	$\overline{\nu}_2^F$
	0	.4	.218	.332	.144	.212	.099	.144
I	.2	.4	.492	.575	.394	.476	.286	.345
	.4	.8	.217	.317	.144	.221	.099	.145
	.7	1	.326	.421	.234	.309	.166	.220
	0	.4	.155	.179	.099	.117	.072	.088
II	.2	.4	.267	.272	.196	.194	.164	.165
	.4	.8	.157	.180	.100	.120	.073	.096
	.7	1	.229	.247	.158	.170	.120	.142
	0	.4	.139	.170	.095	.111	.068	.075
III	.2	.4	.190	.204	.157	.167	.136	.135
	.4	.8	.144	.160	.097	.110	.069	.086
	.7	1	.215	.223	.158	.166	.116	.132

TABLE 4.138

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
I	0	.4	.153	.211	.123	.188	.107	.171	.226	.289	.185	.257	.183	.229
	.2	.4	.225	.258	.218	.251	.208	.236	.316	.333	.305	.326	.292	.301
	.4	.8	.152	.218	.134	.186	.116	.151	.228	.271	.199	.242	.188	.205
	.7	1	.209	.229	.192	.209	.173	.183	.282	.283	.264	.270	.261	.251
II	0	.4	.157	.211	.135	.173	.126	.162	.231	.283	.207	.255	.198	.232
	.2	.4	.206	.217	.194	.220	.212	.223	.298	.297	.275	.292	.295	.320
	.4	.8	.197	.232	.170	.211	.151	.168	.290	.301	.258	.279	.221	.239
	.7	1	.293	.297	.303	.288	.283	.267	.378	.366	.385	.383	.355	.340
III	0	.4	.112	.167	.118	.156	.125	.139	.182	.231	.189	.222	.197	.237
	.2	.4	.152	.179	.175	.192	.194	.211	.227	.268	.254	.276	.268	.290
	.4	.8	.197	.233	.194	.226	.158	.181	.284	.305	.278	.308	.247	.269
	.7	1	.344	.348	.351	.346	.331	.321	.422	.417	.445	.434	.409	.401

TABLE 4.139

EMPIRICAL SIZES OF W_2^I AND W_2^F FOR $\rho = 0$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F
I	0	.4	.191	.271	.147	.227	.130	.194	.283	.350	.228	.290	.210	.270
	.2	.4	.342	.360	.301	.349	.289	.304	.419	.441	.394	.422	.370	.372
	.4	.8	.194	.265	.145	.219	.131	.176	.281	.332	.232	.279	.207	.246
	.7	1	.264	.281	.220	.250	.193	.213	.349	.348	.304	.309	.284	.288
II	0	.4	.230	.274	.197	.237	.161	.211	.311	.345	.257	.306	.247	.285
	.2	.4	.313	.319	.292	.325	.317	.332	.410	.384	.380	.398	.390	.397
	.4	.8	.246	.284	.201	.232	.169	.194	.330	.343	.272	.305	.253	.285
	.7	1	.331	.329	.302	.304	.270	.270	.411	.399	.394	.378	.357	.352
III	0	.4	.176	.237	.177	.224	.177	.211	.260	.300	.251	.295	.253	.299
	.2	.4	.228	.253	.253	.278	.279	.284	.319	.337	.346	.375	.371	.378
	.4	.8	.239	.258	.209	.243	.183	.209	.322	.332	.289	.319	.270	.316
	.7	1	.348	.354	.346	.340	.314	.310	.432	.436	.423	.432	.397	.397

TABLE 4.140

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.172	.263	.151	.240	.111	.209	.247	.335	.228	.298	.192	.284
I	.2	.4	.275	.336	.283	.327	.263	.311	.362	.416	.354	.398	.357	.381
	.4	.8	.170	.242	.161	.240	.108	.194	.244	.314	.232	.297	.199	.264
	.7	1	.222	.275	.221	.263	.193	.228	.308	.345	.292	.341	.277	.299
	0	.4	.273	.343	.263	.361	.232	.320	.361	.425	.346	.443	.326	.400
II	.2	.4	.434	.473	.490	.549	.518	.536	.511	.550	.578	.622	.588	.605
	.4	.8	.252	.325	.236	.343	.206	.300	.329	.399	.317	.422	.294	.383
	.7	1	.339	.398	.380	.448	.344	.406	.430	.476	.470	.527	.423	.485
	0	.4	.418	.488	.407	.481	.365	.419	.521	.571	.502	.569	.462	.515
III	.2	.4	.618	.646	.665	.701	.682	.706	.694	.708	.741	.768	.747	.764
	.4	.8	.322	.392	.301	.401	.261	.349	.412	.473	.388	.476	.348	.432
	.7	1	.444	.488	.451	.513	.417	.480	.514	.561	.529	.607	.506	.548

TABLE 4.141

EMPIRICAL SIZES OF W_2^I AND W_2^F FOR $\rho = .5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F
	0	.4	.206	.312	.158	.274	.121	.242	.287	.391	.246	.336	.210	.309
I	.2	.4	.352	.430	.322	.394	.300	.353	.455	.507	.412	.475	.384	.431
	.4	.8	.197	.312	.162	.267	.121	.228	.286	.380	.247	.333	.211	.295
	.7	1	.275	.333	.231	.285	.207	.259	.362	.414	.298	.364	.275	.336
	0	.4	.262	.347	.221	.356	.196	.321	.338	.433	.308	.438	.281	.402
II	.2	.4	.439	.495	.447	.529	.446	.504	.513	.573	.522	.606	.522	.574
	.4	.8	.262	.354	.227	.371	.192	.310	.346	.446	.304	.444	.281	.398
	.7	1	.344	.415	.332	.441	.306	.394	.433	.499	.421	.523	.381	.460
	0	.4	.329	.448	.276	.412	.220	.339	.412	.526	.358	.488	.310	.424
III	.2	.4	.563	.623	.582	.647	.562	.613	.636	.691	.649	.703	.631	.672
	.4	.8	.292	.400	.246	.395	.204	.336	.379	.484	.339	.472	.286	.411
	.7	1	.415	.492	.404	.497	.341	.440	.502	.566	.488	.571	.441	.528

TABLE 4.142

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.177	.295	.134	.248	.134	.219	.263	.353	.228	.318	.209	.293
I	.2	.4	.294	.351	.265	.331	.264	.313	.372	.434	.361	.405	.342	.386
	.4	.8	.162	.285	.141	.241	.136	.203	.262	.350	.236	.310	.208	.278
	.7	1	.241	.294	.207	.264	.201	.231	.330	.364	.301	.336	.281	.304
	0	.4	.268	.372	.269	.366	.252	.313	.351	.441	.363	.440	.336	.402
II	.2	.4	.469	.512	.517	.550	.515	.557	.549	.584	.588	.630	.595	.631
	.4	.8	.242	.352	.255	.362	.224	.291	.328	.430	.339	.438	.301	.360
	.7	1	.368	.416	.389	.450	.347	.400	.449	.496	.464	.528	.432	.485
	0	.4	.410	.475	.400	.485	.357	.405	.502	.574	.494	.563	.446	.495
III	.2	.4	.637	.659	.695	.705	.677	.707	.710	.733	.750	.765	.736	.767
	.4	.8	.310	.415	.310	.408	.254	.331	.408	.494	.396	.489	.345	.416
	.7	1	.463	.501	.460	.524	.424	.477	.545	.578	.532	.585	.493	.540

