Indifference Pricing with Uncertainty Averse Preferences

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others in which case the extent of any work carried out jointly by me and any other person is clearly identified in it. The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgment is made. This thesis may not be reproduced without the prior written consent of the author. I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

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Foreword

I spent the first year of my PhD working on a completely different research project whose results I did not include in this dissertation as I preferred submitting a unified work. The paper resulting from this research project is entitled "A Semiparametric Model for the Systematic Factors of Portfolio Credit Risk Premia" and it is published in the Journal of Empirical Finance, Volume 16, Issue 4, September 2009, pp. 655-670.

The paper investigates the empirical relationship between the daily returns of a Credit Default Swap (CDS) index and stock returns, stock price volatility, and interest rates. Analogous empirical analyses were previously conducted in the literature by Bystrom (2008) and Alexander and Kaeck (2008). Bystrom (2008) estimated several linear regression models finding a negative relationship between the daily returns of various CDS indexes, current stock returns, and lagged stock returns. Alexander and Kaeck (2008) estimated a Markov-switching model with a low-volatility regime and a high-volatility regime finding that interest rates, stock returns, and stock volatility have a stronger linear relationship with various CDS indexes in the high-volatility regime.

Both in Bystrom (2008) and Alexander and Kaeck (2008) the relationship between the daily returns of a CDS index and its determinants is assumed to be well approximated by a unique parametric model over a rather extended period of time. However, the parametric model describing the relationship between the daily returns of a CDS index and its determinants is likely to be affected by instability and sudden shifts over a relatively long period of time. Instability and sudden shifts in the regression function could result, for instance, from the smooth evolution of the economic scenario, or from extreme and unexpected negative developments in the economy. For this reason, in our paper the relationship between the daily returns of a CDS index and its determinants is described by a semiparametric model which accounts for a nonlinear regression function characterized by inhomogeneous smoothness properties and unknown number and locations of jumps. The model is estimated by the adaptive nonparametric techniques introduced by Spokoiny (1998) and further developed by Čižek et al. (2009) which consist in locally approximating a regression function by a simple parametric model and in selecting the degree of locality of the parametric approximation by a multiscale local change point analysis.

Our estimation results indicate that from November 2004 to January 2008 the relationships between the daily returns of the considered CDS index and stock returns, stock price volatility, and interest rates[1] were characterized by relatively long phases of stability interrupted by several sudden and extreme jumps. The jumps were associated to the downgrade of Ford and General Motors in 2005, to the slowdown of the US housing market in 2006, and to the credit crisis started in 2007. Our estimation results also suggest that in normal economic conditions the relationship between the daily returns of the considered CDS index and its determinants tends to be relatively weak but coherent with economic intuition and with earlier empirical findings, while during periods of economic instability the relationship between the daily returns of the CDS index and its determinants tends to be stronger but often inconsistent with common economic intuition and with earlier em-

[1]The considered CDS index is the iTraxx Europe index. The iTraxx Europe is the benchmark credit index in Europe and it is constructed as an equally weighted portfolio of 125 liquidly traded single-name CDS's. The considered stock returns are the daily returns of the Dow Jones Euro Stoxx 50 index. The Dow Jones Euro Stoxx 50 is a blue-chip index and it is constructed as a portfolio of 50 stocks weighted according to their market capitalization and representative of different Eurozone countries. The considered proxy for stock volatility is the Dow Jones VStoxx 50 index. The Dow Jones VStoxx 50 index measures the volatility implied in options on the Dow Jones Euro Stoxx 50 index. The considered interest rate variable is the Euro swap rate versus Euribor for 1-year maturity.

pirical findings as it tends to reflect the prevailing circumstances of economic distress.

Abstract

In this dissertation we study the indifference buyer's price and the indifference seller's price of an uncertainty averse decision-maker and the characterization of a decision-maker's attitudes toward uncertainty.

In the first part of the dissertation we study the properties fulfilled by the indifference buyer's price and by the indifference seller's price of an uncertainty averse decision-maker. We find that the indifference buyer's price is a quasiconvex risk measure and that the indifference seller's price is a cash-additive convex risk measure. We identify the acceptance family of the indifference buyer's price as well as the acceptance set of the indifference seller's price. We characterize the dual representations of the indifference buyer's price and of the indifference seller's price both in terms of probability charges and in terms of probability measures.

In the second part of the dissertation we study the characterization of a decision-maker's attitudes toward uncertainty in terms of the indifference buyer's price and of the indifference seller's price. We find that a decision-maker is more uncertainty averse than another if and only if her indifference buyer's price and her indifference seller's price are larger than for the other. We find that a decision-maker is increasingly (respectively, decreasingly, constantly) uncertainty averse if and only if her indifference buyer's price and her indifference seller's price are increasing (respectively, decreasing, constant) functions of her constant initial wealth.

In the last part of the dissertation we further develop the characterization of increasing, decreasing, and constant uncertainty aversion and we derive a technical condition that allows to immediately verify whether an uncertainty averse representation of preferences exhibits increasing, decreasing, or constant uncertainty aversion. We find that this technical condition allows

to classify a large class of uncertainty averse representations of preferences into increasingly, decreasingly, and constantly uncertainty averse.

Contents

Chapter 1

Introduction

The indifference prices are the boundaries delimiting the prices of a contract that would be agreed to by an individual who prefers more money to less money and who endeavors to maximize the relative desirability of her monetary endowment. The technique of indifference pricing was introduced by Bernoulli (1738) contextually with the prediction that an individual chooses, among alternative monetary endowments, the one providing a maximum of expected utility. The consistency of the paradigm of expected utility maximization and, accordingly, of the resulting indifference prices, with criteria of logic and rationality, was established by von Neumann and Morgenstern (1953) in a framework in which future events are assigned objective probabilities, and extended by Savage (1972) to a framework in which future events are assigned subjective probabilities (see also Ramsey (1931) and de Finetti (1964)).

The technique of indifference pricing was further developed by Pratt (1964) in relation to the characterization of an individual's attitudes toward risk. In Pratt (1964) risk is intended as the variability of the outcomes of a monetary prospect, irrespective of whether the different possible outcomes of the monetary prospect are described by objective probabilities as in von Neumann and Morgenstern (1953) or by subjective probabilities as in Savage (1972). Pratt (1964) found that an individual is more risk averse than another if and only if the maximum price that she would offer to avoid a risky monetary prospect is larger than for the other, and that an individual is increasingly (respectively, decreasingly, constantly) risk averse if and only if the maximum price that she would offer to avoid a risky monetary prospect is an increasing (respectively, decreasing, constant) function of her constant initial wealth. Pratt (1964) further observed that an individual is more risk averse than another if and only if the degree of relative convexity of her utility function (de Finetti (1952), Arrow (1970) and Pratt (1964)) is larger than for the other, and that an individual is increasingly (respectively, decreasingly, constantly) risk averse if and only the degree of relative convexity of her utility function is an increasing (respectively, decreasing, constant) function of her constant initial wealth. Pratt (1964) showed that the characterization of increasing, decreasing and constant risk aversion in terms of the degree of relative convexity of a utility function (de Finetti (1952)) or Arrow-Pratt coefficient of risk aversion (Arrow (1970) and Pratt (1964)) allows to immediately classify the different possible specifications of a utility function into increasingly, decreasingly, and constantly risk averse. The indifference prices of an expected utility maximizer, intended indifferently either in the sense of von Neumann and Morgenstern (1953) or in the sense of Savage (1972), have been extensively applied in the actuarial mathematics literature on premium calculation principles (see, among others, Bühlmann (1970) and Deprez and Gerber (1985)), and in the financial mathematics literature on pricing in incomplete markets (see, for instance, Carmona (2009) and the references therein). Ellsberg (1961) observed that, however, individuals do not always act consistently with the maximization of expected utility[1]. Ellsberg (1961) observed that, specifically, if an individual considers herself considerably ignorant of the relative frequencies of future events, and if she dislikes her state of considerable ignorance of the relative frequencies of future events, then "*it is impossible to find probability numbers in terms of which [... her] choices could be described - even roughly or approximately - as maximizing the mathematical expectation of utility*". Schmeidler (1989) indicated that the violation of the paradigm of expected utility maximization in the particular situations described by Ellsberg (1961) is consistent with a disfavor for the choices involving subjective rather than objective probabilities. Schmeidler (1989) designated an individual's disfavor for the choices involving subjective rather than objective probabilities as uncertainty aversion, and showed that the choices of an uncertainty averse individual could be described as maximizing an objective function which is more general than the mathematical expectation of utility. Schmeidler (1989) showed that, in particular, the choices of an uncertainty averse individual could be described as maximizing the integral of the utility with respect to a capacity or non-additive probability. Since the seminal paper of Schmeidler (1989), several other objective functions have been proposed in the literature to describe the choices of an uncertainty averse individual. Examples are the multiple priors (Gilboa and Schmeidler (1989)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)) and the variational

[1]Other deviations from the paradigm of expected utility maximization different from the one described by Ellsberg (1961) were discovered, for instance, by Allais (1953). This dissertation is concerned, however, only with the violations of expected utility theory observed by Ellsberg (1961), and not also with the ones observed by Allais (1953).

(Maccheroni et al. (2006)) representations of preferences. Cerreia Vioglio et al. (2011a) showed that, however, many objective functions which describe the choices of an uncertainty averse individual are particular cases of a more general objective function which, because of its unifying character, was denominated by Cerreia Vioglio et al. (2011a) the uncertainty averse representation of preferences.

In this dissertation we study the indifference prices defined by the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) and their relationship with the characterization of an individual's attitudes toward uncertainty.

In Chapter 2 we introduce the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) along with its particular specifications corresponding to the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)) and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences.

In Chapter 3 we study the indifference prices defined by the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a). We define the indifference buyer's price as the maximum price that an uncertainty averse individual would offer to avoid an uncertain monetary prospect, and the indifference seller's price as the minimum price that an uncertainty averse individual would demand to accept an uncertain monetary prospect. We show that the indifference buyer's price is a quasiconvex risk measure, and that the indifference seller's price is a cash-additive convex risk measure. We study the relationship between the indifference buyer's price and its acceptance family, as well as the relationship between the indifference seller's price and its acceptance set. We provide the dual representations of the indifference buyer's price and of the indifference seller's price both on probability charges and on probability measures. We further develop the dual representations on probability measures of the indifference buyer's price and of the indifference seller's price defined in terms of the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)) and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences.

In Chapter 4 we study the characterization of an individual's attitudes toward uncertainty in terms of the indifference buyer's price and of the indifference seller's price defined by the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a). We show that a decision-maker is more uncertainty averse than another if and only if her indifference buyer's price and her indifference seller's price are larger than for the other, and that a decision-maker is increasingly (respectively, decreasingly, constantly) uncertainty averse if and only if her indifference buyer's price and her indifference seller's price are increasing (respectively, decreasing, constant) functions of her constant initial wealth. We find a correspondence between increasing, decreasing, and constant uncertainty aversion and the additive properties that the indifference buyer's price satisfies with respect to the positive constants (e.g. cash-subadditivity (El Karoui and Ravanelli (2009))), and we show that these additive properties fulfilled by the indifference buyer's price allow to immediately establish various inequalities between the indifference buyer's price and the indifference seller's price. We find a correspondence between increasing, decreasing, and constant uncertainty aversion and the multiplicative properties that the uncertainty index appearing in the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) satisfies with respect to the positive constants (e.g. star-shapedness (Cerreia Vioglio et al. (2010))), and we show that these multiplicative properties fulfilled by the uncertainty index allow to immediately classify the different possible specifications of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) into increasingly, decreasingly, and constantly uncertainty averse. We find that the variational (Maccheroni et al. (2006)) and, as a consequence, the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), representations of preferences are decreasingly uncertainty averse, and that the multiple priors (Gilboa and Schmeidler (1989)) representation of preferences is constantly uncertainty averse.

Chapter 2

Decision-Theoretic Framework

2.1 Notations and Basic Concepts

2.1.1 Mathematical Notations

The pair (S, Σ) is a measurable space where S is a set of future states of nature and Σ is a σ-algebra of subsets of S representing future events.

The set $\mathcal{X} = B(S, \Sigma)$ is the set of all bounded, real-valued, Σ-measurable functions X on S, including the constant functions $X(s) = x \in \mathbb{R}$ for all $s \in S$. The subset of constant functions in $\mathcal{X}$ is identified with $\mathbb{R}$ and every equality or inequality involving elements of $\mathcal{X}$ is intended as holding for all $s \in S$.

The set $\mathcal{X}^* = \text{ba}(S, \Sigma)$ is the set of all bounded, finitely additive, real-valued set functions P on Σ and $\mathcal{X}^*_\sigma = \text{ca}(S, \Sigma) \subset \mathcal{X}^*$ is its subset of countably additive elements. A set function P on Σ is finitely additive if $P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i)$ for every finite family $\{E_i\}_{i=1}^n$ of pairwise disjoint sets in Σ

and it is countably additive if $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ for every countable family $\{E_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets in Σ. The set of positive normalized set functions in $\mathcal{X}^*$ is indicated by,

$$\Delta := \{P \in \mathcal{X}^* \mid P(E) \geq 0 \; \forall \, E \in \Sigma, \, P(S) = 1\}$$

and the subset of countably additive elements in $\Delta \subset \mathcal{X}^*$ is indicated by $\Delta^{\sigma} \subset \mathcal{X}_{\sigma}^*$.

The sets $\mathcal{X}$ and $\mathcal{X}^*$ endowed, respectively, with the supremum norm[1] and with the total variational norm[2], are Banach spaces. The space $\mathcal{X}^*$ is identified with the dual space of $\mathcal{X}$ and the evaluation duality is given by,

$$E_P[X] := \int_S X(s) P(\mathrm{d}s)$$

for all $(X, P) \in \mathcal{X} \times \mathcal{X}^*$. Unless otherwise specified, $\mathcal{X}$ is endowed with its norm topology, $\mathcal{X}^*$ is endowed with its weak* topology, and product spaces are endowed with their product topology. For the definitions of the different topologies see Aliprantis and Border (2006, Chapter 2).

2.1.2 A Note on Terminology

In the terminology of measure theory the elements of $\Delta \subset \mathcal{X}^*$ are *probability charges* and the elements of $\Delta^{\sigma} \subset \mathcal{X}_{\sigma}^*$ are *probability measures*. In this dissertation we adopt, however, the terminology of decision-theory, in which the elements of $\Delta \subset \mathcal{X}^*$ are *finitely additive probabilities*, and the elements of $\Delta^{\sigma} \subset \mathcal{X}_{\sigma}^*$ are *countably additive probabilities*.

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[1] That is, $||X||_{\infty} := \sup_{s \in S} |X(s)|$ for all $X \in \mathcal{X}$.

[2] For every $E \in \Sigma$ the total variation of a P on E is defined as $||P|| = \sup \sum_{i=1}^{n} |P(E_i)|$ where the supremum is taken over all finite sequences $\{E_i\}$ of disjoint sets in Σ with $E_i \subseteq E$ (see Dunford and Schwartz (1988, III.1.4)).

2.1.3 Decision-Theoretic Concepts

The considered decision-theoretic set-up is a Savage (1972) framework. The elements of $\mathcal{X}$ are interpreted as *monetary payoffs*, that is as the alternative courses of actions that are available to an individual whose consequences are money payments.

The non-constant monetary payoffs in $\mathcal{X}$ are interpreted as entailing "unmeasurable" *uncertainty* (Knight (1921)) in the sense of having as their "[...] *consequences a set of possible specific outcomes, but where the probabilities of these outcomes are completely unknown or are not even meaningful*" (Luce and Raiffa (1989)).

The constant monetary payoffs in $\mathcal{X}$ are instead interpreted as *certain* as they "[...] *lead invariably to a specific outcome*" (Luce and Raiffa (1989)).

2.2 Background on Uncertainty Aversion

In this section we present the notions and concepts of decision under uncertainty which were relevant to the study and development of uncertainty averse preferences. In Subsection 2.2.1 we briefly introduce Savage's (1972) expected utility. In Subsection 2.2.2 we describe the violation of Savage's (1972) expected utility known as Ellsberg's (1961) paradox. In Subsection 2.2.3 we describe Schmeidler's (1989) rationalization of Ellsberg's (1961) paradox in terms of a preference for mixtures, averages, or randomizations.

2.2.1 Subjective Expected Utility

According to Savage's (1972) *expected utility theory* an individual whose choices are consistent with some essential principles of logic and are not intrinsically contradictory evaluates the relative desirability of alternative monetary payoffs by a function $U^{u,Q} : \mathcal{X} \to \mathbb{R}$ of the form,

$$U^{u,Q}(X) = E_Q[u(X)] \tag{2.1}$$

for all $X \in \mathcal{X}$. The finitely additive probability $Q \in \Delta$ in Equation (2.1) is a *subjective probability*, *prior*, or *belief* (see also Ramsey (1931) and de Finetti (1964)) reflecting the decision-maker's personal opinion on the relative likelihoods of future events. The function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.1) is a *utility function* reflecting the value, utility, or advantage, that the decision-maker derives from particular monetary outcomes (see Bernoulli (1738) and von Neumann and Morgenstern (1953)).

2.2.2 Ellsberg's Paradox

The representation in Equation (2.1) implies that an individual whose choices are consistent with the normative principles established by Savage (1972) acts as if she assigned probabilities to future events and as if she chose, among alternative monetary payoffs, the one providing a maximum of expected utility. Ellsberg (1961) observed that there are, however, some particular circumstances in which an individual perceives her information on the relative likelihoods of future events as considerably opaque, deceitful, or insufficient and, in contrast with the predictions of Savage's (1972) expected utility theory, she does not assign, or act as if she assigned, "meaningful probabilities" to future events.

