SEASONAL AND CYCLICAL LONG-MEMORY IN TIME SERIES [1]

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Abstract

This thesis deals with the issue of persistence, focusing on economic time series, and extending the subject to seasonal and cyclical long memory time series. Such processes are defined. In the frequency domain they are characterized by spectral poles/zeros at some frequency ω between 0 and π.

First we review some of the work done to date on seasonality and long memory, and we focus on research that try to link both issues. One of the limitations of the existing work is the imposition of asymptotic symmetry in the spectral density around ω. We describe some processes that allow for spectral asymmetry around the frequency ω where the pole/zero occurs. They are naturally described in the frequency domain, and they imply two possibly different persistence parameters describing the behaviour of the spectrum to the right and left of ω. Two semiparametric methods of estimating the persistence parameters in the frequency domain, which have been proposed for the symmetric case $\omega = 0$ and are based on a partial knowledge of the spectral density around ω, are extended to $\omega \neq 0$ and their asymptotic properties are analysed. These are the log-periodogram regression and the local Whittle or Gaussian semiparametric estimates. Their performance in finite samples is studied via Monte Carlo analysis.

Some semiparametric Wald and LM type tests on the symmetry of the spectral density at ω and on the equality of persistence parameters at different frequencies are proposed, showing their good asymptotic properties. Their performance in finite samples is analysed through a small Monte Carlo study.

All these techniques are applied to a monthly UK inflation series from January 1915 to April 1996, where we test not only the symmetry of the spectral poles but also the equality of persistence parameters across seasonal frequencies.

Finally some concluding remarks and possible extensions are suggested.

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Chapter 1

GENERAL INTRODUCTION AND LITERATURE REVIEW

1.1 INTRODUCTION

The evolution of economic time series is often determined by different phenomena, such as weather, calendar events (e.g. Christmas, Easter) or timing decisions (industry vacations, tax years), which have a regular or quasi-regular behaviour (Hylleberg (1992)) that cause the cyclical movement we observe in many economic time series. Some of these phenomena are fixed and repeat exactly along time (e.g. Christmas) so that they are completely deterministic in the sense that they can be forecast with zero mean square error. Others, although varying over time, are also deterministic because they can be forecast perfectly (e.g. Easter). But many of them are time-varying and not perfectly predictable (e.g. weather), although their variation is small so that we can talk about quasi-regular behaviour. Different processes have been proposed to model different cyclical movements like those mentioned above. Some of them are described in Sections 1.2 and 1.3. But before going any further we define some basic concepts that will be used in this chapter and over the whole thesis.

1.1.1 Definitions and concepts

Let $\{x_t, t = 0, \pm 1, ...\}$ be a real and discrete covariance stationary process with mean μ and lag-j autocovariance γ_j,

$$\mu = Ex_t \quad , \quad \gamma_j = E(x_{t+j} - \mu)(x_t - \mu)$$

where E denotes the mathematical expectation. If x_t shows a regular or quasi-regular behaviour that causes cycles, then we call *period* of the cycle the time period (or number of observations if they are equally spanned) needed to complete the cycle. If we denote the period by z, then the frequency (in radians) of the cycle is $2\pi/z$. This implies that the lag-kz autocovariances, where k is an integer, are high in relation to neighbour autocovariances. Thus, cyclical behaviour of x_t will be reflected in the movements of γ_j with j. However a visual inspection of the autocovariances may not be very informative about the period or the frequency of the cycle, specially if the repetitive evolution of the series is not very regular, which is the typical case in most economic time series. When analyzing cyclical time series, the frequency domain is a more adequate framework than the time domain, since it reflects the cycle more clearly. The basic tool is the spectral distribution function which, although containing the same information as the autocovariances, gives a clearer view of the period and frequency that define the cycle. The relation between autocovariances, γ_j, and spectral distribution function, $F(\lambda)$, can be written in terms of Stieltjes integrals

$$\gamma_j = \int_{-\pi}^{\pi} \cos(j\lambda)\mathrm{d}F(\lambda) \quad , \quad j = 0, \pm 1, ... \tag{1.1}$$

where $F(\lambda)$ is a monotonically non-decreasing function, with symmetric increments (i.e. $\mathrm{d}F(\lambda) = \mathrm{d}F(-\lambda)$), with $F(-\pi) = 0$ and continuous from the right. Note that $\gamma_0 = \int_{-\pi}^{\pi} \mathrm{d}F(\lambda)$ so that $F(\lambda)$ gives a decomposition of the overall variance into components, each describing the variance due to a different frequency λ. The relationship (1.1) exists for all covariance stationary processes. When $F(\lambda)$ is absolutely continuous, there exists an even and continuous function, $f(\lambda)$, such that $\mathrm{d}F(\lambda) = f(\lambda)\mathrm{d}\lambda$. $f(\lambda)$ is the spectral density function, also called power spectrum or spectrum. Since, for x_t discrete, $f(\lambda)$ is a periodic function of period 2π such that $f(\lambda) = f(2\pi + \lambda)$, and, for x_t real, it is symmetric around $\lambda = 0$, $f(\lambda)$ is usually defined at frequencies $\lambda \in [0, \pi]$. The cyclical behaviour is reflected in $f(\lambda)$ by a peak at frequency ω, which defines the cycle such that the period is $2\pi/\omega$. Thus, the location of the spectral peak determines the cycle. In relation to this spectral characteristic arising from cyclical behaviour, Nerlove (1964) defined seasonality as "that characteristic of a time series that gives rise to spectral peaks at seasonal frequencies". Seasonal frequencies are defined as $2\pi j/s$ for $j = 1, 2, ..., [s/2]$, where $[s/2]$ is the integer part of $s/2$, that is $s/2$ if s is even and $(s-1)/2$ if s is odd, and s is the number of observations per year. This is the definition we adopt throughout the whole thesis. Thus seasonality implies the existence of $[s/2]$ cycles of periods s/j, $j = 1, 2, ..., [s/2]$.

We say that a process has long memory if its spectral density satisfies

$$f(\omega + \lambda) \sim C|\lambda|^{-2d} \quad \text{as} \quad \lambda \to 0 \tag{1.2}$$

where $0 < C < \infty$, the memory or persistence parameter, d, is different from zero, ω is a frequency in the interval $[0, \pi]$ and $a \sim b$ means that $a/b \to 1$. Using the notation in Engle et al. (1989) we also call a process with such a spectrum integrated of order d at frequency ω, and we denote it by $I_\omega(d)$. Stationarity entails $d < 1/2$ (note that if $d \geq 1/2$ $f(\lambda)$ is not integrable) and $d > -1/2$ is required for invertibility so that the persistence parameter, d, is often restricted to be between $-1/2$ and $1/2$. Although processes satisfying (1.2) are considered long memory as long as $d \neq 0$, more rigorously we say that the process has long memory or persistence if $d > 0$, short memory if $d = 0$, and antipersistence if $d < 0$ (some surveys on long memory are Robinson (1994d), Baillie (1996) and Beran (1992, 1994a)). Long memory literature has traditionally focused on $I_0(d)$ processes satisfying (1.2) with $\omega = 0$. When $\omega \in (0, \pi]$ we say that the process has cyclical long memory with period $2\pi/\omega$. The most common case is seasonal long memory that occurs when the spectral density satisfies (1.2) for every seasonal frequency. However, for non-seasonal time series (for example with annual data) we can have a cyclical behaviour such that (1.2) holds for a single ω. Since (1.2) only restrict the behaviour of $f(\lambda)$ around one specific frequency, ω, and does not impose any other condition far from ω (in particular, in the seasonal case, (1.2) only refers to one of the $[s/2]$ seasonal frequencies) we feel it appropriate to use the terminology Seasonal/Cyclical Long Memory (SCLM) to denote $I_\omega(d)$ processes with spectral density satisfying (1.2) for some $\omega \in (0, \pi]$.

Under mild conditions (see for example Yong (1974) and Chapter 2 in this thesis) the spectral relation (1.2) translates in the time domain to autocovariances that are

$O(j^{2d-1})$ as $j \to \infty$. In particular, when $d > 0$ and $\omega = 0$ the autocovariances are not summable and when $\omega \neq 0$ the absolute values of the autocovariances are not summable but the raw values are summable as long as the spectrum is bounded at the origin.

When $d \geq 1/2$ the process is not stationary and the spectral distribution function (and thus the spectral density) does not exist. In those cases we consider $f(\lambda)$ in (1.2) represents the pseudospectrum which we define in the following manner. Let x_t be a non-stationary process such that $u_t = \tau(L)x_t$ is stationary with absolutely continuous spectral distribution function, and $\tau(L)$ is a polynomial in the lag operator L ($L^k x_t = x_{t-k}$ for k integer). If $f_u(\lambda)$ is the spectrum of u_t, then the pseudospectrum of x_t is $f(\lambda) = |\tau(e^{i\lambda})|^{-2} f_u(\lambda)$, where $|z|^2 = z \times \bar{z}$ and $\bar{z}$ is the complex conjugate of z. In this way we allow the definition of SCLM in (1.2) be valid for stationary and non-stationary processes.

In Section 1.2 we review traditional methods of modelling cyclical and seasonal behaviour, as well as some methods of adjusting for seasonality. We dedicate Section 1.3 to the description of some parametric SCLM processes satisfying (1.2). Section 1.4 pays attention to the estimation of the persistence parameter in long memory time series. Several tests on seasonal integration and cointegration are briefly reviewed in Section 1.5. Whereas $f(\lambda)$ has to be symmetric around frequency zero, it need not be symmetric around a ω different from $0, mod(\pi)$. This broadens the scope of modelling SCLM time series. In Section 1.6 we introduce the possibility of asymmetric spectral poles or zeros. Some effects of this asymmetry are analysed throughout the whole thesis.

1.2 MODELLING SEASONALITY AND CYCLES

Seasonality has traditionally been considered a nuisance, and several seasonal adjustment procedures have been proposed. They are typically based on the idea that a time series $\{x_t, t = 0, \pm 1, ...\}$, possibly after logarithmic transformation, is additively composed of three different components, the trend-cycle, T_t, the seasonal, S_t, and the irregular component, I_t,

$$x_t = T_t + S_t + I_t. \tag{1.3}$$

Traditionally T_t includes also the possibility of a cyclical component. The reason for this is that the cycle in economics has usually been considered a periodic component with period larger than the number of observations per year. This implies a spectral or pseudospectral peak at some frequency between zero and $2\pi/s$, where s is the number of observations per year. This phenomenon may be indistinguishable from a stochastic trend, characterized by a pseudo-spectral pole at the origin. However, there may be cycles of period different from the seasonal ones, s/j, for $j = 1, 2, ..., [s/2]$. To allow for this behaviour we can include a cyclic component, C_t, in the model (1.3),

$$x_t = T_t + C_t + S_t + I_t. \tag{1.4}$$

The additive form in (1.3) and (1.4) (perhaps after taking logarithms) is often known as Unobserved Component (UC) or Structural Time Series model. The seasonally adjusted series is obtained by subtracting an estimate of S_t. We group the different methods of estimation of S_t and adjustment of x_t in two classes, "model-free" and "model-based" adjusting procedures. The "model-free" techniques do not take into account the possible form of the seasonality and the same procedure is essentially applied irrespective of the series. They are basically based on the application of a succession of moving averages to produce seasonally adjusted data. The most widely used is the US Bureau of the Census X-11 procedure (Shiskin et al. (1967)). This technique is based on the application of a series of two-sided filters to the series. Clearly it is not possible to apply a two-sided filter at the end of the series. Instead a one-sided filter must be applied and the latest adjusted figures must be revised as new observations become available and it becomes possible to apply a two-sided filter. The X-11 ARIMA (Dagum (1980)) allows the application of two sided filters by fitting an ARIMA model to the series, forecasting future values and seasonally adjusting the whole series, actual and predicted, by X-11. Revisions are still necessary as observations come in to replace the predicted values, but they should be smaller than before.

The "model-based" seasonal adjustment procedures are made according to the characteristic of each series. They are based on the estimation of parametric models that fit the seasonal behaviour of the series. Some of these models are described below.

Seasonal adjustment procedures have been criticized for causing undesirable effects such as spectral dips at seasonal frequencies or distortion of the spectral density at other frequencies (see Nerlove (1964) or Bell and Hillmer (1984)). Furthermore, the UC models in (1.3) and (1.4) suppose that each component in x_t can be specified separately and independently of the rest of components. This is not always the case. Often the same model includes two or more of the components in x_t (for example the stochastic seasonal processes classified as b), c) and d) below include an irregular component). The existence of these models and the undesirable effects caused by traditional methods of seasonal adjustment have given rise to the use of seasonally unadjusted data.

Most of the processes we describe in this section are seasonal, so that they model a specific cyclical behaviour. However, other cyclic patterns can be modelled similarly by suitably choosing the dummy variables, cosinusoids or lag operators in the models described below.

One of the earliest attempts to model seasonality assumes that the series repeats the cyclical behaviour in a regular manner, and uses seasonal dummies , D_{kt}, to construct the deterministic model,

$$x_t = \sum_{k=1}^{s} a_k D_{kt} \tag{1.5}$$

where $D_{kt} = 1$ if $t - k$ is a multiple of s (the number of observations per year) and 0 otherwise. It is usually assumed that

$$\sum_{k=1}^{s} a_k = 0$$

since we may achieve this, if it is not so, by subtracting a constant from the original series in such a way that the seasonal movement is not affected. We can express (1.5) as a function of sine and cosine waves via the equivalent formula

$$x_t = \sum_{h=1}^{[\frac{s}{2}]} \Psi_{h,t} \tag{1.6}$$

where

$$\begin{aligned}\Psi_{h,t} &= \alpha_h \cos(\omega_h t) + \beta_h \sin(\omega_h t) \ , \ \omega_h = \frac{2\pi h}{s}, \\ \alpha_h &= \frac{2}{s}\sum_{k=1}^{s} a_k \cos(k\omega_h), \\ \beta_h &= \frac{2}{s}\sum_{k=1}^{s} a_k \sin(k\omega_h),\end{aligned} \tag{1.7}$$

for $1 \leq h < s/2$, and if s is even $\beta_{\frac{s}{2}} \sin(\omega_{\frac{s}{2}} t)$ is zero and

$$\alpha_{\frac{s}{2}} = \frac{1}{s}\sum_{k=1}^{s} a_k \cos(k\omega_{\frac{s}{2}})$$

(see Hannan (1963)). x_t in (1.6) can be equally written $x_t = \sum_{h=1}^{[s/2]} r_h \cos(\omega_h t - \theta_h)$ where $r_h = \sqrt{\alpha_h^2 + \beta_h^2}$ is the h-th amplitude and $\theta_h = \arctan(\beta_h/\alpha_h)$ is the h-th phase. It is rarely plausible that time series have such a rigid deterministic behaviour as (1.5) or (1.6) impose, so a stochastic error is often added. If this irregular component is well behaved and the frequencies ω_h are known, then α_h and β_h in (1.7) or a_k in (1.5) can be estimated through simple regression methods.

The processes (1.5) and (1.6) are completely deterministic, and if α_h, β_h are fixed parameters they are non-stationary so that it does not make sense to speak of spectral distribution function or spectral density. However the spectral behaviour of stochastic seasonal time series will give us relevant information about the characteristics of the process. According to spectral characteristics we distinguish four classes of stochastic seasonal/cyclical processes:

a) Stationary with spectral distribution function with jumps and thus not absolutely continuous.

b) Stationary with absolutely continuous spectral distribution function everywhere and smooth, positive, spectral density.

c) Stationary with absolutely continuous spectral distribution function but spectral density with one or more singularities or zeros.

d) Non-stationary so that no spectral distribution function exists.

a) Stationary processes with jumping spectral distribution. This kind of process is defined by (1.6) and (1.7) but we make $\Psi_{h,t}$ stochastic by allowing α_h and β_h be random variables satisfying

$$\begin{array}{l} E[\alpha_h] = E[\beta_h] = 0 \ , \ E[\alpha_h^2] = E[\beta_h^2] = \sigma_h^2 \ \text{ for all } h \\ E[\alpha_h \alpha_i] = E[\beta_h \beta_i] = 0 \ \ h \neq i \ , \ E[\alpha_h \beta_i] = 0 \ \text{ for all } h, i. \end{array} \tag{1.8}$$

Under (1.8), x_t is covariance stationary with lag-j autocovariance

$$\gamma_j = E(x_t x_{t-j}) = \sum_{h=1}^{[\frac{s}{2}]} \sigma_h^2 \cos(\omega_h j) = \int_{-\pi}^{\pi} \cos(j\lambda) \mathrm{d}F(\lambda) \quad j = 0, \pm 1 \ldots .$$

Although α_h and β_h are random variables, they are fixed in any particular realization. Thus, although $\Psi_{h,t}$ is stationary, the model is still deterministic; only two observations are necessary to determine α_h and β_h, and once this has been done the remaining points in the series can be forecast with zero mean square error. In practice, therefore, the only difference between the non stationary model, (1.7), and the corresponding stationary model, (1.7) and (1.8), is the interpretation of the parameters. The spectral distribution function, $F(\lambda)$, is a step function consisting of jumps of magnitude $\sigma_h^2/2$ at frequencies $-\omega_h$ and ω_h, for $h = 1, ..., [s/2]$. Since $F(\lambda)$ is not continuous the spectral density does not exist. However, in a similar manner as Stieltjes integration is carried out, we can define the so-called *line or discrete spectrum*, which is a discrete function with values $\sigma_h^2/2$ at frequencies $-\omega_h$ and ω_h for $h = 1, ..., [s/2]$. The line spectrum at ω_h gives the relative importance of a cycle of period s/h in the variance of x_t.

b) Stationary processes with absolutely continuous spectral distribution and smooth spectral density. The models in (1.5) and (1.6) assume that the cyclic behaviour in x_t is constant over time and does not change its form. However, economic systems are evolving over time and the seasonal/cyclical behaviour is likely to change across time. Of course the variation must be slow (otherwise we cannot speak of seasonality or cycle) in such a way that the periodical structure seems to persist and the series has a quasi-periodic behaviour. Hannan (1964) allows for this behaviour by modelling x_t as the seasonal process,

$$x_t = \sum_{h=1}^{[\frac{s}{2}]} \Psi_{h,t} \ , \ \Psi_{h,t} = \alpha_{h,t} \cos(\omega_h t) + \beta_{h,t} \sin(\omega_h t), \tag{1.9}$$

where $\omega_h = 2\pi h/s$ are seasonal frequencies and $\alpha_{h,t}$ and $\beta_{h,t}$ are not constant but evolve with time. Hannan (1964) assumed

$$\begin{array}{l} E[\alpha_{h,t}] = E[\beta_{h,t}] = 0 \quad \text{for all } h \text{ and all } t \ , \\ E[\alpha_{h,t}\alpha_{h,t-j}] = E[\beta_{h,t}\beta_{h,t-j}] = c_h\rho_h^j, |\rho_h| < 1, \\ E[\alpha_{h,t}\alpha_{i,s}] = E[\beta_{h,t}\beta_{i,s}] = 0 \quad \text{for } h \neq i \text{ and all } t, s, \\ E[\alpha_{h,t}\beta_{i,s}] = 0 \quad \text{for all } h, i \text{ and all } t, s. \end{array} \tag{1.10}$$

Thus the lag-j autocovariance of $\Psi_{h,t}$ is

$$E[\Psi_{h,t}\Psi_{h,t-j}] = c_h\rho_h^j \cos(\omega_h j). \tag{1.11}$$

Stationarity of $\Psi_{h,t}$ entails $|\rho_h| < 1$. However, ρ_h has to be close to 1 to avoid quickly changing behaviour of $\Psi_{h,t}$. When $|\rho_h| < 1$, $\Psi_{h,t}$ is stationary and non deterministic with absolutely continuous spectral distribution and smooth spectral density,

$$\begin{aligned} f_h(\lambda) &= \frac{c_h}{2\pi} \sum_{j=-\infty}^{\infty} \rho_h^j \cos(\omega_h j)\cos(\lambda j) \\ &= \frac{c_h}{4\pi} \left\{ \frac{1-\rho_h^2}{1+\rho_h^2 - 2\rho_h\cos(\lambda-\omega_h)} + \frac{1-\rho_h^2}{1+\rho_h^2 - 2\rho_h\cos(\lambda+\omega_h)} \right\} \end{aligned} \tag{1.12}$$

which, for ρ_h near to unity, will be concentrated around $\lambda = \omega_h$. Hannan et al. (1970) considered a parameterization of $\alpha_{h,t}$ and $\beta_{h,t}$ obeying (1.10)

$$\alpha_{h,t} = \rho_h\alpha_{h,t-1} + \varepsilon_{h,t} \ , \quad \beta_{h,t} = \rho_h\beta_{h,t-1} + \varepsilon_{h,t}^{\dagger} \ , \quad |\rho_h| < 1 \tag{1.13}$$

where $\varepsilon_{h,t}$ and $\varepsilon_{h,t}^{\dagger}$ have zero mean and common variance σ_h^2, and all correlations between ε, $\varepsilon^{\dagger}$ and between two time points and for differing values of h vanish. Substituting (1.13) in $\Psi_{h,t}$ in (1.9), we get that $\Psi_{h,t}$ is an ARMA(2,1) process

$$(1 - 2\rho_h\cos(\omega_h)L + \rho_h^2L^2)\Psi_{h,t} = \eta_{h,t} - \rho_h\cos(\omega_h)\eta_{h,t-1} - \rho_h\sin(\omega_h)\eta_{h,t-1}^{\dagger} \tag{1.14}$$

where

$$\begin{aligned} \eta_{h,t} &= \varepsilon_{h,t}\cos(\omega_h t) + \varepsilon_{h,t}^{\dagger}\sin(\omega_h t) \\ \eta_{h,t}^{\dagger} &= \varepsilon_{h,t}\sin(\omega_h t) - \varepsilon_{h,t}^{\dagger}\cos(\omega_h t) \end{aligned}$$

are thus zero mean random variables with variance σ_h^2 and they inherit the uncorrelatedness properties of $\varepsilon_{h,t}$ and $\varepsilon_{h,t}^{\dagger}$. The lag-$j$ autocovariance and spectral density of

$\Psi_{h,t}$ are (1.11) and (1.12) with $c_h = \sigma_h^2/(1-\rho_h^2)$. Consequently the spectrum of x_t is a smooth function

$$f(\lambda) = \sum_{h=1}^{[\frac{s}{2}]} f_h(\lambda) \tag{1.15}$$

which shows peaks (sharper the closer ρ_h is to 1) around seasonal frequencies ω_h, $h = 1, 2, ..., [s/2]$. An equivalent manner of allowing for a time-evolving behaviour of $\Psi_{h,t}$ is via the recursion (see Harvey (1989)),

$$\begin{bmatrix} \Psi_{h,t} \\ \Psi_{h,t}^{\dagger} \end{bmatrix} = \rho_h \begin{bmatrix} \cos\omega_h & \sin\omega_h \\ -\sin\omega_h & \cos\omega_h \end{bmatrix} \begin{bmatrix} \Psi_{h,t-1} \\ \Psi_{h,t-1}^{\dagger} \end{bmatrix} + \begin{bmatrix} \eta_{h,t} \\ \eta_{h,t}^{\dagger} \end{bmatrix} \quad t = 1, 2, ..., T \tag{1.16}$$

where $\eta_{h,t}$ and $\eta_{h,t}^{\dagger}$ are two uncorrelated white noise sequences with the same variance, σ_h^2, and ρ_h is a damping factor, $0 \leq \rho_h \leq 1$. If $\Psi_{h,0} = \alpha_h$, $\Psi_{h,0}^{\dagger} = \beta_h$ and $\rho_h = 1$, the recursion in (1.16), apart from the stochastic vector including $\eta_{h,t}$ and $\eta_{h,t}^{\dagger}$, gives the value of $\Psi_{h,t}$ in (1.7). Note that $\Psi_{h,t}^{\dagger}$ only appears by construction to form $\Psi_{h,t}$ and is of no intrinsic importance. When we include the stochastic terms we see that $\Psi_{h,t}$ is the ARMA(2,1) process in (1.14) with spectral density (1.12) with $c_h = \sigma_h^2/(1-\rho_h)$. Thus the spectrum of x_t is (1.15) and for ρ_h close to 1 it will be concentrated around frequencies ω_h.

In addition to the specific ARMA in (1.14) we can use many other ARMA processes to model a changing cyclical behaviour. In particular, if the spectrum of an AR(2), $(1-\phi_1 L - \phi_2 L^2)x_t = \varepsilon_t$, contains a peak at frequency λ^* within the range $0 < \lambda^* < \pi$, its exact position is

$$\lambda^* = \cos^{-1}\left[\frac{-\phi_1(1-\phi_2)}{4\phi_2}\right].$$

For example the spectrum of the AR part in (1.14) has a peak at

$$\lambda^* = \cos^{-1}\left[\frac{(1+\rho_h^2)\cos\omega_h}{2\rho_h}\right]$$

so that λ^* is closer to ω_h the closer ρ_h is to 1. We can also use the seasonal lag operator, L^s, ($L^s x_t = x_{t-s}$) to define the seasonal ARMA(1,1) model

$$(1-\phi_s L^s)x_t = (1+\theta_s L^s)\varepsilon_t \tag{1.17}$$

where ε_t is white noise with variance σ^2. When ϕ_s and θ_s are inside the unit circle,

x_t is stationary and invertible with smooth spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1 + \theta_s^2 + 2\theta_s \cos(\lambda s)}{1 + \phi_s^2 - 2\phi_s \cos(\lambda s)}.$$

If $\phi_s > 0$ and $\theta_s > 0$, $f(\lambda)$ exhibits peaks at the seasonal harmonic frequencies, $\omega_h = 2\pi h/s$, $h = 1, 2, ..., [s/2]$, as well as at zero. More general seasonal ARMA processes can be defined as

$$\Phi_s(L^s)x_t = \Theta_s(L^s)\varepsilon_t \tag{1.18}$$

where $\Phi_s(L^s)$ and $\Theta_s(L^s)$ are polynomials in the seasonal lag operator with zeros outside the unit circle (see Box and Jenkins (1976)).

c) Stationary processes with absolutely continuous spectral distribution and spectral density with singularities or zeros. The structure of α_h and β_h in (1.13) may generate a relatively rapid change in the seasonal pattern, whereas the definition of seasonality implies a regular or quasi-regular behaviour. The closer ρ_h is to 1 the more regular the movement of $\Psi_{h,t}$. In fact we can choose $\rho_h = 1$, but in this case $\Psi_{h,t}$ ceases to be stationary. Instead we can assume that $\alpha_{h,t}$ and $\beta_{h,t}$ evolve as

$$(1-L)^{d_h}\alpha_{h,t} = \varepsilon_{h,t} \quad , \quad (1-L)^{d_h}\beta_{h,t} = \varepsilon^{\dagger}_{h,t} \tag{1.19}$$

where $\varepsilon_{h,t}$ and $\varepsilon^{\dagger}_{h,t}$ are defined as in (1.13). Thus $\alpha_{h,t}$ and $\beta_{h,t}$ are fractional ARIMA(0,d_h,0) processes and they are stationary if $d_h < 1/2$ and invertible if $d_h > -1/2$ (see Hosking (1981)). The slowly changing behaviour necessary for seasonality requires $d_h > 0$ and stationarity entails $d_h < 1/2$. Under these circumstances $\alpha_{h,t}$ and $\beta_{h,t}$ are $I_0(d_h)$ with spectral density, $f_0(\lambda)$, diverging at the origin, and lag-j autocovariances

$$\gamma^{\dagger}_{h,j} = E[\alpha_{h,t}\alpha_{h,t-j}] = E[\beta_{h,t}\beta_{h,t-j}] = \sigma_h^2 \frac{\Gamma(1-2d_h)\Gamma(j+d_h)}{\Gamma(d_h)\Gamma(1-d_h)\Gamma(j+1-d_h)}.$$

Thus the lag-j autocovariance of $\Psi_{h,t}$ is

$$E[\Psi_{h,t}\Psi_{h,t-j}] = \gamma^{\dagger}_{h,j}\cos(j\omega_h)$$

and its spectral density is

$$\begin{aligned} f_h(\lambda) &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma^{\dagger}_{h,j}\cos(j\omega_h)e^{-i\lambda j} = \frac{1}{4\pi}\sum_{j=-\infty}^{\infty}\gamma^{\dagger}_{h,j}e^{-i\lambda j}(e^{ij\omega_h} + e^{-ij\omega_h}) \\ &= \frac{1}{2}f_0(\lambda - \omega_h) + \frac{1}{2}f_0(\lambda + \omega_h). \end{aligned}$$

The multiplication of $\alpha_{h,t}$ by $\cos(\omega_h t)$ and $\beta_{h,t}$ by $\sin(\omega_h t)$ produces a phase shift such that the spectral pole moves from zero in $\alpha_{h,t}$ and $\beta_{h,t}$ to ω_h in $\Psi_{h,t}$. Thus, the process (1.9) and (1.19) has an absolutely continuous spectral distribution but its spectral density is not smooth, but goes to ∞ (if $d_h > 0$) or is zero (if $d_h < 0$), at frequencies $\pm\omega_h$ as described in (1.2). This is the SCLM property that characterizes the processes we analyze in this thesis. A more detailed description of existing models conforming to this property is given in the next section.

d) Non-stationary and non-deterministic stochastic seasonal processes. If $\alpha_{h,t}$ and $\beta_{h,t}$ are determined by the fractional ARIMA processes in (1.19) but with $d_h \geq 1/2$, then they are not stationary and thus $\Psi_{h,t}$ in (1.9) is clearly non-stationary. In this case there does not exist a spectral distribution. Nevertheless, the frequency domain is still an adequate framework to detect seasonality using the pseudospectrum. For example, if $d_h = 1$ in (1.19) or equivalently $\rho_h = 1$ in (1.13) or (1.16), then $\Psi_{h,t}$ is a non-stationary ARMA(2,1) process

$$\tau_h(L)\Psi_{h,t} = \eta_{h,t} - \cos(\omega_h)\eta_{h,t-1} - \sin(\omega_h)\eta^{\dagger}_{h,t-1}$$

where $\tau_h(L) = 1 - 2\cos(\omega_h)L + L^2$. The non-stationarity comes from the fact that the AR polynomial, $\tau_h(L)$, has zeros at $\cos\omega_h \pm \sqrt{\cos^2\omega_h - 1}$, with modulus one. However $\tau_h(L)\Psi_{h,t}$ is a stationary MA(1). Since $|\tau_h(e^{i\lambda})|^{-2} = (2(\cos\omega_h - \cos\lambda))^{-2}$ diverges at $\lambda = \pm\omega_h$, then the pseudospectrum of $\Psi_{h,t}$ goes to infinity at frequencies $\pm\omega_h$, reflecting a strong cyclical pattern with period $2\pi/\omega_h = s/h$. Hannan et al. (1970) used this model and estimated it using optimal methods for the extraction of a signal (i.e. the seasonal component) extended to allow for non-stationary signal (Hannan (1967)).

In the Box-Jenkins framework we can define the seasonal ARIMA(P,D,Q) time series

$$\Phi_s(L^s)(1 - L^s)^D x_t = \Theta_s(L^s)\varepsilon_t \tag{1.20}$$

where the ε_t are white noise $(0, \sigma^2)$, $\Phi_s(L^s)$ and $\Theta_s(L^s)$ are polynomials in the lag operator with zeros outside the unit circle, and D is a positive integer (Box and Jenkins (1976)). We can also consider a fractional D. In this case (1.20) defines the fractional seasonal ARIMA(P,D,Q), which is stationary if $D < 1/2$ and non-stationary if $D \geq 1/2$. The spectrum ($D < 1/2$) or pseudospectrum (if $D \geq 1/2$) of x_t is

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\Theta_s(e^{i\lambda s})|^2}{|\Phi_s(e^{i\lambda s})|^2} \left[4 \sin^2 \frac{\lambda s}{2}\right]^{-D} \tag{1.21}$$

and diverges if $D > 0$ or is zero if $D < 0$ at frequencies $\omega_h = 2\pi h/s$, $h = 0, ..., [s/2]$, that is at the origin and at seasonal frequencies. The seasonal difference operator, $(1 - L^s)$, can be written as the product of the difference operator, $(1 - L)$, and the seasonal summation operator, $S(L) = (1 + L + ... + L^{s-1})$, such that the pole in (1.21) at the origin corresponds to the operator $(1 - L)$, and the spectral poles at seasonal frequencies are due to $S(L)$. Thus $(1 - L^s)$ includes a stochastic trend in addition to the seasonal factor. This is why sometimes (e.g. Harvey (1989)), $S(L)$ is used instead of $(1 - L^s)$ to model the seasonal component of a UC time series model as described in (1.3) and (1.4).

A different class of non-stationarity may be due to the fact that the autocovariances are not time invariant, for example because there is a different data generating process for each season. This phenomenon is often modelled via the Periodic ARIMA process that allows for a different behaviour in each season of the cycle (e.g. Troutman (1979), Tiao and Grupe (1980), Osborn (1991), Franses and Ooms (1995)). This kind of model can be written as

$$\Phi_q(L)(1 - L^s)^{d_q} x_T^q = \Theta_q(L)\varepsilon_T^q \quad q = 1, ..., s, \quad T = 1, 2, ..., \tag{1.22}$$

where ε_T^q is white noise with variance σ_q^2, the index q indicates the season or situation of the observation in the cycle (for example different months) and T represents the year such that $x_T^q = x_{(T-1)s+q}$. The lag operator, L, retards the observation one period such that $Lx_T^q = Lx_{(T-1)s+q} = x_{(T-1)s+q-1} = x_T^{q-1}$ and for any integer k, $L^{ks} x_T^q = x_{T-k}^q$. Thus, (1.22) allows for s different models, one for every season (month in case $s = 12$). When the zeros of $\Phi_q(L)$ lie outside the unit circle, and $d_q < 1/2$, then (1.22) is stationary for every $q = 1, 2, ..., s$. Although x_t^q may be stationary, the variable x_t is clearly non-stationary if some of the parameters corresponding to different qs are different. In this case the variance and autocovariances of x_t depend on

q and therefore are not time invariant. Thus there does not exist a spectral distribution function and we cannot use frequency domain techniques to explore the characteristics of such a process. The analysis of this kind of process is usually performed in a multivariate setup using a vector ARMA representation. Define the $s \times 1$ vector $z_T = (x_T^1, ..., x_T^s)' = (x_{(T-1)s+1}, ..., x_{Ts})'$. The periodic process in (1.22) can be written in vector ARMA form as

$$A(L^*)C(L^*)z_T = B(L^*)u_T \qquad T = 1, 2, ..., \tag{1.23}$$

where $u_T = (\varepsilon_T^1, ..., \varepsilon_T^q)'$, $C(L^*) = diag\{(1 - L^*)^{d_q}\}$, $A(L^*)$ and $B(L^*)$ are matrix polynomials in L^*, and the operator L^* is the lag operator for the index T, $L^* z_T = z_{T-1}$. This implies seasonal difference in the elements of z_T, $L^* x_T^q = L^s x_{(T-1)s+q} = x_{(T-2)s+q} = x_{T-1}^q$. The vector z_T is stationary if $d_q < 1/2$ for $q = 1, ..., s$, and $|A(z)|$ has zeros outside the unit circle, and invertible if $d_q > -1/2$ for $q = 1, ..., s$, and the zeros of $|B(z)|$ lie outside the unit circle. Under stationarity z_T has a spectral density matrix $f_z(\lambda)$. We have already pointed out that x_t is not stationary and consequently, it does not have a spectral distribution function. However, the expectation of the sample autocovariances of x_t converge to the autocovariances of a stationary process with spectral density function

$$f(\lambda) = \frac{1}{s} R(e^{i\lambda})' f_z(s\lambda) R(e^{-i\lambda}) \tag{1.24}$$

where $R(r)$ is a $s \times 1$ vector with k-th element r^k (Tiao and Grupe (1980)). Of course the spectrum of x_t is not (1.24), but asymptotically we can use (1.24) to classify periodic processes in the same way we have done before for non-periodic seasonal models.

1.3 SEASONAL/CYCLICAL LONG MEMORY PROCESSES

In this thesis we focus on the analysis of seasonal/cyclical stationary processes with spectral density satisfying (1.2). These are the class c) of the stochastic seasonal time series models introduced in the previous section, processes with absolutely continuous spectral distribution but non-smooth spectral density function. In particular the spectral density satisfies (1.2) so that it diverges (if $d > 0$) or is zero (if $d < 0$) at some frequency $\omega \in (0, \pi]$. Since (1.2) only restricts the behaviour of $f(\lambda)$ around ω, and ω can be any frequency between 0 and π (seasonal or not), then we say that a process with spectrum satisfying (1.2) has SCLM or is integrated of order d at ω, $I_\omega(d)$. This notation covers the stationary case ($d < 1/2$) as well as the non-stationary one ($d \geq 1/2$). In the latter, $f(\lambda)$ in (1.2) represents the pseudospectrum.

Though (1.2) is a semiparametric condition and only imposes knowledge of $f(\lambda)$ around ω, it is interesting to describe parametric processes satisfying (1.2), specifying short memory as well as long memory components of x_t, for example for the purpose of Monte Carlo simulations. Some examples have been introduced in the previous section (e.g. (1.9) and (1.19) or (1.20)). In case of Gaussian series it suffices to specify $f(\lambda)$ for all $\lambda \in (-\pi, \pi]$, and the mean, μ, to have an absolute knowledge of the process. Since the spectral density and the autocovariances give the same information, we could equivalently specify the autocovariances, γ_j, for all j. A characteristic of autocovariances of SCLM processes is that they have a slow decay typical of long memory but they also have oscillations that depend on the frequency ω. Often $\gamma_j = O(j^{2d-1})$ as $j \to \infty$ but with sine oscillations depending on the frequency ω, and if $d > 0$ then $\sum |\gamma_j| = \infty$, although $\sum \gamma_j$ may be finite if $f(\lambda)$ is bounded at zero frequency. Complete parameterization of μ and $f(\lambda)$ or γ_j permits the simulation of Gaussian series satisfying (1.2). Non-Gaussian x_t are not fully described by μ and $f(\lambda)$ or γ_j, but nevertheless, assuming they have finite variance, they could have spectrum or autocovariances of the type we discuss, and so will be SCLM as far as second moment properties are concerned. There remains the possibility that x_t may not exhibit long memory in second moments but in some other way (for example x_t^2 could have long memory) as discussed in Chapter 8.

Two different types of parametric SCLM models have been stressed in the literature. They are natural extensions of the processes used to model standard long memory at zero frequency , ((1.2) with $\omega = 0$), namely the fractional noise and the fractional ARIMA in the Box-Jenkins setup.

1.3.1 Seasonal Fractional Noise

This kind of stationary process is characterized by a spectral density

$$f(\lambda) = C|1 - \cos(s\lambda)| \sum_{j=-\infty}^{\infty} \left|\lambda + \frac{2\pi}{s}j\right|^{-2(1+d)} \tag{1.25}$$

and lag-j autocovariance

$$\gamma_j = \frac{Ex_1^2}{2}(|\frac{j}{s} + 1|^{2d+1} - 2|\frac{j}{s}|^{2d+1} + |\frac{j}{s} - 1|^{2d+1}) \tag{1.26}$$

where s is the number of observations per year, C is a positive constant and $d < 1/2$ (see Jonas (1983), Carlin and Dempster (1989) or Ooms (1995)). The spectrum in (1.25) satisfies (1.2) for $\omega = 2\pi h/s$, $h = 0, 1, ..., [s/2]$, and the γ_j in (1.26) have slow and oscillating decay as $j \rightarrow \infty$, and if $d > 0$ they are not absolutely summable. This kind of process is a generalization of the fractional noise described by Mandelbrot and Van Ness (1968) that is characterized by (1.25) and (1.26) with $s = 1$, and has the typical long memory behaviour at frequency zero.

1.3.2 SCLM in the Box-Jenkins setup

Gray et al. (1989, 1994) analysed the so-called Gegenbauer process, first proposed by Hosking (1981), which is of the form

$$(1 - 2L\cos\omega + L^2)^d x_t = u_t \tag{1.27}$$

where u_t is a process with positive, finite and continuous spectrum, $f_u(\lambda)$, and d can be any real number. For example when u_t is a stationary and invertible ARMA(p,q) (1.27) is called GARMA (Gegenbauer ARMA). The spectral density of x_t in (1.27) is

$$f(\lambda) = (4(\cos\omega - \cos\lambda)^2)^{-d} f_u(\lambda) \tag{1.28}$$

and it satisfies (1.2) so that x_t has SCLM at frequency ω. For $\omega \neq 0, \pi$, the process in (1.27) is stationary if $d < 1/2$ and invertible if $d > -1/2$. When $\omega = 0$ and u_t is a stationary and invertible $ARMA(p,q)$, then (1.27) is the fractional $ARIMA(p, 2d, q)$, $(1 - L)^{2d}x_t = u_t$, so that x_t is stationary if $d < 1/4$ and invertible when $d > -1/4$. If $\omega = \pi$ the spectrum of x_t has a pole at frequency π and it is stationary if $d < 1/4$

and invertible when $d > -1/4$. When the u_t's in (1.27) are $iid(0, \sigma^2)$ and $d < 1/2$ the autocovariances of x_t are

$$\gamma_j = \frac{\sigma^2}{2\sqrt{\pi}}\Gamma(1-2d)(2\sin\omega)^{\frac{1}{2}-2d}[P_{j-\frac{1}{2}}^{2d-\frac{1}{2}}(\cos\omega) + (-1)^j P_{j-\frac{1}{2}}^{2d-\frac{1}{2}}(-\cos\omega)] \qquad (1.29)$$

where $P_a^b(z)$ are associated Legendre functions (Chung(1996a)). The asymptotic behaviour of γ_j in (1.29) is

$$\gamma_j \sim K\cos(j\omega)j^{2d-1} \quad \text{as } j \to \infty \qquad (1.30)$$

where K is a finite constant that depends on d but not on j (see Gray et al. (1989) or Chung (1996a)). We observe that the autocovariances of x_t in (1.27) have the slow and oscillating decay typical of SCLM.

Porter-Hudak (1990) and Ray (1993) among others, proposed the use of the fractional seasonal difference operator, $(1-L^s)^d$, where d can be any real number. Porter-Hudak (1990) used the operator $(1-L^{12})^d$ in monthly monetary USA aggregates and Ray (1993) used $(1-L^3)^{d_3}(1-L^{12})^{d_{12}}$ for monthly IBM revenue data. Note that $(1-L^s)^d$ can be decomposed into the product of some operators $(1-2L\cos\omega + L^2)^d$. For instance if $s = 4$,

$$(1-L^4)^d = (1-2L\cos\omega_0 + L^2)^{\frac{d}{2}}(1-2L\cos\omega_1 + L^2)^d(1-2L\cos\omega_2 + L^2)^{\frac{d}{2}} \qquad (1.31)$$

for $\omega_0 = 0$, $\omega_1 = \pi/2$ and $\omega_2 = \pi$. Thus the process x_t in $(1-L^4)^d x_t = u_t$ is $I_0(d)$, $I_{\frac{\pi}{2}}(d)$ and $I_\pi(d)$.

In order to allow for different persistence parameters across different frequencies, Chan and Wei (1988), Chan and Terrin (1995), Giraitis and Leipus (1995) and Robinson (1994a) used the model

$$(1-L)^{d_0}\{\prod_{j=1}^{h-1}(1-2L\cos\omega_j + L^2)^{d_j}\}(1+L)^{d_h}x_t = u_t \qquad (1.32)$$

where ω_j can be any frequency between 0 and π and u_t has continuous, positive and bounded spectrum. Thus x_t in (1.32) is $I_{\omega_j}(d_j)$ for $j = 0, 1, 2, ..., h$, where $\omega_0 = 0$ and $\omega_h = \pi$. When u_t is a stationary and invertible ARMA, Giraitis and Leipus (1995) used the terminology ARUMA to denote a process satisfying (1.32). When $|d_j| < 1/2$

for $j = 0, 1, ..., h$, (1.32) can be expressed

$$\sum_{j=0}^{\infty} \pi_j x_{t-j} = u_t$$

or

$$x_t = \sum_{j=0}^{\infty} \psi_j u_{t-j}$$

where $\pi_0 = \psi_0 = 1$ and

$$\pi_j = \sum_{\substack{0 \leq k_0, ..., k_h \leq j \\ k_0 + ... + k_h = j}} C_{k_0}^{(-d_0/2)}(\eta_0) C_{k_1}^{(-d_1)}(\eta_1) ... C_{k_{h-1}}^{(-d_{h-1})}(\eta_{h-1}) C_{k_h}^{(-d_h/2)}(\eta_h) \quad (1.33)$$

for $j = 1, ...,$ where $\eta_i = \cos \omega_i$, $i = 0, 1, ..., h$, and $C_k^{(d)}(x)$ are orthogonal Gegenbauer polynomials. Similarly ψ_j is (1.33) with $d_0, ..., d_h$ instead of $-d_0, ..., -d_h$ (see Giraitis and Leipus (1995)). The weights π_j in (1.33) have the asymptotic behaviour

$$\pi_j \sim K[j^{-1-d_0} + (-1)^j j^{-1-d_h} + \sum_{k=1}^{h-1} j^{-d_k-1}(\cos(\omega_k j) + v_k)] \quad (1.34)$$

where K is a finite constant and v_k is a constant depending on $d_0, ..., d_h$ and ω_k. Similarly the ψ_j behave asymptotically as (1.34) with $d_0, ..., d_h$ instead of $-d_0, ..., -d_h$.

The complicated form of the ARUMA model in (1.32) makes it difficult to calculate an explicit formula for the autocovariances. As a matter of fact, they have only been obtained for the Gegenbauer process in (1.27) (see (1.29)), but if there are more than one spectral pole/zero, only the asymptotic behaviour has been established. Giraitis and Leipus (1995) showed that the autocovariances of the ARUMA process (1.32) satisfy

$$\gamma_j \sim K \sum_{k=0}^{h} j^{2d_k - 1} \cos(j\omega_k) \quad \text{as } j \to \infty$$

where K is a finite constant. Thus π_j, ψ_j and γ_j have slow decay with oscillations that depend on the different ω_k. Eventually it is the largest persistence parameter which governs the behaviour of π_j, ψ_j and γ_j.

The model (1.32) allows for spectral poles/zeros at any frequency $\omega_j \in [0, \pi]$. One particular case occurs when ω_j are seasonal frequencies, $\omega_j = 2\pi j/s$, $j = 1, 2, ..., [s/2]$. Then (1.32) has been called "flexible ARFISMA" (Hassler (1994)) or "flexible (seasonal) ARMA$(p, d, q)_s$" (Ooms (1995)).

1.4 ESTIMATION IN SCLM PROCESSES

Since the analysis of long memory in the flows of the river Nile and the introduction of the rescaled range statistic (R/S) to measure this phenomenon by Hurst (1951), interest in long memory processes has increased significantly. Some applications, analysis and extensions of the R/S statistic are Mandelbrot (1972,1975), Mandelbrot and Wallis (1968), Mandelbrot and Taqqu (1979), Taqqu (1975,1977), Davies and Harte (1987) and Lo (1991). The interest in the analysis of long memory in economics has its origin in Granger (1966) who observed that most economic variables have an estimated spectrum which is consistent with the behaviour of long memory processes.

Estimation and statistical inference in long memory processes can be done using parametric or semiparametric techniques. Parametric methods are generally more efficient if they are based on a correct and complete specification of $f(\lambda)$. However, parametric estimation of the persistence parameter, d in (1.2), can be inconsistent if $f(\lambda)$ is misspecified at frequencies far from ω. Semiparametric techniques, that only assume partial knowledge of $f(\lambda)$ around a known frequency (like in (1.2)), guarantee consistency under this type of misspecification. The price to pay is a loss of efficiency with respect to parametric methods when the model is correctly specified.

Since R/S analysis several techniques have been developed. Some of them, like R/S itself, are not suited for SCLM processes (see Ooms (1995)). In this section we review methods proposed to estimate SCLM processes and propose extensions of those techniques that have been developed for the standard long memory case at frequency zero.

1.4.1 Parametric Estimation

Consider the covariance stationary process, x_t, satisfying

$$\phi(L)(x_t - \mu) = \varepsilon_t \tag{1.35}$$

where

$$\phi(z) = 1 - \sum_{j=1}^{\infty} \phi_j z^j \quad , \quad \sum_{j=1}^{\infty} \phi_j^2 < \infty \quad , \tag{1.36}$$

μ is the population mean of x_t and the ε_t have zero mean and are uncorrelated with variance σ^2, for all t. All the stationary and invertible processes described in previous sections can be written as (1.35) satisfying (1.36). Suppose that the ϕ_j and σ^2, as well as μ are unknown, but we know a function

$$\phi(z;\theta) = 1 - \sum_{j=1}^{\infty} \phi_j(\theta) z^j$$

where θ is an unknown $k \times 1$ parameter vector such that there exists θ_0 for which $\phi_j(\theta_0) = \phi_j$ for all j, and therefore $\phi(z;\theta_0) = \phi(z)$. The spectral density of x_t is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi} |\phi(e^{i\lambda})|^{-2} \quad , \quad -\pi < \lambda \leq \pi \tag{1.37}$$

and the lag-j autocovariance by

$$\gamma_j = \int_{-\pi}^{\pi} f(\lambda) \cos(j\lambda) \mathrm{d}\lambda.$$

Writing $f(\lambda)$ and γ_j as a function of the unknown parameters, we have

$$\begin{aligned} f(\lambda;\theta,\sigma^2) &= \frac{\sigma^2}{2\pi} h(\lambda;\theta) \\ \gamma_j(\theta,\sigma^2) &= \sigma^2 \gamma_j(\theta) \end{aligned}$$

where

$$\gamma_j(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\lambda;\theta) \cos(j\lambda) \mathrm{d}\lambda$$

and $h(z;\theta) = |\phi(e^{iz};\theta)|^{-2}$. Thus the parameter vector θ_0 describes the autocorrelation properties of x_t. In this section we consider the so-called Gaussian estimates, although Gaussianity is not required to achieve good asymptotic properties. First, denote by $\Delta(\theta)$ the $n \times n$ Toeplitz matrix with (i,j)-th element $\gamma_{i-j}(\theta)$, by $\mathbf{1}$ the $n \times 1$ vector of ones and by x the $n \times 1$ vector of observations $(x_1, x_2, ..., x_n)'$. Consider the function

$$L_a(\theta,\mu,\sigma^2) = \frac{1}{2}\log\sigma^2 + \frac{1}{2}\log|\Delta(\theta)| + \frac{1}{2\sigma^2}(x-\mu\mathbf{1})'\Delta(\theta)^{-1}(x-\mu\mathbf{1}) \tag{1.38}$$

where μ and σ^2 are scalars. Define

$$(\hat{\theta}_a, \hat{\mu}_a, \hat{\sigma}_a^2) = \arg\min_{\theta,\mu,\sigma^2} L_a(\theta,\mu,\sigma^2)$$

where the minimization is carried out over an appropriate range of values of the unknown parameters. In case the ε_t in (1.35) (and therefore x_t) are Gaussian, $\hat{\theta}_a$ is a maximum likelihood estimate of θ_0.

As in other minimization problems introduced below, σ^2 and μ can be estimated in closed form and the nonlinear optimization carried out only with respect to θ. Under regularity conditions (e.g. Theil (1971) 8.5) $\hat{\theta}_a$ is consistent and

$$\sqrt{n}(\hat{\theta}_a - \theta_0) \xrightarrow{d} N_k(0, \Omega^{-1}) \tag{1.39}$$

where $\xrightarrow{d}$ means convergence in distribution, $N_k(\cdot,\cdot)$ is a k-variate normal and

$$\Omega = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial\theta}\log h(\lambda;\theta_0)\frac{\partial}{\partial\theta'}\log h(\lambda;\theta_0)\mathrm{d}\lambda. \tag{1.40}$$

Since the function $h(z;\theta)$ is known, Ω can be consistently estimated by, for example, substituting θ_0 in (1.40) by a consistent estimate (e.g. $\hat{\theta}_a$). These asymptotic properties do not rely on x_t being Gaussian, though under Gaussianity $\hat{\theta}_a$ is also asymptotically efficient.

We can approximate $L_a(\theta,\mu,\sigma^2)$ by

$$L_b(\theta,\mu,\sigma^2) = \frac{1}{2}\log\sigma^2 + \frac{1}{2\sigma^2}\sum_{t=1}^{n}\varepsilon_t^2(\theta,\mu) \tag{1.41}$$

where $\varepsilon_t(\theta,\mu) = \phi(L;\theta)(x_t - \mu)$ and we take $x_t = 0$ for $t \le 0$. We call

$$(\hat{\theta}_b, \hat{\mu}_b, \hat{\sigma}_b^2) = \arg\min_{\theta,\mu,\sigma^2} L_b(\theta,\mu,\sigma^2)$$

a (nonlinear) least squares estimate of (θ_0,μ,σ^2), where the minimization carries out over an appropriate range of values (see Box and Jenkins (1976)). Under regularity conditions, $\hat{\theta}_b$ has the same asymptotic properties as those of $\hat{\theta}_a$, that is $\hat{\theta}_b$ is consistent with asymptotic distribution (1.39), and if the ε_t are Gaussian it is asymptotically efficient.

Next define the centered periodogram

$$I_n(\lambda;\mu) = \frac{1}{2\pi n}|\sum_{t=1}^{n}(x_t - \mu)e^{it\lambda}|^2. \tag{1.42}$$

Whittle (1953) proposed to approximate $L_a(\theta, \mu, \sigma^2)$ by the frequency domain approximate likelihood

$$L_c(\theta, \mu, \sigma^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \log \sigma^2 h(\lambda; \theta) + \frac{I_n(\lambda; \mu)}{\sigma^2 h(\lambda; \theta)} \right\} \mathrm{d}\lambda. \tag{1.43}$$

The so-called Whittle estimates are

$$(\hat{\theta}_c, \hat{\mu}_c, \hat{\sigma}_c^2) = \arg \min_{\theta, \mu, \sigma^2} L_c(\theta, \mu, \sigma^2).$$

Under regularity conditions, $\hat{\theta}_c$ has the same asymptotic properties as $\hat{\theta}_a$ and $\hat{\theta}_b$.

Finally define the (uncentered) periodogram

$$I_n(\lambda) = \frac{1}{2\pi n} |\sum_{t=1}^{n} x_t e^{it\lambda}|^2. \tag{1.44}$$

Define the Fourier or harmonic frequencies $\lambda_j = 2\pi j/n$, and consider the discrete approximation to $L_c(\theta, \mu, \sigma^2)$ (see Hannan (1973c))

$$L_d(\theta, \sigma^2) = \frac{1}{n} \sum_j{}' \left\{ \log \sigma^2 h(\lambda_j; \theta) + \frac{I_n(\lambda_j)}{\sigma^2 h(\lambda_j; \theta)} \right\} \tag{1.45}$$

where $\sum_j'$ runs over all $j = 1, ..., n-1$, such that $0 < h(\lambda_j; \theta) < \infty$ for all admissible θ. By omitting $j = 0$ and n we avoid the need to estimate μ. Let

$$(\hat{\theta}_d, \hat{\sigma}_d^2) = \arg \min_{\theta, \sigma^2} L_d(\theta, \sigma^2)$$

where the minimization is over a compact subset of R^{k+1}. Then $\hat{\theta}_d$ has the same asymptotic properties as $\hat{\theta}_a$, $\hat{\theta}_b$ and $\hat{\theta}_c$ described above.

The relative computational needs of $\hat{\theta}_a$, $\hat{\theta}_b$, $\hat{\theta}_c$ and $\hat{\theta}_d$, which we call Gaussian estimates, depend on the parameterization we impose. In general, $\hat{\theta}_b$ is more easily calculated than $\hat{\theta}_a$ since it avoids the matrix inversion in (1.38). Despite its representation in terms of $h(\lambda; \theta)$, (1.43) avoids the matrix inversion in (1.38) as well as the linear transformation in (1.41). Furthermore, in many cases $h(\lambda; \theta)$ is more easily written down than $\gamma_j(\theta)$. Thus $\hat{\theta}_d$ has computational advantages, especially because it can make use of the fast Fourier transform.

The above discussion has made no reference to long memory or SCLM models, and in fact $\hat{\theta}_a$, $\hat{\theta}_b$, $\hat{\theta}_c$ and $\hat{\theta}_d$ and their asymptotic properties were originally obtained for short memory time series models such as stationary and invertible ARMAs (see for example Whittle (1953) or Hannan (1973c)). However the discussion also seems relevant to long memory and SCLM. In fact, for long memory models with a spectral pole/zero only at the origin, Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990) and Heyde and Gay (1993) provide asymptotic properties for $\hat{\theta}_c$ which are consistent with those earlier obtained for short memory processes, namely consistency, asymptotic normality and efficiency under Gaussianity. Li and McLeod (1986) and Sowell (1986, 1992) discuss computational aspects and Yajima (1985) asymptotic properties of $\hat{\theta}_a$ for fractional ARIMA processes

$$\Phi(L)(1-L)^d(x_t-\mu) = \Theta(L)\varepsilon_t \tag{1.46}$$

where d can be any real number and the zeros of $\Phi(z)$ and $\Theta(z)$ lie outside the unit circle. Beran (1994b) proposed a modified version of $\hat{\theta}_b$ for long memory processes that is robust against the presence of outliers. Asymptotic theory for $\hat{\theta}_d$ has not been considered explicitly for long memory models with a spectral pole at zero frequency but it seems it can be done by avoiding the spectral singularity with the omission of frequencies close to the origin in $L_d(\theta,\sigma^2)$. In the long memory case, $\hat{\theta}_d$ appears to have an extra advantage over $\hat{\theta}_a$, $\hat{\theta}_b$ and $\hat{\theta}_c$, because it does not require estimation of μ. When there is a spectral pole/zero at zero frequency, $\hat{\mu}_a$, $\hat{\mu}_b$ and $\hat{\mu}_c$ converge more slowly than $\sqrt{n}$ (see Vitale (1973), Adenstedt (1974) and Samarov and Taqqu (1988)) which can affect the finite sample properties of $\hat{\theta}_a$, $\hat{\theta}_b$ and $\hat{\theta}_c$ as discussed by Cheung and Diebold (1994) via Monte Carlo analysis.

The discussion of Gaussian estimates is also relevant to SCLM models with spectral poles/zeros at known frequencies different from zero. In case of processes satisfying (1.2) we have to obtain the function $h(\lambda;\theta)$ or $\gamma_j(\theta)$ and then the same optimization procedures can be applied to get $\hat{\theta}_a$, $\hat{\theta}_b$, $\hat{\theta}_c$ and $\hat{\theta}_d$. Nevertheless some comments are needed. The complicated form of the autocovariances (when they can be obtained in an explicit form) in many SCLM processes (see for example (1.29)) makes the matrix inversion in (1.38) rather difficult to calculate, and therefore obtaining $\hat{\theta}_a$ may be rather complicated. Chung (1996a,b) considered the estimate $\hat{\theta}_b$ for the GARMA process

$$\Phi(L)(1-2L\cos\omega+L^2)^d(x_t-\mu)=\Theta(L)\varepsilon_t \qquad (1.47)$$

where the ε_t are white noise $(0,\sigma^2)$, $|d|<1/2$ and $\Phi(z)$ and $\Theta(z)$ are polynomials of order p and q respectively with zeros outside the unit circle. Chung (1996a,b) claimed that $\hat{\theta}_b$ for (1.47) is $\sqrt{n}$-consistent and asymptotically normal.

Due to the natural expression of SCLM in the frequency domain, a more elegant manner of estimating SCLM processes like (1.47) seems to be $\hat{\theta}_c$ and $\hat{\theta}_d$. For (1.47)

$$h(\lambda;\theta)=\left|\frac{\Theta(e^{i\lambda})}{\Phi(e^{i\lambda})}\right|^2(4(\cos\lambda-\cos\omega)^2)^{-d}$$

where $\theta=(\Phi_1,...,\Phi_p,\Theta_1,...,\Theta_q,d)'$. More generally consider the stationary ARUMA model (assume $\mu=0$)

$$\Phi(L)\prod_{j=0}^{h}(1-2L\cos\omega_j+L^2)^{d_j}x_t=\Theta(L)\varepsilon_t \qquad (1.48)$$

where $d_j>0$ for all j, and $d_j<1/2$ if $\omega_j\neq 0,\pi$ and $d_j<1/4$ if $\omega_j=0,\pi$, $\Theta(z)$ and $\Phi(z)$ have their roots outside the unit circle and the ε_t are $iid(0,\sigma^2)$. In this case

$$h(\lambda;\theta)=\left|\frac{\Theta(e^{i\lambda})}{\Phi(e^{i\lambda})}\right|^2\prod_{j=0}^{h}(4(\cos\lambda-\cos\omega_j)^2)^{-d_j}$$

where $\theta=(\Phi_1,...,\Phi_p,\Theta_1,...,\Theta_q,d_0,...,d_h)'$. Giraitis and Leipus (1995) obtain consistency of $\hat{\theta}_c$ but they do not establish the asymptotic distribution, although a non-Gaussian limit distribution is conjectured.

As a matter of fact, Chung (1996a,b) and Giraitis and Leipus (1995) consider also the estimation of the frequencies ω_j, $j=0,...,h$, where the spectral poles/zeros occur. Since in this thesis we consider ω_j fixed and known we do not discuss the estimation of ω_j in this introductory chapter. However, a section of the concluding chapter is dedicated to review the literature on estimating frequency, and in particular on estimating ω_j in SCLM processes.

Hosoya (1996a,b) considered x_t in (1.35) a vector instead of a scalar. Allowing for spectral poles at a finite number of known frequencies, Hosoya proposed a multivariate extension of $L_c(\theta,\mu,\sigma^2)$ in (1.43)

$$L_c'(\theta,\mu,\sigma^2)=\int_{-\pi}^{\pi}\left\{\log|\sigma^2h(\lambda;\theta)|+tr\left[\frac{1}{\sigma^2}h^{-1}(\lambda;\theta)I_n(\lambda,\mu)\right]\right\}\mathrm{d}\lambda \qquad (1.49)$$

where $h(\lambda;\theta)$ is now a matrix defining the spectra and cross-spectra and $I_n(\lambda;\mu)$ is a matrix with the periodogram and cross-periodogram of the elements of the vector series x_t. Without assuming Gaussianity, Hosoya showed that the arguments that minimize (1.49) are asymptotically normal.

A different approach focuses attention on the semiparametric specification of the spectral density in (1.2). The estimation is based on a regression of the logarithm of the periodogram onto the logarithm of Fourier frequencies and known as log-periodogram regression. Since this technique is basically semiparametric we describe it more thoroughly in the next subsection where we review semiparametric methods of estimation. We mention it here because Kashyap and Eom (1988) proposed the use of all harmonic frequencies in the log-periodogram regression performed to obtain an estimate of d. This technique is used by Ray (1993) to estimate d_3 and d_{12} in the SCLM process

$$\phi_0(L)\phi_3(L^3)\phi_{12}(L^{12})(1-L^3)^{d_3}(1-L^{12})^{d_{12}}x_t = \theta_0(L)\theta_3(L^3)\theta_{12}(L^{12})\varepsilon_t \qquad (1.50)$$

where the ε_t are white noise. Ray (1993) uses these estimates of d_3 and d_{12} as a first step in the estimation of the complete model (1.50) for monthly IBM revenues.

1.4.2 Semiparametric Estimation

When we are only interested in the estimation of the persistence parameter, d in (1.2), we only need to specify $f(\lambda)$ around ω in order to obtain consistent estimates of d that we call semiparametric. This is a clear advantage with respect to parametric estimates that need a complete and correct specification of $f(\lambda)$ over the whole band of Nyquist frequencies, though in the event of such specification the parametric estimates have the competing advantage of converging faster. The semiparametric estimates we describe in this subsection are consistent even if we do not have any knowledge about the behaviour of $f(\lambda)$ at frequencies far from ω, whereas the parametric methods may be inconsistent if $f(\lambda)$ is misspecified at those frequencies. In this section we only assume that x_t is a process whose spectral density satisfies (1.2) around a known frequency ω.

Due to their simplicity, perhaps the most popular semiparametric procedures are variants of the one introduced by Geweke and Porter-Hudak (1983). This methodology, known as log-periodogram regression, has gained great popularity among empirical researchers, and often, a semiparametric estimate of d is used as a first step prior to a complete parametric fit of the model (see for example Geweke and Porter-Hudak (1983) or Diebold and Rudebusch (1989)). This class of estimate is based on a least squares regression of $\log I_n(\omega + \lambda_j)$ on $-2\log\lambda_j$ and an intercept, where $I_n(\lambda)$ is the periodogram defined in (1.44) and $\lambda_j = 2\pi j/n$ are Fourier frequencies. The regression is carried out for $j = 1, ..., m$, where m is an integer between 1 and $n/2$, called the bandwidth, satisfying at least

$$\frac{1}{m} + \frac{m}{n} \to 0 \quad \text{as } n \to \infty. \tag{1.51}$$

The original version, due to Geweke and Porter-Hudak (1983), uses instead of $-2\log\lambda_j$ the regressor $-\log\{4\sin^2(\lambda_j/2)\}$, but as indicated by Robinson (1995a), use of the simpler $-2\log\lambda_j$, which corresponds more naturally to (1.2), leads to equivalent asymptotic properties. This class of estimates was originally proposed for the standard long memory at zero frequency, (1.2) with $\omega = 0$. Note that in that case $I_n(\lambda_j)$ is an even function, so regression of $\log I_n(\lambda_j)$ on $-2\log|\lambda_j|$ for $j = \pm1, .., \pm m$ is equivalent to using frequencies for $j = 1, ..., m$. When $\omega \neq 0, \pi$, use of information on both sides of the pole/zero makes a substantial difference. Thus the log-periodogram estimate for such a ω is

$$\hat{d} = -\frac{1}{2}\frac{\sum_{j=\pm1}^{\pm m} v_j \log I_n(\omega + \lambda_j)}{\sum_{j=\pm1}^{\pm m} v_j^2} \tag{1.52}$$

where $v_j = \log|j| - \frac{1}{m}\sum_1^m \log l$. Work on estimating (1.2) with $\omega = 0$ suggests two possible modifications to this scheme. Due to anomalous behaviour of the periodogram very close to a spectral pole/zero (see Robinson (1995a), Kunsch (1986) and Hurvich and Beltrao (1993,1994)), Kunsch (1986) and Robinson (1995a) trimmed out some frequencies close to ω (the proofs of the asymptotics without trimming in Geweke and Porter-Hudak (1983) and Hassler (1993a,b) are incomplete as pointed out in Robinson (1995a)). The second type of modification is an efficiency improvement suggested by Robinson (1995a) and based on pooling adjacent periodogram ordinates.

Incorporating these two suggestions we have the estimate

$$\hat{d}^{(J)} = -\frac{1}{4}\frac{\sum_k' v_k[y_k^{(J)} + \tilde{y}_k^{(J)}]}{\sum_k' v_k^2} \tag{1.53}$$

where $y_k = \log(\sum_{j=1}^J I_n(\omega+\lambda_{k+j-J}))$, $\tilde{y}_k^{(J)} = \log(\sum_{j=1}^J I_n(\omega-\lambda_{k+j-J}))$, J is a positive integer (the pooling number) and $\sum_k'$ is a sum over $k = l + J, l + 2J, ..., m$. When the pooling number, $J = 1$, and the trimming number, $l = 0$, then (1.53) reduces to (1.52). When $\omega = 0$, and for symmetry of the periodogram at the origin we only use $m - l$ frequencies in the estimation, Robinson (1995a) proved that under Gaussianity

$$\sqrt{m}(\hat{d}^{(J)} - d) \xrightarrow{d} N\left(0, \frac{J\psi'(J)}{4}\right) \quad \text{as } n \to \infty$$

where $\psi'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\psi(z)$ and $\psi(z)$ is the digamma function defined as $\frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$ where $\Gamma(z)$ is the gamma function. In Chapter 3 we show that the same asymptotics follow for $\omega \neq 0$ in a more general spectral specification. Velasco (1997c) relaxes the assumption of Gaussianity and only imposes boundness of the fourth moments of the ε_t in (1.35). Using a tapered periodogram (cosine bell or hanning taper) he obtains consistency and asymptotic normality with variance $3J\psi'(J)/4$. Note that tapering increases the variance. Assuming Gaussianity Velasco (1997a) proved consistency of $\hat{d}^{(J)}$ for the non-stationary case $d \in [1/2, 1)$. Velasco (1997a) also shows that $\hat{d}^{(J)}$ is asymptotically normal with variance $J\psi'(J)/4$ for the non-tapered estimate if $d \in [1/2, 3/4)$, and $3J\psi'(J)/4$ for $d \in [1/2, 3/2)$ if the tapered periodogram is used. The good properties in finite samples of $\hat{d}^{(1)}$ for $d \in [1/2, 1)$ are shown in Hurvich and Ray (1995).

A variant of (1.52) has been proposed by Reisen (1994) and Chen et al. (1994). They used a smoothed periodogram instead of the raw periodogram in (1.52) trying to soften the anomalous behaviour of $I_n(\lambda)$ close to ω. Janacek (1982) introduced an alternative method to estimate d through estimation of the Fourier coefficients of $\log f(\lambda)$ using the log-periodogram. Although originally this estimate was proposed for long memory at zero frequency, Janacek claimed that this method can be naturally extended to SCLM time series.

Related to the parametric Gaussian estimates described in the previous section, Kunsch (1987) and Robinson (1995b) considered a semiparametric approximation of

$L_d(\theta, \sigma^2)$ in (1.45) assuming just the partial knowledge of $f(\lambda)$ described in (1.2). The estimate, $\tilde{d}$, is the argument that minimizes

$$Q(C,d) = \frac{1}{2m} \sum_{j=\pm 1}^{\pm m} \left\{ \log C|\lambda_j|^{-2d} + \frac{|\lambda_j|^{2d}}{C} I_n(\omega + \lambda_j) \right\} \tag{1.54}$$

where the bandwidth number, m, is an integer satisfying at least (1.51). The estimate $\tilde{d}$ has received the names of Gaussian semiparametric or local Whittle estimate. When $\omega = 0$ only frequencies on one side of ω are used, due to the symmetry of $I_n(\lambda)$ at the origin. But if $\omega \in (0, \pi)$, periodogram ordinates on both sides of ω are informative and should be used in the estimation. Without requiring Gaussianity, Robinson (1995b) obtained consistency and asymptotic normality for the case $\omega = 0$ such that

$$\sqrt{m}(\tilde{d} - d) \xrightarrow{d} N(0, 1/4).$$

Note that $\tilde{d}$ is asymptotically more efficient than $\hat{d}^{(J)}$ because $J\psi'(J) \downarrow 1$ as $J \to \infty$. The same asymptotics are shown to hold for $\omega \neq 0$ in Chapter 4 in a more general setup. Velasco (1997b) extended Robinson's results to non-stationary processes obtaining consistency for $d \in [1/2, 1)$ and asymptotic normality when $d \in [1/2, 2/3)$ ($d \in [1/2, 3/4)$ under Gaussianity).

Lobato (1995) extended Robinson's Gaussian semiparametric technique to a stationary long memory multivariate setup. Lobato considered x_t in (1.35) a $r \times 1$ vector with a-th element x_t^a and with spectral density matrix

$$f(\lambda) \sim \Lambda^0 G_0 \Lambda^0 \quad \text{as} \quad \lambda \to 0^+$$

where $\Lambda^0 = diag\{\lambda^{-d_a}\}$ for $a = 1, ..., r$, and G_0 is a positive definite Hermitian matrix. Note that under this specification every x_t^a has a spectrum

$$f_a(\lambda) \sim g_{aa} \lambda^{-2d_a} \quad \text{as} \quad \lambda \to 0^+$$

where g_{aa} is the a-th element in the diagonal of G_0. The objective function to minimize is a discrete semiparametric version of (1.49)

$$Q(G,d) = \sum_{j=1}^{m} \{\log|\Lambda_j G \Lambda_j| + tr[\Lambda_j^{-1} G^{-1} \Lambda_j^{-1} I_n(\lambda_j)]\} \tag{1.55}$$

where $\Lambda_j = diag\{\lambda_j^{-d_a}\}$. Under conditions similar to those in Robinson (1995b) for the univariate case, Lobato obtained consistency and asymptotic normality of the estimate of the vector of persistence parameters $d^0 = (d_1, ..., d_r)$,

$$\sqrt{m}(\tilde{d} - d^0) \xrightarrow{d} N_r(0, E^{-1})$$

where $E = 2I_r + 2Re(G_0 * (G_0^{-1})')$, I_r is the $r \times r$ identity matrix, Re denotes "the real part of" and $*$ is the Hadamard product. A generalization of this method to SCLM processes is analyzed in Chapter 4.

Robinson (1994c) proposed an alternative technique to estimate d when the spectral density satisfies

$$f(\omega + \lambda) \sim L\left(\frac{1}{|\lambda|}\right) |\lambda|^{-2d} \quad \text{as } \lambda \to 0 \tag{1.56}$$

where $L(z)$ is a slowly varying function, that is a positive measurable function satisfying

$$\frac{L(tz)}{L(z)} \to 1 \quad \text{as } z \to \infty \quad \text{for all } t > 0.$$

Note that (1.56) specializes to (1.2) when the function $L(z)$ is a constant. The proposed estimate is

$$\hat{d}_{qm\omega} = \frac{1}{2} - \frac{\log\{\hat{F}(q\lambda_m)/\hat{F}(\lambda_m)\}}{2\log q} \tag{1.57}$$

where

$$\hat{F}(\lambda) = \frac{2\pi}{n} \sum_{j=\pm 1}^{\pm[\lambda n/2\pi]} I_n(\omega + \lambda_j), \tag{1.58}$$

$\lambda_j = 2\pi j/n$, $q \in (0, 1)$ is a user chosen number and m is again a bandwidth parameter satisfying at least (1.51). With only second moment restrictions and without requiring Gaussianity, Robinson (1994c) showed the consistency of $\hat{d}_{qm\omega}$ for $\omega = 0$. In this case only periodogram ordinates on one side of zero frequency are used to construct (1.58). Assuming Gaussianity, Lobato and Robinson (1996) obtained the asymptotic distribution of $\hat{d}_{qm\omega}$ for $\omega = 0$. This is normal for $d \in (0, 1/4)$ and non-normal (related to Rosenblatt processes) for $d \in (1/4, 1/2)$. The same properties are likely to hold for $\omega \neq 0$.

Finally, based on an original idea of Parzen (1986) an alternative estimate of d in (1.2) has been proposed by Hidalgo and Yajima (1997),

$$\hat{d}^* = \frac{1}{m}\sum_{p=1}^{m} \hat{d}_p \tag{1.59}$$

where $\hat{d}_p = a_1/a_2$ and

$$\begin{aligned} a_1 &= \frac{1}{p}\sum_{l=1}^{p} w(l)\log \hat{f}_p(\lambda_l) - \left(\frac{1}{p}\sum_{l=1}^{p} w(l)\right)\log \hat{f}_p(\lambda_{p+1}) \\ a_2 &= -2\int_0^1 w(u)\log u du \end{aligned}$$

where $w(l) = (l/p)^{\frac{1}{c}} - (l/p)^{\frac{1}{c+1}}$, $c > 1$ and $\hat{f}_p(\lambda_l)$ is a particular moving average of periodogram ordinates at frequencies close to ω. Under some regularity conditions, but without assuming Gaussianity

$$\sqrt{m}(\hat{d}^* - d) \xrightarrow{d} N\left(0, \left(\frac{(1+c)(2+c)}{2(1+4c+2c^2)}\right)^2\right).$$

The variance of $\hat{d}^*$ is, for $c > 1$, smaller than that of $\tilde{d}$ so that a gain in asymptotic efficiency is achieved with respect to previous semiparametric estimates.

1.5 TESTING ON SEASONAL/CYCLICAL INTEGRATION AND COINTEGRATION

The characteristics of the process generating the series depend strongly on the value of the persistence parameter, d. In particular, d determines if the process has persistence (stationary or non-stationary), short memory or antipersistence (invertible or non-invertible). Some interesting situations that may require a rigorous test are

a) $d = 0$ (short memory) against $d > 0$ (persistence or long memory) or $d < 0$ (antipersistence),

b) $d = 1/2$ ("just" non-stationarity) against $d > 1/2$ (non-stationarity) or $d < 1/2$ (stationarity),

c) $d = -1/2$ ("just" non-invertibility) against $d > -1/2$ (invertibility) or $d < -1/2$ (non-invertibility).

The hypotheses involved in a) can be tested using simple t methods based on the estimates and their asymptotic distributions described in Section 1.4. t-tests of b) and c) can be carried out using those estimates whose limit distributional properties hold for non-stationary or non-invertible processes.

Traditionally, interest has focused on testing the possibility of unit roots where d in (1.2) is an integer. Some early work is due to Dickey, Hasza and Fuller (1984) who test the possibility of a seasonal unit root of the form

$$(1 - L^s)x_t = \varepsilon_t \quad t = 1, 2, ...$$

where the ε_t are iid $(0,\sigma^2)$ random variables, against the alternative

$$x_t = \alpha x_{t-s} + \varepsilon_t$$

with $|\alpha| < 1$. They provide percentiles for the proposed test statistic. One of the limitations of this procedure is that it is a joint test for unit roots at the origin and seasonal frequencies, $\omega_h = 2\pi h/s$, $h = 1, 2, ..., [s/2]$ (see (1.31) for the case $s = 4$). Furthermore the alternative is a specified form of s-th order autoregressive process. Hylleberg et al. (1990), using quarterly data, extended this procedure allowing for an individual test at zero and at every seasonal frequency that is robust to behaviour at other frequencies. Some extensions of this procedure to monthly data are Beaulieu and Miron (1993) and Franses (1991). The null hypothesis in all of them is pure integrability ($I_\omega(1)$) and the alternative is pure stationarity or short memory ($I_\omega(0)$). Canova and Hansen (1995) extended the test of Kwiatkowsky et al. (1992) to the seasonal case, testing the null of stationarity ($I_\omega(0)$) against the alternative of pure integration ($I_\omega(1)$). Bearing in mind the properties of these two types of tests, that basically differ in the specification of the null and alternative, the simultaneous use of both procedures has been advised in order to test for pure integrability. If the conclusion in both types of test is the same (i.e. one rejects and the other does not reject the null), then we conclude that there is strong evidence to accept the result implied by both procedures. If one test contradicts the other, then we need a more thorough analysis. In this case we may have fractional integration. A general test, based on the parametric model (1.32) and allowing for fractional and integer $I_\omega(d)$ as null and alternative, has been recently proposed by Robinson (1994a) and applied to quarterly macroeconomic data by Gil-Alaña and Robinson (1997). The procedure considers a scalar real-valued sequence satisfying

$$\begin{aligned} \phi(L)x_t &= u_t \qquad t = 1, 2, ..., \\ x_t &= 0 \qquad t \leq 0, \end{aligned}$$

where u_t is a short memory covariance stationary sequence with zero mean, and $\phi(z)$ is a known function. Consider the function $\phi(z;\vartheta)$ where ϑ is a p-dimensional vector of real valued parameters such that $\phi(z;\vartheta) = \phi(z)$ if and only if

$$H_0 : \vartheta = 0. \tag{1.60}$$

The hypotheses of principal interest entail ϕ of the form

$$\phi(L;\vartheta) = (1-L)^{d_0+\vartheta_{i_0}}\{\prod_{j=1}^{h-1}(1-2L\cos\omega_j + L^2)^{d_j+\vartheta_{i_j}}\}(1+L)^{d_h+\vartheta_{i_h}} \tag{1.61}$$

where for each j, $\vartheta_{i_j} = \vartheta_l$ for some l and for each l there is at least one j such that $\vartheta_{i_j} = \vartheta_l$. The null hypothesis to test is that the $p \times 1$ vector $(p \leq h+1)$ $\vartheta = (\vartheta_1, \vartheta_2, ..., \vartheta_p)$ is equal to a vector of zeros. Thus fractional seasonal and cyclical integration is allowed in the null and alternative. This is a new feature with respect to previous unit root tests that usually consider stationary AR and integrated processes of order one as null and alternative. To avoid estimation of the persistence parameters, Robinson (1994a) used a score test although undoubtedly the same asymptotic behaviour can be expected of Wald and likelihood ratio tests. When u_t is white noise the proposed test statistic is

$$R = \frac{n}{\tilde{\sigma}^4}\tilde{a}'\tilde{A}^{-1}\tilde{a}$$

where $\tilde{\sigma}^2 = \frac{1}{n}\sum_1^n u_t^2$, $u_t = \phi^{-1}(L;0)x_t$, $\tilde{a} = -\frac{2\pi}{n}\sum_j' \Psi(\lambda_j)I_u(\lambda_j)$, $I_u(\lambda)$ is the periodogram of u_t defined in (1.44), $\Psi(\lambda) = Re\{\frac{\partial}{\partial\vartheta}\log\phi(e^{i\lambda};0)\}$ and $\tilde{A} = \frac{2}{n}\sum_j' \Psi(\lambda_j)\Psi(\lambda_j)'$ where the primed sum is over $\lambda_j \in M = \{\lambda : -\pi < \lambda < \pi, \lambda \notin (\omega_l - \lambda_1, \omega_l + \lambda_1), l = 0, 1, ..., h\}$ and ω_l are the distinct poles of $\Psi(\lambda)$ on $(-\pi, \pi]$. Asymptotically equivalent expressions for $\tilde{a}$ and $\tilde{A}$ can be found in Robinson (1994a), as well as a time domain test statistic. Robinson (1994a) also proposed a modification of R that allows for parametric weak correlation in u_t as long as its spectrum is bounded and bounded away from zero and of known parametric form (although a fixed number of parameters may be unknown). Unlike the techniques earlier described these procedures have the advantage of being standard in the sense that the test statistic has a well known χ_p^2 limit distribution under the null and a limiting non-central χ_p^2 distribution against Pitman or local alternatives of the form

$$H_1 : \vartheta = \frac{\delta}{\sqrt{n}}$$

where δ is any $p \times 1$ vector. Furthermore they are asymptotically most powerful against those local alternatives.

Also of interest is the test of the hypothesis of equality of persistence parameters across different frequencies. This is done in Chapter 5 of this thesis on a semiparametric basis.

As far as cointegration is concerned, Hylleberg et al. (1990) considered the possibility of seasonal cointegration and defined this concept as

> A pair of series each of which are integrated at frequency ω are said to be cointegrated at that frequency if a linear combination of the series is not integrated at ω.

Hylleberg et al. (1990) pointed out that if the series present several spectral poles (as for example x_t in (1.32)) the procedure in Engle and Granger (1987) to test for cointegration at zero frequency is invalid, so that prior to any test for cointegration we have to filter the data in such a way that only the pole at the frequency where we suspect the cointegration occurs remains. For instance, if we want to test for cointegration at the origin, we have first to remove seasonal roots, for example by applying the seasonal summation operator, $S(L) = (1 + L + ... + L^{s-1})$, to the original series and then perform a standard cointegration test such as those discussed in Engle and Granger (1987).

Engle and Granger (1987) and Hylleberg et al. (1990) focus on pure cointegration, that is they only consider the possibility of a linear combination of $I_\omega(1)$ processes be $I_\omega(0)$. But our definition of SCLM or $I_\omega(d)$ processes allows for the possibility of fractional integration (fractional d) and cointegration. In this sense Engle et al. (1989) define cyclical cointegration in the following manner,

> A vector of series x_t, each component $I_\omega(d)$ (integrated of order d at frequency ω), may be said to be cointegrated at that frequency if there exists a vector α_ω such that $z_t^\omega = \alpha_\omega' x_t$ is integrated of lower order at ω.

As in the definition of SCLM, if the series are $I_\omega(d)$ at every seasonal frequency we are in the case of seasonal cointegration, but in general ω can be any frequency between 0 and π, both inclusive.

