Quantum waves in configuration space

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Abstract

The thesis deals with issues in the foundations of quantum mechanics, having to do with configuration space, the physical reality of quantum waves, additive conservation and EPR experiments.

After a historical sketch of optical theories, concentrating on the dual nature of light, the passage from Hamilton's optico-mechanical analogy to wave mechanics is looked at. The wave-particle duality of de Broglie's *théorie de la double solution* is favoured after comparison with some of Schrödinger's views. Three experiments are considered which support that realist duality by indicating corpuscular and undulatory properties. If wave and particle coexist and the wave guides the particle along its trajectory, the wave must have a physical reality. The issue is whether such a wave can propagate in a fictitious configuration space.

Features of quantum-mechanical interference are represented on the Riemann sphere. The treatment is generalized to infinite dimensions and then to tensor product spaces. 'Entanglement' is defined; certain states of composite systems cannot be broken up in such a way that every subsystem has a (pure) state. Entanglement is shown to be always empirically visible in principle; for every entangled state $|\Psi\rangle$ there exists a 'sensitive' observable which can tell $|\Psi\rangle$ apart from any mixture of factorizable states. Observables represented by functions of tensor products of operators cannot, however, tell the difference.

Additive conservation is considered separately from interference, and is related to Schmidt's theorem and Bertlmann's socks in cases involving two subsystems. The treatment is then generalized to N subsystems. Interference and additive conservation are combined in two examples: the violation of Bell's inequality, and the theorem of Wigner, Araki and Yanase. Schrödinger's cat is made to 'oscillate.'

Interpretations of quantum waves in configuration space are assessed and Furry's hypothesis discarded. The distinction is drawn between *weak* Bell inequalities deduced from local realism alone, and *strong* inequalities which involve physically unreasonable additional assumptions. It is shown that, as long as inefficient detectors are employed, photons can only be used to violate strong inequalities. Kaons are almost always detected and can be used to discriminate between quantum mechanics and local realism, and determine whether quantum waves really propagate in configuration space.

Wie alles sich zum Ganzen webt,
Eins in dem andern wirkt und lebt!

Faust

Contents

Introduction

Several well-known difficulties of quantum theory—the violation of Bell's inequality, Schrödinger's cat and so forth—have a common structure, inasmuch as they involve entangled states. In fact the problems *reside* in that structure and in that sense represent the same issue, despite their different appearances. They are expressions of a difficulty which can be traced back to 1926 when Schrödinger, who suspected something was wrong, allowed quantum waves to propagate in configuration space.

Quantum mechanics is problematic even in ordinary space, but there at least one can conceive a physical reality corresponding to the quantum-mechanical description. In configuration space it is much harder to work out an ontology that makes sense. How can a *wave* propagate in a fictitious configuration space?

If one allows the status of quantum waves to be determined by their propagation in configuration space, there will not be much to them; acceptance of the configuration space description usually leads to a dismissal of quantum waves as unreal mathematical fictions; without explicit reference to waves the question then assumes such abstract forms as $2\sqrt{2} > 2$. Alternatively the status of quantum waves can be determined in *ordinary* space, where it is easier to accept their reality. There they manifestly behave like physical waves by interfering and diffracting, and even appear to guide particles. Certain experiments indicate undulatory *and* corpuscular behaviour; if wave and corpuscle coexist, the wave presumably guides the particle; and to guide the particle—possibly exchanging energy and momentum with it—the wave must be more than empty mathematics.

If quantum waves are indeed real, one has to wonder about their propagation in configuration space. Perhaps the quantum-mechanical description breaks down with separation, and quantum waves in fact only propagate in ordinary space. Experience, which can decide the matter, supports that description more firmly when the particles are close together than when they are far apart.

The photons used to test quantum mechanics for widely separated particles are not detected often enough to settle the matter. Kaons on the other hand are almost always detected, and can be used to establish whether quantum waves do propagate in configuration space.

Doubts concerning quantum waves in configuration space can be based not only on ontological prejudices founded in ordinary space, or on specific experiments, but also on an abstract examination of composite systems, tensor multiplication and superposition. The existence of sensitive observables, the violation[1] of Bell's inequality, the measurement problem, or the theorem of Wigner, Araki and Yanase can cast doubts on

[1]The merely mathematical, rather than the experimental violation.

the configuration space description. So parts of the thesis are devoted to more formal 'configuration space' issues, in which explicit reference need seldom be made to waves, physical reality or experiments.

A position I consider, then, is that real physical waves guide particles in ordinary space, and that such waves do not propagate in configuration space. The position can partly be founded on the coexistence of wave and particle—which leads to a belief in the guidance of the particle and hence in the reality of the wave—and partly on the tension between additive conservation and the propagation of quantum waves in configuration space.

The plan of the thesis is as follows. To show how quantum waves ended up in configuration space (I. 1) I begin with a brief historical sketch of optical theories (I. 1.1): light was usually taken to be a stream of particles until the nineteenth century, when it became a wave with the work of Young, Fresnel and Maxwell. Einstein, while accepting the statistical validity of the wave theory, pointed out that individual emissions and absorptions could only be explained in corpuscular terms, and hence introduced the *Lichtquantum* (I. 1.2). Both trajectories and wave propagation had already been associated with light *and* matter in Hamilton's optico-mechanical analogy (I. 1.3), which was duly generalized by de Broglie and Schrödinger into an analogy between physical optics and *wave* mechanics (I. 1.4). Schrödinger's waves propagated in configuration space, unlike de Broglie's.

After a brief discussion of configuration space in analytical and quantum mechanics (I. 2), a picture of real waves guiding particles in ordinary space is considered (I. 3). De Broglie believed that real physical waves were at issue, but had trouble characterizing them exactly; soon they would be forgotten, and the merely mathematical waves we are more familiar with today would replace them (I. 3.1). He extended the notion of *trajectories orthogonal to wave surfaces* from geometrical optics and analytical mechanics, where it certainly belongs, to physical optics and wave mechanics, where according to Schrödinger it does not (I. 3.2). To clarify the matter I appeal to three experiments which support de Broglie's picture of a wave guiding a particle by indicating undulatory *and* corpuscular behaviour (I. 4). Admittedly it can be argued in all three cases that particles are *created* by measurement, as wave and particle do not in fact manifest themselves at the same time. It is more natural to assume, however, that position measurement reveals the presence of a particle that was there all along, describing a trajectory.

The treatment in Part I is 'ontological.' The issues are expressed in terms of waves, physical reality and experiments; but they can also be given more abstract reformulations. Even if explicit reference to waves is avoided and a different language is adopted, however, the 'configuration space' problem remains and emerges in other forms, such as violations of inequalities or of conservation principles. This is seen in Parts III and IV, where the question of quantum waves in configuration space is reformulated in terms of sensitive observables, Bell's inequality and measurement theory. Interference is first looked at, however, in cases that do not involve tensor multiplication (II).

The arguments of complex numbers somehow represent the phases of quantum waves. The simplest Hilbert space in which to explore their significance is the two-dimensional complex space H, so I begin there (II. 1). Rank one projection operators or rays (pure states) can be represented by points on the Riemann sphere, non-idempotent statistical operators (non-trivial mixtures) by internal points, and equivalence classes of commuting maximal normal operators by diameters. Two ways of exploring the statistical significance of phase are considered: its *removal* and its *transformation*. Both are defined with respect to a particular basis, say $\mathfrak{b}' = \{|\uparrow\rangle, |\downarrow\rangle\}$. Phase can be 'removed' from the state

$$|\Psi\rangle\langle\Psi| = |c_\uparrow|^2|\uparrow\rangle\langle\uparrow| + |c_\downarrow|^2|\downarrow\rangle\langle\downarrow| + (c_\uparrow c_\downarrow^*|\uparrow\rangle\langle\downarrow| + c_\downarrow c_\uparrow^*|\downarrow\rangle\langle\uparrow|),$$

by leaving out the terms in parentheses—the 'interference operator'—and taking the statistical operator represented by the other terms. Geometrically this corresponds to the orthogonal projection of a point on the sphere to a point on the diameter $\mathfrak{D}_{\mathfrak{b}'}$ corresponding to $\mathfrak{b}'$. Phase can be 'transformed' with respect to $\mathfrak{b}'$ by applying a unitary operator

$$U_{\mathfrak{b}'} = e^{i\theta_\uparrow}|\uparrow\rangle\langle\uparrow| + e^{i\theta_\downarrow}|\downarrow\rangle\langle\downarrow|$$

with eigenbasis $\mathfrak{b}'$, which gives rise to a rotation around a circle perpendicular to $\mathfrak{D}_{\mathfrak{b}'}$. One such unitary transformation will be time evolution, which produces a precession around a circle perpendicular to the 'energy' eigendiameter shared by all conserved quantities.

The statistics of observables with eigenbasis $\mathfrak{b}'$ are given by orthogonal projection onto $\mathfrak{D}_{\mathfrak{b}'}$. Such observables will not notice the transformation or removal of phase with respect to $\mathfrak{b}'$, which can, however, affect the statistics of observables with another eigenbasis $\mathfrak{b}$. If $\mathfrak{b}'$ is the energy eigenbasis, for instance, measurements corresponding to $\mathfrak{D}_{\mathfrak{b}}$ can give rise to beats.

I then generalize from H to infinite dimensions (II. 2), introduce tensor multiplication, and show that certain vectors cannot be factorized (III. 1.1). Not all observables can tell the difference between such non-factorizable vectors and products, or even mixtures of products; some are 'indifferent' to quantum waves in configuration space, in the sense that they cannot distinguish between an essentially undulatory case and one that could have nothing to do with waves (III. 1.3). Functions of tensor products of operators, for instance, represent indifferent observables. 'Entanglement' is always statistically visible, however, in the sense that for every non-factorizable vector $|\Psi\rangle$ there is a (bounded) self-adjoint operator A such that $\mathrm{Tr}\big(|\Psi\rangle\langle\Psi|A\big) \neq \mathrm{Tr}(\rho A)$ for every statistical operator ρ whose eigenvectors are tensor products (III. 1.2).

Schmidt's theorem (III. 2.1) concerning biorthogonal expansions provides a basis for the reality criterion and argument of Einstein *et al.* (III. 2.2), and for the expression of additive conservation laws. Such laws are also looked at in the general case involving several subsystems (III. 2.3). Multiorthogonal expansions with respect to energy eigenvectors indicate correlations too strong to be due to additive conservation (III. 2.4). It is as though, if Bertlmann had three feet and as many socks, one could tell which sock is on the third foot by looking only at the first.

Having dealt with interference and additive conservation separately, in Part IV I consider examples in which they are at work together. Bell's inequality (IV. 1.3) can be viewed as a consequence of four additive conservation laws, and its violation as manifesting a tension between additive conservation and interference in configuration space. A similar tension is expressed in the theorem of Wigner, Araki and Yanase (IV. 2.2), according to which 'pre-measurement' interactions described in IV. 2.1 can violate additive conservation laws.

I then return to a more ontological treatment in V. 1, and consider what may happen to quantum waves in configuration space. Perhaps they only really propagate in ordinary space, and something breaks down with separation. It is natural to attempt a representation of such a break-down *within* the quantum formalism. As the arguments of expansion coefficients represent the phases of quantum waves, it is sometimes suggested that separation 'removes' them, and turns an entangled state into a mixture of products represented by a statistical operator v. Though this solution can make some sense for observables with the same eigenstates as v, it does not work with others (V. 2). Non-factorizable vectors are therefore the best *quantum-mechanical* descriptions of certain physical states, but may be empirically inadequate nonetheless.

Bell's inequality can be used to test the configuration space description; if it is really violated in nature, *something*—perhaps empty mathematics—propagates in configuration space. In VI. 1 I show that photons, which are seldom detected, can only be used to violate *strong inequalities*, deduced with the help of physically unreasonable additional assumptions. *Weak inequalities*, deduced from local realism alone, have never been violated experimentally. Again, kaons are almost always detected, and can be used to establish whether quantum waves really propagate in configuration space (VI. 2).

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I. History, wave-particle duality and three experiments

1
History

We begin by seeing *how* quantum waves ended up in configuration space.

The world, at the beginning of the century, was made of matter and light,[2] described respectively by analytical mechanics and optics, say geometrical optics for the time being. Mechanics was governed by Maupertuis' principle of least action, optics by Fermat's principle of least time. Hamilton had pointed out their similarity, and unified the treatments of matter and light: mechanical trajectories are orthogonal to surfaces of equal action, which look like wavefronts; and optical rays, which resemble trajectories, to surfaces of equal time. But light is accurately described by geometrical optics only where the index of refraction varies slowly on the scale of the wavelength. Otherwise a wave theory should be used.

Light had been a stream of particles[3] before becoming a wave in the nineteenth century; Einstein pointed out that it must be both to explain the undulatory averages and the individual emissions and absorptions. Extending wave-particle duality to matter, and ascribing the peculiarities of the new quantum theory and atomic mechanics to waves, de Broglie suggested that analytical mechanics also approximated a wave theory. Undertaking a unification of matter and light more general than Hamilton's, de Broglie developed a rudimentary *wave mechanics*, refining Hamilton's analogy with Planck's relation $E = h\nu$ and the new relation $p = h/\lambda$. The waves propagated in four-dimensional space-time, or just in ordinary three-dimensional space. Surprisingly de Broglie did not arrive at Schrödinger's equation, which with hindsight can seem an almost inevitable outcome of his unification programme. Once the equation existed, however, its historical roots indicated a natural interpretation: the particle, though subject to the undulations of the wave, still described a trajectory orthogonal to the surfaces of equal action, which now really were wavefronts.

1.1 The dual nature of light[4]

Light, which brought wave-particle duality to the attention of physicists, was first thought to be made of particles.[3] Indeed its properties most obvious to the naked eye, and hence first observed—reflection, refraction, and rectilinear propagation—are more

[2]By 'light' I mean electromagnetic waves of all frequencies.
[3]This is a simplification. Although corpuscular theories were certainly popular before Huygens—and indeed remained so until the nineteenth century—there were others as well. See Lindberg (1978a) and Whittaker (1951).
[4]The treatment here will sometimes follow de Broglie (1937) pp.133-177.

easily explained in corpuscular than undulatory terms. Reflection is like bouncing, and can be attributed to a similar elastic mechanism. Refraction can also be explained in corpuscular terms: particles are deflected as resistance to their motion changes. Light clearly travels very fast; if it is indeed a stream of particles, their very speed would make one expect a more or less rectilinear motion. It is not surprising, then, that almost all of Newton's predecessors, to whom undulatory properties were hardly visible, should have viewed light as being made of particles. Despite noticing interference effects,[5] Newton (1931) also favoured a corpuscular picture. His contemporary Huygens, discarding the possibility of light particles, proposed an undulatory theory modelled on sound and water waves.

> ... quand on considere l'extreme vitesse dont la lumiere s'étend de toutes parts, & que quand elle vient de differents endroits ... elles se traversent l'une l'autre sans s'empecher; on comprend bien que quand nous voyons un objet lumineux, ce ne sçauroit estre par le transport d'une matiere, qui depuis cet objet s'en vient jusqu'à nous, ainsi qu'une bale ou une fleche traverse l'air: car assurément cela repugne trop à ces deux qualités de la lumiere .... C'est donc d'une autre maniere qu'elle s'étend, & ce qui nous peut conduire à la comprendre c'est la connoissance que nous avons de l'extension du Son dans l'air.
>
> Nous sçavons que par le moyen de l'air ... le Son s'étend tout à l'entour du lieu où il a esté produit, par un mouvement qui passe successivement d'une partie de l'air à l'autre .... Or il n'y a point de doute que la lumiere ne parvienne aussi depuis le corps lumineux jusqu'à nous par quelque mouvement imprimé à la matière qui est entre deux: puisque nous avons deja veu que ce ne peut pas estre par le transport d'un corps qui passeroit de l'un à l'autre. Que si avec cela la lumiere employe du temps à son passage ... il s'ensuira que ce mouvement imprimé à la matière est successif, & que par consequent il s'etend, ainsi que celui du Son, par des surfaces & des ondes spheriques: car je les appelle ondes à la ressemblances de celles que l'on voit se former dans l'eau quand on y jette une pierre, qui representent une telle extension successive en rond, quoique provenant d'une autre cause, & seulement dans une surface plane.[6]

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[5]Newton's rings

[6]Huygens (1690). Translation: ... when we consider the extreme speed with which light extends itself in all directions, and that when it comes from different places ... they cross one another without obstruction; one understands that when we see a luminous object, it would not be through the transport of a substance, which from this object reaches us, just as a bullet or arrow crosses the air: for assuredly this is in contrast with these two qualities of light .... It is therefore in another way that it extends itself, & what can lead us to understand it is the knowledge we have of the extension of Sound in air.

We know that through the medium of air ... Sound extends itself all round the place where it was produced, by a movement which passes successively from one part of the air to another .... Now there is no doubt that light also gets from the luminous body to us by some movement impressed on the substance which is between the two: as we have already seen that it cannot be by the transport of a body that would pass from one to the other. That if, with this, light takes time in its passage ... it will follow that this movement impressed on matter is successive, & as a consequence it extends itself, much as that of Sound, by surfaces & spherical waves: for I call them waves from the resemblance to those seen to be formed in water when one throws in a pebble, which represent such a successive round extension, although deriving from another cause, & only on a plane surface.

Despite accounting for reflection and refraction, Huygens' theory offered no explanation of rectilinear propagation, and reflection was perhaps dealt with more naturally in corpuscular terms.[7] So the granular picture, supported by Newton's authority, prevailed until the work of Young (1855) and Fresnel in the early nineteenth century led to an acceptance of light waves. Young's experiments on interference and diffraction concerned undulatory phenomena that could not be explained by a purely corpuscular theory. Once Fresnel accounted for rectilinear propagation in undulatory terms, all phenomena then known could be explained with waves. The success of Maxwell's theory, in which light was just another electromagnetic wave, gave further support to an undulatory view.

1.2 *Lichtquanten*

Effects involving the emission and absorption of light were, however, incompatible with a purely undulatory description. There was the photoelectric effect, for instance.

A metal can be made to release electrons by shining light on it. If the light frequency exceeds a threshold ν_0, electrons are emitted, with kinetic energy $E = h(\nu - \nu_0)$ and in an amount proportional to the intensity of the light. This was hard to account for with Maxwell's theory, in which light is a continuous wave whose properties are spread smoothly over the volume occupied. Since energy and momentum in that theory are distributed according to the description given by the Poynting vector, an increase in the intensity of the wave leads to an increase in the moduli of the electric and magnetic vectors, and hence also of the density of electromagnetic energy. The part of the wave that strikes a particular electron of the surface of the metal should release, for a given time of exposure, an amount of energy proportional to the intensity of the incident wave. So the kinetic energy of the photoelectron ought to depend on the intensity of the light, and yet no such dependence is manifested.

Furthermore the energy of photoelectrons should not depend on the frequency of the light; being free, they can vibrate at different frequencies, and resonate with the incident light wave as they absorb its energy. So it is not clear, within a purely undulatory theory, why only frequencies above a certain threshold give rise to the emission of electrons, and why their kinetic energy increases with this frequency.

Einstein (1905) recognized the value of the *Undulationstheorie des Lichtes*, but claimed that its validity was only statistical and limited to *Mittelwerte*:

> Die mit kontinuierlichen Raumfunktionen operierende Undulationstheorie des Lichtes hat sich zur Darstellung der rein optischen Phänomene vortrefflich bewährt und wird wohl nie durch eine andere Theorie ersetzt werden. Es ist jedoch im Auge zu behalten, daß sich die optischen Beobachtungen auf zeitliche

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[7]Until the velocities of light in different media were accurately compared by the experiments of Fizeau and Foucault, refraction could be dealt with either way.

> Mittelwerte, nicht aber auf Momentanwerte beziehen, und es ist trotz der vollständigen Bestätigung der Theorie der Beugung, Reflexion, Brechung, Dispersion etc. durch das Experiment wohl denkbar, daß die mit kontinuierlichen Raumfunktionen operierende Theorie des Lichtes zu Widersprüchen mit der Erfahrung führt, wenn man sie auf die Erscheinungen der Lichterzeugung und Lichtverwandlung anwendet.
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> Es scheint mir nun in der Tat, daß die Beobachtungen über die „schwarze Strahlung", Photolumineszenz, die Erzeugung von Kathodenstrahlen durch ultraviolettes Licht

in other words the photoelectric effect,

> und andere die Erzeugung bez. Verwandlung des Lichtes betreffende Erscheinungsgruppen besser verständlich erscheinen unter der Annahme, daß die Energie des Lichtes diskontinuierlich im Raume verteilt sei. Nach der hier ins Auge zu fassenden Annahme ist bei Ausbreitung eines von einem Punkte ausgehenden Lichtstrahles die Energie nicht kontinuierlich auf größer und größer werdende Räume verteilt, sondern es besteht dieselbe aus einer endlichen Zahl von in Raumpunkten lokalisierten Energiequanten, welche sich bewegen, ohne sich zu teilen und nur als Ganze absorbiert und erzeugt werden können.[8]

By attributing a frequency $\nu = E/h$ to each corpuscle he moreover brought undulations down from the familiar macroscopic level of Maxwell's theory, where of course they remained, to the individual *Energiequanten* or '*Lichtquanten*,' as he was to call them.

According to Einstein, then, every light particle is made up of oscillating energy of some sort. When such a corpuscle strikes an electron, it gets absorbed and turned into kinetic energy. The electron starts moving into the photosensitive material (in the same direction as the incident light corpuscle), but after one or more elastic collisions with atoms in the crystalline lattice it can reverse its motion and leave the metal.

Einstein assumed that a photoelectron absorbed exactly one *Lichtquantum*. The intensity of the light—the amount of *Lichtquanten* per unit volume, not the frequency—is proportional to the number of *Lichtquanten* incident on a unit area of the metal per unit time. Hence the amount of electrons extracted, in other words the intensity of the photoelectric current, must be proportional to the intensity of the incident light.

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[8]Translation: The wave theory of light, which uses continuous spatial functions, has proved very successful for the representation of purely optical phenomena and will never be replaced by another theory. One must nevertheless bear in mind that optical observations refer rather to time averages than to instantaneous values, and that despite the absolute experimental confirmation of the theory of diffraction, reflection, refraction, dispersion, *etc.*, it remains entirely conceivable that the theory of light operating with continuous spatial functions might lead to contradictions with experience, if one applies it to the phenomena of light emission and transformation. Indeed it seems to me that observations concerning "black body radiation," photoluminescence, the emission of cathode rays by means of ultraviolet light and other groups of phenomena regarding the emission or transformation of light appear more intelligible under the assumption of a discontinuous spatial distribution of the energy of light. According to the assumption in question the energy of a beam of light propagating out from a point is not distributed continuously over larger and larger spaces, but is made up of a finite number of spatially localized energy quanta, which move without splitting, and can only be absorbed and emitted as wholes.

A certain amount w_0 of work is required to release an electron, as contact forces between metal and atmosphere have to be overcome. Only electrons whose frequency exceeds $\nu_0 = w_0/h$ can therefore leave the metal, and indeed $\nu > \nu_0$ is the escape condition observed.

Other important results regarding the photoelectric effect were obtained by Mayer and Gerlach in 1914, with the experimental examination of fine particles of metal powder suspended in an electric field, illuminated and observed under a microscope. The escape of a photoelectron from a grain of powder was indicated by the sudden acceleration of the grain. The acceleration sometimes took place after a very short exposure to light, even when the illumination was so weak that a couple of hours would have been necessary for the electron to accumulate enough energy to escape if the electromagnetic energy had been uniformly distributed over the wave. This too was easily explained by Einstein's corpuscular description of light.

What if radiation propagated in concentric spheres[9] whose entire energy could be suddenly condensed by contact with an electron? Einstein showed this was impossible in *Zur Quantentheorie der Strahlung* (1917) by considering a gas of molecules which interacted through the emission and absorption of electromagnetic radiation. He assumed that transitions between the discrete energy states he attributed to the molecules were associated with the emission or absorption of radiation. Both could take place under the influence of the radiative field, and emission even without being induced by external causes, or 'spontaneously' as we now say. He finally assumed that the distribution of the energy states in the molecular gas could be deduced from the canonical distribution of statistical mechanics. The results obtained were derived from Planck's formula and Bohr's relation

$$\nu_{ij} = \frac{E_i - E_j}{h},$$

where ν_{ij} is the frequency of the radiation, E_i and $E_j < E_i$ the energy levels.

When a molecule emits or absorbs radiation, its momentum changes unless the radiation is spherically symmetrical. The difference is greatest when all the energy E is exchanged in a single direction, in which case it is equal to E/c. Einstein proved that Maxwell's well-verified distribution for molecular velocities could only be obtained by assuming that all the energy was *precisely directed*[10] in every interaction between

[9]Wavefronts are spherical in homogeneous media, elsewhere they may not be.

[10]Cartwright (1983) considers a related case: "A particle with a fixed direction and a fixed energy bombards a target and is scattered. The state of the scattered particle is represented by an outgoing spherical wave .... After scattering the particle travels in no fixed direction; its outgoing state is a superposition of momentum states in all directions. We may circle the target with a ring of detectors." She concludes, however, that "Even without the detectors, the particle must be travelling one way or another far away from the target."

radiation and matter.

The accuracy of Maxwell's distribution for velocities imparted to the molecules by the emission or absorption of electromagnetic radiation can be understood bearing the following in mind: that Maxwell's original calculation of the distribution of molecular velocities takes account only of molecular collisions; that it is valid experimentally; that matter emits and absorbs radiation constantly. Such radiation would have given rise to different statistics if the interaction of radiation and matter had not been dealt with correctly. Hence Einstein's article can be considered a kind of theoretical Compton effect:

> Bewirkt ein Strahlenbündel, daß ein von ihm getroffenes Molekül die Energiemenge $h\nu$ in Form von Strahlung durch einen Elementarprozeß aufnimmt oder abgibt (Einstrahlung), so wird stets der Impuls $h\nu/c$ auf das Molekül übertragen, und zwar bei Energieaufnahme in der Fortpflanzungsrichtung des Bündels, bei Energieabgabe in der Entgegensetzten Richtung.[11]

Einstein considered this last result the most important conclusion of his article, because it shed light on the real nature of electromagnetic radiation.

So light, which was first made of particles, then of waves, turned out to be *both*. This dual nature would soon be extended to matter (see Section I. 1.4).

1.3 The optico-mechanical analogy

Hamilton[12] noticed a formal analogy[13] between analytical mechanics, to which he attributed a kind of wave motion, and geometrical optics. As it was through this optico-mechanical analogy that wave mechanics inherited configuration space from analytical mechanics, it will be worth looking at.

Hamilton's equation for the energy of a mechanical system can be written

$$\frac{\partial W}{\partial t} + T\left(q_k, \frac{\partial W}{\partial q_k}\right) + V(q_k, t) = 0, \tag{1}$$

where the action function W is the integral

$$W(P_2, t) = \int_{P_1}^{P_2} (T - V)dt$$

[11]Translation: A radiation bundle has the following effect: a molecule reached by it absorbs or emits (outgoing radiation), through an elementary process, an amount of energy $h\nu$ in the form of radiation; thus an impulse of $h\nu/c$ will be conveyed to the molecule, in the direction of propagation of the bundle in the case of absorption, in the opposite direction in the case of emission.

[12]Hamilton (1833) p.795; Hamilton (1931) volume I, pp.1-294

[13]See Schrödinger (1926); see also Lanczos (1970) pp.262-80, Tarozzi (1992) pp.18-20 and Arnold (1992) pp.245-54.

(viewed as a function of the final position P_2 and of time) of Lagrange's function $T - V$ over the actual path. The kinetic energy T is a function of the generalized coordinates q_k and momenta

$$p_k = \frac{\partial W(q_k, t)}{\partial q_k} = \frac{\partial S(q_k)}{\partial q_k},$$

Jacobi's S-function being a section of W at a given time. Although the potential V generally depends on time, we will assume it not to, so that energy $E = -\partial W/\partial t$ is conserved and (1) becomes

$$-E + \frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + V(x, y, z) = 0$$

for a single particle of mass m and rectangular coordinates.

As the momentum $\boldsymbol{p} = \operatorname{grad} W$ is orthogonal to the surfaces of equal action, the path of the particle is too. This is an expression of Maupertuis' principle

$$\delta \int_{P_1}^{P_2} \sqrt{2m(E - V)}ds = 0$$

of least action, where ds is an infinitesimal element of the path. To see this, suppose a particle crosses neighbouring surfaces Σ and Σ' of equal action at points P and P'. Since P' is closer to P than to any other point on Σ', the actual path, orthogonal to both surfaces because $\boldsymbol{p} = \operatorname{grad} W$, will correspond to less action than any other path through P and Σ', for the action difference dW is equal to

$$p\,ds = \sqrt{2m(E - V)}ds,$$

where p is the modulus of the momentum and ds the distance between the surfaces. This infinitesimal argument can be extended to longer paths by integration (up to the kinetic focus, beyond which ambiguities can arise).

Given an arbitrary surface Σ_0 corresponding to action W_0, a neighbouring surface of action $W_0 + dW_0$ can be constructed by laying off, at every point of Σ, the infinitesimal distance

$$ds = \frac{dW_0}{\sqrt{2m(E - V)}}$$

perpendicular to Σ. Enough iterations will fill the space and yield the S-function.

To construct the W-function, account must be taken of time evolution, given by $W = -Et + S$. As the system is conservative, the surfaces of equal action will remain unchanged as time passes, retaining the same shape and position. The action values can be viewed as moving through them so that the surface corresponding to W_0 at time t will correspond to $W_0 - Et'$ at $t + t'$, while W_0 will have moved along, during the same interval, to the surface that corresponded to $W_0 + Et'$ at t. Alternatively the surfaces can be seen as moving with their respective action values, deforming themselves in accordance with the local variations of the potential. *This is like a wave motion*, in which the surfaces of equal action correspond to wavefronts and action to phase. The normal velocity of the wavefronts, which varies with the potential, is given by

$$u = \frac{ds}{dt} = \frac{E}{\sqrt{2m(E-V)}}. \tag{2}$$

Already we have a kind of geometrical optics, where the index of refraction is inversely proportional to the normal velocity u and Fermat's principle assumes the form

$$\delta \int_{P_1}^{P_2} \frac{ds}{u} = \delta \int_{P_1}^{P_2} \frac{ds\sqrt{2m(E-V)}}{E} = 0.$$

Otherwise the index of refraction $n(x,y,z) = c/v(x,y,z)$ can be introduced without reference to a potential, $v(x,y,z) \leq c$ being the local speed of light.[14] We can again begin with an arbitrary basic surface Σ, to which we assign the time $t = 0$, and construct a neighbouring surface corresponding to the infinitesimal time $t = \varepsilon$ by laying off the distance $\sigma = v\varepsilon$ orthogonally from every point of Σ. Iterations yield the φ-function, which is a solution of the equation

$$\|\text{grad}\,\varphi\|^2 = \frac{n^2}{c^2} = \frac{1}{v^2}$$

and has a role similar to Jacobi's S-function. It gives the time light takes to travel from the basic surface $t = 0$ to the point x, y, z.

The normal velocity u of the *mechanical* wavefronts was inversely proportional to the speed $\|\text{grad}\,W\|/m$ of the particle; where the surfaces were close together, their propagation was slow, whereas the motion of the particle was fast. Here, on the other hand, the speed $v = 1/\|\text{grad}\,\varphi\|$, so light propagates rapidly where the gradient is small and the surfaces far apart. Still, light rays are orthogonal to the surfaces $\varphi(x,y,z) = C$ of equal time, and hence satisfy Fermat's principle

$$\delta \int \frac{ds}{v} = \delta \int \frac{c}{n} ds = 0.$$

Introducing the frequency $\nu(x,y,z)$,

$$\|\text{grad}\,\varphi\|^2 = \frac{\nu^2 n^2}{c^2}.$$

Accordingly the wavelength

---

[14]Equality holds *in vacuo*.

$$\lambda(x, y, z) = \frac{c}{n\nu} = \frac{v}{\nu}.$$

So far, then, a wave motion has been formally associated with mechanical trajectories, and rays with the wavefronts of geometrical optics.

1.4 Wave mechanics

Analytical mechanics was, at the beginning of this century, having trouble accounting for many 'quantum' phenomena, and could only do so with additional restrictions. In such restrictions—Bohr's quantum conditions on the orbits of electrons, for instance—often featured integers reminiscent of resonance and waves. What if waves *were* involved? After all light, which was once particles, then waves, could best be explained if it was in fact both. Maybe everything, matter and light, was corpuscular and undulatory. It was well known that geometrical optics is valid only when the index of refraction varies slowly on the scale of the wavelength, and that otherwise a wave optics should be used. With the scheme

wave optics	
geometrical optics	analytical mechanics

in mind, de Broglie filled in the empty box with *wave mechanics*. The relationship between geometrical optics and analytical mechanics was provided by Hamilton's analogy; for the corresponding relationship between wave optics and wave mechanics: *Il est dans l'esprit de la théorie des quanta de poser* $E = h\nu$.[15] So de Broglie set $E/\nu = W/\phi = S/\varphi = h$, where

$$\phi(x, y, z, t) = \nu t - \varphi(x, y, z).$$

As

$$\|\operatorname{grad} S\|^2 = 2m[E - V(x, y, z)],$$

$S = h\varphi$ led to de Broglie's relation

$$\lambda = \frac{1}{\|\operatorname{grad} \varphi\|} = \frac{h}{\|\operatorname{grad} S\|} = \frac{h}{\sqrt{2m[E - V(x, y, z)]}} = \frac{h}{p}.$$

A wave theory—this is the step de Broglie did not take—requires a wave equation, say

$$\operatorname{div} \operatorname{grad} \psi + \frac{4\pi^2}{\lambda^2(x, y, z)} \psi = 0. \tag{3}$$

[15] de Broglie (1956a)

To arrive at the wave equation of mechanics, all de Broglie had to do was substitute

$$\lambda = \frac{h}{\sqrt{2m[E - V(x, y, z)]}}$$

into (3). This would have yielded the equation

$$\operatorname{div}\operatorname{grad}\psi + \frac{8\pi^2 m}{h^2}[E - V(x, y, z)]\psi = 0,$$

which Schrödinger (1926) found instead:

> In welcher Weise wird man nun bei der undulatorischen Ausgestaltung der Mechanik in den Fällen, wo sie sich als notwendig erweist, vorzugehen haben? Man muß statt von den Grundgleichungen der Mechanik von einer Wellengleichung für den q-Raum

—we are already in configuration space—

> ausgehen und die Mannigfaltigkeit der *nach ihr* möglichen Vorgänge betrachten. ... Das einzige Datum zu ihrer Aufstellung ist die durch
>
> (4) $$u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E - V)}}$$
>
> oder
>
> (4′) $$u = \frac{h\nu}{\sqrt{2(h\nu - V)}}$$
>
> als Funktion des mechanischen Energieparameters bzw. der Frequenz gegebene *Wellengeschwindigkeit* und durch dieses Datum ist die Wellengleichung selbstverständlich nicht eindeutig festgelegt.[16]

So far only (2) and the relation $E = h\nu$ have been used to determine a wave velocity. Next Schrödinger introduces a wave equation:

> Es ist gar nicht ausgemacht, daß sie gerade von der zweiten Ordnung sein muß, nur das Bestreben nach Einfachheit veranlaßt dazu, es zunächst einmal damit zu versuchen. Man wird dann für die Wellenfunktion ψ ansetzen

[16]Translation: How, then, is one to develop this undulatory version of mechanics, for cases in which it is necessary? One must start with a wave equation for q-space instead of the fundamental equations of mechanics, and consider the manifold of processes that are possible according to it. ... The only thing that can guide us is the *wave velocity*

(3) $$u = \frac{ds}{dt} = \frac{E}{\sqrt{2(E - V)}}$$

or

(3′) $$u = \frac{h\nu}{\sqrt{2(h\nu - V)}}$$

given as a function of the mechanical energy, or of the frequency; but this is evidently not enough to determine the wave equation uniquely.

$$\operatorname{div}\operatorname{grad}\psi - \frac{1}{u^2}\ddot{\psi} = 0,$$

gültig für Vorgänge, welche von der Zeit nur durch einen Faktor $e^{2\pi i\nu t}$ abhängen. Das heißt also, mit Beachtung von (4), (4′) und $\nu = \frac{E}{h}$

$$\operatorname{div}\operatorname{grad}\psi + \frac{8\pi^2}{h^2}(h\nu - V)\psi = 0, \tag{5}$$

bzw.

$$\operatorname{div}\operatorname{grad}\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0. \tag{5'}$$[17]

Recapitulating, light was originally a stream of particles, but became a wave in the nineteenth century. The photoelectric effect and other electromagnetic phenomena were then given a corpuscular interpretation by Einstein. Matter, meanwhile, was having problems of its own; its motions at atomic levels were restricted by quantization conditions that brought resonance and waves to mind. Hamilton's optico-mechanical analogy, partly founded on the similarity between Fermat's principle of least time and Maupertuis' principle of least action, was duly generalized into an analogy between wave optics and wave mechanics, which involved an extension of wave-particle duality from light to matter. The waves of this generalized mechanics, however, propagated in configuration space.

It is surprising that a natural synthesis of staple classical theories should have produced so radical a departure from classical physics.

[17]Translation: It is not even clear that it will be exactly of second order; only our desire for simplicity leads us to try such an equation to start with. One would then have for the wave equation ψ

$$\operatorname{div}\operatorname{grad}\psi - \frac{1}{u^2}\ddot{\psi} = 0,$$

which is valid for processes which depend on time through a factor of $e^{2\pi i\nu t}$. This means therefore, bearing (4), (4′) and $\nu = \frac{E}{h}$ in mind, that

$$\operatorname{div}\operatorname{grad}\psi + \frac{8\pi^2}{h^2}(h\nu - V)\psi = 0, \tag{5}$$

or

$$\operatorname{div}\operatorname{grad}\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0. \tag{5'}$$

2
Configuration space

Just as the position of a classical particle is represented by a point in a three-dimensional space, the configuration of an N-particle system can be represented by a point in a $3N$-dimensional space. One advantage of this is theoretical economy; it is simpler to have a single point describing a trajectory subject to a single variational principle, than for each particle to obey its own. It is also easier to imagine the motion of one representative point than of many particles.

> The pictorial language of n-dimensional geometry makes it possible to extend the mechanics of a single mass-point to arbitrarily complicated mechanical systems. Such a system may be replaced by a single point for the study of its motion. But the space which carries this point is no longer the ordinary physical space. It is an abstract space with as many dimensions as the nature of the problem requires.[18]

The use of configuration space in analytical mechanics is, however, merely representative, and has no more fundamental significance: the configuration of a classical system can always be described in ordinary space. A *wavefunction* in a 'many-dimensional' configuration space cannot, on the other hand, necessarily be broken up into N factors in N ordinary spaces. This 'irreducible' propagation in a fictitious configuration space was disturbing from the beginning. Schrödinger, for instance, expressed a preference[19] for de Broglie's four-dimensional wave mechanics, which *das Wesen der Sache besser trifft*:

> ... zwar verwenden wir hier die „Wellenmechanik" in der dort fast ausschließlich bearbeiteten *viel*dimensionalen Form ... nicht in jener *vier-* ... welche der ursprünglichen De Broglieschen Konzeption entspricht und möglicherweise das Wesen der Sache besser trifft, aber vorläufig nur Programm ist, weil man das *Mehr*elektronenproblem nach ihr noch nicht zu formulieren versteht.[20]

Again, at the 1927 Solvay conference:

> On développe actuellement sous ce nom ['mécanique des ondes'], l'une à côté de l'autre, deux théories qui sont, il est vrai, étroitement liées, mais ne sont pas cependant identiques. Dans l'une, qui se rattache directement aux importantes thèses de M. de Broglie, il s'agit d'ondes dans l'espace à trois dimensions. Eu égard au traitement strictement relativiste, qui est suivi dès le début dans cette façon d'envisager le problème, nous l'appellerons la mécanique ondulatoire

[18]Lanczos (1970)

[19]Michel Bitbol assures me that Schrödinger changed his mind on this point.

[20]Schrödinger (1927b). Translation: We apply 'wave mechanics' in its *many*-dimensional form ... not in the *four*-[dimensional form] which corresponds to the original conception of de Broglie and possibly better reaches the essence of the matter, but for the time being represents no more than a programme, as one has not yet understood how to formulate the *several*-electron problem in that form.

quadridimensionnelle. L'autre théorie s'écarte d'avantage des idées originales de M. de Broglie, en ce sens qu'elle prend comme base un phénomène ondulatoire dans l'espace des coordonnées de situation (espace *q*) d'un système mécanique quelconque. C'est pourquoi nous l'appellerons la mécanique ondulatoire *polydimensionnelle*. Il va de soi que cet emploi de l'espace *q* ne doit être considéré que comme un artifice mathématique, qui trouva, d'ailleurs, souvent son application dans l'ancienne mécanique; en dernière analyse on décrira ici aussi un événement dans l'espace et dans le temps.

Perhaps Schrödinger thought the propagation of quantum waves in configuration space—*un artifice mathématique*—could always be re-expressed in ordinary space, much as the configuration of a system in analytical mechanics can be described without appealing to configuration space. We return to this question in Section III. 1.2. Continuing,

> Mais en réalité on n'a pas encore réussi à établir une conciliation parfaite des deux points de vue. Tout ce qui dépasse le mouvement d'un électron unique n'a pu être traité jusqu'ici que dans la manière de voir *polydimensionnelle* .... C'est pourquoi je place cette manière de voir en première ligne et j'espère pouvoir mieux mettre en lumière de cette façon les difficultés caracteristiques qui sont inhérentes à la conception quadridimensionnelle, plus belle en elle-même.[21]

Again, *tout ce qui dépasse le mouvement d'un électron unique* is intractable in three or four dimensions; configuration space seems to be indispensable, whatever reservations one may have about it.

De Broglie was *beaucoup scandalisé* by the use of configuration space:

> L'idée de M. Schrödinger de définir l'onde Ψ d'un système dans l'espace de configuration m'avait au début beaucoup scandalisé parce que, l'espace de configuration étant purement fictif, cette conception enlève à l'onde toute réalité physique: pour moi, l'onde de la Mécanique ondulatoire devait évoluer dans l'espace physique à trois dimensions.[22]

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[21]*Electrons et Photons*. Translation: We currently develop under this name ['wave mechanics'], one alongside the other, two theories which are, admittedly, closely related, but which are not identical. In one, which is directly related to the important theses of Mr. de Broglie, waves in three-dimensional space are at issue. In reference to the strictly relativistic treatment, which is followed from the beginning in this way of envisaging the problem, we will call it *quadridimensional* wave mechanics. The other theory deviates more from the original ideas of Mr. de Broglie, in the sense that it is based on an undulatory phenomenon in the space of the position coordinates (*q* space) of any mechanical system. For that reason we will call it *polydimensional* wave mechanics. It goes without saying that this use of *q* space has to be viewed as a mathematical artifice, which was often applied in the old mechanics; ultimately an event in space and time will be described here too. But in fact one has not yet managed to reconcile the two points of view completely. Anything beyond the motion of a single electron has only been tractable with the *polydimensional* approach .... This is why I give priority to this way of viewing things and hope thus to shed light on the characteristic difficulties which are inherent in the quadridimensional conception, which is in itself more beautiful.

[22]de Broglie (1956a). Translation: At first I was appalled at Mr. Schrödinger's idea of defining the Ψ wave of a system in configuration space because, configuration space being purely fictitious, this conception removes all physical reality from the wave: for me, the wave of Wave mechanics had to evolve in three-dimensional physical space.

Schrödinger's configuration space description was nevertheless adopted because *man das* Mehr*elektronenproblem nach ihr noch nicht*—this still applies—*zu formulieren versteht.* These issues will be returned to in Part III.

Configuration space can be used to express correlations between subsystems. Suppose two free classical particles with a total momentum of $\boldsymbol{p}$ collide with one another. The conservation of momentum determines *correlated pairs* $(\boldsymbol{p}_n^1, \boldsymbol{p}_n^2)$ of momenta by allowing us to infer the momentum $\boldsymbol{p} - \boldsymbol{p}_n^\sigma$ of one particle from the momentum $\boldsymbol{p}_n^\sigma$ of the other ($\sigma = 1$ or 2; $n = 1, 2, \dots$). The corresponding trajectories $\mathfrak{T}_n$ in configuration space are obtained varying the initial conditions $\mathfrak{C}$: conditions $\mathfrak{C}_n \in \mathfrak{C}$ give rise to trajectory $\mathfrak{T}_n$ and momenta $(\boldsymbol{p}_n^1, \boldsymbol{p}_n^2)$, $n = 1, 2, \dots$. This already brings to mind the situation described by Einstein, Podolsky and Rosen (see Section III. 2.2).