TABLE 4.143

EMPIRICAL SIZES OF W_2^I AND W_2^F FOR $\rho = -.5$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F
	0	.4	.191	.340	.161	.290	.144	.254	.285	.401	.242	.362	.220	.332
I	.2	.4	.349	.422	.311	.405	.299	.376	.431	.498	.397	.472	.375	.445
	.4	.8	.190	.328	.160	.286	.140	.238	.287	.392	.244	.356	.217	.318
	.7	1	.270	.344	.236	.301	.214	.267	.359	.403	.317	.367	.295	.352
	0	.4	.249	.391	.226	.360	.188	.300	.331	.463	.304	.434	.282	.378
II	.2	.4	.437	.504	.460	.520	.449	.499	.516	.576	.544	.602	.524	.574
	.4	.8	.246	.373	.222	.369	.187	.292	.334	.463	.311	.436	.277	.367
	.7	1	.358	.431	.347	.441	.305	.377	.435	.518	.450	.525	.399	.455
	0	.4	.304	.456	.276	.417	.220	.342	.400	.524	.359	.503	.316	.409
III	.2	.4	.569	.623	.601	.650	.547	.601	.640	.689	.672	.707	.611	.669
	.4	.8	.267	.407	.260	.390	.215	.308	.358	.488	.348	.464	.306	.384
	.7	1	.420	.492	.415	.485	.361	.426	.500	.563	.480	.554	.439	.507

TABLE 4.144

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.214	.365	.167	.341	.154	.329	.303	.423	.249	.418	.223	.392
I	.2	.4	.390	.464	.372	.466	.385	.484	.455	.536	.450	.528	.488	.551
	.4	.8	.209	.356	.159	.336	.152	.337	.294	.427	.250	.409	.220	.390
	.7	1	.294	.405	.262	.384	.255	.378	.378	.490	.344	.464	.338	.453
	0	.4	.459	.560	.504	.595	.455	.541	.562	.637	.594	.681	.554	.625
II	.2	.4	.748	.786	.840	.866	.841	.866	.818	.849	.879	.897	.891	.914
	.4	.8	.373	.516	.366	.545	.316	.508	.457	.591	.460	.616	.399	.591
	.7	1	.511	.622	.531	.666	.502	.627	.594	.679	.617	.726	.578	.700
	0	.4	.768	.806	.750	.803	.691	.735	.835	.861	.819	.854	.770	.805
III	.2	.4	.946	.944	.964	.963	.960	.968	.963	.967	.972	.975	.976	.981
	.4	.8	.544	.647	.507	.653	.433	.597	.622	.708	.590	.717	.518	.668
	.7	1	.645	.723	.664	.749	.617	.726	.699	.767	.730	.805	.686	.784

TABLE 4.145

EMPIRICAL SIZES OF W_2^I AND W_2^F FOR $\rho = .75$, $\phi_i = \psi_i = 0$, $i = 1, 2$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F	W_2^I	W_2^F
	0	.4	.209	.407	.148	.362	.124	.348	.303	.486	.234	.440	.202	.416
I	.2	.4	.393	.521	.345	.484	.317	.482	.466	.574	.432	.555	.406	.538
	.4	.8	.212	.402	.148	.370	.124	.357	.299	.475	.237	.443	.202	.411
	.7	1	.299	.435	.233	.392	.197	.371	.381	.505	.309	.470	.281	.432
	0	.4	.333	.520	.283	.497	.228	.446	.416	.589	.378	.560	.324	.525
II	.2	.4	.646	.734	.689	.783	.654	.758	.698	.788	.751	.829	.720	.802
	.4	.8	.301	.514	.237	.506	.216	.473	.389	.574	.332	.566	.292	.544
	.7	1	.427	.604	.392	.613	.360	.560	.515	.657	.478	.672	.440	.631
	0	.4	.546	.695	.483	.649	.357	.548	.631	.757	.551	.714	.455	.623
III	.2	.4	.891	.901	.894	.926	.845	.902	.916	.919	.924	.945	.885	.927
	.4	.8	.410	.587	.339	.580	.255	.516	.500	.657	.437	.644	.349	.576
	.7	1	.568	.697	.538	.711	.471	.646	.631	.752	.615	.761	.550	.719

Chapter 5

Testing for the equality of orders of integration

5.1 Introduction

From our results in Chapters 2, 3 and 4, it can be inferred that the wider modelling framework that fractional co-integration (with possibly unknown integration orders) allows, enjoys several important advantages over the traditional $CI\,(1,1)$ setting, where the risk of misspecification in optimal Gaussian estimation is not negligible. However, it is clear that this new methodology introduces additional challenges, as in practice those, generally noninteger, orders of integration are unknown. Among other issues, it seems that the traditional way of testing for co-integration, based on ideas like the ones of Dickey and Fuller (1979) or Phillips and Perron (1988) needs to be revised. For example, given two observable series, y_t and x_t, a necessary condition for these processes to be co-integrated is that their orders of integration, say δ_y and δ_x, be equal, so that a necessary preliminary step in order to test for co-integration between two series is to check for the equality of their orders. Thus, we devote this final chapter of the thesis to address this problem, choosing a point of view which differs substantially from usual testing procedures proposed in the literature. As will be seen, our procedure offers several important advantages over those well known procedures.

Several tests involving linear restrictions among memory parameters of multivariate time series have been developed, mainly assuming the processes (more general than fractionally integrated processes) to be covariance stationary, and being based on different estimates of the memory parameters of given series. In the parametric setting, rigorous asymptotic theory has been developed, assuming the vector process considered to be covariance stationary, by Heyde and Gay (1993) and Hosoya (1997). In the semiparametric setting, under only local assumptions, Wald tests of linear restrictions on memory parameters have been proposed for the stationary case by Robinson (1995a) and Lobato (1999), but results in Robinson (1995b) and Lobato (1996) suggest also the use of Lagrange Multiplier and Likelihood Ratio tests, see Marinucci and Robinson (2001). These semiparametric tests enjoy standard asymptotics (feature also shared by the parametric ones), but suffer from a serious drawback, as they are invalid in case there exists co-integration among the series. The reason is that the test statistics involve inversion of a matrix tending in probability to a singular matrix. This problem was acknowledged by Marinucci and Robinson (2001), and Robinson and Yajima (2002) offered a sensible solution at cost of introducing an additional user-chosen number.

We propose a test procedure for the equality of orders of integration of two possibly fractionally integrated (see Definitions 1.2 and 1.3) with arbitrary non-negative orders of integration time series (as this is the most relevant case in practice), which is valid irrespectively of whether the time series are co-integrated or not. Its computation only requires estimates of the different orders of integration involved in the null hypothesis and of the spectral density of the short memory input series which originate the fractionally integrated processes. The different test statistics we propose are based on partial sums of certain fractionally differenced processes and can be computed under semiparametric or parametric assumptions. The kind of assumptions made will determine the type of the estimates of the orders of integration and spectral density used in obtaining the test statistics. These statistics enjoy standard asymptotic theory under the null hypothesis of equality of orders assuming very mild conditions on our estimates of the nuisance parameters, which, in fact, are very similar to those presented in Chapters 2 and 4. Partial sums are not expected to be very informative about memory parameters (see Robinson, 1993, for a unit root test based also on partial sums), but although low power could have been predicted, our test seems to perform relatively well in finite samples. Our test procedure can be easily extended to the multivariate framework, and also can be interpreted as a test for the size of the gap of co-integration (difference between the order of integration of the observables and the one of the co-integrating error) once the pretest of equality of orders has been performed. Inference about the co-integrating gap seems very relevant, because it heavily affects the asymptotic properties of different estimates of the co-integrating parameter, as it is clear from our results in Chapters 2, 3 and 4 (see also Kim and Phillips, 2000, Velasco, 2000).

In the next section we present our testing procedure, which is rigorously justified in the Appendix 5. Section 5.3 includes a Monte Carlo study of finite-sample behavior.