1.6 INTRODUCTION TO ASYMMETRIC SCLM

The research on SCLM reviewed in this chapter is based upon the semiparametric specification of the spectral density about ω described in (1.2). This definition imposes an asymptotic symmetry (that is for frequencies very close to ω) of $f(\lambda)$ around ω. Of course, this has to happen for $\omega = 0, mod(\pi)$ and real x_t. However, when $\omega \neq (0, \pi)$, $f(\lambda)$ need not be symmetric and can behave like

$$f(\omega + \lambda) \sim \begin{cases} C\lambda^{-2d_1} & \text{as} \quad \lambda \to 0^+ \\ D|\lambda|^{-2d_2} & \text{as} \quad \lambda \to 0^- \end{cases} \tag{1.62}$$

where $0 < C, D < \infty$, and we permit

$$d_1 \neq d_2 \qquad \text{and/or} \qquad C \neq D \tag{1.63}$$

so that we have two (possibly different) persistence parameters, d_1 and d_2 at the same frequency ω. In case $d_1 \neq d_2$ we say that x_t has asymmetric SCLM. Clearly (1.62) nests (1.2) as a special case. In Chapter 2 we analyze some parametric asymmetric SCLM processes that satisfy (1.62). Since its definition is naturally done in the frequency domain, we found the time domain parameterization of processes satisfying (1.62) and (1.63) rather difficult. Instead we analyze in Chapter 2 the behaviour of the autocovariances, and in most cases we are only able to give their asymptotic behaviour.

The asymmetry in (1.62) has not been considered to our knowledge in any work on seasonal or cyclical long memory done to date. This possibility will have important consequences on the estimation procedures described in Section 1.4. In particular, assuming (without loss of generality) that $d_2 > d_1$, the periodogram ordinates just before ω exert a relatively serious effect on those just after, contaminating in many cases the estimation of d_1 if these frequencies are used. In Chapters 3 and 4 we analyze the effects of this possible asymmetry on two semiparametric estimates, namely the log-periodogram in (1.53) and the Gaussian semiparametric or local Whittle based on the minimization of (1.54). Chapter 5 proposes some tests of the hypothesis of symmetry $d_1 = d_2$ in (1.62) as well as of the equality of persistence parameters across different frequencies, showing their good and standard asymptotic properties. The behaviour of these estimates and tests procedures in finite samples is studied in Chapter 6 via Monte Carlo analysis. In Chapter 7 we apply the techniques developed in earlier chapters to a monthly UK inflation series. Finally Chapter 8 suggests some possible uses and extensions of SCLM. In particular we include one section that reviews existing work on estimating frequency. We place this section in the concluding chapter because throughout the whole thesis we assume ω is known. Estimating ω, in case it is unknown, in SCLM processes, symmetric or asymmetric, is a rather difficult task and further research seems worthwhile.

Chapter 2

SEASONAL/CYCLICAL ASYMMETRIC LONG MEMORY

2.1 INTRODUCTION

Various parametric processes have been proposed to model seasonal/cyclical long memory defined by a spectral density that satisfies (1.2) for some ω different from zero. Some of them are described in Chapter 1. Perhaps the more general form is the process used for instance by Robinson (1994a) and Giraitis and Leipus (1995) that we introduced in (1.32) and we rewrite here,

$$D(L)x_t = u_t \quad \text{for} \quad t = 1, 2, ... \tag{2.1}$$
$$x_t = 0 \quad \text{if} \quad t \leq 0$$

where

$$D(z) = (1-z)^{d_0}(1+z)^{d_h} \prod_{j=1}^{h-1} (1 - 2z\cos\omega_j + z^2)^{d_j},$$

L is the lag operator such that $L^k x_t = x_{t-k}$ and u_t is a process with positive and bounded continuous spectral density (e.g. a stationary and invertible ARMA(p,q) process, $\Phi(L)u_t = \Theta(L)\varepsilon_t$, where ε_t is white noise $(0, \sigma^2)$ and the zeros of $\Phi(z)$ and $\Theta(z)$ lie outside the unit circle).

This general specification covers many cases studied by several authors in the Box-Jenkins setup. Some examples are the following:

1. $D(z) = (1-z)^d$. This parameterization corresponds to "fractional ARIMA" processes introduced by Hosking (1981) and Granger and Joyeux (1980).

2. $D(z) = (1+z)^d$. This process has a spectral pole/zero at frequency π , useful to model cycles with period two (for example many half-yearly series).

3. $D(z) = (1 - 2z\cos\omega + z^2)^d$. These are the Gegenbauer processes introduced by Hosking (1981) and extended and analysed in Gray et al.(1989) and Andel (1986), modelling a cyclical behaviour at any frequency ω between 0 and π.

4. $D(z) = (1-z)^{d_0}(1+z)^{d_{\frac{s}{2}}}\prod_{j=1}^{\frac{s}{2}-1}(1 - 2z\cos\frac{2\pi j}{s} + z^2)^{d_j}$. This model is called "flexible ARFISMA" (Hassler (1994)) or "flexible (seasonal) ARMA(p,d,q)$_s$" (Ooms (1995)), and allows for different persistence parameters at frequency zero and at each seasonal frequency $\frac{2\pi j}{s}$, $j = 1, 2, ..., s/2$, where s (that here we assume to be even) is the number of observations per year.

The spectral density function of the process in (2.1) is:

$$
\begin{aligned}
f(\lambda) &= |D(e^{i\lambda})|^{-2} f_u(\lambda) \qquad (2.2)\\
&= (4\sin^2\frac{\lambda}{2})^{-d_0}(4\cos^2\frac{\lambda}{2})^{-d_h}\prod_{j=1}^{h-1}\{4(\cos\omega_j - \cos\lambda)^2\}^{-d_j} f_u(\lambda)\\
&= (4\sin^2\frac{\lambda}{2})^{-d_0}(4\cos^2\frac{\lambda}{2})^{-d_h}\prod_{j=1}^{h-1}(4\sin^2\frac{\lambda+\omega_j}{2})^{-d_j}(4\sin^2\frac{\lambda-\omega_j}{2})^{-d_j} f_u(\lambda)
\end{aligned}
$$

where $f_u(\lambda)$ is positive and bounded (e.g. $f_u(\lambda) = \frac{\sigma^2}{2\pi}\frac{|\Theta(e^{i\lambda})|^2}{|\Phi(e^{i\lambda})|^2}$ for u_t an ARMA process). The second specification of $f(\lambda)$ will be useful in subsequent analysis.

As pointed out in Chapter 1, all these manners of modelling SCLM suffer the drawback of imposing the same memory parameter on either side of the possible spectral poles/zeros at frequencies $\omega \in (0, \pi)$. In Section 1.6 of Chapter 1 we introduced the notion of asymmetric SCLM and we defined it by saying that x_t is an asymmetric SCLM process if its spectral density satisfies (1.62) and (1.63). This implies two (possibly different) persistence parameters at each frequency between 0 and π with a spectral pole/zero. The possibility of spectral asymmetry imposes a serious difficulty when trying to parameterize a process with such a $f(\lambda)$ in the time domain. In fact we do not propose any time domain parametric model, like those in (2.1), with a spectral density satisfying (1.62) and (1.63). Instead we study the behaviour of the autocovariances of parametric asymmetric SCLM processes defined via a complete specification of its spectral density. In Section 2.2 we calculate the autocovariances of some asymmetric SCLM processes with only one spectral pole/zero. Section 2.3 generalizes the results obtained in Section 2.2 allowing for the possibility of a finite number of spectral poles. In this case no explicit form for the autocovariances is obtained, but only their asymptotic behaviour can be offered. Finally Section 2.4 analyses the asymptotic bias of the periodogram as an estimate of the spectral density in a general asymmetric SCLM with only one spectral pole/zero. This bias will be relevant when explaining the behaviour of different semiparametric estimates of d_1 and d_2 in (1.62) in following chapters.

2.2 ASYMMETRIC GEGENBAUER PROCESS

Let $\{x_t\}$ be a stationary SCLM process with spectral density

$$\begin{aligned} f(\lambda) &= \frac{\sigma_1^2}{2\pi}|1-2e^{i\lambda}\cos\omega+e^{i2\lambda}|^{-2d_1} \text{ if } \omega<\lambda\leq\pi \\ &= \frac{\sigma_2^2}{2\pi}|1-2e^{i\lambda}\cos\omega+e^{i2\lambda}|^{-2d_2} \text{ if } 0\leq\lambda\leq\omega. \end{aligned} \tag{2.3}$$

By analogy with the Gegenbauer processes analysed by Gray et al. (1989), we call a process with spectral density (2.3) *asymmetric Gegenbauer process.* $f(\lambda)$ in (2.3) can be written

$$\begin{aligned} f(\lambda) &= \frac{\sigma_1^2}{2\pi}2^{-4d_1}\{\sin^2\frac{\lambda+\omega}{2}\}^{-d_1}\{\sin^2\frac{\lambda-\omega}{2}\}^{-d_1} \text{ if } \omega<\lambda\leq\pi \\ &= \frac{\sigma_2^2}{2\pi}2^{-4d_2}\{\sin^2\frac{\lambda+\omega}{2}\}^{-d_2}\{\sin^2\frac{\lambda-\omega}{2}\}^{-d_2} \text{ if } 0\leq\lambda\leq\omega \end{aligned}$$

and thus its asymptotic behaviour around the frequency ω is

$$\begin{aligned} f(\lambda) &\sim \frac{\sigma_1^2}{2\pi}2^{-2d_1}\{\sin\frac{\lambda+\omega}{2}\}^{-2d_1}|\lambda-\omega|^{-2d_1} \text{ as } \lambda\downarrow\omega \\ &\sim \frac{\sigma_2^2}{2\pi}2^{-2d_2}\{\sin\frac{\lambda+\omega}{2}\}^{-2d_2}|\lambda-\omega|^{-2d_2} \text{ as } \lambda\uparrow\omega. \end{aligned}$$

Now we extend the Lemma in Gray et al. (1989) to the asymmetric case. This extension, stated in Lemma 1, will be useful when investigating the behaviour of the autocovariances of a process with spectral density (2.3).

Lemma 1 *Let $R(k) = \int_0^\pi f(w)\cos(kw)dw$ for an integer k. Let $w_0 \in (0,\pi)$ and suppose that $f(w)$ can be expressed as:*

$$\begin{aligned} f(w) &= b_1(w)|w-w_0|^{-\beta_1} \text{ if } w \in (w_0,\pi] \\ &= b_2(w)|w-w_0|^{-\beta_2} \text{ if } w \in [0,w_0] \end{aligned}$$

where $0 < \beta_1, \beta_2 < 1$, $b_2(w)$ is a function of bounded variation in $(0, w_0-\varepsilon)$ and slowly varying from the left at w_0 and $b_1(w)$ is a function of bounded variation in $(w_0+\varepsilon,\pi)$ and slowly varying at w_0 from the right, where $\varepsilon > 0$. Then when $k \to \infty$:

$$\begin{aligned} R(k) &\sim k^{\beta_1-1}\sin(\frac{\pi}{2}\beta_1 - kw_0)b_1(w_0+\frac{1}{k})\Gamma(1-\beta_1) \\ &+ k^{\beta_2-1}\sin(\frac{\pi}{2}\beta_2 + kw_0)b_2(w_0-\frac{1}{k})\Gamma(1-\beta_2) \end{aligned}$$

where $a \sim b$ if $\frac{a}{b} \to 1$ and $\Gamma(\cdot)$ is the gamma function.

Proof: Write

$$\begin{aligned} R(k) &= \int_0^\pi f(w)\cos(kw)\mathrm{d}w \\ &= \int_0^{w_0} b_2(w)|w-w_0|^{-\beta_2}\cos(kw)\mathrm{d}w + \int_{w_0}^\pi b_1(w)|w-w_0|^{-\beta_1}\cos(kw)\mathrm{d}w \\ &= S_a + S_b. \end{aligned}$$

Using the change of variable $x = w - w_0$ the integral S_a can be expressed

$$\begin{aligned} S_a &= \int_0^{w_0} b_2(w)|w-w_0|^{-\beta_2}\cos(kw)\mathrm{d}w \\ &= \int_{-w_0}^0 |x|^{-\beta_2}b_2(x+w_0)\cos(k[x+w_0])\mathrm{d}x \\ &= \int_{-w_0}^0 (-x)^{-\beta_2}b_2(x+w_0)\cos(kx)\cos(kw_0)\mathrm{d}x \\ &- \int_{-w_0}^0 (-x)^{-\beta_2}b_2(x+w_0)\sin(kx)\sin(kw_0)\mathrm{d}x \\ &= S_{a1} - S_{a2}. \end{aligned}$$

Now

$$S_{a1} = \int_0^{w_0} x^{-\beta_2}b_2(w_0-x)\cos(kw_0)\cos(kx)\mathrm{d}x = \cos(kw_0)\int_0^\pi x^{-\beta_2}b_2^1(x)\cos(kx)\mathrm{d}x$$

where

$$\begin{aligned} b_2^1(x) = & \quad b_2(w_0 - x) \quad \text{if } x \in (0, w_0] \\ = & \quad\quad 0 \quad\quad\quad \text{if } x \in (w_0, \pi] \end{aligned}$$

$b_2^1(x)$ is of bounded variation in any interval (ε, π), $\varepsilon > 0$, (because $b_2(w)$ is of bounded variation in $(0, w_0 - \varepsilon)$) and slowly varying as $x \to 0^+$ ($b_2(w)$ is slowly varying from the left at w_0). Then applying Theorem 2.24 of Zygmund (1977, Chapter 5),

$$S_{a1} \sim k^{\beta_2 - 1} b_2^1(\frac{1}{k}) \Gamma(1 - \beta_2) \sin \frac{\pi \beta_2}{2} \cos(k w_0) \text{ as } k \to \infty.$$

Similarly

$$\begin{aligned} S_{a2} &= -\sin(kw_0) \int_0^{w_0} x^{-\beta_2} b_2(w_0 - x) \sin(kx) \mathrm{d}x \\ &= -\sin(kw_0) \int_0^{\pi} x^{-\beta_2} b_2^1(x) \sin(kx) \mathrm{d}x \end{aligned}$$

and applying the same theorem,

$$S_{a2} \sim -k^{\beta_2 - 1} \sin(kw_0) b_2^1(\frac{1}{k}) \Gamma(1 - \beta_2) \cos \frac{\pi \beta_2}{2} \quad \text{as } k \to \infty.$$

Thus

$$\begin{aligned} S_a &\sim k^{\beta_2 - 1} \Gamma(1 - \beta_2) b_2^1(\frac{1}{k}) \left\{ \sin \frac{\pi \beta_2}{2} \cos(kw_0) + \sin(kw_0) \cos \frac{\pi \beta_2}{2} \right\} \\ &= k^{\beta_2 - 1} \Gamma(1 - \beta_2) b_2^1(\frac{1}{k}) \sin(\frac{\pi \beta_2}{2} + kw_0) \quad \text{as } k \to \infty. \end{aligned}$$

With respect to S_b,

$$\begin{aligned} S_b &= \int_{w_0}^{\pi} b_1(w) |w - w_0|^{-\beta_1} \cos(kw) \mathrm{d}w \\ &= \int_0^{\pi - w_0} |x|^{-\beta_1} b_1(x + w_0) \cos(k[x + w_0]) \mathrm{d}x \\ &= \int_0^{\pi - w_0} (x)^{-\beta_1} b_1(x + w_0) \cos(kx) \cos(kw_0) \mathrm{d}x \\ &- \int_0^{\pi - w_0} (x)^{-\beta_1} b_1(x + w_0) \sin(kx) \sin(kw_0) \mathrm{d}x \\ &= S_{b1} - S_{b2}. \end{aligned}$$

As before

$$S_{b1} = \cos(kw_0) \int_0^{\pi} x^{-\beta_1} b_1^1(x) \cos(kx) \mathrm{d}x$$

where

$$b_1^1(x) = \quad b_1(w_0 + x) \quad \text{if } x \in (0, \pi - w_0]$$
$$= \quad 0 \quad \text{if } x \in (\pi - w_0, \pi].$$

Since $b_1^1(x)$ is of bounded variation in any interval (ε, π), $\varepsilon > 0$ ($b_1(w)$ is of bounded variation in $(w_0 + \varepsilon, \pi)$) and slowly varying as $x \to 0^+$ ($b_1(w)$ is slowly varying from the right at w_0) we again apply Theorem 2.24 of Zygmund (1977, Chapter 5) and obtain,

$$S_{b1} \sim k^{\beta_1 - 1} \cos(kw_0) b_1^1(\frac{1}{k}) \Gamma(1 - \beta_1) \sin \frac{\pi \beta_1}{2} \quad \text{as } k \to \infty.$$

Similarly

$$\begin{aligned} S_{b2} &= \int_0^\pi x^{-\beta_1} b_1^1(x) \sin(kx) \sin(kw_0) \mathrm{d}x \\ &\sim k^{\beta_1 - 1} \sin(kw_0) b_1^1(\frac{1}{k}) \Gamma(1 - \beta_1) \cos \frac{\pi \beta_1}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} S_b &= S_{b1} - S_{b2} \\ &\sim k^{\beta_1 - 1} b_1^1(\frac{1}{k}) \Gamma(1 - \beta_1) \{\cos(kw_0) \sin \frac{\pi \beta_1}{2} - \sin(kw_0) \cos \frac{\pi \beta_1}{2}\} \\ &= k^{\beta_1 - 1} b_1^1(\frac{1}{k}) \Gamma(1 - \beta_1) \sin(\frac{\pi \beta_1}{2} - kw_0) \end{aligned}$$

and

$$\begin{aligned} R(k) &\sim k^{\beta_1 - 1} b_1^1(\frac{1}{k}) \Gamma(1 - \beta_1) \sin(\frac{\pi \beta_1}{2} - kw_0) \\ &+ k^{\beta_2 - 1} b_2^1(\frac{1}{k}) \Gamma(1 - \beta_2) \sin(\frac{\pi \beta_2}{2} + kw_0) \quad \text{as } k \to \infty \quad \square \end{aligned}$$

Using Lemma 1 we show in the following theorem that the autocovariances of a process with spectral density (2.3) not only decrease in a hyperbolic rate typical of long-range dependent data but also exhibit the cyclic behaviour of the sine function with a period depending on ω.

Theorem 1 *Let x_t be a stationary process with spectral density function*

$$\begin{aligned} f(\lambda) &= \frac{\sigma_1^2}{2\pi} |1 - 2e^{i\lambda} \cos \omega + e^{i2\lambda}|^{-2d_1} \quad \textit{if } \omega < \lambda \leq \pi \\ &= \frac{\sigma_2^2}{2\pi} |1 - 2e^{i\lambda} \cos \omega + e^{i2\lambda}|^{-2d_2} \quad \textit{if } 0 \leq \lambda \leq \omega \end{aligned}$$

where $\omega \neq 0$ and $0 < d_1, d_2 < 1/2$, and denote $\gamma_j = E[x_t - Ex_1][x_{t-j} - Ex_1]$ the lag-j autocovariance. Then

$$\gamma_j \approx j^{2d_1-1}\sin(\pi d_1 - j\omega) + j^{2d_2-1}\sin(\pi d_2 + j\omega) \quad as\ j \to \infty.$$

where $a \approx b$ if $\frac{a}{b} \to C$ where C is a finite non-zero constant.

Proof: Due to symmetry of $f(\lambda_{\gamma_j} = 2\int_0^\pi f(\lambda)\cos(j\lambda)d\lambda.$

$$\gamma_j = 2\int_0^\pi f(\lambda)\cos(j\lambda)d\lambda.$$

Now $f(\lambda)$ is

$$\frac{\sigma_1^2}{2\pi}2^{-4d_1}\left\{\sin^2(\frac{\lambda+\omega}{2})\right\}^{-d_1}\left\{\left(\frac{\sin\frac{\lambda-\omega}{2}}{\lambda-\omega}\right)^2\right\}^{-d_1}|\lambda-\omega|^{-2d_1} \quad \text{if} \quad \omega < \lambda \leq \pi$$

$$\frac{\sigma_2^2}{2\pi}2^{-4d_2}\left\{\sin^2(\frac{\lambda+\omega}{2})\right\}^{-d_2}\left\{\left(\frac{\sin\frac{\lambda-\omega}{2}}{\lambda-\omega}\right)^2\right\}^{-d_2}|\lambda-\omega|^{-2d_2} \quad \text{if} \quad 0 \leq \lambda \leq \omega$$

that is,

$$\begin{aligned} f(\lambda) &= b_1(\lambda)|\lambda-\omega|^{-2d_1} \quad \text{if } \omega < \lambda \leq \pi \\ &= b_2(\lambda)|\lambda-\omega|^{-2d_2} \quad \text{if } 0 \leq \lambda \leq \omega \end{aligned}$$

In order to apply the previous lemma we have to show:

a) $b_2(\lambda)$ is of bounded variation in $(0, \omega - \varepsilon)$.
$b_1(\lambda)$ is of bounded variation in $(\omega + \varepsilon, \pi)$.

b) $b_2(\lambda)$ is slowly varying from the left at ω.
$b_1(\lambda)$ is slowly varying from the right at ω.

The proof of a) is clear from the form of $b_1(\lambda)$ and $b_2(\lambda)$. In order to show b) we say that $b_2(\lambda)$ is slowly varying from the left at ω if :

1. $(\omega - \lambda)^\delta b_2(\lambda)$ is decreasing,

2. $(\omega - \lambda)^{-\delta} b_2(\lambda)$ is increasing

in some left-hand neighbourhood of ω, $(\lambda < \omega)$, for $\delta > 0$.

1. Define the function

$$\begin{aligned}
\phi_{21}(\lambda) &= (\omega-\lambda)^{\delta} b_2(\lambda) \\
&= \frac{\sigma_2^2}{2\pi}(\omega-\lambda)^{\delta+2d_2} 2^{-4d_2} \left\{\sin\frac{\lambda+\omega}{2}\right\}^{-2d_2} \left\{\sin\frac{\lambda-\omega}{2}\right\}^{-2d_2} \\
&= \frac{\sigma_2^2}{2\pi} 4^{-d_2}(\omega-\lambda)^{\delta+2d_2}(\cos\lambda-\cos\omega)^{-2d_2}.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{21}(\lambda) &= \frac{\sigma_2^2}{2\pi} 4^{-d_2}(\omega-\lambda)^{\delta+2d_2-1}(\cos\lambda-\cos\omega)^{-2d_2-1} \\
&\times \{-(\delta+2d_2)(\cos\lambda-\cos\omega)+2d_2\sin\lambda(\omega-\lambda)\}.
\end{aligned}$$

The terms outside the braces are positive because $\lambda<\omega$. The expression within braces is negative if

$$\frac{2d_2(\omega-\lambda)\sin\lambda}{(\delta+2d_2)(\cos\lambda-\cos\omega)}<1.$$

By L'Hopital it tends to $\frac{2d_2}{\delta+2d_2}<1$ as $\lambda\to\omega$ so that 1 holds for λ close enough to ω.

2. Define

$$\phi_{22}(\lambda)=(\omega-\lambda)^{-\delta} b_2(\lambda)=\frac{\sigma_2^2}{2\pi}4^{-d_2}(\omega-\lambda)^{2d_2-\delta}(\cos\lambda-\cos\omega)^{-2d_2}.$$

Differentiating with respect to λ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{22}(\lambda) &= \frac{\sigma_2^2}{2\pi} 4^{-d_2}(\omega-\lambda)^{2d_2-\delta-1}(\cos\lambda-\cos\omega)^{-2d_2-1} \\
&\times \{-(2d_2-\delta)(\cos\lambda-\cos\omega)+2d_2(\omega-\lambda)\sin\lambda\}
\end{aligned}$$

that is positive for $\lambda<\omega$ if $2d_2(\omega-\lambda)\sin\lambda>(2d_2-\delta)(\cos\lambda-\cos\omega)$. As $\lambda\to\omega$ we have that

$$\frac{2d_2(\omega-\lambda)\sin\lambda}{(2d_2-\delta)(\cos\lambda-\cos\omega)}\to\frac{2d_2}{2d_2-\delta}>1$$

for a small enough δ. Note that if $(\omega-\lambda)^{-\delta}b_2(\lambda)$ is increasing for a small enough δ, the same holds for any $a>\delta$ because $(\omega-\lambda)^{-a}b_2(\lambda)=(\omega-\lambda)^{-a+\delta}(\omega-\lambda)^{-\delta}b_2(\lambda)$ so that $(\omega-\lambda)^{a}b_2(\lambda)$ is the product of two positive (since $\omega>\lambda$) and increasing functions and thus is itself increasing. Thus 2 is proved.

Now we have to show that $b_1(\lambda)$ is slowly varying from the right at ω. In order to do that we will check that

1. $(\lambda-\omega)^{\delta}b_1(\lambda)$ is increasing

2. $(\lambda-\omega)^{-\delta}b_1(\lambda)$ is decreasing

in some right-hand neighbourhood of ω, $(\lambda > \omega)$, with $\delta > 0$.

1. Define the function

$$\phi_{11}(\lambda) = (\lambda-\omega)^{\delta}b_1(\lambda) = \frac{\sigma_1^2}{2\pi}4^{-d_1}(\lambda-\omega)^{\delta+2d_1}(\cos\lambda - \cos\omega)^{-2d_1}.$$

Now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{11}(\lambda) &= \frac{\sigma_1^2}{2\pi}4^{-d_1}(\lambda-\omega)^{\delta+2d_1-1}(\cos\lambda-\cos\omega)^{-2d_1-1}\\ &\times \{(\delta+2d_1)(\cos\lambda-\cos\omega)+2d_1\sin\lambda(\lambda-\omega)\}.\end{aligned}$$

Since $\lambda > \omega$ the term outside the braces is negative and thus $\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{11}(\lambda)$ is positive for λ close enough to ω because

$$\frac{(\delta+2d_1)(\cos\omega-\cos\lambda)}{2d_1(\lambda-\omega)\sin\lambda} \to \frac{\delta+2d_1}{2d_1} > 1 \text{ as } \lambda\to\omega.$$

2. Define

$$\phi_{12}(\lambda) = (\lambda-\omega)^{-\delta}b_1(\lambda) = \frac{\sigma_1^2}{2\pi}4^{-d_1}(\lambda-\omega)^{2d_1-\delta}(\cos\lambda-\cos\omega)^{-2d_1}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{12}(\lambda) &= \frac{\sigma_1^2}{2\pi}4^{-d_1}(\lambda-\omega)^{2d_1-\delta-1}(\cos\lambda-\cos\omega)^{-2d_1-1}\\ &\times \{(2d_1-\delta)(\cos\lambda-\cos\omega)+2d_1(\lambda-\omega)\sin\lambda\}\end{aligned}$$

that is negative for λ close to ω because

$$\frac{2d_1(\lambda-\omega)\sin\lambda}{(2d_1-\delta)(\cos\omega-\cos\lambda)} \to \frac{2d_1}{2d_1-\delta} > 1 \text{ as } \lambda\to\omega.$$

Thus if we apply Lemma 1 we obtain

$$\begin{aligned}\gamma_j &\sim j^{2d_1-1}2b_1(\omega+\frac{1}{j})\Gamma(1-2d_1)\sin(\pi d_1 - j\omega)\\ &+ j^{2d_2-1}2b_2(\omega-\frac{1}{j})\Gamma(1-2d_2)\sin(\pi d_2 + j\omega)\end{aligned}$$

as $k \to \infty$, what concludes the proof because $b_2(\omega - \frac{1}{j})$ and $b_1(\omega + \frac{1}{j})$ tend to finite non-zero constants as $j \to \infty$. □

Remark 1: If $d_1 = d_2 = d$ and $\sigma_1^2 = \sigma_2^2$ then $b_1(\lambda) = b_2(\lambda) = b(\lambda)$ and we obtain the same result as in Gray et al. (1989, 1994) and Chung (1996a and b), because in that case γ_j behaves as $j \to \infty$,

$$\begin{aligned}\gamma_j &\sim 2j^{2d-1}\Gamma(1-2d)[\cos(j\omega)\sin(\pi d)\{b(\omega-\frac{1}{j})+b(\omega+\frac{1}{j})\}\\ &\quad + \sin(j\omega)\cos(\pi d)\{b(\omega-\frac{1}{j})-b(\omega+\frac{1}{j})\}]\end{aligned}$$

and thus $\gamma_j \approx j^{2d-1}\cos(j\omega)$ because $b(\omega - \frac{1}{j}) + b(\omega + \frac{1}{j})$ approaches a finite non-zero constant and $b(\omega - \frac{1}{j}) - b(\omega + \frac{1}{j})$ approaches zero as $j \to \infty$.

Remark 2: Some of the heuristic approaches to estimate the persistence parameter at the origin that use the sample autocovariances (for a description of some of these techniques see Delgado and Robinson (1994)) are not valid to estimate the persistence parameter at frequencies $\omega \neq 0$ because they require γ_j to be eventually positive, a condition that does not hold if $\omega \neq 0(\mod 2\pi)$.

Remark 3: Parallel to and independent of this work, Chung (1996a) has obtained an explicit expression for the autocovariances of the GARMA(0,0) process of the form $(1 - 2L\cos\omega + L^2)^d x_t = \varepsilon_t$ with ε_t white noise. These autocovariances have the form described in (1.29) in Chapter 1. With a slight modification of Chung's proof and using equation 3.663.1 in Gradshteyn and Ryzhik (1980) we obtain the following exact expression for the autocovariances of a process with spectral density function (2.3) but with d_1 and d_2 constrained only to be less than 1/2 for stationarity,

$$\begin{aligned}\gamma_j &= \frac{\sigma_2^2}{2\sqrt{\pi}}(2\sin\omega)^{\frac{1}{2}-2d_2}\Gamma(1-2d_2)P_{j-\frac{1}{2}}^{2d_2-\frac{1}{2}}(\cos\omega)\\ &+ (-1)^j\frac{\sigma_1^2}{2\sqrt{\pi}}(2\sin\omega)^{\frac{1}{2}-2d_1}\Gamma(1-2d_1)P_{j-\frac{1}{2}}^{2d_1-\frac{1}{2}}(-\cos\omega) \qquad (2.4)\end{aligned}$$

where $P_a^b(x)$ are associated Legendre functions. Applying equation 8.721.3 in Gradshteyn and Ryzhik (1980), namely

$$P_a^b(\cos\theta) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(a+b+1)}{\Gamma(a+\frac{3}{2})} \frac{\cos[(a+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{b\pi}{2}]}{\sqrt{2\sin\theta}} \left[1 + O\left(\frac{1}{a}\right)\right] \tag{2.5}$$

and Stirling's formula to (2.4) we obtain the result stated in Theorem 1 using the fact that $-\cos\omega = \cos(\pi - \omega)$. Note that while Theorem 1 is only valid for the persistent and stationary case ($0 < d_1, d_2 < 1/2$), using expressions (2.4) and (2.5) we obtain the asymptotic behaviour of the autocovariances stated in Theorem 1 as long as $d_1, d_2 < 1/2$.

In case $\omega = \pi/2$, (2.4) can be enormously simplified and the autocovariances γ_j have a much simpler form. This fact is stated in the following proposition.

Proposition 1 *Let $\{x_t\}$ be a stationary process with spectral density*

$$\begin{aligned} f(\lambda) &= \frac{\sigma_1^2}{2\pi}|1+e^{i2\lambda}|^{-2d_1} \quad \text{if } \frac{\pi}{2} < \lambda \le \pi \\ &= \frac{\sigma_2^2}{2\pi}|1+e^{i2\lambda}|^{-2d_2} \quad \text{if } 0 \le \lambda \le \frac{\pi}{2} \end{aligned}$$

where $d_1, d_2 < 1/2$. Then the lag-j autocovariance, γ_j, is equal to

$$\frac{\sigma_2^2}{2} \frac{\Gamma(1-2d_2)}{\Gamma(1-d_2-\frac{j}{2})\Gamma(1-d_2+\frac{j}{2})} + (-1)^j \frac{\sigma_1^2}{2} \frac{\Gamma(1-2d_1)}{\Gamma(1-d_1-\frac{j}{2})\Gamma(1-d_1+\frac{j}{2})} \tag{2.6}$$

for $j = 0, \pm 1, \pm 2, \ldots$

Proof: The proof is shown in two different ways.

First we can use expression (2.4) and equation 8.756.1 in Gradshteyn and Ryzhik (1980),

$$P_a^b(0) = \frac{2^b\sqrt{\pi}}{\Gamma(\frac{a-b}{2}+1)\Gamma(\frac{-a-b+1}{2})}$$

and we obtain the desired result.

Secondly, without using (2.4) and directly from the form of the spectral density, we can prove the proposition in the following manner. Write

$$\begin{aligned} \gamma_j &= 2\int_0^{\pi} f(\lambda)\cos(j\lambda)d\lambda \\ &= \frac{\sigma_2^2}{\pi}\left\{\int_0^{\frac{\pi}{2}} 2^{-2d_2}(\cos\lambda)^{-2d_2}\cos(j\lambda)d\lambda\right\} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sigma_1^2}{\pi}\left\{\int_{\frac{\pi}{2}}^{\pi} 2^{-2d_1}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda\right\} \\
&= \frac{\sigma_2^2}{\pi}2^{-2d_2}S_2 + \frac{\sigma_1^2}{\pi}2^{-2d_1}S_1.
\end{aligned}
$$

Now

$$
S_2 = \int_0^{\frac{\pi}{2}}(\cos\lambda)^{-2d_2}\cos(j\lambda)\mathrm{d}\lambda = \frac{\pi\Gamma(1-2d_2)}{2^{1-2d_2}\Gamma(1-d_2+\frac{j}{2})\Gamma(1-d_2-\frac{j}{2})}
$$

applying formula 3.631.9 in Gradshteyn and Ryzhik (1980). Also

$$
S_1 = \int_{\frac{\pi}{2}}^{\pi}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda = \int_0^{\frac{\pi}{2}}[\cos(\lambda+\frac{\pi}{2})]^{-2d_1}\cos[j(\lambda+\frac{\pi}{2})]\mathrm{d}\lambda.
$$

Now

$$
\begin{aligned}
\cos(\lambda+\frac{\pi}{2}) &= -\sin\lambda \\
\cos[j(\lambda+\frac{\pi}{2})] &= \cos j\lambda\cos\frac{j\pi}{2} - \sin j\lambda\sin\frac{j\pi}{2} \\
&= (-1)^{\frac{j}{2}}\cos j\lambda \quad \text{if } j \text{ even} \\
&= (-1)^{\frac{j+1}{2}}\sin j\lambda \quad \text{if } j \text{ odd}.
\end{aligned}
$$

Thus if j is even

$$
S_1 = (-1)^{\frac{j}{2}}\int_0^{\frac{\pi}{2}}(\sin\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda = (-1)^{\frac{j}{2}}\left[\int_0^{\pi} - \int_{\frac{\pi}{2}}^{\pi}\right]
$$

and

$$
\begin{aligned}
&\int_{\frac{\pi}{2}}^{\pi}(\sin\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda \\
&= \int_0^{\frac{\pi}{2}}(\sin(\lambda+\frac{\pi}{2}))^{-2d_1}\cos(j(\lambda+\frac{\pi}{2}))\mathrm{d}\lambda \\
&= (-1)^{\frac{j}{2}}\int_0^{\frac{\pi}{2}}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda.
\end{aligned}
$$

Then

$$
\begin{aligned}
S_1 &= (-1)^{\frac{j}{2}}\int_0^{\pi}(\sin\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda - \int_0^{\frac{\pi}{2}}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda \\
&= \frac{(-1)^{\frac{j}{2}}\pi\cos(\frac{j\pi}{2})\Gamma(1-2d_1)}{2^{-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})} - \frac{\pi\Gamma(1-2d_1)}{2^{1-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})} \\
&= \frac{\pi\Gamma(1-2d_1)}{2^{1-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})}
\end{aligned}
$$

by formulae 3.631.8 and 3.631.9 in Gradshteyn and Ryzhik (1980).

Thus if j is even

$$\gamma_j = \frac{\sigma_2^2}{2}\frac{\Gamma(1-2d_2)}{\Gamma(1-d_2-\frac{j}{2})\Gamma(1-d_2+\frac{j}{2})} + \frac{\sigma_1^2}{2}\frac{\Gamma(1-2d_1)}{\Gamma(1-d_1-\frac{j}{2})\Gamma(1-d_1+\frac{j}{2})}.$$

If j is odd,

$$S_1 = (-1)^{\frac{j+1}{2}}\int_0^{\frac{\pi}{2}}(\sin\lambda)^{-2d_1}\sin(j\lambda)\mathrm{d}\lambda = (-1)^{\frac{j+1}{2}}\left[\int_0^{\pi} - \int_{\frac{\pi}{2}}^{\pi}\right]$$

and the integral between $\pi/2$ and π is equal to

$$\int_0^{\frac{\pi}{2}}(\sin(\lambda+\frac{\pi}{2}))^{-2d_1}\sin(j(\lambda+\frac{\pi}{2}))\mathrm{d}\lambda = (-1)^{\frac{j-1}{2}}\int_0^{\frac{\pi}{2}}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda.$$

Then S_1 is equal to

$$\begin{aligned}
&(-1)^{\frac{j+1}{2}}\int_0^{\pi}(\sin\lambda)^{-2d_1}\sin(j\lambda)\mathrm{d}\lambda - (-1)^j\int_0^{\frac{\pi}{2}}(\cos\lambda)^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda \\
= \;& \frac{(-1)^{\frac{j+1}{2}}\pi\sin(\frac{j\pi}{2})\Gamma(1-2d_1)}{2^{-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})} - (-1)^j\frac{\pi\Gamma(1-2d_1)}{2^{1-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})} \\
= \;& -\frac{\pi\Gamma(1-2d_1)}{2^{1-2d_1}\Gamma(1-d_1+\frac{j}{2})\Gamma(1-d_1-\frac{j}{2})}
\end{aligned}$$

by formulae 3.631.1 and 3.631.9 in Gradshteyn and Ryzhik (1980). Thus when j is odd

$$\gamma_j = \frac{\sigma_2^2}{2}\frac{\Gamma(1-2d_2)}{\Gamma(1-d_2-\frac{j}{2})\Gamma(1-d_2+\frac{j}{2})} - \frac{\sigma_1^2}{2}\frac{\Gamma(1-2d_1)}{\Gamma(1-d_1-\frac{j}{2})\Gamma(1-d_1+\frac{j}{2})}$$

and the proposition is proved. □

Applying Stirling's formula to γ_j we obtain that when $\omega = \frac{\pi}{2}$,

$$\begin{aligned}
\gamma_j &\approx (-1)^{\frac{j}{2}}j^{2d_2-1} + (-1)^{\frac{j}{2}}j^{2d_1-1} \quad \text{if } j \text{ even} \\
&\approx (-1)^{\frac{j-1}{2}}j^{2d_2-1} - (-1)^{\frac{j-1}{2}}j^{2d_1-1} \quad \text{if } j \text{ odd}
\end{aligned}$$

as $j \to \infty$, which corresponds to the result obtained in Theorem 1.

When $\omega = \pi/2$ we can construct a parametric process with autocovariances (2.6) in the following manner. Suppose we have two independent series x_{it}, $i = 1,2$, of quarterly data, each of them formed from two half-yearly series, y_{it}^1 and y_{it}^2, such that $x_{it} = y_{i\frac{t}{2}}^1$ if t is even and $x_{it} = y_{i\frac{t+1}{2}}^2$ if t is odd. Assume that the different y_{it}^k come from $I_\pi(d_i)$ processes,

$$(1+L)^{d_i}y_{it}^k = \varepsilon_{it}^k \quad i = 1,2, \;\; k = 1,2$$

where the different ε_{1t}^k and ε_{2t}^k are independent white noises $(0,\sigma_1^2)$ and $(0,\sigma_2^2)$ respectively. Thus the lag-j autocovariance of y_{it}^k, $i = 1,2$ and $k = 1,2$, is

$$\gamma_{yj}^i = \sigma_i^2 \frac{\Gamma(1-d_i)}{\Gamma(1-d_i-j)\Gamma(1-d_i+j)} \quad j = 0, \pm 1, \dots . \tag{2.7}$$

Note that we have the same autocovariances for y_{it}^1 and y_{it}^2. Although we suppose that ε_{1t}^k and ε_{2t}^k are independent we assume a certain covariance between ε_{it}^1 and ε_{it}^2 such that the covariance between y_{it}^1 and y_{it-j}^2 is $\gamma_{y\frac{2j+1}{2}}^i$ defined in (2.7). Then the lag-j autocovariance of x_{it}, for $i = 1,2$, is

$$\gamma_{xj}^i = \gamma_{y\frac{j}{2}}^i \quad k = 0, \pm 1, ...$$

Now let x_t be

$$x_t = \frac{1}{\sqrt{2}} x_{2t} + (-1)^t \frac{1}{\sqrt{2}} x_{1t}. \tag{2.8}$$

Since x_{1t} and x_{2t} are independent (because ε_{1t}^k and ε_{2t}^k are independent), then the lag-j autocovariance of x_t in (2.8) is (2.6) in Proposition 1.

We observe a similarity between these type of processes and periodic ARIMA processes described in Chapter 1, which are of the form

$$\alpha_q(L) x_T^q = \beta_q(L) \varepsilon_T^q \qquad q = 1, ..., s,$$

where s is the number of periods and the sequences ε_T^q, $T = 1,2,....$, for $q = 1,...,s$, are white noise with variance σ_q^2. In our case we have

$$\begin{aligned} (1+L)^{d_1} y_{1T}^1 &= \varepsilon_{1T}^1 \\ (1+L)^{d_1} y_{1T}^2 &= \varepsilon_{1T}^2 \end{aligned}$$

and we form the quarterly series, x_{1t}, from these two half-yearly series with a specific correlation between them. We proceed similarly for x_{2t} so that x_t is a linear combination of periodic fractionally integrated processes.

2.3 FINITELY MANY POLES IN THE SPECTRUM

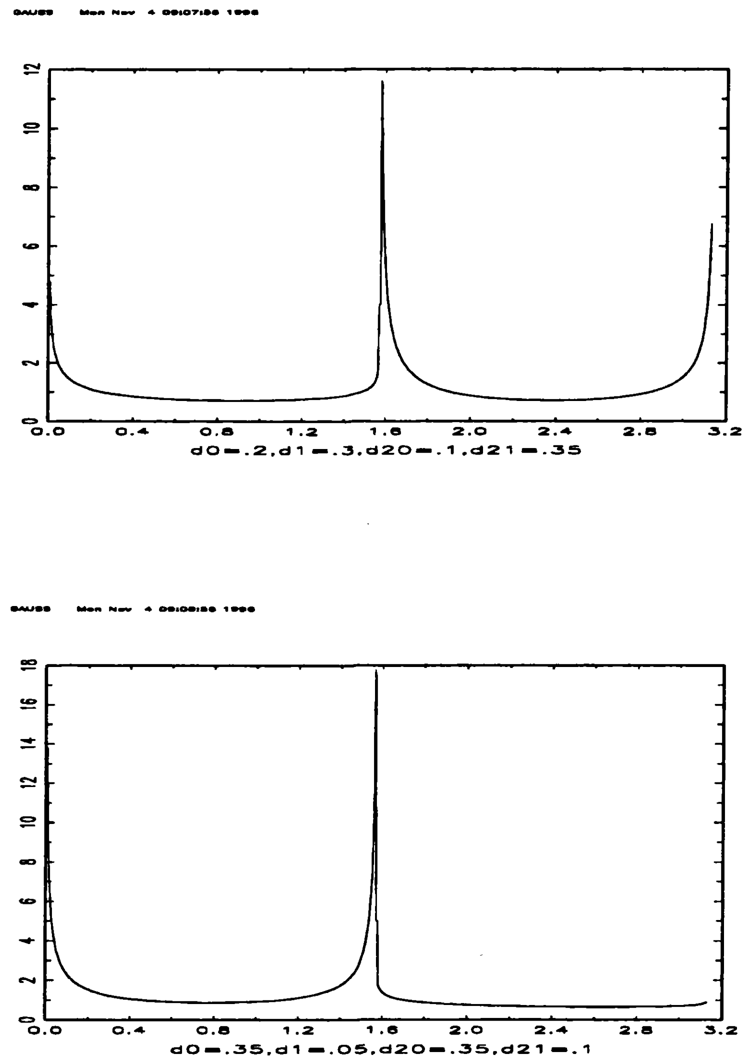
Consider the process $\{x_t\}$ with spectral density function

$$f(\lambda) = \begin{cases} \frac{\sigma_1^2}{2\pi} |1-e^{i\lambda}|^{-2d_0} |1+e^{i\lambda}|^{-2d_1} |1-2e^{i\lambda}\cos\omega + e^{i2\lambda}|^{-2d_{21}} & \text{if } \omega < \lambda \le \pi \\ \frac{\sigma_2^2}{2\pi} |1-e^{i\lambda}|^{-2d_0} |1+e^{i\lambda}|^{-2d_1} |1-2e^{i\lambda}\cos\omega + e^{i2\lambda}|^{-2d_{22}} & \text{if } 0 \le \lambda \le \omega \end{cases} \tag{2.9}$$

with $0 < d_0, d_1, d_{21}, d_{22} < 1/2$. This specification of the spectral density allows for different poles at 0, π and to the right and left at ω. $f(\lambda)$ can be equally written

$$\begin{cases} \frac{\sigma_1^2}{2\pi}(4\sin^2\frac{\lambda}{2})^{-d_0}(4\cos^2\frac{\lambda}{2})^{-d_1}\{4\sin^2(\frac{\lambda+\omega}{2})\}^{-d_{21}}\{4\sin^2(\frac{\lambda-\omega}{2})\}^{-d_{21}} & \text{if } \omega < \lambda \leq \pi \\ \frac{\sigma_2^2}{2\pi}(4\sin^2\frac{\lambda}{2})^{-d_0}(4\cos^2\frac{\lambda}{2})^{-d_1}\{4\sin^2(\frac{\lambda+\omega}{2})\}^{-d_{22}}\{4\sin^2(\frac{\lambda-\omega}{2})\}^{-d_{22}} & \text{if } 0 \leq \lambda \leq \omega \end{cases} \quad (2.10)$$

Figure 2.1: Spectra with asymmetric pole at $\frac{\pi}{2}$



An example of two different spectra like (2.9), for the persistent case with $\omega = \pi/2$, can be seen in Figure 2.1. Note that $f(\lambda)$ is in fact infinity at the origin, at π and at

$\pi/2$. In Figure 2.1 those frequencies are not considered and only frequencies around them are plotted. This is enough to give some intuition of the behaviour of the spectral density (2.9).

If there is more than one spectral pole/zero we can not obtain an explicit expression for the autocovariances like in the simple GARMA(0,0) with only one pole. Nevertheless, in the persistent case (no zeros in the spectral density), we can achieve some knowledge of the asymptotic behaviour of γ_j when $j \to \infty$. In Theorem 2 we state that if there is more than one (possibly asymmetric) spectral pole, then the autocovariances have the expected hyperbolic decay, but the possible cyclic pattern depends on the magnitude of the different persistence parameters, such that if d_0 is the biggest, then the autocovariances will eventually have a monotonic decrease without any cyclical movement.

Theorem 2 *If x_t has a spectral density function (2.9) then the lag-j autocovariance, γ_j, behaves*

$$\gamma_j \approx j^{2d_0-1} + j^{2d_{22}-1}\sin(\pi d_{22} + j\omega) + j^{2d_{21}-1}\sin(\pi d_{21} - j\omega) + j^{2d_1-1}(-1)^j$$

as $j \to \infty$.

Proof:

$$\gamma_j = 2\int_0^\pi f(\lambda)\cos(j\lambda)\mathrm{d}\lambda = 2\left[\int_0^{\delta_1} + \int_{\delta_1}^{\omega} + \int_{\omega}^{\delta_2} + \int_{\delta_2}^{\pi}\right]$$

where $0 < \delta_1 < \omega < \delta_2 < \pi$. Now we study the behaviour of the four integrals as $j \to \infty$. The integral between 0 and δ_1 is equal to

$$\begin{aligned} & \frac{\sigma_2^2}{2\pi}\int_0^{\delta_1}(4\sin^2\frac{\lambda}{2})^{-d_0}(4\cos^2\frac{\lambda}{2})^{-d_1} \\ \times & \ \{4\sin^2(\frac{\lambda+\omega}{2})\}^{-d_{22}}\{4\sin^2(\frac{\lambda-\omega}{2})\}^{-d_{22}}\cos(j\lambda)\mathrm{d}\lambda \\ = & \ G_1\int_0^{\delta_1}(2\sin\frac{\lambda}{2})^{-2d_0}\cos(j\lambda)\mathrm{d}\lambda \end{aligned} \qquad (2.11)$$

where G_1 depends on $\delta_1, \sigma_2^2, d_1, d_{22}$. The integral in (2.11) can be written

$$\int_0^{\delta_1}\lambda^{-2d_0}\left(\frac{2\sin\frac{\lambda}{2}}{\lambda}\right)^{-2d_0}\cos(j\lambda)\mathrm{d}\lambda = \int_0^{\pi}\lambda^{-2d_0}b_1(\lambda)\cos(j\lambda)\mathrm{d}\lambda$$

where

$$b_1(\lambda) = \begin{cases} \left(\frac{2\sin\frac{\lambda}{2}}{\lambda}\right)^{-2d_0} & \text{if} \quad 0 \le \lambda \le \delta_1 \\ 0 & \text{if} \quad \delta_1 < \lambda \le \pi. \end{cases}$$

Now $b_1(\lambda)$ is of bounded variation in any interval $(\varepsilon, \pi]$ and we have to show that it is slowly varying as $\lambda \to 0^+$ in order to apply Theorem 2.24 in Zygmund (1977, Chapter 5). Two conditions have to hold:

1. $\lambda^\delta b_1(\lambda)$ is increasing

2. $\lambda^{-\delta} b_1(\lambda)$ is decreasing

for $\delta > 0$, in some right hand neighbourhood of 0.

1. Define $\phi_{11}(\lambda) = \lambda^\delta b_1(\lambda) = \lambda^{\delta+2d_0}(2\sin\frac{\lambda}{2})^{-2d_0}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{11}(\lambda) = \lambda^{\delta+2d_0-1}(2\sin\frac{\lambda}{2})^{-2d_0-1}\{(\delta+2d_0)2\sin\frac{\lambda}{2} - 2d_0\lambda\cos\frac{\lambda}{2}\}$$

   is positive if $\frac{(\delta+2d_0)2\sin\frac{\lambda}{2}}{2d_0\lambda\cos\frac{\lambda}{2}} > 1$. This fraction tends to $\frac{\delta+2d_0}{2d_0} > 1$ as $\lambda \to 0$ and 1 holds for $\delta > 0$.

2. Similarly let $\phi_{12}(\lambda) = \lambda^{-\delta} b_1(\lambda) = \lambda^{2d_0-\delta}(2\sin\frac{\lambda}{2})^{-2d_0}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{12}(\lambda) = \lambda^{2d_0-\delta-1}(2\sin\frac{\lambda}{2})^{-2d_0-1}\{(2d_0-\delta)2\sin\frac{\lambda}{2} - 2d_0\lambda\cos\frac{\lambda}{2}\}$$

   is negative for a small enough δ because

$$\frac{2d_0\lambda\cos\frac{\lambda}{2}}{(2d_0-\delta)2\sin\frac{\lambda}{2}} \to \frac{2d_0}{2d_0-\delta} > 1 \text{ as } \lambda \to 0$$

Thus we can apply Theorem 2.24 in Zygmund (1977, Chapter 5) and we have that as $j \to \infty$,

$$\int_0^\pi \lambda^{-2d_0} b_1(\lambda)\cos(j\lambda)\mathrm{d}\lambda \sim j^{2d_0-1} b_1(\frac{1}{j})\Gamma(1-2d_0)\sin\pi d_0.$$

The second integral is

$$\int_{\delta_1}^\omega f(\lambda)\cos(j\lambda)\mathrm{d}\lambda = G_2\int_{\delta_1}^\omega \left\{4\sin^2\frac{\lambda-\omega}{2}\right\}^{-d_{22}}\cos(j\lambda)\mathrm{d}\lambda \qquad (2.12)$$

where G_2 depends on $\delta_1, \omega, \sigma_2^2, d_0, d_1$ and d_{22}. The integral in (2.12) can be written

$$\begin{aligned}
&\int_{\delta_1}^{\omega} (\omega-\lambda)^{-2d_{22}} \left\{ \frac{2\sin(\frac{\omega-\lambda}{2})}{\omega-\lambda} \right\}^{-2d_{22}} \cos(j\lambda)\mathrm{d}\lambda \\
= &\int_0^{\omega-\delta_1} x^{-2d_{22}} \left\{ \frac{2\sin\frac{x}{2}}{x} \right\}^{-2d_{22}} \cos(j(\omega-x))dx \\
= &\int_0^{\pi} x^{-2d_{22}} b_2(x) \cos(j(\omega-x))\mathrm{d}x
\end{aligned}$$

where

$$b_2(x) = \begin{cases} \left(\frac{2\sin\frac{x}{2}}{x}\right)^{-2d_{22}} & \text{if} \quad 0 \le x \le \omega-\delta_1 \\ 0 & \text{if} \quad \omega-\delta_1 < x \le \pi. \end{cases}$$

(2.12) is then equal to

$$\begin{aligned}
&\cos(j\omega)\int_0^{\pi} x^{-2d_{22}} b_2(x)\cos(jx)\mathrm{d}x + \sin(j\omega)\int_0^{\pi} x^{-2d_{22}} b_2(x)\sin(jx)\mathrm{d}x \\
\sim\ & j^{2d_{22}-1} b_2(\frac{1}{j})\Gamma(1-2d_{22})\sin(\pi d_{22}+j\omega) \quad \text{as } j \to \infty,
\end{aligned}$$

applying Theorem 2.24 in Zygmund (1977, Chapter 5) because $b_2(x)$, like $b_1(\lambda)$, is of bounded variation in any (ε, π) and slowly varying as $x \to 0^+$.

Similarly,

$$\int_{\omega}^{\delta_2} f(\lambda)\cos(j\lambda)\mathrm{d}\lambda = G_3 \int_{\omega}^{\delta_2} \left\{ 2\sin\frac{\lambda-\omega}{2} \right\}^{-2d_{21}} \cos(j\lambda)\mathrm{d}\lambda$$

where G_3 depends on $\delta_2, \omega, \sigma_1^2, d_0, d_1$ and d_{21} and the integral is equal to

$$\begin{aligned}
&\int_{\omega}^{\delta_2} (\lambda-\omega)^{-2d_{21}} \left\{ \frac{2\sin(\frac{\lambda-\omega}{2})}{\lambda-\omega} \right\}^{-2d_{21}} \cos(j\lambda)\mathrm{d}\lambda \\
= &\int_0^{\delta_2-\omega} x^{-2d_{21}} \left\{ \frac{2\sin\frac{x}{2}}{x} \right\}^{-2d_{21}} \cos(j(\omega+x))\mathrm{d}x \\
= &\cos(j\omega)\int_0^{\pi} x^{-2d_{21}} b_3(x)\cos(jx)\mathrm{d}x - \sin(j\omega)\int_0^{\pi} x^{-2d_{21}} b_3(x)\sin(jx)\mathrm{d}x \\
\sim\ & j^{2d_{21}-1} b_3(\frac{1}{j})\Gamma(1-2d_{21})\sin(\pi d_{21}-j\omega) \text{ as } j \to \infty,
\end{aligned}$$

applying once more Theorem 2.24 in Zygmund (1977, Chapter 5) because the function

$$b_3(x) = \begin{cases} \left(\frac{2\sin\frac{x}{2}}{x}\right)^{-2d_{21}} & \text{if} \quad 0 \le x \le \delta_2-\omega \\ 0 & \text{if} \quad \delta_2-\omega < x \le \pi \end{cases}$$

is of bounded variation in any (ε, π) and slowly varying as $x \to 0^+$.

Finally

$$\int_{\delta_2}^{\pi} f(\lambda)\cos(j\lambda)\mathrm{d}\lambda = G_4 \int_{\delta_2}^{\pi} (2\cos\frac{\lambda}{2})^{-2d_1} \cos(j\lambda)\mathrm{d}\lambda$$

and the integral is

$$\begin{aligned}
&\int_{\delta_2}^{\pi}(\pi-\lambda)^{-2d_1}\left\{\frac{2\cos\frac{\lambda}{2}}{\pi-\lambda}\right\}^{-2d_1}\cos(j\lambda)\mathrm{d}\lambda \\
=\;& \int_0^{\pi-\delta_2} x^{-2d_1}\left\{\frac{2\cos\frac{\pi-x}{2}}{x}\right\}^{-2d_1}\cos(j(\pi-x))\mathrm{d}x \\
=\;& \cos(j\pi)\int_0^{\pi} x^{-2d_1}b_4(x)\cos(jx)\mathrm{d}x+\sin(j\pi)\int_0^{\pi} b_4(x)\sin(jx)\mathrm{d}x \\
\sim\;& j^{2d_1-1}b_4(\tfrac{1}{j})\Gamma(1-2d_1)\sin(\pi d_1+j\pi) \quad \text{as } j\to\infty \\
=\;& j^{2d_1-1}b_4(\tfrac{1}{j})\Gamma(1-2d_1)\sin(\pi d_1)(-1)^j
\end{aligned}$$

applying the same Theorem in Zygmund (1977, Chapter 5) because the function

$$b_4(x)=\begin{cases}\left(\frac{2\cos\frac{\pi-x}{2}}{x}\right)^{-2d_1} & \text{if } \; 0\le x\le\pi-\delta_2 \\ 0 & \text{if } \; \pi-\delta_2<x\le\pi\end{cases}$$

is of bounded variation in any (ε,π) and slowly varying as $x\to 0^+$. To prove this note that

1. $\phi_{21}=x^{\delta}b_4(x)=x^{\delta+2d_1}(2\cos\frac{\pi-x}{2})^{-2d_1}$ is increasing in some right hand neighbourhood of 0, since

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_{21}=x^{\delta+2d_1-1}(2\cos\frac{\pi-x}{2})^{-2d_1-1}\{(\delta+2d_1)2\cos\frac{\pi-x}{2}-2d_1x\sin\frac{\pi-x}{2}\}$$

   is positive for x close enough to 0 because

$$\frac{(\delta+2d_1)2\cos\frac{\pi-x}{2}}{2d_1x\sin\frac{\pi-x}{2}}\to\frac{\delta+2d_1}{2d_1}>1 \quad \text{as } x\to 0^+,$$

   and similarly

2. $\phi_{22}=x^{-\delta}b_4(x)=x^{2d_1-\delta}(2\cos\frac{\pi-x}{2})^{-2d_1}$ is decreasing in some right-hand neighbourhood of 0.

Thus the proof of the theorem is completed because $b_1(\frac{1}{j})$, $b_2(\frac{1}{j})$, $b_3(\frac{1}{j})$ and $b_4(\frac{1}{j})$ tend to non-zero constants as $j\to\infty$. □

Remark 1: The result obtained in the previous theorem can be generalized to finitely many spectral poles in the interval $[0,\pi]$, possibly with different persistence parameters across different frequencies as well as on each side of the poles at frequencies in $(0,\pi)$. Let x_t have a spectral density function $f(\lambda)$. Suppose $f(\lambda)$ has poles at frequencies $0 < w_1 < w_2 ... < w_{r-1} < \pi$ and possibly at $w_0 = 0$ and $w_r = \pi$. Let $S_j = (w_{j-1}, w_j]$ for $j = 1, ..., r$. Write

$$\begin{aligned} h_0(\lambda) &= |1 - e^{i\lambda}|^{-d_0}, \\ h_r(\lambda) &= |1 + e^{i\lambda}|^{-d_r}, \\ h_{j,k}(\lambda) &= |1 - 2e^{i\lambda}\cos w_j + e^{i2\lambda}|^{-d_{j,k}} \quad , \quad j = 1, ..., r-1 \ , \ k = 1, 2, \\ g(\lambda) &= h_0(\lambda)^2 h_r(\lambda)^2 \prod_{j=1}^{r-1} h_{j,1}(\lambda) h_{j,2}(\lambda), \end{aligned}$$

where $d_0, d_r, d_{jk} \in (0, 1/2)$. Let $g_j(\lambda)$, $j = 1, ..., r$, be even, positive and bounded functions in $[-\pi, \pi]$. Now specify the spectral density function as

$$f(\lambda) = \begin{cases} g_1(\lambda) g(\lambda) \frac{h_{12}(\lambda)}{h_{11}(\lambda)} & \text{if } \lambda \in S_1, \\ g_j(\lambda) g(\lambda) \frac{h_{j-1,1}(\lambda) h_{j,2}(\lambda)}{h_{j-1,2}(\lambda) h_{j,1}(\lambda)} & \text{if } \lambda \in S_j, \qquad j = 2, 3, ..., r-1, \\ g_r(\lambda) g(\lambda) \frac{h_{r-1,1}(\lambda)}{h_{r-1,2}(\lambda)} & \text{if } \lambda \in S_r. \end{cases} \tag{2.13}$$

If we take $g_j(\lambda) = f_u(\lambda)$ for all j, and $d_{j1} = d_{j2} = d_j$, we have (2.2) for the case of symmetric spectral poles.

For a process with a spectral density (2.13), the lag-j autocovariance is

$$\gamma_j = 2\int_0^{\pi} f(\lambda)\cos(j\lambda) \mathrm{d}\lambda = 2\sum_{k=0}^{r-1} \int_{w_k}^{w_{k+1}} f(\lambda)\cos(j\lambda)\mathrm{d}\lambda.$$

Proceeding as in Theorem 2, we get that as $j \to \infty$ the autocovariances are

$$\begin{aligned} \gamma_j &\approx j^{2d_0-1} + j^{2d_r-1}(-1)^j \\ &+ \sum_{k=1}^{r-1} \{ j^{2d_{k1}-1} \sin(\pi d_{k1} - j w_k) + j^{2d_{k2}-1} \sin(\pi d_{k2} + j w_k) \} \end{aligned}$$

with behaviour finally governed by the highest d. A similar result has been found by Giraitis and Leipus (1995) for the case of symmetric poles.

Remark 2: The autocovariances of a process with spectral density like those studied in this section are not summable if $d_0 > 0$ and are not absolutely summable whenever any of the d's is positive. This fact corresponds to the long memory property in the time domain. We also observe that the asymptotic behaviour of the autocovariances is finally governed by the highest d, with hyperbolic decay and cyclical behaviour if this d corresponds to a positive frequency.

Remark 3: Remark 2 in the previous section applies in this more general case. Furthermore the time domain techniques that use the sample autocorrelations for long lags j are only asymptotically valid (if they are at all) for the highest d.

2.4 ASYMPTOTIC RELATIVE BIAS OF THE PERIODOGRAM

The periodogram has traditionally been used to estimate the spectral density. It is thus important to achieve some knowledge (at least asymptotically) about the relationships between periodogram and spectral density function in the different situations we analyse in this thesis, and in particular the effects the existence of SCLM may have on these relationships. One important and early work is due to Hannan (1973b) who showed that the periodograms evaluated at Fourier frequencies close to a fixed frequency λ are asymptotically independent and identically distributed as $\frac{f(\lambda)}{2}\chi_2^2$ where $f(\lambda)$ is the spectral density at λ and χ_2^2 is the chi-square distribution with two degrees of freedom. However his assumptions rule out the possibility of long-range dependence. Yajima (1989) allowed for the possibility of long memory and gave the joint asymptotic distribution of the periodogram when evaluated at a set of fixed frequencies not depending on n, the sample size, so that Fourier frequencies are not considered. These results have led many authors (e.g. Geweke and Porter-Hudak (1983) based their proof on Hannan's theorem) to conclude that the log-periodogram estimator proposed by Geweke and Porter-Hudak is asymptotically normal with variance $\pi^2/6$. However, the log-periodogram regression performed to obtain the estimate is based on Fourier frequencies $\lambda_j = 2\pi j/n$ for $j = 1, 2..., g(n)$, where $g(n)$ is an integer smaller than $n/2$. Consequently these frequencies do change with n, so that Yajima's result can not be applied. As far as Hannan's result is concerned, his assumptions rule out the possibility of persistence. For $d < 0$ Hannan (1973b) stated that the periodogram evaluated at a finite number of Fourier frequencies close to the origin converges in probability to zero. However, when we normalize with the spectral density the remainder is divided by a quantity which approaches zero, and therefore does not need to be negligible. These facts have been noted in Hurvich and Beltrao

(1993) and Robinson (1995a). Hurvich and Beltrao (1993) considered the asymptotic distribution of the periodogram normalized by the spectral density function of weakly stationary time series with zero mean and spectral density

$$f(\lambda) = |1 - e^{-i\lambda}|^{-2d} f^*(\lambda)$$

where $|d| < .5$ and $f^*(\lambda)$ is an even, positive, bounded and continuous function on $[-\pi, \pi]$. They studied the behaviour of the normalized periodogram at Fourier frequencies, $\lambda_j = 2\pi j/n$, where j is fixed and $n \to \infty$, and they showed that they are not asymptotically identically distributed. In fact $\lim_{n\to\infty} E[I_n(\lambda_j)/f(\lambda_j)]$ depends on j and d and is typically greater than 1, implying positive asymptotic relative bias in the periodogram as estimate of $f(\lambda)$. Hurvich and Ray (1995) extended the results in Hurvich and Beltrao (1993) to the case when d falls outside the range $(-1/2, 1/2)$. They proved that when $d < -0.5$, $E[I_n(\lambda_j)/f(\lambda_j)]$ tends to infinity as $n \to \infty$, when $d \in [0.5, 1)$ the asymptotic relative bias of the periodogram is finite and decreases with j, if $d = 1$ it is constant for all j and when $d \in (1, 1.5)$ it increases with j. In this section we extend these results to the SCLM case allowing for the possibility of asymmetric spectral poles/zeros like those described in earlier sections.

Let $\{x_t\}$ be a stationary process with spectral density function

$$\begin{aligned} f(\lambda) &= \begin{cases} |1 - 2e^{i\lambda}\cos\omega + e^{i2\lambda}|^{-2d_1} g_1(\lambda) & \omega < \lambda \le \pi \\ |1 - 2e^{i\lambda}\cos\omega + e^{i2\lambda}|^{-2d_2} g_2(\lambda) & 0 \le \lambda \le \omega \end{cases} \qquad (2.14) \\ &= \begin{cases} g_1(\lambda)\left\{4^2 \sin^2\left(\frac{\lambda+\omega}{2}\right)\left(\frac{\sin\frac{\lambda-\omega}{2}}{\lambda-\omega}\right)^2\right\}^{-d_1} |\lambda-\omega|^{-2d_1} & \omega < \lambda \le \pi \\ g_2(\lambda)\left\{4^2 \sin^2\left(\frac{\lambda+\omega}{2}\right)\left(\frac{\sin\frac{\omega-\lambda}{2}}{\omega-\lambda}\right)^2\right\}^{-d_2} |\omega-\lambda|^{-2d_2} & 0 \le \lambda \le \omega \end{cases} \end{aligned}$$

where $g_1(\lambda)$ and $g_2(\lambda)$ are even, positive and bounded continuous functions on $[-\pi, \pi]$. We can write (2.14) as

$$f(\lambda) = \begin{cases} f_1^*(\lambda)|\lambda-\omega|^{-2d_1} & \omega < \lambda \le \pi \\ f_2^*(\lambda)|\omega-\lambda|^{-2d_2} & 0 \le \lambda \le \omega \end{cases} \qquad (2.15)$$

where $f_2^*(\lambda)$ and $f_1^*(\lambda)$ are positive and bounded continuous functions on $[0, \omega]$ and $(\omega, \pi]$ respectively. Let $I_n(\lambda) = |W_n(\lambda)|^2 = \frac{1}{2\pi n}|\sum_{t=1}^n x_t e^{it\lambda}|^2$ be the (uncentered) periodogram of x_t at frequency λ. Then the result stated in the following theorem follows.

Theorem 3 *Let x_t have spectral density (2.14) with $-1/2 < d_1, d_2 < 1/2$. The asymptotic relative bias of the periodogram as estimate of $f(\lambda)$ at Fourier frequencies just after ω, $\omega + \lambda_j$, where $\lambda_j = \frac{2\pi j}{n}$ and j is fixed, is*

$$L_j(d_1, d_2) = \lim_{n\to\infty} E\left[\frac{I_n(\omega+\lambda_j)}{f(\omega+\lambda_j)}\right].$$

a) *If $d_2 < d_1$*

$$L_j(d_1, d_2) = \int_0^\infty \frac{2}{\pi}\frac{\sin^2\frac{\lambda}{2}}{(2\pi j-\lambda)^2}\left|\frac{\lambda}{2\pi j}\right|^{-2d_1} d\lambda,$$

b) *if $d_2 = d_1 = d$*

$$L_j(d_1, d_2) = \int_0^\infty \frac{2}{\pi}\frac{\sin^2\frac{\lambda}{2}}{(2\pi j-\lambda)^2}\left|\frac{\lambda}{2\pi j}\right|^{-2d} d\lambda + \frac{f_2^*(\omega)}{f_1^*(\omega)}\int_{-\infty}^0 \frac{2}{\pi}\frac{\sin^2\frac{\lambda}{2}}{(2\pi j-\lambda)^2}\left|\frac{\lambda}{2\pi j}\right|^{-2d} d\lambda,$$

c) *and if $d_1 < d_2$*

$$\lim_{n\to\infty} n^{2(d_1-d_2)} E\left[\frac{I(\omega+\lambda_j)}{f(\omega+\lambda_j)}\right] = \frac{f_2^*(\omega)}{f_1^*(\omega)}|2\pi j|^{2d_1}\int_{-\infty}^0 \frac{2}{\pi}\frac{\sin^2\frac{\lambda}{2}}{(2\pi j-\lambda)^2}|\lambda|^{-2d_2} d\lambda.$$

Proof: Since $d_1, d_2 < 0.5$ we can write the expectation as

$$E\left[\frac{I_n(\omega+\lambda_j)}{f(\omega+\lambda_j)}\right] = \int_{-\pi}^{\pi} g_n(\lambda)\mathrm{d}\lambda \qquad (2.16)$$

where

$$g_n(\lambda) = K_n(\omega+\lambda_j-\lambda)\frac{f(\lambda)}{f(\omega+\lambda_j)}$$

and $K_n(\cdot)$ is Fejer's kernel

$$K_n(\lambda) = \frac{1}{2\pi n}\left|\sum_{t=1}^n e^{it\lambda}\right|^2 = \frac{\sin^2(\frac{\lambda}{2}n)}{2\pi n\sin^2\frac{\lambda}{2}}.$$

The integral (2.16) can be decomposed into

$$\left\{\int_{-\pi}^{-\omega-n^{-\alpha}} + \int_{-\omega-n^{-\alpha}}^{-\omega+n^{-\alpha}} + \int_{-\omega+n^{-\alpha}}^{0} + \int_0^{\omega-n^{-\alpha}} + \int_{\omega-n^{-\alpha}}^{\omega+n^{-\alpha}} + \int_{\omega+n^{-\alpha}}^{\pi}\right\} g_n(\lambda)\mathrm{d}\lambda$$

for some $\alpha \in (0, 0.5]$.

The integral over $[-\pi, -\omega - n^{-\alpha}]$ can be written

$$
\begin{aligned}
&\int_{-\pi+\omega}^{-n^{-\alpha}} K_n(2\omega + \lambda_j - \lambda)\frac{f(\lambda-\omega)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\
= &\int_{-\pi+\omega}^{-n^{-\alpha}} \frac{\sin^2(\frac{2\omega+\lambda_j-\lambda}{2}n)}{2\pi n\sin^2(\frac{2\omega+\lambda_j-\lambda}{2})}\frac{f_1^*(\lambda-\omega)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_1}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda \qquad (2.17)
\end{aligned}
$$

by symmetry of the spectral density around zero. Now $f_1^*(\lambda)$ is bounded and positive and $\sin^2(\frac{2\omega+\lambda_j-\lambda}{2}) \neq 0$ for $-\pi+\omega \leq \lambda \leq -n^{-\alpha}$ and n sufficiently large. Consequently (2.17) is

$$
O(n^{-1-2d_1}\int_{-\pi+\omega}^{-n^{-\alpha}} |\lambda|^{-2d_1}\mathrm{d}\lambda) = O(n^{-1-2d_1}) = o(1)
$$

as $n \to \infty$ for $d_1 > -0.5$ and $\alpha > 0$. Similarly the integral over $[-\omega + n^{-\alpha}, 0]$ is equal to

$$
\begin{aligned}
&\int_{n^{-\alpha}}^{\omega} K_n(2\omega + \lambda_j - \lambda)\frac{f(\lambda-\omega)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\
= &\int_{n^{-\alpha}}^{\omega} \frac{\sin^2(\frac{2\omega+\lambda_j-\lambda}{2}n)}{2\pi n\sin^2(\frac{2\omega+\lambda_j-\lambda}{2})}\frac{f_2^*(\lambda-\omega)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_2}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda. \qquad (2.18)
\end{aligned}
$$

Since $0 < 2\omega + \lambda_j - \lambda < 2\pi$ for $\lambda \in [n^{-\alpha}, \omega]$ and a large enough n, and f_1^* and f_2^* are positive and bounded functions, we have that (2.18) is

$$
O(\int_{n^{-\alpha}}^{\omega} n^{-1-2d_1}|\lambda|^{-2d_2}\mathrm{d}\lambda) = O(n^{-1-2d_1}) = o(1)
$$

for $d_1, d_2 > -0.5$, ω fixed and $\alpha > 0$.

Now the integral over $[0, \omega - n^{-\alpha}]$ is equal to

$$
\begin{aligned}
&\int_{-\omega}^{-n^{-\alpha}} K_n(\lambda_j - \lambda)\frac{f(\omega+\lambda)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\
= &\int_{-\omega}^{-n^{-\alpha}} \frac{\sin^2(\frac{\lambda_j-\lambda}{2}n)}{2\pi n\sin^2(\frac{\lambda_j-\lambda}{2})}\frac{f_2^*(\omega+\lambda)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_2}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda. \qquad (2.19)
\end{aligned}
$$

Since f_1^* and f_2^* are positive and bounded, and

$$
\inf_{-\omega\leq\lambda\leq-n^{-\alpha}} \sin^2\left(\frac{\lambda_j-\lambda}{2}\right) = \sin^2\left(\frac{\lambda_j+n^{-\alpha}}{2}\right)
$$

for a sufficiently large n, then (2.19) is bounded by

$$
const. \int_{-\omega}^{-n^{-\alpha}} n^{-1-2d_1}\sin^{-2}\left(\frac{\lambda_j+n^{-\alpha}}{2}\right)|\lambda|^{-2d_2}\mathrm{d}\lambda. \qquad (2.20)
$$

Since $\sin^2\lambda = \lambda^2 + O(\lambda^4)$ then $\sin^{-2}\lambda = \lambda^{-2} + O(1)$ as $\lambda \to 0$ and (2.20) is

$$\begin{aligned} & O\left(\frac{n^{-1-2d_1}(\omega^{1-2d_2} + n^{-\alpha(1-2d_2)})}{(\lambda_j + n^{-\alpha})^2}\right) \\ = \; & O(n^{-1-2d_1} n^{2\alpha}) \\ = \; & o(1) \end{aligned}$$

for $d_1, d_2 > -0.5$ and $0 < \alpha < 0.5 + d_1$. Similarly the integral between $\omega + n^{-\alpha}$ and π is

$$\begin{aligned} & \int_{n^{-\alpha}}^{\pi-\omega} K_n(\lambda_j - \lambda)\frac{f(\omega+\lambda)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\ = \; & \int_{n^{-\alpha}}^{\pi-\omega} \frac{\sin^2(\frac{\lambda_j-\lambda}{2}n)}{2\pi n \sin^2(\frac{\lambda_j-\lambda}{2})}\frac{f_1^*(\omega+\lambda)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_1}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda & (2.21) \\ \leq \; & const.n^{-1-2d_1}\sin^{-2}\left(\frac{\lambda_j - n^{-\alpha}}{2}\right)[(\pi-\omega)^{1-2d_1} + n^{-\alpha(1-2d_1)}] \\ = \; & O(n^{-1-2d_1} n^{2\alpha}) \\ = \; & o(1) & (2.22) \end{aligned}$$

for a large enough n, such that $\lambda_j - n^{-\alpha} < 0$, $d_1 > -0.5$ and $0 < \alpha < 0.5 + d_1$.

Now the integral over $[-\omega - n^{-\alpha}, -\omega]$ is

$$\begin{aligned} & \int_{-n^{-\alpha}}^{0} K_n(2\omega + \lambda_j - \lambda)\frac{f(\lambda-\omega)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\ = \; & \int_{-n^{-\alpha}}^{0} \frac{\sin^2(\frac{2\omega+\lambda_j-\lambda}{2}n)}{2\pi n \sin^2(\frac{2\omega+\lambda_j-\lambda}{2})}\frac{f_1^*(\lambda-\omega)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_1}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda \\ = \; & O(n^{-1-2d_1}\int_{-n^{-\alpha}}^{0} |\lambda|^{-2d_1}\mathrm{d}\lambda) \\ = \; & O(n^{-1-2d_1} n^{-\alpha(1-2d_1)}) \\ = \; & o(1) \end{aligned}$$

as $n \to \infty$, for $\alpha > 0$ and $d_1 > -0.5$. Similarly

$$\begin{aligned} & \int_{-\omega}^{-\omega+n^{-\alpha}} g_n(\lambda)\mathrm{d}\lambda \\ = \; & \int_{0}^{n^{-\alpha}} K_n(2\omega + \lambda_j - \lambda)\frac{f(\lambda-\omega)}{f(\omega+\lambda_j)}\mathrm{d}\lambda \\ = \; & \int_{0}^{n^{-\alpha}} \frac{\sin^2(\frac{2\omega+\lambda_j-\lambda}{2}n)}{2\pi n \sin^2(\frac{2\omega+\lambda_j-\lambda}{2})}\frac{f_2^*(\lambda-\omega)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_2}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
&= O(n^{-1-2d_1}\int_0^{n^{-\alpha}} |\lambda|^{-2d_2}\mathrm{d}\lambda) \\
&= O(n^{-1-2d_1}n^{-\alpha(1-2d_2)}) \\
&= o(1)
\end{aligned}
$$

as $n \to \infty$, for $\alpha > 0$ and $d_2, d_1 > -0.5$. Thus as $n \to \infty$ and j fixed

$$
E\left[\frac{I_n(\omega+\lambda_j)}{f(\omega+\lambda_j)}\right] = \int_{\omega-n^{-\alpha}}^{\omega+n^{-\alpha}} g_n(\lambda)\mathrm{d}\lambda + o(1). \tag{2.23}
$$

Since the behaviour of the spectral density (2.14) is different to the right and left of ω, we split the integral in (2.23) into two. First

$$
\begin{aligned}
&\int_{\omega}^{\omega+n^{-\alpha}} g_n(\lambda)\mathrm{d}\lambda \\
&= \int_0^{n^{-\alpha}} \frac{\sin^2(\frac{\lambda_j-\lambda}{2}n)}{2\pi n\sin^2(\frac{\lambda_j-\lambda}{2})}\frac{f_1^*(\omega+\lambda)}{f_1^*(\omega+\lambda_j)}\frac{|\lambda|^{-2d_1}}{|\lambda_j|^{-2d_1}}\mathrm{d}\lambda \\
&= \int_0^{n^{1-\alpha}} \frac{\sin^2(\frac{2\pi j-\lambda}{2})}{2\pi n^2\sin^2(\frac{2\pi j-\lambda}{2n})}\frac{f_1^*(\omega+\frac{\lambda}{n})}{f_1^*(\omega+\lambda_j)}\left|\frac{\lambda}{2\pi j}\right|^{-2d_1}\mathrm{d}\lambda \\
&= \int_0^{\infty} h_n^1(\lambda)\mathrm{d}\lambda
\end{aligned}
$$

where

$$
h_n^1(\lambda) = \frac{\sin^2(\frac{2\pi j-\lambda}{2})}{2\pi n^2\sin^2(\frac{2\pi j-\lambda}{2n})}\frac{f_1^*(\omega+\frac{\lambda}{n})}{f_1^*(\omega+\lambda_j)}\left|\frac{\lambda}{2\pi j}\right|^{-2d_1}\chi_{[0,n^{1-\alpha}]}
$$

and $\chi_{[0,n^{1-\alpha}]}$ is the indicator function of the interval $[0, n^{1-\alpha}]$. As $n \to \infty$ we have that $h_n^1(\lambda) \to h_1(\lambda)$ where

$$
h_1(\lambda) = \frac{2}{\pi}\frac{\sin^2(\frac{2\pi j-\lambda}{2})}{(2\pi j-\lambda)^2}\left|\frac{\lambda}{2\pi j}\right|^{-2d_1}
$$

for $0 \le \lambda < \infty$. Proceeding like in the proof of Theorem 1 in Hurvich and Beltrao (1993) we see that $h_n^1(\lambda)$ is dominated by an integrable function. Thus we can use Lebesgue's dominated convergence theorem (see for instance Temple (1971), Theorem 9.3.7) and we have that

$$
\lim_{n\to\infty}\int_0^{\infty} h_n^1(\lambda)\mathrm{d}\lambda = \int_0^{\infty} h_1(\lambda)\mathrm{d}\lambda = \int_0^{\infty}\frac{2}{\pi}\frac{\sin^2(\frac{\lambda}{2})}{(2\pi j-\lambda)^2}\left|\frac{\lambda}{2\pi j}\right|^{-2d_1}\mathrm{d}\lambda
$$

using the fact that $\sin^2(\frac{2\pi j-\lambda}{2}) = \sin^2(\frac{\lambda}{2})$ for j an integer.

Finally scale the integral over $[\omega - n^{-\alpha}, \omega]$ multiplying it by $n^{2(d_1-d_2)}$. Then we have

$$
\begin{aligned}
& n^{2(d_1-d_2)} \int_{\omega-n^{-\alpha}}^{\omega} g_n(\lambda)\mathrm{d}\lambda \\
= \;& n^{2(d_1-d_2)} \int_{-n^{-\alpha}}^{0} \frac{\sin^2(\frac{\lambda_j-\lambda}{2}n)}{2\pi n \sin^2(\frac{\lambda_j-\lambda}{2})} \frac{f_2^*(\omega+\lambda)}{f_1^*(\omega+\lambda_j)} \frac{|\lambda|^{2d_2}}{|\lambda_j|^{-2d_1}} \mathrm{d}\lambda \\
= \;& n^{-2d_2} \int_{-n^{1-\alpha}}^{0} \frac{\sin^2(\frac{2\pi j-\lambda}{2})}{2\pi n^2 \sin^2(\frac{2\pi j-\lambda}{2n})} \frac{f_2^*(\omega+\frac{\lambda}{n})}{f_1^*(\omega+\lambda_j)} |2\pi j|^{2d_1} \left|\frac{\lambda}{n}\right|^{-2d_2} \mathrm{d}\lambda \\
= \;& \int_{-\infty}^{0} h_n^2(\lambda)\mathrm{d}\lambda
\end{aligned}
$$

where

$$
h_n^2(\lambda) = |2\pi j|^{2d_1} \frac{\sin^2(\frac{2\pi j-\lambda}{2})}{2\pi n^2 \sin^2(\frac{2\pi j-\lambda}{2n})} \frac{f_2^*(\omega+\frac{\lambda}{n})}{f_1^*(\omega+\lambda_j)} |\lambda|^{-2d_2} \chi_{[-n^{1-\alpha},0]}
$$

and $\chi_{[-n^{1-\alpha},0]}$ is the indicator function of the interval $[-n^{1-\alpha}, 0]$. Proceeding as before we see that as $n \to \infty$, $h_n^2(\lambda) \to h_2(\lambda)$ where

$$
h_2(\lambda) = |2\pi j|^{2d_1} \frac{2}{\pi} \frac{\sin^2(\frac{\lambda}{2})}{(2\pi j - \lambda)^2} \frac{f_2^*(\omega)}{f_1^*(\omega)} |\lambda|^{-2d_2}
$$

for $0 \leq \lambda < \infty$ so that proceeding as before we see that

$$
\lim_{n\to\infty} \int_{-\infty}^{0} h_n^2(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{0} h_2(\lambda)\mathrm{d}\lambda
$$

using Lebesgue's dominated convergence theorem. Thus if $d_1 > d_2$ then $\int_{\omega-n^{-\alpha}}^{\omega} g_n(\lambda)\mathrm{d}\lambda \to 0$ as $n \to \infty$ and consequently a) is proved. When $d_1 = d_2$ we obtain the result stated in b). If $d_1 < d_2$ then $n^{2(d_1-d_2)} \to 0$ as $n \to \infty$ so that if we multiply the integrals with a finite or zero limit by $n^{2(d_1-d_2)}$ the only integral with a limit different from zero is

$$
n^{2(d_1-d_2)} \int_{\omega-n^{-\alpha}}^{\omega} g_n(\lambda)\mathrm{d}\lambda
$$

so that c) is obtained. □

When $d_1 = d_2$ and $g_1(\lambda) = g_2(\lambda)$ for all $\lambda \in [-\pi, \pi]$, the spectral density function (2.14) is the one analysed by Hurvich and Beltrao (1993), and the same result as their Theorem 1 is obtained. However, when $d_2 < d_1$, the asymptotic relative bias, although depending on d_1 and j, reduces with respect to that obtained by Hurvich and Beltrao at zero frequency. Finally, when $d_1 < d_2$ the asymptotic relative bias of the periodogram as estimate of $f(\lambda)$ increases without limit as $n \to \infty$. This feature will affect the behaviour of those estimates analysed in Chapters 3 and 4. A more exhaustive comment on this fact will be done when studying the performance of those estimates in finite samples in Chapter 6.

