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Path-dependent functionals of Constant Elasticity of Variance and related processes: distributional results and applications in Finance.

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Abstract

The present thesis provides an analysis of some path-dependent functionals of Constant Elasticity of Variance (CEV) processes. More precisely, we study the continuous arithmetic average of the process over time, plain or sometimes multiplied by a knock-out indicator. We start by describing its mathematical properties and provide new distributional results (moments, densities, moment generating function among others). Some of these results also pertain to the joint distribution of the integral and the process itself. The versatility of the process enables us to consider diverse financial applications: fixed and floating strike Asian options on equities, European vanilla options on equity in the presence of stochastic volatility as well as zero-coupon bonds, guaranteed endowment options and average-rate claims under stochastic interest rates.

We devote a great part of the present work to the square-root process and the geometric Brownian motion, two important subcases of the CEV process. For both these nested diffusions, a number of mathematical and financial quantities have been solved for in the literature in closed-form, in terms of Laplace transforms. In this thesis, we derive these quantities in a fully explicit form, which is advantageous both from a theoretical point of view, to gain insight in their mathematical structure and from a practical stand, as the numerical evaluation of our formulae appear more robust and efficient than other numerical methods for some ranges of parameters. In the general CEV case, for which the integrated process has scarcely been considered in the literature, we derive semi-closed form expressions.

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Introduction

Option pricing theory - an endeavor to reduce the likelihood of being *fooled by randomness*, as Nassim Taleb would claim, but also a fascinating field with a relatively long history. By 1900, Bachelier first suggested a fair game approach to a world driven by a gaussian underlying security price, suspicious idea when examined under the utility and risk-aversion theory light. Shouldn't an option be worth less than its fair value for a risk-averse individual? This puzzle was solved only seventy-three years later by Black and Scholes who proved that a call option can be perfectly replicated by a self-financing portfolio of cash and asset under the assumption that the security follows a geometric Brownian motion with constant volatility, that there is no short-selling constraint and no transaction cost and finally that absence of arbitrage opportunity prevails. Their work represents the cornerstone of financial mathematics and the ulterior option pricing results derived under the same set of assumptions have generally - respectfully or affectionately - referred to this model as the Black-Scholes environment. Most of the subsequent research has been heading towards either relaxing one or more of these basic assumptions, specially the distributional hypothesis (jump diffusion: Merton [55], etc., constant elasticity of variance processes: Cox and Ross [16], Beckers [6], etc., stochastic volatility models: Hull and White [41], Stein and Stein [67], Heston [40], etc., Levy processes, etc.) or towards valuing more complex exotic or path-dependent instruments (barrier options: Merton [54], etc., lookback options: Goldman and al. [36], occupation time derivatives: Dassios [20], etc.).

In the present work, we focus on path-dependent functionals of constant elasticity of variance (called CEV hereafter) processes, more precisely on temporal integrals of these processes, either unrestrained or constrained by a knock-out condition, i.e. multiplied by the indicator of the underlying process not reaching a given high level before some maturity date. The versatility of the CEV process enables us to develop applications in different main branches of mathematical finance, since it has been used to model equities, interest rates and other stochastic volatilities.

Our analysis, i.e. this thesis, is constituted of four chapters. The first and most general one collects a summary of the distributional attributes and a thorough analysis of the CEV processes. We first deal with the square-root process as it could be hailed as the root or rather the common ancestor of all the processes encountered in this thesis. Indeed, both the CEV process of elasticity strictly less than unit and the geometric Brownian motion originate from the square-root process through a power transformation and respectively a random time-change. We therefore devote a major part of the chapter to the square-root case. After getting better acquainted with the Feller [29] process itself by studying its different mathematical properties, we present various new results pertaining to the integrated process, starting with its moments. The moments have been used in the literature, typically for stochastic volatility models (see Ball and Roma [5] for example). However, they are usually obtained by successive differentiation of the moment generating function of the integral - which can prove tedious- and thus quoted only up the fourth-order. We show that the joint moments of the process and its integral are expressible as a polynomial of exponentials and powers of time whose coefficient can be computed with a simple recursion. We also show that knowing these moments is of paramount importance since they determine the joint distribution. Though this result theoretically enables us to express a great number of quantities related to the integral, we prefer a more direct approach to the issue of determining the joint distribution and to this effect, employ the joint moment generating function (abbreviated as MGF throughout the thesis) of the process and its integral as our starting point. It turns out that the problem boils down to considering the - relatively simpler - corresponding squared Bessel process, since we prove that a general square-root process can be brought back to its squared Bessel counterpart with an appropriate change of measure. Other links between these processes have been provided in the literature. Yet, our change of measure result is stronger in the sense that it allows to study path-dependent properties in a simple way. It noticeably enables us to derive the joint distribution as an explicit series, both for the mean-reverting and the non mean-reverting case, which needs a careful treatment to account for the absorption at the origin. This formulation turns to be quite simple for the non mean-reverting case. However, a simpler expression can be obtained for the marginal distribution of the integral in the mean-reverting case from a different construction of the inverse Laplace transform of the moment generating function. All of these results are novel to the best of our knowledge and have a number of applications, some of which are discussed in the following chapters. Having thoroughly examined the square-root case, we move to another important subcase, the geometric Brownian motion case and then to the general CEV process. For the Geometric Brownian motion, we will only present an overview of the research developed so far, as we will directly look at this process with the motivation of pricing Asian options rather than deriving results on the already well-studied distribution itself. All our new results concerning this process are therefore left to be presented in the second chapter. We finally study the general CEV case for elasticities strictly between 0 and 1. Showing how they relate to the square-root process, we deduce a number of properties, among which a generalisation of the change of measure which simplifies the diffusion equation in the square-root case. We then attempt to characterise the distribution of the integrated process through its moment generating function. Under the least restrictive hypothesis that the elasticity is a rational number, we provide an expression for the Laplace transform with respect to time of the moment generating function resulting from a second-order inhomogeneous differential equation. This approach has never been attempted before, to the best of our knowledge, and helps us to gain insight into the mathematical structure of the problem.

The second chapter focuses on the pricing of Asian options on equities, derivatives of considerable importance both for market practitioners and financial theorists. Asian type of options are advantageous treasury management tools as well as safer structures for thinly-traded assets whose price could be hugely impacted by one large enough market participant. They are also renowned as one of the most difficult to evaluate path-dependent options as testified by the concentration of research and diversity of approaches explored for this problem. We concentrate on continuous arithmetic averages, i.e. underlyings related to the temporal integral of the process. Given the richness of the field, a review of the various results and main techniques proposed in the literature appears a compulsory step and is carried out first. We then explore the most commonly adopted model, the geometric Brownian motion, i.e. the Black and Scholes environment. Our contribution consists in the derivation of fully explicit forms for the prices of both floating and fixed-strike Asian options.

We provide as well a synthesis of different formulations reducing the problem to a one-factor Markovian one, either by time-reversal arguments or by working under the asset-numeraire. Along these lines, we present a simple alternative derivation of the *Geman and Yor* [77] Laplace transform of the Asian call price, methodology later generalised to handle jump processes. We then invert analytically this Laplace transform by contour integration for different types of options. A second approach, which is to bound the state space by absorbing the process at a high level, enables us to derive these prices as an eigenfunction series instead of an integral. We finally show how these Laplace transforms can be modified under the extended model including multiplicative jumps on the underlying, which completes our study of the geometric Brownian motion case. In the last part of this section, we show that simpler explicit solutions can be obtained under the alternative square-root model of Cox and Ross [16] as an application of the results derived in the first chapter. This makes this model quite interesting for testing and risk-management purposes.

Interest rates derivatives are another main branch of application and the subject of Chapter 3. The Cox Ingersoll Ross [15] (CIR) model based on a square-root process for the instantaneous rate constitutes a benchmark model for its combined tractability and adequacy to what is expected from a short rate process evolution. This model is generally labeled as tractable for the existence of an explicit formulation for bond prices. Yet, a number of other derivatives can only be obtained in term of Laplace transforms. The distributional results derived in the first chapter allows us to further the analysis of this model and provide explicit prices for guaranteed endowment options as well digital and regular average-rate claims. As any interest rate derivative depends on the cumulated rate, other applications could be considered as well. A second model is also treated in this chapter, the Chan, Karolyi, Longstaff and Sanders [13] popular and empirically validated model featuring the short rate as a CEV process. Unlike in the CIR case, no explicit solutions could be given. We however extend the results of the first chapter on equities CEV and propose a semi-closed expression for the Laplace transform of the zero-coupon bonds prices with respect to the maturity. A fast Fourier inversion of this transform would be an effective solution when a whole yield-curve is needed, as often happens in practice.

The fourth chapter takes us back to the equity world. However, instead of considering elaborate path-dependent derivatives, we will study standard vanilla options under more complex models including stochastic volatility. The classical stochastic volatility processes considered in the financial literature are typical subcases of the CEV process: Geometric Brownian motion for the Hull-White model [41], Ornstein-Uhlenbeck for the Stein and Stein [67] model and square-root process for the Heston [40] model. Synthetising these original models first, we propose an intuitive way of using moment generating and characteristic functions to recover very simply the characteristic function of the log-asset under the last two models. We also extend this methodology to derive novel close-form solutions for the Hull-White case. We then focus on moment-based approximates for the options prices, first by extending and comparing several approximation previously proposed for the Hull and White model and then by finding convergent series based on polynomial expansions.

A couple of remarks should be added before finishing this introduction. Except for the distributional hypothesis in some cases, all the standard assumptions of the Black and Scholes world hold throughout this thesis, namely, no short-selling constraint, no transaction cost and no free-lunch. We work on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and always place ourselves under the risk-neutral measure unless otherwise specified. W_t (or (W_t^1, W_t^2) in bivariate cases) denote a standard Brownian motion under this measure. The work of Harrison and Kreps [39] then equates the value at time t of any derivative with the expectation of its payoff conditionally on $\mathcal{F}_t$ under the risk-neutral measure. All the option prices handled in this thesis are derived as such expectations.

We finally wish to point out some abbreviations used throughout this thesis, some of them have already been defined in this introduction. We abbreviate Constant Elasticity of Variance by CEV, Moment Generating Function by MGF, Stochastic Differential Equation by SDE, Partial Differential Equation by PDE and Ordinary Differential Equation by ODE.

Chapter 1

Distributional results

The popularity of the CEV process in all main branches of financial modelling can be explained by its desirable property of positivity and its richness of behaviour: indeed, depending on its parameters, the process can be mean-reverting or exploding, the boundary at 0 can be absorbing, reflecting ... It has been used to model equities, interest rates, stochastic volatility and other financial quantities. Our goal, here, is to derive some properties and distributional results for these processes.

We first consider the square-root process, which is the most tractable since it allows for fully explicit formulations for various quantities. It is also closely linked with the more general CEV process, as explained in the introduction. We start by presenting some of the main properties of the square-root process given in the literature. We then derive the joint moments of the process and its average. The average moments have indeed an important informational content, since they are proven to actually determine the distribution of the average. A more direct approach yet allows us to determine in a more efficient way the joint distribution of the process and its integral by analytic Laplace transform inversion, using a simplifying measure change relating the square-root process to its square Bessel process counterpart. We then proceed to find a simpler expression for the marginal distribution of the average in the mean-reverting case. This completes our study of the square-root case. We then present the main classical results concerning the distribution of the temporal integral of a geometric Brownian motion. Finally, we turn to the general CEV process for elasticities strictly between 0 and 1 and attempt to solve for its moment generating function under the non restrictive assumption that the elasticity is a rational number.

1.1 The square-root process

Defining first the notations used in this section, X_t will represent a smooth version of the square-root process following the stochastic differential equation:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t \tag{1.1}$$

with $(a, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $b \in \mathbb{R}$.

The next part of this section collects a number of important properties of X_t, some of which are well-known[1] but need to be recalled to provide a deeper understanding of the process structure. X_t may sometimes be referred to in the following as the spot (for spot-rate or spot-equity) whereas the temporal integral $Y_t = \int_0^t X_s ds$ will often be called the integrated process.

1.1.1 Study of the process

The first and foremost issue to consider when studying a diffusion remains the actual existence and uniqueness of such a process.

[1]References and sources will always be provided for those.

i. Strong solution

Proposition 1.1.1. *(see Feller [29], Lamberton-Lapeyre [46]) For any positive x_0, there exists a unique continuous adapted process X_t satisfying the stochastic differential equation (1.1) and the initial condition $X_0 = x_0$.*

Proof. Though the usual Lipschitz condition is not satisfied by the local volatility of the equation (1.1), the square-root function is locally hölderian. Adding that the drift coefficient is locally Lipschitzian, existence and uniqueness of a smooth version are ensured (see the references in Lamberton and Lapeyre [46]). □

ii. The zero-boundary

Positivity is a very well-known (and for modelling purposes, a generally appreciated and useful) property of the square-root process. But, for a proper understanding of the process, its behaviour at 0 needs to be analysed.

Proposition 1.1.2. *(see Lamberton-Lapeyre [46]) If $a \geq \frac{\sigma^2}{2}$, 0 is an entrance boundary, i.e almost surely, the process will not reach 0 in finite time[2]. If $a = 0$, 0 is an absorbing boundary. Then, if $b \geq 0$, the process will almost surely get absorbed in finite time. If $b < 0$, the absorption probability lies strictly between 0 and 1.*

Proof. A proof of all these results can be found in Lamperton and Lapeyre [46]. □

iii. The limit-distribution in the mean-reverting case

It appears from the previous result that the process behaviour depends crucially on the sign of b. Indeed, a strictly positive b induces the process to mean-revert to

[2] *Except when $X_0 = 0$, in which case the process automatically departs from 0 and never get back to it in finite time*

the level $\frac{a}{b}$ and convergence with time to a stationary distribution can be expected. Therefore, whenever positive and non-null, b will be thereafter denominated the mean-reversion strength.

Proposition 1.1.3. *(see Shreve [66]) When b is strictly positive, the process converges with time to the gamma distribution*

$$\left(\frac{2b}{\sigma^2}\right)^{\frac{2a}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2a}{\sigma^2}\right)} x^{\frac{2a-\sigma^2}{\sigma^2}} e^{-\frac{2bx}{\sigma^2}} \tag{1.2}$$

Proof. If such a equilibrium-distribution p_∞ exists, it has to be the solution of

$$\begin{cases} -\frac{\sigma^2}{2}\frac{\partial^2 p_\infty(x)}{\partial x^2} + (a - \sigma^2 - bx)\frac{\partial p_\infty(x)}{\partial x} - bp_\infty(x) = 0 \\ \int_0^\infty p_\infty(x)dx = 1 \end{cases} \tag{1.3}$$

with the the constraint $p_\infty(x) \geq 0, \forall y \geq 0$.

(1.3) arises from taking the limit when time tends to infinity of the Kolmogorov forward equation

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial}{\partial x}((a - bx)p(t,x)) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 x p(t,x)) = 0$$

given that for a distributional limit to exist, the following condition should be satisfied

$$\lim_{t\to\infty} \frac{\partial p(t,x)}{\partial t} = 0$$

Note that the forward equation represents here a natural choice compared to the backward one, since the limit distribution should be independent of the starting values, i.e. the backward variables x_0 and t_0, implying that taking the limit in time of the Kolmogorov backward equation would provide no information.

The basis for the vector space of solutions being

$$\left(p_1 = \phi\left(1, 2 - \frac{2a}{\sigma^2}, \frac{-2bx}{\sigma^2}\right), p_2 = \left(\frac{2bx}{\sigma^2}\right)^{\frac{2a}{\sigma^2}-1} \phi\left(\frac{2a}{\sigma^2}, \frac{2a}{\sigma^2}, \frac{-2bx}{\sigma^2}\right)\right)$$

The second function, which can be rewritten as $p_2 = \left(\frac{2bx}{\sigma^2}\right)^{\frac{2a}{\sigma^2}-1} e^{\frac{-2bx}{\sigma^2}}$ is the only integrable solution over $\mathbb{R}^+$ and is also clearly positive. Normalising it by its integral leads to (1.2).

This convergence in distribution can be actually proven by taking the limit in infinite time of the moment generating function of X_t which will be given in the next subsection in Proposition 1.1.4. □

Remark. When b is negative, on the other hand, the MGF does not converge and the first moments of the process are easily shown to grow to infinity with time.

iv. The joint moment-generating function

The first and actually main results concerning the temporal integral $Y_t = \int_0^t X_s ds$ have been derived by Cox and al. [15] in their computation of the price of a zero-coupon bond. Their result can easily be generalised to obtain the moment generating function of Y_t. Without heavy complication, the same method actually produces the MGF of the joint distribution (X_t, Y_t) as observed, for example, by Lamberton and Lapeyre [46]. The (relative) simplicity of these functions comes from the additivity property of the process. This same tractability of solutions of linear (or rather affine) differential equations have given rise to the so-called affine models.

Proposition 1.1.4. *The moment generating function of the joint distribution of (X_t, Y_t) has the exponential form*

$$\mathcal{L}^{X,Y}(\lambda, \mu) = E\left(e^{-\lambda X_t} e^{-\mu \int_0^t X_s ds} \middle| X_0 = x_0\right) = e^{-a\Theta(t) - x_0 \Upsilon(t)} \tag{1.4}$$

with

$$\Upsilon(t) = \frac{\lambda((\gamma - b) + e^{-\gamma t}(\gamma + b)) + 2\mu(1 - e^{-\gamma t})}{\sigma^2 \lambda(1 - e^{-\gamma t}) + (\gamma + b) + e^{-\gamma t}(\gamma - b)} \tag{1.5}$$

and

$$\Theta(t) = \frac{-2}{\sigma^2} \ln \left(\frac{2\gamma e^{\frac{(b-\gamma)t}{2}}}{\sigma^2 \lambda (1 - e^{-\gamma t}) + (\gamma + b) + e^{-\gamma t}(\gamma - b)} \right) \tag{1.6}$$

where

$$\gamma = \sqrt{b^2 + 2\mu\sigma^2} \tag{1.7}$$

Proof. This proof is included both because of the importance of the result and because of the development of the proof itself, which might be useful to be compared with related results we will derive later in this thesis for the CEV process.

Consider $\tilde{L}(t, x)$, the bounded solution of the following partial differential equation

$$\frac{\partial \tilde{L}}{\partial t} = \frac{\sigma^2}{2} x \frac{\partial^2 \tilde{L}}{\partial^2 x^2} + (a - bx) \frac{\partial \tilde{L}}{\partial x} - \mu x \tilde{L}$$

subject to the initial condition

$$\tilde{L}(0, x) = e^{-\lambda x}$$

The process

$$M_t = e^{-\mu \int_0^t X_s ds} \tilde{L}(T - t, X_t)$$

is then a martingale and this implies

$$\tilde{L}(T, X_T) = M_0 = E(M_T) = E\left(e^{-\mu \int_0^T X_s} e^{-\lambda X_T}\right) = L(\lambda, \mu)$$

As in Corollary (1.3), Chapter 11 in Revuz and Yor [62] (See also Pitman and Yor [60] and Shiga and Watanabe [70]), the solution has the form $L(\lambda, \mu) = e^{-a\Theta(t) - x_0 \Upsilon(t)}$. The sum of two independent square-root process with parameters a^1, b, σ and a^2, b, σ respectively and initial values x_0^1 and x_0^2 respectively is a square-root process[3] of

[3]It follows from the corresponding result for square Bessel processes (see Revuz-Yor [62]) and the fact that a square-root process is the product of a deterministic function and a time-changed square Bessel process.

parameters $a^1 + a^2, b, \sigma$ and initial value $x_0^1 + x_0^2$. The joint MGF for the process with parameters a, b, σ and x_0 is hence the product of the joint MGF for a square-root process X^1 with parameters $0, b, \sigma$ and x_0 and the joint MGF for an independent square-root process X^2 with parameters a, b, σ and initial value 0. Each of these two MGF are multiplicative and equal to 1 at 0.

The differential equation then becomes

$$-a\Theta'(t) + a\Upsilon(t) = x\left(\frac{\sigma^2}{2}\Upsilon^2(t) + \Upsilon'(t) + b\Upsilon(t) - \mu\right)$$

giving the following system

$$\left\{\begin{array}{lcl} \Theta'(t) & = & \Upsilon(t) \\ \Upsilon'(t) & = & -\frac{\sigma^2}{2}\Upsilon^2(t) - b\Upsilon(t) + \mu \end{array}\right. \tag{1.8}$$

with the initial conditions $\Theta(0) = 0$ and $\Upsilon(0) = \lambda$.

The last equation in (1.8) is a ordinary Riccati differential equation, with the constant particular solution: $\Upsilon_0 = \frac{-b+\sqrt{b^2+2\mu\sigma^2}}{\sigma^2}$. With γ defined as in (1.7), the change of variables: $h(t) = \frac{1}{\Upsilon(t)-\Upsilon_0}$ leads to

$$h'(t) = \frac{\sigma^2}{2} + (\sigma^2\Upsilon_0 + b)h(t)$$

and

$$h(0) = \frac{1}{\lambda - \Upsilon_0}$$

which result in the expressions (1.5) and (1.6). □

v. The density of the spot-process

Although the MGF completely characterises a distribution in theory, its density remains in practice most desirable since needed, for instance, to compute the expectation of any non-analytic function.

Theorem 1.1.1. *The square-root process density is an infinite weighted average of gamma densities. In the non-absorption case,*

$$f^X(x) = e^{-x_0 Be^{-bt}} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} e^{-bnt} B^{2n+\frac{2a}{\sigma^2}} \left(\frac{x^{n+\frac{2a}{\sigma^2}-1} e^{-xB}}{\Gamma(n+\frac{2a}{\sigma^2})} \right) \tag{1.9}$$

where

$$B = \frac{2b}{\sigma^2(1-e^{-bt})} \tag{1.10}$$

Feller [29] represents it in the equivalent form

$$f^X(x) = Be^{-(x_0 e^{-bt}+x)B} \left(\frac{x}{x_0 e^{-bt}} \right)^{\frac{a}{\sigma^2}-\frac{1}{2}} I_{\frac{2a}{\sigma^2}-1}\left(2\sqrt{x_0 e^{-bt} x} B\right) \tag{1.11}$$

where $I_{\frac{2a}{\sigma^2}-1}$ is the modified Bessel function of the first kind of order $\frac{2a}{\sigma^2}-1$. This density corresponds to a non-central chi-square with $\frac{4a}{\sigma^2}$ degrees of freedom and parameter of non-centrality $2Bx_0e^{-bt}$.

Remark. Besides the intrinsic importance of this result, the proof given below is of specific interest as it illustrates the analytic Laplace transform inversion method which consists in decomposing the transform into a sum or a series of elementary analytically invertible terms, method widely used in this thesis.

