Financial and Actuarial Valuation of Insurance Derivatives

by

ALEXANDER MÜRMANN

The London School of Economics and Political Science

Thesis submitted for the degree of
Doctor of Philosophy
at the University of London

Abstract

This dissertation looks into the interplay of financial and insurance markets that is created by securitization of insurance related risks. It comprises four chapters on both the common ground and different nature of actuarial and financial risk valuation.

The first chapter investigates the market for catastrophe insurance derivatives that has been established at the Chicago Board of Trade in 1992. Modeling the underlying index as a compound Poisson process the set of financial derivative prices that exclude arbitrage opportunities is characterized by the market prices of frequency and jump size risk. Fourier analysis leads to a representation of price processes that separates the underlying stochastic structure from the contract's payoff and allows derivation of the inverse Fourier transform of price processes in closed form. In a market with a representative investor, market prices of frequency and jump size risk are uniquely determined by the agent's coefficient of absolute risk aversion which consequently fixes the price process on the basis of excluding arbitrage strategies.

The second chapter analyzes a model for a price index of insurance stocks that is based on the Cramér-Lundberg model used in classical risk theory. It is shown that price processes of basic securities and derivatives can be expressed in terms of the market prices of risk. This parameterization leads to formulae in closed form for the inverse Fourier transform of prices and the conditional probability distribution. Financial spreads are examined in more detail as their structure resembles the characteristics of stop loss reinsurance treaties. The equivalence between a representative agent approach and the Esscher transform is shown and the financial price process that is robust to these two selection criteria is determined. Finally, the analysis is generalized to allow for risk processes that are perturbed by diffusion.

In the third chapter an integrated market is introduced containing both insurance and financial contracts. The calculation of insurance premia and financial derivative prices is presented assuming the absence of arbitrage opportunities. It is shown that in contrast to financial contracts, there exist infinitely many market prices of risk that lead to the same premium process. Thereafter a link between financial and actuarial prices is established based on the requirement that financial prices should be consistent with actuarial valuation. This connection is investigated in more detail under certain premium calculation principles.

The starting point of the final chapter is the Fourier technique developed in Chapters 1 and 2. It is the aim of this chapter to generalize the analysis to underlying Lévy processes. Expressions for the conditional moments and probabilities based on these processes are derived and their inverse Fourier transforms are obtained in closed form. The representation of conditional moments and probabilities separates the stochastic structure from the deterministic dependence on the underlying Lévy processes.

Acknowledgements

I am deeply indebted to my supervisor, Pierre Mella-Barral, for his constant encouragement and support throughout different stages of my life as a Ph.D. student. I benefited enormously from his sharp and intuitive analysis of ideas that we discussed in long, and sometimes for me exhausting, meetings. I am very thankful for his endeavor to direct my attention to the economic questions that underlie analytical concepts.

During my Ph.D., I have profited continuously from the stimulating research environment at the Financial Markets Group. I am thankful to David Webb for devoting his energy to the development of the group and to Margaret Bray and Bob Nobay for running a fruitful Ph.D. seminar and for their valuable comments. Thanks to Tara Lehane, Pippa Maddicott, and Sarah Vink for always being supportive and caring.

I am indebted to Dieter Sondermann for initiating my research interests with a lecture that I attended at the University of Bonn and for being supportive throughout the years. Many thanks also to Urs Schweizer for organizing the European Doctoral Program at the University of Bonn where I spent the first year of my doctorate and was first exposed to economic theory.

Recalling my time visiting the Economics Department at the European University Institute in Florence I would like to express my gratitude to Søren Johansen for enlightening discussions and stimulating comments. Furthermore I feel grateful to the Department for their hospitality and welcoming atmosphere.

I would like to thank Tomas Bjørk, Markus Brunnermeier, Lucien Foldes, Rüdiger Frey, Herakles Polemarchakis, and Alexander Stremme for their valuable comments on parts of this dissertation.

In the last years, I was surrounded by a great group of fellow students. Many thanks for numerous discussions, friendship, and help from Heski Bar-Isaac, Eric Ben-Hamou, Andrea Caggese, Paola Conconi, Vicente Cuñat, Andrew Ellul, Simon Emrich, Sylvain Friederich, Urs Hägler, Philip Kalmus, Niko Matouschek, Ceema Namazie, Richard Payne, Paolo Ramezzana, and Daniel Sturm.

In particular, I would like to thank Sönje Reiche with whom I shared all different facets of my life in London and whose life path exhibits a surreal similar pattern to mine. She has been both my most severe critic and best friend.

My greatest debt is to my family. Their support is far from being describable in words or measurable in any scale. I am particularly grateful for initiating my interests in various subjects.

Finally, I would like to express my gratitude and love to Ann who through her continuous support and warm-heartedness has emerged as an important pillar of my life.

Parts of this work have been presented at EFA 99 (Helsinki), EEA 99 (Santiago de Compostela), Symposium on Finance, Banking, and Insurance 99 (Karlsruhe), ASSET 2000 (Lisbon), Workshop on Mathematical Finance 2000 (Berlin), Wharton School, University of Alberta, CEMFI, Carlos III, Universitat Pompeu Fabra, European University Institute, University of Munich, University of Heidelberg, Stockholm School of Economics, Heriot-Watt University Edinburgh, and Imperial College London. I thank seminar participants for their valuable comments and relevant feedback.

Contents

Chapter 0

Introduction

In the past decade the convergence of capital and insurance markets has emerged as one of the most important phenomena in risk management. This overlap manifests itself in the growing number of products coming onto one of the markets and containing a component of the other market. Insurers are developing policies that depend on the performance of financial indicators such as indices and interest rates. Financial contracts are introduced that encompass insurance risk such as natural catastrophes. Both approaches aim to create new investment opportunities and hedging instruments for global risk management. In order to tailor these new products optimally to the needs of market participants, both financial as well as actuarial risk valuation must be reconsidered and further developed, explicitly taking into account their similarities and differences.

This dissertation consists of four chapters that investigate the valuation of financial instruments that are based on insurance related risk, the connection between actuarial and financial valuation in an integrated market, and the class of stochastic processes that comprises insurance processes used in risk theory. The following sections summarize the idea for each chapter, the techniques and methods applied, and the main results.

0.1 Pricing Catastrophe Insurance Derivatives

This chapter focuses on the financial valuation of catastrophe insurance derivatives that have been introduced at the Chicago Board of Trade in 1992.

Catastrophe insurance derivatives are financial securities whose payments depend on the value of an underlying index that reflects estimated insured property losses due to natural catastrophes. They are traded as European call, put, and spread options with the aim of providing an alternative to reinsurance contracts and attracting additional capital sources from financial investors.

The underlying loss index is modeled as an compound Poisson process, a stochastic jump process that is used in classical risk theory to model aggregate losses.

First, solely the assumption of absence of arbitrage opportunities is imposed and the set of consistent derivative prices is parameterized by the market prices of frequency and jump size risk. Fourier analysis is used to deduce a representation of financial derivative prices that separates the underlying stochastic structure from the contract's payoff. The first component is captured by the characteristic function of the underlying loss index, the latter by the inverse Fourier transform of the payoff structure. This representation makes it possible to derive the inverse Fourier transform of derivative prices in closed form.

Second, a representative agent is introduced whose preferences are represented by a utility function. The investor's preferences determine the market prices of frequency and jump size risk and consequently the unique price process of the catastrophe insurance derivative.

The analysis and results developed in this chapter suggest to calibrate the model to market data. Since we derived the inverse Fourier transform of derivative prices in closed form, it is suggested that there is much to be gained by using Fast Fourier Transform as an efficient algorithm for the calculation of prices.

0.2 Asset Valuation in Risk Theory

This chapter builds on the previous one and investigates the interplay of insurance risk theory and financial valuation through the securitization of insurance related risk.

The Cramér-Lundberg model is a classic model in insurance risk theory, used to model the surplus of an insurance company with income from premia arriving at a constant rate and outflow in the form of claims. The exponential of this surplus process is used to describe the dynamics of a basic financial security that could reflect a price index of a portfolio of insurance stocks. In addition, the basic security serves as an underlying process for derivative securities.

In this framework, the Fourier technique developed in Chapter 1 is applied to describe the dynamics of securities' price processes that do not allow for arbitrage strategies. Similar to the result in Chapter 1, the martingale property of price processes alongside the Fourier analysis techniques leads to a representation of prices that separates the underlying uncertainty in the market from the specification of the financial contract. The set of no-arbitrage derivative prices is parameterized by the market price of jump size risk and their inverse Fourier transform is derived in closed form. The valuation of spread options is examined in more detail as their payoff structure reflects the specifications of stop loss reinsurance treaties. As well as the representation of financial prices of these contracts, the conditional probability of the surplus being between two boundaries is analyzed and its inverse Fourier transform derived in closed form.

Next, a representative agent is considered and the unique price process is determined that is consistent with the investor's preferences. In addition, the Esscher transform that has been introduced as a premium calculation principle in actuarial science is used as an alternative price process selection criterion. Financial prices that are robust with respect to this criterion are derived and the analogy to the representative agent approach is shown.

Finally, the analysis is generalized to allow the underlying surplus process to be perturbed by an independent diffusion process that might reflect additional market noise. Similar analysis applies and the corresponding results are derived.

0.3 Financial and Actuarial Valuation in an Integrated Market

Chapters 1 and 2 discuss a market that consists of financial securities based on insurance related risk. In these chapters, the interplay between insurance and capital markets is established through financial valuation techniques applied to insurance models of risk processes.

In this chapter, however, an integrated market is considered in which both insurance and financial contracts are available for trading. The consistency and the consequent relationship between insurance premia and financial prices is examined.

Analogous to Chapter 1, the underlying uncertainty in the economy is related to insurance risk. Therefore, a compound Poisson process is used to describe the dynamics of the fundamental process since it exhibits characteristics similar to aggregate claim processes, namely random loss sizes at random points in time. Both an insurance contract and a financial derivative, each of which is based on the same fundamental risk, are traded. The insurance contract specifies a premium process for which the remaining risk can be sold off, whereas the payoff of the financial contract depends on the realization of the underlying process at maturity of the contract.

Premium calculation principles can be understood as insurance prices that arise in a no-arbitrage framework. Financial prices are assumed not only to exclude arbitrage strategies but additionally to be consistent with the actuarial valuation of the same underlying risk. The Fourier techniques developed in Chapters 1 and 2 are used to derive a representation of financial price processes that are robust to these two selection criteria. The concept of actuarially consistency of financial prices provides the analytical ground for linking financial prices with insurance premia. It is shown that this link is inherent in the characteristic function of the underlying risk process.

Finally, certain commonly used premium calculation principles and their corresponding financial prices are investigated in more detail, taking into account actuarial consistency.

0.4 Conditional Moments Based on Lévy Processes

This chapter generalizes the analytical techniques developed in Chapters 1 and 2 to investigate conditional moments of random variables that are based on stochastic processes with stationary and independent increments.

The stochastic processes that are used in the previous chapters to model the dynamics of insurance related risk belong to a more general class - stochastic processes with stationary and independent increments, the so-called Lévy processes. Stationarity and independence of increments are the only properties used to derive the representation of financial prices in previous chapters. Therefore, the same Fourier technique is applicable to a set-up in which the underlying process on which random variables depend is a general Lévy process.

Furthermore, financial prices in a no-arbitrage market can be expressed as the conditional expected payoff under an appropriate equivalent probability measure. It is shown that the same techniques can be applied to conditional moments of arbitrary order.

Analogously, a representation of conditional moments based on Lévy processes is derived that consists of two components. One factor captures the complete stochastic structure in the form of the characteristic function of the underlying Lévy process. The other factor contains solely the dependence of the random variable on the underlying Lévy process in form of the inverse Fourier transform. This representation leads to a closed form expression for the inverse Fourier transform of the conditional moments. In addition, an expression for the probability of the Lévy process taking values between certain boundaries is derived in a similar manner.

Chapter 1

Pricing Catastrophe Insurance Derivatives

1.1 Introduction

In recent years there has been an ongoing economic and political debate on whether financial markets should be used to insure risk that has been traditionally hedged through other channels. Famous examples include the discussion about the change to a funded pension scheme, equity-linked life insurance contracts, and insurance derivatives. This need for an alternative way of insurance resulted in a growing number of insurance products coming onto the market and containing a financial component of some sort. In order to tailor these new financial products optimally to the needs of the different markets, both finance experts as well as actuaries will have to get to know the other expert's field better. This overlap suggests that combining the methods used in both areas, insurance mathematics and mathematical finance should prove indispensable. The objective of this chapter is to model the risk involved in insurance markets by the appropriate class of stochastic processes and to focus on the problem of price determination for catastrophe insurance derivatives that have been introduced at the Chicago Board of Trade in December 1992. These are traded financial securities based on an underlying index that encompasses insurance losses due to natural catastrophes.

Most models that have been proposed in mathematical finance include a continuity assumption on the evolution of prices, i.e. the underlying risk is predictable. In the presence of enough securities, Black and Scholes [12] and Merton [59] have shown how to determine prices of derivatives relying only on the absence of arbitrage opportunities. An arbitrage opportunity is a trading strategy that with probability one yields a positive return without any initial investment.

However, when being exposed to insurance related risk - e.g. earthquake, windstorm, or flood - one necessarily has to include unpredictable movements of the underlying index reflecting the risk involved. This leads in a natural way to the class of stochastic processes including jumps at random time points. We therefore model the dynamics of the index that underlies catastrophe insurance derivatives as a compound Poisson process, a stochastic process that is used in risk theory to model aggregate losses.

In the context of catastrophic risk, the valuation of such derivatives proves to be more problematic compared to the Black and Scholes setup [12] for two reasons. First, valuation based on arbitrage arguments make sense only when all underlying assets are explicitly defined. However, the current generation of catastrophe derivatives is based on underlying loss indices that are not traded on the market. Second, stochastic jump sizes of the underlying index 'create' an incomplete market. It is thus not possible to perfectly duplicate the movement and consequent payoffs of insurance derivatives by continuously trading in other securities. Both problems are inherently related to the fact that price processes of insurance derivatives cannot be uniquely determined solely on the basis of excluding arbitrage opportunities.

Cummins and Geman [21] were the first to investigate the valuation of catastrophe futures and derivative on futures. These securities were the first generation of traded contracts at the Chicago Board of Trade. The authors model the increments of the underlying index as a geometric Brownian motion plus a jump process that is assumed to be a Poisson process with fixed loss sizes. Because of the nonrandomness in jump sizes the model can be nested into the Black and Scholes framework [12]. Since the futures' price, the basis for catastrophe derivatives, is traded on the market the market is complete and unique pricing is possible solely based on assuming absence of arbitrage opportunities. While the completeness of the market is convenient, the assumption of constant loss sizes is questionable in the context of insurance related risk. Furthermore, futures and derivatives on futures did not generate enough interest and ceased to be traded in 1995. They were replaced by options and spread options that are based on an underlying loss index that is not traded itself. The market is thus incomplete even with constant jump sizes of the underlying index.

Geman and Yor [37] examine the valuation of options that are based on the non-traded underlying loss index. In the paper, the underlying index is directly modeled as a geometric Brownian motion plus a Poisson process with constant jump sizes. The authors base their arbitrage arguments on the existence of a vast class of layers of reinsurance with different attachment points to guarantee completeness of the insurance derivative market. An Asian options approach is used to obtain semi-analytical solutions for call option prices in form of their Laplace transform. In addition to the assumption of constant jump sizes, the existence of a liquid catastrophe reinsurance market is questionable since coverage and premium rates in catastrophe reinsurance are individually negotiated and depend on the insurance company's past loss experience. Furthermore, the observed loss index exhibits no change in value between catastrophic events except from adjustments in loss amounts. These rare and small adjustments of the loss index do not justify dynamics with infinite variation that are inherent to a Brownian motion.

Aase [1] and [2] takes a different, more realistic modeling approach and uses a compound Poisson process with random jump sizes to describe the dynamics of the underlying index. The author investigates the valuation of catastrophe futures and derivatives on futures that ceased to be traded in 1995. Since the underlying futures' price is traded on the market the incompleteness in his setup does not arise out of the fact that the underlying index is not traded - as in Geman and Yor [37] - but from the randomness in jump sizes. The author specifies the preferences of market participants by a utility function and determines unique price processes within the framework of partial equilibrium theory under uncertainty. Closed pricing formulae are derived under the assumption of negative exponential utility function and Gamma distributed loss sizes.

In this chapter, we fill the gap in the literature by investigating the valuation of current catastrophe insurance derivatives based on a non-traded underlying loss index that is modeled as a compound Poisson process with stochastic jump sizes. We therefore examine the actually traded derivatives - as in Geman and Yor [37] - while using a model that is more accurate in this actuarial context - as in Aase [1] and [2].

The derivation of prices purely based on no-arbitrage arguments is very attractive as prices arise independent of investors' preferences. The disadvantage, however, is the indeterminacy of price processes since the insurance derivative market is incomplete.

In this chapter, we tackle this problem in the following way. Without imposing any preferences, except that agents prefer more to less, we apply Fourier analysis to derive a representation of the class of possible price processes solely on the basis of excluding arbitrage strategies. This set of no-arbitrage price processes is parameterized by market prices of frequency and jump size risk. For every fixed pair of market prices of risk, our approach enables us to derive the inverse Fourier transform of price processes in closed form. We allow for a very general class of financial contracts - including the currently traded catastrophe derivatives - and do not impose any assumptions on the distribution of jump sizes. Building upon this characterization, we show that the set of price processes excluding arbitrage opportunities and the set of market prices of frequency and jump size risk are one-to-one connected. In a liquid insurance derivative market, it is therefore possible to obtain the market prices of risk as implied parameters from observed derivative prices.

In the context of a market with a representative agent, market prices of frequency and jump size risk are determined by the preferences of the representative agent. The principle of utility maximization thus determines the unique price process of the insurance derivative.

An additional nice feature of our approach is that the representation of no-arbitrage price processes separates the underlying stochastic structure from the financial contract's specification. The stochastic structure is captured by the characteristic function of the underlying index, the contract's specification by the inverse Fourier transform of payoffs. In a fixed stochastic environment, this separation allows for faster calculation of derivative prices. The characteristic function has to be derived once and, thereafter, the calculation of derivative prices is reduced to the derivation of the inverse Fourier transform of the contract's payoff structure.

The remainder of this chapter is organized as follows: in Section 1.2 we discuss the catastrophe insurance market with emphasis on the current generation of catastrophe insurance options. Section 1.3 presents the model that describes the economic environment, the dynamics of the underlying catastrophe index and the change between equivalent probability measures. In Section 1.4 we investigate the pricing mechanism, first solely based on an arbitrage approach, then by adding a representative agent. Section 1.5 concludes.

1.2 Catastrophe Insurance Derivatives

This section presents the main ideas behind the development of the catastrophe insurance market and describes the structure and specification of existing derivatives related to catastrophic risk.

1.2.1 Alternative Risk Transfer

The experience of major natural catastrophes in the nineties - e.g. Hurricane Andrew in 1992, Northridge California earthquake in 1994, earthquake in Kobe in 1995 - resulted in a widespread concern among insurance and reinsurance companies that there might not be enough allocated capital to meet their underwriting goals. This fear provoked a growing demand for additional capital sources and accelerated interest in using financial markets to spread catastrophic risk.

The standardization and securitization of insurance related risk provides an alternative to reinsurance contracts that traditionally have been purchased to manage catastrophe exposure. Catastrophe reinsurance is a highly customized business, where coverage and rates are individually negotiated. Premium rates vary depending on a specific company's present and past loss exposure, the layers covered, and current market conditions. On the contrary, financial contracts are not negotiated and contract specifications do not vary over time. In addition to the integrity and protection of standardized, exchange-traded instruments, price transparency also attracts investors and capital from outside the insurance industry.

Let us summarize the main attractions for buyers and sellers of catastrophe insurance derivatives:

First, insurance derivatives can be used by insurers and reinsurers to buy standardized protection against catastrophic risk. Alternatively, gaps in existing reinsurance contracts can be filled since financial protection can be provided between a lower desired retention level and the attachment point currently offered. In addition, these derivatives can offer an opportunity to synthetically exchange one layer for another without the need to enter costly negotiations.

Second, securitization of catastrophic risk turns catastrophes into tradeable commodities. Investors thus have the opportunity to invest indirectly in risk that traditionally has been addressed by the insurance industry only. Since catastrophic risk should prove highly uncorrelated to any other financial risk that underlies stock or bond price movements trading in catastrophes provides an additional way to diversify the investors' portfolio.

1.2.2 ISO Futures and Options

The first generation of catastrophe insurance derivatives was developed by the Chicago Board of Trade (CBoT) and trading started in December 1992. Futures and options on futures were launched based on an index that should reflect accumulated claims caused by catastrophes. The index consisted of the ratio of quarterly settled claims to total premium reported by approximately 100 insurance companies to the statistical agent Insurance Service Office (ISO). The CBoT announced the estimated total premium and the list of the reporting companies before the beginning of the trading period. A detailed description of the structure of these contracts can be found in Aase [1] and Meister [58]. Due to the low trading volume in these derivatives trading was given up in 1995.

One major concern was a moral-hazard problem involved in the way the index was constructed: the fact that a reporting company could trade conditional on its past loss information could have served as an incentive to delay reporting in correspondence with the company's insurance portfolio. Even if the insurance company reported promptly and truthfully, the settlement of catastrophe claims might be extensive and the incurred claims might not be included in the final settlement value of the appropriate contract. This problem occurred with the Northridge earthquake which was a late quarter catastrophe of the March 1994 contract. The settlement value was too low and did not entirely represent real accumulated losses of the industry.

Since options based on these futures had more success - especially call option spreads - they were replaced by a new generation of options called PCS options.

1.2.3 PCS Catastrophe Insurance Options

PCS Catastrophe Insurance Options were introduced at the CBoT in September 1995. They are standardized, exchange-traded contracts that are based on catastrophe loss indices provided daily by Property Claim Services (PCS) - a US industry authority which estimates catastrophic property damage since 1949. The PCS indices reflect estimated insured industry losses for catastrophes that occur over a specific period. Only cash options on these indices are available; no physical entity underlies the contracts. They can be traded as calls, puts, or spreads; futures are no longer listed for trading. Most of the trading activity occurs in call spreads, since they essentially work like aggregate excess-of-loss reinsurance agreements or layers of reinsurance that provide limited risk profiles to both the buyer and seller.

By definition, a catastrophe is an event that causes in excess of \$5 million of insured property damage and affects a significant number of policyholders and insurance companies. PCS assigns a serial number to each catastrophe for identification throughout the industry. It also compiles estimates of insured property damage using a combination of procedures, including a general survey of insurers, its National Insurance Risk Profile, and, where appropriate, its own on-the-ground survey. PCS estimates take into account both the expected dollar loss and the projected number of claims to be filed. If a catastrophe causes more than $250 million according to preliminary estimates, PCS will continue to survey loss information to determine whether its estimate should be adjusted.

PCS Options offer flexibility in geographical diversification, in the amount of aggregate losses to be included, in the choice of the loss period and to a certain extent in the choice of the contracts' expiration date. Let us describe the contracts' specifications in more detail:

PCS provides nine geographically diverse loss indices to the CBoT: a National index; five regional indices covering Eastern, Northeastern, Southeastern, Midwestern, and Western exposures; and three state indices covering catastrophe-prone Florida, Texas, and California.

The CBoT lists PCS Options both as "small cap" contracts, which limit the amount of aggregate industry losses that can be included under the contract to $20 billion, and as "large cap" contracts, which track losses from $20 billion to $50 billion.

Furthermore, most PCS Options track calendar quarters to allow insurers and reinsurers to focus financial coverage towards those times when they might be particularly exposed to catastrophe risk. A catastrophic event must occur during that loss period in order for resulting losses to be included in a particular index. During the loss period, PCS provides loss estimates as catastrophes occur. The PCS indices that best cover hurricane risk - Eastern, Southeastern, Florida, and Texas - all track quarterly loss periods, as do the National, Northeastern, and Midwestern indices. The California and Western indices track annual loss periods, since the catastrophe most common in that region - earthquake - is not seasonal. Insurers and reinsurers that want broader protection can buy PCS Options in one-year strips, covering an entire year of risk in one transaction.