5.2 Testing the equality of fractional difference parameters

Consider the bivariate process $z_t = (y_t, x_t)'$, $t \in Z$, where

$$\begin{aligned} y_t &= \Delta^{-\delta_y} \{v_{1t} 1(t>0)\}, \;\; y_t = 0,\, t \leq 0, && (5.1) \\ x_t &= \Delta^{-\delta_x} \{v_{2t} 1(t>0)\}, \;\; x_t = 0,\, t \leq 0, && (5.2) \end{aligned}$$

with

$$\delta_x, \delta_y \geq 0. \qquad (5.3)$$

We introduce

Assumption 5.1. *The process $v_t = (v_{1t}, v_{2t})'$, $t \in Z$, has representation*

$$v_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}, \; \sum_{j=0}^{\infty} j \, \|A_j\|^2 < \infty, \tag{5.4}$$

where

(i) ε_t are independent and identically distributed vectors with mean zero, positive definite covariance matrix Ω, $E \|\varepsilon_t\|^q < \infty$, $q > 2$;

(ii) $g_{ii}(0) > 0$, $i = 1, 2$, where $g_{ij}(0)$ is the (i, j) element of the spectral density of v_t, denoted by $g(\lambda)$.

In view of Definition 1.1, noting that (5.4) implies that $g(\lambda)$ is $Lip(\varkappa)$, $\varkappa > 0$, by Assumption 5.1, v_{1t}, v_{2t}, are $I(0)$ processes. Consider also certain estimates $\widehat{\delta}_x$, $\widehat{\delta}_y$, $\widehat{g}(0)$ of δ_x, δ_y, $g(0)$ respectively, such that

Assumption 5.2. *As $n \to \infty$,*

$$\widehat{g}(0) \to_p g(0), \tag{5.5}$$

and for any $\kappa > 0$ and $K < \infty$,

$$\widehat{\delta}_x = \delta_x + O_p\left(n^{-\kappa}\right), \widehat{\delta}_y = \delta_y + O_p\left(n^{-\kappa}\right), \tag{5.6}$$

where

$$\left|\widehat{\delta}_x\right| + \left|\widehat{\delta}_y\right| \leq K. \tag{5.7}$$

This assumption, although not primitive, is very mild and, with respect to the estimates of the orders of integration, very similar to Assumptions 2.3 and 4.1 of Chapters 2 and 4 respectively. It is repeated here for readability, but note that it now refers to estimates of integration orders of observable series. As in previous chapters, under some parametric structure for v_t, $\sqrt{n}$-consistent estimates of the orders of integration and $g(0)$ could be achievable by a multivariate extension of the results in Robinson (2002), which extended results in Velasco and Robinson (2000) in the univariate case to cover our type of nonstationarity. Of course, this rate is far better than needed, so we might be content by assuming some weak conditions of smoothness of the spectral density of v_t around frequency zero, and estimate the orders and $g(0)$ semiparametrically. For example, the estimates in Robinson (1994c, 1995a,b), Velasco (1999a,b), justified by Robinson (2002) for our type of nonstationarity, satisfy Assumption 5.2. Also, given estimates $\widehat{\delta}_x$, $\widehat{\delta}_y$, a nonparametric estimate of $g(0)$ could be based on weighted averages of the periodogram of the proxy $\widehat{v}_t = (y_t(\widehat{\delta}_y), x_t(\widehat{\delta}_x))'$ of v_t. The validity of such estimate can be justified by similar techniques as the ones in the proof of Theorem 5.1 below, or the ones already employed in Chapters 2 and 4.

Now, for any non-stochastic 2×1 vector $a = (a_1, a_2)'$, such that $a'g(0)\,a > 0$, noting (1.44), (2.2), (2.3), we could define the class of test statistics

$$\widehat{t}(a) = \frac{a'w_{z(\widehat{\delta}_x,\widehat{\delta}_y)}(0)}{\left(a'\widehat{g}(0)\,a\right)^{1/2}}, \tag{5.8}$$

for testing $H_0 : \delta_x = \delta_y$ against the alternative $H_1 : \delta_x \neq \delta_y$.

Theorem 5.1. *Let (5.1), (5.2), (5.3) and Assumptions 5.1, 5.2 hold. Then, for any 2×1 deterministic vector a such that $a'g(0)\,a > 0$, as $n \to \infty$,*

$$\widehat{t}(a) \rightarrow_d N(0,1) \text{ under } H_0, \tag{5.9}$$

$$\widehat{t}(a) \sim n^{|\delta_x-\delta_y|} \text{under } H_1, \tag{5.10}$$

where " $\sim$ " means now exact rate of convergence.

The proof of the theorem is left to the Appendix 5.

Remark 5.1. The test statistic has standard asymptotic distribution under the null and a rate of divergence increasing exponentially with the difference $|\delta_x - \delta_y|$ under fixed alternatives. Although we presented results and proofs just for the Type II fractionally integrated process, this was simply motivated by the uniform treatment of any value of δ_x and δ_y this definition allows, the same result holding also for Type I processes. Noting from (1.8) that for any $j \geq 1$,

$$a_j(d-1) = a_j(d) - a_{j-1}(d), \tag{5.11}$$

with $a_0(d) \equiv 1$, in case for example that $\delta_x - \delta_y > 0$,

$$S_n = \sum_{t=1}^{n} x_t(\delta_y) = \sum_{t=1}^{n} v_{2t} \sum_{j=0}^{n-t} a_j(\delta_x - \delta_y) = \sum_{t=1}^{n} a_{n-t}(\delta_x - \delta_y + 1)\, v_{2t}, \tag{5.12}$$

so S_n is a Type II fractionally integrated process $I(\delta_x - \delta_y + 1)$. Hence, Theorem 1 and Corollary 1 of Marinucci and Robinson (2000) would straightforwardly imply that

$$n^{-(\delta_x-\delta_y)}\widehat{t}(a) \rightarrow {}_dN\left(0, \frac{a_2^2 2\pi g_{22}(0)}{a'g(0)\,a\Gamma(\delta_x-\delta_y+1)\left(2(\delta_x-\delta_y)+1\right)}\right), \text{ if } \delta_x - \delta_y > 0, \tag{5.13}$$

noting that $\max\{2, 2/(2(\delta_x - \delta_y) + 1)\} = 2$, so that our Assumption 5.1(i) implies Assumption B in Marinucci and Robinson (2000). Similarly, it is straightforward to show

$$n^{-(\delta_y-\delta_x)}\widehat{t}(a) \rightarrow {}_dN\left(0, \frac{a_1^2 2\pi g_{11}(0)}{a'g(0)\,a\Gamma(\delta_y-\delta_x+1)\left(2(\delta_y-\delta_x)+1\right)}\right), \text{ if } \delta_y - \delta_x > 0, \tag{5.14}$$

hence (5.10) is justified.

Note also that we could have proposed instead

$$\widehat{t}^2(a) = \frac{a' I_{z(\widehat{\delta}_x, \widehat{\delta}_y)}(0)\, a}{a' \widehat{g}(0)\, a}, \tag{5.15}$$

where by straightforward application of the continuous mapping theorem and Theorem 5.1,

$$\begin{aligned} \widehat{t}^2(a) &\rightarrow {}_d\chi_1^2 \text{ under } H_0, & (5.16)\\ \widehat{t}^2(a) &\sim n^{2|\delta_x - \delta_y|} \text{ under } H_1. & (5.17) \end{aligned}$$

Remark 5.2. This test procedure is also valid in case the processes y_t and x_t are co-integrated. The main consequence of co-integration can be analysed in the simple model (1.25), (1.26), which has been discussed in all previous chapters. This model implies that

$$\begin{pmatrix} \Delta^\delta y_t \\ \Delta^\delta x_t \end{pmatrix} = \begin{pmatrix} (1-L)^{\delta-\gamma} & \nu \\ 0 & 1 \end{pmatrix} \{u_t 1(t>0)\}, \tag{5.18}$$

where the spectral density (not constant over t) of the bivariate asymptotically stationary process on the right hand side of (5.18) is singular at frequency zero. Thus, the main implication of co-integration between two $I(\delta)$ processes is that the bivariate process resulting from $\delta-$differencing both time series has singular spectral density matrix at frequency zero, and this is precisely the reason why the different semiparametric tests considered in the literature are not valid with co-integration, as they require inversion of a matrix which tends in probability to a singular matrix, which is the equivalent to $g(0)$ in a more general framework. Fortunately, in our case, although $g(0)$ could be singular, this does not prevent the condition $a'g(0)\,a > 0$ from holding for a certain deterministic vector a.