Proof of Theorem 1.1.1. Taking $\mu = 0$ in (1.6) and (1.5), leads to the MGF of X_t

$$\mathcal{L}^X(\lambda) = \left(\frac{2b}{\lambda\sigma^2(1-e^{-bt})+2b} \right)^{\frac{2a}{\sigma^2}} e^{-x_0 \frac{2\lambda b e^{-bt}}{\lambda\sigma^2(1-e^{-bt})+2b}} \tag{1.12}$$

(1.12) can be rewritten as

$$\begin{aligned} \mathcal{L}^X(\lambda) &= \left(\frac{B}{\lambda+B} \right)^{\frac{2a}{\sigma^2}} e^{-x_0 e^{-bt} B} e^{x_0 B \frac{B}{\lambda+B}} \\ &= \sum_{n=0}^{\infty} e^{-x_0 Be^{-bt}} \frac{x_0^n}{n!} (Be^{-bt}) \left(\frac{B}{\lambda+B} \right)^{n+\frac{2a}{\sigma^2}} \end{aligned} \tag{1.13}$$

Given that $(\lambda + \alpha)^{-\nu}$ is the MGF of the gamma distribution of density $\frac{x^{\nu-1}e^{-\alpha x}}{\Gamma(\nu)}$, the linearity property of Laplace transforms along with the Beppo Levi monotone convergence theorem applied to this weighted sum of (positive) densities allow the inversion (1.9). The equivalence with (1.11) follows from the series representation of the Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k} \tag{1.14}$$

□

This is the first result in this chapter involving Y_t. Our interest for this quantity is rooted in its importance as a financial underlying. But, if, for some derivatives, the price depends only on the marginal distribution of the average, the spot enters as well the payoff of some other derivatives, in which case the joint density might be needed [4]. We will therefore first study the joint distributional properties of (X_t, Y_t), starting with their moments.

1.1.2 The joint moments of (X_t, Y_t)

In the literature concerning stochastic volatility, the moments of the average have been used either to compute approximations for the price (see Ball and Roma [5]) or to gain insight in the distribution of the stock (see Das [19]). However, only the four first moments have been given in these texts since they were computed through successive differentiation of the moment-generating function. Though it is theoretically possible to obtain all these moments through repeated differentiation, this method remains tedious and even with formal calculus packages like Mathematica or Maple, only the first ones can be handled in this quite time-consuming way. We show here that it is

[4] This is also the reason why we chose here to present the joint and not the marginal moment generating function.

actually possible to obtain all of them analytically, since it turns out that they have a relatively simple form.

i. The joint moments in explicit form

Theorem 1.1.2. *The joint moments of X_t and Y_t are given by*

$$M_{m,n}(t) = E(Y_t^m X_t^{n-m}) = \sum_{j=0}^{n} e^{-jbt}\left(\sum_{i=1}^{I_j^{m,n}} \alpha_{j,i}^{m,n} \frac{t^{i-1}}{(i-1)!}\right) \tag{1.15}$$

where

$$I_j^{m,n} = \min(n+1-j, m+1) \tag{1.16}$$

The coefficients $\alpha_{j,i}^{m,n}$ can be obtained by recursion through the relations

- *For $j \neq n-m$*

$$\alpha_{j,i}^{m,n} = m \sum_{i'=i}^{I_j^{m-1,n}} \frac{(-1)^{i'-i}\alpha_{j,i'}^{m-1,n}}{((n-m-j)b)^{i'-i+1}}$$

$$+(n-m)\left(a+(n-m-1)\frac{\sigma^2}{2}\right) \sum_{i'=i}^{I_j^{m,n-1}} \frac{(-1)^{i'-i}\alpha_{j,i'}^{m,n-1}}{((n-m-j)b)^{i'-i+1}} \tag{1.17}$$

- *For $j = n-m$*
  - *For $i = 1$*

$$\alpha_{n-m,1}^{m,n} = c_{m,n} + m\sum_{j=0}^{n} \sum_{i=1}^{I_{n-m}^{m-1,n}} \frac{\alpha_{j,i}^{m-1,n}}{((j-n+m)b)^i}$$

$$+(n-m)\left(a+(n-m-1)\frac{\sigma^2}{2}\right)\sum_{j=0}^{n-1} \sum_{i=1}^{I_{n-m}^{m,n-1}} \frac{\alpha_{j,i}^{m,n-1}}{((j-n+m)b)^i} \tag{1.18}$$

  - *For $i > 1$*

$$\alpha_{n-m,i}^{m,n} = m\alpha_{n-m,i-1}^{m-1,n} + (n-m)\left(a+(n-m-1)\frac{\sigma^2}{2}\right)\alpha_{n-m,i-1}^{m,n-1} \tag{1.19}$$

where

$$c_{m,n} = r_0^n 1_{\{m=0\}} \tag{1.20}$$

- Initial condition

$$\alpha_{0,1}^{0,0} = 1 \tag{1.21}$$

Remark. The marginal moments of Y_t are obtained with $n = m$, since $E(Y_t^n) = M_{n,n}(t)$.

Proof of Theorem 1.1.2. Assessing first the issue of existence, the joint MGF given in Proposition 1.1.4 is infinitely differentiable in a neighbourhood of $(0,0)$, implying the joint moments of X_t and Y_t exist for any positive order for any finite t.

Therefore, applying the Itô formula to $Y_t^m X_t^k$, taking the expectation of the result and differentiating it with respect to t gives

$$\begin{aligned}\frac{dE(Y_t^m X_t^k)}{dt} &= mE(Y_t^{m-1}X_t^{k+1}) + akE(Y_t^m X_t^{k-1}) \\ &-bkE(Y_t^m X_t^k) + k(k-1)\frac{\sigma^2}{2}E(Y_t^m X_t^{k-1})\end{aligned} \tag{1.22}$$

since the stochastic integral appearing when applying the Ito lemma is a martingale, due to the square-integrability property of the integrand formed by power functions. It should be noticed that the computation of positive order moments does not actually involve the moments of the reciprocals of either X_t or Y_t. When $m = 0$ or respectively $k = 0$, the use of $mE(Y_t^{m-1}X_t^{k+1})$ and respectively $kE(Y_t^m X_t^{k-1})$ are mere notations let for simplicity while those expectations do not in fact appear in the equation and the terms quoted are simply null.

Now, denoting $\widehat{M}_{m,n}(\zeta)$ the Laplace transform of $M_{m,n}(t) = E(Y_t^m X_t^{n-m})$ with respect to time for $\zeta \in \mathbb{R}^+$, $\widehat{M}_{m,n}(\zeta) = \int_0^\infty e^{-\zeta t} M_{m,n}(t)dt$, the ordinary differential equation (1.22) becomes

$$\widehat{M}_{m,n}(\zeta)[\zeta + b(n-m)] - M_{m,n}(0) = m\widehat{M}_{m-1,n}(\zeta) + d(n,m)\widehat{M}_{m,n-1}(\zeta) \tag{1.23}$$

with

$$d(n,m) = (n-m)\left(a + (n-m-1)\frac{\sigma^2}{2}\right) \tag{1.24}$$

(1.15) can then be shown trough induction. Assuming that for a given $n > 0$ and for all integers $m < n$, the joint moments have the form

$$\widehat{M}_{m,n-1} = \sum_{j=0}^{n-1} \sum_{i=1}^{I_j^{m,n-1}} \frac{\alpha_{j,i}^{m,n-1}}{(\zeta + jb)^i} \tag{1.25}$$

The $\widehat{M_{k,n}}$ are also assumed the corresponding form for all k strictly below a given integer $m \geq 0$, whenever $m > 0$. As in (1.20), defining the variable $c_{m,k}$ by $c_{0,k} = x_0$ and $c_{m,k} = 0$ if $m > 0$, (1.23) implies that[5]

$$\sum_{j=0}^{n} \sum_{i=1}^{I_j^{m,n}} \frac{\alpha_{j,i}^{m,n}}{(\zeta + jb)^i} - \frac{c_{m,n}}{\zeta + b(n-m)} = m \sum_{j=0}^{N} \sum_{i=1}^{I_j^{m-1,n}} \frac{\alpha_{j,i}^{m-1,n}}{(\zeta + jb)^i(\zeta + b(n-m))}$$

$$+d(n,m) \sum_{j=0}^{n-1} \sum_{i=1}^{I_j^{m,n-1}} \frac{\alpha_{j,i}^{m-1,n}}{(\zeta + jb)^i(\zeta + b(n-m))} \tag{1.26}$$

Given that, for $p \neq q$ positive integers,

$$\frac{1}{(\zeta + qb)^i(\zeta + pb)} = \frac{\beta_{1,p}^i}{\zeta + pb} + \sum_{l=1}^{i} \frac{\beta_{l,q}^i}{(\zeta + qb)^l}$$

with

$$\begin{cases} \beta_{1,p}^i & = & \frac{1}{(q-p)b} \\ \beta_{l,p}^i & = & \frac{(-1)^{i-l}}{((q-p)b)^{i-l+1}} \end{cases} \tag{1.27}$$

(1.26) and (1.27) leads to the recursions (1.17), (1.18) and (1.19).

Since (1.25) is clearly initially satisfied for $(m,n) = (0,0)$ with $\widehat{M_{0,0}} = \frac{1}{\zeta}$ (explaining (1.21)), the $\widehat{M_{0,0}} = \frac{1}{\zeta}$ have the form given in (1.25).

[5]Once again, whenever $m = 0$ or resp. $n = m$, the sum following m or resp. $d(n,m)$, which are null in those cases, is a mere simplifying notation

The classical result

$$\int_0^\infty e^{-\zeta t - jbt} \frac{t^{i-1}}{(i-1)!} dt = \frac{1}{(\zeta + jb)^i}$$

along with the linearity property of the Laplace transform operator induce the result (1.15). □

Corollary 1.1.1. *Y_t possesses the following mean and non-centered 2^{nd} moment*

$$\mu_1^{Y_t} = \Big(e^{-bt}\{a - bx_0\} + \{abt + (bx_0 - a)\}\Big)\tfrac{1}{b^2}$$

$$E(Y_t^2) = \Big(\{(2x_0\sigma^2 b - 4ax_0 b - 5a\sigma^2 + 2a^2 + 2x_0^2 b^2) + 2(\sigma^2 - 2a + 2bx_0)abt$$
$$+2a^2b^2t^2\} + e^{-bt}\{4(a\sigma^2 - a^2 - x_0^2 b^2 + 2ax_0 b) + 4(a\sigma^2 - ax_0 b^2 - x_0 b^2 \sigma^2$$
$$+a^2 b)t\} + e^{-2bt}\{(2x_0^2 b^2 + 2a^2 + a\sigma^2 - 4ax_0 b - 2x_0\sigma^2 b)\}\Big)\tfrac{1}{2b^4}$$

ii. Importance of those moments

To complete this moments study, it should be noticed that the information conveyed by them is total, in the sense that they fully characterise the distribution:

Theorem 1.1.3. *The joint distribution of (X_t, Y_t) is determined by its moments, the same being true for each marginal distribution.*

Proof. The analytic expression for the moment-generating function of (X_t, Y_t) exists for some negative values. More precisely, it exists for $\mu > -\frac{b^2}{2\sigma^2}$ and for $\lambda \geq \bar{\lambda}_\mu$ with $\bar{\lambda}_\mu = -\frac{\gamma + b + e^{-\gamma t}(\gamma - b)}{\sigma^2(1 - e^{-\gamma t})}$, where γ defined as in (1.7) is a function of μ. It actually also exists for $\mu = -\frac{b^2}{2\sigma^2}$ and λ greater than or equal to the lower bound:

$$\bar{\lambda}_{-\frac{b^2}{2\sigma^2}} = \lim_{\mu \to -\frac{b^2}{2\sigma^2}} -\frac{\gamma + b + e^{-\gamma t}(\gamma - b)}{\sigma^2(1 - e^{-\gamma t})} = -\frac{2 + tb}{t\sigma^2} \tag{1.28}$$

By the application of the Beppo Levi theorem, $E\Big(e^{-\mu \int_0^t X_s ds} \big| X_0 = x_0\Big)$ exists for $\mu \geq -\frac{b^2}{2\sigma^2}$ and is given by the value taken by the analytic expression at these points.

Similarly, $E\left(e^{-\lambda X_t} e^{-\mu \int_0^t X_s ds} \middle| X_0 = x_0\right)$ exists for $\mu \geq -\frac{b^2}{2\sigma^2}$ and for $\lambda \geq \bar{\lambda}_\mu$. The distribution is hence doubly subexponential and, as a consequence, determined by its moments. □

Therefore, the moments could be used to approximate the distribution: Laguerre-polynomials expansion method (see Dufresne [26]), Edgeworth expansion, etc. The coefficients of the moments can be computed given a specific set of parameters a, b and σ. Evaluating the moments for different values of time is then straightforward, which makes this procedure all the more interesting, since it would enable us to evaluate options of different maturities and strikes at a reduced level of additional computation. However, given the actual form of the marginal density derived later in Part 1.1.4, these moments-based approximations are not likely to be very fast-converging. Laguerre polynomials expansions, for example, do not converge in 60 steps[6] for our selections of parameters. The problem inherent to this expansion method is that the parameters of the expansion play a critical role in the convergence speed of the series, while there is no specific selection criterium or algorithm for these parameters, except for proceeding by trial and error. Edgeworth expansions around a normal might be faster in the mean-reverting case for very large values of time, t, since the distribution of the average $\frac{Y_t}{t}$ tends to a gaussian (see Fouque and al. [31]).

1.1.3 The joint density of (X_t, Y_t)

If the preceding results allow us to state explicitly the density in an analytical form, possibly involving some free parameters, these expansion methods[7] in terms

[6] Computing 60 moments, although they are expressed as simply as they can, requires quite a number of operations.

[7] Some of these methods are unidimensional, but might be generalised or adapted.

of moments remain quite general and their actual efficiency depends on the specific distribution. In this part we will derive a better specific explicit series form for this density, arising from the exploitation of a change of measure under which the spot process and hence its integral follows a simpler diffusion, resulting in a joint moment generating function itself simpler than in Proposition 1.1.4.

i. An equivalent measure result

Theorem 1.1.4. *The following process L is a martingale*

$$L_t = e^{\frac{b^2 Y_t}{2\sigma^2} + \frac{b(X_t - x_0)}{\sigma^2} - \frac{abt}{\sigma^2}} \tag{1.29}$$

Proof. From the SDE defining X,

$$\int_0^t \frac{b}{\sigma}\sqrt{X_u}dW_u = \frac{b}{\sigma^2}\Big(- X_0 - \int_0^t (a - bX_u)du + X_t\Big)$$

which implies

$$L_t = e^{\int_0^t \frac{b}{\sigma}\sqrt{X_u}dW_u - \int_0^t \frac{b^2}{2\sigma^2}X_u du}$$

Since the Novikov condition $E\Big(e^{\frac{b^2}{2\sigma^2}Y_t}\Big) < \infty$ is verified (see proof of Theorem 1.1.3), L is an exponential martingale with mean 1. □

Theorem 1.1.5. *Under the measure Q^* given by the Radon-Nykodim derivative $\frac{dQ^*}{dQ} = L(T)$, the process $X(t)$ follows a.s. the SDE*

$$dX_t = adt + \sigma\sqrt{X_t}dW_t^* \tag{1.30}$$

W_t^* being a Brownian motion under the Q^*-measure.

Proof. From Girsanov's theorem and the previous results on L, the process W^* defined by $W_t^* = W_t - \int_0^t \frac{b}{\sigma}\sqrt{X_u}dW_u$ is a Brownian motion under Q^*. □

Remark. X_t is then a multiple of (precisely $\frac{\sigma^2}{4}$ times) a squared Bessel process of index $\frac{4a}{\sigma^2}$ under Q^*. If X_t might also be connected with Bessel processes through other transformations, the change of measure proposed here is a simple result, easy to manipulate and suited to the analysis of the path-dependent integral Y_t which requires path properties to be exploitable.

This result allows us to work under the Q^*-measure, finding the Q^*-joint density of (X_t, Y_t) and then coming back to the Q-measure through the Radom-Nykodim derivative.

ii. The case $a > 0$

As showed in Part 1.1.1, the behaviour of the process crucially depends on the sign of a. We will start with the mean-reverting case.

Theorem 1.1.6. *Denoting $\alpha = \frac{\sigma^2}{8}$ and D_ν the parabolic cylinder function of order ν, the joint density of X_t and Y_t (under Q) is given by*

$$f^{X,Y}(x,y) = \frac{\left(\frac{x}{\sqrt{2}}\right)^{\frac{2a}{\sigma^2}-1}}{2\sqrt{\pi}(\sqrt{y\alpha})^{\frac{2a}{\sigma^2}+2}} e^{-\frac{b^2 y}{2\sigma^2} - \frac{b(x-x_0)}{\sigma^2} + \frac{abt}{\sigma^2}} \sum_{n=0}^{\infty} \frac{n!\alpha}{\Gamma(n+\frac{2a}{\sigma^2})} N_n(y) \tag{1.31}$$

with the term $N_n(y)$ defined as

$$\sum_{p=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-p}}{p!} \left(\frac{-x}{\sqrt{2y\alpha}}\right)^p \sum_{q=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-q}}{q!} \left(\frac{-x_0}{\sqrt{2y\alpha}}\right)^q D_\nu\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right) e^{-\frac{\alpha_n^2}{8y\alpha}} \tag{1.32}$$

where

$$\alpha_n = \frac{x + x_0 + (a + n\sigma^2)t}{2} \tag{1.33}$$

$$\nu = p + q + \frac{2a}{\sigma^2} + 1 \tag{1.34}$$

Proof. Under the Q^* measure, the joint MGF becomes, with $\gamma = \sqrt{2\sigma^2\mu}$,

$$\mathcal{L}^{*X,Y}(\lambda,\mu) = \left(\frac{2\gamma e^{-\frac{\gamma t}{2}}}{\sigma^2\lambda(1-e^{-\gamma t}) + \gamma(1+e^{-\gamma t})}\right)^{\frac{2a}{\sigma^2}} e^{-x_0 \frac{\lambda\gamma(1+e^{-\gamma t}) + 2\mu(1-e^{-\gamma t})}{\sigma^2\lambda(1-e^{-\gamma t}) + \gamma(1+e^{-\gamma t})}}$$

$$= \left(\frac{2\gamma e^{-\frac{\gamma t}{2}}}{\sigma^2(1-e^{-\gamma t})}\right)^{\frac{2a}{\sigma^2}} e^{-x_0\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} \frac{\left(\frac{4e^{-\gamma t}\gamma^2}{\sigma^4(1-e^{-\gamma t})^2}\right)^n}{\left(\lambda + \frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}\right)^{n+\frac{2a}{\sigma^2}}} \tag{1.35}$$

The integral $l^*_{Y_t}(\mu) = \int_0^\infty e^{-\mu y} f^{X,Y}(x,y)dy$ can be calculated by inverting this Laplace transform with respect to λ. Observing that (1.35) is a weighted average of gamma distribution MGFs, it can be inverted[8] just like (1.13) in Theorem 1.1.1

$$l^*_{Y_t}(\mu) = e^{-(x_0+x)\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} \left(\frac{4e^{-\gamma t}\gamma^2}{\sigma^4(1-e^{-\gamma t})^2}\right)^{n+\frac{a}{\sigma^2}} \frac{x^{n+\frac{2a}{\sigma^2}-1}}{\Gamma(n+\frac{2a}{\sigma^2})}$$

Noting $\zeta = \frac{2a}{\sigma^2} - 1$, $l^*_{Y_t}(\mu)$ can be linked with the modified Bessel function of the first kind of index ζ through (1.14)

$$l^*_{Y_t}(\mu) = \left(\frac{2\gamma e^{-\frac{\gamma t}{2}}}{\sigma^2(1-e^{-\gamma t})}\right)\left(\frac{x}{x_0}\right)^{\frac{\zeta}{2}} I_\zeta\left(\frac{2\sqrt{4xx_0e^{-\gamma t}\gamma^2}}{\sigma^2(1-e^{-\gamma t})}\right) e^{-(x+x_0)\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}}$$

where I_ζ is the modified Bessel function of the first kind of index ζ. It can be rewritten[9] as

$$x^\zeta \left(\frac{2\gamma}{\sigma^2}\right)^{\zeta+1} e^{-(x+x_0)\frac{\gamma}{\sigma^2} - \frac{\gamma a t}{\sigma^2}} \sum_{n=0}^{\infty} \frac{n! L_n^\zeta\left(\frac{2x\gamma}{\sigma^2}\right) L_n^\zeta\left(\frac{2x_0\gamma}{\sigma^2}\right) e^{-\gamma n t}}{\Gamma(n+\zeta+1)}$$

where L_n^ζ refers to the Laguerre polynomial of order n and index ζ, i.e.
$L_n^\zeta(z) = \sum_{m=0}^n (-1)^m \binom{n+\zeta}{n-m} \frac{z^m}{m!}$.

It is known that the inverse of $\left(\frac{2\gamma}{\sigma^2}\right)^{2\kappa} e^{-\frac{2q\gamma}{\sigma^2}}$ for q and κ positive is (see Gradshteyn and Ryzhik [37])

$$\alpha\sqrt{\frac{2}{\pi}}(2y\alpha)^{-\kappa-1} e^{-\frac{q^2}{8y\alpha}} D_{2\kappa+1}\left(\frac{q}{\sqrt{2y\alpha}}\right) \tag{1.37}$$

---

[8] That the series of the inverses converges to the inverse of the series can be proved by using Beppo Levi theorem, again as in Theorem 1.1.1.

[9] For any $|z| < 1$, we have the relation (see Gradshteyn and Ryzhik [37])

$$\frac{(xyz)^{-\frac{\alpha}{2}}}{1-z} e^{-z\frac{x+y}{1-z}} I_\alpha\left(2\frac{\sqrt{xyz}}{1-z}\right) = \sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x) L_n^\alpha(y) z^n}{\Gamma(n+\alpha+1)} \tag{1.36}$$

where $\alpha = \frac{\sigma^2}{8}$ appears because of the scaling property of the Laplace transform and D_ς is the parabolic cylinder function of index ς (see Appendix A), related to the degenerate hypergeometric function ϕ through

$$D_\varsigma(z) = 2^{\frac{\varsigma}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\varsigma}{2}\right)} \phi\left(-\frac{\varsigma}{2}, \frac{1}{2}, \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{\varsigma}{2}\right)} \phi\left(\frac{1-\varsigma}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \right\}$$

This result, along with the linearity of the inverse Laplace transformation operator and the fact that dividing by $L(t)$ transfers the density back to the Q-measure[10], completes the proof. □

Remark. For computational purposes, it should be noted that most of the mathematical/statistical packages possess quick built-in routines to compute the special function D_ν and that once it is computed for the first two indexes, its value for the subsequent indexes can be deduced from the relation

$$D_{\nu+2}(z) = zD_{\nu+1}(z) - (\nu + 1)D_{\nu+2}(z)$$

iii. The case $a = 0$

For the Cox-Ross equity process , the results are slightly different because of the absorption at zero, implying a mass at that point. More precisely, the case $X_t > 0$ and $X_t = 0$ have to be treated separately.