Ellsberg (1961) discussed, for instance, the following example. Consider an urn known to contain 30 red balls and 60 black and green balls, the latter in unknown proportion. One ball is to be drawn at random from the urn. The "objective" probabilities of a red ball being drawn (R) and of a red ball not being drawn (R^C) are "completely known" and equal to 1/3 and 2/3 respectively. In contrast, the "objective" probabilities of a black ball being drawn (B) and of a green ball being drawn (G) are "significantly ignored". They are not "completely ignored" because they are known to lie in the interval $[0, 2/3]$; there is, however, only little information to judge their precise values. Ellsberg (1961) investigated how an individual chooses among the alternative monetary payoffs described in Table 2.1 whose outcomes are contingent on the colour of the ball drawn from the urn described above. Ellsberg (1961) observed that most individuals prefer X to Y and Y' to X'

	30	60	
	R	B	G
X	\$100	\$0	\$0
Y	\$0	\$100	\$0
X'	\$100	\$0	\$100
Y'	\$0	\$100	\$100

Table 2.1: Urn Example I.

and that "*it is impossible to find probability numbers in terms of which these choices could be described - even roughly or approximately - as maximizing the mathematical expectation of utility*". Consider, in fact, an individual who maximizes the expected utility in Equation (2.1) and assume, without loss of generality[3], that $u(0) = 0$, and that $u(100) = 1$. This individual strictly prefers X to Y if and only if $Q(R) > Q(B)$ and strictly prefers Y' to X' if and only if $Q(R^C) > Q(B^C)$. Thus, the choices of this individual reveal that her subjective probability satisfies both $Q(R) > Q(B)$ and $Q(R) < Q(B)$, which is impossible. For this reason, these findings, as well

[3]The utility function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.1) is unique up to positive affine transformations. Thus, there is no loss of generality in replacing $u : \mathbb{R} \to \mathbb{R}$ by $\tilde{u} : \mathbb{R} \to \mathbb{R}$ where $\tilde{u}(x) = \beta + \alpha u(x)$ for all $x \in \mathbb{R}$ with $\beta \in \mathbb{R}$ and $\alpha \in (0, +\infty)$ such that $\alpha = 1/(u(100) - u(0))$ and $\beta = 1 - u(100)/(u(100) - u(0))$ provided that $u(100) > u(0)$.

as other analogous findings collected by Ellsberg (1961), are referred to as the *Ellsberg's paradox*.

Ellsberg (1961) further observed that the choice of X over Y and of Y' over X' is motivated by a preference for the monetary payoffs whose outcomes are realized on events whose "objective" probabilities are "completely known", as opposed to the monetary payoffs whose outcomes are realized on events whose "objective" probabilities are "significantly ignored".

2.2.3 Uncertainty Aversion

Schmeidler (1989) noticed that an individual's violation of the paradigm of expected utility maximization in the circumstances described by Ellsberg (1961) is consistent with a preference for smoothing, or averaging, alternative monetary payoffs across states of nature, and that this preference for mixtures reflects an endeavor to "objectify" the probabilities of the future events on which the outcomes of the monetary payoffs are realized (see Klibanoff (2001)). Consider again the urn described in Subsection 2.2.2 and assume that an individual is asked to choose among the alternative monetary payoffs described in Table 2.2. An individual who considers herself consid-

	30	60	
	R	B	G
Y	\$0	\$100	\$0
Z	\$0	\$0	\$100
$\frac{1}{2}Y + \frac{1}{2}Z$	\$0	\$50	\$50

Table 2.2: Urn Example II.

erably ignorant of the relative likelihoods of the events on which a blue ball is drawn (B) and a green ball is drawn (G) would be indifferent between Y and Z, but would prefer their "average" $\frac{1}{2}Y + \frac{1}{2}Z$ to either of them alone:

while the probability that either Y or Z pays \$100 is "significantly ignored", as it is only known to lie in the interval $[0, 2/3]$, the probability that their "average" $\frac{1}{2}Y + \frac{1}{2}Z$ pays \$50 is "completely known", as it is known to be equal to 2/3.

Schmeidler (1989) designated an individual who prefers a mixture of equally desirable monetary payoffs to either of them alone as *uncertainty averse* to indicate that her choices reveal a preference for the monetary payoffs whose outcomes are realized on the future events to which probabilities are assigned "objectively" and a disfavour for the monetary payoffs whose outcomes are realized on the future events to which probabilities are to be assigned subjectively (see also Klibanoff (2001)).

2.3 Uncertainty Averse Preferences

Although whether uncertainty aversion should be considered a normative principle of decision under uncertainty or a possibly implausible and irrational trait of particular individuals is still the object of debate (see Al-Najjar and Weinstein (2009)) several models of choice under uncertainty have been developed in the economic literature which explicitly postulate uncertainty aversion of the decision-maker. Examples are the *multiple priors* (Gilboa and Schmeidler (1989)), the *multiplier* (Hansen and Sargent (2001), Strzalecki (2011)), and the *variational* (Maccheroni et al. (2006)) representations of preferences. Other models of choice under uncertainty which allow for uncertainty aversion of the decision-maker are, among others, the *Choquet expected utility* (Schmeidler (1989)) and the *smooth ambiguity* (Klibanoff et al. (2005)) representations of preferences.

Cerreia Vioglio et al. (2011a) showed that all the decision-theoretic models which characterize uncertainty aversion through Schmeidler's (1989) preference for mixtures represent particular cases of a more fundamental class of preferences which, because of their great generality, were denominated by Cerreia Vioglio et al. (2011a) *uncertainty averse* preferences[4]. The uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ of Cerreia Vioglio et al. (2011a) is given by,

$$U^{u,G}(X) = \inf_{P \in \Delta} G(E_P[u(X)], P) \tag{2.2}$$

for all $X \in \mathcal{X}$. The function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.2) is a utility function (as in Equation (2.1)) reflecting the decision-maker's attitudes toward risk. The function $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2.2) is an *uncertainty index* reflecting the decision-maker's attitudes toward uncertainty. The smaller the uncertainty index G, the larger the decision-maker's uncertainty aversion. The uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2.2) is increasing on $\mathbb{R}$ for each $P \in \Delta$, lower semi-continuous and quasi-convex on $\mathbb{R} \times \Delta$, normalized, that is such that $\inf_{P \in \Delta} G(y, P) = y$ for all $y \in \mathbb{R}$, and such that $G(., P)$ is extended-valued continuous on $\mathbb{R}$ for each $P \in \Delta$[5].

The representation in Equation (2.2) implies that an uncertainty averse decision-maker whose choices are consistent with the principles established by Cerreia Vioglio et al. (2011a) evaluates the relative desirability of an uncertain monetary payoff in $\mathcal{X}$ as if, by the function G, she appraised its expected utility under each probabilistic scenario in Δ, and as if she summarized her appraisal by considering exclusively the worst probabilistic scenario in Δ.

[4]As the Choquet expected utility model of Schmeidler (1989) and the smooth ambiguity model of Klibanoff et al. (2005) do not require a priori that the decision-maker is uncertainty averse, in this dissertation they will not be treated as special cases of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a).

[5]That is, $\lim_{x \to x_0} G(x, P) = G(x_0, P) \in (-\infty, +\infty]$ for all $x_0 \in \mathbb{R}$ and $P \in \Delta$.

In the following subsections we illustrate that the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences are obtained as particular cases of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) under suitable specifications of the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2.2).

2.3.1 Variational Preferences

The *variational representation of preferences* $U^{u,c} : \mathcal{X} \to \mathbb{R}$ of Maccheroni et al. (2006) is given by,

$$U^{u,c}(X) = \inf_{P \in \Delta} \Big(\mathbb{E}_P[u(X)] + c(P) \Big) \tag{2.3}$$

for all $X \in \mathcal{X}$. The function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.2) is a utility function (as in Equation (2.1)) reflecting the decision-maker's attitudes toward risk. The function $c : \Delta \to (-\infty, +\infty]$ in Equation (2.3) is an *ambiguity index* reflecting the decision-maker's attitudes toward uncertainty. The smaller the ambiguity index c, the larger the decision-maker's uncertainty aversion. The ambiguity index $c : \Delta \to (-\infty, +\infty]$ is convex, lower semi-continuous, and normalized, that is such that $\inf_{P \in \Delta} c(P) = 0$.

The representation in Equation (2.3) implies that an uncertainty averse decision-maker whose choices are consistent with the principles established by Maccheroni et al. (2006) evaluates the relative desirability of an uncertain monetary payoff in $\mathcal{X}$ as if, by the function c, she applied a correction to its expected utility under each probabilistic scenario in Δ, and as if she summarized her appraisal by considering exclusively the worst probabilistic scenario in Δ.

The variational representation of preferences is a particular case of the uncertainty averse representation of preferences in Equation (2.2) which is obtained when the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ satisfies,

$$G(x, P) = x + c(P) \tag{2.4}$$

for all $(x, P) \in \mathbb{R} \times \Delta$.

2.3.2 Multiplier Preferences

The *multiplier representation of preferences* $U^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$ introduced by Hansen and Sargent (2001) and axiomatized by Strzalecki (2011) is given by,

$$U^{u,\theta,\mathcal{R},\mathbb{P}^\star}(X) = \inf_{P \in \Delta} \Big(\mathbb{E}_P[u(X)] + \theta \mathcal{R}(P \,||\, \mathbb{P}^\star) \Big) \tag{2.5}$$

for all $X \in \mathcal{X}$ with $\theta \in (0, +\infty]$ and $\mathbb{P}^\star \in \Delta^\sigma$. The function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.2) is a utility function (as in Equation (2.1)) reflecting the decision-maker's attitudes toward risk. The constant $\theta \in (0, +\infty]$ in Equation (2.2) is a parameter reflecting the decision-maker's attitudes toward uncertainty. The smaller the parameter $\theta \in (0, +\infty]$, the larger the decision-maker's uncertainty aversion. The function $\mathcal{R}(. \,||\, \mathbb{P}^\star) : \Delta \to [0, +\infty]$ in Equation (2.2) is the *relative entropy* with respect to $\mathbb{P}^\star \in \Delta^\sigma$ which is given by,

$$\mathcal{R}(P \,||\, \mathbb{P}^\star) = \begin{cases} E_P\Big[\ln\Big(\dfrac{dP}{d\mathbb{P}^\star}\Big)\Big] & \text{if } P \in \Delta^\sigma(\mathbb{P}^\star) \\ +\infty & \text{otherwise} \end{cases}$$

for all $P \in \Delta$. The set $\Delta^\sigma(\mathbb{Q}) \subset \Delta^\sigma$ is the set of all probability measures on (S, Σ) which are absolutely continuous with respect to $\mathbb{P}^\star \in \Delta^\sigma$. The probability measure $\mathbb{P}^\star \in \Delta^\sigma$ is interpreted as the decision-maker's best guess of the "right" probability on (S, Σ) and it is designated as the decision-maker's *reference probability* (see Strzalecki (2011)).

The representation in Equation (2.5) implies that an uncertainty averse decision-maker whose choices are consistent with the principles established by Strzalecki (2011) evaluates the relative desirability of an uncertain monetary payoff in $\mathcal{X}$ as if she applied a correction to its expected utility under each probabilistic scenario in $P \in \Delta$, the correction depending on the "distance" $\mathcal{R}(P \,||\, \mathbb{P}^\star)$ of the considered probabilistic scenario $P \in \Delta$ from the reference scenario $\mathbb{Q} \in \Delta^\sigma$, and on the relevance $\theta \in (0, +\infty]$ that the decision-maker assigns to this "distance", and as if she summarized her appraisal by considering exclusively the worst probabilistic scenario in Δ.

The multiplier representation of preferences is a particular case of the uncertainty averse representation of preferences in Equation (2.2) which is obtained when the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ satisfies,

$$G(x, P) = x + \theta \mathcal{R}(P \,||\, \mathbb{P}^\star) \tag{2.6}$$

for all $(x, P) \in \mathbb{R} \times \Delta$.

2.3.3 Multiple Priors Preferences

The *multiple priors representation of preferences* $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ of Gilboa and Schmeidler (1989) is given by,

$$U^{u,\mathcal{P}}(X) = \inf_{P \in \mathcal{P}} \mathbb{E}_P[u(X)] \tag{2.7}$$

for all $X \in \mathcal{X}$. The function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.2) is a utility function (as in Equation (2.1)) reflecting the decision-maker's attitudes toward risk. The set $\mathcal{P} \subset \Delta$ is a *set of priors* reflecting the decision-maker's attitudes toward uncertainty. The larger the set of priors $\mathcal{P}$, the larger the decision-maker's uncertainty aversion.

The representation in Equation (2.7) implies that an uncertainty averse decision-maker whose choices are consistent with the principles established by Gilboa and Schmeidler (1989) evaluates the relative desirability of an uncertain monetary payoff in $\mathcal{X}$ as if she appraised its expected utility under each probabilistic scenario in $\mathcal{P} \subset \Delta$ and as if she summarized her appraisal by considering exclusively the worst probabilistic scenario in $\mathcal{P}$.

The multiple priors representation of preferences is a particular case of the uncertainty averse representation of preferences in Equation (2.2) which is obtained when the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ satisfies,

$$G(x, P) = x + \delta(P \,|\, \mathcal{P}) \tag{2.8}$$

for all $(x, P) \in \mathbb{R} \times \Delta$ and where $\delta(. \,|\, \mathcal{P}) \to [0, +\infty]$ is defined by,

$$\delta(P \,|\, \mathcal{P}) = \begin{cases} 0 & \text{if } P \in \mathcal{P} \\ +\infty & \text{otherwise} \end{cases}$$

for all $P \in \Delta$.

2.4 Remarks, Assumptions, and Continuity Concepts

In this section we discuss various technical aspects of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a). In section 2.4.2 we clarify the relationship between the Savage (1972) framework considered in this dissertation and the generalized Anscombe and Aumann (1963) framework originally considered by Cerreia Vioglio et al. (2011a). In Section 2.4.2 we clearly state the assumptions on the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) which allow to obtain the results presented in this dissertation. In Section 2.4.3 we discuss some continuity concepts which allow to characterize the situations in which the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) can be equivalently expressed in terms of countably additive probabilities.

2.4.1 Remarks

The representations of preferences of Cerreia Vioglio et al. (2011a), Strzalecki (2011), Maccheroni et al. (2006) and Gilboa and Schmeidler (1989) were originally obtained in a generalized Anscombe and Aumann (1963) framework in which the objects of choice are *uncertain acts*. An uncertain act is a Σ-measurable *simple*[6] function f on S taking values in a convex subset C of a vector space. C could be specified, for instance, as the set of all probability measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with *finite* support, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. A probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has finite support if the smallest closed set $B \in \mathcal{B}(\mathbb{R})$ such that $\mu(B^c) = 0$ is finite.

Let $\mathcal{F} := \mathcal{F}(C)$ be the set of all uncertain acts. Let $\tilde{u} : C \to \mathbb{R}$ be a non-constant affine function. Let $\tilde{u}(C) := \{\tilde{u}(c), c \in C\} \subseteq \mathbb{R}$. Let $B_0(\Sigma, \tilde{u}(C)) := B_0(S, \Sigma; \tilde{u}(C))$ be the set of all real-valued Σ-measurable *simple* functions on S with values in $\tilde{u}(C) \subseteq \mathbb{R}$. Observe that, if $f \in \mathcal{F}$, then $\tilde{u}(f) \in B_0(\Sigma, \tilde{u}(C))$. Let $\tilde{G} : \tilde{u}(C) \times \Delta \to (-\infty, +\infty]$ be such that $\tilde{G}(., P)$ is increasing on $\tilde{u}(C) \subseteq \mathbb{R}$ for each $P \in \Delta$, lower semi-continuous and quasi-convex on $\tilde{u}(C) \times \Delta$, normalized, that is such that $\inf_{P \in \Delta} \tilde{G}(y, P) = y$ for all $y \in \tilde{u}(C)$, and such that $\tilde{G}(., P)$ is extended-valued continuous on $\tilde{u}(C) \subseteq \mathbb{R}$ for each $P \in \Delta$. An uncertainty averse representation of preferences (Cer-

[6] A function is said to be simple if it takes on only finitely many values.

reia Vioglio et al. (2011a, Theorem 3)) is a function $V^{\tilde{u},\tilde{G}} : \mathcal{F} \to \mathbb{R}$ defined by,

$$V^{\tilde{u},\tilde{G}}(f) := \inf_{P\in\Delta} \tilde{G}(E_P[\tilde{u}(f)], P) \tag{2.9}$$

for all $f \in \mathcal{F}$. As in Section 2.3, the function $\tilde{G} : \tilde{u}(C) \times \Delta \to (-\infty, +\infty]$ in Equation (2.9) is an uncertainty index describing the decision-maker's attitudes toward uncertainty. Differently from Section 2.3, the function $\tilde{u} : C \to \mathbb{R}$ in Equation (2.9) is not a utility function[7] but a *utility index* describing the decision-maker's attitudes toward risk. If C is specified, for instance, as the set of all probability measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite support, then the utility index $\tilde{u} : C \to \mathbb{R}$ in Equation (2.9) is related to the utility function $u : \mathbb{R} \to \mathbb{R}$ in Equation (2.2) by the following relationship,

$$\tilde{u}(f(s)) = \int_{\mathbb{R}} u(x) \mathrm{d}\, f(s, x)$$

for all $s \in S$. Although the uncertainty averse representation of preferences was defined and characterized by Cerreia Vioglio et al. (2011a) on the set $\mathcal{F}$ of all Σ-measurable simple function f on S with values in C, it admits a continuous extension $V^{\tilde{u}',\tilde{G}'} : \mathcal{F}_b \to \mathbb{R}$ to the set $\mathcal{F}_b := \mathcal{F}_b(C)$ of all *bounded* Σ-measurable functions f_b on S with values in C (see Ghirardato and Siniscalchi (2009)). As a result of this extension, C can be specified as the set $\mathcal{M}_b := \mathcal{M}_b(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of all probability measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with *bounded* support. A probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has bounded support if $\mu([-b, b]) = 1$ for some $b \geq 0$. Denote by $\tilde{\mathcal{X}} := \mathcal{F}_b(\mathcal{M}_b)$ the set of

[7]As indicated in Section 2.2.1 in this dissertation a *utility function* is intended as a *utility function for money* $u : \mathbb{R} \to \mathbb{R}$. Thus, in this dissertation a utility function is not, in general, an affine function.

all bounded Σ-measurable functions $\tilde{X} := f_b$ on S with values in $\mathcal{M}_b$. Then,

$$\tilde{u}'(\tilde{X}(s)) = \int_{\mathbb{R}} u(x) \mathrm{d}\, \tilde{X}(s,x) \tag{2.10}$$

for all $s \in S$. Let $X \in \mathcal{X}$ and let $\delta_X \in \tilde{\mathcal{X}}$ be such that $\delta_{X(s)}(.)$ is a Dirac measure on $\mathcal{B}(\mathbb{R})$ for each $s \in S$, that is,

$$\delta_{X(s)}(B) = \begin{cases} 1 & \text{if } X(s) \in B \\ 0 & \text{otherwise} \end{cases}$$

for all $B \in \mathcal{B}(\mathbb{R})$. Föllmer and Schied (2004, Section 2.5) observed that the mapping,

$$X \in \mathcal{X} \quad \mapsto \delta_X \in \tilde{\mathcal{X}}$$

is an embedding. It follows that the space $\mathcal{X}$ of all bounded, real-valued, Σ-measurable functions on S can be identified with the set of all the elements of $\tilde{\mathcal{X}}$ which are Dirac measures on $\mathcal{B}(\mathbb{R})$ for each $s \in S$. Let $U^{u,G'} : \mathcal{X} \to \mathbb{R}$ be the function defined by,

$$U^{u,G'}(X) := V^{\tilde{u}',\tilde{G}'}(\delta_X) \tag{2.11}$$

for all $X \in \mathcal{X}$. Thus, by Equation (2.11), Equation (2.9), and Equation (2.10),

$$\begin{aligned} U^{u,G'}(X) &= \inf_{P \in \Delta} G'(E_P[\tilde{u}'(\delta_X)], P) \\ &= \inf_{P \in \Delta} G'(E_P\Big[\int_{\mathbb{R}} u(x) \mathrm{d}\, \delta_X(x)\Big], P) \\ &= \inf_{P \in \Delta} G'(E_P[u(X)], P) \end{aligned}$$

as in Equation (2.2).