Remark 5.3. There is an element of arbitrariness in the test procedure due to the choice of a, as different a's could lead to different decisions in a given situation. We consider this arbitrariness to be similar to certain extent to the one present in Robinson and Yajima (2002) related to their choice of the additional bandwidth $h(n)$ to account for possible co-integration between the series (see Section 2.5 of Chapter 2). As it will become clear in the Monte Carlo section, a sensible approach in order to improve the power of the test is to give less relative weight to the overdifferenced process. This basic idea is captured in a certainly radical way by the choice for a

$$\widetilde{a} = \left(1(\widehat{\delta}_x - \widehat{\delta}_y \le n^{-\eta}),\, 1(\widehat{\delta}_x - \widehat{\delta}_y > n^{-\eta})\right)', \tag{5.19}$$

for a certain $0 < \eta < \kappa$, although, due to the stochastic nature of $\widetilde{a}$, the asymptotics in Theorem 5.1 are not directly applicable to this choice of a.

Theorem 5.2. *Let (5.1), (5.2), (5.3) and Assumptions 5.1, 5.2 hold. Then, as* $n \rightarrow \infty$

$$\widehat{t}(\widetilde{a}) = \frac{w_{y(\widehat{\delta}_x)}(0)}{\widehat{g}_{11}^{1/2}(0)} + o_p\left(n^{\delta_y - \delta_x}\right) \text{ if } \delta_x < \delta_y, \tag{5.20}$$

$$= \frac{w_{y(\widehat{\delta}_x)}(0)}{\widehat{g}_{11}^{1/2}(0)} + o_p(1) \;\; if\, \delta_x = \delta_y, \tag{5.21}$$

$$= \frac{w_{x(\widehat{\delta}_y)}(0)}{\widehat{g}_{22}^{1/2}(0)} + o_p\left(n^{\delta_x - \delta_y}\right) \;\; if\, \delta_x > \delta_y. \tag{5.22}$$

The proof of this theorem is left to the Appendix 5. In view of Theorem 5.1, the main consequence of this result is that $\widehat{t}(\widetilde{a}) \to_d N(0,1)$ under H_0 and $\widehat{t}(\widetilde{a}) \sim n^{|\delta_x - \delta_y|}$ under H_1, but $\widehat{t}(\widetilde{a})$ has also other desirable properties. Roughly speaking, under H_1, the test statistic is going to be based simply on the underdifferenced processes $y_t(\widehat{\delta}_x)$ or $x_t(\widehat{\delta}_y)$ depending on whether $\delta_x < \delta_y$ or $\delta_x > \delta_y$ respectively. Under H_0, the test statistic is asymptotically equivalent to $w_{y(\widehat{\delta}_x)}(0) / \widehat{g}_{11}^{1/2}(0)$, but it could have been equally based on $w_{x(\widehat{\delta}_y)}(0) / \widehat{g}_{22}^{1/2}(0)$, just considering

$$\overline{a} = \left(1(\widehat{\delta}_y - \widehat{\delta}_x \leq n^{-\eta}),\, 1(\widehat{\delta}_y - \widehat{\delta}_x > n^{-\eta})\right)', \tag{5.23}$$

instead of $\widetilde{a}$.

Remark 5.4. In case the series y_t and x_t are not co-integrated, an alternative test statistic to consider is

$$\widetilde{t} = w'_{z(\widehat{\delta}_x, \widehat{\delta}_y)}(0)\, \widehat{g}(0)^{-1}\, w_{z(\widehat{\delta}_x, \widehat{\delta}_y)}(0).$$

Theorem 5.3. *Let (5.1), (5.2), (5.3) and Assumptions 5.1, 5.2 hold. Then, if* $g(0)$ *is nonsingular, as* $n \to \infty$,

$$\widetilde{t} \;\to\; {}_d\chi_2^2 \text{under } H_0, \tag{5.24}$$

$$\widetilde{t} \;\sim\; n^{2|\delta_x - \delta_y|} \text{under } H_1. \tag{5.25}$$

The proof is omitted as it is a straightforward application of the continuous mapping theorem and the proof of Theorem 5.1. Using $\widetilde{t}$ instead of $\widehat{t}^2(a)$ we avoid the arbitrariness due to the choice of a, but the test is invalid in case the series are co-integrated. In fact, the introduction of the user-chosen number $h(n)$ in Robinson and Yajima (2002) was also due to the possible co-integration between the series, as this was making invalid the standard test procedures based on normalized estimates of $\delta_x - \delta_y$.

Furthermore, $\widetilde{t}$ has an interesting interpretation in terms of our original class of test statistics as $\widetilde{t} = \widehat{t}^2(\widehat{a})$ with

$$\widehat{a} = \widehat{g}(0)^{-1}\, w_{z(\widehat{\delta}_x, \widehat{\delta}_y)}(0), \tag{5.26}$$

being the asymptotic distributions of $\widehat{t}^2(\widehat{a})$ and $\widehat{t}^2(a)$ for any deterministic a under H_0 different due to the randomness of $\widehat{a}$. Also, for any $a \neq 0$,

$$\widetilde{t} - \widehat{t}^2(a) = \widehat{b}'\widehat{b} \geq 0, \tag{5.27}$$

with

$$\widehat{b}=\left[I_n-\widehat{g}(0)^{\frac{1}{2}}a\left(a'\widehat{g}(0)a\right)^{-1}a'\widehat{g}(0)^{\frac{1}{2}}\right]\widehat{g}(0)^{-\frac{1}{2}}w_{z(\widehat{\delta}_x,\widehat{\delta}_y)}(0), \tag{5.28}$$

so $\widehat{a}$ is the particular value of a that maximizes $\widehat{t}^2(a)$, but we could not refer to $\widehat{a}$ as the "optimal" choice of a (in terms of maximizing the power of the test), as $\widetilde{t}$ and $\widehat{t}^2(a)$ are not directly comparable due to their different asymptotic distributions under the null.

Remark 5.5. In case we want to perform a test of equality of the orders against one-sided alternatives

$$H_1^1:\delta_x>\delta_y,\text{ or }H_1^2:\delta_x<\delta_y, \tag{5.29}$$

we could use instead the test statistics

$$\widehat{t}_1=\frac{w_{x(\widehat{\delta}_y)}(0)}{\widehat{g}_{22}^{1/2}(0)},\ \widehat{t}_2=\frac{w_{y(\widehat{\delta}_x)}(0)}{\widehat{g}_{11}^{1/2}(0)}, \tag{5.30}$$

respectively, where no choice of the weighting vector a is required. It is straightforward to show that, as $n\to\infty$,

$$\begin{aligned}\widehat{t}_1,\widehat{t}_2 &\to {}_dN(0,1)\text{ under }H_0, &(5.31)\\ \widehat{t}_1 &\sim n^{\delta_x-\delta_y}\text{ under }H_1^1, &(5.32)\\ \widehat{t}_2 &\sim n^{\delta_y-\delta_x}\text{ under }H_1^2. &(5.33)\end{aligned}$$

Remark 5.6. Our test procedure could be also applied to test for the dimension of the co-integrating gap (defined as $\delta-\gamma$ in (1.25), (1.26)). Here, most of the literature has been based on the case where $\delta=1$, $\gamma=0$ in similar models to (1.25), (1.26). In the more general framework of fractional co-integration, in view of Chapters 2, 3, 4 of this thesis, and also Kim and Phillips (2000),Velasco (2000), it seems to be very relevant for estimation and testing whether the co-integrating gap is "big" (more precisely $\delta-\gamma>1/2$) or "small" (with $\delta-\gamma<1/2$). We have denoted these two situations as strong and weak fractional co-integration respectively. For example, a test for "strong" co-integration could be set in our framework as

$$H_0:\delta-\gamma=\frac{1}{2},\text{ against }H_1:\delta-\gamma>\frac{1}{2}, \tag{5.34}$$

noting that we face an additional problem here, as u_{1t} in (1.25) is not observable, although we could use instead a proxy like $y_t-\widehat{\nu}x_t$ for certain estimate $\widehat{\nu}$ of ν, which could be for example the OLS or the NBLS estimate, whose asymptotic properties were discussed in Chapter 1.