Theorem 1.1.7. *With the same notation as above, the joint density of X_t and Y_t, for $X_t > 0$ under Q is given by*

$$f^{X,Y}(x, y) = \frac{x_0 \alpha}{2\sqrt{2\pi}} \frac{1}{(y\alpha)^2} e^{-\frac{b^2 y}{2\sigma^2} - \frac{b(x-x_0)}{\sigma^2}} \sum_{n=0}^{\infty} \frac{O_n^{(X,Y)}(y)}{n+1} \tag{1.38}$$

[10]Linearity is actually not sufficient since an infinite series is dealt with here. Yet, convergence and even uniform convergence of the series of inverses can be proven using uniform convergence properties of the Laplace transform series. The same kind of arguments will be used later for the marginal density of Y_t in the mean-reverting case and we refer to this analysis as exactly the same applies here.

with the terms $O_n^{(X,Y)}(y)$ *defined as*

$$\sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!}\left(\frac{-x}{\sqrt{2y\alpha}}\right)^p \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!}\left(\frac{-x_0}{\sqrt{2y\alpha}}\right)^q He_{p+q+3}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right) e^{-\frac{\alpha_n^2}{4y\alpha}} \tag{1.39}$$

and

$$\alpha_n = \frac{x + x_0 + ((n+1)\sigma^2 t)}{2} \tag{1.40}$$

Proof. The absorption point at zero changes slightly the joint MGF (still under the Q^*-measure)

$$\mathcal{L}^{*X,Y}(\lambda,\mu) = e^{-x_0\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}} \sum_{n=0}^{\infty} \frac{x_0^n}{n!} \frac{\left(\frac{4e^{\gamma t}\gamma^2}{\sigma^4(1-e^{-\gamma t})^2}\right)^n}{\left(\lambda + \frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}\right)^n} \tag{1.41}$$

Inverting this MGF with respect to λ as we did for (1.35) leads to

$$\delta_x(0) + I_1\left(\frac{2\sqrt{4xx_0e^{\gamma t}\gamma^2}}{\sigma^2(1-e^{-\gamma t})}\right)\sqrt{\frac{x_0}{x}}\frac{2\gamma e^{-x_0\frac{\gamma t}{2}}}{\sigma^2(1-e^{-\gamma t})}e^{-(x+x_0)\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}}$$

since (1.41) is the sum of a constant and weighted gamma MGFs. $\delta_x(0)$ stands here for the Dirac delta function which is null everywhere except at 0 where it is infinite.

For non-null x, this can be rewritten with (1.36) as

$$x_0\frac{4\gamma^2e^{-\gamma t}}{\sigma^4}e^{-(x+x_0)\frac{\gamma}{\sigma^2}}\sum_{n=0}^{\infty}\frac{n!L_n^1(\frac{2x\gamma}{\sigma^2})L_n^1(\frac{2x_0\gamma}{\sigma^2})e^{-\gamma nt}}{(n+1)!}$$

Inverting this expression with respect to μ as in the previous result (with the same convergence argument) leads to the formula (1.38), as for any $n \in \mathbb{N}$

$$D_n(z) = e^{-\frac{z^2}{4}}He_n(z)$$

where He_n is the n$^{\text{th}}$ Hermite polynomial: $He_n(z) = (-1)^n e^{\frac{z^2}{2}}\frac{d^n}{dz^n}\left[e^{-\frac{z^2}{2}}\right]$. □

The joint MGF containing all the information needed, the distribution of Y_t can also be deduced for the case $X_t = 0$.

Theorem 1.1.8. *Under Q, the density of Y_t conditional on $X_t = 0$ is $\frac{f_0^Y(y)}{P_Q(X_t=0)}$ where:*

$$f_0^Y(y) = \frac{e^{-\frac{b^2 y}{2\sigma^2}+\frac{bx_0}{\sigma^2}}}{y\sqrt{2\pi}} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{\binom{n}{n-p}}{p!}\left(\frac{-x_0}{\sqrt{2y\alpha}}\right)^p \left(He_{p+1}\left(\frac{\beta_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\beta_n^2}{4y\alpha}}\right.$$
$$\left.-He_{p+1}\left(\frac{\beta_{n+1}}{\sqrt{2y\alpha}}\right)e^{-\frac{\beta_{n+1}^2}{4y\alpha}}\right) \tag{1.42}$$

with

$$\beta_n = \frac{x_0 + nt\sigma^2}{2} \tag{1.43}$$

and

$$P_Q(X_t = 0) = e^{\frac{x_0 b(1+e^{bT})}{\sigma^2(1-e^{bT})}} \tag{1.44}$$

Proof. Since $e^{-\mu Y_t - \lambda X_t} = e^{-\mu Y_t - \lambda X_t} I_{\{X_t>0\}} + e^{-\mu Y_t} I_{\{X_t=0\}}$, taking the limit of the joint MGF at $\lambda \to \infty$ gives

$$\lim_{\lambda\to\infty} \mathcal{L}^{*X,Y}(\lambda,\mu) = E^{Q^*}(e^{-\mu Y_t} I_{\{X_t=0\}}) = e^{-x_0\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}}$$

It can be reexpressed as

$$E^{Q^*}\left(e^{-\mu Y_t} I_{\{X_t=0\}}\right) = e^{-x_0\frac{\gamma}{\sigma^2}}(1-e^{-\gamma t})\sum_{n=0}^{\infty} L_n\left(\frac{2x_0\gamma}{\sigma^2}\right)e^{-\gamma n t} \tag{1.45}$$

since (See Gradshteyn and Ryzhik [37]), for $|z| < 1$,

$$\frac{1}{1-z}e^{-\frac{xz}{z-1}} = \sum_{n=0}^{\infty} L_n(x) z^n \tag{1.46}$$

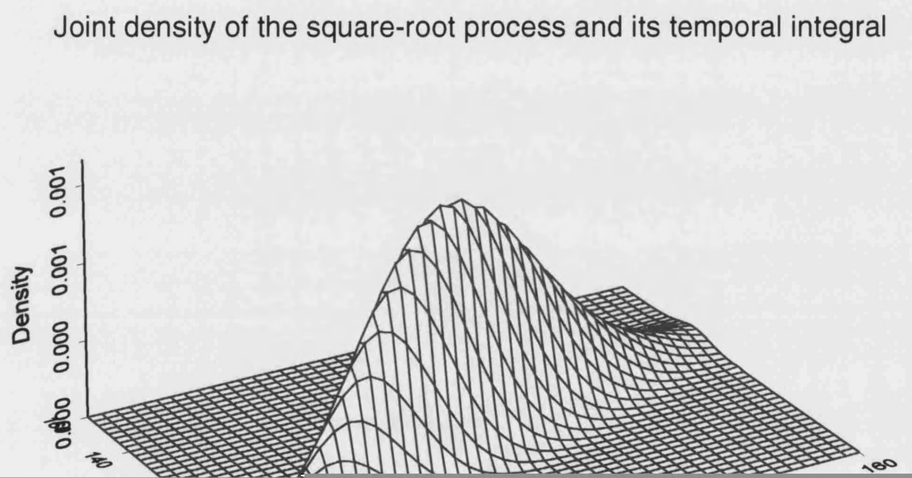
$L_n(x)$ being the Laguerre polynomial of order n and index 0.

The formula (1.45) expanded and inverted as previously gives the result. □

Remark. This series is fast-converging, as the leading term is roughly of order $e^{-\frac{\sigma^2 t^2}{2y}n^2}$.

iv. Numerical illustrations

We choose to illustrate this series method with an adaptation of the textbook Black-Scholes regular example $S_0^{\mathrm{BS}} = 100$, $r^{\mathrm{BS}} = 0.05$ and $\sigma^{\mathrm{BS}} = 20\%$. A square-root process with comparable parameters would be $S_0 = 100$, $a = 0$, $b = -0.05$ and $\sigma = 2$. With this choice of parameters, Figure 1.1 draws the joint density surface of (X_1, Y_1) when no absorption occurred.



In the following tables, N represents the number of terms needed for the absolute difference[11] between the limit (series truncated at 50 terms) and the series truncated at N terms or more to be less than 10^{-4}.

We thus observe in Table 1.1 that N increases with y and the evolution is indeed rather quadratic than linear in N.

Y	80	90	100	110	120	130	140	150	160
N	1	20	31	37	42	46	51	55	57

Table 1.1: Evolution with y at $x = 100$.

For $y = E(Y_1) \approx 102.54$, N plunges extremely quickly with increasing maturities, as shown in Table 1.2.

T	1	1.1	1.2
N	33	22	1

Table 1.2: Evolution with t at $x = 100$ and $y = 102.54$.

For increasing volatilities ($y = E(Y_1) \approx 102.54$), the decrease in N is also pronounced but still less violent.

σ	2	3	4	5	6
N	31	12	6	4	1

Table 1.3: Evolution with σ at $x = 100$ and $y = 102.54$.

But, we would have expected the series to react to changes in T and to changes in σ in roughly the same way. This fulgurant evolution with maturity can be accounted

[11] We prefer holding the absolute difference as the stopping criteria rather than the relative difference, since the density can reach values quite close to 0.

for by the fact that the density itself vanishes as $y = E(Y_1) \approx 102.5$ moves further and further away from $E(Y_t)$. A better understanding of the series behaviour can hence be obtained by studying the density at the moving point $y = E(Y_t) = X_0\frac{e^{rt}-1}{r}$.

t	1	2	3	4
$y = X_0\frac{e^{rt}-1}{r}$	102.54	210.34	323.66	442.80
N	33	15	9	1

Table 1.4: Evolution with t at $x = 100$ and $y = E(Y_t)$.

Table 1.4 indeed shows the expected deceleration in the decrease.

1.1.4 The marginal density of the integral in the mean-reverting case

Though the formulae presented in the preceding results prove simpler in the case $a = 0$, they appear complex when it comes to integrating out the marginal density of Y_t when $a > 0$. Indeed, taking the expectation of the joint density (1.31) with respect to x leads to (using the same notation as in Theorem 1.1.6)

$$f^Y(y) = \frac{\sqrt{2}}{\sqrt{\pi}(\sqrt{2y\alpha})^{\frac{2a}{\sigma^2}+2}} e^{-\frac{b^2y}{2\sigma^2}+\frac{bx_0}{\sigma^2}+\frac{abt}{\sigma^2}} \sum_{n=0}^{\infty} \frac{n!\alpha}{\Gamma(n+\frac{2a}{\sigma^2})} N_n^Y(y) \tag{1.47}$$

with $N_n^Y(y)$ defined as

$$\sum_{p=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-p}}{p!} \left(\frac{-1}{\sqrt{2y\alpha}}\right)^p \sum_{q=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-q}}{q!} \left(\frac{-x_0}{\sqrt{2y\alpha}}\right)^q D^Y(n,p)(y) \tag{1.48}$$

and

$$D^Y(n,p)(y) = \int_0^\infty \left(\frac{x}{\sqrt{2}}\right)^{p+\frac{2a}{\sigma^2}-1} e^{-\frac{bx}{\sigma^2}-\frac{\alpha_n^2(x,y)}{8y\alpha}} D_\nu\left(\frac{\alpha_n(x,y)}{\sqrt{2y\alpha}}\right) dx \tag{1.49}$$

in which we have altered the notations to $\alpha_n(x,y)$ to emphasise the dependence on x and y. $D_\nu(\cdot)$ represents the parabolic cylinder function of index ν.

Recursions could be used for the $D^Y(n,p)(y)$. Yet, the initial values for those terms would still not be that easy to compute and would be needed for each new n. We thus present here another approach: a direct analytical inversion of the marginal moment generating function of the integral process, resulting in a tractable and easier to evaluate formula for its density.

i. Relating the two approaches

Before developing this alternative approach, we first show here how to retrieve (1.47) by a direct inversion of the marginal MGF of Y_t. From (1.1.4),

$$\mathcal{L}^Y(\mu) = \left(\frac{2\gamma e^{\frac{bt}{2}} e^{-\frac{\gamma t}{2}}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}\right)^{\frac{2a}{\sigma^2}} e^{-x_0\frac{2\mu(1-e^{-\gamma t})}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}} \tag{1.50}$$

Decomposing it thanks to the relation (1.46),

$$\mathcal{L}^Y(\mu) = e^{-\frac{x_0+at}{\sigma^2}(\gamma-b)}\left(\frac{\gamma}{\gamma+b}\right)^{\frac{2a}{\sigma^2}} \sum_{n=0}^{\infty} L_n^{\frac{2a}{\sigma^2}-1}\left(\frac{2\gamma x_0}{\sigma^2}\right)\left(\frac{b-\gamma}{\gamma+b}\right)^n e^{-\gamma n t} \tag{1.51}$$

Recalling (1.37), the inverse of $\frac{\gamma^{k+\frac{2a}{\sigma^2}}}{(\gamma+b)^{n+\frac{2a}{\sigma^2}}} e^{-q\gamma} = \int_0^\infty \frac{e^{-\gamma(q+u)}\gamma^{k+\frac{2a}{\sigma^2}}}{\Gamma\left(n+\frac{2a}{\sigma^2}\right)} u^{n+\frac{2a}{\sigma^2}-1} e^{-bu} du$ is[12]

$$\beta\frac{e^{-b^2 y\beta}}{\Gamma\left(n+\frac{2a}{\sigma^2}\right)}\int_0^\infty \frac{2^{-\frac{a}{\sigma^2}-\frac{k}{2}} e^{-\frac{(u+q)^2}{8y\beta}}}{\sqrt{2\pi}(y\beta)^{\frac{a}{\sigma^2}+\frac{k}{2}+1}} D_{\frac{2a}{\sigma^2}+k+1}\left(\frac{u+q}{\sqrt{2y\beta}}\right) u^{n+\frac{2a}{\sigma^2}-1} e^{-bu} du \tag{1.52}$$

where the term β comes from rescaling the Laplace transform. More precisely,

$$\beta = \frac{1}{2\sigma^2} \tag{1.53}$$

Inverting (1.51) term by term, we recognise the terms appearing in (1.47).

---

[12]Taking the Laplace transform of the expression, Fubini-Tonelli theorem allows us to interchange the order of integration thanks to the absolute convergence of the double integral.

The sole purpose of this result is to show the consistency of the two approaches presented in this thesis to compute the marginal density of the integral process. The marginal MGF can actually be manipulated in a slightly different manner to lead to a series representation in which the numerical integration of parabolic cylinder functions is not required. We will first present a complete formula, in which each term is explicitly written down, but is - for that very reason - unduly complex. The purpose of this first representation is to show that it is indeed possible to express the density in a totally explicit form although in practice, it is more efficient to evaluate the terms recursively. We therefore present afterwards a reformulation of this series, constructing its inner terms by recursion. For the square-root process, all the formulae and applications following this result in the rest of the thesis will be given in a recursive form. But, it should be kept in mind that all of them can be expressed in a completely explicit manner. For this reason, we call them fully-explicit, given that the really complete series formulation is a straightforward corollary.

ii. The complete formula for the density.

Definitions and notations. The density depends on the following functions:

- He_k is the k$^{\text{th}}$ Hermite polynomial (see Appendix A)

$$He_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!} \tag{1.54}$$

- $\widetilde{He}_k$ is a polynomial of order k given by (see Appendix A)

$$\widetilde{He}_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!} \tag{1.55}$$

- For $q \in \mathbb{N}$ and $\varpi \in \mathbb{R}^+$, the function $J^{\mathrm{m}}_{q,\varpi}(y)$ is defined as

$$J^{\mathrm{m}}_{q,\varpi}(y) = \frac{1}{2\sqrt{\pi y^3}} He_{q+1}\left(\frac{\varpi}{\sqrt{2y}}\right) \frac{e^{-\frac{\varpi^2}{4y}-b^2 y}}{(\sqrt{2y})^{q-1}} \tag{1.56}$$

- For $l \in \mathbb{N}$ and $\varpi \in \mathbb{R}^+$, the function $J^{\mathrm{d}}_{l,\varpi}(y)$ is as follows

Case $l = 0$

$$J^{\mathrm{d}}_{0,\varpi}(y) = \frac{\varpi}{2\sqrt{\pi y^3}} e^{-\frac{\varpi^2}{4y}-b^2 y} \tag{1.57}$$

Case $l = 1$

$$J^{\mathrm{d}}_{1,\varpi}(y) = I_1(y) = \frac{1}{\sqrt{\pi y}} e^{-\frac{\varpi^2}{4y}-b^2 y} \tag{1.58}$$

Case $l > 1$

$$J^{\mathrm{d}}_{l,\varpi}(y) = (-\sqrt{2y})^{l-2}\sqrt{\frac{2}{\pi}} e^{-b^2 y}\left\{ \widetilde{He}_{l-2}\left(\frac{\varpi}{\sqrt{2y}}\right) \int_{\frac{\varpi}{\sqrt{2y}}}^{\infty} e^{-\frac{h^2}{2}} dh \right.$$

$$\left. - \sum_{j=0}^{l-3} \widetilde{He}_j^{(l-3-j)}\left(\frac{\varpi}{\sqrt{2y}}\right) e^{-\frac{\varpi^2}{4y}} \right\} \tag{1.59}$$

Theorem 1.1.9. *The density of Y_t is given by (β defined as in j(1.53))*

$$f^Y(y) = \beta e^{b\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{n!} \frac{x_0^n}{\sigma^{2n} 2^{k-n}} \binom{k + \frac{2a}{\sigma^2} - 1}{k-n} G_{k,n}(y) \tag{1.60}$$

with

$$G_{k,n}(y) = \sum_{i=0}^{k-n} \binom{k-n}{i} (-1)^i \left(\sum_{j=0}^{k-n} \binom{k}{j} (-b)^{k-j} J^d_{k-n-j,\varpi_{n,i}}(y\beta) \right.$$

$$\left. + \sum_{j=k-n+1}^{n} \binom{k}{j} (-b)^{k-j} J^m_{j-k+n,\varpi_{n,i}}(y\beta) \right) \tag{1.61}$$

and

$$\varpi_{n,i} = \frac{at + x_0}{\sigma^2} + (n+i)t \tag{1.62}$$

Proof. See Appendix 1.5.1. □

Remarks.

1. The proof of the uniform convergence of the series (1.60) presented in Appendix 1.5.1 is of importance, since it is also applicable for most of the other expansions derived in the square-root process context. Appendix A also contains recursion formulae (see (A.3) and (A.6)) useful to compute numerically the polynomials $He_k(x)$ and $\widetilde{He_k}(x)$.

2. The integrals appearing in (1.59) are only (up to a multiplicative constant) the complementary error function, which has been widely studied in the literature. There exists a good number of algorithms to compute it numerically with accuracy, some of which extremely fast and not taking more than thrice the time of an exponential evaluation to get it to machine precision. Most mathematical software packages have their own built-in routines to compute it. It can therefore be considered just as another standard arithmetic function like exp(·) or cos(·) .

(1.60) provides a full analytical expression for $f^Y(y)$, which enables us to gain a better insight into this density. Yet, as mentioned earlier, employing simplifying recursions is more efficient than using the explicit formulation of the $G_{k,n}(y)$, since it cuts down the amount of calculations by an order of magnitude proportional to N^2.

iii. A recursive formulation

Definitions and notations. For $\varpi \in \mathbb{R}^+\backslash\{0\}$, we construct a sequence $I_{p,q}(\varpi)$ in the following recursive way for positive integers p and q

• For $q = 0$

$$I_{p+1,0}(y,\varpi) = I_{p,0}(y,\varpi) - \sqrt{\frac{2}{\pi}}\frac{(b\varpi + p + 1)}{\sqrt{(2y\beta)^{p+3}}}He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}} \tag{1.63}$$

- For $q = 1$

$$I_{p,1}(y,\varpi) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+1}}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) \tag{1.64}$$

- For $q = 2$

$$I_{p+1,2}(y,\varpi) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\varpi^2}{4y\beta}}}{(\sqrt{2y\beta})^{p+1}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) - bI_{p,2}(y,\varpi) \tag{1.65}$$

- For $q = 3$

$$I_{p,3}(y,\varpi) = p1_{\{p>0\}} I_{p-1,2}(y,\varpi) - \varpi I_{p,2}(y,\varpi) + \frac{e^{-\frac{\varpi^2}{4y\beta}}\sqrt{2}}{\sqrt{\pi(2y\beta)^{p-1}}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) \tag{1.66}$$

- For $q > 3$

$$I_{p,q}(y,\varpi) = \frac{p1_{\{p>0\}} I_{p-1,q-1}(y,\varpi) + 2y\beta I_{p,q-2}(y,\varpi) - \varpi I_{p,q-1}(y,\varpi)}{q-2} \tag{1.67}$$

from the only two initial conditions needed:

$$\begin{cases} I_{0,0}(y,\varpi) &= \frac{\varpi}{2\sqrt{\pi(y\beta)^3}} e^{-\frac{\varpi^2}{4y\beta}} \\ I_{0,2}(y,\varpi) &= \operatorname{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \end{cases} \tag{1.68}$$

Remarks.

1. The term constructed with the previous formulae is either $I_{p,q}$ or $I_{p+1,q}$. This differentiation is meant at emphasising whether the formula holds for $p = 0$ or whether an initial condition is needed.

2. The Hermite polynomials appearing in the following recursions are to be themselves computed by recursion, see Appendix 1.5.1

Theorem 1.1.10. *The marginal density of the integral Y_t can be rewritten as*

$$f^Y(y) = \beta e^{b\frac{at+x_0}{\sigma^2} - b^2 y\beta} \sum_{k=0}^{\infty} \frac{f_k^Y(y)}{2^k} \tag{1.69}$$

where

$$f_k^Y(y) = \sum_{n=0}^{k}\sum_{m=n}^{k}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\binom{k-n}{m-n}\frac{(-2x_0)^n}{n!\sigma^{2n}}(-1)^m I_{k,k-n}(y,\varpi_m) \tag{1.70}$$

and

$$\varpi_m = \frac{at+x_0}{\sigma^2} + mt \tag{1.71}$$

Proof. See Appendix 1.5.2. □

Remark. For programming purposes, it might be simpler to use (with $K \geq 0$, the number of terms included to compute the series)

$$\sum_{k=0}^{K}\frac{f_k^Y(y)}{2^k} = \sum_{m=0}^{K}\sum_{k=m}^{K}\sum_{n=k-m}^{k} u_{k,n,m} I_{k,n}(y,\varpi_m) \tag{1.72}$$

where $u_{k,n,m} = (-1)^m \frac{\left(-\frac{x_0}{\sigma^2}\right)^{k-n}}{2^n(k-n)!}\binom{k+\frac{2a}{\sigma^2}-1}{n}\binom{n}{m+n-k}$ can also be simply recursively computed with

$$u_{k,n+1,m} = \frac{(n-k)(2a+(k-n-1)\sigma^2)}{2x_0(m+n+1-k)}u_{k,n,m}$$

$$u_{k+1,k+1-m,m} = \frac{\left(k+\frac{2a}{\sigma^2}\right)}{2(k+1-m)}u_{k,k-m,m}$$

$$u_{m+1,0,,m+1} = \frac{x_0}{\sigma^2}\frac{u_{m,0,m}}{m+1}$$

and the initial condition

$$u_{0,0,0} = 1$$

iv. Numerical applications

We illustrate the numerical implementation of the series on the set of parameters used later in Chapter 3 for interest rate derivatives in the Cox Ingersoll Ross model, parameters taken for comparison purposes from the reference cases presented in Chacko and Das [12]. The base case corresponds to $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ and $X_0 = 0.1$. In all the tables of this section, N represents the first integer for which the relative error is less than 10^{-4}, i.e. $|f_N^Y(y) - f^Y(y)| < 10^{-4} f^Y(y)$.