2.4.2 Assumptions

Throughout this dissertation we will implicitly assume that the utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and concave and that the monotonicity of the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in its first argument is strict.

The assumption that the utility function u is increasing ensures that the preferences that the decision-maker expresses through the indifference prices are consistent with the compelling principle of *rationality* (see Cerreia Vioglio et al. (2010)). The assumption that the utility function u is concave ensures that the preferences that the decision-maker expresses through the indifference prices are consistent with the fundamental principle that *diversification* does not increase "risk" (see Cerreia Vioglio et al. (2010)).

The assumptions that the utility function u and the uncertainty index G are strictly increasing guarantee that the indifference prices are uniquely defined.

Assumption 2.1. The utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and concave.

Assumption 2.2. The uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ is strictly increasing in its first argument.

Observe that, as u is concave and finite on all of $\mathbb{R}$, it is necessarily continuous (see Rockafellar (1970, Corollary 10.1.1)) and that the continuity of the utility function u, combined with the different continuity properties of the uncertainty index G, guarantees that the indifference prices exist.

Overall, Assumption 2.1 and Assumption 2.2 ensure that the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ in Equation (2.2) is strictly increasing, quasiconcave, and continuous with respect to the sup-norm $||.||_\infty$.

2.4.3 Continuity Concepts

The theories of decision under uncertainty discussed in Section 2.3 describe a decision-maker's subjective probabilities, priors, or beliefs in terms of finitely additive probabilities. The conditions of positivity, normalization, and finite additivity are in fact sufficient for a set function to be interpretable as an individual's "coherent" judgment of probabilities (de Finetti, Chapter 3 (1970)). An individual's judgment of probabilities is said to be "coherent", acceptable, or admissible if it is not intrinsically contradictory (de Finetti, Chapter 3 (1970)).

Although from the decision-theoretic perspective the condition of countable additivity is objectionable (see Ramsey (1931), de Finetti (1964), and Savage (1972)), from the mathematical perspective it is a convenient simplification. The axiom with which an individual's choices must be consistent for her preferences to be represented in terms of countably additive probabilities is the axiom of *monotone continuity* (Arrow (1970)). The axiom of monotone continuity was introduced by Arrow (1970) in the framework of Savage's (1972) expected utility and was employed, among others, by Cerreia Vioglio et al. (2011a), Maccheroni et al. (2006) and Chateauneuf et al. (2005) to express, respectively, the uncertainty averse (Cerreia Vioglio et al. (2011a)), the variational (Maccheroni et al. (2006)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences in terms of countably additive probabilities. An equivalent technical condition, known as *continuity from below*, is employed in the financial mathematics literature to describe the situation in which a quasiconvex (see Cerreia Vioglio et al. (2010)) or convex (see, for instance, Föllmer and Schied (2004, Chapter 4)) risk measure admits a dual representation on probability measures.

For simplicity in this dissertation we will assume, however, the stronger condition of *continuity with respect to bounded point-wise convergence*, also known as the *Lebesgue property* (see Jouini et al. (2006)).

Definition 2.1. An uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be *continuous with respect to bounded point-wise convergence* if $U^{u,G}(X_n) \to U^{u,G}(X)$ whenever $(X_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence[8] in $\mathcal{X}$ such that $X_n(s) \to X(s)$ for every $s \in S$.

Proposition 2.1 shows that the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ in Equation (2.2) is continuous with respect to bounded point-wise convergence if and only if the set of finitely additive probabilities $\Delta \subset \mathcal{X}^*$ in Equation (2.2) can be equivalently replaced by its subset of countably additive elements $\Delta^\sigma \subset \mathcal{X}^*_\sigma$ as the elements of $\Delta \subset \mathcal{X}^*$ which are not countably additive do not contribute to the formation of the minimum in Equation (2.2). In what follows the set of all elements of $\Delta \subset \mathcal{X}^*$ which do not belong to $\Delta^\sigma \subset \mathcal{X}^*_\sigma$ is denoted by,

$$\Delta \setminus \Delta^\sigma = \{P \in \mathcal{X}^* : P \in \Delta \text{ and } P \notin \Delta^\sigma\}$$

The proof of Proposition 2.1 is an immediate application of Proposition 4.3 and Proposition 4.5 in Cerreia Vioglio et al. (2010).

Proposition 2.1. *An uncertainty averse representation of preferences* $U^{u,G} : \mathcal{X} \to \mathbb{R}$ *is continuous with respect to bounded point-wise convergence if and only if,*

$$G(x, P) = +\infty$$

for all $(x, P) \in \mathbb{R} \times (\Delta \setminus \Delta^\sigma)$.

[8] That is, there exists $M \in \mathbb{R}$ such that $||X_n||_\infty \leq M$ for all $n \in \mathbb{N}$.

Proof of Proposition 2.1. Observe that Equation (2.2) can be equivalently written as,

$$U^{u,G}(X) = -\rho^R(u(X)) \quad \forall\, X \in \mathcal{X}$$

where $\rho^R : \mathcal{X} \to \mathbb{R}$ is a quasiconvex risk measure which is continuous with respect to the sup-norm $||.||_\infty$ and which is represented by the maximal risk function $R : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ given by,

$$R(x, Q) = -G(-x, Q)$$

for all $(x, Q) \in \mathbb{R} \times \Delta$. Observe also that $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if $\rho^R : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence. Thus, the statement follows directly from Proposition 4.3 and Proposition 4.5 in Cerreia Vioglio et al. (2010). □

Chapter 3

Indifference Pricing with Uncertainty Averse Preferences

3.1 Indifference Buyer's Price

Consider an uncertainty averse decision-maker who is endowed with a constant monetary payoff $w_0 \in \mathbb{R}$ and an uncertain monetary payoff $X \in \mathcal{X}$. The uncertainty averse decision-maker contemplates a transaction which allows her to transfer the uncertain component of her wealth $X \in \mathcal{X}$ in exchange for paying a constant amount of money $m \in \mathbb{R}$. Accepting the agreement would make her wealth constant and equal to $w_0 - m \in \mathbb{R}$. The uncertainty averse decision-maker is therefore in the position of a buyer of a policy (or insured) and the *maximum price* (or insurance premium) $m \in \mathbb{R}$ which, from the perspective of her uncertainty averse preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$, makes the the constant monetary payoff $w_0 - m \in \mathbb{R}$ more desirable than the uncertain monetary payoff $w_0 + X \in \mathcal{X}$, corresponds precisely to the price which makes them equally desirable. For this reason, this maximum price is denominated *indifference buyer's price*. In Subsection 3.1.1 we introduce its definition, we derive its properties, and we identify its acceptance family. In Subsection 3.1.2 we characterize its dual representation on finitely additive probabilities and on countably additive probabilities. In Subsection 3.1.3 we provide more explicit characterizations of its dual representation on countably additive probabilities in terms of the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences.

3.1.1 Definition, Properties, and Acceptance Family

3.1.1.1 Definition

The indifference buyer's price, which in this dissertation is considered from an actuarial perspective, is defined as a function $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ yielding the *maximum price* that a decision-maker with uncertainty averse preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ and with constant initial wealth $w_0 \in \mathbb{R}$ would offer to avoid an uncertain monetary prospect in $\mathcal{X}$ (e.g. to receive insurance).

Definition 3.1. A function $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be an *indifference buyer's price* if it satisfies,

$$u(w_0 - \pi_{w_0}^{u,G}(X)) = U^{u,G}(w_0 + X) \tag{3.1}$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$.

3.1.1.2 Properties

Proposition 3.1 asserts that the indifference buyer's price is *monotone decreasing*, *quasiconvex*, and *normalized.* As a consequence of these properties, the indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is a *quasiconvex risk measure.* Quasiconvex risk measures were introduced in the financial mathematics literature by Cerreia Vioglio et al. (2010). A quasiconvex risk measure is a function representing the ordering of alternative monetary payoffs in $\mathcal{X}$ based on their relative "risk", where the term risk is "*[...] used in a loose way to refer to any sort of uncertainty viewed from the standpoint of the unfavorable contingency*" (Knight (1921)).

Proposition 3.1. *The indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ satisfies the following properties for all $X, Y \in \mathcal{X}$ and $w_0 \in \mathbb{R}$.*

(i) Monotonicity: If $X \geq Y$, then $\pi_{w_0}^{u,G}(X) \leq \pi_{w_0}^{u,G}(Y)$.

(ii) Quasiconvexity: $\pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\}$ for all $\lambda \in [0,1]$.

(iii) Normalization: $\pi_{w_0}^{u,G}(m) = -m$ for all $m \in \mathbb{R}$.

Proof of Proposition 3.1. (i) Let $X, Y \in \mathcal{X}$. If $X \geq Y$, then by Definition 3.1 and by the increasing monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$,

$$u(w_0 - \pi_{w_0}^{u,G}(X)) = U^{u,G}(w_0 + X) \geq U^{u,G}(w_0 + Y) = u(w_0 - \pi_{w_0}^{u,G}(Y))$$

and the increasing monotonicity of $u : \mathbb{R} \to \mathbb{R}$ yields $\pi_{w_0}^{u,G}(X) \leq \pi_{w_0}^{u,G}(Y)$. Thus, $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is monotone decreasing.
(ii) If $\lambda \in [0,1]$, then by Definition 3.1, by the quasiconcavity of $U^{u,G} : \mathcal{X} \to$

$\mathbb{R}$ and by the increasing monotonicity of $u : \mathbb{R} \to \mathbb{R}$,

$$
\begin{aligned}
u(w_0 - \pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y)) \\
&= U^{u,G}(w_0 + \lambda X + (1-\lambda)Y) \\
&= U^{u,G}(\lambda(w_0+X) + (1-\lambda)(w_0+Y)) \\
&\geq \min\{U^{u,G}(w_0+X), U^{u,G}(w_0+Y)\} \\
&= \min\{u(w_0 - \pi_{w_0}^{u,G}(X)), u(w_0 - \pi_{w_0}^{u,G}(Y))\} \\
&= u(w_0 - \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\})
\end{aligned}
$$

and the increasing monotonicity of $u : \mathbb{R} \to \mathbb{R}$ yields,

$$\pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\}$$

for all $\lambda \in [0,1]$. Thus, $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is quasiconvex.

(iii) If $m \in \mathbb{R}$, then by Definition 3.1,

$$u(w_0 - \pi_{w_0}^{u,G}(m)) = u(w_0 + m)$$

and the strict monotonicity of $u : \mathbb{R} \to \mathbb{R}$ yields $\pi_{w_0}^{u,G}(m) = -m$. Thus, $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is normalized. □

Decreasing monotonicity implies that the decision-maker is willing to offer higher prices to avoid higher losses. Quasiconvexity implies that the maximum price that the decision-maker is willing to offer to avoid a portfolio of uncertain monetary payoffs is lower than the highest of the prices that she is inclined to offer to avoid its constituents. Normalization implies that the maximum price that the decision-maker is willing to offer to avoid a certain loss is exactly equal to its amount.

3.1.1.3 Acceptance Family

The indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ can be equivalently studied in terms of an appropriately defined *acceptance family*. Acceptance families were introduced in the mathematical finance literature on quasiconvex risk measures by Drapeau and Kupper (2010). An acceptance family is a collection of *acceptance sets*. We define the *acceptance set* of an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$ at level $m \in \mathbb{R}$ as the subset $\mathcal{A}_{w_0,m}^{u,G}$ of $\mathcal{X}$ given by,

$$\mathcal{A}_{w_0,m}^{u,G} := \{X \in \mathcal{X} \,|\, U^{u,G}(w_0 + X) \geq u(w_0 - m)\} \tag{3.2}$$

The acceptance set $\mathcal{A}_{w_0,m}^{u,G}$ defined by Equation (3.2) is the set of uncertain monetary payoffs in $\mathcal{X}$ that a an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$ prefers to the constant monetary payoff $-m \in \mathbb{R}$. We call $(\mathcal{A}_{w_0,m}^{u,G})_{m\in\mathbb{R}}$ the *acceptance family* of an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$. Proposition 3.2 asserts that the acceptance family $(\mathcal{A}_{w_0,m}^{u,G})_{m\in\mathbb{R}}$ is monotone, convex, and normalized.

Proposition 3.2. *The acceptance family* $(\mathcal{A}_{w_0,m}^{u,G})_{m\in\mathbb{R}}$ *satisfies the following properties for all* $X, Y \in \mathcal{X}$ *and* $m, n \in \mathbb{R}$.

(i) Monotonicity:

- *(a) If* $X \in \mathcal{A}_{w_0,m}^{u,G}$ *and* $Y \geq X$*, then* $Y \in \mathcal{A}_{w_0,m}^{u,G}$.
- *(b) If* $m \leq n$*, then* $\mathcal{A}_{w_0,m}^{u,G} \subseteq \mathcal{A}_{w_0,n}^{u,G}$.

(ii) Convexity: If $X, Y \in \mathcal{A}_{w_0,m}^{u,G}$*, then* $\lambda X + (1-\lambda)Y \in \mathcal{A}_{w_0,m}^{u,G}$ *for all* $\lambda \in [0, 1]$.

(iii) Normalization: $\inf\{x \in \mathbb{R} \,|\, x \in \mathcal{A}_{w_0,m}^{u,G}\} = -m$.

Proof. Let $X, Y \in \mathcal{X}$ and $m, n \in \mathbb{R}$.

(i-a) Let $X \in \mathcal{A}^{u,G}_{w_0,m}$ and $Y \geq X$. By the increasing monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$,

$$U^{u,G}(w_0 + Y) \geq U^{u,G}(w_0 + X) \geq u(w_0 - m)$$

Thus, $Y \in \mathcal{A}^{u,G}_{w_0,m}$.

(i-b) Let $n \geq m$ and $X \in \mathcal{A}^{u,G}_{w_0,m}$. By the increasing monotonicity of $u : \mathbb{R} \to \mathbb{R}$,

$$U^{u,G}(w_0 + X) \geq u(w_0 - m) \geq u(w_0 - n)$$

Thus, $X \in \mathcal{A}^{u,G}_{w_0,n}$.

(ii) Let $X, Y \in \mathcal{A}^{u,G}_{w_0,m}$. By the quasiconcavity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$,

$$\begin{aligned} U^{u,G}&(w_0 + \lambda X + (1 - \lambda)Y) \\ &= U^{u,G}(\lambda(w_0 + X) + (1 - \lambda)(w_0 + Y)) \\ &\geq \min\{U^{u,G}(w_0 + X), U^{u,G}(w_0 + Y)\} \\ &\geq u(w_0 - m) \end{aligned}$$

for all $\lambda \in [0, 1]$. Thus, $\lambda X + (1 - \lambda)Y \in \mathcal{A}^{u,G}_{w_0,m}$ for all $\lambda \in [0, 1]$.

(iii) By the increasing monotonicity of $u : \mathbb{R} \to \mathbb{R}$,

$$\inf\{x \in \mathbb{R} \mid u(w_0 + x) \geq u(w_0 - m)\} = -m.$$

□

Observe that the acceptance set $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ corresponds to the set of all uncertain monetary prospects for which an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$ would agree to buy protection, or insurance, at a price less than $m \in \mathbb{R}$, that is,

$$\mathcal{A}^{u,G}_{w_0,m} = \{X \in \mathcal{X} \,|\, \pi^{u,G}_{w_0}(X) \leq m\} \tag{3.3}$$

for all $m \in \mathbb{R}$. Observe also that the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ satisfies,

$$\pi^{u,G}_{w_0}(X) = \inf\{m \in \mathbb{R} \,|\, X \in \mathcal{A}^{u,G}_{w_0,m}\}$$

for all $X \in \mathcal{X}$. It follows from Equation (3.3) that Proposition 3.1 could be equivalently obtained combining Proposition 3.2 with Theorem 1.7 in Drapeau and Kupper (2010). It also follows from Equation (3.3) that the indifference buyer's price inherits directly all the different continuity properties of the uncertainty averse representation of preferences because $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ and $U^{u,G} : \mathcal{X} \to \mathbb{R}$ have the same level sets.
Let ${\mathcal{A}^{u,G}_{w_0,m}}^C$ be the subset of $\mathcal{X}$ given by,

$${\mathcal{A}^{u,G}_{w_0,m}}^C := \{X \in \mathcal{X} \,|\, U^{u,G}(w_0 + X) < u(w_0 - m)\} \tag{3.4}$$

for every $m \in \mathbb{R}$. Remark 3.1, when combined with Equation (3.3), asserts that the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ inherits the continuity of the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with respect to the sup-norm $||.||_\infty$.

Remark 3.1. The sets $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ and ${\mathcal{A}^{u,G}_{w_0,m}}^C \subset \mathcal{X}$ are closed with respect to convergence in sup-norm $||.||_\infty$ for all $m \in \mathbb{R}$.

Remark 3.2, when combined with Equation (3.3), asserts that the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ inherits the continuity with respect to bounded point-wise convergence of the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$.

Remark 3.2. $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if the sets $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ and $\mathcal{A}^{u,G\ C}_{w_0,m} \subset \mathcal{X}$ are closed with respect to bounded point-wise convergence for all $m \in \mathbb{R}$.

3.1.2 Dual Representation

3.1.2.1 Finitely Additive Probabilities

As a consequence of its monotonicity, quasiconvexity, and continuity with respect to the sup-norm $||.||_\infty$, the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ admits a representation in term of the set $\Delta \subset \mathcal{X}^*$ of all finitely additive probabilities. The illustration of the representation results is considerably simplified by introducing an appropriate notation for the support function[1] of the acceptance set $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ which, consistently with the terminology adopted in the mathematical finance literature, we designate as *minimal penalty function* (see Drapeau and Kupper (2010) and Föllmer and Schied (2002, 2004)).

Definition 3.2. The *minimal penalty function* $r^{u,G}_{w_0} : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ of the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ is defined by,

$$r^{u,G}_{w_0}(m, Q) := \sup_{X \in \mathcal{A}^{u,G}_{w_0,m}} E_Q[-X] \tag{3.5}$$

for all $(m, Q) \in \mathbb{R} \times \Delta$.