Remark 5.7. The idea of testing for the equality of the orders of integration of a bivariate fractionally integrated process can be easily extended to test for the validity of any linear restriction among the orders of integration of the elements of any $p\times 1$ vector $q_t=(q_{1t},...,q_{pt})'$ where $q_{it}(\delta_i)=v_{it}$, $\delta_i>0$, $i=1,...,p$, and $v_t=(v_{1t},...,v_{pt})'$ is an $I(0)$ vector process for which the equivalent of Assumption 5.1 holds. Denoting by $\delta=(\delta_1,\delta_{(1)}')'$, where $\delta_{(1)}=(\delta_2,...,\delta_p)'$, without loss of generality, we could test for any linear restriction among two or more elements of δ by means of $H_0 : b'\delta = r$ against $H_1 : b'\delta \neq r$, where $b = (1, b'_{(1)})'$, $b_{(1)}$ being certain $(p-1) \times 1$ deterministic vector and r a real number. Thus, denoting

$$q(c_1, ..., c_p) = (q_1(c_1), ..., q_p(c_p))', \tag{5.35}$$

given certain estimate $\widehat{\delta}$ of δ satisfying the equivalent of Assumption 5.2, for any $p \times 1$ deterministic vector a, defining now

$$\widehat{t}\,(a) = \frac{a' w_{q(r - b'_{(1)}\widehat{\delta}_{(1)}, \widehat{\delta}_2 + b'\widehat{\delta}, ..., \widehat{\delta}_p + b'\widehat{\delta})}\,(0)}{\left(a'\widehat{g}\,(0)\,a\right)^{1/2}}, \tag{5.36}$$

it is straightforward to show, as in Theorem 5.1, that

$$\begin{aligned} \widehat{t}\,(a) &\rightarrow {}_dN\,(0,1)\; under\; H_0, & (5.37)\\ \widehat{t}\,(a) &\sim n^{|b'\delta - r|} under\; H_1. & (5.38) \end{aligned}$$

5.3 Monte Carlo evidence

With the aim of assessing for the finite sample behavior of the test procedure presented in the previous section, we performed a small Monte Carlo experiment. We generated y_t and x_t as in (5.1), (5.2), v_t being a bivariate Gaussian white noise with covariance matrix

$$\Omega = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \tag{5.39}$$

where results are displayed for $\rho = 0, .5, -.5, 1$, reflecting this last case the situation where y_t and x_t are co-integrated. We computed the test statistic $\widehat{t}^2\,(\cdot)$ evaluated at five different weighting vectors: $a_1 = (1,1)'$, $a_2 = (1,4)'$, $a_3 = (1,.25)'$, $\widehat{a}$ as in (5.26) and $\widetilde{a}$ given by (5.19) with $\eta = 0.3$. Note that the test statistic with $\widehat{a}$ for the case $\rho = 1$ is invalid, so we do not report results for this situation. All those test statistics are obtained under parametric or nonparametric assumptions. In the former case, we assumed knowledge of the white noise condition of v_t (but of course not of Ω), and the orders δ_x, δ_y and $g\,(0)$ were estimated parametrically by means of the procedure described in Beran (1995) and

$$\widehat{g}\,(0) = \frac{1}{2\pi n} \sum_{t=1}^{n} \widehat{v}_t \widehat{v}'_t, \tag{5.40}$$

respectively. In the nonparametric case (without considering v_t to be white noise), we computed the estimates of the orders as the Robinson's (1995a) version of the Geweke and Porter-Hudak (1983) log-periodogram estimate without pooling or trimming, with bandwidth parameter m taking values to be described subsequently. The nonparametric estimate of $g\,(0)$ was

$$\widehat{g}\,(0) = \frac{1}{2b+1} \sum_{j=-b}^{b} I_{\widehat{v}}\,(\lambda_j)\,, \tag{5.41}$$

for a certain bandwidth b. We gave results for 1000 replications and three different sample sizes $n = 64, 128, 256$, for which we chose bandwidths $m = 20, 30, 60$ and $b = 3, 6, 10$ respectively.

We computed the empirical sizes of the different test statistics described above corresponding to the nominal ones $\alpha = .01, .05, .10$, for different combinations of δ_x, δ_y. Denoting $\phi = \delta_y - \delta_x$, we considered only the case $\delta_y = 0.4$, without loss of generality, as our test procedure is invariant to the particular values of δ_x, δ_y, depending only on ϕ, this difference taking values $\phi = 0, 0.1, 0.2, 0.3$.

Results are reported in Tables 5.1-5.8. In terms of comparison of the empirical sizes with the nominal ones, looking at $\phi = 0$, the first thing to be noted is that there exists certain differences on the performance of the statistics depending on whether the weighting vector is deterministic (a_i, $i = 1, 2, 3$) or stochastic. In the former case, the behavior is reasonably good and quite similar for the three vectors considered, with empirical sizes being on average too large (except for the parametric statistic with $\rho = 1$), but in general moving in the appropriate direction as the sample size increases. The behavior for the statistics with the two stochastic weighting vectors is also relatively similar to each other for $\rho = 0, 0.5$, being for these cases in general sizes bigger than for the statistics with deterministic weighting, but clearly the one based on $\widetilde{a}$ behaving better than the one based on $\widehat{a}$, tests based on $\widetilde{a}$ being comparable to the ones based on deterministic a for the cases $\rho = -0.5$, 1. As expected, sizes are closer to the nominal ones for statistics derived from parametric estimates, than for the nonparametric ones, being generally this difference accentuated the large ρ is. Also, sizes are larger for smaller ρ for the deterministically weighted statistics. Finally, it is remarkably clear that the smallest sizes correspond to the case of co-integration between x_t and y_t.

We also looked at the power related to the different test statistics by means of letting $\phi \neq 0$. The power increases as ϕ and n grow, being $\widehat{t}^2(\widetilde{a})$ the most powerful one followed by $\widehat{t}^2(\widehat{a})$ and $\widehat{t}^2(a_3)$ which behave quite similarly. A very striking feature of the experiment is the importance of the choice of a in order to obtain a test with good power. As mentioned before, a sensible approach is to give more weight to the process we believe is underdifferenced. If the contrary happens, the effect is dramatic, as can be observed in the results for $\widehat{t}^2(a_2)$. Here, in most of the cases, the test have negligible power, with the exception of the case $\phi = 0.3$. Very noticeable are the similarities in power of the parametric and nonparametric test statistics and also the increase in power as ρ decreases.

Overall, it seems that the best test statistic is the one based on $\widetilde{a}$, as it behaves reasonably well in terms of empirical sizes, being in general the best in terms of power. This statistic is also the most realistic one, as our decision about to which process give more weight will be based on the comparison of the estimates of the orders of integration, and this additional randomness source is not taken into account by the statistics based on deterministic weighting vectors.

5.4 Appendix 5

Proof of Theorem 5.1. Defining

$$t(a) = \frac{a' w_{z(\delta_x,\delta_y)}(0)}{(a' g(0) a)^{\frac{1}{2}}}, \tag{5.42}$$

it is clear that under Assumption 5.1, by Central Limit Theorem (see Hannan, 1970)

$$t(a) \to_d N(0,1) \text{ under } H_0, \tag{5.43}$$

(5.10) following by Marinucci and Robinson (2000) as in Remark 5.1, noting that the overdifferenced process under H_1 has smaller order.