We first consider the process at different ages.

$\frac{y}{t}$	Density	N	Density	N	Density	N	Density	N
	$t = 0.1$		$t = 0.5$		$t = 1$		$t = 2$	
0.08	65.9406	54	28.1437	21	14.4597	15	7.4378	11
0.09	257.4734	50	39.0138	21	18.0505	14	9.3976	13
0.1	364.6207	54	39.9561	22	17.7163	16	9.1161	12
0.11	205.9241	64	32.1209	23	14.4371	16	7.2715	13
0.12	69.1999	70	21.1870	21	10.1401	13	4.9905	13

Table 1.5: Evolution with time (base parameters: $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ and $X_0 = 0.1$).

We observe in Table 1.5 that the series converges more and more rapidly as t increases. Globally, the series converges in around or less than 20 terms for dates greater than half-a-year and in around or less than 15 terms for a year, less than 10 terms for more than 5 years (see Table 1.6).

All the density values presented in this section have been tested and confirmed by numerical Laplace transform inversion by the Abate and Whitt method, one of the most popular inversion method. Whenever this method failed, we still managed to check the results against a numerical integration of (B.1) using the *NIntegrate* built-in routine of Mathematica, which allows for very precise and fine integration.

All the numerical Laplace transform inversion schemes involve the selection of some free parameters. The analytical formula we propose for the density in Theorem 1.1.10 has the important advantage of allowing a systematic implementation: the only parameter N can be selected with a stopping criteria ensuring convergence is obtained; N could actually be theoretically analytically chosen using the same type of bounds exploited in Appendix 1.5.2 to prove the convergence of the serie but an empirical method is more advisable as such a bound overestimates the number of terms needed. On the contrary, the selection of the free parameters requires a great deal of care for numerical inversions. For example, in the Abate and Whitt method described in Appendix B, the value and the speed of convergence can depend on the value of the parameter A in a critical way. To determine the right value for the inverse, a plateau of stability should be reached, i.e. an interval of A for which the numerical inverse remains the same. Table 1.6 points out this interval denoted $[\mathrm{A}_{\min}, \mathrm{A}_{\max}]$ can become tight and quite variable across parameters. N still represents the number of terms needed for the series (1.69).

$\frac{y}{t}$	Density	N	$\mathrm{A}_{\min}$	$\mathrm{A}_{\max}$	Density	N	$\mathrm{A}_{\min}$	$\mathrm{A}_{\max}$
	$t = 5$				$t = 10$			
0.08	2.7501	8	3.1	3.9	0.7094	8	9.34	9.45
0.09	4.5641	9	1.7	3.1	2.6151	8	9.59	9.68
0.1	4.7297	9	0.1	2.2	3.1546	7	9.92	9.99
0.11	3.4902	9	0	1.3	2.0543	8	10.39	10.45
0.12	2.0004	10	0	0.5	0.8454	9	11.01	11.06

Table 1.6: Inversion for large t (base parameters: $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ and $X_0 = 0.1$).

These $[\mathrm{A}_{\min}, \mathrm{A}_{\max}]$ are actually computed as the ones for which the numerical inverse agrees with series (1.69) and a precise numerically integration of (B.1), not as the ones for which we detected stability. In fact, without a-priori information, the Abate-Whitt is so unstable in some of those cases that the numerical inverse cannot be determined.

Figure 1.2 shows the values for the Abate-Whitt numerical inverse for different A for the case $\frac{y}{t} = 0.1$ and $t = 10$. There is no apparent stability or change of slope around $[\mathrm{A}_{\min}, \mathrm{A}_{\max}] = [9.92, 9.99]$ which contains the actual density. In other cases, a plateau

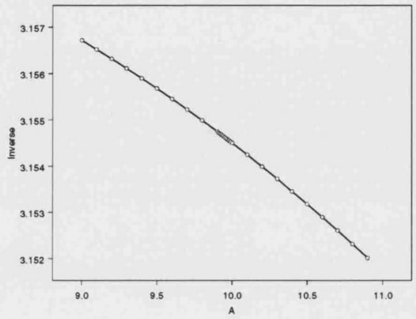


Figure 1.2: *Instability of Abate-Whitt numerical inverse.*

of stability can be found around the right inverse but the intervals $[\mathrm{A}_{\min}, \mathrm{A}_{\max}]$ vary a lot across different parameters, making it necessary to scan a number of values for A as no intuition of the location of the stability plateau is available.

The numerical inversion is inefficient for large times whereas the analytical series actually converges very fast in those regions. As a major application of this form of the square-root process is interest rate modelling, it should be pointed out that interest rate products can be very long-dated and maturities of ten, twenty years or more are not all unusual in this context. In the same way, the Abate-Whitt method has difficulties to cope with high volatilities whereas the analytical series converges more and more quickly as σ increases, as can be seen from Table 1.7.

$\frac{y}{t}$	Density	N	$\mathrm{A}_{\min}$	$\mathrm{A}_{\max}$
	$\sigma = 0.3$			
0.06	7.7615	9	-	-
0.08	12.4710	12	5.7	6.3
0.09	12.6671	12	-	-
0.1	11.6966	11	2.6	4.3
0.11	10.0330	10	1.5	4
0.12	8.1133	13	0	3.8

Table 1.7: Inversion for $\sigma = 0.3$ and $t = 1$.

If a numerical integration of (B.1) works where the Abate and Whitt method fails, it should be added that a number of free parameters are also involved here and the degree of precision required to perform the integration for these specific cases would not be usually chosen. The numerical integration is, in general, more involved than the Abate and Whitt inversion and is used in this thesis only for checking purposes.

With this numerical results, we conclude this section by the observation that the analytical expressions we derived here are a robust and efficient way to evaluate the corresponding densities. We now turn to the geometric Brownian motion case.

1.2 The geometric Brownian motion

The geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1.73}$$

is the basis of option pricing theory. The study of its temporal integral $Y_t = \int_0^t S_u du$ is both of great global interest for various branches of science and of specific importance in finance due to the popularity of the so-called Asian options. A considerable amount of research (see the references within Dufresne [27]) has consequently been devoted to this temporal integral. In this thesis, we prefer to analyse the temporal integral directly in the context of Asian options and with the motivation of evaluating them rather than to study the distribution of Y_t per se. We will therefore present all of our new results on this model in the next chapter, in which the actual study of these Asian options will be treated. In this section are only recalled known properties and main classical results pertaining to Y_t.

Geman and Yor [77] exploited advanced probabilistic techniques and the representation of the geometric Brownian motion as a random-time changed squared Bessel process to derive the Laplace transform with respect to maturity of a multiple of the fixed-strike Asian option price of payoff $(Y_T - K)^+$.

They first normalise the problem, noticing that

$$E\left(e^{-rT}\left(\frac{Y_T}{T} - K\right)^+\right) = \frac{e^{-rT}}{T}\frac{4S_0}{\sigma^2}C^\nu(h,q)$$

where

$$C^\nu(h,q) = E(\left(\int_0^h e^{2(\nu u+W_u)}du - q\right)^+ \tag{1.74}$$

with the normalisation parameters $h = \frac{\sigma^2 T}{4}$, $q = \frac{\sigma^2}{4S_0}KT$ and $\nu = \frac{2r}{\sigma^2} - 1$.

They then obtain the Laplace transform $LC^\nu(\lambda, q)$ as

$$\begin{aligned}
LC^\nu(\lambda,q) &= \int_0^\infty e^{-\lambda h}C^\nu(h,q)dh \\
&= \frac{\int_0^1 e^{-\frac{w}{2q}}w^{\frac{1}{2}(\mu-\nu)-2}(1-w)^{\frac{1}{2}(\mu+\nu)+1}dw}{(2q)^{\frac{1}{2}(\mu-\nu)-1}\lambda(\lambda-2-2\nu)\Gamma(\frac{1}{2}(\mu-\nu)-1)} \\
&= \frac{1}{\lambda(\lambda-2\nu-2)}\sum_{n=0}^\infty \frac{1}{n!}\left(-\frac{1}{2q}\right)^{\frac{\mu-\nu}{2}+n-1}\frac{\Gamma(\frac{\mu-\nu}{2}-1+n)\Gamma(\frac{\mu+\nu}{2}+2)}{\Gamma(\mu+1+n)\Gamma(\frac{\mu-\nu}{2}-1)} \qquad (1.75)
\end{aligned}$$

where $\mu = \sqrt{2\lambda + \nu^2}$.

This expression for the Laplace transform , main discovery in the Asian options field, results from Yor's analysis of the integral Y_h or rather of the normalised quantity $D_h^\nu = \int_0^h e^{2(\nu s+W_s)}ds$. Yor showed in 92 that D_h taken at an exponentially distributed random time T_γ, independent of W, with mean $\frac{1}{\gamma}$ is characterised by

$$2D_{T_\gamma}^\nu \cong \frac{B_{1,\alpha}}{G_\gamma}$$

where $B_{1,\alpha}$ represents a Beta distributed variable of parameters 1 and $\alpha = \frac{\nu+\mu}{2}$ and G_γ a Gamma distributed variable of parameters $\beta = \alpha - \nu$ and 1. He also shows that

$$P\Big[D_h^\nu \in [u, u+du]\Big|W_t + \nu t = x\Big] = \frac{\sqrt{2\pi t}}{u}e^{\frac{x^2}{2t}-\frac{1}{2u}(1+e^{2x})}\theta_{\frac{e^x}{u}}du$$

where $\theta_r(t) = I_0(r)f_r(t)$, $f_r(t)$ being the Hartman-Watson density. More precisely,

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\frac{\pi^2}{2t}} \int_0^\infty e^{-\frac{y^2}{2t} - r\cosh(y)} \sinh(y) \sin\left(\frac{\pi y}{t}\right) dy$$

Dufresne [27], [25] and Schröder [65] also contributed to deepen the mathematical analysis of the density of this integral by proposing sophisticated serial and integral forms for it. The references within these papers present other related prior works in non-financial fields. Chapter 2 provides a review of the other approaches carried out in the literature for the treatment of Asian options. Only the works presented in this chapter focus on an analysis of the exact distribution of Y_T.

1.3 The general CEV process

The general Constant Elasticity of Variance process is defined as a process of the form

$$dS_t = (a - bS_t)dt + \sigma S_t^\alpha dW_t \tag{1.76}$$

Using the same terminology as in Section 1.1, we will be considering only the non mean-reverting form in this chapter, the mean-reverting case being analysed in Chapter 3 as it is most importantly used for modelling interest rates. This allows us to present our results in a simpler case first, before extending them in Chapter 3.

1.3.1 The equity CEV model

In this section, we will hence more precisely focus on the CEV processes used to model the evolution of equities.

$$dS_t = rS_t dt + \sigma S_t^\alpha dW_t, \qquad X_0 = x_0 \tag{1.77}$$

This model was introduced by Cox and Ross [16] as an alternative for the Black-Scholes log-Brownian model. This process allows us to model many desirable features. Firstly, the volatility changes stochastically and the inverse type of relationship between the evolution of the asset and the local volatility $\sigma S_t^{\alpha-1}$ of the instantaneous return $\frac{dS_t}{S_t}$ is an empirically well-proven property of financial equities and indices time series. The main drawback in this construction is that the absolute correlation between the asset and its volatility is full. Yet, the CEV process remains the most tractable model allowing for stochastic volatility. Secondly, the absorption at the origin provides a convenient way to account for industrial bankruptcy, which was an almost impossible event in the Black-Scholes world. Finally, the extra parameter α enables us to better fit the distribution of assets and capture more information, as their skewness.

We will restrict ourselves to the case $0 < \alpha < 1$, for this is the case initially studied in the original Cox [14] model and it enables us to restrain ourselves to well-defined problems, avoid the possibility of encountering explosions, etc. Our results could though be extended and generalised, provided extra care is taken.

With these restrictions, it turns out that the CEV process is closely linked to the square-root process.

Proposition 1.3.1. $Z_t = X_t^{2(1-\alpha)}$ *follows*

$$dZ_t = (\sigma^2(1-\alpha)(1-2\alpha) + 2r(1-\alpha)Z_t)dt + 2\sigma(1-\alpha)\sqrt{Z_t}dW_t \tag{1.78}$$

Proof. This a simple application of Itô Lemma. □

This result has been often used in the literature (see Davydov and Linetsky [21] for example) since it considerably simplifies the analysis of the spot-equity process. Indeed, it is simply a power function of the square-root process, which is a well-studied process. From the results presented in Section 1.1, it is possible to deduce a number of properties for X_t.

The behaviour of the process at the zero-boundary can be also be inferred from Proposition 1.1.2 type of results and Feller classification. For $\alpha \geq \frac{1}{2}$, 0 is an exit boundary. For $\alpha < \frac{1}{2}$, the origin is regular boundary point and we adjoin a killing boundary condition so that the process is an absorption boundary at 0 in any case. The process has no limit-distribution given that (1.78) has the non mean-reverting form. The density of the spot-equity is a straightforward transform of the corresponding density given in Theorem 1.1.1. Analytics to compute vanilla options prices have been produced in the literature by integration against this density. Exotic options such as barrier and lookback options on this process have also been given some importance very recently thanks to the works of Boyle and Tian [8], Davydov and Linetsky [21] and Lo and al. [52]. Yet, Asian types of derivative depending on the continuous average of the process have not received much consideration till now although they constitute the next step to be taken to deepen our understanding of the derivatives market under this model. We will therefore try to derive here some properties concerning the distribution of the temporal integral $Y_t = \int_0^T X_t dt$.

Before beginning our analysis of Y_T in the next subsection, a last property of interest for X_T should be noticed as it involves another path-dependent integral functional of X_t. The relation between the CEV process and its non-drifted (martingale) counterpart involves the integral $\int_0^T X_t^{2(1-\alpha)} dt$.

Theorem 1.3.1. *The change of measure defined by* $\frac{dQ^*}{dQ} = \hat{L}(T)$, *with the exponential Radon-Nikodym derivative*

$$\hat{L}(t) = e^{\frac{r^2}{2\sigma^2}\int_0^t X_s^{2(1-\alpha)} ds - \frac{r}{2\sigma^2(1-\alpha)}\left(X_t^{2(1-\alpha)} - x_0^{2(1-\alpha)}\right) + \frac{r}{2}(1-2\alpha)t} \tag{1.79}$$

cancels the drift of X_t, i.e.

$$dX_t = \sigma X_t^\alpha dW_t^{Q^*} \tag{1.80}$$

$W_t^{Q^*}$ being a Brownian motion under Q^*.

Proof. This change of measure is an application and the counterpart of Theorem 1.1.5. Indeed, Proposition 1.3.1 establishes $Z_t = X_t^{2(1-\alpha)}$ as a square-root process. From Theorem 1.1.4 and 1.1.5, $\hat{L}(t)$ is then a martingale and under Q^*,

$$dZ_t = \sigma^2(1-\alpha)(1-2\alpha)dt + 2\sigma(1-\alpha)\sqrt{Z_t}dW_t^{Q^*}$$

Applying Itô Lemma to $X_t = Z_t^{\frac{1}{2(1-\alpha)}}$ leads to the stochastic differential equation (1.80) under Q^*. □

Remark. Although this change of measure is interesting from a theoretical point of view, it does not immediately help to simplify the problem of evaluating the moment generating function of $Y_T = \int_0^T X_t dt$, unlike in the square-root case. Indeed, computing $E(e^{-\mu Y_T}) = E^{Q^*}(e^{-\mu Y_T} L^{-1}(T))$ using the Q^*-measure requires solving a partial differential equation even more complicated.

1.3.2 The integrated process

To characterise the distribution of $Y_t = \int_0^T X_t dt$, we would like to obtain its moment-generating function as this random variable is positive.

Theorem 1.3.2. *Any function belonging to $C^{1,2}(\mathbb{R}^+, \mathbb{R}^+)$, bounded with bounded derivatives for its second variable, bounded on any compact interval for its first variable and satisfying the partial differential equation*

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - \mu x f \tag{1.81}$$

with the initial condition

$$\forall x_0 \geq 0, \quad f(0, x_0) = 1 \tag{1.82}$$

is equal to the moment generating function $E(e^{-\mu Y_T})$ at time T.

Proof. Applying the Itô lemma to

$$M_t = e^{-\mu \int_0^t X_s ds} f(T-t, X_t)$$

proves to that M_t is a martingale.
Therefore, $E(M_T) = E(e^{-\mu \int_0^T X_t dt}) = M_0 = f(T, X_0)$ □

Given the complexity of the easier subcase constituted by the Geometric Brownian motion, the simpler result we expect to obtain is an elaborated function, typically in series or integral form, for the Laplace transform with respect to T of the MGF of Y_T. We hence define $g(\lambda, x) = \int_0^\infty e^{-\lambda t} f(t, x) dt$ and look for it as a bounded solution to

$$\lambda g - 1 = \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 g}{\partial x^2} + rx \frac{\partial g}{\partial x} - \mu x g \tag{1.83}$$

To obtain a solution for this equation, we will work under the following least-constraining but useful assumption.

Assumption. For the remainder of this section, we will assume that α is a rational number, i.e. $\alpha = \frac{p}{q}$. For simplicity, p and q will denote the integers such that $\frac{p}{q}$ is irreducible.

This assumption is not restrictive at all in practice. Indeed, the finite precision of computers bounds us anyway to use rational numbers for any sort of numerical calculation. Statistical estimates for α will always be rational just as the α input in any numerical routines related to this model.

We should notice that the condition $0 < \alpha < 1$ implies $0 < p < q$.

i. The homogeneous equation

In this framework, it is worthwhile considering first the homogeneous equation for two reasons. Firstly, it enables us to understand the structure of the affine space of solutions to the inhomogeneous equation; given any particular solution, g_p, of the inhomogeneous equation, this space is the set of linear combinations of two independent solutions to the homogeneous equation added to g_p. Secondly, this analysis throws some light on the possibility\ feasibility of an eigenfunction expansion.

For this purpose, we start by setting some definitions.

Definitions and notations. Given $l = 2(q-p)$, $A = \frac{\sigma^2}{2}\frac{1-q}{q^2}$, $B = \frac{r}{q}$ and $C = \frac{\sigma^2}{2q^2}$, $h^{\mu,\lambda}(\nu_0, \nu_1, x)$ refers, in this section, to the power series

$$h^{\mu,\lambda}(\nu_0, \nu_1, x) = \sum_{k=0}^{\infty} u_k^{\mu,\lambda}(\nu_0, \nu_1) x^{\frac{k}{q}}$$

$$\left\{\begin{array}{lcll} u_0^{\mu,\lambda}(\nu_0,\nu_1) & = & \nu_0 & \text{for } k=0 \\ u_k^{\mu,\lambda}(\nu_0,\nu_1) & = & 0 & \text{for } 0<k<l \\ u_k^{\mu,\lambda}(\nu_0,\nu_1) & = & \frac{\lambda - B(k-l)}{k[A+C(k-1)]} u_{k-l}^{\mu,\lambda}(\nu_0,\nu_1) & \text{for } l \le k < q+l,\ k \neq q \\ u_q^{\mu,\lambda}(\nu_0,\nu_1) & = & \nu_1 & \text{for } k=q \\ u_k^{\mu,\lambda}(\nu_0,\nu_1) & = & \frac{(\lambda - B(k-l)) u_{k-l}^{\mu,\lambda}(\nu_0,\nu_1) + \mu u_{k-q-l}^{\mu,\lambda}(\nu_0,\nu_1)}{k[A+C(k-1)]} & \text{for } q+l \le k \end{array}\right. \tag{1.84}$$

Theorem 1.3.3. *The vectorial space of solutions to the homogeneous differential equation on* $]0,\infty[$

$$\lambda g = \frac{\sigma^2}{2} x^{2\frac{p}{q}} \frac{\partial^2 g}{\partial x^2} + r x \frac{\partial g}{\partial x} - \mu x g \tag{1.85}$$

is generated by the basis $\left(h_1^{\mu,\lambda}(x) = h^{\mu,\lambda}(1,0,x), h_2^{\mu,\lambda}(x) = h^{\mu,\lambda}(0,1,x)\right)$ *except in the case* $\alpha = \frac{p}{q} = \frac{2k'+1}{2(k'+1)}$ *for some* $k' \in \mathbb{N}$.

Proof. The change of variables $y = x^{\frac{1}{q}}$, $h(\lambda, y) = g(\lambda, x)$ gives

$$0 = Cy^2 \frac{\partial^2 h}{\partial y^2} + [A + By^l] y \frac{\partial h}{\partial y} - \mu y^{q+l} h - \lambda y^l h$$

A classical method to find a solution to an homogeneous second-order differential equation is the Frobenius or analytic coefficient method. Assuming a power series form $h = \sum_0^\infty u_k y^k$ leads to the relations (1.84), leaving u_0 and u_q as free parameters whenever $\forall k' \in \mathbb{N},\ (2k' + 1)q \neq 2(k' + 1)p$. □

If there exists some $k' \in \mathbb{N}$ such that the integral parameters are related through $2(k' + 1)p = (2k' + 1)q$, which includes the square-root case $(p = 1,\ q = 2)$, then all the solutions to the homogeneous equation are not analytic.

Theorem 1.3.4. *When $(2k'+1)q = 2(k'+1)p$ for some $k' \in \mathbb{N}$, $h_2^{\mu,\lambda}(x) = h^{\mu,\lambda}(0, 1, x)$ is still a solution. But, $h^{\mu,\lambda}(1, 0, x)$ is no more.*

Proof. The relation

$$k[A + C(k-1)] u_k^{\mu,\lambda}(\nu_0, \nu_1) = [\lambda - B(k-l)] u_{k-l}^{\mu,\lambda}(\nu_0, \nu_1) \quad \text{for } l \leq k < q + l$$

together with $q[A + C(q-1)] = 0$ imply that the terms $u_{Q-kl}^{\mu,\lambda}(\nu_0, \nu_1)$, for $k \leq k' + 1$ should remain null. These terms include u_0, removing hence one degree of freedom. All the analytic solutions become therefore multiples of $h_2^{\mu,\lambda}(x)$.

An independent solution could formally be calculated with

$$h_1^{\mu,\lambda}(x) = \int_0^x e^{-\int_0^x \frac{(h_2^{\mu,\lambda})'(y)}{h_2^{\mu,\lambda}(y)} \frac{2\sigma^2 y^{2\frac{p}{q}}}{\sigma^2 y^{2\frac{p}{q}} + ry} dy}$$

□

ii. The inhomogeneous equation

The preceding discussion enables us to deduce the following result.