After the contract specific loss period, PCS Option users can choose either a six-month or a twelve-month development period. The development period is the time during which PCS estimates and reestimates for catastrophes that occurred during the loss period and continue to affect the PCS indices. The contract expires at the end of the chosen development period and settles in cash, even though PCS loss estimates may continue to change. The exercise style of PCS Options is European. The following table clarifies the time structure of the insurance contracts:

Contract Month	**Loss Period**	**Development Period**		**Settlement Date**	
		Six Month	Twelve Month	Six Month	Twelve Month
March	Jan-Mar	Apr 1-Sep 30	Apr 1-Mar 31	Sep 30	Mar 31
June	Apr-Jun	Jul 1-Dec 31	Jul 1-Jun 30	Dec 31	Jun 30
September	Jul-Sep	Oct 1-Mar 31	Oct 1-Sep 30	Mar 31	Sep 30
December	Oct-Dec	Jan 1-Jun 30	Jan 1-Dec 31	Jun 30	Dec 31
Annual	Jan-Dec	Jan 1-Jun 30	Jan 1-Dec 31	Jun 30	Dec 31

Each PCS loss index represents the sum of then-current PCS estimates for insured catastrophic losses in the area and loss period divided by $100 million. The indices are quoted in points and tenths of a point and each index point equals $200 cash value as indicated in the chart below:

PCS Loss Index Value	**PCS Options Cash Equivalent**	**Industry Loss Equivalent**
0.1	$20	$10 million
1.0	$200	$100 million
50.0	$10,000	$5 billion
200.0	$40,000	$20 billion (small cap limit)
250.0	$50,000	$25 billion
350.0	$70,000	$35 billion
500.0	$100,000	$50 billion (large cap limit)

Strike values are listed in integral multiples of five points. For small cap contracts, strike values range from 5 to 195. For large cap contracts, strike values range from 200 to 495.

In the next section we introduce the stochastic fundamentals, the model for the dynamics of the underlying loss index, and an investigation of changing equivalent probability measures.

1.3 The Economic Environment

Uncertainty in the insurance market is modeled by a complete probability space $(\Omega, \mathcal{F}, P)$ on which all following random variables will be defined. Ω is the set of all states of the world ω and $\mathcal{F}$ is the σ-algebra of possible events on Ω. The economy has finite horizon $T < \infty$ where T represents the maturity of the insurance derivative.

Let the stochastic process $X = (X_t)_{0 \le t \le T}$ represent the PCS loss index, i.e. we assume that X_t reflects aggregated insured industry losses resulting from catastrophes up to and including time t. Let us suppose that all investors in this market observe the past evolution of the loss index including the current value. Therefore, the flow of information is given by the augmented filtration $(\mathcal{F}_t)_{0 \le t \le T}$ of σ-algebras generated by the process X with $\mathcal{F}_T = \mathcal{F}$. Let us assume that the usual hypotheses hold, that is the filtration is right-continuous and $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

The market consists of one risky European insurance derivative with payoff depending on the value X_T of the loss index at maturity T. We also assume the existence of a risk-free asset with price process $B = (B_t)_{0 \le t \le T}$, i.e.

$$dB_t = r_t B_t dt, \tag{1.1}$$

where r is the deterministic short rate of interest. Without loss of generality, we express the price process of the insurance derivative in discounted terms, i.e. we set $r \equiv 0$.

1.3.1 Modeling the PCS Loss Index

The classical approach of modeling the dynamics of financial stock prices assumes that news in the market causes an infinitesimal change in corresponding prices. Black and Scholes [12], for example, modeled the stock price as a geometric Brownian motion, i.e. as a continuous stochastic process. In actuarial risk models, however, claims cause sudden movements in the affected processes. Particularly in the context of catastrophes, losses cannot be considered as being infinitesimal. Hence we assume that catastrophic events cause unpredictable jumps in the specific PCS index at random time points. Therefore, we model the underlying index X of a PCS contract by a stochastic process of the form

$$X_t = \sum_{\{k|T_k \leq t\}} Y_k = \sum_{k=1}^{N_t} Y_k, \tag{1.2}$$

where T_k is the random time point of occurrence of the kth catastrophe that causes a jump of size Y_k in the underlying index and N_t is a random variable counting catastrophic events up to time t. We shall assume that $X = (X_t)_{0 \leq t \leq T}$ is a compound Poisson process, i.e. the counting process $N = (N_t)_{0 \leq t \leq T}$ is a Poisson process with intensity λ, and $Y_1, Y_2, ...$ are nonnegative, independent and identically distributed random variables, all independent of the counting process N. Let G be the distribution function of Y_k with support $[0, \infty)$. The parameters $(\lambda, dG(y))$ are called the characteristics of the process X.

Under our assumption, the index X of a PCS contract thus is a time-homogeneous process with independent increments. Actuarial studies (see Levi and Partrat [55]) have shown that these assumptions are reasonable in the context of losses arising from windstorm, hail and flood. Earthquakes are described as events arising from a superposition of events caused by several independent sources. The PCS index therefore approximates a compound Poisson process. The assumption on time-homogeneity is questionable for the case of hurricanes which occur seasonally. However, the indices of regions, that are exposed to hurricane risk, all track quarterly loss periods to account for seasonal effects.

Remark 1 *Filtrations that are generated by compound Poisson processes and completed by P-null sets of $\mathcal{F}$ satisfy the usual hypotheses, i.e. they are right-continuous* *(see Protter [67] p. 22).*

1.3.2 Change of Equivalent Measures

In this section we examine the change between equivalent probability measures and the change in the characteristics that it induces on compound Poisson processes. We restrict the set of equivalent probability measures to the subset of probability measures under which the structure of the underlying process X is preserved, i.e. under which the index remains a compound Poisson process. This subset has been characterized by Delbaen and Haezendonck [25] as follows:

Let P denote the physical probability measure in the insurance market under which the compound Poisson process X has characteristics $(\lambda, dG(y))$. A probability measure Q is equivalent to P, and X is a compound Poisson process under Q if and only if there exists a nonnegative constant κ and a nonnegative, measurable function $v : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_0^\infty v(y)\, dG(y) = 1,$$

such that the associated density process $\xi_t = \mathbf{E}^P[\xi_T \mid \mathcal{F}_t]$ of the Radon-Nikodym derivative $\xi_T = \frac{dQ}{dP}$ is given by

$$\begin{aligned} \xi_t &= \left(\prod_{k=1}^{N_t} \kappa v(Y_k)\right) \cdot \exp\left(\int_0^t \int_0^\infty (1 - \kappa v(y))\, \lambda dG(y)\, ds\right) \\ &= \exp\left(\sum_{k=1}^{N_t} \ln(\kappa v(Y_k)) + \lambda(1-\kappa)\, t\right), \end{aligned} \tag{1.3}$$

for any $0 \le t \le T$. $\mathbf{E}^P[\cdot]$ denotes the expectation operator under the probability measure P.

Under the new measure Q the process X has characteristics $\left(\lambda^Q, dG^Q(y)\right) = (\lambda\kappa, v(y)\, dG(y))$.

Let us denote the measure Q corresponding to the constant κ and the function $v(\cdot)$ by $P^{\kappa,v}$ and the corresponding distribution function G^Q by G^v. Hence, for all $A \in \mathcal{B}_+$

$$G^v(A) = \int_A v(y)\, dG(y), \tag{1.4}$$

and

$$\mathbf{E}^{P^{\kappa,v}}[N_1] = \lambda\kappa, \tag{1.5}$$

where $\mathcal{B}_+$ represents the Borel σ-algebra on $\mathbb{R}_+$ and $\mathbf{E}^{P^{\kappa,v}}[\cdot]$ denotes the expectation operator under the measure $P^{\kappa,v}$.

Remark 2 *In an economic sense, κ can be interpreted as a premium of frequency risk and $v(\cdot)$ as a premium of claim size risk.*

Remark 3 *Meister [58] generalized the result of Delbaen and Haezendonck [25] to mixed Poisson and doubly stochastic Poisson processes.*

In the following Lemma we show that the correspondence between the set of parameters κ, $v(\cdot)$ and the set of equivalent measures $P^{\kappa,v}$ is one-to-one.

Lemma 1 *Define $K \times V \equiv \left\{(\kappa, v(\cdot)) \in \mathbb{R}_+ \times \mathbf{L}^1(\mathbb{R}_+, G) \,|\mathbf{E}^P[v(Y_1)] = 1\right\}$. Then the mapping*

$$(\kappa, v(\cdot)) \in K \times V \rightarrow P^{\kappa,v}$$

is injective.

Proof. Let $(\kappa, v(\cdot))$ and $(\kappa', v'(\cdot))$ belong to $K \times V$ with $P^{\kappa,v} = P^{\kappa',v'}$. Then $\mathbf{E}^{P^{\kappa,v}}[N_1] = \mathbf{E}^{P^{\kappa',v'}}[N_1]$ and thus $\kappa = \kappa'$. Furthermore, for all $A \in \mathcal{B}_+$

$$\int_A v(y)\, dG(y) = \int_A v'(y)\, dG(y),$$

and so $v \equiv v'$ G-a.s. ■

1.4 Pricing of Insurance Derivatives

The aim of this section is to investigate the price determination of insurance derivatives that are based on PCS indices under the assumption of the previous section that the underlying index is a compound Poisson process. First, we review the equivalence between the existence of equivalent martingale measures and the absence of arbitrage opportunities in the market. Then by solely imposing absence of arbitrage possibilities we derive the inverse Fourier transform of price processes in closed form. Thereafter, we will be more restrictive and assume the existence of a representative investor in the market whose preferences determine uniquely the price of derivatives.

1.4.1 The Fundamental Theorem of Asset Pricing

The equivalence between the existence of equivalent martingale measures and the absence of arbitrage opportunities in the market plays a central role in mathematical finance. An equivalent martingale measure is a probability measure that is equivalent to the "reference" measure P and under which discounted price processes are martingales. It is important to be aware of the specifications of the model in which this equivalence is used since arbitrage has to be differently defined to guarantee the existence of equivalent martingale measures.

Harrison and Kreps [45], and Harrison and Pliska [46] were the first to establish an equivalence result in a model based on a finite state space Ω. In a discrete infinite or continuous world, the absence of arbitrage is not a sufficient condition for the existence of an equivalent martingale measure. Other definitions of arbitrage opportunity or restricting conditions on the dynamics of price processes have been derived to guarantee the existence of martingale measures. Fritelli and Lakner [35] give a definition of arbitrage, called "free lunch", under which the equivalence result is derived with high level of generality. The only mathematical condition that is imposed on asset prices is that they are adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which is a natural requirement.

As asset price processes are not a priori assumed to be semimartingales stochastic integrals that reflect achievable gains from continuous trading strategies are not well-defined. To circumvent this problem, the set of trading strategies is restricted to permit trading at either deterministic times or stopping times. The "no free lunch" condition then postulates that the set of achievable gains contains no positive random variables. In a continuous time setting closure of the set of gains has to be considered which essentially depends on the topology on this set. Under a topology that makes use of certain dualities, Fritelli and Lakner [35] prove that there is "no free lunch" with trading strategies at deterministic times if and only if there exists an equivalent martingale measure. Furthermore, if every underlying process is right-continuous, then this result holds additionally for trading strategies at stopping times.

Henceforth, we assume "no free lunch" in the market as outlined above, so that the existence of an equivalent martingale measure is guaranteed.

1.4.2 Representation of No-Arbitrage Prices

In this subsection we deduce a representation of prices solely on the basis of excluding arbitrage opportunities as defined above. We will present two possible methods of deriving prices:

- the first relies on risk neutral valuation and is simply a calculation of the expected payoff under the appropriate probability measure;
- the second method makes use of the infinitesimal generator of the underlying process X to derive prices as solutions of the appropriate integro-differential equation that represents the corresponding pricing equation.

In the catastrophe insurance market, the underlying index X is not traded. Thus it is not possible to construct a hedging portfolio based on X and hence the price of a derivative cannot be uniquely determined by the assumption of "no free lunch" in the market. However, assuming "no free lunch" guarantees the existence of an equivalent probability measure $Q \sim P$ under which discounted price processes of insurance derivatives are martingales. In addition, our model exhibits a second source of incompleteness arising from stochastic jump sizes of the underlying PCS index.

Let us suppose that we choose and fix an arbitrary equivalent martingale measure Q such that the index process $X = (X_t)_{0 \leq t \leq T}$ remains a compound Poisson process after the change to the probability measure Q with characteristics $\left(\lambda^Q, dG^Q(y)\right)$. The set of equivalent probability measures that preserve the structure of X has been characterized by Delbaen and Haezendonck [25] and presented in Section 1.3.2, p. 23 of this chapter.

First Method (Risk neutral valuation)

Assuming "no free lunch" in the market, a consistent price process of an insurance derivatives that pays out $\phi(X_T)$ at maturity can be expressed as

$$\pi_t^Q = \mathbf{E}^Q \left[\exp\left(-\int_t^T r_s ds \right) \phi(X_T) \,|\mathcal{F}_t \right]. \tag{1.6}$$

π_t^Q is of the form $f^Q(X_t, t)$ since we have assumed that r is deterministic ($r \equiv 0$ without loss of generality), $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by X, and X is a Markov process under Q. The stochastic process $\left(f^Q(X_t, t)\right)_{0 \leq t \leq T}$ reflects the consistent price process under the probability measure Q with payoff $f^Q(X_T, T) = \phi(X_T)$ at maturity T.

Let us assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{C} \text{ measurable} \mid \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty \right\}$ for some $k \in \mathbb{R}$. This assumption is satisfied by all catastrophe insurance derivatives that are traded at the CBoT. Notice that the payoff of all call options is capped at either \$20 billion or \$50 billion. We will now make use of Fourier analysis to calculate the expected payoff in (1.6).

The Fourier transformation is a one-to-one mapping of $\mathbf{L}^2(\mathbb{R})$ onto itself. In other words, for every $g \in \mathbf{L}^2(\mathbb{R})$ there corresponds one and only one $f \in \mathbf{L}^2(\mathbb{R})$ such that the Fourier transform of f is the function g, that is

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} g(x)\, dx \tag{1.7}$$

is the inverse Fourier transform of g.

Applying the Fourier transform, and thereafter the inverse Fourier transform, to the function $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ we deduce

$$\phi(x) - k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iux} e^{-iuz} (\phi(z) - k)\, dz du. \tag{1.8}$$

With respect to (1.6) we get

$$\begin{aligned} \pi_t^Q &= f^Q(X_t, t) = \mathbf{E}^Q [\phi(X_T) \,|\mathcal{F}_t] \\ &= \mathbf{E}^Q [\phi(X_T) - k \,|\mathcal{F}_t] + k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\mathbf{E}^Q\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iuX_T}e^{-iuz}\left(\phi\left(z\right)-k\right)dzdu\left|\mathcal{F}_t\right.\right]+k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{E}^Q\left[e^{iuX_T}\left|\mathcal{F}_t\right.\right]e^{-iuz}\left(\phi\left(z\right)-k\right)dzdu+k \\
&= \int_{-\infty}^{\infty}\mathbf{E}^Q\left[e^{iuX_T}\left|\mathcal{F}_t\right.\right]\check{\varphi}\left(u\right)du+k,
\end{aligned}
$$

where we applied Fubini's theorem and $\check{\varphi}\left(\cdot\right)$ denotes the inverse Fourier transform of $\phi\left(\cdot\right)-k$, i.e.

$$
\check{\varphi}\left(u\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iuz}\left(\phi\left(z\right)-k\right)dz. \tag{1.9}
$$

Since a compound Poisson process is a Markov process with stationary and independent increments, we have

$$
\begin{aligned}
\mathbf{E}^Q\left[e^{iuX_T}\left|\mathcal{F}_t\right.\right] &= e^{iuX_t}\mathbf{E}^Q\left[e^{iu\left(X_T-X_t\right)}\left|X_t\right.\right] \\
&= e^{iuX_t}\mathbf{E}^Q\left[e^{iuX_{T-t}}\left|X_t\right.\right] \\
&= e^{iuX_t}\mathbf{E}^Q\left[e^{iuX_{T-t}}\right].
\end{aligned}
$$

$\mathbf{E}^Q\left[e^{iuX_{T-t}}\right]$ is the characteristic function of the random variable X_{T-t} under the probability measure Q and given by

$$
\chi_{T-t}^{Q}\left(u\right)=\exp\left(\lambda^Q\left(\int_0^{\infty}e^{iuy}dG^Q\left(y\right)-1\right)\left(T-t\right)\right) \tag{1.10}
$$

(see for example Karlin and Taylor [52] p.428).

Hence, the price at time t of the catastrophe insurance derivative is given by

$$
\begin{aligned}
f^Q\left(X_t,t\right) &= \int_{-\infty}^{\infty}e^{iuX_t}\chi_{T-t}^{Q}\left(u\right)\check{\varphi}\left(u\right)du+k \qquad (1.11)\\
&= \int_{-\infty}^{\infty}e^{iuX_t}\exp\left(\lambda^Q\left(\mathbf{E}^Q\left[e^{iuY_1}\right]-1\right)\left(T-t\right)\right)\check{\varphi}\left(u\right)du+k.
\end{aligned}
$$

The inverse Fourier transform can be explicitly calculated for the catastrophe derivatives that are traded at the CBoT, i.e. for spreads, call and put options.

This representation of no-arbitrage price processes enables us to derive the inverse Fourier transform of the price process in closed form. For a given value of the loss index $X_t=x$, we have

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iux}\left(f^Q\left(x,t\right)-k\right)dx=\chi_{T-t}^{Q}\left(u\right)\cdot\check{\varphi}\left(u\right). \tag{1.12}
$$

Our result can be summarized as follows:

Proposition 2 *Let X be a compound Poisson process with characteristics $\left(\lambda^Q, dG^Q(y)\right)$ under the probability measure Q, let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$, and let $\left(f^Q(X_t, t)\right)_{0 \le t \le T}$ be a stochastic process defined through*

$$f^Q(X_t, t) = \mathbf{E}^Q\left[\phi(X_T) \,|\mathcal{F}_t\right].$$

Then the function $f^Q : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ defining the process $\left(f^Q(X_t, t)\right)_{0 \le t \le T}$ can be represented by

$$f^Q(x, t) = \int_{-\infty}^{\infty} e^{iux} \chi^Q_{T-t}(u)\, \check{\varphi}(u)\, du + k,$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$ and $\chi^Q_{X_{T-t}}(\cdot)$ is the characteristic function of X_{T-t} under the probability measure Q, i.e.

$$\chi^Q_{T-t}(u) = \exp\left(\lambda^Q \left(\int_0^{\infty} e^{iuy} dG^Q(y) - 1\right)(T-t)\right).$$

Therefore, the inverse Fourier transform of $f^Q(\cdot, t) - k$ is given by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(f^Q(x, t) - k\right) dx = \chi^Q_{T-t}(u) \cdot \check{\varphi}(u).$$

Remark 4 *It is interesting to observe that the ratio*

$$\frac{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(\mathbf{E}^Q\left[\phi(X_T) \,|X_t = x\right] - k\right) dx}{\mathbf{E}^Q\left[e^{iuX_{T-t}}\right]} = \check{\varphi}(u) \tag{1.13}$$

does not depend on the probability measure that we choose. Hence, for any two equivalent probability measures Q_1 and Q_2 we have

$$\frac{\int_{-\infty}^{\infty} e^{-iux} \left(\mathbf{E}^{Q_1}\left[\phi(X_T) \,|X_t = x\right] - k\right) dx}{\int_{-\infty}^{\infty} e^{-iux} \left(\mathbf{E}^{Q_2}\left[\phi(X_T) \,|X_t = x\right] - k\right) dx} = \frac{\mathbf{E}^{Q_1}\left[e^{iuX_{T-t}}\right]}{\mathbf{E}^{Q_2}\left[e^{iuX_{T-t}}\right]}. \tag{1.14}$$

One question we would like to answer is whether different equivalent probability measures will lead to different prices for a given payoff $\phi(X_T)$ at maturity. To be consistent with the notation used in Section 1.3.2, p. 23, let us characterize the equivalent probability measure Q by the parameters $(\kappa, v(\cdot))$ that reflect the change in the local characteristics of the compound Poisson process X. Recall that the local characteristics of the process X under the probability measure $Q = P^{\kappa,v} \sim P$ are given by $\lambda^Q = \lambda\kappa$ and $dG^Q(y) = v(y)\, dG(y)$. Let us denote the price process that corresponds to the probability measure $P^{\kappa,v}$ by $\left(f^{\kappa,v}\left(X_t,t\right)\right)_{0\le t\le T}$, i.e. $f^{\kappa,v}$ is given by

$$f^{\kappa,v}\left(x,t\right)=\int_{-\infty}^{\infty}e^{\left(\lambda\kappa\left(\int_0^{\infty}e^{iuy}v(y)dG(y)-1\right)(T-t)+iu(x-z)\right)}\check{\varphi}\left(u\right)du+k. \tag{1.15}$$

Lemma 3 *Assume that the payoff function ϕ is non-constant. Then the mapping*

$$\left(\kappa,v\left(\cdot\right)\right)\in K\times V\rightarrow f^{\kappa,v}\in \mathbf{C}^{0,1}\left(\mathbb{R}\times\left[0,T\right]\right)$$

is injective where $f^{\kappa,v}$ is given by the formula (1.15) *and*
$K\times V\equiv\left\{\left(\kappa,v\left(\cdot\right)\right)\in\mathbb{R}_{+}\times\mathbf{L}^1\left(\mathbb{R}_{+},G\right)|\mathbf{E}^P\left[v\left(Y_1\right)\right]=1\right\}$.

Proof. Assume that $f^{\kappa,v}\left(x,t\right)=f^{\kappa',v'}\left(x,t\right)$ for all $x\ge0$ and $0\le t\le T$ for some $\left(\kappa,v\left(\cdot\right)\right),\left(\kappa',v'\left(\cdot\right)\right)\in K\times V$. From the formula for $f^{\kappa,v}$ and $f^{\kappa',v'}$ we deduce that for all x and t

$$\begin{aligned}0&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{iu(x-z)}\left(\phi\left(z\right)-k\right)\\&\quad\times\left(e^{\lambda\kappa\mathbf{E}^P\left[e^{iuY_1}\cdot v(Y_1)-1\right](T-t)}-e^{\lambda\kappa'\mathbf{E}^P\left[e^{iuY_1}\cdot v'(Y_1)-1\right](T-t)}\right)dzdu.\end{aligned}$$

We observe that the double integral is the Fourier transform of

$$\begin{aligned}&\frac{1}{2\pi}\left(e^{\lambda\kappa\mathbf{E}^P\left[e^{iuY_1}\cdot v(Y_1)-1\right](T-t)}-e^{\lambda\kappa'\mathbf{E}^P\left[e^{iuY_1}\cdot v'(Y_1)-1\right](T-t)}\right)\\&\qquad\qquad\times\int_{-\infty}^{\infty}e^{-iuz}\left(\phi\left(z\right)-k\right)dz.\end{aligned}$$

The Fourier transform is a one-to-one mapping of $\mathbf{L}^2\left(\mathbb{R}\right)$ onto itself. Since it is assumed that ϕ is non-constant, for all u and t we have

$$\lambda\kappa\mathbf{E}^P\left[e^{iuY_1}\cdot v\left(Y_1\right)-1\right]\left(T-t\right)=\lambda\kappa'\mathbf{E}^P\left[e^{iuY_1}\cdot v'\left(Y_1\right)-1\right]\left(T-t\right).$$

For $u\rightarrow\infty$ we deduce $\kappa=\kappa'$ and hence

$$\mathbf{E}^P\left[e^{iuY_1}\cdot v\left(Y_1\right)\right]=\mathbf{E}^P\left[e^{iuY_1}\cdot v'\left(Y_1\right)\right],$$

for all u. Again, since the Fourier transform is a one-to-one mapping we can conclude that

$$v\equiv v'.$$

■

This result is important as it shows that the market price of frequency risk κ and jump size risk $v(\cdot)$ can be uniquely obtained as implied parameters from observed derivative prices. However, the result does not carry over to actuarial valuation in a similar "no-arbitrage" context as introduced by Delbaen and Haezendonck [25]. In fact there are many equivalent probability measures that lead to the same insurance premium. We refer to Chapter 3, Section 3.4.1, p. 87 on actuarial valuation in a no-arbitrage insurance market.

Before investigating spreads, call and put options in more depth, we present an alternative method of deriving the pricing formula (1.11) that can be reconciled with the first method presented.

Second Method (Pricing equation)

This method exploits the fact that discounted price processes in the insurance market are martingales under an equivalent martingale measure. To characterize martingales based on the underlying PCS loss index X we make use of the concept of an infinitesimal generator associated with a Markov process. In fact, it is possible to define the infinitesimal generator by the following martingale property (see e.g. Davis [23] for further details):

The infinitesimal generator $\mathcal{A}$ associated with a Markov process $X = (X_t)_{0 \le t \le T}$ is an operator on the set of functions $f : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ in its domain, for which the process $M = (M_t)_{0 \le t \le T}$ with

$$M_t = f(X_t, t) - f(X_0, 0) - \int_0^t \mathcal{A}(f)(X_s, s)\, ds \tag{1.16}$$

is a martingale under Q. Let $\mathcal{D}(\mathcal{A})$ denote the domain of the infinitesimal generator.