Thus, the main task is to prove that

$$\begin{aligned} \widehat{t}(a) - t(a) &= o_p(1) \text{ under } H_0, & (5.44)\\ &= o_p\left(n^{|\delta_x-\delta_y|}\right) \text{ under } H_1, & (5.45)\end{aligned}$$

which, as by Assumption 5.2, $\widehat{g}(0)$ is a consistent estimate of $g(0)$, follows immediately from proving

$$\begin{aligned} w_{z(\widehat{\delta}_x,\widehat{\delta}_y)}(0) - w_{z(\delta_x,\delta_y)}(0) &= o_p(1) \text{ under } H_0, & (5.46)\\ &= o_p\left(n^{|\delta_x-\delta_y|}\right) \text{ under } H_1. & (5.47)\end{aligned}$$

Now, by Taylor's expansion, for certain constant R to be defined subsequently, the left hand side of (5.46) is

$$\begin{aligned} &\frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_{r=1}^{R-1} \frac{1}{r!} \begin{pmatrix} (\delta_x - \widehat{\delta}_x)^r & 0 \\ 0 & (\delta_y - \widehat{\delta}_y)^r \end{pmatrix} \sum_{t=2}^{n} g^{(r)}(v_t; \delta_y - \delta_x, \delta_x - \delta_y) \\ &+ \frac{1}{(2\pi n)^{\frac{1}{2}} R!} \begin{pmatrix} (\delta_x - \widehat{\delta}_x)^R & 0 \\ 0 & (\delta_y - \widehat{\delta}_y)^R \end{pmatrix} \sum_{t=2}^{n} g^{(R)}(v_t; \delta_y - \overline{\delta}_x, \delta_x - \overline{\delta}_y), \end{aligned} \tag{5.48}$$

where $|\overline{\delta}_x - \delta_y| \le |\widehat{\delta}_x - \delta_y|$, $|\overline{\delta}_y - \delta_x| \le |\widehat{\delta}_y - \delta_x|$, for any scalar or vector sequence ψ_t and any real b,

$$g^{(r)}(\psi_t; b) = \sum_{s=1}^{t-1} a_s^{(r)}(b)\, \psi_{t-s}, \tag{5.49}$$

with

$$a_s^{(r)}(b) = \frac{d^r a_s(b)}{db^r}, \tag{5.50}$$

and for any p-dimensional vector ξ_t and real $b_1, ..., b_p$,

$$g^{(r)}(\xi_t; b_1, ..., b_p) = \left(g^{(r)}(\xi_{1t}; b_1), ..., g^{(r)}(\xi_{pt}; b_p)\right)'. \tag{5.51}$$

First, we prove (5.46). Now

$$Var\left(\sum_{t=2}^{n} g^{(r)}\left(v_t;0\right)\right)=\int_{-\pi}^{\pi}\sum_{t=2}^{n}\sum_{j=1}^{t-1}\sum_{s=2}^{n}\sum_{k=1}^{s-1} a_j^{(r)}\left(0\right)a_k^{(r)}\left(0\right)e^{i(t-j-(s-k))\mu}g\left(\mu\right)d\mu, \tag{5.52}$$

which by Assumption 5.1 is bounded in norm by

$$K\int_{-\pi}^{\pi}\left|\sum_{t=2}^{n}\sum_{j=1}^{t-1} a_j^{(r)}\left(0\right)e^{i(t-j)\mu}\right|^2 d\mu \leq K\sum_{t=2}^{n}\sum_{j=1}^{t-1}\sum_{k=1}^{n+j-t}\left|a_j^{(r)}\left(0\right)a_k^{(r)}\left(0\right)\right|. \tag{5.53}$$

Now, by the bounds in Lemma 2.D.4, (5.53) is bounded by

$$K\sum_{t=2}^{n}\sum_{j=1}^{t-1}\sum_{k=1}^{n+j-t}\frac{\left(\log\left(j+k\right)\right)^{r-1}}{jk}\leq Kn\left(\log n\right)^{2r}, \tag{5.54}$$

implying that

$$\sum_{t=2}^{n} g^{(r)}\left(v_t;0\right)=O_p\left(n^{\frac{1}{2}}\left(\log n\right)^r\right), \tag{5.55}$$

and therefore the first term in (5.48) is $O_p\left(n^{-\kappa}\log n\right)$.

Next, by Lemma 2.C.4

$$g^{(R)}\left(v_t;\delta_y-\overline{\delta}_x,\delta_x-\overline{\delta}_y\right)=O_p\left(t^{\frac{1}{2}}\right), \tag{5.56}$$

so that the second term in (5.48) is $O_p\left(n^{3/2-R\kappa}\right)$, and choosing $R>(1+\kappa)/\kappa$, (5.48) is $O_p\left(n^{-\kappa}\log n\right)$, and we conclude (5.46) by Assumption 5.2.

Regarding (5.47), by previous arguments $Var\left(\sum_{t=2}^{n} g^{(r)}\left(v_t;\delta_y-\delta_x,\delta_x-\delta_y\right)\right)$ is bounded in norm by

$$\begin{aligned}&K\sum_{t=2}^{n}\sum_{j=1}^{t-1}\sum_{k=1}^{n+j-t}\left\{\left|a_j^{(r)}\left(\delta_y-\delta_x\right)\right|+\left|a_j^{(r)}\left(\delta_x-\delta_y\right)\right|\right\}\\&\times\left\{\left|a_k^{(r)}\left(\delta_y-\delta_x\right)\right|+\left|a_k^{(r)}\left(\delta_x-\delta_y\right)\right|\right\},\end{aligned} \tag{5.57}$$

which by Lemma 2.C.1 is bounded by

$$K\sum_{t=2}^{n}\sum_{j=1}^{t-1}\sum_{k=1}^{n+j-t}\left(\log\left(j+k\right)\right)^r\left(jk\right)^{\left|\delta_x-\delta_y\right|-1}\leq K\left(\log n\right)^{2r}n^{2\left|\delta_x-\delta_y\right|+1}, \tag{5.58}$$

and therefore

$$\sum_{t=2}^{n} g^{(r)}\left(v_t;\delta_y-\delta_x,\delta_x-\delta_y\right)=O_p\left(\left(\log n\right)^r n^{\left|\delta_x-\delta_y\right|+\frac{1}{2}}\right), \tag{5.59}$$

noting that by a straightforward modification of this lemma and the Stirling's approximation, for any $c<0$, $s\geq 1$,

$$\left|a_s^{(r)}\left(c\right)\right|\leq K\left(\log s\right)^r s^{c-1}. \tag{5.60}$$

Next, as in Lemma 2.C.4 of Chapter 2, for any $\epsilon > 0$,

$$E\left\|g^{(R)}\left(v_t; \delta_y - \overline{\delta}_x, \delta_x - \overline{\delta}_y\right)\right\|^2 \leq K \sum_{s=1}^{t} (\log s)^{2R} s^{2(|\delta_x - \delta_y| + \epsilon - 1)}, \tag{5.61}$$

implying that

$$g^{(R)}\left(v_t; \delta_y - \overline{\delta}_x, \delta_x - \overline{\delta}_y\right) = O_p\left(t^{\max\left\{|\delta_x - \delta_y|, \frac{1}{2}\right\} + \epsilon}\right), \tag{5.62}$$

and choosing $R > \left(\max\left\{|\delta_x - \delta_y|, \frac{1}{2}\right\} + 1/2 + \kappa\right)/\kappa$, (5.48) is $O_p\left(n^{|\delta_x - \delta_y| - \kappa} \log n\right)$ under H_1 to conclude the proof of (5.47).