Theorem 1.3.5. *Any solution to the inhomogeneous equation candidate to be the Laplace transform of the MGF should verify*

$$i^{\mu,\lambda}(\nu,x) = \sum_{k=0}^{\infty} v_k^{\mu,\lambda}(\nu) x^{\frac{k}{q}} \tag{1.86}$$

$$\begin{cases} v_0^{\mu,\lambda}(\nu) &= \frac{1}{\lambda} & \textit{for } k=0 \\ v_k^{\mu,\lambda}(\nu) &= 0 & \textit{for } 0<k<q+l,\ k\neq q \\ v_q^{\mu,\lambda}(\nu) &= \nu & \textit{for } k=q \\ v_k^{\mu,\lambda}(\nu) &= \frac{[\lambda - B(k-l)]v_{k-l}^{\mu,\lambda}(\nu) + \mu v_{k-q-l}^{\mu,\lambda}(\nu)}{k[A+C(k-1)]} & \textit{for } q+l\leq k \end{cases} \tag{1.87}$$

The Laplace transform of the MGF is more precisely the transform among this class of functions which satisfies

$$\lim_{x\longrightarrow\infty} i^{\mu,\lambda}(\nu,x) = 0$$

Proof. The same change of variable as previously leads to

$$0 = Cy^2\frac{\partial^2 h}{\partial y^2} + [A+By^l]y\frac{\partial h}{\partial y} - \mu y^{q+l}h - \lambda y^l h + y^l$$

As in the homogeneous case, we take the Frobenius approach and look for a particular solution in power series. We obtain the same recursions as in (1.84) with the difference

$$v_l^{\mu,\lambda}(\nu) = \frac{\lambda v_0^{\mu,\lambda} - 1}{l[A+C(l-1)]}$$

$i^{\mu,\lambda}(0,x)$ is thus a particular solution of the inhomogeneous equation with $v_0^{\mu,\lambda}(\nu) = \frac{1}{\lambda}$ and $v_q^{\mu,\lambda}(\nu) = 0$. $(h_1^{\mu,\lambda}(x), h_2^{\mu,\lambda}(x))$ being the basis of the vector space, the Laplace transform of the MGF can then be expressed as $\hat{f}^{\mu}(\lambda,x) = i^{\mu,\lambda}(0,x) + \phi h_1^{\mu,\lambda}(x) + \psi h_2^{\mu,\lambda}(x)$.

The absorption at 0 of the CEV process implies that $E(e^{-\mu Y_T}|x_0=0) = 1$ and $\hat{f}^{\mu}(\lambda,0) = \frac{1}{\lambda}$. This condition imposes $\phi = 0$ because $h_2^{\mu,\lambda}(0) = 0$. Thus the result. □

iii. Absorption

To obtain the exact expression for the transform, i.e. the right value for ν, we would need to understand the asymptotic behaviour of $i^{\mu,\lambda}(\nu, x)$, which seems tricky from the recursion formulae.

One solution would be to numerically find the limit of $\frac{i^{\mu,\lambda}(0,x)}{h_2^{\mu,\lambda}(x)}$ by approximating it by $\frac{i^{\mu,\lambda}(0,B)}{h_2^{\mu,\lambda}(B)}$ for a sufficiently large B. This approach might be difficult to handle numerically as well as risky, since uniform convergence in λ is not guaranteed.

A better - i.e. theoretically valid and safer - approach is to consider the approximate model in which the CEV process gets absorbed at a level B when it is reached. We will call it restrained model. The MGF under this model, denoted $\mathcal{L}^{\alpha,B}(\mu)$, converges towards the MGF under the standard assumptions (unrestrained model) $\mathcal{L}^{\alpha}(\mu)$ as B grows to infinity. The density under the restrained model converges too towards the density under the standard assumptions. This convergence remains also true for other related quantities such as moments, bounded functionals, increasing and other functionals of Y_T. Thus, the restrained model stands as a good approximation for the unrestrained model for B sufficiently large, although the adverb needs to be defined relatively to the specific problem considered. In our case, a condition to ensure the absolute error is less than ϵ for any μ (and any λ) is to bound the probability $P(1_{\{\tau_B<T\}}) < \epsilon$ where τ_B denotes the first hitting time of B by X_t. Although, numerically, we would practically just analyse the direct convergence to decide the right value of B, it can give more intuition and better understanding of the results to compute this probability for some value. For this reason and for the sake of rigour, we first study this hitting time.

Theorem 1.3.6. *The MGF of the hitting time is*

$$E(e^{-\lambda\tau_B}1_{\{\tau_B<\infty\}}|X_0=x_0)=\frac{x_0^\alpha\phi\Big(\frac{\lambda}{2r(\alpha-1)}-\frac{1-2\alpha}{2(1-\alpha)}+1,2-\frac{1-2\alpha}{2(1-\alpha)},\frac{rx_0^{2(1-\alpha)}}{\sigma^2(\alpha-1)}\Big)}{B^\alpha\phi\Big(\frac{\lambda}{2r(\alpha-1)}-\frac{1-2\alpha}{2(1-\alpha)}+1,2-\frac{1-2\alpha}{2(1-\alpha)},\frac{rB^{2(1-\alpha)}}{\sigma^2(\alpha-1)}\Big)} \tag{1.88}$$

Proof. To determine the hitting-time of any positive level B, we consider the following ordinary differential equation, transformed of a Kummer equation,

$$\frac{\sigma^2}{2}x^\alpha f''(x)+rxf'(x)=\lambda f(x) \tag{1.89}$$

for which $x^\alpha\phi\Big(\frac{\lambda}{2r(\alpha-1)}-\frac{1-2\alpha}{2(1-\alpha)}+1,2-\frac{1-2\alpha}{2(1-\alpha)},\frac{rx^{2(1-\alpha)}}{\sigma^2(\alpha-1)}\Big)$ is the only solution null at 0. The boundedness of this function on $[0,B]$ permits $E(e^{-\lambda(t\wedge\tau_B)}f(X_{(t\wedge\tau_B)}))$ to be a martingale. In the limit, we then obtain (1.88). □

Now, we just have to solve for the Laplace transform of the MGF in the restrained model.

Theorem 1.3.7. *The Laplace transform of the MGF under the restrained model is*

$$g_B^\mu(\lambda,x)=\left[i^{\mu,\lambda}(0,x)-\frac{i^{\mu,\lambda}(0,B)}{i^{\mu,\lambda}(1,B)}i^{\mu,\lambda}(1,x)\right]\left(1-\frac{i^{\mu,\lambda}(0,B)}{i^{\mu,\lambda}(1,B)}\right)^{-1} \tag{1.90}$$

Proof. The moment generating function $E(e^{-\mu Y_T}1_{\{\tau_B>T\}})$ is the solution $f\in\mathcal{C}^{1,2}(\mathbb{R}^+,\mathbb{R}^+)$ to the partial differential equation

$$\frac{\partial f}{\partial t}=\frac{\sigma^2}{2}x^{2\alpha}\frac{\partial^2 f}{\partial x^2}+rx\frac{\partial f}{\partial x}-\mu xf \tag{1.91}$$

with the initial condition

$$\forall x_0\geq 0,\quad f(0,x_0)=1$$

as well as the boundary condition

$$\forall 0\leq t\leq T,\quad f(t,B)=0$$

and which is bounded on $K\times\mathbb{R}$ for any compact $K\subset\mathbb{R}$. The previous results help to identify this function as (1.90). □

This completes the approximation.

1.3.3 Numerical illustrations

We preferred to focus on only one illustration but see it in detail to gain insight in the behaviour of the algorithm. We chose the base case $r = 0.05$, $\sigma = 0.1$, $p = 1$, $q = 4$.

The first question concerning this method arising in our mind: is how fast the series are?

x	f1	n1	f2	n2
0.25	1.000051	9	1.004006	4
0.5	1.065834	28	1.192474	28
0.75	41.954773	74	83.540390	74
1	4345753.182	186	8732160.599	186
1.25	9.74605E+15	594	1.95833E+16	594
1.5	3.23941E+32	1697	6.50913E+32	1704
1.75	1.23651E+60	4104	2.48459E+60	4104

Table 1.8: Evolution of the basis functions with respect to r_0 .

Table 1.8 shows that, not surprisingly, the power series converge faster for small x. Both the basis functions grow to infinity, which was expected as well.

B	1.25	1.75
const	0.497672	0.497672
x	g_B^μ	
0.25	0.996133	0.996133
0.5	0.940368	0.940368
0.75	0.754586	0.754586
1	0.494434	0.494434
1.25		0.287738
1.5		0.163842

Table 1.9: Evolution of the approximate Laplace transform for different B .

Table 1.9 shows that the approximate Laplace transform has satisfactorily converged for $B = 1.25$ for initial values S_0 less than 1.

Figure 1.3 draws the evolution of the Laplace transform with respect to μ at the point $\lambda = 1$. We observe a decrease with increasing μ as expected.

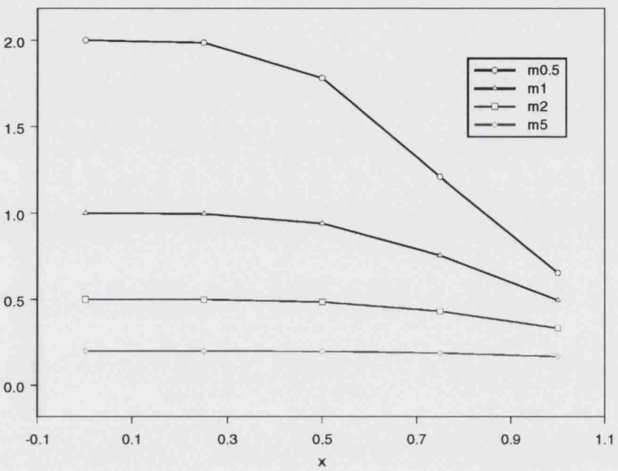


Figure 1.3: *Evolution of the transform with respect to* μ *at* $\lambda = 1$.

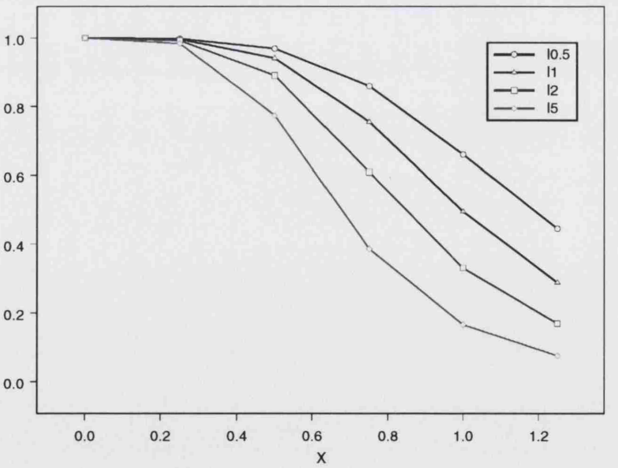


Figure 1.4: *Evolution of the transform with respect to* λ *at* $\mu = 1$.

Figure 1.4 shows the decreasing family of curves for increasing λ at $\mu = 1$. The transform is correctly decreasing from $\frac{1}{\lambda}$ at $x = 0$.

1.4 Conclusion

In this chapter, we aimed at a thorough analysis of the distributional properties of temporal integrals of CEV processes. Special attention has been devoted to the square-root process, for which we derived a number of properties from the joint MGF of the process and its integral. This results built in explicit series form- moments, densities, etc. - pertain both to the marginal and to the joint distribution. Implementing numerically these series, it appeared that they are practical as they do not involve any free-parameter selection and more efficient than numerical inversion methods in regions of high volatilities or maturities. These probabilistic results will be applied in the next chapter to the pricing of financial derivatives.

We also presented a way to compute the Laplace transform with respect to time of the MGF of the integral of the general CEV process. An adaptation to interest rate will be furnished in Chapter 3.

1.5 Appendices to this chapter

1.5.1 Proof of Theorem 1.1.9

i. Expansion

Reexpressing the marginal MGF of Y_t (1.50) in such a manner that we can separate an exponential term with the same argument (up to a multiplicative constant) as the one in the power function

$$\mathcal{L}^Y(\mu) = e^{\frac{atb - x_0(\gamma-b)}{\sigma^2}} \left(\frac{2\gamma e^{-\frac{\gamma t}{2}}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}} \right)^{\frac{2a}{\sigma^2}} e^{\frac{x_0(\gamma-b)}{\sigma^2} \frac{2\gamma e^{-\gamma t}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}}$$

we expand it

$$\mathcal{L}^Y(\mu) = e^{(b-\gamma)\frac{at+x_0}{\sigma^2}} \sum_{n=0}^{\infty} \frac{(x_0(\gamma-b)e^{-\gamma t})^n}{\sigma^{2n} n!} \left(\frac{2\gamma}{(\gamma+b)+(\gamma-b)e^{-\gamma t}} \right)^{n+\frac{2a}{\sigma^2}} \tag{1.92}$$

Now, expanding $\left(\frac{2\gamma}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}\right)^{n+\frac{2a}{\sigma^2}} = \left(1 - \frac{(\gamma-b)(1-e^{-\gamma t})}{2\gamma}\right)^{-n-\frac{2a}{\sigma^2}}$, we obtain

$$\begin{aligned} \mathcal{L}^Y(\mu) = e^{-(\gamma-b)\frac{at+x_0}{\sigma^2}} & \left[\sum_{n=0}^{\infty} \frac{1}{n!} \frac{x_0^n}{\sigma^{2n}} (\gamma-b)^n e^{-n\gamma t} \right. \\ & \left. \times \left(\sum_{k=0}^{\infty} \binom{n+k+\frac{2a}{\sigma^2}}{k} \left(\frac{(\gamma-b)^k (1-e^{-\gamma t})^k}{(2\gamma)^k} \right) \right) \right] \end{aligned} \tag{1.93}$$

where the generalised binomial coefficient is defined as

$$\binom{n+k+\frac{2a}{\sigma^2}}{k} = \frac{(n+\frac{2a}{\sigma^2})\dots(n+\frac{2a}{\sigma^2}+k-1)}{k!}, \ k > 0$$

and is conventionally equal to 1 when $k = 0$.

Noticing that the sums involved in (1.93) are absolutely convergent, we can modify the order of summation

$$\mathcal{L}^Y(\mu) = e^{-(\gamma-b)\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k+\frac{2a}{\sigma^2}-1}{k-n} \frac{x_0^n (\gamma-b)^k (1-e^{-\gamma t})^{k-n} e^{-\gamma n t}}{n! \sigma^{2n} (2\gamma)^{k-n}} \tag{1.94}$$

To obtain (1.60), we will actually need to go a bit further (though it would not be needed to compute the density through recursions) in the decomposition in order to invert the MGF in terms of elementary Laplace inverses and split the term $\frac{(\gamma-b)^k(1-e^{-\gamma t})^{k-n}e^{-\gamma nt}}{(\gamma)^{k-n}}$ appearing in (1.94) into

$$\sum_{i=0}^{k-n} \sum_{j=0}^{k} \binom{k-n}{i} (-1)^i \binom{k}{j} (-b)^{k-j} e^{-\gamma t(n+i)} \gamma^{j-k+n}$$

Before going in the details of the analytic inversion of the elementary terms $e^{-\gamma t(n+i)}\gamma^{j-k+n}$, we will first study the convergence of the global MGF inverse resulting from this decomposition. In the next subsection, we point out that for each of the series expansions used here, we are in a closed bounded set belonging to the convergence disk of the involved entire series, implying they are not only convergent, but uniformly convergent.

ii. Uniform convergence of the inverse

This time, the elementary terms in the expansion are not Laplace transforms of positive functions, so that we cannot appeal to the Beppo Levi theorem as for the X_t-density. But, the convergence still holds and is even uniform, as will be proven here.

To this effect, we will need to use Bromwich integral representations. For f being either the marginal density $f^Y(y)$ itself or any of the internal terms or sum of terms in the series (1.94), its inverse Laplace transform can be written in the form[13]

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} e^{ys}f(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy\omega}f(-i\omega)d\omega \tag{1.95}$$

since 0 is to the right of all the singularities of f in every case.

We will denote $C = e^{b\frac{at+x_0}{\sigma^2}}$ and

$$\mathcal{L}_k^Y(-i\omega) = \sum_{n=0}^{k} \frac{1}{n!}\frac{x_0^n}{\sigma^{2n}}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\frac{(\gamma-b)^k(1-e^{-\gamma t})^{k-n}e^{-\gamma nt}}{(2\gamma)^{k-n}}$$

We observe that, though $\gamma = \sqrt{b^2+2\mu\sigma^2}$ is multivalued, the value of the MGF remains the same no matter which branch of γ is chosen. We hence arbitrarily take the branch with positive real part

$$\gamma = \sqrt{\frac{b^2+\sqrt{b^4+4\sigma^4\omega^2}}{2}} - i\frac{\sqrt{2}\sigma^2\omega}{\sqrt{b^2+\sqrt{b^4+4\sigma^4\omega^2}}}$$

[13]This form also highlights the link with the characteristic function $\mathcal{L}_k^Y(-i\omega)$ and Fourier inversion.

We deduce that $|\gamma - b|e^{-\gamma t}$ is bounded by some positive constant M for any $\omega \in \mathbb{R}$ and that

$$\left|\frac{(1-e^{-\gamma t})^{k-n}}{2^{k-n}}\right| \leq \frac{1+e^{-bt}}{2} \tag{1.96}$$

Having as well - since $b > 0$ - the inequality

$$\left|\frac{(\gamma - b)^{k-n}}{\gamma^{k-n}}\right| \leq 1 \tag{1.97}$$

it follows that

$$\left|\mathcal{L}_k^Y(-i\omega)\right| \leq \sum_{n=0}^{k} \frac{1}{n!}\frac{x_0^n}{\sigma^{2n}}\binom{k+\frac{2a}{\sigma^2}-1}{k-n} M^n \left(\frac{1+e^{-bt}}{2}\right)^{k-n}$$

Therefore,

$$\mathcal{L}^Y(-i\omega)e^{\gamma(\frac{at+x_0}{\sigma^2})} \leq C\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(x_0M)^n}{n!\sigma^{2n}}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\left(\frac{1+e^{-bt}}{2}\right)^{k-n} \tag{1.98}$$

The series in (1.98) being convergent, the expansion (1.94) is uniformly convergent $\forall \epsilon > 0, \exists K > 0, \forall \tilde{K} \geq K, \forall \omega \in \mathbb{R}$,

$$\left|\sum_{k=0}^{\tilde{K}} \mathcal{L}_k^Y(-i\omega)e^{\gamma(\frac{at+x_0}{\sigma^2})} - \mathcal{L}^Y(-i\omega)e^{\gamma(\frac{at+x_0}{\sigma^2})}\right| < \epsilon$$

which, for any positive y, leads to

$$\left|\sum_{k=0}^{\tilde{K}}\int_{-\infty}^{\infty}\mathcal{L}_k^Y(-i\omega)e^{-i\omega y}d\omega - \int_{-\infty}^{\infty}\mathcal{L}^Y(-i\omega)e^{-i\omega y}d\omega\right| \leq$$

$$\int_{-\infty}^{\infty}\epsilon\left|e^{-\gamma(\frac{at+x_0}{\sigma^2})}\right|d\omega \leq 2\epsilon\int_0^{\infty}e^{-(\frac{at+x_0}{\sigma})\sqrt{\omega}}d\omega$$

This proves that the sum of the inverses of the $\tilde{\mathcal{L}}_k^Y(\mu)$ converges uniformly to the inverse of $\tilde{\mathcal{L}}^Y(\mu)$, i.e. to the density of $\int_0^t X_s ds$.

iii. A slightly stronger result

We can actually go a bit further than the previous convergence result. Setting $\tilde{\gamma} = \sqrt{\frac{b^2}{4} + 2\sigma^2\mu}$, if we replace γ by $\tilde{\gamma}$ in (1.6) and (1.5), we obtain the Laplace transform of $e^{\frac{3b^2}{8\sigma^2}y} f^Y(y)$. Expanding it exactly in the same way as (1.94), we keep the uniform convergence property. Indeed, (1.97) still holds, $|\gamma - b|e^{-\gamma t}$ remains bounded, but by a different M', and $\frac{1+e^{-bt}}{2}$ can replaced by $\frac{1+e^{-\frac{bt}{2}}}{2}$ in (1.96). This normal convergence then implies

$\forall \epsilon > 0, \exists K > 0, \forall \tilde{K} \geq K, \forall y \geq 0,$

$$\left| f^Y(y) - \sum_{k=0}^{\tilde{K}} f_k^Y(y) \right| \leq \epsilon e^{-\frac{3b^2}{8\sigma^2}y}$$

if we note $f_k^Y(y)$ the inverse of the $\mathcal{L}_k^Y(\mu)$.

With this result, we can for instance interchange sum and integral when computing the expectation of $g(Y_t)$, where g is a bounded function. Indeed, in that case,

$$\left| \int_0^\infty f^Y(y) g(y) dy - \sum_{k=0}^{\tilde{K}} \int_0^\infty f_k^Y g(y) dy \right| \leq ||f||_\infty \int_0^\infty |f^Y(y) - \sum_{k=0}^{\tilde{K}} f_k^Y(y)| dy$$

$$\leq \epsilon ||f||_\infty \frac{8\sigma^2}{3b^2}$$

Hence, the sum of integrals converges towards the integral of the sum, which is an useful result whenever we want to compute the expectation of bounded functions of Y_t, for pricing issues, etc.

The same type of arguments apply for all the series decompositions for densities, probabilities and prices under the square-root model (both mean-reverting model and equity model) and will not hence be explicitly stated in the corresponding proofs.

iv. Inversion of the elementary terms

From (1.94), it follows that only two kinds of elementary inverses appear in the decomposition of $f^Y(y)$. Thanks to the scaling property of Laplace transforms, we only need to find for $\varpi > 0$ the inverse $J^{\mathrm{d}}_{p,\varpi}(y)$ of $\frac{e^{-\varpi\sqrt{b^2+\mu}}}{(\sqrt{b^2+\mu})^p}$ for $p \in \mathbb{N}$ and $J^{\mathrm{m}}_{q,\varpi}(y)$ the inverse of $(\sqrt{\mu+b^2})^q e^{-\varpi\sqrt{\mu+b^2}}$ for $q \in \mathbb{N}\backslash\{0\}$.

First kind of elementary inverses

From the inverse gaussian distribution theory[14], we can infer that, for any $\mu \geq 0$,

$$\int_0^\infty J^{\mathrm{d0}}_{\varpi}(y,\beta,\eta)e^{-\mu y}dy = e^{-\varpi\sqrt{b^2+\mu}}e^{-\beta(\sqrt{b^2+\mu}+\eta)}\frac{(-1)^{p-1}}{(p-1)!}$$

for

$$J^{\mathrm{d0}}_{\varpi}(y,\beta,\eta) = \frac{1}{2\sqrt{\pi y^3}}(\varpi+\beta)e^{-\frac{(\varpi+\beta)^2}{4y}}e^{-b^2y-\eta\beta}\frac{(-1)^{p-1}}{(p-1)!}$$

the parameters being subjected to the constraints $\eta > -b$, $\varpi > 0$ and $\beta > 0$.