With a (time-independent) potential V, even though we cannot appeal as directly to conservation, we can invoke Lagrange's principle

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0$$

of least action, valid for systems whose total energy $E = T + V$ is conserved. The principle can be viewed as producing correlations by determining a mapping $\mathfrak{L}(V, E) : \mathfrak{C} \to \mathfrak{T}$ from the set $\mathfrak{C}$ of initial conditions to the set $\mathfrak{T}$ of trajectories in configuration space. Suppose there are just two particles in the system, which interact, not necessarily by colliding. Suppose furthermore that all the initial conditions are fixed, with the exception of a parameter $\mathfrak{C}'$ which is allowed to assume the values $\mathfrak{C}_n'$, $n = 1, 2, \dots$. Again, $\mathfrak{C}_n'$ will give rise to trajectory $\mathfrak{T}_n$, $n = 1, 2, \dots$. But $\mathfrak{T}_n$ can be broken up uniquely, for all n, into trajectory $\mathfrak{T}_n^1$ for the first particle and trajectory $\mathfrak{T}_n^2$ for the second. So Lagrange's principle can be viewed as determining a mapping

$$\mathfrak{L}(V, E) : \mathfrak{T}^1 \to \mathfrak{T}^2$$

from the set of trajectories of the first particle to those of the second. If we find the first particle on $\mathfrak{T}_n^1$, we know the second is on $\mathfrak{T}_n^2$.[23]

The corresponding quantum-mechanical case will be examined more closely in Chapter III. 2, but we can look at it briefly here. If subsystems $\mathbf{S}^1$ and $\mathbf{S}^2$ are described in Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$, the composite system $\mathbf{S} = \mathbf{S}^1 + \mathbf{S}^2$ will be described in their tensor product $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$. For any state

$$|\Xi\rangle = \sum_{mn} c_{mn} |\zeta_m^1\rangle |\zeta_n^2\rangle$$

[23]We can assume that $\mathfrak{T}_m^1$ has no point in common with $\mathfrak{T}_n^1$ for $m \neq n$, and that the mapping $\mathfrak{L}(V, E)$ is bijective.

of **S** there will be bases $\{|\varphi_r^1\rangle\} \subset \mathcal{H}^1$ and $\{|\varphi_r^2\rangle\} \subset \mathcal{H}^2$ such that

$$|\Xi\rangle = \sum_r b_r |\varphi_r^1\rangle|\varphi_r^2\rangle.$$

If we find the first system in state $|\varphi_m^1\rangle$, we know the second will be in the corresponding state $|\varphi_m^2\rangle$.

3
Waves and trajectories

Now let us try to work out an ontology. We might, as we have already reached configuration space, be led to question the reality of quantum waves, as anything propagating in a fictitious configuration space should also be fictitious.

> Was meint man, wenn man die Wellen in Konfigurationsraum »wirklich« oder »real« nennt? Dieser Konfigurationsraum ist ein sehr abstrakter Raum. Das Wort »real« aber geht zurück zu dem lateinischen Wort »res« und bedeutet »Sache« oder »Ding«. Die Dinge aber sind im gewöhnlichen dreidimensionalen Raum, nicht in einem abstrakten Konfigurationsraum.[24]

But let us forget about configuration space for the time being, and try to establish an ontology in ordinary space, where it is easier to accept the reality of quantum waves. Doubts concerning configuration space can then be based on realist prejudices founded in ordinary space.

The approach here will first be historical, and focus on the ontological preoccupations of de Broglie and Schrödinger. The reality of quantum waves and the status of particles will be central issues. The question of trajectories, on which de Broglie and Schrödinger disagreed, has evolved into that of 'completeness' and 'hidden variables': trajectories cannot exist if ordinary quantum mechanics is complete; the position of the particle along the trajectory represents a hidden variable.

3.1 The reality of de Broglie's waves

Guided by the dual nature of light and the optico-mechanical analogy, de Broglie extended wave-particle duality to matter. He associated the propagation of a wave of frequency $\nu = E/h$ and wavelength $\lambda = h/p$ with the motion of a particle of energy E and momentum p. He certainly had a *real* wave in mind, but had trouble pinning it down.

> Mais qu'est-ce que cette vibration dont la propagation constitue l'onde Ψ associée à une particule materielle? Comme la vibration électromagnétique, elle parait suspendue dans le vide et ne correspondre à aucune image physique concrète.[25]

As he pointed out in his thesis, it was

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[24]Heisenberg (1959). Translation: What does one mean, when one calls waves in configuration space 'real'? This configuration space is a very abstract space. But the word 'real' goes back to the Latin word 'res' and means 'thing.' But things are in the usual three-dimensional space, not in an abstract configuration space.

[25]de Broglie (1960). Translation: But what is this vibration whose propagation constitutes the Ψ wave associated with a material particle? As the electromagnetic vibration, it appears to be suspended in emptiness and to correspond to no concrete physical image.

> … d'une nature encore à préciser. … J'ai intentionnellement laissé assez vagues les définitions de l'onde de phase et du phénomène périodique dont elle serait en quelque sorte la traduction …. La présente théorie doit donc plutôt être considérée comme une forme dont le contenu physique n'est pas entièrement précisé que comme une doctrine homogène définitivement constituée.[26]

It remained *encore à préciser*, however, over thirty years later:

> Puis [la Physique] a retrouvé la vibration à des niveaux beaucoup plus cachés de la réalité physique, dans la structure profonde du rayonnement et de la matière: mais ici, n'ayant pas trouvé d'images appropriées à sa representation concrète, si elle peut affirmer que la vibration est là encore présente, elle ne peut plus dire à l'heure actuelle exactement ce qu'elle est.[27]

Prudence prevented de Broglie from saying too much, and making unwarranted claims about the physical nature of quantum waves. His realism kept him from being too timid, however, and reducing quantum waves to mathematical fictions.[28] Exaggerated prudence could lead to excessive *abstraction*. *Trahere* is 'to draw,' *abstrahere* 'to remove.' Quantum waves have an objective 'skeleton' capable of rigorous description, which can be identified with the wavefunction, or at least represented by it; the issue is what to do with whatever other characteristics they may have.[29] By prudently 'removing' them, one is left with the abstract skeleton of bare (statistical) certainties which is so scanty one does not know what to make of it. The rest—whatever it may be—is, however, hard to reveal experimentally or even describe.

> Parce qu'il est difficile de définir la nature physique de l'onde associée aux particules, beaucoup de théoriciens de la Physique, entraînés peut-être par la tendance abstraite de leur esprit, sont portés à considérer cette onde comme une pure expression mathématique qui servirait uniquement à prévoir la probabilité de certains phénomènes. Personnellement, il me semble qu'il y a là quelque exagération: la vibration Ψ, dont l'existence est si clairement inscrite dans des

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[26]de Broglie (1925). Translation: … of a nature which remained to be specified. … I have deliberately left the definition of phase-wave and of the periodic phenomenon of which it would be a kind of translation rather vague …. The present theory must therefore be viewed rather as a form whose physical content is not entirely specified than as a definitively constituted homogeneous doctrine.

[27]de Broglie (1960). Translation: Then [Physics] found vibration at much more hidden levels of physical reality, in the deep structure of radation and of matter: but here, as it has not found images appropriate for its concrete representation, if it can assert that vibration is still present, it can no longer say at present exactly what it is.

[28]*Cf.* Russell (1925) on *matter*: "… I suggested what may be called a minimum definition of matter, that is to say, one in which matter has, so to speak, as little 'substance' as is compatible with the truth of physics. In adopting a definition of this kind, we are playing for safety: our tenuous matter will exist, even if something more beefy also exists. We tried to make our definition of matter, like Isabella's gruel in Jane Austen, 'thin but not too thin'. We shall, however, fall into error if we assert positively that matter is nothing more than this. Leibnitz thought that a piece of matter is really a colony of souls. There is nothing to show that he was wrong, though there is also nothing to show that he was right: we know no more about it either way than we do about the flora and fauna of Mars."

[29]I think it was Mahler who said that "the score contains everything except what matters." Both score and wavefunction no doubt contain much that matters, but not everything. The fact that music has been reduced to 'the score' much as quantum mechanics has been reduced to the bare formalism suggests that perhaps the general *Zeitgeist* is to blame.

> phénomènes observables entièrement analogues à ceux de l'Optique, doit avoir une signification plus réelle et plus concrète que beaucoup ne le pensent aujourd'hui. S'il serait certainement trop naïf de se représenter les ondes électromagnétiques et les ondes associées aux particules comme des vibrations qui se propageraient dans un milieu élastique analogue à un corps matériel, il serait cependant conforme au réalisme scientifique de penser qu'elles sont constituées par une sorte de frissonnement de nature encore inconnue qui se propage dans l'espace au cours du temps. Nous ne devons pas insister ici d'avantage sur des problèmes difficiles que l'avenir seul pourra élucider . . . .[30]

If they are indeed real, quantum waves presumably propagate in a medium. I shall, however, not discuss the nature of the medium—whether it is the æther Einstein did away with, or whether electrons and photons share the same medium—for it is imperfectly understood, and its existence seldom acknowledged. Without a satisfactory understanding of the medium, however, there are limits to what can be said about its perturbations or undulations.

Endowing quantum waves with such physical characteristics as energy and momentum is a natural way of attributing reality to them. If the wave does indeed guide the particle, one presumes their interaction would involve the exchange of energy and momentum; and to transmit and absorb quantities a wave should also be able to carry them. If account is taken of the particle alone, diffraction through a hole in a screen would appear to involve a creation of momentum out of nowhere, and hence a violation of conservation; perhaps the particle exchanges momentum with the wave and screen, and momentum *is* conserved if such interactions with the environment are taken into account.[31]

De Broglie accordingly gave the wave a small part of the energy and momentum of the whole quantum object.[32] But this was not enough to save his wave, which, without

[30]de Broglie (1960). Translation: As it is difficult to define the physical nature of wave associated with particles, many theoreticians of Physics, perhaps on account of the abstract tendency of their minds, are inclined to consider this wave as a pure mathematical expression which would merely serve to predict the probabilities of certain phenomena. That is going a bit too far, or at least so it appears to me personally: the Ψ-vibration, whose existence is so clearly manifested in the observable phenomena entirely similar to those of Optics, must have a meaning more real and concrete than many people think nowadays. Although it would certainly be too naïve to represent to oneself electromagnetic waves and the waves associated with particles as vibrations which propagate in an elastic medium analogous to a material body, it would, however, be consistent with scientific realism to think that they are made up of a kind of shiver of a yet unknown nature which propagates in space as time passes. We should not insist any more on these difficult problems which only the future can elucidate . . . .

[31]*Cf.* Poincaré (1902): On croit toucher l'éther du doigt. On peut concevoir cependant des expériences qui nous le feraient toucher de plus près encore. Supposons que le principe de Newton, de l'égalité de l'action et de la réaction, ne soit plus vrai si on l'applique à la matière *seule* et que l'on vienne à le constater. La somme géométrique de toutes les forces appliquées à toutes les molécules matérielles ne serait plus nulle. Il faudrait bien, si on ne voulait changer toute la mécanique, introduire l'éther, pour que cette action que la matière paraîtrait subir fût contrebalancée par la réaction de la matière sur quelque chose.

[32]Selleri, who ascribes a very tenuous reality to quantum waves, does not attribute *any* energy or momentum to them.

the 'protection' of a definite physical description, began to fade once others got hold of it. First Schrödinger did away with the particle, or rather tried to identify it with the wave:

> Die Tatsache läßt sich nun dazu benützen, um eine viel innigere Verbindung zwischen Wellenausbreitung und Bildpunktbewegung herzustellen als bisher geschehen. Man kann versuchen, eine Wellengruppe aufzubauen, welche in allen Richtungen relativ kleine Abmessungen hat. Eine solche Wellengruppe wird dann voraussichtlich dieselben Bewegungsgesetze befolgen wie ein einzelner Bildpunkt des mechanischen Systems. Sie wird sozusagen ein *Ersatz* des Bildpunkts abgeben können, solange man sie als angenähert punktförmig ansehen kann, d. h. solange man ihre Ausdehnung vernachlässigen darf gegenüber den Dimensionen der Systembahn.[33]

With Born's proposal that the squared modulus of an expansion coefficient be interpreted as a probability, the wave would soon disappear into the mathematics and become little more than a probability distribution. The whole business about waves had been an unfortunate misunderstanding, in fact the very use of the word was misleading:

> The assumption of superposition relationships between the states leads to a mathematical theory in which the equations that define a state are linear in the unknowns. In consequence of this, people have tried to establish analogies with systems in classical mechanics, such as vibrating strings or membranes, which are governed by linear equations and for which, therefore, a superposition principle holds. Such analogies have led to the name 'Wave Mechanics' being sometimes given to quantum mechanics. It is important to remember, however, that *the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in classical theory* .... The analogies are thus liable to be misleading.[34]

Observing the alarming attenuation[35] of his wave—soon there would be nothing left—de Broglie responded with his *théorie de la double solution*. The probabilistic interpretation of the normalized ψ-function seemed accurate, and was kept. But a ψ normalized to comply with a probabilistic convention could not really represent a physical wave. So alongside ψ de Broglie introduced his u-function, which represented both wave and particle. When the theory was first formulated in 1927, u contained a point singularity representing the particle. Then in the the fifties u became the sum

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[33]Schrödinger (1926). Translation: This result can be used to establish a much closer connection between wave propagation and the motion of the representative point. One can try to build a wave group with relatively small dimensions in all directions. Such a wave group will then evidently follow the same laws of motion as a single representative point of the mechanical system. As long as one can view it as being approximately punctual, that is, as long as one can neglect its extension with respect to the dimensions of the system's trajectory, one can view the wave group as *replacing* the representative point, as it were.

[34]Dirac (1958)

[35]I may have given the impression that the disappearance of quantum waves was a continuous process. Of course there were discontinuities, one of the most abrupt of which occurred at the 1927 Solvay conference, after which de Broglie, feeling hopelessly outnumbered, thought it prudent to 'retreat into the formalism' and forget about his waves.

$u = u_0 + v$ of a small *région singulière* u_0 representing the particle, and v, the *onde réelle*.[36] The u-function assumed very high values in the particle or 'singular region' u_0, whose substance—de Broglie did not characterize it exactly—differed more in intensity than in kind from that of v. As there was no real boundary between wave and particle, just a vague 'halo' of continuous transition around u_0, ambiguities concerning the distribution of energy and momentum between wave and particle were thus incorporated in the formalism itself.

Another issue was the relationship between the statistical instrument ψ and the real wave v, which presumably had much in common. Attributing reality to wavelengths and frequencies, which manifest themselves experimentally, de Broglie gave v the same phase as ψ. The relative amplitudes of ψ—the ratios $|\psi(x)| : |\psi(x')| : \ldots$ between moduli at different points $x, x', \ldots$—are also observable, so it made sense to have v proportional[37] to ψ. Dynamically the particle u_0 and wave v are related by the guidance condition inherited from the optico-mechanical analogy: as $\boldsymbol{p} = \operatorname{grad} S$ in analytical mechanics and $\|\operatorname{grad}\varphi\|^2 = 1/v^2$ in geometrical optics, trajectories are determined by $\boldsymbol{p} = -\operatorname{grad}\varphi$ in the *double solution*, where $\varphi = S/h$. Again, they are orthogonal to the wave-surfaces of equal action.

3.2 Trajectories

The matter of trajectories and guidance has a relevance to the configuration space question. Only a real physical wave can do something as physical as guiding a particle along a trajectory; but it is implausible that such a wave should propagate in configuration space, and guide the point representing the positions of arbitrarily many particles.

Schrödinger disagreed that rays or trajectories could figure in a wave theory:

> Trotzdem in den vorstehenden Überlegungen von Wellenflächen, Fortpflanzungsgeschwindigkeit, Huygensschem Prinzip die Rede ist, hat man dieselben doch eigentlich nicht als eine Analogie der Mechanik mit der *Wellen*optik, sondern mit der *geometrischen* Optik anzusehen. Denn der Begriff der *Strahlen*, auf den es für die Mechanik dann hauptsächlich ankommt, gehört der *geometrischen* Optik an, er ist nur *ihr* ein scharfer Begriff.[38]

Whereas for de Broglie most of the electron's properties were concentrated in the particle, which was guided by the wave along a single privileged trajectory, Schrödinger's electron was smeared out over all available trajectories.

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[36] See de Broglie (1956a,b), Ben-Dov (1989).

[37] De Broglie (1956a,b) subsequently developed doubts about this exact proportionality.

[38] Schrödinger (1926). Translation: Although it has so far been a matter of wave-surfaces, velocity of propagation, Huygens' principle, one should view the analogy as being between mechanics and *geometrical* optics, not *wave* optics. For the concept of rays, which applies so fundamentally to mechanics, belongs to *geometrical* optics, and is only a sharp concept *there*.

Aber die Wellenflächen, selbst wenn man nur kleine Stückchen davon in Betracht ziehen will ... fassen doch mindestens ein schmales *Bündel* möglicher Bahnkurven zusammen, zu denen allen sie in der gleichen Beziehung stehen. Nach der alten Auffassung ist eine von ihnen als die „wirklich durchlaufene" vor allen übrigen „bloß möglichen" im konkreten Einzelfall ausgezeichnet, nach der neuen Auffassung aber nicht. ... Vom Standpunkte der Wellenmechanik wäre die unendliche Schar der möglichen Punktebahnen nur etwas Fiktives, keine davon hätte vor den übrigen das Prärogative, die im Einzelfall wirklich durchlaufene zu sein.[39]

The same applied to light: *Lichtstrahlen* were only a feature of geometrical optics, and ceased to make sense in conditions that required the use of wave optics.

Nach der Wellentheorie des Lichtes haben die Lichtstrahlen eigentlich nur fiktive Bedeutung. Sie sind nicht physische Bahnen irgendwelcher Lichtteilchen, sondern eine mathematische Hilfskonstruktion, die sogenannten Orthogonaltrajektorien der Wellenflächen, gleichsam gedachte Führungslinien, die an jeder Stelle in die Richtung senkrecht zur Wellenfläche weisen, in der letztere fortschreitet....[40]

Trajectories are determined by orthogonality to wavefronts, but according to Schrödinger the very idea of wave-surfaces breaks down in all essentially undulatory cases.

Yet the longitudinal and the transversal linkage—*Wirkungszusammenhang* I call it in German—are not sharply delimited, nay they are ever sharply delimited against one another, because, as everybody knows, the wave-surfaces and the wave normals (the rays) are *never* sharply defined. It is true that in some cases one can stipulate an artificial sharp definition; e.g. a complex scalar wave function $\psi(\mathbf{x}, t)$ can be uniquely written:

$$\psi(\mathbf{x}, t) = A(\mathbf{x}, t)e^{i\varphi(\mathbf{x},t)}$$

with A and φ real, and you may call the surfaces $\varphi = \text{const.}$ the wave surfaces. But this has a good meaning only when A varies (in space and time) slowly compared with φ.

Now I believe everybody agrees that the *path* or world-line of a particle can be given no *other* meaning than that of a *ray* or (orthogonal) trajectory of the family of wave surfaces. Since these rays are never sharply defined, the paths are at any rate, to say the least, never sharply defined. ... there are cases when the

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[39]Schrödinger (1934). Translation: But the wave surfaces, even if one only wants to consider little pieces of them, determine at least a thin *bundle* of possible trajectories, all of which bear the same relation to the wave surfaces. According to the old conception one of these is, in each individual case, distinguished among all the other 'merely possible' trajectories as the one that is 'really described,' but not according to the new conception. From the point of view of wave mechanics the multitude of possible paths would only be something fictitious, none of them would have, among the others, the prerogative of being really described in the individual case.

[40]Schrödinger (1934). Translation: Light rays, according to the wave theory of light, have only a fictitious meaning. They are not the physical paths of some or other light particles, but a mathematical auxiliary construction, the so-called orthogonal trajectories of the wave surfaces, viewed as guidance lines, which always point in the direction orthogonal to the wave surfaces, along which these advance.

> notions of wave-surfaces and wave-normals break down entirely, and … the chief interest of wave mechanics is concentrated on these cases.[41]

Surfaces of equal phase, and hence trajectories orthogonal to them, can only be defined if phases at different points can be compared. Though he does not say so explicitly, perhaps Schrödinger was unwilling to make such comparisons.

Attached as he was to trajectories, de Broglie disagreed that guidance no longer works in essentially undulatory cases, and that "there are cases when the notions of wave-surfaces and wave-normals break down entirely":

> Le théorème du guidage est valable dans le cas général où la propagation de l'onde ne s'opère pas nécessairement à l'approximation de l'optique géométrique.[42]

Trajectories are undeniably a feature of analytical mechanics, where the *wavefronts* are the *mathematische Hilfskonstruktion.* In geometrical optics it is the other way around; the wavefronts are primary, and rays are determined by the orthogonality condition. De Broglie's sense of the symmetry between matter and light, indeed his intention to unify them, led him to disregard the distinction: analytical mechanics and geometrical optics both involved paths and wave surfaces, in more or less the same way. And once trajectories belonged to the lower level of the scheme

wave optics	wave mechanics
geometrical optics	analytical mechanics

it seemed natural to extend them to the upper level. This was the step Schrödinger refused to take.

3.3 Wave-particle duality

So is there a wave guiding a particle, as de Broglie thought, or just a wave that can turn into a particle on measurement?

Experiments can reveal waves or particles, but it is often claimed that no experiment can reveal both. Although both aspects figure in the experiments to be considered in Chapter I. 4, *only one is manifested at a time*. Is it that both wave and particle are there at the same time, but cannot be manifested together for experimental reasons? Or do they never appear at the same time because they are never even present together in nature?

Where a wave is split into two parts which are then recombined and made to interfere—as in the experiment of Janossy and Naray (Section I. 4.2)—it could be that a position measurement turns a wave without a particle into a particle without a wave. In the "experiment to throw more light on light" (Section I. 4.3) perhaps the *two* waves that

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[41] Schrödinger (1952)

[42] de Broglie (1959). Translation: The guidance theorem is valid in the general case where the propagation of the wave cannot necessarily be approximated by geometrical optics.

emerge from the prism become a *single* particle at one of the detectors (even if these are arbitrarily far apart). Maybe *no* experiment can distinguish between a wave guiding a particle whose presence can be manifested on measurement, and a wave that can be abruptly condensed by measurement. The former picture has the advantage of plausibility, the latter perhaps of economy. Dispensing with empirically superfluous mathematical objects may represent theoretical economy. It is implausible, however, that measurement should condense all the properties of a quantum object—which could in principle, if they are not already concentrated in a particle, be spread over the whole universe—into an arbitrarily small region, instantaneously. This would involve infinite forces and accelerations towards the region in question. Measurement furthermore can at best only account for corpuscular *absorption*; but we have seen in Einstein's *Zur Quantentheorie der Strahlung* (1917) that *emission*, in which measurement is clearly not involved, is also corpuscular. Particles therefore appear to be there anyway, without having to be created by measurement.

So a possibility worth considering is that, in ordinary space, the wave guides the particle along a trajectory. But would the usual normalized ψ-function do the guiding, or does another wave have to be provided, as in de Broglie's *double solution*?

Giving the abstract probability wave ψ a role as physical as guidance confuses matters. De Broglie therefore left the ψ-wave to its statistical functions, and got a more explicitly physical wave to guide the particle. As ψ and v are proportional, they are the same as far as phase is concerned, so it makes no mathematical difference which one guides. But it is physically clearer to have, alongside the normalized ψ-function, a function $u = u_0 + v$ representing a particle u_0 embedded in a real wave v.

Doubts concerning configuration space can be partly founded on the above wave-particle duality; for it is not clear how a real physical wave can propagate in configuration space, and how such a wave can guide the configuration-point representing the positions of particles that could be far apart. Let us now see how this picture of a wave guiding a particle fares in the following three experiments.

4
Experiments

The following experiments do not determine the exact nature of wave-particle duality in ordinary space, but they at least reveal both waves and particles. The first two are 'self-interference' experiments. Any quantum object can in principle be made to interfere with itself, by splitting its wave $\mathfrak{W}$ into $\mathfrak{W}_1$ and $\mathfrak{W}_2$, which are then brought back together again. This has been done with photons, but also with electrons, neutrons and even atoms. Once $\mathfrak{W}_1$ and $\mathfrak{W}_2$ have been recombined, localized *corpuscular* detections reveal the *undulatory* interference pattern. The particle can also be looked for before the waves are recombined; it is always found all in one piece, and never in more than one place at a time.

The "experiment to throw more light on light" reveals the undulatory nature of single photon states by tunnelling rather than interference.

4.1 Electron interferometry

The dual nature of electrons appears clearly in an experiment performed by Matteucci and Pozzi.[43] A source accelerates electrons in an electric field; a quartz filament then splits each electron's wave into two parts which, once recombined, give rise to an interference pattern of dark and bright stripes on a screen Σ covered with a fluorescent substance that produces a flash at the point of impact. The flashes are distributed very irregularly, as can be seen if they are filmed. If the intensity of the source—the number of electrons emitted per second—is increased, the frequency of the flashes rises until they become almost continuous, while remaining distinct. The flashes are not randomly distributed, but are concentrated on several parallel stripes, which means that an electron has a higher probability of falling on the light stripes than on the dark ones. Even at high intensities simultaneous flashes never occur, so electrons appear indeed to be indivisible.

Let π be the point of impact, and Σ' the part of the screen reached by the wave, in other words the set of points that have a nonzero probability of flashing. It could be that a wave without a particle is divided by the filament, and turns into a particle at π. But then all other parts $\Sigma' - \pi$ of the screen have to be informed instantaneously not to flash. It is customary to associate energy with the transmission of information; but how can energy be transmitted instantaneously from π to parts of Σ' which could, in principle, be light years away? Even if what is being transmitted is disembodied information and not energy or anything else, the propagation remains instantaneous, which seems implausible.

[43] See Matteucci (1990). Matteucci and Pozzi made a movie, which has been described to me, of the experiment in question.

Let us remove the filament, and suppose the electron leaves the source in an energy eigenstate belonging to the eigenvalue E. The energy must somehow be carried by the electron, since the same E can be found farther on; surely the energy does not disappear before reappearing on measurement.[44] But is the energy of the electron spread out, in transit, over the whole wave—perhaps distributed according to the squared modulus of the wavefunction—or concentrated in a particle describing a trajectory? We know that the electron can be *found* within an arbitrarily small region τ, presumably with most of its properties, including energy. If those properties were not already concentrated before measurement, they would have to be condensed instantaneously by contact with the apparatus, which would involve infinite forces and accelerations towards τ. It therefore appears that most of the electron is concentrated in a particle.

4.2 Janossy and Naray

The experiment of Janossy and Naray (1958) is similar, but involves photons instead: single photon states of frequency ν which are partly reflected and partly transmitted by a semitransparent mirror M are recombined and made to interfere in a Michelson interferometer. If the photons are looked for just beyond M, before the waves are recombined, the entire energy $h\nu$ and momentum $h\nu/c$ of the photon are found undivided on one path *or* the other. This means that there is an energy- and momentum-bearing particle which is either reflected or transmitted, but not both. The interference pattern shows, however, that part of each single photon state *is* divided by the mirror. The part in question must be a wave because interference is an undulatory effect; the fringe width, for instance, depends on the frequency of the light. The wave is, moreover, extended longitudinally, because the interference pattern remains even if the lengths l_r, l_t of the transmitted and reflected paths are varied by changing the configuration of the apparatus. Beyond a certain difference $|l_r - l_t| = \mathsf{d}$, however, the interference pattern disappears; $\mathsf{d} = c\tau$ represents the (longitudinal coherence) length of the wave, and can be worked out independently from the known lifetime τ of the excited level and the velocity of light c. The wave also has extension perpendicular to the direction of propagation, revealed by the interference pattern.

All this is consistent with the wave-particle duality of the *double solution*: the corpuscle chooses between transmission and reflection, while the wave is split into two parts. Once these are recombined, they both act on the corpuscle and generate a distribution in the interferometer determined by the intensity of the combined wave. Hence the probability of arrival is characterized by parallel stripes, in which the alternations of probability are represented. A beam with many photons produces the well

[44]Conservation must be more than just a matter of nonlocal agreement between apparatuses.

known bright and dark lines. One can, however, always say that the particle is not there until the interferometer is reached, and the wave collapses to a point.

4.3 "An experiment to throw more light on light"

The undulatory and corpuscular aspects of light both appear in the "experiment to throw more light on light" proposed by Ghose, Home and Agarwal[45] (1991) and performed by Mizobuchi and Ohtaké (1992).

Single photon states directed at a 45° prism are all totally internally reflected from the oblique face, and reach detector D_r. Introducing another prism, in contact with the first, all the photons tunnel straight through and reach detector D_t instead. Intermediate distributions can be obtained by varying the gap d between the prisms, which changes the probability of tunnelling; the larger the gap, the more photons are reflected. When d is a tenth of the wavelength λ of the incident photons, D_r clicks about as often as D_t. Tunnelling ceases once the gap exceeds the wavelength. The detectors never register coincidences, which indicates the corpuscular nature of the radiation. Tunnelling, and dependence of its probability on d/λ, indicates the undulatory nature.

This experiment also favours the wave-particle duality of the *double solution*, according to which something like the following happens: When the prisms are far apart, the particle reaches the oblique surface of the first prism and gets reflected every time. As the prisms are brought together, the wave accompanying the particle eventually 'bridges the gap' between the prisms, and allows the undulatory phenomenon of tunnelling. The wave can be seen as 'helping the particle across'; the smaller the separation, the easier this is. Eventually, with contact, reflection becomes impossible.

Undulatory behaviour is therefore manifested at the prisms, corpuscular behaviour at the detectors. What if there is *just* a wave at the prisms, without a particle? Whenever reflection and transmission are both possible,[46] the wave is divided and goes both ways, since the transmitted and reflected beams could, in principle, be recombined and made to interfere with one another. One of the detectors would then have to condense *two* waves, that could be arbitrarily far apart, into a single particle. Even if the waves are just disembodied information (with the capacity of producing energy, momentum *etc.* when required), the one that decides to make its detector click has to inform the other not to, instantaneously, which is a problem. Any previous agreement between the waves would suggest the incompleteness of the formalism, and the presence of a particle describing a trajectory.

The above experiments can be taken to indicate—though admittedly not unambiguously—the coexistence of waves and particles. If the particle *is* always present,

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[45]See also Ghose, Home and Agarwal (1992).
[46]Whenever, in other words, neither detector has a vanishing probability of clicking.

it describes a trajectory. Experiments that split the wave $\mathfrak{W}$ into an 'empty' wave $\mathfrak{W}_1$ and a wave $\mathfrak{W}_2$ accompanying the particle indicate that $\mathfrak{W}_1$ can influence the trajectory; for the interference pattern depends on the availability of $\mathfrak{W}_1$, which can be obstructed right after it has been separated from $\mathfrak{W}_2$. So if wave and particle coexist, it seems the wave must guide the particle. The natural guidance formula to adopt—more accurate ones may be much more complicated—is the one inherited from the optico-mechanical analogy, namely $\boldsymbol{p} = -\mathrm{grad}\,\varphi$.

A plausible ontology for ordinary space is therefore the following: guided by the wave, the particle describes a trajectory given by $\boldsymbol{p} = -\mathrm{grad}\,\varphi$. Only a real physical wave could have such an influence on the particle.

I shall, in the course of the thesis, approach the configuration space question in several very different ways. Historically the first expression of the problem was explicitly ontological: 'How can a wave propagate in a fictitious configuration space?' Despite remaining the most *direct* formulation—it goes to the heart of the matter, without mathematical circumlocution—it cannot be addressed experimentally in exactly that form; and admittedly it is not entirely clear what is meant by a 'wave.'

Quantum waves have since gone out of fashion. They are hard to pin down; their features—phases, amplitudes—most susceptible of rigorous description do figure in the mathematical theory, but may not be enough to constitute a 'wave' on their own; and the rest—the æther in which they presumably propagate *etc.*—is too nebulous to be worthy of serious scientific consideration. As the formalism has developed an autonomous algorithmic existence of its own, the undulatory reality it was once supposed to represent[47] has been largely forgotten. The matrix mechanics of Heisenberg, Born and Jordan was already a pure *calculus*, behind which it was hard to see waves, or even particles. Then came the abstract and rigorous Hilbert space formalism of von Neumann, which, with all its geometrical niceties, was even more remote from physical reality.

So the problem of quantum waves in configuration space is no longer formulated in terms of waves; nowadays one speaks of the violation of Bell's inequality, and above all of the measurement problem. These, however, are particular expressions. The *general* problem can be given a more abstract formulation, less dependent on a naïve and obsolete wave language; one can say, for instance, that there is something wrong with the superposition of tensor products. But there is nothing *mathematically* wrong with such superposition. It only becomes a problem when an ontology behind the symbols is sought, and then we are back to the old worries of de Broglie and Schrödinger. As the general formulation is not mathematically problematic, the aforementioned special ones have arisen. These appeal to precise formal criteria, such as inequalities involving

[47]At least in the undulatory form given it by de Broglie and Schrödinger.

measurable quantities, rather than to primitive intuitions about what a wave can or cannot do.

As there may well be an undulatory reality behind the symbols, and hence an underlying ontological problem, I shall be just as interested in the general abstract expression of that problem as in the particular ones. To arrive at that expression, however, I begin with a two-dimensional space, generalize to infinite dimensions, then introduce tensor multiplication.

II. Interference and incompatibility

1
The Riemann sphere

So far there has been *explicit* reference to waves, physical reality and experiments; the language and tone may not have seemed too remote from those of Huygens, or Thomas Young (1855):

> Neither series of waves will destroy the other, but their effects will be combined: if ... the elevations of one series coincide with those of the other, they must together produce a series of greater joint elevations; but if the elevations of one series are so situated as to correspond to the depression of the other, they must exactly fill up those depressions, and the surface of the water must remain smooth. ... Now I maintain that similar effects take place whenever two portions of light are thus mixed; and this I call the general law of the interference of light.

I shall now leave the 'ontological' treatment of Part I and adopt a more formal and abstract approach. Interference—expressed in quantum theory by the addition of complex numbers, and no longer in terms of 'elevations' and 'depressions'—is a fundamental issue here, and should be looked at on its own.

A complex number $z = |z|e^{i\arg z}$ has two features, the modulus $|z|$ and phase factor $e^{i\arg z}$ or argument $\arg z$. Presumably these represent corresponding characteristics of quantum waves, but exactly what they are is not clear. The square of the modulus is just a probability in orthodox quantum mechanics, but is also the intensity[48] of the undulatory perturbation in the realist theories of de Broglie (1956b) and Bohm (1952a,b).

In a wave represented by the function $\psi = \cos\omega t$, the modulus $|\psi|$ and phase ωt are closely related, since the phase partly determines the modulus. Complex moduli and phases, on the other hand, are independent in the sense that phase can be changed without affecting the modulus. But they are connected by superposition, since the modulus of the sum $z = z_1 + z_2$ depends on the arguments of z_1 and z_2.

The rules for adding complex numbers express more or less all we know with certainty about the phases of quantum waves. What those phases represent in nature one can only conjecture. Rather than speculating on the physical meaning of arguments and moduli using a natural language less suited than the mathematical formalism—which, for all its neutrality, is at least reliable—to the description of circumstances so remote from our everyday experience, in the next chapters I shall look at interference in purely formal terms.

To emphasize the connection with Part I it would be possible, for instance, to provide hidden variables throughout, or to multiply spinors explicitly by wavefunctions.

[48]The coincidence of intensities and probabilities is a mystery, and has to be postulated.

Hidden variables could assume the form of particle-singularities in position space, guided according to $\boldsymbol{p} = -\mathrm{grad}\,\varphi$, or of hidden-variable points on the Riemann sphere;[49] but it seems superfluous to complicate the theory with additional hidden-variable elements once we know they *can* be provided. Explicit multiplication by the spatial part $\psi_{+}(x, y, z)$ could serve to bring out the undulatory character of the spinor u_{+}, which otherwise is hidden: $u_{+}\psi_{+}(x, y, z)$ or even $|\uparrow\rangle \otimes |\psi\rangle$ look more like waves than just u_{+} or $|\uparrow\rangle$. For simplicity, however, I shall leave the multiplication implicit, and just write the spinor. Even if the position representative $\varphi(x, y, z) = U|\varphi\rangle$ is more explicitly undulatory than the ket $|\varphi\rangle$, both can be viewed as representing the same wave.

I begin with the simplest Hilbert space in which interesting interference effects appear, namely a two-dimensional complex space H. Here an intuitive geometrical treatment is possible, with stereographic projection onto the Riemann sphere. As there is some loss of generality in a consideration of only two dimensions, I then look at interference in infinite dimensions in Chapter II. 2, and introduce tensor multiplication in Section III. 1.1. Instances of interference involving tensor multiplication—the violation of Bell's inequality and the measurement problem—will then be considered in Part IV.

I show on the Riemann sphere that the phase difference between the coefficients in an expansion with respect to a basis $\mathfrak{b} = \big(|\varsigma\rangle, |\varsigma_{\perp}\rangle\big)$ will be statistically meaningless to an observable represented by an operator with eigenvectors $|\varsigma\rangle$, $|\varsigma_{\perp}\rangle$; and that it remains meaningless, in time, if $|\varsigma\rangle$ and $|\varsigma_{\perp}\rangle$ are energy eigenvectors. Appeal to another basis is therefore necessary to 'see' phase at a given moment; alternatively, probability beats can be seen with respect to an observable incompatible with the Hamiltonian. Both possibilities can be intuitively illustrated in H, with stereographic projection onto the Riemann sphere.

The statistical significance of phase, both static and dynamic, will be explored in two ways: by comparing states in which it is present, namely superpositions, with states from which it has been removed, namely mixtures; and by considering the effects of phase transformations, in other words of unitary operators. The 'removal' of phase projects a state orthogonally from the surface of the sphere onto a diameter. A unitary operator instead moves pure states around a circle perpendicular to its 'eigendiameter.'

It will be worth dwelling on the Riemann sphere at some length, as almost all the essential features of interference can be illustrated on it geometrically. It also serves as a useful introduction to the physics of kaons.

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[49]The projection of a hidden-variable point onto the measurement's 'eigendiameter' gives the outcome. If the diameter is vertical, 'up' will be the result when the point is in the northern hemisphere, and otherwise 'down.' Hidden-variable points get redistributed by measurement in such a way as to preserve quantum-mechanical statistics.

1.1 One basis

We map spinors of a two-dimensional complex Hilbert space H to the Riemann sphere[50]

$$\Sigma(\mathbb{R}^3) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

by projecting stereographically from a point $\mathbf{P} \in \Sigma(\mathbb{R}^3)$ which determines an Argand plane $\mathbb{C}_\mathbf{P} \perp \mathbf{P}$. The sphere $\Sigma(\mathbb{R}^3)$ is cut in half by $\mathbb{C}_\mathbf{P}$, whose origin is the centre of $\Sigma(\mathbb{R}^3)$.

We begin with the unitary transformation $\upsilon_\mathfrak{b} : \mathsf{H} \to \mathbb{C}^2$ which assigns to $|\varPsi\rangle$ the expansion coefficients $\upsilon_\mathfrak{b}|\varPsi\rangle = (\langle\varsigma|\varPsi\rangle, \langle\varsigma_\perp|\varPsi\rangle)$ with respect to an orthonormal pair $\mathfrak{b} = (|\varsigma\rangle, |\varsigma_\perp\rangle) \subset \mathsf{H}$. Next the map[51] $\tau : \omega \to \mathbb{C}$ from $\omega = \{(\alpha, \beta) \in \mathbb{C}^2 : \alpha \neq 0\}$ to $\mathbb{C}$ assigns the quotient $\tau(\alpha, \beta) = \beta/\alpha$, wherever defined, to the pair (α, β). Then the stereographic projection $\pi_\mathbf{P} : \mathbb{C}_\mathbf{P} \to \Sigma(\mathbb{R}^3)$ from the point $\mathbf{P} \in \Sigma(\mathbb{R}^3)$ assigns to $\zeta \in \mathbb{C}_\mathbf{P}$ a point $\pi_\mathbf{P}(\zeta) \in \Sigma(\mathbb{R}^3)$ collinear with ζ and $\mathbf{P}$.

The composition $\pi_\mathbf{P} \circ \tau \circ \upsilon_\mathfrak{b} : \mathsf{H} \to \Sigma(\mathbb{R}^3)$ therefore takes spinors in H to points on the sphere according to:

$$\boldsymbol{\varPsi} = \pi_\mathbf{P} \circ \tau \circ \upsilon_\mathfrak{b}|\varPsi\rangle = \begin{cases} \pi_\mathbf{P}(\langle\varsigma_\perp|\varPsi\rangle/\langle\varsigma|\varPsi\rangle) & \text{if } \langle\varsigma|\varPsi\rangle \neq 0 \\ \mathbf{P} & \text{otherwise,} \end{cases}$$

where $\boldsymbol{\varPsi}$ is the image on the Riemann sphere of the spinor $|\varPsi\rangle$.

The first basic spinor $|\varsigma\rangle = 1|\varsigma\rangle + 0|\varsigma_\perp\rangle$ is represented by the origin $0/1 \in \mathbb{C}_\mathbf{P}$, and hence by the pole $\mathbf{Q} = \pi_\mathbf{P}(0)$ diametrically opposite $\mathbf{P}$. Representation of the second basic spinor $|\varsigma_\perp\rangle = 0|\varsigma\rangle + 1|\varsigma_\perp\rangle$ is less straightforward, since the second coefficient cannot be divided by the first; but in this case we have just said that

$$\mathbf{P} = \pi_\mathbf{P}(\infty) = \pi_\mathbf{P} \circ \tau \circ \upsilon_\mathfrak{b}|\varsigma_\perp\rangle$$

is the result, including the projection pole in the mapping 'by hand.' Suppose $|\varPsi\rangle$ is normalized, and we make it tend to $|\varsigma_\perp\rangle$; as $c_\perp$ approaches the unit circle, c begins to vanish, ζ becomes infinitely large, and its projection on the sphere approaches $\mathbf{P}$. Once $|\varsigma_\perp\rangle$ has been reached, however—when c has vanished altogether and $c_\perp/c$ ceases to be defined—there is no point on the Argand plane to project onto the sphere. Since the inverse image $\pi_\mathbf{P}^{-1}(\mathbf{P})$ is empty, the correspondence $|\varsigma_\perp\rangle \mapsto \mathbf{P}$ has to be established independently of stereographic projection.

Taking the Argand plane

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[50]See Penrose and Rindler (1984), Penrose (1989) pp.341-6, Penrose (1995) pp.270-7; *cf.* Hughes (1989) pp.139-41, Beltrametti (1985), Beltrametti and Cassinelli (1981), Hilbert and Cohn-Vossen (1932) pp.218-28.

[51]In a sense another mapping is required to take a complex number $\zeta \in \mathbb{C}$ to the associated point $\zeta \in \mathbb{C}_\mathbf{P}$ on the Argand plane, but I shall not take account of it explicitly.

$$\mathbb{C}_{\mathrm{S}} = \left(\mathbb{R}^3\right)_{x,y} = \left\{(x,y,z) \in \mathbb{R}^3 : z = 0\right\}$$

perpendicular to the z-axis, we project from the south pole **S**. The symbols $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the orthonormal spinors mapped to the north and south poles:

$$\begin{aligned}\pi_{\mathrm{S}} \circ \tau \circ v_{\mathfrak{b}'}|\uparrow\rangle &= \pi_{\mathrm{S}} \circ \tau(1,0) = \pi_{\mathrm{S}}(0) = \mathbf{N} = (0,0,1) \in \mathbb{R}^3 \\ \pi_{\mathrm{S}} \circ \tau \circ v_{\mathfrak{b}'}|\downarrow\rangle &= \pi_{\mathrm{S}} \circ \tau(0,1) = \pi_{\mathrm{S}}(\infty) = \mathbf{S} = (0,0,-1) \in \mathbb{R}^3,\end{aligned}$$

where $\mathfrak{b}' = \left(|\uparrow\rangle, |\downarrow\rangle\right)$. The quotient

$$\tau \circ v_{\mathfrak{b}'}|\Psi\rangle = \tau\left(\langle\uparrow|\Psi\rangle, \langle\downarrow|\Psi\rangle\right) = \frac{\langle\downarrow|\Psi\rangle}{\langle\uparrow|\Psi\rangle} = \zeta' \in \mathbb{C}_{\mathrm{S}}$$

has coordinates $(\operatorname{Re}\zeta', \operatorname{Im}\zeta', 0)$ in $\mathbb{R}^3$, and

$$\pi_{\mathrm{S}}(\zeta') = \left(\frac{2\operatorname{Re}\zeta'}{|\zeta'|^2+1}, \frac{2\operatorname{Im}\zeta'}{|\zeta'|^2+1}, \frac{1-|\zeta'|^2}{1+|\zeta'|^2}\right) \in \Sigma\left(\mathbb{R}^3\right).$$

The projection π_{S} takes numbers $\{\zeta' \in \mathbb{C} : |\zeta'| < 1\}$ whose modulus is less than one to the northern hemisphere, numbers $\{\zeta' \in \mathbb{C} : |\zeta'| > 1\}$ from outside the unit circle to the southern hemisphere, and leaves numbers on the equator where they are:

$$\pi_{\mathrm{S}}\left(e^{i\varphi}\right) = (\cos\varphi, \sin\varphi, 0) \in \mathbb{R}^3.$$

The spinors

$$\begin{aligned}|\rightarrow\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \\ |\leftarrow\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)\end{aligned} \tag{6}$$

are represented on $\mathbb{C}_{\mathrm{S}}$ by $(1,0)$ and $(-1,0)$, and on $\Sigma\left(\mathbb{R}^3\right)$ by the 'same' antipodal points $(1,0,0)$ and $(-1,0,0)$. Indeed any spinor of the form $\left(|\uparrow\rangle + e^{i\beta}|\downarrow\rangle\right)/\sqrt{2}$ will end up on the equator $\mathbb{C}_{\mathrm{S}} \cap \Sigma\left(\mathbb{R}^3\right)$.