The underlying PCS index X is a Markov process as it is a stochastic process with stationary increments that are independent of the past. The infinitesimal generator of X with local characteristics $\left(\lambda^Q, dG^Q(y)\right)$ can be represented as

$$\mathcal{A}\left(f^Q\right)(x,t) = \frac{\partial}{\partial t} f^Q(x,t) + \lambda^Q \cdot \int_0^\infty \left(f^Q(x+y,t) - f^Q(x,t)\right) dG^Q(y), \quad (1.17)$$

for all $f^Q \in \mathcal{D}(\mathcal{A})$ (see Davis [23]).

Dassios and Embrechts [22] proved that if f is a measurable function, and

$$\mathbf{E}^Q\left[\sum_{T_i \leq t} \left|f\left(X_{T_i}, T_i\right) - f\left(X_{T_i -}, T_i\right)\right|\right] < \infty, \quad (1.18)$$

for all $0 \leq t \leq T$ then f belongs to the domain of the infinitesimal generator.

Since the discounted price process of an insurance derivative is a martingale under the measure Q, we are interested in characterizing the set of martingales that can be constructed as a function of the underlying index X for a particular contract. In the following Proposition we present a necessary and sufficient condition, in form of an integro-differential equation, for a process $\left(f^Q(X_t,t)\right)_{0\leq t\leq T}$ to be a martingale under Q. This equation can also be derived by using the change of variable formula as described by Barfod and Lando [8].

Proposition 4 *Let X be a compound Poisson process with local characteristics $\left(\lambda^Q, dG^Q(y)\right)$ under the measure Q and let $f^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ belong to the domain of the infinitesimal generator $\mathcal{A}$ of X. Then $\left(f^Q(X_t,t)\right)_{0\leq t\leq T}$ is a martingale under Q if and only if f^Q satisfies the integro-differential equation*

$$\frac{\partial}{\partial t} f^Q(x,t) = \lambda^Q \cdot f^Q(x,t) - \lambda^Q \cdot \int_0^\infty f^Q(x+y,t)\, dG^Q(y), \quad (1.19)$$

for all given values $X_t = x \geq 0$ and $0 \leq t \leq T$.

Proof. Suppose f^Q satisfies the integro-differential equation (1.19), i.e. $\mathcal{A}\left(f^Q\right) \equiv 0$ by (1.17). Therefore, we know from (1.16) that $\left(f^Q(X_t,t)\right)_{0\leq t\leq T}$ is a martingale under Q.

Now suppose that $\left(f^Q(X_t,t)\right)_{0\leq t\leq T}$ is a martingale under Q with mean $f^Q(X_0,0)$. Applying the martingale property to the martingale M in (1.16) we can deduce that the process

$$\left(\int_0^t \mathcal{A}\left(f^Q\right)(X_s,s)\, ds\right)_{0\leq t\leq T}$$

is a zero-mean martingale under Q. Furthermore

$$\begin{aligned}
&\int_0^{\cdot} \mathcal{A}\left(f^Q\right)(X_s,s)\,ds \\
&= \int_0^{\cdot} \left(\frac{\partial}{\partial s} f^Q(X_s,s) + \lambda^Q \int_0^{\infty} \left(f^Q(X_s+y,s) - f^Q(X_s,s)\right) dG^Q(y)\right) ds
\end{aligned}$$

is a continuous process of finite variation. Therefore, it has to be constant (see Revuz and Yor [70] p.120) and equal to zero, i.e.

$$\int_0^t \left(\frac{\partial}{\partial s} f^Q(X_s,s) + \lambda^Q \int_0^{\infty} \left(f^Q(X_s+y,s) - f^Q(X_s,s)\right) dG^Q(y)\right) ds = 0.$$

For a given value $X_s = x$, differentiation with respect to t leads to the integro-differential equation (1.19). ∎

In order to prove the uniqueness of the solution of this integro-differential equation for a given boundary condition it is useful to transform the integro-differential equation (1.19) into an integral equation using variation of constants.

Corollary 5 *f^Q satisfies the integro-differential equation* (1.19) *if and only if*

$$\begin{aligned}
f^Q(x,t) &= e^{-\lambda^Q(T-t)} f^Q(x,T) \\
&\quad + \lambda^Q \cdot \int_t^T \int_0^{\infty} e^{-\lambda^Q(s-t)} f^Q(x+y,s)\,dG^Q(y)\,ds, \qquad (1.20)
\end{aligned}$$

for $0 \leq t \leq T$ and $x \geq 0$.

Proof. Define $h^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ through

$$f^Q(x,t) = e^{\lambda^Q t} \cdot h^Q(x,t).$$

Substitution into the integro-differential equation (1.19) leads to

$$\frac{\partial}{\partial t} h^Q(x,t) = -\lambda^Q \cdot \int_0^{\infty} h^Q(x+y,t)\,dG^Q(y).$$

By integrating we obtain

$$h^Q(x,t) = \lambda^Q \cdot \int_t^T \int_0^{\infty} h^Q(x+y,s)\,dG^Q(y)\,ds + h^Q(x,T).$$

Resubstitution leads to the integral equation (1.20) for f^Q. ∎

In the following Proposition we provide a solution of the integro-differential equation (1.19) and prove uniqueness for an arbitrary but fixed boundary condition. In the context of the insurance market, we thus derive the unique price of an insurance derivative for a fixed martingale measure and payoff structure at maturity. The solution coincides with the pricing formula (1.11) derived through risk neutral valuation.

Proposition 6 *Let $G^Q : \mathbb{R} \to [0,1]$ be a distribution function with support $[0,\infty)$, $\lambda^Q \in \mathbb{R}_+$, and $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$. Then the integro-differential equation*

$$\frac{\partial}{\partial t} f^Q(x,t) = \lambda^Q \cdot f^Q(x,t) - \lambda^Q \cdot \int_0^\infty f^Q(x+y,t)\, dG^Q(y) \tag{1.21}$$

with the boundary condition $f^Q(x,T) = \phi(x)$ has the unique solution

$$f^Q(x,t) = \int_{-\infty}^{\infty} \exp\left(\lambda^Q \left(\int_0^\infty e^{iuy} dG^Q(y) - 1\right)(T-t) + iux\right) \check{\varphi}(u)\, du + k, \tag{1.22}$$

in the space of all measurable functions $f^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ that are differentiable with respect to the second variable. $\check{\varphi}(\cdot)$ denotes the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (\phi(z) - k)\, dz.$$

Proof. First, we prove uniqueness by using the Gronwall inequality that states the following:

Let v be a nonnegative function such that

$$v(t) \leq C + A \cdot \int_0^t v(s)\, ds, \qquad \text{for all } 0 \leq t \leq T, \tag{1.23}$$

for some constants C and A. Then

$$v(t) \leq C \cdot \exp(A \cdot t) \qquad \text{for all } 0 \leq t \leq T. \tag{1.24}$$

Suppose now that $f_1^Q, f_2^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{C}$ are solutions of (1.21) with the same boundary condition ϕ, i.e. $f_1^Q(x,T) = f_2^Q(x,T) = \phi(x)$. Define the function

$h^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ by $h^Q(x,t) = \left| f_1^Q(x,t) - f_2^Q(x,t) \right|$. Then $h^Q \geq 0$ and by the integral representation (1.20) of solutions given in Corollary 5, p. 33 we deduce that

$$\begin{aligned} & h^Q(x,t) \\ = & \left| \lambda^Q \cdot \int_t^T \int_0^\infty e^{-\lambda^Q(s-t)} \left(f_1^Q(x+y,s) - f_2^Q(x+y,s) \right) dG^Q(y)\, ds \right| \\ \leq & \ \lambda^Q \cdot \int_t^T \int_0^\infty e^{-\lambda^Q(s-t)} h^Q(x+y,s)\, dG^Q(y)\, ds. \end{aligned}$$

Let us revert time by defining the function $\bar{h}^Q : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ by $\bar{h}^Q(x,t) = h^Q(x, T-t)$. Hence $\bar{h}^Q \geq 0$ and

$$\begin{aligned} \bar{h}^Q(x,t) & \leq \lambda^Q \cdot \int_{T-t}^T \int_0^\infty e^{-\lambda^Q(s-T+t)} \bar{h}^Q(x+y, T-s)\, dG^Q(y)\, ds \\ & = \lambda^Q \cdot \int_0^t \int_0^\infty e^{-\lambda^Q(t-s)} \bar{h}^Q(x+y,s)\, dG^Q(y)\, ds. \end{aligned}$$

Since G^Q is a distribution function we derive for $0 \leq t \leq T$ and $x \geq 0$

$$\begin{aligned} \bar{h}^Q(x,t) & \leq \sup_{x \geq 0} \lambda^Q \cdot \int_0^t \bar{h}^Q(x,s)\, ds \\ & \leq \lambda^Q \cdot \int_0^t \sup_{x \geq 0} \bar{h}^Q(x,s)\, ds. \end{aligned}$$

As this inequality holds for all $x \geq 0$ it is satisfied for the supremum, i.e.

$$\sup_{x \geq 0} \bar{h}^Q(x,t) \leq \lambda^Q \cdot \int_0^t \sup_{x \geq 0} \bar{h}^Q(x,s)\, ds, \qquad \text{for all } 0 \leq t \leq T.$$

If we define the function v by $v(t) = \sup_{x \geq 0} \bar{h}^Q(x,t)$ we have thus shown that

$$v(t) \leq \lambda^Q \cdot \int_0^t v(s)\, ds, \qquad \text{for all } 0 \leq t \leq T,$$

and therefore condition (1.23) for applying the Gronwall inequality is satisfied for $C = 0$ and $A = \lambda^Q$. From (1.24) we deduce that

$$v(t) = \sup_{x \geq 0} \bar{h}^Q(x,t) \leq 0, \qquad \text{for all } 0 \leq t \leq T.$$

Since $\bar{h}^Q$ is a nonnegative function it follows that $\bar{h}^Q \equiv 0$ and thus $h^Q \equiv 0$. Given the definition of h^Q, uniqueness of the solution is proved.

Existence is proven by the explicit solution given in (1.22). ■

Given that the price can be expressed as an expected value of the real-valued random variable $\phi(X_T)$ (see first method) it follows that the solution (1.22) is a real-valued function which we may confirm as follows:

We observe that the first integral term in the solution

$$g(x,t) = \int_{-\infty}^{\infty} \exp\left(\lambda^Q \left(\int_0^{\infty} e^{iuy} dG^Q(y) - 1\right)(T-t) + iux\right) \breve{\varphi}(u)\, du$$

is the Fourier transform of the function

$$\breve{g}(u,t) = \exp\left(\lambda^Q \left(\int_0^{\infty} e^{iuy} dG^Q(y) - 1\right)(T-t)\right) \cdot \breve{\varphi}(u).$$

Remember that $\breve{\varphi}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}(\phi(z) - k)\, dz$ is the inverse Fourier transform of $\phi(\cdot) - k$.

In the situation in which the boundary function ϕ is real-valued, we know that $\breve{\varphi}(-u) = \overline{\breve{\varphi}(u)}$. Therefore

$$\begin{aligned} \breve{g}(-u,t) &= \overline{\exp\left(\lambda^Q \left(\int_0^{\infty} e^{iuy} dG^Q(y) - 1\right)(T-t)\right)} \cdot \overline{\breve{\varphi}(u)} \\ &= \overline{\breve{g}(u,t)}. \end{aligned}$$

The Fourier transform of a function with this property is real-valued. Hence we conclude that the solution (1.22) defines a real-valued function.

Risk Premium

The risk premium in insurance economics is defined as the difference between the market price of an insurance contract and the expected payoff under the contract. In our analysis the financial market determines the risk premium that is thus defined as

$$f^Q(X_t,t) - \mathbf{E}^P[\phi(X_T)|\mathcal{F}_t] = \mathbf{E}^Q[\phi(X_T)|\mathcal{F}_t] - \mathbf{E}^P[\phi(X_T)|\mathcal{F}_t], \qquad (1.25)$$

for a fixed equivalent martingale measure Q.

From our pricing formula (1.11) we conclude that the risk premium can be represented in the form

$$\int_{-\infty}^{\infty} e^{iuX_t} \left(\chi_{T-t}^{Q}(u) - \chi_{T-t}^{P}(u) \right) \check{\varphi}(u)\, du, \tag{1.26}$$

where $\chi_{T-t}^{Q}(u) = \exp\left(\lambda^{Q}\left(\int_0^\infty e^{iuy} dG^{Q}(y) - 1\right)(T-t)\right)$ is the characteristic function of X_{T-t} under the probability measure Q, and $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$. The inverse Fourier transform of the risk premium is thus given by

$$\left(\chi_{T-t}^{Q}(u) - \chi_{T-t}^{P}(u) \right) \cdot \check{\varphi}(u). \tag{1.27}$$

In the next subsections, we explicitly calculate the Fourier inverse of $\phi(\cdot) - k$ in the situation of call options, put options, and spreads. Thus under a fixed equivalent martingale measure, we will give a closed-form expression of the inverse Fourier transform of PCS option prices.

Call Spreads

A call spread on the index is a capped call option and can be created by buying a call option with strike price K_1, and selling at the same time a call option with the same maturity but with strike price $K_2 > K_1$. Hence the payoff function $\phi_{CS}(x)$ depends on the index value x at maturity in the following way

$$\phi_{CS}(x) = \begin{cases} 0 & \text{if } 0 \le x \le K_1 \\ x - K_1 & \text{if } K_1 < x \le K_2 \\ K_2 - K_1 & \text{if } x > K_2. \end{cases} \tag{1.28}$$

As $X_T \geq 0$ it is sufficient that $(\phi_{CS}(\cdot) - k) \cdot \mathbf{1}_{[0,\infty)}(\cdot) \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$ where $\mathbf{1}_A(\cdot)$ denotes the indicator function on a Borel set A. The integrability condition is satisfied for $k = K_2 - K_1$ and the inverse Fourier transform is given by

$$\begin{aligned} \check{\varphi}_{CS}(u) &= \frac{1}{2\pi} \int_0^\infty e^{-iux} \left(\phi_{CS}(x) - (K_2 - K_1)\right) dx \\ &= \frac{1}{2\pi} \frac{1}{u^2} \left(e^{-iuK_2} - e^{-iuK_1} + iu(K_2 - K_1) \right). \end{aligned}$$

Hence, under the equivalent martingale measure Q the price at time t of a call spread with underlying PCS index value $X_t = x$ and strike prices $K_1 < K_2$ is

$$\begin{aligned}
f_{CS}^{Q}(x,t) &= \int_{-\infty}^{\infty} e^{iux}\chi_{T-t}^{Q}(u)\,\check{\varphi}_{CS}(u)\,du + K_2 - K_1 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{u^2}\chi_{T-t}^{Q}(u)\,e^{iux}\left(e^{-iuK_2} - e^{-iuK_1} + iu\left(K_2 - K_1\right)\right)du \\
&\quad + K_2 - K_1.
\end{aligned}$$

Equivalently, applying the inverse Fourier transform

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-iux}\left(f_{CS}^{Q}(x,t) - (K_2 - K_1)\right)dx \\
&= \chi_{T-t}^{Q}(u)\cdot\frac{1}{u^2}\left(e^{-iuK_2} - e^{-iuK_1} + iu\left(K_2 - K_1\right)\right).
\end{aligned}$$

Remember that $\chi_{T-t}^{Q}(u) = \exp\left(\lambda^{Q}\left(\int_0^{\infty} e^{iuy}dG^{Q}(y) - 1\right)(T-t)\right)$ is the characteristic function of the process X_{T-t} under the measure Q.

Put Spreads

A put spread is a capped put option and thus the payoff ϕ_{PS} is given by

$$\phi_{PS}(x) = \begin{cases} K_2 - K_1 & \text{if } 0 \le x \le K_1 \\ K_2 - x & \text{if } K_1 < x \le K_2 \\ 0 & \text{if } x > K_2. \end{cases} \tag{1.29}$$

We observe that $\phi_{PS}(\cdot)\cdot\mathbf{1}_{[0,\infty)}(\cdot) \in \mathbf{L}^2(\mathbb{R})$ and $\phi_{PS}(x) = -\left(\phi_{CS}(x) - (K_2 - K_1)\right)$. Therefore

$$\begin{aligned}
\check{\varphi}_{PS}(u) &= -\check{\varphi}_{CS}(u) \\
&= -\frac{1}{2\pi}\frac{1}{u^2}\left(e^{-iuK_2} - e^{-iuK_1} + iu\left(K_2 - K_1\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
f_{PS}^{Q}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{u^2}\chi_{T-t}^{Q}(u)\,e^{iux}\left(e^{-iuK_1} - e^{-iuK_2} - iu\left(K_2 - K_1\right)\right)du \\
&= -f_{CS}^{Q}(x,t) + K_2 - K_1.
\end{aligned}$$

We have thus shown that our pricing formula fulfills the put-call parity under every equivalent martingale measure Q.

Call Options

Since every PCS index is capped at either \$20 billion or \$50 billion, a call option with strike price K is in fact a call spread with "lower strike price" $K_1 = K$ and "upper strike price" K_2 =\$20 billion or \$50 billion. Hence we can use the pricing formula for call spreads.

Put Options

A put option on a PCS index with strike price K can also be understood as a put spread with "lower strike price" $K_1 = 0$ and "upper strike price" $K_2 = K$. Due to this observation we can again apply the pricing formula for put spreads, i.e.

$$f_P^Q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{u^2}\chi_{T-t}^Q(u)\,e^{iux}\left(1 - e^{-iuK} - iuK\right)du,$$

or

$$\int_{-\infty}^{\infty} e^{-iux} f_P^Q(x,t)\,dx = \chi_{T-t}^Q(u)\cdot\frac{1}{u^2}\left(1 - e^{-iuK} + iuK\right)$$

Characteristic Function of Parameterized Distributions

Let us review some parameterized distributions with support $[0,\infty)$ and their characteristic function. We assume that the parameters are already determined under the equivalent martingale measure Q.

- The Gamma distribution $\Gamma(c,\gamma)$ is defined by its density function

$$\frac{d\Gamma(c,\gamma)(y)}{dy} = \frac{c^{\gamma}e^{-cy}y^{\gamma-1}}{\Gamma(\gamma)},$$

with mean γ/c and variance γ/c^2 where $\Gamma(\cdot)$ is the Gamma function, and $0 < c, \gamma < \infty$. The characteristic function is given by

$$\int_0^{\infty} e^{iuy}d\Gamma(c,\gamma)(y) = \left(\frac{c}{c-iu}\right)^{\gamma}.$$

The characteristic function of X_{T-t} under Q is thus,

$$\chi_{T-t}^Q(u) = \exp\left(\lambda^Q\left(\left(\frac{c}{c-iu}\right)^{\gamma} - 1\right)(T-t)\right). \tag{1.30}$$

- The inverse Gaussian distribution $IG(\mu,\sigma)$ has density function

$$\frac{dIG(\mu,\sigma)(y)}{dy} = \sqrt{\frac{\sigma}{2\pi y^3}} \cdot \exp\left(\frac{-\sigma(y-\mu)^2}{2\mu^2 y}\right),$$

with mean is μ and variance μ^3/σ for $\mu \in \mathbb{R}$ and $\sigma > 0$. The characteristic function is given by

$$\int_0^\infty e^{iuy} dIG(\mu,\sigma)(y) = \exp\left(\sigma/\mu - \sqrt{(\sigma/\mu)^2 - 2\sigma iu}\right).$$

Therefore,

$$\chi_{T-t}^Q(u) = \exp\left(\lambda^Q\left(\exp\left(\sigma/\mu - \sqrt{(\sigma/\mu)^2 - 2\sigma iu}\right) - 1\right)(T-t)\right). \tag{1.31}$$

- The distribution Pareto mixtures of exponentials $PME(\delta)$ belongs to the class of distributions with heavy tails. Their density function is given by

$$\frac{dPME(\delta)}{dy} = \int_{(\delta-1)/\delta}^\infty \delta^{-\delta+1}(\delta-1)^\delta z^{-(\delta+1)} z^{-1} e^{-y/z} dz,$$

with mean 1 and variance $1 + 2/\delta(\delta-2)$ for $\delta > 1$. The characteristic function is given by

$$\int_0^\infty e^{iuy} PME(\delta)(y) = \frac{\int_0^{\frac{\delta}{\delta-1}} \frac{z^\delta}{z-iu} dz}{\frac{\delta^{\delta-1}}{(\delta-1)^\delta}}.$$

Therefore,

$$\chi_{T-t}^Q(u) = \exp\left(\lambda^Q\left(\frac{\int_0^{\frac{\delta}{\delta-1}} \frac{z^\delta}{z-iu} dz}{\frac{\delta^{\delta-1}}{(\delta-1)^\delta}} - 1\right)(T-t)\right). \tag{1.32}$$

In this paragraph, we investigated the valuation of catastrophe insurance derivatives for an arbitrary but fixed equivalent martingale measure. However, in the setup of our insurance market there exist an infinite number of equivalent martingale measures, and hence an infinite collection of prices that are consistent with the no-arbitrage assumption on the bond market. Therefore, we need to be more specific on the preferences of market participants. We follow an approach suggested by Aase [1] and [2] who uses the framework of partial equilibrium theory under uncertainty. The next section includes a brief outline of the economic theory as it is presented in Duffie [28], Chapter 10.

1.4.3 Representative Agent's Valuation

Let us characterize the insurance companies $i = 1, 2, ..., I$ that are affected by catastrophes under a specific PCS contract by net reserves $S^i = (S_t^i)_{t\geq 0}$ and utility functions $U^i : L_+ \to \mathbb{R}$ defined on the consumption space L_+. We assume that L_+ is the set of nonnegative, adapted processes C with $\mathbf{E}^P \left[\int_0^T C_t^2 dt\right] < \infty$ and smooth-additivity of utility functions, i.e.

$$U^i\left(C^i\right) = \mathbf{E}^P\left[\int_0^T u^i\left(C_t^i, t\right) dt\right], \tag{1.33}$$

for $C^i \in L_+$. Furthermore, smooth-additivity requires that for all $i \in \{1, 2, ..., I\}$ $u^i : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is smooth on $(0, \infty) \times [0, T]$ and, for each $0 \leq t \leq T$, $u^i(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ is increasing, strictly concave, with an unbounded derivative $u_c^i(\cdot, t) = \frac{\partial}{\partial c} u^i(\cdot, t)$ on $(0, \infty)$.

An Arrow-Debreu equilibrium is a collection $\left(\Pi, C^1, C^2, ..., C^I\right)$ such that C^i solves insurance company i's maximization problem

$$\sup_{C \in L_+} U^i(C) \qquad \text{subject to } \Pi(C) \leq \Pi\left(S^i\right), \tag{1.34}$$

where $\left(C^1, C^2, ..., C^I\right)$ is a feasible allocation, i.e. $\sum_{i=1}^{I} C^i \leq \sum_{i=1}^{I} S^i \equiv S$, where S is the aggregated net reserves, and $\Pi : L \to \mathbb{R}$ is a linear price function that describes the price at time 0 for a consumption process in L. Furthermore, if Π is strictly increasing, then there is a unique, strictly positive process $\pi \in L_+$ such that

$$\Pi(C) = \mathbf{E}^P\left[\int_0^T \pi_t C_t dt\right] \qquad \text{for } C \in L. \tag{1.35}$$

Since U^i is strictly increasing any Arrow-Debreu equilibrium price function Π is strictly increasing. The representation (1.35) is known as the Riesz representation of $\Pi(\cdot)$ (see Duffie [28] p. 221).

For incomplete markets, there is yet no set of conditions that is sufficient for the existence of an Arrow-Debreu equilibrium. However, with negative exponential utility functions a Pareto efficient outcome can be achieved and is characterized by a linear risk-sharing rule. This implies that every investor holds a certain fraction of the aggregate risk.

Let us therefore assume that preferences of investors can be described by negative exponential utility functions, i.e.

$$u_c^i(c,t) = e^{-\alpha^i c - \rho^i t}, \tag{1.36}$$

for some $\alpha^i > 0$, $\rho^i > 0$. α^i represents the intertemporal coefficient of absolute risk aversion and ρ^i the time impatience rate of agent i.

Under these assumptions, there exists a representative agent in the market with utility function

$$U(C) = \mathbf{E}^P\left[\int_0^T u(C_t,t)\,dt\right], \tag{1.37}$$

where u is of the form

$$u_c(c,t) = e^{-\alpha c - \rho t}, \tag{1.38}$$

with intertemporal coefficient of absolute risk aversion $\alpha > 0$ and time impatience rate $\rho > 0$ in the market. Furthermore, the Riesz representation π of $\Pi(\cdot)$ is given by

$$\pi_t = u_c(S_t,t), \tag{1.39}$$

with aggregated net reserves $S_t = \sum_{i=1}^{I} S^i$.