Theorem 3 focuses on the behaviour of the scaled periodogram at Fourier frequencies just after ω. A similar result is obtained for frequencies just before the spectral pole/zero. In particular, the asymptotic relative bias evaluated at those frequencies diverges as $n \to \infty$ when $d_1 > d_2$.

Chapter 3

LOG-PERIODOGRAM REGRESSION

3.1 INTRODUCTION

Let $\{x_t, t = 0, \pm 1, ...\}$ be a real valued and scalar covariance stationary process with absolutely continuous spectral distribution function and spectral density satisfying (1.62). When $C = D$, $d_1 = d_2 = d$ and $\omega \neq 0$ we say that x_t has symmetric SCLM. Some parametric processes which are in accord with this property have been mentioned in Chapter 1. Several parametric and semiparametric methods to estimate d in symmetric SCLM processes were also described in Chapter 1. Many of them were originally proposed, and their properties derived, for the standard long memory case at zero frequency where $f(\lambda)$ is always symmetric for x_t real. The same properties are likely to hold for any $\omega \in (0, \pi]$ as long as $f(\lambda)$ is symmetric around ω. Of course this symmetry holds for $\omega = \pi$ in addition to $\omega = 0$. However, when $0 < \omega < \pi$ there exists the possibility of what we called asymmetric SCLM in Chapter 1. In this case we have two (possibly different) persistence parameters at the same frequency ω, and the relationship between periodogram and spectral density at frequencies close to ω depends on the difference between both parameters (see Theorem 3 in Chapter 2 of this thesis). This dependence will affect the properties of those estimates proposed for symmetric long memory processes. In this chapter we analyse the log-periodogram estimate proposed by Robinson (1995a). In Chapter 1 we described this technique as well as a more efficient (at least asymptotically) method of estimation, namely the

Gaussian semiparametric or local Whittle estimate proposed by Robinson (1995b). This latter technique will be analyzed in Chapter 4. There are other estimates of d based on a semiparametric specification similar to (1.2), like the averaged periodogram introduced by Robinson(1994c), but its complicated asymptotic distribution makes it less preferable than the two methods studied in this thesis (see Chapter 1).

The different semiparametric methods to estimate C and d_1[1] are based on an approximate knowledge of $f(\lambda)$ at frequencies just after ω,

$$f(\omega+\lambda) \sim C\lambda^{-2d_1} \text{ as } \lambda \to 0^+ \tag{3.1}$$

for $C \in (0,\infty)$ and $d_1 \in (-1/2, 1/2)$. Taking logarithms in (3.1) and substituting $f(\lambda)$ for the periodogram, we get a simple linear relationship between $\log I_n(\lambda)$ and $-2\log\lambda$. The log-periodogram estimate of d_1, $\hat{d}_1$, is obtained by applying least squares to

$$\log I_n(\omega+\lambda_j) = c + d_1(-2\log\lambda_j) + u_j \quad j = 1, ..., m, \tag{3.2}$$

where m is the bandwidth such that $\frac{m}{n} \to 0$ as $n \to \infty$, $\lambda_j = \frac{2\pi j}{n}$, $I_n(\lambda) = |W_n(\lambda)|^2$ is the periodogram and $W_n(\lambda) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} x_t e^{it\lambda}$ is the discrete Fourier transform of x_t. The simplicity of this approach makes it very easy to implement and that is why log-periodogram and its variants have become the most used methods of estimating d in applied work (see for example Diebold and Rudebush (1989), Porter-Hudak (1990), Shea (1991), Cheung and Lai (1993) or Hassler and Wolters (1995)). The original version of this approach, due to Geweke and Porter-Hudak (1983), uses the regressor $-\log\{4\sin^2(\lambda_j/2)\}$, but as indicated by Robinson (1995a), use of the simpler $-2\log\lambda_j$, which corresponds more naturally to (3.1), leads to equivalent asymptotic results. The good properties of these estimates hold if the u_j are uncorrelated and homoscedastic. However, if $c = \log C - \eta$, where $\eta = 0.5772...$ is Euler's constant, u_j can be considered, for frequencies close to ω, as

$$u_j = \log(\frac{I_n(\omega+\lambda_j)}{C\lambda_j^{-2d_1}}) + \eta$$

[1] Estimating D and d_2 is equivalent and only changes in the use of frequencies just before ω instead of those just after.

and for $d_1 \neq 0$ they are not asymptotically uncorrelated nor identically distributed as $n \to \infty$ and j fixed (see Theorem 1 in Robinson (1995a) for $\omega = 0$). In order to obtain the asymptotic properties, Robinson (1995a) introduced a trimming number, l, such that the number of frequencies used in the regression (3.2) is from $j = l+1$ to $j = m$. Clearly l has to go to infinity more slowly than m such that $\frac{l}{m} \to 0$ as $n \to \infty$. Under Gaussianity and some other mild conditions, Robinson(1995a) showed that when $\omega = 0$ (and because of the symmetry of the spectral density around 0, $d_1 = d_2$), $\sqrt{m}(\hat{d}_1 - d_1) \xrightarrow{d} N(0, \frac{\pi^2}{24})$. A gain in efficiency is obtained by pooling J adjacent frequencies and regressing

$$y_k^{(J)} = c^{(J)} + d_1(-2\log\lambda_k) + u_k^{(J)} \qquad k = l+J, l+2J, ..., m. \tag{3.3}$$

where $y_k^{(J)} = \log(\sum_{j=1}^{J} I_n(\omega + \lambda_{k+j-J}))$ and J is fixed and assumed that $m - l$ is a multiple of J (if this condition does not hold the effect on the asymptotic properties is negligible because J is fixed and $\frac{m}{l} \to \infty$). Note that, even if we use the pooling of J adjacent periodogram ordinates, every frequency from $\omega + \lambda_{l+1}$ up to $\omega + \lambda_m$ is used in the estimation so that there is no loss of efficiency. In this case the asymptotic distribution of the least squares estimate of d_1 in (3.3), $\hat{d}_1^{(J)}$, is $\sqrt{m}(\hat{d}_1^{(J)} - d_1) \xrightarrow{d} N(0, \frac{J\psi'(J)}{4})$, where $\psi'(z) = \frac{d}{dz}\psi(z)$, ψ is the digamma function, $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ and Γ is the gamma function. The gain in efficiency comes about because $\psi'(1) = \pi^2/6$ and $J\psi'(J)$ decreases in J and goes to 1 as $J \to \infty$.

In regression (3.3), $u_k^{(J)}$ can be considered,

$$u_k^{(J)} = \log\{\sum_{j=1}^{J} \frac{I_n(\omega + \lambda_{k+j-J})}{C\lambda_{k+j-J}^{-2d_1}}\} - \psi(J) \qquad k = l+J, l+2J, ..., m. \tag{3.4}$$

If the $u_k^{(J)}$ are uncorrelated and homoscedastic with zero mean, least squares in (3.3) provides the best linear unbiased estimates of $c^{(J)}$ and d_1. The disturbances in (3.3) do not have those properties, but Robinson (1995a) showed that, when $\omega = 0$, $\hat{d}_1^{(J)}$ has the same limiting distributional behaviour as if such properties held. In this chapter we prove that this fact holds for $\omega \neq 0$, allowing for asymmetric SCLM.

3.2 ASYMPTOTIC DISTRIBUTION

Let $\{x_{gt}, t = 0, \pm1, \pm2, ...\}$ and $\{x_{ht}, t = 0, \pm1, \pm2, ...\}$ be two real valued scalar processes with spectral density functions $f_g(\lambda)$ and $f_h(\lambda)$ respectively, integrables over $[-\pi, \pi]$, and cross-spectral density $f_{gh}(\lambda)$. Let us state the following assumptions:

A.1: For a frequency $\omega \in (0, \pi)$ there exists $\alpha \in (0, 2]$ such that as $\lambda \to 0^+$,

$$\begin{aligned} f_s(\omega + \lambda) &= C_s \lambda^{-2d_{1s}}(1 + O(\lambda^\alpha)) \\ f_s(\omega - \lambda) &= D_s \lambda^{-2d_{2s}}(1 + O(\lambda^\alpha)) \end{aligned}$$

for $s = g, h$, where $C_s, D_s \in (0, \infty)$ and $d_{1s}, d_{2s} \in (-1/2, 1/2)$.

A.2: In a neighbourhood $(-\varepsilon, 0) \cup (0, \varepsilon)$ of ω f_{gh} is differentiable and as $\lambda \to 0^+$,

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} f_{gh}(\omega + \lambda) \right| &= O(\lambda^{-1-2d_1}) \\ \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} f_{gh}(\omega - \lambda) \right| &= O(\lambda^{-1-2d_2}) \end{aligned}$$

where $2d_i = d_{ig} + d_{ih}$, $i = 1, 2$.

A.3: For some $\beta \in (0, 2]$:

$$|R_{gh}(\omega + \lambda) - R_{gh}(\omega)| = O(\lambda^\beta) \qquad \text{as } \lambda \to 0^+$$

where $R_{gh}(\lambda) = \frac{f_{gh}(\lambda)}{\sqrt{f_g(\lambda) f_h(\lambda)}}$ is the coherency between x_{gt} and x_{ht} .

The two main assumptions on the spectral density used in our univariate analysis are **A.1** and **A.2** (for $g = h$), but we introduce **A.3** to allow an easy multivariate extension of the results obtained in the univariate case. These assumptions hold with $\alpha = \beta = 2$ in the cases studied in Chapter 2 (note that $\sin(\omega - \lambda)^{-2d} = (\omega - \lambda)^{-2d}(1 + O((\omega - \lambda)^2))$ as $\lambda \to \omega$). Assumption **A.1** could be generalized allowing for different α's before and after ω but this increase in the number of parameters would complicate the notation and the results we obtain hereafter would be similar.

Let $W_{ns}(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} x_{st} e^{it\lambda}$ be the discrete Fourier transform of x_{st} ($s = g, h$), $t = 1, 2, ..., n$, where correction for an unknown mean of x_{st} is not necessary because $W_{ns}(\lambda)$ is computed only at frequencies $\lambda_j = \frac{2\pi j}{n}$ for $j = 1, ..., m$, where m is an integer less than $n/2$. Introduce the scaled discrete Fourier transform $v_s(\omega + \lambda) = \frac{W_{ns}(\omega+\lambda)}{C_s^{\frac{1}{2}} \lambda^{-d_{1s}}}$ and denote $\bar{v}_s(\lambda)$ the complex conjugate of $v_s(\lambda)$.

Theorem 4 *Let assumptions* **A.1-A.3** *hold and let* $k = k(n)$ *and* $j = j(n)$ *be two sequences of positive integers such that* $j > k$ *and* $\frac{j}{n} \to 0$ *as* $n \to \infty$. *Then as* $n \to \infty$,

a) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega+\lambda_j)] = R_{gh}(\omega) + O\left(\frac{\log j}{j}\lambda_j^{-2(d_i-d_1)} + \left(\frac{j}{n}\right)^{\min(\alpha,\beta)}\right)$

b) $E[v_g(\omega+\lambda_j)v_h(\omega+\lambda_j)] = O\left(\frac{\log j}{j}\lambda_j^{-2(d_i-d_1)}\right)$

c) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega+\lambda_k)] = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-(d_{ig}-d_{1g})}\lambda_k^{-(d_{ih}-d_{1h})}\right)$

d) $E[v_g(\omega+\lambda_j)v_h(\omega+\lambda_k)] = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-(d_{ig}-d_{1g})}\lambda_k^{-(d_{ih}-d_{1h})}\right)$

e) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega-\lambda_j)] = O\left(\frac{\log j}{j}(\lambda_j^{d_{1g}-d_{2g}} + \lambda_j^{d_{2h}-d_{1h}})\right)$

f) $E[v_g(\omega+\lambda_j)v_h(\omega-\lambda_j)] = O\left(\frac{\log j}{j}\right)$

g) $E[v_g(\omega+\lambda_j)\bar{v}_h(\omega-\lambda_k)] = O\left(\frac{\log j}{\sqrt{jk}}(\lambda_j^{d_{1g}-d_{2g}} + \lambda_k^{d_{2h}-d_{1h}})\right)$

h) $E[v_g(\omega+\lambda_j)v_h(\omega-\lambda_k)] = O\left(\frac{\log j}{\sqrt{jk}}(\lambda_j^{d_{1g}-d_{2g}} + \lambda_k^{d_{2h}-d_{1h}})\right)$

where $i = 1$ *if* $d_1 \geq d_2$ *in a) and b) and if* $d_{1s} \geq d_{2s}$, $s = g, h$ *in c) and d) and* $i = 2$ *otherwise.*

Proof: See Appendix A.

If $d_1 \geq d_2$ the results in a), b), c) and d) are basically those obtained by Robinson (1995a, Theorem 2) when $\omega = 0$. We focus on frequencies just after ω because Theorem 4 will be useful when studying the properties of semiparametric estimates of C and d_1 in (1.62), which describe the behaviour of the spectrum after ω. If we aim to estimate D and d_2 in (1.62) equivalent results to a),b),c) and d) in Theorem 4 would be obtained for the scaled discrete Fourier transforms evaluated at $\omega - \lambda_j$ and $\omega - \lambda_k$.

Remark: Even in the case $d_2 > d_1$, b) and d) are $O(\frac{\log j}{j})$ and $O(\frac{\log j}{k})$ respectively if $1/2 - d_{2s} + d_{1s} \geq 0$ for $s = g, h$ and $1/2 - d_{2s} + 2d_1 \geq 0$ for $s = g$ or h (see proof of Theorem 4 in Appendix A). These conditions hold if $d_{1s} \geq 0$ for $s = g, h$ irrespective of the values of d_{2s}. In Appendix A we also show that if $d_{2h} \geq d_{1s}$ for $s = g$ or h then h) is $O(\frac{\log j}{\sqrt{jk}})$.

Hereafter we focus on the estimation of C and d_1 in the univariate case, so that $g = h$ and $d_{ig} = d_{ih} = d_i$, $i = 1, 2$, in **A.1** and **A.2**. In order to obtain the asymptotic distribution of the least squares estimates in (3.3) two further assumptions are needed.

A.4: $\{x_t, t = 0, \pm 1, \pm 2, ...\}$ is a Gaussian process.

A.5:

$$\frac{\sqrt{m} n^{2(d_i - d_1)} \log m}{l^{1+2(d_i-d_1)}} + \frac{l(\log n)^2}{m} + \frac{m^{1+\frac{1}{2\alpha}}}{n} \to 0 \text{ as } n \to \infty,$$

where $i = 1$ if $d_1 \geq d_2$ and $i = 2$ if $d_1 < d_2$.

m and l are the bandwidth and trimming numbers respectively such that the estimation is carried out using frequencies $\omega + \frac{2\pi j}{n}$ for $l < j \leq m$. If $d_1 \geq d_2$ **A.5** is Assumption 6 in Robinson(1995a) and the proof of the asymptotic normality of the least squares estimates in (3.3) is basically the same, noting Theorem 4. However when $d_1 < d_2$ a stronger condition needs to be imposed on the bandwidth and trimming numbers. In this case there is a "trade-off" between assumptions **A.1** and **A.5** in the sense that the larger the difference $d_2 - d_1$ the larger the lower bound of α. For example if $d_2 - d_1 \geq 1/2$ **A.5** can only hold if $\alpha > 1$, because in that case

$$\frac{\sqrt{m} n^{2(d_2 - d_1)} \log m}{l^{1+2(d_2-d_1)}} \geq \frac{\sqrt{m} n \log m}{l^2} = \frac{n}{m^{1+\frac{1}{2\alpha}}} \frac{m^{\frac{3}{2}+\frac{1}{2\alpha}} \log m}{l^2}. \quad (3.5)$$

The first fraction goes to ∞ under **A.5**, so that the whole expression in (3.5) can go to zero only if the second fraction converges to 0. Since $l/m \to 0$ under **A.5**, then that can only happen if $\alpha > 1$. Consider for example $m \sim n^\theta$, $l \sim n^\phi$. In this case **A.5** entails

$$2(d_i - d_1) + \frac{1}{2}\theta - \phi(1 + 2(d_i - d_1)) < 0 \ , \ \phi < \theta \ , \ \theta\left(1 + \frac{1}{2\alpha}\right) < 1. \quad (3.6)$$

The first two conditions imply $\theta > \phi > 4(d_i - d_1)/(1 + 4(d_i - d_1))$, and incorporating the last condition in (3.6) we have that $\alpha > 2(d_i - d_1)$ has to hold. Because $|d_2 - d_1| < 1$, **A.4** can be satisfied for any d_1, d_2 if $\alpha = 2$. We also observe that the larger d_2 with respect to d_1 the larger m and l needed to get rid of the influence of the periodogram at frequencies just before ω on the estimation of d_1. This is so because, according to Theorem 4, the scaled discrete Fourier transforms are asymptotically homoscedastic and uncorrelated if $\frac{n^{d_i-d_1}\log j}{j^{\frac{1}{2}+(d_i-d_1)}k^{\frac{1}{2}+(d_i-d_1)}} \to 0$ as $n \to \infty$, so that although $\frac{k}{j}+\frac{j}{n} \to 0$ as $n \to \infty$ the increase in k and j has to be faster with respect to n the larger $d_2 - d_1$ is.

Define $v(\lambda) = \frac{W_n(\omega+\lambda)}{C^{\frac{1}{2}}\lambda^{-d_1}} = v_R(\lambda) + iv_I(\lambda)$, where $v_R(\lambda)$ and $v_I(\lambda)$ are the real and imaginary parts of $v(\lambda)$. Thus the $u_k^{(J)}$ in (3.4) can be written

$$u_k^{(J)} = \log[\sum_{j=1}^{J}\{v_R^2(\lambda_{k+j-J}) + v_I^2(\lambda_{k+j-J})\}e^{-\psi(J)}]. \tag{3.7}$$

Introduce the 2×1 vector $\nu(\lambda) = (v_R(\lambda), v_I(\lambda))$. The second moments of the elements of $\nu(\lambda_j)$ and $\nu(\lambda_k)$ can be deduced from those of $v(\lambda_j)$ and $v(\lambda_k)$ and their complex conjugates. Theorem 4 indicates that the different $\nu(\lambda_j)$ for j increasing adequately slowly with n can be regarded as approximately uncorrelated with zero mean (because $W_n(\lambda) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}(x_t - Ex_1)e^{it\lambda}$) and covariance matrix $\frac{1}{2}I_2$, where I_2 is the 2×2 identity matrix. Assumption **A.4** implies that the $\nu(\lambda_j)$ are Gaussian and thus the approximate uncorrelation can be interpreted as approximate independence. Introduce the two dimensional vector

$$V_j \sim NID(0, \frac{1}{2}I_2) \quad j = l+1, \ldots\ldots, m \tag{3.8}$$

where $V_j = (V_{1,j}, V_{2,j})$, and the variates

$$w_k^{(J)} = \log[\sum_{j=1}^{J}\{V_{1,k+j-J}^2 + V_{2,k+j-J}^2\}e^{-\psi(J)}] \ , k = l+J, l+2J, \ldots, m. \tag{3.9}$$

It follows that $\sum_{j=1}^{J}(V_{1,k+j-J}^2 + V_{2,k+j-J}^2) \sim \frac{1}{2}\chi_{2J}^2$ for each k. Thus (see Johnson and Kotz (1970) pg.167 and 181) $E[w_k^{(J)}] = 0$ and $w_k^{(J)}$ has finite moments of all orders and variance $\psi'(J)$, where $\psi'(z) = \frac{d}{dz}\psi(z)$ is the first derivative of the digamma function. Further, independence of the V_j implies independence of $w_{l+J}^{(J)}, w_{l+2J}^{(J)}, \ldots, w_m^{(J)}$. Consequently if the $u_k^{(J)}$ in (3.3) can be replaced by $w_k^{(J)}$ without affecting the limit distribution of the centered and adequately scaled least squares estimates in (3.3), we can apply the Lindeberg-Feller CLT and we will obtain the result stated in the following theorem.

Theorem 5 *Let* **A.1**, **A.2** *(with g=h),* **A.4** *and* **A.5** *hold. Then as $n \to \infty$,*

$$\begin{bmatrix} \frac{\sqrt{m}}{\log n}(\hat{c}^{(J)} - c^{(J)}) \\ 2\sqrt{m}(\hat{d}_1^{(J)} - d_1) \end{bmatrix} \xrightarrow{d} N\left[0, J\psi'(J)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]. \tag{3.10}$$

Proof: For $d_1 \geq d_2$ the assumptions and proof are equal to those in Robinson(1995a) for $\omega = 0$, noting Theorem 4. When $d_1 < d_2$ **A.5** differs from Assumption 6 in Robinson. Anyway the steps followed in the proof are quite similar and therefore they will be presented very briefly, paying attention to the steps where **A.5** takes part. The proof is based on showing that each moment of the variates on the left-hand side of (3.10) converges to the corresponding moments of the normal distribution implied by the right-hand side, and then appeal to the Frechet-Shohat "moment convergence theorem" (Loeve(1977), pg.187) and the unique determination of the normal distribution by its moments. We use Theorem 4 to show that the moments differ negligibly from those which would arise if instead of $u_k^{(J)}$ we had $w_k^{(J)}$ and then apply the Lindeberg-Feller CLT.

The least squares estimates in equation (3.3) are

$$\begin{bmatrix} \hat{c}^{(J)} \\ \hat{d}_1^{(J)} \end{bmatrix} = (Z'Z)^{-1}Z'Y \tag{3.11}$$

where Y is a $\frac{m-l}{J} \times 1$ column vector such that $Y_i = y_k^{(J)}$, and Z is a $\frac{m-l}{J} \times 2$ matrix with the first column a vector of ones and the components of the second column are $z_i = -2\log\lambda_k$, $k = l + iJ$, $i = 1, 2, ..., (m-l)/J$. Then

$$\begin{bmatrix} \hat{c}^{(J)} - c^{(J)} \\ \hat{d}_1^{(J)} - d_1 \end{bmatrix} = (Z'Z)^{-1}Z'U$$

where U is a $\frac{m-l}{J} \times 1$ column vector such that $U_i = u_k^{(J)}$ for $k = l+iJ$, $i = 1, 2, ..., (m-l)/J$. By approximation of sums by integrals we have as $n \to \infty$[2] (see Robinson (1995a)),

$$\sum_k z_k = \frac{2m}{J}[\log n + 1] + O(l \log n)$$

$$\sum_k z_k^2 = \frac{4m}{J}[(\log n)^2 + 2\log n + 2] + O(l(\log n)^2)$$

and thus

$$|Z'Z| = 4\frac{m-l}{J}\sum_k(\log\lambda_k)^2 - 4(\sum_k \log\lambda_k)^2 = 4\frac{m^2}{J^2} + O(lm(\log n)^2). \tag{3.12}$$

Under **A.5**,

$$\frac{\sqrt{m}}{\log n}\bar{z} = 2\sqrt{m}(1 + O(\log n)^{-1}) \tag{3.13}$$

[2]The sum $\sum_k$ is over $k = l + J, l + 2J, ..., m$.

where $\bar{z} = \frac{J}{m-l}\sum_k z_k = -2\frac{J}{m-l}\sum_k \log\lambda_k$, and we also have

$$\frac{J}{m-l}|Z'Z| = 4\frac{m}{J} + O(l(\log n)^2). \tag{3.14}$$

Now,

$$(Z'Z)^{-1} = \frac{1}{|Z'Z|}\frac{m-l}{J}\begin{bmatrix}\bar{z}\\ -1\end{bmatrix}[\bar{z}\ -1] + \frac{J}{m-l}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

and

$$[\bar{z}\ -1]Z'U = [\bar{z}\ -1]\begin{bmatrix}\sum_k u_k^{(J)}\\ \sum_k z_k u_k^{(J)}\end{bmatrix} = 2\sum_k(\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k)u_k^{(J)}$$

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} Z'U = \begin{bmatrix}1\\ 0\end{bmatrix}\sum_k u_k^{(J)}.$$

Now define the matrix

$$\Delta = \begin{bmatrix}\frac{\sqrt{m}}{\log n} & 0\\ 0 & 2\sqrt{m}\end{bmatrix}.$$

We have that

$$\begin{aligned}\Delta(Z'Z)^{-1}Z'U &= J^{\frac{1}{2}}(\frac{J}{m+o(m)})^{\frac{1}{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\sum_k(\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k)u_k^{(J)}\\ &+ \frac{J}{m-l}\begin{bmatrix}\frac{\sqrt{m}}{\log n}\\ 0\end{bmatrix}\sum_k u_k^{(J)}.\end{aligned} \tag{3.15}$$

The proof of the theorem is completed if as $n \to \infty$,

a) $(\frac{J}{m})^{\frac{1}{2}}\sum_k(\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k)u_k^{(J)} \xrightarrow{d} N(0, \psi'(J))$

b) $\frac{1}{\sqrt{m}\log n}\sum_k u_k^{(J)} \xrightarrow{p} 0.$

In order to prove a) and b) we claim that

$$(\frac{J}{m})^{\frac{1}{2}}\sum_k a_k u_k^{(J)} \xrightarrow{d} N(0, \psi'(J)) \tag{3.16}$$

for any triangular array $a_{kn} = a_k$ satisfying as $n \to \infty$,

$$\max_k |a_k| = o(m)\ ,\ \sum_k a_k^2 \sim \frac{m}{J} \text{ and } \sum_k |a_k|^p = O(m) \text{ for all } p \geq 1. \tag{3.17}$$

For b) $a_k = 1$ and for a) $a_k = \log k - \frac{J}{m-l}\sum_k \log k$ and (3.17) holds for both of them (see Robinson(1995a), pg.1067). Thus if we can verify our claim (3.16) the proof is completed. If instead of $u_k^{(J)}$ in (3.16) we have $w_k^{(J)}$ a direct application of the Lindeberg-Feller CLT shows that (3.16) holds under (3.17). Thus we have to prove that the moments of $(\frac{J}{m})^{\frac{1}{2}} \sum_k a_k u_k^{(J)}$ differ negligibly from those of $(\frac{J}{m})^{\frac{1}{2}} \sum_k a_k w_k^{(J)}$ and then we use the Frechet-Shohat "moment convergence theorem" (Loeve (1977)).

Write $\chi_k = (\frac{J}{m})^{\frac{1}{2}} a_k u_k^{(J)}$. Fix an integer N, $E[\sum_k \chi_k]^N$ is a sum of finitely many terms of the form

$$\sum_{k_1} \cdots \sum_{k_M} E(\prod_{i=1}^{M} \chi_{k_i}^{N_{k_i}}) \tag{3.18}$$

where $N_{k_1}, N_{k_2}, ..., N_{k_M}$ are all positive and sum to N and $1 \le M \le N$. Fix such M and $N_{k_1}, ..., N_{k_M}$, and introduce the $2J \times 1$ vector $\nu_k^* = (\nu(\lambda_{k+j-J})', ..., \nu(\lambda_k)')'$ and the $2JM \times 1$ vector $\nu^* = (\nu_{k_1}^{*}{}', ..., \nu_{k_M}^{*}{}')'$. Under **A.4** ν^* is normally distributed with zero mean and Theorem 4 implies that for $d_2 > d_1$,

$$\begin{aligned} E[\nu_j^* \nu_k^{*\prime}] &= \frac{1}{2} I_{2J} + O\left((\frac{j}{n})^\alpha + \frac{\log j}{j} \lambda_j^{-2(d_2-d_1)} \right) \text{ if } j = k \\ &= O\left(\frac{\log j}{\sqrt{jk}} \lambda_j^{-(d_2-d_1)} \lambda_k^{-(d_2-d_1)} \right) \text{ if } j > k. \end{aligned}$$

as $n \to \infty$. It follows from **A.5** that

$$\begin{aligned} \Sigma = E[\nu^* \nu^{*\prime}] &= \frac{1}{2} I_{2JM} + O\left((\frac{m}{n})^\alpha + \frac{\log m}{l} \lambda_l^{-2(d_2-d_1)} \right) && (3.19) \\ &= \frac{1}{2} I_{2JM} + o(m^{-\frac{1}{2}}) && (3.20) \end{aligned}$$

as $n \to \infty$. Thus $\Sigma^{-1} = \Psi$ exists for a large enough n. If φ_p is the density function of a p-dimensional standard normal variate, (3.18) is

$$\sum_{k_1} \cdots \sum_{k_M} |\Psi|^{\frac{1}{2}} \int (\prod_{i=1}^{M} \chi_{k_i}^{N_{k_i}}) \varphi_{2JM}(\Psi^{\frac{1}{2}} \nu^*) d\nu* \tag{3.21}$$

for n sufficiently large. Robinson(1995a) has proved that the difference between (3.21) and

$$\sum_{k_1} \cdots \sum_{k_M} E\left[\prod_{i=1}^{M} ((\frac{J}{m})^{\frac{1}{2}} a_{k_i} w_{k_i})^{N_{k_i}} \right] \tag{3.22}$$

is negligible (tends to zero as $n \to \infty$) which proves the theorem. □

Remark 1: $\hat{c}^{(J)}$ converges more slowly than $\hat{d}_1^{(J)}$ and there exists perfect negative correlation in the limiting joint distribution of $\hat{c}^{(J)}$ and $\hat{d}_1^{(J)}$. This distribution, as we could expect, is equal to that obtained by Robinson(1995a) for the case $\omega = 0$ and it only differs in a stronger condition on the bandwidth, m, and trimming, l, in order to get asymptotic uncorrelation of the scaled discrete Fourier transforms of x_t.

Remark 2: C can be estimated from $\hat{c}^{(J)}$, $\hat{C}^{(J)} = exp(\hat{c}^{(J)} - \psi(J))$, and a simple application of the "delta method" provides the asymptotic distribution of $\hat{C}^{(J)}$:

$$\frac{\sqrt{m}}{\log n}(\hat{C}^{(J)} - C) \xrightarrow{d} N(0, C^2 J\psi'(J)).$$

3.3 MULTIVARIATE EXTENSIONS

The results obtained in the previous section can be easily generalized to the multivariate case where x_t is a $G \times 1$ vector where all the components, x_{st}, for $s = 1, 2...G$, have a spectral pole or zero at frequency ω. Noting Theorem 4 and our assumption **A.5** the asymptotic distribution can be obtained as in Robinson (1995a).

We can also consider the possibility of simultaneous estimation of d_1, C and d_2, D but as we will see in Chapter 5, if d_1, C and d_2, D are functionally unrelated there is no gain in asymptotic efficiency because the estimates of the parameters before and after ω are asymptotically independent. However if we test and do not reject the hypothesis $d_1 = d_2 = d$, frequencies on both sides of ω are informative in order to estimate d (of course the estimates of $c^{(J)} = \log C - \psi(J)$ and $\delta^{(J)} = \log D - \psi(J)$ can be incorrect using frequencies on both sides of ω if $C \neq D$). The log-periodogram estimate of d is

$$\hat{d}^{(J)} = \frac{\hat{d}_1^{(J)} + \hat{d}_2^{(J)}}{2}$$

and since $\hat{d}_1^{(J)}$ and $\hat{d}_2^{(J)}$ are asymptotically independent (see Chapter 5), then

$$2\sqrt{2m}(\hat{d}^{(J)} - d) \xrightarrow{d} N(0, J\psi'(J)). \tag{3.23}$$

The same result is obtained in the multivariate setup followed by Robinson (1995a) imposing the restriction $d_1 = d_2$. Let

$$\begin{aligned} y_k^{(J)} &= c^{(J)} - d_1(2\log\lambda_k) + u_k^{(J)} \\ \tilde{y}_k^{(J)} &= \delta^{(J)} - d_2(2\log\lambda_k) + \tilde{u}_k^{(J)} \end{aligned}$$

where

$$y_k^{(J)} = \log(\sum_{j=1}^{J} I_n(\omega + \lambda_{k+j-J}))$$
$$\tilde{y}_k^{(J)} = \log(\sum_{j=1}^{J} I_n(\omega - \lambda_{k+j-J}))$$

for $k = l + J, l + 2J, ..., m$, $u_k^{(J)}$ is defined in (3.4) and $\tilde{u}_k^{(J)}$ differs only in the use of frequencies before ω like in the definition of $y_k^{(J)}$ and $\tilde{y}_k^{(J)}$. Denote $(X)_i$ the i-th row of the matrix X, and write

$$(Y)_i = (y_k^{(J)}, \tilde{y}_k^{(J)})$$
$$(Z)_i = (1, -2\log\lambda_k)$$
$$(U)_i = (u_k^{(J)}, \tilde{u}_k^{(J)})$$
$$A = \begin{pmatrix} c^{(J)} & \delta^{(J)} \\ d_1 & d_2 \end{pmatrix}$$

for $k = li + J$, $i = 1, 2, ..., (m - l)/J$. Then

$$Y = ZA + U.$$

Consider the restriction $d_1 = d_2 = d$,

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} d = Pd.$$

The restricted estimate in Robinson (1995a) is

$$\begin{bmatrix} \hat{c}^{(J)} \\ \hat{\delta}^{(J)} \\ \hat{d}^{(J)} \end{bmatrix} = \{Q'(Z'Z \otimes \Omega^{-1})Q\}^{-1} Q' vec(\Omega^{-1} Y' Z) \quad (3.24)$$

where

$$Q = \begin{pmatrix} I_2 & 0 \\ 0 & P \end{pmatrix}$$

and $\Omega = \psi'(J) I_2$. From (3.24) the estimate of d is

$$\hat{d}^{(J)} = \frac{\sum_k \log\lambda_k \sum_k (y_k^{(J)} + \tilde{y}_K^{(J)}) - \frac{m-l}{J} \sum_k \log\lambda_k (y_k^{(J)} + \tilde{y}_k^{(J)})}{4\frac{m-l}{J}\sum_k (\log\lambda_k)^2 - 4(\sum_k \log\lambda_k)^2}$$

the same as the estimate we obtain from the regression

$$z_k^{(J)} = c - d(2\log|\lambda_k|) + \bar{u}_k^{(J)} \quad k = -m, -m + J, ..., -l - J, l + J, l + 2J, ..., m, \quad (3.25)$$

where $z_k^{(J)} = \tilde{y}_{-k}^{(J)}$, $\bar{u}_k^{(J)} = \tilde{u}_k^{(J)}$ if k is negative and $z_k^{(J)} = y_k^{(J)}$, $\bar{u}_k^{(J)} = u_k^{(J)}$ for positive k. Not surprisingly the asymptotic distribution obtained in Theorem 4 by Robinson (1995a) is (3.23), the same as the one we obtained by applying least squares to (3.25).

It is also interesting to study the multivariate extension in the case of a $G \times 1$ vector series, x_t, whose elements, x_{st}, $s = 1, 2, ..., G$, have a spectral pole/zero at different frequencies. If for all $s = 1, ..., G$, **A.1** holds substituting the fixed ω for a different ω_s for every x_{st}, if x_t is a Gaussian process and if assumptions **A.2** (with $g = h$) and **A.5** hold for every d_{1s}, d_{2s} then

$$\begin{bmatrix} \frac{\sqrt{m}}{\log n}(\hat{c}^{(J)} - c^{(J)}) \\ 2\sqrt{m}(\hat{d}^{(J)} - d) \end{bmatrix} \xrightarrow{d} N\left(0, J\psi'(J)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes I_G\right) \tag{3.26}$$

where now $c^{(J)} = (c_1^{(J)}, ..., c_G^{(J)})'$, $d = (d_{11}, ..., d_{1G})'$ and the estimates are obtained from

$$\begin{bmatrix} \hat{c}^{(J)} \\ \hat{d}^{(J)} \end{bmatrix} = vec\{Y^{(J)\prime}Z^{(J)}(Z^{(J)\prime}Z^{(J)})^{-1}\}$$

where $Y^{(J)} = (Y_1^{(J)}, ..., Y_G^{(J)})$, $Y_s^{(J)} = (y_{s,l+j}^{(J)}, y_{s,l+2J}^{(J)}, ..., y_{s,m}^{(J)})'$, $y_{s,k}^{(J)} = \log(\sum_{j=1}^{J} I_s(\omega_s + \lambda_{k+j-J}))$, $Z^{(J)} = (z_{l+J}, z_{l+2J}, ..., z_m)'$ and $z_k = (1, -2\log\lambda_k)$. The proof is similar to that in Robinson (1995a) noting that now the vectors $v(\lambda_j) = (v_1^R(\lambda_j), ..., v_G^R(\lambda_j), v_1^I(\lambda_j), ..., v_G^I(\lambda_j))$ where $v_s(\lambda) = \frac{W_{ns}(\omega_s+\lambda)}{C_s^{0.5}\lambda^{-d_{1s}}} = v_s^R(\lambda) + iv_s^I(\lambda)$ can be considered, for j increasing suitably slowly with n, approximately uncorrelated with mean zero and covariance matrix $\frac{1}{2}I_{2G}$. This differs from the case $\omega_s = \omega$ for all $s = 1, ..., G$, when the covariance matrix of $v(\lambda_j)$ can be regarded as

$$R = \frac{1}{2}\begin{bmatrix} R_R & -R_I \\ R_I & R_R \end{bmatrix}$$

where R_R and R_I are the real and imaginary parts of $R(\omega)$, the matrix of coherencies at frequency ω ($[R(\omega)]_{gh} = R_{gh}(\omega)$, $g, h = 1, ..., G$), and $R(\omega)$ is assumed to be nonsingular. In case of different ω_ss we do not need any condition on the cross-spectral densities between the different x_{st}s because it can be shown in the same way as in Theorem 4 that $E[v_g(\omega_g + \lambda_j)\bar{v}_h(\omega_h + \lambda_j)] = O(j^{-1}\log j\lambda_j^{-2(d_i - d_1)})$ and of course the rest of statements in Theorem 4 hold if we allow for different ω_ss. Thus the $u_{sk}^{(J)} = \log[\sum_{j=1}^{J}((v_s^R(\omega_s + \lambda_{k+j-J}))^2 + (v_s^I(\omega_s + \lambda_{k+j-J}))^2 e^{-\psi(J)}]$ for $s = 1, ..., G$, can be considered approximately independent for all $k = l + J, l + 2J, ..., m$, which differs from the case with equal ω_ss where only the vectors $U_k^{(J)} = (u_{1k}^{(J)}, ..., u_{Gk}^{(J)})$ can be considered asymptotically independent but not the intravector variates. Taking this fact into account, the proof of (3.26) is straightforward following the steps in Robinson (1995a) and noting Theorem 4 and assumption **A.5** for d_{1s}, d_{2s}, $s = 1, ..., G$.

In (3.26) we focus on the estimation of the different d_{1s}s, that is, we study the behaviour of the spectral density matrix at frequencies just after the different ω_ss. The same result would be obtained for $d = (d_{21}, ..., d_{2G})'$ as long as the frequencies ω_s are different for all $s = 1, ...G$.

Finally we can consider the case $\omega_s = \omega$ if $s \in H$, and H is a subset of $\{1, 2, ..., G\}$, that is , the case when only some of the ω_ss are equal. In this case we need to introduce assumptions **A.2** and **A.3** for those $g, h \in H$ and substitute α in **A.5** by $\min(\alpha, \beta)$ and we obtain the same result as in Robinson (1995a), namely

$$\begin{bmatrix} \frac{\sqrt{m}}{\log n}(\hat{c}^{(J)} - c) \\ 2\sqrt{m}(\hat{d}^{(J)} - d) \end{bmatrix} \xrightarrow{d} N\left(0, J\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \Omega^{(J)}\right]$$

where the diagonal elements of $\Omega^{(J)}$ are $\psi'(J)$ and $\Omega_{gh}^{(J)}$ for $g \neq h$ is zero if $\omega_g \neq \omega_h$ and some finite figure if $\omega_g = \omega_h$.

3.4 APPENDIX A: PROOF OF THEOREM 4

The proof of Theorem 4 is based on that of Theorem 2 in Robinson (1995a) for the case $\omega = 0$.

a) In order to show a) we see first that

$$E[W_g(\omega + \lambda_j)\bar{W}_h(\omega + \lambda_j)] = f_{gh}(\omega + \lambda_j) + O\left(\frac{\log j}{j}\lambda_j^{-2d_i}\right) \tag{3.27}$$

and then that

$$f_{gh}(\omega + \lambda_j) - C_g^{\frac{1}{2}} C_h^{\frac{1}{2}} \lambda_j^{-2d_1} R_{gh}(\omega) = O(\lambda_j^{\min(\alpha,\beta) - 2d_1}). \tag{3.28}$$

To prove (3.27) first write the left hand side of the equality as

$$\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(t-s)e^{is(\omega+\lambda_j)}e^{-it(\omega+\lambda_j)} = \int_{-\pi}^{\pi} f_{gh}(\lambda)K(\omega + \lambda_j - \lambda)\mathrm{d}\lambda$$

where $\gamma_{gh}(t-s)$ is the covariance between x_{gt} and x_{hs} and $K(\lambda) = \frac{1}{2\pi n}\sum_t \sum_s e^{i(t-s)\lambda}$ is Fejer's kernel.

Since $\int_{-\pi}^{\pi} K(\omega+\lambda_j-\lambda)\mathrm{d}\lambda = 1$ we have to study the order of magnitude of

$$\int_{-\pi}^{\pi} \{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_j)\}K(\omega+\lambda_j-\lambda)\mathrm{d}\lambda. \tag{3.29}$$

Due to assumptions **A.1** and **A.2** we can chose so small a ε that for some $C_\varepsilon < \infty$,

$$\begin{aligned} |f_{gh}(\omega+\lambda)| &\leq f_g^{\frac{1}{2}}(\omega+\lambda)f_h^{\frac{1}{2}}(\omega+\lambda) \leq C_\varepsilon \lambda^{-2d_1} \\ \left|\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{gh}(\omega+\lambda)\right| &\leq C_\varepsilon \lambda^{-1-2d_1} \end{aligned}$$

for $\lambda \in (0,\varepsilon)$ and $2d_1 = d_{g1} + d_{h1}$, and

$$\begin{aligned} |f_{gh}(\omega+\lambda)| &\leq C_\varepsilon |\lambda|^{-2d_2} \\ \left|\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{gh}(\omega+\lambda)\right| &\leq C_\varepsilon |\lambda|^{-1-2d_2} \end{aligned}$$

for $\lambda \in (-\varepsilon, 0)$ and $2d_2 = d_{g2} + d_{h2}$.

Because $\omega \in (0,\pi)$ and $\frac{j}{n} \to 0$ as $n \to \infty$, we can choose $\varepsilon > 0$ such that for a large enough n

$$\begin{aligned} \varepsilon &> 2\lambda_j \\ 2\omega \pm \lambda_j - \varepsilon &> 0 \\ 2\omega + \lambda_j + \varepsilon &< 2\pi \end{aligned} \tag{3.30}$$

which will be necessary for subsequent analysis. For such a ε we have that the absolute value of part of the integral (3.29) is

$$\begin{aligned} &\left|\int_{-\pi}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\pi}\right| \\ \leq\ & \{\max_{\lambda\in\Omega} K(\omega+\lambda_j-\lambda)\} \int_{-\pi}^{\pi} \{|f_{gh}(\lambda)| + |f_{gh}(\omega+\lambda_j)|\}\mathrm{d}\lambda \\ =\ & O(n^{-1}(1+\lambda_j^{-2d_1})) = O(\frac{1}{j}\lambda_j^{-2d_1}) \end{aligned}$$

where $\Omega = [-\pi, \omega-\varepsilon] \cup [\omega+\varepsilon, \pi]$. The first equality comes from the following facts which will be useful in subsequent analysis:

$$K(\lambda) = \frac{|D(\lambda)|^2}{2\pi n} \tag{3.31}$$

$$|D(\lambda)| = |\sum_t e^{it\lambda}| \le \frac{1}{|\sin\frac{\lambda}{2}|} \quad \text{if } 0 < \lambda < 2\pi \tag{3.32}$$

$$|K(\lambda)| = O(n^{-1}\lambda^{-2}) \quad \text{for } 0 < |\lambda| < \pi \tag{3.33}$$

$$|f_{gh}(\lambda)| \le f_g^{\frac{1}{2}}(\lambda) f_h^{\frac{1}{2}}(\lambda) \text{ and} \int_{-\pi}^{\pi} f_i(\lambda)\mathrm{d}\lambda = var(x_{it}) < \infty, i = g, h, \tag{3.34}$$

and the second one because

$$n^{-1} = O\left(\left(\frac{j}{n}\right)^{1+2d_1} \frac{1}{j}\lambda_j^{-2d_1}\right) \text{ and } 1 + 2d_1 > 0. \tag{3.35}$$

Now decompose the remainder of the integral,

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} = \int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}} + \int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}} + \int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{\omega+2\lambda_j}^{\omega+\varepsilon}. \tag{3.36}$$

The first integral in (3.36) is bounded in modulus by

$$\begin{aligned}
& \{\max_{\omega-\varepsilon\le\lambda\le\omega-\frac{\lambda_j}{2}} |f_{gh}(\lambda)|\} \int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}} K(\omega+\lambda_j-\lambda)\mathrm{d}\lambda \\
+ & |f_{gh}(\omega+\lambda_j)| \int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}} K(\omega+\lambda_j-\lambda)\mathrm{d}\lambda \\
= & \{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon} |f_{gh}(\omega-\lambda)|\} \int_{\frac{\lambda_j}{2}}^{\varepsilon} K(\lambda_j+\lambda)\mathrm{d}\lambda + |f_{gh}(\omega+\lambda_j)| \int_{\frac{\lambda_j}{2}}^{\varepsilon} K(\lambda_j+\lambda)\mathrm{d}\lambda \\
\le & \left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon+\lambda_j} \frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2h}}}\right\} \int_{\frac{\lambda_j}{2}}^{\varepsilon+\lambda_j} K(\lambda)\lambda^{\frac{1}{2}-d_{2h}}\mathrm{d}\lambda \\
+ & |f_{gh}(\omega+\lambda_j)| \int_{\frac{\lambda_j}{2}}^{\varepsilon+\lambda_j} K(\lambda)\mathrm{d}\lambda \\
= & O(n^{-1}\lambda_j^{-1-2d_2} + n^{-1}\lambda_j^{-1-2d_1}) = O(j^{-1}\lambda_j^{-2d_i})
\end{aligned}$$

because of (3.33). Similarly the last integral in (3.36) is bounded in absolute value by

$$\begin{aligned}
& \{\max_{2\lambda_j\le\lambda\le\varepsilon} |f_{gh}(\omega+\lambda)|\} \int_{2\lambda_j}^{\varepsilon} K(\lambda_j-\lambda)\mathrm{d}\lambda + |f_{gh}(\omega+\lambda_j)| \int_{2\lambda_j}^{\varepsilon} K(\lambda_j-\lambda)\mathrm{d}\lambda \\
= & O(j^{-1}\lambda_j^{-2d_1}).
\end{aligned}$$

Now, using the mean value theorem,

$$\begin{aligned}
\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right| &= \left|\int_{\frac{\lambda_j}{2}}^{2\lambda_j} \{f_{gh}(\omega+\lambda) - f_{gh}(\omega+\lambda_j)\} K(\lambda_j-\lambda)\mathrm{d}\lambda\right| \\
&\le \{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j} |f'_{gh}(\omega+\lambda)|\} \int_{\frac{\lambda_j}{2}}^{2\lambda_j} |\lambda-\lambda_j| K(\lambda_j-\lambda)\mathrm{d}\lambda \\
&= O(n^{-1}\lambda_j^{-1-2d_1} \int_{\frac{\lambda_j}{2}}^{2\lambda_j} |D(\lambda_j-\lambda)|\mathrm{d}\lambda) = O\left(\frac{\log j}{j}\lambda_j^{-2d_1}\right)
\end{aligned}$$

because of (3.31) and

$$|D(\lambda)| \leq 2|\lambda|^{-1}\ ,\ 0 < |\lambda| < \pi \tag{3.37}$$

$$\int_{-C\lambda_j}^{C\lambda_j} |D(\lambda)| \mathrm{d}\lambda = O(\log j) \text{ for } C < \infty \tag{3.38}$$

For the property (3.37) on Dirichlet's kernel $D(\lambda)$ see Zygmund (1977), pages 49-51, (3.38) is Lemma 5 of Robinson (1994b).

To complete the proof of (3.27), $\int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}}$ in (3.36) is bounded in absolute value by

$$\begin{aligned}
&\{\max_{-\frac{\lambda_j}{2}\leq\lambda\leq\frac{\lambda_j}{2}} K(\lambda_j-\lambda)\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}\{|f_{gh}(\omega+\lambda)|+|f_{gh}(\omega+\lambda_j)|\}\mathrm{d}\lambda\\
= &\ O(n^{-1}\lambda_j^{-2}\lambda_j^{1-2d_i}) = O(j^{-1}\lambda_j^{-2d_i}).
\end{aligned}$$

Now the left hand side of (3.28) is dominated by:

$$f_g^{\frac{1}{2}} f_h^{\frac{1}{2}} \left| 1 - \frac{C_g^{\frac{1}{2}} C_h^{\frac{1}{2}} \lambda_j^{-2d_1}}{f_g^{\frac{1}{2}} f_h^{\frac{1}{2}}} \right| + |R_{gh}(\omega+\lambda_j) - R_{gh}(\omega)| C_g^{\frac{1}{2}} C_h^{\frac{1}{2}} \lambda_j^{-2d_1}$$

where the spectral densities are evaluated at $\omega + \lambda_j$. This is

$$O(\lambda_j^{\alpha-2d_1}) + O(\lambda_j^{\beta-2d_1})$$

under assumption **A.1** and **A.3**.

b) To prove b) write

$$\begin{aligned}
&E[W_g(\omega+\lambda_j)W_h(\omega+\lambda_j)]\\
= &\ \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(t-s)e^{it(\omega+\lambda_j)}e^{is(\omega+\lambda_j)}\\
= &\ \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega+\lambda_j-\lambda)\mathrm{d}\lambda.
\end{aligned}$$

Decompose the integral into

$$\begin{aligned}
&\int_{-\pi}^{-\omega-\varepsilon} + \int_{-\omega-\varepsilon}^{-\omega-2\lambda_j} + \int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}} + \int_{-\omega-\frac{\lambda_j}{2}}^{-\omega+\frac{\lambda_j}{2}} + \int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon} + \int_{-\omega+\varepsilon}^{\omega-\varepsilon}\\
+ &\int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}} + \int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}} + \int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{\omega+2\lambda_j}^{\omega+\varepsilon} + \int_{\omega+\varepsilon}^{\pi}.
\end{aligned}$$

The integral over $\Omega = [-\pi, -\omega-\varepsilon] \cup [-\omega+\varepsilon, \omega-\varepsilon] \cup [\omega+\varepsilon, \pi]$ is bounded in absolute value by

$$\begin{aligned} & \frac{1}{2\pi n}\{\max_{\lambda\in\Omega}|D(\omega+\lambda_j+\lambda)||D(\omega+\lambda_j-\lambda)|\}\int_{-\pi}^{\pi}|f_{gh}(\lambda)|\mathrm{d}\lambda \\ = \ & O(n^{-1}) = O(j^{-1}\lambda_j^{-2d_1}) \end{aligned}$$

using (3.30), (3.32), (3.34) and (3.35). Now $|\int_{-\omega-\varepsilon}^{-\omega-2\lambda_j}|$ is bounded by

$$\begin{aligned} & \frac{1}{2\pi n}\{\max_{2\lambda_j\le\lambda\le\varepsilon}|f_{gh}(-\lambda-\omega)|\}\int_{2\lambda_j}^{\varepsilon}|D(\lambda_j-\lambda)||D(2\omega+\lambda_j+\lambda)|\mathrm{d}\lambda \\ \le \ & \frac{1}{2\pi n}\left\{\max_{\lambda_j\le\lambda\le\varepsilon}\frac{|f_{gh}(-\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{1h}}}\right\}\left\{\max_{\lambda_j\le\lambda\le\varepsilon}\frac{1}{|\sin\frac{2\omega+\lambda_j+\lambda}{2}|}\right\}\int_{\lambda_j}^{\varepsilon}2\lambda^{-\frac{1}{2}-d_{1h}}\mathrm{d}\lambda \\ = \ & O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{1g}}) = O(\frac{1}{j}\lambda_j^{-2d_1}(\frac{j}{n})^{\frac{1}{2}+d_{1h}}) = O(j^{-1}\lambda_j^{-2d_1}) \end{aligned}$$

the first inequality because of (3.30), (3.32) and (3.37) and the last equality because $1/2+d_{1g} > 0$. Similarly

$$\left|\int_{\omega+2\lambda_j}^{\omega+\varepsilon}\right| = O(j^{-1}\lambda_j^{-2d_1}).$$

Proceeding in the same manner the integral over $[-\omega+\frac{\lambda_j}{2}, -\omega+\varepsilon]$ is bounded in modulus by

$$\begin{aligned} & \frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon}|f_{gh}(-\omega+\lambda)|\}\int_{\frac{\lambda_j}{2}}^{\varepsilon}|D(\lambda_j+\lambda)||D(2\omega+\lambda_j-\lambda)|\mathrm{d}\lambda \\ \le \ & \frac{1}{2\pi n}\{\max|f_{gh}(-\omega+\lambda)|\}\left\{\max\frac{1}{|\sin\frac{2\omega+\lambda_j-\lambda}{2}|}\right\}2\int_{\frac{\lambda_j}{2}}^{\varepsilon}\frac{1}{|\lambda_j+\lambda|}\mathrm{d}\lambda \\ \le \ & \frac{1}{2\pi n}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon+\lambda_j}\frac{|f_{gh}(-\omega+\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon}\frac{1}{|\sin\frac{2\omega+\lambda_j-\lambda}{2}|}\right\}2\int_{\frac{\lambda_j}{2}}^{\varepsilon+\lambda_j}\lambda^{-\frac{1}{2}-d_{2g}}\mathrm{d}\lambda \\ = \ & O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2h}}) = O(j^{-1}\lambda_j^{-2d_2}) \end{aligned}$$

and under the conditions in the remark to Theorem 4 this is

$$O(j^{-1}\lambda_j^{-2d_1}\lambda_j^{\frac{1}{2}-d_{2h}+2d_1}) = O(j^{-1}\lambda_j^{-2d_1}).$$

Similarly

$$\left|\int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}}\right| = O(j^{-1}\lambda_j^{-2d_2})$$

and $O(j^{-1}\lambda_j^{-2d_1})$ under the conditions in the remark to Theorem 4.

Now

$$\begin{aligned}
\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right| &\leq \frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq 2\lambda_j}|f_{gh}(\omega+\lambda)|\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(2\omega+\lambda_j+\lambda)||D(\lambda_j-\lambda)|\mathrm{d}\lambda \\
&\leq \frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq 2\lambda_j}|f_{gh}(\omega+\lambda)|\}\left\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq 2\lambda_j}\frac{1}{|\sin\frac{2\omega+\lambda_j+\lambda}{2}|}\right\}\int_{-\lambda_j}^{\lambda_j}|D(\lambda)|\mathrm{d}\lambda \\
&= O(n^{-1}\lambda_j^{-2d_1}\log j) = O(\frac{\log j}{j}\lambda_j^{-2d_1})
\end{aligned}$$

the second inequality because of (3.30) and (3.32) and the first equality due to (3.38). Similarly

$$\left|\int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}\right| = O(\frac{\log j}{j}\lambda_j^{-2d_1}).$$

To complete the proof of b), the integral over $[\omega-\frac{\lambda_j}{2},\omega+\frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\begin{aligned}
&\frac{1}{2\pi n}\{\max_{-\frac{\lambda_j}{2}\leq\lambda\leq\frac{\lambda_j}{2}}|D(2\omega+\lambda_j+\lambda)|\}\{\max_{-\frac{\lambda_j}{2}\leq\lambda\leq\frac{\lambda_j}{2}}|D(\lambda_j-\lambda)|\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}|f_{gh}(\omega+\lambda)|\mathrm{d}\lambda \\
&= O(n^{-1}\lambda_j^{-1}(\lambda_j^{1-2d_1}+\lambda_j^{1-2d_2})) = O(j^{-1}\lambda_j^{-2d_i})
\end{aligned}$$

the first equality because of (3.30), (3.34) and (3.37) and under the conditions in the remark this is

$$O(j^{-1}\lambda_j^{-2d_1}(\lambda_j+\lambda_j^{1-2d_2+2d_1})) = O(j^{-1}\lambda_j^{-2d_1}).$$

The analysis for the integral over $[-\omega\pm\frac{\lambda_j}{2}]$ is similar and this concludes the proof of b).

c) To prove c) write

$$\begin{aligned}
&E[W_g(\omega+\lambda_j)\bar{W}_h(\omega+\lambda_k)] \\
&= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(s-t)e^{it(\omega+\lambda_j)}e^{-is(\omega+\lambda_k)} \\
&= \int_{-\pi}^{\pi}f_{gh}(\lambda)E_{jk}(\lambda)\mathrm{d}\lambda
\end{aligned}$$

where $E_{jk}=\frac{1}{2\pi n}D(\omega+\lambda_j-\lambda)D(\lambda-\omega-\lambda_k)$. Since $\int_{-\pi}^{\pi}e^{i(s-t)\lambda}\mathrm{d}\lambda=0$ for $s\neq t$ and 2π for $s=t$, and $\sum_{t=1}^{n}e^{it(\lambda_j-\lambda_k)}=0$ for $0<j-k<n/2$, then

$$\int_{-\pi}^{\pi}E_{jk}(\lambda)\mathrm{d}\lambda = 0. \tag{3.39}$$

Thus we can expand the integral as

$$\left\{\int_{-\pi}^{\omega+\frac{\lambda_k}{2}} + \int_{\omega+2\lambda_j}^{\pi}\right\}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_j)\}E_{jk}(\lambda)\mathrm{d}\lambda \tag{3.40}$$

$$+ \int_{\omega+\frac{\lambda_k+\lambda_j}{2}}^{\omega+2\lambda_j}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_j)\}E_{jk}(\lambda)\mathrm{d}\lambda \tag{3.41}$$

$$+ \int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k+\lambda_j}{2}}\{f_{gh}(\lambda) - f_{gh}(\omega+\lambda_k)\}E_{jk}(\lambda)\mathrm{d}\lambda \tag{3.42}$$

$$- \{f_{gh}(\omega+\lambda_j) - f_{gh}(\omega+\lambda_k)\}\int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k+\lambda_j}{2}} E_{jk}(\lambda)\mathrm{d}\lambda. \tag{3.43}$$

Now (3.41) is bounded by

$$\begin{aligned}
&\frac{1}{\pi n}\{\max_{(\lambda_k+\lambda_j)/2\le\lambda\le 2\lambda_j}|f'_{gh}(\omega+\lambda)|\}\int_{\frac{\lambda_j+\lambda_k}{2}}^{2\lambda_j}|D(\lambda-\lambda_k)|\mathrm{d}\lambda \\
= \;& O(n^{-1}\lambda_j^{-1-2d_1}\log j) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}(\frac{k}{j})^{\frac{1}{2}+d_{1h}}\right) \\
= \;& O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)
\end{aligned}$$

for $j > k$. The absolute value of (3.42) is bounded by

$$\begin{aligned}
&\frac{1}{\pi n}\{\max_{\lambda_k/2\le\lambda\le(\lambda_j+\lambda_k)/2}|f'_{gh}(\omega+\lambda)|\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_k+\lambda_j}{2}}|D(\lambda_j-\lambda)|\mathrm{d}\lambda \\
= \;& O\left(n^{-1}\lambda_k^{-1-2d_1}\log j\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}(\frac{j}{k})^{\frac{1}{2}+d_{1g}}\right) \\
= \;& O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)
\end{aligned}$$

if $j/2 \le k < j$, and when $k < j/2$ (3.42) is bounded by

$$\begin{aligned}
&\frac{1}{\pi n}\{\max_{\lambda_k\le 2\lambda\le(\lambda_j+\lambda_k)}|f_{gh}(\omega+\lambda)| + |f_{gh}(\omega+\lambda_j)|\}(\lambda_j-\lambda_k)^{-1}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_k+\lambda_j}{2}}|D(\lambda_j-\lambda)|\mathrm{d}\lambda \\
= \;& O((\lambda_j^{-2d_1} + \lambda_k^{-2d_1})(j-k)^{-1}\log j) \\
= \;& O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\frac{j}{j-k}\left((\frac{k}{j})^{\frac{1}{2}+d_{1h}} + (\frac{k}{j})^{\frac{1}{2}-d_{1g}}\right)\right) \\
= \;& O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).
\end{aligned}$$

Now (3.43) is bounded by

$$\begin{aligned}
&\frac{1}{2\pi n}\{\max_{\lambda_k\le\lambda\le\lambda_j}|f'_{gh}(\omega+\lambda)|\}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_j+\lambda_k}{2}}|D(\lambda-\lambda_k)|\mathrm{d}\lambda \\
= \;& O(n^{-1}\lambda_k^{-1-d_1}\log j) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)
\end{aligned}$$

if $k \geq j/2$, and when $k < j/2$ (3.43) is bounded by

$$\frac{1}{\pi n}\{|f_{gh}(\omega+\lambda_j)| + |f_{gh}(\omega+\lambda_k)|\}(\lambda_j-\lambda_k)^{-1}\int_{\frac{\lambda_k}{2}}^{\frac{\lambda_j+\lambda_k}{2}} |D(\lambda-\lambda_k)|\mathrm{d}\lambda$$
$$= O((\lambda_j^{-2d_1}+\lambda_k^{-2d_1})(j-k)^{-1}\log j) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

as in the evaluation of (3.42). Now the integral in (3.40) can be decomposed into

$$\int_{-\pi}^{\omega-\varepsilon} + \int_{\omega-\varepsilon}^{\omega-\lambda_j} + \int_{\omega-\lambda_j}^{\omega-\frac{\lambda_k}{2}} + \int_{\omega-\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k}{2}} + \int_{\omega+2\lambda_j}^{\omega+\varepsilon} + \int_{\omega+\varepsilon}^{\pi}.$$

As in **a)**, the integral over $[-\pi,\omega-\varepsilon]\cup[\omega+\varepsilon,\pi]$ is $O(n^{-1}(1+\lambda_j^{-2d_1})) = O((jk)^{-\frac{1}{2}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}})$. The integral over $[\omega-\varepsilon,\omega-\lambda_j]$ is bounded by

$$\frac{1}{2\pi n}|f_{gh}(\omega+\lambda_j)|\int_{\lambda_j}^{\varepsilon}|D(\lambda_j+\lambda)||D(-\lambda-\lambda_k)|\mathrm{d}\lambda$$
$$+ \frac{2}{\pi n}\left\{\max_{\lambda_j\leq\lambda\leq 2\varepsilon}\frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\int_{\lambda_j}^{2\varepsilon}\lambda^{-\frac{3}{2}-d_{2g}}\mathrm{d}\lambda$$
$$= O(n^{-1}\lambda_j^{-2d_1}\lambda_j^{-1}+n^{-1}\lambda_j^{-1-2d_2}) = O(j^{-1}\lambda_j^{-2d_i}) = O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}).$$

Similarly for the integral on $[\omega+2\lambda_j,\omega+\varepsilon]$ we obtain the upper bound

$$O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

The integral on $[\omega-\lambda_j,\omega-\lambda_k/2]$ is bounded by

$$\frac{1}{2\pi n}\{\max_{\lambda_k/2\leq\lambda\leq\lambda_j}|f_{gh}(\omega-\lambda)| + |f_{gh}(\omega+\lambda_j)\}\int_{\frac{\lambda_k}{2}}^{\lambda_j}|D(\lambda_j+\lambda)||D(-\lambda-\lambda_k)|\mathrm{d}\lambda$$
$$= O(n^{-1}(\lambda_j^{-2d_i}+\lambda_k^{-2d_2})\lambda_j^{-1}\log j) = O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}})$$

as in the evaluation in (3.42). Finally the integral on $[\omega\pm\lambda_k/2]$ is bounded in absolute value by

$$\frac{1}{2\pi n}\{\max_{-\lambda_k\leq 2\lambda\leq\lambda_k}|D(\lambda_j-\lambda)D(\lambda-\lambda_k)|\int_{-\frac{\lambda_k}{2}}^{\frac{\lambda_k}{2}}\{|f_{gh}(\omega+\lambda)|+|f_{gh}(\omega+\lambda_j)|\}\mathrm{d}\lambda$$
$$= O(n^{-1}\lambda_j^{-1}\lambda_k^{-1}(\lambda_k^{1-2d_i}+\lambda_k\lambda_j^{-2d_1})) = O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}})$$

and this completes the proof of **c)**.

d) Write

$$\begin{aligned}
& E[W_g(\omega+\lambda_j)W_h(\omega+\lambda_k)] \\
= \ & \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{gh}(t-s)e^{it(\omega+\lambda_j)}e^{is(\omega+\lambda_k)} \\
= \ & \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega+\lambda_k-\lambda)\mathrm{d}\lambda
\end{aligned}$$

and split the integral into

$$\begin{aligned}
& \int_{-\pi}^{-\omega-\varepsilon}+\int_{-\omega-\varepsilon}^{-\omega-2\lambda_j}+\int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}+\int_{-\omega-\frac{\lambda_j}{2}}^{-\omega+\frac{\lambda_j}{2}}+\int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon}+\int_{-\omega+\varepsilon}^{\omega-\varepsilon} \\
+ & \int_{\omega-\varepsilon}^{\omega-\frac{\lambda_j}{2}}+\int_{\omega-\frac{\lambda_j}{2}}^{\omega-\frac{\lambda_k}{2}}+\int_{\omega-\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k}{2}}+\int_{\omega+\frac{\lambda_k}{2}}^{\omega+2\lambda_k}+\int_{\omega+2\lambda_k}^{\omega+\varepsilon}+\int_{\omega+\varepsilon}^{\pi}.
\end{aligned}$$

Doing the same as in b),

$$\begin{aligned}
\left|\int_{-\pi}^{-\omega-\varepsilon}+\int_{-\omega+\varepsilon}^{\omega-\varepsilon}+\int_{\omega+\varepsilon}^{\pi}\right| &= O(n^{-1}) = O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}+d_{1g}}\lambda_k^{\frac{1}{2}+d_{1h}}\right) \\
&= O\left(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).
\end{aligned}$$

Now the integral over $[-\omega-\varepsilon, -\omega-2\lambda_j]$ is bounded in absolute value by

$$\begin{aligned}
& \frac{1}{2\pi n}\{\max_{2\lambda_j\le\lambda\le\varepsilon}|f_{gh}(-\omega-\lambda)|\}\int_{2\lambda_j}^{\varepsilon}|D(\lambda_j-\lambda)||D(2\omega+\lambda_k+\lambda)|\mathrm{d}\lambda \\
\le \ & \frac{1}{\pi n}\left\{\max_{2\lambda_j\le\lambda\le\varepsilon}\frac{1}{|\sin\frac{2\omega+\lambda_k+\lambda}{2}|}\right\}\left\{\max_{\lambda_j\le\lambda\le\varepsilon}\frac{|f_{gh}(-\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{1h}}}\right\}\int_{\lambda_j}^{\varepsilon}\lambda^{-\frac{1}{2}-d_{1h}}\mathrm{d}\lambda \\
= \ & O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{1g}}) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_k^{\frac{1}{2}+d_{1h}}\right) \\
= \ & O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)
\end{aligned}$$

and similarly

$$\left|\int_{\omega+2\lambda_k}^{\omega+\varepsilon}\right| = O(n^{-1}\lambda_k^{-\frac{1}{2}-d_{1h}}) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}+d_{1g}}\right) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

The integral over $[-\omega-2\lambda_j, -\omega-\frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\begin{aligned}
& \frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j}|f_{gh}(-\omega-\lambda)|\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(\lambda_j-\lambda)||D(2\omega+\lambda_k+\lambda)|\mathrm{d}\lambda \\
= \ & O(n^{-1}\lambda_j^{-2d_1}\log j) = O\left(\frac{\log j}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}-d_{1h}}\lambda_k^{\frac{1}{2}+d_{1h}}\right) \\
= \ & O\left(\frac{\log j}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)
\end{aligned}$$

and similarly

$$\left|\int_{\omega+\frac{\lambda_k}{2}}^{\omega+2\lambda_k}\right| = O(n^{-1}\lambda_k^{-2d_1}\log k) = O(\frac{\log j}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}).$$

The integral over $[-\omega \pm \frac{\lambda_j}{2}]$ is bounded in absolute value by

$$\begin{aligned} &\frac{1}{2\pi n}\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}} |D(\lambda_j+\lambda)||D(2\omega+\lambda_k-\lambda)|\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}} |f_{gh}(-\omega+\lambda)|\mathrm{d}\lambda \\ = \ & O(n^{-1}\lambda_j^{-1}\lambda_j^{1-2d_i}) = O(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}) \end{aligned}$$

and under the conditions stated in the remark this is

$$O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}[\lambda_j^{\frac{1}{2}-d_{1h}}\lambda_k^{\frac{1}{2}+d_{1h}} + \lambda_j^{\frac{1}{2}-2d_2+d_{1g}}\lambda_k^{\frac{1}{2}+d_{1h}}]) = O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}).$$

We obtain similarly the same result for the integral over $[\omega \pm \frac{\lambda_k}{2}]$.

The integral over $[-\omega + \frac{\lambda_j}{2}, -\omega + \varepsilon]$ is bounded in absolute value by

$$\begin{aligned} &\frac{1}{\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon} |D(2\omega+\lambda_k-\lambda)|\}\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon} |f_{gh}(\lambda-\omega)|\}\int_{\frac{\lambda_j}{2}}^{\varepsilon}\frac{1}{|\lambda_j+\lambda|}\mathrm{d}\lambda \\ \le\ & \frac{1}{\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le\varepsilon} |D(2\omega+\lambda_k-\lambda)|\}\{\max_{\lambda_j\le\lambda\le\lambda_j+\varepsilon}\frac{|f_{gh}(\lambda-\omega)|}{\lambda^{\frac{1}{2}-d_{2h}}}\}\int_{\lambda_j}^{\varepsilon+\lambda_j}\lambda^{-\frac{1}{2}-d_{2h}}\mathrm{d}\lambda \\ = \ & O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2g}}) = O(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}) \end{aligned}$$

and under the conditions in the remark

$$\begin{aligned} & O(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{-d_{2g}+d_{1g}}\lambda_k^{\frac{1}{2}+d_{1h}}) \\ = \ & O(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}-d_{2g}+2d_1}) = O(\frac{1}{\sqrt{kj}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}). \end{aligned}$$

We obtain similarly the same upper bound for the integral over $[\omega-\varepsilon, \omega-\frac{\lambda_j}{2}]$.

Finally the absolute value of the integral over $[\omega - \frac{\lambda_j}{2}, \omega - \frac{\lambda_k}{2}]$ is bounded by

$$\begin{aligned} &\frac{1}{2\pi n}\{\max_{\frac{\lambda_k}{2}\le\lambda\le\frac{\lambda_j}{2}} |D(2\omega+\lambda_j-\lambda)|\}\left\{\max_{\lambda_k\le\lambda\le\lambda_j}\frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\int_{\lambda_k}^{\lambda_j}\lambda^{-\frac{1}{2}-d_{2g}}\mathrm{d}\lambda \\ = \ & O(n^{-1}\lambda_k^{-\frac{1}{2}-d_{2h}}\lambda_j^{\frac{1}{2}-d_{2g}}) = O(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}) \end{aligned}$$

and under the conditions in the remark this is

$$O(k^{-1}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\lambda_j^{\frac{1}{2}-d_{2g}+d_{1g}}\lambda_k^{\frac{1}{2}-d_{2h}+d_{1h}}) = O(\frac{1}{k}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}})$$

so that d) is proved.

e) Write

$$E[W_g(\omega+\lambda_j)\bar{W}_h(\omega-\lambda_j)] = \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j-\lambda)D(\lambda-\omega+\lambda_j)\mathrm{d}\lambda$$

and split the integral up,

$$\int_{-\pi}^{\omega-\varepsilon}+\int_{\omega-\varepsilon}^{\omega-2\lambda_j}+\int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_j}{2}}+\int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}}+\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}+\int_{\omega+2\lambda_j}^{\omega+\varepsilon}+\int_{\omega+\varepsilon}^{\pi} \tag{3.44}$$

with ε such that (3.30) holds. As before the integral over $[-\pi,\omega-\varepsilon]\cup[\omega+\varepsilon,\pi]$ is $O(n^{-1})=O(j^{-1}\lambda_j^{-2d_1})$.

The integral over $[\omega-\varepsilon,\omega-2\lambda_j]$ is bounded in absolute value by

$$\begin{aligned}
&\frac{1}{2\pi n}\{\max_{2\lambda_j\le\lambda\le\varepsilon}|f_{gh}(\omega-\lambda)|\}\int_{2\lambda_j}^{\varepsilon}|D(\lambda_j+\lambda)||D(\lambda_j-\lambda)|\mathrm{d}\lambda\\
\le\ &\frac{2}{\pi n}\left\{\max_{\lambda_j\le\lambda\le\infty}\frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\int_{\lambda_j}^{\infty}\frac{\lambda^{\frac{1}{2}-d_{2g}}}{\lambda^2}\mathrm{d}\lambda\\
=\ &O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2h}}\lambda_j^{-\frac{1}{2}-d_{2g}})=O\left(\frac{1}{j}\lambda_j^{-2d_2}\right).
\end{aligned}$$

Similarly

$$\left|\int_{\omega+2\lambda_j}^{\omega+\varepsilon}\right|=O\left(\frac{1}{j}\lambda_j^{-2d_1}\right).$$

Now the integral over $[\omega-2\lambda_j,\omega-\frac{\lambda_j}{2}]$ is bounded in modulus by

$$\begin{aligned}
&\frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j}|f_{gh}(\omega-\lambda)|\}\{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j}|D(\lambda_j+\lambda)|\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(\lambda_j-\lambda)|\mathrm{d}\lambda\\
&=O(n^{-1}\lambda_j^{-2d_2}\lambda_j^{-1}\log j)=O\left(\frac{\log j}{j}\lambda_j^{-2d_2}\right)
\end{aligned}$$

and similarly

$$\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right|=O\left(\frac{\log j}{j}\lambda_j^{-2d_1}\right).$$

Finally the absolute value of the integral over $[\omega\pm\frac{\lambda_j}{2}]$ is bounded by

$$\begin{aligned}
&\frac{1}{2\pi n}\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|D(\lambda_j-\lambda)|\}\{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|D(\lambda_j+\lambda)|\}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}f_{gh}(\omega+\lambda)\mathrm{d}\lambda\\
&=O\left(\frac{1}{j}(\lambda_j^{-2d_2}+\lambda_j^{-2d_1})\right)
\end{aligned}$$

which proves e).

f) To prove f) write

$$E[W_g(\omega+\lambda_j)W_h(\omega-\lambda_j)] = \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega-\lambda_j-\lambda)\mathrm{d}\lambda$$

and split the integral into

$$\int_{-\pi}^{-\omega-\varepsilon}+\int_{-\omega-\varepsilon}^{-\omega-2\lambda_j}+\int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}+\int_{-\omega-\frac{\lambda_j}{2}}^{-\omega+\frac{\lambda_j}{2}}+\int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon}+\int_{-\omega+\varepsilon}^{\omega-\varepsilon}$$
$$+\int_{\omega-\varepsilon}^{\omega-2\lambda_j}+\int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_j}{2}}+\int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}}+\int_{\omega+\frac{\lambda_j}{2}}^{\omega+\varepsilon}+\int_{\omega+\varepsilon}^{\pi}.$$

Since the proof is like that in b) we present it in a more abbreviated form.

$$\left|\int_{-\pi}^{-\omega-\varepsilon}+\int_{-\omega+\varepsilon}^{\omega-\varepsilon}+\int_{\omega+\varepsilon}^{\pi}\right| = O(n^{-1}) = O\left(\frac{1}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right)$$

for an ε such that (3.30) holds. Now

$$\left|\int_{-\omega-\varepsilon}^{-\omega-2\lambda_j}\right| = O\left(\frac{1}{n}\lambda_j^{-\frac{1}{2}-d_{1g}}\right) = O\left(\frac{1}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\lambda_j^{\frac{1}{2}+d_{2h}}\right) = O\left(\frac{1}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right)$$

and the same upper bound is obtained for the modulus of the integral over $[\omega+\frac{\lambda_j}{2},\omega+\varepsilon]$. Now

$$\left|\int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}\right| = O\left(\frac{\log j}{n}\lambda_j^{-2d_1}\right) = O\left(\frac{\log j}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right)$$

and

$$\left|\int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}}\right| = O\left(\frac{1}{n}(\lambda_j^{-2d_1}+\lambda_j^{-2d_2})\right) = O\left(\frac{1}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right)$$

and we obtain the same bound for the absolute value of the integral over $[-\omega\pm\frac{\lambda_j}{2}]$. The rest of integrals are

$$\left|\int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon}\right| = O\left(\frac{1}{n}\lambda_j^{-\frac{1}{2}-d_{2h}}\right) =)\left(\frac{1}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right)$$

and similarly the same bound is obtained for the modulus of the integral over $[\omega-\varepsilon,\omega-2\lambda_j]$. Finally

$$\left|\int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_j}{2}}\right| = O\left(\frac{\log j}{n}\lambda_j^{-2d_2}\right) = O\left(\frac{\log j}{j}\lambda_j^{-d_{1g}}\lambda_j^{-d_{2h}}\right).$$

g) Write

$$E[W_g(\omega+\lambda_j)\bar{W}_h(\omega-\lambda_k)] = \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j-\lambda)D(\lambda-\omega+\lambda_k)\mathrm{d}\lambda.$$

The modulus of the integral over $[-\pi,\omega-\varepsilon]\cup[\omega+\varepsilon,\pi]$ is $O(n^{-1})$. The rest of integrals are

$$\int_{\omega-\varepsilon}^{\omega-2\lambda_j}+\int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_k}{2}}+\int_{\omega-\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k}{2}}+\int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_j}{2}}+\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}+\int_{\omega+2\lambda_j}^{\omega+\varepsilon}.$$

Now

$$\begin{aligned}\left|\int_{\omega-\varepsilon}^{\omega-2\lambda_j}\right| &\le \frac{1}{2\pi n}\left\{\max_{\lambda_j\le\lambda\le 2\varepsilon}\frac{|f_{gh}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_{2g}}}\right\}\int_{\lambda_j}^{2\varepsilon}\lambda^{-\frac{3}{2}-d_{2g}}\mathrm{d}\lambda\\ &= O\left(\frac{1}{n}\lambda_j^{-1-2d_2}\right)=O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}(\frac{k}{j})^{\frac{1}{2}+d_{2h}}\right)\\ &= O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}\right)\end{aligned}$$

and similarly

$$\left|\int_{\omega+2\lambda_j}^{\omega+\varepsilon}\right|=O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

The integral over $[\omega-2\lambda_j,\omega-\frac{\lambda_k}{2}]$ is bounded in modulus by

$$\begin{aligned}&\frac{1}{2\pi n}\{\max_{\frac{\lambda_k}{2}\le\lambda\le2\lambda_j}|f_{gh}(\omega-\lambda)|\}\int_{\frac{\lambda_k}{2}}^{2\lambda_j}|D(\lambda_j+\lambda)||D(\lambda_k-\lambda)|\mathrm{d}\lambda\\ &=O\left(\frac{1}{n}(\lambda_k^{-2d_2}+\lambda_j^{-2d_2})\lambda_j^{-1}\log j\right)\\ &=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}\sqrt{\frac{k}{j}}\left[(\frac{k}{j})^{-d_{2g}}+(\frac{k}{j})^{d_{2h}}\right]\right)\\ &=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}}\right)\end{aligned}$$

and similarly

$$\left|\int_{\omega+\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_j}{2}}\right|=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right).$$

Now the integral over $[\omega+\frac{\lambda_j}{2},\omega+2\lambda_j]$ is bounded in modulus by

$$\begin{aligned}&\frac{1}{2\pi n}\{\max_{\frac{\lambda_j}{2}\le\lambda\le2\lambda_j}|f_{gh}(\omega+\lambda)|\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|D(\lambda_j-\lambda)||D(\lambda_k+\lambda)|\mathrm{d}\lambda\\ &=O\left(\frac{\log j}{n}\lambda_j^{-2d_1}\lambda_j^{-1}\right)=O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)\end{aligned}$$

and finally the absolute value of the integral over $[\omega\pm\frac{\lambda_k}{2}]$ is bounded by

$$\begin{aligned}&\frac{1}{2\pi n}\{\max_{-\frac{\lambda_k}{2}\le\lambda\le\frac{\lambda_k}{2}}|D(\lambda_j-\lambda)||D(\lambda+\lambda_k)|\}\int_{-\frac{\lambda_k}{2}}^{\frac{\lambda_k}{2}}|f_{gh}(\omega+\lambda)|\mathrm{d}\lambda\\ &=O\left(\frac{1}{n}\lambda_j^{-1}\lambda_k^{-1}\lambda_k^{1-2d_i}\right)=O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}\right)\end{aligned}$$

and thus g) is proved.

h) The proof of h) is fairly similar to that of d) so we present it in a more abbreviated manner. Write

$$E[W_g(\omega+\lambda_j)W_h(\omega-\lambda_k)] = \frac{1}{2\pi n}\int_{-\pi}^{\pi} f_{gh}(\lambda)D(\omega+\lambda_j+\lambda)D(\omega-\lambda_k-\lambda)\mathrm{d}\lambda.$$

Now split the integral up

$$\int_{-\pi}^{-\omega-\varepsilon} + \int_{-\omega-\varepsilon}^{-\omega-2\lambda_j} + \int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}} + \int_{-\omega-\frac{\lambda_j}{2}}^{-\omega+\frac{\lambda_j}{2}} + \int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon} + \int_{-\omega+\varepsilon}^{\omega-\varepsilon}$$
$$+ \int_{\omega-\varepsilon}^{\omega-2\lambda_j} + \int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_k}{2}} + \int_{\omega-\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k}{2}} + \int_{\omega+\frac{\lambda_k}{2}}^{\omega+\lambda_j} + \int_{\omega+\lambda_j}^{\omega+\varepsilon} + \int_{\omega+\varepsilon}^{\pi}.$$

Like in d),

$$\left|\int_{-\pi}^{-\omega-\varepsilon} + \int_{-\omega+\varepsilon}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\pi}\right| = O(n^{-1})$$
$$\left|\int_{-\omega-\frac{\lambda_j}{2}}^{-\omega+\frac{\lambda_j}{2}}\right| = O\left(\frac{1}{n}\lambda_j^{-1}\lambda_j^{1-2d_i}\right) = O\left(\frac{1}{\sqrt{jk}}(\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}} + \lambda_j^{-d_{2g}}\lambda_k^{-d_{2h}})\right)$$
$$\text{and } O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{2h}}\right)$$
$$\left|\int_{-\omega-2\lambda_j}^{-\omega-\frac{\lambda_j}{2}}\right| = O\left(\frac{\log j}{n}\lambda_j^{-2d_1}\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{oh}}\right)$$
$$\left|\int_{-\omega-\varepsilon}^{-\omega-2\lambda_j}\right| + \left|\int_{\omega+\lambda_j}^{\omega+\varepsilon}\right| = O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{1g}}) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{oh}}\right)$$
$$\left|\int_{-\omega+\frac{\lambda_j}{2}}^{-\omega+\varepsilon}\right| + \left|\int_{\omega-\varepsilon}^{\omega-2\lambda_j}\right| = O(n^{-1}\lambda_j^{-\frac{1}{2}-d_{2h}}) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{og}}\lambda_k^{-d_{2h}}\right)$$
$$\left|\int_{\omega-2\lambda_j}^{\omega-\frac{\lambda_k}{2}}\right| = O\left(\frac{\log j}{n}(\lambda_j^{-2d_2} + \lambda_k^{-2d_2})\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{og}}\lambda_k^{-d_{2h}}\right)$$

where o can be either 1 or 2. Now

$$\left|\int_{\omega-\frac{\lambda_k}{2}}^{\omega+\frac{\lambda_k}{2}}\right| = O(n^{-1}\lambda_k^{-1}\lambda_k^{1-2d_i}) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{ig}}\lambda_k^{-d_{ih}}\right)$$

where $d_i = \max\{d_1, d_2\}$, and

$$O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{2h}}\lambda_j^{\frac{1}{2}+d_{1g}}\lambda_k^{\frac{1}{2}+d_{2h}-2d_i}\right) = O\left(\frac{1}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{2h}}\right)$$

if $d_{2h} \geq d_{1s}$ for $s = g$ or h.