The representation of the indifference buyer's price $\pi^{u,G}_{w_0}$ in Proposition 3.3 is an application of the duality for quasiconcave functions introduced by de Finetti (1949), extended by Cerreia Vioglio et al. (2011b) and further developed by Drapeau and Kupper (2010).

[1] For support functions see §13 in Rockafellar (1970).

Proposition 3.3. *The indifference buyer's price* $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ *has the following representation,*

$$\pi^{u,G}_{w_0}(X) = \sup_{Q \in \Delta} R(E_Q[-X], Q) \tag{3.6}$$

for all $X \in \mathcal{X}$*. The* maximal risk function $R^{u,G}_{w_0} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ *for which the representation in Equation* (3.6) *holds is unique and defined by,*

$$R^{u,G}_{w_0}(x, Q) := \inf\{m \in \mathbb{R} \,|\, r^{u,G}_{w_0}(m, Q) \geq x\} \tag{3.7}$$

for all $(x, Q) \in \mathbb{R} \times \Delta$*. In particular, if* $R : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ *is any other function satisfying the representation in Equation* (3.6)*, then* $R(x, Q) \leq R^{u,G}_{w_0}(x, Q)$ *for all* $(x, Q) \in \mathbb{R} \times \Delta$*.*

Proof of Proposition 3.3. By Proposition 3.1 and Remark 3.1 $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ is monotone decreasing, quasiconvex, and continuous with respect to the sup-norm $||.||_\infty$. By Theorem 4 in Cerreia Vioglio et al. (2011b) a monotone decreasing and quasiconvex function $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to the sup-norm $||.||_\infty$ has the following representation,

$$\pi^{u,G}_{w_0}(X) = \sup_{Q \in \Delta} R^{u,G}_{w_0}(E_Q[-X], Q)$$

for all $X \in \mathcal{X}$, where $R^{u,G}_{w_0} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ is defined by,

$$R^{u,G}_{w_0}(x, Q) := \inf_{X \in \mathcal{X} :\, E_Q[-X] \geq x} \pi^{u,G}_{w_0}(X)$$

for all $(x, Q) \in \mathbb{R} \times \Delta$, and can be rewritten as,

$$R^{u,G}_{w_0}(x, Q) = \inf\{\pi^{u,G}_{w_0}(X) \in \mathbb{R} \,|\, X \in \mathcal{X} :\, E_Q[-X] \geq x\}$$

$$= \inf\{m \in \mathbb{R} \,|\, \exists X \in \mathcal{X} : \, \pi_{w_0}^{u,G}(X) \leq m \, , \, E_Q[-X] \geq x\}$$

$$= \inf\{m \in \mathbb{R} \,|\, \sup_{X \in \mathcal{X} : \pi_{w_0}^{u,G}(X) \leq m} E_Q[-X] \geq x\}$$

$$= \inf\{m \in \mathbb{R} \,|\, r_{w_0}^{u,G}(m, Q) \geq x\}$$

for all $(x, Q) \in \mathbb{R} \times \Delta$. By Corollary 2 in Cerreia Vioglio et al. (2011b), if $R : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ is increasing in the first component, quasiconcave, upper semi-continuous, and such that $R(., Q)$ is extended-valued continuous on $\mathbb{R}$ for each $Q \in \Delta$, then the function $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ defined by,

$$\pi_{w_0}^{u,G}(X) := \sup_{Q \in \Delta} R(E_Q[-X], Q)$$

for all $X \in \mathcal{X}$, is monotone decreasing, quasiconvex and continuous with respect to the sup-norm $||.||_\infty$. Moreover, for all $(\bar{x}, \bar{Q}) \in \mathbb{R} \times \Delta$ and $X \in \mathcal{X}$ such that $E_{\bar{Q}}[-X] \geq \bar{x}$,

$$\pi_{w_0}^{u,G}(X) = \sup_{Q \in \Delta} R(E_Q[-X], Q) \geq R(E_{\bar{Q}}[-X], \bar{Q}) \geq R(\bar{x}, \bar{Q})$$

Thus[2],

$$R_{w_0}^{u,G}(\bar{x}, \bar{Q}) = \inf_{X \in \mathcal{X} : \, E_{\bar{Q}}[-X] \geq \bar{x}} \pi_{w_0}^{u,G}(X) \geq R(\bar{x}, \bar{Q})$$

for all $(\bar{x}, \bar{Q}) \in \mathbb{R} \times \Delta$. □

The representation in Equation (3.6) implies that an uncertainty averse decision-maker evaluates the maximum price that she would pay to avoid an uncertain monetary payoff in $\mathcal{X}$ as if, by the function $R_{w_0}^{u,G}$, she appraised its expected loss under each probabilistic scenario in Δ, the appraisal $R_{w_0}^{u,G}$ depending on her risk attitudes u, on her uncertainty attitudes G and on her initial wealth $w_0 \in \mathbb{R}$ and as if, by the function $\pi_{w_0}^{u,G}$, she summarized

[2] See also Cerreia Vioglio et al. (2011a, Lemma 51)

her appraisal by considering exclusively the worst probabilistic scenario in Δ.

3.1.2.2 Countably Additive Probabilities

Proposition 3.4 shows that an uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if the set of all finitely additive probabilities $\Delta \subset \mathcal{X}^*$ in Equation (3.6) can be equivalently replaced by its subset of countably additive elements $\Delta^\sigma \subset \mathcal{X}_\sigma^*$.
Proposition 3.4 is a direct consequence of Remark 3.2 and of Proposition 4.3 and Proposition 4.5 in Cerreia Vioglio et al. (2010).

Proposition 3.4. *An uncertainty averse representation of preferences* $U^{u,G}$: $\mathcal{X} \to \mathbb{R}$ *is continuous with respect to bounded point-wise convergence if and only if,*

$$R(x, Q) = -\infty$$

for all $(x, Q) \in \mathbb{R} \times (\Delta \setminus \Delta^\sigma)$.

3.1.3 Examples

In this subsection we characterize the dual representations on countably additive probabilities of the indifference buyer's price defined in terms of the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences. In what follows $u^* : \mathbb{R} \to \mathbb{R}$ denotes the *convex conjugate* of the strictly increasing and concave utility function $u : \mathbb{R} \to \mathbb{R}$, that is,

$$u^*(\lambda) := \sup_{x \in \mathbb{R}} (u(x) - \lambda x) \tag{3.8}$$

for all $\lambda \in \mathbb{R}$. All the examples presented in this subsection are direct applications of various representation results collected in the Appendix at the end of this chapter.

Example 3.1. The indifference buyer's price $\pi_{w_0}^{u,c} : \mathcal{X} \to \mathbb{R}$ defined by a variational representation of preferences $U^{u,c} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence has the representation in Proposition 3.3 with,

$$R_{w_0}^{u,c}(x, \mathbb{Q}) = w_0 - u^{-1}\Big(\inf_{\lambda \in (0,+\infty)} \Big\{ \lambda(w_0 - x) + \inf_{\mathbb{P} \in \Delta^\sigma} \Big(c(\mathbb{P}) + \mathbb{E}_\mathbb{P}\Big[u^*\Big(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big]\Big)\Big\}\Big) \tag{3.9}$$

for all $(x, \mathbb{Q}) \in \mathbb{R} \times \Delta^\sigma$. Equation (3.9) follows directly from Proposition 3.13 and Theorem 3.1 in the Appendix.

Example 3.2. The indifference buyer's price $\pi_{w_0}^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$ defined by a multiplier representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$ has the representation in Proposition 3.3 with,

$$R_{w_0}^{u,\theta,\mathcal{R},\mathbb{P}^\star}(x, \mathbb{Q}) =$$
$$w_0 - u^{-1}\Big(\inf_{\lambda \in (0,+\infty)} \Big\{ \lambda(w_0 - x) + \inf_{\mathbb{P} \in \Delta^\sigma(\mathbb{P}^\star)} \Big(\theta\mathcal{R}(\mathbb{P}||\mathbb{P}^\star) + \mathbb{E}_\mathbb{P}\Big[u^*\Big(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big]\Big)\Big\}\Big) \tag{3.10}$$

for all $(x, \mathbb{Q}) \in \mathbb{R} \times \Delta^\sigma$ with $\theta \in (0, +\infty]$ and $\mathbb{P}^\star \in \Delta^\sigma$. Equation (3.10) follows directly from Proposition 3.13 and Corollary 3.1 in the Appendix.

Example 3.3. The indifference buyer's price $\pi_{w_0}^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ defined by a multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence has the representation in Proposition 3.3 with,

$$R_{w_0}^{u,\mathcal{P}}(x,\mathbb{Q}) = w_0 - u^{-1}\Big(\inf_{\lambda\in(0,+\infty)}\Big\{\lambda(w_0-x) + \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big]\Big\}\Big) \quad (3.11)$$

for all $(x,\mathbb{Q}) \in \mathbb{R}\times\Delta^\sigma$ with $\mathcal{P}\subset\Delta^\sigma$. Equation (3.11) follows directly from Proposition 3.13 and Corollary 3.2 in the Appendix.

3.2 Indifference Seller's Price

Consider an uncertainty averse decision-maker who is endowed with a constant monetary payoff $w_0 \in \mathbb{R}$. The uncertainty averse decision-maker is offered a constant amount of money $m \in \mathbb{R}$ in exchange for accepting an uncertain monetary payoff $X \in \mathcal{X}$. Agreeing to the transaction would make her wealth uncertain and equal to $w_0 + X + m \in \mathcal{X}$. The uncertainty averse decision-maker is therefore in the position of a seller of a contract (or insurer) and the *minimum price* (or insurance premium) $m \in \mathbb{R}$ which, from the perspective of her uncertainty averse preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$, makes the uncertain monetary payoff $w_0 + X + m \in \mathcal{X}$ more desirable than the constant monetary payoff $w_0 \in \mathbb{R}$, corresponds precisely to the price which makes them equally desirable. For this reason, this minimum price is denominated *indifference seller's price*. In Subsection 3.2.1 we introduce its definition, we derive its properties, and we identify its acceptance set. In Subsection 3.2.2 we characterize its dual representation on finitely additive probabilities and on countably additive probabilities. In Subsection 3.2.3 we provide more explicit characterizations of its dual representation on countably additive probabilities in terms of the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences.

3.2.1 Definition, Properties, and Acceptance Set

3.2.1.1 Definition

The indifference seller's price, which in this dissertation is considered from an actuarial perspective, is defined as the *minimum price* that a decision-maker with uncertainty averse preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ and with constant initial wealth $w_0 \in \mathbb{R}$ would demand to accept an uncertain monetary prospect in $\mathcal{X}$ (e.g. to provide insurance).

Definition 3.3. A function $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be an *indifference seller's price* if it satisfies,

$$u(w_0) = U^{u,G}(w_0 + X + \varphi_{w_0}^{u,G}(X)) \tag{3.12}$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$.

3.2.1.2 Properties

Proposition 3.5 asserts that the indifference seller's price is *monotone decreasing*, *convex*, *cash-additive*, and *normalized*. As a result of these properties, the indifference seller's price $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is a *cash-additive convex risk measure*. Cash-additive convex risk measures were introduced by Deprez and Gerber (1985) in the actuarial mathematics literature and by Frittelli and Rosazza Gianin (2002) and Föllmer and Schied (2002) in the financial mathematics literature. A cash-additive convex risk measure is a function yielding the minimum constant amount of money $m \in \mathbb{R}$ that must be added to an uncertain monetary payoff in $X \in \mathcal{X}$ such that the adjusted uncertain position $X + m \in \mathcal{X}$ becomes acceptable[3] to a decision-maker.

Proposition 3.5. *The indifference seller's price* $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ *satisfies the following properties for all* $X, Y \in \mathcal{X}$*.*

(i) Monotonicity: If $X \geq Y$*, then* $\varphi_{w_0}^{u,G}(X) \leq \varphi_{w_0}^{u,G}(Y)$*.*

(ii) Convexity: $\varphi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \lambda \varphi_{w_0}^{u,G}(X) + (1-\lambda)\varphi_{w_0}^{u,G}(Y)$ *for all* $\lambda \in [0,1]$*.*

(iii) Cash-additivity: $\varphi_{w_0}^{u,G}(X + m) = \varphi_{w_0}^{u,G}(X) - m$ *for all* $m \in \mathbb{R}$*.*

(iv) Normalization: $\varphi_{w_0}^{u,G}(0) = 0$*.*

Proof of Proposition 3.5. (i) Let $X, Y \in \mathcal{X}$. If $X \geq Y$, then by Definition 3.3 and by the increasing monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$,

$$
\begin{aligned}
u(w_0) = U^{u,G}(w_0 + Y + \varphi_{w_0}^{u,G}(Y)) \\
= U^{u,G}(w_0 + X + \varphi_{w_0}^{u,G}(X)) \\
\geq U^{u,G}(w_0 + Y + \varphi_{w_0}^{u,G}(X))
\end{aligned}
$$

and the increasing monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$ yields $\varphi_{w_0}^{u,G}(X) \leq \varphi_{w_0}^{u,G}(Y)$. Thus, $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is monotone decreasing.

(ii) If $m \in \mathbb{R}$, then by Definition 3.3,

$$
\begin{aligned}
u(w_0) = U^{u,G}(w_0 + X + \varphi_{w_0}^{u,G}(X)) \\
= U^{u,G}(w_0 + X + m + \varphi_{w_0}^{u,G}(X + m)) \\
= u(w_0)
\end{aligned}
$$

[3]The criterion of acceptability is subjectively determined by the decision-maker depending on the situation and on the problem under consideration.

and the strict monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$ yields,

$$\varphi_{w_0}^{u,G}(X) = m + \varphi_{w_0}^{u,G}(X + m)$$

Thus, $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is cash-additive.

(iii) If $\lambda \in [0, 1]$, then by the quasiconcavity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$ and by Definition 3.3,

$$\begin{aligned}
&U^{u,G}(w_0 + \lambda X + (1-\lambda)Y + \lambda\varphi_{w_0}^{u,G}(X) + (1-\lambda)\varphi_{w_0}^{u,G}(Y)) \\
&\quad = U^{u,G}(\lambda(w_0 + X + \varphi_{w_0}^{u,G}(X)) + (1-\lambda)(w_0 + Y + \varphi_{w_0}^{u,G}(Y))) \\
&\qquad \geq \min\{U^{u,G}(w_0 + X + \varphi_{w_0}^{u,G}(X)), U^{u,G}(w_0 + Y + \varphi_{w_0}^{u,G}(Y))\} \\
&\qquad\quad = U^{u,G}(w_0 + \lambda X + (1-\lambda)Y + \varphi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y)) \\
&\qquad\quad = u(w_0)
\end{aligned}$$

and the increasing monotonicity of $U^{u,G} : \mathcal{X} \to \mathbb{R}$ yields,

$$\varphi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \lambda\varphi_{w_0}^{u,G}(X) + (1-\lambda)\varphi_{w_0}^{u,G}(Y)$$

Thus, $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is convex.

(iv) As $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing,

$$u(w_0 + \varphi_{w_0}^{u,G}(0)) = u(w_0)$$

if and only if $\varphi_{w_0}^{u,G}(0) = 0$. Thus, $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is normalized. □

Decreasing monotonicity implies that a decision-maker would demand higher prices to accept higher losses. Convexity implies that the minimum price that a decision-maker would demand to accept of a portfolio of uncertain monetary payoffs is lower than the convex combinations of the prices that she would demand to accept its constituents. Cash-additivity implies that adding a constant amount of money to an uncertain monetary payoff decreases the minimum price that a decision-maker would demand to accept the uncertain prospect exactly by this constant amount. Normalization implies that a decision-maker would not pay any money to receive a monetary payoff which is certainly equal to zero[4].

3.2.1.3 Acceptance Set

The indifference seller's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ can be equivalently studied in terms of the *acceptance set* $\mathcal{A}_{w_0,0}^{u,G} \subset \mathcal{X}$ of an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$ at level zero. This is the subset $\mathcal{A}_{w_0,0}^{u,G}$ of $\mathcal{X}$ given by,

$$\mathcal{A}_{w_0,0}^{u,G} = \{X \in \mathcal{X} \,|\, U^{u,G}(w_0 + X) \geq u(w_0)\} \tag{3.13}$$

and it corresponds to the set of uncertain monetary payoffs in $\mathcal{X}$ that a decision-maker with uncertainty averse preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ and with constant initial wealth $w_0 \in \mathbb{R}$ finds more desirable than nothing.

Proposition 3.6 asserts that the acceptance set $\mathcal{A}_{w_0,0}^{u,G} \subset \mathcal{X}$ is monotone, convex, and normalized. Proposition 3.6 is a direct consequence of Proposition 3.2 in Subsection 3.2.1.3 and its proof is not provided.

Proposition 3.6. *The acceptance set $\mathcal{A}_{w_0,0}^{u,G} \subset \mathcal{X}$ satisfies the following properties for all $X, Y \in \mathcal{X}$.*

(i) Monotonicity: If $X \in \mathcal{A}_{w_0,0}^{u,G}$ and $Y \geq X$, then $Y \in \mathcal{A}_{w_0,0}^{u,G}$.

(ii) Convexity: If $X, Y \in \mathcal{A}_{w_0,0}^{u,G}$, then $\lambda X + (1-\lambda) Y \in \mathcal{A}_{w_0,0}^{u,G}$ for all $\lambda \in [0,1]$.

[4]In other words, a decision-maker would demand a non-negative price to accept a monetary payoff which is certainly equal to zero.

(iii) Normalization: $\inf\{x \in \mathbb{R} \mid x \in \mathcal{A}^{u,G}_{w_0,0}\} = 0$.

Remark 3.3 clarifies the relationship between the acceptance set $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ at level $m \in \mathbb{R}$ and the acceptance set $\mathcal{A}^{u,G}_{w_0,0}$ at level zero of an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$.

Remark 3.3. The acceptance set $\mathcal{A}^{u,G}_{w_0,m} \subset \mathcal{X}$ satisfies $\mathcal{A}^{u,G}_{w_0,m} = \mathcal{A}^{u,G}_{w_0-m,0} + m$ for all $m \in \mathbb{R}$.

Observe that the acceptance set $\mathcal{A}^{u,G}_{w_0,0} \subset \mathcal{X}$ corresponds to the set of all uncertain monetary prospects for which an uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with constant initial wealth $w_0 \in \mathbb{R}$ would agree to sell protection, or insurance, in exchange of nothing, that is,

$$\mathcal{A}^{u,G}_{w_0,0} = \{X \in \mathcal{X} \mid \varphi^{u,G}_{w_0}(X) \leq 0\} \tag{3.14}$$

Observe also that the indifference seller's price $\varphi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ satisfies,

$$\varphi^{u,G}_{w_0}(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}^{u,G}_{w_0,0}\}$$

for all $X \in \mathcal{X}$. It follows from Equation (3.14) that Proposition 3.1 could be equivalently obtained combining Proposition 3.6 with Proposition 4.7 in Föllmer and Schied (2004). It also follows from Equation (3.14) that the indifference seller's price inherits directly all the different continuity properties of the uncertainty averse representation of preferences because $\varphi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ and $U^{u,G} : \mathcal{X} \to \mathbb{R}$ have the same level sets.