The orthogonal spinors $|\uparrow\rangle$ and $|\downarrow\rangle$, then, correspond to antipodal points **N** and **S** respectively, and the spinors

$$\begin{aligned}|\beta\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\beta}|\downarrow\rangle) \\ |\beta_\perp\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle - e^{i\beta}|\downarrow\rangle)\end{aligned}$$

to antipodal points

$$\begin{aligned}\pi_{\mathrm{S}} \circ \tau \circ v_{\mathfrak{b}'}|\beta\rangle &= \pi_{\mathrm{S}} \circ \tau\left(1, e^{i\beta}\right) = (\cos\beta, \sin\beta, 0) \in \mathbb{R}^3 \\ \pi_{\mathrm{S}} \circ \tau \circ v_{\mathfrak{b}'}|\beta_\perp\rangle &= \pi_{\mathrm{S}} \circ \tau\left(1, -e^{i\beta}\right) = (-\cos\beta, -\sin\beta, 0) \in \mathbb{R}^3.\end{aligned}$$

We can prove that orthogonality in H *always* corresponds to antipodality in $\Sigma(\mathbb{R}^3)$ as follows. If the unit spinors $|\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ and $|\phi'\rangle = c|\uparrow\rangle + d|\downarrow\rangle$ are orthogonal, then $a^*c = -b^*d$. So we have to establish the antipodality of the stereographic projections $\mathfrak{y} = \pi_S(\eta)$ and $\mathfrak{y}' = \pi_S(\eta')$ of the points

$$\eta = \frac{b}{a} = \tau \circ v_{\mathfrak{b}'}|\phi\rangle$$
$$\eta' = \frac{d}{c} = -\frac{a^*}{b^*} = \tau \circ v_{\mathfrak{b}'}|\phi\rangle.$$

As the arguments of the numbers

$$\frac{b}{a} = \frac{|b|}{|a|}e^{i(\arg b - \arg a)} \text{ and } -\frac{a^*}{b^*} = \frac{|a|}{|b|}e^{i(\arg b - \arg a + \pi)}$$

differ by π, so will the azimuths of their stereographic projections, which settles the azimuthal part of antipodality. For the polar part we can write $p = |b|/|a|$ and $q = 1/p = |a|/|b|$. Since $\Sigma(\mathbb{R}^3)$ is a unit circle, the lengths p and q determine a right angle between the segments $\overline{Sp}$ and $\overline{Sq}$, which therefore cut the Riemann sphere at antipodal points.

Conversely if the points $\mathfrak{y}$ and $\mathfrak{y}'$ on the Riemann sphere are antipodal, the points $\eta = \pi_S^{-1}(\mathfrak{y})$ and $\eta' = \pi_S^{-1}(\mathfrak{y}')$ on the Argand plane are related by $\eta = -1/\eta'$. Then

$$v_{\mathfrak{b}}^{-1} \circ \tau^{-1}(\eta) = v_{\mathfrak{b}}^{-1}(kb/a) = |k\phi\rangle$$
$$v_{\mathfrak{b}}^{-1} \circ \tau^{-1}(\eta') = v_{\mathfrak{b}}^{-1}(-k'a^*/b^*) = |k'\phi'\rangle,$$

where k and k' are constants. As $\langle k\phi|k'\phi'\rangle = 0$, the antipodal points $\mathfrak{y}$ and $\mathfrak{y}'$ on the Riemann sphere are the images of *orthogonal* spinors, *q.e.d.*

The polar angle θ and azimuth φ of the stereographic projection

$$\boldsymbol{\Psi} = \pi_S \circ \tau \circ v_{\mathfrak{b}'}|\Psi\rangle$$

can in fact be directly derived from the coefficients $c_\uparrow = \langle\uparrow|\Psi\rangle$ and $c_\downarrow = \langle\downarrow|\Psi\rangle$. If $|\Psi\rangle$ is normalized, the polar angle θ will be equal to

$$2\arccos|c_\uparrow| = 2\arcsin|c_\downarrow|.$$

The azimuth φ of $\boldsymbol{\Psi}$ is the argument of the quotient $c_\downarrow/c_\uparrow$, or the difference $\varphi = \arg c_\downarrow - \arg c_\uparrow$. To bisect the angle φ with the real axis, we can rotate $c_\uparrow$ and $c_\downarrow$ by half of $\chi = \arg c_\downarrow + \arg c_\uparrow$, and write

$$|\Psi\rangle = e^{i\chi/2}\left(\cos\frac{\theta}{2}e^{-i\varphi/2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\varphi/2}|\downarrow\rangle\right)$$
$$= \left(|c_\uparrow|e^{i(\chi-\varphi)/2}|\uparrow\rangle + |c_\downarrow|e^{i(\chi+\varphi)/2}|\downarrow\rangle\right).$$

The coordinates of the point $\boldsymbol{\Psi} = \pi_{\mathrm{S}} \circ \tau \circ \upsilon_{\mathfrak{b}'}|\Psi\rangle$ can be expressed directly in terms of the coefficients $c_{\uparrow}$ and $c_{\downarrow}$ (even if either one vanishes) as

$$\boldsymbol{\Psi} = \frac{1}{\langle\Psi|\Psi\rangle}\left(2\,\mathrm{Re}\,[c_{\uparrow}^{*}c_{\downarrow}], 2\,\mathrm{Im}\,[c_{\uparrow}^{*}c_{\downarrow}], |c_{\uparrow}|^2 - |c_{\downarrow}|^2\right) \in \Sigma(\mathbb{R}^3).$$

Since $\pi_{\mathrm{S}} \circ \tau \circ \upsilon_{\mathfrak{b}}|\Psi\rangle = \pi_{\mathrm{S}} \circ \tau \circ \upsilon_{\mathfrak{b}}|\alpha\Psi\rangle$ for every complex number α and every nonvanishing spinor $|\Psi\rangle$ in H, it is in fact the ray $[\Psi]$ determined by $|\Psi\rangle$ that matters, in other words $\pi_{\mathrm{S}} \circ \tau \circ \upsilon_{\mathfrak{b}}$ can be viewed as a map on the set $R(\mathsf{H})$ of the rays of H.

Let us now turn to operators. A normal operator in H is characterized, aside from its two eigenvalues, by the corresponding eigenspinors $|\mathfrak{a}\rangle$ and $|\mathfrak{a}_{\perp}\rangle$, and hence by the projectors

$$P_{[\mathfrak{a}]} = |\mathfrak{a}\rangle\langle\mathfrak{a}|, \quad P_{[\mathfrak{a}_{\perp}]} = |\mathfrak{a}_{\perp}\rangle\langle\mathfrak{a}_{\perp}| = I - P_{[\mathfrak{a}]}$$

in its spectral decomposition $A = \lambda P_{[\mathfrak{a}]} + \lambda_{\perp} P_{[\mathfrak{a}_{\perp}]}$. Varying the eigenvalues, we obtain a class of commuting normal operators. Leaving out multiples of the identity (which commute with everything) by requiring that $\lambda \neq \lambda_{\perp}$, we obtain the *equivalence class* $\mathfrak{A}$ of commuting normal operators determined by the eigenspinors $|\mathfrak{a}\rangle$, $|\mathfrak{a}_{\perp}\rangle$, and hence by the antipodal points $\boldsymbol{\mathfrak{a}}, \boldsymbol{\mathfrak{a}}_{\perp}$ and the diameter $\mathfrak{D}_{\mathfrak{A}}$ on the Riemann sphere. The class $\mathfrak{Z}$ of normal operators with eigenprojectors $P_{\uparrow} = |\uparrow\rangle\langle\uparrow|$, $P_{\downarrow} = |\downarrow\rangle\langle\downarrow|$, for instance, will be represented by the north-south diameter $\mathfrak{D}_3$.

Orthogonal projection onto a diameter provides a probability measure, in fact the very one we need. If $\boldsymbol{\Psi} \in \Sigma(\mathbb{R}^3)$ is the stereographic projection of $\zeta' = \pi_{\mathrm{S}}^{-1}(\boldsymbol{\Psi}) \in \mathbb{C}_{\mathrm{P}}$ on the Riemann sphere, we can define the orthogonal projection

$$\Pi_3\boldsymbol{\Psi} = \left(0, 0, \frac{1 - |\zeta'|^2}{1 + |\zeta'|^2}\right) \in \mathbb{R}^3$$

of $\boldsymbol{\Psi}$ onto $\mathfrak{D}_3$. As we are dealing with a *unit* sphere, the Euclidean distances

$$\rho(\mathrm{N}, \Pi_3\boldsymbol{\Psi}) + \rho(\Pi_3\boldsymbol{\Psi}, \mathrm{S}) = \rho(\mathrm{N}, \mathrm{S}) = 2.$$

It turns out that $\rho(\mathrm{N}, \Pi_3\boldsymbol{\Psi})$ and $\rho(\Pi_3\boldsymbol{\Psi}, \mathrm{S})$, when halved to take account of the length of the diameter, give the right probabilities. In other words

$$\frac{1}{2}\rho(\Pi_3\boldsymbol{\Psi}, \mathrm{S}) = \frac{1}{2}(1 + \cos\theta) = |c_{\uparrow}|^2 = \|P_{\uparrow}|\Psi\rangle\|^2$$
$$\frac{1}{2}\rho(\mathrm{N}, \Pi_3\boldsymbol{\Psi}) = \frac{1}{2}(1 - \cos\theta) = |c_{\downarrow}|^2 = \|P_{\downarrow}|\Psi\rangle\|^2,$$

where θ is again the polar angle of $\boldsymbol{\Psi}$.

Let us now see the effect of phase transformations with respect to $|\uparrow\rangle$ and $|\downarrow\rangle$. Consider the set of states

$$\mathfrak{Y} = \{|\Psi_{U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)}\rangle = U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)|\Psi\rangle : \theta_\uparrow,\theta_\downarrow \in [0,2\pi)\}$$

obtained by applying to $|\Psi\rangle$ the two-parameter unitary group

$$U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow) = e^{i\theta_\uparrow}P_\uparrow + e^{i\theta_\downarrow}P_\downarrow$$

of phase transformations. Expanding $|\Psi_{U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)}\rangle$ with respect to the eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$ of $U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)$ we have

$$|\Psi_{U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)}\rangle = e^{i\theta_\uparrow}c_\uparrow|\uparrow\rangle + e^{i\theta_\downarrow}c_\downarrow|\downarrow\rangle.$$

Sometimes we will just write U rather than $U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)$.

All states in the set $\mathfrak{Y}$ have the same probabilities with respect to the projectors $P_\uparrow$ and $P_\downarrow$, and hence with respect to all (real) linear combinations $Z = \lambda_\uparrow P_\uparrow + \lambda_\downarrow P_\downarrow$ as well. Algebraically this can be seen as follows. As Z commutes with the phase transformations U with eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$, we have that

$$\langle\Psi_U|Z|\Psi_U\rangle = \langle\Psi|U^\dagger ZU|\Psi\rangle = \langle\Psi|U^\dagger UZ|\Psi\rangle = \langle\Psi|Z|\Psi\rangle$$

and

$$\mathrm{Tr}(P_{[\Psi_U]}Z) = \mathrm{Tr}(UP_{[\Psi]}U^\dagger Z) = \mathrm{Tr}(P_{[\Psi]}UU^\dagger Z) = \mathrm{Tr}(P_{[\Psi]}Z). \tag{7}$$

We can also write

$$\frac{\partial}{\partial\theta_\uparrow}\mathrm{Tr}(P_{[\Psi_U]}Z) = \frac{\partial}{\partial\theta_\downarrow}\mathrm{Tr}(P_{[\Psi_U]}Z) = 0 \tag{8}$$

to express the invariance under U of probabilities associated with $P_\uparrow$ and $P_\downarrow$.

This has the following geometrical expression. The quotients

$$\{\tau(e^{i\theta_\uparrow}c_\uparrow, e^{i\theta_\downarrow}c_\downarrow) = e^{i(\theta_\downarrow-\theta_\uparrow)}c_\downarrow/c_\uparrow : \theta_\uparrow,\theta_\downarrow \in [0,2\pi)\}$$

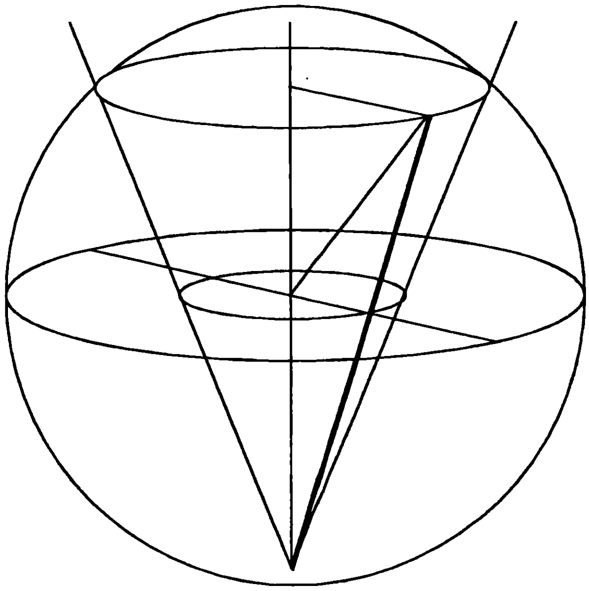
of the expansion coefficients

$$\{v_{\mathfrak{b}'}|\Psi_{U_{\mathfrak{b}'}(\theta_\uparrow,\theta_\downarrow)}\rangle = (e^{i\theta_\uparrow}c_\uparrow, e^{i\theta_\downarrow}c_\downarrow) : \theta_\uparrow,\theta_\downarrow \in [0,2\pi)\}$$

form a circle of radius $|c_\downarrow/c_\uparrow|$ and centre 0 on the Argand plane, whose stereographic projection is a circle

$$\mathsf{C} = \{\pi_{\mathbf{P}} \circ \tau(e^{i\theta_\uparrow}c_\uparrow, e^{i\theta_\downarrow}c_\downarrow) : \theta_\uparrow,\theta_\downarrow \in [0,2\pi)\} \subset \Sigma(\mathbb{R}^3).$$

As C is perpendicular to the diameter $\mathfrak{D}_3$, all points $\boldsymbol{\Psi}_U \in \mathsf{C}$ will have the same projection $\Pi_3\boldsymbol{\Psi}_U = \Pi_3\boldsymbol{\Psi}$ onto $\mathfrak{D}_3$, and hence the same probabilities with respect to $P_\uparrow$ and $P_\downarrow$.



To illustrate with a *precession* around C we can define the unitary group

$$e^{-iHt/\hbar} = e^{-iE_{\uparrow}t/\hbar}P_{\uparrow} + e^{-iE_{\downarrow}t/\hbar}P_{\downarrow}$$

of time-evolution operators, whose self-adjoint infinitesimal generator H has the spectral resolution $E_{\uparrow}P_{\uparrow} + E_{\downarrow}P_{\downarrow}$. The corresponding wave equation

$$i\hbar\frac{d|\Psi(t)\rangle}{dt} = H|\Psi(t)\rangle$$

can be broken up into the equations

$$i\hbar\frac{dc_{\uparrow}(t)}{dt} = E_{\uparrow}c_{\uparrow}(t), \quad i\hbar\frac{dc_{\downarrow}(t)}{dt} = E_{\downarrow}c_{\downarrow}(t),$$

with solutions

$$c_{\uparrow}(t) = e^{-iE_{\uparrow}t/\hbar}c_{\uparrow}, \quad c_{\downarrow}(t) = e^{-iE_{\downarrow}t/\hbar}c_{\downarrow},$$

where

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle = c_{\uparrow}(t)|\uparrow\rangle + c_{\downarrow}(t)|\downarrow\rangle.$$

Again, for every observable $Z \in \mathfrak{Z}$,

$$\mathrm{Tr}(P_{[\Psi(t)]}Z) = \mathrm{Tr}\big(e^{-iHt/\hbar}P_{[\Psi]}e^{iHt/\hbar}Z\big) = \mathrm{Tr}(P_{[\Psi]}Z) \tag{9}$$

is invariant under the group $e^{-iHt/\hbar}$ because $[Z, H] = 0$.

Pure states, then, are on the surface of the sphere. Mixtures, represented by statistical operators,[52] correspond to internal points. A statistical operator on H can be

[52] In H these are self-adjoint operators whose trace is one.

written as a convex sum $\boldsymbol{\nu} = wP_{[\mathfrak{n}]} + (1-w)P_{[\mathfrak{n}_\perp]}$, $w \in [0,1]$, of orthogonal projectors $P_{[\mathfrak{n}]} = |\mathfrak{n}\rangle\langle\mathfrak{n}|$ and $P_{[\mathfrak{n}_\perp]} = |\mathfrak{n}_\perp\rangle\langle\mathfrak{n}_\perp|$. We have seen that two such projectors determine a diameter $\mathfrak{D}_{\mathfrak{N}}$, with endpoints $\boldsymbol{\mathfrak{n}}$ and $\boldsymbol{\mathfrak{n}}_\perp$. If we then require that

$$w : (1-w) = \rho(\boldsymbol{\nu}, \boldsymbol{\mathfrak{n}}_\perp) : \rho(\boldsymbol{\mathfrak{n}}, \boldsymbol{\nu}),$$

a point $\boldsymbol{\nu} \in \mathfrak{D}_{\mathfrak{N}}$ inside $\Sigma(\mathbb{R}^3)$ representing the mixture will be singled out. The probabilities for a mixture are also given by orthogonal projection onto a diameter.

Any spinor in the ray $[\Psi]$ generated by $|\Psi\rangle$ can be represented by the statistical operator $P_{[\Psi]} = |\Psi\rangle\langle\Psi|$. Expressing $P_{[\Psi]}$ with respect to the basis $|\uparrow\rangle$, $|\downarrow\rangle$ we have

$$P_{[\Psi]} = |c_\uparrow|^2 P_\uparrow + |c_\downarrow|^2 P_\downarrow + \big(c_\uparrow c_\downarrow^* |\uparrow\rangle\langle\downarrow| + c_\downarrow c_\uparrow^* |\downarrow\rangle\langle\uparrow|\big).$$

The first two terms give the statistical operator $\rho_3 = |c_\uparrow|^2 P_\uparrow + |c_\downarrow|^2 P_\downarrow$ from which '$\mathfrak{Z}$-phase,' represented by the 'interference operator' $c_\uparrow c_\downarrow^* |\uparrow\rangle\langle\downarrow| + c_\downarrow c_\uparrow^* |\downarrow\rangle\langle\uparrow|$, has been removed. As ρ_3 is represented by the point $\boldsymbol{\rho_3} = \Pi_3 \boldsymbol{\Psi} \in \mathfrak{D}_3$ on the plane $\mathcal{P} \perp \mathfrak{D}_3$ determined by C, it will have the same '$\mathfrak{Z}$-statistics' as $|\Psi\rangle$:

$$\mathrm{Tr}(\rho_3 Z) = \mathrm{Tr}(P_{[\Psi]} Z). \tag{10}$$

In fact all statistical operators in $\mathsf{R} = \{\rho : \boldsymbol{\rho} \in \mathcal{P}\}$ will have the same $\mathfrak{Z}$-statistics: $\mathrm{Tr}(\rho Z) = \mathrm{Tr}(P_{[\Psi]} Z)$ for all $\rho \in \mathsf{R}$. Furthermore

$$\mathrm{Tr}(U\rho U^\dagger Z) = \mathrm{Tr}(\rho Z) \tag{11}$$

for all $\rho \in \mathsf{R}$ since U and Z commute.

Observables represented by operators in $\mathfrak{Z}$ will not, then, notice the transformation or removal of phase with respect to $\mathfrak{Z}$.

1.2 Another basis

Let us now see what happens with respect to the left-right or '$\mathfrak{X}$'-basis $\mathfrak{b}'' = (|\leftarrow\rangle, |\rightarrow\rangle)$, and how the statistics of self-adjoint operators belonging to

$$\mathfrak{X} = \{X = \lambda_\rightarrow P_\rightarrow + \lambda_\leftarrow P_\leftarrow : \lambda_\rightarrow \neq \lambda_\leftarrow,\ P_\rightarrow \text{ and } P_\leftarrow \text{ fixed}\}$$

are influenced by the transformation or removal of $\mathfrak{Z}$-phase.

We again compose three mappings:

$$\begin{aligned} &v_{\mathfrak{b}''} : \mathsf{H} \to \mathbb{C}^2, \\ &\tau : \omega \to \mathbb{C}, \\ &\pi_{\mathbf{R}} : \mathbb{C}_{\mathbf{R}} \to \Sigma(\mathbb{R}^3). \end{aligned}$$

First we map the expansion coefficients

$$c_{\rightarrow} = \langle \rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}}(\langle \uparrow |\Psi\rangle + \langle \downarrow |\Psi\rangle) = \frac{1}{\sqrt{2}}(c_{\uparrow} + c_{\downarrow})$$
$$c_{\leftarrow} = \langle \leftarrow |\Psi\rangle = \frac{1}{\sqrt{2}}(\langle \uparrow |\Psi\rangle - \langle \downarrow |\Psi\rangle) = \frac{1}{\sqrt{2}}(c_{\uparrow} - c_{\downarrow})$$

to their quotient $\eta = \tau(c_{\rightarrow}, c_{\leftarrow}) = c_{\leftarrow}/c_{\rightarrow}$ on the Argand plane

$$\mathbb{C}_{\mathbf{R}} = \left(\mathbb{R}^3\right)_{y,z} = \left\{(x, y, z) \in \mathbb{R}^3 : x = 0\right\}$$

perpendicular to the x-axis. We then compose τ with the stereographic projection $\pi_{\mathbf{R}} : \mathbb{C}_{\mathbf{R}} \rightarrow \Sigma(\mathbb{R}^3)$ from the 'right pole' $\mathbf{R} = (1, 0, 0) \in \mathbb{R}^3$ to the Riemann sphere. The 'left pole' $\pi_{\mathbf{R}}(0) = \mathbf{L} = (-1, 0, 0) \in \mathbb{R}^3$ corresponds to $| \leftarrow \rangle$ by projection, and we will have to establish the correspondence between $| \rightarrow \rangle$ and $\mathbf{R}$ 'by hand,' much as before.

Since

$$\begin{aligned}
\pi_{\mathbf{R}}(\eta) &= \left(\frac{|\eta|^2 - 1}{|\eta|^2 + 1}, \frac{2\,\mathrm{Im}\,\eta}{|\eta|^2 + 1}, \frac{-2\,\mathrm{Re}\,\eta}{|\eta|^2 + 1}\right) \\
&= \left(\frac{2\,\mathrm{Re}\,\zeta}{|\zeta|^2 + 1}, \frac{2\,\mathrm{Im}\,\zeta}{|\zeta|^2 + 1}, \frac{1 - |\zeta|^2}{1 + |\zeta|^2}\right) \\
&= \pi_{\mathbf{S}}(\zeta) = \boldsymbol{\Psi} \in \Sigma(\mathbb{R}^3),
\end{aligned}$$

$\zeta = \langle \downarrow |\Psi\rangle / \langle \uparrow |\Psi\rangle$ and $\eta = \langle \leftarrow |\Psi\rangle / \langle \rightarrow |\Psi\rangle$ get projected stereographically to the same point on the sphere. In fact for any orthonormal pair there will be a corresponding projection pole and Argand plane that give rise to the same stereographic projection $\boldsymbol{\Psi}$. So there is an invariant mapping $\pi \circ \tau \circ \upsilon : \mathsf{H} \rightarrow \Sigma(\mathbb{R}^3)$ from H to the Riemann sphere which does not depend on the basis chosen. We can write $|\Psi\rangle \mapsto \pi \circ \tau \circ \upsilon |\Psi\rangle = \boldsymbol{\Psi}$ without indices, to indicate the invariance (which is also useful for diameters and hence normal operators).

Let us apply the two-parameter unitary group U to $|\Psi\rangle$, and represent the resulting states $|\Psi_U\rangle = e^{i\theta_{\uparrow}} c_{\uparrow} | \uparrow \rangle + e^{i\theta_{\downarrow}} c_{\downarrow} | \downarrow \rangle$ with respect to $\mathfrak{b}''$. Inverting equations (6), we have that

$$\begin{aligned}
|\Psi_U\rangle &= \frac{1}{\sqrt{2}}\left\{e^{i\theta_{\uparrow}} c_{\uparrow}\left(| \rightarrow \rangle + | \leftarrow \rangle\right) + e^{i\theta_{\downarrow}} c_{\downarrow}\left(| \rightarrow \rangle - | \leftarrow \rangle\right)\right\} \\
&= \frac{1}{\sqrt{2}}\left\{\left(e^{i\theta_{\uparrow}} c_{\uparrow} - e^{i\theta_{\downarrow}} c_{\downarrow}\right)| \leftarrow \rangle + \left(e^{i\theta_{\uparrow}} c_{\uparrow} + e^{i\theta_{\downarrow}} c_{\downarrow}\right)| \rightarrow \rangle\right\}.
\end{aligned}$$

Suppose the angles $\theta_{\uparrow}$ and $\theta_{\downarrow}$ both vanish, so that $e^{i\theta_{\uparrow}} = e^{i\theta_{\downarrow}} = 1$; then

$$|\langle \leftarrow |\Psi_U\rangle|^2 = \left|(c_{\uparrow} - c_{\downarrow})/\sqrt{2}\right|^2.$$

By changing $\theta_\downarrow$ to π, however, $e^{i\theta_\downarrow} = -1$, and $|\langle \leftarrow |\Psi_U\rangle|^2$ becomes $\left|(c_\uparrow + c_\downarrow)/\sqrt{2}\right|^2$. The action of U, which self-adjoint operators in $\mathfrak{Z}$ do not notice, can therefore produce an observable difference with respect to $\mathfrak{X}$.

This can also be seen in terms of time evolution. Whereas all points $\boldsymbol{\Psi}_U \in \mathsf{C}$ had the same projection $\Pi_3\boldsymbol{\Psi}$ onto $\mathfrak{D}_3$, the precession around C generated by $e^{-iHt/\hbar}$, projected onto $\mathfrak{D}_\mathfrak{X}$, gives rise to probability beats. Expanding $|\Psi(t)\rangle$ with respect to $| \rightarrow \rangle$ and $| \leftarrow \rangle$, we obtain the oscillating probabilities

$$\|P_\leftarrow|\Psi(t)\rangle\|^2 = \left|\frac{1}{\sqrt{2}}\left(e^{i\theta_\uparrow t}c_\uparrow + e^{i\theta_\downarrow t}c_\downarrow\right)\right|^2 = \frac{1}{2}\{1 + 2|c_\uparrow||c_\downarrow|\cos(\theta_\uparrow - \theta_\downarrow)t\}$$

$$\|P_\rightarrow|\Psi(t)\rangle\|^2 = \left|\frac{1}{\sqrt{2}}\left(e^{i\theta_\uparrow t}c_\uparrow - e^{i\theta_\downarrow t}c_\downarrow\right)\right|^2 = \frac{1}{2}\{1 - 2|c_\uparrow||c_\downarrow|\cos(\theta_\uparrow - \theta_\downarrow)t\}.$$

Let us now compare the mixture ρ_3 with the class of $\mathfrak{Z}$-equivalent superpositions $|\Psi_U\rangle$, whose stereographic projections $\boldsymbol{\Psi}_U$ make up C. Since $\Pi_\mathfrak{X}\boldsymbol{\rho}_3$ is null, a superposition $|\Psi_U\rangle$ will be $\mathfrak{X}$-equivalent to ρ_3 if $e^{i(\theta_\downarrow-\theta_\uparrow)}c_\downarrow/c_\uparrow$ is purely imaginary; that is, when $\boldsymbol{\Psi}_U$ lies on the vertical plane $\mathbb{C}_\mathbf{R}$ perpendicular to $\mathfrak{D}_\mathfrak{X}$, and only then. As $\mathsf{C} \cap \mathbb{C}_\mathbf{R}$ is practically empty (just two elements), ρ_3 is $\mathfrak{X}$-distinguishable from almost all[53] states $|\Psi_U\rangle$.

Although an observable with eigendiameter $\mathfrak{D}_3$ will not notice the transformation or removal of phase with respect to the corresponding basis $\mathfrak{b}'$, an observable with a different basis $\mathfrak{b}''$ almost always will (at least in H).

1.3 Kaons

Neutral K-mesons[54] or 'kaons'[55] are a useful example. Two quantities will interest us: charge-parity, represented by the operator

$$CP = |K_\mathrm{S}\rangle\langle K_\mathrm{S}| - |K_\mathrm{L}\rangle\langle K_\mathrm{L}|$$

and assumed to be conserved,[56] and strangeness, represented by[57]

$$S = |K\rangle\langle K| - |\overline{K}\rangle\langle\overline{K}|.$$

[53]The matter can be completely settled by a third diameter.

[54]About kaons see Pais (1986) pp.515-42, Gell-Mann and Pais (1955), Pais and Piccioni (1955).

[55]The term 'kaon' can in fact be applied either to *any* K-meson—to a $|K_\mathrm{S}\rangle$ or a $|\overline{K}\rangle$, for instance—or, in particular, to the strangeness eigenstate $|K\rangle$ (as opposed to the 'antikaon' $|\overline{K}\rangle$). Context should be enough to eliminate ambiguities.

[56]The eigenvectors of the CP operator are in fact slightly oblique, as charge-parity is only approximately conserved. Violations occur seldom enough, however, to justify the assumption of conservation. The differences at issue in Chapter VI. 2, which discriminate between local realism and quantum mechanics, are of another order. Decay also produces effects of another order: if kaons were stable, Bell's limit 2 would be *abundantly* exceeded by the quantum-mechanical prediction $2\sqrt{2}$, regardless of CP-violation; but even if decay is uneliminable (see VI. 2.6), CP-violation is not significant enough to compensate its effects.

Expressing the kaon $|K\rangle$ and antikaon $|\overline{K}\rangle$ in terms of CP eigenstates, we have

$$|K\rangle = \frac{1}{\sqrt{2}}(|K_{\mathrm{S}}\rangle + |K_{\mathrm{L}}\rangle) \qquad (12)$$
$$|\overline{K}\rangle = \frac{1}{\sqrt{2}}(|K_{\mathrm{S}}\rangle - |K_{\mathrm{L}}\rangle).$$

One must always be able to say *Tische, Stühle und Bierseidel*—tables, chairs and beermugs, rather than *Punkten, Geraden und Ebenen*—points, lines and planes. Thus Hilbert expressed his formalist attitude to geometry: the objects themselves matter much less than their relations. Here we are reminded of spin and can adopt a similar attitude, extending spin-half geometrical intuitions to the physics of kaons with the aid of stereographic projection. One can associate $|K_{\mathrm{S}}\rangle$ and $|K_{\mathrm{L}}\rangle$ with $|\uparrow\rangle$ and $|\downarrow\rangle$, for instance, without being overly distracted by the very different physics.

The parity or 'space inversion' operator P changes the sign of the spatial coordinates:

$$P\hat{\boldsymbol{x}}P = -\hat{\boldsymbol{x}}, \qquad P\psi(\boldsymbol{x}) = \psi(-\boldsymbol{x}),$$

where $\hat{\boldsymbol{x}}$ is the position operator and $\psi(\boldsymbol{x})$ a wavefunction. The charge-conjugation operator C carries particles into their antiparticles, so

$$C|\pi^{\pm}\rangle = |\pi^{\mp}\rangle, \quad C|\mu^{\pm}\rangle = |\mu^{\mp}\rangle$$

for π- and μ-mesons. It turns a kaon into an antikaon $|\overline{K}\rangle = C|K\rangle$ and *vice-versa*: $C|\overline{K}\rangle = |K\rangle$. The charge-parity operator CP is obtained composing C with P.

The eigenstate $|K_{\mathrm{S}}\rangle$ represents a 'short kaon' and $|K_{\mathrm{L}}\rangle$ a 'long kaon,' so called because they decay weakly—into two and three pions, respectively—according to the factors $e^{-\frac{1}{2}\gamma_{\mathrm{S}}t}$, $e^{-\frac{1}{2}\gamma_{\mathrm{L}}t}$, and the decay rate $\gamma_{\mathrm{S}} \gg \gamma_{\mathrm{L}}$. The kaon $|K\rangle$ and antikaon $|\overline{K}\rangle$ can be told apart by strong interactions. An antikaon $|\overline{K}\rangle$ can react with ordinary matter (protons and neutrons) to produce a hyperon—a Λ, for instance—whereas a kaon $|K\rangle$ will never do so, at least at moderate energies. Alternatively, if one directs a superposition of $|K\rangle$ and $|\overline{K}\rangle$ at a suitable slab, the kaons go through whereas the antikaons are absorbed.[58]

The time evolution of a kaon state $|\mathfrak{k}(t)\rangle$ is governed by the equation

$$i\frac{d|\mathfrak{k}(t)\rangle}{dt} = M'|\mathfrak{k}(t)\rangle,$$

[57]The standard notation here is in fact $|K^0\rangle$ for the kaon (strangeness $+1$) and $|\overline{K}^0\rangle$ for the antikaon (strangeness -1). I have done away with the '0' superscripts which indicate vanishing charge or 'neutrality.'

[58]In fact the antikaons have a nonvanishing probability of crossing the slab, and the kaons of being absorbed, but a suitable slab will let through almost all the kaons and absorb almost all the antikaons.

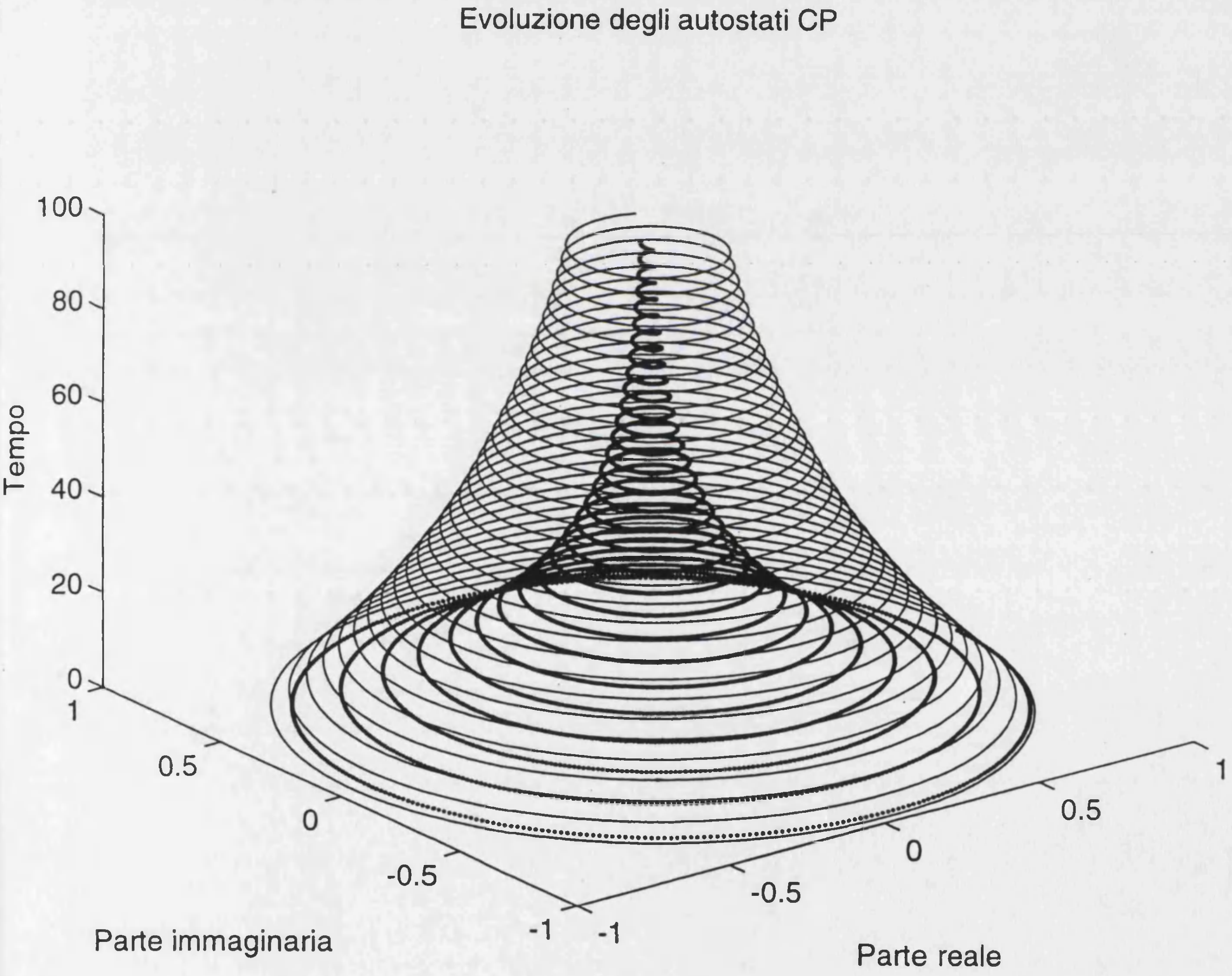
where the infinitesimal generator M' is the sum $M + i\frac{1}{2}\Gamma$, and the positive operators

$$M = m_{\mathrm{S}}|K_{\mathrm{S}}\rangle\langle K_{\mathrm{S}}| + m_{\mathrm{L}}|K_{\mathrm{L}}\rangle\langle K_{\mathrm{L}}|$$
$$\Gamma = \gamma_{\mathrm{S}}|K_{\mathrm{S}}\rangle\langle K_{\mathrm{S}}| + \gamma_{\mathrm{L}}|K_{\mathrm{L}}\rangle\langle K_{\mathrm{L}}|$$

represent mass and decay respectively ($c = \hbar = 1$). As the decay rates $\gamma_{\mathrm{S}}, \gamma_{\mathrm{L}} > 0$, the time evolution operator

$$U'(t) = e^{-iM't} = e^{-(im_{\mathrm{S}}+\frac{1}{2}\gamma_{\mathrm{S}})t}|K_{\mathrm{S}}\rangle\langle K_{\mathrm{S}}| + e^{-(im_{\mathrm{L}}+\frac{1}{2}\gamma_{\mathrm{L}})t}|K_{\mathrm{L}}\rangle\langle K_{\mathrm{L}}|$$

shortens vectors as time passes. The evolutions of the charge-parity eigenstates are shown in the figure below.[59]



In the figure I have adhered to the relations $\gamma_{\mathrm{S}} > \gamma_{\mathrm{L}}$ and $m_{\mathrm{L}} > m_{\mathrm{S}}$, but not to the ratios $\gamma_{\mathrm{S}} : \gamma_{\mathrm{L}}$ and $m_{\mathrm{S}} : m_{\mathrm{L}}$ (for graphical reasons).

[59]Produced with the help of Fasma Diele and Stefania Ragni, *Istituto per Ricerche di Matematica Applicata, CNR*, Bari.

The normal operator $U'(t)$ can be expressed as the product $U(t)e^{-\frac{1}{2}\Gamma t}$ of the unitary operator

$$U(t) = e^{-iMt} = e^{-im_{\mathrm{S}}t}|K_{\mathrm{S}}\rangle\langle K_{\mathrm{S}}| + e^{-im_{\mathrm{L}}t}|K_{\mathrm{L}}\rangle\langle K_{\mathrm{L}}|$$

and the positive operator

$$e^{-\frac{1}{2}\Gamma t} = e^{-\frac{1}{2}\gamma_{\mathrm{S}}t}|K_{\mathrm{S}}\rangle\langle K_{\mathrm{S}}| + e^{-\frac{1}{2}\gamma_{\mathrm{L}}t}|K_{\mathrm{L}}\rangle\langle K_{\mathrm{L}}|.$$

For the time being let us assume that Γ vanishes and that hence $e^{-\frac{1}{2}\Gamma t} = I$, as we are less interested in decay than in the undulatory evolution represented by the wave equation

$$i\frac{d|\mathfrak{k}(t)\rangle}{dt} = M|\mathfrak{k}(t)\rangle$$

and therefore by $U(t)$.

When $|K_{\mathrm{S}}\rangle$ and $|K_{\mathrm{L}}\rangle$ are superposed, their phase difference will vary in time and give rise to beats known as 'strangeness oscillations.' The evolution

$$|K(t)\rangle = U(t)|K\rangle = \frac{1}{\sqrt{2}}\left(e^{-im_{\mathrm{S}}t}|K_{\mathrm{S}}\rangle + e^{-im_{\mathrm{L}}t}|K_{\mathrm{L}}\rangle\right),$$

for instance, becomes

$$\begin{aligned} |K(t)\rangle &= \frac{1}{\sqrt{2}}\left\{e^{-im_{\mathrm{S}}t}\frac{1}{\sqrt{2}}\left(|K\rangle - |\overline{K}\rangle\right) + e^{-im_{\mathrm{L}}t}\frac{1}{\sqrt{2}}\left(|K\rangle + |\overline{K}\rangle\right)\right\} \\ &= \frac{1}{2}\left\{\left(e^{-im_{\mathrm{S}}t} + e^{-im_{\mathrm{L}}t}\right)|K\rangle + \left(e^{-im_{\mathrm{L}}t} - e^{-im_{\mathrm{S}}t}\right)|\overline{K}\rangle\right\} \end{aligned}$$

with respect to strangeness. The probabilities

$$\begin{aligned} |\langle K_{\mathrm{S}}|K(t)\rangle|^2 &= \left|\frac{1}{2}\left(e^{-im_{\mathrm{S}}t} + e^{-im_{\mathrm{L}}t}\right)\right|^2 = \frac{1}{2}\{1 + \cos(m_{\mathrm{S}} - m_{\mathrm{L}})t\} \\ |\langle K_{\mathrm{L}}|K(t)\rangle|^2 &= \left|\frac{1}{2}\left(e^{-im_{\mathrm{S}}t} - e^{-im_{\mathrm{L}}t}\right)\right|^2 = \frac{1}{2}\{1 - \cos(m_{\mathrm{S}} - m_{\mathrm{L}})t\} \end{aligned}$$

clearly oscillate.

If we place the charge-parity eigenstates on the north-south diameter, the evolution $|K(t)\rangle$ will be represented by a precession around the equator. More generally, for any state $|\mathfrak{k}\rangle = c_{\mathrm{S}}|K_{\mathrm{S}}\rangle + c_{\mathrm{L}}|K_{\mathrm{L}}\rangle$, the evolution

$$|\mathfrak{k}(t)\rangle = U(t)|\mathfrak{k}\rangle = e^{-im_{\mathrm{S}}t}c_{\mathrm{S}}|K_{\mathrm{S}}\rangle + e^{-im_{\mathrm{L}}t}c_{\mathrm{L}}|K_{\mathrm{L}}\rangle$$

will give rise to a precession of the quotient $\zeta(t) = e^{i(m_{\mathrm{S}}-m_{\mathrm{L}})t}c_{\mathrm{L}}/c_{\mathrm{S}}$ around a circle of radius $|c_{\mathrm{L}}/c_{\mathrm{S}}|$ on the Argand plane, with an angular velocity proportional to the difference $\Delta m = m_{\mathrm{L}} - m_{\mathrm{S}}$ of the masses. Its stereographic projection will be a precession around the 'parallel' $\mathfrak{P}_\vartheta$ determined by the polar angle

$$\vartheta = \arccos|c_S| = \arcsin|c_L|.$$

The probabilities

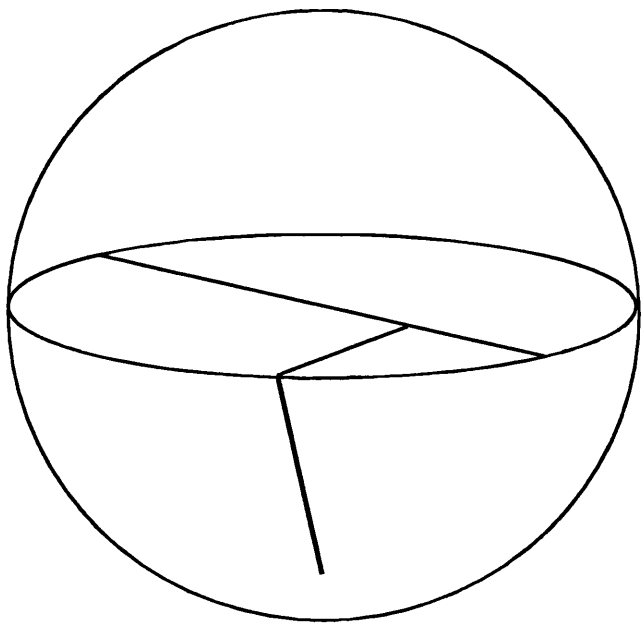
$$|c_S|^2 = \frac{1}{2}\rho\big(\Pi_{CP}\mathfrak{k}, \mathbf{S}\big) = \frac{1}{2}\{1 + \cos\theta(\mathbf{N}, \mathfrak{k})\}$$
$$|c_L|^2 = \frac{1}{2}\rho\big(\mathbf{N}, \Pi_{CP}\mathfrak{k}\big) = \frac{1}{2}\{1 - \cos\theta(\mathbf{N}, \mathfrak{k})\},$$

where $\Pi_{CP}\mathfrak{k}$ is the orthogonal projection of $\mathfrak{k} = \pi \circ \tau \circ v|\mathfrak{k}\rangle$ on the diameter corresponding to CP, $\rho(\mathfrak{x}, \mathfrak{y})$ the distance between the points $\mathfrak{x}$ and $\mathfrak{y}$, and ϑ the polar angle of $\mathfrak{k}$.