Coming back to the martingale approach, π is not only the Riesz representation of Π but also the gradient of U (see Duffie [28] p. 300) and defines a state-price deflator. Furthermore, this state-price deflator determines an equivalent martingale measure Q through the Radon-Nikodym density process

$$\begin{aligned}\xi_t &= \exp\left(\int_0^t r_u du\right)\cdot\frac{\pi_t}{\pi_0} \\ &= \exp\left(\int_0^t r_u du\right)\cdot\frac{u_c(S_t,t)}{u_c(S_0,0)}.\end{aligned} \tag{1.40}$$

In addition, we know from the last section (see (1.3)) that ξ can be represented by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} \ln\left(\kappa v\left(Y_k\right)\right) + \int_0^t \int_0^\infty \left(1 - \kappa v\left(y\right)\right) \lambda dG\left(y\right) ds\right), \tag{1.41}$$

for some nonnegative constant κ and nonnegative function v.

Remark 5 *This equivalent martingale measure $Q = P^{\kappa,v}$ can be interpreted as the one under which the representative agent calculates prices in the insurance market. Hence, the corresponding local characteristics κ and $v\left(\cdot\right)$ reflect the representative agent's market price of frequency risk and claim size risk respectively.*

We follow the classical Cramér-Lundberg model and assume that aggregate net reserves in the insurance industry is represented by a stochastic process $S = (S_t)_{0 \le t \le T}$ of the form

$$\begin{aligned} S_t &= s_0 + pt - X_t \\ &= s_0 + pt - \sum_{k=1}^{N_t} Y_k, \end{aligned} \tag{1.42}$$

where s_0 represents aggregate initial capital in the market by time 0, X_t is the PCS index at time t for a specific contract, and p is total premium of the industry for a unit time interval within the loss period of the contract. Hence, the process S represents the surplus of those companies that are affected by catastrophe losses reflected in the particular PCS index X. For example, the net reserves of an insurance company in Florida would not be included if we consider the California index.

By equating the two representations (1.40) and (1.41) of ξ and putting $r \equiv 0$, we deduce

$$-\left(\alpha p + \rho\right) t + \sum_{k=1}^{N_t} \alpha Y_k = \lambda\left(1 - \kappa\right) t + \sum_{k=1}^{N_t} \ln\left(\kappa v\left(Y_k\right)\right), \tag{1.43}$$

for $0 \le t \le T$. Therefore

$$\kappa v(y) = e^{\alpha y}, \tag{1.44}$$

for $y \geq 0$. Since $\int_0^\infty v(y)\,dG(y) = 1$

$$\kappa = \mathbf{E}^P\left[e^{\alpha Y_1}\right] \tag{1.45}$$

$$v(y) = \frac{e^{\alpha y}}{\mathbf{E}^P\left[e^{\alpha Y_1}\right]}. \tag{1.46}$$

Additionally, equation (1.43) imposes the following restriction on the parameters of the model:

$$\alpha p + \rho = \lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right). \tag{1.47}$$

This leads to the following corollary:

Corollary 7 *Consider a market containing a risk averse representative agent as outlined above. Then the coefficient of absolute risk aversion α is uniquely determined by the equilibrium relation* (1.47) *for a given premium rate p and time impatience rate ρ in the market.*

Proof. We only consider risk aversion, i.e. we assume $\alpha > 0$. The same argument holds for a risk loving agent. We have to prove the existence of a unique $\alpha^* > 0$ satisfying (1.47).

Define the function $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(\alpha) = \lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right) - \alpha p - \rho, \tag{1.48}$$

for given $p, \rho > 0$. We deduce $h(0) = -\rho < 0$ and

$$\frac{d^2}{d\alpha^2} h(\alpha) = \lambda \mathbf{E}^P\left[Y_1^2 e^{\alpha Y_1}\right] > 0,$$

i.e. h is a convex function.

If the distribution function G is sufficiently regular then

$$h(\alpha) \to +\infty \qquad \text{for } \alpha \to \infty,$$

and there exists a unique $\alpha^* > 0$ such that $h(\alpha^*) = 0$. ■

Alternatively, for a given degree of absolute risk aversion α the premium rate p is of the form

$$p = \frac{1}{\alpha}\left(\lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right) - \rho\right). \tag{1.49}$$

The first factor $1/\alpha$ reflects the representative agent's risk tolerance whereas the second can be interpreted as the difference between the frequency risk premium $\lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right)$ and the time impatience rate ρ. The agent's risk tolerance and frequency risk premium are positively related to the premium rate p contrary to the time impatience rate.

Remark 6 *Under risk aversion, i.e. $\alpha > 0$ we observe that $\kappa v(y) > 1$ for all $y > 0$. As $v(\cdot)$ is a density, it follows that $\kappa > 1$. We conclude that in a risk-averse insurance market the risk-adjusted frequency $\lambda\kappa$ is larger than the physical frequency λ.*

The coefficient of absolute risk aversion α determines uniquely the market prices of frequency risk κ and of jump size risk $v(\cdot)$ and thus the equivalent martingale measure $P^{\kappa,v} = P^\alpha$, the local characteristics of the underlying PCS loss index under P^α, and the price process of catastrophe insurance derivatives as follows:

- prices are calculated under the equivalent measure $P^\alpha \sim P$ that is defined through its density process

$$\begin{aligned}
\xi_t &= \exp\left(\sum_{k=1}^{N_t}\alpha Y_k + \int_0^t\int_0^\infty\left(1 - e^{\alpha y}\right)\lambda dG(y)\,ds\right) \\
&= \exp\left(\alpha X_t + \int_0^t\int_0^\infty\left(1 - e^{\alpha y}\right)\lambda dG(y)\,ds\right),
\end{aligned} \tag{1.50}$$

- X is a compound Poisson process under P^α with local characteristics

$$\mathbf{E}^{P^\alpha}[N_1] = \lambda\cdot\mathbf{E}^P\left[e^{\alpha Y_1}\right] \tag{1.51}$$

$$dG^\alpha(y) = \frac{e^{\alpha y}}{\mathbf{E}^P\left[e^{\alpha Y_1}\right]}dG(y), \tag{1.52}$$

- the unique price process $\left(f^\alpha(X_t,t)\right)_{0\le t\le T}$ of an insurance derivative with payoff function ϕ is given by

$$f^{\alpha}(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \chi^{P\alpha}_{X_{T-t}}(u)\, \check{\varphi}(u)\, du + k, \tag{1.53}$$

where

$$\chi^{P\alpha}_{T-t}(u) = \exp\left(\lambda \int_0^{\infty} e^{\alpha y}\left(e^{iuy} - 1\right) dG(y)\,(T-t)\right), \tag{1.54}$$

and $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$.

The parameters of the model are restricted by the equilibrium relation

$$\alpha p + \rho = \lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right).$$

Let us finish this chapter with the following remark.

Remark 7 *For given parameters α, p, and ρ the characteristic function is of the form*

$$\chi^{P\alpha}_{T-t}(u) = \exp\left(\frac{\mathbf{E}^P\left[e^{\alpha Y_1}\left(e^{iuY_1} - 1\right)\right]}{\mathbf{E}^P\left[e^{\alpha Y_1} - 1\right]} \cdot (\alpha p + \rho) \cdot (T-t)\right).$$

We have thus established a link between the premium rate p and the price process $(f^{\alpha}(X_t, t))_{0 \le t \le T}$ through the characteristic function. The connection between actuarial and financial prices will be introduced and examined in Chapter 3.

1.5 Conclusion

In this chapter we examined the valuation of catastrophe insurance derivatives in a model in which the underlying, non-traded loss index is a compound Poisson process, a stochastic process used to describe aggregate losses in risk theory. Initially, we only imposed the absence of arbitrage strategies and showed how to structure the market's incompleteness by exploiting the fact that prices under specific probability measures are martingales. This structure was built on parameters that capture the market prices of frequency and loss size risk.

We introduced a new technique based on Fourier analysis that allowed us to deduce a representation of the set of no-arbitrage price processes. This representation enabled us to derive the inverse Fourier transform of derivative prices in closed form and to separate the underlying stochastic environment from the derivative's payoff structure. Furthermore, it was shown that the set of no-arbitrage prices and the set of market prices of frequency and loss size risk is one-to-one connected.

In the preference based equilibrium model the utility function of a representative agent determines uniquely the market prices of frequency and jump size risk. Building upon our representation of derivative prices and their link to market prices of risk, we determined the agent's attitude towards catastrophic risk and thus the unique price for the representative agent.

The analysis and results developed in this chapter suggest to calibrate the model to market data, i.e. to obtain the market prices of risk as implied parameters from observed derivative prices. Since we derived the inverse Fourier transform of derivative prices in closed form, it is moreover suggested that there is much to be gained by using Fast Fourier Transform as an efficient algorithm for the calculation of prices.

Chapter 2

Asset Valuation in Risk Theory

2.1 Introduction

Within the last decade the securitization of insurance related risk has evolved to one of the most important phenomena in risk management initiated by the fact that risks in the insurance business have become more apparent and severe. Examples include the introduction of catastrophe insurance derivatives due to global warming, funded pension schemes and equity-linked life insurance contracts due to changes in the population statistics. The main idea behind spreading insurance related risk through financial markets is twofold: diversification and use of additional capital sources.

A combined financial and insurance market offers an increased set of products that financial investors, insurance and reinsurance companies might use to diversify financial and insurance related risk. The merger of both markets can thus be understood as an essential step towards a complete Arrow-Debreu market. In addition, connecting both markets opens up new capital sources for the insurance and reinsurance business addressing the growing concern in the insurance world that reinsurance companies might not have allocated sufficient capital to cover huge losses caused by events such as Hurricane Andrew. A financial contract spreading insurance related risk thus represents an alternative to the traditional insurance or reinsurance treaty. With additional market participants these financial contracts offer higher liquidity compared to the customized insurance business. Therefore, they enhance the possibility to quickly react to changes in the economic environment by synthetically exchanging one layer for another without entering into costly negotiations.

One issue that arises in this context and that will be investigated in this chapter is the determination of financial security prices, the dynamics of which evolve due to insurance risk. With this aim in view, it is essential to take into account the characteristics that are specific for insurance related risk, e.g. random occurrences of events and changes in the economic environment that cannot be regarded as being infinitesimal. We will therefore consider dynamics of the underlying state variable that have been used in risk theory, such as the classical Cramér-Lundberg model. This model describes the surplus process of an insurance company with income from premia arriving at a constant rate and outflow in the form of claims occurring stochastically of random size.

The consequence of events causing unpredictable movements of random size to the underlying process is that assets in this market are non-redundant. Hence, even by adding more and more basic securities to the market, it is impossible to perfectly hedge against the risk that is inherent in the market. As regards the determination of prices the exclusion of arbitrage strategies is consequently not a sufficient condition for unique valuation of derivative securities.

There exists a vast literature on stochastic processes used in risk theory and questions arising in the context of insurance risk such as the probability of ruin, the distribution of the risk process immediately before ruin and at the time of ruin (see e.g. Paulsen [65], Rolski et al. [73], or Wang and Wu [78]). Only recently, the literature started to investigate these models in the context of financial markets.

Gerber and Shiu [40] examine an underlying asset price that is based on a stochastic process with deterministic, negative drift and jumps occurring randomly with constant, positive jump size. This is a simplified version of the 'reversed' Cramér-Lundberg model seeing that the only source of uncertainty in this model stems from the random moments of jump occurrence. In this framework, a self-financing portfolio can be constructed that perfectly replicates the derivative's payoff. Under absence of arbitrage opportunities in the market, the derivative price is thus uniquely determined by the initial investment of the portfolio. Furthermore, it is shown how option prices evolve from the Esscher transform of underlying processes, a method that can be justified by the existence of a representative agent maximizing expected utility. The Esscher transform is additionally applied to the valuation of perpetual American options in a setup in which the underlying process is a two-dimensional Wiener process without a jump component.

Neither one of the models seems to be accurate to be applied to an economic environment in which market uncertainty arises from insurance risk. The latter model with an underlying Wiener process does not capture the random occurrence of claims causing unpredictable movements in the asset price. In the former model, the results rely on the assumption of constant jump sizes. The corresponding market is complete and allows for unique valuation of derivatives based solely on the absence of arbitrage opportunities. While a complete market is convenient for price determination the assumption of constant jump sizes is rather unrealistic in an actuarial context.

Gerber and Landry [39] investigate the classical Cramér-Lundberg model that is perturbed by an independent Wiener process for the surplus process of an insurance company. The authors derive a renewal equation that is satisfied by the expected discounted value of a penalty that has to be paid at ruin and depends on the level of deficit. The results are then applied to determine the optimal exercise boundary for perpetual put options that exponentially depend on the perturbed Cramér-Lundberg model.

In a later paper by Gerber and Shiu [41], the pricing of reset guarantees for a mutual fund and perpetual put options is examined in a model in which the logarithm of the underlying asset price follows the classical Cramér-Lundberg model. This seems to be a reasonable model for the stock price of an insurance or reinsurance company or the price index of a portfolio of insurance stocks. The authors derive pricing formulae for these derivatives from analyzing the time of ruin in terms of its Laplace transform. However, the multiplicity of no-arbitrage prices arising from stochastic jump sizes is not investigated.

In both papers, Gerber and Landry [39] and Gerber and Shiu [41], the authors consider financial contracts with an infinite expiration time. Perpetuity in the model removes the dependence on time and adds thereby tractability. These models thus approximate a financial market in which contracts are traded that expire after a finite time period.

In this chapter, we contribute to the existing literature by addressing the valuation of European contracts based on an underlying asset price, the logarithm of which follows the classical Cramér-Lundberg model. In addition, we investigate and structure the multiplicity of no-arbitrage derivative prices by linking the set of prices with the set of market prices of uncertainty.

The technique that we introduce to tackle these questions is based on Fourier analysis and generalizes the technique developed in Chapter 1. It enables us to derive a representation of derivative prices that separates the uncertainty which underlies the market from the contract's payoff. Furthermore, the inverse Fourier transform of derivative prices is obtained in closed form. The multiplicity of prices that are solely assumed to exclude arbitrage opportunities is captured by two parameters, the market price of frequency and jump size risk.

Our approach is applicable within a very general framework, where there are no distributional assumptions or restriction to specific contracts. It also allows for an analogous analysis of risk processes that are perturbed by diffusion to include independent financial market risk. On top of that, the method presented in this chapter is applicable to a model in which the logarithm of the stock price follows the 'reversed' Cramér-Lundberg model, i.e. with deterministic outflow at a constant rate and stochastic income at random points in time. This model could capture the price index of a portfolio of annuities or the stock price of a company that continuously invests in research and development and discovers randomly new inventions.

We examine the valuation of spread options in more detail since these derivatives capture characteristics of stop loss reinsurance treaties and therefore provide an alternative way of reinsurance. As a by-product we derive a closed-form expression for the inverse Fourier transform of the conditional probability that the insurance company's surplus at maturity takes values in between a given lower and upper value.

As a selection criterion for price processes we consider a representative agent and determine the unique price process that is consistent with the agent's preferences. The Esscher transform is used as a second criterion and the analogy to the representative agent approach is shown.

The remainder of the chapter is structured as follows: Section 2.2 presents the economic environment, the dynamics of asset prices, and the change between equivalent probability measures. In Section 2.3, we investigate the valuation of derivative securities in the absence of arbitrage opportunities. Section 2.4 examines a market in which prices are determined by a representative investor and in Section 2.5 the results are compared to prices derived through Esscher transformation. In Section 2.6, we generalize the model to allow for perturbed classical risk processes. Section 2.7 concludes.

2.2 The Basic Model

In this section we introduce the structure of the market and the model that we use to describe the dynamics of the underlying fundamental and asset prices.

2.2.1 The Economic Environment

We consider an economy of finite horizon $T < \infty$ with an underlying complete probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P\right)$ on which all random variables will be defined. The state space Ω consists of all possible realizations ω of the economy and the σ-algebra $\mathcal{F}$ is the set of all possible events on Ω. We assign probabilities to all events in $\mathcal{F}$ through the probability measure P. The flow of information in the market is described by an increasing sequence of σ-algebras $\mathcal{F}_t$. We assume that the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right-continuous, $\mathcal{F}_0$ contains all the events in $\mathcal{F}$ that are of P-measure zero and $\mathcal{F}_T = \mathcal{F}$.

In this framework, we assume the existence of a fundamental process $X = (X_t)_{0 \leq t \leq T}$ that drives all financial products and generates the information observable in the market. The filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is thus assumed to be generated by the process X.

A priori, the market consists of one traded risky and one traded risk-free asset. The price process of the risky asset is denoted by $S = (S_t)_{0\leq t\leq T}$ and the price process of the risk-free asset by $B = (B_t)_{0\leq t\leq T}$. The risk-free asset is assumed to yield a deterministic return process r, i.e.

$$dB_t = r_t B_t dt. \tag{2.1}$$

Without loss of generality we set $r \equiv 0$.

2.2.2 Insurance Risk Models

To investigate the valuation of securities whose uncertain movements are related to insurance risk it is necessary to take into account the characteristics of these sources. As opposed to financial market risk that can be regarded as the cause of approximately infinitesimal changes in prices, insurance risk is based on random occurrences of events such as accidents or natural disasters. These events unavoidably give rise to unpredictable jump movements of the underlying state variable X.

More specifically, we assume two sources of uncertainty: the moment of the event and the magnitude of the impact the event has on the underlying process X. They are described as sequences of random variables. Let

- T_1, T_2, T_3, ... denote the moments of the first, second, third, ... event,

and

- Y_1, Y_2, Y_3, ... the magnitude of corresponding events measured in real-valued sizes.

To motivate the additional structure that we put on the fundamental process X, we think of X as reflecting the surplus of an insurance company. In general, surplus of a company is given by initial capital + income - outflow.

In our setup we examine the classical model for surplus of an insurance company, the Cramér-Lundberg model:

- income in this model is defined as the total premium paid by all policyholders of the company which is assumed to be deterministic at a constant premium rate $p > 0$;
- outflow of an insurance company is defined as the total amount of capital the company has to pay to their policyholders due to occurrences of claims. Let us assume that the points in time of claim occurrences and settlements of these claims coincide and that claim sizes are independent and identically distributed. Therefore, Y_1, Y_2, Y_3, ... is a collection of iid random variables, having a common distribution function G with support $[0, \infty)$;
- as regards occurrences of claims, let us consider the process $N = (N_t)_{0 \le t \le T}$ counting the number of claim arrivals up to time $t \le T$, i.e.

$$N_t = \sup \{k \ge 1 \,|T_k \le t\}. \tag{2.2}$$

In the classical Cramér-Lundberg model N is a homogeneous Poisson process with frequency parameter λ, i.e.

$$P[N_t - N_s = n] = \frac{(\lambda\,(t-s))^n}{n!} \cdot e^{-\lambda(t-s)}, \tag{2.3}$$

for $0 \le s \le t \le T$. It is assumed that claim sizes are independent of the counting process N. λ represents the expected number of claims occurring within a unit time interval.

In summary, the surplus process $X = (X_t)_{0 \le t \le T}$ of an insurance company in the classical Cramér-Lundberg model is given by

$$X_t = x_0 + p \cdot t - \sum_{k=1}^{N_t} Y_k, \tag{2.4}$$

where $x_0 \ge 0$ denotes the initial capital stock.

A different model for the underlying stochastic process would be one with contrasting properties to the Cramér-Lundberg model, i.e. with interchanged roles of claims and premium. This model could be applied to a portfolio of annuities with deterministic outflow at rate p and random income reflecting the reserve that becomes free whenever a policyholder dies.

The fundamental process $X = (X_t)_{0 \le t \le T}$ for the 'reversed' Cramér-Lundberg model is thus given by

$$X_t = x_0 - p \cdot t + \sum_{k=1}^{N_t} Y_k, \tag{2.5}$$

for $p > 0$ and $0 \le t \le T$.

One major object of interest in risk theory is the moment of ruin, i.e. the point in time when X crosses zero, and the probability of the moment of ruin being finite. The two models differ enormously with respect to the tractability of this problem for the simple reason that the surplus at ruin in the latter model is zero whereas it could well be that the surplus at ruin in the classical Cramér-Lundberg model is strictly negative. This reflects the possibility of undershoot and gives rise to the difficulties in determining the probability of ruin. However, as regards valuation of derivatives analogous results can be derived by changing the appropriate signs.

Remark 8 *Both stochastic processes exhibit stationary and independent increments, i.e. they belong to the class of Lévy processes. Filtrations generated by Lévy processes and completed by of all P-null sets are right-continuous (see e.g. Protter [67] p. 22).*

2.2.3 Model for Insurance Stocks

The financial market consists of one risk-free asset with short rate of interest $r \equiv 0$ and one risky asset with price process $S = (S_t)_{0 \le t \le T}$. The stochastic evolution of the risky asset's price process is assumed to be solely based on the fundamental risk process X, i.e. we consider a basic security that is traded on a financial market but for which the risk is purely insurance related. Later, we will extend the framework to allow for additional noise driven by a diffusion process that can be interpreted as independent financial market risk.

To preserve the properties of the underlying risk the price process S should be a monotonic transformation of the underlying process X such that asset prices are positive. For simplicity we consider that S is of exponential form, i.e.

$$S_t = \exp\left(X_t\right), \tag{2.6}$$

for $0 \leq t \leq T$.

However, we will see that the technique we use for valuation of derivatives that are based on S does not require the specific exponential form of insurance stock prices.

2.2.4 Equivalent Probability Measures

Changing the probability measure is a crucial tool for determining financial prices that do not allow for arbitrage strategies, i.e. for portfolios which generate almost surely positive gains without risks. In this section we briefly review the change between equivalent probability measures and the impact it has on the underlying stochastic process X.

In both models for the fundamental process X underlying the insurance market the stochastic component of X is a compound Poisson process. As we mentioned above, uncertainty of these processes can be decomposed into two sources, the moment of occurrence and the magnitude of events. In the previous section, these two factors have been parameterized by λ, the expected number of events within a unit time interval, and G, the distribution function of jump sizes. Let us call the pair $(\lambda, dG\left(y\right))$ the characteristics of the process X under our "reference" measure P. We are interested in the change in characteristics the switch to a different probability measure induces.

We restrict the set of all probability measures on $\mathcal{F}$ to those that preserve two properties under the change. First, we only consider probability measures that are equivalent to the "reference" measure P, i.e. the fact of attributing zero likelihood to an event in $\mathcal{F}$ is to be invariant under the change. We further assume that the structure of the underlying process X is invariant under the change of probability measures, that is X under the new probability measure can be characterized by a time-constant, deterministic frequency rate and a time-constant distribution function of jump sizes. The subset of equivalent probability measures that preserve the structure of compound Poisson processes under a change of measure has been described by Delbaen and Haezendonck [25] as follows:

A probability measure Q is equivalent to P and the structure of the compound Poisson process under Q is preserved if and only if there exists a nonnegative constant κ and a nonnegative, measurable function $v : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_0^\infty v(y)\, dG(y) = 1,$$

such that the associated density process $\xi_t = \mathbf{E}^P[\xi_T \mid \mathcal{F}_t]$ of the Radon-Nikodym derivative $\xi_T = \frac{dQ}{dP}$ is given by

$$\begin{aligned}\xi_t &= \left(\prod_{k=1}^{N_t} \kappa v(Y_k)\right) \cdot \exp\left(\int_0^t \int_0^\infty (1 - \kappa v(y))\, \lambda dG(y)\, ds\right) \\ &= \exp\left(\sum_{k=1}^{N_t} \ln(\kappa v(Y_k)) + \lambda(1-\kappa)\, t\right), \qquad (2.7)\end{aligned}$$

for any $0 \leq t \leq T$. $\mathbf{E}^P[\cdot]$ denotes the expectation operator under the probability measure P.

Under the new measure Q the process X has characteristics $\left(\lambda^Q, dG^Q(y)\right)$ $= (\lambda\kappa, v(y)\, dG(y))$. Let us recall that the set of parameters κ, $v(\cdot)$ is in one-to-one correspondence to the set of equivalent measures (see Lemma 1, p. 24 in the previous chapter). We therefore denote the measure Q corresponding to the constant κ and the function $v(\cdot)$ by $P^{\kappa,v}$ and the corresponding distribution function G^Q by G^v. Hence, for all $A \in \mathcal{B}_+$

$$G^v(A) = \int_A v(y)\, dG(y), \qquad (2.8)$$

and

$$\mathbf{E}^{P^{\kappa,v}}[N_1] = \lambda\kappa, \qquad (2.9)$$

where $\mathcal{B}_+$ denotes the Borel σ-algebra on $\mathbb{R}_+$ and $\mathbf{E}^{P^{\kappa,v}}[\cdot]$ the expectation operator under the measure $P^{\kappa,v}$.