Let ${\mathcal{A}^{u,G}_{w_0,0}}^C$ be the subset of $\mathcal{X}$ given by,

$${\mathcal{A}^{u,G}_{w_0,0}}^C := \{X \in \mathcal{X} \mid U^{u,G}(w_0 + X) < u(w_0)\} \tag{3.15}$$

Remark 3.4, when combined with Equation (3.14), asserts that the indifference seller's price $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ inherits the continuity of the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ with respect to the sup-norm $||.||_\infty$.

Remark 3.4. The sets $\mathcal{A}_{w_0,0}^{u,G} \subset \mathcal{X}$ and $\mathcal{A}_{w_0,0}^{u,G\,C} \subset \mathcal{X}$ are closed with respect to convergence in sup-norm $||.||_\infty$.

Remark 3.5, when combined with Equation (3.14), asserts that the indifference seller's price $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ inherits the continuity with respect to bounded point-wise convergence of the uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$.

Remark 3.5. $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if the sets $\mathcal{A}_{w_0,0}^{u,G} \subset \mathcal{X}$ and $\mathcal{A}_{w_0,0}^{u,G\,C} \subset \mathcal{X}$ are closed with respect to bounded point-wise convergence.

Observe that, as a result of decreasing monotonicity and cash-additivity, the indifference seller's price $\phi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is even Lipschitz continuous with respect to the supremum norm $||.||_\infty$ (see Föllmer and Schied (2004, Lemma 4.3)).

3.2.2 Dual Representation

3.2.2.1 Finitely Additive Probabilities

As a consequence of its monotonicity, convexity, and cash-additivity, the indifference seller's price $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ admits a representation in terms of the set $\Delta \subset \mathcal{X}^*$ of all finitely additive probabilities. The following proposition is a direct application of Proposition 3.5 and of Föllmer and Schied (2004, Theorem 4.15).

Proposition 3.7. *The indifference seller's price* $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ *has the following representation,*

$$\varphi_{w_0}^{u,G}(X) = \sup_{Q \in \Delta} \Big(E_Q[-X] - \alpha(Q) \Big) \tag{3.16}$$

for all $X \in \mathcal{X}$. *The* minimal penalty function $\alpha_{w_0}^{u,G} : \Delta \to (-\infty, +\infty]$ *for which the representation in Equation* (3.16) *holds is unique and defined by,*

$$\alpha_{w_0}^{u,G}(Q) := \sup_{X \in \mathcal{A}_{w_0,0}^{u,G}} E_Q[-X] \tag{3.17}$$

for all $Q \in \Delta$. *In particular, if* $\alpha : \Delta \to (-\infty, +\infty]$ *is any other function satisfying the representation in Equation* (3.16)*, then* $\alpha_{w_0}^{u,G}(Q) \leq \alpha(Q)$ *for all* $Q \in \Delta$.

The representation in Equation (3.16) implies that an uncertainty averse decision-maker evaluates the minimum price that she would demand to accept an uncertain monetary payoff in $\mathcal{X}$ as if, by the function $\alpha_{w_0}^{u,G}$, she applied a correction to its expected loss under each probabilistic scenario in Δ, the correction $\alpha_{w_0}^{U}$ depending on her risk attitudes u, on her uncertainty attitudes G and on her initial wealth $w_0 \in \mathbb{R}$ and as if, by the function $\varphi_{w_0}^{u,G}$, she summarized her appraisal by considering exclusively the worst probabilistic scenario in Δ.

3.2.2.2 Countably Additive Probabilities

Proposition 3.4 shows that an uncertainty averse representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if the set of all finitely additive probabilities $\Delta \subset \mathcal{X}^*$ in Equation (3.16) can be equivalently replaced by its subset of countably additive elements $\Delta^\sigma \subset \mathcal{X}_\sigma^*$.

Proposition 3.8 is a direct consequence of Remark 3.5, of Proposition 4.5 in Cerreia Vioglio et al. (2010), and of Proposition 3 in Krätschmer (2005).

Proposition 3.8. *An uncertainty averse representation of preferences $U^{u,G}$: $\mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if,*

$$\alpha(Q) = +\infty$$

for all $Q \notin \Delta^\sigma$.

3.2.3 Examples

In this subsection we derive the dual representations on countably additive probabilities of the indifference seller's price defined in terms of the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences. As in subsection 3.1.3, in what follows $u^* : \mathbb{R} \to \mathbb{R}$ will denote the convex conjugate of the strictly increasing and concave utility function $u : \mathbb{R} \to \mathbb{R}$. All the examples presented in this subsection are direct applications of various representation results collected in the Appendix at the end of this chapter.

Example 3.4. The indifference seller's price $\varphi_{w_0}^{u,c} : \mathcal{X} \to \mathbb{R}$ defined by a variational representation of preferences $U^{u,c} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence has the representation in Proposition 3.7 with,

$$\alpha_{w_0}^{u,c}(\mathbb{Q}) = w_0 + \inf_{\lambda \in (0,+\infty)} \frac{1}{\lambda} \Big\{ \inf_{\mathbb{P} \in \Delta^\sigma} \Big(c(\mathbb{P}) + \mathbb{E}_\mathbb{P}\Big[u^*\Big(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big]\Big) - u(w_0) \Big\} \tag{3.18}$$

for all $\mathbb{Q} \in \Delta^\sigma$. Equation (3.18) follows directly from Remark 3.6 and

Theorem 3.1 in the Appendix.

Example 3.5. The indifference seller's price $\varphi_{w_0}^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$ defined by a multiplier representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$ has the representation in Proposition 3.7 with,

$$\alpha_{w_0}^{u,\theta,\mathcal{R},\mathbb{P}^\star}(\mathbb{Q}) = \\ w_0 + \inf_{\lambda \in (0,+\infty)} \frac{1}{\lambda} \Big\{ \inf_{\mathbb{P} \in \Delta^\sigma(\mathbb{P}^\star)} \Big(\theta \mathcal{R}(\mathbb{P}||\mathbb{P}^\star) + \mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big]\Big) - u(w_0) \Big\} \quad (3.19)$$

for all $(x, \mathbb{Q}) \in \mathbb{R} \times \Delta^\sigma$ with $\theta \in (0, +\infty]$ and $\mathbb{P}^\star \in \Delta^\sigma$. Equation (3.19) follows directly from Remark 3.6 and Corollary 3.1 in the Appendix.

Example 3.6. The indifference seller's price $\varphi_{w_0}^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ defined by a multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence has the representation in Proposition 3.7 with,

$$\alpha_{w_0}^{u,\mathcal{P}}(\mathbb{Q}) = w_0 + \inf_{\lambda \in (0,+\infty)} \frac{1}{\lambda} \Big\{ \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] - u(w_0) \Big\} \quad (3.20)$$

for all $(x, \mathbb{Q}) \in \mathbb{R} \times \Delta^\sigma$ with $\mathcal{P} \subset \Delta^\sigma$. Equation (3.20) follows directly from Remark 3.6 and Corollary 3.1 in the Appendix.

3.3 Appendix

The dual representations of the indifference buyer's price and of the indifference seller's price defined by the variational, the multiplier, and the multiple priors representations of preferences presented in Subsection 3.1.3 and Subsection 3.2.3 are special cases of the dual representations of the indifference buyer's price and of the indifference seller's price defined by a general strictly increasing, concave, and continuous function. In Subsection 3.3.1 of this Appendix we describe the dual representation of a general strictly increasing, concave, and continuous function while in Subsection 3.3.2 of this Appendix we characterize the maximal risk function and the minimal penalty function representing the indifference buyer's price and the indifference seller's price defined by a general strictly increasing, concave, and continuous function.

3.3.1 Dual Representation of a Concave Preference Functional

In this section we describe the dual representation of a general strictly increasing, concave, and continuous function $U : \mathcal{X} \to \mathbb{R}$ in terms of its *convex conjugate function* $U^* : \mathcal{X}^* \to (-\infty, +\infty]$ defined by,

$$U^*(\mu) := \sup_{X \in \mathcal{X}} \Big(U(X) - \mathbb{E}_\mu[X] \Big) \tag{3.21}$$

for all $\mu \in \mathcal{X}^*$.

Proposition 3.9. *A strictly increasing, concave, and continuous function $U : \mathcal{X} \to \mathbb{R}$ has the following representation,*

$$U(X) = \inf_{Q \in \Delta} \inf_{\lambda \in (0,+\infty)} \Big(E_Q[\lambda X] + U^*(\lambda Q) \Big) \tag{3.22}$$

for all $X \in \mathcal{X}$.

Proof. By Proposition 3.3 a monotone increasing, quasiconcave, and continuous function $U : \mathcal{X} \to \mathbb{R}$ has the following representation,

$$U(X) = \inf_{Q \in \Delta} V(E_Q[X], Q) \tag{3.23}$$

for all $X \in \mathcal{X}$. The *minimal value function* $V^\star : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ for which the representation in Equation (3.23) holds is unique and defined by,

$$V^\star(x,Q) := \sup_{X\in\mathcal{X}:\, E_Q[X]\leq x} U(X)$$

for all $(x,Q)\in\mathbb{R}\times\Delta$. In particular, if $V:\mathbb{R}\times\Delta\to(-\infty,+\infty]$ is any other function satisfying the representation in Equation (3.23), then $V(x,Q)\geq V^\star(x,Q)$ for all $(x,Q)\in\mathbb{R}\times\Delta$. The minimal value function $V^\star:\mathbb{R}\times\Delta\to(-\infty,+\infty]$ can be rewritten as,

$$\begin{aligned}
V^\star(x,Q) &= \sup\{U(X)\in\mathbb{R} \mid X\in\mathcal{X}:\ E_Q[-X]\geq -x\}\\
&= \sup\{m\in\mathbb{R} \mid \exists X\in\mathcal{X}:\ U(X)\geq m\,,\ E_Q[-X]\geq -x\}\\
&= \sup\{m\in\mathbb{R} \mid \sup\nolimits_{X\in\mathcal{X}:U(X)\geq m} E_Q[-X]\geq -x\}
\end{aligned}$$

for all $(x,Q)\in\mathbb{R}\times\Delta$. As $U:\mathcal{X}\to\mathbb{R}$ is concave[5] and $U^*(0)=+\infty$, by Theorem 13.5 and Theorem 9.7 in Rockafellar (1970),

$$\sup_{X\in\mathcal{X}:U(X)\geq m} E_Q[-X] = \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(U^*(\lambda Q)-m\Big) \tag{3.24}$$

for all $(m,P)\in\mathbb{R}\times\Delta$. Thus,

$$\begin{aligned}
V^\star(x,Q) &= \sup\Big\{m\in\mathbb{R}\Big|\ \inf_{\lambda\in(0,+\infty)}\frac{1}{\lambda}\Big(U^*(\lambda Q)-m\Big)\geq -x\Big\}\\
&= \sup\Big\{m\in\mathbb{R}\Big|\ \inf_{\lambda\in(0,+\infty)}\Big(\lambda x+U^*(\lambda Q)\geq m\Big)\Big\}\\
&= \inf_{\lambda\in(0,+\infty)}\Big(\lambda x+U^*(\lambda Q)\Big) \qquad (3.25)
\end{aligned}$$

for all $(x,P)\in\mathbb{R}\times\Delta$. See also Cerreia Vioglio et al. (2011b, Example 3.2) and Cerreia Vioglio et al. (2011a, Corollary 38). □

[5] Observe that a concave function is quasiconcave, while a quasiconcave function is not necessarily concave.

Proposition 3.10 characterizes the situation in which the set of finitely additive probabilities $\Delta \subset \mathcal{X}^*$ in Equation (3.22) can be equivalently replaced by its subset of countably additive elements $\Delta^\sigma \subset \mathcal{X}_\sigma^*$. Proposition 3.10 is a direct consequence of Proposition 2.1.

Proposition 3.10. *A strictly increasing, concave, and continuous function $U : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if,*

$$U^*(\lambda Q) = +\infty$$

for all $Q \notin \Delta^\sigma$ and $\lambda \in (0, +\infty)$.

A classic example of strictly increasing, concave, and continuous function $U : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence is the expected utility $U^{u,\mathbb{P}} : \mathcal{X} \to \mathbb{R}$ defined in terms of a countably additive probability $\mathbb{P} \in \Delta^\sigma$ (see Equation (2.1)). Consistently with the assumptions and with the notation employed throughout this dissertation, in what follows $u : \mathbb{R} \to \mathbb{R}$ denotes a strictly increasing and concave function, and $u^* : \mathbb{R} \to \mathbb{R}$ denotes its convex conjugate function (see Equation (3.8)).

Proposition 3.11. *Let $U : \mathcal{X} \to \mathbb{R}$ be the function defined by,*

$$U(X) = \mathbb{E}_\mathbb{P}[u(X)] \tag{3.26}$$

for all $X \in \mathcal{X}$ where $\mathbb{P} \in \Delta^\sigma$. Then $U : \mathcal{X} \to \mathbb{R}$ has the representation in Proposition 3.9 with,

$$U^*(\lambda Q) = \begin{cases} \mathbb{E}_\mathbb{P}\Big[u^*\Big(\lambda \dfrac{dQ}{d\mathbb{P}}\Big)\Big] & \text{if } Q \in \Delta^\sigma(\mathbb{P}) \\ +\infty & \text{otherwise} \end{cases}$$

for all $\lambda \in (0, +\infty)$.

Proof of Proposition 3.11. By Proposition 4.104 and Theorem 4.106 in Föllmer and Schied (2004),

$$\sup_{X\in\mathcal{X}:\mathbb{E}_{\mathbb{P}}[u(X)]\geq m} E_Q[-X] = \begin{cases} \inf\limits_{\lambda\in(0,+\infty)} \dfrac{1}{\lambda}\Big(\mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda\dfrac{dQ}{d\mathbb{P}}\Big)\Big] - m\Big) & \text{if } Q\in\Delta^\sigma(\mathbb{P}) \\ +\infty & \text{otherwise} \end{cases}$$

Thus, the statement follows from Theorem 13.5 and Theorem 9.7 in Rockafellar (1970) (see Equation (3.24)). □

Theorem 3.1 describes the convex conjugate function of a variational representation of preferences $U^{u,c}:\mathcal{X}\to\mathbb{R}$ which is continuous with respect to bounded point-wise convergence. Recall that a variational representation of preferences $U^{u,c}:\mathcal{X}\to\mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if $c(P)=+\infty$ for all $P\notin\Delta^\sigma$ (see Proposition 2.1).

Theorem 3.1. *Let $U:\mathcal{X}\to\mathbb{R}$ be the function defined by,*

$$U(X) = \inf_{\mathbb{P}\in\Delta^\sigma}\Big(\mathbb{E}_{\mathbb{P}}[u(X)] + c(\mathbb{P})\Big)$$

for all $X\in\mathcal{X}$. Then $U:\mathcal{X}\to\mathbb{R}$ has the representation in Proposition 3.9 with,

$$U^*(\lambda Q) = \begin{cases} \inf\limits_{\mathbb{P}\in\Delta^\sigma}\Big(c(\mathbb{P}) + \mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda\dfrac{dQ}{d\mathbb{P}}\Big)\Big]\Big) & \textit{if } Q\in\Delta^\sigma \\ +\infty & \textit{otherwise} \end{cases}$$

for all $\lambda\in(0,+\infty)$.

Proof of Theorem 3.1. Observe that a variational representation of preferences $U:\mathcal{X}\to\mathbb{R}$ which is continuous with respect to bounded point-wise convergence can be equivalently written as,

$$U(X) = -\rho^c(u(X)) \quad \forall\, X\in\mathcal{X}$$

where $\rho^c : \mathcal{X} \to \mathbb{R}$ is a cash-additive convex risk measure represented by the minimal penalty function $c : \Delta \to (-\infty, +\infty]$ such that $c(P) = +\infty$ for all $P \notin \Delta^\sigma$. Let $\lambda \in (0, +\infty)$ and $Q \in \Delta$. By decreasing monotonicity of $\rho^c : \mathcal{X} \to \mathbb{R}$,

$$\begin{aligned} U^*(\lambda Q) &= \sup_{X \in \mathcal{X}} \Big(\mathbb{E}_Q[-\lambda X] - \rho^c(u(X)) \Big) \\ &= \sup_{(X,Z) \in \mathcal{X} \times \mathcal{X} : u(X) \geq Z} \Big(\mathbb{E}_Q[-\lambda X] - \rho^c(Z) \Big) \\ &= \sup_{(X,Z) \in \mathcal{X} \times \mathcal{X}} \Big(\mathbb{E}_Q[-\lambda X] - \rho^c(Z) - \delta(X, Z \,|\, \mathcal{U}) \Big) \end{aligned}$$

where $\mathcal{U} \subset \mathcal{X} \times \mathcal{X}$ is the convex set defined by,

$$\mathcal{U} := \{(X, Z) \in \mathcal{X} \times \mathcal{X} : u(X) \geq Z\}$$

and $\delta(.\,|\,\mathcal{U}) : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is the convex function defined by,

$$\delta(X, Z \,|\, \mathcal{U}) := \begin{cases} 0 & \text{if } (X, Z) \in \mathcal{U} \\ +\infty & \text{otherwise} \end{cases}$$

Let $g : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be the concave function defined by,

$$g(X, Z) := E_Q[-\lambda X] - \rho^c(Z)$$

for all $(X, Z) \in \mathcal{X} \times \mathcal{X}$. Let $f : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ be the convex function defined by,

$$f(X, Z) := \delta(X, Z \,|\, \mathcal{U})$$

for all $(X, Z) \in \mathcal{X} \times \mathcal{X}$. By *Fenchel's Duality Theorem* (see Rockafellar

(1970, Theorem 31.1)),

$$\sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}}\Big(g(X,Z)-f(X,Z)\Big)=\inf_{(\bar{P},\bar{Q})\in\mathcal{X}^*\times\mathcal{X}^*}\Big(g^*(\bar{P},\bar{Q})+f^*(\bar{P},\bar{Q})\Big)$$