FIRST STEP: INTEGRATION

We have

$$\int_0^\infty\int_0^\infty J^{\mathrm{d0}}_{\varpi}(y,\beta,\eta)e^{-\mu y}dyd\beta = \int_0^\infty e^{-\varpi\sqrt{b^2+\mu}-\beta(\sqrt{\mu+b^2}+\eta)}\frac{(-1)^{p-1}}{(p-1)!}d\beta \tag{1.101}$$

$\int_0^\infty\int_0^\infty |J^{\mathrm{d0}}_{\varpi}(y,\beta,\eta)e^{-\mu y}|dyd\beta$ is clearly well-defined. Hence, by the Tonelli test, $J^{\mathrm{d0}}_{\varpi}(y,\beta,\eta)e^{-\mu y}$ as a function of y and β is in $L(\mathbb{R}^2)$ and by Fubini's theorem, we can interchange the order of integration.

[14]The inverse gaussian distribution has for density

$$f^{\mathrm{IG}}(y) = \sqrt{\frac{\kappa}{2\pi y^3}}e^{-\frac{\kappa(y-\omega)^2}{2\omega^2 y}} \tag{1.99}$$

and for moment generating function

$$\mathcal{L}^{\mathrm{IG}}(y) = e^{\frac{\kappa}{\omega}\left(1-\sqrt{1+\frac{2\omega^2\mu}{\kappa}}\right)} \tag{1.100}$$

with the parameters $\kappa > 0$ and μ real.

$J_{\varpi}^{\mathrm{d1}}(y,\eta) = \int_0^{\infty} J_{\varpi}^{\mathrm{d0}}(y,\beta,\eta)d\beta$ is therefore the inverse of

$$\frac{e^{-\varpi\sqrt{b^2+\mu}}(-1)^{p-1}}{(\sqrt{b^2+\mu}+\eta)(p-1)!}$$

Integration by part leads to

$$J_{\varpi}^{\mathrm{d1}}(y,\eta) = \frac{(-1)^{p-1}e^{-b^2y}}{\sqrt{\pi y}(p-1)!}\left\{e^{-\frac{\varpi^2}{4y}} - \eta\sqrt{2y}e^{\eta^2y+\varpi\eta}\int_A^{\infty} e^{-\frac{h^2}{2}}dh\right\}$$

where we have defined

$$A = \frac{\varpi + 2y\eta}{\sqrt{2y}}$$

Second Step: differentiation

Now, assuming $p \geq 2$, differentiating $(p-1)$ times (1.101) with respect to η gives

$$\frac{\partial^{p-1}}{\partial\eta^{p-1}}\bigg|_{\eta=0}\int_0^{\infty} J_{\varpi}^{\mathrm{d1}}(y,\eta)e^{-\mu y}dy = \frac{e^{-\varpi\sqrt{b^2+\mu}}}{(\sqrt{b^2+\mu})^p}$$

which is our desired Laplace transform.

We first define the following notations

$$J_{\varpi}^{\mathrm{d1,a}}(y) = \frac{(-1)^{p-1}}{\sqrt{\pi y}(p-1)!}e^{-\frac{\varpi^2}{4y}-b^2y}$$

$$J_{\varpi}^{\mathrm{d1,b}}(y,\eta) = \eta\sqrt{\frac{2}{\pi}}\frac{(-1)^p}{(p-1)!}e^{\eta^2y+\varpi\eta-b^2y}\int_A^{\infty} e^{-\frac{h^2}{2}}dh$$

$$J_{\varpi}^{\mathrm{d1,c}}(y,\eta) = e^{\frac{A^2}{2}}\int_A^{\infty} e^{-\frac{h^2}{2}}dh$$

and

$$\frac{\partial^k e^{\frac{x^2}{2}}}{\partial x^k} = \widetilde{He}_k(x)e^{\frac{x^2}{2}}$$

The $\widetilde{He}_k$ are then the polynomials defined in (1.55).

Then, differentiating $J_{\varpi}^{\mathrm{d1,c}}(y,\eta)$

$$\frac{\partial^q J_{\varpi}^{\mathrm{d1,c}}}{\partial\eta^q} = \left[\widetilde{He}_q(A)e^{\frac{A^2}{2}}\int_A^{\infty} e^{-\frac{h^2}{2}}dh - \sum_{j=0}^{q-1}\widetilde{He}_j^{(q-1-j)}(A)\right](\sqrt{2y})^q \tag{1.102}$$

where the superscript in $\widetilde{He}_j^{(q-1-j)}(A)$ represents the $(q-1-j)$ derivative of the polynomial, with the convention $\widetilde{He}_k^{(0)}(A) = \widetilde{He}_k(A)$.

Coming now to the derivative of $J_{\varpi}^{\mathrm{d1,b}}(y,\eta)$, we obtain

$$\frac{\partial^{p-1} J_{\varpi}^{\mathrm{d1,c}}}{\partial \eta^{p-1}}(y,\eta)\Big|_{\eta=0} = \sqrt{\frac{2}{\pi}}\frac{(-1)^p e^{-b^2 y}}{(p-2)!}e^{-\frac{\varpi^2}{4y}}\left\{\frac{\partial^{p-2} J_{\varpi}^{\mathrm{d1,c}}}{\partial \eta^{p-2}}\Big|_{\eta=0}(y,\eta)\right\}$$

Hence, from equation (1.102),

$$\frac{\partial^{p-1} J_{\varpi}^{\mathrm{d1,c}}}{\partial \eta^{p-1}}\Big|_{\eta=0}(y,\eta) = \left\{\widetilde{He}_{p-2}\left(\frac{\varpi}{\sqrt{2y}}\right)\int_{\frac{\varpi}{\sqrt{2y}}}^{\infty} e^{-\frac{h^2}{2}}dh\right.$$

$$\left.-\sum_{j=0}^{p-3}\widetilde{He}_j^{(p-3-j)}\left(\frac{\varpi}{\sqrt{2y}}\right)e^{-\frac{\varpi^2}{4y}}\right\}(\sqrt{2y})^{p-2}\sqrt{\frac{2}{\pi}}\frac{(-1)^p}{(p-2)!}e^{-b^2 y} \tag{1.103}$$

Since $J_{\varpi}^{\mathrm{d1}} = J_{\varpi}^{\mathrm{d1,a}} + J_{\varpi}^{\mathrm{d1,b}}$ and from (1.103), $J_{\varpi}^{\mathrm{d}} = \frac{\partial^{p-1} J_{\varpi}^{\mathrm{d1}}}{\partial \eta^{p-1}}\Big|_{\eta=0}$ is indeed uniformly bounded by an integrable function of y for η in a neighborhood of 0, the differentiation under the integral sign is justified, implying that

$$\int_0^{\infty} J_{p,\varpi}^{\mathrm{d}}(y)e^{-\mu y}dy = \frac{e^{-\varpi\sqrt{\mu+b^2}}}{(\sqrt{\mu+b^2})^p}$$

with, for $p > 0$

$$J_{\varpi}^{\mathrm{d}} = 1_{\{p=1\}}(J_{\varpi}^{\mathrm{d1,a}}) + 1_{\{p>1\}}\frac{\partial^{p-1} J_{\varpi}^{\mathrm{d1,b}}}{\partial \eta^{p-1}}\Big|_{\eta=0}$$

This result leads to (1.58) and (1.57).

The case $p = 0$ now is directly given by the inverse gaussian density multiplied by the constant $e^{\varpi b}$ (see (1.57)).

Second kind of elementary inverse

We now want to find the inverse Laplace transform of the term $(\sqrt{\mu+b^2})^q e^{-\varpi\sqrt{\mu+b^2}}$ for $q > 0$ (and $\varpi > 0$).

For this purpose, consider first J^{m0} such that

$$\int_0^{\infty} J^{\mathrm{m0}}(y,\rho)e^{-\mu y}dy = e^{-\rho(\sqrt{\mu+b^2})} \tag{1.104}$$

Once again, from the inverse gaussian distribution,

$$J^{\mathrm{m}0}(y,\rho) = \frac{1}{2\sqrt{\pi y^3}}\rho e^{-\frac{\rho^2}{4y}-b^2 y}$$

Differentiating both sides of equation (1.104) with respect to ρ at the point ϖ, we obtain

$$\frac{\partial^q}{\partial \rho^q}\bigg|_{\rho=\varpi}\int_0^\infty J^{\mathrm{m}0}(y,\rho)e^{-\mu y}dy = (-1)^q(\sqrt{\mu+b^2})^q e^{-\varpi\sqrt{\mu+b^2}}$$

The right-hand side expression being almost our desired transform, we have our inverse on condition the left-side expression is differentiable under the sum sign. But,

$$\frac{\partial^q J^{\mathrm{m}0}}{\partial \rho^q} = \frac{1}{2\sqrt{\pi y^3}}\left(\rho(-1)^q He_q\left(\frac{\rho}{\sqrt{2y}}\right)\frac{e^{-\frac{\rho^2}{4y}}}{(\sqrt{2y})^q}\right.$$
$$\left. +q(-1)^{q-1}He_{q-1}\left(\frac{\rho}{\sqrt{2y}}\right)\frac{e^{-\frac{\rho^2}{4y}}}{(\sqrt{2y})^{q-1}}\right) \tag{1.105}$$

where $He_k(x)$ refers to the k^{th} Hermite polynomial (1.54)

With (A.3), it is possible to rewrite (1.105) as (1.56). The derivative (1.105) being uniformly bounded by an integrable function of y for ρ in a neighborhood of ϖ, the differentiation under the integral sign is justified and thus the result follows.

1.5.2 Proof of Theorem 1.1.10

Our aim, here, is to build the inverse Laplace transform of $\frac{(\gamma-b)^p}{\gamma^q}e^{-\varpi\gamma}$ in a recursive manner for any strictly positive real number ϖ. This inverse will be denoted $I_{p,q}(y,\varpi)$. Since this term is defined differently for $q > 0$ and $q = 0$, we need to treat these cases separately.

<u>CASE $q > 0$</u>

From (1.100), the inverse in this case can be expressed as

$$I_{p,q}(y,\varpi) = (-1)^p \frac{\partial^p}{\partial \zeta^p}\bigg|_{\zeta=0} \left[\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta} - b^2 y\beta + b\zeta} (\varpi+\zeta+u) u^{q-1}\beta}{2\sqrt{\pi (y\beta)^3}(q-1)!} du \right] \quad (1.106)$$

since the previous Appendix 1.5.1 shows that when taking the Laplace transform of (1.107), it is licit to interchange the order of integration and differentiate under the sum sign at the point $\zeta = 0$.

From (1.106), we are naturally lead to define the simpler expression

$$\tilde{J}_{p,q}(y,\varpi) = (-1)^p \beta \frac{\partial^p}{\partial \zeta^p} \int_0^\infty e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta} - b^2 y\beta} \frac{e^{b\zeta}}{2\sqrt{\pi (y\beta)^3}} \frac{u^{q-1}}{(q-1)!} du \quad (1.107)$$

Differentiating it once under the integral sign, we obtain

$$\tilde{J}_{p+1,q}(y,\varpi) = \frac{\partial^p}{\partial \zeta^p} \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta} - b^2 y\beta + b\zeta} (\varpi+\zeta+u) u^{q-1}}{(-1)^p 4\sqrt{\pi (y\beta)^3}(q-1)! y} du - b\tilde{J}_{p,q}(y,\varpi)$$

Integrating by parts the integral on the right-hand side and setting $\zeta = 0$ leads to

$$\tilde{J}_{p+1,q}(y,\varpi) = -b\tilde{J}_{p,q}(y,\varpi) + 1_{\{q>1\}} \tilde{J}_{p,q-1}(y,\varpi)$$

$$+1_{\{q=1\}} \left\{ \frac{e^{-\frac{(\varpi-2yb\beta)^2}{4y\beta} - \varpi b}}{(\sqrt{2y\beta})^p} \frac{\beta}{2\sqrt{\pi (y\beta)^3}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}} \right) \right\} \quad (1.108)$$

For $q > 1$, (1.108) actually provides an alternative formulation for the recursive construction of the sequence $\tilde{J}_{p,q}(y,\varpi)$, which can be computed in different equivalent ways.

Plugging $q = 1$ in (1.108) results (with appropriate transformations) in

$$J_{p,1}(y,\varpi) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\varpi^2}{4y\beta} - b^2 y\beta}}{(\sqrt{2y\beta})^{p+2}} He_{p-1}\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}} \right) - bJ_{p-1,1}(y,\varpi) \quad (1.109)$$

Trying now to relate $\tilde{J}_{p,q+1}(y,\varpi)$ with the preceding terms, we replace u^q by $(2y\beta)\frac{(\varpi+\zeta+u)}{2y\beta} u^{q-1} - (\varpi+\zeta) u^{q-1}$ in the integro-differential form (1.107).

Integrating then by part the first term coming from this replacement gives

$$\tilde{J}_{p,q+1}(y,\varpi) = 1_{\{q=1\}} 2y\beta \left\{ \frac{e^{-\frac{(\varpi-2yb\beta)^2}{4y\beta} - \varpi b}\beta\sqrt{2}}{\sqrt{\pi(2y\beta)^{p+3}}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right)\right\}$$

$$+1_{\{q>1\}} \frac{2y\beta}{q} \tilde{J}_{p,q-1}(y,\varpi) - \frac{(\varpi+\zeta)}{q} \tilde{J}_{p,q}(y,\varpi) + 1_{\{p>0\}} \frac{p}{q} \tilde{J}_{p-1,q}(y,\varpi) \tag{1.110}$$

The initial condition

$$J_{0,1}(y,\varpi) = \frac{e^{-b^2 y\beta}}{2y} \operatorname{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{1.111}$$

can straightforwardly be worked out from the corresponding equation (1.58) defined in the complete formulation of the marginal density of Y_t.

Coming back to the original $I_{p,q}(y,\varpi)$, we still need to relate it to the $\tilde{J}_{p,q+1}(y,\varpi)$ to complete the inversion for $q > 0$

$$I_{p,q}(y,\varpi) = \int_0^\infty (\varpi + \zeta + u)(-1)^p \frac{\partial^p}{\partial \zeta^p} \left[\frac{\beta e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta} - b^2 y\beta + b\zeta} u^{q-1}}{2\sqrt{\pi (y\beta)^3}(q-1)!} \right] du$$

$$-1_{\{p>0\}} \left\{ p(-1)^{p-1} \int_0^\infty \frac{\partial^{p-1}}{\partial \zeta^{p-1}} \left[\frac{\beta e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta} - b^2 y\beta + b\zeta} u^{q-1}}{2\sqrt{\pi (y\beta)^3}(q-1)!} \right] du \right\}$$

which can be reexpressed as

$$I_{p,q}(y,\varpi) = \varpi J_{p,q}(y,\varpi) + q J_{p,q+1}(y,\varpi) - 1_{\{p>0\}} \{ p J_{p-1,q}(y,\varpi) \} \tag{1.112}$$

<u>CASE $q = 0$</u>

Now, for this case, the integro-differential formula (1.106) is not valid anymore and is to be replaced by the differential formula

$$I_{p,0}(y) = (-1)^p \frac{\partial^p}{\partial \zeta^p} \left[e^{-\frac{(\varpi+\zeta)^2}{4y\beta} - b^2 y\beta + b\zeta} \frac{\beta(\varpi+\zeta)}{2\sqrt{\pi (y\beta)^3}} \right] \tag{1.113}$$

since, again from Appendix 1.5.1, it is justified to interchange differentiation and integration when taking the Laplace transform of the right-hand side of (1.113).

It follows from (A.1) that

$$I_{p,0}(y,\varpi) = \left\{ \frac{\varpi}{(2y\beta)^p} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}} \right) \right.$$

$$\left. - \frac{p}{(2y\beta)^{p-1}} He_{p-1}\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}} \right) \right\} \beta \frac{e^{-\frac{\varpi^2}{4y\beta} - b^2 y\beta}}{2\sqrt{\pi (y\beta)^3}}$$

expression from which it is possible to derive (1.63) and

$$I_{0,0}(y,\varpi) = \frac{\varpi\beta}{2\sqrt{\pi (y\beta)^3}} e^{-\frac{\varpi^2}{4y\beta} - b^2 y\beta} \tag{1.114}$$

Combining together these different ingredients enables us to express the density in the form given in Theorem 1.1.10.

Chapter 2

Asian options on stocks

Asian options, whose payoff is based on the average of the underlying equity over some period of time, have known a very large success over the past years, specially in their most popular form, the arithmetic Asian option and have been used in many different markets[1]: equity, interest rate, commodity, energy, etc. This success can be explained by several economic reasons. Firstly, Asian options provide a better hedge for firms having a stream of positions over time and thus constitute a natural hedging toolbox for managing the financial exposure (typically foreign exchange or commodity exposure) over an accounting period for corporate treasury. The Asian option will indeed be cheaper than the corresponding portfolio of plain vanilla options although, as noted in Geman and Eydeland [35], a single Asian option is not always cheaper than the standard European option with the same characteristics. Secondly, these derivatives reduce the exposition to near-the-maturity market manipulation. Indeed, for thinly traded underlyings, large market participants may have a weight

[1]Though the title emphasizes the fact the processes studied in this chapter are mainly used for equity modelling, the results concerning Asian options are still valid for other markets and types of underlying on condition that the lognormal or the square-root processes provide a good representation for these underlyings.

sufficient to impact the price movements in their favor when the option maturity comes close. This kind of market manipulation have a much smaller effect on Asian-type derivatives.

Asian options are among the most difficult to price path-dependent options, as mentioned in Geman and Eydeland [35]. This fact, along with the practical importance of this financial instrument, accounts for the amount of research dedicated to this specific pricing problem.

In this work, we will only consider European-type - i.e. exercisable at maturity only - Asian options on continuous arithmetic averages. The first part of this chapter synthetises the main results that have been derived and the main techniques that have been applied to this problem in the literature. In the second part are presented the analytic pricing formulae for fixed and floating strike seasoned and unseasoned options that we obtain by inverting the Laplace transform with respect to time of these options, in the classical Black-Scholes lognormal equity model. In the last part, we derive slightly simpler analytic formulae for an alternative equity model: the square-root CEV process of Cox and Ross [16], extending first the results of the previous chapter to the non mean-reverting case and then applying them to this option pricing.

2.1 Literature review of Asian options

2.1.1 The different types of Asian options

Different types of Asian options have been studied in the literature, although all of them are not actually traded. Typically, the continuous-time averaging option, the most studied Asian option, is an "ideal" option, since real continuous reporting is not possible. It yet remains of great importance to develop intuition on discrete options and approximate well the frequently marked ones. We will expose here these different varieties of Asian options.

To present a fairly general definition, Asian options are derivatives whose payoff is a function of the values their underlying asset assumes over a part or the whole of their life-time. Rogers and Shi [63] define more precisely this payoff $f(I_T)$ as a call $(f(I_T) = (I_T - K)^+)$ or put $(f(I_T) = (K - I_T)^+)$ on the variable I_T defined as

$$I_T = \int_0^T S_u \mu(du) \tag{2.1}$$

where μ is a measure on $[0, T]$, T being the maturity of the option and S_u the underlying asset, which has mainly be modelled so far as a geometric Brownian motion[2], i.e. $dS_u = rdu + \sigma dW_u$.

More commonly, four main types of averaging are considered:

- Discrete arithmetic average

$$I_T = \frac{1}{N+1} \sum_{n=0}^{N} S_{T_n}$$

- Discrete geometric average

$$I_T = \left(\prod_{n=0}^{N} S_{T_n} \right)^{\frac{1}{N+1}}$$

- Continuous arithmetic average

$$I_T = \frac{1}{T - T_0} \int_{T_0}^{T} S_u du$$

- Continuous geometric average

$$I_T = \exp\left\{ \frac{1}{T - T_0} \int_{T_0}^{T} \ln(S_u) du \right\}$$

[2]With the notable exception of average options on interest rates, where square-root processes and affine models have been considered. But, since this chapter focuses on the equity case, they are not included in this review.

where $\{T_i,\ i = 0..N\}$ is a set of different ordered dates such that $T_N = T$ is the maturity and T_0 the averaging starting time.

According to their payoff, those derivatives[3] can moreover be

- Fixed-strike options

$$\text{payoff} = (I_T - K)^+$$

- Floating-strike or average-strike options

$$\text{payoff} = (S_T - I_T)^+$$

Finally, another differentiation lays the emphasis on the role of the pricing date. Indeed, these options being path-dependent, the date t at which the option is evaluated can change the nature and complexity of the pricing problem. We hence classify these options as

- Forward-starting: $T_0 > t$

The averaging has not yet started, which basically means that the price is the expectation of the starting option over the distribution of the asset at the starting-date.

$$E(f(I_T)|S_t) = E\Big(E(f(I_T)|S_{T_0})|S_t\Big) \tag{2.2}$$

This option will not be much considered here, since the main difficulty in this valuation problem is the actual starting option pricing.

- Starting: $T_0 = t$

The option is priced at the very moment the averaging is activated.

- Backward-started: $T_0 < t$

The option is priced while the averaging has already started and some information is therefore already available. This can either simplify a lot the pricing or complicate

[3]The payoffs given here are those of call options, the corresponding put options payoff can be easily inferred.

it or not really matter. The case a simplification occurs is the fixed-strike option which turns to already be in the money: this option will almost surely be exercised at maturity and its price is simply $\frac{E(I_T - I_t|S_t)+I_t}{T} - K$, expectation known in a simple closed-form thanks to Fubini theorem. In the case we are dealing with an out-of-the-money fixed-strike option, the price is simply $\frac{T-t}{T}$ times the starting option price with maturity $T-t$ and strike $(K - \frac{I_t}{T})\frac{T}{T-t}$. As for the backward-started floating-strike option, it cannot be brought back to the same form as a floating-strike starting option; the structure of the problem is thus more complex for a backward-started option. As a final remark, it should be noticed that these backward-started derivatives are also called seasoned as opposed the unseasoned starting options.

2.1.2 The different approaches carried out

Geometric averages both discrete and continuous have simple dynamics under the assumption of lognormality of the asset, which constitutes the main reason why these options have been studied, along with the fact that they are somehow - but in a complex way - linked with the arithmetic option: they provide bounds (arithmetic mean always dominating the corresponding geometric average) and also prove to be useful as control-variate in Monte-Carlo pricing of arithmetic options since they are highly correlated with the arithmetic average.

However, the valuation of arithmetic average Asian options turns out to be far more complex and has been triggering the interest of financial mathematicians for over a decade. Research has been evolving towards an intricate interplay between theoretical and computational approaches. The following review, though not exhaustive, presents the main approaches and attempts proposed in the literature.