Finally

$$\left|\int_{\omega+\frac{\lambda_k}{2}}^{\omega+\lambda_j}\right| = O\left(\frac{\log j}{n}(\lambda_j^{-2d_1}+\lambda_k^{-2d_1})\right)$$
$$= O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{1h}}\right)$$

and

$$O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{2h}}\frac{\sqrt{jk}}{n}(\lambda_j^{-d_{1h}}\lambda_k^{d_{2h}}+\lambda_j^{d_{1g}}\lambda_k^{d_{2h}-2d_1})\right) = O\left(\frac{\log j}{\sqrt{jk}}\lambda_j^{-d_{1g}}\lambda_k^{-d_{2h}}\right)$$

if $d_{2h} \geq d_{1s}$ for $s = g$ or h. □

Chapter 4

GAUSSIAN SEMIPARAMETRIC ESTIMATION

4.1 INTRODUCTION

This method of estimation has its origin in the approximation of the Gaussian likelihood function suggested by Whittle (1953) and described in Chapter 1. Whittle (1953) proposed the maximization of the approximate frequency domain likelihood function (1.43), so that absolute knowledge of the spectral density up to a vector of parameters is assumed. This technique was originally proposed for short memory processes with a smooth spectral density function. The application of this methodology to standard long memory processes with a spectral pole only at the origin has been analyzed by Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990) and Heyde and Gay (1993). They showed that the good asymptotic properties of the estimates obtained for short memory hold for standard long memory. In particular the estimates are $\sqrt{n}$-consistent, asymptotically normal and, when x_t is actually Gaussian, asymptotically efficient. However these properties depend strongly on correct specification of $f(\lambda)$ over $(-\pi, \pi]$ and if some kind of misspecification occurs the estimates will in general be inconsistent. In particular, the estimates of long memory parameters will be inconsistent if short memory components are misspecified. To overcome this inconvenience Kunsch (1987) and Robinson (1995b) considered a semiparametric discrete version of (1.43) so that his estimate, $\tilde{d}$, is obtained by minimizing

(1.54). Robinson (1995b) called this estimate *Gaussian semiparametric.* For obvious reasons it has also received the name of *local Whittle* estimation. Here we use both terms indistinguishably. The advantages of this estimate over the one obtained by log-periodogram regression analysed in Chapter 3, $\hat{d}^{(J)}$, are that we neither need to trim out frequency components close to the spectral pole/zero (at least under symmetric SCLM) nor the user-chosen number J, much weaker assumptions than Gaussianity are imposed and we gain asymptotic efficiency in the sense that $\tilde{d}$ has a lower asymptotic variance than $\hat{d}^{(J)}$. The main disadvantage is that $\tilde{d}$, unlike $\hat{d}^{(J)}$, is not defined in a closed form. Nevertheless, despite the non-linearity of the objective function to minimize, its higher asymptotic efficiency and less restrictive assumptions (gaussianity is not needed despite its name) makes the Gaussian semiparametric a very interesting estimate to study. Furthermore it has been found in the univariate case (Robinson (1995b) for $\omega = 0$ and Chapter 6 of this thesis for $\omega \neq 0$) and in the multivariate extension (Lobato (1995)) that with very simple iterative procedures the estimates converge quickly, which makes the Gaussian semiparametric a very attractive method of estimation.

The drawback of this estimate with respect to the parametric one studied by Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990) and Heyde and Gay (1993) is that only $\sqrt{m}$-consistency is achieved because only m frequencies are used such that $\frac{m}{n} \to 0$, and the proportion of the frequency band $(-\pi, \pi]$ involved in the estimation degenerates relatively slowly to 0 as n increases. Therefore the semiparametric estimate is much less efficient than that based on a complete and correct specification of $f(\lambda)$. This loss in efficiency is the price to pay for guaranteeing consistency under misspecification of the spectral density at frequencies far from the one we are interested in.

In this chapter we study the properties of the Gaussian semiparametric estimate analysed by Robinson (1995b) when $f(\lambda)$ satisfies (1.62) around a positive frequency, ω, and we aim to estimate the parameters C and d_1 (the procedure is similar for D and d_2 and it only differs in the use of frequencies just before ω). The case of asymmetric SCLM has a peculiarity with respect to the analysis at zero frequency where $f(\lambda)$ is symmetric for real x_t. If the parameter we want to estimate, d_1, is such that $d_1 \geq d_2$, the asymptotic normality follows directly from the analysis in Robinson (1995b). However when $d_1 < d_2$ we trim out some frequencies close to ω in the same way as in log-periodogram regression in order to get rid of the influence of periodogram ordinates just before ω, where the spectral density is governed by the parameter d_2 (see Theorem 4).

Let $\{x_t, t = 0, \pm 1, \pm 2, ...\}$ be a real-valued covariance stationary process with spectral density function $f(\lambda)$. Assume we only know that $f(\lambda)$ satisfies (1.62) as λ approaches ω and it is integrable over $(-\pi, \pi]$ (necessary for covariance stationarity). The Gaussian semiparametric estimates of d_1 and C are obtained by minimizing

$$Q(C,d) = \frac{1}{m-l} \sum_{j=l+1}^{m} \left\{ \log C\lambda_j^{-2d} + \frac{\lambda_j^{2d}}{C} I_j \right\} \tag{4.1}$$

where $\lambda_j = \frac{2\pi j}{n}$, $I_j = I_n(\omega + \lambda_j) = |W_n(\omega + \lambda_j)|^2$ is the (uncentered) periodogram of x_t at frequency $\omega + \lambda_j$, $l = 0$ if $d_1 \geq d_2$ and $l \to \infty$ more slowly than m as $n \to \infty$ if $d_1 < d_2$.

Concentrating C out of the objective function we have that minimizing (4.1) is equivalent to minimizing

$$R(d) = \log \tilde{C}(d) - 2d \frac{1}{m-l} \sum_{l+1}^{m} \log \lambda_j \tag{4.2}$$

where

$$\tilde{C}(d) = \frac{1}{m-l} \sum_{l+1}^{m} \lambda_j^{2d} I_j. \tag{4.3}$$

Then the procedure consists in obtaining an estimate of d_1, $\tilde{d}_1 = \arg\min_{d \in \Theta} R(d)$, where $\Theta = [\Delta_1, \Delta_2]$ is the set of admissible values for d_1, and then plug $\tilde{d}_1$ in (4.3) to obtain an estimate of C, $\tilde{C}(\tilde{d}_1)$.

4.2 CONSISTENCY

In order to prove the consistency of $\tilde{d}_1$ we need to make the following assumptions:

B.1: For $\alpha \in (0,2]$ and $\omega \in (0,\pi)$, as $\lambda \to 0^+$,

$$
\begin{aligned}
f(\omega+\lambda) &= C\lambda^{-2d_1}(1+O(\lambda^\alpha)) \\
f(\omega-\lambda) &= D\lambda^{-2d_2}(1+O(\lambda^\alpha))
\end{aligned}
$$

where $C, D \in (0,\infty)$, $|d_2| < 1/2$ and $d_1 \in \Theta = [\Delta_1, \Delta_2]$ where $-1/2 < \Delta_1 < \Delta_2 < 1/2$. The choice of Δ_1 and Δ_2 reflects prior knowledge on d_1, for example if we know that $f(\omega+\lambda) \not\to 0$ as $\lambda \to 0^+$ a reasonable choice is $\Delta_1 = 0$.

B.2: In a neighbourhood $(-\delta, 0) \cup (0, \delta)$ of ω, $f(\lambda)$ is differentiable and

$$
\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f(\omega \pm \lambda) = O(\lambda^{-1}) \qquad \text{as } \lambda \to 0^+.
$$

B.3: $x_t - Ex_1 = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$ and $\sum_{j=0}^{\infty} \alpha_j^2 < \infty$ where $E[\varepsilon_t|F_{t-1}] = 0$, $E[\varepsilon_t^2|F_{t-1}] = 1$ for $t = 0, \pm1, \pm2, \ldots$, F_t is the σ-field generated by ε_s, $s \leq t$ and there exists a random variable ε such that $E\varepsilon^2 < \infty$ and for all $\eta > 0$ and some $\kappa < 1$, $P(|\varepsilon_t| > \eta) \leq \kappa P(|\varepsilon| > \eta)$.

B.4: If $d_1 \geq d_2$ then $l = 0$ and

$$
\frac{1}{m} + \frac{m}{n} \to 0 \text{ as } n \to \infty
$$

and if $d_1 < d_2$,

$$
\frac{m}{n} + \frac{l}{m}\log m + \frac{n^{d_2-d_1}}{l^{\frac{1}{2}+(d_2-d_1)}}(\log m)^{\frac{3}{2}} \to 0 \text{ as } n \to \infty.
$$

Assumption **B.1** is just (1.62) with d_1 contained in the interval of admissible estimates $\Theta = [\Delta_1, \Delta_2]$ and a rate of convergence is imposed as in **A.1**, while **B.2** is equivalent to **A.2** in the log-periodogram analysis with $g = h$. Assumption **B.3** says that the innovations in the Wold decomposition of x_t are a square integrable martingale difference sequence that satisfies a milder homogeneity restriction than strict stationarity. Assumption **B.4** differentiates the cases $d_1 \geq d_2$ from $d_1 < d_2$. In the former, m tends to ∞ (necessary for consistency) but more slowly than n (due to our semiparametric specification of $f(\lambda)$). In the latter we introduce the trimming number, l, which has to go to infinity with n at a slower rate than m (for consistency) but its rate of divergence, i.e. the velocity at which it goes to infinity as $n \to \infty$, is higher the larger the difference $d_2 - d_1$ is. This is due to the fact that the higher d_2 with respect to d_1 the more influential the periodogram at frequencies just before ω (where the spectral density is governed by d_2) in the estimation of d_1 (see Theorem 4) such that a larger trimming is needed. The more restrictive case comes up when $d_2 - d_1$ approaches 1. In that case $\frac{n}{l^{\frac{3}{2}}}(\log m)^{\frac{3}{2}} \to 0$ as $n \to \infty$ so that if $m \sim n^\theta$ and $l \sim n^\phi$, then $1 > \theta > \phi > 2/3$ ensures **B.4**.

The following theorem establishes the consistency of the Gaussian semiparametric estimate of d_1 when the spectral density satisfies (1.62). We only focus on the case $d_1 < d_2$. The proof when $d_1 \geq d_2$ is almost equal to that when $\omega = 0$ in Robinson (1995b), with the modification of some minor steps because of the positiveness of ω, steps that can be deduced from the proof of the former case.

Theorem 6 *Let assumptions* **B.1-B.4** *hold. Then as* $n \to \infty$

$$\tilde{d}_1 \xrightarrow{p} d_1.$$

Proof: $\tilde{d}_1 = \arg\min_\Theta R(d)$ where $R(d)$ is defined in (4.2). Write $S(d) = R(d) - R(d_1)$, $N_\delta = \{d : |d - d_1| < \delta\}$ for $0 < \delta < 1/4$ and $\bar{N}_\delta = (-\infty, \infty) - N_\delta$. Then

$$P(|\tilde{d}_1 - d_1| \geq \delta) = P(\tilde{d}_1 \in \bar{N}_\delta \cap \Theta) = P(\inf_{\bar{N}_\delta \cap \Theta} R(d) \leq \inf_{N_\delta \cap \Theta} R(d)) \leq P(\inf_{\bar{N}_\delta \cap \Theta} S(d) \leq 0)$$

because $d_1 \in N_\delta \cap \Theta$. Now define the following subsets of the set of admissible estimates Θ,

$$\Theta_1 = \{d : \Delta \leq d \leq \Delta_2\} \text{ such that } \begin{cases} \Delta = \Delta_1 & \text{if} \quad d_1 < \Delta_1 + \frac{1}{2} \\ d_1 \geq \Delta > d_1 - \frac{1}{2} & \text{if} \quad d_1 \geq \Delta_1 + \frac{1}{2} \end{cases}$$

$$\Theta_2 = \begin{cases} \{d : \Delta_1 \leq d < \Delta\} & \text{if } d_1 \geq \Delta_1 + \frac{1}{2} \\ \emptyset & \text{otherwise} \end{cases} .$$

Thus

$$P(|\tilde{d}_1 - d_1| \geq \delta) \leq P(\inf_{\bar{N}_\delta \cap \Theta_1} S(d) \leq 0) + P(\inf_{\Theta_2} S(d) \leq 0).$$

Write $S(d) = U(d) - T(d)$ where $U(d)$ is the deterministic part of $S(d)$ obtained by replacing I_j by $C\lambda_j^{-2d_1}$ and sums by integrals, and $T(d)$ is the remainder.

$$U(d) = 2(d - d_1) - \log\{2(d - d_1) + 1\}$$

$$T(d) = \log\left\{\frac{\tilde{C}(d_1)}{C}\right\} - \log\left\{\frac{\tilde{C}(d)}{C(d)}\right\} - \log\left\{\frac{1}{m-l}\sum_{j=l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d-d_1)}\{2(d-d_1)+1\}\right\}$$
$$+ \quad 2(d-d_1)\left\{\frac{1}{m-l}\sum_{j=l+1}^{m}\log j - \log(m-l) + 1\right\}$$

where

$$C(d) = C\frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2(d-d_1)}. \tag{4.4}$$

Note that $U(d)$ achieves a unique minimum in Θ_1 for $d = d_1$. Now,

$$P(\inf_{\bar{N}_\delta \cap \Theta_1} S(d) \leq 0) = P(\inf_{\bar{N}_\delta \cap \Theta_1} (U(d) - T(d)) \leq 0) \leq P(\sup_{\Theta_1} |T(d)| \geq \inf_{\bar{N}_\delta \cap \Theta_1} U(d)).$$

Using the mean value theorem we have

$$\log(1+x) \leq x - \frac{1}{8}x^2$$
$$-\log(1-x) \geq x + \frac{1}{2}x^2$$

for $0 < x < 1$. It follows that

$$\inf_{\bar{N}_\delta \cap \Theta_1} U(d) \geq \min(2\delta - \log\{2\delta + 1\}, -2\delta - \log\{1 - 2\delta\}) \geq \frac{\delta^2}{2}. \tag{4.5}$$

On the other hand, from the inequality $|\log(1+x)| \leq 2|x|$ for $|x| \leq \frac{1}{2}$, we deduce that for any nonnegative random variable y, $P(2|y-1| \leq \varepsilon) \leq P(|\log y| \leq \varepsilon)$ for $\varepsilon \leq 1/2$ and

$$P\left\{\left|\log\left(\frac{\tilde{C}(d)}{C(d)}\right)\right| > \varepsilon\right\} \leq P\left\{\left|\frac{\tilde{C}(d) - C(d)}{C(d)}\right| > \frac{\varepsilon}{2}\right\}$$

and thus $\sup_{\Theta_1} |T(d)| \xrightarrow{p} 0$ if

a) $\sup_{\Theta_1} | \frac{\tilde{C}(d) - C(d)}{C(d)} | = o_p(1)$

b) $\sup_{\Theta_1} | \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} (\frac{j}{m-l})^{2(d-d_1)} - 1 | = o(1)$

c) $| \frac{1}{m-l} \sum_{l+1}^{m} \log j - \log(m-l) + 1 | = o(1)$.

If $d_1 \leq d_2$ ($l = 0$) the left hand sides of b) and c) are $O(m^{-1-2(\Delta-d_1)})$ and $O(\frac{\log m}{m})$ from Lemmas 1 and 2 in Robinson (1995b). If $d_1 > d_2$ ($l \to \infty$) the left hand sides of b) and c) are $O((\frac{l}{m})^{1+2(\Delta-d_1)})$ and $O(\frac{l \log m}{m})$ from Lemmas 2 and 3 in Appendix B. Since $1 + 2(\Delta - d_1) > 0$ in Θ_1, condition **B.4** implies that b) and c) hold.

In order to prove a) write,

$$\frac{\tilde{C}(d) - C(d)}{C(d)} = \frac{A(d)}{B(d)}$$

where

$$A(d) = \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{2(d-d_1)} \left(\frac{I_j}{g_j} - 1\right)$$

$$B(d) = \frac{2(d-d_1)+1}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{2(d-d_1)}$$

for $g_j = C\lambda_j^{-2d_1}$. Since $B(d) + |B(d) - 1| \geq 1$ it follows that

$$\inf_{\Theta_1} B(d) \geq 1 - \sup_{\Theta_1} |B(d) - 1| \geq \frac{1}{2} \tag{4.6}$$

for all sufficiently large m using Lemma 2. Now, by summation by parts, $A(d)$ is bounded in absolute value by,

$$\frac{3}{m-l} \left| \sum_{r=l+1}^{m-1} \left\{ \left(\frac{r}{m-l}\right)^{2(d-d_1)} - \left(\frac{r+1}{m-l}\right)^{2(d-d_1)} \right\} \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right| \tag{4.7}$$

$$+ \frac{3}{m-l} \left(\frac{m}{m-l}\right)^{2(d-d_1)} \left| \sum_{j=l+1}^{m} \left(\frac{I_j}{g_j} - 1\right) \right|. \tag{4.8}$$

Using the mean value theorem we have that for $r \geq 1$,

$$\left| \left(1 + \frac{1}{r}\right)^{2(d-d_1)} - 1 \right| \leq \frac{2|(d-d_1)|}{r} \max_{r \geq 1} \left(1 + \frac{1}{r}\right)^{2(d-d_1)-1} \leq \frac{4}{r}$$

in Θ. Thus the supremum in Θ_1 of (4.7) is bounded by

$$\sup_{\Theta_1} \frac{3}{m-l} \left| \sum_{r=l+1}^{m-1} \left(\frac{r}{m-l}\right)^{2(d-d_1)} \left\{ 1 - \left(\frac{r+1}{r}\right)^{2(d-d_1)} \right\} \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right|$$

$$\leq 12 \left(\frac{m}{m-l}\right)^{2(\Delta_2-d_1)+1} \sum_{r=l+1}^{m-1} \left(\frac{r}{m}\right)^{2(\Delta-d_1)+1} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right|.$$

Since $(\frac{m}{m-l})^{\alpha} \to 1$ for all α we focus on the analysis of

$$\sum_{r=l+1}^{m-1} \left(\frac{r}{m}\right)^{2(\Delta-d_1)+1} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1\right) \right|. \tag{4.9}$$

Now,

$$\frac{I_j}{g_j} - 1 = \left(1 - \frac{g_j}{f_j}\right) \frac{I_j}{g_j} + \frac{1}{f_j}[I_j - |\alpha_j|^2 I_{\varepsilon j}] + (2\pi I_{\varepsilon j} - 1) \tag{4.10}$$

where $I_{\varepsilon j} = I_\varepsilon(\omega + \lambda_j) = |W_\varepsilon(\omega + \lambda_j)|^2$, $W_\varepsilon(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \varepsilon_t e^{it\lambda}$, $f_j = f(\omega + \lambda_j)$, $I_j = I_n(\omega + \lambda_j)$ and $\alpha_j = \alpha(\omega + \lambda_j) = \sum_{k=0}^\infty \alpha_k e^{ik(\omega+\lambda_j)}$. Assumption **B.1** implies that

$$\left|1 - \frac{g_j}{f_j}\right| = O(\lambda_j^\alpha). \tag{4.11}$$

Assumptions **B.1** and **B.2** and Theorem 4 imply that for n sufficiently large,

$$E\left|\frac{I_j}{g_j}\right| = 1 + O\left(\left(\frac{j}{n}\right)^\alpha + \frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right). \tag{4.12}$$

Thus

$$\begin{aligned}
& E\left\{\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\left(1-\frac{g_j}{f_j}\right)\frac{I_j}{g_j}\right|\right\} \\
= {} & O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\sum_{l+1}^{r}\left(\frac{j}{n}\right)^\alpha\left(1+\left(\frac{j}{n}\right)^\alpha+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right) \\
= {} & O\left(\left(\frac{m}{n}\right)^\alpha\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left(r+\frac{r^{\alpha+1}}{n^\alpha}+\frac{n^{2(d_2-d_1)}\log r}{l^{2(d_2-d_1)}}\right)\right) \\
= {} & O\left(\left(\frac{m}{n}\right)^\alpha\left(1+\left(\frac{m}{n}\right)^\alpha+\frac{n^{2(d_2-d_1)}\log m}{l^{2(d_2-d_1)}m^{2(\Delta-d_1)+1}}(m^{2(\Delta-d_1)}\log m + l^{2(\Delta-d_1)})\right)\right) \\
= {} & O\left(\left(\frac{m}{n}\right)^\alpha\left(1+\left(\frac{m}{n}\right)^\alpha+\frac{n^{2(d_2-d_1)}(\log m)^2}{ml^{2(d_2-d_1)}}+\frac{n^{2(d_2-d_1)}\log m}{l^{1+2(d_2-d_1)}}\right)\right) \\
= {} & O\left(\left(\frac{m}{n}\right)^\alpha\right) = o(1)
\end{aligned}$$

in Θ_1 under **B.4**.

Since $||a|^2 - |b|^2| = |Re\{(a-b)(\bar{a}+\bar{b})\}| \leq |(a-b)(\bar{a}+\bar{b})| \leq |a-b||a+b|$ and applying the Cauchy-Schwarz inequality we have that $E|I_j - |\alpha_j|^2 I_{\varepsilon j}|$ is bounded by

$$\begin{aligned}
& E[|W_j - \alpha_j W_{\varepsilon j}||W_j + \alpha_j W_{\varepsilon j}|] \\
\leq {} & \{EI_j - \alpha_j EW_{\varepsilon j}\bar{W}_j - \bar{\alpha}_j E\bar{W}_{\varepsilon j}W_j + |\alpha_j|^2 EI_{\varepsilon j}\}^{\frac{1}{2}} \\
\times {} & \{EI_j + \alpha_j EW_{\varepsilon j}\bar{W}_j + \bar{\alpha}_j E\bar{W}_{\varepsilon j}W_j + |\alpha_j|^2 EI_{\varepsilon j}\}^{\frac{1}{2}}.
\end{aligned} \tag{4.13}$$

In view of the proof of Theorem 4, as $n \to \infty$, $l < j \leq m$ and $d_1 < d_2$, we have that

$$EI_j = f_j + O\left(\frac{\log j}{j}\lambda_j^{-2d_2}\right), \quad EW_j\bar{W}_{\varepsilon j} = \frac{\alpha_j}{2\pi} + O\left(\frac{\log j}{j}\lambda_j^{-2d_2}\right)$$

because $I_{\varepsilon j} = \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n}\varepsilon_t\varepsilon_s e^{i(t-s)(\omega+\lambda_j)}$ and under **B.3** $EI_{\varepsilon j} = \frac{1}{2\pi}$. Thus under **B.4** and $d_2 > d_1$, (4.13) is $O((\frac{\log j}{j})^{\frac{1}{2}}\lambda_j^{-(d_2+d_1)})$. Now

$$E\left\{\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}\frac{1}{f_j}(I_j - |\alpha_j|^2 I_{\varepsilon j})\right|\right\}$$

$$= O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\sum_{l+1}^{r}\left(\frac{\log j}{j}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{d_2-d_1}\right). \tag{4.14}$$

We distinguish the cases $d_2 - d_1 > 1/2, = 1/2, < 1/2$. When $d_2 - d_1 < 1/2$, (4.14) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m}r^{2(\Delta-d_1)-(d_2-d_1)-\frac{1}{2}}\right). \tag{4.15}$$

Now if $2(\Delta - d_1) - (d_2 - d_1) - 1/2 \geq -1$, (4.15) is

$$O\left(\frac{n^{d_2-d_1}m^{2(\Delta-d_1)-(d_2-d_1)+\frac{1}{2}}(\log m)^{\frac{3}{2}}}{m^{2(\Delta-d_1)+1}}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{3}{2}}}{m^{\frac{1}{2}+d_2-d_1}}\right) = o(1)$$

under **B.4**, and if $2(\Delta - d_1) - (d_2 - d_1) - 1/2 < -1$, (4.15) is

$$O\left(\frac{n^{d_2-d_1}l^{2(\Delta-d_1)-(d_2-d_1)+\frac{1}{2}}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right) = o(1)$$

in Θ_1 because of **B.4**. Now if $d_2 - d_1 = 1/2$, (4.14) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m}r^{2(\Delta-d_1)-1}(\log r)\right). \tag{4.16}$$

In case $2(\Delta - d_1) \geq 0$, (4.16) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{5}{2}}}{m}\right) = O\left(\frac{\sqrt{n}(\log m)^{\frac{3}{2}}}{l}\frac{l}{m}(\log m)\right) = o(1)$$

under **B.4**, and if $2(\Delta - d_1) < 0$, (4.16) is

$$O\left(\frac{\sqrt{n}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}l^{2(\Delta-d_1)}\right) = O\left(\frac{\sqrt{n}(\log m)^{\frac{1}{2}}}{l}\frac{l^{2(\Delta-d_1)+1}}{m^{2(\Delta-d_1)+1}}\right) = o(1)$$

in Θ_1 and because of **B.4**. Finally when $d_2 - d_1 > 1/2$, (4.14) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}}l^{\frac{1}{2}-(d_2-d_1)}\sum_{l+1}^{m}r^{2(\Delta-d_1)-1}\right), \tag{4.17}$$

if $2(\Delta - d_1) \geq 0$, (4.17) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{3}{2}}}{m^{2(\Delta-d_1)+1}}l^{\frac{1}{2}-(d_2-d_1)}m^{2(\Delta-d_1)}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{3}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right)$$

and if $2(\Delta - d_1) < 0$, (4.17) is

$$O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{m^{2(\Delta-d_1)+1}} l^{2(\Delta-d_1)-(d_2-d_1)+\frac{1}{2}}\right) = O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right)$$

and consequently (4.14) is $o(1)$ in Θ_1 under **B.4**.

The final contribution to (4.9) comes from the term involving $2\pi I_{\varepsilon j} - 1$. Write

$$\begin{aligned} 2\pi I_{\varepsilon j} - 1 &= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}\varepsilon_t\varepsilon_s e^{i(t-s)(\omega+\lambda_j)} - 1 \\ &= \frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2 - 1) + \frac{1}{n}\sum\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\varepsilon_t\varepsilon_s \end{aligned}$$

because $\sin(\lambda) = -\sin(-\lambda)$. Thus

$$\begin{aligned} &\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\left|\sum_{j=l+1}^{r}(2\pi I_{\varepsilon j} - 1)\right| \\ \leq &\left|\frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2 - 1)\right|\sum_{l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{r-l}{r^2} \qquad (4.18) \\ + &\sum_{l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r^2}\frac{1}{n}\sum_{l+1}^{r}\left|\sum\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\varepsilon_t\varepsilon_s\right|. \qquad (4.19) \end{aligned}$$

Under **B.3**, $\frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2) \xrightarrow{p} 1$ from Theorem 1 in Heyde and Seneta (1972) and

$$\sum_{r=l+1}^{m-1}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{r-l}{r^2} = O\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{1}{r}\right) = O(1)$$

and thus (4.18) is $o_p(1)$ in Θ_1. Assumption **B.3** also implies that

$$\begin{aligned} &E[\sum_s\sum_{t\neq s}\varepsilon_t\varepsilon_s\sum_{j=l+1}^{r}\cos\{(t-s)(\omega+\lambda_j)\}]^2 \\ = &\ 2\sum_s\sum_{t\neq s}[\sum_{l+1}^{r}\cos\{(t-s)(\omega+\lambda_j)\}]^2 \\ = &\ 2\sum_{j=l+1}^{r}\sum_{k=l+1}^{r}[\sum_s\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\}] \\ = &\ 2\sum_{j=l+1}^{r}\sum_{k=l+1}^{r}[\sum_{t=1}^{n}\sum_{s=1}^{n}\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\} - n] \\ = &\ (r-l)n^2 - 2(r-l)^2 n \qquad (4.20) \end{aligned}$$

for r such that $0 < \omega + \lambda_r < \pi$. To prove (4.20) write

$$\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\} = a_{ts} + b_{ts} + c_{ts} + d_{ts}$$

where

$$a_{ts} = \cos[s(\omega+\lambda_j)]\cos[s(\omega+\lambda_k)]\cos[t(\omega+\lambda_j)]\cos[t(\omega+\lambda_k)]$$
$$b_{ts} = \cos[s(\omega+\lambda_j)]\sin[s(\omega+\lambda_k)]\cos[t(\omega+\lambda_j)]\sin[t(\omega+\lambda_k)]$$
$$c_{ts} = \sin[s(\omega+\lambda_j)]\cos[s(\omega+\lambda_k)]\sin[t(\omega+\lambda_j)]\cos[t(\omega+\lambda_k)]$$
$$d_{ts} = \sin[s(\omega+\lambda_j)]\sin[s(\omega+\lambda_k)]\sin[t(\omega+\lambda_j)]\sin[t(\omega+\lambda_k)]. \qquad (4.21)$$

Now

$$\sum_{t=1}^{n}\sum_{s=1}^{n} a_{ts} = \sum_{t}\cos[t(\omega+\lambda_j)]\cos[t(\omega+\lambda_k)]\frac{n}{2} \begin{array}{l} = \frac{n^2}{4} \text{ if } k=j \\ = 0 \text{ otherwise} \end{array}$$

$$\sum_{t=1}^{n}\sum_{s=1}^{n} b_{ts} = \sum_{t=1}^{n}\sum_{s=1}^{n} c_{ts} = 0$$

and

$$\sum_{t=1}^{n}\sum_{s=1}^{n} d_{ts} \begin{array}{l} = \frac{n^2}{4} \text{ if } k=j \\ = 0 \text{ otherwise} \end{array}$$

which proves (4.20), and thus (4.19) is

$$O_p\left(\sum_{l+1}^{m}\left(\frac{r}{m}\right)^{2(\Delta-d_1)+1}\frac{(r-l)^{\frac{1}{2}}}{r^2}\right) = O_p\left(\frac{1}{m^{2(\Delta-d_1)+1}}\sum_{l+1}^{m} r^{2(\Delta-d_1)-\frac{1}{2}}\right)$$
$$= O_p\left(\frac{1}{m^{2(\Delta-d_1)+1}}\left(m^{2(\Delta-d_1)+\frac{1}{2}}\log m + l^{2(\Delta-d_1)+\frac{1}{2}}\right)\right)$$
$$= O_p\left(\frac{\log m}{\sqrt{m}} + \frac{1}{\sqrt{l}}\left(\frac{l}{m}\right)^{2(\Delta-d_1)+1}\right)$$
$$= o_p(1)$$

as $n \to \infty$. In the second equality the cases $2(\Delta - d_1) \geq -1/2$ and $2(\Delta - d_1) < -1/2$ are distinguished and the last equality comes from the definition of Θ_1 and assumption **B.4**. Thus we have proved that $\sup_{\Theta_1}$(4.7)$= o_p(1)$.

Now $\sup_{\Theta_1}$(4.8) is bounded by

$$\frac{3}{m-l}\left(\frac{m}{m-l}\right)^{2(\Delta_2-d_1)}\left|\sum_{j=l+1}^{m}\left(\frac{I_j}{g_j}-1\right)\right|. \qquad (4.22)$$

Because $(\frac{m}{m-l})^\alpha \to 1$ for all α as $n \to \infty$ we focus on $\frac{1}{m}|\sum_{l+1}^m(\frac{I_j}{g_j} - 1)|$ and use (4.10) to show in the same manner as above that $\sup_{\Theta_1}(4.8)$ is $o_p(1)$. When $d_1 < d_2$,

$$\begin{aligned}
& E\left\{\frac{1}{m}\sum_{l+1}^{m}\left|1-\frac{g_j}{f_j}\right|\left|\frac{I_j}{g_j}\right|\right\} \\
&= O\left(\frac{1}{m}\sum_{l+1}^{m}\left(\frac{j}{n}\right)^{\alpha}\left(1+\left(\frac{j}{n}\right)^{\alpha}+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right) \\
&= O\left(\frac{m^\alpha}{n^\alpha}+\frac{m^{2\alpha}}{n^{2\alpha}}+\frac{m^\alpha}{n^\alpha}\frac{n^{2(d_2-d_1)}\log m}{l^{1+2(d_2-d_1)}}\right) \\
&= O\left(\frac{m^\alpha}{n^\alpha}\right) = o(1)
\end{aligned}$$

under **B.4**. On the other hand,

$$\begin{aligned}
& E\left\{\frac{1}{m}\sum_{l+1}^{m}\frac{1}{f_j}|I_j-|\alpha_j|^2 I_{\varepsilon j}|\right\} \\
&= O\left(\frac{1}{m}\sum_{l+1}^{m}\frac{n^{d_2-d_1}(\log j)^{\frac{1}{2}}}{j^{\frac{1}{2}+d_2-d_1}}\right) \\
&= O\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right) = o(1)
\end{aligned}$$

under **B.4**, and finally ,

$$\begin{aligned}
& \frac{1}{m}\sum_{l+1}^{m}|2\pi I_{\varepsilon j}-1| \\
&\leq \frac{m-l}{m}\left|\frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2-1)\right| + \left|\frac{1}{m}\sum_{l+1}^{m}\frac{1}{n}\sum_{t}\sum_{s\neq t}\cos\{(t-s)(\omega+\lambda_j)\}\varepsilon_t\varepsilon_s\right| \\
&= o_p(1)+O_p\left(\frac{1}{\sqrt{m}}\right) = o_p(1).
\end{aligned}$$

We have shown that $\sup_{\Theta_1}|A(d)| \xrightarrow{p} 0$ and thus

$$\sup_{\Theta_1}\left|\frac{\tilde{C}(d)}{C(d)}-1\right| = \sup_{\Theta_1}\left|\frac{A(d)}{B(d)}\right| \leq \frac{\sup_{\Theta_1}|A(d)|}{\inf_{\Theta_1}|B(d)|} \xrightarrow{p} 0$$

and the proof is completed in the case $d_1 < \Delta_1 + \frac{1}{2}$. But if $d_1 \geq \Delta_1 + \frac{1}{2}$, Θ_2 is not an empty set and $P(\inf_{\Theta_2} S(d) \leq 0)$ may be different from zero. However we will see that in fact $P(\inf_{\Theta_2} S(d) \leq 0) \to 0$ as $n \to \infty$ when $d_1 < d_2$ (the proof when $d_1 \geq d_2$ and $l = 0$ is the same as that in Robinson (1995b)). Write $p = p_m = \exp(\frac{1}{m-l}\sum_{l+1}^m \log j)$ and $S(d) = \log\{\frac{\hat{D}(d)}{\hat{D}(d_1)}\}$ where $\hat{D}(d) = \frac{1}{m-l}\sum_{l+1}^m(\frac{j}{p})^{2(d-d_1)}j^{2d_1}I_j$. Since $l+1 \leq p \leq m$

then,

$$\inf_{\Theta_2}\left(\frac{j}{p}\right)^{2(d-d_1)} \geq \left(\frac{j}{p}\right)^{2(\Delta-d_1)} \text{ for } l+1 \leq j \leq p$$
$$\inf_{\Theta_2}\left(\frac{j}{p}\right)^{2(d-d_1)} \geq \left(\frac{j}{p}\right)^{2(\Delta_1-d_1)} \text{ for } p < j \leq m.$$

It follows that

$$\inf_{\Theta_2}\hat{D}(d) \geq \frac{1}{m-l}\sum_{l+1}^{m} a_j j^{2d_1} I_j$$

where

$$a_j = \begin{cases} \left(\frac{j}{p}\right)^{2(\Delta-d_1)} & \text{for } l+1 \leq j \leq p \\ \left(\frac{j}{p}\right)^{2(\Delta_1-d_1)} & \text{for } p < j \leq m \end{cases}.$$

Thus

$$P(\inf_{\Theta_2} S(d) \leq 0) \leq P(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)j^{2d_1}I_j \leq 0).$$

Under **B.4**,

$$\begin{aligned} p &\sim \exp\left(\frac{1}{m-l}\{m[\log m-1]-l[\log l-1]\}\right) \\ &= \exp(-1+\log m)\exp\left(\frac{l\log m}{m-l}-\frac{l\log l}{m-l}\right) \\ &= \frac{m}{e}(1+o(1)) \sim \frac{m}{e}, \end{aligned}$$

and

$$\sum_{j=l+1}^{p} a_j \sim p^{2(d_1-\Delta)}\int_l^p x^{2(\Delta-d_1)}dx = \frac{p}{2(\Delta-d_1)+1} - \frac{l^{2(\Delta-d_1)+1}}{(2(\Delta-d_1)+1)p^{2(\Delta-d_1)}}.$$

Thus

$$\begin{aligned} &\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1) \geq \frac{1}{m-l}\sum_{l+1}^{p}a_j - 1 \\ \sim\ & \frac{1}{e(2(\Delta-d_1)+1)}\left(1+\frac{l}{m-l}-\frac{(le)^{2(\Delta-d_1)+1}}{(m-l)m^{2(\Delta-d_1)}}\right)-1 \\ =\ & \frac{1}{e(2(\Delta-d_1)+1)}-1+o(1). \end{aligned}$$

Choosing $\Delta < d_1 - 1/2 + 1/(4e)$ (which can be done without loss of generality because $d_1 - 1/2 \geq \Delta_1$ in Θ_2) we have that for m sufficiently large and $\frac{l}{m} \to 0$ as $m \to \infty$, $\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1) \geq 1$ and

$$P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)j^{2d_1}I_j \leq 0\right)$$

$$
\begin{aligned}
&= P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\frac{\lambda_j^{2d_1}I_j}{C}+1\le 1\right)\\
&\le P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\frac{I_j}{g_j}+1\le \frac{1}{m}\sum_{l+1}^{m}(a_j-1)\right)\\
&= P\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\left(\frac{I_j}{g_j}-1\right)\le -1\right)\\
&\le P\left(\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\left(\frac{I_j}{g_j}-1\right)\right|\ge 1\right).
\end{aligned}
$$

Since $\sum_{l+1}^{m} a_j \sim p^{2(d_1-\Delta)}\int_l^p x^{2(\Delta-d_1)}dx + p^{2(d_1-\Delta_1)}\int_p^m x^{2(\Delta_1-d_1)}dx = O(m)$ it follows that

$$
\begin{aligned}
&\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\left(1-\frac{g_j}{f_j}\right)\frac{I_j}{g_j}\right|\\
&= O_p\left(\frac{1}{m}\sum_{l+1}^{m}(a_j+1)\left(\frac{j}{n}\right)^{\alpha}\left(1+\left(\frac{j}{n}\right)^{\alpha}+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right)\\
&= O_p\left(\left(\frac{m}{n}\right)^{\alpha}+\left(\frac{m}{n}\right)^{2\alpha}+\left(\frac{m}{n}\right)^{\alpha}\frac{n^{2(d_2-d_1)}\log m}{l^{1+2(d_2-d_1)}}\right)\\
&= O_p\left(\left(\frac{m}{n}\right)^{\alpha}\right)=o_p(1)
\end{aligned}
$$

as $n \to \infty$, under **B.4** and because $\alpha > 0$. On the other hand,

$$
\begin{aligned}
&\left|\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\frac{1}{f_j}[I_j-|\alpha_j|^2 I_{\varepsilon j}]\right|\\
&O_p\left(\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\frac{n^{d_2-d_1}(\log j)^{\frac{1}{2}}}{j^{\frac{1}{2}+d_2-d_1}}\right)\\
&O_p\left(\frac{n^{d_2-d_1}(\log m)^{\frac{1}{2}}}{l^{\frac{1}{2}+d_2-d_1}}\right)=o_p(1)
\end{aligned}
$$

as $n \to \infty$ under **B.4**, and finally

$$
\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)(2\pi I_{\varepsilon j}-1)=
$$

$$
\frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2-1)\frac{1}{m-l}\sum_{l+1}^{m}(a_j+1) \tag{4.23}
$$

$$
+\ \frac{1}{n}\sum_{t}\sum_{s\neq t}\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\cos\{(t-s)(\omega+\lambda_j)\}\varepsilon_t\varepsilon_s. \tag{4.24}
$$

Since $\frac{1}{m-l}\sum(a_j-1)=O(1)$, (4.23) is $o_p(1)$. Now (4.24) has variance

$$\begin{aligned}
&\frac{2}{n^2}\sum_t\sum_{s\neq t}\left[\frac{1}{m-l}\sum_{l+1}^{m}(a_j-1)\cos\{(t-s)(\omega+\lambda_j)\}\right]^2\\
=\ &\frac{2}{n^2}\frac{1}{(m-l)^2}\sum_{j=l+1}^{m}(a_j-1)\sum_{k=l+1}^{m}(a_k-1)(\sum_{t=1}^{n}\sum_{s=1}^{n}(a_{ts}+b_{ts}+c_{ts}+d_{ts})-n)
\end{aligned}$$

where a_{ts}, b_{ts}, c_{ts} and d_{ts} are defined in (4.21). Thus the variance of (4.24) is

$$\frac{2}{n^2}\frac{1}{(m-l)^2}\sum_{l+1}^{m}(a_j-1)^2\frac{n^2}{2}-\frac{2}{n(m-l)^2}(\sum_{l+1}^{m}(a_j-1))^2. \tag{4.25}$$

The second term of (4.25) is $O(n^{-1})$ because $\sum_{l+1}^{m}(a_j-1)=O(m)$. Now

$$\begin{aligned}
\sum_{l+1}^{m}a_j^2 &= \sum_{l+1}^{p}\left(\frac{j}{p}\right)^{4(\Delta-d_1)}+\sum_{p+1}^{m}\left(\frac{j}{p}\right)^{4(\Delta_1-d_1)}\\
&= O\left(p\log p+l\left(\frac{l}{p}\right)^{4(\Delta-d_1)}+m\log m\left(\frac{m}{p}\right)^{4(\Delta_1-d_1)}\right)\\
&= O\left(m\log m+l\left(\frac{l}{m}\right)^{4(\Delta-d_1)}\right)
\end{aligned}$$

and thus

$$\frac{1}{(m-l)^2}\sum_{l+1}^{m}a_j^2=O\left(\frac{\log m}{m}+\frac{l^{4(\Delta-d_1)+1}}{m^{4(\Delta-d_1)+2}}\right)=o(1)$$

because $4(\Delta-d_1)+2>0$. Thus (4.24) is $o_p(1)$ and consequently $P(\inf_{\Theta_2}S(d)\leq 0)\rightarrow 0$ so that the proof is completed. □

4.3 ASYMPTOTIC DISTRIBUTION

In this section we show that under some conditions stronger than those needed for consistency but milder than the assumptions imposed in the log-periodogram regression, in the sense that Gaussianity is not needed,

$$\sqrt{m}(\tilde{d}_1-d_1)\xrightarrow{d}N(0,\frac{1}{4}).$$

The constancy of the asymptotic variance of $\tilde{d}_1$ makes easy the use of approximate rules of inference. We also observe a gain in efficiency with respect to log-periodogram regression where the asymptotic variance has an upper bound of $\pi^2/24$ and a lower bound of $1/4$, but this lower bound is only achieved when the pooling number J is ∞, so that the asymptotic variance $1/4$ is not attainable by this class of estimates.

The assumptions needed in this section are the following:

C.1: **B.1** in previous section holds.

C.2: In a neighbourhood $(-\delta, 0) \cup (0, \delta)$ of ω, $\alpha(\lambda) = \sum_{k=0}^{\infty} \alpha_k e^{ik\lambda}$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha(\omega \pm \lambda) = O\left(\frac{|\alpha(\omega \pm \lambda)|}{\lambda}\right) \quad \text{as } \lambda \to 0^+.$$

C.3: Assumption **B.3** holds and

$$E(\varepsilon_t^3 | F_{t-1}) = \mu_3 \quad \text{and} \quad E(\varepsilon_t^4 | F_{t-1}) = \mu_4, \quad t = 0, \pm 1, ...,$$

for finite constants μ_3 and μ_4.

C.4: If $d_1 \geq d_2$,

$$\frac{1}{m} + \frac{m^{1+2\alpha}(\log m)^2}{n^{2\alpha}} \to 0 \quad \text{as } n \to \infty$$

and if $d_1 < d_2$,

$$\frac{(\log m)^3}{l^2} + \frac{l^3}{m}(\log m)^4 + \frac{n^{2(d_2-d_1)}}{l^{1+2(d_2-d_1)}}\log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

Assumption **C.2** implies **B.2** because $f(\lambda) = \frac{\sigma^2}{2\pi}|\alpha(\lambda)|^2$. **C.3** implies that x_t is fourth order stationary and holds if the ε_t are independent and identically distributed with finite fourth moments, and **C.4** is Assumption A.4' in Robinson (1995b) if $d_1 \geq d_2$ but when $d_1 < d_2$ we use a trimming as in the proof of consistency. Taking $m \sim n^{\theta}$ and $l \sim n^{\phi}$ we have that in case $d_1 \geq d_2$, $\theta < 2\alpha/(1+2\alpha)$ suffices, but when $d_2 > d_1$, **C.4** can only be satisfied if $d_2 - d_1 < \alpha/(3+4\alpha)$. For instance when $\alpha = 2$, $d_2 - d_1$ has to be smaller than $2/11$. However we can relax **C.4** by strengthening **C.3**. We thus consider:

C.5: The fourth cumulant of ε_t is zero for all t.

C.6: If $d_1 \geq d_2$ **C.4** holds and when $d_1 < d_2$

$$\frac{(\log m)^3}{l^2} + \frac{l^2}{m}(\log m)^2 + \frac{n^{2(d_2-d_1)}}{l^{1+2(d_2-d_1)}}\log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

Assumption **C.5** is implied by Gaussianity and **C.6** entails $(d_2 - d_1) < \alpha/2(1+\alpha)$, where the upper bound is 1/3 when $\alpha = 2$. This requirement is not much stronger than $d_2 - d_1 < 1/2$ which is satisfied if there is both a left and right (stationary) spectral pole at ω.

Theorem 7 *Under* **C.1-C.3** *and either* **C.4** *or* **C.5** *and* **C.6**,

$$\sqrt{m}(\tilde{d}_1 - d_1) \xrightarrow{d} N(0, \frac{1}{4}) \quad as \quad n \to \infty.$$

Proof: Like in the proof of consistency we focus on the case $d_2 > d_1$, the proof with $d_2 \leq d_1$ is a straightforward extension of that in Robinson(1995b). Since $\tilde{d}_1$ is consistent under the conditions in Theorem 7, then with probability approaching 1 as $n \to \infty$, $\tilde{d}_1$ satisfies

$$0 = \frac{\mathrm{d}R(\tilde{d}_1)}{\mathrm{d}d} = \frac{\mathrm{d}R(d_1)}{\mathrm{d}d} + \frac{\mathrm{d}^2 R(\bar{d})}{\mathrm{d}d^2}(\tilde{d}_1 - d_1) \tag{4.26}$$

where $|\bar{d} - d_1| \leq |\tilde{d}_1 - d_1|$. Write

$$\tilde{C}_k(d) = \frac{1}{m-l} \sum_{j=l+1}^{m} \lambda_j^{2d} (\log \lambda_j)^k I_j.$$

Then

$$\begin{aligned} \frac{\mathrm{d}R(d)}{\mathrm{d}d} &= 2\frac{\tilde{C}_1(d)}{\tilde{C}(d)} - \frac{2}{m-l}\sum_{l+1}^{m} \log \lambda_j \\ \frac{\mathrm{d}^2 R(d)}{\mathrm{d}d^2} &= \frac{4\{\tilde{C}_2(d)\tilde{C}(d) - \tilde{C}_1^2(d)\}}{\tilde{C}^2(d)}. \end{aligned}$$

Define also

$$\tilde{F}_k = \frac{1}{m-l} \sum_{j=l+1}^{m} (\log j)^k \lambda_j^{2d} I_j \quad , \quad \tilde{E}_k(d) = \frac{1}{m-l} \sum_{j=l+1}^{m} (\log j)^k j^{2d} I_j,$$

thus

$$\frac{\mathrm{d}^2 R(d)}{\mathrm{d}d^2} = \frac{4\{\tilde{F}_2(d)\tilde{F}_0(d) - \tilde{F}_1^2(d)\}}{\tilde{F}_0^2(d)} = \frac{4\{\tilde{E}_2(d)\tilde{E}_0(d) - \tilde{E}_1^2(d)\}}{\tilde{E}_0^2(d)}.$$

Fix $\zeta > 0$ and choose n such that $2\zeta < (\log m)^2$. On the set $M = \{d : (\log m)^3 |d - d_1| \leq \zeta\}$

$$\begin{aligned} |\tilde{E}_k(d) - \tilde{E}_k(d_1)| &\leq \frac{1}{m-l}\sum_{l+1}^{m} |j^{2(d-d_1)} - 1| j^{2d_1} (\log j)^k I_j \\ &\leq 2e|d - d_1|\tilde{E}_{k+1}(d_1) \\ &\leq 2e\zeta(\log m)^{k-2}\tilde{E}_0(d_1) \end{aligned}$$

where the second inequality comes from the fact that

$$\frac{|j^{2(d-d_1)}-1|}{2|d-d_1|} \leq (\log j) m^{2|d-d_1|} \leq m^{\frac{1}{\log m}} \log j = e \log j$$

on M. Thus for $\eta > 0$,

$$\begin{aligned} & P\left(|\tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1)| > \eta(\frac{2\pi}{n})^{-2d_1}\right) \\ \leq\ & P\left(2e\zeta(\log m)^{k-2}\tilde{E}_0(d_1) > \eta(\frac{2\pi}{n})^{-2d_1} | \bar{d} \in M\right) \\ +\ & P\left(|\tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1)| > \eta(\frac{2\pi}{n})^{-2d_1} | \bar{d} \notin M\right) \\ \leq\ & P(\tilde{C}(d_1) > \frac{\eta}{2e\zeta}(\log m)^{2-k}) + P((\log m)^3 |\bar{d} - d_1| > \zeta). \end{aligned} \tag{4.27}$$

Since from the proof of Theorem 6, $\tilde{C}(d_1) \xrightarrow{p} C \in (0, \infty)$, the first probability in (4.27) tends to zero for ζ sufficiently small and $k = 0, 1, 2$. The second probability is bounded by

$$\begin{aligned} & P((\log m)^3 |\tilde{d}_1 - d_1| > \zeta) \\ & \leq P(\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} S(d) \leq 0) + P(\inf_{\Theta_1 \cap \bar{N}_\delta} S(d) \leq 0) + P(\inf_{\Theta_2} S(d) \leq 0) \end{aligned} \tag{4.28}$$

where $\bar{M} = (-\infty, \infty) - M$. We have already shown in the proof of Theorem 6 that the last two probabilities in (4.28) tend to zero. The first probability is bounded by

$$P(\sup_{\Theta_1 \cap N_\delta} |T(d)| \geq \inf_{\Theta_1 \cap N_\delta \cap \bar{M}} U(d)). \tag{4.29}$$

As in the proof of Theorem 6,

$$\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} U(d) \geq \frac{\zeta^2}{(\log m)^6}.$$

Call $\gamma = 2(\Delta - d_1) + 1$. On Θ_1, $\gamma > 0$. Consider $(\frac{l}{m})^\gamma (\log m)^6 = \frac{l^\gamma}{m^{\gamma - a}} \frac{(\log m)^6}{m^a}$ where $0 < a < \gamma$. Now $\frac{(\log m)^6}{m^a} \to 0$ as $m \to \infty$ and

$$\left(\frac{l^\gamma}{m^{\gamma-a}}\right)^{\frac{2}{\gamma}} = \frac{l^2}{m} \frac{1}{m^{1-\frac{2a}{\gamma}}}.$$

Under **C.6** (and of course **C.4**), $\frac{l^2}{m} \to 0$ as $n \to \infty$. Chose $a < \frac{\gamma}{2}$, which can always be done because $\gamma > 0$. Then $(\frac{l}{m})^\gamma (\log m)^6 \to 0$ as $n \to \infty$. Thus noting the form of

$T(d)$ and the orders of magnitude obtained in Lemmas 2 and 3 it follows that under **C.4** or **C.6** (4.29) tends to 0 if

$$\sup_{\Theta_1 \cap N_\delta} \left| \frac{\tilde{C}(d) - C(d)}{C(d)} \right| = o_p((\log m)^{-6}).$$

Using the notation in the proof of Theorem 6, and because of (4.6),

$$\inf_{\Theta_1 \cap N_\delta} B(d) \geq \inf_{\Theta_1} B(d) \geq \frac{1}{2}$$

for all large enough m. Thus it remains to prove that $\sup_{\Theta_1 \cap N_\delta} |A(d)| = o_p((\log m)^{-6})$. Now

$$\begin{aligned} & \sup_{\Theta_1 \cap N_\delta} |A(d)| \\ \leq & \sup_{\Theta_1 \cap N_\delta} \left\{ 12 \left(\frac{m}{m-l} \right)^{2(d-d_1)+1} \sum_{r=l+1}^{m-1} \left(\frac{r}{m} \right)^{2(d-d_1)+1} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1 \right) \right| \right. \\ + & \left. \frac{3}{m-l} \left(\frac{m}{m-l} \right)^{2(d-d_1)} \left| \sum_{j=l+1}^{m} \left(\frac{I_j}{g_j} - 1 \right) \right| \right\}. \end{aligned}$$

Since $(\frac{m}{m-l})^\alpha \to 1$ for all α we focus on

$$\begin{aligned} & \sup_{\Theta_1 \cap N_\delta} \left\{ \sum_{l+1}^{m-1} \left(\frac{r}{m} \right)^{2(d-d_1)+1} \frac{1}{r^2} \left| \sum_{l+1}^{r} \left(\frac{I_j}{g_j} - 1 \right) \right| + \frac{1}{m} \left| \sum_{l+1}^{m} \left(\frac{I_g}{g_j} - 1 \right) \right| \right\} \\ \leq & \sum_{r=l+1}^{m} \left(\frac{r}{m} \right)^{1-2\delta} \frac{1}{r^2} \left| \sum_{j=l+1}^{r} \left(\frac{I_j}{g_j} - 1 \right) \right| + \frac{1}{m} \left| \sum_{l+1}^{m} \left(\frac{I_j}{g_j} - 1 \right) \right|. \end{aligned} \tag{4.30}$$

Now, using Lemmas 4 and 6, the first part of (4.30) is

$$\begin{aligned} & \sum_{l+1}^{m} \left(\frac{r}{m} \right)^{1-2\delta} \frac{1}{r^2} \left| \sum_{l+1}^{r} \left(\frac{I_j}{g_j} - 2\pi I_{\varepsilon j} + 2\pi I_{\varepsilon j} - 1 \right) \right| \\ = & O_p \left(\sum_{l+1}^{m} \left(\frac{r}{m} \right)^{1-2\delta} \frac{1}{r^2} \left(\frac{r^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}} r^{\frac{1}{4}} + r^{\frac{1}{2}} \right) \right) \\ = & O_p \left(\left(\frac{m}{n} \right)^\alpha + \left(\frac{l}{m} \right)^{\frac{3}{4}} \log m + \left(\frac{l}{m} \right)^{1-2\delta} + \frac{\log m}{\sqrt{m}} \right) \end{aligned} \tag{4.31}$$

under **C.4** or **C.6**. Since $\delta < \frac{1}{4}$ then (4.31) is $o_p((\log m)^{-6})$. Similarly the second part of (4.30) is

$$O_p \left(\frac{1}{m} (\sqrt{m} + \frac{m^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}} m^{\frac{1}{4}}) \right) = o_p((\log m)^{-6})$$

under **C.4** or **C.6**. Thus $P(\inf_{\Theta_1 \cap N_\delta \cap \bar{M}} S(d) \le 0) \to 0$ and

$$P\left(|\tilde{E}_k(\bar{d}) - \tilde{E}_k(d_1)| > \eta(\frac{2\pi}{n})^{-2d_1}\right) \to 0$$

as $n \to \infty$. Consequently

$$\begin{aligned}
&\frac{\mathrm{d}^2 R(\bar{d})}{\mathrm{d}d^2} \\
&= \frac{4[\{\tilde{E}_2(d_1) + o_p(n^{2d_1})\}\{\tilde{E}_0(d_1) + o_p(n^{2d_1})\} - \{\tilde{E}_1(d_1) + o_p(n^{2d_1})\}^2]}{\{\tilde{E}_0(d_1) + o_p(n^{2d_1})\}^2} \\
&= \frac{4[\tilde{F}_2(d_1)\tilde{F}_0(d_1) - \tilde{F}_1^2(d_1)]}{\tilde{F}_0^2(d_1)} + o_p(1) \qquad \text{as} \quad n \to \infty. \qquad (4.32)
\end{aligned}$$

Now for $k = 0, 1, 2$,

$$\begin{aligned}
&\left|\tilde{F}_k(d_1) - C\frac{1}{m-l}\sum_{l+1}^{m}(\log j)^k\right| \\
&= \left|\frac{C}{m-l}\sum_{l+1}^{m}(\log j)^k\left(\frac{I_j}{g_j} - 1\right)\right| \le (\log m)^k \frac{C}{m-l}\sum_{l+1}^{m}\left|\frac{I_j}{g_j} - 1\right| \\
&= O_p\left(\frac{(\log m)^2}{\sqrt{m}} + \left(\frac{m}{n}\right)^\alpha (\log m)^2 + \left(\frac{l}{m}\right)^{\frac{3}{4}} (\log m)^2\right) \\
&= o_p(1) \qquad (4.33)
\end{aligned}$$

under **C.4** or **C.6**. Thus from (4.32) and (4.33)

$$\begin{aligned}
&\frac{\mathrm{d}^2 R(\bar{d})}{\mathrm{d}d^2} \\
&= \frac{4[\{\frac{C}{m-l}\sum_{l+1}^{m}(\log j)^2 + o_p(1)\}\{C + o_p(1)\} - \{\frac{C}{(m-l)}\sum_{l+1}^{m}\log j + o_p(1)\}^2]}{\{C + o_p(1)\}^2} + o_p(1) \\
&= 4\left\{\frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2 - \left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2\right\}(1 + o_p(1)) + o_p(1) \xrightarrow{p} 4
\end{aligned}$$

as $n \to \infty$. Now since $\tilde{C}(d_1) \xrightarrow{p} C$

$$\begin{aligned}
\sqrt{m}\frac{\mathrm{d}R(d_1)}{\mathrm{d}d} &= 2\frac{\sqrt{m}}{m-l}\sum_{l+1}^{m}\left(\frac{\lambda_j^{2d_1} I_j \log\lambda_j}{\tilde{C}(d_1)} - \log\lambda_j\right) \\
&= 2\frac{\sqrt{m}}{m-l}\frac{1}{C + o_p(1)}\sum_{l+1}^{m}(\log\lambda_j - \frac{1}{m-l}\sum_{l+1}^{m}\log\lambda_j)\lambda_j^{2d_1} I_j \\
&= \frac{2}{\sqrt{m}}(1 + o(1))\left(1 - \frac{o_p(1)}{C + o_p(1)}\right)\sum_{l+1}^{m} v_j \frac{I_j}{g_j} \\
&= \frac{2}{\sqrt{m}}\sum_{l+1}^{m} v_j\left(\frac{I_j}{g_j} - 2\pi I_{\varepsilon j}\right)(1 + o_p(1)) \qquad (4.34) \\
&+ \frac{2}{\sqrt{m}}\sum_{l+1}^{m} v_j 2\pi I_{\varepsilon_j}(1 + o_p(1)) \qquad (4.35)
\end{aligned}$$

where $v_j = \log j - \frac{1}{m-l}\sum_{l+1}^m \log j$ satisfies $\sum_{l+1}^m v_j = 0$. Since $|v_j| = O(\log m)$, using Lemma 6 we have that (4.34) is

$$O_p\left(\frac{m^{\alpha+\frac{1}{2}}}{n^\alpha}\log m + \frac{l^{\frac{3}{4}}}{m^{\frac{1}{4}}}\log m\right)$$

under **C.4** and

$$O_p\left(\frac{m^{\alpha+\frac{1}{2}}}{n^\alpha}\log m + \frac{l}{\sqrt{m}}\log m\right)$$

under **C.5** and **C.6**. In both cases (4.34) is $o_p(1)$. Apart from the $o_p(1)$ terms, (4.35) is

$$\begin{aligned}
&\frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^m v_j \sum_{t=1}^n\sum_{s=1}^n \varepsilon_t\varepsilon_s e^{i(t-s)(\omega+\lambda_j)} \\
&= \frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^m v_j 2\sum_{t=2}^n\sum_{s=1}^{t-1} \varepsilon_t\varepsilon_s \cos\{(t-s)(\omega+\lambda_j)\} \\
&= 2\sum_{t=1}^n z_t
\end{aligned}$$

where $z_1 = 0$ and

$$\begin{aligned}
z_t &= \varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s} \quad \text{for} \quad t = 2,3,\ldots,n, \\
c_s &= \frac{2}{n}\frac{1}{\sqrt{m}}\sum_{l+1}^m v_j\cos\{s(\omega+\lambda_j)\}. \qquad (4.36)
\end{aligned}$$

The z_t form a zero-mean martingale difference array and from a standard martingale CLT (Hall and Heyde (1980), section 3.2) $\sum_{t=1}^n z_t$ converges in distribution to a $N(0,1)$ random variable if

$$\begin{aligned}
&\textbf{a)}\ \sum_1^n E(z_t^2|F_{t-1}) - 1 \xrightarrow{p} 0 \\
&\textbf{b)}\ \sum_1^n E(z_t^2 I(|z_t| > \delta)) \to 0 \quad \text{for all } \delta > 0.
\end{aligned}$$

To prove **a)** write

$$\begin{aligned}
&\sum_1^n E(z_t^2|F_{t-1}) - 1 \\
&= \sum_2^n E[\varepsilon_t^2(\sum_{s=1}^{t-1}\varepsilon_s c_{t-s})^2|F_{t-1}] - 1 = \sum_2^n(\sum_{s=1}^{t-1}\varepsilon_s c_{t-s})^2 - 1 \\
&= \{\sum_{t=2}^n\sum_{s=1}^{t-1}\varepsilon_s^2 c_{t-s}^2 - 1\} + \sum_{t=2}^n\sum\sum_{r\neq s}^{t-1}\varepsilon_r\varepsilon_s c_{t-r}c_{t-s}.
\end{aligned}$$

The term in braces is

$$\{\sum_{t=1}^{n-1}(\varepsilon_t^2-1)\sum_{s=1}^{n-t}c_s^2\}+\{\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s^2-1\}. \tag{4.37}$$

Now $\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s^2$ is equal to

$$\begin{aligned}
&\frac{4}{n^2}\frac{1}{m}\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}(\sum_{l+1}^{m}v_j\cos\{s(\omega+\lambda_j)\})^2\\
=\;&\frac{4}{n^2}\frac{1}{m}\sum_{j=l+1}^{m}\sum_{k=l+1}^{m}v_jv_k\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos\{s(\omega+\lambda_j)\}\cos\{s(\omega+\lambda_k)\}\\
=\;&\frac{4}{mn^2}\sum_{j=l+1}^{m}v_j^2\sum_{1}^{n-1}\sum_{1}^{n-t}\cos^2\{s(\omega+\lambda_j)\} &(4.38)\\
+\;&\frac{2}{mn^2}\sum_{l+1}^{m}\sum_{j\neq k}v_jv_k\sum_{1}^{n-1}\sum_{1}^{n-t}[\cos\{s(2\omega+\lambda_j+\lambda_k)\}+\cos\{s(\lambda_j-\lambda_k)\}]. &(4.39)
\end{aligned}$$

From formula (4.18) in Robinson (1995b), namely,

$$\sum_{r=1}^{q-1}\sum_{t=1}^{q-r}\cos(\theta t)=\frac{\cos\theta-\cos(q\theta)}{4\sin^2\frac{\theta}{2}}-\frac{q-1}{2} \tag{4.40}$$

for $\theta\neq 0,\mathrm{mod}(2\pi)$, we have that for j such that $0<\omega+\lambda_j<\pi$ (which holds for n large enough),

$$\begin{aligned}
&\sum_{t=1}^{n}\sum_{s=1}^{n-t}\cos^2\{s(\omega+\lambda_j)\}=\frac{1}{2}\sum_{1}^{n-1}\sum_{1}^{n-t}(1+\cos\{2s(\omega+\lambda_j)\})\\
=\;&\frac{1}{2}\sum_{t=1}^{n-1}(n-t)+\frac{1}{2}\frac{\cos\{2(\omega+\lambda_j)\}-\cos\{2n(\omega+\lambda_j)\}}{4\sin^2(\omega+\lambda_j)}-\frac{n-1}{4}\\
=\;&\frac{(n-1)^2}{4}+O(1).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{m-l}\sum_{l+1}^{m}v_j^2 &= \frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2-\left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2\\
&= 1+O\left(\frac{(\log m)^2}{m}\right)
\end{aligned}$$

we have that (4.38) is

$$\begin{aligned}
&4\frac{m-l}{mn^2}\left(1+O(\frac{(\log m)^2}{m})\right)\left(\frac{(n-1)^2}{4}+O(1)\right)\\
=\;&\frac{4}{n^2}\left(\frac{(n-1)^2}{4}+O(\frac{n^2}{m}(\log m)^2)\right)\to 1 \quad \text{as } n\to\infty.
\end{aligned}$$

Now for j,k such that $0 < 2\omega + \lambda_j + \lambda_k < 2\pi$ (which always holds for a large enough n) and $j \neq k$ we can again apply formula (4.40) and we get that

$$\begin{aligned}
& \sum_{t=1}^{n-1}\sum_{s=1}^{n-t}[\cos\{s(2\omega+\lambda_j+\lambda_k)\} + \cos\{s(\lambda_j-\lambda_k)\}] \\
= \ & -n + O(1)
\end{aligned}$$

so that (4.39) is

$$O\left(\frac{1}{n}\sum_{l+1}^{m} v_j^2\right) = O\left(\frac{m}{n}\right) = o(1).$$

Thus the second term in (4.37) tends to zero as $n \to \infty$. The first term has mean zero and its variance is

$$O(\sum_{t=1}^{n-1}(\sum_{s=1}^{n-t} c_s^2)^2).$$

Now

$$|c_s| \leq \frac{2}{\sqrt{m}}\frac{1}{n}\sum_{l+1}^{m}|v_j| = O\left(\frac{\sqrt{m}\log m}{n}\right) \tag{4.41}$$

and for $1 \leq s \leq n/2$, by summation by parts, $|c_s|$ is

$$\begin{aligned}
& \left|\frac{2}{n\sqrt{m}}\sum_{l+1}^{m} v_j \cos\{s(\omega+\lambda_j)\}\right| \\
= & \left|\frac{2}{n\sqrt{m}}\sum_{r=l+1}^{m-1}(v_r - v_{r+1})\sum_{j=l+1}^{r}\cos\{s(\omega+\lambda_j)\} + \frac{2}{n\sqrt{m}} v_m \sum_{l+1}^{m}\cos\{s(\omega+\lambda_j)\}\right| \\
= & \left|\frac{2}{n\sqrt{m}}\sum_{r=l+1}^{m-1}(\log r - \log(r+1))\sum_{j=l+1}^{r}\cos\{s(\omega+\lambda_j)\} + \frac{2}{n\sqrt{m}} v_m \sum_{l+1}^{m}\cos\{s(\omega+\lambda_j)\}\right| \\
= & \ O\left(\frac{1}{n\sqrt{m}}\sum_{r=l+1}^{m-1}\log(1+\frac{1}{r})\frac{n}{s} + \frac{1}{n\sqrt{m}}\log m \frac{n}{s}\right) \\
= & \ O\left(\frac{\log m}{s\sqrt{m}}\right)
\end{aligned} \tag{4.42}$$

because $\sum_{l+1}^{m}\cos\{s(\omega+\lambda_j)\} = O(ns^{-1})$ for $1 \leq s \leq n/2$ (see proof of Lemma 4) and $|\log(1+\frac{1}{r})| \leq 1/r$ for $r \geq 1$. The bound in (4.42) is at least as good as that in (4.41) for $n/m \leq s \leq n/2$. Consider ω a harmonic frequency (which can always be done for n sufficiently large), then $c_s = c_{n-s}$ and from (4.41) and (4.42)

$$\begin{aligned}
\sum_{s=1}^{n} c_s^2 &= O\left(\frac{n}{m}\frac{m(\log m)^2}{n^2} + \frac{(\log m)^2}{m}\sum_{s>\frac{n}{m}} s^{-2}\right) \\
&= O\left(\frac{(\log m)^2}{n}\right)
\end{aligned} \tag{4.43}$$

and the variance of the first part of (4.37) is $O(\frac{(\log m)^4}{n})$. Thus (4.37) is $o_p(1)$. In order to prove **a)** it remains to show that

$$\sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1}\varepsilon_r\varepsilon_s c_{t-r}c_{t-s} = o_p(1). \tag{4.44}$$

The left hand side of (4.44) has mean zero and variance

$$\begin{aligned}
&\sum_{t=2}^{n}\sum_{u=2}^{n}\sum\sum_{r\neq s}^{t-1}\sum\sum_{p\neq q}^{u-1} E[\varepsilon_r\varepsilon_s\varepsilon_p\varepsilon_q]c_{t-r}c_{t-s}c_{u-p}c_{u-q}\\
=\;& 2\sum_{t=2}^{n}\sum_{u=2}^{n}\sum\sum_{t\neq s}^{\min(t-1,u-1)} c_{t-r}c_{t-s}c_{u-r}c_{u-s}\\
=\;& 2\sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1} c_{t-r}^2 c_{t-s}^2 + 4\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum\sum_{r\neq s}^{u-1} c_{t-r}c_{t-s}c_{u-r}c_{u-s}.
\end{aligned} \tag{4.45}$$

The first part of (4.45) is $O((\log m)^4 n^{-1})$ from (4.43). The second part is bounded in absolute value by

$$4\sum_{t=3}^{n}\sum_{u=2}^{t-1}(\sum_{r=1}^{u-1} c_{t-r}^2 \sum_{s=1}^{u-1} c_{u-s}^2) \leq 4(\sum_{1}^{n} c_t^2)(\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{t-u+1}^{t-1} c_r^2). \tag{4.46}$$

Now

$$\begin{aligned}
&\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{t-u+1}^{t-1} c_r^2 = \sum_{j=1}^{n-2} j(n-j-1)c_{j+1}^2 \leq n\sum_{1}^{n} jc_{j+1}^2\\
&\leq n\sum_{2}^{[nm^{-\frac{2}{3}}]} jc_j^2 + n\sum_{[nm^{-\frac{2}{3}}]+1}^{n} jc_j^2\\
&= O\left(n\frac{n}{m^{\frac{2}{3}}}\frac{n}{m^{\frac{2}{3}}}\frac{m(\log m)^2}{n^2} + n^2\frac{(\log m)^2}{m}\frac{m^{\frac{2}{3}}}{n}\right)\\
&= O\left(\frac{n(\log m)^2}{m^{\frac{1}{3}}}\right)
\end{aligned}$$

using (4.41) and (4.42). Thus noting (4.43) we see that (4.46) is

$$O\left(\frac{n(\log m)^2}{m^{\frac{1}{3}}}\frac{(\log m)^2}{n}\right) = O\left(\frac{(\log m)^4}{m^{\frac{1}{3}}}\right) = o(1)$$

and (4.44), and thus **a)**, are proved.

In order to prove **b)** we check the sufficient condition

$$\sum_{t=1}^{n} E[z_t^4] \to 0 \quad \text{as} \quad n\to\infty.$$

Now

$$\sum_1^n E[z_t^4] = \sum_2^n E[\varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s c_{t-s}]^4 = \mu_4 \sum_2^n E[\sum_1^{t-1} \varepsilon_s c_{t-s}]^4$$

$$\leq \mu_4 \sum_2^n E[\sum_s \sum_r \sum_p \sum_{q=1}^{t-1} \varepsilon_s \varepsilon_r \varepsilon_p \varepsilon_q c_{t-s} c_{t-r} c_{t-p} c_{t-q}]$$

$$\leq \mu_4^2 \sum_2^n (\sum_{s=1}^n c_s^4) + 3\mu_4 \mu_2^2 \sum_{t=2}^n \sum_s \sum_{r=1}^{t-1} c_{t-s}^2 c_{t-r}^2$$

$$= O(n(\sum_1^n c_t^2)^2) = O\left(\frac{(\log m)^4}{n}\right) = o(1)$$

in view of (4.43) and this concludes the proof of the theorem. □

4.4 MULTIVARIATE EXTENSION

The multivariate case is important in order to analyse the interrelations between different variables. Lobato (1995) proposed the joint estimation of the memory parameters at the origin of a vector series x_t of r elements x_t^a, $a = 1, ..., r$, by minimizing the local discrete approximation to the Whittle likelihood function (1.55) in Chapter 1. Considering the correlation structure among the elements of x_t Lobato obtained an asymptotic efficiency improvement with respect to the individual estimation of the persistence parameters in each x_t^a. The methodology and properties of this semiparametric multivariate Whittle estimation are mentioned in Chapter 1.

The extension from $\omega = 0$ to $\omega \neq 0$ introduces two possible modifications with respect to the case at the origin. First, the frequency, ω_a, where the pole zero occurs can be different for every x_t^a, $a = 1, ..., r$. Secondly, as in the univariate case, we can allow for the possibility of asymmetric poles/zeros for $\omega_a \neq 0 (mod\, \pi)$ which forces us to trim out some corresponding frequencies. Thus, in order to estimate the persistence parameters just after the spectral pole, the objective function to minimize is

$$Q(C,d) = \sum_{j=l+1}^{m} \{\log |\Lambda_j C \Lambda_j| + \text{tr}[\Lambda_j^{-1} C^{-1} \Lambda_j^{-1} I_j]\} \tag{4.47}$$

where $\Lambda_j = diag\{\lambda_j^{-d_a}\}$ for $a = 1, ..., r$, $d = (d_1, ..., d_r)$ is any admissible value of the vector of memory parameters, C is a $r \times r$ matrix and I_j is a $r \times r$ matrix with diagonal elements $I_{aa}(\omega_a + \lambda_j) = |W_a(\omega_a + \lambda_j)|^2 = \frac{1}{2\pi n} |\sum_{t=1}^n x_t^a e^{-it(\omega_a + \lambda)}|^2$, the periodogram of x_t^a, $a = 1, ..., r$, at frequency $\omega_a + \lambda_j$ and the off-diagonal elements are 0 if $\omega_a \neq \omega_b$ and the cross periodogram, $I_{ab}(\omega + \lambda_j) = W_a(\omega + \lambda_j)W_b^*(\omega + \lambda_j)$, if $\omega_a = \omega_b = \omega$, where * indicates conjugation and transposition. The trimming number will be zero only if $d_{a2} \leq d_{a1}$ for all $a = 1, ..., r$, and will go to ∞ more slowly than m if $d_{a1} < d_{a2}$ for some a. Thus, one disadvantage of this multivariate set up with respect to the univariate case is that we need the trimming even in the cases $d_{a1} \geq d_{a2}$ as long as there exists at least one b in $\{1, ..., r\}$ such that $d_{b2} > d_{b1}$.

In Lobato (1995) $f(\lambda)$ represents the $r \times r$ spectral density matrix of the vector x_t, where $f_{aa}(\lambda)$ is the spectrum of x_t^a and $f_{ab}(\lambda)$ is the cross spectral density between x_t^a and x_t^b for $a, b = 1, ..., r$. All the elements in $f(\lambda)$ are evaluated at the same frequency λ. In the multivariate SCLM case, where the poles/zeros can appear at different frequencies, we denote $f(\lambda)$ the matrix whose diagonal elements are the spectra $f_{aa}(\omega_a + \lambda)$ and the off-diagonal figures represent the cross spectral densities $f_{ab}(\omega_a + \lambda)$ if $\omega_a = \omega_b$ and are 0 otherwise. Note that in the case of different ω_a's $f(\lambda)$ is not the spectral density matrix of x_t. Modifying Lobato (1995) to allow $\omega_a \neq 0$ and $\omega_a \neq \omega_b$, $a, b = 1, ..., r$, we introduce the following assumptions,

J.1: For $\alpha \in (0, 2]$,

$$f(\lambda) \sim \Lambda_1 C \Lambda_1 (1 + O(\lambda^\alpha)) \quad \text{as} \quad \lambda \to 0^+$$

$$f(\lambda) \sim \Lambda_2 D \Lambda_2 (1 + O(|\lambda|^\alpha)) \quad \text{as} \quad \lambda \to 0^-$$

where C, D are two Hermitian positive definite matrices with typical elements C_{ab}, D_{ab} if $\omega_a = \omega_b$ and 0 otherwise, and $\Lambda_i = diag\{|\lambda|^{-d_{ai}}\}$, $a = 1, ..., r$, $i = 1, 2$, where $0 < d_{a2} < 0.5$ and $d_{a1} \in \Theta = [\Delta_1, \Delta_2]$ where $0 < \Delta_1 < \Delta_2 < 0.5$.

J.2: In a neighbourhood $(-\delta, 0) \cup (0, \delta)$ of ω_a, $f_{aa}(\lambda)$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f_{aa}(\omega_a \pm \lambda) = O(\lambda^{-1}) \quad \text{as} \quad \lambda \to 0^+$$

for $a = 1, ..., r$.

J.3: $x_t = Ex_0 + \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}$ where ε_t is a martingale difference sequence with $E|\varepsilon_t| < \infty$, $E[\varepsilon_t \varepsilon_t' | F_{t-1}] = R$ where the diagonal elements of R are equal to 1, F_{t-1} is the σ-field generated by ε_s, $s \leq t - 1$ and ε_t and $\varepsilon_t \varepsilon_t' - R$ are uniformly integrable.

J.4: If $d_{a1} \geq d_{a2}$ for all $a = 1, ..., r$, then $l = 0$ and

$$\frac{1}{m} + \frac{m}{n} \to 0 \quad \text{as } n \to \infty$$

and if $d_{a1} < d_{a2}$ for some a

$$\frac{m}{n} + \frac{l}{m} \log m + \frac{n^{2A}}{l^{1+2A}} (\log m)^3 \to 0 \quad \text{as } n \to \infty$$

where $A = \max_a \{d_{a2} - d_{a1}\}$.

Assumption **J.1** is similar to C5.1' in Lobato (1995) but we allow for different ω's and asymptotic asymmetry of the spectral density at each of those frequencies. As in the case $\omega_a = 0$ studied by Lobato, we only focus on positive values of the different persistence parameters. **J.2** is **B.2** in the univariate case for every a. Assumption **J.3** implies that the typical element of x_t is

$$x_t^a = Ex_0^a + \sum_{j=0}^{\infty} \alpha_{aj} \varepsilon_{t-j}$$

where α_{aj} is the $1 \times r$ a-th row of A_j. It also implies

$$E[\varepsilon_t \varepsilon_u'] = 0 \quad \text{for } t \neq u$$

and

$$\frac{1}{n} \sum_{t=1}^{n} \varepsilon_t \varepsilon_t' \xrightarrow{p} R$$

(see Lobato (1995)).

Let $d^1 = (d_{11}, ..., d_{r1})'$ be the vector of actual memory parameters just after the spectral poles, $d = (d_1, ..., d_r)$ any vector of admissible values and $\tilde{d}^1$ the local Whittle estimate of d^1. Concentrating C out of the objective function (4.47), we have that $\tilde{d}^1 = \arg\min_{d \in \Theta} R(d)$ where

$$R(d) = -2 \sum_{a=1}^{r} d_a \frac{1}{m-l} \sum_{j=l+1}^{m} \log \lambda_j + \log |\tilde{C}(d)| \tag{4.48}$$

and

$$\tilde{C}(d) = \frac{1}{m-l} \sum_{j=l+1}^{m} \{\Lambda_j^{-1} I_j \Lambda_j^{-1}\}. \tag{4.49}$$

As in the univariate case the procedure consists in obtaining $\tilde{d}^1$ by minimizing (4.48) over a closed set of admissible values and then plug $\tilde{d}^1$ into (4.49) to get an estimate of the matrix C.

Theorem 8 *Under* **J.1-J.4,**

$$\tilde{d}^1 \xrightarrow{p} d^1 \quad as \ n \to \infty.$$

Proof: The proof is similar to that in Lobato (1995) for $\omega_a = 0$ noting the results we had in the univariate case so that it will be described here in an abbreviated manner. Write $R(d) - R(d^1) = U(d) - T(d)$ where

$$\begin{aligned} U(d) &= 2\sum_{a=1}^{r}(d_a - d_{a1}) - \sum_{a=1}^{r}\log\{1 + 2(d_a - d_{a1})\} \\ T(d) &= 2\sum_{a=1}^{r}(d_a - d_{a1})\left[\frac{1}{m-l}\sum_{l+1}^{m}\log j - \log(m-l) + 1\right] \\ &+ \log|\Gamma^{-1}\tilde{C}(d^1)| - \log|Z\Gamma^{-1}M\tilde{C}(d)| \end{aligned}$$

where

$$M = diag\{\lambda_{(m-l)}^{2(d_{a1}-d_a)}\}\ , \ \ \Gamma = diag\{C_{aa}\}\ , \ \ Z = diag\{1 + 2(d_a - d_{a1})\}.$$

Now

$$\inf_{\bar{N}_\delta \cap \Theta} U(d) \geq \frac{r\delta^2}{2} > 0$$

where $\bar{N}_\delta = R^r - N_\delta$, $N_\delta = \{d :\| d - d^1 \| < \delta\}$ and $\| B \| = \max_i(|B_i|)$. Thus it only remains to show that $\sup_\Theta |T(d)| \xrightarrow{p} 0$. Now $\sup_\Theta |T(d)|$ is bounded by

$$2r\left[\frac{1}{m-l}\sum_{l+1}^{m}\log j - \log(m-l) + 1\right] \tag{4.50}$$

$$+ \ 2\sup_\Theta|\log|Z\Gamma^{-1}M\tilde{C}(d)||. \tag{4.51}$$

By Lemma 3, (4.50) is $o(1)$ and using $\log|A| \leq \text{tr}(A-I)$, $\log|Z\Gamma^{-1}M\tilde{C}(d)|$ is bounded by

$$\begin{aligned} &\text{tr}(Z\Gamma^{-1}M\tilde{C}(d) - I_r) \\ &= \text{tr}\left(\frac{1}{m-l}\sum_j ZM\Lambda_j^{-2}\Psi_j^{-1}\Psi_j\Gamma_j^{-1}I_j - I_r\right) \\ &= \text{tr}\left(\frac{1}{m-l}\sum_j Z\Phi_j\Psi_j\Gamma_j^{-1}I_j - I_r\right) \\ &= \text{tr}\left(\frac{1}{m-l}Z\sum_j \Phi_j[\Psi_j\Gamma_j^{-1}I_j - I_r]\right) \\ &+ \text{tr}\left(\frac{1}{m-l}\sum_j(Z\Phi_j - I_r)\right) \end{aligned} \tag{4.52}$$

where $\Psi_j = diag\{\lambda_j^{2d_{a1}}\}$, $\Phi = diag\{(\frac{j}{m-l})^{2(d_a-d_{a1})}\}$ and because $M\Lambda_j^{-2}\Psi_j^{-1}\Phi_j^{-1} = I_r$. Then $\text{tr}(Z\Gamma^{-1}M\tilde{C}(d) - I_r)$ is equal to

$$\sum_{a=1}^{r}(1+2(d_a-d_{a1}))\frac{1}{m-l}\sum_{j=l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d_a-d_{a1})}[\lambda_j^{2d_{a1}}C_{aa}^{-1}I_{aa}(\omega_a+\lambda_j)-1] \quad (4.53)$$

$$+\sum_{a=1}^{r}(1+2(d_a-d_{a1}))\frac{1}{m-l}\sum_{j=l+1}^{m}\left[\left(\frac{j}{m-l}\right)^{2(d_a-d_{a1})}-1\right] \quad (4.54)$$

and (4.54) is $o(1)$ by Lemma 2 and the fact that (4.53) is $o_p(1)$ has been shown in the proof of Theorem 6 for a single a. □

Note that the proof of the consistency does not use any information about the correlation structure of the different elements of x_t (reflected in the cross spectra in our frequency domain set up). This information will be needed for the asymptotic distribution and it will produce a gain in asymptotic efficiency in the same way as when $w_a = 0$ for all $a = 1, ..., r$.