Proof of Theorem 5.2. The result follows immediately from showing

$$\begin{aligned} \widetilde{a} &\rightarrow {}_p (1,0)' \text{ for } \delta_x \leq \delta_y, & (5.63)\\ \widetilde{a} &\rightarrow {}_p (0,1)' \text{ for } \delta_x > \delta_y, & (5.64) \end{aligned}$$

which as

$$1(\widehat{\delta}_x - \widehat{\delta}_y \leq n^{-\eta}) + 1(\widehat{\delta}_x - \widehat{\delta}_y > n^{-\eta}) = 1, \tag{5.65}$$

follows from proving

$$\begin{aligned} 1(\widehat{\delta}_x - \widehat{\delta}_y &> n^{-\eta}) \rightarrow_p 0 \text{ for } \delta_x \leq \delta_y, & (5.66)\\ 1(\widehat{\delta}_x - \widehat{\delta}_y &\leq n^{-\eta}) \rightarrow_p 0 \text{ for } \delta_x > \delta_y. & (5.67) \end{aligned}$$

First, for $\delta_x \leq \delta_y$,

$$\begin{aligned} 1(\widehat{\delta}_x - \widehat{\delta}_y &> n^{-\eta}) = 1(\widehat{\delta}_x - \widehat{\delta}_y - (\delta_x - \delta_y) > n^{-\eta} - (\delta_x - \delta_y)) \\ &\leq \frac{\left|\widehat{\delta}_x - \widehat{\delta}_y - (\delta_x - \delta_y)\right|}{n^{-\eta} - (\delta_x - \delta_y)}, & (5.68) \end{aligned}$$

so that $1(\widehat{\delta}_x - \widehat{\delta}_y > n^{-\eta}) = O_p\left(n^{\eta - \kappa}\right)$ to conclude for (5.66) as $\eta < \kappa$.

Next, for $\delta_x > \delta_y$, noting that

$$1(\widehat{\delta}_x - \widehat{\delta}_y \leq n^{-\eta}) = 1(\left|\widehat{\delta}_x - \widehat{\delta}_y\right| \leq n^{-\eta}) + 1(\widehat{\delta}_y - \widehat{\delta}_x > n^{-\eta}), \tag{5.69}$$

the left side of (5.69) is bounded by

$$\begin{aligned} &\frac{n^{-\eta}}{\left|\widehat{\delta}_x - \widehat{\delta}_y\right|} + 1\left(\widehat{\delta}_y - \widehat{\delta}_x - (\delta_y - \delta_x) > n^{-\eta} - (\delta_y - \delta_x)\right) \\ \leq\ &\frac{n^{-\eta}}{\left|\widehat{\delta}_x - \widehat{\delta}_y\right|} + \frac{\left|\widehat{\delta}_y - \widehat{\delta}_x - (\delta_y - \delta_x)\right|}{n^{-\eta} + \delta_x - \delta_y}, & (5.70) \end{aligned}$$

so that $1(\widehat{\delta}_x - \widehat{\delta}_y \leq n^{-\eta}) = O_p\left(n^{-\eta} + n^{-\kappa}\right)$ to conclude the proof.

TABLE 5.1

EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = 0$, PARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
.01	a_1	.036	.054	.143	.279	.034	.078	.189	.354	.015	.064	.223	.434
	a_2	.045	.010	.010	.023	.033	.004	.008	.036	.021	.003	.003	.058
	a_3	.047	.127	.259	.420	.047	.145	.305	.501	.021	.150	.366	.585
	$\widehat{a}$	.062	.095	.203	.364	.052	.122	.257	.447	.024	.121	.305	.526
	$\widetilde{a}$	.067	.140	.268	.432	.055	.148	.317	.516	.027	.163	.381	.592
.05	a_1	.106	.147	.250	.405	.096	.170	.300	.477	.070	.154	.346	.569
	a_2	.093	.037	.023	.050	.093	.024	.028	.089	.068	.008	.019	.125
	a_3	.094	.220	.381	.523	.110	.236	.428	.615	.076	.259	.475	.679
	$\widehat{a}$	.133	.177	.312	.449	.115	.186	.356	.550	.076	.203	.406	.613
	$\widetilde{a}$	.117	.233	.390	.531	.122	.255	.452	.620	.083	.266	.485	.694
.10	a_1	.175	.228	.337	.476	.175	.235	.371	.557	.123	.231	.418	.632
	a_2	.155	.071	.051	.076	.147	.065	.054	.151	.121	.032	.052	.189
	a_3	.159	.292	.439	.585	.158	.312	.503	.682	.133	.333	.561	.731
	$\widehat{a}$	.188	.249	.371	.507	.189	.255	.416	.593	.141	.259	.457	.661
	$\widetilde{a}$	.184	.311	.460	.596	.165	.326	.518	.686	.145	.355	.566	.735

TABLE 5.2

EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = 0$, NONPARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
.01	a_1	.066	.081	.159	.268	.066	.106	.216	.353	.045	.075	.221	.432
	a_2	.075	.036	.024	.033	.077	.036	.021	.051	.055	.012	.015	.060
	a_3	.083	.157	.271	.395	.084	.182	.318	.477	.047	.153	.340	.578
	$\widehat{a}$	.140	.175	.257	.374	.134	.180	.309	.446	.084	.133	.300	.533
	$\widetilde{a}$	.131	.182	.290	.409	.134	.216	.334	.497	.074	.165	.359	.586
.05	a_1	.131	.157	.249	.387	.145	.189	.304	.460	.104	.168	.329	.565
	a_2	.133	.075	.053	.064	.133	.068	.055	.113	.107	.037	.044	.119
	a_3	.136	.239	.368	.500	.141	.280	.424	.579	.097	.253	.482	.662
	$\widehat{a}$	.220	.243	.340	.465	.216	.273	.374	.530	.146	.209	.399	.619
	$\widetilde{a}$	.201	.269	.382	.523	.204	.310	.443	.588	.132	.265	.497	.668
.10	a_1	.195	.243	.332	.468	.199	.266	.373	.532	.162	.225	.420	.624
	a_2	.164	.106	.084	.112	.179	.101	.103	.161	.159	.075	.073	.186
	a_3	.188	.303	.435	.559	.204	.333	.487	.644	.146	.323	.559	.715
	$\widehat{a}$	.274	.308	.396	.518	.266	.324	.437	.582	.197	.283	.469	.651
	$\widetilde{a}$	.248	.347	.456	.573	.268	.366	.512	.652	.175	.340	.560	.725

TABLE 5.3

EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = .5$, PARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.011	.024	.094	.216	.008	.035	.123	.289	.009	.027	.164	.358
	a_2	.028	.005	.004	.009	.025	.002	.002	.017	.016	.000	.000	.026
.01	a_3	.028	.099	.228	.382	.023	.108	.275	.469	.011	.122	.314	.545
	$\widehat{a}$	.068	.129	.248	.417	.049	.114	.301	.513	.030	.137	.349	.587
	$\widetilde{a}$	.048	.136	.265	.435	.036	.131	.329	.526	.017	.158	.366	.602
	a_1	.054	.096	.207	.337	.068	.117	.241	.420	.047	.112	.271	.485
	a_2	.067	.025	.009	.034	.071	.013	.013	.060	.050	.013	.015	.107
.05	a_3	.083	.199	.346	.495	.080	.210	.409	.591	.059	.225	.442	.660
	$\widehat{a}$	.139	.202	.338	.504	.116	.203	.409	.593	.078	.224	.452	.670
	$\widetilde{a}$	.113	.233	.393	.544	.094	.259	.458	.633	.078	.264	.496	.688
	a_1	.133	.176	.283	.430	.139	.187	.320	.499	.111	.186	.352	.573
	a_2	.131	.052	.027	.071	.131	.050	.044	.118	.105	.033	.047	.164
.10	a_3	.137	.276	.425	.568	.134	.293	.489	.661	.124	.299	.516	.703
	$\widehat{a}$	.203	.264	.413	.560	.178	.279	.471	.652	.148	.285	.505	.701
	$\widetilde{a}$	.169	.306	.459	.615	.140	.339	.534	.694	.138	.344	.567	.740