Remark 9 *As mentioned in the last chapter, we interpret κ as the premium of frequency risk and $v(\cdot)$ as the premium of claim size risk.*

2.3 Risk-Neutral Valuation

In this section, we examine the valuation of securities in our market based on the fundamental theorem of asset pricing that establishes the equivalence between the absence of arbitrage opportunities and the existence of a so-called martingale measure. Based on this equivalence we investigate the dynamics of the basic security S under the martingale measure before turning the attention to price processes of derivatives that are written on S. For the latter, Fourier analysis proves to be a useful tool to represent price processes.

The equivalence between absence of arbitrage strategies and the existence of equivalent probability measures under which discounted price processes are martingales plays a central role in mathematical finance. It is important to be aware of the specifications of the model in which this equivalence is used since arbitrage has to be differently defined to guarantee the existence of equivalent martingale measures.

In this chapter, we adopt the definition given by Fritelli and Lakner [35], called "no free lunch", under which the equivalence result is derived with high level of generality. The only mathematical condition that is imposed on asset prices is that they are adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which is a natural requirement. As discussed in the previous chapter, the "no free lunch" condition postulates that the set of gains that can be achieved by trading at deterministic or stopping times contains no positive random variable. In a continuous time setting closure of the set of gains has to be considered which essentially depends on the topology on this set. Under a topology that makes use of certain dualities Fritelli and Lakner [35] prove that there is no free lunch with trading strategies at deterministic times if and only if there exists an equivalent martingale measure. Furthermore, if every underlying process is right-continuous, then this result holds additionally for trading strategies at stopping times.

Henceforth, we assume absence of "free lunch" in the market as outlined above, i.e. the existence of an equivalent martingale measure is guaranteed.

2.3.1 The Basic Security

In our market, we assume the basic security to be defined through its price process $S = (S_t)_{0 \leq t \leq T}$ of the form

$$S_t = \exp(X_t), \tag{2.10}$$

for $0 \leq t \leq T$.

For both the classical and the 'reversed' Cramér-Lundberg model the results on the determination of security prices are the same modulo changes of the appropriate signs. We therefore assume from now on that the fundamental process X follows the classical Cramér-Lundberg model, i.e.

$$X_t = x_0 + p \cdot t - \sum_{k=1}^{N_t} Y_k, \tag{2.11}$$

for $p > 0$ and $0 \leq t \leq T$.

The condition that discounted price processes are martingales under appropriate probability measures can be seen as a further restriction on the set of equivalent martingale measures under consideration. Due to the one-to-one correspondence between the set of equivalent probability measures and the set of market prices of frequency and magnitude risk, this restriction can be transferred to the latter set. If the market were complete the remaining set would be a singleton, i.e. the condition would pin down an unique equivalent martingale measure. Let us now state the following proposition:

Proposition 8 *Let the underlying process X have characteristics $(\lambda, dG(y))$ under P. Let us characterize a probability measure $Q = P^{\kappa, v(\cdot)}$ that is equivalent to P and preserves the structure of the underlying process X by the parameters $\kappa > 0$ and $v : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which reflect the changes in characteristics of X. Then the price process $S = (S_t)_{0 \leq t \leq T}$ is a martingale under $P^{\kappa, v(\cdot)}$ if and only if*

$$p + \lambda\kappa \left(\mathbf{E}^P \left[v(Y_1) e^{-Y_1} \right] - 1 \right) = 0, \tag{2.12}$$

with $\mathbf{E}^P\left[v\left(Y_1\right)\right]=1.$

Proof. In the proof we make use of the infinitesimal generator $\mathcal{A}$ of the underlying process X. For a given function f in the domain of the generator and an equivalent probability measure $P^{\kappa,v(\cdot)}$ under which X has characteristics $(\lambda\kappa, v(y)\,dG(y))$ the infinitesimal generator is given by

$$\mathcal{A}(f)(x)=p\cdot\frac{d}{dx}f(x)+\lambda\kappa\cdot\int_0^\infty(f(x-y)-f(x))\,v(y)\,dG(y). \tag{2.13}$$

A stochastic process $(f(X_t))_{0\le t\le T}$ is a martingale under $P^{\kappa,v(\cdot)}$ if and only if

$$\mathcal{A}(f)(x)=0,$$

that is

$$p\cdot\frac{d}{dx}f(x)+\lambda\kappa\cdot\int_0^\infty(f(x-y)-f(x))\,v(y)\,dG(y)=0, \tag{2.14}$$

for all given values $X_t=x$ (see e.g. Proposition 4, p. 32 in Chapter 1).

For the asset price process $\left(e^{X_t}\right)_{0\le t\le T}$ of the basic security we consider the function

$$f(x)=e^x, \tag{2.15}$$

and a necessary and sufficient condition for the discounted price process to be a martingale under $P^{\kappa,v(\cdot)}$ is consequently

$$p+\lambda\kappa\cdot\left(\mathbf{E}^P\left[v\left(Y_1\right)e^{-Y_1}\right]-1\right)=0.$$

■

The restricting equation (2.12) is the well known Lundberg fundamental equation that plays a crucial role in determining upper bounds for the probability of ruin in the classical Cramér-Lundberg model. In our context, the equation puts a restriction on the set of market prices of risk under consideration, namely for every market price of jump size risk $v(\cdot)$ with $\mathbf{E}^P\left[v\left(Y_1\right)\right]=1$ the market price of frequency risk is given by

$$\kappa=\frac{p}{\lambda\left(1-\mathbf{E}^P\left[v\left(Y_1\right)e^{-Y_1}\right]\right)}. \tag{2.16}$$

Alternatively, for a given jump size distribution $v(\cdot)\, dG(\cdot)$ under $P^{\kappa,v(\cdot)}$ the frequency of events is given by

$$\lambda\kappa = \frac{p}{1 - \mathbf{E}^P\left[v(Y_1)\, e^{-Y_1}\right]}. \tag{2.17}$$

2.3.2 Derivative Securities

We now extend the set of available securities by introducing a derivative that is written on the existing risky asset. In this section, we examine the valuation of this derivative assuming absence of free lunch in the market.

We assume the new derivative to be of European style and written on the asset S. The new derivative is thus defined through its payoff at expiration date T that depends on the realization S_T, and therefore on the realization of X_T. We represent the specifications of the contract by a payoff function $\phi : \mathbb{R} \to \mathbb{R}$, i.e. the buyer of the contract receives $\phi(X_T)$ at maturity.

As the payoff depends on the realization of the economy at T we are interested in determining the value, i.e. the price of the contract for all $0 \leq t \leq T$. Imposing the absence of free lunch in the market guarantees that discounted price processes are martingales under appropriate probability measures. The martingale property is a powerful tool in determining the price of such derivatives as will be seen in the following proposition:

Proposition 9 *Let us fix an arbitrary equivalent probability measure $P^{\kappa,v(\cdot)}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that*

$$\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}), \tag{2.18}$$

for some $k \in \mathbb{R}$. Suppose $\left(f^{\kappa,v(\cdot)}(X_t, t)\right)_{0 \leq t \leq T}$ is a martingale under $P^{\kappa,v(\cdot)}$ with $f^{\kappa,v(\cdot)}(X_T, T) = \phi(X_T)$. Then the function $f^{\kappa,v(\cdot)} : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ defining the martingale $\left(f^{\kappa,v(\cdot)}(X_t, t)\right)_{0 \leq t \leq T}$ can be represented by

$$f^{\kappa,v(\cdot)}(x,t) = \int_{-\infty}^{\infty} e^{iux} \chi_{T-t}^{\kappa,v(\cdot)}(u)\, \check{\varphi}(u)\, du + k, \tag{2.19}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\left(\phi(z) - k\right)dz, \tag{2.20}$$

and $\chi_{T-t}^{\kappa,v(\cdot)}(\cdot)$ *is the characteristic function of* X_{T-t} *under the probability measure* $P^{\kappa,v(\cdot)}$, *i.e.*

$$\chi_{T-t}^{\kappa,v(\cdot)}(u) = \mathbf{E}^{P^{\kappa,v(\cdot)}}\left[e^{iuX_{T-t}}\right]. \tag{2.21}$$

Before we proceed to the proof let us point out that the characteristic function is given by

$$\begin{aligned}\chi_{T-t}^{\kappa,v(\cdot)}(u) &= e^{iu(x_0+p(T-t))}\cdot\mathbf{E}^{P^{\kappa,v(\cdot)}}\left[\exp\left(-iu\sum_{k=1}^{N_{T-t}}Y_k\right)\right]\\ &= \exp\left(ix_0u + \left(\lambda\kappa\left(\mathbf{E}^P\left[v(Y_1)e^{-iuY_1}\right] - 1\right) + ipu\right)(T-t)\right).\end{aligned} \tag{2.22}$$

Proof. The proof is analogous to Proposition 2, p. 29 in Chapter 1.

The Fourier transform is a one-to-one mapping of $\mathbf{L}^2(\mathbb{R})$ onto itself. In other words, for every $g \in \mathbf{L}^2(\mathbb{R})$ there corresponds one and only one inverse Fourier transform that is of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}g(x)\,dx \tag{2.23}$$

First, we apply the Fourier and thereafter the inverse Fourier transform to $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$, i.e.

$$\phi(x) - k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux}e^{-iuz}\left(\phi(z) - k\right)dzdu.$$

Using the martingale property under $P^{\kappa,v(\cdot)}$ and Fubini we deduce that

$$\begin{aligned}f^{\kappa,v(\cdot)}(X_t,t) &= \mathbf{E}^{P^{\kappa,v(\cdot)}}\left[\phi(X_T)\,|\mathcal{F}_t\right]\\ &= \mathbf{E}^{P^{\kappa,v(\cdot)}}\left[\phi(X_T) - k\,|\mathcal{F}_t\right] + k\\ &= \frac{1}{2\pi}\mathbf{E}^{P^{\kappa,v(\cdot)}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iuX_T}e^{-iuz}\left(\phi(z) - k\right)dzdu\,|\mathcal{F}_t\right] + k\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{E}^{P^{\kappa,v(\cdot)}}\left[e^{iuX_T}\,|\mathcal{F}_t\right]e^{-iuz}\left(\phi(z) - k\right)dzdu + k\\ &= \int_{-\infty}^{\infty}\mathbf{E}^{P^{\kappa,v(\cdot)}}\left[e^{iuX_T}\,|\mathcal{F}_t\right]\check{\varphi}(u)\,du + k,\end{aligned}$$

where $\check{\varphi}(\cdot)$ denotes the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (\phi(z) - k)\, dz.$$

The underlying process X is a Markov process with stationary and independent increments. Therefore

$$\begin{aligned} \mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iuX_T} | \mathcal{F}_t \right] &= e^{iuX_t} \mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iu(X_T - X_t)} | X_t \right] \\ &= e^{iuX_t} \mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iuX_{T-t}} | X_t \right] \\ &= e^{iuX_t} \mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iuX_{T-t}} \right]. \end{aligned}$$

$\mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iuX_{T-t}} \right]$ is the characteristic function of the random variable X_{T-t} under the probability measure $P^{\kappa,v(\cdot)}$.

Hence, we deduce

$$f^{\kappa,v(\cdot)}(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \mathbf{E}^{P^{\kappa,v(\cdot)}} \left[e^{iuX_{T-t}} \right] \check{\varphi}(u)\, du + k.$$

■

In the context of derivative pricing, we derived a representation of discounted price processes in a market without free lunch for every fixed martingale measure. This representation enables us to deduce the inverse Fourier transform of prices in closed form. For a given value $X_t = x$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(f^{\kappa,v(\cdot)}(x,t) - k \right) dx = \chi_{T-t}^{P^{\kappa,v(\cdot)}}(u) \cdot \check{\varphi}(u). \tag{2.24}$$

Hence the technique presented here splits the inverse Fourier transform of derivative prices into two components, the characteristic function of the underlying state variable X and the inverse Fourier transform of the contract's payoff. Interestingly, the indeterminacy of equivalent martingale measures is wholly captured by the characteristic function whereas the contract's specification are reflected in the second factor. The ratio

$$\frac{\int_{-\infty}^{\infty} e^{-iux} \left(f^{\kappa,v(\cdot)}(x,t) - k \right) dx}{\chi_{T-t}^{P^{\kappa,v(\cdot)}}(u)} \tag{2.25}$$

is therefore independent of the equivalent martingale measure $P^{\kappa,v(\cdot)}$.

Remark 10 *Let us note that the representation of derivative prices in Proposition 9, p. 61 could be analogously derived for asset price processes S that are not of exponential form.*

So far, we investigated the valuation of derivative securities neglecting the fact that the underlying risk X in the market is traded itself through the basic security $S = \exp(X)$. In Proposition 8, p. 59 of the previous section we derived an equation reflecting the restriction on the set of all equivalent probability measures under consideration. The following corollary characterizes the set of derivative prices that are consistent with the dynamics of underlying asset S in the market.

Corollary 10 *Suppose the market contains one risk-free asset with return $r \equiv 0$, one basic security $S = \exp(X)$ and one derivative with payoff $\phi(X_T)$ at maturity T of the contract such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$. If there is no free lunch in the market, then the function $f^{v(\cdot)} : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ defining the price process $\left(f^{v(\cdot)}(X_t, t)\right)_{0 \le t \le T}$ of the derivative for a given market price of jump size risk $v : \mathbb{R}_+ \to \mathbb{R}$ can be represented by*

$$f^{v(\cdot)}(x,t) = \int_{-\infty}^{\infty} e^{iux} \chi_{T-t}^{v(\cdot)}(u)\, \check{\varphi}(u)\, du + k, \tag{2.26}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$ and the characteristic function is given by

$$\chi_{T-t}^{v(\cdot)}(u) = \exp\left(i x_0 u - \left(\frac{\mathbf{E}^P\left[v(Y_1)\, e^{-iuY_1}\right] - 1}{\mathbf{E}^P\left[v(Y_1)\, e^{-Y_1}\right] - 1} \cdot p - ipu \right)(T-t) \right). \tag{2.27}$$

Proof. We combine the representation of derivative price processes (2.19) with the restriction on the market prices of frequency and jump size risk, reflecting the fundamental Lundberg equation (2.12). ■

The fact that the underlying risk is traded itself through the basic security puts a further restriction on the set of possible equivalent probability measures, and therefore the set of market prices of frequency and jump size risk. In Corollary 10, p. 64 we used this consistency requirement in such a way that derivative prices appear to be independent of the "physical" frequency rate λ.

Remark 11 *Corollary 10, p. 64 establishes a link between the premium rate p and derivative prices. It provides the motivation for Chapter 3 in which the connection between actuarial and financial prices will be investigated in more detail.*

The analysis presented in this section achieves a high level of generality as no distributional assumptions need to be imposed. For certain parameterized distributions and their characteristic functions we refer to Section 1.4.2, p. 39 of Chapter 1. The condition we imposed on the payoff structure of contracts is very mild. For call, put, and spread options the inverse Fourier transform of ϕ has been derived explicitly in Section 1.4.2, p. 37 of Chapter 1.

In the following section we examine a class of derivatives that captures certain features that are inherent in reinsurance treaties.

2.3.3 Stop Loss Reinsurance and Financial Spreads

One typical reinsurance contract is called stop loss reinsurance or aggregate excess of loss cover contract. The reinsurer under such a contract pays the excess of an agreed limit amount M of the cedent's aggregate claim amount accumulated during a certain time period. In practice aggregate excess of loss reinsurance cover is usually limited. This means that the reinsurer's liability is capped at a level A, i.e. the treaty covers the layer A in excess of M.

If C_T denotes total claims aggregated over the time period $[0,T]$ then the payoff of such a contract is given by

$$\min\left(A,(C_T-M)^+\right), \tag{2.28}$$

where $(C_T-M)^+=\max(C_T-M,0)$.

Let us now transfer the specifications of stop loss reinsurance treaties to financial markets by securitizing the underlying insurance risk. Hence we would like to design a financial contract that can be used by insurance companies as an alternative to the traditional reinsurance contract. An insurance company with surplus process $X=(X_t)_{0\leq t\leq T}$ might be interested in reinsuring itself against large losses, i.e. it would like to be compensated if the surplus of the company at T drops below a certain level $M + A$ for example. However, the counterpart would not be interested in bearing an unlimited risk, i.e. the liability should be limited at a certain level M for example. A financial derivative with these specifications is a put spread or bear spread. A bear spread can be created by selling a put option on the asset price S with a certain strike price and buying a put option on the same underlying with a higher strike price. Therefore, the payoff function ϕ is given by

$$\phi(X_T) = \min\left(A, (M + A - X_T)^+\right), \tag{2.29}$$

which reflects the payoff of a bear spread based on $S = \exp(X)$ with lower strike price e^M and higher strike price e^{M+A}.

For this contract we observe that there does not exists any $k \in \mathbb{R}$ such that the integrability condition

$$\int_{-\infty}^{\infty} |\phi(x) - k|^2 \, dx < \infty \tag{2.30}$$

is satisfied. Let us therefore follow the idea to decompose the function ϕ into

$$\phi(x) = \phi(x) \cdot \mathbf{1}_{(-\infty,M]}(x) + \phi(x) \cdot \mathbf{1}_{(M,\infty)}(x), \tag{2.31}$$

where $\mathbf{1}_A(\cdot)$ denotes the indicator function on a Borel set A.

For the bull spread outlined above we get

$$\phi(x) = A \cdot \mathbf{1}_{(-\infty,M)}(x) + (M + A - x)^+ \cdot \mathbf{1}_{[M,\infty)}(x), \tag{2.32}$$

where the advantage is that the second function in the decomposition satisfies the integrability condition (2.30).

The price process $\left(f^{\kappa,v(\cdot)}(X_t,t)\right)_{0\le t\le T}$ under an equivalent martingale measure $P^{\kappa,v(\cdot)}$ can therefore be written as

$$\begin{aligned} f^{\kappa,v(\cdot)}(X_t,t) &= \mathbf{E}^{P^{\kappa,v(\cdot)}}\left[\phi(X_T)\,|X_t\right] \\ &= A \cdot P^{\kappa,v(\cdot)}\left[X_T < M\,|X_t\right] \\ &\quad + \mathbf{E}^{P^{\kappa,v(\cdot)}}\left[(M + A - X_T)^+ \cdot \mathbf{1}_{(M,\infty)}(X_T)\,|X_t\right] \\ &= A \cdot P^{\kappa,v(\cdot)}\left[X_T < M\,|X_t\right] + \int_{-\infty}^{\infty} e^{iuX_t} \chi_{T-t}^{\kappa,v(\cdot)}(u)\, \check{\varphi}(u)\, du, \end{aligned} \tag{2.33}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $(M+A-\cdot)^{+}\cdot\mathbf{1}_{[M,\infty)}(\cdot)$ and can be derived explicitly as

$$\begin{aligned}\check{\varphi}(u) &= \frac{1}{2\pi}\int_{M}^{M+A} e^{-iuz}(M+A-z)\,dz \\ &= \frac{1}{2\pi}\cdot\frac{1}{u^2}\cdot e^{-iuM}\cdot\left(1-iuA-e^{-iuA}\right).\end{aligned}$$

Suppose we decompose the payoff function ϕ of a bull spread in a different way, e.g.

$$\phi(x) = A\cdot\mathbf{1}_{(-\infty,L]}(x) + \min\left(A,(M+A-x)^{+}\right)\cdot\mathbf{1}_{(L,\infty)}(x),$$

for some real number $L \leq M$. The second term of this decomposition too satisfies the required integrability condition (2.30).

However, this decomposition leads to the following representation

$$f^{\kappa,v(\cdot)}(X_t,t) = A\cdot P^{\kappa,v(\cdot)}[X_T<L\,|X_t] + \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}^{\kappa,v(\cdot)}(u)\,\check{\varphi}^{L}(u)\,du, \quad (2.34)$$

where the appropriate inverse Fourier transform $\check{\varphi}^{L}(\cdot)$ is

$$\begin{aligned}\check{\varphi}^{L}(u) &= \frac{A}{2\pi}\int_{L}^{M} e^{-iuz}dz + \frac{1}{2\pi}\int_{M}^{M+A} e^{-iuz}(M+A-z)\,dz \\ &= \check{\varphi}(u) - \frac{A}{2\pi}\cdot\frac{1}{iu}\cdot\left(e^{-iuM}-e^{-iuL}\right).\end{aligned}$$

Equating the two representations (2.33) and (2.34) leads to

$$\begin{aligned}P^{\kappa,v(\cdot)}[X_T<M\,|X_t] &= P^{\kappa,v(\cdot)}[X_T<L\,|X_t] \\ &\quad -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iuX_t}}{iu}\left(e^{-iuM}-e^{-iuL}\right)\chi_{T-t}^{\kappa,v(\cdot)}(u)\,du,\end{aligned}$$

and thus

$$P^{\kappa,v(\cdot)}[L\leq X_T<M\,|X_t] = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iuX_t}}{iu}\left(e^{-iuM}-e^{-iuL}\right)\chi_{T-t}^{\kappa,v(\cdot)}(u)\,du,$$

or alternatively

$$\int_{-\infty}^{\infty} e^{-iux}P^{\kappa,v(\cdot)}[L\leq X_T<M\,|X_t=x]\,dx = \frac{e^{-iuL}-e^{-iuM}}{iu}\chi_{T-t}^{\kappa,v(\cdot)}(u),$$

For $L=-M$ we derive

$$P^{\kappa,v(\cdot)}\left[-M \leq X_T < M \,| X_t\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{iuX_t}\frac{\sin\left(Mu\right)}{u}\chi_{T-t}^{\kappa,v(\cdot)}\left(u\right)du,$$

or alternatively

$$\int_{-\infty}^{\infty} e^{-iux}P^{\kappa,v(\cdot)}\left[-M \leq X_T < M \,| X_t = x\right]dx = 2\frac{\sin\left(Mu\right)}{u}\chi_{T-t}^{\kappa,v(\cdot)}\left(u\right).$$

We have thus derived a closed-form expression for the inverse Fourier transform of the probability that the surplus of the insurance company at T takes values in between a given lower and upper value.

2.4 Representative Investor

In the previous section, we derived a representation of derivative prices that are consistent with absence of free lunch in the market and the dynamics of the traded, underlying asset. However, due to random jump sizes the no free lunch condition does not determine uniquely the market price of risk and consequently the derivative price process. We therefore have to be more specific about preferences of investors. One approach is to assume the existence of a representative investor whose preferences are defined through a utility function. We refer to Section 1.4.3, p. 41 in the last chapter and to Duffie [28], Chapter 10 for a detailed outline of the economic theory.

We assume that consumption preferences of agents can be represented by negative exponential utility functions. We then obtain a representative agent with a utility function U of the form

$$U\left(C\right) = \mathbf{E}^P\left[\int_0^T u\left(C_t, t\right)dt\right], \tag{2.35}$$

where u is of the form

$$\frac{\partial}{\partial c}u\left(c,t\right) = e^{-\alpha c-\rho t}, \tag{2.36}$$

with intertemporal coefficient of absolute risk aversion $\alpha > 0$ and time impatience rate $\rho > 0$ in the market.

The representative investor maximizes her utility over the set of nonnegative, adapted consumption processes $C = \left(C_t\right)_{0\leq t\leq T}$ with $\mathbf{E}^P\left[\int_0^T C_t^2 dt\right] < \infty$ subject to

$$\Pi(C) \leq \Pi(X), \tag{2.37}$$

where $X = (X_t)_{0 \leq t \leq T}$ reflects aggregated net reserves and Π is a linear price function that describes the price at time 0 for a consumption process. If Π is strictly increasing, then the unique, strictly positive process $\pi = (\pi_t)_{0 \leq t \leq T}$ such that

$$\Pi(C) = \mathbf{E}^P \left[\int_0^T \pi_t C_t dt \right], \tag{2.38}$$

is given by

$$\pi_t = \frac{\partial}{\partial c} u(X_t, t). \tag{2.39}$$

π is the Riesz representation of Π that defines a state-price deflator and therefore an equivalent martingale measure through the Radon-Nikodym density process

$$\begin{aligned} \xi_t &= \frac{\pi_t}{\pi_0} \\ &= \frac{\frac{\partial}{\partial c} u(X_t, t)}{\frac{\partial}{\partial c} u(X_0, 0)}. \end{aligned} \tag{2.40}$$

In addition, we know from Section 2.2.4, p. 56 that $\xi = (\xi_t)_{0 \leq t \leq T}$ can be represented by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} \ln(\kappa v(Y_k)) + \int_0^t \int_0^\infty (1 - \kappa v(y)) \lambda dG(y) \, ds \right), \tag{2.41}$$

for some nonnegative constant κ and nonnegative function v.