Since the estimates of the different memory parameters d_{a1}, d_{b1} for $a, b = 1, ..., r$ are asymptotically independent when $\omega_a \neq \omega_b$ (this can be shown in the same way as we prove the asymptotic independence of $\tilde{d}_{a1}$ and $\tilde{d}_{a2}$ in the univariate case in Chapter 5), the efficiency improvement will only occur in those cases when $\omega_a = \omega_b = \omega$. Thus we study the asymptotic distribution when the spectral poles/zeros in every element of x_t are at the same frequency ω. As in Lobato (1995) and the univariate case, we need the following assumptions,

J.5: Assumption **J.1** holds.

J.6: Let

$$A(\lambda) = \begin{bmatrix} A_1(\omega+\lambda) \\ \vdots \\ A_r(\omega+\lambda) \end{bmatrix}$$

where $A_a(\lambda) = \sum_{j=0}^{\infty}\alpha_{aj}e^{ij(\lambda)} = (A_a^1(\lambda), ..., A_a^r(\lambda))$. Assume

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}A_a^k(\omega\pm\lambda) = O\left(\frac{A_a^k(\omega\pm\lambda)}{\lambda}\right) \quad \text{as } \lambda \to 0^+$$

for $a, k = 1, ..., r$.

J.7: **J.3** holds and

$$E[\varepsilon_a(t)\varepsilon_b(t)\varepsilon_c(t)|F_{t-1}] = \mu_{abc} \quad |\mu_{abc}| < \infty$$
$$E[\varepsilon_a(t)\varepsilon_b(t)\varepsilon_c(t)\varepsilon_d(t)|F_{t-1}] = 3 + \kappa_{abcd} \quad |\kappa_{abcd}| < \infty$$

for $a, b, c, d = 1, ..., r$.

J.8: If $d_{a1} \geq d_{a2}$ for all $a = 1, ...r$, then $l = 0$ and

$$\frac{1}{m} + \frac{m^{1+2\alpha}(\log m)^2}{n^{2\alpha}} \to 0 \quad \text{as } n \to \infty$$

and if $d_{a1} < d_{a2}$ for some a,

$$\frac{(\log m)^3}{l^2} + \frac{l^3}{m}(\log m)^4 + \frac{n^{2A}}{l^{1+2A}} \log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

Theorem 9 *Under* **J.5-J.8**

$$\sqrt{m}(\tilde{d}^1 - d^1) \xrightarrow{d} N_r(0, E^{-1})$$

as $n \to \infty$, where $E = 2I_r + 2Re(C * (C^{-1})')$ and $*$ is the Hadamard product so that the typical element of E is

$$E_{ab} = \begin{cases} 2 + 2C_{aa}C^{aa} & \text{if} \quad a = b \\ 2ReC_{ab}C^{ba} & \text{if} \quad a \neq b \end{cases}$$

where C_{ab} and C^{ab} are typical elements of C and C^{-1} respectively.

Proof: The proof is similar to that of Theorem 5.2 in Lobato (1995) noting the possible asymmetry as in the univariate case, therefore it will be presented in an abbreviated form. In particular, since $\tilde{d}^1$ estimates consistently d^1 we have to show that

$$\frac{\partial^2 R(\bar{d}^1)}{\partial d_a \partial d_b} \xrightarrow{p} E_{ab} \tag{4.55}$$

$$\sqrt{m} \sum_{a=1}^{r} \eta_a \frac{\partial R(d^1)}{\partial d_a} \xrightarrow{d} N(0, \eta' E \eta) \tag{4.56}$$

for any $r \times 1$ vector η and $||\bar{d}^1 - d^1|| \leq ||\tilde{d}^1 - d^1||$. Now

$$\frac{\partial R(d^1)}{\partial d_a} = -\frac{2}{m-l} \sum_{j=l+1}^{m} \log \lambda_j + \text{tr}\left(\tilde{C}^{-1}(d^1) \frac{\partial \tilde{C}(d^1)}{\partial d_a}\right) \tag{4.57}$$

$$\frac{\partial^2 R(\bar{d}^1)}{\partial d_a \partial d_b} = \text{tr}\left(-\tilde{C}^{-1}(\bar{d}^1) \frac{\partial \tilde{C}(\bar{d}^1)}{\partial d_b} \tilde{C}^{-1}(\bar{d}^1) \frac{\partial \tilde{C}(\bar{d}^1)}{\partial d_a} + \tilde{C}^{-1}(\bar{d}^1) \frac{\partial^2 \tilde{C}(\bar{d}^1)}{\partial d_a \partial d_b}\right) \tag{4.58}$$

The proof of (4.55) is similar to that in Lobato (1995). Noting that the sums are from $j = l + 1$ to m, the behaviour of l and m in assumption **J.8** and the results obtained in the univariate case (and in particular Theorem 4) we can see that (4.58) is asymptotically equivalent to

$$\operatorname{tr}\left(-C^{-1}C_{\underline{b}}C^{-1}C_{\underline{a}}\left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2+C^{-1}C_{\underline{ab}}\frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2\right)$$

where $C_{\underline{s}} = Ci_s + i_sC$, $s = a, b$, and $C_{\underline{ab}} = i_a i_b C + i_a C i_b + i_b C i_a + C i_a i_b$ where i_s is a matrix of zeros except a 1 in the $s \times s$ element. Since

$$\frac{1}{m-l}\sum_{l+1}^{m}(\log j)^2-\left(\frac{1}{m-l}\sum_{l+1}^{m}\log j\right)^2\to 1 \quad \text{as } n\to\infty$$

and $\operatorname{tr}[C^{-1}C_{\underline{b}}C^{-1}C_{\underline{a}}] = \operatorname{tr}[C^{-1}C_{\underline{ab}}]$ then (4.55) follows.

Now since $\tilde{C}^{-1}(d^1) = C^{-1} + o_p(1)$ then (4.57) is asymptotically equivalent to

$$-\frac{2}{m-l}\sum_{l+1}^{m}\log\lambda_j+\operatorname{tr}\left[C^{-1}\frac{\partial\tilde{C}(d^1)}{\partial d_a}\right]. \tag{4.59}$$

Omitting the $o_p(1)$ terms we have that $\sqrt{m}\sum_{a=1}^{r}\eta_a\frac{\partial R(d^1)}{\partial d_a}$ is

$$\frac{2}{\sqrt{m}}\sum_{a=1}^{r}\eta_a\sum_{j=l+1}^{m}\log j\{Re[\sum_{k=1}^{r}C^{ak}\lambda_j^{d_a+d_k}I_{ka}(\omega+\lambda_j)]-1\}. \tag{4.60}$$

Now

$$\sum_{k=1}^{r}C^{ak}\frac{1}{m-l}\sum_{j=l+1}^{m}\lambda_j^{d_a+d_k}I_{ka}(\omega+\lambda_j)$$
$$=\sum_{k=1}^{r}C^{ak}\tilde{C}_{ka}=\sum_{k=1}^{r}C^{ak}(C_{ka}+o_p(1))=1+o_p(1)$$

and then (4.60) is asymptotically equivalent to

$$\frac{2}{\sqrt{m}}\sum_{a=1}^{r}\eta_a\sum_{j=l+1}^{m}v_j\{Re[\sum_{k=1}^{r}C^{ak}\lambda_j^{d_a+d_k}I_{ka}(\omega+\lambda_j)]-1\} \tag{4.61}$$

where $v_j = \log j - \frac{1}{m-l}\sum_{l+1}^{m}\log j$ such that $\sum_{l+1}^{m}v_j = 0$. Then proceeding like in Lobato (1995) and taking into account the results we obtained in the univariate case we have

$$\sqrt{m}\sum_{a=1}^{r}\eta_a\frac{\partial R(d^1)}{\partial d_a}=\sum_{t=1}^{n}z_t(1+o_p(1))$$

where $z_t = \varepsilon_t'\sum_{s=1}^{t-1}\Gamma_{t-s}^{m}\varepsilon_s$ and

$$\Gamma_{t-s}^{m}=\frac{1}{\pi\sqrt{mn}}\sum_{j=l+1}^{m}v_j\sum_{a=1}^{r}\eta_a Re[\sum_{k=1}^{r}C^{ak}\lambda_j^{d_{a1}+d_{k1}}(A_k'\bar{A}_a+A_a'\bar{A}_k)]\cos[(t-s)(\omega+\lambda_j)]$$

where the different A_s are evaluated at $\omega + \lambda_j$ and the overline indicates conjugation. Thus z_t is a martingale difference so that the Theorem is proved if

$$1) \sum_{t=1}^{n} E[z_t^2|F_{t-1}] - \sum_{a=1}^{r}\sum_{b=1}^{r} \eta_a \eta_b E_{ab} \xrightarrow{p} 0$$

$$2) \sum_{t=1}^{n} E[z_t^2 I(|z_t| > \delta)] \to 0 \quad \text{for all } \delta > 0.$$

Proceeding as in Lobato (1995) we can see that $\sum_{t=1}^{n} E(z_t^2|F_{t-1})$ is asymptotically equivalent to $\sum_{t=1}^{n}\sum_{s=1}^{t-1} \mathrm{tr}(\Gamma_{t-s}^{m*} R \Gamma_{t-s}^{m} R)$ which tends to $\sum_{a=1}^{r}\sum_{b=1}^{r} \eta_a \eta_b E_{ab}$. To prove 2) it suffices to show that $\sum_{t=1}^{n} E z_t^4 \to 0$ which can be proved in the same way as in Lobato (1995). □

Remark 1: We observe that, unlike the log periodogram regression, the multivariate extension of the local Whittle estimate produces a gain in asymptotic efficiency when $\omega_a = \omega_b$ for $a, b = 1, ..., r$, because the variances of the joint estimates of $d_{11}, ..., d_{r1}$ are lower than the variances of the individual estimates (see Lobato (1995)).

Remark 2: Assumption **J.8** implies that $A = \max_a\{d_{a2} - d_{a1}\} < \alpha/(3 + 4\alpha)$. As in the univariate case this restriction can be relaxed to $A < \alpha/2(1 + \alpha)$ by imposing a stronger condition in the distribution of the ε_t, namely that their fourth cumulant is zero for all t.

Remark 3: Since E depends only on the different elements of C and C^{-1}, and C can be estimated consistently using (4.49), then E can be consistently estimated, which is useful for statistical inference.

4.5 APPENDIX B: TECHNICAL LEMMAS

Lemma 2 *For $\varepsilon \in (0, 1]$ and $\kappa \in (\varepsilon, \infty)$, when $l \to \infty$ and $\frac{l}{m} \to 0$,*

$$\sup_{\varepsilon \le \gamma \le \kappa} \left| \frac{\gamma}{m-l} \sum_{j=l+1}^{m} \left(\frac{j}{m-l}\right)^{\gamma-1} - 1 \right| = O\left(\left(\frac{l}{m}\right)^{\varepsilon}\right). \tag{4.62}$$

Proof: For $\gamma > 0$

$$\left| \frac{\gamma}{m-l} \sum_{l+1}^{m} \left(\frac{j}{m-l}\right)^{\gamma-1} - 1 \right|$$

$$= \left|\gamma\int_0^{\frac{1}{m-l}}\left\{\left(\frac{l+1}{m-l}\right)^{\gamma-1} - x^{\gamma-1}\right\}dx + \gamma\sum_{l+2}^{m}\int_{\frac{j-l-1}{m-l}}^{\frac{j-l}{m-l}}\left\{\left(\frac{j}{m-l}\right)^{\gamma-1} - x^{\gamma-1}\right\}dx\right|$$

$$\leq \gamma\int_0^{\frac{1}{m-l}}\left\{\left(\frac{l+1}{m-l}\right)^{\gamma-1} + x^{\gamma-1}\right\}dx + \gamma\sum_{l+2}^{m}\left|\int_{\frac{j-l-1}{m-l}}^{\frac{j-l}{m-l}}\left\{\left(\frac{j}{m-l}\right)^{\gamma-1} - x^{\gamma-1}\right\}dx\right|$$

$$\leq \frac{\gamma}{m-l}\left(\frac{l+1}{m-l}\right)^{\gamma-1} + \frac{1}{(m-l)^{\gamma}} + \frac{l\gamma|\gamma-1|}{(m-l)^2}\sum_{l+1}^{m}\left(\frac{j}{m-l}\right)^{\gamma-2} \tag{4.63}$$

using the mean value theorem. The first term is $O(m^{-\gamma}l^{\gamma-1})$, the second is $O(m^{-\gamma})$ and the third is $O(m^{-1}l)$ for $\gamma > 1$, zero for $\gamma = 1$ and $O(m^{-\gamma}l^{\gamma})$ if $\gamma < 1$. Thus (4.63) is $O((\frac{l}{m})^{\gamma} + \frac{l}{m})$ and the right hand side of (4.62) is $O((\frac{l}{m})^{\varepsilon})$ because $\varepsilon \in (0,1]$. □

Lemma 3 *Let $l \to \infty$ and $\frac{l}{m} \to 0$ as $m \to \infty$. Then,*

$$\left|\frac{1}{m-l}\sum_{j=l+1}^{m}\log j - \log(m-l) + 1\right| = O\left(\frac{l}{m}\log m\right).$$

Proof:

$$\left|\frac{1}{m-l}\sum_{l+1}^{m}\log j - \log(m-l) + 1\right|$$

$$= \left|\frac{1}{m-l}\sum_{l+2}^{m}\int_{j-l-1}^{j-l}\log\left(\frac{j}{x}\right)dx + \frac{1}{m-l} + \frac{1}{m-l}\log(l+1)\right|$$

$$\leq \frac{1}{m-l}\sum_{l+2}^{m}\int_{j-l-1}^{j-l}|x-j|\frac{1}{j-l-1}dx + \frac{1}{m-l} + \frac{1}{m-l}\log(l+1)$$

$$\leq \frac{|l+1|}{m-l}\sum_{1}^{m-l-1}\frac{1}{j} + \frac{1}{m-l}(1+\log(l+1))$$

$$= O\left(\frac{l}{m}\log m + \frac{1}{m} + \frac{1}{m}\log l\right) = O\left(\frac{l}{m}\log m\right). \quad □$$

Lemma 4 *Let $r > l$ and $I_{\varepsilon j}$ defined in (4.10) and* **C.3** *hold. Then*

$$\sum_{j=l+1}^{r}(2\pi I_{\varepsilon j} - 1) = O_p(r^{\frac{1}{2}}).$$

Proof: Write

$$\begin{aligned}
2\pi I_{\varepsilon j} &= 2\pi|W_\varepsilon(\omega+\lambda_j)|^2 = \frac{1}{n}\left|\sum_{t=1}^{n}\varepsilon_t e^{it(\omega+\lambda_j)}\right|^2 \\
&= \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2 + \frac{1}{n}\sum_{s}\sum_{t\neq s}\varepsilon_t\varepsilon_s\cos\{(\omega+\lambda_j)(t-s)\} \\
&= \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2 + \frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s\cos\{(\omega+\lambda_j)(t-s)\}.
\end{aligned}$$

Thus

$$\sum_{j=l+1}^{r}(2\pi I_{\varepsilon j}-1)=\frac{r-l}{n}\sum_{t=1}^{n}(\varepsilon_t^2-1)+\sum_{t=2}^{n}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s d_{t-s} \qquad (4.64)$$

where $d_s=\frac{2}{n}\sum_{l+1}^{r}\cos\{(\omega+\lambda_j)s\}$. If ω is a harmonic frequency of the form $\omega=\frac{2\pi w}{n}$, where w is an integer then $d_s=d_{n-s}$. Since in our analysis $n\to\infty$, we can express any frequency $\omega\in(0,\pi]$ as a harmonic frequency for a large enough n. Also $|d_s|\le\frac{2r}{n}$ and for $1\le s\le n/2$, $|d_s|\le\frac{6}{\pi s}+\frac{6}{n}$. This last inequality can be proved in the following way. Write

$$d_s=\frac{2}{n}\sum_{l+1}^{r}\cos(\omega s)\cos(s\lambda_j)-\frac{2}{n}\sum_{l+1}^{r}\sin(\omega s)\sin(s\lambda_j),$$

then

$$|d_s|\le\frac{2}{n}|\sum_{l+1}^{r}\cos(s\lambda_j)|+\frac{2}{n}|\sum_{l+1}^{r}\sin(s\lambda_j)|.$$

By formulae (5.10) and (5.11) in Zygmund (1977, Chapter 2),

$$|\frac{1}{2}+\sum_{v=1}^{r}\cos tv-\frac{1}{2}\cos rt|\le\frac{1}{t}\quad\text{for}\quad 0<t\le\pi$$

and

$$|\sum_{v=1}^{r}\sin tv-\frac{1}{2}\sin rt|\le\frac{2}{t}.$$

Thus

$$\begin{aligned}|\sum_{l+1}^{r}\cos(\lambda_j s) &\le |\sum_{1}^{r}\cos(\lambda_j s)|+|\sum_{1}^{l}\cos(\lambda_j s)|\\ &\le |\frac{1}{2}+\sum_{1}^{r}\cos(\lambda_j s)-\frac{1}{2}\cos\lambda_r s|+|\frac{1}{2}-\frac{1}{2}\cos\lambda_r s|\\ &+ |\frac{1}{2}+\sum_{1}^{l}\cos(\lambda_j s)-\frac{1}{2}\cos\lambda_l s|+|\frac{1}{2}-\frac{1}{2}\cos\lambda_l s|\\ &\le \frac{n}{\pi s}+2\quad\text{for}\quad 1\le s\le\frac{n}{2}\end{aligned}$$

and

$$\begin{aligned}|\sum_{l+1}^{r}\sin(\lambda_j s)| &\le |\sum_{1}^{r}\sin(\lambda_j s)|+|\sum_{1}^{l}\sin(\lambda_j s)|\\ &\le |\sum_{1}^{r}\sin(\lambda_j s)-\frac{1}{2}\sin(\lambda_r s|+|\frac{1}{2}\sin(\lambda_r s)|\\ &+ |\sum_{1}^{l}\sin(\lambda_j s)-\frac{1}{2}\sin(\lambda_l s|+|\frac{1}{2}\sin(\lambda_l s)|\\ &\le \frac{2n}{\pi s}+1\end{aligned}$$

and thus $|d_s| \le \frac{6}{\pi s} + \frac{6}{n}$.

Both terms on the right hand side of (4.64) have zero mean and variance respectively $O(\frac{r^2}{n})$ and

$$\begin{aligned} O(n\sum_1^n d_s^2) &= O\left(n\sum_1^{[\frac{n}{r}]}\left(\frac{r}{n}\right)^2 + n\sum_{[\frac{n}{r}]}^n\left(\frac{1}{\pi s}+\frac{1}{n}\right)^2\right) \\ &= O\left(\frac{n^2}{r}\left(\frac{r}{n}\right)^2 + n\sum_{[\frac{n}{r}]}^n \frac{1}{s^2}\right) \\ &= O(r) \end{aligned}$$

which concludes the proof. □

Lemma 5 *Let j be a sequence of integers such that $\frac{j}{n} \to 0$ as $n \to \infty$. Then under* **C.1** *and* **C.2**,

$$\begin{aligned} \int_{-\pi}^{\pi}\left|\frac{\alpha(\lambda)}{\alpha(\omega+\lambda_j)} - 1\right|^2 K(\lambda-\lambda_j-\omega)d\lambda &= O\left(\frac{1}{j}\right) \quad \textit{if } d_1 \ge d_2 \\ &= O\left(\frac{1}{j}\left[\frac{n}{j}\right]^{2(d_2-d_1)}\right) \quad \textit{if } d_1 < d_2. \end{aligned}$$

Proof: **C.1** and **C.2** imply that we can pick $\delta \in (2\lambda_j, \pi)$ such that for some $C < \infty$,

$$\begin{aligned} |\alpha(\omega+\lambda)| &\le C\lambda^{-d_1} \\ |\alpha(\omega-\lambda)| &\le C\lambda^{-d_2} \end{aligned}$$

and

$$\begin{aligned} |\alpha'(\omega+\lambda)| &\le C\lambda^{-d_1-1} \\ |\alpha'(\omega-\lambda)| &\le C\lambda^{-d_2-1} \end{aligned}$$

for $0 < \lambda < \delta$. Now split the integral up into,

$$\int_{-\pi}^{\omega-\delta} + \int_{\omega-\delta}^{\omega-\frac{\lambda_j}{2}} + \int_{\omega-\frac{\lambda_j}{2}}^{\omega+\frac{\lambda_j}{2}} + \int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j} + \int_{\omega+2\lambda_j}^{\omega+\delta} + \int_{\omega+\delta}^{\pi}.$$

Write $\alpha_j = \alpha(\omega+\lambda_j)$ and $f_j = f(\omega+\lambda_j)$. The first integral is equal to

$$\frac{1}{|\alpha_j|^2}\int_{-\pi}^{\omega-\delta}\{|\alpha(\lambda)|^2 - \alpha(\lambda)\bar{\alpha}_j - \alpha_j\bar{\alpha}(\lambda) + |\alpha_j|^2\}K(\lambda-\lambda_j-\omega)\mathrm{d}\lambda$$

and this is bounded in absolute value by

$$
\begin{aligned}
& \frac{1}{f_j}\{\max_{-\pi\le\lambda\le\omega-\delta} K(\lambda-\lambda_j-\omega)\}\left\{\int_{-\pi}^{\pi} f(\lambda)\mathrm{d}\lambda+\frac{|\bar{\alpha}_j|}{2\pi}\int_{-\pi}^{\pi}|\alpha(\lambda)|\mathrm{d}\lambda\right. \\
+ & \left.\frac{|\alpha_j|}{2\pi}\int_{-\pi}^{\pi}\bar{\alpha}(\lambda)\mathrm{d}\lambda\right\}+\int_{-\pi}^{\omega-\delta}K(\lambda-\lambda_j-\omega)\mathrm{d}\lambda \\
= & \ O\left(\frac{j^{2d_1}}{n^{1+2d_1}}+\frac{j^{d_1}}{n^{1+d_1}}+n^{-1}\right)=O(j^{-1})
\end{aligned}
$$

using (3.33) and (3.34). Similarly

$$\left|\int_{\omega+\delta}^{\pi}\right|=O(j^{-1}).$$

Using again (3.33) the integral over $[\omega-\delta,\omega-\lambda_j/2]$ has an absolute value bounded by

$$
\begin{aligned}
& \frac{1}{f_j}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{f(\omega-\lambda)}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\mathrm{d}\lambda \\
+ & \frac{|\bar{\alpha}_j|}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{|\alpha(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\mathrm{d}\lambda \\
+ & \frac{|\alpha_j|}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\le\lambda\le\delta}\frac{|\bar{\alpha}(\omega-\lambda)|}{\lambda^{\frac{1}{2}-d_2}}\right\}\int_{\frac{\lambda_j}{2}}^{\delta}\lambda^{\frac{1}{2}-d_2}K(-\lambda-\lambda_j)\mathrm{d}\lambda \\
+ & \int_{\frac{\lambda_j}{2}}^{\delta}K(-\lambda-\lambda_j)\mathrm{d}\lambda \\
= & \ O(\lambda_j^{2d_1}n^{-1}\lambda_j^{-1-2d_2}+\lambda_j^{d_1}n^{-1}\lambda_j^{-1-d_2}+n^{-1}\lambda_j^{-1}) \\
= & \ O\left(\frac{1}{j}\right) \quad \text{if} \quad d_1\ge d_2 \\
= & \ O\left(\frac{1}{j}\left[\frac{n}{j}\right]^{2(d_2-d_1)}\right) \quad \text{if} \quad d_1<d_2.
\end{aligned}
$$

Proceeding similarly we get that

$$\left|\int_{\omega+2\lambda_j}^{\omega+\delta}\right|=O(j^{-1}).$$

Now the integral over $[\omega\pm\frac{\lambda_j}{2}]$ is bounded in modulus by

$$
\begin{aligned}
& \{\max_{-\frac{\lambda_j}{2}\le\lambda\le\frac{\lambda_j}{2}}|K(\lambda-\lambda_j)|\left\{\frac{1}{f_j}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}f(\omega+\lambda)\mathrm{d}\lambda\right. \\
+ & \left.\frac{|\alpha_j|}{|\alpha_j|^2}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}|\bar{\alpha}(\omega+\lambda)|\mathrm{d}\lambda+\frac{|\bar{\alpha}_j|}{|\alpha_j|^2}\int_{-\frac{\lambda_j}{2}}^{\frac{\lambda_j}{2}}|\alpha(\omega+\lambda)|\mathrm{d}\lambda+\lambda_j\right\} \\
= & \ O(n^{-1}\lambda_j^{-2}[\lambda_j^{2d_1}\lambda_j^{1-2d_i}+\lambda_j^{d_1}\lambda_j^{1-d_i}+\lambda_j]) \qquad (4.65)
\end{aligned}
$$

where $i = 1$ if $d_1 \geq d_2$ and $i = 2$ if $d_2 > d_1$. Thus (4.65) is $O(j^{-1})$ if $d_1 \geq d_2$ and $O(\frac{1}{j}[\frac{n}{j}]^{2(d_2-d_1)})$ if $d_2 > d_1$.

Finally using the mean value theorem

$$\begin{aligned}
\left|\int_{\omega+\frac{\lambda_j}{2}}^{\omega+2\lambda_j}\right| &= \frac{1}{|\alpha_j|^2}\left|\int_{\frac{\lambda_j}{2}}^{2\lambda_j} |\alpha(\omega+\lambda)-\alpha(\omega+\lambda_j)|^2 K(\lambda-\lambda_j)\mathrm{d}\lambda\right| \\
&\leq \frac{1}{|\alpha_j|^2}\left\{\max_{\frac{\lambda_j}{2}\leq\lambda\leq 2\lambda_j}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha(\omega+\lambda)\right|^2\right\}\int_{\frac{\lambda_j}{2}}^{2\lambda_j}|\lambda-\lambda_j|^2 K(\lambda-\lambda_j)\mathrm{d}\lambda \\
&= O(\lambda_j^{2d_1}\lambda_j^{-2-2d_1}n^{-1}\lambda_j) = O(j^{-1})
\end{aligned}$$

using (3.33) which concludes the proof. □

Lemma 6 *Let $0 < l < r \leq m$. Let* **C.1-C.3** *hold and $d_2 > d_1$. Then*

$$\sum_{j=l+1}^{r}\left(\frac{I_j}{g_j} - 2\pi I_{\varepsilon j}\right) = O_p\left(\frac{r^{\alpha+1}}{n^\alpha} + l^{\frac{3}{4}}r^{\frac{1}{4}}\right) \quad \text{under } \mathbf{C.4} \tag{4.66}$$

$$= O_p\left(\frac{r^{\alpha+1}}{n^\alpha} + l\right) \quad \text{under } \mathbf{C.5} \text{ and } \mathbf{C.6} \tag{4.67}$$

where I_j is the periodogram of $x_t = Ex_1 + \sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j}$ at $(\omega+\lambda_j)$, $I_{\varepsilon j}$ is the periodogram of ε_t at frequency $(\omega+\lambda_j)$ and $g_j = C\lambda_j^{-2d_1}$.

Proof: From Theorem 4 and **C.1**,

$$\begin{aligned}
E\left|\sum_{l+1}^{r}\left(\frac{I_j}{g_j}-\frac{I_j}{f_j}\right)\right| &= E\left|\sum_{l+1}^{r}\left(1-\frac{g_j}{f_j}\right)\left(\frac{I_j}{g_j}\right)\right| \\
&= O\left(\sum_{l+1}^{r}\left(\frac{j}{n}\right)^\alpha\left(1+(\frac{j}{n})^\alpha+\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right)\right) \\
&= O\left(\frac{r^{\alpha+1}}{n^\alpha}+\frac{r^{\alpha+1}}{n^\alpha}\frac{n^{2(d_2-d_1)}\log r}{l^{1+2(d_2-d_1)}}\right) = O\left(\frac{r^{\alpha+1}}{n^\alpha}\right)
\end{aligned}$$

under **C.4** or **C.6**.

Write $u_j = \sqrt{2\pi}\frac{W_j}{|\alpha_j|}$ and $v_j = \sqrt{2\pi}W_{\varepsilon j}$ where W_j and $W_{\varepsilon j}$ are discrete Fourier transforms of x_t and ε_t respectively at frequency $\omega+\lambda_j$, and $\alpha_j = \alpha(\omega+\lambda_j) = \sum_{k=0}^{\infty}\alpha_k e^{ik(\omega+\lambda_j)}$. Then

$$E\left\{\sum_{l+1}^{r}\left(\frac{I_j}{f_j}-2\pi I_{\varepsilon j}\right)\right\}^2 = E\{\sum_{l+1}^{r}(|u_j|^2-|v_j|^2)\}^2 = a+b$$

where

$$\begin{aligned} a &= \sum_{j=l+1}^{r} (E|u_j|^4 + E|v_j|^4 - 2E|u_jv_j|^2) \\ b &= 2\sum_{j=l+1}^{r}\sum_{k>j}(E|u_ju_k|^2 - E|u_jv_k|^2 - E|u_kv_j|^2 + E|v_jv_k|^2). \end{aligned}$$

Since for any zero mean random variables u, v, w, z,

$$E(uvwz) = E(uv)E(wz) + E(uw)E(vz) + E(uz)E(vw) + cum(u, v, w, z)$$

where $cum(u, v, w, z)$ is the joint cumulant of u, v, w and z, we can decompose a and b into $a_1 + a_2$ and $b_1 + b_2$ where

$$\begin{aligned} a_1 &= \sum_{l+1}^{r}\{2(E|u_j|^2)^2 + |E(u_j^2)|^2 - 2|E(u_jv_j)|^2\} \\ &\quad -2|E(u_j\bar{v}_j)|^2 - 2E|u_j|^2E|v_j|^2 + 2(E|v_j|^2)^2 + |E(v_j^2)|^2\} \\ a_2 &= \sum_{l+1}^{r}\{cum(u_j, u_j, \bar{u}_j, \bar{u}_j) - 2cum(u_j, v_j, \bar{u}_j, \bar{v}_j) + cum(v_j, v_j, \bar{v}_j, \bar{v}_j)\} \\ b_1 &= 2\sum_{j=l+1}^{r}\sum_{k>j}\{E|u_j|^2E|u_k|^2 + |E(u_ju_k)|^2 + |E(u_j\bar{u}_k)|^2 - E|u_j|^2E|v_k|^2 \\ &\quad -|E(u_jv_k)|^2 - |E(u_j\bar{v}_k)|^2 - E|u_k|^2E|v_j|^2 - |E(u_kv_j)|^2 - |E(u_k\bar{v}_j)|^2 \\ &\quad +E|v_j|^2E|v_k|^2 + |E(v_jv_k)|^2 + |E(v_j\bar{v}_k)|^2\} \\ b_2 &= 2\sum_{j=l+1}^{r}\sum_{k>j}\{cum(u_j, u_k, \bar{u}_j, \bar{u}_k) - cum(u_j, v_k, \bar{u}_j, \bar{v}_k) \\ &\quad -cum(u_k, v_j, \bar{u}_k, \bar{v}_j) + cum(v_j, v_k, \bar{v}_j, \bar{v}_k)\}. \end{aligned}$$

Now because $E|v_j|^2 = 1$ and Theorem 4,

$$\begin{aligned} a_1 &= \sum_{l+1}^{r}\{2(E|u_j|^2 - 1)^2 + 2(E|u_j|^2 - 1) + |E(u_j^2)|^2 - 2|E(u_jv_j)|^2 \\ &\quad -2|E(u_j\bar{v}_j) - 1|^2 - 2(E(u_j\bar{v}_j - 1) - 2(E(\bar{u}_jv_j) - 1) + |E(v_j^2)|^2\} \\ &= O\left(\sum_{l+1}^{r}\frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right) = O\left(\frac{n^{2(d_2-d_1)}}{l^{2(d_2-d_1)}}\log r\right) \\ b_1 &= 2\sum_{l+1}^{r}\sum_{k>j}\{(E|u_j|^2 - 1)(E|u_k|^2 - 1) + |E(u_ju_k)|^2 + |E(u_j\bar{u}_k)|^2 - |E(u_jv_k)|^2 \\ &\quad -|E(u_j\bar{v}_k)|^2 - |E(u_kv_j)|^2 - |E(u_k\bar{v}_j)|^2 + |E(v_jv_k)|^2 + |E(v_j\bar{v}_k)|^2\} \\ &= O\left(\sum_{j=l+1}^{r}\sum_{k>j}\frac{n^{4(d_2-d_1)}(\log k)^2}{k^{1+2(d_2-d_1)}j^{1+2(d_2-d_1)}}\right) \\ &= O\left(\frac{n^{4(d_2-d_1)}}{l^{4(d_2-d_1)}}(\log r)^2\right) \end{aligned}$$

and under **C.4** or **C.6**, a_1 is $O(l)$ and b_1 is $O(l^2)$. Now applying formula (2.6.3) of Brillinger (1975),

$$cum(u_j, v_k, \bar{u}_j, \bar{v}_k) = \iiint_{-\pi}^{\pi} f_{u_j,v_k,\bar{u}_j,\bar{v}_k}(\lambda, \mu, \zeta) d\lambda d\mu d\zeta$$

where $f_{u_j,v_k,\bar{u}_j,\bar{v}_k}$ is the fourth order cumulant spectrum, and by formula (2.10.3) in Brillinger (1975), we have that

$$cum(u_j, v_k, \bar{u}_j, \bar{v}_k) = \iiint_{-\pi}^{\pi} \frac{\kappa}{(2\pi)^3} A_{u_j}(-\lambda-\mu-\zeta) A_{v_k}(\lambda) A_{\bar{u}_j}(\mu) A_{\bar{v}_k}(\zeta) d\lambda d\mu d\zeta$$

where κ is the fourth cumulant of ε_t, $\kappa = \mu_4 - 3$, and $A_{u_j}, A_{v_k}, A_{\bar{u}_j}, A_{\bar{v}_k}$ are transfer functions of the filters implied in the definition of u_j and v_j,

$$\begin{aligned} u_j &= \frac{1}{|\alpha_j|} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{it(\omega+\lambda_j)} \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k} \\ v_j &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{it(\omega+\lambda_j)} \varepsilon_t \end{aligned}$$

so that if $\alpha(\lambda) = \sum_{k=0}^{\infty} \alpha_k e^{ik\lambda}$,

$$\begin{aligned} A_{u_j}(\lambda) &= \frac{1}{|\alpha_j|} \frac{1}{\sqrt{n}} \alpha(-\lambda) \sum_{t=1}^{n} e^{it(\omega+\lambda_j+\lambda)} \\ A_{\bar{u}_j}(\lambda) &= \frac{1}{|\alpha_j|} \frac{1}{\sqrt{n}} \alpha(-\lambda) \sum_{t=1}^{n} e^{it(\lambda-\omega-\lambda_j)} \\ A_{v_k}(\lambda) &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{it(\omega+\lambda_k+\lambda)} \\ A_{\bar{v}_k}(\lambda) &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{it(\lambda-\omega-\lambda_k)}. \end{aligned}$$

Since $\kappa = 0$ under **C.5**, then $a_2 = b_2 = 0$, and (4.67) follows. In any other case $cum(u_j, v_k, \bar{u}_j, \bar{v}_k)$ is equal to

$$\frac{\kappa}{(2\pi)^3} \frac{1}{n^2} \iiint_{-\pi}^{\pi} \frac{\alpha(\lambda+\mu+\zeta)\alpha(-\mu)}{|\alpha_j|^2} E_{jk}(\lambda, \mu, \zeta) d\lambda d\mu d\zeta \tag{4.68}$$

where $E_{jk}(\lambda, \mu, \zeta) = D(\omega + \lambda_j - \lambda - \mu - \zeta) D(\omega + \lambda_k + \lambda) D(\mu - \omega - \lambda_j) D(\zeta - \omega - \lambda_k)$ and $D(\lambda) = \sum_{t=1}^{n} e^{it\lambda}$ is Dirichlet's kernel. Doing the same with the other cumulants in b_2 we see that the summand of $(2\pi)^3 b_2$ is

$$\frac{\kappa}{n^2} \iiint_{-\pi}^{\pi} \left\{ \frac{\alpha(\lambda+\mu+\zeta)\alpha(-\mu)}{|\alpha_j|^2} - 1 \right\} \left\{ \frac{\alpha(-\lambda)\alpha(-\zeta)}{|\alpha_k|^2} - 1 \right\} E_{jk}(\lambda, \mu, \zeta) d\lambda d\mu d\zeta. \tag{4.69}$$

Since

$$(c_1c_2-1)(c_3c_4-1)=\prod_{j=1}^{4}(c_j-1)+\sum_{i=1}^{4}\prod_{j\neq i}(c_j-1)+\sum_{i=1}^{2}\sum_{j-1}^{2}(c_i-1)(c_{j+2}-1)$$

then (4.69) has components of three types. The first one is

$$\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\mu)}{\bar{\alpha}_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}\left\{\frac{\alpha(-\zeta)}{\bar{\alpha}_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta. \tag{4.70}$$

Proceeding as in Robinson (1995b) we have that because of the Schwarz inequality and by periodicity, (4.70) is bounded in absolute value by $\kappa(2\pi)^3P_jP_k$ where

$$P_j=\int_{-\pi}^{\pi}\left|\frac{\alpha(\lambda)}{\alpha_j}-1\right|^2K(\lambda-\omega-\omega_j)\mathrm{d}\lambda$$

and $K(\lambda)=\frac{|D(\lambda)|^2}{2\pi n}$ is Fejer's kernel.

The second component is

$$\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\mu)}{\bar{\alpha}_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta. \tag{4.71}$$

As before, (4.71) is bounded in absolute value by $\kappa(2\pi)^3P_jP_k^{\frac{1}{2}}$.

An example of the third type component is

$$\begin{aligned}
&\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\lambda+\mu+\zeta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}E_{jk}(\lambda,\mu,\zeta)\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\zeta\\
=&\frac{\kappa}{n^2}\iiint_{-\pi}^{\pi}\left\{\frac{\alpha(\theta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}E_{jk}(\lambda,\theta-\lambda-\zeta,\zeta)\mathrm{d}\lambda\mathrm{d}\theta\mathrm{d}\zeta\\
=&\frac{\kappa2\pi}{n^2}\iint_{-\pi}^{\pi}\left\{\frac{\alpha(\theta)}{\alpha_j}-1\right\}\left\{\frac{\alpha(-\lambda)}{\alpha_k}-1\right\}\\
\times\ &D(\omega+\lambda_j-\theta)D(\omega+\lambda_k+\lambda)D(\theta-2\omega-\lambda-\lambda_j-\lambda_k)\mathrm{d}\lambda\mathrm{d}\theta
\end{aligned} \tag{4.72}$$

because

$$\int_{-\pi}^{\pi}D(u+\lambda)D(v-\lambda)\mathrm{d}\lambda=2\pi D(u+v).$$

Thus the absolute value of (4.72) is bounded by $\frac{\kappa(2\pi)^3}{\sqrt{n}}P_j^{\frac{1}{2}}P_k^{\frac{1}{2}}$.

Now since the summand of a_2 is that of b_2 with $j=k$, applying Lemma 5 we have when $d_1<d_2$,

$$a_2 = O\left(\sum_{l+1}^{r}\left\{\frac{n^{4(d_2-d_1)}}{j^{2+4(d_2-d_1)}}+\frac{n^{3(d_2-d_1)}}{j^{\frac{3}{2}+3(d_2-d_1)}}+\frac{n^{2(d_2-d_1)}-\frac{1}{2}}{j^{1+2(d_2-d_1)}}\right\}\right)$$

$$
\begin{aligned}
&= O\left(\frac{n^{4(d_2-d_1)}}{l^{1+4(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}}{l^{\frac{1}{2}+3(d_2-d_1)}} + \frac{n^{2(d_2-d_1)} - \frac{1}{2}}{l^{2(d_2-d_1)}}\right) \\
&= O(l) \\
b_2 &= O\left(\sum_{j=l+1}^{r}\sum_{k=l+1}^{r}\left\{\frac{n^{4(d_2-d_1)}}{j^{1+2(d_2-d_1)}k^{1+2(d_2-d_1)}}\right.\right. \\
&+ \left.\left.\frac{n^{3(d_2-d_1)}}{j^{1+2(d_2-d_1)}k^{\frac{1}{2}+(d_2-d_1)}} + \frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{j^{\frac{1}{2}+(d_2-d_1)}k^{\frac{1}{2}+(d_2-d_1)}}\right\}\right) \\
&= O\left(\frac{n^{4(d_2-d_1)}}{l^{4(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}\log r}{l^{2(d_2-d_1)}r^{-\frac{1}{2}+(d_2-d_1)}} + \frac{n^{3(d_2-d_1)}}{l^{-\frac{1}{2}+3(d_2-d_1)}}\right. \\
&+ \left.\frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{l^{-1+2(d_2-d_1)}} + \frac{1}{\sqrt{n}}\frac{n^{2(d_2-d_1)}}{r^{-1+2(d_2-d_1)}\log r}\right) \\
&= O(l^{\frac{3}{2}}r^{\frac{1}{2}})
\end{aligned}
$$

under **C.4** which completes the proof of the lemma. □

Chapter 5

SEMIPARAMETRIC TESTS ON ASYMMETRIC SPECTRA

5.1 LOG-PERIODOGRAM WALD TEST

The simplicity of the asymptotic distribution of the log-periodogram estimates of the memory parameters on either side of a spectral pole/zero, $\hat{d}_1^{(J)}$ and $\hat{d}_2^{(J)}$, (see Theorem 5 in Chapter 3), allows us to suggest a very simple Wald type test of the hypothesis of spectral symmetry, $d_1 = d_2$, that we will state in Theorem 11. In order to analyse the properties of this Wald test we need first to investigate the possible asymptotic dependence between $\hat{d}_1^{(J)}$ and $\hat{d}_2^{(J)}$. In the next theorem we calculate the joint distribution of the log-periodogram estimates of the parameter vector $\vartheta = (c^{(J)}, \delta^{(J)}, d_1, d_2)$ where $c^{(J)} = \log C + \psi(J)$ and $\delta^{(J)} = \log D + \psi(J)$ in the semiparametric specification of the spectral density function (1.62). We obtain asymptotic independence of $(\hat{c}^{(J)}, \hat{d}_1^{(J)})$ and $(\hat{\delta}^{(J)}, \hat{d}_2^{(J)})$ under similar assumptions to those used in Chapter 3. Since we do not know which parameter, d_1 or d_2, is larger we impose the same condition on the trimming and bandwidth numbers on both sides of ω. This condition is stated in the following assumption.

A.6: As $n \to \infty$

$$\frac{\sqrt{m}n^{2|d_1-d_2|}\log m}{l^{1+2|d_1-d_2|}} + \frac{l(\log n)^2}{m} + \frac{m^{1+\frac{1}{2\alpha}}}{n} \to 0.$$

If we take $m \sim n^{\theta}$ and $l \sim n^{\phi}$, **A.6** entails $\alpha > 2|d_1 - d_2|$. Since $|d_1 - d_2| < 1$, **A.6** can be satisfied for any d_1, d_2, if $\alpha = 2$. Had we some information about the relationship between d_2 and d_1 we could use different bandwidths and trimmings on the estimation of ($c^{(J)}$, d_1) and ($\delta^{(J)}$, d_2), so that **A.6** would only hold for the smaller d and a weaker condition would be needed for the higher parameter (namely Assumption 6 in Robinson (1995a)). Imposing **A.6** we guarantee that the results obtained in Chapter 3 for the parameters describing the behaviour of the spectral density on one side of ω hold for both sides of the spectral pole/zero.

Introduce the matrix

$$\Delta = \begin{bmatrix} \frac{\sqrt{m}}{\log n} I_2 & 0 \\ 0 & 2\sqrt{m} I_2 \end{bmatrix}.$$

Theorem 10 *Let $\vartheta = (c^{(J)}, \delta^{(J)}, d_1, d_2)'$. Under assumptions* **A.1**, **A.2***(with g=h),* **A.4** *and* **A.6**

$$\Delta(\hat{\vartheta} - \vartheta) \xrightarrow{d} N\left(0, J\psi'(J) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes I_2\right)$$

as $n \to \infty$.

Proof: Write $Y = (\bar{Y}, \tilde{Y})$, $[\bar{Y}]_k = y_k^{(J)}$, $[\tilde{Y}]_k = \tilde{y}_k^{(J)}$, where $y_k^{(J)} = \log(\sum_{j=1}^J I_n(\omega + \lambda_{k+j-J}))$ and $\tilde{y}_k^{(J)} = \log(\sum_{j=1}^J I_n(\omega - \lambda_{k+j-J}))$, $[Z]_k = (1, -2\log\lambda_k)$ and $[U]_k = (u_k^{(J)}, \tilde{u}_k^{(J)})$ where $u_k^{(J)}$ is defined in (3.4) and

$$\tilde{u}_k^{(J)} = \log\left(\sum_{j=1}^J \frac{I_n(\omega - \lambda_{k+j-J})}{D\lambda_{k+j-J}^{-2d_2}}\right) - \psi(J) \tag{5.1}$$

for $k = l + J, l + 2J, ..., m$. Then $\hat{\vartheta} = vec(Y'Z(Z'Z)^{-1})$ and

$$\hat{\vartheta} - \vartheta = vec(U'Z(Z'Z)^{-1}) = ((Z'Z)^{-1} \otimes I_2)vec(U'Z).$$

Proceeding as in Theorem 5 we get that under **A.6**, $\Delta((Z'Z)^{-1} \otimes I_2)vec(U'Z)$ is equal to

$$J^{\frac{1}{2}}\left(\frac{J}{m + o(m)}\right)^{\frac{1}{2}} \begin{bmatrix} I_2 \\ -I_2 \end{bmatrix} \sum_k (\log\lambda_k - \frac{J}{m-l}\sum_k \log\lambda_k)U_k$$
$$+ \frac{J}{m-l} \begin{bmatrix} \frac{\sqrt{m}}{\log n} I_2 \\ 0 \end{bmatrix} \sum_k U_k$$

where $U_k = (u_k^{(J)}, \tilde{u}_k^{(J)})'$. Thus the proof is completed if for any 2×1 vector $\eta \neq 0$

$$\left(\frac{J}{m}\right)^{\frac{1}{2}} \sum_k a_k \eta' U_k \xrightarrow{d} N(0, \psi'(J)\eta' I_2 \eta) \text{ as } n \to \infty,$$

for any triangular array $a_{kn} = a_k$ satisfying (3.17). Now write

$$u_k^{(J)} = \log[\sum_{j=1}^{J}(v_R^2(\omega + \lambda_{k+j-J}) + v_I^2(\omega + \lambda_{k+j-J}))e^{-\psi(J)}]$$

$$\tilde{u}_k^{(J)} = \log[\sum_{j=1}^{J}(v_R^2(\omega - \lambda_{k+j-J}) + v_I^2(\omega - \lambda_{k+j-J}))e^{-\psi(J)}]$$

for $k = l + J, l + 2J, ..., m$, where

$$v(\omega + \lambda) = \frac{W(\omega + \lambda)}{C^{\frac{1}{2}}\lambda^{-d_1}} = v_R(\omega + \lambda) + iv_I(\omega + \lambda)$$

$$v(\omega - \lambda) = \frac{W(\omega - \lambda)}{D^{\frac{1}{2}}\lambda^{-d_2}} = v_R(\omega - \lambda) + iv_I(\omega - \lambda)$$

for positive λ. Introduce the vector

$$v(\lambda) = (v_R(\omega + \lambda), v_R(\omega - \lambda), v_I(\omega + \lambda), v_I(\omega - \lambda)).$$

From Theorem 4 we can consider the $v(\lambda_j)$, for j increasing suitably slowly with n, as approximately uncorrelated (independent under **A.4**) with mean zero and covariance matrix $\frac{1}{2}I_4$. Now consider the 4-dimensional vector variates

$$V_j \sim NID(0, \frac{1}{2}I_4) \qquad j = l + 1, ..., m.$$

where $V_j = (V_{1,j}, V_{2,j}, V_{3,j}, V_{4,j})'$ and introduce the variates

$$w_k^{(J)} = \log[\sum_{j=1}^{J}(V_{1,k+j-J}^2 + V_{2,k+j-J}^2)e^{-\psi(J)}]$$

$$\tilde{w}_k^{(J)} = \log[\sum_{j=1}^{J}(V_{3,k+j-J}^2 + V_{4,k+j-J}^2)e^{-\psi(J)}]$$

for $k = l + J, l + 2J, ..., m$. Arguing as in Theorem 5, $w_k^{(J)}$ and $\tilde{w}_k^{(J)}$ have mean zero and variance $\psi'(J)$ and they are independent and independent of the rest of $w_k^{(J)}$, $\tilde{w}_k^{(J)}$ $k = l + J, l + 2J, ..., m$. Write $w_k = (w_k^{(J)}, \tilde{w}_k^{(J)})'$. Since as $n \to \infty$

$$E[\sum_k(\frac{J}{m})^{\frac{1}{2}}\eta' a_k U_k]^N = E[\sum_k(\frac{J}{m})^{\frac{1}{2}}\eta' a_k w_k]^N + o(1)$$

for fixed N, as proved in Robinson(1995a), a simple application of the Lindeberg-Feller CLT concludes the proof. □

Once we have obtained the asymptotic independence of the estimates on each side of ω we propose the following simple Wald test, where $\chi_{1,\alpha}^2$ denotes the critical value of a χ_1^2 distribution at $100\alpha\%$ significance level.

Theorem 11 *Let assumptions* **A.1**, **A.2***(with g=h),* **A.4** *and* **A.6** *hold. Then under the hypothesis* $H_0 : d_1 = d_2 + c$, *where* $c \in (-1, 1)$,

$$\hat{W} = \frac{2m}{J\psi'(J)}(\hat{d}_1^{(J)} - \hat{d}_2^{(J)} - c)^2 \xrightarrow{d} \chi_1^2 \tag{5.2}$$

as $n \to \infty$. *The test based on rejecting* H_0 *at* $100\alpha\%$ *significance level when* $\hat{W} > \chi_{1,\alpha}^2$ *is consistent.*

Proof: From Theorem 10 we have that under H_0

$$\sqrt{m}(\hat{d}_1^{(J)} - \hat{d}_2^{(J)} - c) \xrightarrow{d} N\left(0, \frac{J\psi'(J)}{2}\right) \tag{5.3}$$

and thus $\hat{W} \xrightarrow{d} \chi_1^2$. In order to prove the consistency of the test we have to show that

$$P(\hat{W} > \theta | d_1 - d_2 - c = \kappa) \to 1$$

as $m \to \infty$, for $\kappa \neq 0$ and for all $\theta > 0$. Under $H_1 : d_1 - d_2 - c = \kappa$,

$$\sqrt{m}(\hat{d}_1^{(J)} - \hat{d}_2^{(J)} - c - \kappa) \xrightarrow{d} N(0, \frac{J\psi'(J)}{2})$$

so that $\sqrt{m}(\hat{d}_1^{(J)} - \hat{d}_2^{(J)} - c) = \sqrt{m}\kappa + O_p(1)$ and $\hat{W} \xrightarrow{p} \infty$ as $m \to \infty$. □

Remark 1: A similar analysis can be done in order to test the hypothesis $H_0 : C = \rho D$. Since $c^{(J)} = \log C + \psi(J)$ and $\delta^{(J)} = \log D + \psi(J)$, this null is equivalent to the hypothesis $H_0 : c^{(J)} = \delta^{(J)} + \log \rho$. From asymptotic independence obtained in Theorem 10 it is easy to see that under H_0,

$$\hat{W}_c = \frac{m}{2J\psi'(J) \log n}(\hat{c}^{(J)} - \hat{\delta}^{(J)} - \log \rho)^2 \xrightarrow{d} \chi_1^2 \tag{5.4}$$

and the test based on rejecting H_0 at the $100\alpha\%$ significance level when $\hat{W}_c > \chi_{1,\alpha}^2$ is consistent.

Remark 2: If we are interested in testing, for instance, the hypothesis $d_1 < d_2$ (the same can be done with C and D) we can use the asymptotically normal statistic from (5.3),

$$\hat{W}_N = \sqrt{\frac{2m}{J\psi'(J)}}(\hat{d}_1^{(J)} - \hat{d}_2^{(J)}) \xrightarrow{d} N(0, 1)$$

as $m \to \infty$ under the hypothesis $d_1 = d_2$, and compare it with the critical value obtained from a standard normal distribution at $100\alpha\%$ significance level, z_α. We do not reject that $d_1 < d_2$ ($d_1 > d_2$) if $\hat{W}_N < -z_\alpha$ ($\hat{W}_N > z_\alpha$).

Remark 3: Under symmetric spectral poles, $d_1 = d_2 = d$, the estimate of the persistence parameter is $\hat{d}^{(J)} = (\hat{d}_1^{(J)} + \hat{d}_2^{(J)})/2$ and it can be used to construct the operator $(1-2L\cos\omega+L^2)^{\hat{d}^{(J)}}$ to seasonally (or cyclically) adjust series with stochastic seasonality or any other cyclical behaviour. This is a more flexible alternative than the typical fractional seasonal difference operator, $(1-L^s)^d$, or the summation operator $S(L) = 1 + L + ... + L^{s-1}$, which impose the same persistence at every seasonal frequency (as well as at the origin in $(1-L^s)^d$). The limit distributional properties of $\hat{d}^{(J)}$ are easily deduced from Theorem 10.

Remark 4: It may also be interesting to test the hypothesis $H_0 : d_1 - d_2 = \frac{1}{2}$ against $H_1 : d_1 - d_2 > \frac{1}{2}$. The rejection of the null suggests evidence of persistence on one side of ω and antipersistence on the other which seems rather unrealistic.

5.2 GAUSSIAN SEMIPARAMETRIC WALD TEST

We can also use the Gaussian semiparametric estimates, $\tilde{d}_1$ and $\tilde{d}_2$, to perform Wald type tests on the relationship between d_1 and d_2. As in the log-periodogram case, the properties of this test will depend on the asymptotic independence of the estimates on each side of the spectral pole/zero. In the next theorem we obtain the joint distribution of $(\tilde{d}_1, \tilde{d}_2)$ showing their asymptotic independence. Arguing as in the previous section we impose the same trimming and bandwidth in both estimates such that the restriction on those numbers is now,

C.4': If $d_1 \neq d_2$,

$$\frac{(\log m)^3}{l^2} + \frac{l^3}{m}(\log m)^4 + \frac{n^{2|d_1-d_2|}}{l^{1+2|d_1-d_2|}}\log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$, and if $d_1 = d_2$ then $l = 0$ and

$$\frac{1}{m} + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

If we have some knowledge about the relationship between d_1 and d_2 such that we know which one is bigger, we can use the previous trimming and bandwidth when estimating the smallest parameter but we do not need any trimming in the estimation of the largest one and the bandwidth is restricted only to the first part of **C.4'** for the smallest d and the second one for the largest. Taking $m \sim n^{\theta}$ and $l \sim n^{\phi}$ again, we see that **C.4'** can only be satisfied if $|d_1 - d_2| < \alpha/(3+4\alpha)$, where the upper bound is $2/11$ for $\alpha = 2$. However, assumption **C.4'** can be relaxed in the same way as in Chapter 4 imposing condition **C.5** which restricts the fourth cumulant of ε_t, the variates in the Wold decomposition of x_t, to be zero. Imposing this condition (which holds under Gaussianity) it is possible to find suitable m and l for a larger range of distant d_2 and d_1 as pointed out in Theorem 7. Thus if **C.5** is assumed the restriction on the bandwidth and trimming numbers is

C.6': As $n \to \infty$, if $d_2 \neq d_1$

$$\frac{(\log m)^3}{l^2} + \frac{l^2}{m}(\log m)^2 + \frac{n^{2|d_2-d_1|}}{l^{1+2|d_2-d_1|}}\log m + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

and if $d_1 = d_2$, **C.4'** holds.

For $m \sim n^{\theta}$ and $l \sim n^{\phi}$, **C.6'** entails $|d_1 - d_2| < 1/3$ for $\alpha = 2$. This requirement is not much stronger than $|d_1 - d_2| < 1/2$, which is implied by a left and right stationary spectral pole. Consider also the following condition:

C.1': For $\alpha \in (0,2]$ and $\omega \in (0,\pi)$, as $\lambda \to 0^+$,

$$\begin{aligned} f(\omega+\lambda) &= C\lambda^{-2d_1}(1+O(\lambda^{\alpha})) \\ f(\omega-\lambda) &= D\lambda^{-2d_2}(1+O(\lambda^{\alpha})) \end{aligned}$$

where $C, D \in (0,\infty)$, $d_1, d_2 \in \Theta = [\Delta_1, \Delta_2]$ and $-1/2 < \Delta_1 < \Delta_2 < 1/2$.

Theorem 12 *Let* $d = (d_1, d_2)'$ *and* $\tilde{d} = (\tilde{d}_1, \tilde{d}_2)'$. *Under assumptions* **C.1'**, **C.2**, **C.3** *and either* **C.4'** *or* **C.5** *and* **C.6'**,

$$\sqrt{m}(\tilde{d} - d) \xrightarrow{d} N(0, \frac{1}{4}I_2) \quad as \quad n \to \infty.$$

Proof: In order to show this result we follow the multivariate setup used in Theorem 5.2 in Lobato (1995). Using Taylor's series expansions

$$\sqrt{m}\begin{bmatrix} \tilde{d}_1 - d_1 \\ \tilde{d}_2 - d_2 \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}^2 R_1(\bar{d}_1)}{\mathrm{d}d^2} & 0 \\ 0 & \frac{\mathrm{d}^2 R_2(\bar{d}_2)}{\mathrm{d}d^2} \end{bmatrix}^{-1} \sqrt{m}\begin{bmatrix} \frac{\mathrm{d}R_1(d_1)}{\mathrm{d}d} \\ \frac{\mathrm{d}R_2(d_2)}{\mathrm{d}d} \end{bmatrix} = A^{-1}\sqrt{m}B$$

where $|\bar{d}_1 - d_1| \leq |\tilde{d}_1 - d_1|$, $|\bar{d}_2 - d_2| \leq |\tilde{d}_2 - d_2|$, $R_1(d)$ is $R(d)$ in (4.2) and

$$R_2(d) = \log \tilde{D}(d) - \frac{2d}{m-l}\sum_{l+1}^{m} \log \lambda_j \tag{5.5}$$

where $\tilde{D}(d) = \frac{1}{m-l}\sum_{l+1}^{m} \lambda_j^{2d}\tilde{I}_j$ and $\tilde{I}_j = I_n(\omega - \lambda_j) = |W_n(\omega - \lambda_j)|^2$.

Since

$$A \xrightarrow{p} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

(see the proof of Theorem 7) and $\tilde{d}_1$ and $\tilde{d}_2$ estimate consistently d_1 and d_2, it only remains to show that

$$\sqrt{m}B \xrightarrow{d} N(0, 4I_2)$$

as $n \to \infty$, that is for every 2×1 vector $\eta = (\eta_1, \eta_2)' \neq 0$, $\sqrt{m}\eta' B \xrightarrow{d} N(0, 4\eta_1^2 + 4\eta_2^2)$. Proceeding as in the proof of Theorem 7 we get

$$\begin{aligned} \sqrt{m}\frac{\mathrm{d}R_1(d_1)}{\mathrm{d}d} &= 2\sum_{t=2}^{n} \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s c_{t-s} + o_p(1) \\ \sqrt{m}\frac{\mathrm{d}R_2(d_2)}{\mathrm{d}d} &= 2\sum_{t=2}^{n} \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s \tilde{c}_{t-s} + o_p(1) \end{aligned}$$

where

$$\begin{aligned} c_s &= \frac{2}{n}\frac{1}{\sqrt{m}}\sum_{l+1}^{m} v_j \cos(s(\omega + \lambda_j)) \\ \tilde{c}_s &= \frac{2}{n}\frac{1}{\sqrt{m}}\sum_{l+1}^{m} v_j \cos(s(\omega - \lambda_j)) \end{aligned}$$

and $v_j = \log j - \frac{1}{m-l}\sum_{l+1}^{m} \log j$. Thus

$$\sqrt{m}\eta' B = 2\sum_{t=2}^{n} \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s [\eta_1 c_{t-s} + \eta_2 \tilde{c}_{t-s}] + o_p(1) = 2\sum_{t=1}^{n} z_t + o_p(1)$$

where $z_1 = 0$, $z_t = \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s b_{t-s}$ for $t \geq 2$ and $b_s = \eta_1 c_s + \eta_2 \tilde{c}_s$. The z_t form a zero mean martingale difference array. Then $\sqrt{m}\eta' B \xrightarrow{d} N(0, 4\eta_1^2 + 4\eta_2^2)$ if

a) $\sum_{t=1}^n E[z_t^2|F_{t-1}] - \eta_1^2 - \eta_2^2 \xrightarrow{p} 0$

b) $\sum_{t=1}^n E[z_t^2 I(|z_t| > \delta)] \to 0$ for all $\delta > 0$

where F_{t-1} is the σ-field generated by ε_s, $s < t$, and $I(\cdot)$ is here the indicator function. To prove a) write

$$\begin{aligned}
&\sum_{t=1}^{n} E[z_t^2|F_{t-1}] - \eta_1^2 - \eta_2^2 && (5.6)\\
=\ &\sum_{t=2}^{n}(\sum_{s=1}^{t-1}\varepsilon_s b_{t-s})^2 - \eta_1^2 - \eta_2^2 \\
=\ &\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{r=1}^{t-1}\varepsilon_s\varepsilon_r b_{t-s}b_{t-r} - \eta_1^2 - \eta_2^2 \\
=\ &\eta_1^2[\sum_{t=2}^{n}\sum_{s=1}^{t-1}\varepsilon_s^2 c_{t-s}^2 - 1] + \eta_1^2\sum_{t=2}^{n}\sum_{r}\sum_{s\neq r}^{t-1}\varepsilon_s\varepsilon_r c_{t-s}c_{t-r} && (5.7)\\
+\ &\eta_2^2[\sum_{t=2}^{n}\sum_{s=1}^{t-1}\varepsilon_s^2 \tilde{c}_{t-s}^2 - 1] + \eta_2^2\sum_{t=2}^{n}\sum_{r}\sum_{s\neq r}^{t-1}\varepsilon_s\varepsilon_r \tilde{c}_{t-s}\tilde{c}_{t-r} && (5.8)\\
+\ &2\eta_1\eta_2\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{r=1}^{t-1}\varepsilon_s\varepsilon_r c_{t-r}\tilde{c}_{t-s}. && (5.9)
\end{aligned}$$

Proceeding as in the proof of the asymptotic normality of $\tilde{d}_1$ in Theorem 7 we have that (5.7) and (5.8) are $o_p(1)$. Thus it remains to show that

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{r=1}^{t-1}\varepsilon_s\varepsilon_r c_{t-r}\tilde{c}_{t-s} = o_p(1). \tag{5.10}$$

The mean of the left side of the equality in (5.10) is

$$\begin{aligned}
&\sum_{t=2}^{n}\sum_{s=1}^{t-1} c_{t-s}\tilde{c}_{t-s} = \sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s\tilde{c}_s \\
=\ &\frac{4}{n^2 m}\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\sum_{j=l+1}^{m} v_j\cos(s(\omega+\lambda_j))\sum_{k=l+1}^{m} v_k\cos(s(\omega-\lambda_k)) \\
=\ &\frac{2}{n^2 m}\sum_{j=l+1}^{m}\sum_{k=l+1}^{m} v_j v_k \sum_{t=1}^{n-1}\sum_{s=1}^{n-t}[\cos(s(2\omega+\lambda_j-\lambda_k)) + \cos(s(\lambda_j+\lambda_k))] && (5.11)
\end{aligned}$$

Since from (4.40)

$$\sum_{r=1}^{q-1}\sum_{t=1}^{q-r}\cos(\theta t) = \frac{\cos\theta - \cos(q\theta)}{4\sin^2\frac{\theta}{2}} - \frac{q-1}{2}$$

then the absolute value of (5.11) is bounded by

$$\frac{2}{n^2 m}\sum_{j=l+1}^{m}\sum_{k=l+1}^{m}|v_j v_k|(O(1)+n) = O\left(\frac{m}{n}(\log m)^2\right) = o(1).$$

The variance of the left side of (5.10) is

$$E[\sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{r}^{t-1}\sum_{s}^{t-1}\sum_{p}^{u-1}\sum_{q}^{u-1}\varepsilon_r\varepsilon_s\varepsilon_p\varepsilon_q c_{t-r}\tilde{c}_{t-s}c_{u-p}\tilde{c}_{u-q}]$$

$$= \mu_4\sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{s}^{\min(t-1,u-1)} c_{t-s}\tilde{c}_{t-s}c_{u-s}\tilde{c}_{u-s} \tag{5.12}$$

$$+ \sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{r}^{t-1}\sum_{s\neq r}^{u-1} c_{t-r}\tilde{c}_{t-r}c_{u-s}\tilde{c}_{u-s} \tag{5.13}$$

$$+ \sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{s}\sum_{r\neq s}^{\min(t-1,u-1)} c_{t-r}\tilde{c}_{t-s}c_{u-r}\tilde{c}_{u-s} \tag{5.14}$$

$$+ \sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{s}\sum_{r\neq s}^{\min(t-1,u-1)} c_{t-r}\tilde{c}_{t-s}c_{u-s}\tilde{c}_{u-r}. \tag{5.15}$$

Now (5.12) is

$$\mu_4\sum_{t=2}^{n}\sum_{s=1}^{t-1} c_{t-s}^2\tilde{c}_{t-s}^2 + 2\mu_4\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=1}^{u-1} c_{t-s}\tilde{c}_{t-s}c_{u-s}\tilde{c}_{u-s}. \tag{5.16}$$

Since $|c_s|$ and $|\tilde{c}_s|$ are $O(\frac{\sqrt{m}\log m}{n})$, the first part of (5.16) is

$$O\left(\sum_{t=2}^{n}\sum_{s=1}^{t-1}\frac{m^2(\log m)^4}{n^4}\right) = O\left(\frac{m^2(\log m)^4}{n^2}\right) = o(1).$$

The second block of (5.16) is bounded in absolute value by

$$2\mu_4\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=1}^{t-1}|c_{t-s}\tilde{c}_{t-s}|\sum_{s=1}^{u-1}|c_{u-s}\tilde{c}_{u-s}|$$

$$\leq 2\mu_4(\sum_{s=1}^{n}|c_s\tilde{c}_s|)\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=t-u+1}^{t-1}|c_s\tilde{c}_s|. \tag{5.17}$$

Since $|c_r|$ and $|\tilde{c}_r|$ are $O(n^{-1}\sqrt{m}\log m)$, and for $1 \leq r \leq n/2$ they are $O(\frac{\log m}{r\sqrt{m}})$ (see the proof of Lemma 4) and $c_r = c_{n-r}$ we have that

$$\sum_{r=1}^{n}|c_r\tilde{c}_r| = O\left(\frac{n}{m}\frac{m(\log m)^2}{n^2} + \sum_{s>[\frac{n}{m}]}\frac{(\log m)^2}{s^2 m}\right) = O\left(\frac{(\log m)^2}{n}\right). \tag{5.18}$$

Now

$$\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=t-u+1}^{t-1}|c_s\tilde{c}_s|$$

$$= \sum_{j=1}^{n-2} j(n-j-1)|c_{j+1}\tilde{c}_{j+1}|$$

$$\leq 2n\sum_{1}^{[\frac{n}{2}]} j|c_{j+1}\tilde{c}_{j+1}|$$

$$= O\left(\frac{m(\log m)^2}{n}\sum_{1}^{\left[\frac{n}{m^{\frac{2}{3}}}\right]} j + n^2 \sum_{j>\left[\frac{n}{m^{\frac{2}{3}}}\right]} \frac{(\log m)^2}{ms^2}\right)$$

$$= O\left(\frac{n(\log m)^2}{m^{\frac{1}{3}}}\right).$$

Thus (5.17) is

$$O\left(\frac{(\log m)^2}{n}\frac{n(\log m)^2}{m^{\frac{1}{3}}}\right) = O\left(\frac{(\log m)^4}{m^{\frac{1}{3}}}\right) = o(1)$$

and (5.12) is $o(1)$.

Now (5.13) is equal to

$$\sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{r}^{t-1}\sum_{s\neq r}^{u-1} c_{t-r}\tilde{c}_{t-r}c_{u-s}\tilde{c}_{u-s}$$

$$= (\sum_{t=2}^{n}\sum_{s=1}^{t-1} c_{t-s}\tilde{c}_{t-s})^2 - \sum_{t=2}^{n}\sum_{u=2}^{n}\sum_{s=1}^{\min(t-1,u-1)} c_{t-s}\tilde{c}_{t-s}c_{u-s}\tilde{c}_{u-s}. \quad (5.19)$$

The term in braces is $o(1)$ as in the proof of (5.10), and the other term is (5.12) divided by the constant μ_4 and we have already proved that this is $o(1)$. Thus (5.13) is $o(1)$.

Now (5.14) is

$$\sum_{t=2}^{n}\sum_{s}\sum_{r\neq s}^{t-1} c_{t-r}^2\tilde{c}_{t-s}^2 + 2\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s}\sum_{r\neq s}^{u-1} c_{t-r}\tilde{c}_{t-s}c_{u-r}\tilde{c}_{u-s}. \quad (5.20)$$

The first part of (5.20) is bounded by

$$\sum_{t=2}^{n}\sum_{r=1}^{t-1} c_{t-r}^2 \sum_{s=1}^{t-1}\tilde{c}_{t-s}^2 \leq (\sum_{s=1}^{n}\tilde{c}_s^2)\sum_{t=2}^{n}\sum_{r=1}^{t-1} c_{t-r}^2.$$

Now $\sum_1^n \tilde{c}_s^2 = O(n^{-1}(\log m)^2)$ and from the proofs in (4.38) and (4.39) $\sum_{t=2}^{n}\sum_{r=1}^{t-1} c_{t-r}^2 = \sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s^2 = O(1)$. Thus the first part of (5.20) is $o(1)$. The second block of (5.20) is bounded in absolute value by

$$2\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum\sum_{r\neq s}^{u-1} |c_{t-r}\tilde{c}_{t-s}c_{u-r}\tilde{c}_{u-s}|$$

$$\leq 2\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{r=1}^{u-1} |c_{t-r}c_{u-r}| \sum_{s=1}^{u-1} |\tilde{c}_{u-s}\tilde{c}_{t-s}|$$

$$= O\left(\frac{\sqrt{m}\log m}{n}\sum_{s=1}^{n}|\tilde{c}_s|\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{r=1}^{u-1}|c_{t-r}c_{u-r}|\right). \quad (5.21)$$

Now

$$\begin{aligned}\sum_{s=1}^{n}|\tilde{c}_s| &= O\left(\sum_{s=1}^{\frac{n\log n}{m}}\frac{\sqrt{m}\log m}{n}+\sum_{s>\frac{n\log n}{m}}\frac{\log m}{s\sqrt{m}}\right)\\ &= O\left(\frac{\log n\log m}{\sqrt{m}}\right)\end{aligned}$$

and

$$\begin{aligned}&\sum_{t=3}^{u}\sum_{u=2}^{t-1}\sum_{r=1}^{u-1}|c_{t-r}c_{u-r}|\\ &= \sum_{t=1}^{n-2}\sum_{u=t+1}^{n-1}(n-u)|c_tc_u|\\ &\le n\sum_{u=1}^{n}|c_u|\sum_{t=1}^{n}|c_t|\\ &= O\left(\frac{n(\log n)^2(\log m)^2}{m}\right).\end{aligned}$$

Thus (5.21) is $O(\frac{(\log m)^4(\log n)^3}{m})$ which is $o(1)$ if for example $m\sim Cn^{\alpha}$ as $n\to\infty$ for $C\in(0,\infty)$ and $0<\alpha<1$, and we get that (5.14) is $o(1)$.

Finally (5.15) is equal to

$$\sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1}c_{t-r}\tilde{c}_{t-s}c_{t-s}\tilde{c}_{t-r} \tag{5.22}$$

$$+\ 2\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum\sum_{r\neq s}^{u-1}c_{t-r}\tilde{c}_{t-s}c_{u-s}\tilde{c}_{u-r}. \tag{5.23}$$

Now (5.22) is bounded in absolute value by

$$\begin{aligned}&\sum_{t=2}^{n}\sum_{r=1}^{t-1}|c_{t-r}\tilde{c}_{t-r}|(\sum_{s=1}^{n}|c_s\tilde{c}_s|)\\ &= O\left(n^2\frac{m(\log m)^2}{n^2}\frac{(\log m)^2}{n}\right)=O\left(\frac{m}{n}(\log m)^4\right)=o(1)\end{aligned}$$

using (5.18) and because $|c_s|$ and $|\tilde{c}_s|$ are $O(n^{-1}\sqrt{m}\log m)$. The absolute value of (5.23) is bounded by

$$\begin{aligned}&2\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{r=1}^{u-1}|c_{t-r}\tilde{c}_{u-r}|\sum_{s=1}^{u-1}|c_{u-s}\tilde{c}_{t-s}| \qquad (5.24)\\ &= O\left(\frac{\sqrt{m}\log m}{n}\sum_{s=1}^{n}|c_s|\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{r=1}^{t-1}|c_{t-r}\tilde{c}_{u-r}|\right).\end{aligned}$$

Now

$$\sum_{s=1}^{n} |c_s| = O\left(\frac{\log n \log m}{\sqrt{m}}\right)$$

and

$$\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{r=1}^{u-1} |c_{t-r}\tilde{c}_{u-r}| = \sum_{t=1}^{n-2}\sum_{u=t+1}^{n-1} (n-u)|\tilde{c}_t c_u|$$

$$\leq n\sum_{u=1}^{n}|c_u|\sum_{t=1}^{n}|\tilde{c}_t| = O\left(\frac{n}{m}(\log m)^2(\log n)^2\right).$$

Thus (5.24) is $O(m^{-1}(\log n)^3(\log m)^4) = o(1)$, (5.15) is $o(1)$ and a) is proved. In order to prove the Lindeberg condition stated in b) we show that the Liapounov condition, $\sum_1^m E|z_t|^4 \to 0$, holds, which is sufficient to prove b). Write

$$\begin{aligned}
&\sum_1^n E[z_t^4] \\
= \; & \sum_{t=2}^{n} E[\varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s b_{t-s}]^4 \\
= \; & \mu_4 \sum_{t=2}^{n} E[\sum_r\sum_s\sum_p\sum_q^{t-1} \varepsilon_r\varepsilon_s\varepsilon_p\varepsilon_q b_{t-r}b_{t-s}b_{t-p}b_{t-q}] \\
= \; & \mu_4^2 \sum_{t=2}^{n}\sum_{r=1}^{t-1} b_{t-r}^4 + 3\mu_4\mu_2^2 \sum_{t=2}^{n}\sum\sum_{r\neq s}^{t-1} b_{t-r}^2 b_{t-s}^2 \\
= \; & O(n(\sum_{t=1}^{n} b_t^2)^2).
\end{aligned} \qquad (5.25)$$

Since $b_t^2 = \eta_1^2 c_t^2 + \eta_2^2 \tilde{c}_t^2 + 2\eta_1\eta_2 c_t \tilde{c}_t$ then

$$\sum_{t=1}^{n} b_t^2 = O\left(\frac{(\log m)^2}{n}\right)$$

and (5.25) is $O(n^{-1}(\log m)^4) = o(1)$ which concludes the proof. □

Perhaps the most interesting situation we can test is the hypothesis of spectral symmetry, $d_1 = d_2$. In this case no trimming is needed to obtain the asymptotic distribution under the null. However, to prove the consistency of the test we need the consistency of the estimates under the alternative and that condition requires trimming out some frequencies close to ω. We introduce now the following condition on the bandwidth and trimming numbers.

D.4: If $d_1 \neq d_2$,

$$\frac{l}{m}\log m + \frac{n^{2|d_2-d_1|}}{l^{1+2|d_2-d_1|}}(\log m)^3 + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$, and if $d_1 = d_2$ then $l = 0$ and

$$\frac{1}{m} + \frac{m^{1+2\alpha}}{n^{2\alpha}}(\log m)^2 \to 0$$

as $n \to \infty$.

If we consider $m \sim n^\theta$ and $l \sim n^\phi$ we have that **D.4** entails $|d_1 - d_2| < \alpha$, so that **D.4** holds if $\alpha \geq 1$ for any $d_1, d_2 \in \Theta$. Based on the asymptotic independence of $\tilde{d}_1$ and $\tilde{d}_2$ we propose the following simple Wald type test.

Theorem 13 *Let assumptions* **C.1', C.2, C.3** *and* **D.4** *hold. Under the hypothesis* $H_0 : d_1 - d_2 = 0$,

$$\tilde{W} = 2m(\tilde{d}_1 - \tilde{d}_2)^2 \xrightarrow{d} \chi_1^2$$

as $n \to \infty$ *and the test based on rejecting* H_0 *at 100α% significance level whenever* $\tilde{W} > \chi^2_{1,\alpha}$ *is a consistent test.*

Proof: The asymptotic distribution is easily deduced from the asymptotic independence of $\tilde{d}_1$ and $\tilde{d}_2$ obtained in the previous theorem. Note that no trimming of frequencies close to ω is needed to obtain this result because under the null $d_1 = d_2$. But in order to prove the consistency we have to show that

$$\sqrt{m}(\tilde{d}_1 - \tilde{d}_2) \xrightarrow{p} \pm\infty \tag{5.26}$$

under the hypothesis $H_1 : d_1 - d_2 = \theta \neq 0$. In these circumstances

$$\begin{aligned}\sqrt{m}(\tilde{d}_1 - \tilde{d}_2) &= \sqrt{m}(\tilde{d}_1 - d_1) + \sqrt{m}(\tilde{d}_2 - d_2) + \sqrt{m}\theta \\ &= o_p(\sqrt{m}) + o_p(\sqrt{m}) + \sqrt{m}\theta\end{aligned}$$

because of consistency of $\tilde{d}_1$ and $\tilde{d}_2$ under the conditions in the theorem (see Theorem 6) and thus trimming out l frequencies close to ω. Then (5.26) and the consistency of the test are proved. □

Remark 1: As in the log-periodogram Wald test we can use the one-tailed test $\sqrt{2m}(\tilde{d}_1 - \tilde{d}_2)$, which has a standard normal limit distribution under the null, for the hypotheses $d_1 > d_2$ or $d_2 > d_1$.

Remark 2: In view of the local character of $\tilde{d}_1$ and $\tilde{d}_2$ (the same remark follows for $\hat{d}_1^{(J)}$ and $\hat{d}_2^{(J)}$) we can similarly estimate the right and left memory parameters at each of several known spectral poles/zeros, ω_j, as permitted in the modelling of Chapter 2. It is clear from Theorem 12 that the asymptotic properties of the left and right d estimates will not vary across the ω_j, and moreover the estimates will be asymptotically independent across the ω_j so that we can readily construct statistics for testing hypotheses across the ω_j, for example of equality of all right or left memory parameters. In the interests of parsimony this would be a useful preliminary to parametric modelling.

Remark 3: Since $J\psi'(J) > 1$ it seems by comparison with Theorem 11 that $\tilde{d}_1 - \tilde{d}_2$ produces a locally more powerful test of spectral symmetry than $\hat{d}_1^{(J)} - \hat{d}_2^{(J)}$ for any J. The finite sample performance of both tests will be analysed in Chapter 6. However $\tilde{d}_1$, $\tilde{d}_2$, unlike $\hat{d}_1^{(J)}$, $\hat{d}_2^{(J)}$, are not defined in closed form. It is possible to alleviate this problem by means of a score test which entails only estimation of a single parameter under the null hypothesis. This procedure is described in the next section.

5.3 GAUSSIAN SEMIPARAMETRIC LM TEST

In this section we consider a score or Lagrange Multiplier type test of the hypothesis of spectral symmetry, $d_1 = d_2$. Unlike the Wald tests, this procedure only requires one estimation of $d_1 = d_2$, using frequencies on both sides of ω. Consider the following objective function, which is a semiparametric discrete version of the Whittle approximate likelihood function,

$$Q(C,D,d_1,d_2)=\frac{1}{2(m-l)}\left[\sum_{j=l+1}^{m}\left\{\log C\lambda_j^{-2d_1}+\frac{\lambda_j^{2d_1}}{C}I_j\right\}+\sum_{j=l+1}^{m}\left\{\log D\lambda_j^{-2d_2}+\frac{\lambda_j^{2d_2}}{D}\tilde{I}_j\right\}\right] \tag{5.27}$$

where I_j and $\tilde{I}_j$ are the periodogram ordinates of x_t, $t = 1, ..., n$, at frequencies $\omega + \lambda_j$ and $\omega - \lambda_j$ respectively. If C, D, d_1 and d_2 are functionally unrelated, minimization of Q implies using frequencies just after ω in the estimation of C and d_1 and those before ω for D and d_2. Also if we assume $d_1 = d_2 = d$ we can estimate d using frequencies on both sides of ω without assuming equality of C and D. Concentrating C and D out of the objective function we have that minimizing $Q(C, D, d_1, d_2)$ is equivalent to minimizing

$$R(d_1, d_2) = \frac{1}{2}\log \tilde{C}(d_1) + \frac{1}{2}\log \tilde{D}(d_2) - \frac{d_1}{m-l}\sum_{l+1}^{m}\log \lambda_j - \frac{d_2}{m-l}\sum_{l+1}^{m}\log \lambda_j$$

where

$$\tilde{C}(d) = \frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2d} I_j \quad (5.28)$$

$$\tilde{D}(d) = \frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2d} \tilde{I}_j. \quad (5.29)$$

We restrict our analysis to the persistent case, that is $d_1, d_2 \in \Theta$ where $\Theta = [\Delta_1, \Delta_2]$ and $0 < \Delta_1 < \Delta_2 < 1/2$. If we assume wrongly that $d_1 = d_2 = d$ and we estimate d using frequencies on both sides of ω we will obtain a value, $\tilde{d}$, which consistently estimates some value, d_0, that will be different from d_1 and d_2. Intuitively d_0 will be between d_1 and d_2 and closer to the highest one due to the larger influence of periodogram ordinates at frequencies where the highest parameter define the behaviour of the spectral density. In order to prove this fact we modify assumption **B.1** (imposed for the consistency of $\tilde{d}_1$ in Theorem 6) in the following manner.

B.1': For $\alpha \in (0, 2]$ and $\omega \in (0, \pi)$, as $\lambda \to 0^+$

$$f(\omega + \lambda) = C\lambda^{-2d_1}(1 + O(\lambda^\alpha))$$

$$f(\omega - \lambda) = D\lambda^{-2d_2}(1 + O(\lambda^\alpha))$$

where $C, D \in (0, \infty)$, $d_1, d_2 \in \Theta = [\Delta_1, \Delta_2]$ and $0 < \Delta_1 < \Delta_2 < 1/2$.

We also need the following condition on the bandwidth and trimming numbers.

B.4': If $d_1 \neq d_2$,

$$\frac{m}{n} + \frac{l}{m}\log m + \frac{n^{|d_2-d_1|}}{l^{\frac{1}{2}+|d_2-d_1|}}(\log m)^{\frac{3}{2}} \to 0 \quad \text{as } n \to \infty$$

and if $d_1 = d_2$ then $l = 0$ and

$$\frac{1}{m} + \frac{m}{n} \to 0 \quad \text{as } n \to \infty.$$

Assumption **B.1'** is a restriction of **B.1** in the sense that we only focus on the persistent case when both d_1 and d_2 are positive. **B.4'** is similar to assumption **B.4** but we take into account that we are using frequencies on both sides of ω. In the worst case, when $|d_1 - d_2|$ approaches 1/2, the last summand in **B.4'** is bounded by $\frac{n^{0.5}}{l}(\log m)^{\frac{3}{2}}$ which goes to zero if, for example, $l = n^{\alpha}$ and $\alpha > 1/2$ so that we can always find some m and l such that **B.4'** holds.

Theorem 14 *Let $\tilde{d} = \arg\min_{\Theta} R(d, d)$. Under* **B.1'**, **B.2**, **B.3** *and* **B.4'**

$$\tilde{d} \xrightarrow{p} d_0 = \frac{1}{4}[2d_1 + 2d_2 - 1 + \sqrt{4(d_1 - d_2)^2 + 1}]$$

as $n \to \infty$.

Note that only if $d_1 = d_2 = d$, $\tilde{d}$ estimates the memory parameter consistently, in any other case d_0 is between d_1 and d_2.