TABLE 5.4

EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = .5$, NONPARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.029	.058	.121	.201	.053	.073	.152	.294	.035	.067	.157	.361
	a_2	.060	.029	.014	.024	.058	.025	.014	.039	.046	.010	.010	.048
.01	a_3	.050	.135	.229	.344	.064	.148	.290	.456	.046	.132	.312	.540
	$\widehat{a}$	.141	.181	.279	.400	.139	.184	.329	.480	.088	.164	.346	.572
	$\widetilde{a}$	.122	.183	.279	.399	.122	.187	.355	.496	.073	.164	.365	.573
	a_1	.111	.152	.207	.324	.112	.148	.258	.425	.084	.137	.276	.486
	a_2	.120	.062	.047	.059	.111	.054	.057	.095	.100	.038	.043	.111
.05	a_3	.124	.212	.329	.462	.122	.235	.407	.554	.100	.226	.440	.620
	$\widehat{a}$	.209	.259	.359	.485	.201	.266	.420	.570	.148	.245	.442	.648
	$\widetilde{a}$	.194	.263	.379	.507	.174	.288	.450	.610	.133	.269	.490	.675
	a_1	.186	.204	.288	.394	.179	.208	.326	.481	.146	.187	.354	.555
	a_2	.164	.096	.080	.111	.163	.092	.094	.146	.148	.079	.082	.160
.10	a_3	.188	.284	.394	.538	.174	.322	.471	.633	.137	.306	.516	.688
	$\widehat{a}$	.278	.319	.413	.545	.261	.330	.463	.623	.212	.299	.501	.693
	$\widetilde{a}$	.256	.321	.444	.583	.230	.363	.507	.671	.183	.344	.550	.722

TABLE 5.5

EMPIRICAL SIZES OF $\widehat{t}^2(\cdot)$ FOR $\rho = -.5$, PARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.061	.099	.224	.388	.058	.119	.285	.479	.026	.121	.329	.576
	a_2	.048	.008	.007	.017	.036	.005	.006	.035	.029	.003	.003	.066
.01	a_3	.045	.139	.302	.455	.044	.157	.339	.547	.025	.173	.407	.628
	$\widehat{a}$	.062	.103	.236	.406	.051	.129	.291	.492	.030	.132	.336	.580
	$\widetilde{a}$	.040	.117	.264	.427	.040	.140	.306	.512	.016	.155	.362	.595
	a_1	.142	.184	.343	.512	.118	.213	.396	.613	.087	.216	.468	.677
	a_2	.094	.036	.020	.044	.099	.024	.020	.096	.062	.007	.019	.139
.05	a_3	.100	.241	.411	.569	.111	.266	.470	.666	.078	.283	.530	.716
	$\widehat{a}$	.127	.191	.337	.495	.119	.207	.376	.590	.078	.219	.446	.652
	$\widetilde{a}$	.092	.222	.378	.532	.104	.247	.435	.645	.071	.272	.499	.686
	a_1	.204	.265	.409	.572	.181	.286	.463	.673	.152	.289	.535	.725
	a_2	.151	.067	.032	.075	.149	.041	.039	.133	.115	.025	.039	.200
.10	a_3	.147	.323	.486	.615	.157	.339	.551	.711	.137	.353	.598	.753
	$\widehat{a}$	.185	.257	.408	.551	.187	.272	.445	.652	.138	.277	.504	.704
	$\widetilde{a}$	.150	.301	.456	.589	.149	.312	.507	.691	.129	.337	.572	.736

TABLE 5.6

EMPIRICAL SIZES OF $\widehat{t}^2(\cdot)$ FOR $\rho = -.5$, NONPARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.078	.106	.216	.360	.078	.153	.280	.452	.058	.127	.329	.572
	a_2	.092	.044	.021	.033	.085	.027	.023	.058	.061	.012	.011	.077
.01	a_3	.089	.173	.292	.422	.100	.212	.355	.505	.059	.193	.412	.619
	$\widehat{a}$	.148	.187	.275	.404	.149	.205	.327	.467	.091	.163	.356	.576
	$\widetilde{a}$	.140	.178	.277	.392	.138	.213	.333	.474	.068	.177	.375	.580
	a_1	.143	.188	.327	.472	.168	.236	.381	.574	.119	.206	.453	.666
	a_2	.137	.071	.047	.060	.137	.058	.045	.111	.099	.031	.033	.135
.05	a_3	.154	.259	.378	.529	.165	.299	.456	.614	.111	.278	.527	.703
	$\widehat{a}$	.236	.262	.375	.489	.230	.287	.405	.565	.150	.241	.454	.650
	$\widetilde{a}$	.198	.264	.367	.507	.215	.293	.432	.587	.131	.265	.499	.674
	a_1	.209	.254	.393	.536	.228	.303	.448	.631	.170	.274	.535	.714
	a_2	.177	.102	.072	.094	.186	.090	.076	.155	.152	.047	.065	.194
.10	a_3	.192	.316	.450	.585	.208	.361	.513	.670	.162	.358	.594	.741
	$\widehat{a}$	.287	.322	.417	.545	.299	.335	.457	.615	.202	.292	.514	.693
	$\widetilde{a}$	.252	.325	.447	.569	.267	.357	.495	.643	.180	.339	.558	.725

TABLE 5.7
EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = 1$, PARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.000	.000	.015	.118	.000	.000	.048	.240	.001	.006	.084	.294
	a_2	.000	.000	.000	.000	.001	.000	.000	.000	.002	.000	.000	.002
.01	a_3	.000	.021	.147	.347	.000	.045	.243	.449	.001	.062	.277	.509
	$\widetilde{a}$	.000	.060	.235	.435	.001	.089	.319	.542	.001	.111	.369	.607
	a_1	.013	.039	.118	.264	.022	.051	.197	.367	.030	.065	.220	.433
	a_2	.017	.000	.000	.000	.023	.002	.000	.014	.035	.003	.003	.051
.05	a_3	.023	.128	.297	.475	.027	.176	.376	.576	.036	.180	.403	.633
	$\widetilde{a}$	.028	.189	.394	.583	.029	.244	.472	.649	.037	.257	.490	.697
	a_1	.077	.109	.218	.369	.094	.140	.278	.464	.083	.139	.309	.511
	a_2	.081	.015	.006	.029	.092	.016	.017	.068	.077	.024	.033	.121
.10	a_3	.084	.222	.397	.563	.091	.267	.469	.636	.087	.273	.488	.686
	$\widetilde{a}$	.089	.279	.481	.646	.089	.332	.552	.708	.085	.345	.575	.739

TABLE 5.8
EMPIRICAL SIZES OF $\widehat{t^2}(\cdot)$ FOR $\rho = 1$, NONPARAMETRIC ESTIMATION

	n	64				128				256			
α	ϕ	0	.1	.2	.3	0	.1	.2	.3	0	.1	.2	.3
	a_1	.005	.030	.088	.174	.029	.049	.124	.254	.016	.041	.130	.300
	a_2	.025	.005	.001	.000	.028	.008	.006	.020	.016	.004	.004	.032
.01	a_3	.026	.092	.194	.337	.030	.121	.265	.431	.021	.116	.286	.488
	$\widetilde{a}$	.042	.123	.263	.391	.041	.156	.337	.503	.025	.160	.368	.568
	a_1	.101	.129	.184	.292	.100	.130	.226	.373	.077	.114	.244	.418
	a_2	.090	.031	.015	.032	.092	.036	.038	.078	.081	.026	.036	.106
.05	a_3	.089	.190	.317	.438	.109	.220	.379	.533	.080	.214	.402	.599
	$\widetilde{a}$	.098	.231	.376	.522	.115	.275	.449	.619	.080	.270	.483	.667
	a_1	.172	.192	.271	.363	.156	.200	.298	.445	.129	.185	.314	.491
	a_2	.165	.091	.066	.100	.145	.084	.081	.136	.133	.068	.078	.162
.10	a_3	.162	.267	.380	.513	.156	.297	.453	.607	.135	.288	.483	.652
	$\widetilde{a}$	.152	.309	.443	.592	.155	.337	.517	.684	.137	.340	.547	.723

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