- Analytical approaches and Laplace transform analysis

As early as 1992, Yor [75] formulated the price as a triple integral, which, though theoretically of great value, remains difficult to compute numerically (see Schröder [65] for some numerical illustrations). He also initiated with Geman [76] the Laplace transform approach, expressing the Laplace transform with respect to time of the option as a product of special functions, exploiting the connection with Bessel processes. Most of the work along this line has afterwards consisted in numerically inverting this transform. Some of the main contributions on this topic are: Geman and Eydeland [35] who proceeded through quasi-fast fourier methods, Craddock, Heath and Platen [17] who compared the efficiency of different numerical Laplace inversion techniques and Fu and al. [32] who compared Monte-Carlo and Abate-Whitt methods. However, the recent work of Schröder [65] provides an innovative analytical inversion of this Laplace transform.

- Pseudo-analytical approximations

This approach mainly consists in finding quick and simple approximations. The principal technique used to this effect is to approximate the distribution of I_T with a known distribution sharing a number of higher moments, typically at least the first two[4]. One of the first such attempts is Levy's [48] who used a lognormal distribution for I_T and a bivariate normal distribution for $(\ln(I_T), \ln(S_T))$, which is computationally very simple but provides a poor approximation, given that a sum of lognormals is far from being a lognormal. Milevsky and Posner [57] provide an approximation for long-dated options, using the limit distribution of I_T, which is a reciprocal gamma distribution whenever the interest rate is less than half the square-volatility. They later used Johnson distribution of type I and II in [61], since those allow for the fitting

[4]This notably allows the conservation of the important call-put parity relation.

of more higher moments. Turnbull and Wakeman [71] chose a different way to include more moments by opting for an Edgeworth expansion around a lognormal.

Those approximations based on the moments of I_T share the common pitfall that the distribution of I_T might not be determined by its moments, determinacy problem which has remained unresolved. Other approaches less simple but more theoretically justified are using variables linked to I_T which, unlike I_T, are proven to be determined by their moments: Dufresne [26] proposed a Laguerre expansion using the reciprocal $\frac{1}{I_T}$ while Fusai and Tagliani [34] used different methods (normal approximation, Edgeworth series around a normal, generalized beta of second kind approximation which allows for a greater flexibility since this distribution depends on more characteristic parameters, etc.) to approximate the law of $\ln(I_T)$. The main difficulty in these two approaches is the actual computation of the moments, requiring to solve a numerical differential-difference equation[5] for $\frac{1}{I_T}$ or the inversion of a Laplace transform for $\ln(I_T)$.

Other simple approximations not based on moments are the linearisation argument used by Bouaziz, Briys and Crouhy [7] who approximated $\int_0^T e^{\nu s+\sigma W_s} ds$ to the first order by the normal variable $\frac{e^{\nu T}}{\nu} + \sigma \int_0^T e^{\nu s} W_s ds$ for small volatilities and the bounds and approximations given by Curran [18], Rogers and Shi [63], deduced by conditioning on the variable $\int_0^T W_u du$. These last bounds have shown to be surprisingly tight and have even be further tighten by the work of Thompson [68].

[5]Work has been done on a easier computation of the reciprocal moments, but they still need involved numerical procedure.

• <u>Numerical PDE</u>

The first attempt to evaluate the fixed-strike Asian call option through the numerical resolution of a partial differential equation was the three-dimensional initial-boundary value problem given by (See Ingersoll [43]):

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{S}{T-T_0}\frac{\partial C}{\partial I} - rC + \frac{\partial C}{\partial t} = 0 \tag{2.3}$$

$$C(T, S_T, I_T) = (I_T - K)^+ \tag{2.4}$$

$$C(t, S_t, K) = \frac{S_t}{r(T-T_0)}(1 - e^{-r(T-t)}) \tag{2.5}$$

$$C(t, 0, I_t) = e^{-r(T-t)}(I_t - K)^+ \tag{2.6}$$

Similar equations can be deduced for puts and for floating-strike options.

Ingersoll [43] first noticed that the floating-strike option valuation can be reduced to a two-dimensional problem by using homogeneity properties. Rogers and Shi [63] and Alziary and al. [1] extended this result to the fixed-strike option using the variable $Z_t = \frac{K-I_t}{S_t}$ (see also section 2.2.1):

$$\frac{1}{2}\sigma^2 Z^2 \frac{\partial^2 \tilde{C}}{\partial Z^2} - \left(rZ + \frac{1}{T-T_0}\right)\frac{\partial \tilde{C}}{\partial Z} + \frac{\partial \tilde{C}}{\partial t} = 0 \tag{2.7}$$

$$\tilde{C}(Z,T) = Z^- \tag{2.8}$$

$$\tilde{C}(0,t) = \frac{1}{r(T-T_0)}(1 - e^{-r(T-t)}) \tag{2.9}$$

$$\lim_{Z\to\infty} \tilde{C} = 0 \tag{2.10}$$

A large part of the research has then focused on how to improve the numerical handling of this PDE: Zvan [79], Vecer [73] and how the PDE can be extended for slightly different assumption: Andreasen [2]. These reference are far from being exhaustive.

- Monte Carlo simulation

Monte-Carlo pricing remains in general the most flexible (allowing for example more complex modelling of the underlying asset, including dividends, etc.) but also usually the most time-consuming numerical method. In the case of Asian options, this approach has been considered by Kemna and Vorst [45] and a good number of other authors. Notably, Fu and al. [32] explained the inefficiency of naive simulations and the great care that should be taken when using control variates.

- Tree and lattices

Trees are well-known not to be a suitable method for pricing Asian options, since the latters depend on the whole path of the asset during the averaging time and require to keep track of all the averages possible at a node, the cardinal of these possibilities growing explosively. A number of works have though considered this numerical approach: Hull and White [42], Neave and Turnbull [58] and sophisticated methods have been developed to try and circumvent the problem, the main advantage of tree and lattice methods being the extension to American-type of Asian options, which can be exercised at any time in a predefined discrete or continuous window of dates including the maturity of the option.

- Miscellaneous numerical methods

Other miscellaneous methods that cannot be classified in the previous categories have been explored, as for example: Caverhill and Clewlow [11] who use Fourier transforms for discrete options.

2.2 The classical lognormal case

In this section, we assume that the asset X_t follows a lognormal diffusion

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{2.11}$$

under the risk-neutral measure Q. All the options studied in this section depend on the random temporal integral $Y_T = \int_0^T S_t dt$.

The recent developments in Asian option pricing are mainly based on the fact that, in the lognormal case, the SDE followed by the integral can be fully specified independently - in a loose sense - of the one followed by S_t. Hence, the dimensionality of the problem can be reduced, thanks to the properties of the following linear SDE. We here present the different forms of reduction possible, showing how they relate, how they have been used so far and how they can be used, synthetic point of view yet lacking in the literature - to the best of our knowledge.

2.2.1 A linear SDE of interest

Theorem 2.2.1. *The following linear SDE*

$$dX_t = (\alpha(t) - \beta(t)X_t)dt + \sigma(t)X_t dW_t \tag{2.12}$$

has the unique strong solution

$$X_t = L_t\left(X_0 + \int_0^t \frac{\alpha(s)}{L_s} ds\right) \tag{2.13}$$

where

$$L_t = e^{\int_0^t (\beta(s) - \frac{\sigma^2(s)}{2})ds + \int_0^t \sigma(s)dW_s} \tag{2.14}$$

Proof. The existence and uniqueness of the strong solution come from the fact that the coefficients satisfy the Lipschitz condition. Now, looking at solutions of the form

$$X_t = L_t Z_t$$

natural candidates are

$$dL_t = \beta(t)L_t dt + \sigma(t)L_t dW_t$$

and

$$dZ_t = \frac{\alpha(t)}{L_t}dt$$

Applying Ito lemma to this guess with $Y_0 = X_0$ and $Z_0 = 1$ shows that it is the strong solution. $\square$

Corollary 2.2.1. *The SDE*

$$dX_t = (\alpha - \beta X_t)dt + \sigma X_t dW_t \tag{2.15}$$

has the unique strong solution

$$X_t = e^{-(\beta+\frac{\sigma^2}{2})t+\sigma W_t}\left(X_0 + \alpha\int_0^t e^{(\beta+\frac{\sigma^2}{2})s-\sigma W_s}ds\right) \tag{2.16}$$

Proof. This straightforwardly follows from the previous result. $\square$

This result allows us to express both the floating and fixed-strike Asian pricing issue in function of a one-dimensional process following a SDE of the type (2.15). In fact, different reformulations of the problem are possible.

The standard reformulation

For the floating-strike Asian option, this formulation first appeared in Ingersoll [43]. For the fixed-strike option, this appeared in Rogers and Shi [63] and Alziary and al. [1].

They are all based on the use of the following equivalent probability measure

$$\frac{dQ^S}{dQ} = \frac{S_T}{E^Q(S_T)} \tag{2.17}$$

The fixed strike option price can then be reexpressed as

$$e^{-rT}E^Q\left(\frac{Y_T}{T} - K\right)^+ = S_0 E^{Q^S}\left(\frac{K}{S_T} - \frac{Y_T}{TS_T}\right)^- \tag{2.18}$$

The real underlying process $X_t = \frac{K}{S_t} - \frac{Y_t}{TS_t}$ with the starting value $X_0 = \frac{K}{S0}$ satisfies the SDE

$$dX_t = \left(-\frac{1}{T} - (r - \sigma^2)X_t \right)dt - \sigma X_t dW_t^Q = \left(-\frac{1}{T} - rX_t \right)dt - \sigma X_t dW_t^{Q^S} \quad (2.19)$$

W_t^Q (resp. $W_t^{Q^S}$) being a Brownian motion under Q (resp. Q^S).

The floating strike price takes the similar corresponding form

$$e^{-rT}E^Q\left(S_T - \frac{Y_T}{T}\right)^- = S_0 E^{Q^S}\left(1 - \frac{Y_T}{TS_T}\right)^- \quad (2.20)$$

with $X_t = -\frac{Y_t}{TS_t}$ following the same SDE as (2.19) but the starting value $X_0 = 0$.

Another reformulation

We present here a second type of reformulation, which will be useful in the next part of this section.

Theorem 2.2.2. *If the process X_t follows the SDE*

$$dX_t = (\alpha S_0 + rX_t)dt + \sigma X_t dW_t^Q \quad (2.21)$$

with the initial value $X_0 = x_0 S_0$, then X_T has the same distribution as $x_0 S_T + \alpha Y_T$ with S_t a lognormal process of parameters (r, σ) starting at S_0.

Proof. As demonstrated above, the SDE has the unique strong solution

$$X_t = x_0 S_0 e^{(r-\frac{\sigma^2}{2})t - \sigma W_t^Q} + \alpha S_0 \int_0^t e^{(\frac{\sigma^2}{2}-r)(s-t)-\sigma(W_t^Q - W_s^Q)} ds$$

$$= x_0 S_0 e^{(r-\frac{\sigma^2}{2})t - \sigma W_t^Q} + \alpha S_0 \int_0^t e^{(r-\frac{\sigma^2}{2})u - \sigma(W_t^Q - W_{t-u}^Q)} du$$

The rightmost part of this equality follows from the change of variable $u = t - s$. But, $B(t) = W_t^Q - W_{t-u}^Q$ is a Brownian motion under Q since its paths are continuous and the increments $B(u_2) - B(u_1)$ are normal of mean 0 and variance $u_2 - u_1$ and independent of the past $(B(u), u \leq u_1)$. Hence, by time-reversal,

$$X_t \cong x_0 S_0 e^{(r-\frac{\sigma^2}{2})t-\sigma W_t^Q} + \alpha S_0 \int_0^t e^{(r-\frac{\sigma^2}{2})u-\sigma W_u^Q} du$$

□

A special case of this result can be found in Dufresne [25]. This result will be used in the following part to produce a simple derivation of the Laplace transform with respect to maturity of the Asian option and to propose another form for the transform.

2.2.2 Hitting time MGF

The hitting time of X_t following the linear SDE (2.15) is a random quantity of great value in the Asian option problem. Indeed, since the integral Y_t is an increasing process, the hitting time being less than maturity or not determines whether the option ends in the money or not.

Theorem 2.2.3. *Considering a process X_t defined as the strong solution of the SDE (2.15) with $X_0 = x_0 > 0$ and $\alpha < 0$, the hitting time of $a \neq 0$ defined as $\tau_a = inf\{t > 0 : X_t = a\}$ has for moment generating function*

$$E(e^{-\lambda\tau_a}) = \frac{\frac{1}{x_0^{\mu^+}}\phi(\mu^+, 2\mu^+ + \frac{2\beta}{\sigma^2} + 2, \frac{2\alpha}{\sigma^2 x_0})}{\frac{1}{a^{\mu^+}}\phi(\mu^+, 2\mu^+ + \frac{2\beta}{\sigma^2} + 2, \frac{2\alpha}{\sigma^2 a})} \tag{2.22}$$

with

$$\mu^+ = \frac{-(\frac{\sigma^2}{2} + \beta) + \sqrt{(\frac{\sigma^2}{2} + \beta)^2 + 2\sigma^2\lambda}}{\sigma^2} \tag{2.23}$$

where the complex square-root function is here defined as the branch with positive real part and ϕ represents the Kummer confluent hypergeometric function of the first kind.

Proof. Applying the Itô lemma to $e^{-\lambda t} f(X_t)$ prompts us to search the smooth functions f satisfying the PDE

$$\frac{\sigma^2}{2} x^2 f''(x) + (\alpha - \beta x) f'(x) = \lambda f(x) \tag{2.24}$$

We attempt a Frobenius or analytic coefficient approach and assume that the solution can be written under the form $f(x) = \sum_0^\infty a_n x^{\mu+n}$. Transposing this into the equation (2.24) gives a simple form for the coefficients but leads to a diverging series. The form of the coefficients suggests that using the variable $z = 1/x$ would preserve a simple form for the coefficients and at the same time lead to a convergent series. So, letting $g(z) = f\left(\frac{1}{z}\right)$, the equation (2.24) is transformed into

$$\frac{\sigma^2}{2} z^2 g''(z) + z(\sigma^2 + \beta - \alpha z) g'(z) - \lambda g(z) = 0 \tag{2.25}$$

Now, assuming $g(z) = \sum_0^\infty a_n z^{\mu+n}$, (2.25) becomes

$$\sum_{n=0}^{\infty} z^{\mu+n} a_n \left((n+\mu)(n+\mu-1)\frac{\sigma^2}{2} + (\sigma^2+\beta)(n+\mu) - \lambda\right) = \sum_{n=1}^{\infty} z^{\mu+n} a_{n-1}(n+\mu-1)\alpha$$

μ is determined by the lower order coefficient[6]

$$\frac{\sigma^2}{2}\mu(\mu-1) + (\sigma^2+\beta)\mu = \lambda$$

Hence, μ can take the two possible values μ^+ and μ^-

$$\mu^+_- = \frac{-\left(\frac{\sigma^2}{2}+\beta\right) \pm \sqrt{\left(\frac{\sigma^2}{2}+\beta\right)^2 + 2\sigma^2\lambda}}{\sigma^2}$$

And the coefficients follow the recursion

$$a_n = \frac{\alpha(n+\mu-1)}{\frac{\sigma^2}{2} n \left(n + 2\mu + 1 + \frac{2\beta}{\sigma^2}\right)} a_{n-1} = \left(\frac{2\alpha}{\sigma^2}\right)^n \frac{(\mu)_n}{n!\left(2\mu + \frac{2\beta}{\sigma^2} + 2\right)_n} a_0$$

[6]This PDE has been solved in the literature in some other contexts by transforming it to a classical known PDE (See Lewis [49], for example). Yet, the proof given here starts from the roots of the equation and naturally explains the intuition behind the change of variable made in Lewis [49].

where

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

denotes the Pochhammer symbol. Hence, two independent solutions of (2.24) are

$$\varphi_-^+ = \frac{1}{x^{\mu_-^+}}\phi\left(\mu_-^+, 2\mu_-^+ + \frac{2\beta}{\sigma^2} + 2, \frac{2\alpha}{\sigma^2 x}\right) \tag{2.26}$$

For $\alpha < 0$, φ^+, the solution with μ^+, is the only one bounded in $(0, \infty)$. Its two first derivatives are bounded as well on $\mathbb{R}^+$. After prolonging this function to $\mathbb{R}^-$ so that the prolonged function $\tilde{\varphi}^+$ is $\mathbf{C}^2$ on $\mathbb{R}$, the Itô lemma can be applied. Then, after stopping the relation obtained at τ_a,

$$e^{-\lambda\tau_a}\tilde{\varphi}^+(X_{\tau_a}) - \tilde{\varphi}^+(X_0) = \int_0^{\tau_a} \frac{d\tilde{\varphi}^+}{dx}(X_s)dW_s$$

Therefore,

$$E(e^{-\lambda\tau_a}) = \frac{\tilde{\varphi}^+(x_0)}{\tilde{\varphi}^+(a)} = \frac{\varphi^+(x_0)}{\varphi^+(a)}$$

□

Corollary 2.2.2. *The hitting time of the origin, τ_0, possesses the MGF*

$$E(e^{-\lambda\tau_0}) = \frac{1}{x_0^{\mu^+}}\phi\left(\mu^+, 2\mu^+ + \frac{2\beta}{\sigma^2} + 2, \frac{2\alpha}{\sigma^2 x_0}\right)\frac{\Gamma(\mu^+ + \frac{2\beta}{\sigma^2} + 2)}{\Gamma(2\mu^+ + \frac{2\beta}{\sigma^2} + 2)}\left(-\frac{2\alpha}{\sigma^2}\right)^{\mu^+} \tag{2.27}$$

Proof. By dominated convergence, the result is the limit of the expression (2.22) when $a \to 0^+$. The asymptotic behavior of the confluent hypergeometric function of the first kind (See Erdelyi [28]) when $x \to -\infty$,

$$\phi(a, c, x) = \frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}[1 + \mathcal{O}(|x|^{-1})] \tag{2.28}$$

implies that

$$\varphi(a) \to \frac{\Gamma(2\mu^+ + \frac{2\beta}{\sigma^2} + 2)}{\Gamma(\mu^+ + \frac{2\beta}{\sigma^2} + 2)}\left(-\frac{\sigma^2}{2\alpha}\right)^{\mu^+}$$

□

This last result enables a simple alternative derivation of the Laplace transform of Asian options given in Geman and Yor [77].

2.2.3 An alternative derivation of the Laplace transform of fixed-strike Asian options

The following simple proof does not directly appeal to the techniques like conditioning on exponential random time or Green function methods employed by Yor in different papers ([77], [22], etc.) to deduce the Laplace transform. Placing ourselves in the framework defined in 1.2, Chapter 1 and using the same notations, consider the Laplace transform

$$LC^{\nu}(\lambda, q) = \int_0^{\infty} e^{-\lambda h} C^{(\nu)}(h, q) dh \tag{2.29}$$

of the normalised fixed-strike Asian option

$$C^{(\nu)}(h, q) = E\left(\int_0^h e^{2(\nu u + W_u)} du - q \right)^+ \tag{2.30}$$

and X_t the process following

$$dX_t = (-1 - (2\nu - 2)X_t)dt - 2X_t dW_t \tag{2.31}$$

with $X_0 = \frac{\sigma^2 KT}{4S_0} = q$.

From the results of section 2.2.1,

$$C^{(\nu)}(h, q) = E^{Q^S}(X_h)^- \tag{2.32}$$

under the S_t-numeraire measure.

Yet, the calculation of the Laplace transform (2.29) can be conducted under the risk-neutral measure itself while still using the one-factor process X_t. Indeed, a quite generic relation between a vanilla option and the derivative of the corresponding digital option with respect to the strike gives (see for example Alziary and al. [1]):

$$C^{(\nu)}(h,q) - C^{(\nu)}(h,q_0) = \int_0^{q_0} Q(X_h < 0, X_0 = k)dk \tag{2.33}$$

The properties of the process X_t are such that once it crosses 0 (starting from a strictly positive value), it cannot become positive again. This is easily seen by writing the strong solution and implies that

$$Q(X_h < 0|X_0 = k) = Q(\tau_0 < h|X_0 = k) \tag{2.34}$$

Moreover, the Laplace transform of a distribution function can be deduced from the moment generating function

$$\int_0^{\infty} e^{-\lambda h} Q(\tau_0 < h|X_0 = k)dh = \frac{E(e^{-\lambda\tau_0}|X_0 = k)}{\lambda}$$

Taking $q_0 \to \infty$[7] in (2.33), the celebrated classical formula (1.75) can be recovered from

$$LC^{\nu}(\lambda,q) = \int_q^{\infty} \frac{E(e^{-\lambda\tau_0}|X_0 = k)}{\lambda} dk$$

$$= \frac{\Gamma(\mu^+ + \nu + 1)}{\Gamma(2\mu^+ + \nu + 1)} \left(\frac{1}{2q}\right)^{\mu^+} \sum_{n=0}^{\infty} \frac{(\mu^+)_n \left(-\frac{1}{2q}\right)^{n-1}}{(2\mu^+ + \nu + 1)_n n! (\mu^+ + n - 1)} \tag{2.35}$$

given that, here, with this parametrisation,

$$\mu^+ = \frac{-\nu + \sqrt{\nu^2 + 2\lambda}}{2} \tag{2.36}$$

In the next section, this Laplace transform will be analytically inverted using complex analysis and contour integration.

[7] In the limit $q_0 \to \infty$, the call value vanishes.

2.2.4 Analytical inversion of the Asian Laplace transform: Bromwich integral and residue calculus.

We keep using the notations of Section 1.2, Chapter 1.