Proof: The proof is quite similar to that of Theorem 6, therefore it will be presented in a more abbreviated manner. Write $S(d) = R(d, d) - R(d_0, d_0) = U(d) - T(d)$ where $U(d)$ is the deterministic part of $S(d)$ and $T(d)$ is the remainder,

$$\begin{aligned}
U(d) &= 2(d - d_0) + \frac{1}{2}\log[2(d_0 - d_1) + 1] + \frac{1}{2}\log[2(d_0 - d_2) + 1] \\
&\quad -\frac{1}{2}\log[2(d - d_1) + 1] - \frac{1}{2}\log[2(d - d_2) + 1] \\
T(d) &= \frac{1}{2}\log\left(\frac{\tilde{C}(d_0)}{C(d_0)}\right) + \frac{1}{2}\log\left(\frac{\tilde{D}(d_0)}{D(d_0)}\right) && (5.30) \\
&- \frac{1}{2}\log\left(\frac{\tilde{C}(d)}{C(d)}\right) - \frac{1}{2}\log\left(\frac{\tilde{D}(d)}{D(d)}\right) && (5.31) \\
&- \frac{1}{2}\log\left\{\frac{1}{m-l}\sum_{l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d-d_1)}\{2(d - d_1) + 1\}\right\} && (5.32) \\
&+ \frac{1}{2}\log\left\{\frac{1}{m-l}\sum_{l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d_0-d_1)}\{2(d_0 - d_1) + 1\}\right\} && (5.33) \\
&- \frac{1}{2}\log\left\{\frac{1}{m-l}\sum_{l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d-d_2)}\{2(d - d_2) + 1\}\right\} && (5.34) \\
&+ \frac{1}{2}\log\left\{\frac{1}{m-l}\sum_{l+1}^{m}\left(\frac{j}{m-l}\right)^{2(d_0-d_2)}\{2(d_0 - d_2) + 1\}\right\} && (5.35) \\
&+ 2(d - d_0)\left\{\frac{1}{m-l}\sum_{l+1}^{m}\log j - \log(m - l) + 1\right\} && (5.36)
\end{aligned}$$

where $\tilde{C}(d)$ and $\tilde{D}(d)$ are (5.28) and (5.29), $C(d)$ is defined in (4.4) and

$$D(d) = D\frac{1}{m-l}\sum_{l+1}^{m}\lambda_j^{2(d-d_2)}. \tag{5.37}$$

Note that d_0 is the unique minimum of $U(d)$ in Θ because the other local minimum $0.25[2(d_1+d_2)-1-\sqrt{4(d_1-d_2)^2+1}]$ does not belong to Θ if $d_1, d_2 \in \Theta$. Thus $U(d_0) = \min_\Theta U(d) = 0$ and since $U(d)$ is a convex function for all d (the second derivative is positive for all d) we have that

$$\inf_{|d-d_0|>\delta} U(d) \geq \eta > 0$$

for all $\delta > 0$ and some $\eta > 0$. Thus it remains to show that $\sup_\Theta |T(d)| \xrightarrow{p} 0$. Since we focus on positive values of d_1 and d_2 we have that $(d_0-d_i) > -1/2$ and $(d-d_i) > -1/2$ on Θ for $i = 1, 2$. Consequently the supremum on Θ of the absolute value of (5.32), (5.33), (5.34) and (5.35) are $o(1)$ due to Lemma 2. Applying Lemma 3 we also see that $\sup_\Theta |(5.36)| = o(1)$. Then it remains to show that

$$\sup_\Theta \left|\frac{\tilde{C}(d)-C(d)}{C(d)}\right| = o_p(1) \tag{5.38}$$

$$\sup_\Theta \left|\frac{\tilde{D}(d)-D(d)}{D(d)}\right| = o_p(1). \tag{5.39}$$

Since (5.38) has been shown when proving Theorem 6, and (5.39) can be demonstrated similarly the proof is concluded. □

Note that although we are using frequencies on both sides of ω we still need to trim out some points close to ω to get rid of the bad behaviour of the periodogram evaluated at those frequencies. This trimming seems necessary to obtain the value d_0 defined in the theorem but it seems that if we do not use the trimming, the estimate $\tilde{d}$ will converge to a value different from d_1 and d_2 and even closer to the highest one than the trimmed $\tilde{d}$.

A similar result is obtained for the log-periodogram regression

$$\log(\sum_{j=1}^{J} I_n(\omega+\lambda_{k+j+J})) = \alpha^{(J)} + d(-2\log|\lambda_k|) + \bar{u}_k$$

for $k = \pm(l+J), \pm(l+2J), ..., \pm m$ where $\bar{u}_k = \tilde{u}_k$ in (5.1) if $k < 0$ and $\bar{u}_k = u_k$ in (3.4) if $k > 0$. The least squares estimates of α and d are

$$\begin{bmatrix} \hat{\alpha}^{(J)} \\ \hat{d}^{(J)} \end{bmatrix} = \frac{1}{2}(Z'Z)^{-1}[Z'Y + Z'\tilde{Y}] = \frac{1}{2}\begin{bmatrix} \hat{c}^{(J)} \\ \hat{d}_1^{(J)} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \hat{\delta}^{(J)} \\ \hat{d}_2^{(J)} \end{bmatrix}$$

where Z, Y and $\tilde{Y}$ are defined in the proof of Theorem 10. Then under the conditions in Theorem 10

$$\hat{d} \xrightarrow{p} \frac{d_1 + d_2}{2}.$$

The result obtained in Theorem 14 will be useful when proving the consistency of the score tests we propose in the next theorem. But prior to stating the theorem we modify the notation to facilitate the understanding of the test procedure. Write $\theta = d_1 - d_2$. Then

$$R(\theta, d_2) = \frac{1}{2}\log \tilde{C}(\theta + d_2) + \frac{1}{2}\log \tilde{D}(d_2) - \frac{\theta}{m-l}\sum_{l+1}^{m}\log\lambda_j - \frac{2d_2}{m-l}\sum_{l+1}^{m}\log\lambda_j. \quad (5.40)$$

The hypothesis $d_1 = d_2$ in $R(d_1, d_2)$ is equivalent to $\theta = 0$ in $R(\theta, d_2)$ and under this hypothesis, minimizing $R(0, d_2)$ we obtain an estimate of $d_1 = d_2$ using frequencies on both sides of ω. Call d_{10}, d_{20} and θ_0 the true unknown parameters and d_1, d_2 and θ any admissible value. Now

$$\begin{aligned} \frac{\partial R(\theta, d_2)}{\partial \theta} &= G(\theta + d_2) \\ \frac{\partial R(\theta, d_2)}{\partial d_2} &= G(\theta + d_2) + H(d_2) \end{aligned}$$

where

$$G(\theta + d_2) = \frac{\tilde{C}_1(\theta + d_2)}{\tilde{C}_0(\theta + d_2)} - \frac{1}{m-l}\sum_{l+1}^{m}\log\lambda_j \quad (5.41)$$

$$H(d_2) = \frac{\tilde{D}_1(d_2)}{\tilde{D}_0(d_2)} - \frac{1}{m-l}\sum_{l+1}^{m}\log\lambda_j. \quad (5.42)$$

and

$$\tilde{C}_k(d) = \frac{1}{m-l}\sum_{l+1}^{m}(\log\lambda_j)^k \lambda_j^{2d} I_j \quad (5.43)$$

$$\tilde{D}_k(d) = \frac{1}{m-l}\sum_{l+1}^{m}(\log\lambda_j)^k \lambda_j^{2d} \tilde{I}_j. \quad (5.44)$$

Now

$$\frac{\partial^2 R(\theta,d_2)}{\partial\theta^2} = \frac{\partial^2 R(\theta,d_2)}{\partial\theta\partial d_2} = \frac{\partial^2 R(\theta,d_2)}{\partial d_2\partial\theta} = \frac{2[\tilde{C}_2(\theta+d_2)\tilde{C}(\theta+d_2) - \tilde{C}_1^2(\theta+d_2)]}{\tilde{C}_0^2(\theta+d_2)}$$

$$\frac{\partial^2 R(\theta,d_2)}{\partial d_2^2} = \frac{\partial^2 R(\theta,d_2)}{\partial\theta^2} + \frac{2[\tilde{D}_2(d_2)\tilde{D}(d_2) - \tilde{D}_1^2(d_2)]}{\tilde{D}_0^2(d_2)}.$$

Write $\psi_0 = [\theta_0, d_{20}]$ the vector of true parameters where $\theta_0 = d_{10} - d_{20}$. Since $\sqrt{m}G(\theta_0 + d_{20})$ and $\sqrt{m}H(d_{20})$ are asymptotically uncorrelated, as has been proved in Theorem 12, then under **C.1'**, **C.2**, **C.3** and **C.4'**

$$\sqrt{m}\begin{bmatrix} \frac{\partial R}{\partial\theta}\Big|_{\psi_0} \\ \frac{\partial R}{\partial d_2}\Big|_{\psi_0} \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right).$$

Since $d_1 = \theta + d_2$ we know that under **C.1'**, **C.2**,**C.3** and **C.4'**

$$\frac{\partial^2 R}{\partial\psi^2}\Big|_{(\bar{\theta},\bar{d}_2)} \xrightarrow{p} \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

for $|\bar{\theta} - \theta_0| \leq |\tilde{\theta} - \theta_0|$ and $|\bar{d}_2 - d_{20}| \leq |\tilde{d}_2 - d_{20}|$. Thus the estimates of θ_0 and d_{20} obtained minimizing $R(\theta, d_2)$ are asymptotically distributed as

$$\sqrt{m}\begin{bmatrix} \tilde{\theta} - \theta_0 \\ \tilde{d}_2 - d_{20} \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}\right) \tag{5.45}$$

as $n \to \infty$ and under assumptions **C.1'**, **C.2**, **C.3** and **C.4'**. That is what we would expect because $\theta = d_1 - d_2$ and the estimates of d_1 and d_2 are asymptotically independent.

Now we propose some score tests of the hypothesis of symmetric spectral poles ($H_0 : d_1 = d_2$) in the environment described above. The advantage of these tests with respect to the Wald type one is that only one estimation around ω is needed. In order to prove the asymptotic distribution and consistency of the tests, condition **D.4** on the trimming and bandwidth has to be assumed. This is a weaker condition than that needed in Theorem 7 for the asymptotic normality of $\tilde{d}_1$. As a matter of fact, no trimming is needed in order to prove the asymptotic distribution of the test under the null. However the consistency is obtained trimming out l frequencies close to ω because we use the result stated in Theorem 14.

Three different statistics, depending on the alternative hypothesis we use, are proposed in order to test the null of spectral symmetry, $d_1 = d_2$:

$$\begin{array}{lll} LM_1 = \sqrt{2m}\tilde{e}_1 & \text{if the alternative is} & H_1 : d_1 > d_2 \\ LM_2 = \sqrt{2m}\tilde{e}_2 & \text{if the alternative is} & H_2 : d_1 < d_2 \\ LM_3 = 2m\tilde{e}_3^2 & \text{if the alternative is} & H_3 : d_1 \neq d_2 \end{array} \tag{5.46}$$

where $\tilde{e}_1 = \frac{L_1}{L_0}$, $\tilde{e}_2 = \frac{M_1}{M_0}$, $\tilde{e}_3 = \frac{N_1}{N_0}$, $L_k = \frac{1}{m}\sum_1^m v_j^k \lambda_j^{2\tilde{d}} I_j$, $M_k = \frac{1}{m}\sum_1^m v_j^k \lambda_j^{2\tilde{d}} \tilde{I}_j$, $N_k = \frac{1}{m-l}\sum_{l+1}^m w_j^k \lambda_j^{2\tilde{d}} I_j$, $v_j = \log j - \frac{1}{m}\sum_1^m \log k$, $w_j = \log j - \frac{1}{m-l}\sum_{l+1}^m \log k$, $I_j = I_n(\omega + \lambda_j)$, $\tilde{I}_j = I_n(\omega - \lambda_j)$ and $\tilde{d}$ is the estimate of $d_1 = d_2 = d$ under the null, that is using frequencies on both sides of ω but trimming out the l nearest frequencies.

Theorem 15 *Under* **B.1', C.2, C.3** *and* **D.4** *and the hypothesis* $H_0 : d_1 = d_2$*,*

$$LM_1 \xrightarrow{d} N(0,1) \quad LM_2 \xrightarrow{d} N(0,1) \quad LM_3 \xrightarrow{d} \chi_1^2 \quad \text{as } m \to \infty.$$

The tests based on rejecting the null in favour of the respective alternatives whenever $LM_1 < -z_\alpha$, $LM_2 < -z_\alpha$ *or* $LM_3 > \chi^2_{1\alpha}$ *at* $100\alpha\%$ *significance level (where* z_α *and* $\chi^2_{1\alpha}$ *are the corresponding critical values from a standard normal and a chi-square with one degree of freedom) are consistent.*

Proof: Note that

$$\tilde{e}_1 = \left.\frac{\partial R(\theta, d_2)}{\partial \theta}\right|_{(0,\tilde{d})}$$

where $R(\theta, d_2)$ is now (5.40) with $l = 0$ and $\tilde{d}$ is the estimate of $d_1 = d_2 = d_0$ trimming out l frequencies close to ω. Since $\tilde{d} \xrightarrow{p} d_0$ we have that under the null

$$\tilde{e}_1 = \left.\frac{\partial R}{\partial \theta}\right|_{(0,d_0)} + \left.\frac{\partial^2 R}{\partial \theta \partial d_2}\right|_{(0,\bar{d})} (\tilde{d} - d_0)$$

$$0 = \left.\frac{\partial R}{\partial d_2}\right|_{(0,\tilde{d})} = \left.\frac{\partial R}{\partial d_2}\right|_{(0,d_0)} + \left.\frac{\partial^2 R}{\partial d_2^2}\right|_{(0,\bar{d})} (\tilde{d} - d_0)$$

for $|\bar{d} - d_0| \leq |\tilde{d} - d_0|$ so that $\sqrt{m}\tilde{e}_1$ is

$$\left[1 \quad - \left.\frac{\partial^2 R}{\partial \theta \partial d_2}\right|_{(0,\bar{d})} \left(\left.\frac{\partial^2 R}{\partial d_2^2}\right|_{(0,\bar{d})} \right)^{-1} \right] \sqrt{m} \left[\begin{array}{c} \left.\frac{\partial R}{\partial \theta}\right|_{(0,d_0)} \\ \left.\frac{\partial R}{\partial d_2}\right|_{(0,d_0)} \end{array} \right]$$

$$\xrightarrow{d} N\left[0, (1 \quad -\frac{1}{2}) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}\right]$$

$$= N(0, \frac{1}{2})$$

under **B.1'**, **C.2**, **C.3**, **D.4** and the hypothesis $H_0 : d_1 = d_2$. This result is proved in the same way as Theorem 7 in Chapter 4.

A similar proof can be done for LM_2 noting that

$$\tilde{e}_2 = \left.\frac{\partial R(d_1,\theta_1)}{\partial \theta_1}\right|_{(\tilde{d},0)}$$

where $\theta_1 = d_2 - d_1$, and

$$R(d_1,\theta_1) = \frac{1}{2}\log \tilde{C}(d_1) + \frac{1}{2}\log \tilde{D}(\theta_1 + d_1) - \frac{\theta_1}{m}\sum_1^m \log \lambda_j - \frac{2d_1}{m}\sum_1^m \log \lambda_j$$

where $\tilde{C}(d)$ and $\tilde{D}(d)$ are defined as in (5.28) and (5.29) with $l = 0$.

Note that $\tilde{e}_3$ is $\tilde{e}_1$ but introducing the trimming so that $\sqrt{2m}\tilde{e}_3 \xrightarrow{d} N(0,1)$ and $LM_3 \xrightarrow{d} \chi_1^2$ under the null.

The proof of the consistency is based on the following relations based on Theorem 4,

$$E\left|\frac{I_j}{g_j}\right| = 1 + O\left(\frac{\log j}{j} + \left(\frac{j}{n}\right)^{\alpha}\right) \quad \text{if } d_1 \geq d_2 \tag{5.47}$$

$$E\left|\frac{I_j}{g_j}\right| = 1 + O\left(\left(\frac{j}{n}\right)^{\alpha} + \frac{n^{2(d_2-d_1)}\log j}{j^{1+2(d_2-d_1)}}\right) \quad \text{if } d_1 < d_2 \tag{5.48}$$

$$E\left|\frac{\tilde{I}_j}{h_j}\right| = 1 + O\left(\frac{\log j}{j} + \left(\frac{j}{n}\right)^{\alpha}\right) \quad \text{if } d_1 \leq d_2 \tag{5.49}$$

$$E\left|\frac{\tilde{I}_j}{h_j}\right| = 1 + O\left(\left(\frac{j}{n}\right)^{\alpha} + \frac{n^{2(d_1-d_2)}\log j}{j^{1+2(d_1-d_2)}}\right) \quad \text{if } d_1 > d_2 \tag{5.50}$$

under **B.1'** and **C.2** where $g_j = C\lambda_j^{-2d_1}$ and $h_j = D\lambda_j^{-2d_2}$. Although only the results corresponding to $I_j = I_n(\omega + \lambda_j)$ are rigorously proved in Theorem 4, the properties concerning $\tilde{I}_j = I_n(\omega - \lambda_j)$ can be similarly deduced. Note also that as $m \to \infty$,

$$\sum_1^m \log j \sim m[\log m - 1] \tag{5.51}$$

$$\sum_1^m j^{\alpha} \sim \frac{m^{1+\alpha}}{1+\alpha} \tag{5.52}$$

$$\sum_1^m (\log j) j^{\alpha} \sim \left(\log m - \frac{1}{1+\alpha}\right)\frac{m^{\alpha+1}}{1+\alpha} \tag{5.53}$$

$$\sum_{l+1}^m \log j \sim m[\log m - 1] - l[\log l - 1] \tag{5.54}$$

$$\sum_{l+1}^m j^{\alpha} \sim \frac{m^{1+\alpha}}{1+\alpha} - \frac{l^{1+\alpha}}{1+\alpha} \tag{5.55}$$

$$\sum_{l+1}^{m}(\log j)j^{\alpha} \sim \left(\log m - \frac{1}{1+\alpha}\right)\frac{m^{\alpha+1}}{1+\alpha} - \left(\log l - \frac{1}{1+\alpha}\right)\frac{l^{\alpha+1}}{1+\alpha} \qquad (5.56)$$

for $\alpha > -1$. From Theorem 14 we know that $\tilde{d} \xrightarrow{p} d_0$ so that

$$L_1 \overset{p}{\sim} \frac{1}{m}\sum_{1}^{m} v_j \lambda_j^{2d_0} I_j = \frac{C}{m}\sum_{1}^{m} v_j \lambda_j^{2(d_0-d_1)}\frac{I_j}{g_j} \qquad (5.57)$$

where $a \overset{p}{\sim} b$ means that $\frac{a}{b} \xrightarrow{p} 1$. From the proof of Theorem 6

$$\frac{I_j}{g_j} = 1 + \left(1 - \frac{g_j}{f_j}\right)\frac{I_j}{g_j} + \frac{1}{f_j}[I_j - |\alpha_j|^2 I_{\varepsilon j}] + (2\pi I_{\varepsilon j} - 1). \qquad (5.58)$$

Now

$$\begin{aligned} & E\left\{\frac{C}{m}\sum_{j=1}^{m} v_j \lambda_j^{2(d_0-d_1)}\left|1 - \frac{g_j}{f_j}\right|\left|\frac{I_j}{g_j}\right|\right\} \\ = & \; O\left(\frac{\log m}{m}\sum_{j=1}^{m}\left(\frac{j}{n}\right)^{2(d_0-d_1)}\left(\frac{j}{n}\right)^{\alpha}\left(1 + \left(\frac{j}{n}\right)^{\alpha} + \frac{\log j}{j}\right)\right) \qquad (5.59) \end{aligned}$$

from (4.11) and (5.47) if $d_1 \geq d_2$. Then (5.59) is

$$\begin{aligned} & O\left(\frac{\log m}{mn^{2(d_0-d_1)+\alpha}}(m^{2(d_0-d_1)+\alpha+1})\right) \\ = & \; O(\lambda_m^{2(d_0-d_1)+\alpha}\log m) \qquad (5.60) \end{aligned}$$

because $2(d_0 - d_1) > -1$ and $\alpha > 0$. Since $E|I_j - |\alpha_j|^2 I_{\varepsilon j}| = O(f_j(\log j/j)^{\frac{1}{2}})$ for $d_1 \geq d_2$ we have that

$$\begin{aligned} & \frac{C}{m}\sum_{j=1}^{m} v_j \lambda_j^{2(d_0-d_1)}\frac{1}{f_j}[I_j - |\alpha_j|^2 I_{\varepsilon j}] \\ = & \; O_p\left(\frac{\log m}{m}\sum_{j=1}^{m}\lambda_j^{2(d_0-d_1)}\left(\frac{\log j}{j}\right)^{\frac{1}{2}}\right) \\ = & \; O_p\left(\frac{(\log m)^{\frac{3}{2}}}{mn^{2(d_0-d_1)}}\sum_{1}^{m} j^{2(d_0-d_1)-\frac{1}{2}}\right) \\ = & \; O_p\left(\frac{(\log m)^{\frac{5}{2}}}{\sqrt{m}}\lambda_m^{2(d_0-d_1)} + \frac{(\log m)^{\frac{3}{2}}}{m^{1+2(d_0-d_1)}}\lambda_m^{2(d_0-d_1)}\right). \qquad (5.61) \end{aligned}$$

Now

$$\begin{aligned} & \frac{C}{m}\sum_{j=1}^{m} v_j \lambda_j^{2(d_0-d_1)}(2\pi I_{\varepsilon j} - 1) \\ = & \; \frac{C}{m}\sum_{j=1}^{m} v_j \lambda_j^{2(d_0-d_1)}\frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t^2 - 1) \qquad (5.62) \\ + & \; \frac{C}{m}\sum_{j=1}^{m} v_j \lambda_j^{2(d_0-d_1)}\frac{1}{n}\sum_{s}\sum_{t\neq s}\cos\{(t-s)(\omega + \lambda_j)\}\varepsilon_t\varepsilon_s. \qquad (5.63) \end{aligned}$$

Since under **C.3**, $\frac{1}{n}\sum_1^n(\varepsilon_t^2) \xrightarrow{p} 1$ then (5.62) is

$$o_p\left(\frac{C}{m}\sum_1^m v_j\lambda_j^{2(d_0-d_1)}\right). \tag{5.64}$$

Note that $E[(5.63)^2]$ is

$$\frac{2C^2}{m^2n^2}\sum_{j=1}^m\sum_{k=1}^m v_jv_k\lambda_j^{2(d_0-d_1)}\lambda_k^{2(d_0-d_1)}\sum_s^n\sum_{t\neq s}\cos\{(t-s)(\omega+\lambda_j)\}\cos\{(t-s)(\omega+\lambda_k)\}$$
$$= \frac{2C^2}{m^2n^2}\sum_1^m\sum_1^m v_jv_k\lambda_j^{2(d_0-d_1)}\lambda_k^{2(d_0-d_1)}[\sum_{t=1}^n\sum_{s=1}^n(a_{ts}+b_{ts}+c_{ts}+d_{ts})-n] \tag{5.65}$$

where a_{ts}, b_{ts}, c_{ts} and d_{ts} are defined in (4.21). Proceeding as in the proof of Theorem 6, (5.65) is

$$\frac{C^2}{m^2n^2}\sum_{j=1}^m v_j^2\lambda_j^{4(d_0-d_1)}n^2 - \frac{2C^2}{m^2n}(\sum_{j=1}^m v_j\lambda_j^{2(d_0-d_1)})^2$$
$$= O\left(\frac{(\log m)^2}{m^2n^{4(d_0-d_1)}}(m^{1+4(d_0-d_1)}\log m+1)+\frac{(\log m)^2}{m^2n^{1+4(d_0-d_1)}}m^{2+4(d_0-d_1)}\right)$$
$$= O\left(\frac{(\log m)^3}{m}\lambda_m^{4(d_0-d_1)}+\frac{(\log m)^2}{m^{2+4(d_0-d_1)}}\lambda_m^{4(d_0-d_1)}+\frac{(\log m)^2}{n}\lambda_m^{4(d_0-d_1)}\right)$$

so that (5.63) is

$$O_p\left(\lambda_m^{2(d_0-d_1)}\left(\frac{(\log m)^{\frac{3}{2}}}{\sqrt{m}}+\frac{\log m}{m^{1+2(d_0-d_1)}}+\frac{\log m}{\sqrt{n}}\right)\right). \tag{5.66}$$

Now using (5.51), (5.52) and (5.53) ,

$$\frac{C}{m}\sum_{j=1}^m v_j\lambda_j^{2(d_0-d_1)} \sim C\lambda_m^{2(d_0-d_1)}\frac{2(d_0-d_1)}{(1+2(d_0-d_1))^2} \tag{5.67}$$

as $n\to\infty$ so that noting (5.57), (5.58) and the orders of magnitude in (5.60), (5.61), (5.64) and (5.66) we conclude from (5.67)

$$L_1 \overset{p}{\sim} C\lambda_m^{2(d_0-d_1)}\frac{2(d_0-d_1)}{(1+2(d_0-d_1))^2} \quad \text{as } n\to\infty.$$

Similarly

$$L_0 \overset{p}{\sim} C\lambda_m^{2(d_0-d_1)}\frac{1}{1+2(d_0-d_1)} \quad \text{as } n\to\infty$$

so that

$$\tilde{e}_1 \overset{p}{\sim} \frac{2(d_0-d_1)}{1+2(d_0-d_1)} \quad \text{as } n\to\infty \quad \text{if } d_1\geq d_2$$

and $\sqrt{m}\tilde{e}_1 \xrightarrow{p} -\infty$ unless $d_1 = d_0$ which only happens if $d_1 = d_2$.

In the same way, and noting (5.49), we can show that

$$\tilde{e}_2 \overset{p}{\sim} \frac{2(d_0 - d_2)}{1 + 2(d_0 - d_2)} \quad \text{as } n \to \infty \quad \text{if } d_2 \geq d_1$$

and $\sqrt{m}\tilde{e}_2 \overset{p}{\to} -\infty$ unless $d_1 = d_2$.

Finally, noting (5.47), (5.48) and (5.54), (5.55), (5.56), assumption **D.4** and the fact that in $\tilde{e}_3$ we trim out the lowest l frequencies, we can show, using the same type of calculations, that

$$\tilde{e}_3 \overset{p}{\sim} \frac{2(d_0 - d_1)}{1 + 2(d_0 - d_1)} \quad \text{as } n \to \infty$$

so that $2m\tilde{e}_3^2 \overset{p}{\to} \infty$ unless $d_1 = d_0$ which only happens when $d_1 = d_2$. □

Remark 1: The consistency of the previous score tests has been obtained only for the persistent case, i.e. $d_1, d_2 \in \Theta = [\Delta_1, \Delta_2]$ where $0 < \Delta_1 < \Delta_2 < 1/2$. This is so in order to use the consistency of $\tilde{d}$ obtained in Theorem 14. However, the same result is achieved as long as $d_1, d_2 \in \Theta = [\Delta_1, \Delta_2]$ and $\Delta_2 - \Delta_1 < 1/2$ so that the antipersistent case and a mixture of persistence and antipersistence is also covered by the previous test procedures.

Remark 2: The statistic $\sqrt{m}\tilde{e}_3$ provides also a consistent test of the hypotheses $d_1 > d_2$ or $d_1 < d_2$. However $\tilde{e}_3$ implies a trimming of l points which reduces the number of observations used in the construction of the statistic. That is why we propose the untrimmed LM_1 and LM_2 to perform the one-tailed tests. Furthermore it is not clear how important the trimming is in finite samples, or if it is only a theoretical device needed to obtain the asymptotic properties derived above. Note also that in the three statistics we use the trimmed estimation of d under the null. This is so in order to use the convergence of this estimate proved in Theorem 14. The effect of the trimming on the performance of the different test procedures in finite samples will be analysed via a small Monte Carlo study in Chapter 6.

5.4 TESTING EQUALITY ACROSS FREQUENCIES

The procedures described in previous sections focus on testing the hypothesis of spectral symmetry at one known frequency ω. Thus, only spectral behaviour around that frequency is considered. However there may be several spectral poles/zeros at different frequencies. For example many economic time series are likely to have spectral poles at every seasonal frequency as well as at the origin as described in Chapters 1 and 2 (see for example UK monthly inflation in Chapter 7). Thus it is interesting to investigate the possible equality of persistence parameters at different frequencies, trying to find a parsimonious and reliable model.

Throughout this section we impose symmetry of every spectral pole in the sense that the spectral density function satisfies the following condition:

E.1 : For $\alpha \in (0,2]$,

$$f(\omega_i \pm \lambda) = C_i \lambda^{-2d_i}(1 + O(\lambda^\alpha)) \quad \text{as} \quad \lambda \to 0^+$$

where $\omega_i \in [0,\pi]$, $C_i \in (0,\infty)$, $d_i \in [\Delta_1, \Delta_2]$ and $0 < \Delta_1 < \Delta_2 < 1/2$ for $i = 0, ..., H$.

The hypothesis we want to test is the equality of the persistence parameters at frequencies $0 \leq \omega_0 < \omega_1 < ... < \omega_H \leq \pi$, $d_0 = d_1 = ... = d_H$. Note that $f(\lambda)$ can have other spectral poles/zeros in addition to those in **E.1**. Taking into account the asymptotic independence of the log-periodogram or Gaussian semiparametric estimates at different frequencies, Wald tests can be easily constructed. An application of these tests to UK monthly inflation is performed in Chapter 7. The main inconvenient of Wald tests is that we need to perform $H + 1$ estimations. Based on the Gaussian semiparametric or local Whittle procedure we suggest a score test that only requires one estimation using frequencies around all ω_i, $i = 0, 1, ..., H$. As in Section 5.3 we first propose the following objective function to obtain local Whittle estimates of $C_0, ..., C_H$, and $d_0, ..., d_H$,

$$Q(C_0, ..., C_H, d_0, ..., d_H) = \frac{1}{m2H} \sum_{i=0}^{H} \sum_{j=1}^{m} \left\{ \delta_i \log C_i \lambda_j^{-2d_i} + \frac{\lambda_j^{2d_i}}{C_i} F_{ij} \right\} \qquad (5.68)$$

where $\delta_i = 1$ if $\omega_i = 0, \pi$, and $\delta_i = 2$ otherwise, $F_{ij} = I_n(\lambda_j)$ if $\omega_i = 0$, $F_{ij} = I_n(\pi - \lambda_j)$ if $\omega_i = \pi$ and $F_{ij} = I_n(\omega_i + \lambda_j) + I_n(\omega_i - \lambda_j)$ if $\omega_i \in (0, \pi)$. Since we are interested in inference on $d_0, ..., d_H$, we concentrate $C_0, ..., C_H$, out of (5.68) so that the estimates of d_i, $i = 0, ..., H$, are obtained by minimizing

$$R(d_0, ..., d_H) = \sum_{i=0}^{H} \delta_i \left\{ \log \tilde{C}_i^0(d_i) - \frac{2d_i}{m} \sum_{j=1}^{m} \log \lambda_j \right\} \tag{5.69}$$

where now

$$\tilde{C}_i^k(d_i) = \frac{1}{\delta_i m} \sum_{j=1}^{m} \lambda_j^{2d_i} F_{ij} (\log \lambda_j)^k.$$

The hypothesis we want to test is the equality of $H+1$ persistence parameters, H_0 : $d_0 = d_1 = ... = d_H$, against the alternative that at least one of the equalities does not hold. Therefore there are H restrictions.

Since we are assuming symmetric spectral poles we do not trim out any frequency close to ω_i in the estimation of d_i. Thus we only need to impose Assumption A4' in Robinson (1995b) that we rewrite here.

E.4: As $n \to \infty$

$$\frac{1}{m} + \frac{m^{1+2\alpha}(\log m)^2}{n^{2\alpha}} \to 0.$$

This bandwidth is enough to guarantee the properties of the score test procedure we describe in the following theorem.

Theorem 16 *Let* **E.1**, **C.2**, **C.3** *and* **E.4** *hold. Then under the hypothesis* $H_0 : d_0 = d_1 = ... = d_H$,

$$LM_H = m\tilde{e}' A^{-1} \tilde{e} \xrightarrow{d} \chi_H^2 \quad as \quad m \to \infty$$

where $\tilde{e}$ *is a* $H \times 1$ *vector with i-th element* $[\tilde{e}]_i = 2\delta_i \frac{T_i^1(\tilde{d}_0)}{T_i^0(\tilde{d}_0)}$, $T_i^k(d) = \sum_1^m v_j^k \lambda_j^{2d} F_{ij}$, $\tilde{d}_0$ *is the joint estimate under the null (using frequencies around* $\omega_0, ..., \omega_H$*),* $v_j = \log j - \frac{1}{m} \sum_1^m \log l$ *and* A *is a* $H \times H$ *matrix with elements*

$$\begin{aligned} A_{ii} &= \delta_i^2 \varphi_i - \frac{\delta_i^3 \varphi_i}{\sum_{j=0}^{H} \delta_j} \qquad i = 1, ..., H \\ A_{ij} &= -\frac{\delta_i^2 \delta_j \varphi_i}{\sum_{j=0}^{H} \delta_j} \qquad j, i = 1, ..., H \quad i \neq j \end{aligned}$$

where $\varphi_i = 4$ *if* $\omega_i = 0, \pi$ *and* $\varphi_i = 2$ *otherwise. The test based on rejecting the null against the alternative that at least one of the equalities does not hold if* $LM_H > \chi_{H,\alpha}^2$ *at* $100\alpha\%$ *significance level is consistent.*

Proof: Call $\theta_i = d_i - d_0$, $i = 0, ..., H$, (note that instead of d_0 we can take any other d_i as reference parameter) such that $\theta_0 = 0$. We are going to test $H_0 : d_0 = d_1 = ... = d_H$ by testing $H_0 : \theta_1 = ... = \theta_H = 0$. In order to do this define

$$R(\theta_1, ..., \theta_H, d_0) = \sum_{i=0}^{H} \delta_i \left\{ \log \tilde{C}_i^0(\theta_i + d_0) - \frac{2(\theta_i + d_0)}{m} \sum_{j=1}^{m} \log \lambda_j \right\}.$$

Thus

$$\begin{aligned} \frac{\partial R(\theta_1, ..., \theta_H, d_0)}{\partial d_0} &= 2 \sum_{i=0}^{H} \delta_i G_i(\theta_i + d_0) \\ \frac{\partial R(\theta_1, ..., \theta_H, d_0)}{\partial \theta_i} &= 2\delta_i G_i(\theta_i + d_0) \qquad i = 1, ..., H, \end{aligned}$$

where

$$G_i(z) = \frac{\tilde{C}_i^1(z)}{\tilde{C}_i^0(z)} - \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j.$$

Let θ_i^o, d_0^o be the actual parameters and θ_i, d_0 any admissible value. Proceeding as in the proof of Theorem 12 we get that the different $G_i(\theta_i^o + d_0^o)$ are asymptotically independent and

$$\begin{aligned} \sqrt{m} \frac{\partial R(\theta_1^o, ..., \theta_H^o, d_0^o)}{\partial d_0} &\xrightarrow{d} N(0, \sum_{i=0}^{H} \delta_i^2 \varphi_i) \\ \sqrt{m} \frac{\partial R(\theta_1^o, ..., \theta_H^o, d_0^o)}{\partial \theta_i} &\xrightarrow{d} N(0, \delta_i^2 \varphi_i) \quad i = 1, 2, ..., H. \end{aligned}$$

The second derivatives are

$$\begin{aligned} \frac{\partial^2 R}{\partial \theta_i \partial \theta_j} &= 0 \qquad j \neq i \\ \frac{\partial^2 R}{\partial d_0^2} &= 4 \sum_{i=0}^{H} \delta_i G_i^1(\theta_i + d_0) \\ \frac{\partial^2 R}{\partial d_0 \partial \theta_i} &= \frac{\partial^2 R}{\partial \theta_i^2} = 4\delta_i G_i^1(\theta_i + d_0) \end{aligned}$$

where

$$G_i^1(z) = \frac{\tilde{C}_i^2(z)\tilde{C}_i^0(z) - (\tilde{C}_i^1(z))^2}{(\tilde{C}_i^0(z))^2}.$$

We have already shown in the proof of Theorem 7 in Chapter 4 that under the null, $G_i^1(0 + \bar{d}_0) \xrightarrow{p} 1$ for $|\bar{d}_0 - d_0| \leq |\tilde{d}_0 - d_0|$ and $\tilde{d}_0$ a consistent estimate of d_0. Call

$\theta = (\theta_1, ..., \theta_H)'$. Thus under $H_0 : \theta = 0$,

$$\begin{aligned} \sqrt{m}\tilde{e} &= \sqrt{m}\frac{\partial R}{\partial \theta}\bigg|_{(\bar{d}_0,0)} \\ &= \sqrt{m}\frac{\partial R}{\partial \theta}\bigg|_{(d_0^o,0)} - \frac{\partial^2 R}{\partial \theta \partial d_0}\bigg|_{(\bar{d}_0,0)} \left(\frac{\partial^2 R}{\partial d_0^2}\bigg|_{(\bar{d}_0,0)} \right)^{-1} \sqrt{m}\frac{\partial R}{\partial d_0}\bigg|_{(d_0^o,0)} \\ &\xrightarrow{d} N(0, A) \end{aligned}$$

which proves the asymptotic distribution under the null. The consistency of the test procedure can be shown in the same way as Theorem 15 for $H = 1$ using a similar result to Theorem 14. We conjecture that the same result follows for $H > 1$. □

Remark : In this section we assume spectral symmetry at the frequencies ω_i, $i = 0, 1, ..., H$. Of course this symmetry can be relaxed modifying the objective function (5.68) in the same manner as we did in the previous section. Thus we can perform similar tests of the equality of all or some right and/or left memory parameters across different frequencies.

Chapter 6

SMALL SAMPLE BEHAVIOUR

6.1 SIMULATION PROCEDURE

In this chapter we study via Monte Carlo analysis the performance of the different methods of estimation and test procedures proposed in Chapters 3, 4 and 5. In order to do that we generate a process with a spectral density (2.3) in Chapter 2 in the following manner. Let $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$ be two independent Gaussian processes with zero mean and lag-j autocovariances

$$\begin{aligned} \gamma_j^1 &= \sigma_1^2 \left(\delta_{j0} - \frac{\sin(j\omega)}{\pi j} \right), \\ \gamma_j^2 &= \sigma_2^2 \frac{\sin(j\omega)}{\pi j}, \end{aligned}$$

respectively, where $\delta_{j0} = 1$ if $j = 0$ and 0 otherwise. Now let $x_{1,t}$ and $x_{2,t}$ be formed as

$$(1 - 2L\cos\omega + L^2)^{d_k} x_{k,t} = \varepsilon_{k,t} \quad , \quad k = 1,2 \quad , \quad t = 0, \pm 1, \pm 2 ... \tag{6.1}$$

and call $x_t = x_{1,t} + x_{2,t}$. Since the spectral density of $\{\varepsilon_{k,t}\}$ is

$$f_k(\lambda) = \frac{\gamma_0^k}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{\infty} \gamma_j^k \cos(j\lambda)$$

then using formula 1.441.1 in Gradshteyn and Ryzhik (1980), namely

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \qquad 0 < x < 2\pi,$$

and the fact that $2\sin(j\omega)\cos(j\lambda) = \sin(j(\omega - \lambda)) + \sin(j(\omega + \lambda))$, we have that the spectral density of $\{x_t\}$ is

$$f(\lambda) = \begin{cases} \frac{\sigma_1^2}{2\pi} (4(\cos\omega - \cos\lambda)^2)^{-d_1} & \text{if} \quad \omega < \lambda \leq \pi \\ \frac{\sigma_2^2}{2\pi} (4(\cos\omega - \cos\lambda)^2)^{-d_2} & \text{if} \quad 0 \leq \lambda \leq \omega \end{cases} . \tag{6.2}$$

The filter in (6.1) implies an infinite sum of the form

$$\sum_{s=0}^{\infty} C_s^{(d_k)}(\cos\omega)x_{k,t-s} = \varepsilon_{k,t} \tag{6.3}$$

where the Gegenbauer polynomials, $C_s^{(d)}(\eta)$, are of the form

$$C_s^{(d)}(\eta) = \sum_{j=0}^{[s/2]} \frac{(-1)^j \Gamma(s-j-d)(2\eta)^{s-2j}}{\Gamma(j+1)\Gamma(s-2j+1)\Gamma(-d)}$$

where $[s/2]$ is the integer part of $s/2$ (see Gray et al.(1989)). We truncate the sum in (6.3) so that the series generated are

$$x_{k,t} = -\sum_{s=1}^{1500} C_s^{(d_k)}(\cos\omega)x_{k,t-s} + \varepsilon_{k,t} \quad , \quad k = 1, 2,$$

where we put $x_{k,t} = 0$ for $t \leq 0$, and the Gegenbauer functions are obtained via the recursion

$$C_s^{(d)}(\eta) = 2\eta\left(\frac{-d+s-1}{s}\right)C_{s-1}^{(d)}(\eta) - \left(\frac{-2d+s-2}{s}\right)C_{s-2}^{(d)}(\eta)$$

(see formula 8.933.1 in Gradshteyn and Ryzhik (1980)). This method permits the approximate generation of Gegenbauer processes with an asymmetric spectral pole/zero at any frequency between 0 and π. A more direct generation procedure, without the truncation used above, is the application of some algorithm (e.g. Davies and Harte (1987)) to the autocovariances obtained in Proposition 1 in Chapter 2 for $\omega = \pi/2$. However this method is only valid for that specific frequency. That is why we use the more general procedure described earlier despite the truncation it implies. Furthermore, comparison of the exact and approximate procedures on the basis of actual and sample autocovariances plots indicated little difference in performance.

In the Monte Carlo study reported, $\omega = \pi/2, \sigma_1^2 = \sigma_2^2 = 1$ and $d_1, d_2 \in \{-0.4, -0.2, 0, 0.2, 0.4\}$. Note that the processes generated satisfy **C.1(A.1)** with $\alpha = 2$, **C.2(A.2)**, **C.3** and **C.5** since ε_{1t} and ε_{2t} are Gaussian. In order to obtain log-periodogram estimates, the explicit formula from the least squares method of estimation is used and we consider only the case $J = 1$. Gaussian semiparametric estimates are obtained by applying a simple golden section search to the first derivative of the objective function (4.2). The minimization is carried out over the closed set $\Theta = [-0.499, 0.499]$.

The chosen sample sizes are $n = 64, 128, 256$ and 512 and for each three different bandwidths are tried, $m = n/16, n/8$ and $n/4$. The effect of the trimming number is only analysed for $n = 128, 256$ and 512 and three different trimmings are used, $l = n/128, n/64$ and $n/32$. The number of replications was 1000 and all the calculations were done using GAUSS-386i VM version 3.2.8.

6.2 LOG-PERIODOGRAM AND GAUSSIAN SEMIPARAMETRIC ESTIMATION

The results we present in this section are bias and mean square error (MSE) of the two methods of estimation described in Chapters 3 and 4 (for an analysis of the effects of short memory components on the bias of the log-periodogram estimate at zero frequency see Agiakloglou et al. (1993)). We only consider the estimation of d_1, that of d_2 is equivalent and only differs in the utilization of periodogram ordinates situated just before the frequency where the spectral pole or zero occurs. In all the tables the number within parentheses correspond to the log-periodogram estimate, $\hat{d}_1$.

6.2.1 Bias

The bias of the untrimmed estimates of d_1 for different n and m is described in tables 6.1-6.4. We can observe that the bias of $\tilde{d}_1$ tends to decrease from $m = n/16$ to $m = n/8$ and to increase thereafter. The tendency of $\hat{d}_1$ is of a greater increase with all m. However when d_2 is quite large with respect to d_1 the bias of both estimates tends to decrease with m. This is what we would expect because the more frequencies we use in the estimation the less important the influence of periodogram ordinates close to ω that are "contaminated" by d_2.

The bias tends to be positive for negative values of d_1 and when $d_1 < d_2$ and negative for positive d_1 and when $d_1 > d_2$, although a positive bias is more pervasive in the log-periodogram method of estimation. We also observe that when d_2 is higher than d_1 the bias of the estimates increases with n for fixed m. These facts can be explained noting the results obtained in Theorem 3 and that the spectral density (6.2) is of the form (2.14) where $g_1(\lambda)$ and $g_2(\lambda)$ are constants $\sigma_1^2/2\pi$ and $\sigma_2^2/2\pi$. According to this

$$\log I_n(\omega + \lambda_j) = \log f_1^*(\omega + \lambda_j) - 2d_1 \log|\lambda_j| + \log\left(\frac{I_n(\omega + \lambda_j)}{f(\omega + \lambda_j)}\right) \qquad (6.4)$$

where $f_1^*(z)$ can be deduced from (2.14), (2.15) and (6.2). We have from Theorem 3 that $E[I_n(\omega + \lambda_j)/f(\omega + \lambda_j)]$ decreases as j increases for n sufficiently large (see also Hurvich and Beltrao (1993)). Then it is plausible that $E[\log(I_n(\omega + \lambda_j)/f(\omega + \lambda_j))]$ decreases as j increases producing a positive bias in $\hat{d}_1$. Furthermore, since the relative bias of $I_n(\omega + \lambda_j)$ as estimate of $f(\omega + \lambda_j)$ increases with n when $d_1 < d_2$, then this fact can explain the increase of the bias of $\hat{d}_1$ that we observe for fixed m (for instance when $m = 16$) and an increasing n ($n = 64, 128, 256$), in tables 6.1-6.3, when d_2 is higher than d_1. The same behaviour occurs for $\tilde{d}_1$ and the cause of it is also likely to be the increase of the relative bias of the periodogram, although an intuitive explanation like that for $\hat{d}_1$ can not be applied here.

In case $d_1 > d_2$ the bias is much smaller than when $d_1 < d_2$. This is so due to the influence of periodogram ordinates just before ω on those ordinates just after (see Theorem 4) where the behaviour of the spectral density function is governed by d_1.

When we introduce the trimming we observe in tables 6.5-6.13 that the bias reduces in those cases where the difference $d_2 - d_1$ is positive and large. In the rest of the cases the bias tends to increase, mainly when the number of frequencies used in $\tilde{d}_1$ is small. However the bias of $\hat{d}_1$ reduces in some cases when m is small, even if $d_2 < d_1$.

We also observe that when $d_2 > d_1$ the decrease of the bias due to the trimming is more important for $\hat{d}_1$ than for $\tilde{d}_1$. As a matter of fact the bias of $\tilde{d}_1$ finally increases with a large trimming, whereas that of $\hat{d}_1$ tends to decrease for d_2 large with respect to d_1. For instance when $d_1 = -0.4$ and $d_2 = 0.4$ the bias of $\hat{d}_1$ decreases with the trimming l in all situations except the case $m = n/4$ when it increases from $l = n/64$ to $l = n/32$, whereas that of $\tilde{d}_1$ has a clearer increasing tendency when l passes from $n/64$ to $n/32$.

6.2.2 Mean Square Error

We observe in tables 6.14-6.21 that the mean square error (MSE) of both untrimmed estimates decreases with m and n. It also tends to increase with the difference $d_2 - d_1$ when this is positive. When we introduce the trimming we use fewer frequencies for the same bandwidth, m, and the MSE in both estimates tends to increase. Only when d_2 is quite large with respect to d_1 the MSE decreases. This behaviour can be seen in tables 6.16-6.21 for $n = 256$ and $n = 512$.

We also report the efficiency of the Gaussian semiparametric estimate, $\tilde{d}_1$, with respect to the log-periodogram one, $\hat{d}_1$. The entries in tables 6.22-6.29 are to be compared with the asymptotic relative efficiency 0.608 obtained from the asymptotic distributions in Theorems 5 and 7. When $d_1 \geq d_2$ the ratios of MSEs tend to that figure from below as m and n increase. However when $d_2 > d_1$ the ratio tends to be higher than 0.608 for $n \geq 128$ and is higher than one when $d_1 = -0.4$, $d_2 = 0.4$ and $n = 512$, $m = 64, 128$. This fact reflects a greater sensitivity of the MSE of $\tilde{d}_1$ to the difference $d_2 - d_1$, which is in accordance with the stronger trimming we needed to obtain the asymptotic distribution of $\tilde{d}_1$. When the trimming is introduced we observe that the efficiency is always below one and decreases with l. Only the cases $n = 128$ and $n = 256$ are reported in tables 6.23-6.28. The behaviour of the efficiency when $n = 512$ is similar and can be deduced from the MSE in tables 6.19-6.21.

6.3 TESTS ON THE SYMMETRY OF THE SPECTRUM

6.3.1 Symmetry at the same frequency

In tables 6.30-6.33 we present a small Monte Carlo study of the Wald tests of the hypothesis of spectral symmetry at $\pi/2$ introduced in Chapter 5. The significance level is 5%. The test statistics are calculated through the estimates obtained in the previous section. Only the untrimmed test statistics are included. The trimmed versions perform quite worse than the untrimmed ones with a higher size in all cases and not a higher power so that the type I error increases and the type II error does not tend to decrease with the trimming.

The entries in the tables are size (along the NW-SE diagonal) and power obtained with 1000 replications. The numbers within parentheses correspond to the log-periodogram test and the other figures are size and power of the Gaussian semiparametric Wald test. As expected, in both test procedures the power tends to increase and the size to decrease with m and n. The Gaussian Wald test performs better in the sense that the power tends to be higher and the size lower than the test based on log-periodogram estimates. In both cases the size is higher than 0.05 (the nominal size) and tends to that number as m and n increase.

The behaviour of the different score tests of the hypothesis of asymptotic spectral symmetry at $\pi/2$ is described in tables 6.34-6.51 for $n = 128, 256, 512$, $m = n/4, n/8, n/16$ and $l = n/128, n/64, n/32$. The trimming concerns only the estimation of the persistence parameter under the null, i.e. using frequencies on both sides of $\pi/2$. Although the consistency of the tests is only rigorously proved in Chapter 5 when $|d_2 - d_1| < 1/2$, we present the results also for the cases $|d_2 - d_1| \geq 1/2$, and we see that the good properties of these tests are likely to hold also for those cases. We only report power and size for the LM_2 and LM_3 tests at 5% significance level, those of LM_1 are similar to the powers and sizes of LM_2 for the corresponding null and alternative. The trimming in the LM_3 test statistic concerns only the estimation of d so that the statistic used is that presented in Theorem 15 in Chapter 5 with $l = 0$. The reason for this is that performance of the tests was found to worsen with the exclusion of frequencies close to ω.

We observe that powers and sizes tend to increase with d_1 and d_2, being higher for positive values of both parameters than for negative ones, reflecting a more conservative behaviour of the tests under antipersistence than under persistence. The size increases with the trimming applied to the joint estimation under the null. The power of LM_2 tends to increase with l, mainly when the difference $d_2 - d_1$ is not very large and/or when m is small. The power behaviour of the LM_3 test is similar with respect to the positive difference $d_1 - d_2$. However, when $d_2 > d_1$ the power of LM_3 tends to decrease for $m > 16$ with the introduction of the trimming. This behaviour can be explained from the expression obtained in the proof of Theorem 15

$$\tilde{e}_3 \overset{p}{\sim} \frac{2(d_0 - d_1)}{1 + 2(d_0 - d_1)} \quad \text{as } m \to \infty.$$

In an intuitive manner we can argue that the joint estimate, $\tilde{d}$, will estimate a value, d_0, which is closer to the highest d (d_2 in this case), the smaller the trimming. Then the difference $d_0 - d_1$ will decrease with the introduction of the trimming and the LM_3 statistic will be lower reducing its power in finite samples. The same type of intuitive explanation can be applied for the increase of power with the trimming in the LM_2 test since

$$\tilde{e}_2 \overset{p}{\sim} \frac{2(d_0 - d_2)}{1 + 2(d_0 - d_2)} \quad \text{as } m \to \infty$$

and under the alternative $d_2 > d_1$, $\tilde{d}$ will estimate some value closer to d_2 the smaller the trimming.

If we compare the results for the score and Wald tests when no trimming is used we observe that the score tests tend to be more conservative, with generally lower sizes and powers than the two Wald type tests analysed here. However, as n and m increase, although the LM sizes remain lower than those corresponding to the Wald procedures, the powers tend to be similar and in some cases higher than those of the Wald tests.

6.3.2 Equality across frequencies

In this section we analyse the performance in finite samples of the LM_H test of the equality of persistence parameters across different frequencies introduced in Section 5.4. In order to do this we generate a Gaussian process with symmetric spectral poles/zeros at 0 and $\pi/2$ by adding two independent Gaussian Gegenbauer processes generated using the truncation described in Section 6.1. For each spectral singularity five different persistence parameters are used, $-0.4, -0.2, 0, 0.2, 0.4$, corresponding to antipersistence $(-0.4, -0.2)$, short memory (0) and long memory or persistence $(0.2, 0.4)$. Three sample sizes are analysed, $n = 128, 256, 512$, and for each of them three bandwidths are used, $m = n/16, n/8, 3n/16$. We only use until $m = 3n/16$ in order to avoid the use of frequencies close to a spectral pole/zero different to those used in the construction of the test statistic as well as the twofold use of the same frequency in the estimation of the persistence parameter under the null. The number of replications is 1000.

The null hypothesis to test is $H_0 : d_0 = d_1$, where d_0 and d_1 are the persistence parameters at the origin and $\pi/2$ respectively. From Theorem 16 in Chapter 5 the test statistic is

$$LM_H = \frac{3m}{8}\tilde{e}^2$$

where $\tilde{e} = 4T_1^1(\tilde{d})/T_1^0(\tilde{d})$, $T_1^k(d) = \sum_1^m v_j^k \lambda_j^{2d}[I_n(\frac{\pi}{2}+\lambda_j) + I_n(\frac{\pi}{2}-\lambda_j)]$, $v_j = \log j - \frac{1}{m}\sum_1^m \log l$ and $\tilde{d}$ is the estimate under the null, i.e. using frequencies around 0 and $\pi/2$.

Tables 6.52-6.54 show powers and sizes of the LM_H test for the different sample sizes and bandwidths. Power and size increase with m and n and tend to be higher under persistence than under antipersistence. We also observe a large size for the extreme cases $d_0 = d_1 = -0.4, 0.4$, and a larger power for $d_0 > d_1$ than for $d_0 < d_1$. This latter fact occurs because we use d_0 as reference parameter in the test procedure. Thus we construct the statistic LM_H using frequencies around $\pi/2$. This implies that if equality does not hold

$$\tilde{e} \overset{p}{\sim} 4\frac{2(d^* - d_1)}{1 + 2(d^* - d_1)} = 4 - \frac{8}{2(d_0 - d_1) + 1 + \sqrt{4(d_1 - d_0)^2 + 1}}, \tag{6.5}$$

as $n \to \infty$, which can be shown in the same way as the proof of the consistency of the score tests of spectral symmetry at one known frequency in Section 5.3, and using a similar result to Theorem 14 concerning the convergence in probability of the joint estimate to d^*. Thus (6.5) is larger when $d_0 > d_1$ than when $d_1 > d_0$, even if the distance between d_0 and d_1 is the same. This fact is reflected in higher power of the test against $d_0 > d_1$ than against $d_0 < d_1$. The opposite occurs when we use d_1 as a reference parameter and construct the LM_H statistic using frequencies around 0. In this case

$$\tilde{e} \overset{p}{\sim} 4\frac{2(d^* - d_0)}{1 + 2(d^* - d_0)} = 4 - \frac{8}{2(d_1 - d_0) + 1 + \sqrt{4(d_1 - d_0)^2 + 1}},$$

as $n \to \infty$, and this LM_H test is more powerful against $d_1 > d_0$ than against $d_1 < d_0$ (we do not report results for this case due to the similarity with the tables reported).

6.4 TABLES

6.4.1 Bias

Table 6.1: Bias of the Gaussian (log-periodogram) estimates of d_1, n=64

			$m = 4$		
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.1561 (0.0535)	0.1597 (0.0601)	0.1707 (0.0802)	0.1978 (0.1288)	0.2573 (0.2305)
-0.2	0.0469 (0.0162)	0.0482 (0.0201)	0.0544 (0.0336)	0.0727 (0.0655)	0.1137 (0.1269)
0	-0.0427 (-0.0040)	-0.0412 (0.0030)	-0.0378 (0.0136)	-0.0283 (0.0336)	-0.0018 (0.0686)
0.2	-0.1171 (-0.0058)	-0.1168 (-0.0012)	-0.1155 (0.0045)	-0.1129 (0.0158)	-0.1030 (0.0292)
0.4	-0.1912 (0.0071)	-0.1919 (0.0079)	-0.1932 (0.0086)	-0.1949 (0.0086)	-0.1961 (0.0088)
			$m = 8$		
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0820 (0.0479)	0.0852 (0.0521)	0.0938 (0.0651)	0.1154 (0.0925)	0.1644 (0.1565)
-0.2	0.0055 (0.0161)	0.0087 (0.0197)	0.0163 (0.0286)	0.0334 (0.0462)	0.0688 (0.0857)
0	-0.0356 (-0.0045)	-0.0337 (-0.0001)	-0.0295 (0.0062)	-0.0208 (0.0180)	0.0011 (0.0402)
0.2	-0.0648 (-0.0140)	-0.0635 (-0.0116)	-0.0614 (-0.0074)	-0.0576 (-0.0003)	-0.0484 (0.0095)
0.4	-0.1093 (-0.0111)	-0.1088 (-0.0102)	-0.1083 (-0.0096)	-0.1074 (-0.0091)	-0.1052 (-0.0066)
			$m = 16$		
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0802 (0.1124)	0.0831 (0.1152)	0.0899 (0.1228)	0.1047 (0.1400)	0.1386 (0.1795)
-0.2	0.0204 (0.0681)	0.0222 (0.0701)	0.0267 (0.0750)	0.0374 (0.0862)	0.0618 (0.1122)
0	-0.0203 (0.0299)	-0.0188 (0.0338)	-0.0159 (0.0373)	-0.0095 (0.0440)	0.0058 (0.0603)
0.2	-0.0524 (-0.0000)	-0.0514 (0.0018)	-0.0496 (0.0057)	-0.0461 (0.0089)	-0.0377 (0.0174)
0.4	-0.0889 (-0.0220)	-0.0882 (-0.0215)	-0.0873 (-0.0203)	-0.0859 (-0.0192)	-0.0827 (-0.0146)

Table 6.2: Bias of the Gaussian (log-periodogram) estimates of d_1, n=128

	$m=8$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0765 (0.0378)	0.0840 (0.0503)	0.1040 (0.0809)	0.1543 (0.1503)	0.2726 (0.3051)
-0.2	0.0028 (0.0041)	0.0077 (0.0091)	0.0196 (0.0264)	0.0512 (0.0682)	0.1317 (0.1682)
0	-0.0338 (-0.0080)	-0.0315 (-0.0073)	-0.0257 (-0.0006)	-0.0091 (0.0204)	0.0372 (0.0741)
0.2	-0.0549 (-0.0117)	-0.0540 (-0.0097)	-0.0517 (-0.0065)	-0.0449 (0.0046)	-0.0248 (0.0299)
0.4	-0.0943 (-0.0033)	-0.0939 (-0.0027)	-0.0928 (-0.0004)	-0.0902 (0.0050)	-0.0839 (0.0143)
	$m=16$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0402 (0.0351)	0.0450 (0.0423)	0.0575 (0.0602)	0.0907 (0.1023)	0.1730 (0.1978)
-0.2	-0.0090 (0.0077)	-0.0060 (0.0108)	0.0020 (0.0212)	0.0232 (0.0464)	0.0787 (0.1079)
0	-0.0280 (-0.0065)	-0.0264 (-0.0056)	-0.0226 (-0.0017)	-0.0119 (0.0108)	0.0187 (0.0440)
0.2	-0.0368 (-0.0151)	-0.0361 (-0.0137)	-0.0344 (-0.0117)	-0.0297 (-0.0055)	-0.0153 (0.0104)
0.4	-0.0555 (-0.0158)	-0.0553 (-0.0152)	-0.0546 (-0.0136)	-0.0529 (-0.0103)	-0.0480 (-0.0042)
	$m=32$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0635 (0.0840)	0.0665 (0.0880)	0.0744 (0.0978)	0.0966 (0.1224)	0.1514 (0.1793)
-0.2	0.0214 (0.0456)	0.0234 (0.0471)	0.0281 (0.0534)	0.0409 (0.0684)	0.0750 (0.1047)
0	-0.0094 (0.0146)	-0.0085 (0.0153)	-0.0061 (0.0182)	0.0003 (0.0256)	0.0193 (0.0455)
0.2	-0.0352 (-0.0127)	-0.0348 (-0.0113)	-0.0336 (-0.0103)	-0.0304 (-0.0063)	-0.0209 (0.0043)
0.4	-0.0599 (-0.0337)	-0.0597 (-0.0331)	-0.0592 (-0.0319)	-0.0577 (-0.0297)	-0.0536 (-0.0258)

Table 6.3: Bias of the Gaussian (log-periodogram) estimates of d_1, n=256

	$m=16$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0408 (0.0338)	0.0477 (0.0420)	0.0705 (0.0725)	0.1395 (0.1523)	0.2999 (0.3293)
-0.2	-0.0085 (0.0059)	-0.0051 (0.0097)	0.0065 (0.0206)	0.0431 (0.0568)	0.1490 (0.1742)
0	-0.0231 (-0.0032)	-0.0217 (-0.0029)	-0.0174 (0.0004)	-0.0024 (0.0129)	0.0521 (0.0736)
0.2	-0.0271 (-0.0053)	-0.0266 (-0.0059)	-0.0248 (-0.0040)	-0.0192 (0.0012)	0.0021 (0.0238)
0.4	-0.0446 (0.0013)	-0.0444 (0.0012)	-0.0439 (0.0016)	-0.0423 (0.0032)	-0.0368 (0.0105)
	$m=32$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0224 (0.0244)	0.0275 (0.0297)	0.0429 (0.0484)	0.0896 (0.0956)	0.2014 (0.2050)
-0.2	-0.0080 (0.0023)	-0.0055 (0.0049)	0.0021 (0.0121)	0.0259 (0.0341)	0.0961 (0.1057)
0	-0.0195 (-0.0083)	-0.0185 (-0.0078)	-0.0156 (-0.0052)	-0.0057 (0.0028)	0.0293 (0.0404)
0.2	-0.0253 (-0.0141)	-0.0248 (-0.0144)	-0.0237 (-0.0128)	-0.0198 (-0.0095)	-0.0053 (0.0055)
0.4	-0.0335 (-0.0139)	-0.0332 (-0.0138)	-0.0328 (-0.0135)	-0.0314 (-0.0124)	-0.0268 (-0.0071)
	$m=64$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0519 (0.0634)	0.0550 (0.0664)	0.0649 (0.0772)	0.0949 (0.1046)	0.1699 (0.1699)
-0.2	0.0187 (0.0301)	0.0201 (0.0316)	0.0244 (0.0358)	0.0387 (0.0488)	0.0827 (0.0913)
0	-0.0084 (0.0038)	-0.0078 (0.0040)	-0.0060 (0.0055)	0.0001 (0.0105)	0.0218 (0.0334)
0.2	-0.0317 (-0.0193)	-0.0314 (-0.0196)	-0.0307 (-0.0187)	-0.0281 (-0.0164)	-0.0188 (-0.0068)
0.4	-0.0522 (-0.0385)	-0.0520 (-0.0383)	-0.0517 (-0.0384)	-0.0507 (-0.0376)	-0.0471 (-0.0335)

Table 6.4: Bias of the Gaussian (log-periodogram) estimates of d_1, n=512

	$m = 32$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0200 (0.0201)	0.0294 (0.0312)	0.0609 (0.0677)	0.1539 (0.1609)	0.3422 (0.3492)
-0.2	-0.0086 (0.0019)	-0.0051 (0.0049)	0.0071 (0.0178)	0.0517 (0.0631)	0.1762 (0.1902)
0	-0.0160 (-0.0050)	-0.0149 (-0.0043)	-0.0111 (-0.0012)	0.0044 (0.0155)	0.0646 (0.0784)
0.2	-0.0167 (-0.0059)	-0.0163 (-0.0055)	-0.0153 (-0.0047)	-0.0107 (-0.0009)	0.0109 (0.0210)
0.4	-0.0208 (-0.0015)	-0.0207 (-0.0018)	-0.0205 (-0.0020)	-0.0193 (-0.0012)	-0.0137 (0.0052)
	$m = 64$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0165 (0.0201)	0.0229 (0.0270)	0.0436 (0.0490)	0.1064 (0.1047)	0.2429 (0.2224)
-0.2	-0.0029 (0.0038)	-0.0007 (0.0059)	0.0069 (0.0139)	0.0348 (0.0410)	0.1189 (0.1189)
0	-0.0121 (-0.0055)	-0.0114 (-0.0049)	-0.0090 (-0.0027)	0.0006 (0.0072)	0.0393 (0.0456)
0.2	-0.0172 (-0.0109)	-0.0170 (-0.0105)	-0.0162 (-0.0100)	-0.0134 (-0.0075)	0.0005 (0.0059)
0.4	-0.0202 (-0.0123)	-0.0201 (-0.0124)	-0.0200 (-0.0125)	-0.0192 (-0.0119)	-0.0152 (-0.0079)
	$m = 128$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0476 (0.0539)	0.0514 (0.0580)	0.0639 (0.0710)	0.1035 (0.1037)	0.1988 (0.1746)
-0.2	0.0188 (0.0256)	0.0201 (0.0269)	0.0245 (0.0316)	0.0411 (0.0475)	0.0959 (0.0941)
0	-0.0056 (0.0013)	-0.0051 (0.0017)	-0.0037 (0.0031)	0.0021 (0.0090)	0.0264 (0.0320)
0.2	-0.0274 (-0.0204)	-0.0272 (-0.0201)	-0.0267 (-0.0198)	-0.0249 (-0.0182)	-0.0161 (-0.0100)
0.4	-0.0466 (-0.0395)	-0.0465 (-0.0395)	-0.0464 (-0.0395)	-0.0458 (-0.0391)	-0.0430 (-0.0364)

Table 6.5: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 128, m=8

	$l = 1$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.1300 (0.0472)	0.1331 (0.0487)	0.1412 (0.0582)	0.1609 (0.0955)	0.2176 (0.1834)
-0.2	0.0459 (0.0200)	0.0481 (0.0223)	0.0541 (0.0287)	0.0693 (0.0512)	0.1087 (0.1105)
0	-0.0180 (0.0054)	-0.0160 (0.0060)	-0.0118 (0.0123)	-0.0020 (0.0291)	0.0221 (0.0636)
0.2	-0.0774 (-0.0005)	-0.0762 (-0.0010)	-0.0735 (0.0016)	-0.0680 (0.0117)	-0.0548 (0.0375)
0.4	-0.1510 (0.0065)	-0.1504 (0.0078)	-0.1491 (0.0111)	-0.1463 (0.0184)	-0.1406 (0.0308)
	$l = 2$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.1949 (0.0486)	0.1975 (0.0502)	0.2025 (0.0611)	0.2133 (0.0866)	0.2408 (0.1464)
-0.2	0.0919 (0.0360)	0.0942 (0.0386)	0.0979 (0.0474)	0.1054 (0.0649)	0.1253 (0.1053)
0	-0.0030 (0.0355)	-0.0014 (0.0372)	0.0015 (0.0427)	0.0068 (0.0544)	0.0200 (0.0775)
0.2	-0.0935 (0.0350)	-0.0927 (0.0332)	-0.0913 (0.0352)	-0.0884 (0.0424)	-0.0827 (0.0629)
0.4	-0.1916 (0.0437)	-0.1913 (0.0438)	-0.1907 (0.0469)	-0.1900 (0.0525)	-0.1880 (0.0649)
	$l = 4$				
$d_1 \backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.3009 (0.0170)	0.3027 (0.0146)	0.3065 (0.0159)	0.3106 (0.0208)	0.3162 (0.0801)
-0.2	0.1499 (0.0108)	0.1511 (0.0096)	0.1527 (0.0116)	0.1548 (0.0133)	0.1608 (0.0554)
0	-0.0029 (0.0055)	-0.0014 (0.0066)	0.0000 (0.0104)	0.0019 (0.0134)	0.0064 (0.0353)
0.2	-0.1555 (0.0011)	-0.1540 (0.0018)	-0.1526 (0.0030)	-0.1513 (0.0128)	-0.1472 (0.0290)
0.4	-0.3085 (0.0129)	-0.3080 (0.0128)	-0.3073 (0.0183)	-0.3067 (0.0251)	-0.3046 (0.0427)

Table 6.6: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n=128, m=16

			$l=1$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0655 (0.0483)	0.0672 (0.0493)	0.0718 (0.0546)	0.0845 (0.0731)	0.1207 (0.1225)
−0.2	0.0156 (0.0267)	0.0169 (0.0273)	0.0203 (0.0303)	0.0298 (0.0421)	0.0569 (0.0755)
0	-0.0101 (0.0111)	-0.0090 (0.0105)	-0.0064 (0.0134)	-0.0001 (0.0222)	0.0174 (0.0409)
0.2	-0.0287 (-0.0007)	-0.0282 (-0.0012)	-0.0269 (-0.0005)	-0.0233 (0.0046)	-0.0135 (0.0182)
0.4	-0.0650 (-0.0037)	-0.0648 (-0.0035)	-0.0643 (-0.0019)	-0.0630 (0.0010)	-0.0594 (0.0077)
			$l=2$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0881 (0.0488)	0.0890 (0.0497)	0.0914 (0.0543)	0.0981 (0.0641)	0.1185 (0.0948)
−0.2	0.0289 (0.0329)	0.0301 (0.0331)	0.0327 (0.0360)	0.0391 (0.0434)	0.0559 (0.0640)
0	-0.0039 (0.0211)	-0.0028 (0.0205)	-0.0008 (0.0223)	0.0038 (0.0273)	0.0153 (0.0384)
0.2	-0.0331 (0.0091)	-0.0325 (0.0082)	-0.0315 (0.0083)	-0.0290 (0.0111)	-0.0223 (0.0200)
0.4	-0.0830 (0.0038)	-0.0828 (0.0034)	-0.0824 (0.0045)	-0.0813 (0.0057)	-0.0784 (0.0109)
			$l=4$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1365 (0.0439)	0.1363 (0.0436)	0.1368 (0.0433)	0.1386 (0.0427)	0.1456 (0.0598)
−0.2	0.0553 (0.0267)	0.0553 (0.0251)	0.0558 (0.0242)	0.0576 (0.0241)	0.0636 (0.0353)
0	-0.0080 (0.0091)	-0.0080 (0.0074)	-0.0078 (0.0068)	-0.0064 (0.0070)	-0.0011 (0.0118)
0.2	-0.0685 (-0.0087)	-0.0684 (-0.0093)	-0.0681 (-0.0103)	-0.0671 (-0.0097)	-0.0632 (-0.0067)
0.4	-0.1442 (-0.0200)	-0.1442 (-0.0204)	-0.1441 (-0.0202)	-0.1435 (-0.0211)	-0.1417 (-0.0185)

Table 6.7: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 128, m=32

			$l=1$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0760 (0.0978)	0.0771 (0.0987)	0.0802 (0.1013)	0.0886 (0.1112)	0.1122 (0.1381)
−0.2	0.0311 (0.0580)	0.0320 (0.0586)	0.0342 (0.0610)	0.0403 (0.0674)	0.0570 (0.0854)
0	-0.0045 (0.0226)	-0.0039 (0.0222)	-0.0024 (0.0249)	0.0016 (0.0292)	0.0125 (0.0398)
0.2	-0.0367 (-0.0105)	-0.0362 (-0.0101)	-0.0352 (-0.0090)	-0.0327 (-0.0069)	-0.0261 (0.0008)
0.4	-0.0707 (-0.0376)	-0.0703 (-0.0370)	-0.0697 (-0.0364)	-0.0683 (-0.0348)	-0.0649 (-0.0306)
			$l=2$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0878 (0.1080)	0.0883 (0.1088)	0.0900 (0.1106)	0.0947 (0.1156)	0.1082 (0.1311)
−0.2	0.0376 (0.0669)	0.0384 (0.0674)	0.0399 (0.0695)	0.0438 (0.0732)	0.0541 (0.0835)
0	-0.0020 (0.0292)	-0.0014 (0.0288)	-0.0001 (0.0310)	0.0027 (0.0330)	0.0100 (0.0392)
0.2	-0.0392 (-0.0080)	-0.0386 (-0.0076)	-0.0377 (-0.0067)	-0.0357 (-0.0061)	-0.0309 (-0.0013)
0.4	-0.0814 (-0.0407)	-0.0809 (-0.0402)	-0.0803 (-0.0400)	-0.0792 (-0.0392)	-0.0767 (-0.0361)
			$l=4$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1078 (0.1270)	0.1079 (0.1275)	0.1084 (0.1270)	0.1100 (0.1273)	0.1151 (0.1345)
−0.2	0.0445 (0.0774)	0.0446 (0.0773)	0.0451 (0.0781)	0.0467 (0.0784)	0.0516 (0.0829)
0	-0.0064 (0.0297)	-0.0062 (0.0290)	-0.0057 (0.0306)	-0.0043 (0.0303)	-0.0002 (0.0330)
0.2	-0.0560 (-0.0174)	-0.0557 (-0.0166)	-0.0553 (-0.0157)	-0.0541 (-0.0165)	-0.0511 (-0.0147)
0.4	-0.1135 (-0.0605)	-0.1133 (-0.0598)	-0.1129 (-0.0603)	-0.1122 (-0.0602)	-0.1104 (-0.0585)

Table 6.8: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 256, m=16

			$l=2$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0717 (0.0146)	0.0732 (0.0167)	0.0785 (0.0246)	0.0953 (0.0571)	0.1540 (0.1544)
-0.2	0.0132 (0.0026)	0.0142 (0.0037)	0.0177 (0.0088)	0.0289 (0.0252)	0.0692 (0.0806)
0	-0.0157 (-0.0029)	-0.0150 (-0.0020)	-0.0130 (-0.0002)	-0.0063 (0.0074)	0.0183 (0.0403)
0.2	-0.0358 (-0.0026)	-0.0354 (-0.0024)	-0.0344 (-0.0021)	-0.0310 (0.0013)	-0.0189 (0.0181)
0.4	-0.0795 (0.0022)	-0.0792 (0.0018)	-0.0787 (0.0015)	-0.0772 (0.0016)	-0.0730 (0.0110)
			$l=4$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.1283 (0.0360)	0.1291 (0.0359)	0.1308 (0.0388)	0.1373 (0.0523)	0.1604 (0.0950)
-0.2	0.0496 (0.0288)	0.0502 (0.0275)	0.0514 (0.0296)	0.0556 (0.0386)	0.0735 (0.0621)
0	-0.0055 (0.0238)	-0.0054 (0.0239)	-0.0048 (0.0243)	-0.0022 (0.0285)	0.0091 (0.0440)
0.2	-0.0570 (0.0205)	-0.0571 (0.0204)	-0.0569 (0.0209)	-0.0555 (0.0228)	-0.0493 (0.0346)
0.4	-0.1257 (0.0197)	-0.1258 (0.0192)	-0.1257 (0.0195)	-0.1250 (0.0206)	-0.1220 (0.0285)
			$l=8$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.2542 (0.0367)	0.2549 (0.0390)	0.2559 (0.0421)	0.2581 (0.0543)	0.2650 (0.0759)
-0.2	0.1267 (0.0276)	0.1273 (0.0300)	0.1277 (0.0342)	0.1285 (0.0357)	0.1326 (0.0510)
0	0.0024 (0.0245)	0.0023 (0.0275)	0.0019 (0.0286)	0.0016 (0.0287)	0.0040 (0.0366)
0.2	-0.1224 (0.0182)	-0.1225 (0.0198)	-0.1229 (0.0211)	-0.1235 (0.0194)	-0.1231 (0.0265)
0.4	-0.2505 (0.0131)	-0.2507 (0.0115)	-0.2508 (0.0091)	-0.2509 (0.0080)	-0.2511 (0.0141)

Table 6.9: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 256, m=32

			$l=2$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0402 (0.0277)	0.0414 (0.0289)	0.0449 (0.0330)	0.0560 (0.0497)	0.0934 (0.1011)
-0.2	0.0073 (0.0129)	0.0080 (0.0131)	0.0104 (0.0158)	0.0181 (0.0245)	0.0444 (0.0549)
0	-0.0055 (0.0018)	-0.0050 (0.0020)	-0.0035 (0.0029)	0.0010 (0.0071)	0.0164 (0.0261)
0.2	-0.0142 (-0.0055)	-0.0139 (-0.0056)	-0.0131 (-0.0055)	-0.0107 (-0.0034)	-0.0022 (0.0066)
0.4	-0.0350 (-0.0095)	-0.0348 (-0.0095)	-0.0344 (-0.0098)	-0.0333 (-0.0098)	-0.0299 (-0.0048)
			$l=4$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0561 (0.0385)	0.0569 (0.0388)	0.0586 (0.0403)	0.0635 (0.0467)	0.0803 (0.0685)
-0.2	0.0173 (0.0241)	0.0178 (0.0235)	0.0190 (0.0246)	0.0227 (0.0287)	0.0357 (0.0423)
0	0.0001 (0.0115)	0.0003 (0.0112)	0.0010 (0.0114)	0.0034 (0.0137)	0.0121 (0.0234)
0.2	-0.0143 (0.0007)	-0.0142 (0.0004)	-0.0138 (0.0005)	-0.0123 (0.0019)	-0.0071 (0.0084)
0.4	-0.0481 (-0.0078)	-0.0481 (-0.0076)	-0.0478 (-0.0077)	-0.0471 (-0.0074)	-0.0445 (-0.0041)
			$l=8$		
$d_1\backslash d_2$	-0.4	-0.2	0	0.2	0.4
-0.4	0.0892 (0.0404)	0.0893 (0.0410)	0.0901 (0.0419)	0.0926 (0.0454)	0.1004 (0.0552)
-0.2	0.0331 (0.0220)	0.0333 (0.0220)	0.0341 (0.0229)	0.0362 (0.0242)	0.0426 (0.0322)
0	0.0001 (0.0054)	0.0003 (0.0054)	0.0009 (0.0056)	0.0025 (0.0070)	0.0072 (0.0131)
0.2	-0.0335 (-0.0097)	-0.0334 (-0.0098)	-0.0330 (-0.0096)	-0.0320 (-0.0088)	-0.0292 (-0.0048)
0.4	-0.0885 (-0.0226)	-0.0884 (-0.0223)	-0.0881 (-0.0230)	-0.0875 (-0.0232)	-0.0862 (-0.0216)

Table 6.10: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 256, m=64

			$l=2$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0651 (0.0799)	0.0658 (0.0805)	0.0679 (0.0826)	0.0748 (0.0914)	0.0981 (0.1196)
−0.2	0.0294 (0.0461)	0.0299 (0.0461)	0.0312 (0.0475)	0.0355 (0.0528)	0.0501 (0.0696)
0	-0.0024 (0.0147)	-0.0021 (0.0145)	-0.0013 (0.0152)	0.0012 (0.0180)	0.0099 (0.0289)
0.2	-0.0324 (-0.0148)	-0.0322 (-0.0148)	-0.0317 (-0.0145)	-0.0303 (-0.0128)	-0.0253 (-0.0070)
0.4	-0.0608 (-0.0415)	-0.0606 (-0.0416)	-0.0604 (-0.0417)	-0.0597 (-0.0414)	-0.0573 (-0.0377)
			$l=4$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0767 (0.0951)	0.0770 (0.0952)	0.0780 (0.0960)	0.0810 (0.0991)	0.0912 (0.1111)
−0.2	0.0364 (0.0576)	0.0367 (0.0572)	0.0373 (0.0577)	0.0393 (0.0606)	0.0465 (0.0680)
0	-0.0003 (0.0214)	-0.0002 (0.0210)	0.0002 (0.0214)	0.0015 (0.0232)	0.0062 (0.0287)
0.2	-0.0356 (-0.0141)	-0.0355 (-0.0141)	-0.0352 (-0.0137)	-0.0344 (-0.0124)	-0.0314 (-0.0088)
0.4	-0.0705 (-0.0471)	-0.0704 (-0.0472)	-0.0703 (-0.0472)	-0.0699 (-0.0466)	-0.0682 (-0.0439)
			$l=8$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0949 (0.1152)	0.0951 (0.1153)	0.0956 (0.1158)	0.0969 (0.1171)	0.1011 (0.1227)
−0.2	0.0443 (0.0687)	0.0445 (0.0686)	0.0449 (0.0689)	0.0460 (0.0705)	0.0496 (0.0745)
0	-0.0017 (0.0235)	-0.0016 (0.0230)	-0.0012 (0.0236)	-0.0004 (0.0249)	0.0022 (0.0281)
0.2	-0.0467 (-0.0216)	-0.0466 (-0.0214)	-0.0463 (-0.0210)	-0.0458 (-0.0198)	-0.0440 (-0.0179)
0.4	-0.0941 (-0.0642)	-0.0941 (-0.0643)	-0.0939 (-0.0645)	-0.0937 (-0.0641)	-0.0927 (-0.0620)

Table 6.11: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 512, m=32

			$l=4$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0394 (0.0109)	0.0408 (0.0139)	0.0446 (0.0203)	0.0573 (0.0382)	0.1058 (0.1016)
−0.2	0.0032 (0.0019)	0.0041 (0.0032)	0.0065 (0.0067)	0.0139 (0.0157)	0.0440 (0.0497)
0	-0.0069 (-0.0026)	-0.0064 (-0.0022)	-0.0052 (-0.0003)	-0.0014 (0.0035)	0.0141 (0.0205)
0.2	-0.0132 (-0.0043)	-0.0128 (-0.0043)	-0.0122 (-0.0040)	-0.0106 (-0.0018)	-0.0035 (0.0063)
0.4	-0.0404 (-0.0025)	-0.0401 (-0.0023)	-0.0398 (-0.0022)	-0.0390 (-0.0013)	-0.0364 (0.0019)
			$l=8$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0779 (0.0121)	0.0785 (0.0141)	0.0803 (0.0191)	0.0858 (0.0309)	0.1081 (0.0621)
−0.2	0.0223 (0.0060)	0.0229 (0.0075)	0.0244 (0.0110)	0.0288 (0.0191)	0.0453 (0.0395)
0	-0.0046 (0.0017)	-0.0041 (0.0023)	-0.0029 (0.0042)	0.0003 (0.0103)	0.0107 (0.0250)
0.2	-0.0290 (-0.0004)	-0.0286 (-0.0002)	-0.0279 (0.0006)	-0.0262 (0.0043)	-0.0203 (0.0124)
0.4	-0.0770 (0.0002)	-0.0767 (0.0003)	-0.0763 (0.0007)	-0.0754 (0.0019)	-0.0727 (0.0055)
			$l=16$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1947 (-0.0027)	0.1953 (-0.0008)	0.1964 (0.0016)	0.1988 (0.0105)	0.2061 (0.0309)
−0.2	0.0903 (-0.0149)	0.0907 (-0.0133)	0.0913 (-0.0111)	0.0930 (-0.0023)	0.0984 (0.0133)
0	-0.0079 (-0.0233)	-0.0077 (-0.0219)	-0.0072 (-0.0201)	-0.0060 (-0.0131)	-0.0019 (-0.0022)
0.2	-0.1052 (-0.0292)	-0.1050 (-0.0284)	-0.1047 (-0.0264)	-0.1036 (-0.0216)	-0.1000 (-0.0175)
0.4	-0.2071 (-0.0332)	-0.2068 (-0.0338)	-0.2064 (-0.0328)	-0.2056 (-0.0297)	-0.2035 (-0.0274)