The function $g^*:\mathcal{X}^*\times\mathcal{X}^*\to(-\infty,+\infty]$ is the convex conjugate function of $g:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$, that is,

$$g^*(\bar{P},\bar{Q})=\sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}}\Big(g(X,Z)-E_{\bar{P}}[X]-E_{\bar{Q}}[Z]\Big)$$

for all $(\bar{P},\bar{Q})\in\mathcal{X}^*\times\mathcal{X}^*$. The function $f^*:\mathcal{X}^*\times\mathcal{X}^*\to(-\infty,+\infty]$ is the convex conjugate function of $f:\mathcal{X}\times\mathcal{X}\to[0,+\infty]$, that is,

$$f^*(\bar{P},\bar{Q})=\sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}}\Big(E_{\bar{P}}[X]+E_{\bar{Q}}[Z]-f(X,Z)\Big)$$

for all $(\bar{P},\bar{Q})\in\mathcal{X}^*\times\mathcal{X}^*$. The function $g^*:\mathcal{X}^*\times\mathcal{X}^*\to(-\infty,+\infty]$ satisfies,

$$\begin{aligned}g^*(\bar{P},\bar{Q})&=\sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}}\Big(E_Q[-\lambda X]-\rho^c(Z)-E_{\bar{P}}[X]-E_{\bar{Q}}[Z]\Big)\\&=\sup_{Z\in\mathcal{X}}\Big(E_{\bar{Q}}[-Z]-\rho^c(Z)\Big)+\sup_{X\in\mathcal{X}}\Big(E_Q[-\lambda X]-E_{\bar{P}}[X]\Big)\\&=c(\bar{Q})+\delta(-\bar{P}\,|\,\lambda Q)\end{aligned}$$

where the function $\delta(.\,|\,\lambda Q):\mathcal{X}^*\to[0,+\infty]$ is defined by,

$$\delta(-\bar{P}\,|\,\lambda Q):=\begin{cases}0 & \text{if } -\bar{P}=\lambda Q\\ +\infty & \text{otherwise}\end{cases}$$

for all $\bar{P}\in\mathcal{X}^*$. It follows that,

$$\inf_{(\bar{P},\bar{Q})\in\mathcal{X}^*\times\mathcal{X}^*}\Big(g^*(\bar{P},\bar{Q})+f^*(\bar{P},\bar{Q})\Big)$$

$$= \inf_{(\bar{P},\bar{Q})\in\mathcal{X}^*\times\mathcal{X}^*} \Big(c(\bar{Q}) + \delta(-\bar{P}\,|\,\lambda Q) + f^*(\bar{P},\bar{Q})\Big)$$

$$= \inf_{\bar{Q}\in\mathcal{X}^*} \Big(c(\bar{Q}) + f^*(-\lambda Q,\bar{Q})\Big)$$

$$= \inf_{\bar{Q}\in\Delta^\sigma} \Big(c(\bar{Q}) + f^*(-\lambda Q,\bar{Q})\Big)$$

where the last equality follows from the fact that $c(\bar{Q}) = +\infty$ for all $\bar{Q} \notin \Delta^\sigma$. The function $f^* : \mathcal{X}^* \times \mathcal{X}^* \to (-\infty, +\infty]$ satisfies,

$$\begin{aligned} f^*(-\lambda Q,\bar{Q}) &= \sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}} \Big(E_Q[-\lambda X] + \mathbb{E}_{\bar{Q}}[Z] - \delta(X,Z\,|\,\mathcal{U})\Big) \\ &= \sup_{(X,Z)\in\mathcal{U}} \Big(\mathbb{E}_{\bar{Q}}[Z] - E_Q[\lambda X]\Big) \\ &= \sup_{(X,Z)\in\mathcal{X}\times\mathcal{X}:u(X)\geq Z} \Big(\mathbb{E}_{\bar{Q}}[Z] - E_Q[\lambda X]\Big) \\ &= \sup_{X\in\mathcal{X}} \Big(\mathbb{E}_{\bar{Q}}[u(X)] - E_Q[\lambda X]\Big) \end{aligned}$$

for all $\bar{Q} \in \Delta^\sigma$. Therefore, by Proposition 3.11,

$$f^*(-\lambda Q,\bar{Q}) = \begin{cases} \mathbb{E}_{\bar{Q}}\Big[u^*\Big(\lambda \frac{dQ}{d\bar{Q}}\Big)\Big] & \text{if } Q \in \Delta^\sigma(\bar{Q}) \\ +\infty & \text{otherwise} \end{cases}$$

for all $\bar{Q} \in \Delta^\sigma$. Thus,

$$\inf_{\bar{Q}\in\Delta^\sigma} \Big(c(\bar{Q}) + f^*(-\lambda Q,\bar{Q})\Big) = \begin{cases} \inf_{\bar{Q}\in\Delta^\sigma} \Big(c(\bar{Q}) + \mathbb{E}_{\bar{Q}}\Big[u^*\Big(\lambda \frac{dQ}{d\bar{Q}}\Big)\Big]\Big) & \text{if } Q \in \Delta^\sigma \\ +\infty & \text{otherwise} \end{cases}$$

□

Corollary 3.1 describes the convex conjugate function of a multiplier representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{P}^\star} : \mathcal{X} \to \mathbb{R}$. Observe that, since $\mathcal{R}(P||\mathbb{P}^\star) = +\infty$ for all $P \notin \Delta^\sigma(\mathbb{P}^\star)$, any multiplier representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{P}^\star}$:

$\mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence (see Proposition 2.1).

Corollary 3.1. *Let $U : \mathcal{X} \to \mathbb{R}$ be the function defined by,*

$$U(X) = \inf_{\mathbb{P} \in \Delta^\sigma(\mathbb{P}^\star)} \Big(\mathbb{E}_\mathbb{P}[u(X)] + \theta \mathcal{R}(\mathbb{P}||\mathbb{P}^\star) \Big)$$

for all $X \in \mathcal{X}$ with $\theta \in (0, +\infty]$ and $\mathbb{P}^\star \in \Delta^\sigma$. Then $U : \mathcal{X} \to \mathbb{R}$ has the representation in Proposition 3.9 with,

$$U^*(\lambda Q) = \begin{cases} \inf_{\mathbb{P} \in \Delta^\sigma(\mathbb{P}^\star)} \Big(\mathcal{R}(\mathbb{P}||\mathbb{P}^\star) + \mathbb{E}_\mathbb{P}\Big[u^*\Big(\lambda \frac{dQ}{d\mathbb{P}}\Big)\Big]\Big) & \textit{if } Q \in \Delta^\sigma \\ +\infty & \textit{otherwise} \end{cases}$$

for all $\lambda \in (0, +\infty)$.

Proof of Corollary 3.1. Follows from Theorem 3.1 setting $c(\mathbb{P}) = \theta \mathcal{R}(\mathbb{P}||\mathbb{P}^\star)$ for all $\mathbb{P} \in \Delta^\sigma$ with $\theta \in (0, +\infty]$ and $\mathbb{P}^\star \in \Delta^\sigma$. □

Corollary 3.2 describes the convex conjugate function of a multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ which is continuous with respect to bounded point-wise convergence. Recall that a multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ is continuous with respect to bounded point-wise convergence if and only if $\mathcal{P} \subset \Delta^\sigma$ (Proposition 2.1).

Corollary 3.2. *Let $U : \mathcal{X} \to \mathbb{R}$ be the function defined by,*

$$U(X) = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_\mathbb{P}[u(X)]$$

for all $X \in \mathcal{X}$ with $\mathcal{P} \subset \Delta^\sigma$. Then $U : \mathcal{X} \to \mathbb{R}$ has the representation in

Proposition 3.9 with,

$$U^*(\lambda Q) = \begin{cases} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[u^*\Big(\lambda\frac{dQ}{d\mathbb{P}}\Big)\Big] & \textit{if } Q \in \Delta^\sigma \\ +\infty & \textit{otherwise} \end{cases}$$

for all $\lambda \in (0, +\infty)$.

Proof of Corollary 3.2. Follows from Theorem 3.1 setting $c(\mathbb{P}) = 0$ if $\mathbb{P} \in \mathcal{P}$ and $c(\mathbb{P}) = +\infty$ if $\mathbb{P} \notin \mathcal{P}$ for all $\mathbb{P} \in \Delta^\sigma$ and with $\mathcal{P} \subset \Delta^\sigma$. □

3.3.2 Dual Representation of the Indifference Buyer's Price Defined by a Concave Preference Functional

In this subsection we characterize the dual representation of the indifference buyer's price and of the indifference seller's price defined in terms of a strictly increasing, concave, and continuous function. As in Subsection 3.3.1, we denote by $U : \mathcal{X} \to \mathbb{R}$ a strictly increasing, concave, and continuous function, and by $U^* : \mathcal{X}^* \to (-\infty, +\infty]$ its convex conjugate function (see Equation (3.21)). In addition, we denote by $u : \mathbb{R} \to \mathbb{R}$ the restriction of $U : \mathcal{X} \to \mathbb{R}$ to the real line, that is,

$$u(x) := U(x)$$

for all $x \in \mathbb{R}$. It follows that $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and concave function.

Proposition 3.12 and Remark 3.6 characterize the minimal penalty functions representing, respectively, the indifference buyer's price and the indifference seller's price, defined by a strictly increasing, concave, and continuous function $U : \mathcal{X} \to \mathbb{R}$.

Proposition 3.12. *Let* $r^U_{w_0} : \Delta \times \mathbb{R} \to (-\infty, +\infty]$ *be the function defined* *by,*

$$r^U_{w_0}(m,Q) := \sup_{X\in\mathcal{X}:U(w_0+X)\geq u(w_0-m)} E_Q[-X]$$

for all $(m,Q)\in\Delta\times\mathbb{R}$*. Then,*

$$r^U_{w_0}(m,Q) = w_0 + \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(U^*(\lambda Q) - u(w_0-m)\Big)$$

for all $(m,Q)\in\mathbb{R}\times\Delta$*.*

Proof of Proposition 3.12. As $U:\mathcal{X}\to\mathbb{R}$ is concave and $U^*(0)=+\infty$, by Theorem 13.5 and Theorem 9.7 in Rockafellar (1970),

$$\begin{aligned}
r^U_{w_0}(m,Q) &= \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(\sup_{X\in\mathcal{X}}\Big(U(w_0+X) - u(w_0-m) - E_Q[\lambda X]\Big)\Big)\\
&= \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(\sup_{X\in\mathcal{X}}\Big(U(w_0+X) - E_Q[\lambda(w_0+X)]\Big) + \lambda w_0 - u(w_0-m)\Big)\\
&= w_0 + \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(U^*(\lambda Q) - u(w_0-m)\Big)
\end{aligned}$$

for all $(m,Q)\in\mathbb{R}\times\Delta$. □

Remark 3.6. Let $\alpha^U_{w_0}:\Delta\to(-\infty,+\infty]$ be the function defined by,

$$\alpha^U_{w_0}(Q) := r^U_{w_0}(0,Q)$$

for all $Q\in\Delta$. Then,

$$\alpha^U_{w_0}(Q) = w_0 + \inf_{\lambda\in(0,+\infty)} \frac{1}{\lambda}\Big(U^*(\lambda Q) - u(w_0)\Big)$$

for all $Q\in\Delta$.

Proposition 3.13 characterizes the maximal risk function representing the indifference buyer's price defined by a strictly increasing, concave, and continuous function $U:\mathcal{X}\to\mathbb{R}$.

Proposition 3.13. *Let* $R^U_{w_0} : \Delta \times \mathbb{R} \to [-\infty, +\infty)$ *be the function defined by,*

$$R^U_{w_0}(x, Q) := \inf\{m \in \mathbb{R} \,|\, r^U_{w_0}(m, Q) \geq x\}$$

for all $(x, Q) \in \Delta \times \mathbb{R}$*. Then,*

$$R^U_{w_0}(x, Q) = w_0 - u^{-1}\Big(\inf_{\lambda \in (0,+\infty)} \Big(\lambda(w_0 - x) + U^*(\lambda Q)\Big)\Big)$$

for all $(x, Q) \in \mathbb{R} \times \Delta$*.*

Proof of Proposition 3.13. By Proposition 3.12,

$$\begin{aligned} R^U_{w_0}(x, Q) &= \inf\Big\{m \in \mathbb{R} \,\Big|\, w_0 + \inf_{\lambda \in (0,+\infty)} \frac{1}{\lambda}\Big(U^*(\lambda Q) - u(w_0 - m)\Big) \geq x\Big\} \\ &= \inf\Big\{m \in \mathbb{R} \,\Big|\, \inf_{\lambda \in (0,+\infty)} \Big(\lambda(w_0 - x) + U^*(\lambda Q)\Big) \geq u(w_0 - m)\Big\} \\ &= \inf\Big\{m \in \mathbb{R} \,\Big|\, m \geq w_0 - u^{-1}\Big(\inf_{\lambda \in (0,+\infty)} \Big(\lambda(w_0 - x) + U^*(\lambda Q)\Big)\Big)\Big\} \\ &= w_0 - u^{-1}\Big(\inf_{\lambda \in (0,+\infty)} \Big(\lambda(w_0 - x) + U^*(\lambda Q)\Big)\Big) \end{aligned}$$

for all $(x, Q) \in \mathbb{R} \times \Delta$. □

Chapter 4

Characterizations of Comparative Uncertainty Attitudes

4.1 Comparative Uncertainty Aversion

In this section we study the comparison of the different extents of uncertainty aversion of different decision-makers at a given level of constant initial wealth. In Section 4.1.1 we present a definition of comparative uncertainty aversion which is consistent with the definition of comparative uncertainty aversion of Ghirardato and Marinacci (2001) and with the definition of comparative risk aversion of Yaari (1969). In Section 4.1.2 we provide various characterizations of comparative uncertainty aversion in terms of the indifference buyer's price and of the indifference seller's price introduced in Chapter 3.

Observe that, for simplicity, all the definitions and all the results are provided in terms of the more general notion of comparative risk *and* uncertainty aversion, and that all the definitions and all the results which characterize comparative uncertainty aversion *only* will be recovered as particular cases under a suitable normalization condition.

4.1.1 Definition

The notion of comparative risk and uncertainty aversion allows to compare the different extents of risk and uncertainty aversion of different decision-makers $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ and $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ endowed with the same constant initial wealth $w_0 \in \mathbb{R}$. The intuition underlying the notion of comparative risk and uncertainty aversion is that if a decision-maker $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ endowed with constant initial wealth $w_0 \in \mathbb{R}$ prefers a constant monetary payoff $x \in \mathbb{R}$ to a stochastic monetary payoff $X \in \mathcal{X}$, then a *more risk and uncertainty averse* decision-maker $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ endowed with the same constant initial wealth $w_0 \in \mathbb{R}$ will do the same.

Definition 4.1. A decision-maker $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is said to be *less risk and uncertainty averse* than another $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if,

$$u_1(w_0 + x) \geq U^{u_1,G_1}(w_0 + X) \;\Rightarrow\; u_2(w_0 + x) \geq U^{u_2,G_2}(w_0 + X) \tag{4.1}$$

for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and $w_0 \in \mathbb{R}$.

Note that the second decision-maker U^{u_2,G_2} may prefer the constant monetary payoff $x \in \mathbb{R}$ either because she is more risk averse than the first decision-maker U^{u_1,G_1}, that is because she dislikes the variability of the outcomes of the stochastic monetary payoff $X \in \mathcal{X}$ more than the first decision-maker U^{u_1,G_1}, or because she is more uncertainty averse than the first decision-maker U^{u_1,G_1}, that is because she dislikes the fact that the probabilities of the different possible outcomes of $X \in \mathcal{X}$ are not objectively determined more than the first decision-maker U^{u_1,G_1}. For this reason, Definition 4.1 is, in general, a definition of comparative risk *and* uncertainty aversion, and not a definition of comparative uncertainty aversion *only*.

For Definition 4.1 to specialize to comparative uncertainty aversion *only*, it is necessary that both decision-makers U^{u_1,G_1} and U^{u_2,G_2} display the same risk attitudes u_1 and u_2, that is that $u_1 = u_2$[1]. This normalization condition ensures, in fact, that different choices of the decision-makers are ascribable only to their different uncertainty attitudes G_1 and G_2, and not also to their different risk attitudes u_1 and u_2.

Remark 4.1. Definition 4.1 can be immediately characterized in terms of the acceptance family $(\mathcal{A}^{u,G}_{w_0,m})_{m\in\mathbb{R}}$ introduced in Section 3.1.1.3. In fact, it follows directly from Definition 4.1 that a decision-maker $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than another $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$\mathcal{A}^{u_2,G_2}_{w_0,m} \subseteq \mathcal{A}^{u_1,G_1}_{w_0,m} \tag{4.2}$$

for all $m \in \mathbb{R}$ and $w_0 \in \mathbb{R}$. Therefore, a more risk and uncertainty averse decision-maker $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ prefers fewer stochastic monetary payoffs $X \in \mathcal{X}$ to a constant monetary payoff $-m \in \mathbb{R}$ at every level of constant initial wealth $w_0 \in \mathbb{R}$.

Note that U^{u_1,G_1} is said to be more risk and uncertainty averse than U^{u_2,G_2} when the implication in Equation (4.1) holds true in the opposite direction and that U^{u_1,G_1} is said to be as risk and uncertainty averse as U^{u_2,G_2} when the implication in Equation (4.1) holds true in both directions. The same considerations apply to the set inclusion in Remark 4.1.

[1] Actually, it is not necessary that u_1 and u_2 are identical, as it is sufficient that u_1 and u_2 are equivalent, that is that either u_1 is a positive affine transformation of u_2, or that u_2 is a positive affine transformation of u_1. Nevertheless, without loss of generality, we can set $u_1 = u_2$. See also Cerreia Vioglio et al. (2011a, Section 3.3).

4.1.2 Characterizations

Theorem 4.1 asserts that a decision-maker is less risk and uncertainty averse than another if and only if her indifference buyer's price and her indifference seller's price are smaller than for the the other at every level of constant initial wealth. Analogous results in term of the indifference buyer's price were obtained by Pratt (1964) in the expected utility framework in relation to the characterization of comparative risk aversion.

Theorem 4.1. *The following statements are equivalent.*

(i) U^{u_1,G_1} *is less risk and uncertainty averse than* U^{u_2,G_2}.

(ii) $\pi_{w_0}^{u_1,G_1} \leq \pi_{w_0}^{u_2,G_2}$ *for all* $w_0 \in \mathbb{R}$.

(iii) $\varphi_{w_0}^{u_1,G_1} \leq \varphi_{w_0}^{u_2,G_2}$ *for all* $w_0 \in \mathbb{R}$.