As all points of the parallel $\mathfrak{P}_\vartheta$ have the same polar angle ϑ, and hence the same orthogonal projection onto the diameter $\mathfrak{D}_{CP}$ of CP, the evolution $U(t)$ will be invisible with respect to conserved quantities, in other words with respect to the self-adjoint operators that share the vertical diameter $\mathfrak{D}_{CP}$. But with respect to strangeness, whose diameter $\mathfrak{D}_S$ lies on the equator by virtue of the relations (12), we will see undulatory beats, the so-called 'strangeness oscillations':

$$|c|^2 = \left|\frac{1}{2}\left(e^{-im_S t} + e^{-im_L t}\right)\right|^2 = \frac{1}{2}(1 + \cos\Delta m t)$$
$$|\bar{c}|^2 = \left|\frac{1}{2}\left(e^{-im_S t} - e^{-im_L t}\right)\right|^2 = \frac{1}{2}(1 - \cos\Delta m t),$$

where $c = \langle K|\mathfrak{k}\rangle$ and $\bar{c} = \langle \overline{K}|\mathfrak{k}\rangle$.



We have seen that pure states, corresponding to rank one projection operators (or rays) on H, are represented by points on the surface of the Riemann sphere. Mixed states, corresponding to statistical operators, are represented by points inside the sphere, and equivalence classes of commuting (maximal) normal operators by diameters. As probabilities for the class $\mathfrak{A}$ are given by projection onto the corresponding diameter $\mathfrak{D}_{\mathfrak{A}}$, all states on a disk perpendicular to $\mathfrak{D}_{\mathfrak{A}}$ will be indistinguishable with respect to $\mathfrak{A}$. Unitary operators belonging to $\mathfrak{A}$ will not affect the statistics of observables represented by self-adjoint operators also in $\mathfrak{A}$, but will change the statistics of observables corresponding to different diameters. The unitary time-evolution group $e^{-iHt/\hbar}$, for instance, will give rise to oscillations in the statistics of observables represented by operators that do not commute with $e^{-iHt/\hbar}$. Kaons do not exhibit beats with respect to charge-parity, which is conserved, but they do with respect to strangeness, which is not. Measuring strangeness on two kaons we will, in principle, be able to see undulatory beats in configuration space (Chapter VI. 2).

Phase can be 'transformed with respect to $\mathfrak{D}_{\mathfrak{A}}$' by a unitary operator belonging to $\mathfrak{A}$, or 'removed with respect to $\mathfrak{D}_{\mathfrak{A}}$' by projecting from the surface of the sphere onto $\mathfrak{D}_{\mathfrak{A}}$, which yields a non-idempotent statistical operator. As long as we restrict measurements to the class $\mathfrak{A}$, neither the transformation nor the removal of phase with respect to $\mathfrak{D}_{\mathfrak{A}}$ will make any statistical difference. An operator belonging to a different class $\mathfrak{B}$ is required to detect the transformation or removal of phase with respect to $\mathfrak{A}$.

2
Infinite dimensions

Before introducing tensor multiplication, we generalize the two-dimensional treatment—leaving behind the sphere and its diameters—and consider the relationship between interference and compatibility in infinite dimensions. Again, the transformation or removal of phase with respect to a basis B will not be noticed by an observable represented by an operator with eigenbasis B. Not all operators have eigenbases in Hilbert space, however, so a different approach is adopted for continuous spectra.

Certain 'configuration space' questions are *potentially* involved at this stage, even if tensor multiplication has yet to be introduced. Anticipating what is to follow (especially in Section III. 1), some of the vectors at issue here can already be viewed as products

$$|\varphi_k\rangle = \bigotimes_{\sigma=1}^{N} |\varphi_k^\sigma\rangle.$$

2.1 One decomposition

Applying the unitary group

$$\mathsf{U} = \sum_k e^{i\theta_k} |\varphi_k\rangle\langle\varphi_k|$$

(θ_k variable, $|\varphi_k\rangle$ fixed for all k) of phase transformations to the vector

$$|\Psi\rangle = \sum_k c_k |\varphi_k\rangle$$

we obtain

$$\mathsf{U}|\Psi\rangle = |\Psi_\mathsf{U}\rangle = \sum_k e^{i\theta_k} c_k |\varphi_k\rangle.$$

Clearly U has no effect on the probabilities $|c_k|^2 = |e^{i\theta_k} c_k|^2$, in other words

$$\frac{\partial}{\partial\theta_k} \mathrm{Tr}(|\Psi_\mathsf{U}\rangle\langle\Psi_\mathsf{U}|\varphi_k\rangle\langle\varphi_k|) = 0$$

for all k. More generally, $\mathrm{Tr}\big(|\Psi_\mathsf{U}\rangle\langle\Psi_\mathsf{U}|A\big) = \mathrm{Tr}\big(|\Psi\rangle\langle\Psi|A\big)$ is invariant under U for any self-adjoint operator

$$A = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|$$

that commutes with U. This generalizes (7) and (8): the arguments of the coefficients in the expansion of a vector with respect to a basis $\{|\varphi_k\rangle\}$ can be changed arbitrarily without affecting the statistics of an operator with eigenvectors $\{|\varphi_k\rangle\}$.

Similarly (10) can be generalized. Expressing $|\Psi\rangle\langle\Psi|$ with respect to the basis $\{|\varphi_k\rangle\}$ we have

$$|\Psi\rangle\langle\Psi| = \left(\sum_k |c_k|^2 |\varphi_k\rangle\langle\varphi_k|\right) + \left(\sum_{m\neq n} c_m c_n^* |\varphi_m\rangle\langle\varphi_n|\right).$$

Doing away with the cross terms or 'interference operator'

$$\varpi = \sum_{m\neq n} c_m c_n^* |\varphi_m\rangle\langle\varphi_n|$$

we are left with the statistical operator

$$\eta = |\Psi\rangle\langle\Psi| - \varpi = \sum_k |c_k|^2 |\varphi_k\rangle\langle\varphi_k|.$$

The average

$$\begin{aligned} \mathrm{Tr}(\eta A) &= \mathrm{Tr}\left(\left\{\sum_k |c_k|^2 |\varphi_k\rangle\langle\varphi_k|\right\}\left\{\sum_n \lambda_n |\varphi_n\rangle\langle\varphi_n|\right\}\right) \\ &= \sum_k |c_k|^2 \lambda_k = \mathrm{Tr}\big(|\Psi\rangle\langle\Psi| A\big) \end{aligned} \tag{13}$$

because η and A commute. So the *removal* of $|\Psi\rangle$'s phase information with respect to the basis $\{|\varphi_k\rangle\}$ has no effect on the statistics of an operator with eigenvectors $\{|\varphi_k\rangle\}$.

Now consider the time evolution $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle$ in this more general case. If the Hamiltonian has the form

$$H = \sum_k E_k |\zeta_k\rangle\langle\zeta_k|,$$

we can write

$$e^{-iHt/\hbar} = \sum_k e^{-iE_k t/\hbar} |\zeta_k\rangle\langle\zeta_k|.$$

Any self-adjoint operator C that commutes with H can be given the same eigenvectors $|\zeta_k\rangle$, and hence the form

$$C = \sum_k \nu_k |\zeta_k\rangle\langle\zeta_k|.$$

If we now expand the state $|\Psi\rangle$ with respect to the $|\zeta_k\rangle$'s, we obtain

$$|\Psi\rangle = \sum_k a_k|\zeta_k\rangle,$$

where $a_k = \langle\zeta_k|\Psi\rangle$. Applying to $|\Psi\rangle$ the operator $e^{-iHt/\hbar}$, we have that

$$\begin{aligned}|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi\rangle &= \sum_k e^{-iE_kt/\hbar}|\zeta_k\rangle\langle\zeta_k|\left(\sum_j a_j|\zeta_j\rangle\right)\\ &= \sum_{kj} e^{-iE_kt/\hbar}a_j\langle\zeta_k|\zeta_j\rangle|\zeta_k\rangle,\end{aligned}$$

which, given the orthonormality of the $|\zeta_k\rangle$'s, is equal to

$$\sum_k e^{-iE_kt/\hbar}a_k|\zeta_k\rangle.$$

But the factors $e^{-iE_kt/\hbar}$, being of unit modulus, will change only the arguments of the coefficients a_k. Their moduli, and hence the probabilities $|a_k|^2$, remain unaffected; in other words $|a_k|^2 = |e^{-iE_kt/\hbar}a_k|^2$, which expresses the 'conservation' of the probabilities associated with the eigenvalues of C. As $e^{-iHt/\hbar}$ commutes with C, therefore, the average $\mathrm{Tr}\big(|\Psi(t)\rangle\langle\Psi(t)|C\big)$ is unaffected by the group $e^{-iHt/\hbar}$, so

$$\frac{d}{dt}\mathrm{Tr}\big(|\Psi(t)\rangle\langle\Psi(t)|C\big) = 0,$$

which generalizes (9). Furthermore if the statistical operator

$$\alpha = \sum_k |a_k|^2|\zeta_k\rangle\langle\zeta_k|,$$

we have that $\mathrm{Tr}\big(|\Psi(t)\rangle\langle\Psi(t)|C\big) = \mathrm{Tr}\big(e^{-iHt'/\hbar}\alpha e^{iHt'/\hbar}C\big)$ for all t, t'.

Again, the transformation or removal of phase with respect to a basis B will have no effect on the statistics of an operator with eigenbasis B. Not all normal operators, however, have an eigenbasis.

In infinite dimensions a self-adjoint operator may not have eigenvectors of finite length. The eigenfunctions of the momentum operator $-i\hbar\partial/\partial x$, for instance, are the monochromatic plane waves $e^{-ipx/\hbar}$, which cannot belong to a Hilbert space, as their squared length

$$\int_{-\infty}^{\infty} e^{-ipx/\hbar}e^{ipx/\hbar}dx = \int_{-\infty}^{\infty} dx$$

is infinite. Such operators can nonetheless be expressed in the general spectral form

$$D = \int_{-\infty}^{\infty} \lambda d\mathsf{E}_\lambda,$$

where the $\mathsf{E}_\lambda = \mathsf{E}(-\infty, \lambda]$ are a family of spectral projectors parametrized by the reals; $\mathsf{E}_{-\infty}$ is the null operator $\widehat{0}$, and

$$\mathsf{E}_\infty = \int_{-\infty}^{\infty} d\mathsf{E}_\lambda = I.$$

A unitary operator similarly has the general form

$$U = \int_{-\infty}^{\infty} e^{i\vartheta(\lambda)} d\mathsf{E}_\lambda.$$

Some ideas generalize easily from the cases, involving discrete spectra, considered so far. Provided U commutes with D we again have that[60] $\mathrm{Tr}\big(U\rho U^\dagger D\big) = \mathrm{Tr}(\rho D)$ for the statistical operator ρ, which generalizes (7), (9) and (11). Whereas $[U, D] = 0$ no longer necessarily means that U and D have a complete set of common eigenvectors in Hilbert space, it does mean that they can be given the same spectral family E_λ.[61] Indeed U can be expressed as a function

$$U = f(D) = \int_{-\infty}^{\infty} f(\lambda) d\mathsf{E}_\lambda$$

if D is maximal.

To attempt a generalization of (10) and (13) we can map vectors $|\psi\rangle$ belonging to the abstract Hilbert space $\mathcal{H}$ to corresponding functions $\psi(x) = \mathbf{U}|\psi\rangle$ in the position space $\mathcal{L}_2(\mathbb{R}, \mathbb{C})$ of square-integrable functions. The particular space $\mathcal{L}_2(\mathbb{R}, \mathbb{C})$ can be characterized by the fact that the unitary representation in question, namely $\mathbf{U} : \mathcal{H} \to \mathcal{L}_2(\mathbb{R}, \mathbb{C})$, turns the position operator

$$\widehat{x} = \int_{-\infty}^{\infty} x d\mathsf{E}_x$$

into the *multiplication* operator $M_x = \mathbf{U}\widehat{x}\mathbf{U}^\dagger$, which has the effect of multiplying functions in $\mathcal{L}_2(\mathbb{R}, \mathbb{C})$ by the independent variable x; $M_x\psi(x) = x\psi(x)$. The spectral projectors E_x get likewise turned into characteristic functions, so in particular

$$\mathbf{U}\mathsf{E}_{x_0}\mathbf{U}^\dagger = \chi_{(-\infty, x_0]}(x).$$

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[60]The expression $\mathrm{Tr}(\rho D)$ makes sense if D is bounded, in other words if there is a finite N such that $\|D|\psi\rangle\| \leq N\||\psi\rangle\|$ for every vector $|\psi\rangle$ in Hilbert space.

[61]The trouble is that the projectors in the spectral family E_λ will not always be expressible as orthogonal sums of rank one projectors (of the form $|\lambda\rangle\langle\lambda|$). According to the spectral theorem, there will be a Hilbert space $\mathcal{L}_2(\mathbb{R}, \mathbb{C})$ in which the projectors belonging to the Borel algebra generated by E_λ become characteristic functions $\chi_\Delta(\lambda)$ that vanish outside the Borel set Δ, in which they are equal to one. If Δ is, say, a nonvanishing interval of the real line, the projector $\chi_\Delta(\lambda)$ will be of infinite rank. If, on the other hand, Δ is a set of measure zero, the images $\chi_\Delta(\lambda)f(\lambda)$ will all belong to the origin, as they only differ from the null function 0 on Δ. The operator $\chi_\Delta(\lambda)$ will then vanish, since its kernel is the whole space. For instance $\chi_{\lambda_0}(\lambda)$ corresponds to the null operator, since the Lebesgue measure of a point $\lambda_0 \in \mathbb{R}$ vanishes.

To prevent convergence from becoming an issue we can multiply M_x and $\mathcal{L}_2(\mathbb{R},\mathbb{C})$ by the characteristic function $\chi_{[a,b]}(x)$ and restrict our attention to the interval from a to b:

$$M_{x[a,b]} = \chi_{[a,b]}(x)\, M_x = \mathbf{U}\widehat{x}_{[a,b]}\mathbf{U}^\dagger$$
$$\mathcal{L}_2([a,b],\mathbb{C}) = \chi_{[a,b]}(x)\mathcal{L}_2(\mathbb{R},\mathbb{C}).$$

The operator $\widehat{x}_{[a,b]} = \mathbf{U}^\dagger M_{x[a,b]}\mathbf{U}$ acts on the abstract Hilbert space

$$\mathcal{H}_{[a,b]} = P_{[a,b]}\mathcal{H} = \mathbf{U}^\dagger\mathcal{L}_2([a,b],\mathbb{C}),$$

where the projector $P_{[a,b]} = \mathbf{U}^\dagger\chi_{[a,b]}(x)\mathbf{U}$.

To generalize (10) and (13) we would like to say that *the removal of position-phase makes no difference to the position operator* $\widehat{x}_{[a,b]}$; but to do so we would have to compare $\psi(x) \in \mathcal{L}_2([a,b],\mathbb{C})$ with a suitable statistical operator. Despite being 'normalized' and self-adjoint, the operator

$$\rho_\psi = \int_a^b \psi^*(x)\psi(x)d\mathsf{E}_x$$

will not do, as it is not compact, and hence cannot be a statistical operator. The trouble is that there is no position basis B_x to work with; if there were, we could expand $|\psi\rangle \in \mathcal{H}_{[a,b]}$ with respect to B_x, and write down the corresponding statistical operator.

There are, however, statistical operators that approximate $|\psi\rangle$ with arbitrary accuracy. First let us approximate the operator $\widehat{x}_{[a,b]}$ with the degenerate operator $\widehat{x}'_{[a,b]}$ whose spectrum is discrete ($\widehat{x}'_{[a,b]}$ will in fact be a function of $\widehat{x}_{[a,b]}$).

A partition $\{\Delta_1,\dots,\Delta_M\}$ of the the interval from a to b determines a resolution of the identity with projectors

$$\mathsf{P}_k = \mathbf{U}^\dagger\chi_{\Delta_k}(x)\mathbf{U},$$

and breaks $\mathcal{H}_{[a,b]}$ up correspondingly into mutually orthogonal subspaces $\mathcal{H}_{[a,b]k} = \mathsf{P}_k\mathcal{H}_{[a,b]}$ such that

$$\mathcal{H}_{[a,b]} = \bigoplus_{k=1}^{M}\mathcal{H}_{[a,b]k}.$$

For every set Δ_k choose a point $x_k \in \Delta_k$, and construct the self-adjoint operator

$$\widehat{x}'_{[a,b]} = \sum_{k=1}^{M} x_k\mathsf{P}_k.$$

Selecting a basis $\{|\eta_{(k)1}\rangle, |\eta_{(k)2}\rangle, \dots\}$ in each eigenspace $\mathcal{H}_{[a,b]k}$, we can also write

$$\widehat{x}'_{[a,b]} = \sum_{k=1}^{M} x_k \left(\sum_{m=1}^{\infty} |\eta_{(k)m}\rangle\langle\eta_{(k)m}| \right) = \sum_{km} x_k |\eta_{(k)m}\rangle\langle\eta_{(k)m}|$$

Expanding $|\psi\rangle$ with respect to the whole basis $\{|\eta_{(k)m}\rangle\}$ we obtain

$$|\psi\rangle = \sum_{km} c_{km} |\eta_{(k)m}\rangle$$

and the statistical operator

$$\varrho = \sum_{km} |c_{km}|^2 |\eta_{(k)m}\rangle\langle\eta_{(k)m}|,$$

where $c_{km} = \langle\eta_{(k)m}|\psi\rangle$.

The operators $\widehat{x}'_{[a,b]}$ and ϱ have been constructed so that

$$\mathrm{Tr}\big(\varrho\widehat{x}'_{[a,b]}\big) = \langle\psi|\widehat{x}'_{[a,b]}|\psi\rangle;$$

the difference $\mathrm{Tr}\big(\varrho\widehat{x}'_{[a,b]}\big) - \langle\psi|\widehat{x}_{[a,b]}|\psi\rangle$ and norm $\big\|\widehat{x}'_{[a,b]} - \widehat{x}_{[a,b]}\big\|$ can be made as small as one wishes by refining the partitions $\{\Delta_k\}$. As the average $\langle\psi|\widehat{x}_{[a,b]}|\psi\rangle$ is approximated with arbitrary accuracy by $\mathrm{Tr}\big(\varrho\widehat{x}'_{[a,b]}\big)$, we can say that *the removal of position-phase makes no difference to the position operator* $\widehat{x}_{[a,b]}$.

Similarly *the transformation of position-phase makes no difference to the position operator* $\widehat{x}_{[a,b]}$, for the average

$$\langle\psi|\widehat{x}_{[a,b]}|\psi\rangle = \int_a^b x\psi^*(x)\psi(x)dx \tag{14}$$

of $\widehat{x}_{[a,b]}$ in state $|\psi\rangle$ does not depend on the phases $e^{i\arg\psi(x)}$ of $|\psi\rangle$ in this position space. In other words the unitary operator

$$\mathfrak{U} = \int_a^b e^{i\vartheta(x)} d\mathsf{E}_x,$$

has no effect on the average

$$\langle\mathfrak{U}\psi|\widehat{x}_{[a,b]}|\mathfrak{U}\psi\rangle = \int_a^b xe^{-i\vartheta(x)}\psi^*(x)e^{i\vartheta(x)}\psi(x)dx = \langle\psi|\widehat{x}_{[a,b]}|\psi\rangle. \tag{15}$$

So a measurement corresponding to the spectral family E_λ will not reveal the effect of phase transformations with respect to E_λ. Similarly a time evolution governed by the Hamiltonian

$$H = \int E_\lambda dP_\lambda$$

and hence by the corresponding unitary group

$$e^{-iHt/\hbar} = \int e^{-iE_\lambda t/\hbar} dP_\lambda$$

will remain invisible with respect to an observable represented by an operator with the same spectral family P_λ.

2.2 Telling states apart

We have seen that the removal or transformation of phase with respect to a basis B will not be noticed by an observable represented by an operator with eigenbasis B. But will that removal or transformation be noticeable *somehow*? In the two-dimensional space H we saw (Section II. 1.2) that states which are indistinguishable with respect to one basis, say $\{|\uparrow\rangle, |\downarrow\rangle\}$, could always be told apart appealing to different bases. But in general? The theorem proved below—according to which different states (statistical operators) can always be told apart—tells us that the transformation or removal of phase can necessarily be made statistically visible. In particular this will mean, once tensor multiplication has been introduced, that the 'irreducible' propagation of quantum waves in configuration space can also be seen in principle.

A measure μ on the set Ω of subspaces (closed linear manifolds) of a Hilbert space is a countably additive function $\mu : \Omega \rightarrow [0, +\infty)$ assigning a nonnegative finite real number to every subspace. Provided the subspaces $\{S_n\}$ are pairwise orthogonal, the sum of their measures is the measure of their direct sum:

$$\sum_n \mu(S_n) = \mu\left(\bigoplus_n S_n\right).$$

The measure of the null subspace vanishes.

Gleason[62] (1957) proved that, if the Hilbert space has at least three dimensions, there is a positive semi-definite self-adjoint operator ρ with finite trace such that $\mathrm{Tr}(\rho P_S) = \mu(S)$ for every measure μ. As we are interested in probability measures, we can assign unit measure to the whole space $\mathcal{H}$; but then ρ has to be a statistical operator, because

$$1 = \mu(\mathcal{H}) = \mathrm{Tr}(\rho P_{\mathcal{H}}) = \mathrm{Tr}(\rho).$$

[62]For discussions of Gleason's theorem see Hughes (1989), pp.146-8, 321-46, Redhead (1987) pp.27-30, van Fraassen (1991), pp.165-77, and Isham (1995) pp.177-8. For a proof see Varadarajan (1968) volume I, pp.145-59.

More importantly for us, ρ is the *only* statistical operator satisfying $\mathrm{Tr}(\rho P_S) = \mu(S)$ for all μ,[63] which can be established as follows. We can first suppose, with little loss of generality, that w_1 is larger than any of the other eigenvalues of the statistical operators

$$\rho = \sum_j w_j |\sigma_j\rangle\langle\sigma_j|, \quad \rho' = \sum_j w'_j |\sigma'_j\rangle\langle\sigma'_j|,$$

where $\langle\sigma_m|\sigma_n\rangle = \langle\sigma'_m|\sigma'_n\rangle = \delta_{mn}$. We will use the projector $|\sigma_1\rangle\langle\sigma_1|$ to tell ρ and ρ' apart, and show that $\mathrm{Tr}\big(\rho'|\sigma_1\rangle\langle\sigma_1|\big)$ is less than

$$w_1 = \mathrm{Tr}\big(\rho|\sigma_1\rangle\langle\sigma_1|\big).$$

The value

$$\mathrm{Tr}\big(\rho'|\sigma_1\rangle\langle\sigma_1|\big) = \sum_j w'_j \mathrm{Tr}\big(|\sigma'_j\rangle\langle\sigma'_j|\sigma_1\rangle\langle\sigma_1|\big) = \sum_j w'_j |\langle\sigma'_j|\sigma_1\rangle|^2$$

reaches its maximum when $|\sigma'_1\rangle\langle\sigma'_1| = |\sigma_1\rangle\langle\sigma_1|$, for then the total of

$$\sum_j |\langle\sigma'_j|\sigma_1\rangle|^2 = 1$$

is all concentrated on the largest eigenvalue w'_1. So $\mathrm{Tr}\big(\rho'|\sigma_1\rangle\langle\sigma_1|\big)$ cannot exceed w'_1, which is less than $\mathrm{Tr}\big(\rho|\sigma_1\rangle\langle\sigma_1|\big)$, and we have told ρ apart from ρ' in this case.

More generally the $k-1$ largest eigenvalues of ρ and ρ' may be the same. Suppose w_k is the largest of the remaining ones, and that the eigenvalues are in descending order, so that $w_k > w_{k+1}$ for all k. If P_k is the rank-k projector

$$\sum_{m=1}^{m=k} |\sigma_m\rangle\langle\sigma_m|,$$

$\mathrm{Tr}(\rho P_k)$ will be equal to

$$w = \sum_{m=1}^{m=k} w_m.$$

Again, $\mathrm{Tr}(\rho' P_k)$ is less than w, which we can see by writing

$$\mathrm{Tr}(\rho' P_k) = \sum_j w'_j \mathrm{Tr}\big(|\sigma'_j\rangle\langle\sigma'_j|P_k\big) = \sum_j \sum_{m=1}^{m=k} w'_j |\langle\sigma'_j|\sigma_m\rangle|^2.$$

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[63] This is sometimes considered part of Gleason's theorem—see for instance Varadarajan (1968) volume I, p.159—though Gleason (1957) only said that "*There exists a positive semi-definite self-adjoint operator T of the trace class such that for all closed subspaces A of $\mathfrak{H}$*

$$\mu(A) = \mathrm{trace}\,(TP_A)$$

where P_A is the orthogonal projection of $\mathfrak{H}$ onto A."

This will reach its maximum when the rank-k projector

$$P'_k = \sum_{m=1}^{m=k} |\sigma'_m\rangle\langle\sigma'_m|$$

is in fact P_k, in which case $\mathrm{Tr}(\rho' P_k)$ is equal to

$$w' = \sum_{m=1}^{m=k} w'_m.$$

As $w > w'$, the statistical operators ρ and ρ' can always be told apart.

2.3 Another decomposition

We saw in Section II. 2.1 that if

$$|\Psi\rangle = \sum_k c_k |\varphi_k\rangle, \qquad \eta = \sum_k |c_k|^2 |\varphi_k\rangle\langle\varphi_k|,$$

$$\mathsf{U} = \sum_k e^{i\theta_k} |\varphi_k\rangle\langle\varphi_k|, \qquad A = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|,$$

then

$$\mathrm{Tr}\big(|\Psi\rangle\langle\Psi|A\big) = \mathrm{Tr}\big(\mathsf{U}|\Psi\rangle\langle\Psi|\mathsf{U}^\dagger A\big) = \mathrm{Tr}(\eta A).$$

Measurement with respect to $\{|\varphi_k\rangle\}$ will not be affected, therefore, by the transformation or removal of phase with respect to the same basis $\{|\varphi_k\rangle\}$. But if we appeal to an operator

$$B = \sum_m \mu_m |\xi_m\rangle\langle\xi_m|$$

which is incompatible with A, we could have

$$\mathrm{Tr}\big(|\Psi\rangle\langle\Psi|B\big) \neq \mathrm{Tr}\big(\mathsf{U}|\Psi\rangle\langle\Psi|\mathsf{U}^\dagger B\big) \neq \mathrm{Tr}(\eta B) \neq \mathrm{Tr}\big(|\Psi\rangle\langle\Psi|B\big).$$

In Section III. 1.3 we will see that if the basis $\{|\varphi_k\rangle\}$ is made up of products

$$|\varphi_k\rangle = \bigotimes_{\sigma=1}^{N} |\varphi_k^\sigma\rangle,$$

it will be of no use for seeing quantum waves in configuration space. A basis of non-factorizable vectors will be required to bring out 'paradoxical' interference effects in configuration space.

Let us now look at time evolution. Suppose B does not commute with the Hamiltonian, and hence does not represent a conserved quantity. We then have the expansion

$$|\Psi\rangle = \sum_k b_k |\xi_k\rangle,$$

and the time evolution

$$\begin{aligned}|\Psi(t)\rangle &= \sum_k e^{-iE_k t/\hbar} |\zeta_k\rangle\langle\zeta_k| \Big(\sum_j b_j |\xi_j\rangle\Big) \\ &= \sum_{jk} e^{-iE_k t/\hbar} b_j |\zeta_k\rangle\langle\zeta_k|\xi_j\rangle.\end{aligned}$$

As $\langle\zeta_k|\xi_j\rangle$ does not vanish (for any values of the indices), cross-terms survive, and beats are produced. Whereas the group $e^{-iHt/\hbar}$ left each energy eigenstate $|\zeta_k\rangle$ in its own ray, it will make the vectors $|\xi_j\rangle$ wobble.

Suppose, for definiteness, the $|\xi_x\rangle$'s are 'position' eigenstates (even if B has a discrete spectrum), so that $\zeta_k(x) = \langle\xi_x|\zeta_k\rangle$ is the kth energy eigenfunction. At any point x_0, the complex numbers

$$e^{-iE_1 t/\hbar}\zeta_1(x_0),\ e^{-iE_2 t/\hbar}\zeta_2(x_0),\ \ldots$$

are superposed. As these rotate at different rates, beats will be produced in this 'position' space. Here therefore

$$\frac{d}{dt}\mathrm{Tr}\big(|\Psi(t)\rangle\langle\Psi(t)|B\big) \neq 0.$$

The average $\mathrm{Tr}\big(e^{-iHt/\hbar}\alpha e^{iHt/\hbar}B\big)$ on the other hand is invariant under $e^{-iHt/\hbar}$, because $e^{-iHt/\hbar}$ and α commute. So the evolutions of $e^{-iHt/\hbar}\alpha e^{iHt/\hbar}$ and $|\Psi(t)\rangle$, which are indistinguishable with respect to an observable compatible with the Hamiltonian, can be told apart by B.

III. Composite systems: interference, additive conservation

1
Interference in configuration space

Now that interference has been looked at in general, it is dealt with in cases involving composite systems. Entanglement is first defined, then shown to be always statistically visible: for every non-factorizable vector $|\Psi\rangle$, there exists an observable sensitive to quantum waves in configuration space, which distinguishes between $|\Psi\rangle$ and any statistical operator with factorizable eigenvectors. The observable used by Bell to violate his inequality is an example.

Such 'sensitive' observables, which are behind many quantum-mechanical paradoxes, can assume all sorts of forms, in fact many more than can be gone through exhaustively. One can, however, consider forms they *cannot* have, in other words certain sufficient conditions for 'indifference' to quantum waves in configuration space. Observables represented by tensor products of operators, or even functions of products, are indifferent: for any such observable A and any non-factorizable vector $|\Psi\rangle$, there exists a statistical operator ρ, whose eigenvectors are factorizable, which has the same average for A as $|\Psi\rangle$.

1.1 Entangled states

If the states of quantum-mechanical subsystems Σ^1 and Σ^2 are described by vectors belonging respectively to Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$, the product $|\alpha^1\rangle \otimes |\alpha^2\rangle$—which can also be written $|\alpha^1 \otimes \alpha^2\rangle$, $|\alpha^1\rangle|\alpha^2\rangle$ or $|\alpha^1\alpha^2\rangle$—indicates that Σ^1 is in state $|\alpha^1\rangle \in \mathcal{H}^1$ and Σ^2 in state $|\alpha^2\rangle \in \mathcal{H}^2$. The set of all such products, the Cartesian product $\mathcal{H}^1 \times \mathcal{H}^2$, is not closed under superposition: sums of products cannot, as we shall see, always be expressed as products. In quantum mechanics, however, the principle of superposition *is extended to products*, which are taken to be vectors of a linear space.

If $\{|\alpha^1_\mu\rangle\} \subset \mathcal{H}^1$ and $\{|\alpha^2_\nu\rangle\} \subset \mathcal{H}^2$ are complete orthonormal systems, $\overline{\mathrm{span}\{|\alpha^1_\mu\rangle|\alpha^2_\nu\rangle\}}$ (the bar denotes closure of the span) is another Hilbert space, which we can call the tensor product[64] $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ of the spaces $\mathcal{H}^1$ and $\mathcal{H}^2$. Vectors of $\mathcal{H}$, even those not in $\mathcal{H}^1 \times \mathcal{H}^2$, then represent states of the composite system $\Sigma = \Sigma^1 + \Sigma^2$. We can write $|\gamma_\kappa\rangle = |\gamma_{\mu\nu}\rangle = |\alpha^1_\mu\rangle|\alpha^2_\nu\rangle$ (where a value of the index κ stands for a μ, ν pair) to 'hide' the tensor product. What was said in II. 2 about interference between generic states, which may or may not be products, therefore applies to composite systems as well.

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[64]See Hughes (1989) 148-9, Redhead (1987) pp.174-5, Isham (1995) pp.143-7 and Jauch (1968) pp.175-8, 273-4.

The product $|\Gamma\rangle = |\alpha^1\rangle|\alpha^2\rangle$ can be written

$$|\Gamma\rangle = \left(\sum_\mu a_\mu|\alpha^1_\mu\rangle\right) \otimes \left(\sum_\nu b_\nu|\alpha^2_\nu\rangle\right)$$

where $a^1_\mu = \langle\alpha^1_\mu|\alpha^1\rangle$ and $a^2_\nu = \langle\alpha^2_\nu|\alpha^2\rangle$, μ, $\nu = 1, 2, \ldots$. We can also write

$$|\Gamma\rangle = \sum_{\mu\nu} a_\mu b_\nu|\alpha^1_\mu\rangle|\alpha^2_\nu\rangle = \sum_{\mu\nu} g_{\mu\nu}|\alpha^1_\mu\rangle|\alpha^2_\nu\rangle.$$

If, however, we begin with an arbitrary vector

$$|\Delta\rangle = \sum_{\mu\nu} c_{\mu\nu}|\alpha^1_\mu\rangle|\alpha^2_\nu\rangle,$$

we will not always be able to find coefficients a_μ and b_ν whose product $a_\mu b_\nu$ is equal to $c_{\mu\nu}$. To see this, take the basis

$$\{|\uparrow\rangle|\uparrow\rangle, |\uparrow\rangle|\downarrow\rangle, |\downarrow\rangle|\uparrow\rangle, |\downarrow\rangle|\downarrow\rangle\},$$

with respect to which a superposition will be of the form

$$c_{\uparrow\uparrow}|\uparrow\rangle|\uparrow\rangle + c_{\uparrow\downarrow}|\uparrow\rangle|\downarrow\rangle + c_{\downarrow\uparrow}|\downarrow\rangle|\uparrow\rangle + c_{\downarrow\downarrow}|\downarrow\rangle|\downarrow\rangle.$$

The vector

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$

for instance, is not a product of vectors in the factor spaces. We can proceed by *reductio ad absurdum*, supposing there exist coefficients $a_\uparrow$, $a_\downarrow$, $b_\uparrow$, $b_\downarrow$ such that

$$a_\uparrow b_\downarrow = -a_\downarrow b_\uparrow = \frac{1}{\sqrt{2}}.$$

As the products $a_\uparrow b_\uparrow$ and $a_\downarrow b_\downarrow$ vanish, so must the product $a_\uparrow a_\downarrow b_\uparrow b_\downarrow$, and hence $a_\uparrow b_\downarrow$ or $a_\downarrow b_\uparrow$ as well. Since this is not so, $|\Sigma\rangle$ cannot be expressed as a product.

So it will not always be possible to express a vector

$$|\Omega\rangle = \sum_{\mu^1\ldots\mu^N} c_{\mu^1\ldots\mu^N} \bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle \in \bigotimes_{\sigma=1}^{N} \mathcal{H}^\sigma,$$

in the form

$$\bigotimes_{\sigma=1}^{N} |\Omega^\sigma\rangle = \sum_{\mu^1\ldots\mu^N} \prod_{\sigma=1}^{N} a^\sigma_{\mu^\sigma} \bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle \in \mathcal{H}^1 \times \cdots \times \mathcal{H}^N,$$

where $\{|\varphi^\sigma_1\rangle, |\varphi^\sigma_2\rangle, \ldots\}$ is a complete orthonormal set in the space $\mathcal{H}^\sigma$ and

$$|\Omega^\sigma\rangle = \sum_{\mu^\sigma} a^\sigma_{\mu^\sigma}|\varphi^\sigma_{\mu^\sigma}\rangle,$$

$\sigma = 1, \ldots, N$.

A state represented by a vector that cannot be written as a product is called 'entangled.'[65]

1.2 Sensitive observables

We have seen Schrödinger's reaction to quantum waves in configuration space:

> Il va de soi que cet emploi de l'espace q ne doit être considéré que comme un artifice mathématique ... en dernière analyse on décrira ici aussi un événement dans l'espace et dans le temps.[66]

It is not unnatural to view the configuration space description as a mathematical accident which somehow describes a process in ordinary space, and perhaps to hope that the propagation of quantum waves in configuration space can be re-expressed in ordinary space. The hope was entertained by de Broglie (1956a):

> Un point important est la justification de la formule du guidage et de la signification statistique de l'onde Ψ dans les cas des systèmes de corpuscules en interaction, cas où l'onde Ψ considérée par la Mécanique ondulatoire usuelle est censée se propager dans l'espace de configuration, espace visiblement fictif. Du point de vue causal de la double solution, il faut démontrer que formule du guidage et interprétation statistique du Ψ résultent des interactions entre les régions singulières d'ondes du type u évoluant dans l'espace physique à trois dimensions. Dans mon article du *Journal de Physique* de mai 1927, j'avais esquissé une démonstration de ce genre *en considérant l'espace de configuration comme formé par les coordonnées des singularités*. Je parvenais ainsi à une représentation du mouvement des corpuscules en interaction comme s'accomplissant dans l'espace physique sans avoir nécessairement à faire appel à l'espace de configuration. Cet espace fictif et la propagation de l'onde Ψ dans cet espace seraient seulement des artifices de calcul commodes pour les prévisions statistiques.[67]

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[65]'Entangled' is apparently—see Bergia and Cannata (1991)—a translation of *verschränkt*, first used by Schrödinger (1935b). See also Schrödinger (1935a).

[66]*Electrons et photons*. Translation: It goes without saying that this use of q-space has to be viewed as a mathematical artifice ... ultimately an event in space and time will be described here too.

[67]An important point is the justification of the guidance formula and of the statistical significance of the Ψ-wave in cases of systems of corpuscles in interaction, in which the Ψ-wave considered by the usual Wave mechanics is supposed to propagate in configuration space, which is a manifestly fictitious space. From the causal point of view of the double solution, it has to be demonstrated that the guidance formula and the statistical interpretation of Ψ result from the interactions between the singular regions of u-type waves evolving in three-dimensional physical space. In my article in the *Journal de Physique* of May 1927, I had sketched a demonstration of this kind *by viewing configuration space as being formed by the coordinates of the singularities*. I thus arrived at a representation of the motion of corpuscles in interaction as taking place in physical space without necessarily having to appeal to configuration space. This fictitious space and the propagation of the Ψ-wave in this space would only be computational artifices which are convenient for statistical predictions.

In other words he believed that configuration space had no real ontological or even statistical significance; since it was just used, much as in analytical mechanics, for reasons of convenience or theoretical economy—*seulement des artifices de calcul commodes*—it could be dispensed with altogether.

> Tout phénomène réel pouvant se représenter dans le cadre de l'espace et du temps, il n'est pas admissible que l'on ne puisse traiter le problème de N corpuscules en interaction qu'en considérant une propagation d'ondes dans l'espace de configuration du système, espace visiblement fictif. On doit donc pouvoir poser ce problème, et même en principe le résoudre, en considérant la propagation dans l'espace physique à trois dimensions de N ondes à singularité s'influençant mutuellement. Mais on devra ensuite pouvoir démontrer que le résultat statistique des interactions est exactement donné par la considération de l'onde Ψ du système dans l'espace de configuration qui, n'étant qu'une représentation de probabilité, peut, elle, n'être représentable que dans un cadre fictif.[68]

We will now see that the configuration space description is statistically indispensable and not just convenient, at least in principle. It is clear that N wave-descriptions in ordinary space (or in N copies of ordinary space) cannot express certain correlations between the subsystems. There are cases, for example, in which description ζ^1 of the first subsystem is associated with description ζ^2 of the second, and η^1 is similarly associated with η^2. But then the conjunction $\zeta^1\&\zeta^2$, for instance, will not do as a description of the composite system, as $\eta^1\&\eta^2$ gets left out. So linear combination of some kind is necessary. Superposition would take us back to quantum waves in configuration space. Convex combination or 'mixture' is also possible, however, and may or may not be equivalent to superposition.

We have seen[69] that certain observables cannot tell the difference between the mixture represented by

$$\rho = \sum_{\kappa} |c_{\kappa}|^2 |\gamma_{\kappa}\rangle\langle\gamma_{\kappa}|$$

and the superposition

$$|\Lambda\rangle = \sum_{\kappa} c_{\kappa} |\gamma_{\kappa}\rangle,$$

[68]de Broglie (1956a). Translation: As every real phenomenon can be represented in space and time, it is unacceptable that the problem of N corpuscles in interaction can only be treated by considering a propagation of waves in the configuration space of the system, which space is manifestly fictitious. One must thus be able to pose the problem, and even in principle to resolve it, by considering the propagation in three-dimensional physical space of N waves with singularities, which influence each other mutually. But one must then be able to demonstrate that the statistical result of the interactions is exactly given by the consideration of the Ψ-wave of the system in configuration space, which, being no more than a representation of probability, can only be represented in a fictitious framework.

[69]Section II. 2.1, equation (13)

in other words that self-adjoint operators A can be found such that $\mathrm{Tr}(\rho A) = \mathrm{Tr}\big(P_{[\Lambda]}A\big)$. If the eigenvectors $|\gamma_\kappa\rangle = |\alpha^1_\mu \otimes \alpha^2_\nu\rangle$ are products, so that

$$\rho = \sum_\kappa |c_\kappa|^2 |\gamma_\kappa\rangle\langle\gamma_\kappa| = \sum_{\mu\nu} |c_{\mu\nu}|^2 |\alpha^1_\mu \otimes \alpha^2_\nu\rangle\langle\alpha^1_\mu \otimes \alpha^2_\nu|,$$

and

$$|\Lambda\rangle = \sum_\kappa c_\kappa |\gamma_\kappa\rangle = \sum_{\mu\nu} c_{\mu\nu} |\alpha^1_\mu \otimes \alpha^2_\nu\rangle,$$

A is called 'indifferent' by Capasso, Fortunato and Selleri (1973), because the arguments of the coefficients $c_{\mu\nu}$ make no difference to it.[70] We know from Section II. 2.2, however, that an observable capable of telling $P_{[\Lambda]}$ apart from any other statistical operator must exist. So for every non-factorizable vector $|\Lambda\rangle$ there will be an observable B such that $\mathrm{Tr}(\rho B) \neq \mathrm{Tr}(P_{[\Lambda]}B)$ for every statistical operator ρ that can be given a complete set of factorizable eigenvectors; such an observable is called 'sensitive' by Capasso *et al.* (1973).[71] An example is $B = P_{[\Lambda]}$, for the maximum $\mathrm{Tr}\big(P^2_{[\Lambda]}\big) = 1$ cannot be reached by any other statistical operator. To see this consider the average $\mathrm{Tr}(P_{[\Lambda]}\upsilon)$ for the statistical operator

$$\upsilon = \sum_\mu w_\mu |\kappa_\mu\rangle\langle\kappa_\mu|.$$

As

$$\mathrm{Tr}(P_{[\Lambda]}\upsilon) = \sum_\mu w_\mu \mathrm{Tr}\big(P_{[\Lambda]}|\kappa_\mu\rangle\langle\kappa_\mu|\big) = \sum_\mu w_\mu \|P_{[\Lambda]}|\kappa_\mu\rangle\|^2$$

will be equal to unity only if there is just one term in the sum and $\upsilon = P_{[\Lambda]}$, the state described by $P_{[\Lambda]}$ can be told apart from any mixture of products.

In undulatory terms this means that a wave propagating irreducibly in configuration space can always be told apart from a mixture of waves in ordinary space; phase relations are always significant, even in configuration space.