Table 6.12: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 512, m=64

			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0250 (0.0206)	0.0258 (0.0222)	0.0282 (0.0255)	0.0359 (0.0353)	0.0665 (0.0692)
−0.2	0.0067 (0.0087)	0.0073 (0.0094)	0.0088 (0.0114)	0.0134 (0.0167)	0.0313 (0.0355)
0	-0.0032 (-0.0007)	-0.0029 (-0.0004)	-0.0021 (0.0007)	0.0004 (0.0032)	0.0099 (0.0134)
0.2	-0.0113 (-0.0085)	-0.0111 (-0.0083)	-0.0106 (-0.0080)	-0.0093 (-0.0066)	-0.0044 (-0.0013)
0.4	-0.0228 (-0.0146)	-0.0227 (-0.0145)	-0.0224 (-0.0142)	-0.0218 (-0.0133)	-0.0196 (-0.0105)
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0332 (0.0244)	0.0336 (0.0252)	0.0347 (0.0273)	0.0383 (0.0327)	0.0519 (0.0482)
−0.2	0.0085 (0.0123)	0.0089 (0.0129)	0.0099 (0.0145)	0.0128 (0.0183)	0.0229 (0.0285)
0	-0.0035 (0.0013)	-0.0032 (0.0017)	-0.0025 (0.0025)	-0.0007 (0.0053)	0.0057 (0.0127)
0.2	-0.0140 (-0.0087)	-0.0137 (-0.0084)	-0.0132 (-0.0079)	-0.0120 (-0.0063)	-0.0081 (-0.0018)
0.4	-0.0346 (-0.0178)	-0.0345 (-0.0177)	-0.0342 (-0.0172)	-0.0336 (-0.0162)	-0.0318 (-0.0134)
			$l=16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0557 (0.0271)	0.0557 (0.0275)	0.0560 (0.0279)	0.0570 (0.0301)	0.0612 (0.0374)
−0.2	0.0131 (0.0106)	0.0133 (0.0109)	0.0137 (0.0114)	0.0149 (0.0135)	0.0192 (0.0188)
0	-0.0069 (-0.0047)	-0.0067 (-0.0043)	-0.0063 (-0.0039)	-0.0053 (-0.0026)	-0.0019 (0.0015)
0.2	-0.0262 (-0.0195)	-0.0260 (-0.0190)	-0.0256 (-0.0187)	-0.0248 (-0.0177)	-0.0224 (-0.0151)
0.4	-0.0649 (-0.0344)	-0.0647 (-0.0344)	-0.0644 (-0.0339)	-0.0638 (-0.0325)	-0.0622 (-0.0302)

Table 6.13: Bias of the trimmed Gaussian (log-periodogram) estimates of d_1, n= 512, m=128

			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0594 (0.0644)	0.0600 (0.0654)	0.0615 (0.0672)	0.0662 (0.0724)	0.0838 (0.0908)
−0.2	0.0282 (0.0336)	0.0286 (0.0341)	0.0294 (0.0351)	0.0319 (0.0380)	0.0420 (0.0484)
0	-0.0017 (0.0041)	-0.0015 (0.0044)	-0.0010 (0.0049)	0.0004 (0.0062)	0.0060 (0.0121)
0.2	-0.0306 (-0.0245)	-0.0305 (-0.0243)	-0.0302 (-0.0242)	-0.0294 (-0.0234)	-0.0263 (-0.0202)
0.4	-0.0582 (-0.0522)	-0.0581 (-0.0521)	-0.0579 (-0.0519)	-0.0575 (-0.0515)	-0.0559 (-0.0498)
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0688 (0.0755)	0.0690 (0.0761)	0.0697 (0.0771)	0.0719 (0.0797)	0.0798 (0.0876)
−0.2	0.0328 (0.0405)	0.0330 (0.0409)	0.0335 (0.0415)	0.0350 (0.0434)	0.0403 (0.0487)
0	-0.0020 (0.0061)	-0.0019 (0.0063)	-0.0015 (0.0067)	-0.0005 (0.0079)	0.0029 (0.0118)
0.2	-0.0362 (-0.0278)	-0.0361 (-0.0276)	-0.0358 (-0.0275)	-0.0352 (-0.0266)	-0.0330 (-0.0242)
0.4	-0.0694 (-0.0613)	-0.0693 (-0.0612)	-0.0691 (-0.0610)	-0.0687 (-0.0606)	-0.0674 (-0.0591)
			$l=16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0858 (0.0940)	0.0859 (0.0944)	0.0862 (0.0946)	0.0869 (0.0954)	0.0897 (0.0986)
−0.2	0.0406 (0.0498)	0.0407 (0.0500)	0.0409 (0.0501)	0.0415 (0.0509)	0.0437 (0.0533)
0	-0.0035 (0.0060)	-0.0034 (0.0063)	-0.0032 (0.0064)	-0.0027 (0.0067)	-0.0011 (0.0085)
0.2	-0.0472 (-0.0376)	-0.0471 (-0.0372)	-0.0470 (-0.0373)	-0.0466 (-0.0369)	-0.0454 (-0.0356)
0.4	-0.0906 (-0.0811)	-0.0905 (-0.0811)	-0.0904 (-0.0809)	-0.0902 (-0.0805)	-0.0893 (-0.0796)

6.4.2 Mean Square Error

Table 6.14: MSE of the Gaussian (log-periodogram) estimates of d_1, n=64

	$m=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1258 (0.3581)	0.1282 (0.3620)	0.1329 (0.3575)	0.1491 (0.3910)	0.1886 (0.4111)
−0.2	0.1249 (0.3511)	0.1253 (0.3525)	0.1269 (0.3552)	0.1311 (0.3518)	0.1433 (0.3894)
0	0.1381 (0.3601)	0.1374 (0.3510)	0.1374 (0.3484)	0.1377 (0.3409)	0.1374 (0.3542)
0.2	0.1462 (0.3599)	0.1464 (0.3559)	0.1465 (0.3530)	0.1470 (0.3497)	0.1455 (0.3671)
0.4	0.1486 (0.3587)	0.1488 (0.3568)	0.1495 (0.3568)	0.1513 (0.3642)	0.1559 (0.3758)
	$m=8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0480 (0.1174)	0.0497 (0.1218)	0.0533 (0.1252)	0.0621 (0.1385)	0.0850 (0.1514)
−0.2	0.0614 (0.1176)	0.0621 (0.1196)	0.0632 (0.1239)	0.0665 (0.1283)	0.0760 (0.1382)
0	0.0749 (0.1231)	0.0751 (0.1211)	0.0756 (0.1231)	0.0772 (0.1225)	0.0787 (0.1256)
0.2	0.0755 (0.1228)	0.0754 (0.1229)	0.0754 (0.1206)	0.0759 (0.1199)	0.0764 (0.1246)
0.4	0.0645 (0.1207)	0.0644 (0.1209)	0.0645 (0.1217)	0.0647 (0.1239)	0.0641 (0.1261)
	$m=16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0277 (0.0579)	0.0283 (0.0603)	0.0299 (0.0626)	0.0342 (0.0710)	0.0457 (0.0860)
−0.2	0.0290 (0.0508)	0.0294 (0.0521)	0.0299 (0.0532)	0.0314 (0.0567)	0.0358 (0.0658)
0	0.0315 (0.0487)	0.0316 (0.0496)	0.0318 (0.0503)	0.0324 (0.0499)	0.0333 (0.0541)
0.2	0.0333 (0.0497)	0.0332 (0.0510)	0.0332 (0.0505)	0.0332 (0.0493)	0.0331 (0.0509)
0.4	0.0322 (0.0492)	0.0320 (0.0487)	0.0319 (0.0496)	0.0317 (0.0503)	0.0313 (0.0521)

Table 6.15: MSE of the Gaussian (log-periodogram) estimates of d_1, n=128

			$m = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0452 (0.1215)	0.0480 (0.1206)	0.0545 (0.1215)	0.0753 (0.1446)	0.1413 (0.2113)
−0.2	0.0590 (0.1177)	0.0596 (0.1182)	0.0611 (0.1122)	0.0661 (0.1159)	0.0877 (0.1374)
0	0.0733 (0.1152)	0.0734 (0.1166)	0.0732 (0.1152)	0.0723 (0.1117)	0.0727 (0.1152)
0.2	0.0725 (0.1216)	0.0724 (0.1174)	0.0723 (0.1172)	0.0711 (0.1129)	0.0676 (0.1129)
0.4	0.0597 (0.1184)	0.0597 (0.1182)	0.0594 (0.1160)	0.0583 (0.1141)	0.0560 (0.1164)
			$m = 16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0190 (0.0459)	0.0198 (0.0459)	0.0220 (0.0456)	0.0298 (0.0536)	0.0586 (0.0861)
−0.2	0.0269 (0.0449)	0.0269 (0.0449)	0.0268 (0.0427)	0.0279 (0.0440)	0.0363 (0.0554)
0	0.0308 (0.0439)	0.0307 (0.0448)	0.0305 (0.0440)	0.0302 (0.0429)	0.0314 (0.0455)
0.2	0.0309 (0.0452)	0.0308 (0.0443)	0.0307 (0.0445)	0.0303 (0.0428)	0.0296 (0.0428)
0.4	0.0250 (0.0448)	0.0249 (0.0447)	0.0247 (0.0441)	0.0243 (0.0434)	0.0233 (0.0441)
			$m = 32$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0143 (0.0269)	0.0148 (0.0276)	0.0163 (0.0292)	0.0209 (0.0356)	0.0372 (0.0557)
−0.2	0.0128 (0.0227)	0.0128 (0.0227)	0.0129 (0.0230)	0.0139 (0.0251)	0.0192 (0.0326)
0	0.0125 (0.0204)	0.0125 (0.0209)	0.0124 (0.0214)	0.0124 (0.0215)	0.0132 (0.0227)
0.2	0.0138 (0.0207)	0.0137 (0.0205)	0.0136 (0.0204)	0.0134 (0.0202)	0.0129 (0.0201)
0.4	0.0148 (0.0214)	0.0148 (0.0216)	0.0147 (0.0212)	0.0144 (0.0210)	0.0139 (0.0208)

Table 6.16: MSE of the Gaussian (log-periodogram) estimates of d_1, n=256, $m = 16$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0192 (0.0446)	0.0206 (0.0466)	0.0257 (0.0494)	0.0468 (0.0718)	0.1298 (0.1673)
−0.2	0.0275 (0.0454)	0.0276 (0.0442)	0.0281 (0.0444)	0.0322 (0.0500)	0.0572 (0.0817)
0	0.0312 (0.0444)	0.0312 (0.0448)	0.0312 (0.0446)	0.0319 (0.0463)	0.0348 (0.0509)
0.2	0.0306 (0.0452)	0.0306 (0.0460)	0.0305 (0.0451)	0.0303 (0.0452)	0.0297 (0.0446)
0.4	0.0229 (0.0450)	0.0228 (0.0448)	0.0227 (0.0447)	0.0224 (0.0451)	0.0214 (0.0452)
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0404 (0.1132)	0.0410 (0.1134)	0.0428 (0.1172)	0.0490 (0.1227)	0.0732 (0.1376)
−0.2	0.0576 (0.1159)	0.0578 (0.1153)	0.0586 (0.1175)	0.0608 (0.1231)	0.0679 (0.1263)
0	0.0710 (0.1154)	0.0711 (0.1166)	0.0713 (0.1177)	0.0718 (0.1198)	0.0715 (0.1215)
0.2	0.0705 (0.1151)	0.0707 (0.1146)	0.0710 (0.1158)	0.0708 (0.1190)	0.0687 (0.1168)
0.4	0.0560 (0.1145)	0.0562 (0.1145)	0.0563 (0.1155)	0.0562 (0.1179)	0.0548 (0.1186)
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0878 (0.2554)	0.0880 (0.2549)	0.0888 (0.2573)	0.0914 (0.2637)	0.1024 (0.2487)
−0.2	0.1026 (0.2517)	0.1025 (0.2517)	0.1024 (0.2538)	0.1028 (0.2586)	0.1058 (0.2577)
0	0.1143 (0.2531)	0.1141 (0.2536)	0.1137 (0.2535)	0.1131 (0.2545)	0.1109 (0.2602)
0.2	0.1068 (0.2593)	0.1066 (0.2567)	0.1063 (0.2527)	0.1055 (0.2517)	0.1031 (0.2514)
0.4	0.0916 (0.2609)	0.0916 (0.2597)	0.0915 (0.2587)	0.0910 (0.2573)	0.0897 (0.2493)
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.2233 (1.5794)	0.2244 (1.5695)	0.2257 (1.5785)	0.2278 (1.5758)	0.2319 (1.5655)
−0.2	0.1886 (1.6139)	0.1894 (1.6054)	0.1901 (1.5904)	0.1903 (1.6104)	0.1918 (1.6161)
0	0.1813 (1.6086)	0.1811 (1.6167)	0.1809 (1.6440)	0.1813 (1.7339)	0.1821 (1.6963)
0.2	0.1956 (1.6083)	0.1957 (1.6225)	0.1960 (1.6248)	0.1967 (1.6711)	0.1995 (1.6708)
0.4	0.2370 (1.6308)	0.2370 (1.6295)	0.2373 (1.6397)	0.2382 (1.6702)	0.2417 (1.6544)

Table 6.17: MSE of the Gaussian (log-periodogram) estimates of d_1, n=256, $m = 32$

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0098 (0.0187)	0.0104 (0.0196)	0.0124 (0.0209)	0.0210 (0.0291)	0.0596 (0.0663)
−0.2	0.0127 (0.0189)	0.0127 (0.0185)	0.0127 (0.0187)	0.0139 (0.0205)	0.0244 (0.0318)
0	0.0131 (0.0187)	0.0131 (0.0189)	0.0130 (0.0189)	0.0131 (0.0195)	0.0143 (0.0203)
0.2	0.0134 (0.0192)	0.0134 (0.0196)	0.0133 (0.0193)	0.0132 (0.0192)	0.0130 (0.0186)
0.4	0.0113 (0.0189)	0.0113 (0.0188)	0.0113 (0.0189)	0.0112 (0.0191)	0.0109 (0.0190)
			$l=2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0142 (0.0330)	0.0144 (0.0328)	0.0149 (0.0333)	0.0167 (0.0357)	0.0244 (0.0423)
−0.2	0.0207 (0.0332)	0.0207 (0.0331)	0.0207 (0.0336)	0.0209 (0.0348)	0.0227 (0.0369)
0	0.0226 (0.0334)	0.0226 (0.0337)	0.0226 (0.0339)	0.0226 (0.0342)	0.0226 (0.0347)
0.2	0.0229 (0.0335)	0.0229 (0.0334)	0.0229 (0.0336)	0.0228 (0.0343)	0.0224 (0.0340)
0.4	0.0187 (0.0329)	0.0187 (0.0330)	0.0187 (0.0333)	0.0186 (0.0340)	0.0182 (0.0345)
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0217 (0.0533)	0.0217 (0.0536)	0.0219 (0.0540)	0.0228 (0.0560)	0.0265 (0.0560)
−0.2	0.0302 (0.0522)	0.0303 (0.0529)	0.0303 (0.0531)	0.0304 (0.0541)	0.0314 (0.0551)
0	0.0330 (0.0525)	0.0330 (0.0524)	0.0330 (0.0525)	0.0329 (0.0530)	0.0328 (0.0545)
0.2	0.0319 (0.0531)	0.0319 (0.0529)	0.0319 (0.0525)	0.0318 (0.0526)	0.0314 (0.0530)
0.4	0.0251 (0.0525)	0.0251 (0.0525)	0.0251 (0.0525)	0.0249 (0.0526)	0.0242 (0.0523)
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0442 (0.1275)	0.0443 (0.1275)	0.0444 (0.1277)	0.0450 (0.1272)	0.0470 (0.1302)
−0.2	0.0582 (0.1272)	0.0582 (0.1276)	0.0583 (0.1274)	0.0584 (0.1287)	0.0593 (0.1321)
0	0.0690 (0.1275)	0.0691 (0.1278)	0.0691 (0.1289)	0.0692 (0.1337)	0.0695 (0.1339)
0.2	0.0666 (0.1277)	0.0667 (0.1293)	0.0669 (0.1303)	0.0673 (0.1325)	0.0677 (0.1331)
0.4	0.0586 (0.1298)	0.0588 (0.1303)	0.0590 (0.1313)	0.0594 (0.1331)	0.0598 (0.1334)

Table 6.18: MSE of the Gaussian (log-periodogram) estimates of d_1, n=256, $m = 64$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0083 (0.0131)	0.0088 (0.0136)	0.0101 (0.0151)	0.0157 (0.0207)	0.0384 (0.0405)
−0.2	0.0062 (0.0099)	0.0063 (0.0099)	0.0065 (0.0102)	0.0076 (0.0117)	0.0140 (0.0185)
0	0.0059 (0.0088)	0.0059 (0.0089)	0.0059 (0.0089)	0.0059 (0.0093)	0.0066 (0.0103)
0.2	0.0068 (0.0092)	0.0068 (0.0094)	0.0068 (0.0093)	0.0066 (0.0091)	0.0062 (0.0089)
0.4	0.0083 (0.0101)	0.0083 (0.0101)	0.0083 (0.0101)	0.0082 (0.0101)	0.0078 (0.0100)
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0104 (0.0168)	0.0106 (0.0169)	0.0109 (0.0175)	0.0121 (0.0189)	0.0164 (0.0241)
−0.2	0.0084 (0.0133)	0.0084 (0.0135)	0.0086 (0.0137)	0.0089 (0.0142)	0.0101 (0.0159)
0	0.0081 (0.0120)	0.0081 (0.0121)	0.0082 (0.0123)	0.0082 (0.0124)	0.0083 (0.0127)
0.2	0.0097 (0.0127)	0.0097 (0.0126)	0.0097 (0.0126)	0.0096 (0.0126)	0.0093 (0.0126)
0.4	0.0124 (0.0147)	0.0124 (0.0146)	0.0123 (0.0147)	0.0123 (0.0147)	0.0119 (0.0146)
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0137 (0.0227)	0.0138 (0.0229)	0.0140 (0.0233)	0.0146 (0.0241)	0.0166 (0.0260)
−0.2	0.0108 (0.0178)	0.0108 (0.0181)	0.0109 (0.0182)	0.0112 (0.0186)	0.0119 (0.0195)
0	0.0100 (0.0157)	0.0101 (0.0157)	0.0101 (0.0159)	0.0102 (0.0163)	0.0103 (0.0167)
0.2	0.0117 (0.0164)	0.0118 (0.0164)	0.0118 (0.0164)	0.0118 (0.0163)	0.0116 (0.0163)
0.4	0.0152 (0.0191)	0.0152 (0.0190)	0.0152 (0.0191)	0.0152 (0.0190)	0.0149 (0.0187)
	$l=8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0216 (0.0371)	0.0216 (0.0371)	0.0217 (0.0374)	0.0220 (0.0373)	0.0231 (0.0381)
−0.2	0.0174 (0.0300)	0.0175 (0.0301)	0.0175 (0.0299)	0.0177 (0.0301)	0.0182 (0.0307)
0	0.0166 (0.0266)	0.0166 (0.0266)	0.0167 (0.0270)	0.0167 (0.0277)	0.0168 (0.0278)
0.2	0.0195 (0.0279)	0.0196 (0.0279)	0.0196 (0.0281)	0.0196 (0.0283)	0.0195 (0.0282)
0.4	0.0252 (0.0332)	0.0252 (0.0329)	0.0252 (0.0332)	0.0252 (0.0332)	0.0250 (0.0331)

Table 6.19: MSE of the Gaussian (log-periodogram) estimates of d_1, n=512, m = 32

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0089 (0.0188)	0.0098 (0.0190)	0.0143 (0.0238)	0.0394 (0.0486)	0.1414 (0.1534)
−0.2	0.0117 (0.0180)	0.0117 (0.0181)	0.0118 (0.0184)	0.0154 (0.0233)	0.0490 (0.0607)
0	0.0122 (0.0176)	0.0122 (0.0178)	0.0120 (0.0180)	0.0118 (0.0181)	0.0174 (0.0251)
0.2	0.0122 (0.0177)	0.0122 (0.0176)	0.0121 (0.0177)	0.0118 (0.0180)	0.0117 (0.0184)
0.4	0.0098 (0.0176)	0.0098 (0.0178)	0.0097 (0.0180)	0.0096 (0.0182)	0.0089 (0.0186)
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0233 (0.0548)	0.0236 (0.0551)	0.0246 (0.0565)	0.0279 (0.0594)	0.0441 (0.0720)
−0.2	0.0327 (0.0538)	0.0327 (0.0540)	0.0329 (0.0547)	0.0337 (0.0539)	0.0382 (0.0578)
0	0.0368 (0.0541)	0.0368 (0.0539)	0.0368 (0.0541)	0.0368 (0.0541)	0.0374 (0.0536)
0.2	0.0348 (0.0548)	0.0348 (0.0548)	0.0347 (0.0548)	0.0345 (0.0545)	0.0339 (0.0532)
0.4	0.0235 (0.0561)	0.0234 (0.0561)	0.0234 (0.0561)	0.0232 (0.0558)	0.0225 (0.0540)
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0480 (0.1277)	0.0481 (0.1277)	0.0485 (0.1308)	0.0499 (0.1274)	0.0568 (0.1286)
−0.2	0.0641 (0.1273)	0.0642 (0.1287)	0.0644 (0.1299)	0.0646 (0.1297)	0.0663 (0.1268)
0	0.0721 (0.1287)	0.0722 (0.1297)	0.0722 (0.1309)	0.0723 (0.1301)	0.0721 (0.1285)
0.2	0.0637 (0.1295)	0.0636 (0.1305)	0.0637 (0.1310)	0.0637 (0.1309)	0.0636 (0.1328)
0.4	0.0475 (0.1315)	0.0475 (0.1317)	0.0475 (0.1319)	0.0476 (0.1319)	0.0476 (0.1327)
			$l = 16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1593 (0.7312)	0.1598 (0.7311)	0.1606 (0.7330)	0.1623 (0.7347)	0.1668 (0.7189)
−0.2	0.1535 (0.7236)	0.1537 (0.7246)	0.1541 (0.7301)	0.1550 (0.7309)	0.1571 (0.7148)
0	0.1556 (0.7209)	0.1557 (0.7240)	0.1561 (0.7319)	0.1570 (0.7333)	0.1585 (0.7177)
0.2	0.1603 (0.7242)	0.1604 (0.7281)	0.1605 (0.7352)	0.1604 (0.7402)	0.1603 (0.7296)
0.4	0.1721 (0.7365)	0.1720 (0.7382)	0.1718 (0.7411)	0.1716 (0.7451)	0.1713 (0.7451)

Table 6.20: MSE of the Gaussian (log-periodogram) estimates of d_1, n=512, $m = 64$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0046 (0.0087)	0.0050 (0.0090)	0.0070 (0.0109)	0.0189 (0.0206)	0.0721 (0.0631)
−0.2	0.0051 (0.0081)	0.0051 (0.0082)	0.0051 (0.0083)	0.0067 (0.0101)	0.0228 (0.0246)
0	0.0052 (0.0079)	0.0052 (0.0079)	0.0051 (0.0080)	0.0050 (0.0080)	0.0074 (0.0104)
0.2	0.0053 (0.0079)	0.0053 (0.0079)	0.0053 (0.0079)	0.0052 (0.0080)	0.0050 (0.0080)
0.4	0.0051 (0.0079)	0.0051 (0.0079)	0.0051 (0.0080)	0.0050 (0.0081)	0.0048 (0.0082)
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0081 (0.0151)	0.0082 (0.0150)	0.0084 (0.0152)	0.0093 (0.0162)	0.0142 (0.0216)
−0.2	0.0095 (0.0145)	0.0095 (0.0145)	0.0095 (0.0145)	0.0096 (0.0142)	0.0108 (0.0162)
0	0.0095 (0.0144)	0.0095 (0.0143)	0.0095 (0.0142)	0.0095 (0.0142)	0.0098 (0.0146)
0.2	0.0096 (0.0144)	0.0096 (0.0144)	0.0096 (0.0144)	0.0096 (0.0145)	0.0097 (0.0147)
0.4	0.0084 (0.0147)	0.0084 (0.0147)	0.0084 (0.0148)	0.0084 (0.0149)	0.0083 (0.0150)
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0120 (0.0254)	0.0120 (0.0252)	0.0121 (0.0252)	0.0124 (0.0250)	0.0141 (0.0274)
−0.2	0.0157 (0.0248)	0.0157 (0.0248)	0.0157 (0.0246)	0.0157 (0.0244)	0.0161 (0.0257)
0	0.0162 (0.0248)	0.0162 (0.0247)	0.0162 (0.0246)	0.0163 (0.0245)	0.0164 (0.0251)
0.2	0.0165 (0.0249)	0.0165 (0.0249)	0.0165 (0.0249)	0.0166 (0.0250)	0.0167 (0.0258)
0.4	0.0140 (0.0256)	0.0140 (0.0256)	0.0141 (0.0257)	0.0141 (0.0257)	0.0142 (0.0261)
			$l = 16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0229 (0.0568)	0.0229 (0.0570)	0.0230 (0.0569)	0.0232 (0.0564)	0.0239 (0.0564)
−0.2	0.0326 (0.0563)	0.0326 (0.0564)	0.0326 (0.0565)	0.0326 (0.0559)	0.0327 (0.0556)
0	0.0373 (0.0563)	0.0373 (0.0564)	0.0372 (0.0565)	0.0371 (0.0565)	0.0369 (0.0559)
0.2	0.0371 (0.0571)	0.0371 (0.0571)	0.0371 (0.0573)	0.0371 (0.0575)	0.0370 (0.0574)
0.4	0.0319 (0.0588)	0.0319 (0.0589)	0.0318 (0.0590)	0.0317 (0.0592)	0.0316 (0.0596)

Table 6.21: MSE of the Gaussian (log-periodogram) estimates of d_1, n=512, $m = 128$

	$l = 0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0046 (0.0067)	0.0050 (0.0072)	0.0066 (0.0091)	0.0143 (0.0154)	0.0464 (0.0369)
−0.2	0.0027 (0.0044)	0.0028 (0.0045)	0.0029 (0.0048)	0.0042 (0.0064)	0.0133 (0.0139)
0	0.0024 (0.0037)	0.0024 (0.0038)	0.0024 (0.0038)	0.0023 (0.0040)	0.0034 (0.0053)
0.2	0.0031 (0.0041)	0.0031 (0.0041)	0.0031 (0.0042)	0.0030 (0.0042)	0.0027 (0.0041)
0.4	0.0046 (0.0053)	0.0046 (0.0053)	0.0045 (0.0053)	0.0045 (0.0053)	0.0043 (0.0053)
	$l = 4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0067 (0.0101)	0.0068 (0.0101)	0.0070 (0.0103)	0.0076 (0.0111)	0.0106 (0.0145)
−0.2	0.0041 (0.0069)	0.0041 (0.0068)	0.0042 (0.0068)	0.0043 (0.0070)	0.0052 (0.0080)
0	0.0034 (0.0056)	0.0034 (0.0055)	0.0035 (0.0056)	0.0035 (0.0056)	0.0035 (0.0058)
0.2	0.0045 (0.0061)	0.0045 (0.0061)	0.0045 (0.0061)	0.0045 (0.0061)	0.0043 (0.0061)
0.4	0.0071 (0.0083)	0.0071 (0.0083)	0.0071 (0.0084)	0.0070 (0.0083)	0.0069 (0.0081)
	$l = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0088 (0.0134)	0.0089 (0.0134)	0.0089 (0.0134)	0.0092 (0.0136)	0.0103 (0.0149)
−0.2	0.0056 (0.0094)	0.0056 (0.0093)	0.0056 (0.0093)	0.0057 (0.0093)	0.0061 (0.0099)
0	0.0048 (0.0079)	0.0048 (0.0079)	0.0048 (0.0079)	0.0048 (0.0078)	0.0049 (0.0079)
0.2	0.0064 (0.0088)	0.0064 (0.0088)	0.0064 (0.0088)	0.0064 (0.0088)	0.0063 (0.0088)
0.4	0.0102 (0.0121)	0.0102 (0.0121)	0.0102 (0.0121)	0.0102 (0.0120)	0.0101 (0.0120)
	$l = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0141 (0.0214)	0.0141 (0.0215)	0.0142 (0.0215)	0.0143 (0.0214)	0.0147 (0.0218)
−0.2	0.0093 (0.0152)	0.0093 (0.0152)	0.0093 (0.0152)	0.0094 (0.0151)	0.0095 (0.0153)
0	0.0082 (0.0128)	0.0082 (0.0129)	0.0082 (0.0129)	0.0082 (0.0130)	0.0082 (0.0130)
0.2	0.0110 (0.0145)	0.0110 (0.0146)	0.0110 (0.0147)	0.0110 (0.0147)	0.0109 (0.0147)
0.4	0.0174 (0.0202)	0.0174 (0.0202)	0.0174 (0.0203)	0.0173 (0.0202)	0.0172 (0.0201)

6.4.3 Efficiency

Table 6.22: eff of the untrimmed Gaussian/log-periodogram estimates of d_1, n=64

	$m=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3513	0.3475	0.3519	0.3217	0.3060
−0.2	0.3568	0.3554	0.3528	0.3562	0.3218
0	0.3815	0.3913	0.3943	0.4029	0.3878
0.2	0.4083	0.4129	0.4163	0.4203	0.4004
0.4	0.4345	0.4368	0.4369	0.4280	0.4148
	$m=8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4089	0.3940	0.3832	0.3466	0.3170
−0.2	0.5279	0.5188	0.5008	0.4839	0.4491
0	0.6145	0.6244	0.6142	0.6173	0.6019
0.2	0.6185	0.6181	0.6294	0.6336	0.6095
0.4	0.5316	0.5304	0.5272	0.5177	0.5089
&	$m=16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4787	0.4599	0.4425	0.3904	0.3225
−0.2	0.5776	0.5638	0.5515	0.5175	0.4461
0	0.6526	0.6414	0.6321	0.6370	0.5871
0.2	0.6664	0.6500	0.6566	0.6722	0.6517
0.4	0.6363	0.6416	0.6304	0.6222	0.5999

Table 6.23: eff of the Gaussian/log-periodogram estimates of d_1, n=128, $m = 8$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3724	0.3978	0.4482	0.5206	0.6689
−0.2	0.5012	0.5040	0.5446	0.5702	0.6384
0	0.6365	0.6294	0.6356	0.6476	0.6311
0.2	0.5960	0.6167	0.6168	0.6295	0.5987
0.4	0.5037	0.5050	0.5119	0.5112	0.4812
			$l = 1$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3330	0.3447	0.3526	0.3816	0.4337
−0.2	0.3694	0.3780	0.3748	0.3922	0.3882
0	0.4273	0.4277	0.4240	0.4148	0.4149
0.2	0.4401	0.4356	0.4322	0.4351	0.4159
0.4	0.4161	0.4148	0.4184	0.4241	0.4329
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.2538	0.2507	0.2497	0.2614	0.2715
−0.2	0.2410	0.2426	0.2365	0.2435	0.2498
0	0.2516	0.2540	0.2559	0.2572	0.2548
0.2	0.2733	0.2730	0.2731	0.2756	0.2756
0.4	0.2979	0.2954	0.2925	0.3067	0.3145
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.0802	0.0816	0.0826	0.0819	0.0856
−0.2	0.0637	0.0629	0.0631	0.0656	0.0665
0	0.0594	0.0591	0.0589	0.0595	0.0620
0.2	0.0657	0.0655	0.0661	0.0669	0.0705
0.4	0.0866	0.0861	0.0873	0.0903	0.0925

Table 6.24: eff of the Gaussian/log-periodogram estimates of d_1, n=128, $m = 16$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4142	0.4304	0.4825	0.5560	0.6800
−0.2	0.5997	0.5988	0.6270	0.6342	0.6556
0	0.7022	0.6843	0.6931	0.7043	0.6902
0.2	0.6836	0.6958	0.6892	0.7072	0.6921
0.4	0.5574	0.5573	0.5606	0.5598	0.5289
			$l = 1$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3871	0.3994	0.4047	0.4410	0.5128
−0.2	0.5427	0.5504	0.5381	0.5550	0.5684
0	0.6575	0.6525	0.6422	0.6274	0.6388
0.2	0.6346	0.6256	0.6169	0.6085	0.5930
0.4	0.5076	0.5010	0.4944	0.4941	0.5003
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3837	0.3817	0.3785	0.3934	0.4284
−0.2	0.4895	0.4879	0.4800	0.4812	0.5139
0	0.5557	0.5501	0.5465	0.5423	0.5620
0.2	0.5385	0.5373	0.5365	0.5331	0.5405
0.4	0.4665	0.4638	0.4584	0.4707	0.4772
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3654	0.3674	0.3728	0.3756	0.3920
−0.2	0.3925	0.3892	0.4022	0.4078	0.4107
0	0.4345	0.4320	0.4347	0.4279	0.4365
0.2	0.4419	0.4427	0.4435	0.4361	0.4424
0.4	0.4404	0.4399	0.4377	0.4423	0.4447

Table 6.25: eff of the Gaussian/log-periodogram estimates of d_1, n=128, $m = 32$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5324	0.5377	0.5587	0.5871	0.6675
−0.2	0.5642	0.5636	0.5628	0.5552	0.5892
0	0.6125	0.5965	0.5808	0.5796	0.5818
0.2	0.6663	0.6714	0.6701	0.6656	0.6438
0.4	0.6904	0.6855	0.6936	0.6882	0.6662
			$l = 1$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5185	0.5297	0.5348	0.5534	0.5910
−0.2	0.5835	0.5924	0.5884	0.5889	0.5873
0	0.6439	0.6389	0.6283	0.6023	0.6012
0.2	0.6845	0.6790	0.6671	0.6603	0.6444
0.4	0.6950	0.6963	0.6968	0.6910	0.6954
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5234	0.5244	0.5224	0.5251	0.5451
−0.2	0.5734	0.5762	0.5755	0.5639	0.5673
0	0.6209	0.6168	0.6102	0.5908	0.5902
0.2	0.6502	0.6513	0.6471	0.6396	0.6381
0.4	0.6630	0.6671	0.6694	0.6769	0.6741
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5122	0.5141	0.5187	0.5162	0.5236
−0.2	0.5452	0.5455	0.5561	0.5552	0.5447
0	0.6104	0.6070	0.6051	0.5875	0.5879
0.2	0.6287	0.6271	0.6245	0.6190	0.6292
0.4	0.6446	0.6513	0.6573	0.6637	0.6667

Table 6.26: eff of the Gaussian/log-periodogram estimates of d_1, n=256, $m = 16$

	$l = 0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4296	0.4417	0.5198	0.6519	0.7761
−0.2	0.6063	0.6248	0.6318	0.6432	0.7001
0	0.7042	0.6971	0.7007	0.6881	0.6826
0.2	0.6766	0.6639	0.6751	0.6713	0.6657
0.4	0.5083	0.5097	0.5082	0.4975	0.4737
	$l = 2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3572	0.3613	0.3654	0.3993	0.5321
−0.2	0.4968	0.5017	0.4989	0.4941	0.5379
0	0.6152	0.6099	0.6056	0.5991	0.5883
0.2	0.6121	0.6173	0.6132	0.5954	0.5881
0.4	0.4888	0.4906	0.4877	0.4766	0.4621
	$l = 4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3439	0.3453	0.3451	0.3466	0.4116
−0.2	0.4076	0.4071	0.4035	0.3977	0.4105
0	0.4515	0.4498	0.4486	0.4444	0.4262
0.2	0.4118	0.4155	0.4208	0.4190	0.4101
0.4	0.3511	0.3528	0.3536	0.3539	0.3599
	$l = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.1414	0.1430	0.1430	0.1446	0.1481
−0.2	0.1168	0.1180	0.1195	0.1182	0.1187
0	0.1127	0.1120	0.1100	0.1046	0.1074
0.2	0.1216	0.1206	0.1206	0.1177	0.1194
0.4	0.1453	0.1454	0.1447	0.1426	0.1461

Table 6.27: eff of the Gaussian/log-periodogram estimates of d_1, n=256, $m = 32$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5257	0.5306	0.5949	0.7223	0.8991
−0.2	0.6733	0.6857	0.6783	0.6783	0.7686
0	0.7015	0.6914	0.6913	0.6706	0.7015
0.2	0.6979	0.6810	0.6895	0.6888	0.7002
0.4	0.6012	0.6015	0.5982	0.5850	0.5712
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4316	0.4393	0.4483	0.4688	0.5774
−0.2	0.6220	0.6236	0.6142	0.5998	0.6153
0	0.6761	0.6707	0.6665	0.6607	0.6512
0.2	0.6831	0.6857	0.6814	0.6647	0.6603
0.4	0.5699	0.5686	0.5615	0.5466	0.5260
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4069	0.4050	0.4063	0.4075	0.4727
−0.2	0.5794	0.5725	0.5703	0.5624	0.5705
0	0.6288	0.6301	0.6289	0.6212	0.6023
0.2	0.6013	0.6038	0.6076	0.6038	0.5915
0.4	0.4785	0.4789	0.4773	0.4729	0.4635
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.3466	0.3472	0.3480	0.3534	0.3610
−0.2	0.4579	0.4563	0.4574	0.4542	0.4491
0	0.5413	0.5403	0.5362	0.5173	0.5188
0.2	0.5217	0.5161	0.5136	0.5081	0.5085
0.4	0.4514	0.4511	0.4496	0.4463	0.4484

Table 6.28: eff of the Gaussian/log-periodogram estimates of d_1, n=256, $m = 64$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.6369	0.6416	0.6687	0.7577	0.9465
−0.2	0.6292	0.6394	0.6376	0.6499	0.7600
0	0.6738	0.6652	0.6613	0.6375	0.6457
0.2	0.7431	0.7248	0.7309	0.7245	0.6975
0.4	0.8258	0.8269	0.8205	0.8062	0.7790
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.6219	0.6264	0.6257	0.6380	0.6780
−0.2	0.6303	0.6265	0.6258	0.6228	0.6369
0	0.6773	0.6721	0.6658	0.6588	0.6505
0.2	0.7585	0.7646	0.7644	0.7596	0.7421
0.4	0.8423	0.8466	0.8415	0.8361	0.8190
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.6048	0.6041	0.6025	0.6076	0.6381
−0.2	0.6024	0.5962	0.5998	0.6007	0.6082
0	0.6367	0.6396	0.6363	0.6261	0.6133
0.2	0.7151	0.7186	0.7210	0.7212	0.7109
0.4	0.7945	0.8003	0.7964	0.7996	0.7942
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5811	0.5828	0.5812	0.5905	0.6044
−0.2	0.5813	0.5798	0.5873	0.5891	0.5922
0	0.6219	0.6229	0.6180	0.6037	0.6043
0.2	0.7015	0.7010	0.6976	0.6916	0.6907
0.4	0.7595	0.7659	0.7616	0.7598	0.7564

Table 6.29: eff of the Gaussian/log-periodogram estimates of d_1, n=512

			$m = 32$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.4738	0.5150	0.5992	0.8120	0.9218
−0.2	0.6504	0.6440	0.6397	0.6630	0.8069
0	0.6913	0.6858	0.6699	0.6524	0.6942
0.2	0.6879	0.6908	0.6839	0.6559	0.6351
0.4	0.5560	0.5515	0.5412	0.5264	0.4805
			$m = 64$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.5304	0.5581	0.6414	0.9174	1.1420
−0.2	0.6307	0.6212	0.6128	0.6638	0.9262
0	0.6577	0.6508	0.6349	0.6239	0.7075
0.2	0.6708	0.6731	0.6678	0.6430	0.6257
0.4	0.6458	0.6410	0.6334	0.6222	0.5891
			$m = 128$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.6886	0.6938	0.7257	0.9262	1.2571
−0.2	0.6166	0.6139	0.6073	0.6623	0.9566
0	0.6427	0.6347	0.6160	0.5822	0.6513
0.2	0.7533	0.7527	0.7427	0.7099	0.6473
0.4	0.8603	0.8579	0.8518	0.8436	0.8084

6.4.4 Wald tests of the symmetry of the spectrum

Table 6.30: Power and size of the Gaussian (log-periodogram) Wald test, n=64

	$m = 4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.159 (0.286)	0.194 (0.298)	0.234 (0.323)	0.289 (0.356)	0.313 (0.382)
−0.2	0.166 (0.301)	0.183 (0.292)	0.227 (0.302)	0.260 (0.324)	0.279 (0.369)
0	0.236 (0.332)	0.237 (0.298)	0.232 (0.290)	0.236 (0.302)	0.241 (0.340)
0.2	0.286 (0.369)	0.274 (0.332)	0.252 (0.302)	0.223 (0.281)	0.191 (0.296)
0.4	0.327 (0.400)	0.298 (0.372)	0.267 (0.345)	0.223 (0.315)	0.181 (0.284)
	$m = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.091 (0.184)	0.153 (0.217)	0.292 (0.302)	0.460 (0.409)	0.598 (0.500)
−0.2	0.163 (0.215)	0.167 (0.192)	0.242 (0.233)	0.357 (0.314)	0.468 (0.412)
0	0.303 (0.298)	0.240 (0.229)	0.221 (0.209)	0.267 (0.240)	0.320 (0.317)
0.2	0.465 (0.398)	0.353 (0.297)	0.252 (0.226)	0.201 (0.198)	0.206 (0.243)
0.4	0.600 (0.501)	0.467 (0.396)	0.334 (0.298)	0.214 (0.219)	0.146 (0.199)
	$m = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.082 (0.166)	0.210 (0.223)	0.427 (0.338)	0.686 (0.511)	0.853 (0.691)
−0.2	0.194 (0.211)	0.146 (0.172)	0.252 (0.234)	0.459 (0.353)	0.688 (0.527)
0	0.413 (0.329)	0.234 (0.207)	0.166 (0.183)	0.261 (0.230)	0.440 (0.357)
0.2	0.672 (0.520)	0.453 (0.346)	0.248 (0.203)	0.169 (0.170)	0.228 (0.220)
0.4	0.848 (0.680)	0.680 (0.519)	0.437 (0.342)	0.228 (0.210)	0.116 (0.173)

Table 6.31: Power and size of the Gaussian (log-periodogram) Wald test, n=128

	$m = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.089 (0.184)	0.159 (0.215)	0.303 (0.271)	0.442 (0.366)	0.470 (0.402)
−0.2	0.160 (0.209)	0.164 (0.188)	0.241 (0.196)	0.341 (0.269)	0.388 (0.363)
0	0.276 (0.282)	0.230 (0.224)	0.193 (0.187)	0.236 (0.201)	0.280 (0.279)
0.2	0.432 (0.352)	0.347 (0.295)	0.252 (0.217)	0.186 (0.164)	0.178 (0.200)
0.4	0.477 (0.408)	0.396 (0.359)	0.298 (0.301)	0.179 (0.217)	0.120 (0.176)
	$m = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.058 (0.135)	0.177 (0.179)	0.445 (0.329)	0.700 (0.536)	0.846 (0.685)
−0.2	0.201 (0.200)	0.127 (0.141)	0.248 (0.178)	0.484 (0.343)	0.690 (0.532)
0	0.498 (0.355)	0.276 (0.205)	0.148 (0.142)	0.249 (0.173)	0.457 (0.358)
0.2	0.735 (0.570)	0.525 (0.373)	0.273 (0.214)	0.139 (0.124)	0.212 (0.184)
0.4	0.862 (0.704)	0.732 (0.580)	0.493 (0.385)	0.213 (0.224)	0.083 (0.125)
	$m = 32$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.076 (0.107)	0.309 (0.224)	0.680 (0.502)	0.927 (0.774)	0.987 (0.924)
−0.2	0.291 (0.213)	0.107 (0.109)	0.310 (0.237)	0.693 (0.512)	0.933 (0.773)
0	0.672 (0.513)	0.311 (0.234)	0.115 (0.108)	0.331 (0.234)	0.698 (0.525)
0.2	0.923 (0.773)	0.687 (0.520)	0.308 (0.226)	0.111 (0.104)	0.328 (0.247)
0.4	0.984 (0.910)	0.926 (0.770)	0.675 (0.513)	0.294 (0.223)	0.081 (0.101)

Table 6.32: Power and size of the Gaussian (log-periodogram) Wald test, n=256

	$m = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.080 (0.128)	0.180 (0.197)	0.439 (0.343)	0.646 (0.501)	0.673 (0.510)
−0.2	0.189 (0.185)	0.141 (0.138)	0.247 (0.195)	0.485 (0.346)	0.616 (0.465)
0	0.447 (0.316)	0.254 (0.192)	0.156 (0.127)	0.261 (0.177)	0.453 (0.347)
0.2	0.661 (0.494)	0.483 (0.349)	0.259 (0.192)	0.142 (0.131)	0.208 (0.182)
0.4	0.693 (0.543)	0.610 (0.467)	0.446 (0.346)	0.207 (0.191)	0.077 (0.129)
	$m = 32$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.067 (0.099)	0.288 (0.232)	0.716 (0.548)	0.935 (0.804)	0.974 (0.894)
−0.2	0.337 (0.223)	0.133 (0.103)	0.320 (0.245)	0.737 (0.561)	0.929 (0.795)
0	0.739 (0.576)	0.360 (0.245)	0.134 (0.112)	0.328 (0.247)	0.739 (0.563)
0.2	0.930 (0.812)	0.756 (0.587)	0.365 (0.259)	0.134 (0.112)	0.313 (0.240)
0.4	0.977 (0.906)	0.929 (0.807)	0.752 (0.570)	0.351 (0.241)	0.087 (0.105)
	$m = 64$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.085 (0.099)	0.460 (0.346)	0.924 (0.788)	0.996 (0.970)	1.000 (0.998)
−0.2	0.468 (0.322)	0.104 (0.097)	0.468 (0.365)	0.928 (0.800)	0.997 (0.969)
0	0.923 (0.771)	0.496 (0.343)	0.097 (0.094)	0.482 (0.371)	0.930 (0.801)
0.2	0.996 (0.964)	0.928 (0.787)	0.496 (0.352)	0.095 (0.091)	0.485 (0.373)
0.4	1.000 (0.994)	0.996 (0.955)	0.925 (0.785)	0.475 (0.336)	0.087 (0.093)

Table 6.33: Power and size of the Gaussian (log-periodogram) Wald test, n=512

	$m = 32$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.057 (0.098)	0.304 (0.232)	0.706 (0.537)	0.874 (0.733)	0.877 (0.738)
−0.2	0.296 (0.217)	0.116 (0.101)	0.341 (0.232)	0.728 (0.536)	0.843 (0.681)
0	0.702 (0.514)	0.337 (0.224)	0.114 (0.089)	0.342 (0.228)	0.698 (0.502)
0.2	0.875 (0.717)	0.699 (0.519)	0.335 (0.240)	0.103 (0.087)	0.313 (0.232)
0.4	0.866 (0.730)	0.847 (0.686)	0.673 (0.511)	0.313 (0.235)	0.065 (0.095)
	$m = 64$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.052 (0.068)	0.501 (0.365)	0.952 (0.847)	0.997 (0.978)	1.000 (0.993)
−0.2	0.534 (0.372)	0.085 (0.070)	0.523 (0.375)	0.953 (0.853)	0.998 (0.970)
0	0.949 (0.837)	0.565 (0.382)	0.085 (0.074)	0.526 (0.366)	0.948 (0.838)
0.2	0.995 (0.980)	0.954 (0.842)	0.574 (0.382)	0.084 (0.069)	0.514 (0.361)
0.4	0.997 (0.995)	0.995 (0.979)	0.959 (0.847)	0.555 (0.377)	0.068 (0.071)
	$m = 128$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.068 (0.070)	0.742 (0.565)	0.999 (0.974)	1.000 (1.000)	1.000 (1.000)
−0.2	0.736 (0.544)	0.074 (0.069)	0.756 (0.578)	0.999 (0.979)	1.000 (1.000)
0	0.996 (0.974)	0.753 (0.546)	0.073 (0.067)	0.762 (0.584)	0.998 (0.975)
0.2	1.000 (1.000)	0.997 (0.976)	0.757 (0.557)	0.068 (0.067)	0.758 (0.581)
0.4	1.000 (1.000)	1.000 (1.000)	0.997 (0.973)	0.756 (0.569)	0.064 (0.066)

6.4.5 LM tests of the symmetry of the spectrum

Table 6.34: Power and size of the LM_3 test, $n = 128$, $m = 8$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.001	0.001	0.002	0.008	0.014
−0.2	0.000	0.000	0.000	0.000	0.007
0	0.002	0.000	0.000	0.000	0.002
0.2	0.010	0.005	0.003	0.001	0.002
0.4	0.025	0.013	0.009	0.005	0.004
	$l=1$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.027	0.021	0.020	0.033	0.052
−0.2	0.069	0.055	0.037	0.043	0.045
0	0.140	0.106	0.081	0.065	0.058
0.2	0.212	0.169	0.133	0.099	0.081
0.4	0.266	0.216	0.173	0.139	0.103
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.038	0.028	0.035	0.057	0.091
−0.2	0.085	0.071	0.061	0.064	0.085
0	0.183	0.154	0.118	0.098	0.097
0.2	0.267	0.235	0.192	0.147	0.121
0.4	0.335	0.298	0.243	0.204	0.165
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.060	0.071	0.082	0.100	0.178
−0.2	0.118	0.111	0.114	0.132	0.178
0	0.213	0.198	0.191	0.191	0.205
0.2	0.313	0.297	0.279	0.260	0.252
0.4	0.399	0.379	0.360	0.332	0.302

Table 6.35: Power and size of the LM_3 test, $n = 128$, $m = 16$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.007	0.015	0.086	0.236	0.422
−0.2	0.011	0.004	0.016	0.090	0.233
0	0.068	0.012	0.005	0.016	0.094
0.2	0.202	0.074	0.013	0.005	0.020
0.4	0.384	0.207	0.082	0.021	0.008
	$l=1$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.057	0.050	0.090	0.197	0.328
−0.2	0.125	0.073	0.054	0.088	0.193
0	0.247	0.138	0.076	0.053	0.108
0.2	0.374	0.253	0.145	0.083	0.053
0.4	0.499	0.385	0.258	0.157	0.091
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.072	0.080	0.120	0.213	0.318
−0.2	0.160	0.102	0.094	0.131	0.216
0	0.291	0.181	0.113	0.086	0.121
0.2	0.438	0.317	0.185	0.113	0.087
0.4	0.571	0.449	0.339	0.207	0.123
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.130	0.157	0.208	0.284	0.354
−0.2	0.221	0.189	0.203	0.224	0.275
0	0.358	0.277	0.221	0.205	0.204
0.2	0.482	0.374	0.287	0.217	0.168
0.4	0.603	0.508	0.393	0.305	0.220

Table 6.36: Power and size of the LM_3 test, $n = 128$, $m = 32$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.009	0.083	0.362	0.719	0.935
−0.2	0.064	0.011	0.103	0.401	0.751
0	0.327	0.081	0.013	0.116	0.433
0.2	0.663	0.357	0.093	0.017	0.125
0.4	0.882	0.684	0.380	0.115	0.023
	$l=1$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.043	0.071	0.333	0.648	0.862
−0.2	0.158	0.049	0.092	0.366	0.658
0	0.368	0.175	0.056	0.111	0.358
0.2	0.666	0.415	0.201	0.077	0.122
0.4	0.853	0.705	0.443	0.247	0.100
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.063	0.107	0.342	0.635	0.808
−0.2	0.185	0.071	0.129	0.343	0.612
0	0.404	0.215	0.092	0.133	0.335
0.2	0.658	0.462	0.249	0.106	0.134
0.4	0.840	0.695	0.525	0.313	0.137
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.115	0.188	0.377	0.594	0.731
−0.2	0.230	0.149	0.187	0.362	0.550
0	0.437	0.266	0.183	0.183	0.330
0.2	0.670	0.501	0.326	0.219	0.189
0.4	0.827	0.730	0.586	0.396	0.259

Table 6.37: Power and size of the LM_3 test, $n = 256$, $m = 16$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.004	0.019	0.075	0.197	0.304
−0.2	0.022	0.006	0.017	0.073	0.186
0	0.078	0.024	0.004	0.016	0.073
0.2	0.204	0.088	0.022	0.002	0.015
0.4	0.306	0.194	0.089	0.026	0.005
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.079	0.064	0.103	0.177	0.240
−0.2	0.190	0.120	0.091	0.110	0.177
0	0.324	0.221	0.121	0.090	0.101
0.2	0.432	0.329	0.225	0.131	0.079
0.4	0.540	0.438	0.332	0.220	0.134
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.119	0.143	0.216	0.303	0.377
−0.2	0.257	0.210	0.216	0.258	0.291
0	0.399	0.308	0.240	0.227	0.232
0.2	0.508	0.419	0.318	0.244	0.195
0.4	0.590	0.511	0.426	0.332	0.235
	$l=8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.321	0.368	0.425	0.522	0.607
−0.2	0.438	0.428	0.456	0.492	0.550
0	0.503	0.475	0.452	0.450	0.452
0.2	0.538	0.498	0.458	0.410	0.383
0.4	0.565	0.527	0.476	0.427	0.374

Table 6.38: Power and size of the LM_3 test, $n = 256$, $m = 32$

			$l=0$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.014	0.098	0.427	0.766	0.917
−0.2	0.081	0.013	0.106	0.439	0.762
0	0.423	0.088	0.012	0.112	0.441
0.2	0.781	0.438	0.093	0.014	0.119
0.4	0.909	0.777	0.436	0.091	0.017
			$l=2$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.093	0.120	0.364	0.657	0.759
−0.2	0.256	0.096	0.127	0.371	0.633
0	0.495	0.265	0.096	0.139	0.363
0.2	0.728	0.504	0.279	0.112	0.137
0.4	0.894	0.744	0.526	0.283	0.118
			$l=4$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.150	0.180	0.372	0.584	0.686
−0.2	0.310	0.171	0.205	0.367	0.565
0	0.540	0.325	0.169	0.205	0.356
0.2	0.732	0.552	0.338	0.173	0.190
0.4	0.863	0.748	0.572	0.361	0.189
			$l=8$		
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.245	0.312	0.468	0.585	0.656
−0.2	0.401	0.324	0.355	0.476	0.571
0	0.553	0.417	0.331	0.350	0.461
0.2	0.693	0.574	0.426	0.328	0.320
0.4	0.816	0.715	0.592	0.423	0.287

Table 6.39: Power and size of the LM_3 test, $n = 256$, $m = 64$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.019	0.252	0.817	0.989	1.000
−0.2	0.225	0.028	0.284	0.845	0.988
0	0.789	0.262	0.031	0.314	0.864
0.2	0.996	0.814	0.300	0.033	0.339
0.4	1.000	0.997	0.838	0.324	0.035
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.074	0.277	0.788	0.974	0.993
−0.2	0.280	0.084	0.291	0.780	0.962
0	0.727	0.329	0.096	0.298	0.769
0.2	0.967	0.772	0.383	0.125	0.292
0.4	0.999	0.971	0.823	0.445	0.154
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.108	0.316	0.754	0.954	0.977
−0.2	0.300	0.122	0.301	0.729	0.938
0	0.687	0.362	0.142	0.290	0.697
0.2	0.937	0.755	0.434	0.170	0.265
0.4	0.996	0.957	0.825	0.532	0.223
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.202	0.429	0.725	0.905	0.947
−0.2	0.311	0.214	0.389	0.665	0.847
0	0.660	0.376	0.232	0.345	0.603
0.2	0.898	0.720	0.481	0.268	0.299
0.4	0.984	0.938	0.810	0.608	0.335

Table 6.40: Power and size of the LM_3 test, $n = 512$, $m = 32$

	$l = 0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.009	0.067	0.376	0.680	0.770
−0.2	0.096	0.010	0.079	0.391	0.666
0	0.438	0.103	0.013	0.082	0.388
0.2	0.737	0.445	0.105	0.014	0.080
0.4	0.784	0.686	0.440	0.107	0.016
	$l = 4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.140	0.160	0.319	0.495	0.541
−0.2	0.338	0.189	0.169	0.329	0.488
0	0.551	0.341	0.198	0.168	0.329
0.2	0.751	0.559	0.337	0.189	0.153
0.4	0.865	0.750	0.567	0.340	0.183
	$l = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.235	0.313	0.430	0.533	0.570
−0.2	0.431	0.339	0.352	0.445	0.522
0	0.581	0.437	0.348	0.348	0.421
0.2	0.724	0.580	0.441	0.330	0.309
0.4	0.821	0.723	0.585	0.429	0.285
	$l = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.471	0.550	0.613	0.710	0.778
−0.2	0.653	0.648	0.675	0.707	0.758
0	0.706	0.678	0.684	0.681	0.716
0.2	0.718	0.671	0.628	0.602	0.581
0.4	0.713	0.644	0.577	0.511	0.446

Table 6.41: Power and size of the LM_3 test, $n = 512$, $m = 64$

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.017	0.287	0.890	0.996	1.000
−0.2	0.327	0.024	0.310	0.895	0.997
0	0.898	0.337	0.023	0.333	0.903
0.2	0.990	0.902	0.340	0.020	0.346
0.4	0.996	0.989	0.906	0.332	0.024
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.132	0.301	0.771	0.937	0.921
−0.2	0.430	0.136	0.326	0.785	0.907
0	0.818	0.430	0.134	0.325	0.751
0.2	0.973	0.833	0.449	0.136	0.298
0.4	0.998	0.975	0.844	0.473	0.159
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.195	0.351	0.740	0.877	0.865
−0.2	0.468	0.212	0.348	0.729	0.848
0	0.776	0.476	0.216	0.352	0.699
0.2	0.941	0.782	0.488	0.217	0.336
0.4	0.986	0.947	0.804	0.524	0.219
			$l=16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.336	0.489	0.696	0.792	0.786
−0.2	0.489	0.371	0.484	0.683	0.750
0	0.712	0.503	0.377	0.480	0.645
0.2	0.886	0.736	0.532	0.390	0.470
0.4	0.970	0.903	0.750	0.551	0.346

Table 6.42: Power and size of the LM_3 test, $n = 512$, $m = 128$

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.023	0.588	0.998	1.000	1.000
−0.2	0.593	0.033	0.630	0.998	1.000
0	0.994	0.629	0.038	0.670	1.000
0.2	1.000	0.996	0.671	0.043	0.698
0.4	1.000	1.000	0.996	0.707	0.047
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.075	0.613	0.988	1.000	1.000
−0.2	0.537	0.083	0.612	0.989	1.000
0	0.968	0.608	0.092	0.598	0.985
0.2	1.000	0.982	0.691	0.132	0.544
0.4	1.000	1.000	0.987	0.767	0.182
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.116	0.649	0.975	0.996	0.996
−0.2	0.476	0.128	0.611	0.966	0.991
0	0.945	0.573	0.149	0.543	0.950
0.2	0.999	0.965	0.695	0.203	0.459
0.4	1.000	0.999	0.981	0.794	0.286
			$l=16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.258	0.703	0.953	0.993	0.990
−0.2	0.408	0.227	0.626	0.924	0.974
0	0.888	0.533	0.243	0.514	0.842
0.2	0.994	0.942	0.682	0.314	0.411
0.4	1.000	0.997	0.980	0.829	0.430

Table 6.43: Power and size of the LM_2 test, $n = 128$, $m = 8$

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.001	0.002	0.018	0.033	0.070
−0.2	0.000	0.000	0.004	0.017	0.044
0	0.000	0.000	0.001	0.007	0.029
0.2	0.000	0.000	0.000	0.003	0.018
0.4	0.000	0.000	0.000	0.002	0.018
			$l=1$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.044	0.106	0.194	0.278	0.342
−0.2	0.033	0.079	0.153	0.234	0.292
0	0.023	0.054	0.092	0.173	0.236
0.2	0.026	0.041	0.071	0.112	0.182
0.4	0.038	0.037	0.053	0.072	0.120
			$l=2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.047	0.122	0.215	0.317	0.401
−0.2	0.043	0.099	0.190	0.271	0.341
0	0.040	0.084	0.153	0.226	0.293
0.2	0.056	0.081	0.124	0.183	0.252
0.4	0.095	0.094	0.109	0.148	0.202
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.064	0.142	0.248	0.360	0.436
−0.2	0.060	0.138	0.244	0.342	0.417
0	0.067	0.135	0.231	0.325	0.400
0.2	0.102	0.152	0.239	0.310	0.382
0.4	0.187	0.204	0.246	0.291	0.341

Table 6.44: Power and size of the LM_2 test, n = 128, m = 16

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.007	0.053	0.191	0.416	0.620
−0.2	0.002	0.011	0.060	0.208	0.407
0	0.001	0.001	0.009	0.066	0.212
0.2	0.000	0.001	0.002	0.011	0.066
0.4	0.000	0.000	0.000	0.003	0.026
			$l=1$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.066	0.176	0.332	0.500	0.617
−0.2	0.038	0.092	0.192	0.343	0.506
0	0.012	0.039	0.098	0.211	0.358
0.2	0.008	0.013	0.041	0.090	0.215
0.4	0.009	0.009	0.016	0.040	0.107
			$l=2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.080	0.210	0.359	0.544	0.669
−0.2	0.054	0.124	0.233	0.393	0.556
0	0.030	0.068	0.137	0.247	0.415
0.2	0.022	0.037	0.068	0.149	0.259
0.4	0.022	0.022	0.034	0.067	0.160
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.115	0.272	0.427	0.547	0.663
−0.2	0.086	0.202	0.319	0.459	0.569
0	0.061	0.128	0.227	0.344	0.478
0.2	0.054	0.094	0.150	0.229	0.375
0.4	0.069	0.079	0.105	0.166	0.257

Table 6.45: Power and size of the LM_2 test, $n = 128$, $m = 32$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.011	0.159	0.516	0.836	0.968
−0.2	0.004	0.016	0.184	0.552	0.859
0	0.000	0.003	0.023	0.203	0.580
0.2	0.000	0.000	0.002	0.031	0.229
0.4	0.000	0.000	0.000	0.001	0.036
	$l=1$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.069	0.220	0.498	0.782	0.933
−0.2	0.023	0.078	0.253	0.543	0.820
0	0.006	0.019	0.091	0.294	0.591
0.2	0.002	0.006	0.026	0.109	0.343
0.4	0.000	0.002	0.006	0.025	0.130
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.085	0.245	0.504	0.750	0.905
−0.2	0.019	0.104	0.284	0.571	0.813
0	0.006	0.027	0.118	0.339	0.616
0.2	0.003	0.008	0.032	0.149	0.390
0.4	0.001	0.002	0.013	0.048	0.193
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.110	0.283	0.491	0.742	0.884
−0.2	0.045	0.144	0.332	0.556	0.792
0	0.017	0.062	0.177	0.399	0.641
0.2	0.008	0.027	0.089	0.224	0.479
0.4	0.008	0.014	0.039	0.122	0.283

Table 6.46: Power and size of the LM_2 test, $n = 256$, $m = 16$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.007	0.053	0.165	0.355	0.461
−0.2	0.000	0.009	0.052	0.179	0.345
0	0.000	0.000	0.008	0.051	0.180
0.2	0.000	0.000	0.000	0.010	0.064
0.4	0.000	0.000	0.000	0.000	0.024
			$l = 2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.084	0.218	0.361	0.512	0.609
−0.2	0.057	0.136	0.242	0.373	0.511
0	0.045	0.082	0.140	0.246	0.381
0.2	0.045	0.043	0.074	0.138	0.258
0.4	0.067	0.052	0.045	0.070	0.145
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.106	0.270	0.414	0.527	0.633
−0.2	0.092	0.206	0.330	0.428	0.542
0	0.095	0.156	0.234	0.341	0.446
0.2	0.123	0.134	0.175	0.247	0.350
0.4	0.178	0.148	0.140	0.176	0.259
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.123	0.302	0.428	0.516	0.586
−0.2	0.126	0.272	0.396	0.465	0.541
0	0.139	0.263	0.360	0.428	0.486
0.2	0.216	0.264	0.329	0.384	0.441
0.4	0.316	0.300	0.312	0.340	0.394

Table 6.47: Power and size of the LM_2 test, $n = 256$, $m = 32$

			$l=0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.021	0.180	0.574	0.869	0.961
−0.2	0.000	0.021	0.203	0.597	0.867
0	0.000	0.000	0.024	0.221	0.604
0.2	0.000	0.000	0.000	0.023	0.215
0.4	0.000	0.000	0.000	0.000	0.027
			$l=2$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.104	0.297	0.576	0.810	0.941
−0.2	0.027	0.113	0.306	0.594	0.811
0	0.006	0.025	0.115	0.317	0.609
0.2	0.007	0.010	0.030	0.122	0.337
0.4	0.008	0.008	0.011	0.030	0.130
			$l=4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.138	0.356	0.582	0.786	0.905
−0.2	0.068	0.177	0.359	0.599	0.805
0	0.032	0.080	0.176	0.379	0.612
0.2	0.021	0.032	0.084	0.181	0.401
0.4	0.028	0.023	0.037	0.087	0.205
			$l=8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.156	0.379	0.574	0.729	0.848
−0.2	0.102	0.232	0.403	0.593	0.751
0	0.077	0.139	0.242	0.427	0.614
0.2	0.066	0.094	0.148	0.271	0.454
0.4	0.095	0.085	0.102	0.163	0.289

Table 6.48: Power and size of the LM_2 test, $n = 256$, $m = 64$

	$l=0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.030	0.380	0.902	0.994	1.000
−0.2	0.000	0.032	0.413	0.924	0.997
0	0.000	0.000	0.035	0.445	0.928
0.2	0.000	0.000	0.000	0.043	0.471
0.4	0.000	0.000	0.000	0.000	0.051
	$l=2$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.073	0.373	0.817	0.977	1.000
−0.2	0.008	0.077	0.430	0.858	0.985
0	0.002	0.008	0.089	0.495	0.897
0.2	0.001	0.001	0.011	0.122	0.555
0.4	0.001	0.001	0.002	0.015	0.163
	$l=4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.092	0.374	0.776	0.957	0.995
−0.2	0.017	0.107	0.438	0.834	0.976
0	0.004	0.024	0.145	0.527	0.870
0.2	0.001	0.002	0.027	0.189	0.613
0.4	0.001	0.001	0.004	0.044	0.242
	$l=8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.100	0.380	0.700	0.926	0.983
−0.2	0.030	0.135	0.445	0.794	0.954
0	0.004	0.049	0.191	0.547	0.865
0.2	0.002	0.012	0.077	0.280	0.642
0.4	0.006	0.008	0.025	0.107	0.374

Table 6.49: Power and size of the LM_2 test, $n = 512$, $m = 32$

			$l = 0$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.016	0.143	0.531	0.803	0.845
−0.2	0.001	0.017	0.156	0.565	0.775
0	0.000	0.001	0.020	0.159	0.573
0.2	0.000	0.000	0.001	0.022	0.165
0.4	0.000	0.000	0.000	0.000	0.029
			$l = 4$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.121	0.332	0.571	0.779	0.885
−0.2	0.063	0.170	0.349	0.576	0.783
0	0.026	0.066	0.161	0.354	0.588
0.2	0.038	0.032	0.061	0.154	0.372
0.4	0.114	0.047	0.035	0.065	0.167
			$l = 8$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.166	0.397	0.571	0.746	0.857
−0.2	0.116	0.245	0.407	0.583	0.750
0	0.087	0.143	0.245	0.414	0.597
0.2	0.118	0.101	0.144	0.247	0.421
0.4	0.241	0.151	0.120	0.161	0.263
			$l = 16$		
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.180	0.422	0.530	0.620	0.703
−0.2	0.176	0.353	0.449	0.538	0.630
0	0.194	0.302	0.374	0.455	0.551
0.2	0.273	0.277	0.326	0.385	0.463
0.4	0.378	0.311	0.295	0.333	0.392

Table 6.50: Power and size of the LM_2 test, $n = 512$, $m = 64$

	$l = 0$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.032	0.418	0.943	1.000	1.000
−0.2	0.001	0.031	0.450	0.952	0.999
0	0.000	0.001	0.029	0.467	0.952
0.2	0.000	0.000	0.001	0.033	0.472
0.4	0.000	0.000	0.000	0.001	0.031
	$l = 4$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.128	0.465	0.868	0.986	0.999
−0.2	0.016	0.122	0.488	0.879	0.988
0	0.000	0.015	0.123	0.523	0.892
0.2	0.000	0.000	0.015	0.138	0.554
0.4	0.003	0.000	0.000	0.022	0.158
	$l = 8$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.164	0.493	0.809	0.965	0.997
−0.2	0.035	0.175	0.502	0.830	0.973
0	0.012	0.038	0.186	0.526	0.846
0.2	0.008	0.017	0.040	0.207	0.561
0.4	0.025	0.008	0.014	0.050	0.234
	$l = 16$				
$d_1 \backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.213	0.483	0.739	0.900	0.973
−0.2	0.087	0.262	0.498	0.766	0.920
0	0.043	0.102	0.286	0.537	0.795
0.2	0.040	0.049	0.114	0.311	0.576
0.4	0.087	0.052	0.061	0.141	0.342

Table 6.51: Power and size of the LM_2 test, $n = 512$, $m = 128$

	$l=0$				
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.030	0.716	1.000	1.000	1.000
−0.2	0.000	0.036	0.753	1.000	1.000
0	0.000	0.001	0.041	0.778	1.000
0.2	0.000	0.000	0.001	0.046	0.808
0.4	0.000	0.000	0.000	0.000	0.049
	$l=4$				
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.068	0.610	0.986	1.000	1.000
−0.2	0.000	0.079	0.686	0.994	1.000
0	0.000	0.001	0.110	0.757	0.997
0.2	0.000	0.000	0.001	0.162	0.832
0.4	0.000	0.000	0.000	0.006	0.217
	$l=8$				
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.070	0.558	0.969	0.999	1.000
−0.2	0.002	0.095	0.656	0.976	0.999
0	0.000	0.004	0.153	0.758	0.991
0.2	0.000	0.000	0.008	0.223	0.840
0.4	0.000	0.000	0.001	0.020	0.333
	$l=16$				
$d_1\backslash d_2$	−0.4	−0.2	0	0.2	0.4
−0.4	0.077	0.456	0.910	0.997	1.000
−0.2	0.007	0.119	0.591	0.956	0.999
0	0.001	0.012	0.210	0.746	0.984
0.2	0.000	0.002	0.028	0.333	0.858
0.4	0.001	0.003	0.005	0.066	0.479

6.4.6 LM test of equal persistence across frequencies

Table 6.52: Power and size of the LM_H test, $n = 128$

	$m = 8$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.002	0.002	0.001	0.001	0.005
−0.2	0.003	0.002	0.001	0.001	0.006
0	0.004	0.003	0.002	0.001	0.006
0.2	0.014	0.008	0.004	0.002	0.006
0.4	0.059	0.052	0.030	0.008	0.007
	$m = 16$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.020	0.007	0.005	0.019	0.131
−0.2	0.024	0.005	0.004	0.010	0.090
0	0.038	0.021	0.006	0.002	0.034
0.2	0.148	0.090	0.031	0.008	0.007
0.4	0.422	0.324	0.216	0.092	0.027
	$m = 24$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.037	0.013	0.021	0.159	0.527
−0.2	0.057	0.016	0.010	0.086	0.421
0	0.135	0.054	0.009	0.026	0.202
0.2	0.381	0.200	0.075	0.021	0.043
0.4	0.802	0.657	0.424	0.201	0.062

Table 6.53: Power and size of the LM_H test, $n = 256$

	$m = 16$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.014	0.012	0.007	0.017	0.124
−0.2	0.017	0.013	0.010	0.011	0.104
0	0.031	0.024	0.012	0.005	0.044
0.2	0.104	0.083	0.042	0.015	0.014
0.4	0.414	0.365	0.238	0.092	0.016
	$m = 32$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.048	0.016	0.028	0.205	0.681
−0.2	0.060	0.021	0.017	0.134	0.581
0	0.149	0.063	0.011	0.044	0.318
0.2	0.474	0.317	0.131	0.018	0.062
0.4	0.882	0.826	0.656	0.322	0.053
	$m = 48$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.090	0.018	0.098	0.563	0.958
−0.2	0.156	0.031	0.045	0.384	0.904
0	0.379	0.120	0.025	0.131	0.658
0.2	0.825	0.567	0.249	0.036	0.189
0.4	0.993	0.963	0.869	0.530	0.112

Table 6.54: Power and size of the LM_H test, $n = 512$

	$m = 32$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.039	0.021	0.022	0.128	0.621
−0.2	0.041	0.025	0.021	0.124	0.595
0	0.080	0.054	0.014	0.051	0.391
0.2	0.343	0.300	0.161	0.021	0.074
0.4	0.852	0.825	0.696	0.325	0.042
	$m = 64$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.109	0.034	0.090	0.581	0.988
−0.2	0.147	0.040	0.051	0.473	0.972
0	0.336	0.137	0.026	0.165	0.844
0.2	0.830	0.692	0.344	0.034	0.239
0.4	0.999	0.992	0.957	0.686	0.105
	$m = 96$				
$d_0 \backslash d_1$	−0.4	−0.2	0	0.2	0.4
−0.4	0.221	0.024	0.264	0.941	1.000
−0.2	0.365	0.046	0.102	0.819	0.999
0	0.724	0.288	0.024	0.348	0.978
0.2	0.992	0.923	0.544	0.045	0.447
0.4	1.000	1.000	0.994	0.884	0.216

Chapter 7

SEASONAL LONG-MEMORY IN UK INFLATION

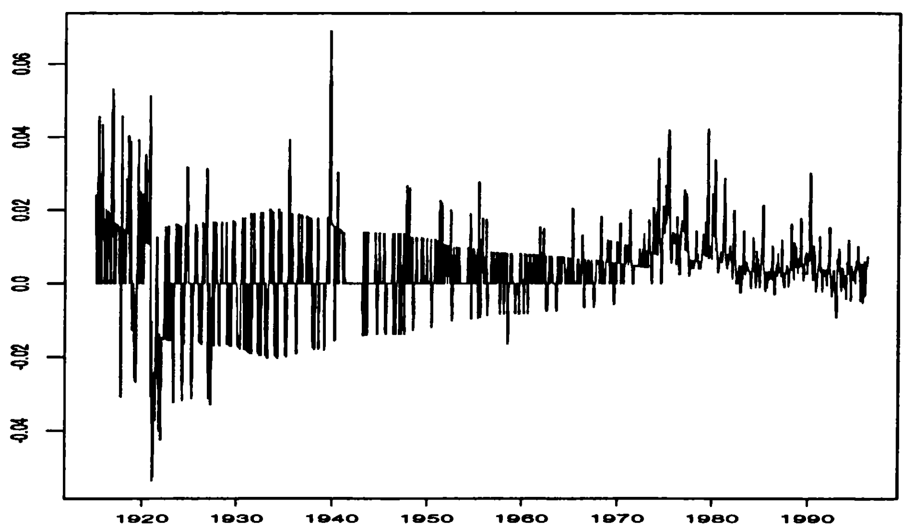
7.1 INTRODUCTION

The characteristic of long-range dependence in various inflation series has been analysed by several authors. Many of these works have focused on testing the PPP (Purchasing Power Parity) (see Cheung and Lai (1993)), Fisher's hypothesis that the one-period nominal rate of interest is the equilibrium real return plus the fully anticipated rate of inflation (e.g. Barsky (1987)) or Friedman's (1977) hypothesis of a positive association between inflation and its uncertainty (e.g. Baillie, Chung and Tieslau (1996)). In order to see if these properties are reflected in the real world we need knowledge of the long-run as well as the short-run behaviour of inflation. The research done to date trying to describe the long-run properties of inflation use seasonally adjusted data (usually via seasonal dummies) or simply pay attention only to low frequency behaviour (as in Delgado and Robinson (1994)). Nevertheless, the seasonal component of economic series such as inflation is important and deserves a more thorough analysis. In that sense Franses and Ooms (1995) model quarterly United Kingdom inflation using the so-called PARFIMA(0,d_s,0) (Periodic Autoregressive Fractionally Integrated Moving Average) that allows for different behaviour in every season since the value of the persistence parameter, d_s, can vary with season, $s = 1, 2, 3, 4$.

In this chapter we analyze UK monthly inflation from 1915 to 1996, investigating the possibility of seasonal as well as low frequency long-range dependence. Long memory in monthly inflation series of the UK and other four industrial countries in the period 1969-1992 has recently been analysed by Hassler and Wolters (1995). They focus only on low frequency behaviour and try to eliminate seasonality by means of seasonal dummies, noticing however that this deterministic seasonal adjustment was not completely adequate. In this chapter we examine the possibility of seasonal long-range dependence in the sense that the spectral density function diverges at seasonal frequencies, in addition to the usual analysis at zero frequency. In order to perform this analysis we avoid restrictive parametric models, instead using the semiparametric methods described in previous chapters.

The series we use is the UK RPI (Retail Price Index) from April 1915 to April 1996 and the inflation series is constructed by first differencing the logarithm of the RPI. Call p_t the RPI at time t, then the series we will analyze is $\pi_t = \log p_t - \log p_{t-1}$ from May 1915 to April 1996 so that we have n =972 observations. All the calculations and figures were done using S-Plus 3.1.

Figure 7.1: UK Inflation from May 1915 to April 1996

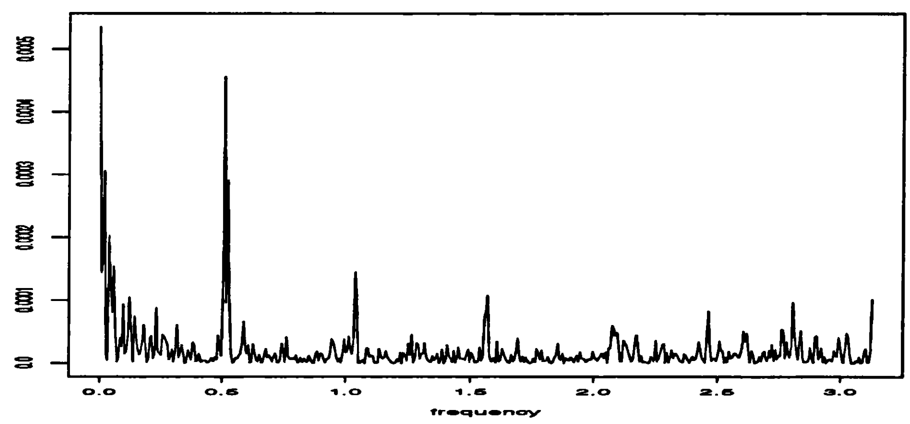


The plot of the inflation series, π_t, in Figure 7.1 suggests different behaviour before and after the late sixties-early seventies. Possible causes of this change of pattern may be the sixties devaluation, the change from fiscal to monetary policy in 1971 focusing on the control of the quantity of money and letting interest rates move freely in the market, and the return to floating exchange rates in 1973. From the technical point of view the apparent structural change can also be related to the inclusion of mortgage interest payments in the RPI since 1974. For a more exhaustive analysis of those facts see among many others Rowlatt (1992) or Joyce (1995). Without attempting to estimate a change point or study its causes, we analyze the existence of long range dependence in the whole series and in two subseries, April 1915-September 1969 and October 1969-April 1996.

7.2 DIFFERENCES ACROSS FREQUENCIES

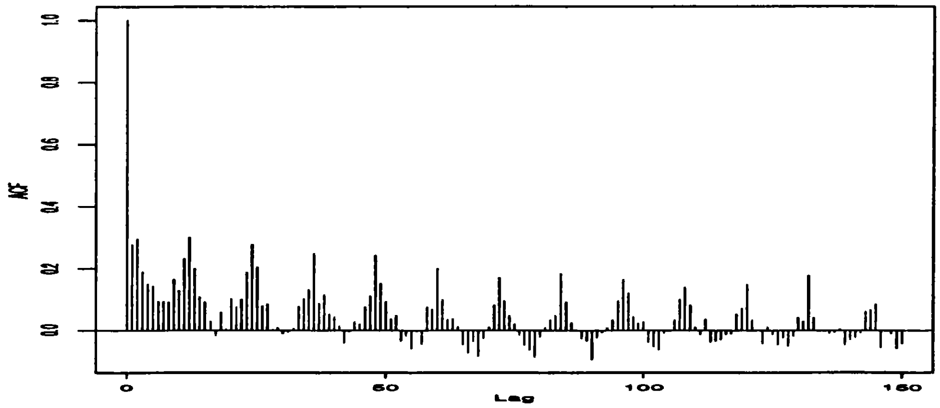
Figure 7.2 displays the periodogram of π_t from 5:1915 to 4:1996. Of course this is not a consistent estimate of the spectral density, but the sharp peaks at the origin and to varying extents at seasonal frequencies, suggest the possibility of low frequency as well as seasonal long memory.

Figure 7.2: Periodogram of UK Inflation (5:1915-4:1996)



We extract the same conclusion from the plot of the first 150 autocorrelations in Figure 7.3. We observe oscillations that decay very slowly, as explained by the theoretical study of seasonal long memory models in Chapters 1 and 2.

Figure 7.3: Sample autocorrelations of UK Inflation (5:1915-4:1996)



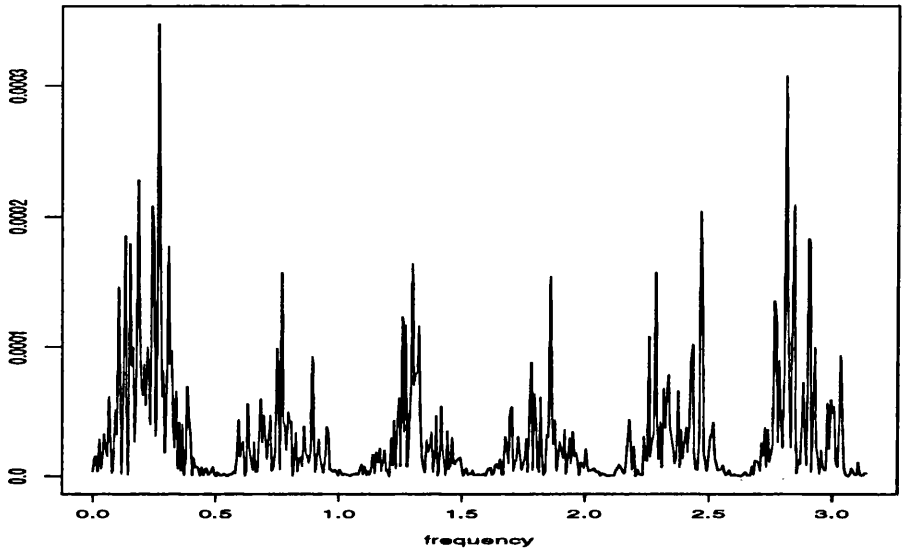
The issue of seasonality has usually been treated either by including seasonal dummies or seasonal differencing. The unsuitability of the former treatment, so far as UK inflation is concerned, has been pointed out by Hassler and Wolters (1995). The latter seems excessive. Figure 7.4 shows the periodogram of the seasonal differenced series $(1 - L^{12})\pi_t$. We observe deep troughs at the origin and at seasonal frequencies suggesting possible overdifferencing. A milder fractional differencing could be more appropriate. Moreover we observe in Figure 7.2 that each of the peaks may be of different magnitude, suggesting the possibility of different persistence parameters at the origin and across seasonal frequencies.

In this section we use the results obtained in Chapter 5 and, assuming symmetry of the spectral poles, we perform Wald and score tests of the equality of the persistence parameters, d_s, across frequencies $\omega_s = 2\pi s/12$, $s = 0, 1, ..., 6$. The first hypothesis we test is

$$H_0 : d_0 = d_1 = ... = d_6 \tag{7.1}$$

against the alternative that one or more of the equalities in (7.1) do not hold. The asymptotic independence of the estimates (log-periodogram and local Whittle) of the persistence parameters across different frequencies (this can be shown in the same

Figure 7.4: Periodogram of Seasonal Differenced UK Inflation



manner as the asymptotic independence of the estimates to the right and left of a known frequency in Chapter 5), provides us with two simple Wald tests statistics

$$\begin{aligned} W_{l1} &= \frac{24m}{\pi^2}(R_1\hat{d}^1)'(R_1H_1R_1')^{-1}(R_1\hat{d}^1) \\ W_{g1} &= 4m(R_1\tilde{d}^1)'(R_1H_1R_1')^{-1}(R_1\tilde{d}^1) \end{aligned}$$

where m is a bandwidth number, $\hat{d}^1$ and $\tilde{d}^1$ are 7×1 vectors with elements the log-periodogram (J=1 and $l = 0$) and local Whittle estimates of $d = (d_0, ..., d_6)$ respectively, R_1 is a 6×7 matrix of zeros except the $[R_1]_{ii}$ and $[R_1]_{i(i+1)}$ elements that are 1 and -1 respectively, for $i = 1, ..., 6$. Thus the null and alternative in (7.1) can be written

$$\begin{aligned} H_0 &: R_1 d = 0 \\ H_1 &: R_1 d \neq 0. \end{aligned}$$

Due to perfect symmetry of the periodogram at $\omega_0 = 0$ and $\omega_6 = \pi$ we only use m frequencies in the estimation of d_0 and d_6. For $d_1, ..., d_5$ we use periodogram ordinates on both sides of $\omega_1, ..., \omega_5$, so that $2m$ frequencies are utilized. This fact is reflected in H_1, that is a 7×7 diagonal matrix such that $[H_1]_{11} = [H_1]_{77} = 1$ and $[H_1]_{ii} = 1/2$ for $i = 2, ..., 6$. If the null is true, both test statistics are asymptotically distributed as a chi square with 6 degrees of freedom (χ^2_6), and the tests based on rejecting the null whenever $W_{l1}, W_{g1} > \chi^2_{6,\alpha}$ at $100\alpha\%$ significance level are consistent.