Aggregate net reserves $X = (X_t)_{0 \leq t \leq T}$ in our insurance market are described by the classical Cramér-Lundberg model, i.e.

$$X_t = x_0 + p \cdot t - \sum_{k=1}^{N_t} Y_k. \tag{2.42}$$

By equating the two representation (2.40) and (2.41) of the Radon-Nikodym density process we deduce

$$(-\alpha p - \rho) t + \sum_{k=1}^{N_t} \alpha Y_k = \lambda (1 - \kappa) t + \sum_{k=1}^{N_t} \ln(\kappa v(Y_k)), \tag{2.43}$$

for $0 \leq t \leq T$. Therefore

$$\kappa v(y) = e^{\alpha y}, \tag{2.44}$$

for $y \geq 0$. Since $\int_0^\infty v(y)\, dG(y) = 1$

$$\begin{aligned} \kappa &= \mathbf{E}^P\left[e^{\alpha Y_1}\right] & (2.45) \\ v(y) &= \frac{e^{\alpha y}}{\mathbf{E}^P\left[e^{\alpha Y_1}\right]}. & (2.46) \end{aligned}$$

Additionally, equation (2.43) imposes a restriction on the coefficient of absolute risk aversion α:

$$\alpha p + \rho = \lambda\left(\mathbf{E}^P\left[e^{\alpha Y_1}\right] - 1\right). \tag{2.47}$$

Let us remember that the martingale property of the discounted asset price process implied another restriction on the parameters of the model, namely

$$p + \lambda\kappa\left(\mathbf{E}^P\left[v(Y_1)\, e^{-Y_1}\right] - 1\right) = 0,$$

i.e.

$$p + \lambda\left(\mathbf{E}^P\left[e^{(\alpha-1)Y_1}\right] - 1\right) = 0. \tag{2.48}$$

Let us summarize the results in the following corollary:

Corollary 11 *Assume a market with a representative investor characterized by a negative exponential utility function with coefficient of absolute risk aversion α and time impatience rate ρ. Then the function $f : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ defining the derivative price process $(f(X_t, t))_{0 \leq t \leq T}$ with payoff $\phi(X_T)$ that is consistent with absence of free lunch and the dynamics of the underlying asset price can be represented by*

$$f(x, t) = \int_{-\infty}^{\infty} e^{iux} \chi_{T-t}(u)\, \check{\varphi}(u)\, du + k, \tag{2.49}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$ and $\chi_{T-t}(\cdot)$ is the characteristic function of X_{T-t} under the corresponding probability measure, i.e.

$$\chi_{T-t}(u) = \exp\left(ix_0 u + \left(\lambda\left(\mathbf{E}^P\left[e^{(\alpha-iu)Y_1}\right] - 1\right) + ipu\right)(T-t)\right). \tag{2.50}$$

Furthermore, the parameters of the model have to satisfy the two equations

$$\alpha p + \rho = \lambda \left(\mathbf{E}^P \left[e^{\alpha Y_1} \right] - 1 \right), \tag{2.51}$$

and

$$p + \lambda \left(\mathbf{E}^P \left[e^{(\alpha-1)Y_1} \right] - 1 \right) = 0. \tag{2.52}$$

Let us investigate the two restricting equations (2.51) and (2.52) in more detail. Positivity of the premium rate p in equation (2.52) requires

$$\alpha < 1, \tag{2.53}$$

i.e. the model only allows for risk-neutral valuation if the representative agent is not too risk-averse.

Let us consider the following two situations:

(1) Assume that the coefficient of absolute risk aversion $\alpha < 1$ and the frequency rate λ are exogenously given. Then the time impatience rate ρ and premium rate p are endogenously determined by equation (2.51) and (2.52), namely

$$\rho = \lambda \left(\mathbf{E}^P \left[e^{\alpha Y_1} - 1 \right] + \alpha \mathbf{E}^P \left[e^{(\alpha-1)Y_1} - 1 \right] \right) \tag{2.54}$$

$$p = \lambda \left(1 - \mathbf{E}^P \left[e^{(\alpha-1)Y_1} \right] \right). \tag{2.55}$$

Hence, we derived endogenously a premium calculation principle that is consistent with the representative agent's valuation of financial contracts.

(2) Assume instead that the frequency rate λ and the premium rate p are exogenously given and define the function $h : (-\infty, 1) \to \mathbb{R}$ by

$$h(\alpha) = \lambda \left(1 - \mathbf{E}^P \left[e^{(\alpha-1)Y_1} \right] \right) - p. \tag{2.56}$$

Equation (2.52) requires $h(\alpha) = 0$ for the coefficient of absolute risk aversion α in equilibrium.

We deduce $h(1) = -p < 0$ and

$$\begin{aligned}
\frac{d}{d\alpha} h(\alpha) &= -\lambda \mathbf{E}^P \left[Y_1 e^{(\alpha-1)Y_1} \right] < 0, \\
\frac{d^2}{d\alpha^2} h(\alpha) &= -\lambda \mathbf{E}^P \left[Y_1^2 e^{(\alpha-1)Y_1} \right] < 0.
\end{aligned}$$

h is therefore a continuous, decreasing and concave function with

$$h(\alpha) \to \lambda - p \qquad \text{for } \alpha \to -\infty.$$

We conclude that if $\lambda \leq p$ then there exists no equilibrium whereas if $\lambda > p$ there exists a unique $\alpha^* \in (-\infty, 1)$ such that $h(\alpha^*) = 0$. Furthermore, we have

$$h(0) = \lambda \left(1 - \mathbf{E}^P \left[e^{-Y_1}\right]\right) - p.$$

If the premium rate is small relative to the frequency of events, i.e.

$$\frac{p}{\lambda} < 1 - \mathbf{E}^P \left[e^{-Y_1}\right]$$

then the representative investor is risk-averse in equilibrium as $h(0) > 0$ implies $\alpha^* > 0$.

If, on the contrary, the premium rate is large relative to the frequency rate, i.e.

$$1 > \frac{p}{\lambda} > 1 - \mathbf{E}^P \left[e^{-Y_1}\right]$$

then the representative agent is risk-loving as $h(0) < 0$ implies $\alpha^* < 0$.

The time impatience rate is then determined by equation (2.51) as

$$\rho = \lambda \left(\mathbf{E}^P \left[e^{\alpha^* Y_1}\right] - 1\right) - \alpha^* p.$$

2.5 Esscher Transform

The Esscher transform has been introduced as a premium calculation principle in actuarial science and reflects a transformation of the original distribution of aggregate claims. More generally, it can be used as a change of probability measure for stochastic processes. If the parameter involved in the Esscher transformation is chosen in such a way that discounted security price processes are martingales, it reflects a selection criterion on the set of equivalent martingale measures. The attractive property of the Esscher transform is that it can be justified on economic grounds with a representative agent that maximizes expected utility. In this section, we examine the analogy between the Esscher and the representative agent approach outlined in the previous section.

In our model, the underlying risk process $X = (X_t)_{0 \leq t \leq T}$ is described by the classical Cramér-Lundberg model (see equation (2.4)), i.e.

$$X_t = x_0 + p \cdot t - \sum_{k=1}^{N_t} Y_k, \tag{2.57}$$

with characteristics $(\lambda, dG(y))$ under the original probability measure P.

To define an equivalent martingale measure by the Esscher transform we need the following lemma:

Lemma 12 *The process*

$$\left(\frac{e^{hX_t}}{\mathbf{E}^P \left[e^{hX_t} \right]} \right)_{0 \leq t \leq T} \tag{2.58}$$

for some $h \in \mathbb{R} \setminus \{0\}$ is a martingale under P with mean 1 if and only if

$$ph + \lambda \cdot \left(\mathbf{E}^P \left[e^{-hY_1} \right] - 1 \right) = 0. \tag{2.59}$$

Proof. The proof is analogous to that of Proposition 8, p. 59. The infinitesimal generator $\mathcal{A}$ of the underlying process X is given by

$$\mathcal{A}(f)(x) = p \cdot \frac{d}{dx} f(x) + \lambda \cdot \int_0^\infty \left(f(x-y) - f(x) \right) dG(y), \tag{2.60}$$

for a given function f in the domain of the generator.

A stochastic process $(f(X_t))_{0\leq t\leq T}$ is a martingale under P if and only if

$$\mathcal{A}(f)(x) = 0,$$

that is

$$p \cdot \frac{d}{dx} f(x) + \lambda \cdot \int_0^\infty (f(x-y) - f(x))\, dG(y) = 0,$$

for all given values $X_t = x$.

Therefore, a necessary and sufficient condition for the process $\left(e^{hX_t}\right)_{0\leq t\leq T}$ to be a martingale under P is

$$ph + \lambda \cdot \left(\mathbf{E}^P\left[e^{-hY_1}\right] - 1\right) = 0.$$

■

If the parameters of the model satisfy Lundberg's fundamental equation (2.59) an equivalent probability measure P^h on $\mathcal{F}$ can be defined through the Radon-Nikodym derivative $\xi_T = \frac{dP^h}{dP}$ with associated density process

$$\xi_t = \frac{e^{hX_t}}{\mathbf{E}^P\left[e^{hX_t}\right]}, \tag{2.61}$$

for any $0 \leq t \leq T$. If we make use of the structure of the underlying process X we deduce

$$\xi_t = \exp\left(-\lambda\left(\mathbf{E}^P\left[e^{-hY_1}\right] - 1\right)t - h\sum_{k=1}^{N_t} Y_k\right). \tag{2.62}$$

Comparing the density process with its general representation in Section 2.2.4, p. 56

$$\xi_t = \exp\left(-\lambda(\kappa - 1)t + \sum_{k=1}^{N_t} \ln\left(\kappa v\left(Y_k\right)\right)\right), \tag{2.63}$$

we derive the following market prices of frequency and jump size risk

$$\kappa = \mathbf{E}^P\left[e^{-hY_1}\right] \tag{2.64}$$

$$v(y) = \frac{e^{-hy}}{\mathbf{E}^P\left[e^{-hY_1}\right]}, \tag{2.65}$$

for a given Esscher parameter $h \in \mathbb{R}\setminus\{0\}$.

We now observe the analogy to the representative agent approach derived in the previous section. Referring to the market prices of frequency risk (2.45) and jump size risk (2.46) of the representative investor, the Esscher parameter h can be interpreted as the negative coefficient of absolute risk aversion.

2.6 Risk Models Perturbed by Diffusion

In this section, we generalize the model by allowing the fundamental risk process $X = (X_t)_{0 \leq t \leq T}$ to be perturbed by diffusion, i.e. we consider the following process

$$X_t = x_0 + pt + \sigma W_t - \sum_{k=1}^{N_t} Y_k, \tag{2.66}$$

where $\sigma > 0$ and $W = (W_t)_{0 \leq t \leq T}$ is a standard Brownian motion that is assumed to be independent of the compound Poisson process. We assume that the probability measure P is already one under which discounted price processes are martingales. Its existence is guaranteed by the "no free lunch" condition.

X is a process with independent and stationary increments and the characteristic function under the probability measure P is given by

$$\begin{aligned} \chi_t^P(u) &= \mathbf{E}^P\left[e^{iuX_t}\right] \\ &= \exp\left(ix_0 u + \left(ipu - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\mathbf{E}^P\left[e^{-iuY_1}\right] - 1\right)\right)t\right). \end{aligned} \tag{2.67}$$

The infinitesimal generator $\mathcal{A}$ of X under the probability measure P for a given function f in the domain of the generator is given by

$$\begin{aligned} \mathcal{A}(f)(x) &= p \cdot \frac{d}{dx} f(x) + \frac{1}{2}\sigma^2 \cdot \frac{d^2}{dx^2} f(x) \\ &\quad + \lambda \cdot \int_0^\infty \left(f(x-y) - f(x)\right) dG(y). \end{aligned} \tag{2.68}$$

The technique we developed in the previous sections can be applied in the same manner and leads to the following results:

- The basic security price process

$$S_t = \exp(X_t) \tag{2.69}$$

is a martingale under the probability measure P if and only if

$$p+\frac{1}{2}\sigma^2+\lambda\left(\mathbf{E}^P\left[e^{-Y_1}\right]-1\right)=0. \tag{2.70}$$

- The function $f:\mathbb{R}_+\times[0,T]\rightarrow\mathbb{R}$ defining the price process $(f(X_t,t))_{0\leq t\leq T}$of a derivative with payoff $\phi(X_T)$ at maturity T of the contract under the probability measure P can be represented by

$$f(x,t)=\int_{-\infty}^{\infty}e^{iux}\chi_{T-t}(u)\,\check{\varphi}(u)\,du+k, \tag{2.71}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot)-k$ and $\chi_{X_{T-t}}(\cdot)$ is the characteristic function of X_{T-t} under P, i.e.

$$\chi_{T-t}(u)=\exp\left(ix_0u+\left(ipu-\frac{1}{2}\sigma^2u^2+\lambda\left(\mathbf{E}^P\left[e^{-iuY_1}\right]-1\right)\right)(T-t)\right).$$

2.7 Conclusion

In this chapter, we investigated the valuation of European financial derivatives that are based on insurance related risk. A combination of tools developed in both actuarial science and financial mathematics proved to be essential. On the one hand, classical models introduced in risk theory have been used for the evolution of underlying risk in the market. On the other hand, absence of arbitrage strategies in capital markets was the essential condition we based financial valuation on.

Using Fourier techniques, the set of derivative price processes that are consistent with the absence of arbitrage opportunities has been characterized by two parameters, the market price of frequency risk and claim size risk. For an arbitrary fixed pair of market prices, the inverse Fourier transform of the corresponding price process has been derived in closed form. The existence of a traded basic security imposes a restriction on the set of parameters of the model. Our technique achieves a high level of generality since no conditions are imposed on the distribution of claim sizes and few on the payoff structure of the contract. We examined the valuation of spreads in more detail since they exhibit the same payoff structure as limited stop loss reinsurance contracts.

In a market with a representative investor, we derived the unique market prices of risk and hence the unique derivative price process that is consistent with the investor's preferences. Equilibrium conditions have been used to endogenously determine a premium calculation principle. Additionally, we outlined the analogy to the selection criterion based on the Esscher transformation of probability measures.

Finally, we generalized the framework to allow the classical risk model to be perturbed by diffusion. The same Fourier technique can be applied and analogous results were derived.

Chapter 3

Financial and Actuarial Valuation in an Integrated Market

3.1 Introduction

The importance of the interface of capital markets and insurance markets has been increasingly emphasized by both the private and public sector. This economic and political debate has its roots in the growing concerns amongst individuals of the long-term risks over the lifecycle as the nature and magnitude of some of these risks have become apparent only recently. In the past 30 years, financial costs from natural catastrophes have risen, risk to social capital and risk of inflation have become more severe. These developments suggest that innovations in risk management would make a valuable contribution in reducing risk over individuals' lifecycle. In response, one major focus in recent years has been the idea of making risks tradeable in financial markets, that were traditionally spread through insurance and reinsurance contracts. This attempt at risk securitization results in the emergence of financial products that capture insurance related risks, e.g. catastrophe insurance derivatives, index-linked life insurance contracts, index-linked debt, or funded pension schemes.

This overlap of insurance and financial markets evokes several questions on risk valuation and suggests to examine the similarities and differences of methods that have been developed in both insurance mathematics and mathematical finance. Let us classify these issues and the related literature by two factors, the specification of the contracts that are available on the market and the source of uncertainty. To be more precise, our classification is based on whether the economy contains

- financial and/or insurance contracts

that are based on

- financial and/or insurance related risk.

The type of contract is related to the concept on which valuation is based on, whereas the type of underlying risk is connected to the appropriate class of stochastic processes that are used to model the evolution of market uncertainty.

Prior to the converge of capital and insurance markets, exclusively either financial contracts based on financial risk or insurance contracts based on insurance related risk have been introduced to the market. Stochastic models for the underlying risk processes and methods for the valuation of the corresponding contracts have been developed separately in mathematical finance and insurance mathematics. We refer to Bjørk [11], Duffie [28], and Musiela and Rutkowski [62] and the references therein for the former field of research, and to Bühlmann [16], Gerber [38], and Grandell [44] for the latter.

In a sequence of papers by Brennan and Schwartz [15], Bacinello and Ortu [6], and Nielsen and Sandmann [63], the pricing equity-linked life insurance contracts is investigated. The benefits of these insurance policies depend on the performance of a reference portfolio that is traded on the capital market. According to our classification, these contracts belong to a market containing insurance contracts that are based on both financial and insurance related risk in form of policyholders' mortality risk.

In Chapters 1 and 2 of this dissertation we looked into the valuation of financial contracts that are based on insurance related risk such as catastrophe insurance derivatives. In Section 2.6, p. 75 of Chapter 2 we added independent capital market noise modeled as a Wiener process. The stochastic processes that we used to model the underlying risk in such an environment are embedded in the class of semimartingales. We refer to Bühlmann et al. [17] for the use of semimartingales in mathematical finance. According to our classification, this economy is one that consists of financial contracts based on both insurance related and financial market risk.

In two articles by Delbaen and Haezendonck [25] and Sondermann [76] the authors show how premium calculation principles for reinsurance contracts can be embedded in a no-arbitrage framework. The important contribution of these papers lies in the construction of an analytical bridge between actuarial and financial valuation. Referring back to our classification, the authors investigate a market that consists of insurance contracts based on insurance related risk.

Recent papers by Schweizer [74] and Møller [60] and [61] consider a capital market in which a risk measure is a priori given that can be interpreted as an actuarial premium. The authors use an indifference argument based on the possibility of trading in financial instruments to transfer the a priori given risk measure into an a posteriori risk measure. The resulting measure can be interpreted as a financial premium.

We conclude that the literature, initiated by the convergence of capital and insurance markets, has separately focused on markets consisting of insurance contracts linked to financial market risk, on markets consisting of financial contracts based on insurance related risk, and on embedding actuarial valuation into a no-arbitrage framework.

In a global economy, in which capital and insurance markets merge, financial investors and insurance companies additionally trade in contracts of the other market with the aim of exploiting new investment opportunities and hedging instruments. It is therefore relevant to consider an economy in which both financial and insurance contracts coexist and to investigate price determination in view of this coexistence. This idea of an integrated market and the valuation therein captures exactly the aim of this chapter and our contribution to the existing literature.

To be more precise, we assume that an investor facing insurance related risk is able to sell off the risk. This possibility reflects the existence of an insurance contract in which the premium to be paid is specified. In addition, we assume the existence of a traded financial contract that securitizes the underlying risk in the form of a European derivative. To come back to our previous classification, we investigate a market consisting of financial and insurance contracts that are both based on insurance related risk.

One major difficulty in valuation of these contracts is the unpredictable nature of insurance related risk. This feature makes it impossible to synthetically provide a completely secure hedge by continuous trading in existing contracts. Our integrated market is thus incomplete and there exists an infinite collection of financial and insurance price processes that exclude arbitrage strategies.

With the aim of tackling the multiplicity of no-arbitrage prices, we require financial prices to be consistent with the actuarial valuation of the same underlying risk. We therefore introduce a new price process selection criterion in incomplete markets that originates in the coexistence of financial and insurance contracts. Additional to exclusion of arbitrage opportunities we thus demand financial prices to be robust with respect to this new selection criteria.

It can be shown that in general there exists still an infinite collection of financial price processes that are consistent with actuarial valuation. However, the additional selection criterion restricts the set of no-arbitrage price processes and we explicitly characterize this remaining set. Building on the representation of price processes that we deduced in Chapters 1 and 2, we show that the connection between financial and actuarial prices, emerging from actuarial consistency, is wholly incorporated in the characteristic function of the underlying risk. These results are valid for a very general class of premium calculation principles. We then pick out some principles that are commonly used by the insurance industry and investigate in more detail the set of financial price processes that correspond to the chosen premium principle.

The remainder of the paper is organized as follows: in Section 3.2 and 3.3 we introduce the fundamentals of the market, the underlying risk process and the contracts that are available in the market. Section 3.4 investigates actuarial and financial valuation and introduces the concept of consistency with insurance premia. In Section 3.5 we examine certain premium calculation principles in more detail before we conclude in Section 3.6.

3.2 The Fundamentals

In this section we introduce the stochastic structure and the underlying process that represents insurance related risk in the market. In addition, we briefly examine the change between equivalent probability measures and the inducing effect on the risk process.

3.2.1 Uncertainty

Uncertainty enters through different possible realizations ω of the world. All realizations are collected in a sample space Ω. An event is defined as a subset of Ω and $\mathcal{F}$ denotes the set of all possible events. We assume that $\mathcal{F}$ forms a σ-algebra. The likelihood of events is represented by a probability measure P that assigns probabilities to every event in $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, P)$ thus describes the stochastic foundation for the market on which all following random variables will be defined.

As we consider the stochastic evolution of prices we need to introduce time and the amount of information that is available to market participants at every point in time. We assume that the economy is of finite horizon $T < \infty$ and the flow of information is modeled by a nondecreasing family of σ-algebras $(\mathcal{F}_t)_{0 \leq t \leq T}$, a filtration. We assume that $\mathcal{F}_T = \mathcal{F}$, each $\mathcal{F}_t$ contains the events in $\mathcal{F}$ that are of P-measure zero, and the filtration is right-continuous, i.e.

$$\mathcal{F}_t = \mathcal{F}_{t+},$$

where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

In the following section we put more structure on the evolution of uncertainty by taking into account the features of insurance related risk.

3.2.2 Risk Process

Risk in an insurance context is caused by single events such as accidents, death, or natural catastrophes. One source of uncertainty is therefore the moment of events. Additionally one has to introduce some variable that measures the impact such an event has on the economy. Let us imagine that this variable measures insured losses and thus claims to be paid by an insurance company. Hence the magnitude of losses represents a second source of uncertainty in the economy.

We model the moments and magnitudes of events as sequences of random variables $(T_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ where T_k denotes the moment of the k-th event causing a corresponding loss of size Y_k. Let us now combine both moment and magnitude risk by introducing a stochastic process $X = (X_t)_{0 \leq t \leq T}$ where for each point in time t the random variable X_t represents the sum of claim amounts incurred in $(0, t]$. Therefore

$$X_t = \sum_{\{k | T_k \leq t\}} Y_k. \tag{3.1}$$

The stochastic process $X = (X_t)_{0 \leq t \leq T}$ is called accumulated claim process, also referred to as risk process in the literature.

We assume that the past evolution and current state of the risk process X is observable by every agent in the economy, i.e. X is assumed to be adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. For simplicity, we shall assume that X generates the flow of information, i.e. $\mathcal{F}_t = \sigma\left(\sigma\left(X_s,\, s \leq t\right) \cup \mathcal{N}\right)$ where $\mathcal{N}$ denotes the events of P-measure zero.

As regards occurrences of events we assume that the counting process $N = (N_t)_{0 \leq t \leq T}$ defined through

$$N_t = \sup\{k \geq 1 \,| T_k \leq t\} \tag{3.2}$$

is a homogeneous Poisson process with frequency parameter $\lambda \in \mathbb{R}_+$. The probability of k events occurring in the time interval $(0, t]$ is therefore

$$P\left[N_t = k\right] = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

with the expected number of events

$$\mathbf{E}^{P}\left[N_t\right] = \lambda t,$$

where $\mathbf{E}^{P}\left[\cdot\right]$ denotes the expectation operator under the measure P.

Furthermore, we assume that loss sizes Y_1, Y_2, Y_3, ... are independent and identically distributed random variables that are independent of the counting process N. Let G denote their common distribution function with support $(0, \infty]$.

In short, we model the risk process X as a compound Poisson process with characteristics $(\lambda, dG(y))$.

3.2.3 Equivalent Probability Measures

In this section, we briefly review the change between equivalent probability measures and the consequent change in characteristics of the process X. Section 1.3.2, p. 23 of Chapter 1 or Section 2.2.4, p. 56 of Chapter 2 provide a more detailed exposition.

Let us examine the set of probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to the "reference" measure P and that preserve the structure of the underlying risk process X, i.e. X is a compound Poisson process under the new probability measure Q. This set can be parameterized by a pair $(\kappa, v(\cdot))$ consisting of a nonnegative constant κ and a nonnegative, measurable function $v : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbf{E}^{P}\left[v(Y_1)\right] = 1$. The density process $\xi_t = \mathbf{E}^{P}\left[\xi_T \mid \mathcal{F}_t\right]$ of the Radon-Nikodym derivative $\xi_T = \frac{dQ}{dP}$ is then given by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} \ln\left(\kappa v\left(Y_k\right)\right) + \lambda\left(1-\kappa\right)t\right), \tag{3.3}$$

for any $0 \leq t \leq T$.