Proof of Theorem 4.1. Let $X \in \mathcal{X}$ and $x \in \mathbb{R}$.
$(i) \Leftrightarrow (ii)$ By Definition 4.1 and Definition 3.1, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$u_1(w_0 + x) \geq u_1(w_0 - \pi_{w_0}^{u_1,G_1}(X)) \Rightarrow u_2(w_0 + x) \geq u_2(w_0 - \pi_{w_0}^{u_2,G_2}(X))$$

that is, since $u_1 : \mathbb{R} \to \mathbb{R}$ and $u_2 : \mathbb{R} \to \mathbb{R}$ are strictly increasing, if and only if,

$$\pi_{w_0}^{u_1,G_1}(X) \geq -x \Rightarrow \pi_{w_0}^{u_2,G_2}(X) \geq -x$$

Thus, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$\pi_{w_0}^{u_1,G_1}(X) \leq \pi_{w_0}^{u_2,G_2}(X).$$

$(i) \Leftrightarrow (iii)$ By Definition 4.1 and Definition 3.3, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$U^{u_1,G_1}(w_0 + x + X + \varphi^{u_1,G_1}_{w_0+x}(X)) \geq U^{u_1,G_1}(w_0 + X) \Rightarrow$$
$$U^{u_2,G_2}(w_0 + x + X + \varphi^{u_2,G_2}_{w_0+x}(X)) \geq U^{u_2,G_2}(w_0 + X)$$

that is, since $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ and $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ are strictly increasing, if and only if,

$$\varphi^{u_1,G_1}_{w_0+x}(X) \geq -x \Rightarrow \varphi^{u_2,G_2}_{w_0+x}(X) \geq -x$$

Thus, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$\varphi^{u_1,G_1}_{w_0+x}(X) \leq \varphi^{u_2,G_2}_{w_0+x}(X).$$

□

Corollary 4.1 provides the dual characterization of uncertainty aversion consistent with Theorem 4.1 and with the representation results in Proposition 3.3 and Proposition 3.7.

Corollary 4.1. *The following statements are equivalent.*

(i) U^{u_1,G_1} *is less risk and uncertainty averse than* U^{u_2,G_2}.

(ii) $R^{u_1,G_1}_{w_0} \leq R^{u_2,G_2}_{w_0}$ *for all* $w_0 \in \mathbb{R}$.

(iii) $\alpha^{u_1,G_1}_{w_0} \geq \alpha^{u_2,G_2}_{w_0}$ *for all* $w_0 \in \mathbb{R}$.

Proof of Corollary 4.1. *(i)* ⇔ *(ii)* By Remark 4.1, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if $\mathcal{A}^{u_2,G_2}_{w_0,m} \subseteq \mathcal{A}^{u_1,G_1}_{w_0,m}$ for all $m \in \mathbb{R}$. By Rockafellar (1970, Corollary 13.1.1),

$$\mathcal{A}^{u_2,G_2}_{w_0,m} \subseteq \mathcal{A}^{u_1,G_1}_{w_0,m} \quad \Leftrightarrow \quad r^{u_2,G_2}_{w_0}(m, Q) \leq r^{u_1,G_1}_{w_0}(m, Q)$$

for all $(m,Q) \in \mathbb{R}\times\Delta$. By Equation (3.7) and by the increasing monotonicity of $r_{w_0}^{u_1,G_1} : \mathbb{R}\times\Delta \to (-\infty,+\infty]$ and $r_{w_0}^{u_2,G_2} : \mathbb{R}\times\Delta \to (-\infty,+\infty]$ in the first argument,

$$r_{w_0}^{u_2,G_2}(m,Q) \leq r_{w_0}^{u_1,G_1}(m,Q) \quad \forall (m,Q) \in \mathbb{R}\times\Delta \quad \Leftrightarrow$$
$$R_{w_0}^{u_1,G_1}(x,Q) \leq R_{w_0}^{u_2,G_2}(x,Q) \quad \forall (x,Q) \in \mathbb{R}\times\Delta$$

Thus, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$R_{w_0}^{u_1,G_1}(x,Q) \leq R_{w_0}^{u_2,G_2}(x,Q)$$

for all $(x,Q) \in \mathbb{R}\times\Delta$.

$(i) \Leftrightarrow (iii)$ By Theorem 4.1 and by Remark 3.3 $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if $\mathcal{A}_{w_0-m,0}^{u_2,G_2} - m \subseteq \mathcal{A}_{w_0-m,0}^{u_1,G_1} - m$ for all $m \in \mathbb{R}$. By Rockafellar (1970, Corollary 13.1.1),

$$\mathcal{A}_{w_0-m,0}^{u_2,G_2} - m \subseteq \mathcal{A}_{w_0-m,0}^{u_1,G_1} - m \quad \Leftrightarrow \quad \alpha_{w_0-m}^{u_2,G_2}(Q) - m \leq \alpha_{w_0-m}^{u_1,G_1}(Q) - m$$

for all $m \in \mathbb{R}$ and $Q \in \Delta$. Thus, $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is less risk and uncertainty averse than $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$\alpha_{w_0-m}^{u_2,G_2}(Q) \leq \alpha_{w_0-m}^{u_1,G_1}(Q)$$

for all $m \in \mathbb{R}$ and $Q \in \Delta$. $\square$

4.2 Increasing, Decreasing, and Constant Uncertainty Aversion

In this section we study the comparison of the different extents of uncertainty aversion of a given decision-maker at different levels of constant initial wealth. In Section 4.2.1 we present a definition of increasing, decreasing, and constant uncertainty aversion which is consistent with the definition of increasing, decreasing, and constant risk aversion in Kreps (1988, Chapter 6, page 75). In Section 4.2.2 we provide various characterizations of increasing, decreasing, and constant uncertainty aversion in terms of the indifference buyer's price and of the indifference seller's price introduced in Chapter 3. Observe that, for simplicity, all the definitions and all the results are provided in terms of the more general notion of increasing, decreasing, and constant risk *and* uncertainty aversion, and that all the definitions and all the results which characterize increasing, decreasing, and constant uncertainty aversion *only* will be recovered as particular cases under a suitable normalization condition.

4.2.1 Definition

The notion of increasing, decreasing, and constant uncertainty aversion allows to compare the different extents of uncertainty aversion of a given decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ at different levels of constant initial wealth $w_1 \in \mathbb{R}$ and $w_2 \in \mathbb{R}$. The intuition underlying the notion of increasing uncertainty aversion is that if a decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ prefers a constant monetary payoff $x \in \mathbb{R}$ to a stochastic monetary payoff $X \in \mathcal{X}$ when her constant initial wealth is $w_1 \in \mathbb{R}$, and if she still prefers the constant monetary payoff $x \in \mathbb{R}$ when her constant initial wealth is increased to $w_2 \in \mathbb{R}$, then she is increasingly risk and uncertainty averse[2].

Definition 4.2. A decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be *increasingly risk and uncertainty averse* if,

$$u(w_1 + x) \geq U^{u,G}(w_1 + X) \Rightarrow u(w_2 + x) \geq U^{u,G}(w_2 + X) \tag{4.3}$$

for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and $w_1, w_2 \in \mathbb{R}$ such that $w_1 \leq w_2$.

Note that the decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ may still prefer the constant monetary payoff $x \in \mathbb{R}$ when her constant initial wealth is increased to $w_2 \in \mathbb{R}$ either because she is increasingly risk averse, that is because she dislikes even more the variability of the outcomes of the stochastic monetary payoff $X \in \mathcal{X}$ when her constant initial wealth is increased to $w_2 \in \mathbb{R}$, or because she is increasingly uncertainty averse, that is because she dislikes even more the fact that the probabilities of the different possible outcomes of $X \in \mathcal{X}$ are not objectively determined when her constant initial wealth is increased to $w_2 \in \mathbb{R}$. For this reason, Definition 4.2 is a definition of increasing risk *and* uncertainty aversion, and not a definition of increasing uncertainty aversion *only*.

For Definition 4.2 to specialize to increasing uncertainty aversion *only*, it is necessary that the decision-maker $U^{u,G}$ displays the same risk aversion at different levels of constant initial wealth $w_1 \in \mathbb{R}$ and $w_2 \in \mathbb{R}$, that is that $u : \mathbb{R} \to \mathbb{R}$ is *constantly absolute risk averse* (CARA). This normalization condition ensures, in fact, that the decision-maker's different choices at different levels of constant initial wealth $w_1 \in \mathbb{R}$ and $w_2 \in \mathbb{R}$ are ascribable only to the way in which her uncertainty aversion changes when her con-

[2]In fact, if this decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ was decreasingly risk and uncertainty averse, than at some high level constant initial wealth $w_2 \in \mathbb{R}$ she would reverse her preferences and choose the stochastic monetary payoff $X \in \mathcal{X}$.

stant initial wealth is increased from $w_1 \in \mathbb{R}$ to $w_2 \in \mathbb{R}$, and not also to the way in which her risk aversion changes when her constant initial wealth is increased from $w_1 \in \mathbb{R}$ to $w_2 \in \mathbb{R}$. Recall that the CARA utility functions are the linear utility function $u(x) = \beta + \alpha x$ for all $x \in \mathbb{R}$ with $\beta \in \mathbb{R}$ and $\alpha \in (0, +\infty)$ and the exponential utility function $u(x) = -\alpha e^{-\theta x}$ for all $x \in \mathbb{R}$ with $\alpha, \theta \in (0, +\infty)$.

Remark 4.2. Definition 4.2 can be immediately characterized in terms of the acceptance family $(\mathcal{A}^{u,G}_{w_0,m})_{m\in\mathbb{R}}$ introduced in Section 3.1.1.3. In fact, it follows directly from Definition 4.2 that a decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$\mathcal{A}^{u,G}_{w_2,m} \subseteq \mathcal{A}^{u,G}_{w_1,m} \tag{4.4}$$

for all $m \in \mathbb{R}$ and $w_1, w_2 \in \mathbb{R}$ such that $w_1 \leq w_2$. Thus, an increasingly risk and uncertainty averse decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ prefers fewer stochastic monetary payoffs $X \in \mathcal{X}$ to the constant monetary payoff $-m \in \mathbb{R}$ when her constant initial wealth is increased from $w_1 \in \mathbb{R}$ to $w_2 \in \mathbb{R}$.

Note that $U^{u,G}$ is said to be decreasingly risk and uncertainty averse when the implication in Equation (4.3) holds true in the opposite direction, and that $U^{u,G}$ is said to be constantly risk and uncertainty averse when the implication in Equation (4.3) holds true in both directions. The same considerations apply to the set inclusion in Remark 4.2.

4.2.2 Characterizations

Theorem 4.2 asserts that a decision-maker is increasingly risk and uncertainty averse if and only if her indifference buyer's price and her indifference seller's price are increasing functions of her constant initial wealth. Analogous results in term of the indifference buyer's price were obtained by Pratt (1964) in the expected utility framework in relation to the characterization of increasing, decreasing, and constant risk aversion.

Theorem 4.2. *The following statements are equivalent.*

(i) $U^{u,G}$ *is increasingly risk and uncertainty averse.*

(ii) $\pi^{u,G}_{w_1} \leq \pi^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

(iii) $\varphi^{u,G}_{w_1} \leq \varphi^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

Proof of Theorem 4.2. Follows from applying the same arguments as in the proof of Theorem 4.1 with $U^{u,G}(w_1 + X) = U^{u_1,G_1}(w_0 + X)$ and $U^{u,G}(w_2 + X) = U^{u_2,G_2}(w_0 + X)$ for all $X \in \mathcal{X}$ and with $w_1, w_2 \in \mathbb{R}$ such that $w_1 \leq w_2$. □

Corollary 4.2 provides a dual characterization of uncertainty aversion consistent with Theorem 4.2 and with the representation results of Proposition 3.3 and Proposition 3.7.

Corollary 4.2. *The following statements are equivalent.*

(i) $U^{u,G}$ *is increasingly risk and uncertainty averse.*

(ii) $R^{u,G}_{w_1} \leq R^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

(iii) $\alpha^{u,G}_{w_1} \geq \alpha^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

Proof of Corollary 4.2. Follows from applying the same arguments as in the proof of Corollary 4.1 with $\mathcal{A}^{u,G}_{w_1,m} = \mathcal{A}^{u_1,G_1}_{w_0,m}$ and $\mathcal{A}^{u,G}_{w_2,m} = \mathcal{A}^{u_2,G_2}_{w_0,m}$ for all $m \in \mathbb{R}$ and with $w_1, w_2 \in \mathbb{R}$ such that $w_1 \leq w_2$. □

4.2.3 Further Characterizations

This section illustrates some further characterizations of increasing, decreasing and constant risk and uncertainty aversion which do not rely on the dependence of the indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ and of the indifference seller's price $\varphi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ on the decision-maker's constant initial wealth $w_0 \in \mathbb{R}$. The characterization results presented in this section rely instead on the observation that the notion of increasing, decreasing, or constant risk and uncertainty aversion describes how a decision-maker's choice between an uncertain monetary payoff $X \in \mathcal{X}$ and a constant monetary payoff $x \in \mathbb{R}$ is altered if a positive constant amount of money $m \in [0, +\infty)$ is added to both alternatives.

Proposition 4.1. *A decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,*

$$u(w_0 + x) \geq U^{u,G}(w_0 + X) \;\Rightarrow\; u(w_0 + x + m) \geq U^{u,G}(w_0 + X + m) \quad (4.5)$$

for all $m \in [0, +\infty)$, $X \in \mathcal{X}$, $x \in \mathbb{R}$, and $w_0 \in \mathbb{R}$.

Proof of Proposition 4.1. Follows from Definition 4.2 taking $w_1 = w_0 \in \mathbb{R}$ and setting $w_2 = w_0 + m \in \mathbb{R}$ with $m \in [0, +\infty)$. □

Proposition 4.2 shows that Proposition 4.1 can be equivalently characterized in terms of the acceptance family $(\mathcal{A}_{w_0,m}^{u,G})_{m \in \mathbb{R}}$.

Proposition 4.2. *A decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,*

$$\mathcal{A}_{w_0,m}^{u,G} \subseteq \mathcal{A}_{w_0,m+n}^{u,G} + n \quad (4.6)$$

for all $m \in \mathbb{R}$, $n \in [0, +\infty)$ and $w_0 \in \mathbb{R}$.

Proof of Proposition 4.2. Let $X \in \mathcal{X}$ and $m \in \mathbb{R}$. By Proposition 4.1 $U^{u,G}$:

$\mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$u(w_0 - m) \geq U^{u,G}(w_0 + X) \; \Rightarrow \; u(w_0 - m - n) \geq U^{u,G}(w_0 + X - n)$$

for all $n \in (-\infty, 0]$, that is, if and only if,

$$\mathcal{A}^{u,G}_{w_0,m} \supseteq \mathcal{A}^{u,G}_{w_0,m+n} + n$$

for all $n \in (-\infty, 0]$. Thus, $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$\mathcal{A}^{u,G}_{w_0,m} \subseteq \mathcal{A}^{u,G}_{w_0,m+n} + n$$

for all $n \in [0, +\infty)$. □

Note that decreasing risk and uncertainty aversion is obtained reversing the direction of the implication in Equation (4.5), and that constant risk and uncertainty aversion is obtained when the implication in Equation (4.5) applies in both directions. The same considerations apply to the set inclusion in Equation (4.6).

4.2.3.1 Cash-Subadditivity

As a consequence of Proposition 4.1, increasing, decreasing and constant risk and uncertainty aversion are equivalently characterized by the additive properties that the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ satisfies with respect to the positive constant monetary payoffs $m \in [0, +\infty)$ for any given $w_0 \in \mathbb{R}$.

Theorem 4.3. *A decision-maker* $U^{u,G} : \mathcal{X} \to \mathbb{R}$ *is increasingly risk and*

uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X+m) \geq \pi_{w_0}^{u,G}(X) - m \tag{4.7}$$

for all $m \in [0, +\infty)$*,* $X \in \mathcal{X}$*, and* $w_0 \in \mathbb{R}$*.*

Proof of Theorem 4.3. Let $X \in \mathcal{X}$, $x \in \mathbb{R}$ and $m \in [0, +\infty)$. By Proposition 4.1 and Definition 3.1 $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$u(w_0 + x) \geq u(w_0 - \pi_{w_0}^{u,G}(X)) \Rightarrow$$
$$u(w_0 + m + x) \geq u(w_0 - \pi_{w_0}^{u,G}(X+m))$$

or, equivalently, as $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, if and only if,

$$\pi_{w_0}^{u,G}(X) \geq -x \Rightarrow \pi_{w_0}^{u,G}(X+m) + m \geq -x$$

Thus, $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X+m) \geq \pi_{w_0}^{u,G}(X) - m.$$

□

Theorem 4.3 asserts that a decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if the indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is *cash-subadditive.* It follows from Proposition 3.1 that the indifference buyer's price of an increasingly risk and uncertainty averse decision-maker is a *cash-subadditive quasiconvex risk measure.* The property of cash-subadditivity was introduced in the mathematical finance literature by El Karoui and Ravanelli (2009) to model the impact of default risk and interest rate ambiguity on the minimal reserve amount that must be added to an uncertain monetary payoff such that it becomes acceptable to a financial regulator or supervisory agency. The property of cash-subadditivity is a weakening of the property of cash-additivity considered by Deprez and Gerber (1985), Frittelli and Rosazza Gianin (2002), and Föllmer and Schied (2002).

It follows from Theorem 4.3 implies that $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is decreasingly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X + m) \leq \pi_{w_0}^{u,G}(X) - m \tag{4.8}$$

for all $m \in [0, +\infty)$ and $X \in \mathcal{X}$, that is if and only if $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is a *cash-superadditive* quasiconvex risk measure, and that $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is constantly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X + m) = \pi_{w_0}^{u,G}(X) - m \tag{4.9}$$

for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$, that is if and only if $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is a *cash-additive convex risk measure* (see Cerreia Vioglio et al. (2010, Proposition 2.1)). In the framework of expected utility preferences $U^{u,Q} : \mathcal{X} \to \mathbb{R}$ the characterization of constant risk aversion in terms of cash-additivity of the indifference buyer's price $\pi_{w_0}^{u,Q} : \mathcal{X} \to \mathbb{R}$ is a direct consequence of the Nagumo-Kolmogorov-de Finetti Theorem (see de Finetti (1931)).

Corollary 4.3 provides a dual characterization of increasing risk and uncertainty aversion consistent with Theorem 4.3 and with the representation result of Proposition 3.3. Proposition 4.3 is a direct application of Proposition 2.11 in Drapeau and Kupper (2010).

Corollary 4.3. *A decision-maker* $U^{u,G} : \mathcal{X} \to \mathbb{R}$ *is increasingly risk and*

uncertainty averse if and only if,

$$R^{u,G}_{w_0}(x - m, Q) \geq R^{u,G}_{w_0}(x, Q) - m \tag{4.10}$$

for all $m \in [0, +\infty)$, $(x, Q) \in \mathbb{R} \times \Delta$, *and* $w_0 \in \mathbb{R}$.