In Section 5.4 of Chapter 5 we introduced a score test procedure for the hypothesis of equality of persistence parameters across different frequencies. In particular, the test statistic for the null in (7.1) is

$$LM_{g1} = m\tilde{e}_1' A_1^{-1} \tilde{e}_1$$

where $\tilde{e}_1$ is a 6×1 vector with elements

$$[\tilde{e}_1]_6 = 2\frac{T_6^1(\tilde{d})}{T_6^0(\tilde{d})} \qquad [\tilde{e}_1]_i = 4\frac{T_i^1(\tilde{d})}{T_i^0(\tilde{d})} \;\; i = 1, ..., 5,$$

$T_i^k(d) = \sum_{j=1}^m v_j^k \lambda_j^{2d}[I_n(\omega_i+\lambda_j)+I_n(\omega_i-\lambda_j)]$ for $i = 1, ..., 5, T_6^k(d) = \sum_{j=1}^m v_j^k \lambda_j^{2d} I_n(\pi-\lambda_j)$, $v_j = \log j - \frac{1}{m}\sum_1^m \log k$, $\tilde{d}$ is the estimate obtained under the null, that is using frequencies around $\omega_0, ..., \omega_6$, and A_1 is a 6×6 matrix with elements $[A_1]_{66} = 11/3$, $[A_1]_{ii} = 20/3$, $[A_1]_{6i} = [A_1]_{i6} = -2/3$ for $i = 1, ..., 5$, and $-4/3$ otherwise. If (7.1) is true, LM_{g1} is asymptotically χ^2_6. Figure 7.5 displays the three test statistics, W_{l1}, W_{g1} and LM_{g1} as a function of the bandwidth, m. We only study the behaviour for $m = 11, ..., 40$, to avoid using the same frequency twice. We reject (7.1) at 5% significance level but the tests are not so conclusive at 1%.

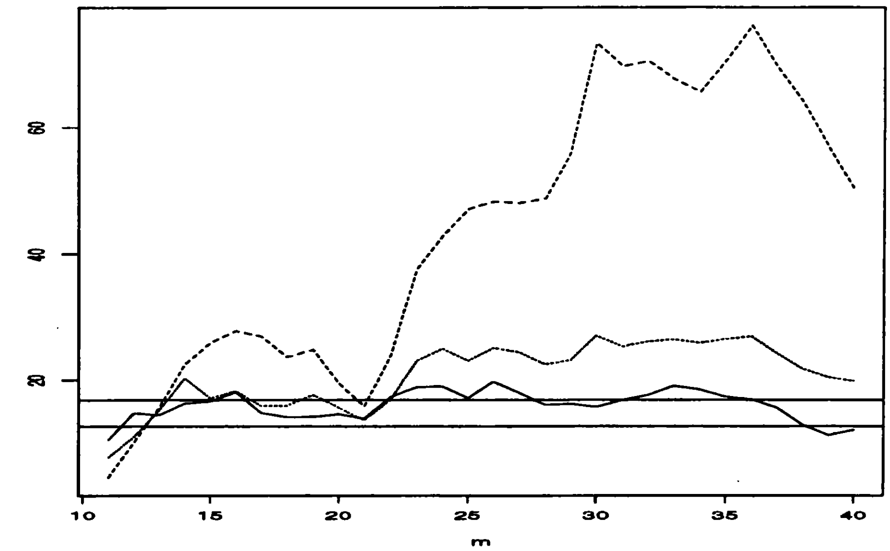
We may also consider the possibility that we may reject (7.1) because of d_0 but the seasonal parameters $d_1, ..., d_6$, are in fact equal. We can test this situation in a similar manner to (7.1). The null hypothesis is now

$$H_0 : d_1 = d_2 = ... = d_6 \tag{7.2}$$

and the alternative is that one or more of the equalities in (7.2) do not hold. The Wald statistics are

$$\begin{aligned} W_{l2} &= \frac{24m}{\pi^2}(R_2\hat{d}^2)'(R_2H_2R_2')^{-1}(R_2\hat{d}^2) \\ W_{g2} &= 4m(R_2\tilde{d}^2)'(R_2H_2R_2')^{-1}(R_2\tilde{d}^2) \end{aligned}$$

Figure 7.5: Tests of equal persistence parameters across all frequencies



Note: The continuous, dotted and dashed lines correspond to W_{l1}, W_{g1} and LM_{g1} respectively. The two horizontal lines are χ_6^2 critical values at 5% (12.6) and 1% (16.8) significance level.

where $\hat{d}^2$ and $\tilde{d}^2$ are 6×1 vectors containing the log periodogram and local Whittle estimates of $d_1, ..., d_6$, R_2 is a 5×6 matrix defined similarly to R_1, and H_2 is a 6×6 diagonal matrix with $[H_2]_{ii} = 1/2$ for $i = 1, ..., 5$ and $[H_2]_{66} = 1$. Under (7.2) W_{l2} and W_{g2} are asymptotically χ_5^2.

The LM statistic for (7.2) is

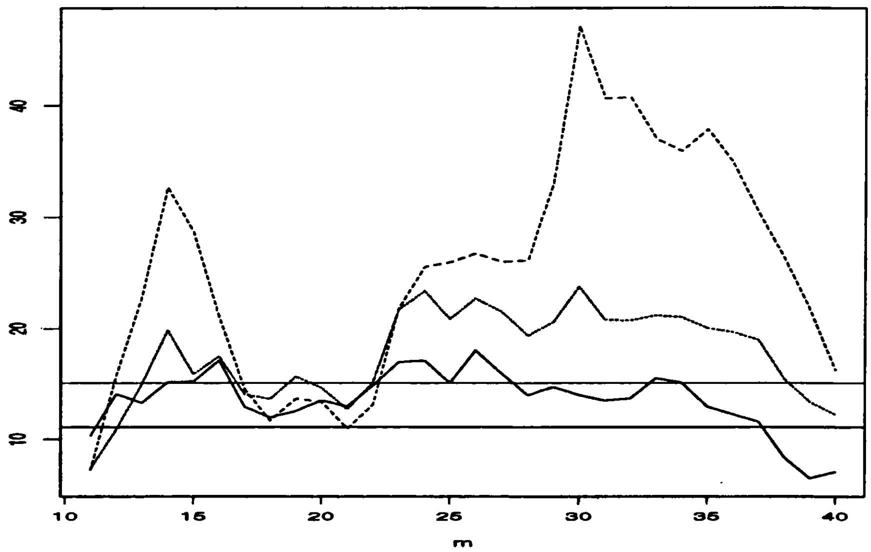
$$LM_{g2} = m\tilde{e}_2' A_2^{-1} \tilde{e}_2$$

where $\tilde{e}_2$ is a 5×1 vector with elements

$$[\tilde{e}_2]_i = 4\frac{T_i^1(\tilde{d}_s)}{T_i^0(\tilde{d}_s)} \quad i = 1, ..., 5,$$

$\tilde{d}_s$ is the estimate under the null, i.e. using frequencies around $\omega_1, ..., \omega_6$, and A_2 is a 5×5 matrix with off-diagonal elements equal to -16/11 and all diagonal elements equal to 72/11. If (7.2) is true, LM_{g2} has a χ_5^2 asymptotic distribution. Figure 7.6 shows W_{l2}, W_{g2} and LM_{g2} in function of the bandwidth m. We see evidence to conclude that the rejection of the hypothesis of equality of all memory parameters is not only due to d_0 but the seasonal d's can not be considered as being equal either.

Figure 7.6: Tests of equal persistence parameters across seasonal frequencies



Note: The continuous, dotted and dashed lines correspond to W_{l2}, W_{g2} and LM_{g2} respectively. The two horizontal lines are χ^2_5 critical values at 5% (11.1) and 1% (15.1) significance level.

However there still exists the possibility of some d's being equal. Thus it may be interesting to test

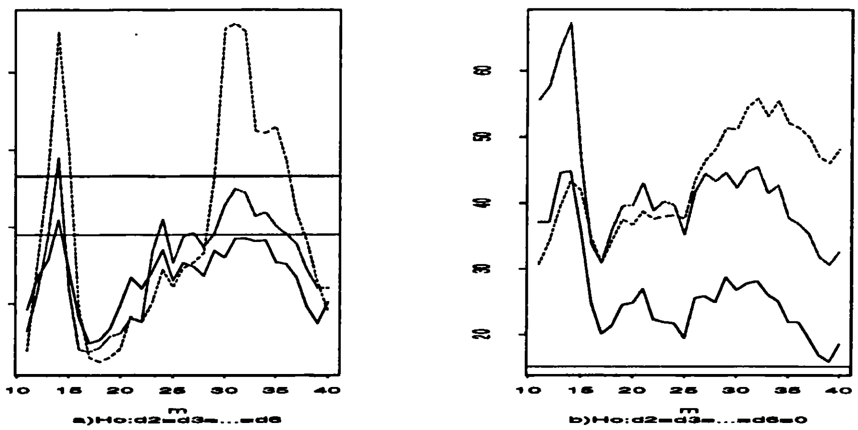
$$H_0 : d_2 = d_3 = ... = d_6 \tag{7.3}$$

against the alternative that one or more of the equalities in (7.3) do not hold. The statistics used are

$$\begin{aligned} W_{l3} &= \frac{24m}{\pi^2}(R_3\hat{d}^3)'(R_3H_3R_3')^{-1}(R_3\hat{d}^3) \\ W_{g3} &= 4m(R_3\tilde{d}^3)'(R_3H_3R_3')^{-1}(R_3\tilde{d}^3) \\ LM_{g3} &= m\tilde{e}_3'A_3^{-1}\tilde{e}_3 \end{aligned}$$

where R_3 is a 4×5 matrix defined in a similar way as R_1 and R_2, H_3 is a 5×5 diagonal matrix with elements $[H_3]_{ii} = 1/2$ for $i = 1,...,4$, and $[H_3]_{55} = 1$, $\hat{d}^3$ and $\tilde{d}^3$ are the vectors of log-periodogram and local Whittle estimates of d_2,...,d_6, respectively, A_3 is a 4×4 matrix with diagonal elements equal to 56/9 and off-diagonal ones equal to -16/9, and $\tilde{e}_3$ is a 4×1 vector with elements $[\tilde{e}_3]_i = 4T^1_{i+1}(\tilde{d}_{s1})/T^0_{i+1}(\tilde{d}_{s1})$, where $\tilde{d}_{s1}$ is the estimate under the null, i.e. using frequencies around ω_2,...,ω_6. The three statistics are shown in Figure 7.7*a*). We observe that the tests tend to accept (7.3) for most values of the bandwidth used.

Figure 7.7: Tests of equal persistence parameters at $\omega_2,\ldots,\omega_6$



Note: The continuous, dotted and dashed lines correspond to W_{l3}, W_{g3} and LM_{g3} in a) and W_{l4}, W_{g4} and LM_{g4} in b) respectively. The horizontal lines are χ^2_4 critical values at 5% (9.49) and 1% (13.3) significance level in a) and χ^2_5 critical value at 1% (15.1) significance level in b).

Once (7.3) is not rejected it may be interesting to study if $d_2,\ldots,d_6$, are in fact zero. The statistics used to test

$$H_0 : d_2 = \ldots = d_6 = 0 \tag{7.4}$$

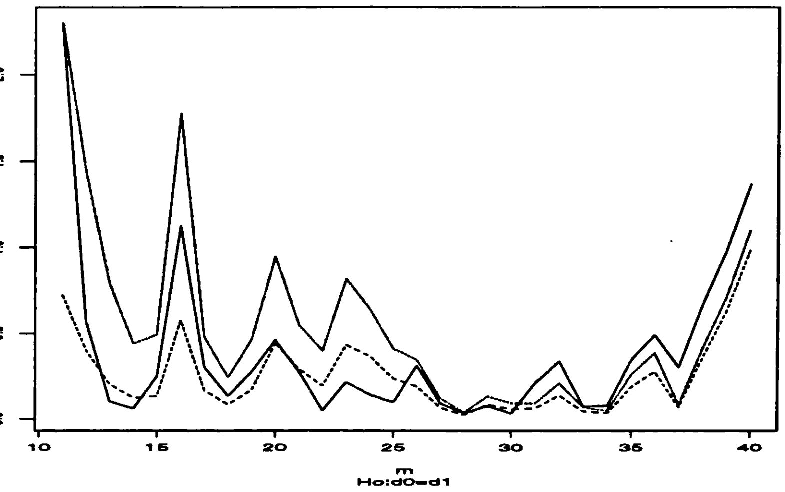
against the alternative that at least one d_i, $i = 2,\ldots,6$, is different from zero, are

$$\begin{aligned} W_{l4} &= \frac{24m}{\pi^2}(\hat{d}^3)' H_3^{-1}(\hat{d}^3) \\ W_{g4} &= 4m(\tilde{d}^3)' H_3^{-1}(\tilde{d}^3) \\ LM_{g4} &= m\tilde{e}_4' A_4^{-1}\tilde{e}_4 \end{aligned}$$

where A_4 is a 5×5 diagonal matrix with elements $[A_4]_{ii} = 8$ for $i = 1,\ldots,4$ and $[A_4]_{55} = 4$, and $\tilde{e}_4$ is a 5×1 vector with elements $[\tilde{e}_4]_i = 4T^1_{i+1}(0)/T^0_{i+1}(0)$ if $i = 1,\ldots,4$, and $[\tilde{e}_4]_5 = 2T^1_6(0)/T^0_6(0)$. The asymptotic properties of LM_{g4} can be proved in the same way as those of the score test of the equality of persistence parameters across different frequencies in Chapter 5. In Figure 7.7*b*) we see that (7.4) is rejected for every m.

Similarly we can test the hypothesis $H_0 : d_0 = d_1$. The three different test statistics are shown in Figure 7.8. We do not reject equality of d_0 and d_1 for any of the bandwidths used (m). The tests of equality of these parameters to zero are not reported but they clearly reject the null for every m.

Figure 7.8: Tests of equal persistence parameters at ω_0 and ω_1



Note: The continuous, dotted and dashed lines correspond to the log-periodogram Wald, Gaussian Wald and LM test statistics respectively.

Assuming symmetry in the spectral poles, we can conclude that there are two different persistence parameters in the UK inflation series from 5:1915 to 4:1996, one describing the spectral behaviour at the origin and at $\pi/6$ (i.e. the long run and the annual movement of π_t) which is around 0.4 (log-periodogram or local Whittle estimates) and the other, closer to 0 (around 0.2 for both, log-periodogram and local Whittle estimates) reflecting the behaviour of the spectrum at $\omega_s = 2\pi s/12$ for $s = 2, ..., 6$ (corresponding to cycles of period of 6, 4, 3, 2.4 and 2 months respectively). However, although there exists spectral symmetry at $\omega_0 = 0$ and $\omega_6 = \pi$, the rest of frequencies may well have asymmetric behaviour as described in previous chapters. This possibility will be formally tested in Section 7.5 using the techniques described in Chapter 5. Some of the statistics we will use require the estimation of the memory parameters on both sides of ω_1,...,ω_5. To this task we dedicate the sections that follow.

7.3 ESTIMATION OF THE PERSISTENCE PARAMETERS

In this section we use the techniques described in Chapters 3 and 4 to estimate the persistence parameters d_{sj}, $j = 1,2$, at frequencies $\omega_s = 2\pi s/12$, $s = 0,1,...,6$, i.e. at the origin and seasonal frequencies. We allow for different behaviour before and after the seasonal frequencies ω_s, $s = 1,2,...,5$, so that the subindex $j = 1$ corresponds to the parameter just after and $j = 2$ just before those frequencies. Of course, by symmetry of the spectral density function, $d_{01} = d_{02} = d_0$ and $d_{61} = d_{62} = d_6$. The tests of spectral symmetry at frequencies ω_s for $s = 1,2,...,5$, will be carried out in Section 7.5, where we use the procedures described in Chapter 5.

In both methods of estimation (Gaussian semiparametric or local Whittle and log-periodogram) we employ the bandwidths $m = 11,12,...,50$. Since we have 81 frequencies between seasonal frequencies, a reasonable choice for m, in order to avoid distorting influence of other spectral poles, seems to be less than 30. However, the influence of neighbour poles will depend on the magnitude of the persistence parameter at those neighbour frequencies, and that is why we analyse the cases up to $m = 50$, although we consider the most relevant results to be those obtained with m between 20 and 30.

Figure 7.9 shows the estimates of d_0 with no trimming ($l = 0$). We see that they stabilize around 0.4. When we introduce trimming (for $l = 1,...,6$) the results only vary for small m (due to the use of fewer periodogram ordinates), and as m increases these estimates also stabilize around that value.

Figure 7.10 shows the periodogram of a truncated version of the fractionally differenced series

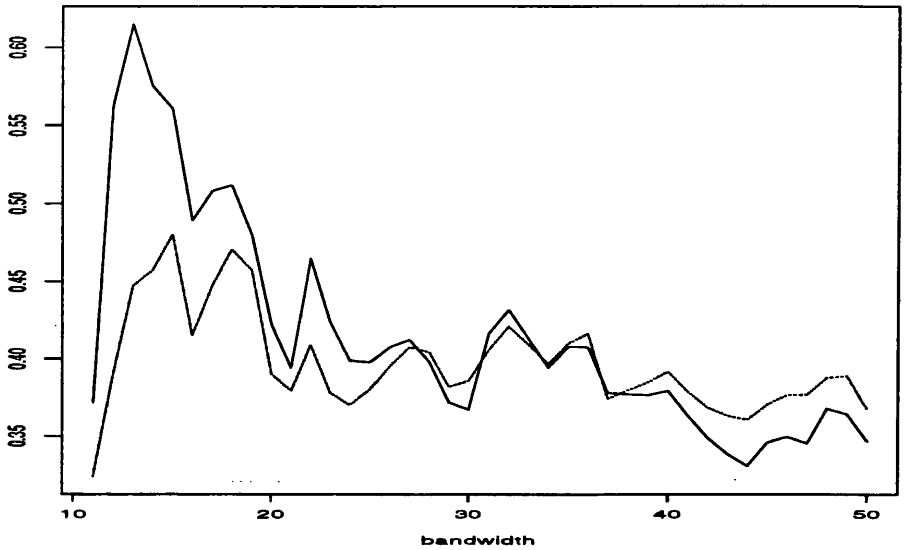
$$(1-L)^{0.4}(\pi_t - \bar{\pi}) = \sum_{k=0}^{\infty} D_k(0.4)(\pi_{t-k} - \bar{\pi}) \tag{7.5}$$

where

$$D_k(d) = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}$$

and $\bar{\pi}$ is the arithmetic mean of π_t. We approximate (7.5) by taking $\pi_t = \bar{\pi}$ for all

Figure 7.9: Estimation at the origin



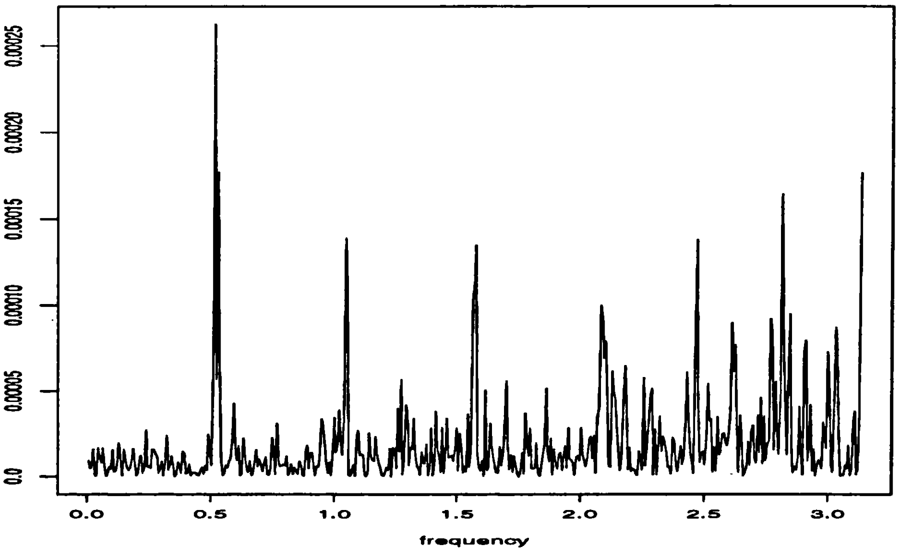
Note: The continuous and dotted line are log-periodogram and local Whittle estimates respectively.

t previous to January 1915 [1]. We observe that fractional differencing removes long-range dependence at frequency zero, but peaks at seasonal frequencies persist and are more noticeable.

Figures 7.11 and 7.12 show the log-periodogram and local Whittle (or Gaussian semiparametric) estimates of d_{s1} and d_{s2} for $s = 1, 2, 3, 4$ with $l = 0$ (no trimming) and $l = 1$ respectively. Throughout this section, the continuous line represents the estimates just after the frequency under study (i.e. d_{s1}) and the dotted line are the estimates just before ω_s (d_{s2}). As we would expect the estimates tend to decrease when we trim out the closest frequency, and as bandwidth increases. The decreasing behaviour of $\hat{d}_{12}$ and $\tilde{d}_{12}$ with m may be generated by the influence of the important peak at the origin. For a bandwidth of around 30 the estimates on either side of the spectral pole are similar and around 0.25. Of interest is the behaviour of the estimates to the right of $\pi/3$ (d_{21}). When we omit the closest frequency these estimates decrease significantly and the difference between $\hat{d}_{21}$, $\tilde{d}_{21}$ and $\hat{d}_{22}$, $\tilde{d}_{22}$ becomes bigger. This

[1] Although we use only data from April 1915 our series starts in January 1915. We do not use the first 4 observations in order to have seasonal frequencies that can be represented as Fourier frequencies of the form $2\pi j/n$ for some integer j.

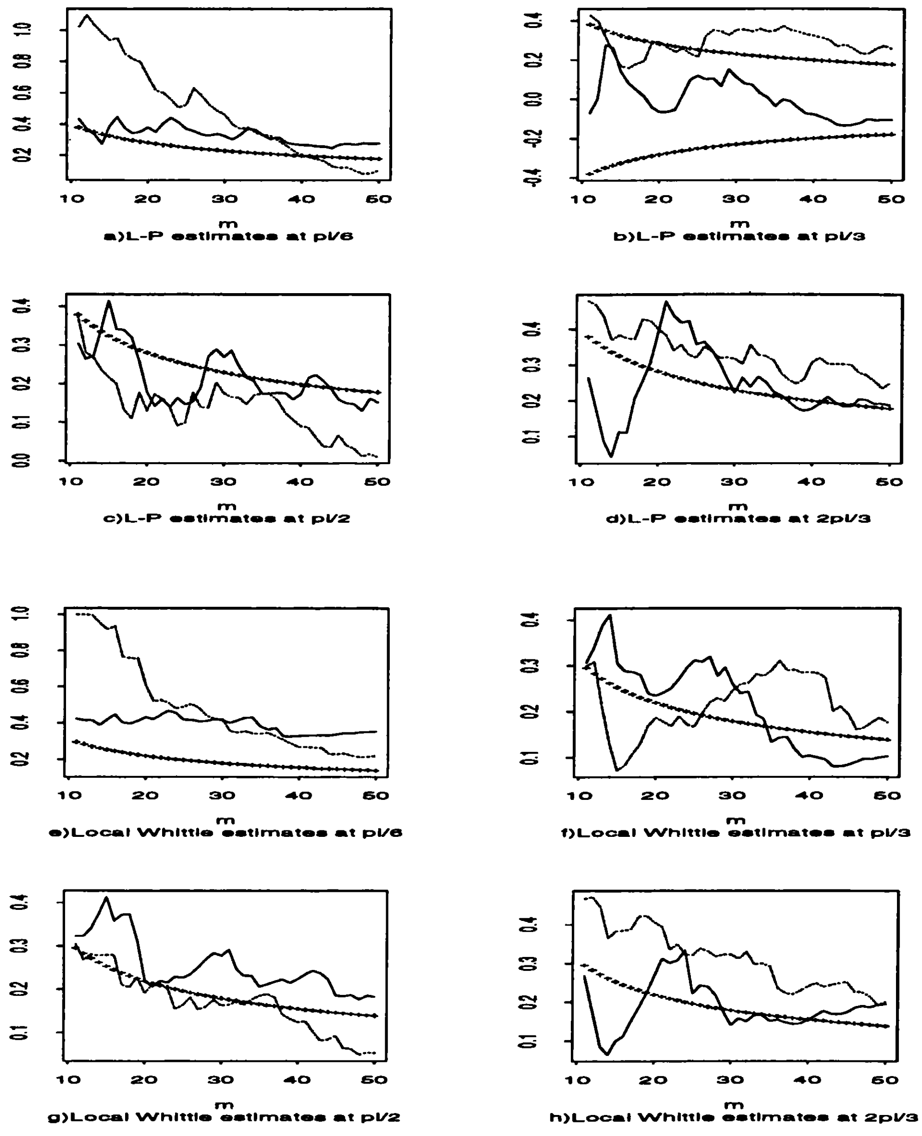
Figure 7.10: Periodogram of $(1-L)^{0.4}(\pi_t - \bar{\pi})$



fact is in accordance with the results obtained theoretically and through simulations in Chapters 3, 4 and 6, where we saw that when the difference between the persistence parameters just before and after the frequency under study is large, trimming seems unavoidable and estimation of the smaller parameter using all frequencies is likely to be positively biased due to the influence of the larger persistence parameter.

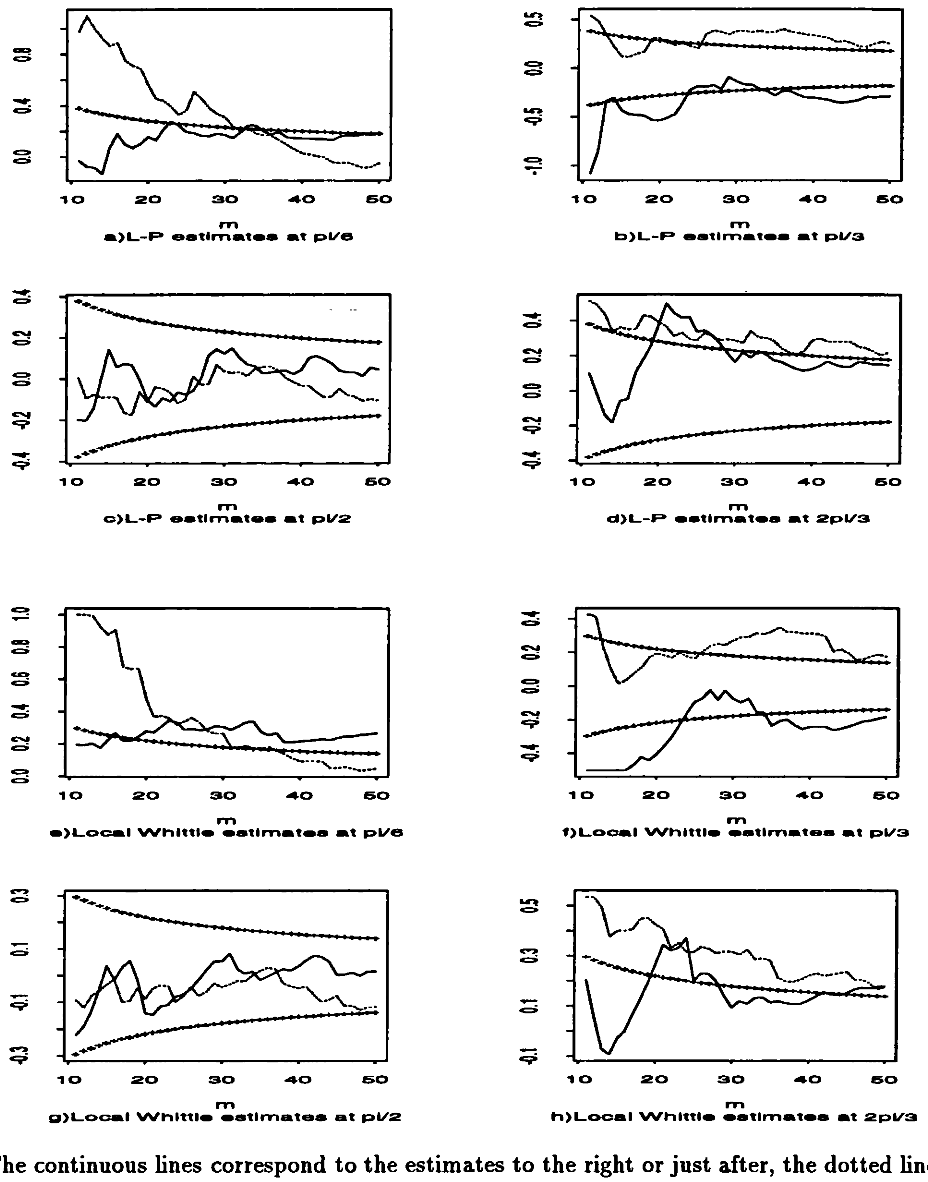
Using the asymptotic distribution of the log-periodogram and Gaussian semiparametric estimators obtained in Chapter 3 and 4 we can test the significance of the different d's. In Figures 7.11 and 7.12 we show the confidence intervals obtained from these asymptotic distributions. The estimates at $5\pi/6$ and π (that we do not report) are not significantly different from zero for almost every m. This is what we would expect, because cycles with period 2.4 and 2 months seem implausible in an inflation series. The rejection or not in the other d's depends on the method of estimation, the trimming and the bandwidth m.

Figure 7.11: Seasonal persistence estimates (l=0)



Note: The continuous lines correspond to the estimates to the right or just after, the dotted lines are the estimates to the left of or just before ω_s and the crossed lines are the bounds of the significance confidence intervals at 5% significance level.

Figure 7.12: Seasonal persistence estimates (l=1)

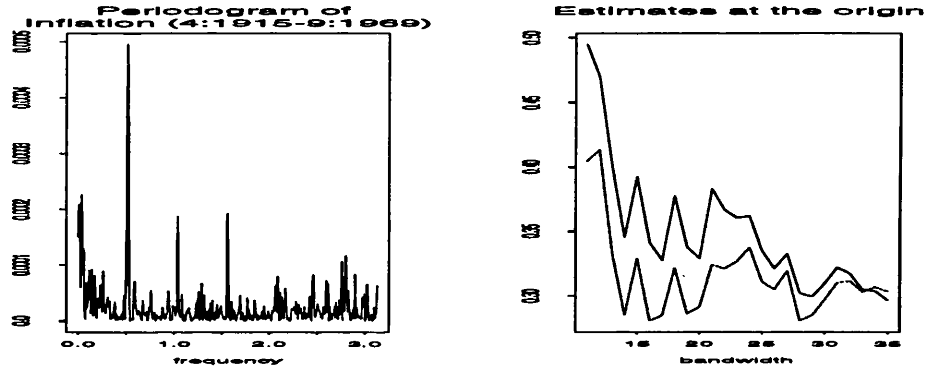


Note: The continuous lines correspond to the estimates to the right or just after, the dotted lines are the estimates to the left of or just before ω_s and the crossed lines are the bounds of the significance confidence intervals at 5% significance level.

7.4 PERSISTENCE IN SUB-SERIES

Figure 7.13 displays the periodogram and estimates of the persistence parameters at zero frequency of the sub-series from April 1915 to September 1969, so that we have n=654 observations. There are at most 55 Fourier frequencies between spectral poles. Arguing as in the estimation in the full series, we only use bandwidths $m = 11, ..., 35$. As we could deduce from a visual inspection of the series in Figure 7.1, the peak in the periodogram and the persistence estimates at the origin are smaller than those in the full series. The estimates of the seasonal persistence parameters without and with trimming can be seen in Figures 7.14 and 7.15.

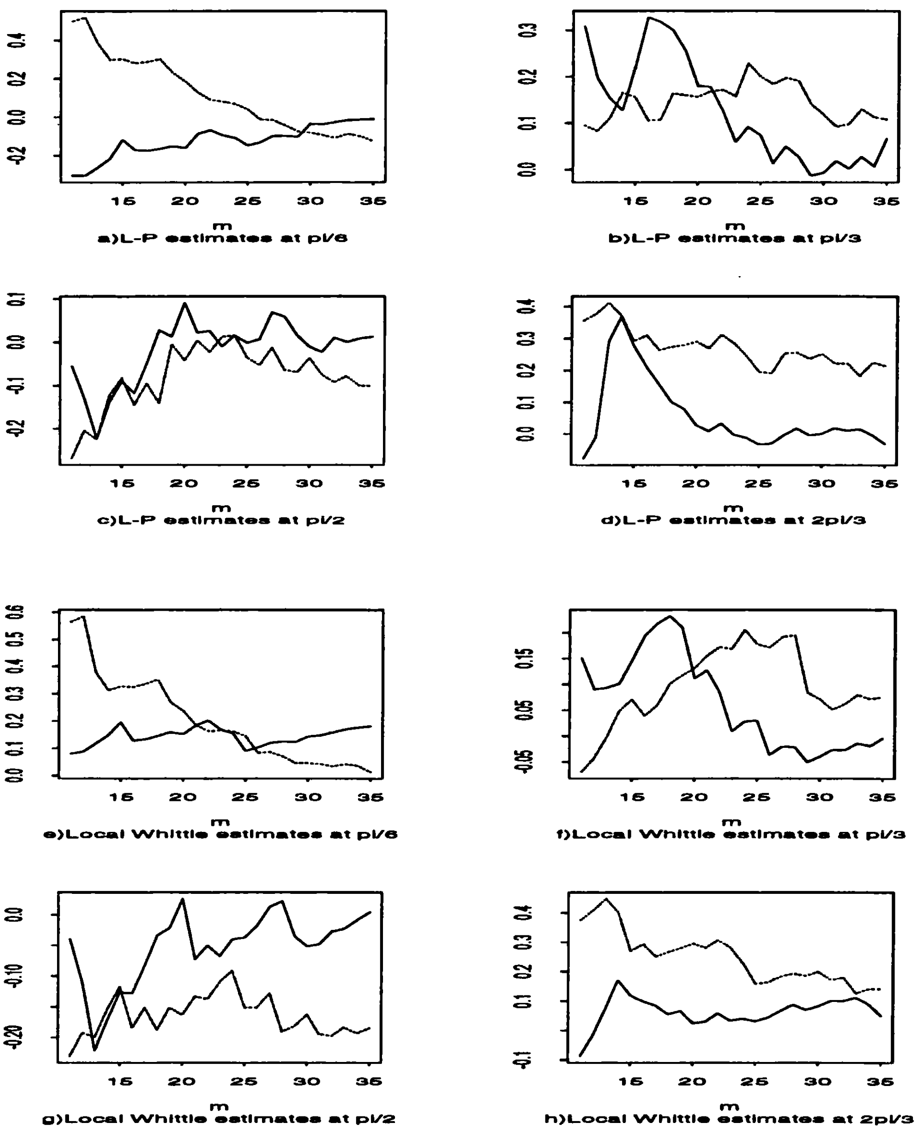
Figure 7.13: Periodogram and estimates at frequency 0 (4:1915-9:1969)



Note: The continuous and dotted lines are the log-periodogram and local Whittle estimates respectively.

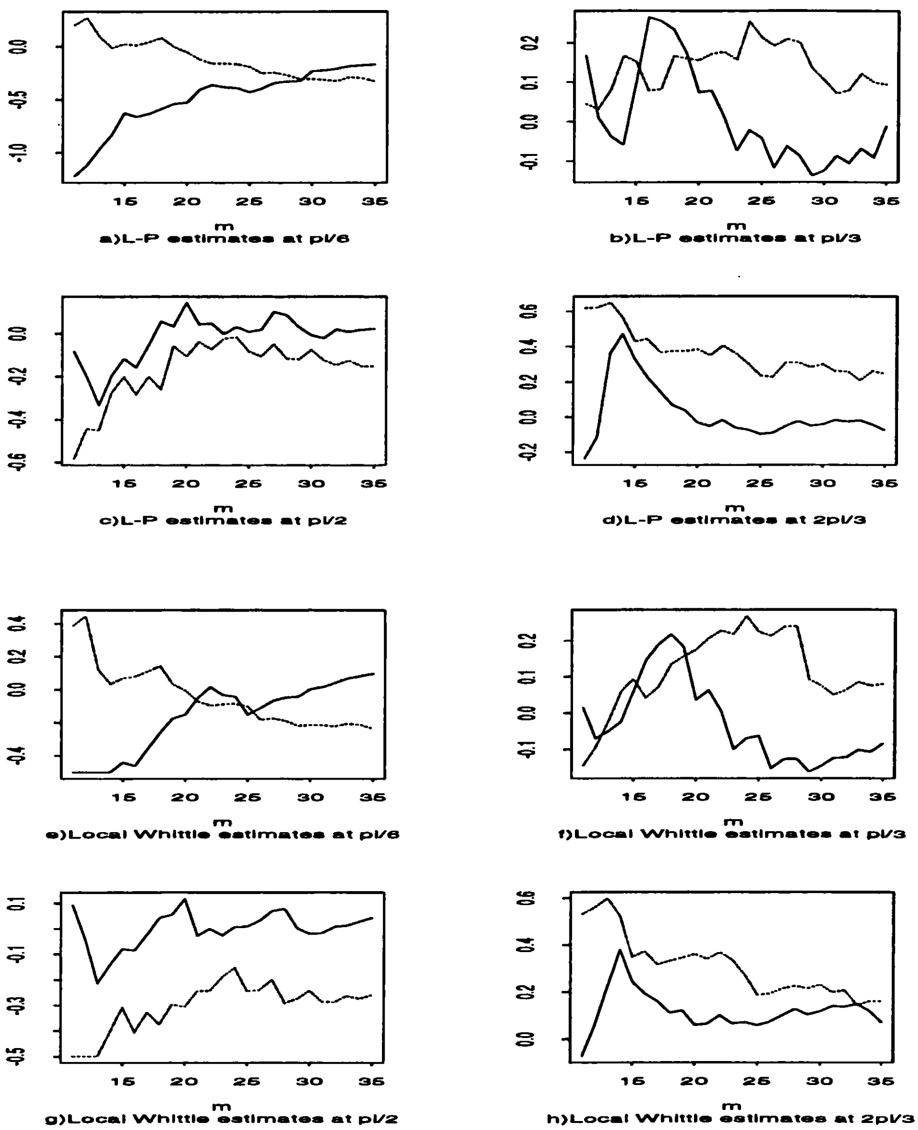
Figures 7.16, 7.17 and 7.18 display the estimates of the various persistence parameters for the second sub-series from October 1969 to April 1996, so that we have n=319 observations. Although there are at most 27 frequencies between different spectral poles we use m=11,...,60, in the estimation of d_0 and m=11,...,40, in the estimation of d_{sj}, $j = 1, 2$, $s = 2, 3, 4$, in order to analyse the effects on the estimates of the use of frequencies around different poles than those under study. We observe that the estimates at the origin, for a reasonable m (say between 10 and 20), are larger than those obtained in the full series which is in accordance with the behaviour of this subsample in Figure 7.1. In fact for $m < 20$ (before the spectral pole at $\pi/6$) we

Figure 7.14: Seasonal persistence estimates, l=0 (4:1915-9:1969)



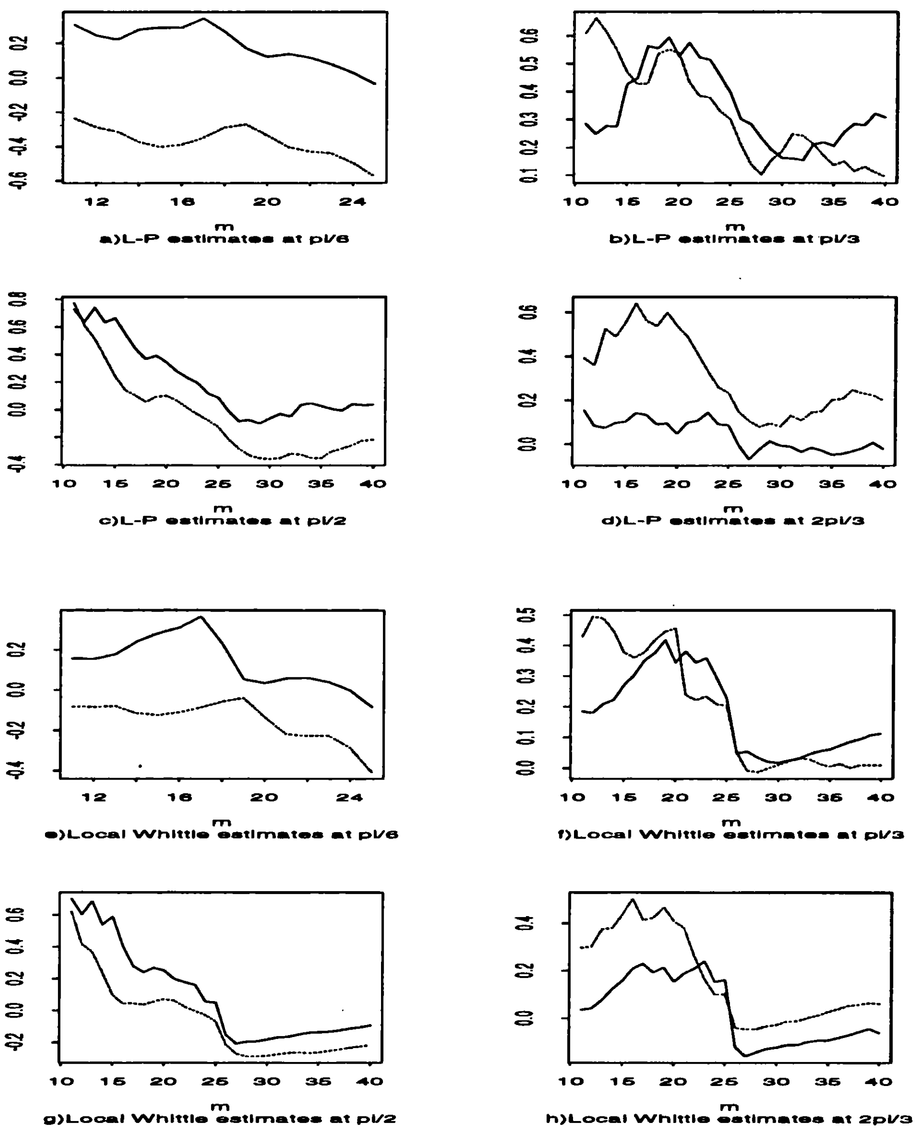
Note: The continuous lines correspond to the estimates to the right or just after and the dotted lines are the estimates to the left of or just before ω_s.

Figure 7.15: Seasonal persistence estimates, l=1 (4:1915-9:1969)



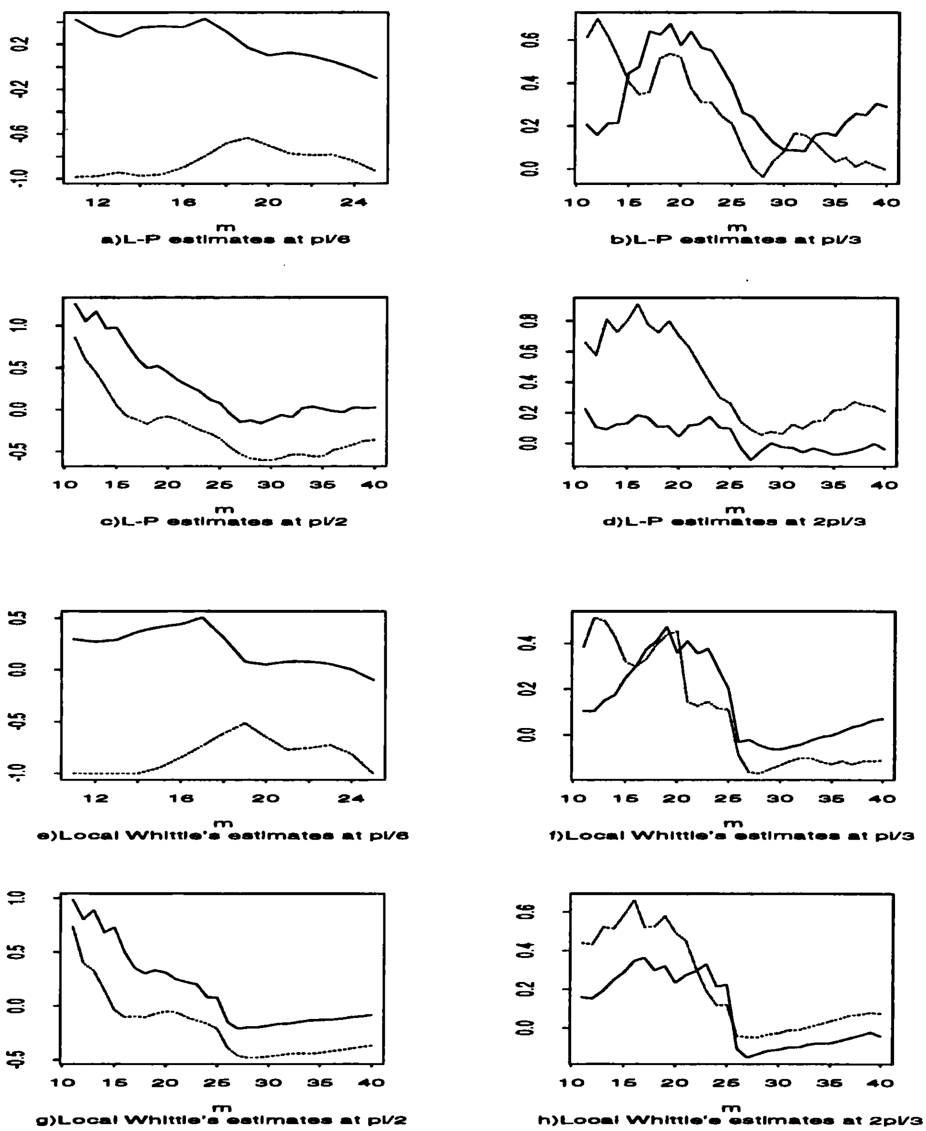
Note: The continuous lines correspond to the estimates to the right or just after and the dotted lines are the estimates to the left of or just before ω_s.

Figure 7.16: Seasonal persistence estimates, l=0 (10:1969-4:1996)



Note: The continuous lines correspond to the estimates to the right or just after and the dotted lines are the estimates to the left of or just before ω_s.

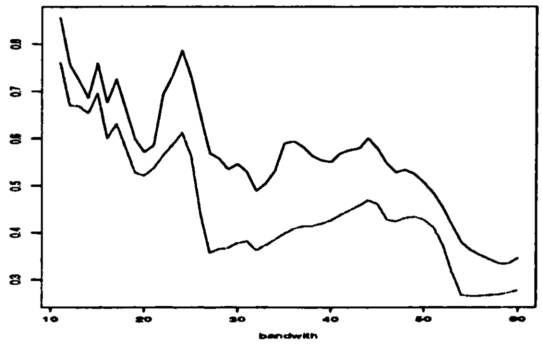
Figure 7.17: Seasonal persistence estimates, l=1 (10:1969-4:1996)



Note: The continuous lines correspond to the estimates to the right or just after and the dotted lines are the estimates to the left of or just before ω_s.

found evidence that $d_0 > 1/2$ reflecting a non-stationary behaviour of this subseries[2]. A similar result is found by Hassler and Wolters (1995). We also see that the estimates of the parameters to the right and left of $\pi/3$ (d_{21} and d_{22}) are closer to each other than those obtained using the complete series. This possible symmetry will be analysed in the next section. The opposite occurs at $\pi/6$ where we observe that when we trim out the closest frequency, the estimates of d_{12} are smaller than -0.5 for every m.

Figure 7.18: Estimates at frequency 0 (10:1969-4:1996)



Note: The continuous and dotted lines are the log-periodogram and local Whittle estimates respectively.

Now we consider the effects caused by the inclusion of periodogram ordinates near a frequency where the spectrum is likely to have a pole on the estimation of the persistence parameter describing the long-memory behaviour at a different frequency. This occurs when we use bandwidths approaching $m = 27$ in the second subseries (from October 1969 until April 1996). We see that the estimates decrease when m includes those frequencies and a sharp fall occurs when the frequency with the spectral pole is used in the estimation. This fact is clearly reflected in the estimation of d_0 in Figure 7.18 where we observe two sharp falls around m=27 and m=54, that is when we include $\omega_1 = \pi/6$ and $\omega_2 = \pi/3$ in the estimation of d_0. We also observe that this

[2]The asymptotic results in Chapters 3 and 4 are only valid for $|d| < 1/2$. Nevertheless Velasco (1997a) has proved that the log-periodogram estimate, $\hat{d}_0$ is consistent for $d_0 \in [1/2, 1)$ and asymptotically normal for $d_0 \in [1/2, 3/4)$. The good properties of $\hat{d}_0$ for $d_0 \in [1/2, 1)$ in finite samples are shown in Hurvich and Ray (1995). As far as the local Whittle estimate is concerned, Velasco (1997b) has demonstrated that it is consistent for $d_0 \in [1/2, 1)$ and asymptotically normal for $d_0 \in [1/2, 2/3)$ ($d_0 \in [1/2, 3/4)$ under Gaussianity) and behaves quite well in finite samples.

effect is stronger in the local Whittle case reflecting a higher sensitivity of this method of estimation to the inclusion of frequencies with important peaks in the periodogram.

7.5 TESTS OF SPECTRAL SYMMETRY

We have seen in Sections 7.3 and 7.4 that some of the estimates of the persistence parameter at some seasonal frequencies are quite different if we use periodogram ordinates before or after the frequency under study. We use now the procedures obtained in Chapter 5 in order to test the possibility of asymmetric spectral poles at ω_s, $s = 1, ..., 5$. We report the results where this asymmetry seems more evident, that is $\pi/6$, $\pi/3$ and $2\pi/3$ for the full series.

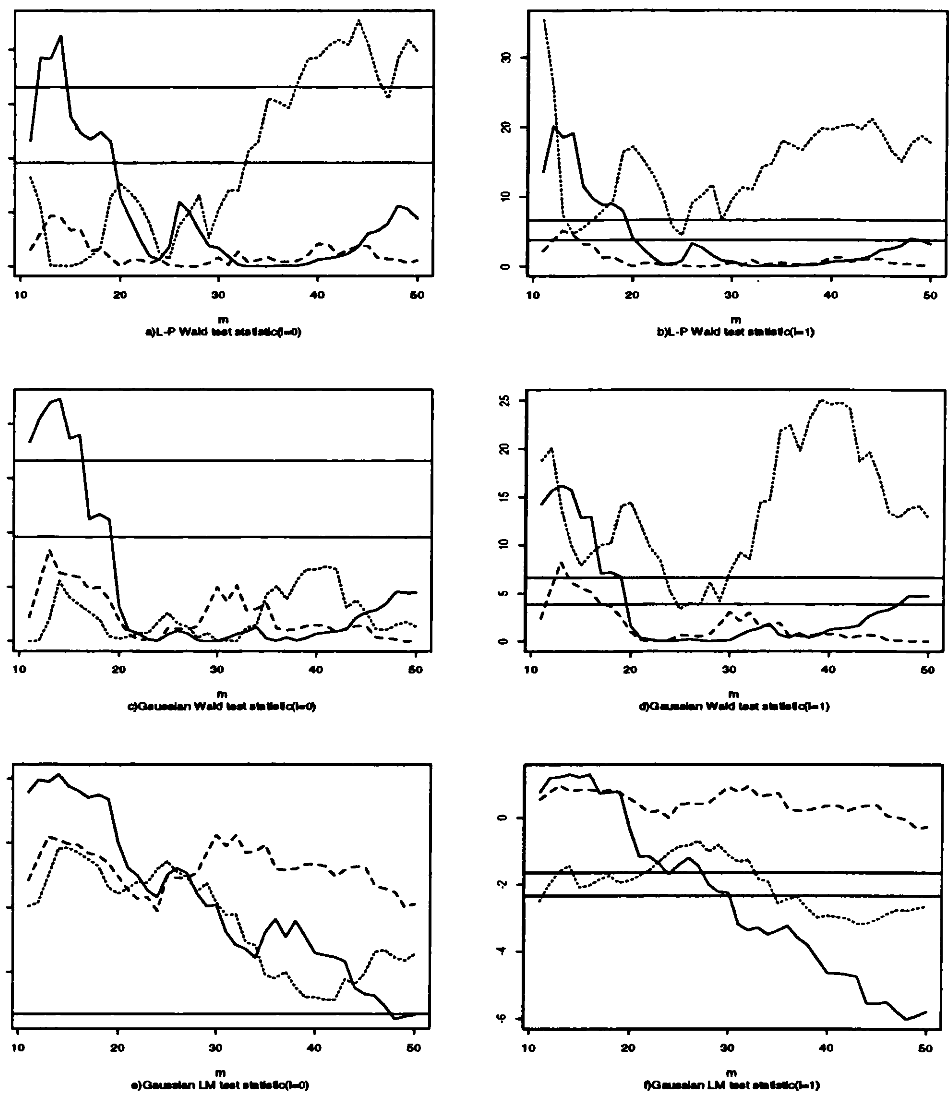
Figures 7.19*a*) and *b*) display the log-periodogram Wald test statistics without trimming ($l = 0$), and trimming out the closest frequency ($l = 1$) for the three seasonal frequencies under study[3]. The most interesting feature is the rejection of the hypothesis of symmetry at $\omega_2 = \pi/3$, specially when $l = 1$. For $\pi/6$ we only reject for small m. A similar behaviour can be observed in the Gaussian Wald test in Figures 7.19c) and d).

Figures 7.19*e*) and *f*) show the LM_1 test of the hypothesis $d_{11} = d_{12}$ (i.e. at $\pi/6$) against the alternative $d_{11} > d_{21}$ (continuous line), the LM_2 statistic to test $d_{21} = d_{22}$ (i.e. at $\pi/3$) against $d_{21} < d_{22}$ (dotted line) and the LM_1 test of the hypothesis $d_{41} = d_{42}$ (i.e. at $2\pi/3$) against the alternative $d_{41} > d_{42}$ (short dashed line)[4]. For a detailed description of these test procedures see Chapter 5. We use one-tailed tests because their theoretical properties do not need trimming and they are more powerful than the corresponding two-tailed tests (see the Monte Carlo study in Chapter 6). We chose LM_1 at $\pi/6$ in order to use frequencies after $\pi/6$ in the construction of the statistic and in this way avoid the influence of the important peak at the origin. Moreover, for a reasonable bandwidth ($m < 30$), the results obtained with LM_1 and LM_2 are complementary in the sense that we do not reject the null

[3]The two straight lines reflect the critical values from a χ_1^2 distribution at 5% and 1% significance level.

[4]The straight lines represent the critical values from a standard normal distribution (N(0,1)) at 5% and 1% significance level. Note that these critical values correspond to one-tailed tests.

Figure 7.19: Tests of spectral symmetry (5:1915-4:1996)



Note: The continuous lines correspond to the different tests at $\pi/6$, the dotted lines are the tests at $\pi/3$ and the dashed lines at $2\pi/3$. The straight lines are N(0,1) and χ_1^2 critical values at 1% and 5% significance level.

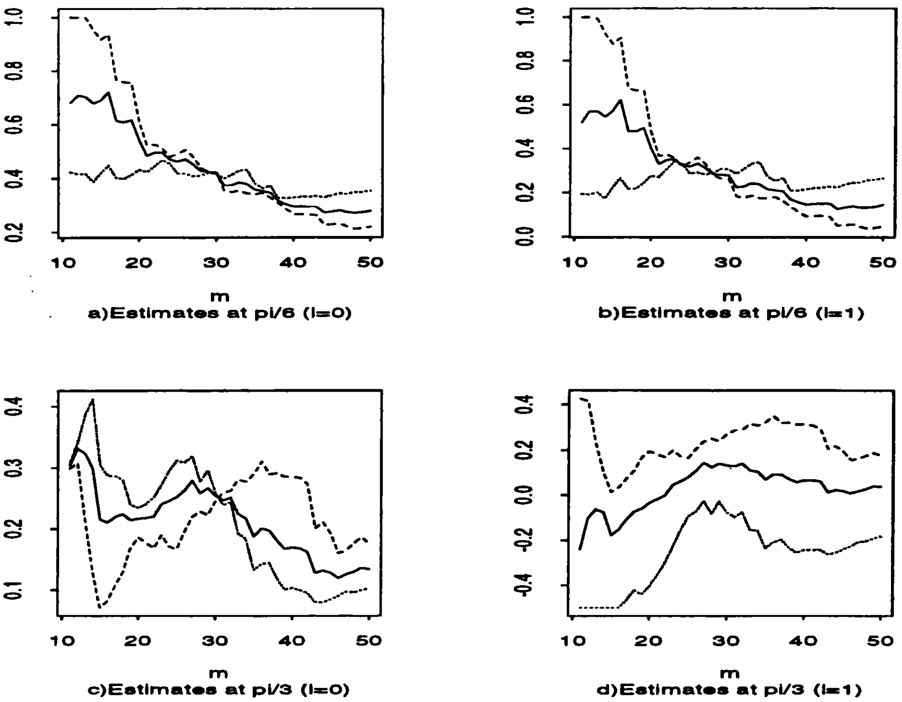
with either procedure. Similarly, the alternative $d_{21} > d_{22}$ at $\pi/3$ is rejected for all m using the LM_1 procedure (we do not report the results). In Figure 7.19 we show the LM_2 statistic at $\pi/3$ and we see that when we do not trim out any frequency in the estimation of the joint parameter under the null we do not reject the hypothesis of spectral symmetry at $\pi/3$ either. The same fact can be observed for $2\pi/3$ using the LM_1 test statistic, although the non rejection is even clearer than for $\pi/3$. The LM_2 test of the hypothesis $d_{41} = d_{42}$ against the alternative $d_{41} < d_{42}$ does not reject the null either (not reported).

When we omit one frequency just before and after the frequency under study in the joint estimation of the persistence parameter under the null we observe that the spectral symmetry at $\pi/6$ and $\pi/3$ is now rejected for a wider range of values of the bandwidth m. This is in accordance with the Monte Carlo results in Chapter 6, where we found that power and size of the different test procedures tend to increase with the introduction of a small trimming in the joint estimation of the persistence parameter under the hypothesis of symmetry.

We also saw in Theorem 14 in Chapter 5 that when $d_{s1} \neq d_{s2}$ then the joint local Whittle estimate converges to a value $[2d_{s1} + 2d_{s2} - 1 + \sqrt{4(d_{s1} - d_{s2})^2 + 1}]/4$ which is between d_{s1} and d_{s2} and closer to the highest one. This behaviour can be seen in Figure 7.20 where we show the joint (continuous line), right (dotted line) and left (dashed line) estimates of the persistence parameters at frequencies where we reject the hypothesis of symmetry in more cases, that is at $\pi/6$ and $\pi/3$. We do not report joint estimation based on log-periodogram regression because it is equal to $(\check{d}_{s1} + \check{d}_{s2})/2$ by definition.

Figures 7.21*a*), *b*), *c*) and *d*) show the log-periodogram and Gaussian semiparametric Wald test statistics for the subseries April 1915 to October 1969 at frequencies $\pi/3$ (continuous line), $\pi/2$ (dotted line) and $2\pi/3$ (dashed line). The behaviour of these tests at $\pi/6$ is similar to when we use the whole series. We do not observe clear evidence of asymmetric spectral behaviour in any of the frequencies analysed, although the trimmed Gaussian Wald test is not conclusive for $\pi/3$ and $\pi/2$ for some

Figure 7.20: Joint, right and left local Whittle estimation



Note: The continuous lines are the joint estimates, the dotted lines are the estimates to the right or just after and the dashed lines the estimates to the left or just before.

of the m used. The score tests do not reject the hypothesis of spectral symmetry at $\pi/2$, $\pi/3$ or $2\pi/3$ for any of the bandwidths. This fact can be seen in Figure 7.21e) and f) where we show the LM_1 statistic for $\pi/2$ and LM_2 for $\pi/3$ and $2\pi/3$.

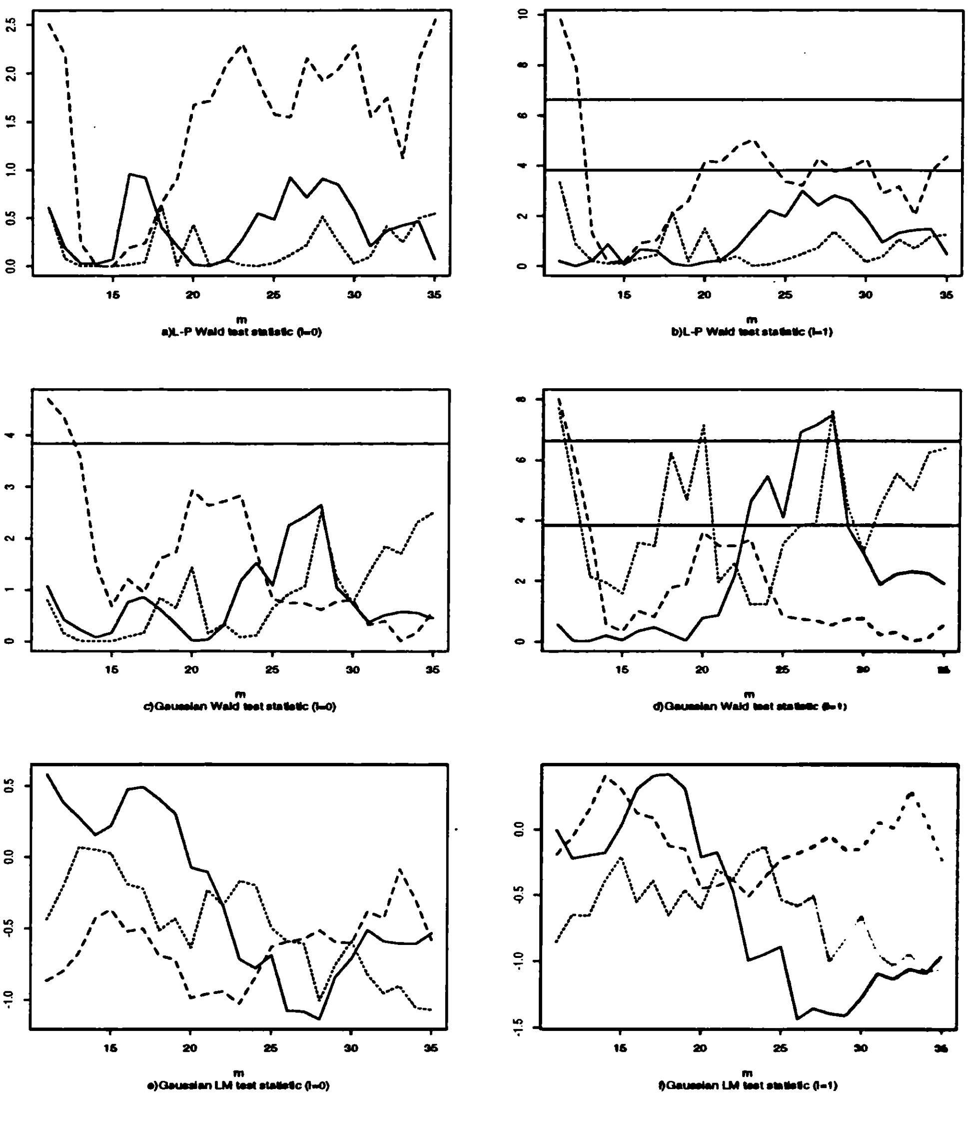
Figures 7.22a), b), c) and d) display the different Wald tests using log-periodogram and local Whittle estimates for the subseries 10:1969-4:1996 for the hypothesis of symmetry at $\pi/6$ (continuous line), $\pi/3$ (dotted line), $\pi/2$ (short dashed line) and $2\pi/3$ (long dashed line). Symmetry is rejected at $\pi/6$. The tests are not conclusive for $\pi/2$ and $2\pi/3$ and they clearly do not reject symmetry at $\pi/3$. Figures 7.22e) and f) show the LM_1 test statistic for the symmetry at $\pi/6$, $\pi/3$ and $\pi/2$ and the LM_2 at $2\pi/3$. We only reject the symmetry at $\pi/6$ for a large bandwidth corroborating the more conservative behaviour of the score tests found in Chapter 6 trough simulations, especially when the sample size is small.

7.6 CONCLUSION

We have found evidence of long memory in the UK monthly inflation series not only at frequency zero (as in Hassler and Wolters (1995)) but also at seasonal frequencies. We have also seen that the persistence parameters are likely to be different at the origin and at seasonal frequencies so that application of the fractional seasonal difference operator $(1 - L^{12})^d$, used in Porter-Hudak (1990), which imposes the same memory parameter at every $\omega_s = 2\pi s/12$, $s = 0, 1, ..., 6$, may lead to distorted conclusions. In fact, assuming spectral symmetry, we have found evidence that there are at least two different memory parameters, one for frequencies 0 and $\pi/6$ and other for $\omega_s = 2\pi s/12$, $s = 2, ..., 6$. Furthermore the spectral poles at some seasonal frequencies may be asymmetric which will cause the joint estimation, using frequencies on both sides of the frequency under study, be incorrect in small and large samples.

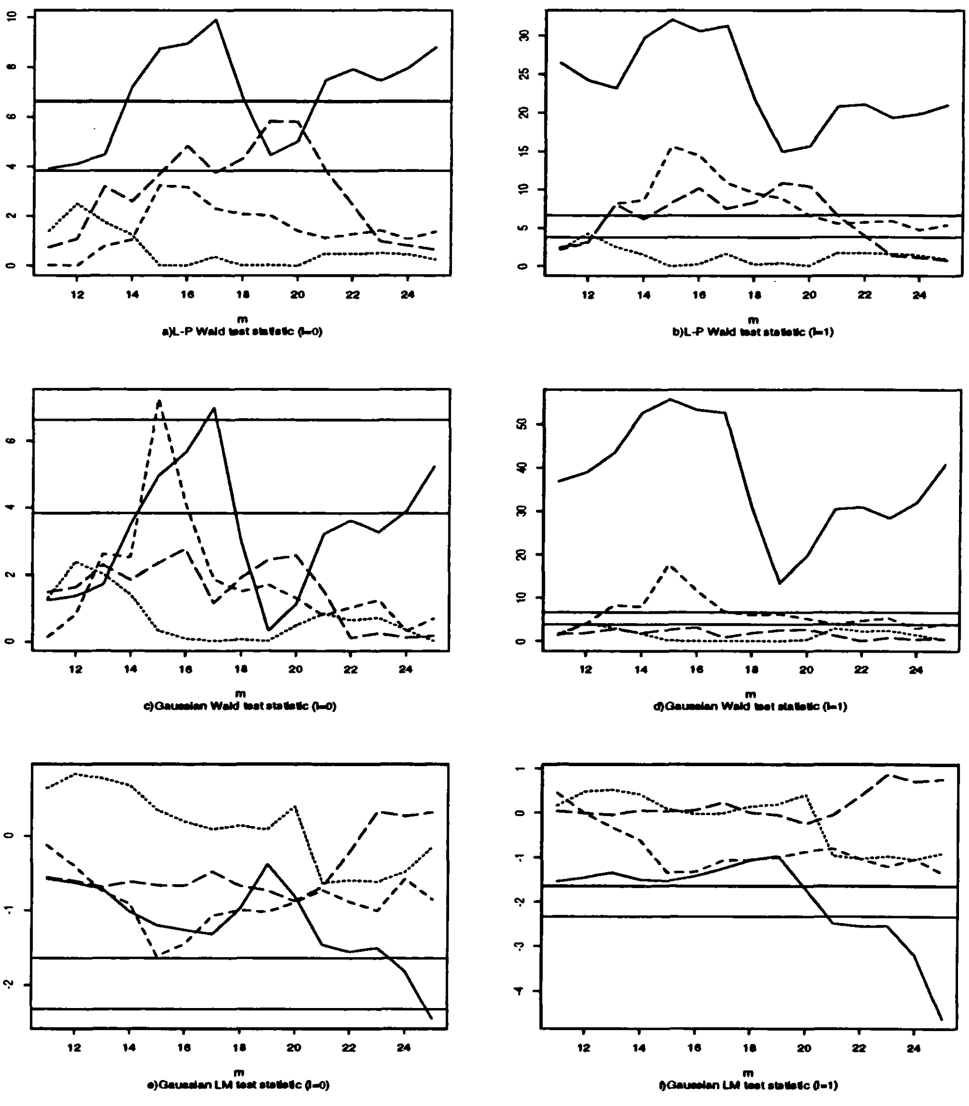
The series suffers a change of pattern in the early seventies. The more relevant feature is the fact that the series from October 1969 presents a stronger persistence in the trend and is likely to be non stationary although "less non stationary" than that caused by a unit root.

Figure 7.21: Tests of spectral symmetry (4:1915-9:1969)



Note: The continuous lines correspond to the different tests at $\pi/3$, the dotted lines are the tests at $\pi/2$ and the dashed lines at $2\pi/3$. The straight lines are N(0,1) and χ_1^2 critical values at 1% and 5% significance level.

Figure 7.22: Tests of spectral symmetry (10:1969-4:1996)



Note: The continuous, dotted, short-dashed and long-dashed lines correspond to the tests at $\pi/6$, $\pi/3$, $\pi/2$ and $2\pi/3$ respectively. The straight lines are N(0,1) and χ_1^2 critical values at 1% and 5% significance level.

Chapter 8

CONCLUSION AND EXTENSIONS

In this thesis we have analysed the possibility of seasonal or cyclical long-range dependence or antipersistence, which is characterized by a spectral (or pseudospectral in the nonstationary case) pole or zero at some frequency ω, reflecting the cycle. One of the originalities of this analysis is that we allow for asymptotic spectral asymmetries at that frequency ω. To date, all extensions of long range dependence to the seasonal or cyclical case impose a symmetric behaviour that is not implied by the definition of spectral density function, as long as $\omega \neq 0, mod(\pi)$. Here, we have tried to relax that condition allowing for a different spectral behaviour before and after the frequency ω.

The analysis of SCLM is naturally done in the frequency domain and we follow this approach throughout the whole thesis. The time domain behaviour (in some cases only asymptotic) of some parametric SCLM (symmetric and asymmetric) models has been described in Chapters 1 and 2.

The possibility of spectral asymmetries at ω has interesting implications on the estimation of the two (possibly different) persistence parameters implied. Some of these implications have been analysed in Chapters 3 and 4, and via simulations in Chapter 6. Consequently, a test of the traditionally assumed symmetry seems necessary, prior to any other analysis. Some semiparametric test procedures of spectral symmetry at one frequency and of equality of persistence parameters across different frequencies have been proposed in Chapter 5, and their finite sample performance analysed in Chapter 6.

Throughout the whole thesis one of the main assumptions is that the frequency ω, where the spectral pole or zero occurs, is known. Of course, this is so in the seasonal long memory case, where those frequencies are the seasonal ones, $\omega_j = 2\pi j/s$, $j = 1, ..., [s/2]$, but in any other case we may need to estimate it. A brief review of the work done to date on this issue is introduced in Section 8.1.

Of course a lot of work remains to be done in the field of seasonal or cyclical long memory, specially taking into account the possible asymmetry of $f(\lambda)$. Some extensions are suggested in Section 8.2.

8.1 ESTIMATION OF THE FREQUENCY ω

Most analyses to date, either to model seasonal/cyclical long memory time series or to estimate the persistence parameters describing that behaviour, are based on the assumption that the frequency ω where the spectral pole occurs is known. Of course, seasonal frequencies are known, but in cyclical time series, the frequency ω may well be unknown and an estimation of it may be required.

The literature on estimating ω in cyclical long memory is of recent date and it is of interest to consider first earlier work on estimating frequency in an alternative model, namely the deterministic periodic time series

$$x_t = \alpha_0 \sin \omega t + \beta_0 \cos \omega t + u_t \tag{8.1}$$

where u_t is a stationary random process with mean zero and spectral density, $f_u(\lambda)$, continuous at ω. Whittle (1952) found that the least squares estimate of ω in (8.1), $\hat{\omega}$, is the periodogram maximizer and has a variance $O(n^{-3})$. Walker (1971) (for u_t white noise) and Hannan (1971, 1973a) extended Whittle's work and, without assuming Gaussianity, found that the asymptotic distribution of $\hat{\omega}$, for $\omega \neq 0, \pi$, is

$$n^{3/2}(\hat{\omega} - \omega) \xrightarrow{d} N\left(0, \frac{48\pi f_u(\omega)}{\alpha_0^2 + \beta_0^2}\right). \tag{8.2}$$

In case $\omega = 0, \pi$, Hannan (1973a) showed that there exists an integer valued random variable, n_0, with $P(n_0 < \infty) = 1$ such that $\hat{\omega} = \omega$ for $n > n_0$, so that $\hat{\omega}$ will be equal to the value it estimates for a large enough sample size. Mackisack and

Poskitt (1989) proposed a different technique based on the minimization of the transfer function calculated by fitting high order autoregressions to x_t. Only $\sqrt{n}$-consistency for $\omega \in (0, \pi)$ is rigorously proved (although it is claimed that the variance of the estimate is $O(n^{-\frac{5}{2}})$ when the order of the autoregression is $O(n^{\frac{1}{2}})$), and their method is computationally intensive. A different approach has been suggested by Quinn and Fernandes (1991). The technique is based on fitting ARMA(2,2) models iteratively and propose a simple algorithm that converges quite quickly to the true parameter ω. The same asymptotic distribution, (8.2), as the maximizer of the periodogram is obtained. A similar procedure with the same asymptotic distribution is described in Truong-Van (1990).

In (8.1) only one sinusoidal component is assumed. However a multiple finite number of components can describe the seasonal or cyclical movement of the series,

$$x_t = \sum_{j=1}^{r} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\} + u_t. \tag{8.3}$$

In this context estimation of r, the number of sinusoidals components, has been treated in Quinn (1989), Kavalieris and Hannan (1994), Hannan (1993) and Wang (1993) among others. Estimation of the different ω_j has been analysed in Chen (1988a,b), Walker (1971) and Kavalieris and Hannan (1994).

Although the behaviour described in (8.1) or (8.3) can be appropriate for some time series in many areas of natural sciences, in economic time series where the cycles or periods have a less regular behaviour, this rigid deterministic periodicity seems implausible. A changing pattern can be generated by stochastic sine and cosine coefficients in (8.1) or (8.3), as in Hannan (1964) (see (1.9)), by seasonal ARMA or ARIMA models (see (1.18) and (1.20)), or more generally using the GARMA process introduced by Gray et al. (1989) (see (1.27)) or the ARUMA (see (1.32)) analysed by Giraitis and Leipus (1995) among others [1]. These processes are characterized by a strong and persistent periodical behaviour, although their amplitude and periodicity can change over time unlike those series generated by deterministic trigonometric polynomials. The estimation of ω in cyclical long memory models may be necessary

[1] For a more exhaustive description of these seasonal models see Sections 2 and 3 in Chapter 1.

to determine the periodicity of the cycle and as a first step prior to the estimation of the persistence parameters or of the complete parametric model. Yajima (1995) proposed an estimate of ω in a process with spectral density function

$$f(\lambda;\omega,\theta)=g(\lambda;\omega,\theta)|\lambda-\omega|^{-2d} \quad \omega\in[0,\pi] \text{ and } 0<d<1/2, d\in\theta \tag{8.4}$$

where θ is a parameter vector of unknown short and long run parameters, and the function $g(\lambda)$ obeys some regularity conditions, such that the GARMA process is a special case of (8.4). The estimate of ω proposed by Yajima is the periodogram maximizer. He only obtains n^{α}-consistency under Gaussianity for any $\alpha\in(0,1)$ and shows that the Whittle estimates of θ obtained by minimizing

$$U_n(\hat{\omega},\theta)=\int_{-\pi}^{\pi}\left\{\log f(\lambda;\hat{\omega},\theta)+\frac{I_n(\lambda)}{2\pi f(\lambda;\hat{\omega},\theta)}\right\}\mathrm{d}\lambda \tag{8.5}$$

are $\sqrt{n}$-consistent and asymptotically normal. Yajima (1995) does not provide any distribution theory for his estimate of ω, but a nonnormal distribution is conjectured.

Hidalgo (1997), without assuming Gaussianity, proposes an alternative semiparametric technique to estimate the frequency ω when the spectral density function behaves around ω as

$$f(\lambda)\sim C|\lambda-\omega|^{-2d} \quad \text{as } \lambda\to\omega$$

for $C\in(0,\infty)$ and $d\in(0,1/2)$. The estimate is the argument that maximizes the estimate of d established in Hidalgo and Yajima (1997) and described in (1.59) in Chapter 1. Asymptotic normality of $nk^{-\frac{1}{2}}(\hat{\omega}-\omega)$, where $k\to\infty$ suitably slowly with n, is obtained.

Chung (1996a) obtained an estimate of $\eta=\cos\omega$ in a simple Gegenbauer process, $(1-2L\eta+L^2)^d x_t=\varepsilon_t$ where ε_t is white noise, by maximizing the conditional sum of squares

$$S(d,\eta)=-\frac{n}{2}(\log 2\pi+1)-\frac{n}{2}\log\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2\right)$$

which clearly is equivalent to minimizing the sum of squared innovations. Chung stated that for $|\eta|<1$ and $d\neq 0$, $n(\hat{\eta}-\eta)$ converges in distribution to a functional of Brownian motions, and for $\eta=1,-1$ and $d\neq 0$, $n^2(\hat{\eta}-\eta)$ converges in distribution to a different functional form of Brownian motions. Chung (1996b) generalizes this result to a general stationary GARMA process

$$\phi(L)(1-2L\eta+L^2)^d(x_t-\mu)=\theta(l)\varepsilon_t$$

where ε_t is white noise, $|\eta|\leq 1$, $|d|<1/2$ and the roots of the ARMA polynomials lie outside the unit circle. The same asymptotic distribution for $\hat{\eta}$ is claimed.

A joint estimation of all the frequencies ω_j, $j=0,1,...,h$, and the rest of long and short memory parameters in the ARUMA model (1.48) is proposed by Giraitis and Leipus (1995). They obtain consistency of the Whittle estimates obtained by minimizing $U_n(\omega,\theta)$ defined in (8.5), but no asymptotic distribution is established.

8.2 FURTHER RESEARCH AND EXTENSIONS

Throughout this thesis we have treated a number of issues concerning seasonal or cyclical long memory. Some other features that may need a more thorough analysis are the following:

1. Obtaining a time domain expression for asymmetric SCLM processes like those in (2.1) for the symmetric case may be useful to empirical researchers. The knowledge of some function $D(z)$ as in (2.1) would facilitate the application of those processes. However the task is not as easy as it appears at first sight. The asymmetry of the spectral density makes the use of techniques similar to those used in the symmetric case inappropriate and some different methods should be used.

2. Related to the previous suggestion, obtaining the AR and MA coefficients for the processes analysed in Chapter 2 may also be useful. The knowledge of the AR coefficients could be interesting for forecasting. Porter-Hudak (1990) used the fractional seasonal differencing operator $(1-L^{12})^d$ to forecast US monetary aggregates and found that it performs better than the usual airline model. However, assuming symmetric spectral poles, if the persistence parameters across different frequencies are different, the ARUMA model in (1.48) could be more

precise. This is described as

$$\prod_{j=0}^{h}(1-2L\cos\omega_j+L^2)^{d_j}x_t=\sum_{j=0}^{\infty}\pi_j x_{t-j}=u_t$$

where the AR coefficients, π_j, are described in (1.33) as functions of Gegenbauer polynomials, $C_k^{(d)}(\eta)$. In Chapter 6 we saw that $C_k^d(\eta)$ can be recursively obtained quite easily. However, the generation of the different π_j's gets more complicated as the number of spectral poles and the sample size increase. A simpler way of generating the π_j's could be useful for applied research.

3. The study of the effects of asymmetric SCLM on the different parametric and semiparametric methods of estimation described in Chapter 1 may reveal interesting features. Of course the parametric estimates of the persistence parameter will be inconsistent if we consider symmetric spectral poles when in fact an asymmetry is present.

4. Analysis of the implications of spectral asymmetries on the tests of fractional integration and fractional cointegration (see Section 1.5 in Chapter 1).

5. Some financial series, such as asset returns, appear to be approximately uncorrelated. However, there are nonlinear transformations, such as squares, that can exhibit autocorrelation as modelled in the extensive ARCH and stochastic volatility literature, following Engle (1982) or Taylor (1986, 1994). GARCH (Generalized Autoregressive Conditional Heterosckedastic) models let the conditional variance be a function of the squares of previous observations and past variances (for a survey on the field see Bollerslev et al. (1992)). Baillie et al. (1996) combined these models with fractional integration to describe long memory and time-dependent heterosckedasticity in the inflation series of ten countries. They proposed the ARFIMA(p,d,q)-GARCH(P,Q) process

$$\phi_p(L)(1-L)^d(\pi_t-\mu-b'x_{1t}-\delta\sigma_t)=\theta_q(L)\varepsilon_t \tag{8.6}$$

$$\varepsilon_t|\Omega_{t-1}\sim(0,\sigma_t^2) \tag{8.7}$$

$$\beta_P(L)\sigma_t^2=w+\alpha_Q(L)\varepsilon_t^2+\gamma'x_{2t} \tag{8.8}$$

where x_{1t} and x_{2t} are vectors of predetermined variables (x_{2t} can include lagged π_t), μ is the mean of the process, $\phi_p(z)$, $\theta_q(z)$, $\beta_P(z)$ and $\alpha_Q(z)$ have roots outside the unit circle and π_t is the monthly inflation at time t. Baillie et al. (1996) noted that seasonality is important for 9 out of the 10 countries analysed and they dealt with it by the inclusion of seasonal coefficients in the ARMA specification of π_t. However we have seen in Chapter 7 that UK inflation is likely to have seasonal long-range dependence. As a result it could be more adequate the use of the ARUMA(p,d_s,q)-GARCH(P,Q) process that can be defined by (8.7), (8.8) and

$$\phi_p(L)\prod_{j=0}^{h}(1-2L\cos\omega_j+L^2)^{d_j}(\pi_t-\mu-b'x_{1t}-\delta\sigma_t)=\theta_q(L)\varepsilon_t$$

where, in the case of the inflation series, the ω_j are seasonal frequencies (of course the ω_j can be any frequency in order to describe any cyclical behaviour).

The long-range dependence can also appear in the volatility of the series. The first model that causes this effect is the general GARCH process proposed by Robinson (1991) who used it as an alternative in testing for no-ARCH. His model is sufficiently general to describe SCLM behaviour in the squares of the series. This effect can also be modelled by specifying σ_t^2 in (8.7) as

$$\prod_{j=0}^{h}(1-2L\cos\omega_j+L^2)^{d_j}\sigma_t^2=w+u_t$$

where w is a constant and u_t is some short memory process (e.g. a stationary and invertible ARMA). This process is a generalization of FIGARCH models and it allows for seasonal long-range dependence in addition to the usual fractional integration at frequency zero.

In a recent paper Henry and Payne (1997) use the "long memory in stochastic volatility" model proposed by Harvey (1993) to describe the possibility of long-range dependence in the volatility of three intra-day foreign exchange data series. The process is described as

$$r_t=\sigma e^{\frac{h_t}{2}}\varepsilon_t \quad , \quad \varepsilon_t\sim N(0,1) \tag{8.9}$$

$$(1-L)^d h_t=u_t \tag{8.10}$$

where u_t is a short memory stationary process. In Henry and Payne (1997) r_t is the intra-day foreign exchange return series. They found that r_t has long memory in the volatility measured by $\log r_t^2$. Henry and Payne (1997) also found strong seasonal behaviour in $\log r_t^2$ and used a Double-Window smoother to remove this seasonality. However, the volatility may have SCLM. In this case a more appropriate model might be the "seasonal long memory in stochastic volatility" that can be defined by (8.9) and

$$\prod_{j=0}^{h}(1 - 2L\cos\omega_j + L^2)^{d_j} h_t = u_t.$$

Similar extensions to cover the possibility of seasonal or cyclical long memory in the volatility (conditional variance) and the mean can be carried out in most models used in Financial Economics.

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