1.3 Compatibility and indifference

Now it has been established that the propagation of quantum waves in configuration space is always, in principle, statistically visible, one can wonder which observables reveal it. I can clearly not go through the whole set of sensitive observables, but can at least point out that none of them are represented by tensor products, or functions

[70] See also Redhead (1987), who calls the observables 'insensitive.'
[71] See Fortunato and Selleri (1975) and Fortunato (1976).

$$A' = f\Big(\bigotimes_\sigma A^\sigma\Big)$$

thereof. The tensor product

$$\begin{aligned} A = A^1 \otimes A^2 &= \Big(\sum_m \lambda_m^1 |\alpha_m^1\rangle\langle\alpha_m^1|\Big) \otimes \Big(\sum_n \lambda_n^2 |\alpha_n^2\rangle\langle\alpha_n^2|\Big) \\ &= \sum_{mn} \lambda_m^1 \lambda_n^2 |\alpha_m^1 \otimes \alpha_n^2\rangle\langle\alpha_m^1 \otimes \alpha_n^2| = \sum_\kappa \lambda_\kappa |\gamma_\kappa\rangle\langle\gamma_\kappa|, \end{aligned}$$

for instance, will be indifferent:

$$\mathrm{Tr}\big(P_{[A]}A\big) = \sum_\kappa \lambda_\kappa |c_\kappa|^2 = \mathrm{Tr}(\rho A),$$

where again

$$\rho = \sum_\kappa |c_\kappa|^2 |\gamma_\kappa\rangle\langle\gamma_\kappa| = \sum_{\mu\nu} |c_{\mu\nu}|^2 |\alpha_\mu^1 \otimes \alpha_\nu^2\rangle\langle\alpha_\mu^1 \otimes \alpha_\nu^2|$$

and

$$|A\rangle = \sum_\kappa c_\kappa |\gamma_\kappa\rangle = \sum_{\mu\nu} c_{\mu\nu} |\alpha_\mu^1 \otimes \alpha_\nu^2\rangle.$$

In fact any real function

$$A' = f(A) = \sum_\kappa f(\lambda_\kappa) |\gamma_\kappa\rangle\langle\gamma_\kappa|$$

of A (indeed any self-adjoint operator that commutes with A) can also be given a complete set of factorizable eigenvectors, and will be indifferent as well:

$$\mathrm{Tr}\big(P_{[A]}A'\big) = \sum_\kappa f(\lambda_\kappa) |c_\kappa|^2 = \mathrm{Tr}(\rho A').$$

The operator $A + C$, for instance, will be indifferent if A and C commute, for then A, C and $A + C$ can all be expressed as functions of a factorizable operator.[72]

A bounded self-adjoint operator $B = B^1 \otimes B^2$ that does not have eigenvectors in Hilbert space can always be approximated by one that does.[73] Represent each factor B^σ in the space $\mathcal{L}_2^\sigma(\mathbb{R}, \mathbb{C}) = U^\sigma \mathcal{H}^\sigma$ in which it becomes the multiplication operator $M_{\lambda^\sigma} = U^\sigma B^\sigma U^{\dagger\sigma}$ $(\sigma = 1, 2)$. Any partition $\{\Delta_1^\sigma, \Delta_2^\sigma, \ldots\}$ of the reals determines a resolution of the identity with projectors $P_k^\sigma = U^{\dagger\sigma} \chi_{\Delta_k^\sigma}(\lambda^\sigma) U^\sigma$, and breaks $\mathcal{H}^\sigma$ up correspondingly into mutually orthogonal subspaces $\mathcal{H}_k^\sigma = P_k^\sigma \mathcal{H}^\sigma$ such that

[72] Capasso *et al.* (1973)

[73] The procedure followed here was suggested to me by Sebastiano Carpi.

$$\mathcal{H}^\sigma = \bigoplus_k \mathcal{H}_k^\sigma.$$

For every set Δ_k^σ choose a point $\lambda_k^\sigma \in \Delta_k^\sigma$ and construct the self-adjoint operator

$$B^{\sigma\prime} = \sum_k \lambda_k^\sigma P_k^\sigma.$$

The operator

$$\begin{aligned} B' &= \left(\sum_{mk} \lambda_k^1 |\eta_{(k)m}^1\rangle\langle\eta_{(k)m}^1|\right) \otimes \left(\sum_{nj} \lambda_j^2 |\eta_{(j)n}^2\rangle\langle\eta_{(j)n}^2|\right) \\ &= \sum_{mknj} \lambda_k^1 \lambda_j^2 |\eta_{(k)m}^1 \otimes \eta_{(j)n}^2\rangle\langle\eta_{(k)m}^1 \otimes \eta_{(j)n}^2| \end{aligned}$$

can then be constructed by choosing a basis $\{|\eta_{(k)1}^\sigma\rangle, |\eta_{(k)2}^\sigma\rangle, \dots\}$ for each eigenspace $\mathcal{H}_k^\sigma$. Expanding $|\Psi\rangle$ with respect to the basis $\{|\eta_{(k)m}^1 \otimes \eta_{(j)n}^2\rangle\}$ we obtain

$$|\Psi\rangle = \sum_{mknj} c_{mknj} |\eta_{(k)m}^1 \otimes \eta_{(j)n}^2\rangle$$

and the statistical operator

$$\rho = \sum_{mknj} |c_{mknj}|^2 |\eta_{(k)m}^1 \otimes \eta_{(j)n}^2\rangle\langle\eta_{(k)m}^1 \otimes \eta_{(j)n}^2|,$$

where $c_{mknj} = \langle\eta_{(k)m}^1 \otimes \eta_{(j)n}^2|\Psi\rangle$. The operators B' and ρ have been constructed so that $\mathrm{Tr}(\rho B') = \langle\Psi|B'|\Psi\rangle$; the difference $\mathrm{Tr}(\rho B') - \langle\Psi|B|\Psi\rangle$ and norm $\|B - B'\|$ can be made as small as one wishes by refining the partitions $\{\Delta_k^\sigma\}$. As the average $\langle\Psi|B|\Psi\rangle$ is approximated with arbitrary accuracy by the statistical operator ρ with factorizable eigenvectors, B is indifferent, despite having no eigenvectors.

So we have seen that quantum waves in configuration space can always be told apart from products of waves in the factor spaces, or even mixtures of them; but that functions of tensor products of operators are of no use for seeing quantum waves in configuration space.

Let us now leave interference and discuss the paradox of Einstein, Podolsky and Rosen, its generalization to N subsystems, and additive conservation. In Part IV interference will be combined with additive conservation in specific examples.

2
Additive conservation: EPR and generalizations

Though the worst problems in configuration space are produced by interference, it did not enter explicitly into the original argument of Einstein, Podolsky and Rosen. Perfect correlations associated with *biorthogonal expansions* were the central issue, as they allowed the identification of elements of reality relating to incompatible observables. I shall ignore interference in this chapter—having dealt with it in Part II and Chapter III. 1—and begin with Schmidt's theorem and the argument of Einstein *et al.*, which depends on the availability of two different biorthogonal expansions. In Bohm's and most subsequent versions of the argument the correlations can be attributed to additive conservation laws, defined in relation to biorthogonal expansions with respect to energy eigenvectors. With more than two subsystems there can be correlations too strong to be ascribed to additive conservation.

Although *waves*, or at least their manifestations, are being deliberately avoided in this chapter, at least we are still in configuration space. Besides which, additive conservation is related to quantum waves in configuration space by leading to 'classical' expectations at odds with interference in configuration space. Indeed the violation of Bell's inequality and a measurement problem considered in Section IV. 2.2 express a tension between the classical expectations derived from additive conservation, and quantum waves in configuration space.

In Part I the guidance of the particle was taken to indicate the physical reality of the wave; such guidance does not take place if the standard formalism of quantum mechanics is complete; but the incompleteness of that formalism is most convincingly established, by the argument of Einstein *et al.*, in a *waveless* configuration space context. Paradoxically the reality of quantum waves in ordinary space is partly indicated by the very trajectories whose existence depends on an incompleteness most easily seen in the absence of interference.

2.1 Schmidt's theorem

By establishing a one-to-one correspondence between bases of the two tensor factor spaces involved, Schmidt's theorem[74] allows an application of the EPR reality criterion, presently to be considered. The theorem states that a biorthogonal expansion is always

[74]Schmidt (1907). See also von Neumann (1932) p.228-31, Schrödinger (1935a), Bergia (1993), Bergia and Cannata (1991) and Peres (1995a).

possible. For any vector $|\Delta\rangle \in \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$, in other words, there will be complete orthonormal sets $\{|\psi^1_\mu\rangle\} \subset \mathcal{H}^1$ and $\{|\psi^2_\mu\rangle\} \subset \mathcal{H}^2$ with respect to which $|\Delta\rangle$ assumes the form

$$|\Delta\rangle = \sum_\mu a_\mu |\psi^1_\mu\rangle \otimes |\psi^2_\mu\rangle. \tag{16}$$

If there are coincidences between the moduli of coefficients in the expansion, other such expansions will be available.

When the system is given the expansion (16), the statistical operators ρ^1 and ρ^2 describing the subsystems[75] have the same spectrum $\{|a_\mu|^2\}$, and their eigenvectors will be $\{|\psi^1_\mu\rangle\}$ and $\{|\psi^2_\mu\rangle\}$:

$$\rho^1|\psi^1_\mu\rangle = |a_\mu|^2|\psi^1_\mu\rangle, \quad \rho^2|\psi^2_\mu\rangle = |a_\mu|^2|\psi^2_\mu\rangle.$$

If the spectrum $\{|a_\mu|^2\}$ is simple, the eigenvectors, and hence the biorthogonal expansion, will be unique. Otherwise there will be freedom in the choice of bases for the multidimensional eigenspaces of ρ^1 and ρ^2.

In finite dimensions we can proceed as follows. Beginning with the arbitrary expansion

$$|\Delta\rangle = \sum_{\mu\nu} c_{\mu\nu} |\alpha^1_\mu\rangle \otimes |\alpha^2_\nu\rangle,$$

we can form the self-adjoint and unitary equivalent matrices

$$\rho^1_{lr} = \sum_j c^*_{jl} c_{jr} = \langle\alpha^1_l|\rho^1|\alpha^1_r\rangle = \langle\Delta|\big(|\alpha^1_l\rangle\langle\alpha^1_r| \otimes I\big)|\Delta\rangle$$
$$\rho^2_{ik} = \sum_j c_{ij} c^*_{kj} = \langle\alpha^2_i|\rho^2|\alpha^2_k\rangle = \langle\Delta|\big(I \otimes |\alpha^2_i\rangle\langle\alpha^2_k|\big)|\Delta\rangle$$

with spectrum $\{|a_\mu|^2\}$. The eigenvectors $\{|\psi^1_\mu\rangle\}$ and $\{|\psi^2_\mu\rangle\}$ of ρ^1 and ρ^2 will diagonalize the representations ρ^1_{lr} and ρ^2_{ik}.

2.2 "Can quantum-mechanical description of reality be considered complete?"

Two different biorthogonal expansions are involved in the argument of Einstein, Podolsky and Rosen (1935), who use the following 'reality criterion' to establish that quantum theory is *incomplete*.

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[75] I will go into the description of subsystems by means of statistical operators in greater detail in Section III. 2.3.

> Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory.* ... The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. ... *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to that physical quantity.*[76]

If a system is in state ψ and $A\psi = a\psi$, "there is an element of physical reality corresponding to the physical quantity A" because the outcome a can be predicted with certainty. As long as the system remains in state ψ, the criterion of Einstein *et al.* can only be used to attribute reality to physical quantities compatible with A; the outcome of measuring B, for instance, cannot be predicted with certainty. The criterion is, however, only sufficient, *and not necessary* for the identification of an element of reality.

> It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.[77]

If it were necessary as well, we might conclude that *when the operators corresponding to two physical quantities do not commute the quantities cannot have simultaneous reality*. As it is not, there could be elements of reality corresponding to both physical quantities, only one of which can be revealed at a time, and the theory would be incomplete.

This can be formalized as follows. Let $\mathfrak{E} = \{\eta\}$ be the set of elements η of the reality of a system $\mathbf{S}$ described by theory $\mathfrak{T}$. To every η there must correspond a 'counterpart' $\kappa(\eta)$ in $\mathfrak{T}$, if $\mathfrak{T}$ is complete. In other words

$$\mathfrak{T} \text{ is complete} \Rightarrow [\forall\eta \in \mathfrak{E} \;\; \exists\kappa(\eta) \in \mathfrak{T}].$$

If we can establish that there exists an element $\eta' \in \mathfrak{E}$ such that $\kappa^{-1}(\eta')$ is empty, then $\mathfrak{T}$ must be incomplete:

$$\left[\exists\eta' \in \mathfrak{E} : \kappa^{-1}(\eta') = \emptyset\right] \Rightarrow \mathfrak{T} \text{ is incomplete.}$$

With a single system $\mathbf{S}$ that cannot be broken up into subsystems, the reality criterion cannot be used to establish that quantum theory is incomplete. A measurement of one physical quantity A on $\mathbf{S}$ cannot determine the outcome of the measurement of another physical quantity B incompatible with A. One can always say that at a given time only one of the incompatible elements of reality can figure in the description of $\mathbf{S}$, which is indeed complete.

---

[76]Einstein, Podolsky and Rosen (1935)
[77]Einstein *et al.* (1935)

So Einstein *et al.* consider a *composite* system $\mathbf{S}'$ made up of two subsystems I and II. Suppose $\mathbf{S}'$ is in state Ψ, which can be given the following two expansions:

$$\Psi(x_1, x_2) = \sum_{n=1}^{\infty} \psi_n(x_2) u_n(x_1) = \sum_{s=1}^{\infty} \varphi_s(x_2) v_s(x_1).$$

"Let $a_1, a_2, a_3, \cdots$ be the eigenvalues of some physical quantity A pertaining to system I and $u_1(x_1), u_2(x_1), u_3(x_1), \cdots$ the corresponding eigenfunctions, where x_1 stands for the variables used to describe the first system." The functions $v_s(x_1)$ are likewise eigenfunctions belonging to eigenvalues $b_1, b_2, b_3, \cdots$ of physical quantity B pertaining to system I, and "$\psi_n(x_2)$ are to be regarded merely as the coefficients of the expansion of Ψ into a series of orthogonal functions $u_n(x_1)$." The same applies to the $\varphi_s(x_2)$'s, which, like the $\psi_n(x_2)$'s, are neither necessarily normalized nor orthogonal. We will assume them *to be* orthogonal—Schmidt's theorem tells us we can—so they are eigenfunctions of physical quantities, say P and Q, pertaining to the second system. If A and B do not commute, nor will P and Q. Summarizing, we have the following eigenvalue relations:

$$\begin{aligned}
&Au_n(x_1) = a_n u_n(x_1) \\
&Bv_s(x_1) = b_s v_s(x_1) \\
&P\psi_n(x_2) = p_n \psi_n(x_2) \\
&Q\varphi_s(x_2) = q_s \varphi_s(x_2),
\end{aligned}$$

$n, s = 1, 2, \ldots$. Two different biorthogonal expansions will be possible if, for instance, the lengths

$$\int \psi_n^*(x_2)\psi_n(x_2)dx_2 = \int u_n^*(x_1)u_n(x_1)dx_1 = 1$$

for all n.

If we now measure A and obtain the eigenvalue a_k corresponding to the eigenfunction $u_k(x_1)$, we can predict with certainty,[78] without in any way disturbing II, that a measurement of P would reveal the eigenvalue p_k corresponding to eigenfunction $\psi_n(x_2)$. So we can attribute the element of reality p_k to II. We can also say, measuring B and obtaining b_r, that q_r is another element of the reality of II. As p_k and q_r are eigenvalues of incompatible physical quantities, Einstein *et al.* conclude that quantum theory is incomplete, *q.e.d.*

'Locality' was hardly an issue, and was just assumed to hold. Einstein did, however, explicitly express his views on the matter elsewhere.

[78]"*Only*," as was pointed out to me by R. I. G. Hughes, "if we assume something equivalent to the Lüders Rule to act on the state of the complex system." To predict that the eigenvalue p_k will be revealed with certainty, the state of the system has to 'collapse' onto $\psi_k(x_2)u_k(x_1)$.

> Aber an *einer* Annahme sollten wir nach meiner Ansicht unbedingt festhalten: Der reale Sachverhalt (Zustand) es Systems S_2 ist unabhängig davon, was mit dem von ihm räumlich getrennten System S_1 vorgenommen wird.[79]

Or as he wrote to Born:

> Wesentlich für diese Einordnung der in der Physik eingeführten Dinge erscheint ferner, daß zu einer bestimmten Zeit diese Dinge eine voneinander unabhängige Existenz beanspruchen, soweit diese Dinge ›in verschiedenen Teilen des Raumes liegen‹. Ohne die Annahme einer solchen Unabhängigkeit der Existenz ... der räumlich distanten Dinge voneinander ... wäre physikalisches Denken in dem uns geläufigen Sinne nicht möglich. Man sieht ohne solche saubere Sonderung auch nicht, wie physikalische Gestze formuliert und geprüft werden könnten. ...
>
> Für die relative Unabhängigkeit räumlich distanter Dinge (A und B) ist die Idee charakteristisch: äußere Beeinflussung von A hat keinen unmittelbaren Einfluß auf B; dies ist als ›Prinzip der Nahewirkung‹ bekannt .... Völlige Aufhebung dieses Grundsatzes würde die Idee von der Existenz (quasi-) abgeschlossener Systeme und damit die Aufstellung empirisch prüfbarer Gesetze in dem uns geläufigen Sinne unmöglich machen.[80]

The state of B cannot depend on the kind of measurement performed on A:

> Da es nur *einen* physikalischen Zustand von B nach der Wechselwirkung geben kann, welcher vernünftigerweise nicht davon abhängig gedacht werden kann, was für Messungen ich an dem von B getrennten System A vornehme, zeigt dies, daß die ψ-Funktion dem physikalischen Zustande *nicht* eindeutig zugeordnet ist.[81]

Measurement on the second subsystem cannot influence *den realen Zustand* of the first through *einer Art unmittelbarer Kopplung räumlich getrennter Dinge*:

> Würde die ψ-Funktion den realen Zustand *vollständig* beschreiben, so würde dies bedeuten, dass die Messung am zweiten Teilsystem den realen Zustand der ersten beeinflusse, was einer Art unmittelbarer Kopplung räumlich getrennter Dinge entspräche. Auch dies wird man intuitiv für ausgeschlossen ansehen müssen.[82]

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[79]Einstein (1949a): Translation: But to *one* assumption we should in my opinion absolutely adhere: the real state of system S_2 is independent of what is done to the spatially separate system S_1.

[80]Einstein (1948). Translation: It furthermore seems essential for this arrangement of the things introduced into physics that at a given time these things claim an existence independent of one another, as long as these things 'are in different parts of space.' Without this assumption of such an independence of the existence ... of spatially distant things from one another ... physical thinking in the usual sense would not be possible. Nor does one see, without such a clean separation, how physical laws could be formulated and tested. ...

The following idea characterizes the relative independence of things (A and B) far apart in space: external influence on A has no immediate influence on B; this is known as the 'principle of close action' .... A complete abandonment of this fundamental principle would render the idea of the existence of (almost) isolated systems impossible, and with it the notion of laws that can be checked empirically in the usual sense.

[81]Einstein (1936). Translation: There can be only *one* physical state of B after the interaction, which cannot reasonably be thought of as depending on the kind of measurements I perform on system A, which is separated from B; this shows that ψ-functions do not correspond bijectively to physical states.

[82]Einstein (1953). Translation: If the ψ-function described the real state *completely*, the measurement on the second subsystem would have to influence the real state of the first, which would imply a kind of

We can now adapt the argument of Einstein, Podolsky and Rosen in two ways. To begin with, quantities A, B, P, Q are not necessarily conserved—Einstein *et al.* mention *position*, for instance—but it will be useful to consider quantities that are. We can also generalize the treatment to N subsystems, rather than just two.

2.3 Additive conserved quantities[83]

Pour a pint of water into three glasses. Once two-thirds of a pint have been found in the first glass, the possibility that the same amount had been poured into either of the others is eliminated *a posteriori*, as volume is conserved. The first glass clearly does not *tell* the others how much they should contain to be consistent; measurement merely reduces *ignorance* as to the distribution of water. This is like Bertlmann's socks: once the pink sock is found on one foot, we know it was not on the other even before. What occurs in quantum mechanics, at least when interference is not involved, is so similar it is hard to believe something fundamentally different is at issue. It is only because this picture—of an objective distribution preceding measurement—can be upset by quantum waves in configuration space that other explanations have been sought.

In the next two sections we want to explore this very picture, so there is no reason to upset it with quantum waves. We saw in Section III. 1.3 that an operator A' expressible as a function $A' = f(A)$ of a factorizable operator

$$A = \bigotimes_{\sigma=1}^{N} A^{\sigma}$$

is indifferent to quantum waves in configuration space, in the sense that it cannot tell an entangled state apart from a mixture of products. If A' is a conserved quantity, furthermore, no undulatory beats will be manifested in configuration space. In this section we will therefore adhere to a single factorizable basis of energy eigenvectors.

Suppose we have a quantity that is conserved in the N subsystems $\{\mathsf{S}^{\sigma}\}$, and hence in the composite system

$$\mathsf{S} = \sum_{\sigma=1}^{N} \mathsf{S}^{\sigma}.$$

In other words we have a basis $\{|\varphi_{\mu}^{\sigma}\rangle\}$ of energy eigenvectors in each factor space $\mathcal{H}^{\sigma}$ of the tensor product space

immediate coupling between spatially separate things. This too one must intuitively view as being excluded.

[83]See Afriat (1995a,b, 1996).

$$\mathcal{H} = \bigotimes_{\sigma=1}^{N} \mathcal{H}^{\sigma}$$

in which states of the system **S** are described. A situation reminiscent of the above glasses can then be constructed, in which measurement seems to *reveal* the distribution of the additive conserved quantity, and not to *create* it by means of transmissions between the subsystems. The probabilistic description would then appear to be incomplete, and to express our ignorance.

Quantum mechanics predicts that if there is an objective distribution of the additive conserved quantity that precedes measurement, the amount possessed by a subsystem can be changed by interference due to a remote 'rotation'; but we will see this in Chapter V. 1, when we discuss the violation of Bell's inequality.

Suppose the self-adjoint operators $\{A^{\sigma} : \mathcal{H}^{\sigma} \to \mathcal{H}^{\sigma}\}$ all represent the same quantity $\mathfrak{A}$ in subsystems $\{\mathbf{S}^{\sigma}\}$, $\sigma = 1, \ldots, N$. The operator[84]

$$A = \sum_{\sigma=1}^{N} \mathbf{A}^{\sigma},$$

where

$$\mathbf{A}^1 = A^1 \otimes I^2 \otimes \cdots \otimes I^N, \ldots, \mathbf{A}^N = I^1 \otimes \cdots \otimes I^{N-1} \otimes A^N,$$

will then represent the same quantity for the whole system. We can also write

$$\mathfrak{A} = \sum_{\sigma=1}^{N} \mathfrak{A}^{\sigma}.$$

To simplify, assume the operators $\{A^{\sigma}\}$ have pure and simple point spectra, so that

$$A^{\sigma} = \sum_{\mu} \alpha^{\sigma}_{\mu} |\varphi^{\sigma}_{\mu}\rangle\langle\varphi^{\sigma}_{\mu}|, \quad \mathbf{A}^{\sigma} = \sum_{\mu^{\sigma}} \alpha^{\sigma}_{\mu^{\sigma}} |\phi^{\sigma}_{\mu^{\sigma}}\rangle\langle\phi^{\sigma}_{\mu^{\sigma}}|,$$

where[85]

$$|\phi^1_{\mu^1}\rangle = |\varphi^1_{\mu^1}\rangle \otimes |\eta^2\rangle \otimes \cdots \otimes |\eta^N\rangle, \ldots, |\phi^N_{\mu^N}\rangle = |\eta^1\rangle \otimes \cdots \otimes |\eta^{N-1}\rangle \otimes |\varphi^N_{\mu^N}\rangle,$$

$\mu^{\sigma} = 1, 2, \ldots,$ and $\sigma = 1, \ldots, N$. So

$$A \bigotimes_{\sigma=1}^{N} |\varphi^{\sigma}_{\mu^{\sigma}}\rangle = \sum_{\sigma=1}^{N} \alpha^{\sigma}_{\mu^{\sigma}} \bigotimes_{\sigma=1}^{N} |\varphi^{\sigma}_{\mu^{\sigma}}\rangle,$$

where again $\mu^{\sigma} = 1, 2, \ldots$. If A also commutes with the Hamiltonian

[84]For the procedure followed here see Bergia (1993), Bergia and Cannata (1991).

[85]The vectors $\{|\eta^{\sigma}\rangle\}$ are unimportant, as they are only acted upon by the identities $\{I^{\sigma}\}$.

$$H = \sum_{\mu^1 \ldots \mu^N} E_{\mu^1 \ldots \mu^N} \left|\varphi^1_{\mu^1} \otimes \cdots \otimes \varphi^N_{\mu^N}\right\rangle\left\langle\varphi^1_{\mu^1} \otimes \cdots \otimes \varphi^N_{\mu^N}\right|$$

of the whole system **S**, we can say that it represents an *additive conserved* quantity.

A state $|\Psi\rangle$ will generally be in a superposition

$$|\Psi\rangle = \sum_{\mu^1 \ldots \mu^N} c_{\mu^1 \ldots \mu^N} \bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle$$

of eigenstates $\left|\varphi^1_{\mu^1}\right\rangle \otimes \cdots \otimes \left|\varphi^N_{\mu^N}\right\rangle$ of A, where

$$c_{\mu^1 \ldots \mu^N} = \left\langle\varphi^1_{\mu^1} \otimes \cdots \otimes \varphi^N_{\mu^N}\right|\Psi\rangle.$$

Suppose we measure $\mathfrak{A}$, find the eigenvalue α, and consequently turn $|\Psi\rangle$ into some vector[86]

$$|\Gamma\rangle = \sum_{\mu^1 \ldots \mu^N} c'_{\mu^1 \ldots \mu^N} \bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle$$

in the corresponding eigenspace $\Xi_\alpha = \{|\xi\rangle : A|\xi\rangle = \alpha|\xi\rangle\}$. As $\mathfrak{A}$ is additive,

$$\Xi_\alpha = \overline{\text{span}\left\{\bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle : \sum_{\sigma=1}^{N} \alpha^\sigma_{\mu^\sigma} = \alpha\right\}},$$

in other words $|\Gamma\rangle$ will be a superposition of products of vectors $\left|\varphi^\sigma_{\mu^\sigma}\right\rangle \in \mathcal{H}^\sigma$ belonging to eigenvalues $\alpha^\sigma_{\mu^\sigma}$ satisfying

$$\sum_{\sigma=1}^{N} \alpha^\sigma_{\mu^\sigma} = \alpha, \tag{17}$$

and of no others. Basic vectors

$$\left\{\bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle : \sum_{\sigma=1}^{N} \alpha^\sigma_{\mu^\sigma} \neq \alpha\right\}$$

violating (17) will not figure in the superposition. The entire Cartesian product

$$\Lambda = \bigtimes_{\sigma=1}^{N} \Lambda^\sigma = \left\{\left(\alpha^1_\nu, \ldots, \alpha^N_\mu\right)\right\}$$

of the spectra $\Lambda^\sigma = \{\alpha^\sigma_1, \alpha^\sigma_2, \ldots\}$, $\sigma = 1, \ldots, N$, will clearly not satisfy (17); only a part

[86] According to the Lüders Rule—see Lüders (1951), Hughes (1989)—$|\Gamma\rangle$ will be the projection (renormalized if necessary) of $|\Psi\rangle$ onto the subspace Ξ_α. Here it is enough that $|\Gamma\rangle \in \Xi_\alpha$.

$$\Lambda_\alpha = \left\{ \left(\alpha^1_{\mu^1}, \ldots, \alpha^N_{\mu^N}\right) : \sum_{\sigma=1}^{N} \alpha^\sigma_{\mu^\sigma} = \alpha \right\} \subset \Lambda$$

of it will.

Additive conservation therefore acts as a selection rule keeping

$$e^{-iHt/\hbar}|\Gamma\rangle = \sum_{\mu^1 \ldots \mu^N} e^{-iE_{\mu^1 \ldots \mu^N} t/\hbar} c'_{\mu^1 \ldots \mu^N} \bigotimes_{\sigma=1}^{N} |\varphi^\sigma_{\mu^\sigma}\rangle$$

out of the orthogonal complement $\Xi^\perp_\alpha$ for all t, by rendering elements of the difference $\Lambda - \Lambda_\alpha$ impossible, and causing the corresponding coefficients to vanish:

$$\sum_{\sigma=1}^{N} \alpha^\sigma_{\mu^\sigma} \neq \alpha_\alpha \text{ implies that } e^{-iE_{\mu^1 \ldots \mu^N} t/\hbar} c'_{\mu^1 \ldots \mu^N} \text{ vanishes for all } t.$$

If we now measure $\mathfrak{A}^1$, then $\mathfrak{A}^2$, $\mathfrak{A}^3$, and so forth, further restrictions will be imposed. The determination of α^1_{m}, by selecting the subspace

$$\Xi_{\alpha - \alpha^1_{\mathrm{m}}} = \overline{\mathrm{span}\left\{ |\varphi^1_{\mathrm{m}}\rangle \otimes |\varphi^2_{\mu^2}\rangle \otimes \cdots \otimes |\varphi^N_{\mu^N}\rangle : \sum_{\sigma=2}^{N} \alpha^\sigma_{\mu^\sigma} = \alpha - \alpha^1_{\mathrm{m}} \right\}}$$

and the subset

$$\Lambda_{\alpha - \alpha^1_{\mathrm{m}}} = \left\{ \left(\alpha^1_{\mu^1}, \ldots, \alpha^N_{\mu^N}\right) : \sum_{\sigma=2}^{N} \alpha^\sigma_{\mu^\sigma} = \alpha - \alpha^1_{\mathrm{m}} \right\} \subset \Lambda_\alpha,$$

eliminates the difference

$$\Lambda_\alpha - \Lambda_{\alpha - \alpha^1_{\mathrm{m}}} = \left\{ \left(\alpha^1_{\mu^1}, \ldots, \alpha^N_{\mu^N}\right) : \sum_{\sigma=2}^{N} \alpha^\sigma_{\mu^\sigma} \neq \alpha - \alpha^1_{\mathrm{m}} \right\},$$

and the orthogonal complement $\Xi^\perp_{\alpha - \alpha^1_{\mathrm{m}}}$. If we now assume A to be positive semi-definite ($\alpha^\sigma_\mu \geq 0$ for all σ, μ), individual eigenvalues and eigenvectors, namely

$$\left\{ \alpha^\sigma_{\mu^\sigma} : \alpha^\sigma_{\mu^\sigma} + \alpha^1_\nu > \alpha \right\} \text{ and } \left\{ |\varphi^\sigma_{\mu^\sigma}\rangle : \alpha^\sigma_{\mu^\sigma} + \alpha^1_\nu > \alpha \right\},$$

can be eliminated by the selection of α^1_{m}. The procedure continues until a complete determination is achieved with a measurement of $\mathfrak{A}^{N-1}$, which reduces the state of $\mathbf{S}^{N-1} + \mathbf{S}^N$ to some product $|\varphi^{N-1}_\eta\rangle \otimes |\varphi^N_\beta\rangle$.

Simplifying, suppose there are just four subsystems $\mathbf{S}^1, \ldots, \mathbf{S}^4$, which have, together, a total of 12 units of the conserved quantity represented by

$$T = \mathbf{T}^1 + \mathbf{T}^2 + \mathbf{T}^3 + \mathbf{T}^4,$$

where

$$\mathbf{T}^1 = T^1 \otimes I \otimes I \otimes I, \ \ldots, \mathbf{T}^4 = I \otimes I \otimes I \otimes T^4,$$

and T^σ has the matrix representation

$$T^\sigma \leftrightarrow \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

with respect to the energy basis $\{|\varphi_\mu^\sigma\rangle\}$, $\sigma = 1, \ldots, 4$. Once we find out that the first subsystem has 5 units of the quantity represented by T, the eigenvalue 8 is eliminated, along with the corresponding eigenvectors, for all three of the other subsystems. If the formalism is complete, however, and the distribution of T is somehow created by measurement and did not precede it, the information that there remain 7 units of T has to be transmitted instantaneously (or at any rate implausibly quickly) to the other three subsystems. Perhaps—again, if conservation is more than just a matter of nonlocal agreement between apparatuses—appropriate amounts of T have to be exchanged at a distance.

I shall now show that quantum-mechanical 'states' can be altered by distant measurements. In a sense we can assign a state to the subsystems $\{\mathbf{S}^\sigma\}$ even when the state of the whole system $\mathbf{S}$ is not a product

$$\bigotimes_{\sigma=1}^{N} |\Omega^\sigma\rangle.$$

Suppose system $\mathbf{S} = \mathbf{S}^1 + \mathbf{S}^2$ is in the pure state

$$|\Upsilon\rangle = \sum_\mu c_\mu |\varphi_\mu^1\rangle \otimes |\varphi_\mu^2\rangle.$$

To any operator of the form $\mathbf{A}^1 = A^1 \otimes I$ or $\mathbf{A}^2 = I \otimes A^2$, where

$$A^1 = \sum_\mu \alpha_\mu^1 |\varphi_\mu^1\rangle\langle\varphi_\mu^1|, \quad A^2 = \sum_\nu \alpha_\nu^2 |\varphi_\nu^2\rangle\langle\varphi_\nu^2|,$$

$P_{[\Upsilon]}$ will be indistinguishable from $v = v^1 \otimes v^2$, where

$$v^1 = \sum_\mu |c_\mu|^2 |\varphi_\mu^1\rangle\langle\varphi_\mu^1|, \quad v^2 = \sum_\nu |c_\nu|^2 |\varphi_\nu^2\rangle\langle\varphi_\nu^2|.$$

In other words

$$\mathrm{Tr}(P_{[\Upsilon]}A^1) = \mathrm{Tr}(vA^1), \quad \mathrm{Tr}(P_{[\Upsilon]}A^2) = \mathrm{Tr}(vA^2).$$

So as long as we just measure A^σ, we can say that $\mathbf{S}^\sigma$ is in state v^σ, in the sense that A^σ cannot tell the difference between $P_{[\Upsilon]}$ and $v^1 \otimes v^2$ ($\sigma = 1$ or 2). An observable of the form $B^1 \otimes I$ (where $[B^1, A^1] \neq 0$) may, however, be able to tell v apart from $P_{[\Upsilon]}$, in which case it would no longer make sense to describe $\mathbf{S}^1$ with v^1, and one would have to come up with another statistical operator for the first subsystem.

More generally $\mathbf{S}$ may be in a state

$$|\Omega\rangle = \sum_{\mu\nu} c_{\mu\nu} |\varphi^1_\mu\rangle \otimes |\varphi^2_\nu\rangle$$

with two indices. This means that if we measure an observable represented by

$$J = \sum_{\mu\nu} \lambda_{\mu\nu} |\varphi^1_\mu \otimes \varphi^2_\nu\rangle\langle\varphi^1_\mu \otimes \varphi^2_\nu|,$$

$\mathbf{S}$ will end up in state $|\varphi^1_\mu\rangle \otimes |\varphi^2_\nu\rangle$ with probability $|c_{\mu\nu}|^2$. So we have to sum over ν to find the probability

$$\sum_\nu |c_{\mu\nu}|^2 = \mathrm{Tr}\left(P_{[\Omega]}\{|\varphi^1_\mu\rangle\langle\varphi^1_\mu| \otimes I\}\right)$$

of $\mathbf{S}^1$ being in state $|\varphi^1_\mu\rangle$ after a measurement of J. The state of $\mathbf{S}^1$ with respect to the basis $\{|\varphi^1_\mu\rangle\}$ will then be given by the statistical operator

$$\omega^1 = \sum_\mu \left(\sum_\nu |c_{\mu\nu}|^2\right) |\varphi^1_\mu\rangle\langle\varphi^1_\mu|$$

with eigenvalues

$$\left\{\sum_\nu |c_{1\nu}|^2, \sum_\nu |c_{2\nu}|^2, \ldots\right\}.$$

These eigenvalues are, incidentally, invariant with respect to rotations of the other basis $\{|\varphi^2_\nu\rangle\}$:

$$\sum_\nu |\langle\varphi^1_\mu \otimes \varphi^2_\nu|\Omega\rangle|^2 = \sum_\nu |\langle\varphi^1_\mu \otimes U\varphi^2_\nu|\Omega\rangle|^2$$

for all μ and all unitary operators U.

Let us now return to the composite system

$$\mathbf{S} = \sum_{\sigma=1}^{N} \mathbf{S}^\sigma$$

in state $|\Gamma\rangle$. Suppose we want to assign a statistical operator ρ^1 with eigenvectors $\{|\varphi^1_\mu\rangle\}$ to the first subsystem $\mathbf{S}^1$. Generalizing the ideas above, we have the statistical operator

$$\rho^1 = \sum_{\mu^1}\left(\sum_{\mu^2\cdots\mu^N}|c'_{\mu^1\mu^2\cdots\mu^N}|^2\right)|\varphi^1_{\mu^1}\rangle\langle\varphi^1_{\mu^1}|,$$

with eigenvalues

$$\left\{\left(\sum_{\mu^2\cdots\mu^N}|c'_{1\mu^2\cdots\mu^N}|^2\right),\left(\sum_{\mu^2\cdots\mu^N}|c'_{2\mu^2\cdots\mu^N}|^2\right),\ldots\right\}.$$

For the second subsystem we can write

$$\rho^2 = \sum_{\mu^2}\left(\sum_{\mu^1\mu^3\cdots\mu^N}|c'_{\mu^1\mu^2\cdots\mu^N}|^2\right)|\varphi^2_{\mu^2}\rangle\langle\varphi^2_{\mu^2}|,$$

and so on. Now we know how to assign descriptions to the subsystems of a composite system **S**, even if the state of **S** is not a product.

As the descriptions ρ^σ will generally change if the squared moduli $|c'_{\mu^1\mu^2\cdots\mu^N}|^2$ do, measurements on one subsystem can alter the descriptions of an arbitrary number of distant subsystems "without in any way disturbing" them. The issue is whether the distant subsystems are actually changed physically. Is the only change in our imperfect *knowledge* of the subsystems, which is supplemented by the measurements performed? As long as interference is not an issue, there is every reason to believe—especially if we appeal explicitly to some of the locality principles quoted at the end of Section III. 2.2—that *ignorance* is what is being eliminated here. The subsystems are not physically altered by the distant measurements, which only *tell us* more about them. The following two possibilities come to mind

> **completeness**: the subsystems somehow find out, *after* measurement, what they have to do (to obey conservation, for instance)
>
> **incompleteness**: the subsystems already knew; it is the observer who finds out.

As **completeness** involves unlikely transmissions between the subsystems, **incompleteness** appears more plausible.

2.4 Beyond additive conservation

So far it has seemed as though quantum correlations, at least those involving energy eigenvectors, may be due to additive conservation. But what if the correlations are *too strong* to be attributed to additive conservation? Suppose Bertlmann has three feet and three socks of different colours, one pink, one green and one red. Once the pink sock is found on one foot, it can be inferred that the red and green socks had been put on the other two; but not which foot the red sock is on. The red sock could be on the second or third foot. There are quantum-mechanical situations, however, in which it is as though the discovery of the pink sock on the first foot tells us not only that the red sock is on one of the other two, but that it is in fact on the third foot.

Pour a pint of water into two glasses; the amount in either one can inferred from a measurement on the other. With three, however, one should not be able to work out the amounts in all three from a measurement on one. In the quantum-mechanical case considered below, however, that is exactly what appears to be happening.

We will be interested in the *multi*orthogonal expansion[87]

$$|\Theta\rangle = \sum_{\mu} c_{\mu} \bigotimes_{\sigma=1}^{N} |\xi^{\sigma}_{\mu}\rangle, \tag{18}$$

where the complete orthonormal sets $\{|\xi^{\sigma}_{1}\rangle, |\xi^{\sigma}_{2}\rangle, \dots\} \subset \mathcal{H}^{\sigma}$, $\sigma = 1, \dots, N$, all have the same cardinality. Such expansions establish a one to one or *bi*jective correspondence between the basic sets $\{|\xi^{1}_{\mu}\rangle\}$ and $\{|\xi^{2}_{\mu}\rangle\}$ where there are two factor spaces; more generally the correspondence can be called

$$\overbrace{\text{one to one to} \dots \text{to one}}^{N \text{ times}}$$

or *multi*jective. To render the correspondence observable and give rise to correlations, we can construct the self-adjoint operator[88]

$$B = \sum_{\sigma=1}^{N} \mathbf{B}^{\sigma}$$

where

$$\mathbf{B}^{1} = B^{1} \otimes I^{2} \otimes \cdots \otimes I^{N}, \dots, \mathbf{B}^{N} = I^{1} \otimes \cdots \otimes I^{N-1} \otimes B^{N} \tag{19}$$

and the maximal operators

$$B^{\sigma} = \sum_{\mu} \lambda^{\sigma}_{\mu} |\xi^{\sigma}_{\mu}\rangle\langle\xi^{\sigma}_{\mu}|,$$

$\sigma = 1, \dots, N$.

By constructing the above observables we establish a bijection between the eigenvalues $\{\lambda^{\sigma}_{\mu}\}$ and the basic vectors $\{|\xi^{\sigma}_{\mu}\rangle\}$, which extends the aforementioned

[87]See Peres (1995b), in which necessary and sufficient conditions for the existence of a Schmidt decomposition involving more than two factor spaces are given.
[88]Again, see Bergia (1993), Bergia and Cannata (1991).

multijection among the basic sets to the spectra $\Lambda^\sigma = \{\lambda_1^\sigma, \lambda_2^\sigma, \dots\}$, $\sigma = 1, \dots, N$. So the discovery of any one of the eigenvalues selects one in each of the other $N-1$ spaces as well. This is especially surprising if we impose, in the choice of the eigenvalues, that

$$\sum_{\sigma=1}^{N} \lambda_\mu^\sigma = \lambda \tag{20}$$

for all μ (which means that $B|\Theta\rangle = \lambda|\Theta\rangle$), as it suggests that the whole system, in state $|\Theta\rangle$, possesses an amount λ of the physical quantity $\mathfrak{B}$ represented by B, and furthermore that the exact distribution of the total λ over all N subsystems can be determined by a measurement on any one of them. This is to be expected with two subsystems, but perhaps not where there are more.

Consider the Cartesian product

$$\Lambda = \mathop{\times}_{\sigma=1}^{N} \Lambda^\sigma = \{ (\lambda_{\mu^1}^1, \dots, \lambda_{\mu^N}^N) \}$$

of the spectra chosen, and the proper subset

$$\Lambda_{(20)} = \left\{ (\lambda_{\mu^1}^1, \dots, \lambda_{\mu^N}^N) : \sum_{\sigma=1}^{N} \lambda_{\mu^\sigma}^\sigma = \lambda \right\} \subset \Lambda$$

of Λ satisfying the additivity requirement (20). The discovery of an eigenvalue $\lambda_\mathfrak{y}^\mathfrak{x}$ will select a proper subset of $\Lambda_{(20)}$,

$$\Lambda_{(20)\mu^\mathfrak{x}=\mathfrak{y}} = \left\{ (\lambda_{\mu^1}^1, \dots, \lambda_{\mu^N}^N) : \sum_{\sigma=1}^{N} \lambda_{\mu^\sigma}^\sigma = \lambda,\ \mu^\mathfrak{x} = \mathfrak{y} \right\} \subset \Lambda_{(20)},$$

which will consist of a single element of Λ only when $N = 2$.

There will also be a subset of $\Lambda_{(20)}$ corresponding to the multiorthogonal decomposition (18), namely $\Lambda_{(18)} = \{ (\lambda_\mu^1, \dots, \lambda_\mu^N) \} \subseteq \Lambda_{(20)}$; of course the index μ selects the right combinations of eigenvalues. Here the discovery of the same eigenvalue $\lambda_\mathfrak{y}^\mathfrak{x}$ will single out $\Lambda_{(18)\mu^\mathfrak{x}=\mathfrak{y}} = (\lambda_\mathfrak{y}^1, \dots, \lambda_\mathfrak{y}^N) \subseteq \Lambda_{(20)\mu^\mathfrak{x}=\mathfrak{y}}$.

The equalities $\Lambda_{(18)} = \Lambda_{(20)}$ and $\Lambda_{(18)\mu^\mathfrak{x}=\mathfrak{y}} = \Lambda_{(20)\mu^\mathfrak{x}=\mathfrak{y}}$ are a peculiarity of the well-known cases involving two subsystems; $\Lambda_{(18)} \subset \Lambda_{(20)}$ and $\Lambda_{(18)\mu^\mathfrak{x}=\mathfrak{y}} \subset \Lambda_{(20)\mu^\mathfrak{x}=\mathfrak{y}}$ where there are more. This means that the correlations produced by the multiorthogonal decomposition are stronger than those required by (20), and hence *cannot be attributed to such an additivity condition.*

We can look at the matter more geometrically. Any vector $|\Phi\rangle$ lying entirely in subspaces corresponding to eigenvalues $\lambda_{\mu^\sigma}^\sigma$ that sum to λ will be an eigenvector of B belonging to λ, in other words the conditions

$$\left(\bigotimes_{\sigma=1}^{N}|\xi^{\sigma}_{\mu^{\sigma}}\rangle\langle\xi^{\sigma}_{\mu^{\sigma}}|\right)|\Phi\rangle = |\Phi\rangle \quad \text{and} \quad \sum_{\sigma=1}^{N}\lambda^{\sigma}_{\mu^{\sigma}} = \lambda$$

together imply that $B|\Phi\rangle = \lambda|\Phi\rangle$. Therefore

$$\overline{\text{span}\left\{\bigotimes_{\sigma=1}^{N}|\xi^{\sigma}_{\mu^{\sigma}}\rangle : \sum_{\sigma=1}^{N}\lambda^{\sigma}_{\mu^{\sigma}} = \lambda\right\}}$$

is the eigenspace belonging to λ.