Let us denote the measure Q corresponding to the constant κ and the function $v(\cdot)$ by $P^{\kappa,v}$. Under the new measure $P^{\kappa,v}$ the process X is a compound Poisson process with characteristics $(\lambda\kappa, v(y)\,dG(y))$. κ can therefore be interpreted as market price of frequency risk, and $v(\cdot)$ as market price of claim size risk.

Changing the probability measure plays a central role in the context of valuation of both insurance and financial contracts. In the following section we introduce the contracts that are available on the market before proceeding to the pricing of these contracts.

3.3 The Market

Suppose an individual or a company is facing the risk process X, e.g. an insurance company that has to pay out claims to their policyholders. The company can make use of three assets that are traded continuously on the market:

- one risk-less bond with price process $B = (B_t)_{0 \leq t \leq T}$ and associated deterministic short rate of interest r. Without loss of generality, we assume $r \equiv 0$, i.e. $B_t \equiv 1$ for all $0 \leq t \leq T$;
- one insurance contract that specifies the premium process of the underlying risk process X;
- one European-style financial contract, i.e. at maturity T the contract's payoff depends on the realization of the risk process X_T only.

Let us define the specifications of the latter two risky assets in more detail.

3.3.1 The Insurance Contract

We consider the setup of Delbaen and Haezendonck [25] in which the insurance (reinsurance) contract allows the individual (insurance company) to sell off the risk of the remaining period. Let p_t denote the premium the individual (insurance company) has to pay at time t to sell the risk $X_T - X_t$ over the remaining period $(t, T]$.

The premium process $p = (p_t)_{0 \leq t \leq T}$ is a stochastic process that is assumed to be predictable, i.e. it is adapted to $(\mathcal{F}_{t-})_{0 \leq t \leq T}$, where $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$.

Remark 12 *Sondermann [76] considers dynamic reinsurance policies, i.e. the insurance company can decide to sell off a certain fraction of their risk and adjust* *their decision continuously. If the insurance company is allowed to only adjust at finitely many times this approach can be embedded in the framework of Delbaen and Haezendonck [25] by defining the maturities of several contracts accordingly.*

3.3.2 The Financial Derivative

We assume that the financial derivative securitizes insurance related risk reflected in the underlying risk process X. The buyer of this contract receives a certain payment at expiry T of the contract that depends solely on the realization of X_T. In exchange the seller of the contract receives a certain price that reflects the value of the payoff. The financial contract is therefore of European-style, i.e. early exercise is not allowed and the contract is path-independent.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function that specifies the buyer's payoff at maturity, i.e. at T the buyer receives $\phi(X_T)$. We shall assume the following integrability condition:

$$\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+), \tag{3.4}$$

from some $k \in \mathbb{R}$ where $\mathbf{L}^2(\mathbb{R}_+)$ is the set of measurable, square-integrable functions with respect to the Lebesgue measure.

Let the random variable π_t denote the price at time t that one has to pay in order to enter into the financial contract. Hence the stochastic process $\pi = (\pi_t)_{0 \le t \le T}$ is the financial price process that reflects the value of the payoff $\phi(X_T)$ at maturity T of the contract.

Financial contracts with a structure that is similar to existing insurance or reinsurance contracts are spread options that cover a certain layer of losses. These contracts with limited liability fulfill the integrability condition specified in equation (3.4).

In the following section we investigate the price process p and π of the insurance and financial contract in more detail.

3.4 No-Arbitrage Valuation

In this section, we examine the valuation of traded assets in the absence of arbitrage strategies. We define "no arbitrage" in the sense of "no free lunch" in Fritelli and Lakner [35] and refer to Section 1.4.1, p. 25 of Chapter 1 for a more detailed exposition.

First, we investigate the valuation of both the insurance and financial contracts under the assumption that the corresponding price processes exclude arbitrage opportunities. Thereafter, we introduce the additional restriction that financial prices should be consistent with actuarial pricing principles.

3.4.1 No-Arbitrage Insurance Premia

One ad-hoc approach of calculating insurance premia would be to take the mathematical expected value of the underlying risk. However, an insurance company charging such a "pure premium" would not be able to survive. Therefore, a sensible insurance premium should be greater than the "pure premium" and the additional increase should reflect the nature of the underlying risk and/or the agents' attitude towards risk. In practice, many different principles are used for calculating insurance premia. The loading factor could be just proportional to the underlying risk or it could take into account higher moments. Another loading factor could depend on agents' preferences that are reflected by some utility function. We refer to Goovaerts et al. [43] for a comprehensive outline of premium calculation principles.

Delbaen and Haezendonck [25] introduced the condition of "no-arbitrage" in an insurance market. Under the additional assumptions of liquidity and divisibility of insurance products, a premium calculation principle is deduced that includes commonly used principles as special cases. In fact, premia are calculated as expected values under a different, equivalent probability measure. A certain loading factor can then be obtained by choosing the equivalent probability measure accordingly.

Albrecht [4], in response to a paper by Venter [77], questions the implications of no-arbitrage in insurance markets, namely that identical risks will be insured at the same price and premium calculation principles have to be additive. The author concludes on p. 251:

> *"Statements on no-arbitrage premiums are completely non-informative for real insurance markets!"*

However, insurance premia can be thought of as if they emerge from a no-arbitrage framework. This standpoint has the advantage of providing a methodological link between financial and actuarial valuation. In this chapter, we deduce results for financial prices that are consistent with specific loading factors. Hence our results do not rely on the "no-arbitrage" framework in the insurance market and can be derived independently for different premium calculation principles. In Section 3.5, p. 93 we examine some commonly used principles. Nevertheless, let us briefly review the setup given by Delbaen and Haezendonck [25]:

Let us assume that the company's liabilities are of the form

$$X_t + p_t, \tag{3.5}$$

for all $0 \leq t \leq T$. The first component X_t denotes accumulated claims up to time t and the second component p_t describes the premium for which the insurance company can sell the risk of the remaining period $(t, T]$.

A trading strategy in this setup means the possibility of 'take-over' and the company's liabilities thus represent the underlying price process. According to the fundamental theorem of asset pricing (see Section 1.4.1, p. 25 of Chapter 1) the absence of arbitrage strategies implies the existence of a probability measure Q that is equivalent to the "reference" measure P and under which price processes are martingales.

If one further assumes that the predictable process $p = (p_t)_{0 \leq t \leq T}$ under Q is linear, i.e. of the form

$$p_t = p\,(Q)\,(T - t)\,, \tag{3.6}$$

then Delbaen and Haezendonck [25] conclude that the existence of sufficiently many reinsurance markets implies that the risk process X under Q is still a compound Poisson process.

As our risk process X has stationary and independent increments the martingale property implies that the premium density takes the form

$$\begin{aligned} p(Q) &= \mathbf{E}^{Q}[X_1] \\ &= \mathbf{E}^{Q}[N_1] \cdot \mathbf{E}^{Q}[Y_1]. \end{aligned} \tag{3.7}$$

In Section 3.2.3, p. 84 the set of equivalent probability measures that preserve the structure of the underlying risk process X has been characterized by the market price of frequency risk κ and the market price of claim size risk $v(\cdot)$. Using the notation $P^{\kappa,v}$ for an equivalent probability measure the premium density corresponding to the pair $(\kappa, v(\cdot))$ is given by

$$\begin{aligned} p(P^{\kappa,v}) &= \mathbf{E}^{P^{\kappa,v}}[X_1] \\ &= \mathbf{E}^{P^{\kappa,v}}[N_1]\,\mathbf{E}^{P^{\kappa,v}}[Y_1] \\ &= \lambda\kappa \cdot \mathbf{E}^{P}[Y_1 \cdot v(Y_1)]. \end{aligned} \tag{3.8}$$

As pointed out and shown in an explicit example by Barfod and Lando [8], the premium density is not in one-to-one correspondence to the set of equivalent measures. This is a crucial difference to the correspondence between financial prices of insurance derivatives and the set of equivalent measures. In Lemma 3, p. 30 of Chapter 1 we have shown that this correspondence is one-to-one for financial valuation.

In fact from the representation of the premium density (3.8) we deduce that there are infinitely many market prices of risk and therefore equivalent probability measures that lead to the same premium process. It is indeed this indeterminacy that does not pin down a unique financial price process under our additional requirement that financial prices should be consistent with actuarial valuation of the same underlying risk.

Before introducing this additional requirement let us review financial valuation of insurance-related risk as derived in Section 1.4.2, p. 26 of Chapter 1:

3.4.2 No-Arbitrage Financial Prices

We denote by π_t the value at time t of a financial contract that pays out $\phi(X_T)$ at maturity T. In the absence of arbitrage strategies the fundamental theorem of asset pricing (see Harrison and Kreps [45], Harrison and Pliska [46], Fritelli and Lakner [35]) implies that the price process $\pi = (\pi_t)_{0 \leq t \leq T}$ is a martingale under an equivalent probability measure $P^{\kappa,v}$. It can therefore be expressed as

$$\pi_t^{\kappa,v} = \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T)\,|\mathcal{F}_t\right], \tag{3.9}$$

for all $0 \leq t \leq T$ where the superscript κ, v states the dependence on the market prices of risk.

As the underlying risk process X is a Markov process and generates the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ π_t is of the form

$$\pi_t^{\kappa,v} = f^{\kappa,v}(X_t, t) = \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T)\,|X_t\right], \tag{3.10}$$

for some function measurable function $f^{\kappa,v}$ with $f^{\kappa,v}(X_T, T) = \phi(X_T)$.

Under the integrability assumption (3.4) $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+)$ we have shown in Proposition 2, p. 29 of Chapter 1 that the price function $f^{\kappa,v}$ can be represented as

$$f^{\kappa,v}(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}^{\kappa,v}(u)\,\check{\varphi}(u)\,du + k, \tag{3.11}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$ and $\chi_{X_{T-t}}^{\kappa,v}(\cdot)$ is the characteristic function of X_{T-t} under the probability measure $P^{\kappa,v}$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\left(\phi(z) - k\right)dz, \tag{3.12}$$

and

$$\chi_{T-t}^{\kappa,v}(u) = \exp\left(\lambda\kappa\left(\int_0^{\infty} e^{iuy}v(y)\,dG(y) - 1\right)(T-t)\right). \tag{3.13}$$

We deduce that the inverse Fourier transform of $f^{\kappa,v}(\cdot, t) - k$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left(f^{\kappa,v}(x,t) - k\right)dx = \chi_{T-t}^{\kappa,v}(u)\cdot\check{\varphi}(u) \tag{3.14}$$

is the product of two factors where the first, the characteristic function, contains the whole stochastic structure and the second solely depends on the contract's payoff. Therefore the characteristic function is the important component in linking financial prices with insurance premia under our concept of consistency that we introduce in the following section.

3.4.3 Actuarially Consistent No-Arbitrage Prices

This section presents an internal consistency requirement that we impose on financial prices in addition to the exclusion of arbitrage strategies. Although the consistency requirement reflects a further restriction on the possible dynamics of financial prices it is not strong enough to pin down a unique price process. Nevertheless, we characterize the remaining set of price processes and derive a connection between financial and actuarial prices.

As outlined above the market consists of an insurance contract and a financial contract that are both written on the same underlying risk process X. The insurance specifies a premium process $(p_t)_{0\leq t\leq T}$ of the linear form

$$p_t = p \cdot (T - t) \tag{3.15}$$

with premium density p for selling the remaining risk $X_T - X_t$. The financial contract specifies a price process $(\pi_t)_{0\leq t\leq T}$ for the payoff $\phi(X_T)$ at maturity.

Internal consistency should require that the financial valuation is consistent with actuarial valuation in the sense that market prices for frequency and claim size risk that lead to the specified premium density are inherent in financial prices.

The following proposition is the main result of this chapter linking financial with actuarial prices on the basis of internal consistency as described above.

Proposition 13 *Let $X = (X_t)_{0\leq t\leq T}$ be a compound Poisson process with characteristics $(\lambda, dG(y))$ and let $(p_t)_{0\leq t\leq T}$ be a linear premium process specified in the insurance contract. Suppose that the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ specifies the payoff of the financial contract at time T and satisfies the integrability condition $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+)$ for some $k \in \mathbb{R}$. Then for a given market price of frequency* *risk* $v : \mathbb{R}_+ \to \mathbb{R}$ *with* $\mathbf{E}^P [v(Y_1)] = 1$ *the function* $f^v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ *defining the financial price process* $(f^v(X_t, t))_{0\leq t \leq T}$ *that excludes arbitrage strategies and is consistent with the premium process can be represented as*

$$f^v(x,t) = \int_{-\infty}^{\infty} e^{iux} \exp\left(p_t \cdot \frac{\mathbf{E}^P\left[e^{iuY_1} \cdot v(Y_1) - 1\right]}{\mathbf{E}^P[Y_1 \cdot v(Y_1)]}\right) \check{\varphi}(u)\, du + k, \qquad (3.16)$$

where $\check{\varphi}(\cdot)$ *is the inverse Fourier transform of* $\phi(\cdot) - k$.

Proof. Internal consistency requires that the market prices of risk characterizing financial no-arbitrage prices lead to the same premium process. This set of market prices of frequency risk κ and claim size risk $v(\cdot)$ can be described by equation (3.8), that is

$$p = \lambda\kappa \cdot \mathbf{E}^P[Y_1 \cdot v(Y_1)],$$

and the corresponding premium process $(p_t)_{0\leq t\leq T}$ is thus given by

$$p_t = \lambda\kappa \cdot \mathbf{E}^P[Y_1 \cdot v(Y_1)] \cdot (T-t).$$

Substituting this expression into the representation (3.11) of no-arbitrage financial prices describes financial prices that are consistent with the specified premium process and completes the proof. ■

If we subtract k and apply the inverse Fourier transform on both sides of equation (3.16) we deduce for every given value $X_t = x$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left(f^v(x,t) - k\right) dx = \exp\left(p_t \cdot \frac{\mathbf{E}^P\left[e^{iuY_1} \cdot v(Y_1) - 1\right]}{\mathbf{E}^P[Y_1 \cdot v(Y_1)]}\right) \cdot \check{\varphi}(u),$$

or alternatively

$$p_t = \ln\left(\frac{\int_{-\infty}^{\infty} e^{-iux}\left(f^v(x,t) - k\right) dx}{\int_{-\infty}^{\infty} e^{-iux}\left(\phi(x) - k\right) dx}\right) \cdot \frac{\mathbf{E}^P[Y_1 \cdot v(Y_1)]}{\mathbf{E}^P\left[e^{iuY_1} \cdot v(Y_1) - 1\right]}.$$

We observe that financial prices under the additional requirement of actuarial consistency can still not be determined uniquely. Nevertheless, the set of prices can be parameterized solely by the market price of claim size risk. The indeterminacy is an implication of the fact that there are many market prices of risk that lead to the same actuarial price.

This is an important difference to financial prices where it is possible to back out market prices of risk from financial prices in a unique way (see Lemma 3, p. 30 of Chapter 1). We therefore conclude that a premium process is uniquely determined by requiring it to be consistent with a given financial price process as it uniquely determines the market prices of risk. The consistent premium density is then determined by equation (3.8).

In the following section, we investigate some premium calculation principles that are commonly used by the insurance industry and derive financial price processes that are actuarially consistent.

3.5 Premium Calculation Principles

As mentioned in the beginning of Section 3.4.1, p. 87 reasonable insurance premia contain a factor in addition to the "pure" mathematical expectation of the underlying risk. The explicit form of this loading factor differs depending on the risk's nature. In the no-arbitrage framework introduced by Delbaen and Haezendonck [25] this is reflected by the fact that the expected value of the underlying risk is taken under different probability measures. The additional factor is thus inherently related to the choice of the equivalent probability measure, i.e. to the market prices of frequency and claim size risk.

We examine three different premium calculation principles and derive a representation of the corresponding market prices of risk. This allows us to represent financial prices that are in line with the respective insurance premia.

3.5.1 Expected Value Principle

Under the expected value principle the premium density is given by

$$p = (1+\delta)\,\mathbf{E}^P[X_1] = (1+\delta)\,\lambda\mathbf{E}^P[Y_1]\,,$$

for some $\delta > 0$. This premium calculation principle is mainly used in life insurance because of the homogeneity of the collectives.

If we choose

$$\kappa = (1+\delta)\frac{\mathbf{E}^P[Y_1]}{\mathbf{E}^P[Y_1 \cdot v(Y_1)]}$$

as a function of $v(\cdot)$ with $\mathbf{E}^P[v(Y_1)] = 1$ we have thus characterized the set of parameters κ and $v(\cdot)$ that correspond to this premium calculation principle.

Furthermore, for any market price of claim size risk $v(\cdot)$ with $\mathbf{E}^P[v(Y_1)] = 1$ the function f^v defining the financial price process that is consistent with the expected value principle can be represented as

$$f^v(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}^v(u)\,\check{\varphi}(u)\,du + k,$$

where the characteristic function is given by

$$\chi_{T-t}^v(u) = \exp\left(\lambda(1+\delta)\cdot\frac{\mathbf{E}^P[Y_1]\cdot\mathbf{E}^P\left[e^{iuY_1}v(Y_1)-1\right]}{\mathbf{E}^P[Y_1 v(Y_1)]}\cdot(T-t)\right).$$

3.5.2 Variance Principle

The variance principle is mostly used in property and casualty insurance. It additionally includes fluctuations of X and the premium density is calculated according to

$$p = \lambda\left(\mathbf{E}^P[Y_1] + \beta\cdot\mathbf{Var}^P[Y_1]\right),$$

for some $\beta > 0$.

To be consistent with this premium density the market price of frequency risk for a given market price of claim size risk $v(\cdot)$ with $\mathbf{E}^P[v(Y_1)] = 1$ has to be determined through

$$\kappa = \frac{\mathbf{E}^P[Y_1] + \beta\cdot\mathbf{Var}^P[Y_1]}{\mathbf{E}^P[Y_1\cdot v(Y_1)]}.$$

The function f^v defining financial price processes that are consistent with this premium calculation principle can be represented as

$$f^v(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}^v(u)\,\check{\varphi}(u)\,du + k,$$

where the characteristic function is given by

$$\chi_{T-t}^v(u) = \exp\left(\frac{\lambda\cdot\left(\mathbf{E}^P[Y_1] + \beta\mathbf{Var}^P[Y_1]\right)\cdot\mathbf{E}^P\left[e^{iuY_1}v(Y_1)-1\right]}{\mathbf{E}^P[Y_1 v(Y_1)]}\cdot(T-t)\right).$$

3.5.3 Esscher Principle

The last example of premium calculation principles we investigate is the so-called Esscher principle that is gaining more and more attention as it can be derived from equilibrium analysis or from the minimization of a particular loss function. It is defined by a premium density of the form

$$p = \lambda \cdot \frac{\mathbf{E}^P\left[Y_1 e^{\gamma Y_1}\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right]},$$

for some $\gamma \in \mathbb{R} \backslash \{0\}$.

Here κ depends on the density function $v(\cdot)$ through

$$\kappa = \frac{\mathbf{E}^P\left[Y_1 e^{\gamma Y_1}\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right] \cdot \mathbf{E}^P\left[Y_1 \cdot v(Y_1)\right]},$$

and the function f^v defining the price process that corresponds to this premium principle for a given market price of claim size risk $v(\cdot)$ can be expressed as

$$f^v(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \chi_{T-t}^v(u)\, \check{\varphi}(u)\, du + k,$$

where the characteristic function is given by

$$\chi_{T-t}^v(u) = \exp\left(\frac{\lambda \cdot \mathbf{E}^P\left[Y_1 e^{\gamma Y_1}\right] \cdot \mathbf{E}^P\left[e^{iuY_1} v(Y_1) - 1\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right] \cdot \mathbf{E}^P\left[Y_1 v(Y_1)\right]} \cdot (T-t)\right).$$

3.6 Conclusion

In this chapter we investigated valuation in a market that contains both insurance and financial contracts written on the same underlying compound Poisson process. We examined both corresponding valuation principles - actuarial and financial - on the basis of excluding arbitrage opportunities and deduced a representation of prices for given market prices of frequency and claim size risk.

We introduced a new concept arising from internal consistency that originates in the coexistence of financial and insurance contracts. Financial prices should be consistent with the actuarial valuation of the insurance contract. Although financial prices cannot be uniquely determined, under this additional restriction on their dynamics, we characterized the set of prices that fulfill both absence of arbitrage and actuarial consistency. Through this characterization we established a link between financial price processes and insurance premia. This connection is wholly incorporated in the characteristic function of the underlying risk process.

We clarified that an important difference between financial and actuarial valuation is contained in the mapping between price processes and market prices of risk. The mapping between financial price processes and market prices of risk is one-to-one whereas there are infinitely many market prices of risk that lead to the same premium process. This implies that premium processes are uniquely determined by assuming them to be consistent with a given financial price process. However, consistency with a given premium process is not strong enough for financial prices to be uniquely determined.

Finally, we examined three premium calculation principles that are widely used by the insurance industry. A representation of financial price processes were derived that are consistent with the respective premium calculation principle.

Chapter 4

Conditional Moments Based on Lévy Processes

4.1 Introduction

The theory of stochastic processes is an approach to the mathematical modeling of changing 'objects' in an environment that encompasses uncertainty. The position of pollen grains in water, the price of tulips, and the number of customers in a queue are famous 'objects'. In an environment with changing 'objects' it is possible to introduce the concept of *time* through which information about past, present and future properties of these 'objects' is generated. Flow of information builds the basis for fundamental concepts to describe certain dependence structures over *time* and to transfer knowledge about past changes into predictions about future properties. Important dependence concepts are adaptation to information, martingales, the Markov property, stationarity, semimartingales, etc. These properties can be used to classify stochastic processes.

In an environment, in which 'objects' take real values, changes can be measured by their increments. We impose two dependence structures on the evolution of these 'objects': independence and stationarity of increments. The former concept describes changes that are independent of the past whereas the latter requires that distribution of changes do only depend on the time difference. The family of stochastic processes that is classified by these two properties is called the class of Lévy processes. This class comprises the Brownian motion and the Poisson process. Amongst many other 'objects', the former stochastic process is used to describe the position of particles or price evolution of stock prices, the latter is applied to an environment with random occurrences of customers or claims.

In Chapters 1 and 2, insured property losses due to natural catastrophe and the surplus of insurance companies were the 'objects' under investigation, changes of which were modeled by specific Lévy processes. In the context of a capital market, these 'objects' were used as building blocks to model the evolution of insurance derivative prices. If the market does not allow for arbitrage strategies, then price processes can be expressed as conditional expectations of the contracts' future payoffs. We then developed a method based on Fourier analysis to derive a representation of derivative prices that separates the stochastic structure of the underlying 'objects' from the contracts' payoff structure that depends in a deterministic way on the realization of our 'objects'.

The aim of this chapter is to generalize the Fourier technique that we developed in Chapters 1 and 2 as follows. The underlying 'objects' are allowed to evolve according to a general Lévy process. We are then interested in deriving an analogous representation for the conditional moments of random variables that are elements of the underlying 'object'.

To be more precise, we are interested in deriving a representation for the conditional moments of random variables that are of the form

$$\phi\left(X_T\right),$$

where the underlying random variable X_T is an element of a Lévy process $X = (X_t)_{t\in\mathbb{R}_+}$ and the function ϕ is assumed to satisfy a certain integrability condition.

We apply Fourier analysis and derive closed form expressions for the inverse Fourier transforms of conditional moments. Similar to the results in Chapters 1 and 2, the inverse Fourier transform is the product of two expressions, one of which comprises the underlying stochastic structure whereas the other depends solely on ϕ. Therefore, the stationarity and independence of increments enables us to decompose conditional moments of $\phi(X_T)$ into a stochastic and a deterministic component. Thereafter we apply the results in such a way that a representation of the conditional probability that the random variable X_T takes values in an interval is derived.

The remainder of this chapter is organized as follows: Section 4.2 introduces the stochastic fundamentals. In Section 4.3 we derive an expression for the conditional moments based on Lévy processes and in Section 4.4 we examine the conditional distribution function. Section 4.5 concludes.

4.2 Stochastic Elements

Let $\left(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P\right)$ be a stochastic basis where Ω is the space of elementary outcomes ω, $\mathcal{F}$ is a collection of subsets $A \subseteq \Omega$ called events and forming a σ-algebra, P is a probability measure on $\mathcal{F}$, and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a non-decreasing family of σ-algebras, called filtration. We assume that the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous, i.e.

$$\mathcal{F}_t = \mathcal{F}_{t+},$$

where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$. Furthermore, we assume that the stochastic basis is complete, i.e. $\mathcal{F}$ is completed by the sets of P-measure zero and each $\mathcal{F}_t$ contains the sets of $\mathcal{F}$ with P-measure zero.