The different additive properties of the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ described by Equation (4.7), Equation (4.8), and Equation (4.9), allow to immediately establish various inequalities between the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ and the indifference seller's price $\varphi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$. The derivation of the various inequalities is based on the useful result of Lemma 4.1.

Lemma 4.1. *The indifference buyer's price* $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ *satisfies,*

$$\pi^{u,G}_{w_0}(X + \varphi^{u,G}_{w_0}(X)) = 0 \tag{4.11}$$

for all $X \in \mathcal{X}$ *and* $w_0 \in \mathbb{R}$.

Proof of Lemma 4.1. Let $X \in \mathcal{X}$. By Definition 3.1 and Definition 3.3,

$$u(w_0 - \pi^{u,G}_{w_0}(X + \varphi^{u,G}_{w_0}(X))) = U^{u,G}(w_0 + X + \varphi^{u,G}_{w_0}(X)) = u(w_0)$$

and the strict monotonicity of $u : \mathbb{R} \to \mathbb{R}$ yields $\pi^{u,G}_{w_0}(X + \varphi^{u,G}_{w_0}(X)) = 0$. □

Lemma 4.1, combined with Theorem 4.3, allows to characterize increasing, decreasing, and constant risk and uncertainty aversion in terms of the inequalities fulfilled by the indifference buyer's price $\pi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ and by the indifference seller's price $\varphi^{u,G}_{w_0} : \mathcal{X} \to \mathbb{R}$ for every $w_0 \in \mathbb{R}$.

Theorem 4.4. *A decision-maker* $U^{u,G} : \mathcal{X} \to \mathbb{R}$ *is increasingly risk and*

uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X) \leq \varphi_{w_0}^{u,G}(X) \tag{4.12}$$

for all $X \in \mathcal{X}$ such that $\varphi_{w_0}^{u,G}(X) \in [0, +\infty)$.

Proof of Theorem 4.4. By Theorem 4.3 and Lemma 4.1 $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$0 = \pi_{w_0}^{u,G}(X + \varphi_{w_0}^{u,G}(X)) \geq \pi_{w_0}^{u,G}(X) - \varphi_{w_0}^{u,G}(X)$$

for all $X \in \mathcal{X}$ such that $\varphi_{w_0}^{u,G}(X) \in [0, +\infty)$. □

It follows from Theorem 4.4 that $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is decreasingly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X) \geq \varphi_{w_0}^{u,G}(X) \tag{4.13}$$

for all $X \in \mathcal{X}$ such that $\varphi_{w_0}^{u,G}(X) \in [0, +\infty)$, and that $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is constantly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X) = \varphi_{w_0}^{u,G}(X) \tag{4.14}$$

for all $X \in \mathcal{X}$.

Lemma 4.2 provides a dual characterization of Lemma 4.1 in terms of the maximal risk function $R_{w_0}^{u,G} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ and of the minimal penalty function $\alpha_{w_0}^{u,G} : \Delta \to (-\infty, +\infty]$.

Lemma 4.2. *The maximal risk function $R_{w_0}^{u,G} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ satisfies,*

$$R_{w_0}^{u,G}(x, Q) \leq 0$$

for all $(x, Q) \in \mathbb{R} \times \Delta$ *such that* $x \leq \alpha^{u,G}_{w_0}(Q)$.

Proof of Lemma 4.2. By the increasing monotonicity of $R^{u,G}_{w_0} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ in its first argument, by Equation (3.6), and by Lemma 4.1,

$$\begin{aligned} R^{u,G}_{w_0}(x, Q) \leq R^{u,G}_{w_0}(E_Q[-X] - \varphi^{u,G}_{w_0}(X), Q) \\ \leq \sup_{Q \in \Delta} R^{u,G}_{w_0}(E_Q[-X] - \varphi^{u,G}_{w_0}(X), Q) \\ = 0 \end{aligned}$$

for all $(x, Q) \in \mathbb{R} \times \Delta$ such that $x \leq E_Q[-X] - \varphi^{u,G}_{w_0}(X)$ for some $X \in \mathcal{X}$, that is for all $(x, Q) \in \mathbb{R} \times \Delta$ such that,

$$\begin{aligned} x \leq \sup_{X \in \mathcal{X}} \left(E_Q[-X] - \varphi^{u,G}_{w_0}(X) \right) \\ = \sup_{X \in \mathcal{A}^{u,G}_{w_0,0}} E_Q[-X] \\ = \alpha^{u,G}_{w_0}(Q) \end{aligned}$$

See also Remark 4.16 point (a) in Föllmer and Schied (2004). Thus, $R(x, Q) \leq 0$ for all $(x, Q) \in \mathbb{R} \times \Delta$ such that $x \leq \alpha^{u,G}_{w_0}(Q)$. □

Lemma 4.2, combined with Corollary 4.3, allows to characterize increasing, decreasing and constant risk and uncertainty aversion in terms of the inequalities that maximal risk function $R^{u,G}_{w_0} : \mathbb{R} \times \Delta \to [-\infty, +\infty)$ and the minimal penalty function $\alpha^{u,G}_{w_0} : \Delta \to (-\infty, +\infty]$ fulfill for every $w_0 \in \mathbb{R}$.

Corollary 4.4. *A decision-maker* $U^{u,G} : \mathcal{X} \to \mathbb{R}$ *is increasingly risk and uncertainty averse if and only if,*

$$R^{u,G}_{w_0}(x, Q) \leq x - \alpha^{u,G}_{w_0}(Q) \tag{4.15}$$

for all $(x, Q) \in \mathbb{R} \times \Delta$ *such that* $x \geq \alpha_{w_0}^{u,G}(Q)$.

Proof. By Lemma 4.2 and Corollary 4.3 $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$0 \geq R_{w_0}^{u,G}(x - (x - \alpha_{w_0}^{u,G}(Q)), Q) \geq R_{w_0}^{u,G}(x, Q) - x + \alpha_{w_0}^{u,G}(Q)$$

for all $(x, Q) \in \mathbb{R} \times \Delta$ such that $x \geq \alpha_{w_0}^{u,G}(Q)$. Thus, $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,

$$R_{w_0}^{u,G}(x, Q) \leq x - \alpha_{w_0}^{u,G}(Q)$$

for all $(x, Q) \in \mathbb{R} \times \Delta$ such that $x \geq \alpha_{w_0}^{u,G}(Q)$ □

4.2.3.2 Star-Shapedness

The uncertainty indexes $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ that are minimally and maximally uncertainty averse consistently with the notion of comparative uncertainty aversion described in Section 4.1 have already been characterized by Cerreia Vioglio et al. (2011a, Section 3.3) who found that a decision-maker $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is more uncertainty averse than another $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if and only if,

$$G_1 \leq G_2$$

provided that $u_1 = u_2 : \mathbb{R} \to \mathbb{R}$. Theorem 4.5 characterizes the uncertainty indexes $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ that are increasingly, decreasingly, and constantly uncertainty averse accordingly with the notion of increasing, decreasing and constant uncertainty aversion described in Section 4.2. The proof of Theorem 4.5 exploits the characterization of a decision-maker's increasing, decreasing, and constant risk and uncertainty aversion in Theorem 4.3 under the normalization condition that the decision-maker's utility function is constantly absolute risk averse (see Subsection 4.1.1).

Theorem 4.5. *A decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly uncertainty averse if and only if,*

$$G(\lambda x + m, P) \leq \lambda G(x, P) + m$$

for all $\lambda \in (0, 1]$, $m \in [0, +\infty)$, and $(x, P) \in \mathbb{R} \times \Delta$, decreasingly uncertainty averse if and only if,

$$G(\lambda x + m, P) \geq \lambda G(x, P) + m$$

for all $\lambda \in (0, 1]$, $m \in [0, +\infty)$, and $(x, P) \in \mathbb{R} \times \Delta$, and constantly uncertainty averse if and only if,

$$G(\lambda x + m, P) = \lambda G(x, P) + m$$

for all $\lambda \in (0, +\infty)$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$.

Proof of Theorem 4.5. The indifference buyer's price $\pi_{w_0}^{\mathcal{L},G} : \mathcal{X} \to \mathbb{R}$ defined in terms of the linear utility function $u(x) = \beta + \alpha x$ for all $x \in \mathbb{R}$ with $\beta \in \mathbb{R}$ and $\alpha \in (0, +\infty)$ is given by,

$$\pi_{w_0}^{\mathcal{L},G}(X) = w_0 + \frac{\beta}{\alpha} - \frac{1}{\alpha} \inf_{P \in \Delta} G(\beta + \alpha w_0 + \alpha E_P[X], P) \tag{4.16}$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$. By Drapeau and Kupper (2010, Proposition 2.11) the function $\pi_0^{\mathcal{L},G} : \mathcal{X} \to \mathbb{R}$ in Equation (4.16) is cash-subadditive if and only if $G(x+m, P) \leq G(x, P)+m$ for all $m \in [0, +\infty)$, cash-superadditive if and only if $G(x+m, P) \geq G(x, P)+m$ for all $m \in [0, +\infty)$, and cash-additive if and only if $G(x+m,P) = G(x,P)+m$ for all $m \in \mathbb{R}$.

The indifference buyer's price $\pi^{\mathcal{E},G}_{w_0} : \mathcal{X} \to \mathbb{R}$ defined in terms of the exponential utility function $u(x) = -\alpha e^{-\theta x}$ for all $x \in \mathbb{R}$ with $\alpha, \theta \in (0,+\infty)$ is given by,

$$\pi^{\mathcal{E},G}_{w_0}(X) = w_0 + \frac{1}{\theta} \ln \Big(-\frac{1}{\alpha} \inf_{P \in \Delta} G(E_P[-\alpha e^{-\theta(w_0+X)}], P) \Big) \tag{4.17}$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$. By Cerreia Vioglio et al. (2010, Proposition 4.1) the function $\pi^{\mathcal{E},G}_{0} : \mathcal{X} \to \mathbb{R}$ in Equation (4.17) is cash-subadditive if and only if $G(\lambda x, P) \leq \lambda G(x,P)$ for all $\lambda \in (0,1]$, cash-superadditive if and only if $G(\lambda x, P) \geq \lambda G(x,P)$ for all $\lambda \in (0,1]$, and cash-additive if and only if $G(\lambda x, P) = \lambda G(x,P)$ for all $\lambda \in (0,+\infty)$.

It follows that $\pi^{\mathcal{L},G}_{w_0} : \mathcal{X} \to \mathbb{R}$ and $\pi^{\mathcal{E},G}_{w_0} : \mathcal{X} \to \mathbb{R}$ are cash-subadditive if and only if $G(\lambda x+m, P) \leq \lambda G(x,P)+m$ for all $\lambda \in (0,1]$ and $m \in [0,+\infty)$, cash-superadditive if and only if $G(\lambda x+m, P) \geq \lambda G(x,P)+m$ for all $\lambda \in (0,1]$ and $m \in [0,+\infty)$, and cash-additive if and only if $G(\lambda x+m, P) = \lambda G(x,P)+m$ for all $\lambda \in (0,+\infty)$ and $m \in \mathbb{R}$.

Thus, the statement follows from Theorem 4.3, together with the normalization condition that the decision-maker's utility function is either linear or exponential discussed in Section 4.2. □

Theorem 4.5 asserts that a decision-maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly uncertainty averse if and only if her uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty,+\infty]$ is *star-shaped*[3] and *cash-subadditive*. The property of star-shapedness was introduced in the mathematical finance literature by Cerreia Vioglio et al. (2010) to model the impact of liquidity risk on the minimal reserve amount that must be added to an uncertain monetary payoff such that it becomes

[3]A function $h : \mathbb{R} \to \mathbb{R}$ is said to be star-shaped if $h(\lambda x) \leq \lambda h(x)$ for all $\lambda \in (0,1]$ and $x \in \mathbb{R}$.

acceptable to a financial regulator or supervisory agency. The property of star-shapedness is a weakening of the property of *positive homogeneity*[4] considered by Artzner et al. (1999) and Delbaen (2002).

The characterization of increasing, decreasing, and constant uncertainty aversion in Theorem 4.5 allows to easily classify the different possible specifications of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) into increasingly, decreasingly, and constantly uncertainty averse.

Example 4.1. By Theorem 4.5, the variational representation of preferences $U^{u,c} : \mathcal{X} \to \mathbb{R}$ of Maccheroni et al. (2006) is decreasingly uncertainty averse. In fact, as $c(P) \geq 0$ for all $P \in \Delta$, the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2.4) satisfies,

$$\begin{aligned} G(\lambda x + m, P) &= \lambda x + m + c(P) \\ &\geq \lambda x + m + \lambda c(P) \\ &= \lambda G(x, P) + m \end{aligned}$$

for all $\lambda \in (0, 1]$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$.

Example 4.2. The multiplier representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{Q}} : \mathcal{X} \to \mathbb{R}$ of Hansen and Sargent (2001) and Strzalecki (2011) is a particular case of the variational representation of preferences of Maccheroni et al. (2006) which is obtained when,

$$c(P) = \theta \mathcal{R}(P || \mathbb{Q})$$

for all $P \in \Delta$ where $\theta \in (0, +\infty)$ and $\mathcal{R}(. \,||\, \mathbb{Q}) : \Delta \to [0, +\infty]$ is the relative entropy with respect to $\mathbb{Q} \in \Delta^\sigma$ (see Subsection 2.3.2). Thus, the multiplier

[4]A function $h : \mathbb{R} \to \mathbb{R}$ is said to be positively homogeneous if $h(\lambda x) = \lambda h(x)$ for all $\lambda \in (0, +\infty)$ and $x \in \mathbb{R}$.

representation of preferences $U^{u,\theta,\mathcal{R},\mathbb{Q}} : \mathcal{X} \to \mathbb{R}$ is decreasingly uncertainty averse.

Example 4.3. By Theorem 4.5, the multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ of Gilboa and Schmeidler (1989) is constantly uncertainty averse. In fact, as for every $P \in \Delta$ either $\delta(P \,|\, \mathcal{P}) = 0$ or $\delta(P \,|\, \mathcal{P}) = +\infty$, the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2.8) satisfies,

$$\begin{aligned} G(\lambda x + m, P) &= \lambda x + m + \delta(P \,|\, \mathcal{P}) \\ &= \lambda x + m + \lambda \delta(P \,|\, \mathcal{P}) \\ &= \lambda G(x, P) + m \end{aligned}$$

for all $\lambda \in (0, +\infty)$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$.

Chapter 5

Conclusion

In this dissertation we studied the problem of indifference pricing in the general decision-theoretic framework of uncertainty averse preferences (Cerreia Vioglio et al. (2011a)).

In the first part of the dissertation we studied the preferences that an uncertainty averse decision-maker expresses through her indifference prices and we found that they are consistent with the basic principles of rationality and diversification (see Cerreia Vioglio et al. (2010)). We found, in particular, that the indifference buyer's price is a quasiconvex risk measure, and that the indifference seller's price is a cash-additive convex risk measure. We found that the acceptance family of the indifference buyer's price as well as the acceptance set of the indifference seller's price are completely characterized by the decision-maker's uncertainty averse preferences and by the decision-maker's constant initial wealth. We found that, as a result, the maximal risk function representing the indifference buyer's price as well as the minimal penalty function representing the indifference seller's price are completely described by the decision-maker's uncertainty averse preferences and by the decision-maker's constant initial wealth. We provided explicit expressions for the maximal risk function and for the minimal penalty functions representing the indifference buyer's price and the indifference seller's price defined by the variational (Maccheroni et al. (2006)), the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), and the multiple priors (Gilboa and Schmeidler (1989)) representations of preferences.

In the second part of the dissertation we studied the different extents of uncertainty aversion that a decision-maker's expresses through her indifference prices. We showed that a decision-maker is more (respectively, less) uncertainty averse than another if and only if her indifference prices are pointwise larger (respectively, smaller) than the other's, and that a decision is as uncertainty averse as another if and only if her indifference prices are pointwise equal to the other's. We also showed that a decision-maker is increasingly (respectively, decreasingly) uncertainty averse if and only if her indifference prices are increasing (respectively, decreasing) functions of her constant initial wealth, and that a decision is constantly uncertainty averse if and only if her indifference prices are constant functions of her constant initial wealth.

We found that a decision-maker is increasingly (respectively, decreasingly) uncertainty averse if and only if her indifference buyer's price is a cash-subadditive (respectively, cash-superadditive) quasiconvex risk measure, and constantly uncertainty averse if and only her indifference buyer's price is a cash-additive convex risk measure. We found that a decision-maker is increasingly (respectively, decreasingly) uncertainty averse if and only if her indifference buyer's price is less than (respectively, greater than) her indifference seller's price whenever the latter is positive, and constantly uncertainty averse if and only if her indifference buyer's price is equal to her indifference seller's price irrespective of whether the latter is positive or negative.

In the last part of the dissertation we derived a technical condition on the uncertainty index appearing in the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) which allows to easily classify the various particular specifications of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) into increasingly, decreasingly, and constantly uncertainty averse. We found that the variational (Maccheroni et al. (2006)) and, as a result, the multiplier (Hansen and Sargent (2001), Strzalecki (2011)), representations of preferences are decreasingly uncertainty averse, and that the multiple priors (Gilboa and Schmeidler (1989)) representation of preferences is constantly uncertainty averse.

Further research might investigate the extension of the analysis of this dissertation to a framework of optimal risk exchange. The problem of optimal risk exchange was studied by Borch (1962), Arrow (1963), and Gerber (1978) in the expected utility framework. The study of the problem of optimal risk exchange was extended by Barrieu and El-Karoui (2005), Jouini et al. (2008) and Filipovic and Kupper (2008) to a more general framework in which the relevant decision-makers evaluate the relative desirability of alternative uncertain monetary endowments by cash-additive convex risk measures. The study of the problem of optimal risk exchange was further developed by Acciaio (2007), who considered decision-makers with concave objective functions which are cash-additive but not necessarily monotone, and by El Karoui and Ravanelli (2009), who considered decision-makers with concave objective functions which are cash-subadditive and monotone. The solution of the problem of optimal risk exchange was characterized under even less restrictive assumptions by Ravanelli and Svindland (2011) who considered decision-makers with objective functions which are only concave and monotone.

All the objective functions previously employed in the literature on optimal risk exchange, with the exception of the non-monotone functions considered by Acciaio (2007), are particular cases of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) and, as shown in Section 4.2.3.2 of this dissertation, they can be classified based on whether they exhibit increasing, decreasing, or constant uncertainty aversion. Thus, future work might investigate the existence and the characterization and the solution of the problem of optimal risk exchange in the general framework of uncertainty averse preferences of Cerreia Vioglio et al. (2011a) and, along the lines of this dissertation, it might examine how the equilibrium depends on the different attitudes toward uncertainty of the decision-makers involved in the exchange.

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