Where $N = 2$, the set $\{B, B^{\mathfrak{g}}\}$ ($\mathfrak{g} = 1$ *or* 2) representing the amounts of $\mathfrak{B}$ possessed by the entire system and by one of the subsystems, respectively, is a *complete* commuting set, since *one* of the products $|\xi^1_\mu\rangle \otimes |\xi^2_\nu\rangle$ would be singled out by measurement of the corresponding observables. The additivity condition represented by $\{B, B^{\mathfrak{g}}\}$ therefore requires neither more nor less correlation than is contained in the multiorthogonal decomposition (18).

Where $N > 2$, however, the set $\{B, B^{\mathfrak{g}}\}$ ($\mathfrak{g} = 1$ *or*... *or* N) is no longer complete, since the subspace

$$\Omega = \overline{\text{span}\left\{\bigotimes_{\sigma=1}^{N}|\xi^{\sigma}_{\mu^{\sigma}}\rangle : \sum_{\sigma=1}^{N}\lambda^{\sigma}_{\mu^{\sigma}} = \lambda,\ \mu^{\mathfrak{g}} = \mathfrak{y}\right\}}$$

it singles out—measurement of $B^{\mathfrak{g}}$ determines the eigenvalue $\lambda^{\mathfrak{g}}_{\mathfrak{y}}$—is larger than just a ray.

If the decomposition is multiorthogonal, the two measurements B and $B^{\mathfrak{g}}$ will identify a single product $|\xi^1_\mu\rangle \otimes \cdots \otimes |\xi^N_\nu\rangle$. The correlations contained in (18) therefore go beyond the additivity requirement represented by the set $\{B, B^{\mathfrak{g}}\}$, which would otherwise have only singled out the larger subspace Ω.

The argument has so far concerned a single instant; one can also wonder about evolution. The multiorthogonal expansion (18) will be preserved if the vectors $|\xi^1_\mu\rangle \otimes \cdots \otimes |\xi^N_\nu\rangle$ represent energy eigenstates, for then the time evolution operator $e^{-iHt/\hbar}$ just leaves them in their rays, changing neither their directions nor their lengths. Then one can speak of conservation, and say that the correlations in question *cannot be attributed to an additive conservation law*.[89]

It is not clear how this should be interpreted. Does it mean that quantum correlations (expressed with respect to a single energy eigenbasis) have nothing to do with additive conservation, or, for that matter, with Bertlmann's socks or the glasses?

[89] Where $N = 2$, the biorthogonal form can be restored whatever happens, but then the observable whose eigenstates give rise to the biorthogonal decomposition becomes a function of time: $B(t)|\Theta(t)\rangle = \lambda|\Theta(t)\rangle$. Even if one can no longer speak of conservation, the same λ can be chosen throughout.

Few vectors can be given multiorthogonal expansions; those that can are non-dense in Hilbert space, and can be found *almost nowhere*.[90] Perhaps they correspond to nothing in nature, or never arise with respect to energy eigenvectors.

[90]See Clifton (1994).

IV. Examples: interference *and* additive conservation

1
Bell

So far interference (II, III. 1) and additive conservation (III. 2) have been considered separately. In the following examples we see them at work together.

Examples are more relevant here as particular instances of the general configuration space problem, than as expressions of distinct individual questions. The measurement problem and the violation of Bell's inequality can be considered essentially the same issue, and not different problems which happen to have structural similarities. For the problem *resides* in that common structure; differences are insignificant when compared with the fundamental formal resemblance. I emphasize the common structure by, for instance, getting Schrödinger's cat to oscillate the way kaons do, and using a state vector which resembles the singlet in a discussion of the Wigner-Araki-Yanase theorem. I also point out the role of additive conservation in the deduction of Bell's inequality *and* in measurement theory.

The emphasis here is more on the basic ontological problem of quantum waves in configuration space, and hence also on the *general* formal problem which expresses it, than on particular formal expressions. Addressing only one particular version of the issue—measurement, for instance—leaves the general problem untouched, with all its other forms. If one believes there is an underlying ontological problem, it will come as no surprise that it should have a general formal expression, and hence several particular ones as well. If, on the other hand, one believes there is no objective physical reality, and therefore no ontological issue, there will be no general formal problem—again, there is nothing *mathematically* wrong with the superposition of products—only the measurement problem, the violation of Bell's inequality and so forth.

I shall now construct a sensitive observable explicitly, namely the one used by Bell to violate his inequality, which I deduce from the perfect anticorrelations that in this case (with two subsystems) are necessary and sufficient for the formulation of additive conservation laws. The inequality is then violated by interference brought out by the sensitive observable constructed.

1.1 Spin-half

The spin-half formalism is useful both here, and for kaons. So I develop it abstractly, without explicit reference to spin (in fact it applies to all two-level quantum-mechanical systems).[91]

Self-adjoint operators on the two-dimensional complex space H form a four-dimensional real vector space spanned by the identity I and the three Pauli operators, which can be represented as

$$\sigma_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$ of σ_3. Any self-adjoint operator on H can therefore be written as a linear combination

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{\sigma} + a_4 I &= (a_1, a_2, a_3)(\sigma_1, \sigma_2, \sigma_3) + a_4 I \\ &= a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + a_4 I \end{aligned}$$

with real coefficients $\{a_i\}$ and matrix representation

$$\begin{pmatrix} a_4 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_4 - a_3 \end{pmatrix}$$

with respect to $|\uparrow\rangle$ and $|\downarrow\rangle$. We will let a_4 vanish, and hence confine ourselves to the subspace spanned by the three Pauli operators. The three remaining coefficients determine a direction $a_1 : a_2 : a_3$ and a squared length $l^2 = a_1^2 + a_2^2 + a_3^2$. If we assume that $a_1^2 + a_2^2 + a_3^2 = 1$, we have the simpler representation

$$\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \leftrightarrow \boldsymbol{a}\cdot\boldsymbol{\sigma}, \tag{21}$$

where the components of the (normalized) vector $\boldsymbol{a}$ are $\sin\theta\cos\varphi$, $\sin\theta\sin\varphi$ and $\cos\theta$. The operators belonging to the set in question, namely

$$\begin{aligned} \mathfrak{V} &= \{\boldsymbol{a}\cdot\boldsymbol{\sigma} : a_1^2 + a_2^2 + a_3^2 = 1, a_4 = 0\} \\ &= \{A \text{ on } \mathsf{H} : A^*A = AA^* = I,\ A = A^*,\ \mathrm{Tr}(A) = 0\}, \end{aligned}$$

are immediate generalizations of the Pauli operators, in the sense that they are self-adjoint unitary operators whose trace vanishes.

[91]Similar treatments are in Hughes (1989) pp.139-41, Beltrametti (1985), Beltrametti and Cassinelli (1981); see also Weingard and Smith (1982).

An operator $\boldsymbol{a}\cdot\boldsymbol{\sigma}\in\mathfrak{V}$ is therefore characterized by a polar angle θ and an azimuth φ, *which takes us back to the sphere of Chapter II. 1.*[92] The eigenvectors $|\psi_\pm(\boldsymbol{a})\rangle$ of $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ have the representations

$$\begin{pmatrix}\langle\uparrow|\psi_\pm(\boldsymbol{a})\rangle\\ \langle\downarrow|\psi_\pm(\boldsymbol{a})\rangle\end{pmatrix}=e^{i\phi_\pm}\begin{pmatrix}\pm\sqrt{\frac{1}{2}(1\pm\cos\theta)}\\ e^{i\varphi}\sqrt{\frac{1}{2}(1\mp\cos\theta)}\end{pmatrix}\tag{22}$$

(the arguments $\phi_\pm$ are unimportant, as we are mainly interested in the directions of $|\psi_\pm(\boldsymbol{a})\rangle$). We can write $\sigma_{\boldsymbol{a}}$ rather than $\boldsymbol{a}\cdot\boldsymbol{\sigma}$.

1.2 Bell's observable

Suppose $\sigma^1_{\boldsymbol{a}}$ acts on the two-dimensional complex Hilbert space H^1, and a similarly defined $\sigma^2_{\boldsymbol{b}}$ on H^2, also two-dimensional and complex. We know from Section III. 1.3 that any product $\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}}$ will represent an indifferent observable, but we can try a linear combination of products. Introducing two further operators $\sigma^1_{\boldsymbol{a}'}$ and $\sigma^2_{\boldsymbol{b}'}$ and taking all products

$$\{\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}},\ \sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}'},\ \sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}},\ \sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}'}\},$$

we can write

$$\Gamma=\rho_1\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}}+\rho_2\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}'}+\rho_3\sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}}+\rho_4\sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}'}.$$

For Γ to represent an observable, the coefficients $\{\rho_i\}$ all have to be real. It is enough for our purposes that $\rho_1,\ldots,\rho_4=\pm1$. A common choice is $\rho_1,\rho_3,\rho_4=+1$, $\rho_2=-1$, so that

$$\Gamma=\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}}-\sigma^1_{\boldsymbol{a}}\otimes\sigma^2_{\boldsymbol{b}'}+\sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}}+\sigma^1_{\boldsymbol{a}'}\otimes\sigma^2_{\boldsymbol{b}'}.$$

From Section III. 1.3 we also know that, if $[\sigma^1_{\boldsymbol{a}},\sigma^1_{\boldsymbol{a}'}]$ and $[\sigma^2_{\boldsymbol{b}},\sigma^2_{\boldsymbol{b}'}]$ both vanish, Γ will be indifferent, for then it will have factorizable eigenvectors. In fact it is enough, for indifference, for *either* commutator—in other words for the product $[\sigma^1_{\boldsymbol{a}},\sigma^1_{\boldsymbol{a}'}][\sigma^2_{\boldsymbol{b}},\sigma^2_{\boldsymbol{b}'}]$—

[92]As the operators of $\mathfrak{V}$ are characterized by a pair of angles, they can be put into correspondence with unit vectors in $\mathbb{R}^3$. For every two operators $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $\boldsymbol{b}\cdot\boldsymbol{\sigma}$ in $\mathfrak{V}$ there is a unitary unimodular operator $U:\mathsf{H}\to\mathsf{H}$ such that $U[\boldsymbol{a}\cdot\boldsymbol{\sigma}]U^\dagger=\boldsymbol{b}\cdot\boldsymbol{\sigma}$, in much the same way that for the corresponding unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ there is an orthogonal operator $B:\mathbb{R}^3\to\mathbb{R}^3$, whose determinant is $+1$, which carries $\boldsymbol{a}$ into $B\boldsymbol{a}=\boldsymbol{b}$. Indeed there is a homomorphism between the special unitary group $SU(2)$ of unitary unimodular operators acting on H and the special orthogonal group $SO(3)$ of orthogonal operators with determinant $+1$ acting on $\mathbb{R}^3$. The coefficients α, β, γ, δ of the matrix representation

$$\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$$

of the operator U are called the 'Cayley-Klein' parameters. See Whittaker (1947) pp.11-3, Goldstein (1980) pp.148-58, Penrose and Rindler (1984) pp.18-24.

to vanish.[93] The operators $\sigma^s_{\boldsymbol{e}}$ and $\sigma^s_{\boldsymbol{e}'}$ commute if $\boldsymbol{e} = \boldsymbol{e}'$, and only then, since $\boldsymbol{e}$ and $\boldsymbol{e}'$ are unit vectors. For Γ to be sensitive, therefore, we must have that $\boldsymbol{a} \neq \boldsymbol{a}'$ *and* $\boldsymbol{b} \neq \boldsymbol{b}'$.

Since a dependence on the azimuth as well as on the polar angle is superfluous for our purposes, we can allow the components a_2, a_2', b_2, b_2' of vectors $\boldsymbol{a}$, $\boldsymbol{a}'$, $\boldsymbol{b}$, $\boldsymbol{b}'$ to vanish, and write σ^1_α, $\sigma^1_{\alpha'}$, σ^2_β, $\sigma^2_{\beta'}$ for $\sigma^1_{\boldsymbol{a}}$, $\sigma^1_{\boldsymbol{a}'}$, $\sigma^2_{\boldsymbol{b}}$, $\sigma^2_{\boldsymbol{b}'}$. The resulting representations with respect to $|\uparrow\rangle$ and $|\downarrow\rangle$ will be

$$\sigma^s_{\mathrm{q}} \leftrightarrow \begin{pmatrix} \cos\mathrm{q} & \sin\mathrm{q} \\ \sin\mathrm{q} & -\cos\mathrm{q} \end{pmatrix}^s,$$

where q stands for the polar angles α, α', β, β' and $s = 1, 2$. Simplifying further, we can assume that the angle α' vanishes—so that $\sigma^1_{\alpha'}$ is the third Pauli operator—and that $\beta = -\beta' = \alpha/2$. What we have called 'Bell's observable' is represented by the operator

$$\Gamma_\alpha = \sigma^1_\alpha \otimes \sigma^2_{\alpha/2} - \sigma^1_\alpha \otimes \sigma^2_{-\alpha/2} + \sigma^1_0 \otimes \sigma^2_{\alpha/2} + \sigma^1_0 \otimes \sigma^2_{-\alpha/2}.$$

As we want to use Γ_α to 'see' a quantum wave in configuration space, we also need a suitable vector $|\Sigma\rangle$ representing such a wave. Suppose biorthogonal form, without which the reality criterion of Einstein, Podolsky and Rosen cannot be applied, is assumed with respect to the $\{|\uparrow\rangle, |\downarrow\rangle\}$ bases, so that

$$|\Sigma\rangle = \eta|\uparrow\rangle|\downarrow\rangle + \zeta|\downarrow\rangle|\uparrow\rangle.$$

But we want a state that can be given biorthogonal expansions with respect to other bases as well, so the moduli of the coefficients η and ζ have to be the same. Normalizing,

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + e^{i\phi}|\downarrow\rangle|\uparrow\rangle).$$

This vector acquires rotational invariance, which is also useful, if it is furthermore assumed that the arguments of η and ζ differ by π, so $\phi = \pi$ and

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$

the 'singlet.' This is the state we will use to violate Bell's inequality.

1.3 Bell's inequality

Let us forget about quantum mechanics for the time being. Suppose a source produces many pairs $\{\mathfrak{O}^1(k), \mathfrak{O}^2(k)\}$ of objects, and that a measuring apparatus M^s is set up to measure, on object $\mathfrak{O}^s(k)$ of the kth pair, the dichotomic property $\underline{\sigma}^s_\varrho(k) = \pm 1$ characterized by the angle ϱ ($s = 1, 2$; $k = 1, 2, \ldots$). Suppose furthermore that

[93] See Landau (1986).

$$\underline{\sigma}_{\varrho}^{1}(k) = \pm 1 \Leftrightarrow \underline{\sigma}_{\varrho}^{2}(k) = \mp 1$$

for each pair. In other words if the same angle ϱ is chosen for both apparatuses and measurement on one side yields ± 1, then ∓ 1 will be found on the other. So the criterion used by Einstein, Podolsky and Rosen for the identification of an element of reality can be applied here too. From a measurement of $\underline{\sigma}_{\varrho}^{1}(k)$, for instance, "we can predict with certainty ... the value of a physical quantity," namely $\underline{\sigma}_{\varrho}^{2}(k)$, "without in any way disturbing" the second object. Einstein *et al.* would conclude that "there exists an element of physical reality corresponding to that physical quantity." To simplify we will identify the values ± 1 with the corresponding elements of reality, which can be assigned indifferently to *both* objects of the kth pair, indeed regardless of whether measurement is in fact undertaken.

Let us define

$$\underline{B}(k) = \underline{\sigma}_{\alpha}^{1}(k)\underline{\sigma}_{\beta}^{2}(k) - \underline{\sigma}_{\alpha}^{1}(k)\underline{\sigma}_{\beta'}^{2}(k) + \underline{\sigma}_{\alpha'}^{1}(k)\underline{\sigma}_{\beta}^{2}(k) + \underline{\sigma}_{\alpha'}^{1}(k)\underline{\sigma}_{\beta'}^{2}(k).$$

So far we know that all four terms are equal to ± 1. If each factor $\underline{\sigma}_{\varrho}^{s}(k)$ were viewed as a function of the adjacent factor's subscript, we could have $(-1)\cdot\underline{\sigma}_{\beta}^{2}(k) - (+1)\cdot\underline{\sigma}_{\beta'}^{2}(k)$ for the first two terms, for instance. The modulus of $\underline{B}(k)$ could then be as large as 4. Let us assume, however, that the value of $\underline{\sigma}_{\varrho}^{1}(k)$ depends only on its own angle ϱ, and not on the angle ϱ' of the adjacent factor $\underline{\sigma}_{\varrho'}^{2}(k)$ (or on any other angle for that matter), for M^2 could be arbitrarily far from $\mathfrak{O}^1(k)$; indeed it would be most surprising if

$$\underline{\sigma}_{\varrho}^{1}(k) = \pm 1 \quad \text{when} \quad \underline{\sigma}_{\varrho}^{2}(k) = \mp 1,$$

but $\underline{\sigma}_{\varrho}^{1}(k) = \mp 1$ if another angle $\varrho' \neq \varrho$ were chosen for M^2. The same applies to each factor of every term; $\underline{\sigma}_{\zeta}^{2}(k)$ is likewise assumed to depend only on ζ, and on no other angle.[94]

This assumption allows us to rewrite $\underline{B}(k)$ as

$$\underline{B}(k) = \underline{\sigma}_{\alpha}^{1}(k)\{\underline{\sigma}_{\beta}^{2}(k) - \underline{\sigma}_{\beta'}^{2}(k)\} + \underline{\sigma}_{\alpha'}^{1}(k)\{\underline{\sigma}_{\beta}^{2}(k) + \underline{\sigma}_{\beta'}^{2}(k)\}.$$

As $\underline{\sigma}_{\beta}^{2}(k)$ and $\underline{\sigma}_{\beta'}^{2}(k)$ are both equal to ± 1, one of the two terms on the right will vanish, and the other will be equal to ± 2.[95] Since $\underline{B}(k) = \pm 2$ for all k, the average

$$\underline{B} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{B}(k)$$

---

[94]This is more or less Redhead's LOC_3: "A sharp value of an observable cannot be changed into another sharp value by altering the setting of a remote piece of apparatus." See Redhead (1987) p.82.
[95]This argument is in Redhead (1987) p.84.

cannot leave the interval running from -2 to 2, which is Bell's inequality. We can also write

$$-2 \leq \underline{B} = \underline{P}(\alpha, \beta) - \underline{P}(\alpha, \beta') + \underline{P}(\alpha', \beta) + \underline{P}(\alpha', \beta') \leq 2,$$

where the correlation function

$$\underline{P}(\mathfrak{a}, \mathfrak{b}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \underline{\sigma}_{\mathfrak{a}}^{1}(k) \underline{\sigma}_{\mathfrak{b}}^{2}(k).$$

The angle $\mathfrak{a}$ can have the two values α and α', the angle $\mathfrak{b}$ the values β and β'.

Returning to quantum mechanics, now suppose the source is in fact producing pairs of spin-half particles in singlet state. Again, from a measurement of σ_{ϱ}^{1} on the first particle "we can predict with certainty ... the value of" the quantity represented by σ_{ϱ}^{2}; so we can assign the 'elements of reality' ± 1 to both subsystems for every angle ϱ and for all k.

Rather than comparing the quantum-mechanical correlation function

$$P(\mathfrak{a}, \mathfrak{b}) = \langle \Sigma | \sigma_{\mathfrak{a}}^{1} \otimes \sigma_{\mathfrak{b}}^{2} | \Sigma \rangle = -\mathfrak{a} \cdot \mathfrak{b}$$

with $\underline{P}(\mathfrak{a}, \mathfrak{b})$, which is unknown, we can compare

$$\underline{B} = \underline{P}(\alpha, \alpha/2) - \underline{P}(\alpha, -\alpha/2) + \underline{P}(0, \alpha/2) + \underline{P}(0, -\alpha/2),$$

whose modulus is bounded by 2, with

$$\langle \Sigma | \Gamma_{\alpha} | \Sigma \rangle = P(\alpha, \alpha/2) - P(\alpha, -\alpha/2) + P(0, \alpha/2) + P(0, -\alpha/2) = 2\sqrt{2}.$$

So Bell's inequality is violated by quantum mechanics. This means—provided it makes sense to apply the criterion of Einstein, Podolsky and Rosen—that either the elements of reality of one subsystem can change when the *other* apparatus is turned, or that quantum mechanics is wrong.

> In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant.
>
> Of course, the situation is different if the quantum mechanical predictions are of limited validity.[96]

The perfect anticorrelations expressed by $\langle \Sigma | \sigma_{\varrho}^{1} \otimes \sigma_{\varrho}^{2} | \Sigma \rangle = -1$ can be ascribed to additive conservation. The criterion of Einstein *et al.* allows the identification of an 'element of reality'—which corresponds to an amount of the conserved quantity σ_{ϱ}—that is there anyway, independently of measurement and the action of apparatuses. This

[96]Bell (1965)

suggests that the distribution of σ_ϱ *preceded* measurement, and indeed was determined at the creation of the pair.

If quantum mechanics and this picture indicated by the reality criterion are both correct, it must be that the rotation of one apparatus can change an element of reality of the other subsystem. It is difficult to attribute such a change to a physical mechanism. If quantum waves did propagate in configuration space, with influences unattenuated by distance, the rotation of the second apparatus could cause the wave to exchange σ_ϱ with the first particle, thus changing the element of reality.

It is implausible that this influence, however it may be communicated, should not be attenuated by separation. I shall therefore consider the possibility, in Part V, that "quantum mechanics may break down when the particles are far enough apart,"[97] and that Bell's inequality is not in fact violated in nature, even though its violation is predicted by quantum mechanics.

[97]Bohm and Aharonov (1957)

2
Measurement

We have seen that the influences conveyed by quantum waves do not in principle depend on the spatial distribution of the entangled objects. Von Neumann's theory of measurement similarly allows a quantum wave to engulf an arbitrarily *large* object, even a cat. Additive conservation will also be an issue in measurement theory.

2.1 Von Neumann's theory of measurement

In von Neumann's theory[98] the apparatus **A** is described by a state vector belonging to a Hilbert space $\mathcal{H}^2$, rather than 'classically,' as wished by Bohr. Such description implicitly involves tensor multiplication and 'configuration space'; for the space

$$\mathcal{H}^2 = \bigotimes_{k=1}^{N} \mathcal{H}_k^2$$

can be broken up into the factors $\mathcal{H}_1^2, \dots, \mathcal{H}_N^2$ in which the N particles making up the apparatus are described. But if quantum waves do not really propagate in configuration space, it makes no sense to assign a state vector to the apparatus, or to any macroscopic body for that matter. Perhaps the problem lies at this level. Penrose (1989), for instance, believes "that one must strongly consider the possibility that quantum mechanics is simply *wrong* when applied to macroscopic bodies ...." But let us assume the apparatus *can* be described quantum-mechanically, and see what happens.

Once there are vectors $\{|\varphi_m\rangle\} \in \mathcal{H}^1$ describing the system **S** and others $\{|\mu_n\rangle\} \in \mathcal{H}^2$ describing the apparatus **A**, it is natural to view *their* products $\{|\varphi_m\rangle|\mu_n\rangle\}$ as elements of another linear space $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ and hence allow superpositions of products, such as $c_1|\varphi_1\rangle|\mu_1\rangle + c_2|\varphi_2\rangle|\mu_2\rangle$.

Before the apparatus has interacted with the system, their state is described by the tensor product $|\Psi\rangle = |\varphi\rangle|\mu\rangle$ of a system state $|\varphi\rangle$ and an apparatus 'ground' state $|\mu\rangle$. Suppose we are interested in the observable $\mathfrak{A}$ represented by the maximal self-adjoint operator

$$A = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|.$$

Expanding $|\varphi\rangle$ with respect to the eigenstates of A, we have

[98] von Neumann (1932) pp.222-37

$$|\Psi\rangle = \sum_k c_k |\varphi_k\rangle|\mu\rangle,$$

where $c_k = \langle\varphi_k|\varphi\rangle$. As we want to know which of the eigenstates $|\varphi_k\rangle$ the system is in after measurement,[99] the apparatus must have states $|\mu_k\rangle$ that can be told apart by the experimenter (and are hence orthogonal[100]) and brought into correspondence with the system states: $|\varphi_k\rangle|\mu_k\rangle$, $k = 1, 2, \ldots$. So if we find the apparatus in state $|\mu_r\rangle$ we know the system is in the corresponding state $|\varphi_r\rangle$.

When system and apparatus interact, the desired correspondence is established and each product $|\varphi_k\rangle|\mu\rangle$ becomes $|\varphi_k\rangle|\mu_k\rangle$. If this 'pre-measurement interaction' $\mathfrak{I}$ takes place in accordance with the Schrödinger equation, there will be a unitary operator U such that $U|\varphi_k\rangle|\mu\rangle = |\varphi_k\rangle|\mu_k\rangle$ for all k, and hence, by linearity

$$|\Psi'\rangle = U|\Psi\rangle = \sum_k c_k U|\varphi_k\rangle|\mu\rangle = \sum_k c_k|\varphi_k\rangle|\mu_k\rangle.$$

So the apparatus gets engulfed in the 'wave' of the system, with which it remains entangled until one of the products $|\varphi_k\rangle|\mu_k\rangle$ is selected.

The superposition $|\Psi'\rangle$ is relatively unproblematic with respect to an observable with eigenstates $|\varphi_k\rangle|\mu_k\rangle$, which cannot distinguish between $|\Psi'\rangle$ and the statistical operator

$$\varrho = \sum_k |c_k|^2 |\varphi_k \otimes \mu_k\rangle\langle\varphi_k \otimes \mu_k|$$

obtained BY removing the phases $e^{i\arg c_k}$. If the composite system $\mathbf{S} + \mathbf{A}$ were described by ϱ, one could attribute, in each single case, a state $|\mu_m\rangle$ to the apparatus and a corresponding state $|\varphi_m\rangle$ to the system. But no unitary evolution can go from the pure state $|\Psi\rangle$ to the mixed state ϱ; and we know that a sensitive observable can be found to tell $|\Psi\rangle$ apart from ϱ. As usual the trouble lies in the arguments $\arg c_k$, which can be viewed as representing the phases of a 'quantum wave in configuration space.'

I shall now return to the tension between additive conservation and quantum waves in configuration space, here expressed by the theorem of Wigner,[101] Araki and Yanase.[102]

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[99]For simplicity the system states will be assumed to remain unaffected by interaction with the apparatus, even if "This restricts," as has been pointed out to me by R. I. G. Hughes, "the measurements to measurements of the first kind (Pauli's terminology)—which is a major restriction. It rules out, for instance, measurements with a standard photographic plate."

[100]An ambiguous 'overlap' between the apparatus states $\{|\mu_n\rangle\}$ would undermine their distinguishability. Orthogonality is required, but is clearly not enough for macroscopic distinguishability.

[101]Wigner (1952)

[102]Araki and Yanase (1960), Yanase (1961)

2.2 The theorem of Wigner, Araki and Yanase

A self-adjoint operator of the form $Q = Q^1 \otimes I + I \otimes Q^2$ that commutes with U represents an 'additive conserved quantity.' For the average of Q we can write:

$$\begin{aligned}\langle\Psi|Q|\Psi\rangle &= \Big\langle\sum_k c_k\varphi_k \otimes \mu\Big|Q\Big|\sum_l c_l\varphi_l \otimes \mu\Big\rangle \\ &= \sum_{kl} c_k^* c_l\big\{\langle\varphi_k \otimes \mu|Q^1 \otimes I|\varphi_l \otimes \mu\rangle + \langle\varphi_k \otimes \mu|I \otimes Q^2|\varphi_l \otimes \mu\rangle\big\} \\ &= \sum_{kl} c_k^* c_l\big\{\langle\varphi_k|Q^1|\varphi_l\rangle\langle\mu|\mu\rangle + \delta_{kl}\langle\mu|Q^2|\mu\rangle\big\},\end{aligned}$$

or alternatively:

$$\begin{aligned}\langle\Psi|Q|\Psi\rangle &= \langle\Psi|U^\dagger UQ|\Psi\rangle = \langle\Psi|U^\dagger QU|\Psi\rangle = \langle\Psi'|Q|\Psi'\rangle \\ &= \sum_{kl} c_k^* c_l\big\{\langle\varphi_k|Q^1|\varphi_l\rangle\langle\mu_k|\mu_l\rangle + \delta_{kl}\langle\mu_k|Q^2|\mu_l\rangle\big\}.\end{aligned}$$

As $\langle\varphi_k|Q^1|\varphi_l\rangle = \langle\varphi_k|Q^1|\varphi_l\rangle\delta_{kl}$, the matrix element $\langle\varphi_k|Q^1|\varphi_l\rangle$ must vanish for $k \neq l$, and therefore

$$[Q^1, A] = [Q, A \otimes I] = 0, \tag{23}$$

where A is again

$$\sum_k \lambda_k|\varphi_k\rangle\langle\varphi_k|.$$

There are different ways to interpret this. Wigner (1952) wrote that

> Die übliche Annahme der statistischen Deutung der Quantenmechanik, daß alle hermitesche Operatoren meßbare Größen darstellen, wird wohl allgemein als eine bequeme mathematische Idealisierung und nicht als ein Ausdruck eines Tatbestandes anerkannt. ... die Gültigkeit von Erhaltungssätzen für gequantelte Größen ..., die die Wechselwirkung von Meßobjekt und Meßapparat beherrschen, die Messung der meisten Operatoren nur als einen Grenzfall gestattet. Insbesondere sind die Bedingungen für die Messung von Operatoren, die mit der Gesamtladung unvertauschbar sind, wahrscheinlich unerfüllbar.
>
> ⋮
>
> Trotzdem wird ξ [the state of the apparatus] eine sehr große Anzahl von Komponenten, also der Meßapparat einen sehr großen Bestand an der Erhaltungsgröße, haben müssen, wenn man eine große Sicherheit haben will, daß die Wechselwirkung zwischen Meßobjekt und Meßapparat zu einer Messung führt.[103]

[103]Translation: The usual assumption of the statistical interpretation of quantum mechanics, that all Hermitian operators represent measurable quantities, will be recognised, in general, to be a convenient mathematical idealisation, and not an expression of an actual state of affairs. ... the validity of conservation laws for quantised quantities ..., which regulate the interaction of measured-object and measuring-apparatus, allows the measurement of most operators only as a limiting case. In particular the

Another interpretation[104] is that (23) is *too restrictive*, for we know there are other additive conserved quantities that do not commute with $A \otimes I$. Conservation in quantum mechanics is usually no more than a matter of compatibility: a quantity is conserved when it is compatible with the Hamiltonian, and only then. But some quantities, like energy or angular momentum, are simply known to be conserved. The theorem of Wigner, Araki and Yanase indicates that the operator U is only compatible with certain additive conserved quantities, for Q^1 and A must commute. This means that von Neumann's pre-measurement interaction $\mathfrak{J}$ changes the statistics of quantities that ought to be conserved.

Suppose the system, before the interaction, is described by the spinor

$$|\uparrow^1\rangle = \frac{1}{\sqrt{2}}(|\rightarrow^1\rangle + |\leftarrow^1\rangle)$$

belonging to the two-dimensional complex space H^1, and the apparatus by the spinor

$$|\xi^2\rangle = a|\rightarrow^2\rangle + b|\leftarrow^2\rangle$$

belonging to the space H^2, also two-dimensional and complex. For my purposes it does not matter that the apparatus is a two-level system described by a spinor. I just want to show, in the simplest terms, that a conservation law is violated by the pre-measurement interaction $\mathfrak{J}$.

System and apparatus are therefore described by the product

$$|\Xi\rangle = |\uparrow^1\rangle|\xi^2\rangle \in \mathsf{H}^1 \otimes \mathsf{H}^2$$

at first. The pre-measurement interaction governed by the unitary operator U then gives rise to the evolutions

$$\begin{aligned}
&U\{|\rightarrow^1\rangle|\xi^2\rangle\} = |\rightarrow^1\rangle|\rightarrow^2\rangle \\
&U\{|\leftarrow^1\rangle|\xi^2\rangle\} = |\leftarrow^1\rangle|\leftarrow^2\rangle \\
&U\left\{\frac{1}{\sqrt{2}}(|\rightarrow^1\rangle + |\leftarrow^1\rangle)\right\}|\xi^2\rangle = |\Xi'\rangle = \frac{1}{\sqrt{2}}(|\rightarrow^1\rangle|\rightarrow^2\rangle + |\leftarrow^1\rangle|\leftarrow^2\rangle).
\end{aligned}$$

Suppose σ_3^s is the difference $|\uparrow^s\rangle\langle\uparrow^s| - |\downarrow^s\rangle\langle\downarrow^s|$, $s = 1, 2$, and that

$$\varsigma_3 = \sigma_3^1 \otimes I + I \otimes \sigma_3^2$$

---

conditions for the measurement of operators, which do not commute with the total charge, probably cannot be satisfied.

⋮

The state ξ of the apparatus will nevertheless have to have a very large number of components, and hence the measuring apparatus a very large amount of the conserved quantity, if one wants to be really sure that the interaction between measured-object and measuring-apparatus leads to a measurement.

[104]See Afriat and Selleri (1995).

represents a quantity *known to be conserved*. While the average $\langle \Xi' | \varsigma_3 | \Xi' \rangle$ of ς_3 after the interaction vanishes, before the interaction we have $\langle \Xi | \varsigma_3 | \Xi \rangle = 1 + \langle \xi | \sigma_3^2 | \xi \rangle$, which will only vanish if $|\xi^2\rangle = |\downarrow^2\rangle$. As the apparatus does not have to be in state $|\downarrow^2\rangle$ before the interaction, ς_3 is not necessarily conserved by U.

So there is again a tension between additive conservation and 'quantum waves in configuration space'; a conservation law is violated if a certain compatibility condition is not satisfied. There almost seems to be an exchange of Q between the quantum wave and the composite system. Perhaps something similar is indicated by the violation of Bell's inequality; namely that the rotation of an apparatus causes the quantum wave to exchange the quantity represented by σ_ϱ with a remote particle (see Section IV. 1.3).

2.3 Schrödinger's oscillating cat

This thought experiment[105] is intended to emphasize the structural similarity between apparently very different 'configuration space' situations; for the cat oscillates in much the same way as kaons do (see Chapter VI. 2).

Bring a two-level system described by the spinor

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \in \mathsf{H}^1$$

into interaction with a cat described in the Hilbert space $\mathcal{H}^2$, such that $|\Xi\rangle = |\rightarrow\rangle|\mathfrak{c}\rangle$ evolves into the entangled state

$$|\Xi'\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\mathfrak{a}\rangle + |\downarrow\rangle|\mathfrak{d}\rangle),$$

where $|\mathfrak{a}\rangle$ means that the cat is alive, $|\mathfrak{d}\rangle$ that it is dead, and $|\mathfrak{c}\rangle$ describes its state before the interaction. If time evolution is governed by the product[106] $\boldsymbol{U} = U^1 \otimes U^2$, where

$$\begin{aligned} U^1 &= e^{-iE_\uparrow t/\hbar}|\uparrow\rangle\langle\uparrow| + e^{-iE_\downarrow t/\hbar}|\downarrow\rangle\langle\downarrow| \\ U^2 &= e^{-iE_\mathfrak{a} t/\hbar}|\mathfrak{a}\rangle\langle\mathfrak{a}| + e^{-iE_\mathfrak{d} t/\hbar}|\mathfrak{d}\rangle\langle\mathfrak{d}|, \end{aligned}$$

the cat will oscillate. We can write

$$|\Xi'(t)\rangle = \boldsymbol{U}|\Xi'\rangle = \frac{1}{\sqrt{2}}\{e^{-i(E_\uparrow + E_\mathfrak{a})t/\hbar}|\uparrow\rangle|\mathfrak{a}\rangle + e^{-i(E_\downarrow + E_\mathfrak{d})t/\hbar}|\downarrow\rangle|\mathfrak{d}\rangle\},$$

or, changing bases,

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[105] See Schrödinger (1935b), van Fraassen (1991) pp.261-4.

[106] Both U^1 and U^2 are consistent with the rules of quantum mechanics, which only require self-adjoint Hamiltonians. Although $|\mathfrak{a}\rangle$ and $|\mathfrak{d}\rangle$ may well be associated with different energies, it is perhaps implausible that the Hamiltonian of either state should be no more than a multiple of the identity.

$$|\Xi'(t)\rangle = \frac{1}{2\sqrt{2}}\Big\{e^{-i(E_\uparrow+E_\mathfrak{a})t/\hbar}\big(|\rightarrow\rangle + |\leftarrow\rangle\big)\otimes\big(|\chi\rangle + |\chi^\perp\rangle\big)$$
$$+\, e^{-i(E_\downarrow+E_\mathfrak{d})t/\hbar}\big(|\rightarrow\rangle - |\leftarrow\rangle\big)\otimes\big(|\chi\rangle - |\chi^\perp\rangle\big)\Big\},$$

where

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\mathfrak{a}\rangle + |\mathfrak{d}\rangle) \quad \text{and} \quad |\chi^\perp\rangle = \frac{1}{\sqrt{2}}(|\mathfrak{a}\rangle - |\mathfrak{d}\rangle).$$

Simplifying,

$$|\Xi'(t)\rangle = \frac{1}{2\sqrt{2}}\{\eta_+(t)|\rightarrow\rangle|\chi\rangle + \eta_-(t)|\rightarrow\rangle|\chi^\perp\rangle$$
$$+\, \eta_-(t)|\leftarrow\rangle|\chi\rangle + \eta_+(t)|\leftarrow\rangle|\chi^\perp\rangle\},$$

where

$$\eta_+(t) = e^{-i(E_\uparrow+E_\mathfrak{a})t/\hbar} + e^{-i(E_\downarrow+E_\mathfrak{d})t/\hbar}$$
$$\eta_-(t) = e^{-i(E_\uparrow+E_\mathfrak{a})t/\hbar} - e^{-i(E_\downarrow+E_\mathfrak{d})t/\hbar}.$$

The cat will participate in the oscillations

$$|\langle\rightarrow\chi|\Xi'(t)\rangle|^2 = |\eta_+(t)/2\sqrt{2}|^2$$
$$|\langle\rightarrow\chi^\perp|\Xi'(t)\rangle|^2 = |\eta_-(t)/2\sqrt{2}|^2$$
$$|\langle\leftarrow\chi|\Xi'(t)\rangle|^2 = |\eta_-(t)/2\sqrt{2}|^2$$
$$|\langle\leftarrow\chi^\perp|\Xi'(t)\rangle|^2 = |\eta_+(t)/2\sqrt{2}|^2.$$

Much as a quantum wave makes a cat oscillate here, it will make a pair of kaons, which could be far apart, oscillate in Chapter VI. 2. Quantum waves in configuration space can therefore, in principle, encompass (1) arbitrarily large objects, and (2) particles that are far apart. Though both are physical problems, in what follows I shall concentrate on the latter question, (2), of separated particles.

V. Separation

1
Interpretations of quantum waves in configuration space

Let us now return to the more explicitly ontological and experimental approach of Part I. In Part IV we saw the tension between the configuration space description and additive conservation (or the criterion of Einstein *et al.*). Here I consider, among other things, what may happen to a 'configuration space' wave as the particles it encompasses separate. Experimental tests involving separated particles are then looked at in Part VI.

In Chapter I. 3 I considered the possibility that, in ordinary space, real quantum waves guide particles, as they do in de Broglie's *théorie de la double solution*. One can wonder, however, what happens in configuration space. There the guidance formula $\boldsymbol{p} = -\operatorname{grad}\varphi$ determines the trajectory, not of a particle, but of a configuration. But how can a single wave guide the point representing the positions of several particles that may be far apart? One has to wonder, furthermore, to what extent something propagating in configuration space should still be called a 'wave.' Perhaps, then, quantum waves are real in ordinary space but not in configuration space. Heisenberg attributed as much reality to quantum waves in ordinary space as to particles or electromagnetic waves, but less to quantum waves in configuration space, which he considered abstract 'probability waves':

> ... nur die Wellen im Konfigurationsraum ... in der üblichen Deutung Wahrscheinlichkeitswellen sind, nicht aber die dreidimensionalen Materie- oder Strahlungswellen. Die letzten sind ebensosehr und ebensowenig objektiv real wie die Teilchen, sie haben unmittelbar nichts mit Wahrscheinlichkeitswellen zu tun, sondern besitzen eine kontinuierliche Energie- und Impulsdichte wie das Maxwellsche Feld.[107]

It seems strange, however, that a single formalism should describe real waves in ordinary space and fake ones in configuration space. If quantum mechanics is correct, quantum waves ought to have the same status in all spaces. "They should either," in the words of R. I. G. Hughes, "be viewed as physical processes in both ordinary space and configuration space, or as mathematical constructions in both."

Suppose a particle u_0 (extending the notation of the *double solution*) accompanied by a wave $v(t)$ decays at time $t = \tau$ into the particles u_0^1 and u_0^2; this could, for instance, be a Φ-meson decaying into a kaon $|K\rangle$ and antikaon $|\overline{K}\rangle$. By continuity, the wave

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[107]Heisenberg (1959). Translation: ... only the waves in configuration space ... are probability waves in the usual interpretation, but not the three-dimensional waves of matter or radiation. For these are just as real as particles, they have nothing to do with probability waves, but possess a continuous energy and momentum density, like the Maxwell field.

$v(\tau + dt)$ right after decay should resemble the wave $v(\tau)$ that accompanied u_0. Perhaps a *single* wave accompanies u_0^1 and u_0^2 when they are still close together (this could be what the configuration space description is somehow getting at). Presumably separation would, however, attenuate the wave and undermine its capacity to convey influences between the particles.

> In fact, Einstein has (in a private communication) actually proposed ... that the current formulation of the many-body problem in quantum mechanics may break down when the particles are far enough apart.[108]

De Broglie had a similar position. Two particles cannot, he claimed, share the same wave when they are far apart:

> Quand deux particules sont sur un même train d'ondes, leurs mouvements, qui dans notre théorie résultent de la loi de guidage et des perturbations subquantiques, sont correlés et c'est cette correlation qui est exprimée par la formule d'antisymétrisation pour les fermions et de symétrisation pour les bosons. Mais, dès que les trains d'ondes se sont séparés, le mouvement de chaque particule dans son train d'ondes devient entièrement indépendant du mouvement que peut avoir l'autre particule dans son train d'ondes éloigné.
>
> La plupart des auteurs qui exposent la Mécanique quantique semblent toujours raisonner comme si les trains d'ondes associés aux particules avaient une longueur infinie. Déjà pour la lumière, si l'on excepte celle qui est émise par les lasers, la longueur des trains d'ondes ne paraît pas de dépasser l'ordre du mètre. Mais, pour les électrons, la longueur des trains d'ondes est de l'ordre du micron ou millionnième de mètre. ...
>
> *En résumé*, M. Bell considère deux électrons qui sont éloignés et portés par un même train d'ondes, mais ces deux hypothèses sont inconciliables.[109]

He appears to suggest, however—*le mouvement de chaque particule dans son train d'ondes devient entièrement indépendant du mouvement que peut avoir l'autre particule dans son train d'ondes éloigné*—that the particles become uncorrelated with separation, which is not the case. Experiments show that the *perfect* correlations indicated by biorthogonal expansions survive separation. For instance, if one of the two kaons described by

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[108]Bohm and Aharonov (1957)

[109]de Broglie (1974). Translation: When two particles are on the same wave packet, their motions, which in our theory result from the guidance formula and from subquantal perturbations, are correlated and it is this correlation which is expressed by the antisymmetrization formula for fermions and the symmetrization formula for bosons. But, once the wave packets have been separated, the motion of each particle in its wave packet becomes entirely independent of any motion of the other in its distant wave packet.

Most authors who deal with Quantum mechanics always seem to reason as though the wave packets associated with the particles had an infinite length. Already for light, if one excepts that which is emitted by lasers, the length of wave packets does not appear to exceed the order of a metre. But for electrons the length of wave packets is of the order of a micron or millionth of a metre. ...

Summing up, Mr. Bell considers two electrons which are far apart and borne along by the same wave packet, but these two hypotheses are unreconcilable.

$$|\kappa\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle|K_L\rangle - |K_L\rangle|K_S\rangle)$$

decays into two pions, the other will always[110] decay into three. Pairs of kaons will be dealt with in Chapter VI. 2.