In this framework, we introduce a stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$ that is adapted to the filtration $\mathbf{F}$ and $X_0 = 0$ P-almost surely with the following properties:

(1) almost all sample paths are right-continuous on $[0, \infty)$ with left limits on $(0, \infty)$,

(2) X has *increments independent of the past*, that is $X_t - X_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$,

(3) X has *stationary increments*, that is $X_t - X_s$ has the same distribution as X_{t-s} for all $0 \leq s < t < \infty$.

A stochastic process with these properties is called a Lévy process and belongs to the class of semimartingales that are Markovian. Well-known Lévy processes are Brownian motions and compound Poisson processes. Indeed, the only continuous Lévy processes are Brownian motions with drift.

It follows that for each $t > 0$ the random variable X_t is infinitely divisible, that is for any $n \geq 1$ it can be represented as the sum of n independent identically distributed random variables. Also the converse is true: any infinitely divisible distribution is the distribution of X_1 for some Lévy process X (see Rogers and Williams [71] Section I.28 or Feller [34] Section XVII.1).

Let $\chi_t(\cdot)$ be the characteristic function of X_t, i.e.

$$\chi_t(u) = \mathbf{E}\left[e^{iuX_t}\right], \tag{4.1}$$

where $\mathbf{E}[\cdot]$ denotes the expectation operator under the probability measure P.

The properties of Lévy processes imply that $\chi_0(u) = 1$, $\chi_{t+s}(u) = \chi_t(u)\cdot\chi_s(u)$, and $\chi_t(u) \neq 0$ for every t, u. As almost all sample paths are right-continuous we conclude that the characteristic function must be of the form

$$\chi_t(u) = e^{t\psi(u)}, \tag{4.2}$$

for some continuous function $\psi(\cdot)$ with $\psi(0) = 0$.

The classical Lévy-Khinchine theorem (see e.g. Rogers and Williams [72] Section VI.2) provides a representation of the cumulant characteristic function $\psi(\cdot)$, namely

$$\begin{aligned}\psi(u) &= i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\{|z|\geq 1\}} \left(e^{iuz} - 1\right)\nu(dz) \\ &\quad + \int_{\{|z|<1\}} \left(e^{iuz} - 1 - iuz\right)\nu(dz),\end{aligned} \tag{4.3}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, and ν is a measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \min\left(z^2, 1\right)\nu(dz) < \infty.$$

The characteristic function of a Lévy process X plays a central role in the representation of conditional moments that we present in the following section.

4.3 Representation of Conditional Moments

In this chapter we are interested in the conditional moments of

$$\phi(X_T),$$

from some $T \in \mathbb{R}_+$, and some function ϕ, i.e. we examine the random variable

$$\mathbf{E}[\phi^n(X_T)|\mathcal{F}_t], \tag{4.4}$$

for $n \geq 1$, $0 \leq t \leq T$, and where $\phi^n(X_T) = (\phi(X_T))^n$.

Lévy processes are Markov processes and the conditional moments M^n are thus of the form

$$M^n(X_t, t) = \mathbf{E}[\phi^n(X_T)|X_t], \tag{4.5}$$

where we assume that $M^n(x,t) < \infty$ for all values $X_t = x$.

Let us now present the main result of this chapter in the following proposition.

Proposition 14 *Let X be a Lévy process with characteristic function $\chi_t(u) = \mathbf{E}\left[e^{iuX_t}\right]$. For $n \geq 1$ let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$\phi^n(\cdot) - k \in \mathbf{L}^2(\mathbb{R}), \tag{4.6}$$

for some $k \in \mathbb{R}$, i.e. $\int_{-\infty}^{\infty} |\phi^n(x) - k|^2 dx < \infty$. Then the conditional n-th moment

$$M^n(X_t, t) = \mathbf{E}[\phi^n(X_T)|X_t]$$

admits the representation

$$M^n(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}(u)\,\check{\varphi}^n(u)\,du + k, \tag{4.7}$$

where $\check{\varphi}^n(\cdot)$ is the inverse Fourier transform of $\phi^n(\cdot) - k$, i.e.

$$\check{\varphi}^n(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}(\phi^n(z) - k)\,dz. \tag{4.8}$$

Proof. The proof is analogous to Proposition 2, p. 29 in Chapter 1 since stationarity and independence of increments were the only properties of the underlying process that were needed. Let us briefly review the technique:

The Fourier transform is a one-to-one mapping of $\mathbf{L}^2(\mathbb{R})$ onto itself and the inverse transform of a function $g \in \mathbf{L}^2(\mathbb{R})$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} g(x)\,dx.$$

Applying the Fourier transform and thereafter the inverse Fourier transform to the function $\phi^n(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ we can deduce

$$\phi^n(x) - k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux} e^{-iuz}\left(\phi^n(z) - k\right) dz du.$$

The n-th conditional moment can be rewritten as

$$\begin{aligned}
M^n(X_t, t) &= \mathbf{E}\left[\phi^n(X_T)\,|X_t\right] \\
&= \mathbf{E}\left[\phi^n(X_T) - k\,|X_t\right] + k \\
&= \frac{1}{2\pi}\mathbf{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iuX_T} e^{-iuz}\left(\phi^n(z) - k\right) dz du\,|X_t\right] + k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}\left[e^{iuX_T}\,|X_t\right] e^{-iuz}\left(\phi^n(z) - k\right) dz du + k \\
&= \int_{-\infty}^{\infty} \mathbf{E}\left[e^{iuX_T}\,|X_t\right] \check{\varphi}^n(u)\,du + k,
\end{aligned}$$

where we applied Fubini's theorem and $\check{\varphi}^n(\cdot)$ is inverse Fourier transform of $\phi^n(\cdot) - k$, i.e.

$$\check{\varphi}^n(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\left(\phi^n(z) - k\right) dz. \tag{4.9}$$

Stationarity and independence of increments of the Lévy process X implies

$$\begin{aligned}
\mathbf{E}\left[e^{iuX_T}\,|X_t\right] &= e^{iuX_t}\mathbf{E}\left[e^{iu(X_T - X_t)}\,|X_t\right] \\
&= e^{iuX_t}\mathbf{E}\left[e^{iuX_{T-t}}\,|X_t\right] \\
&= e^{iuX_t}\mathbf{E}\left[e^{iuX_{T-t}}\right] \\
&= e^{iuX_t}\chi_{T-t}(u).
\end{aligned}$$

Hence we deduce

$$M^n(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}(u)\,\check{\varphi}^n(u)\,du + k. \tag{4.10}$$

From this representation we derive the inverse Fourier transform of $M^n(\cdot,t)-k$ in closed form as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left(M^n(x,t)-k\right)dx = \chi_{T-t}(u)\cdot\check{\varphi}^n(u).$$

■

From the representation (4.7) we derive the inverse Fourier transform of $M^n(\cdot,t)-k$ in closed form as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left(M^n(x,t)-k\right)dx = \chi_{T-t}(u)\cdot\check{\varphi}^n(u). \tag{4.11}$$

We have therefore decomposed the inverse Fourier transform of $M^n(\cdot,t)-k$ into two components. One component, the characteristic function, completely covers the stochastic structure whereas the other depends solely on the function ϕ. The ratio

$$\frac{\int_{-\infty}^{\infty} e^{-iux}\left(M^n(x,t)-k\right)dx}{\chi_{T-t}(u)}$$

is thus deterministic, i.e. independent of the underlying probability measure P.

Setting $t=0$ we deduce a representation for unconditional n-th moment, namely

$$\mathbf{E}\left[\phi^n(X_T)\right] = \int_{-\infty}^{\infty}\chi_T(u)\,\check{\varphi}^n(u)\,du + k. \tag{4.12}$$

4.4 Conditional Distribution Function

In this section we derive a representation for the conditional distribution function of a family of random variables $(X_t)_{t\in\mathbb{R}_+}$ that evolve according to a Lévy process.

Let us fix a time horizon $T<\infty$ and define the conditional distribution function $F_t(\cdot)$ at $t\in\mathbb{R}_+$ as

$$F_t(x) = P\left[X_T < x\,|X_t\right], \tag{4.13}$$

for any $x\in\mathbb{R}$. We are interested in finding an expression for

$$F_t(M)-F_t(L) = P\left[L\le X_T<M\,|X_t\right]. \tag{4.14}$$

for given lower and upper values $-\infty < L < M < \infty$.

With this aim in view, we define a function $\phi : \mathbb{R} \to \mathbb{R}$ through

$$\phi(x) = \mathbf{1}_{[L,M)}(x), \tag{4.15}$$

where $\mathbf{1}_A(\cdot)$ denotes the indicator function on a Borel set A. This function has the property

$$\phi^n(\cdot) = \phi(\cdot) \in \mathbf{L}^2(\mathbb{R}),$$

and the inverse Fourier transform can be derived explicitly as

$$\begin{aligned} \check{\varphi}(u) &= \frac{1}{2\pi}\int_L^M e^{-iuz}dz \\ &= \frac{e^{-iuL} - e^{-iuM}}{2\pi iu}. \end{aligned} \tag{4.16}$$

We now apply representation (4.7) of the conditional moments in Proposition 14, p. 101 to the indicator function on $[L, M)$ and thus derive an expression for the conditional probability function (4.14)

$$\begin{aligned} M^n(X_t, t) &= \mathbf{E}\left[\mathbf{1}_{[L,M)}(X_T)\,|X_t\right] \\ &= P\left[L \le X_T < M\,|X_t\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iuX_t} \cdot \frac{e^{-iuL} - e^{-iuM}}{iu} \cdot \chi_{T-t}(u)\,du. \end{aligned} \tag{4.17}$$

For the inverse Fourier transform we deduce

$$\int_{-\infty}^{\infty} e^{-iux}P\left[L \le X_T < M\,|X_t = x\right]dx = \frac{e^{-iuL} - e^{-iuM}}{iu} \cdot \chi_{T-t}(u). \tag{4.18}$$

Finally, let us consider two special cases:

- For $L = -M$ we deduce

$$\int_{-\infty}^{\infty} e^{-iux}P\left[-M \le X_T < M\,|X_t = x\right]dx = \frac{2\sin(Mu)}{u} \cdot \chi_{T-t}(u). \tag{4.19}$$

- For $t = 0$ we derive an expression for the unconditional probability as

$$\begin{aligned} P\left[L \le X_T < M\right] &= F_0(M) - F_0(L) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iuL} - e^{-iuM}}{iu} \cdot \chi_T(u)\,du. \end{aligned} \tag{4.20}$$

4.5 Conclusion

We derived a representation of conditional moments and the distribution function of a family of random variables $(\phi(X_t))_{t\in\mathbb{R}_+}$ that depends on the flow of a Lévy process $(X_t)_{t\in\mathbb{R}_+}$. The technique developed in this paper is based on Fourier analysis and leads to closed formulae for the inverse Fourier transform of the conditional moments and distribution function. Furthermore, the representation splits the conditional moments and distribution functions into a component that captures the stochastic structure in form of the characteristic function and a component that solely depends on the transformation ϕ.

Bibliography

[1] Aase, K. (1995). "Catastrophe Insurance Futures Contracts", Norwegian School of Economics and Business Administration, Institute of Finance and Management Science, Working Paper 1/95.

[2] Aase, K. (1999). "An Equilibrium Model of Catastrophe Insurance Futures and Spreads", *Geneva Papers on Risk and Insurance Theory* **24**, 69-96.

[3] Aase, K. (2000). "An Equilibrium Asset Pricing Model Based on Lévy Processes: Relations to Stochastic Volatility, and the Survival Hypothesis", *Insurance: Mathematics and Economics* **27**, 345-363.

[4] Albrecht, P. (1992). "Premium Calculation without Arbitrage? A Note on a Contribution by G. Venter", *ASTIN Bulletin* **22**:2, 247-254.

[5] Avram, F., T. Chan, and M. Usabel (1999). "On the Valuation of Constant Barrier Options under Spectrally One Sided Exponential Lévy Models and Carr's Approximation for American Puts", Heriot-Watt University of Edinburgh, Department of Actuarial Mathematics and Statistics, Working Paper.

[6] Bacinello, A. R., and F. Ortu (1993). "Pricing Equity-Linked Life Insurance with Endogenous Minimum Guarantees", *Insurance: Mathematics and Economics* **12**, 245-257.

[7] Balbás, A., I. R. Longarela, J. J. Lucia (1999). "How Financial Theory Applies to Catastrophe-Linked Derivatives - An Empirical Test of Several Pricing Models", *The Journal of Risk and Insurance* **66**:4, 551-582.

[8] Barfod, A. M., and D. Lando (1996). "On Derivative Contracts on Catastrophe Losses", University of Copenhagen, Institute of Mathematical Statistics, Working Paper.

[9] Beals, R. (1987). *Advanced Mathematical Analysis.* (3rd printing) Graduate Texts in Mathematics 12. Springer-Verlag, Berlin, Heidelberg, New York.

[10] Billingsley, P. (1995). *Probability and Measure.* (3rd ed.) John Wiley & Sons, New York.

[11] Bjørk, T. (1998). *Arbitrage Theory in Continuous Time.* Oxford University Press, Oxford, New York.

[12] Black, F., and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy* **81**:3, 637-654.

[13] Bladt, M., and T. H. Rydberg (1998). "An Actuarial Approach to Option Pricing under the Physical Measure and without Market Assumptions", *Insurance: Mathematics and Economics* **22**, 65-73.

[14] Brémaud, P. (1981). *Point Processes and Queues, Martingale Dynamics.* Springer Series in Statistics. Springer-Verlag, Berlin, Heidelberg, New York.

[15] Brennan, M. J., and E. S. Schwartz (1976). "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee", *Journal of Financial Economics* **3**, 195-213.

[16] Bühlmann, H. (1996). *Mathematical Methods in Risk Theory.* (2nd printing) Grundlehren Math. Wiss. Bd. 172. Springer-Verlag, Berlin, Heidelberg, New York.

[17] Bühlmann, H., F. Delbaen, P. Embrechts, and A. N. Shiryaev (1996). "No-Arbitrage, Change of Measure and Conditional Esscher Transform", *CWI Quarterly* **9**:4, 291-317.

[18] Chicago Board of Trade (1995). "PCS Catastrophe Insurance Options: A User's Guide"

[19] Cox, S. H., and R. G. Schwebach (1992). "Insurance Futures and Hedging Insurance Price Risk", *The Journal of Risk and Insurance* **59**, 628-644.

[20] Cummins, J. D. (1990). "Asset Pricing Models and Insurance Ratemaking", *ASTIN Bulletin* **20**:2, 125-166.

[21] Cummins, J. D., and H. Geman (1994). "An Asian Option Approach to Valuation of Insurance Futures Contracts", *Review Futures Markets* **13**, 517-557.

[22] Dassios, A., and P. Embrechts (1989). "Martingales and Insurance Risk", *Communications in Statistical and Stochastic Models* **5**, 181-217.

[23] Davis, M. H. A. (1993). *Markov Models and Optimization.* Monographs on Statistics and Applied Probability 49. Chapman & Hall, London.

[24] Daykin, C. D., T. Pentikäinen, and M. Pesonen (1994). *Practical Risk Theory for Actuaries.* Monographs on Statistics and Applied Probability 53. Chapman & Hall, London.

[25] Delbaen, F., and J. Haezendonck (1989). "A Martingale Approach to Premium Calculation Principles in an Arbitrage-free Market", *Insurance: Mathematics and Economics* **8**, 269-277.

[26] Delbaen, F., and W. Schachermayer (1997). "Non-Arbitrage and the Fundamental Theorem of Asset Pricing: Summary of Main Results", Proceedings of Symposia in Applied Mathematics, Volume 00.

[27] Doherty, N. A. (2000). *Integrated Risk Management: Techniques and Strategies for Reducing Risk.* McGraw-Hill, New York.

[28] Duffie, D. (1996). *Dynamic Asset Pricing Theory.* (2nd ed.) Princeton University Press, Princeton.

[29] Duffie, D., J. Pan, and K. Singleton (2000). "Transform Analysis and Asset Pricing for Affine Jump-Diffusions", *Econometrica* **68**, 1343-1376.

[30] Embrechts, P. (2000). "Actuarial versus Financial Pricing of Insurance", ETH Zürich, Department of Mathematics, Working Paper.

[31] Embrechts, P., R. Frey, and H. Furrer (1998). "Stochastic Processes in Insurance and Finance", ETH Zürich, Department of Mathematics, Working Paper.

[32] Embrechts, P., C. Klüppelberg, and T. Mikosch (1997). *Modelling Extremal Events.* Applications of Mathematics 33. Springer-Verlag, Berlin, Heidelberg, New York.

[33] Embrechts, P., and S. Meister (1997). "Pricing Insurance Derivatives, the Case of CAT-Futures", Proceedings of the 1995 Bowles Symposium on Securitization of Risk, George State University Atlanta, Society of Actuaries, Monograph M-FI97-1, 15-26.

[34] Feller, W. (1971). *An Introduction to Probability Theory and its Applications.* (Volume II, 2nd ed.) John Wiley & Sons, New York.

[35] Fritelli, M., and P. Lakner (1995). "Arbitrage and Free Lunch in a General Financial Market Model; The Fundamental Theorem of Asset Pricing", in M. H. A. Davis *et al.* (eds), *Mathematical Finance.* The IMA Volumes in Mathematics and its Applications Volume 65. Springer-Verlag, Berlin, Heidelberg, New York.

[36] Gasquet, C., and P. Witomski (1999). *Fourier Analysis and Applications.* (translated by R. Ryan) Texts in Applied Mathematics 30. Springer-Verlag, Berlin, Heidelberg, New York.

[37] Geman, H., and M. Yor (1997). "Stochastic Time Changes in Catastrophe Option Pricing", *Insurance: Mathematics and Economics* **21**, 185-193.

[38] Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory.* Huebner Foundation Monographs 8, distributed by R. D. Irwin Inc., Homewood Illinois.

[39] Gerber, H. U., and B. Landry (1998). "On the Discounted Penalty at Ruin in a Jump-Diffusion and the Perpetual Put Option", *Insurance: Mathematics and Economics* **22**, 263-276.

[40] Gerber, H. U., and E. S. W. Shiu (1996). "Actuarial Bridges to Dynamic Hedging and Option Pricing", *Insurance: Mathematics and Economics* **18**, 183-218.

[41] Gerber, H. U., and E. S. W. Shiu (1999). "From Ruin Theory to Pricing Reset Guarantees and Perpetual Put Options", *Insurance: Mathematics and Economics* **24**, 3-14.

[42] Gihman, I. I., and A. V. Skohorod (1979). *The Theory of Stochastic Processes III.* Grundlehren Math. Wiss. Bd. 210, 218, 232. Springer-Verlag, Berlin, Heidelberg, New York.

[43] Goovaerts, M. J., F. de Vylder, and J. Haezendonck (1984). *Insurance Premiums: Theory and Applications.* North-Holland, Amsterdam, New York, Oxford.

[44] Grandell, J. (1991). *Aspects of Risk Theory.* Springer Series in Statistics. Springer-Verlag, Berlin, Heidelberg, New York.

[45] Harrison, J. M., and D. M. Kreps (1979). "Martingales and Arbitrage in Multiperiod Securities Markets", *Journal of Economic Theory* **20**, 381-408.

[46] Harrison, J. M., and S. R. Pliska (1981). "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and their Applications* **11**, 215-260.

[47] Ingersoll, J. E., Jr. (1987). *Theory of Financial Decision Making.* Rowman & Littlefield Publishers, Maryland.

[48] Jacobsen, M. (1999). "Marked Point Processes and Piecewise Deterministic Processes", University of Copenhagen, Department of Theoretical Statistics, Lecture Notes.

[49] Jacod, J., and A. N. Shiryaev (1987). *Limit Theorems for Stochastic Processes.* Grundlehren Math. Wiss. Bd. 288. Springer-Verlag, Berlin, Heidelberg, New York.

[50] Karatzas, I., and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus.* (2nd ed.) Graduate Texts in Mathematics 113. Springer-Verlag, Berlin, Heidelberg, New York.

[51] Karatzas, I., and S. E. Shreve (1998). *Methods of Mathematical Finance.* Applications of Mathematics 39. Springer-Verlag, Berlin, Heidelberg, New York.

[52] Karlin, S., and H. M. Taylor (1981). *A Second Course in Stochastic Processes.* Academic Press, New York.

[53] Klüppelberg, C., and T. Mikosch (1997). "Large Deviations of Heavy-tailed Random Sums with Applications in Insurance and Finance", *Journal of Applied Probability* **34**, 293-308.

[54] Last, G., and A. Brandt (1995). *Marked Point Processes on the Real Line, the Dynamic Approach.* Probability and its Applications. Springer-Verlag, Berlin, Heidelberg, New York.

[55] Levi, Ch., and Ch. Partrat (1991). "Statistical Analysis of Natural Events in the United States", *ASTIN Bulletin* **21**:2, 253-276.

[56] Lipster, R. Sh., and A. N. Shiryaev (1998). "Stochastic Calculus on Filtered Probability Spaces", Chapter 3 in Yu. V. Prokhorov, and A. N. Shiryaev (eds.), *Probability Theory III.* Encyclopaedia of Mathematical Sciences Volume 45. Springer-Verlag, Berlin, Heidelberg, New York.

[57] Mann, S. V., and G. Niehaus (1992). "The Trading of Underwriting Risk; an Analysis of Insurance Futures Contracts and Reinsurance", *The Journal of Risk and Insurance* **59**, 601-627.

[58] Meister, S. (1995). "Contributions to the Mathematics of Catastrophe Insurance Futures", ETH Zürich, Department of Mathematics, Diplomarbeit.

[59] Merton, R. (1973). "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science* **4**, 141-183.

[60] Møller, T. (2001). "On Transformations of Actuarial Valuation Principles", *Insurance: Mathematics and Economics* **28**, 281-303.

[61] Møller, T. (2001). "Indifference Pricing of Insurance Contracts: Applications", University of Copenhagen, Laboratory of Actuarial Mathematics, Working Paper 171.

[62] Musiela, M., and M. Rutkowski (1997). *Martingale Methods in Financial Modelling.* Applications of Mathematics 36. Springer-Verlag, Berlin, Heidelberg, New York.

[63] Nielsen, J. A., and K. Sandmann (1995). "Equity-Linked Life Insurance: A Model with Stochastic Interest Rates", *Insurance: Mathematics and Economics* **16**, 225-253.

[64] O'Brien, T. (1997). "Hedging Strategies Using Catastrophe Insurance Options", *Insurance: Mathematics and Economics* **21**, 153-162.

[65] Paulsen, J. (1998). "Ruin Theory with Compounding Assets - A Survey", *Insurance: Mathematics and Economics* **22**, 3-16.

[66] Pipkin, A. C. (1991). *A Course on Integral Equations.* Texts in Applied Mathematics 9. Springer-Verlag, Berlin, Heidelberg, New York.

[67] Protter, P. (1995). *Stochastic Integration and Differential Equations.* (3rd ed.) Applications of Mathematics 21. Springer-Verlag, Berlin, Heidelberg, New York.

[68] Raible, S. (2000). "Lévy Processes in Finance: Theory, Numerics, and Empirical Facts", University of Freiburg, Department of Mathematics, Ph.D. Thesis.

[69] Rao, B. L. S. Prakasa (1999). *Semimartingales and their Statistical Inference.* Monographs on Statistics and Applied Probability 83. Chapman & Hall, London.

[70] Revuz, D., and M. Yor (1999). *Continuous Martingales and Brownian Motion.* (3rd ed.) Grundlehren Math. Wiss. Bd. 293. Springer-Verlag, Berlin, Heidelberg, New York.

[71] Rogers, L. C. G., and D. Williams (2000). *Diffusions, Markov Processes and Martingales.* (Volume 1) Cambridge University Press, Cambridge.

[72] Rogers, L. C. G., and D. Williams (2000). *Diffusions, Markov Processes and Martingales.* (Volume 2) Cambridge University Press, Cambridge.

[73] Rolski, T., H. Schmidli, V. Schmidt, and J. Teugels (1998). *Stochastic Processes for Insurance and Finance.* John Wiley & Sons, New York.

[74] Schweizer, M. (2001). "From Actuarial to Financial Valuation Principles", *Insurance: Mathematics and Economics* **28**, 31-47.

[75] Singleton, K. (2001). "Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function", forthcoming in *Journal of Econometrics.*

[76] Sondermann, D. (1991). "Reinsurance in Arbitrage-free Markets", *Insurance: Mathematics and Economics* **10**, 191-202.

[77] Venter, G. G. (1991). "Premium Calculation Implications of Reinsurance without Arbitrage", *ASTIN Bulletin* **21**:2, 223-230.

[78] Wang, G., and R. Wu (2000). "Some Distributions for Classical Risk Process that is Perturbed by Diffusion", *Insurance: Mathematics and Economics* **26**, 15-24.