Schrödinger proposed the following *quantum-mechanical* description of separation:

> ... when two systems separate far enough to make it possible to experiment on one of them without interfering with the other, they are bound to pass, during the process of separation, through stages which were beyond the range of quantum mechanics as it stood then. ... It seems worth noticing that the paradox could be avoided by a very simple assumption, namely if the situation after separating were described by the expansion
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> $$\Psi(x, y) = \sum_k a_k g_k(x) f_k(y),$$
>
> but with the additional statement that the knowledge of the phase relations between the complex constants a_k has been entirely lost in consequence of the process of separation. This would mean that not only the parts, but the whole system, would be in the situation of a mixture, not of a pure state.[111]

This is often known as 'Furry's hypothesis,' since Furry (1935) dealt with a similar idea. In the context of neutral kaon pairs it has been considered by Six (1982) and Piccioni and Mehlhop (1991), and addressed experimentally by the CPLEAR Collaboration (1998). Datta and Home (1986) have applied it to the $B^0\overline{B}^0$ system:[112]

> As an example, let us consider Furry's hypothesis in the following form: The wavefunction has the non-separable form ... at the production of the $B^0\overline{B}^0$ pair, but after spatial separation between the two particles the wavefunction becomes an equal mixture (not superposition) of the two independent states $B_H(1)B_L(2)$ and $B_L(1)B_H(2)$. One is tempted to envisage this hypothesis because it enables to avoid the conceptual anomalies arising from the quantum non-separability presented by the EPR paradox.

Separation, then, would turn the pure state described by

$$|\Psi\rangle = \sum_k a_k |g_k \otimes f_k\rangle$$

into the mixture represented by the statistical operator

$$\rho = \sum_k |a_k|^2 |g_k \otimes f_k\rangle\langle g_k \otimes f_k|.$$

---

[110] Or rather, it *always* would if charge-parity were conserved.

[111] Schrödinger (1936)

[112] The $B^0\overline{B}^0$ system is very similar to the $K^0\overline{K}^0$ system discussed in VI. 2, which is essentially described by the same formalism.

The idea can be given the following expression: no wave can survive arbitrary separation; waves are related to phase; if the wave breaks down, maybe phase relations are undermined. Schrödinger was not, however, entirely sure:

> This is a very incomplete description and I would not stand for its adequateness. But I would call it a possible one, until I am told, either why it is devoid of meaning or with which experiments it disagrees. My point is, that in a domain which the present theory does not cover, there is room for new assumptions without necessarily contradicting the theory in that region where it is backed by experiment.

As the configuration space description leads to accurate predictions when the entangled particles are close together ("that region in which it is backed by experiment"), Schrödinger suggests that "there is room for new assumptions," namely that ρ may become a better description than $|\Psi\rangle$, when the particles are far apart ("in a domain which the present theory does not cover"[113]).

Bohm and Aharonov (1957) express a similar belief:

> At first sight it would seem that there exists at present no experimental proof that the paradoxical behavior described by ERP will really occur. If this is so, then we are free to consider the assumption that perhaps the difficulty comes from the yet experimentally unverified extrapolation of the many-body Schrödinger and Dirac equations to the case where the particle's wave functions do not overlap and where the particles do not interact.

They do not go as far as Schrödinger: despite recognizing that the configuration space description may break down for separations at which experimental evidence is lacking, Bohm and Aharonov do not then suggest that the statistical operator ρ becomes a better description than $|\Psi\rangle$ when the particles are far apart.

Having discarded de Broglie's suggestion that the particles become uncorrelated as they separate, we are left with three possibilities: ordinary quantum mechanics always works; quantum mechanics ought to be modified as Schrödinger suggests; or quantum mechanics simply breaks down with separation. I shall consider the possibility that quantum mechanics breaks down and "that perhaps the difficulty comes from the yet experimentally unverified extrapolation of the many-body Schrödinger and Dirac equations to the case where the particles" are far apart. Before suggesting "that there exists at present no experimental proof that the paradoxical behavior described by ERP[114] will really occur," but that experiments with kaons will be able to decide whether ordinary quantum mechanics survives separation or breaks down, I shall look at Furry's hypothesis.

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[113]Perhaps he means a domain that is not as well backed by experiment.
[114]By this I imagine they mean interference in configuration space.

2
Furry's hypothesis

According to Schrödinger's expression of 'Furry's hypothesis,' the statistical operator

$$\rho = \sum_k |a_k|^2 |g_k \otimes f_k\rangle\langle g_k \otimes f_k|$$

becomes, at large separations, a better description of the particles once described by

$$|\Psi\rangle = \sum_k a_k |g_k \otimes f_k\rangle.$$

Experiments tell us, however, that *rotationally invariant* perfect anticorrelations predicted by the singlet

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

(a special case of $|\Psi\rangle$) but not by the statistical operator

$$\chi = \frac{1}{2}(|\uparrow\ \downarrow\rangle\langle\uparrow\ \downarrow| + |\downarrow\ \uparrow\rangle\langle\downarrow\ \uparrow|)$$

(a special case of ρ) *do* in fact survive separation. So the singlet is a better description than χ for the separations in question.

With respect to any operator Q with the same eigenvectors as χ, the states $|\Sigma\rangle$ and χ will be indistinguishable, in other words

$$\mathrm{Tr}(Q|\Sigma\rangle\langle\Sigma|) = \mathrm{Tr}(Q\chi).$$

Neither description, for instance, allows coincidences:

$$\mathrm{Tr}(P|\Sigma\rangle\langle\Sigma|) = \mathrm{Tr}(P\chi) = 0,$$

where the operator

$$P = |\uparrow\ \uparrow\rangle\langle\uparrow\ \uparrow| + |\downarrow\ \downarrow\rangle\langle\downarrow\ \downarrow|$$

projects onto the 'coincidences' subspace spanned by $|\uparrow\ \uparrow\rangle$ and $|\downarrow\ \downarrow\rangle$.

The equality of the moduli $|1/\sqrt{2}|$ and $|-1/\sqrt{2}|$ means that the singlet can be given AN infinite NUMBER OF biorthogonal expansions, which all give rise to the same perfect anticorrelations. Even the phase difference π will be preserved:

$$|\Sigma\rangle = (U \otimes U)|\Sigma\rangle = \frac{1}{\sqrt{2}}(U|\uparrow\rangle \otimes U|\downarrow\rangle - U|\downarrow\rangle \otimes U|\uparrow\rangle)$$

for every unitary operator U. The singlet will, in other words, have the same expansion coefficients with respect to the eigenvectors of any operator with representation

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}.$$

This is the *rotational invariance* referred to above. With respect to the left-right basis, for instance,

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}(|\leftarrow\rangle|\rightarrow\rangle - |\rightarrow\rangle|\leftarrow\rangle).$$

In the corresponding expansion of χ, on the other hand, the diagonal elements

$$\begin{aligned} &\langle\leftarrow\ \leftarrow|\chi|\leftarrow\ \leftarrow\rangle = \langle\leftarrow\ \rightarrow|\chi|\leftarrow\ \rightarrow\rangle \\ &= \langle\rightarrow\ \leftarrow|\chi|\rightarrow\ \leftarrow\rangle = \langle\rightarrow\ \rightarrow|\chi|\rightarrow\ \rightarrow\rangle = \frac{1}{4}, \end{aligned}$$

and hence the corresponding outcomes will be equiprobable. So will left-left or right-right coincidences, and even left-right or right-left anticoincidences:

$$\begin{aligned} &\mathrm{Tr}(\chi P) = \langle\leftarrow\ \leftarrow|\chi|\leftarrow\ \leftarrow\rangle + \langle\rightarrow\ \rightarrow|\chi|\rightarrow\ \rightarrow\rangle = \frac{1}{2} \\ &= \mathrm{Tr}(\chi P^{\perp}) = \langle\leftarrow\ \rightarrow|\chi|\leftarrow\ \rightarrow\rangle + \langle\rightarrow\ \leftarrow|\chi|\rightarrow\ \leftarrow\rangle. \end{aligned}$$

Naturally $\mathrm{Tr}(|\Sigma\rangle\langle\Sigma|P) = 0$ and $\mathrm{Tr}(|\Sigma\rangle\langle\Sigma|P^{\perp}) = 1$.

Representing the more general statistical operator

$$\rho_{\boldsymbol{a}} = \frac{1}{2}|\psi_+(\boldsymbol{a})\psi_-(\boldsymbol{a})\rangle\langle\psi_+(\boldsymbol{a})\psi_-(\boldsymbol{a})| + \frac{1}{2}|\psi_-(\boldsymbol{a})\psi_+(\boldsymbol{a})\rangle\langle\psi_-(\boldsymbol{a})\psi_+(\boldsymbol{a})|$$

with respect to the basis $|\psi_\pm(\boldsymbol{a})\psi_\pm(\boldsymbol{a})\rangle$, both elements corresponding to anticoincidences on the diagonal

$$\begin{aligned} \langle\psi_+(\boldsymbol{a})\psi_+(\boldsymbol{a})|\rho_{\boldsymbol{a}}|\psi_+(\boldsymbol{a})\psi_+(\boldsymbol{a})\rangle &= 0 \\ \langle\psi_+(\boldsymbol{a})\psi_-(\boldsymbol{a})|\rho_{\boldsymbol{a}}|\psi_+(\boldsymbol{a})\psi_-(\boldsymbol{a})\rangle &= \frac{1}{2} \\ \langle\psi_-(\boldsymbol{a})\psi_+(\boldsymbol{a})|\rho_{\boldsymbol{a}}|\psi_-(\boldsymbol{a})\psi_+(\boldsymbol{a})\rangle &= \frac{1}{2} \\ \langle\psi_-(\boldsymbol{a})\psi_-(\boldsymbol{a})|\rho_{\boldsymbol{a}}|\psi_-(\boldsymbol{a})\psi_-(\boldsymbol{a})\rangle &= 0 \end{aligned}$$

vanish, along with $\mathrm{Tr}(\rho_{\boldsymbol{a}}P_{\boldsymbol{a}})$, where

$$P_{\boldsymbol{a}} = |\psi_+(\boldsymbol{a})\psi_+(\boldsymbol{a})\rangle\langle\psi_+(\boldsymbol{a})\psi_+(\boldsymbol{a})| + |\psi_-(\boldsymbol{a})\psi_-(\boldsymbol{a})\rangle\langle\psi_-(\boldsymbol{a})\psi_-(\boldsymbol{a})|.$$

For every other unit vector $\boldsymbol{b}$ the zeros disappear from the diagonal, and with them the perfect anticorrelations, to obtain which the basis $|\psi_\pm(\boldsymbol{a})\psi_\pm(\boldsymbol{a})\rangle$ is therefore sufficient *and necessary*:

$$\mathrm{Tr}\left(\rho_{\boldsymbol{a}} P_{\boldsymbol{b}}\right) = \frac{1}{2}\sin^2\angle \boldsymbol{a}, \boldsymbol{b},$$

where $\angle \boldsymbol{a}, \boldsymbol{b}$ is the angle between the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Naturally $\mathrm{Tr}\left(|\Sigma\rangle\langle\Sigma| P_{\boldsymbol{a}}\right)$ vanishes identically.

Such perfect anticorrelations, which are experimentally known to survive separation with respect to all directions $\boldsymbol{a}$, can therefore only be predicted by *superpositions* of products, at least within quantum mechanics. Mixtures of products cannot reproduce them. So if quantum mechanics breaks down with separation, it simply breaks down. Separation does not, for instance, simply disentangle one quantum-mechanical state to produce another.

As entangled states have done better than alternative quantum-mechanical descriptions, their empirical adequacy has to be judged with respect to other criteria. The predictions of quantum mechanics can be compared experimentally with Bell's inequality, which is derived from principles at odds with the propagation of waves in configuration space. Even though entangled states predict perfect correlations not predicted by mixtures of products, they can still be refuted by an experimental satisfaction of Bell's inequality.

VI. Experiments with correlated particles

1
Photons

So far photons have been used to test the accuracy of the configuration space description for widely separated particles. Bell (1986) expressed reservations, however:

> I always emphasize that the Aspect experiment is too far from the ideal in *many* ways—counter efficiency is only one of them. And I always emphasize that there is therefore a big extrapolation from practical present-day experiments to the conclusion that nonlocality holds. I myself choose to make the extrapolation, for the purpose at least of directing my own future researches. If other people choose differently, I wish them every success and I will watch for their results ....

We can choose differently and try to determine whether quantum waves really propagate in configuration space using kaons rather than photons. In this chapter, however, the 'detector efficiency loophole' of the photon experiments will be considered.

Bell's inequality is a useful criterion here: if it is satisfied when quantum mechanics predicts a violation, the configuration space description is wrong; if it is genuinely violated experimentally, local realism is refuted, and *something*—perhaps empty mathematics—propagates in configuration space. Provided it makes sense to apply the 'reality criterion' of Einstein *et al.*, an experimental violation of Bell's inequality would appear to mean that the rotation of one apparatus can change an element of reality of the other particle. The influence in question has so much to do with the arguments of the entangled state's expansion coefficients[115] that it makes sense to call it an *interference* effect. Indeed it seems as though the influence would be transmitted, if at all, by a quantum wave in configuration space.

As long as inefficient detectors are used, photons can only be used to violate *strong inequalities*,[116] deduced with the help of questionable additional assumptions, rather than *weak inequalities*, deduced from local realism alone. Strong inequalities are typically twenty or thirty times stronger than the weak ones, in the sense that they restrict the same measurable quantity to an interval twenty or thirty times narrower. So far they have always been violated experimentally.[117] Weak inequalities, on the other hand, would disagree with the predictions of quantum mechanics only in the case of nearly perfect instruments, and have never been violated.

The additional assumptions have been given various expressions, all of which have

[115]Bell's inequality cannot be violated by mixtures of factorizable states; see Capasso *et al.* (1973).
[116]This distinction, and much of the theory that follows, is due to Selleri; see Selleri and Lepore (1990), Afriat and Selleri (1998).
[117]At least for all *published* experiments. The results of the Holt-Pipkin (1974) experiment, in which Bell's inequality was not violated, were never published.

the same consequences. Clauser, Horne, Shimony and Holt (1969), for instance, assumed the following:

> Given a pair of photons emerging from two regions of space where two polarizers can be located, the probability of their joint detection from two photomultipliers does not depend on the presence and the orientation of the polarizers.

These additional assumptions are arbitrary and do not rest on any fundamental physical principle. Furthermore there is no experiment to test them independently of local realism.

In this chapter I adopt the language of 'hidden variables,' which are more general than 'elements of reality.' The hidden variable λ, if deterministic, would *fix* the values of the observables in question, in which case one could just as well speak of 'elements of reality.' More generally the hidden variable could determine not outcomes but their probabilities, by adding *something* to the standard quantum-mechanical description, but not enough to fix the results of all measurements. Redhead speaks of 'stochastic' hidden variable theories:

> The idea of such theories is that the 'complete' hidden-variable description of the source does not determine the values of local observables possessed by the two particles in the Bell type of experiment, but only the probabilities for possible values to occur. We can think picturesquely that the values of the spin-components in any given direction are developing in time stochastically, the state of the source controlling only the probabilities that particular values will be revealed when subsequent measurements are performed.[118]

The added generality does not weaken the argument; if one prefers certainties the probabilities can be restricted to the set $\{0, 1\}$. Perfect correlations can give the impression that hidden variables determine outcomes rather than probabilities, and indeed perhaps they do. Such correlations are not exhibited experimentally, however: even if both polarizers have the same alignment it remains likely that *neither* photon will be detected.

Photons that enter detectors do not always make them click. It could be—taking determinism to its extreme consequences—that not only transmission but also detection is determined by the hidden variable λ (which might be changed by the polarizer), in which case the probabilities of transmission, detection, and transmission & detection would be equal to zero or one, the additional assumptions would be inapplicable,[119] and one would end up with experimentally unviolable weak inequalities. We therefore allow probabilities of detection between zero and one, as strong inequalities are also of interest. But then it makes sense to allow intermediate probabilities of transmission as well. The hidden variable should be viewed either as determining whether a photon is transmitted

---

[118] Redhead (1987) pp.98-9

[119] The interval figuring in the Clauser-Horne lemma, presently to be discussed, would run from $-X^1X^2 = -1$ to 0.

and whether it is detected, or as giving the probabilities of *both*. Why should it *determine* whether a photon is transmitted, but only specify the *probability* of detection? The roles of photomultipliers and polarizers are not different enough to deserve treatments as different as they are sometimes given. The anticorrelations predicted by theory when the same axis is chosen on either side—which might be taken to indicate a determinism concerning transmission, but not detection—are after all rendered empirically inaccessible by the inefficiencies of the photomultipliers.

1.1 Preliminaries

The experiments in question involve a source emitting pairs of photons. A polarizer placed on the path of the σth photon can have axis $\mathfrak{x}^\sigma = a^\sigma$ or $\mathfrak{x}^\sigma = b^\sigma$, $\sigma = 1, 2$. One-way polarizers allow *absorption*, which cannot be detected, or transmission, which can. Two-way polarizers instead allow *reflection*, which can be detected, or transmission. The following notation will be used.

> $T^\sigma(\mathfrak{x}^\sigma, \lambda)$ is the probability that the σth photon, described by the hidden variable λ when it left the source, is transmitted by a polarizer with axis $\mathfrak{x}^\sigma$ ($\mathfrak{x}^\sigma = a^\sigma, b^\sigma$; $\sigma = 1, 2$).
>
> $R^\sigma(\mathfrak{x}^\sigma, \lambda)$ is the probability that the σth photon, described by λ when it left the source, is reflected by a two-way polarizer with axis $\mathfrak{x}^\sigma$.
>
> $D^\sigma_T(\mathfrak{x}^\sigma, \lambda)$ is the probability that the σth photon, described by λ when it left the source, is detected by a photomultiplier once it has been transmitted by a polarizer with axis $\mathfrak{x}^\sigma$. For one-way polarizers the subscript T will be dropped.
>
> $D^\sigma_R(\mathfrak{x}^\sigma, \lambda)$ is the probability that the σth photon, described by λ when it left the source, is detected by a photomultiplier once it has been reflected by a polarizer with axis $\mathfrak{x}^\sigma$.
>
> $D^\sigma(\infty, \lambda)$ is the probability that the σth photon, described by λ when it left the source, and on whose trajectory no polarizer is present, is detected by the σth photomultiplier.

Again, $\sigma = 1, 2$; the first polarizer can be aligned along a^1 or b^1, the second along a^2 or b^2. For one-way polarizers $R^\sigma(\mathfrak{x}^\sigma, \lambda) = D^\sigma_R(\mathfrak{x}^\sigma, \lambda) = 0$, which means that the reflected path is not available.

Let us define the following joint detection probabilities for two-way polarizers:

$$\begin{aligned}
\omega(\mathfrak{x}^1_+, \mathfrak{x}^2_+) &= \int d\lambda \rho(\lambda) T^1(\mathfrak{x}^1, \lambda) D^1_T(\mathfrak{x}^1, \lambda) T^2(\mathfrak{x}^2, \lambda) D^2_T(\mathfrak{x}^2, \lambda) \qquad (24) \\
\omega(\mathfrak{x}^1_+, \mathfrak{x}^2_-) &= \int d\lambda \rho(\lambda) T^1(\mathfrak{x}^1, \lambda) D^1_T(\mathfrak{x}^1, \lambda) R^2(\mathfrak{x}^2, \lambda) D^2_R(\mathfrak{x}^2, \lambda) \\
\omega(\mathfrak{x}^1_-, \mathfrak{x}^2_+) &= \int d\lambda \rho(\lambda) R^1(\mathfrak{x}^1, \lambda) D^1_R(\mathfrak{x}^1, \lambda) T^2(\mathfrak{x}^2, \lambda) D^2_T(\mathfrak{x}^2, \lambda) \\
\omega(\mathfrak{x}^1_-, \mathfrak{x}^2_-) &= \int d\lambda \rho(\lambda) R^1(\mathfrak{x}^1, \lambda) D^1_R(\mathfrak{x}^1, \lambda) R^2(\mathfrak{x}^2, \lambda) D^2_R(\mathfrak{x}^2, \lambda),
\end{aligned}$$

where $\rho(\lambda)$ is a normalized density function:

$$\int \rho(\lambda) d\lambda = 1.$$

Let us also define the correlation function

$$P(\mathfrak{r}^1, \mathfrak{r}^2) = \omega(\mathfrak{r}^1_+, \mathfrak{r}^2_+) - \omega(\mathfrak{r}^1_+, \mathfrak{r}^2_-) - \omega(\mathfrak{r}^1_-, \mathfrak{r}^2_+) + \omega(\mathfrak{r}^1_-, \mathfrak{r}^2_-), \tag{25}$$

and the single-particle detection probabilities

$$t^\sigma(\mathfrak{r}^\sigma) = \int d\lambda \rho(\lambda) T^\sigma(\mathfrak{r}^\sigma, \lambda) D^\sigma_T(\mathfrak{r}^\sigma, \lambda)$$
$$r^\sigma(\mathfrak{r}^\sigma) = \int d\lambda \rho(\lambda) R^\sigma(\mathfrak{r}^\sigma, \lambda) D^\sigma_R(\mathfrak{r}^\sigma, \lambda)$$

where t refers to transmission and r to reflection, $\sigma = 1, 2$.

Consider the following lemma, due to Clauser and Horne (1974): *If*

$$0 \leq x^\sigma, y^\sigma \leq X^\sigma, \tag{26}$$

$\sigma = 1, 2$, *then*

$$-X^1 X^2 \leq \Gamma \leq 0, \tag{27}$$

where

$$\Gamma = x^1 x^2 - x^1 y^2 + y^1 x^2 + y^1 y^2 - y^1 X^2 - X^1 x^2.$$

This can be proved as follows. Since Γ is linear in each of the variables x^1, y^1, x^2, y^2 its maximum and minimum will lie on the boundary, where these four variables assume their extreme values. From (26) it is clear the boundary consists of 16 points. The values of Γ at these points are given in Table 1 below. The function Γ assumes the values 0 and $-X^1 X^2$ eight times each, and no other values. So (27) has to be satisfied.

To simplify and illustrate, suppose all transmitted photons are detected, so that

$$D^\sigma_T(a^\sigma, \lambda) = D^\sigma_T(b^\sigma, \lambda) = 1,$$

$\sigma = 1, 2$, or that we are just interested in transmission; what we want is to disregard detection for the time being. Suppose furthermore that the hidden variable λ *determines* whether transmission occurs, so the probabilities

$$T^1(a^1, \lambda),\ T^1(b^1, \lambda),\ T^2(a^2, \lambda),\ T^2(b^2, \lambda)$$

are in fact certainties, and can only assume the values one or zero. Suppose finally, as a specific example, that λ results in the *transmission* of the first photon along a^1, its *absorption* along b^1, and the *transmission* of the second photon along a^2 *and* b^2. Setting

$$x^1 = T^1(a^1, \lambda) = 1, \quad x^2 = T^2(a^2, \lambda) = 1,$$
$$y^1 = T^1(b^1, \lambda) = 0, \quad y^2 = T^2(b^2, \lambda) = 1,$$

we have $X^1 = X^2 = 1$, and indeed

$$\Gamma = (1)(1) - (1)(1) + (0)(1) + (0)(1) - (0)(1) - (1)(1) = -1 \tag{28}$$

lies between $-X^1X^2 = -1$ and 0, as it should according to the lemma.

Table 1

	x^1	y^1	x^2	y^2	Γ
1	0	0	0	0	0
2	0	0	0	X^2	0
3	0	0	X^2	0	$-X^1X^2$
4	0	0	X^2	X^2	$-X^1X^2$
5	0	X^1	0	0	$-X^1X^2$
6	0	X^1	0	X^2	0
7	0	X^1	X^2	0	$-X^1X^2$
8	0	X^1	X^2	X^2	0
9	X^1	0	0	0	0
10	X^1	0	0	X^2	$-X^1X^2$
11	X^1	0	X^2	0	0
12	X^1	0	X^2	X^2	$-X^1X^2$
13	X^1	X^1	0	0	$-X^1X^2$
14	X^1	X^1	0	X^2	$-X^1X^2$
15	X^1	X^1	X^2	0	0
16	X^1	X^1	X^2	X^2	0

Now let us see why the inequality $-1 \leq \Gamma \leq 0$ for this deterministic case is a consequence of local realism. Suppose that, for equal orientations $\mathfrak{r}^1 = \mathfrak{r}^2$ of the polarizers, if one of the photons is transmitted the other is not; in other words that either

$$[T^1(\mathfrak{r}, \lambda) = 1] \Leftrightarrow [T^2(\mathfrak{r}, \lambda) = 0]$$

or

$$[T^1(\mathfrak{r}, \lambda') = 0] \Leftrightarrow [T^2(\mathfrak{r}, \lambda') = 1]$$

for all directions $\mathfrak{r}$. In such cases the criterion of Einstein *et al.* can be applied and elements of reality assigned on either side. In the derivation of Bell's inequality (Section IV 1.3) the elements of reality corresponded to the eigenvalues ± 1, here to the values 0 and 1 taken on by x^1, y^1, x^2, y^2:

$$
\begin{aligned}
+1 &\leftrightarrow [T^\sigma(\mathfrak{x}^\sigma,\lambda)=1] \leftrightarrow \text{transmission}\\
-1 &\leftrightarrow [T^\sigma(\mathfrak{x}^\sigma,\lambda)=0] \leftrightarrow \text{absorption,}
\end{aligned}
$$

$\sigma = 1, 2$.

Locality is again given by Redhead's LOC_3: the values zero and one do not depend on the orientation of the *other* apparatus. So in the first term of (28) the value $x^1 = 1$ can be assigned to the first photon whatever the orientation of the other polarizer. One way of looking at this is that the values are first assigned where the correlations are perfect and the criterion of Einstein *et al.* can be applied; then it is assumed—LOC_3—that each value survives rotations of the other apparatus.

1.2 Weak inequalities

Let us now deduce inequalities using the Clauser-Horne lemma. Take for instance

$$
x^\sigma = T^\sigma(a^\sigma,\lambda)D_T^\sigma(a^\sigma,\lambda), \quad y^\sigma = T^\sigma(b^\sigma,\lambda)D_T^\sigma(b^\sigma,\lambda),
$$

$\sigma = 1, 2$. If nothing else is assumed, x^1, y^1, x^2 and y^2 could all assume the value 1 for some λ. So $X^1 = X^2 = 1$ in (26), and (27) becomes

$$
\begin{aligned}
-1 \le\, & T^1(a^1,\lambda)D_T^1(a^1,\lambda)T^2(a^2,\lambda)D_T^2(a^2,\lambda)\\
&- T^1(a^1,\lambda)D_T^1(a^1,\lambda)T^2(b^2,\lambda)D_T^2(b^2,\lambda)\\
&+ T^1(b^1,\lambda)D_T^1(b^1,\lambda)T^2(a^2,\lambda)D_T^2(a^2,\lambda)\\
&+ T^1(b^1,\lambda)D_T^1(b^1,\lambda)T^2(b^2,\lambda)D_T^2(b^2,\lambda)\\
&- T^1(b^1,\lambda)D_T^1(b^1,\lambda) - T^2(a^2,\lambda)D_T^2(a^2,\lambda) \le 0.
\end{aligned}
$$

Multiplying this last equation by $\rho(\lambda)$ and integrating over λ the Clauser-Horne-Shimony-Holt (1969) inequality is obtained:

$$
\begin{aligned}
-1 \le\, & \omega(a_+^1, a_+^2) - \omega(a_+^1, b_+^2) + \omega(b_+^1, a_+^2)\\
&+ \omega(b_+^1, b_+^2) - t^1(b^1) - t^2(a^2) \le 0,
\end{aligned}
\tag{29}
$$

where the joint detection probabilities were defined in (24). Considering transmission for the first photon and reflection for the second, and taking

$$
\begin{aligned}
x^1 &= T^1(a^1,\lambda)D_T^1(a^1,\lambda)\\
y^1 &= T^1(b^1,\lambda)D_T^1(b^1,\lambda)\\
x^2 &= R^2(a^2,\lambda)D_R^2(a^2,\lambda)\\
y^2 &= R^2(b^2,\lambda)D_R^2(b^2,\lambda)
\end{aligned}
$$

(again $X^1 = X^2 = 1$), we obtain

$$-1 \leq \omega(a_+^1, a_-^2) - \omega(a_+^1, b_-^2) + \omega(b_+^1, a_-^2) + \omega(b_+^1, b_-^2) - t^1(b^1) - r^2(a^2) \leq 0. \tag{30}$$

Taking the reflection channel for the first photon and the transmission channel for the second:

$$-1 \leq \omega(a_-^1, a_+^2) - \omega(a_-^1, b_+^2) + \omega(b_-^1, a_+^2) + \omega(b_-^1, b_-^2) - r^1(b^1) - t^2(a^2) \leq 0. \tag{31}$$

Considering reflection alone for both photons, we have:

$$-1 \leq \omega(a_-^1, a_-^2) - \omega(a_-^1, b_-^2) + \omega(b_-^1, a_-^2) + \omega(b_-^1, b_-^2) - r^1(b^1) - r^2(a^2) \leq 0. \tag{32}$$

The inequalities (29)-(32) are all necessary consequences of local realism.

In this more general case involving probabilities other than just zero or one, hidden variables describe *real propensities,*[120] which belong to the photons and give rise to the specified probabilities. The propensity of one photon to produce a given outcome is assumed not to depend on the orientation of the other polarizer.

We can now deduce an inequality like Bell's for correlation functions, by changing the signs of (30) and (31) and adding the resulting inequalities to (29) and (32):

$$-2 \leq P(a^1, a^2) - P(a^1, b^2) + P(b^1, a^2) + P(b^1, b^2) \leq 2, \tag{33}$$

where the correlation functions involved were defined in (24). For one-way polarizers only (29) is meaningful and (30)-(33) do not correspond to measurable cases.

All of the above inequalities have been deduced from local realism alone. They are called 'weak' to distinguish them from 'strong' inequalities.

1.3 Strong inequalities for one-way polarizers

Again, photons are not reflected in experiments involving one-way polarizers. They are either transmitted, in which case they can be detected, or they are absorbed and not detected. I have adapted my notation accordingly, and dropped the subscript T in the detection probabilities.

The quantum efficiencies η^σ of the photomultipliers will be taken to equal the λ-averages of D^σ:

$$\eta^\sigma = \langle D^\sigma(\mathfrak{r}^\sigma, \lambda)\rangle_\lambda = \int d\lambda \rho(\lambda) D^\sigma(\mathfrak{r}^\sigma, \lambda),$$

$\sigma = 1, 2$. Values assumed in certain experiments are shown in Table 2.

---

[120] See Barretto Bastos Filho and Selleri (1995).

Table 2

	η^1	η^2	$\eta^1\eta^2$
Freedman-Clauser (1972)	0.13	0.28	0.0364
Holt-Pipkin (1974)	0.08	0.27	0.0216
Clauser (1976)	0.26	0.07	0.0182
Aspect (1982b)	0.06	0.25	0.0150
Perrie *et al.* (1985)	0.20	0.20	0.0400

The measurable joint detection probabilities are given by

$$\begin{aligned}
\omega_0 = \omega(\infty,\infty) &= \int d\lambda \rho(\lambda) D^1(\infty,\lambda) D^2(\infty,\lambda) \qquad (34)\\
\omega(\mathfrak{r}^1_+,\infty) &= \int d\lambda \rho(\lambda) T^1(\mathfrak{r}^1,\lambda) D^1(\mathfrak{r}^1,\lambda) D^2(\infty,\lambda)\\
\omega(\infty,\mathfrak{r}^2_+) &= \int d\lambda \rho(\lambda) D^1(\infty,\lambda) T^2(\mathfrak{r}^2,\lambda) D^2(\mathfrak{r}^2,\lambda)\\
\omega(\mathfrak{r}^1_+,\mathfrak{r}^2_+) &= \int d\lambda \rho(\lambda) T^1(\mathfrak{r}^1,\lambda) D^1(\mathfrak{r}^1,\lambda) T^2(\mathfrak{r}^2,\lambda) D^2(\mathfrak{r}^2,\lambda).
\end{aligned}$$

Clauser, Horne, Shimony and Holt (1969) made the assumption

> Given that a pair of photons emerge from two regions of space where two polarizers can be located, the probability of their joint detection from two photomultipliers $D^{12}(\lambda)$ is independent of the presence and the orientation of the polarizers.

which allows one to write

$$D^1(\mathfrak{u},\lambda) D^2(\mathfrak{v},\lambda) = D^{12}(\lambda), \qquad (35)$$

where $\mathfrak{u}$ and $\mathfrak{v}$ can stand for ∞ (no polarizer) or any angles. But the polarizer might *increase* the probability of detection, especially if detection depends on the hidden variable λ, which could be changed by the polarizer. Consider the following example. Let us use the term 'detector' to denote both a horizontally aligned polarizer $\mathfrak{b}$ and a photomultiplier $\mathfrak{c}$ behind it. A 'detection' will therefore involve both objects that make up the detector $\mathfrak{b}+\mathfrak{c}$: a photon is detected when it gets through $\mathfrak{b}$ *and* makes $\mathfrak{c}$ click. As vertically polarized light will never get detected by $\mathfrak{b}+\mathfrak{c}$—its probability of detection vanishes—an oblique polarizer $\mathfrak{a}$ placed in front of $\mathfrak{b}$ will *increase* the probability of detection.

For the inequalities (27) we can write

$$x^\sigma = T^\sigma(a^\sigma,\lambda), \quad y^\sigma = T^\sigma(b^\sigma,\lambda)$$

with $X^\sigma = 1$, $\sigma = 1,2$, so that

$$\begin{aligned}-1 \le\ & T^1(a^1,\lambda)T^2(a^2,\lambda) - T^1(a^1,\lambda)T^2(b^2,\lambda) \\ & + T^1(b^1,\lambda)T^2(a^2,\lambda) + T^1(b^1,\lambda)T^2(b^2,\lambda) \\ & - T^1(b^1,\lambda) - T^2(a^2,\lambda) \le 0.\end{aligned} \tag{36}$$

Because of (35) we can multiply the whole expression by $\rho(\lambda)D^{12}(\lambda)$ and integrate over λ to get

$$\begin{aligned}-\omega_0 \le\ & \omega(a^1_+, a^2_+) - \omega(b^1_+, a^2_+) + \omega(a^1_+, b^2_+) \\ & + \omega(b^1_+, b^2_+) - \omega(b^1_+, \infty) - \omega(\infty, a^2_+) \le 0\end{aligned} \tag{37}$$

which is the basic strong inequality for one-way polarizers.

The difference with respect to the weak inequality (29) is made by the presence of $-\omega_0$ on the left. The quantum prediction $\omega_0 = \eta^1\eta^2$ is neither surprising nor paradoxical, and does not in itself indicate the propagation of quantum waves in configuration space. To see how it 'strengthens' the weak inequality (29), which can be rewritten

$$-1 + t^1(b^1) + t^2(a^2) \le \Omega \le t^1(b^1) + t^2(a^2), \tag{38}$$

the strong inequality (37) can also be rewritten

$$-\omega_0 + \omega(b^1_+, \infty) + \omega(\infty, a^2_+) \le \Omega \le \omega(b^1_+, \infty) + \omega(\infty, a^2_+), \tag{39}$$

where

$$\Omega = \omega(a^1_+, a^2_+) - \omega(b^1_+, a^2_+) + \omega(a^1_+, b^2_+) + \omega(b^1_+, b^2_+).$$

The single-particle probabilities in (38) and the coincidence rates $\omega(b^1_+, \infty)$, $\omega(\infty, a^2_+)$ in (39) have non-paradoxical expressions in quantum theory, which do not in themselves indicate interference effects in configuration space. The interval to which Ω must be restricted has length 1 in (38) but only length $\omega_0 = \eta^1\eta^2 \le 0.04$ in (39), however, as can be seen in Tables 2 and 3.

Table 3

	weak inequality	strong inequality
Freedman-Clauser (1972)	$-0.794 \le \Omega \le 0.206$	$0.000 \le \Omega \le 0.037$
Holt-Pipkin (1974)	$-0.845 \le \Omega \le 0.155$	$-0.002 \le \Omega \le 0.019$
Clauser (1976)	$-0.838 \le \Omega \le 0.162$	$0.000 \le \Omega \le 0.018$
Aspect (1982b)	$-0.845 \le \Omega \le 0.155$	$0.000 \le \Omega \le 0.015$
Perrie *et al.* (1985)	$-0.812 \le \Omega \le 0.188$	$-0.002 \le \Omega \le 0.038$

Clauser and Horne (1974) made the additional assumption

> For every photon in the state λ the probability of detection with a polarizer placed on its trajectory is less than or equal to the detection probability with the polarizer removed.

which means that

$$D^\sigma(\mathfrak{x}^\sigma,\lambda) \le D^\sigma(\infty,\lambda), \tag{40}$$

$\sigma = 1, 2$. To deduce the strong inequality we can use the Clauser-Horne lemma with

$$x^\sigma = T^\sigma(a^\sigma,\lambda)D^\sigma(a^\sigma,\lambda), \quad y^\sigma = T^\sigma(b^\sigma,\lambda)D^\sigma(b^\sigma,\lambda), \quad X^\sigma = D^\sigma(\infty,\lambda),$$

$\sigma = 1, 2$, which clearly satisfy (26). Indeed if (40) holds as it is, it will hold *a fortiori* if $D^\sigma(\mathfrak{x}^\sigma,\lambda)$ is multiplied by a transition probability. Therefore

$$\begin{aligned} -D^1(\infty,\lambda)D^2(\infty,\lambda) \le\ & T^1(a^1,\lambda)D^1(a^1,\lambda)T^2(a^2,\lambda)D^2(a^2,\lambda) \\ & - T^1(a^1,\lambda)D^1(a^1,\lambda)T^2(b^2,\lambda)D^2(b^2,\lambda) \\ & + T^1(b^1,\lambda)D^1(b^1,\lambda)T^2(a^2,\lambda)D^2(a^2,\lambda) \\ & + T^1(b^1,\lambda)D^1(b^1,\lambda)T^2(b^2,\lambda)D^2(b^2,\lambda) \\ & - T^1(b^1,\lambda)D^1(b^1,\lambda)D^2(\infty,\lambda) \\ & - D^1(\infty,\lambda)T^2(a^2,\lambda)D^2(a^2,\lambda) \le 0. \end{aligned}$$

Multiplying by $\rho(\lambda)$, integrating over λ, and remembering definitions (34), we again get the same inequality (37) that was obtained using the additional assumption made by Clauser, Horne, Shimony and Holt.

Aspect (1983) assumed the following:

> The set of detected pairs with a given orientation of the polarizers is an undistorted representative sample of the set of pairs emitted by the source.

This means that every double detection & transmission probability is proportional to its corresponding double transmission probability, and that the proportionality constant is always the same. In order to apply this assumption let us define the following double transmission probabilities:

$$\begin{aligned} \tau(\mathfrak{x}^1_+,\mathfrak{x}^2_+) &= \int d\lambda\rho(\lambda)T^1(\mathfrak{x}^1,\lambda)T^2(\mathfrak{x}^2,\lambda) \\ \tau(\mathfrak{x}^1_+,\infty) &= \int d\lambda\rho(\lambda)T^1(\mathfrak{x}^1,\lambda) \\ \tau(\infty,\mathfrak{x}^2_+) &= \int d\lambda\rho(\lambda)T^2(\mathfrak{x}^2,\lambda) \\ \tau(\infty,\infty) &= \int d\lambda\rho(\lambda) = 1. \end{aligned}$$

Indeed it is natural to assume that $T^1(\infty,\lambda) = T^2(\infty,\lambda) = 1$. We can start with (36), multiply by $\rho(\lambda)$ and integrate over λ to obtain

$$\begin{aligned} -\tau_0 \le\ & \tau(a^1_+,a^2_+) - \tau(a^1_+,b^2_+) + \tau(b^1_+,a^2_+) \\ & + \tau(b^1_+,b^2_+) - \tau(b^1_+,\infty) - \tau(\infty,a^2_+) \le 0. \end{aligned} \tag{41}$$

According to quantum mechanics the double detection probability when no polarizers are placed on the paths of the two photons is $\omega_0 = \eta^1\eta^2$. This allows us to identify $\eta^1\eta^2$ as the universal constant which, according to Aspect's assumption, gives the ratio between a double transmission & detection probability, and the corresponding double transmission probability. We can write

$$\begin{aligned}\omega(\mathfrak{r}^1_+,\mathfrak{r}^2_+) &= \tau(\mathfrak{r}^1_+,\mathfrak{r}^2_+)\eta^1\eta^2\\ \omega(\mathfrak{r}^1_+,\infty) &= \tau(\mathfrak{r}^1_+,\infty)\eta^1\eta^2\\ \omega(\infty,\mathfrak{r}^2_+) &= \tau(\infty,\mathfrak{r}^2_+)\eta^1\eta^2\end{aligned}$$

and $\omega_0 = \tau_0\eta^1\eta^2 = \eta^1\eta^2$. Multiplying (41) by $\eta^1\eta^2$ we again obtain (37).

1.4 Strong inequalities for two-way polarizers

The theory of EPR experiments with two-way polarizers was developed by Garuccio and Rapisarda (1981) and adopted by Aspect *et al.* (1982a) in their second experiment. To obtain a strong inequality Garuccio and Rapisarda made the following assumption:

> The sum of the probability of transmission and detection, and of the probability of reflection and detection in the ordinary and extraordinary beams, respectively, of a two-way polarizer does not depend on the orientation of the polarizer.

The sums

$$F^\sigma(\lambda) = T^\sigma(\mathfrak{r}^\sigma,\lambda)D^\sigma_T(\mathfrak{r}^\sigma,\lambda) + R^\sigma(\mathfrak{r}^\sigma,\lambda)D^\sigma_R(\mathfrak{r}^\sigma,\lambda), \tag{42}$$

therefore do not depend on the directions $\mathfrak{r}^\sigma$, $\sigma = 1,2$. Garuccio and Rapisarda introduced the following definition of a normalized correlation function

$$E(\mathfrak{r}^1,\mathfrak{r}^2) = \frac{P(\mathfrak{r}^1,\mathfrak{r}^2)}{Z(\mathfrak{r}^1,\mathfrak{r}^2)}, \tag{43}$$

where

$$Z(\mathfrak{r}^1,\mathfrak{r}^2) = \omega(\mathfrak{r}^1_+,\mathfrak{r}^2_+) + \omega(\mathfrak{r}^1_+,\mathfrak{r}^2_-) + \omega(\mathfrak{r}^1_-,\mathfrak{r}^2_+) + \omega(\mathfrak{r}^1_-,\mathfrak{r}^2_-). \tag{44}$$

Equations (24), (25) and (44) imply that

$$\begin{aligned}P(\mathfrak{r}^1,\mathfrak{r}^2) &= \int d\lambda\rho(\lambda)[T^1(\mathfrak{r}^1,\lambda)D^1_T(\mathfrak{r}^1,\lambda) - R^1(\mathfrak{r}^1,\lambda)D^1_R(\mathfrak{r}^1,\lambda)]\\ &\qquad\times[T^2(\mathfrak{r}^2,\lambda)D^2_T(\mathfrak{r}^2,\lambda) - R^2(\mathfrak{r}^2,\lambda)D^2_R(\mathfrak{r}^2,\lambda)]\\ Z(\mathfrak{r}^1,\mathfrak{r}^2) &= \int d\lambda\rho(\lambda)[T^1(\mathfrak{r}^1,\lambda)D^1_T(\mathfrak{r}^1,\lambda) + R^1(\mathfrak{r}^1,\lambda)D^1_R(\mathfrak{r}^1,\lambda)]\\ &\qquad\times[T^2(\mathfrak{r}^2,\lambda)D^2_T(\mathfrak{r}^2,\lambda) + R^2(\mathfrak{r}^2,\lambda)D^2_R(\mathfrak{r}^2,\lambda)].\end{aligned} \tag{45}$$

Equations (42) allow us to write

$$Z = \int d\lambda \rho(\lambda) F^1(\lambda) F^2(\lambda)$$

without reference to $\mathfrak{r}^1$ and $\mathfrak{r}^2$. Writing

$$p^\sigma(\mathfrak{r}^\sigma, \lambda) = T^\sigma(\mathfrak{r}^\sigma, \lambda) D_T^\sigma(\mathfrak{r}^\sigma, \lambda) - R^\sigma(\mathfrak{r}^\sigma, \lambda) D_R^\sigma(\mathfrak{r}^\sigma, \lambda),$$

equation (42) leads to

$$|p^\sigma(\mathfrak{r}^\sigma, \lambda)| \le F^\sigma(\lambda) \tag{46}$$

$\sigma = 1, 2$. But (43)-(45) give

$$E(\mathfrak{r}^1, \mathfrak{r}^2) = \frac{1}{Z} \int d\lambda \rho(\lambda) p^1(\mathfrak{r}^1, \lambda) p^2(\mathfrak{r}^2, \lambda),$$

so that

$$\begin{aligned} &\left|E(a^1, a^2) - E(a^1, b^2) + E(b^1, a^2) + E(b^1, b^2)\right| \\ &\le \frac{1}{Z} \int d\lambda \rho(\lambda) \{ |p(a^1, \lambda)| \, |q(a^2, \lambda) - q(b^2, \lambda)| \\ &\qquad + |p(b^1, \lambda)| \, |q(a^2, \lambda) + q(b^2, \lambda)| \}. \end{aligned} \tag{47}$$

Using directions a^1, b^1, a^2, b^2 in (46), (47) becomes

$$2 \le E(a^1, a^2) - E(a^1, b^2) + E(b^1, a^2) + E(b^1, b^2) \le 2.$$

Written in terms of correlations functions $P(\mathfrak{r}^1, \mathfrak{r}^2)$, this new strong inequality becomes

$$-2Z \le P(a^1, a^2) - P(a^1, b^2) + P(b^1, a^2) + P(b^1, b^2) \le 2Z.$$

Experiments have neither refuted local realism nor indicated that quantum waves really propagate in configuration space. Four experimental values of the parameter δ (which is similar to Ω of Table 3) can be compared with both inequalities in the table below. The strong inequality is violated in all cases but one, the weak inequality never.

	weak inequality	strong inequality	δ	δ_{QM}
Freedman-Clauser (1972)	$\delta \le 6.868$	$\delta \le 0.250$	0.300	0.301
Holt-Pipkin (1974)	$\delta \le 11.574$	$\delta \le 0.250$	0.216	0.266
Clauser (1976)	$\delta \le 13.736$	$\delta \le 0.250$	0.286	0.284
Perrie *et al.* (1985)	$\delta \le 6.25$	$\delta \le 0.250$	0.268	0.272

The quantum-mechanical predictions δ_{QM} are based on non-factorizable vectors similar[121] to

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[121] A possible variant, for instance, is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle|y\rangle + |y\rangle|x\rangle)$$

where $|x\rangle$ indicates linear polarization along the x-axis and $|y\rangle$ along the y-axis.

For the time being, then, photons cannot be used to test the configuration space description, or local realism. Rather than waiting for high efficiency detectors, we can turn our attention to kaons, which are almost always detected.

$$|\Psi'\rangle = \frac{-i}{\sqrt{2}}(|x\rangle|y\rangle - |y\rangle|x\rangle).$$

2
Kaons

Much attention has been devoted in recent years to the auxiliary assumptions mentioned above in Chapter VI. 1, and to ways of getting around the 'detector inefficiency' loophole.

> It would be nice to design an experiment which would eliminate the need for such auxiliary assumptions. Such an experiment has been suggested by Lo and Shimony (1981), which employs the coincidence detection of the two dissociation fragments of a metastable molecule. With such a two-body decay, strong angular correlations would obtain, and with Stern-Gerlach analysers and ionization detectors very high efficiencies can in principle be achieved. This experiment appears to be quite possible, and there is no doubt it should be carried out in order to eliminate the 'auxiliary assumptions' loophole in the existing experiments.[122]

Walther and Fry have proceeded along the lines indicated by Lo and Shimony:

> The ^{199}Hg isotope can be used for an experimental realization of Bohm's spin-1/2 EPR-*gedankenexperiment*. The dissociation of dimers of the ^{199}Hg isotopomer using a spectroscopically selective stimulated Raman process leads to the generation of an entangled state between the two spatially separated atoms. The measurement of nuclear spin correlations between these two atoms is achieved by using a spin state selective two photon excitation-ionization scheme that also provides for detection of the atoms. The experiment will ... close the detector efficiency loophole ....
>
> ⋮
>
> The system proposed by Lo and Shimony was the first to use atoms ... and was a precursor to the present experiment. Photoionization schemes for atoms allow very high detection efficiencies for almost all elements of the periodic table .... Therefore, a realization of Bohm's *gedankenexperiment* with atoms rather than photons should be very promising.[123]

I shall instead consider alternative realizations involving kaons—as I am more familiar with them—which can also 'close the detector inefficiency loophole.' The experiments could be performed at the Φ-Factory in Frascati, where electron-positron collisions produce Φ-mesons that can decay into pairs of neutral kaons.

2.1 Bell's inequality

We saw in Chapter II. 3 how to represent the states of single kaons on the Riemann sphere. Alternatively, given the similarities

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[122] Redhead (1987)
[123] Walther and Fry (1997)

$$\begin{array}{ll} CP|K_S\rangle = +|K_S\rangle & \sigma_3|\uparrow\rangle = +|\uparrow\rangle \\ CP|K_L\rangle = -|K_L\rangle & \sigma_3|\downarrow\rangle = -|\downarrow\rangle \\ S|K\rangle = +|K\rangle & \sigma_1|\rightarrow\rangle = +|\rightarrow\rangle \\ S|\overline{K}\rangle = -|\overline{K}\rangle & \sigma_1|\leftarrow\rangle = -|\leftarrow\rangle \\ |K\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle + |K_L\rangle) & |\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \\ |\overline{K}\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle - |K_L\rangle) & |\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \end{array}$$

we can apply the 'spin-half' formalism developed in Section IV 1.1. As σ_2 has no counterpart we are restricted to the subspace spanned by σ_3 and σ_1, or by CP and S in the kaon formalism.

Now we have a *pair* of kaons, and a state like the singlet: for right after the decay of the Φ-meson, the resulting (neutral) kaons are described by the vector[124]

$$|\kappa\rangle = \frac{1}{\sqrt{2}}(|K^1\rangle|\overline{K}^2\rangle - |\overline{K}^1\rangle|K^2\rangle) = \frac{1}{\sqrt{2}}(|K_S^1\rangle|K_L^2\rangle - |K_L^1\rangle|K_S^2\rangle).$$

There are, however, important differences with respect to the well-known cases involving photons or spin-half particles. Kaons are almost always detected; but we have seen that, as they decay, their evolution is not unitary.

Let us now see how kaons can be used to determine whether quantum waves really propagate in configuration space. The single degree of freedom in the 'meridian' (fixed azimuth) determined by the Pauli operator σ_1^n is enough for the construction of observables sensitive to quantum waves in configuration space, $n = 1, 2$. Take the self-adjoint operator

$$\begin{aligned} \Gamma_{\pi/2} &= \sigma_{\pi/2}^1 \otimes \sigma_{\pi/4}^2 - \sigma_{\pi/2}^1 \otimes \sigma_{-\pi/4}^2 + \sigma_0^1 \otimes \sigma_{\pi/4}^2 + \sigma_0^1 \otimes \sigma_{-\pi/4}^2 \\ &= \sigma_1^1 \otimes \sigma_+^2 - \sigma_1^1 \otimes \sigma_-^2 + \sigma_3^1 \otimes \sigma_+^2 + \sigma_3^1 \otimes \sigma_-^2, \end{aligned}$$

where the self-adjoint unitary zero-trace operators $\sigma_\pm^2 = \sigma_{\pm\pi/4}^2$ are the (normalized) sum and difference $\sigma_\pm^2 = \{\sigma_1^2 \pm \sigma_3^2\}/\sqrt{2}$ of the Pauli operators σ_1^2 and σ_3^2. As we know that $\langle\Sigma|\Gamma_{\pi/2}|\Sigma\rangle = 2\sqrt{2}$ for the singlet $|\Sigma\rangle$, the corresponding self-adjoint operator

$$\Xi = S^1 \otimes E_+^2 - S^1 \otimes E_-^2 + CP^1 \otimes E_+^2 + CP^1 \otimes E_-^2$$

will also have an average $\mathsf{B} = \langle\kappa|\Xi|\kappa\rangle$ of $2\sqrt{2}$ for the corresponding vector $|\kappa\rangle$; the self-adjoint unitary zero-trace operators $E_\pm^2$ are equal to $\{CP^2 \pm S^2\}/\sqrt{2}$, and

$$CP^1 \leftrightarrow \sigma_0^1 = \sigma_3^1, \qquad S^1 \leftrightarrow \sigma_{\pi/2}^1 = \sigma_1^1, \qquad E_\pm^2 \leftrightarrow \sigma_{\pm\pi/4}^2 = \sigma_\pm^2.$$

The four operators $E_\pm^n = |\alpha_\pm^n\rangle\langle\alpha_\pm^n| - |\beta_\pm^n\rangle\langle\beta_\pm^n|$ have eigenvectors

[124]See Lipkin (1968).

$$|\alpha^n_\pm\rangle = \cos\Big(\pm\frac{\pi}{4}\Big)|K^n_{\rm S}\rangle + \sin\Big(\pm\frac{\pi}{4}\Big)|K^n_{\rm L}\rangle$$
$$|\beta^n_\pm\rangle = \cos\bigg(\mp\frac{3\pi}{4}\bigg)|K^n_{\rm S}\rangle + \sin\bigg(\mp\frac{3\pi}{4}\bigg)|K^n_{\rm L}\rangle,$$

$n = 1, 2.$

Let us now deduce a Bell inequality (see Section IV. 1.3) applying the criterion used by Einstein *et al.* for the identification of 'elements of reality,' together with a suitable locality condition.

We have seen that the expansion of $|\kappa\rangle$ is biorthogonal with respect to strangeness eigenvectors, and also with respect to charge-parity eigenvectors; but it will be biorthogonal with respect to the eigenvectors of E^1_+, E^2_+ and to those of E^1_-, E^2_- as well:

$$|\kappa\rangle = \frac{1}{\sqrt{2}}\big(|\alpha^1_+\rangle|\beta^2_+\rangle - |\beta^1_+\rangle|\alpha^2_+\rangle\big) = \frac{1}{\sqrt{2}}\big(|\alpha^1_-\rangle|\beta^2_-\rangle - |\beta^1_-\rangle|\alpha^2_-\rangle\big).$$

This allows the application of the 'reality criterion' and the identification of 'elements of reality' $\underline{CP}^m_k, \underline{S}^m_k, \underline{E}^m_{k\pm}$ (for particle m of the kth pair) corresponding to the quantities represented by CP^m, S^m and $E^m_\mp$ ($m = 1, 2;\ k = 1, 2, \ldots$). 'Elements of reality' will be identified with the corresponding eigenvalues ± 1 to simplify, even if there are many such elements for either eigenvalue; so $\underline{CP}^m_k, \underline{S}^m_k, \underline{E}^m_{k+}, \underline{E}^m_{k-} = \pm 1$.

Consider the expression

$$\underline{\mathsf{B}}_k = \underline{S}^1_k\underline{E}^2_{k+} - \underline{S}^1_k\underline{E}^2_{k-} + \underline{CP}^1_k\underline{E}^2_{k+} + \underline{CP}^1_k\underline{E}^2_{k-}.$$

If nothing else is assumed the modulus of $\underline{\mathsf{B}}_k$ could reach, but not exceed, 4. Each factor of every term could be a function depending on the neighbouring factor. We could have $(-1)\underline{E}^2_{k+} - (+1)\underline{E}^2_{k-}$ for the first two terms of $\underline{\mathsf{B}}_k$, for instance; $\underline{S}^1_k = -1$ when it is next to $\underline{E}^2_{k+}$, otherwise $\underline{S}^1_k = +1$.

So far we have considered the kth pair. Summing over k we obtain

$$-4 \le \underline{\mathsf{B}} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\underline{\mathsf{B}}_k = \underline{P}(S, E_+) - \underline{P}(S, E_-) + \underline{P}(CP, E_+) + \underline{P}(CP, E_-) \le 4,$$

where the correlation functions

$$\underline{P}(S, E_\pm) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\underline{S}^1_k\underline{E}^2_{k\pm}$$
$$\underline{P}(CP, E_\pm) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\underline{CP}^1_k\underline{E}^2_{k\pm}.$$

Let us now suppose—the kaons of each pair could after all be far apart—that the various factors of the four terms of $\underline{\mathsf{B}}_k$ do not depend on the adjacent factor. This assumption, which we can call LOC$'_3$, allows us to write

$$\underline{\mathsf{B}}_k = \underline{S}^1_k\{\underline{E}^2_{k+} - \underline{E}^2_{k-}\} + \underline{CP}^1_k\{\underline{E}^2_{k+} + \underline{E}^2_{k-}\}.$$

One of the factors in curly brackets vanishes, the other will be equal to ± 2, and hence $\underline{\mathsf{B}}_k = \pm 2$. As the average of several ± 2's will lie between -2 and 2 we have Bell's inequality $-2 \leq \underline{\mathsf{B}} \leq 2$. So the interval containing $\underline{\mathsf{B}}$ gets *halved* by LOC$'_3$.

We have already seen that Bell's inequality is abundantly violated by the corresponding quantum-mechanical quantity $\mathsf{B} = \langle\kappa|\Xi|\kappa\rangle = 2\sqrt{2}$. But can $E^2_\pm$ be measured?

> We call a real dynamical variable whose eigenstates form a complete set an *observable*. Thus any quantity that can be measured is an observable.
>
> The question now presents itself—Can every observable be measured? The answer theoretically is yes. In practice it may be very awkward, or perhaps even beyond the ingenuity of the experimenter, to devise an apparatus which could measure some particular observable, but the theory always allows one to imagine that the measurement can be made.[125]

All self-adjoint operators may well be assumed to represent observable quantities, but experimenters apparently only know how to measure[126] charge-parity and strangeness. This is not a problem *for quantum mechanics*. Given the linearity of averages—the average of a linear combination is the linear combination of the averages—the quantum mechanical value $\langle\kappa|\Xi|\kappa\rangle$ can be measured measuring only charge-parity and strangeness. The averages

$$\langle\kappa|S^1 \otimes E^2_\pm|\kappa\rangle = \Big\langle\kappa\Big|S^1 \otimes \Big\{(CP^2 \pm S^2)/\sqrt{2}\Big\}\Big|\kappa\Big\rangle,$$

for instance, are equal to

$$\frac{1}{\sqrt{2}}\{\langle\kappa|S^1 \otimes CP^2|\kappa\rangle \pm \langle\kappa|S^1 \otimes S^2|\kappa\rangle\}.$$

But is it legitimate to speak of the elements of reality $\underline{E}^m_{k\pm}$ of the unmeasurable quantities represented by $E^m_\pm$? If they really are unmeasurable, perhaps the reality criterion of Einstein *et al.* should not be applied. We cannot get around the problem by writing something like

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\underline{S}^1_k\underline{E}^2_{k\pm} = \lim_{N\to\infty}\frac{1}{N\sqrt{2}}\left\{\sum_{k=1}^{N}\underline{S}^1_k\underline{CP}^2_k \pm \sum_{k=1}^{N}\underline{S}^1_k\underline{S}^2_k\right\}$$

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[125]Dirac (1958)

[126]See Wigner (1963): "For some observables, in fact for the majority of them (such as xyp_z), nobody seriously believes that a measuring apparatus exists." See also Wigner (1981).

for the equality does not hold. One could argue that eventually a sufficiently ingenious experimenter will work out how to measure the quantities represented by $E^m_{\mp}$, but that we do not have to wait until then to derive an inequality like Bell's involving $\underline{E}^m_{k\pm}$. For the time being, however, let us leave the question open and appeal to time evolution.

2.2 Time evolution

The evolution of a single kaon is given by the normal operator[127]

$$U'(t) = e^{-(iM+\frac{1}{2}\Gamma)t} = e^{-(im_S+\frac{1}{2}\gamma_S)t}|K_S\rangle\langle K_S| + e^{-(im_L+\frac{1}{2}\gamma_L)t}|K_L\rangle\langle K_L|,$$

which can be represented as the product $U(t)e^{-\frac{1}{2}\Gamma t}$ of the unitary operator

$$U(t) = e^{-iMt} = e^{-im_S t}|K_S\rangle\langle K_S| + e^{-im_L t}|K_L\rangle\langle K_L|$$

and the positive operator

$$e^{-\frac{1}{2}\Gamma t} = e^{-\frac{1}{2}\gamma_S t}|K_S\rangle\langle K_S| + e^{-\frac{1}{2}\gamma_L t}|K_L\rangle\langle K_L|,$$

where of course

$$\Gamma = \gamma_S|K_S\rangle\langle K_S| + \gamma_L|K_L\rangle\langle K_L|.$$

For the time being we shall continue to assume (as in Section II. 1.3) that Γ vanishes, as we remain more interested in the unitary factor $U(t)$ than in decay.

Now there are two kaons. Multiplying together the evolutions

$$U^n = e^{-im_S t^n}|K^n_S\rangle\langle K^n_S| + e^{-im_L t^n}|K^n_L\rangle\langle K^n_L|$$

$(n = 1, 2)$ of the single kaons, we obtain the operator

$$\mathbf{U}(t^1, t^2) = U^1(t^1) \otimes U^2(t^2)$$

which describes the evolution of the pair;

$$\begin{aligned}|\kappa(t^1, t^2)\rangle &= \mathbf{U}(t^1, t^2)|\kappa\rangle \\ &= \frac{1}{\sqrt{2}}\left\{e^{-i(m_S t^1 + m_L t^2)}|K^1_S\rangle|K^2_L\rangle - e^{-i(m_L t^1 + m_S t^2)}|K^1_L\rangle|K^2_S\rangle\right\}.\end{aligned}$$

As we will not see beats with respect to CP, we can rotate to the strangeness basis:

[127]Primed quantities in this kaon context *take account of decay*.

$$\begin{aligned}|\kappa(t^1,t^2)\rangle &= \frac{1}{\sqrt{2}}\left\{U^1(t^1)|K^1\rangle U^2(t^2)|\overline{K}^2\rangle - U^1(t^1)|\overline{K}^1\rangle U^2(t^2)|K^2\rangle\right\}\\ &= \frac{1}{2\sqrt{2}}\Big\{e^{-i(m_S t^1 + m_L t^2)}\left(|K^1\rangle + |\overline{K}^1\rangle\right)\left(|K^2\rangle - |\overline{K}^2\rangle\right)\\ &\qquad - e^{-i(m_L t^1 + m_S t^2)}\left(|K^1\rangle - |\overline{K}^1\rangle\right)\left(|K^2\rangle + |\overline{K}^2\rangle\right)\Big\},\end{aligned}$$

in other words

$$\begin{aligned}|\kappa(t^1,t^2)\rangle = \frac{1}{2\sqrt{2}}\big(&\mathfrak{e}_-|K^1\rangle|K^2\rangle + \mathfrak{e}_+|K^1\rangle|\overline{K}^2\rangle\\ &- \mathfrak{e}_+|\overline{K}^1\rangle|K^2\rangle - \mathfrak{e}_-|\overline{K}^1\rangle|\overline{K}^2\rangle\big),\end{aligned} \tag{48}$$

where

$$\begin{aligned}\mathfrak{e}_- &= e^{-i(m_S t^1 + m_L t^2)} - e^{-i(m_L t^1 + m_S t^2)},\\ \mathfrak{e}_+ &= e^{-i(m_S t^1 + m_L t^2)} + e^{-i(m_L t^1 + m_S t^2)}.\end{aligned}$$

The coefficient $\mathfrak{e}_-$, and with it the first and fourth terms of (48), vanish if the strangeness measurements are performed at the same time $t^1 = t^2$ (and would vanish if short and long kaons had the same mass).

The probabilities

$$\begin{aligned}\left|\langle K^1K^2|\kappa(t^1,t^2)\rangle\right|^2 &= \left|\mathfrak{e}_-/2\sqrt{2}\right|^2\\ \left|\langle K^1\overline{K}^2|\kappa(t^1,t^2)\rangle\right|^2 &= \left|\mathfrak{e}_+/2\sqrt{2}\right|^2\\ \left|\langle \overline{K}^1K^2|\kappa(t^1,t^2)\rangle\right|^2 &= \left|-\mathfrak{e}_+/2\sqrt{2}\right|^2\\ \left|\langle \overline{K}^1\overline{K}^2|\kappa(t^1,t^2)\rangle\right|^2 &= \left|-\mathfrak{e}_-/2\sqrt{2}\right|^2\end{aligned}$$

oscillate with the difference $t^1 - t^2$. These beats are produced in configuration space, but may or may not really occur in nature.

2.3 Bell's inequality again

Let us now use time evolution to violate Bell's inequality (see Sections IV. 1.3 and VI. 2.1). First let us deduce the inequality from the reality criterion of Einstein, Podolsky and Rosen, together with a locality condition like Redhead's LOC_3.

The reality criterion of Einstein *et al.* allows the identification of 'elements of reality' associated with strangeness measurements made at equal proper times,[128] when the strangeness-expansion of $|\kappa\rangle$ becomes biorthogonal and measurement results are perfectly anticorrelated. Identifying the elements of reality with their corresponding

[128]For the application of the reality criterion to strangeness see Selleri (1983).

eigenvalues[129] we have

$$\underline{S}_k^1(t_1) = \pm 1 \Leftrightarrow \underline{S}_k^2(t_1) = \mp 1,$$

where $\underline{S}_k^1(t_1)$, $\underline{S}_k^2(t_1)$ are the strangeness values *assumed* at time t_1 by the kth pair. So $\underline{S}_k^1(t_1) = -1$ if a simultaneous strangeness measurement on the other particle gives $+1$. But what if S^2 is measured at a later time $t_2 > t_1$? Can the value $\underline{S}_k^1(t_1) = -1$ depend on *when* S^2 is measured? Of course a measurement of S^2 at time t_2 could, as strangeness is not conserved, yield the different value -1, and hence

$$\underline{S}_k^1(t_2) = +1 \text{ and } \underline{S}_k^2(t_2) = -1.$$

But it would be most surprising if, by waiting and measuring S^2 at time t_2, the value $\underline{S}_k^1(t_1) = -1$ were changed *retroactively* to $+1$. We shall make an assumption similar to Redhead's LOC_3: "A sharp value for an observable cannot be changed into another sharp value by altering the setting of a remote piece of apparatus."[130] Here the unitary time evolution operator that rotates strangeness around the equator assumes, in a sense, the role of the unitary operator representing the physical rotation of the "remote piece of apparatus." So we will assume that *the strangeness value assumed by one kaon does not depend on when the strangeness of the other kaon is measured.*

Consider the expression

$$\underline{\mathfrak{B}}_k = \underline{S}_k^1(t_1^1)\underline{S}_k^2(t_1^2) - \underline{S}_k^1(t_2^1)\underline{S}_k^2(t_1^2) + \underline{S}_k^1(t_1^1)\underline{S}_k^2(t_2^2) + \underline{S}_k^1(t_2^1)\underline{S}_k^2(t_2^2).$$

Taking the first term, $\underline{S}_k^2(t_1^2)$ will generally depend on t_1^2 and $\underline{S}_k^1(t_1^1)$ on t_1^1, but we have assumed that $\underline{S}_k^1(t_1^1)$ does not depend on t_1^2, nor $\underline{S}_k^2(t_1^2)$ on t_1^1. Rewriting $\underline{\mathfrak{B}}_k$ as the difference

$$\underline{S}_k^1(t_1^1)\{\underline{S}_k^2(t_1^2) + \underline{S}_k^2(t_2^2)\} - \underline{S}_k^1(t_2^1)\{\underline{S}_k^2(t_1^2) + \underline{S}_k^2(t_2^2)\}$$

we see that one of the factors in curly brackets will vanish and the other will be equal to ± 2. Therefore $\underline{\mathfrak{B}}_k = \pm 2$ for all t_1^1, t_2^1, t_1^2, t_2^2.

So far we have considered the kth pair. Summing over k we have

$$\underline{\mathfrak{B}} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \underline{\mathfrak{B}}_k = \underline{P}(t_1^1, t_1^2) - \underline{P}(t_2^1, t_1^2) + \underline{P}(t_1^1, t_2^2) + \underline{P}(t_2^1, t_2^2),$$

where the correlation function

$$\underline{P}(t^1, t^2) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \underline{S}_k^1(t^1)\underline{S}_k^2(t^2).$$

[129] All the elements of reality—there could be many—associated with the eigenvalue λ are, in other words, made to correspond to λ.

[130] Redhead (1987) p.82

As the average of many ± 2's will lie between -2 and 2 we have Bell's inequality $|\underline{\mathfrak{B}}| \leq \underline{\Lambda} = 2$.

The quantum-mechanical quantity that corresponds to $\underline{\mathfrak{B}}$ is

$$\mathfrak{B} = P(t_1^1, t_1^2) - P(t_2^1, t_1^2) + P(t_1^1, t_2^2) + P(t_2^1, t_2^2),$$

where the correlation function

$$P(t^1, t^2) = \langle \kappa(t^1, t^2) | S^1 \otimes S^2 | \kappa(t^1, t^2) \rangle = \cos\{\Delta m(t^1 - t^2)\}.$$

At times

$$t_1^1 = \frac{\zeta}{\Delta m} \qquad t_1^2 = \frac{1}{\Delta m}\left(\zeta + \frac{\pi}{4}\right)$$
$$t_2^1 = \frac{1}{\Delta m}\left(\zeta + \frac{\pi}{2}\right) \qquad t_2^2 = \frac{1}{\Delta m}\left(\zeta + \frac{3\pi}{4}\right),$$

the average $\mathfrak{B}$ reaches its maximum of $2\sqrt{2}$, which abundantly violates Bell's inequality. As decay is being ignored, the initial azimuth ζ is arbitrary.

2.4 Decay

We must remember, however, that the time evolution of a kaon is described by

$$U'(t) = e^{-(im_S + \frac{1}{2}\gamma_S)t} |K_S\rangle\langle K_S| + e^{-(im_L + \frac{1}{2}\gamma_L)t} |K_L\rangle\langle K_L|$$

and of a pair by

$$\mathbf{U}'(t^1, t^2) = U^{1\prime}(t^1) \otimes U^{2\prime}(t^2);$$

and that the decay rates γ_S and γ_L do not in fact vanish. We can continue to assume that $\gamma_L = 0$, as the period $\mathfrak{p}$ of strangeness oscillations is very small in comparison with the average lifetime $\tau_L = 1/\gamma_L$. In other words the longevity of the long-lived kaons gives us time to see the effect we are interested in before they start decaying; $\gamma_L t \approx 0$ for all relevant t. It is not physically reasonable, however, to let γ_S vanish (at least for the entire population, but more on this later) since $\mathfrak{p}$ is significant in comparison with the shorter average lifetime $\tau_S = 1/\gamma_S$, which does not give the kaons much time to oscillate. If we use the standard phenomenological value of γ_S, the strangeness oscillations get damped by decay to such an extent that the average

$$\begin{aligned}\mathfrak{B}' \approx\ & e^{-\frac{1}{2}\gamma_S(t_1^1+t_1^2)}\cos\{\Delta m(t_1^1 - t_1^2)\} - e^{-\frac{1}{2}\gamma_S(t_1^1+t_2^2)}\cos\{\Delta m(t_1^1 - t_2^2)\} \\ & + e^{-\frac{1}{2}\gamma_S(t_2^1+t_1^2)}\cos\{\Delta m(t_2^1 - t_1^2)\} + e^{-\frac{1}{2}\gamma_S(t_2^1+t_2^2)}\cos\{\Delta m(t_2^1 - t_2^2)\}\end{aligned}$$

never leaves the interval running from -2 to 2 for all choices of t_1^1, t_2^1, t_1^2, t_2^2.[131] So an evolution that has nothing to do with waves cancels the very wave-mechanical effect we wanted to see. These difficulties have led to the conclusion that

> ... the ϕ-factory facility does not seem to open new ways of testing quantum mechanics versus alternative general schemes of the type which are usually regarded as worth considering in the debate about locality and quantum mechanics.
>
> ... in the case under consideration there is no direct way to discriminate between quantum mechanics and local realistic models.[132]

Damping the oscillations that would otherwise lead to the violation of Bell's inequality, decay in a sense favours local realism.

Suppose we remain interested in times that are very short in comparison with τ_L, that we begin with a large number N of pairs of kaons, and that $(1-e^{-\gamma_S t})N$ pairs have been eliminated by decay at time t. Only the $e^{-\gamma_S t}N$ surviving pairs contribute a product $-1 = (\pm 1)\cdot(\mp 1)$ to the numerator $-e^{-\gamma_S t}N$ of the correlation function

$$P'(t,t) = \frac{e^{-\gamma_S t}N(-1)}{N} = -e^{-\gamma_S t}.$$

The $(1-e^{-\gamma_S t})N$ pairs eliminated by decay attenuate the correlation by increasing the denominator N, by 1 each decay; or alternatively one can say they contribute 0's to a series—the numerator—whose terms would otherwise all be equal to -1. Rather than the perfect anticorrelation

$$\frac{\overbrace{-1-1-\cdots-1}^{e^{-\gamma_S t}N \text{ terms}}}{e^{-\gamma_S t}N} = \frac{e^{-\gamma_S t}N(-1)}{e^{-\gamma_S t}N} = -1,$$

we might have

$$\frac{\overbrace{-1+0-1-1+0-\cdots+0-1}^{N \text{ terms}}}{N} = \frac{-e^{-\gamma_S t}N}{N} = -e^{-\gamma_S t}.$$

For different proper times the quantum-mechanical prediction is

$$P'\left(t^1,t^2\right) = e^{-\frac{1}{2}\gamma_S\left(t^1+t^2\right)}\cos\left\{\Delta m\left(t^1-t^2\right)\right\}$$

and experiment would yield a fraction like

[131] See Ghirardi, Grassi and Weber (1992). Calculations carried out by Franco Selleri, Sebastiano Cobianco, Fasma Diele and myself have yielded no violation (for the standard phenomenological values of γ_S and γ_L).
[132] Ghirardi, Grassi and Weber (1992)

$$\frac{\overbrace{-1+0+1-1+0+1\cdots+0+1}^{N\text{ terms}}}{N}$$

in whose numerator the values ± 1 figure $e^{-\gamma_S t}N$ times.

As decay reduces both the quantum-mechanical average $\mathfrak{B}'$ and the corresponding experimental value, it should, to make the comparison fair, also lower Bell's limit $\underline{\Lambda}$. To take account of decay we could write $|\underline{\mathfrak{B}}'| \leq \underline{\Lambda}'(t_1^1, t_2^1, t_1^2, t_2^2)$, where

$$\underline{\mathfrak{B}}' = \underline{P}'(t_1^1, t_1^2) - \underline{P}'(t_2^1, t_1^2) + \underline{P}'(t_1^1, t_2^2) + \underline{P}'(t_2^1, t_2^2)$$

and

$$\underline{P}'(t^1, t^2) = e^{-\frac{1}{2}\gamma_S(t^1+t^2)}\underline{P}(t^1, t^2).$$

I have not, however, managed to bring the limit $\underline{\Lambda}'(t_1^1, t_2^1, t_1^2, t_2^2)$ below 2, and render it time-dependent.

Alternatively we can adhere to the usual time-independent limit of 2, and ignore decay in quantum theory and in the experimental statistics: account of decay should either be taken on all three fronts—local realism, quantum theory, experiments—or not at all. To ignore it experimentally one would divide a series of ± 1's (without 0's) by the number of surviving pairs (and not by the *total* number of pairs). Quantum-mechanically one would then attribute the unitary evolution $\mathbf{U}(t^1, t^2)$ to the survivors; and we have seen that the average $\mathfrak{B}$, calculated with $\mathbf{U}(t^1, t^2)$, has a maximum of $2\sqrt{2}$ which abundantly violates Bell's inequality (with or without CP invariance).

2.5 Assumptions

Let us briefly reconsider the above assumptions in the simpler context of single kaons. By attributing a unitary evolution $U(t)$ to the survivors I am assuming that in an evolution $U'(t) = U(t)e^{-\frac{1}{2}\Gamma t}$ the unitary factor $U(t)$ applies to every kaon, whereas the positive operator $e^{-\frac{1}{2}\Gamma t}$ applies to ensembles. If the mass difference Δm is indeed a feature of every kaon, the sample of surviving pairs will be representative of the population *as far as strangeness oscillations are concerned* (of course the *longevity* of the sample will not be representative). Those oscillations are, after all, the very effect we are interested in.

The assumption that Δm regards every kaon is strongly supported by both theory and experience. If the kaons do not know when they are going to decay, and do so randomly in every possible (even ontological) sense, then presumably there can be no correlation between lifetimes and masses. But suppose, for the sake of argument, that the lifetime of each kaon, long or short, is determined at its birth. Above I assumed that each long kaon has the same mass m_L, whatever its lifetime, and that each short kaon has the same mass m_S. Let us now assume that short kaons with different lifetimes have different masses, and that long kaons with different lifetimes have different masses as well. The quantity Δm would then be statistical, and apply to ensembles, rather than to single kaons. Suppose the functions $m_S(\mathfrak{t}_S)$ and $m_L(\mathfrak{t}_L)$ of the lifetimes[133] $\mathfrak{t}_S$ and $\mathfrak{t}_L$ are such that the mass difference $\Delta m(\mathfrak{t}^\alpha) < \Delta m(\mathfrak{t}^\beta)$ for $\mathfrak{t}^\alpha < \mathfrak{t}^\beta$, where

$$\Delta m(\mathfrak{t}^\alpha) = m_L(\mathfrak{t}_L^\alpha) - m_S(\mathfrak{t}_S^\alpha), \quad \Delta m(\mathfrak{t}^\beta) = m_L(\mathfrak{t}_L^\beta) - m_S(\mathfrak{t}_S^\beta)$$

and

$$\mathfrak{t}^\alpha = \min\{\mathfrak{t}_L^\alpha, \mathfrak{t}_S^\alpha\}, \quad \mathfrak{t}^\beta = \min\{\mathfrak{t}_L^\beta, \mathfrak{t}_S^\beta\}$$

for any two kaons α and β with different lifetimes. The average frequency of the strangeness oscillations, in other words the frequency of the whole population, would then *increase* with time, as the shorter-lived kaons (with the smaller mass-differences) decay. As no such dependence of strangeness oscillations on lifetimes is observed, the values of the masses in the time-evolution operator

$$U'(t) = e^{-(im_S+\frac{1}{2}\gamma_S)t}|K_S\rangle\langle K_S| + e^{-(im_L+\frac{1}{2}\gamma_L)t}|K_L\rangle\langle K_L|.$$

are time-independent. It would take an unusual dependence of individual masses on lifetimes to yield the (experimentally visible) time-independence of strangeness frequencies. So it seems reasonable to assume that each kaon oscillates in the same way, and that lifetimes and strangeness oscillations are independent.

Then there is the issue of whether the 'contractive' evolution $e^{-\frac{1}{2}\Gamma t}$ applies to individual kaons or to ensembles. To simplify let us again assume that practically all of the long kaons survive and hence $e^{-\frac{1}{2}\gamma_L t} \approx 1$ for all relevant t; the period of strangeness oscillations is very short in comparison with $\tau_L = 1/\gamma_L$. This leaves the other eigenvalue, $e^{-\frac{1}{2}\gamma_S t}$: if something decays, it will almost certainly be a short kaon. Provided $e^{-\frac{1}{2}\gamma_S t}$ bears on populations and not individuals, different values of γ_S will apply to different samples. With a large population, samples can in fact be chosen with any value of γ_S. The decay rate can, in other words, be determined by the selection of the sample. A sample of kaons that decay late, for instance, will have small γ_S, large τ_S and large $e^{-\frac{1}{2}\gamma_S t}$. Here one would discard kaons which decay before the last of the four proper times that figure in Bell's inequality. If the entire sample survives the time range of interest, why bother with the operator $e^{-\frac{1}{2}\Gamma t}$?

The trouble is that kaon and decay products are superposed and not mixed. Suppose we have a beam described at time $t = 0$ by the vector $|\mathfrak{k}(0)\rangle$ belonging to the (neutral) 'kaon' subspace $[\mathfrak{K}] = [K_S] \oplus [K_L]$. The evolution $|\mathfrak{k}(t)\rangle = \mathsf{U}(t)|\mathfrak{k}(0)\rangle$ of the *whole* quantum state $|\mathfrak{k}(t)\rangle$, including decay products, is in fact *unitary*. What gets shorter as

[133] Again these are *predetermined lifetimes*, and not averages.

time passes is the projection $P_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle$ onto $[\mathfrak{K}]$. The projection operator $P_{[\mathfrak{K}]}$ corresponds to the experimental question 'is it still a kaon?,' the eigenvalue 1 to the answer 'it is.' Since

$$P_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle + P^{\perp}_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle = |\mathfrak{k}(t)\rangle \tag{49}$$

we have Pythagoras' relation

$$\|P_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle\|^2 + \|P^{\perp}_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle\|^2 = \||\mathfrak{k}(t)\rangle\|^2,$$

where $\|P_{[\mathfrak{K}]}|\mathfrak{k}(t)\rangle\|^2 = \|e^{-\frac{1}{2}\Gamma t}|\mathfrak{k}(0)\rangle\|^2$. Again, the 'kaon' in $|\mathfrak{k}(t)\rangle$ is not mixed but *superposed* with the decay products: another way of writing (49) for some $t = t_1$ is $|\mathfrak{k}(t_1)\rangle = c_k|k\rangle + c_d|d\rangle$, where $c_k|k\rangle = P_{[\mathfrak{K}]}|\mathfrak{k}(t_1)\rangle$ and $c_d|d\rangle = P^{\perp}_{[\mathfrak{K}]}|\mathfrak{k}(t_1)\rangle$. As $|\mathfrak{k}(t_1)\rangle\langle\mathfrak{k}(t_1)|$ is not the same as the statistical operator

$$\rho_{\mathfrak{k}} = |c_k|^2|k\rangle\langle k| + |c_d|^2|d\rangle\langle d| = |P_{[\mathfrak{K}]}\mathfrak{k}(t)\rangle\langle P_{[\mathfrak{K}]}\mathfrak{k}(t)| + |P^{\perp}_{[\mathfrak{K}]}\mathfrak{k}(t)\rangle\langle P^{\perp}_{[\mathfrak{K}]}\mathfrak{k}(t)|,$$

everything in the beam once described by $|\mathfrak{k}(0)\rangle$ is part kaon and part decay products, rather than *either* kaon *or* decay products, at least until an appropriate measurement is made. Unlikely as it may appear that kaon and decay products influence each other in each individual case, the arguments of c_k and c_d indicate that interference is possible in principle. Cartwright (1983) discusses a different situation, but a similar issue:

> In atomic decay the atom begins in its excited state and the field has no photons in it. Over time the composite atom-plus-field evolves continuously under the Schroedinger equation into a superposition. In one component of the superposition the atom is still in the excited state and there are no photons present; in the other, the atom is de-excited and the field contains one photon of the appropriate frequency. The atom is neither in its outer orbit nor in its inner orbit, and the photon is neither there in the field travelling away from the atom with the speed of light, nor absent. Over time the probability to 'be found' in the state with an excited atom and no photons decays exponentially.

But can kaon and decay products *really* be made to interfere with one another? With respect to any observable represented by an operator that commutes with $P_{[\mathfrak{K}]}$, the arguments of c_k and c_d are meaningless at any given moment (see Sections II. 1.1 and II. 2.1).

Cartwright (1983) considers the possibility of distinguishing between superposition and mixture in the related case of atomic decay:

> ... to distinguish between the two ... we would have to do a correlation experiment on both the atom and its associated photon, and we would have to measure some observable which did not commute with either the energy levels of the atom or the modes of the perturbed field. ... But these kinds of measurements are generally inaccessible to us. ... Still, with ingenuity, we might be able to expose the effects of interference in some more subtle way.

Here it is the time evolution operator $\mathsf{U}(t)$ that 'exposes the effects of interference,' as it does not commute with $P_{[\mathcal{K}]}$. If it did, $P_{[\mathcal{K}]}$ would represent a conserved quantity, and kaons would be stable. In Section II. 1.1 we saw the role of time evolution in exposing the effects of interference between the eigenstates of quantities that are not conserved.

We can compare strangeness oscillations in the following two cases:

> 1) Kaons are stable particles whose evolution is governed by the unitary time evolution operator $U(t)$.
>
> 2) The evolution of kaons is described by the normal operator $U'(t) = U(t)e^{-\frac{1}{2}\Gamma t}$, but we measure strangness (on a renormalized state vector) right after measurement of $P_{[\mathcal{K}]}$ has yielded the eigenvalue 1.

In either case we are only dealing with kaons; in the first because kaons are assumed not to decay, in the second because we only consider kaons and not their decay products. If the operator $e^{-\frac{1}{2}\Gamma t}$ were degenerate—a multiple of the identity—either case would give rise to the same strangeness oscillations. In case 2) the state vector would get shortened by $e^{-\frac{1}{2}\Gamma t}$, but then renormalized after measurement of $P_{[\mathcal{K}]}$. As $e^{-\frac{1}{2}\Gamma t}$ is maximal, however, it damps strangeness oscillations in two ways: by shortening the state vector, *and by shortening the $|K_{\mathrm{S}}\rangle$ component of the superposition more than the $|K_{\mathrm{L}}\rangle$ component*. The strangeness oscillations clearly depend on $|K_{\mathrm{S}}\rangle$'s contribution to them; the shorter the $|K_{\mathrm{S}}\rangle$ component gets—even if $|K_{\mathrm{L}}\rangle$ does not change—the weaker the oscillations will be. As 1) and 2) are statistically distinguishable, kaon and decay products interfere, according to quantum mechanics, in each individual case, even if nothing incompatible with $P_{[\mathcal{K}]}$ is measured. So each kaon becomes part kaon and part decay products—in a sense that in principle is statistically meaningful—and hence it may not be entirely legitimate to ignore decay by attributing a unitary evolution to the survivors and ignoring the others.

It remains possible, however, that *some* variant of Bell's inequality for kaon pairs can be used to discriminate between local realism and quantum mechanics, and indeed establish whether quantum waves really propagate in configuration space.

2.6 Selleri's inequalities

Selleri has proposed another way of discriminating between local realism and quantum mechanics, and indeed testing the configuration space description. On the basis of the assumptions

> 1. If, without in any way disturbing a kaon, one can predict with certainty the value of a physical quantity of that kaon, then there exists an element of reality corresponding to this physical quantity (EPR *reality criterion*).
>
> 2. If two kaons are very far apart, an element of reality belonging to one of them cannot be created by a measurement performed on the other (*locality*).

3. If at a given time t a kaon has an element of reality, the latter cannot be created by measurements on the same kaon performed at time t', if $t' > t$ (*no retroactive causality*).[134]

he deduces[135] the maximum surface

$$P_{LR}^{\max}[\overline{K}(t^1);\overline{K}(t^2)] = \frac{E_L(t^1)E_S(t^2)+E_S(t^1)E_L(t^2)}{4}Q_-(t^2)$$

and minimum surface

$$P_{LR}^{\min}[\overline{K}(t^1);\overline{K}(t^2)] = \frac{E_L(t^1)E_S(t^2)+E_S(t^1)E_L(t^2)}{4}[Q_-(t^2)-Q_-(t^1)]$$

of local realism, where $E_S(t) = e^{-\gamma_S t}$, $E_L(t) = e^{-\gamma_L t}$ and

$$Q_\pm(t) = \frac{1}{2}\left[1 + \frac{2\sqrt{E_L E_S}}{E_L + E_S}\cos\Delta m\, t\right].$$

Both surfaces $P_{LR}^{\max}[\overline{K}(t^1);\overline{K}(t^2)]$ and $P_{LR}^{\min}[\overline{K}(t^1);\overline{K}(t^2)]$ concern the probabilities of *antikaon-antikaon* observations. The corresponding quantum-mechanical surface

$$\begin{aligned}P_{QM}[\overline{K}(t^1);\overline{K}(t^2)] = \frac{1}{8}[&E_S(t^1)E_L(t^2)+E_L(t^1)E_S(t^2)\\ &-2\sqrt{E_S(t^1)E_L(t^2)E_L(t^1)E_S(t^2)}\cos\Delta m(t^1-t^2)]\end{aligned}$$

violates local realism by both exceeding the maximum and going below the minimum. Selleri has calculated the following values for times satisfying $t^2 = 2t^1$, for instance.

$\gamma_S t^1$	$P_{QM}[\overline{K}(t^1),\overline{K}(2t^1)]$	$P_{LR}^{\min}[\overline{K}(t^1),\overline{K}(2t^1)]$	$P_{LR}^{\max}[\overline{K}(t^1),\overline{K}(2t^1)]$
0.2	0.0018	0.0044	0.0052
0.4	0.0051	0.0118	0.0145
0.6	0.0087	0.0171	0.0221
0.8	0.0115	0.0195	0.0258
1.0	0.0133	0.0192	0.0259
1.2	0.0142	0.0174	0.0234
1.4	0.0144	0.0148	0.0198
1.6	0.0139	0.0119	0.0158
1.8	0.0131	0.0083	0.0121

The minimum is violated from $\gamma_S t^1 = 0.2$ to $\gamma_S t^1 = 1.4$, the maximum at $\gamma_S t^1 = 1.8$.

[134]Afriat and Selleri (1998)

[135]Rather than reproducing the complicated procedure by which the surfaces are deduced from the assumptions of local realism, I refer the reader to Selleri (1997), Afriat and Selleri (1998).

Measurements of $P\left[\overline{K}(t^1);\overline{K}(t^2)\right]$ could therefore, with an appeal to Selleri's inequalities, settle the configuration space issue: if the inequalities are always satisfied experimentally, the configuration space description must be incorrect.

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