Theorem 2.2.4. *The normalized fixed-strike Asian call price (2.30) is given by*

$$
\begin{aligned}
C^{(\nu)}(h,q) = \frac{e^{-\frac{\nu^2}{2}h}}{2}\Bigg\{ & \int_0^\infty \frac{(\Xi(\lambda,q))e^{-\frac{\lambda^2 h}{2}-\tilde{q}}}{\pi\tilde{q}^{\frac{\nu}{2}+1}}d\lambda - \frac{L_1^\nu(-\tilde{q})e^{\frac{\nu^2 h}{2}}}{(\nu+1)\tilde{q}} + \frac{e^{\frac{(\nu+2)^2 h}{2}}}{\nu+1} \\
& -1_{\{\nu<0\}}\left[\frac{e^{\frac{\nu^2 h}{2}}}{(\nu+1)}\left(\frac{\phi(-\nu-1,1-\nu,-\tilde{q})}{\Gamma(1-\nu)\tilde{q}^{\nu+1}} - \frac{L_1^\nu(-\tilde{q})}{\tilde{q}}\right)\right] \\
& -1_{\{\nu<-2\}}\left[\left(\frac{\phi(-\nu-2,-1-\nu,-\tilde{q})}{\Gamma(-\nu)\tilde{q}^{\nu+2}} + \frac{1}{\nu+1}\right)e^{\frac{(\nu+2)^2 h}{2}}\right] \\
& +1_{\{\nu\le -4\}}\left[\sum_{n=0}^{N} \frac{16 s_n(-1)^n L_n^{s_n}(\tilde{q})e^{\frac{\lambda_n h}{2}-\tilde{q}}}{\binom{-n-\nu-4}{n} n!\Gamma(s_n+1)(\lambda_n-\nu^2)(\lambda_n-(\nu+2)^2)\tilde{q}^{n+\nu+3}}\right]\Bigg\}
\end{aligned}
\tag{2.37}
$$

where we denote[8] $N=\left[\frac{-\nu-4}{2}\right]$, $\tilde{q}=\frac{1}{2q}$, $\lambda_n=(2n+\nu+4)^2$, $s_n=-2n-\nu-4$ *and*[9]

$$
\Xi(\lambda,q) = \left|\frac{\Gamma(\frac{i\lambda+\nu}{2}+2)}{\Gamma(i\lambda)}\right|^2 \frac{4\tilde{q}^{\frac{i\lambda}{2}}\psi\left(\frac{\nu+i\lambda}{2}+2,1+i\lambda,\tilde{q}\right)}{(\lambda^2+\nu^2)(\lambda^2+(\nu+2)^2)}
$$

$\phi(a,b,\cdot)$ and $\psi(a,b,\cdot)$ represent the Kummer functions of the first and second kind respectively, $L_n^\nu(\cdot)$ the Laguerre polynomial of order n and index ν and $\Gamma(\cdot)$ the Gamma function, all of which are the special functions defined in Appendix A.

Proof. We start by using the shifting and scaling properties of the Laplace transform and define $\tilde{C}^{(\nu)}(h,q)$ as

$$
C^{(\nu)}(h,q) = \tilde{C}^{(\nu)}\left(\frac{h}{2},q\right)\frac{e^{-\frac{\nu^2}{2}h}}{2}
$$

[8] *$[x]$ represents the integer part of x.*

[9] *Although the presence of complex numbers in the definition of $\Xi(\lambda,q)$ casts a doubt as to the nature of $C^{(\nu)}(h,q)$, it actually turns out that $(2q)^{-\frac{i\lambda}{2}}\psi\left(\frac{\nu+i\lambda}{2}+2,1+i\lambda,\frac{1}{2q}\right)$ is reassuringly real valued, which comes from the properties of the Kummer function.*

Therefore, $\tilde{C}^{(\nu)}(h,q)$ has a Laplace transform with respect to h equal to

$$\tilde{LC}^{\nu}(\lambda) = \frac{4(2q)^{1-\frac{\sqrt{\lambda}-\nu}{2}}}{(\lambda-\nu^2)(\lambda-(\nu+2)^2)}\phi\left(\frac{\sqrt{\lambda}-\nu}{2}-1, \sqrt{\lambda}+1, -\frac{1}{2q}\right)\frac{\Gamma(\frac{\sqrt{\lambda}+\nu}{2}+2)}{\Gamma(\sqrt{\lambda}+1)}$$

The usual Laplace inversion integral formula gives

$$\tilde{C}^{(\nu)}(h,q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda h}\tilde{LC}^{\nu}(\lambda, q)d\lambda \tag{2.38}$$

where the real number c can be set anywhere at the right of all the singularities of $\tilde{LC}^{\nu}$ (i.e. any singularity should have a real part inferior to c).

Studying those singularities, it appears that, given that the function $\phi^* = \frac{\phi(\alpha,\gamma,x)}{\Gamma(c)}$ is analytic[10] in both its variable x and its parameters α and γ (see Erdelyi [28]), the only poles of $\tilde{LC}^{\nu}$ are those of $\frac{4}{(\lambda-\nu^2)(\lambda-(\nu+2)^2)}$ and those of $\Gamma(\frac{\sqrt{\lambda}+\nu}{2}+2)$, the Gamma function being well-known (see Gradshteyn and Ryzhik [37] for example) to have only simple poles at the negatives integers $-n$ with the corresponding residues $\frac{(-1)^n}{n!}$. Additionally, the presence of $\sqrt{\lambda}$ makes $\tilde{LC}^{\nu}$ multivalued in the complex plane. But, recalling Theorem 2.2.3, the square-root is here defined as the branch with positive real part. This implies that we need to put a branch cut along the real negative axis to avoid crossing this half axis, which is the only part of the λ-plane where the function is not single-valued.

This analysis prompts us to choose the complex contour $\mathcal{C}$ drawn in Figure 2.1 to calculate the integral. The residue theorem implies that the integral along $\mathcal{C}$ is equal to the sum of the singularities of $e^{\lambda h}\tilde{LC}^{\nu}(\lambda, q)$ lying inside the surface delimited by

[10] In the general case, this follows from the fact that

$$\lim_{\gamma\to -n}\frac{\phi(\alpha,\gamma,z)}{\Gamma(\gamma)} = z^{n+1}\binom{\alpha+n}{n+1}\phi(\alpha+n+1, n+2, z) \tag{2.39}$$

and that the Gamma function has no zero. But, here, the negative values of the parameter $\gamma = 1+\sqrt{\lambda}$ in the above formula would anyway not be attainable given that $\sqrt{\lambda}$ is chosen to be of positive real part.

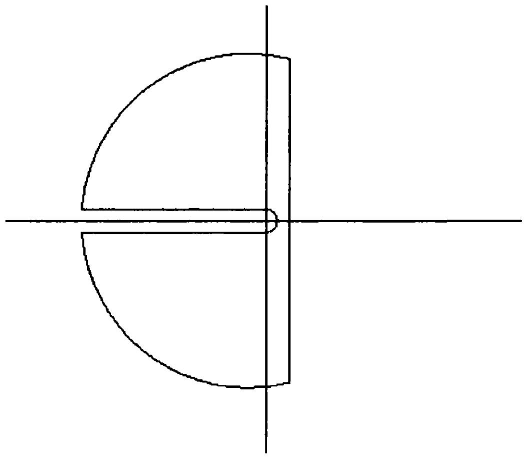


Figure 2.1: *The Bromwich contour*

this closed contour, i.e.

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} e^{\lambda h}\tilde{LC}^{\nu}(\lambda,q)d\lambda = \sum_{\lambda_i\in S(\mathcal{C})} R_i e^{\lambda_i h} \qquad (2.40)$$

where $S(\mathcal{C})$ represents the interior of the surface bounded by $\mathcal{C}$, λ_i the singularities of $\tilde{LC}^{\nu}(\lambda,q)$ and R_i the corresponding residues.

As the radius of the two quarters of circle included in $\mathcal{C}$ is let to infinity, their contributions tend to 0, which can be seen from the asymptotics of the Gamma function for argument with large modulus, a Stirling type of formula

$$\Gamma(z) = z^{z-\frac{1}{2}}e^{-z}\sqrt{2\pi}\{1+O(z^{-1})\}$$

and from

$$\lim_{|\lambda|\to\infty}\phi\left(\frac{\sqrt{\lambda}-\nu}{2}-1,\sqrt{\lambda}+1,-\frac{1}{2q}\right) = e^{-\frac{1}{4q}}$$

Hence, setting c as the abscissa of the vertical line in $\mathcal{C}$ and after a change of variable in the two real integrals, the limit of the integral along $\mathcal{C}$ is equal to the sum

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda h}\tilde{LC}^{\nu}(\lambda)d\lambda + \frac{1}{2\pi i}\int_0^{\infty} e^{-\lambda h}(\tilde{LC}^{\nu}(\lambda e^{i\pi}) - \tilde{LC}^{\nu}(\lambda e^{-i\pi}))d\lambda$$

which means that

$$\tilde{C}^{(\nu)}(h,q) = \sum_{\lambda_i \in S(C_\infty)} R_i e^{\lambda_i h} - \frac{1}{2\pi i}\int_0^\infty e^{-\lambda h}(\tilde{LC}^\nu(\lambda e^{i\pi}) - \tilde{LC}^\nu(\lambda e^{-i\pi}))d\lambda \qquad (2.41)$$

The inversion can be therefore be completed as soon as the residues and the branch cut integral are computed explicitly.

a) Residue at $\lambda = \nu^2$

The residues at the simple poles λ_i are given by

$$R_i = \lim_{\lambda \to \lambda_i} (\lambda - \lambda_i) e^{\lambda h} \tilde{LC}^\nu(\lambda) \qquad (2.42)$$

Therefore, if $\nu \geq 0$ ($\sqrt{\lambda} = \nu$), using $\Gamma(1+z) = z\Gamma(z)$ as well as the relation between the Laguerre polynomials and the confluent hypergeometric functions $L_n^\alpha(z) = \binom{n+\alpha}{n}\phi(-n, \alpha+1, z)$, the residue is $-e^{\nu^2 h}\frac{2q}{\nu+1}L_1^\nu\big(-\frac{1}{2q}\big)$.

If $\nu < 0$, the residue becomes $-\frac{e^{\nu^2 h}}{\nu+1}\frac{(2q)^{\nu+1}}{\Gamma(1-\nu)}\phi\Big(-\nu-1, 1-\nu, -\frac{1}{2q}\Big)$ since $\sqrt{\lambda} = -\nu$.

b) Residue at $\lambda = (\nu+2)^2$

If $\nu + 2 \geq 0$, the residue is $\frac{e^{(\nu+2)^2 h}}{\nu+1}$.

If $\nu + 2 < 0$, it becomes $-e^{(\nu+2)^2 h}\frac{(2q)^{\nu+2}}{\Gamma(-\nu)}\phi\Big(-\nu-2, -1-\nu, -\frac{1}{2q}\Big)$

c) Residue at the poles of $\Gamma(\frac{\sqrt{\lambda}+\nu}{2} + 2)$

Though situated at the negative integer values $-n$ of the argument of the Gamma function, these poles are in finite number because of the condition $Re(\sqrt{\lambda}) \geq 0$, which imposes $n \in [|0, [\frac{-\nu-4}{2}]|]$. Then, for n in this range, using the Kummer relation $\phi(\alpha,\gamma,z) = e^z\phi(\gamma-\alpha,\gamma,-z)$, the residue at $\lambda = (2n+\nu+4)^2$ is

$$e^{\lambda h - \frac{1}{2q}}\frac{L_n^{s_n}\left(\frac{1}{2q}\right)}{\binom{-n-\nu-4}{n}}\frac{(-1)^n}{n!}\frac{(2q)^{n+\nu+3}}{\Gamma(s_n+1)}\frac{16 s_n}{(\lambda-\nu^2)(\lambda-(\nu+2)^2)}$$

d) Branch cut integral

This integral is computed with the complex square-root equal to $i\sqrt{\lambda}$ above the negative axis and $-i\sqrt{\lambda}$ below. Using the Kummer relation and the equality

$\psi(\alpha,\gamma,z)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\phi(\alpha,\gamma,z)+z^{1-\gamma}\frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}\phi(\alpha-\gamma+1,2-\gamma,z)$, as well as the conjugate relation $\overline{\Gamma(z)}=\Gamma(\overline{z})$, it can be established that

$$\phi\Big(\frac{i\sqrt{\lambda}-\nu}{2}-1,i\sqrt{\lambda}+1,-\frac{1}{2q}\Big)\frac{\Gamma(\frac{i\sqrt{\lambda}+\nu}{2}+2)}{\Gamma(i\sqrt{\lambda}+1)}\Big(\frac{1}{2q}\Big)^{\frac{i\sqrt{\lambda}}{2}}$$

$$-\phi\Big(\frac{-i\sqrt{\lambda}-\nu}{2}-1,-i\sqrt{\lambda}+1,-\frac{1}{2q}\Big)\frac{\Gamma(\frac{-i\sqrt{\lambda}+\nu}{2}+2)}{\Gamma(-i\sqrt{\lambda}+1)}\Big(\frac{1}{2q}\Big)^{\frac{-i\sqrt{\lambda}}{2}}$$

$$=\frac{1}{i\sqrt{\lambda}}\left|\frac{\Gamma(\frac{i\sqrt{\lambda}+\nu}{2}+2)}{\Gamma(i\sqrt{\lambda})}\right|^2(2q)^{\frac{\nu-i\sqrt{\lambda}}{2}+1}e^{-\frac{1}{2q}}\psi\Big(\frac{\nu+i\sqrt{\lambda}}{2}+2,1+i\sqrt{\lambda},\frac{1}{2q}\Big)$$

which leads to the result after a change of variable in $\sqrt{\lambda}$. □

2.2.5 Eigenfunction expansion technique: explicit integral form.

Lo and al. [52] showed how the eigenfunction expansion technique is particulary well suited to the interesting problem of pricing barrier options and applied it to a single barrier under the square-root process. Although the eigenfunction expansion is a quite popular and well-known method in other fields, it is relatively new in finance: see Hansen and al. [38] and Florens and al. [30] for some applications in econometrics. Lewis [50] applied the method to option pricing under stochastic volatility. In another remarkable work [49], he also considered two specific problems: the valuation of European-style options on stocks paying a continuous time dividend and the Merton [56] economic growth model, problems which, as will be seen in the next section, are related to the Asian option valuation issue. The work in this section and the following one is mainly inspired by Lewis [49] and Lo and al. [52] research papers and shows how the techniques developed in these texts can be applied to the different types of European-type arithmetic Asian options[11].

The next section will deal with Lo and al. [52] type of countable eigenfunction expansion. In this section, we present an extension of Lewis's result, lifting the constraint of positivity for the interest rate. Though a negative rate would not be expected when modelling vanilla options on stocks, this case will prove to be of importance when considering the standard Asian option case again.

i. Call option on stock paying a continuous absolute dividend

Theorem 2.2.5. *Considering a stock paying the positive constant absolute dividend D following the SDE*

$$dS_t = (rS_t - D)dt + \sigma S_t dW_t \tag{2.43}$$

where $r \in \mathbb{R}$ and defining

$$\beta = \frac{2r}{\sigma^2} \tag{2.44}$$

and

$$\gamma = \frac{2D}{\sigma^2} \tag{2.45}$$

the call option on this stock with strike $K > 0$ and maturity T, of payoff $(S_T - K)^+$ has the value

$$C(S_0, K, T) = 1_{\{\beta>-1\}}\left[S_0 - \frac{D}{r}\left(1 - \frac{(\frac{\gamma}{S_0})^\beta}{\Gamma(\beta+2)}\phi\left(\beta, \beta+2, \frac{-\gamma}{S_0}\right)\right)\right]$$
$$+1_{\{\beta>1\}}\left(\frac{D}{r} - K\right)\left[1 - \frac{(\frac{\gamma}{S_0})^{\beta-1}}{\Gamma(\beta)}\phi\left(\beta-1, \beta, \frac{-\gamma}{S_0}\right)\right]e^{-rT}$$

[11] We note that very recently, in a work carried out independently, Linetsky [51] has also applied the eigenfunction expansion technique to the fixed-strike Asian option problem. But, firstly, we derive the exact integral formulation in a much simpler way by exploiting a slight extension of Lewis's work and a combination of the one-dimensional Markov formulations for the Asian valuation problem. We also express our results in terms of the confluent hypergeometric function rather than the Whittaker function employed by Linetsky [51], a better choice in the asymptotic regions. Finally, we give analytical results and approximations for both unseasoned and seasoned floating strike options which are not at all treated by Linetsky [51].

$$+1_{\{\beta>3\}}\gamma\Big(\frac{\gamma}{S_0}\Big)^{\beta}e^{-\frac{\gamma}{S_0}}\sum_{m=0}^{[\frac{\beta-3}{2}]}\xi_m(S_0,K)e^{-\frac{\sigma^2}{2}T(\delta+\lambda_m)}$$
$$+S_0^{\frac{1-\beta}{2}}K^{\frac{1+\beta}{2}}e^{-\frac{\gamma}{S_0}}\int_0^{\infty}\xi_\mu(S_0,K)e^{-\frac{\sigma^2}{2}T(\delta+\mu^2)}d\mu \tag{2.46}$$

with

$$\xi_m(S_0,K)=\frac{\left(\frac{S_0K}{\gamma^2}\right)^{m+2}}{(c_m)_{m+2}\Gamma(c_m-1)}L_{m+2}^{c_m-1}\Big(\frac{\gamma}{K}\Big)m!L_m^{c_m-1}\Big(\frac{\gamma}{S_0}\Big) \tag{2.47}$$

and

$$\xi_u(S_0,K)=\frac{1}{2\pi}\left|\frac{\Gamma(c_\mu-a_\mu-2)}{\Gamma(2i\mu)}\right|^2\Big(\frac{\gamma^2}{S_0K}\Big)^{i\mu}$$
$$\times\psi\Big(c_\mu-a_\mu-2,c_\mu,\frac{\gamma}{K}\Big)\psi\Big(c_\mu-a_\mu,c_\mu,\frac{\gamma}{S_0}\Big) \tag{2.48}$$

where $\delta=\frac{(\beta+1)^2}{4}$, $\lambda_m=-\frac{(\beta-3)^2}{4}+m(\beta-3-m)$, $a_\mu=\frac{\beta-1}{2}+i\mu$, $c_\mu=1+2i\mu$ *and* $c_m=\beta-2-2m$. *As in Theorem 2.2.4,* ϕ, ψ, Γ, L_n^ν *are the special functions defined in Appendix A.*

Proof. The proof of this result given in Appendix 2.5.1 follows the lines[12] of the one in Lewis [49]. □

ii. Integral formula for the fixed-strike put

Theorem 2.2.6. *The fixed strike Asian option put price, of payoff* $\left(K-\frac{Y_T}{T}\right)^+$, *can be evaluated as*

$$P^{fix}(S_0,K,T)=\left\{1_{\{\frac{2r}{\sigma^2}<1\}}\left[\frac{K}{S_0}+\frac{1}{rT}\Big(1-\frac{(\frac{2S_0}{\sigma^2KT})^{-\frac{2r}{\sigma^2}}}{\Gamma(2-\frac{2r}{\sigma^2})}\phi\Big(-\frac{2r}{\sigma^2},2-\frac{2r}{\sigma^2},-\frac{2S_0}{\sigma^2KT}\Big)\Big)\right]\right.$$
$$-1_{\{\frac{2r}{\sigma^2}<-1\}}\frac{1}{rT}\left[1-\frac{(\frac{2S_0}{\sigma^2KT})^{-\frac{2r}{\sigma^2}-1}}{\Gamma(-\frac{2r}{\sigma^2})}\phi\Big(-\frac{2r}{\sigma^2}-1,-\frac{2r}{\sigma^2},-\frac{2S_0}{\sigma^2KT}\Big)\right]e^{rT}$$

[12]The price has basically the same form as in Lewis [49] with two exceptions: the first residue (first line of the result) and the third line.

$$+1_{\{\frac{2r}{\sigma^2}<-3\}}\frac{2}{\sigma^2 T}\left(\frac{2S_0}{\sigma^2 KT}\right)^{-\frac{2r}{\sigma^2}} e^{-\frac{2S_0}{\sigma^2 KT}} \sum_{m=0}^{[\frac{-\frac{2r}{\sigma^2}-3}{2}]} \xi_m(S_0,K) e^{-\frac{\sigma^2 T}{2}(m+2)(-\frac{2r}{\sigma^2}-1-m)}$$

$$+\left(\frac{K}{S_0}\right)^{\frac{1}{2}+\frac{r}{\sigma^2}}\left(\frac{2}{\sigma^2 T}\right)^{\frac{1}{2}-\frac{r}{\sigma^2}} e^{-\frac{2S_0}{\sigma^2 KT}} \int_0^\infty \xi_\mu(S_0,K) e^{-\frac{\sigma^2 T}{2}(\delta+\mu^2)} d\mu \Big\} S_0 e^{-rT} \tag{2.49}$$

where

$$\xi_m(S_0,K) = \frac{\left(\frac{\sigma^2 KT}{2S_0}\right)^{m+2}(-1)^m L_m^{-\frac{2r}{\sigma^2}-3-2m}\left(\frac{2S_0}{\sigma^2 KT}\right)}{(m+1)(m+2)\Gamma(-\frac{2r}{\sigma^2}-3-2m)(-\frac{2r}{\sigma^2}-2-2m)_{m+2}} \tag{2.50}$$

and

$$\xi_u(S_0,K) = \frac{1}{2\pi}\left|\frac{\Gamma(\frac{r}{\sigma^2}-\frac{1}{2}+i\mu)}{\Gamma(2i\mu)}\right|^2 \left(\frac{2S_0}{\sigma^2 KT}\right)^{i\mu} \psi\left(\frac{3}{2}+\frac{r}{\sigma^2}+i\mu, 1+2i\mu, \frac{2S_0}{\sigma^2 KT}\right) \tag{2.51}$$

and finally $\delta = \frac{(-\frac{2r}{\sigma^2}+1)^2}{4}$.

Proof. With the standard formulation in equation (2.18) of section 2.2.1 using the asset-numeraire defined by the Radon-Nykodym derivative (2.17),

$$e^{-rT}E^Q\left(K-\frac{Y_T}{T}\right)^+ = S_0 E^{Q^S}(X_T^+)$$

where the underlying process $X_t = \frac{K}{S_t} - \frac{Y_t}{TS_t}$ can be defined as the strong solution of the SDE

$$dX_t = \left(-\frac{1}{T} - rX_t\right)dt - \sigma dW_t^{Q^S}$$

with the starting value $X_0 = -\frac{K}{S0}$. The Asian fixed-strike put price is thus worth $S_0 e^{-rT}$ times the vanilla European call option with null strike[13] on a stock paying a dividend $D = \frac{1}{T}$, starting at the initial value $\frac{K}{S_0}$ with the interest rate $-r$. The exponential e^{-rT} in the multiplicative coefficient $S_0 e^{-rT}$ represents the correction for the discounting at the interest rate $-r$, as moving to the asset-numeraire lifts the discounting of the payoff.

[13]The stock having here the possibility to go negative, this call price does not reduce to the expectation of the stock.

Given this, the expression (2.51) follows from the asymptotics (2.28), (2.76) and from the limit:

$$\lim_{x \to \infty} \frac{L_m^\alpha(x)}{x^m} = \frac{(-1)^m}{m!} \tag{2.52}$$

The limit and summation can be inverted for the calculation of the last term of (2.51), since the limit expression is absolutely convergent. This results from the expansion (2.76), the relation (see Gradshteyn and Ryzhik [37]):

$$\forall y \in \mathbb{R}, |\Gamma(iy)|^2 = \frac{\pi}{ysh(\pi y)} \tag{2.53}$$

and from the limit (see Erdelyi [28]):

$$\forall (x,y) \in \mathbb{R}^2, \lim_{|y| \to \infty} |\Gamma(x+iy)| e^{\frac{\pi |y|}{2}} |y|^{\frac{1}{2}-x} = \sqrt{(2\pi)} \tag{2.54}$$

□

iii. Integral formula for the unseasoned floating-strike call

Theorem 2.2.7. *The floating strike Asian option price of payoff* $\left(S_T - \frac{Y_T}{T}\right)^+$ *can be evaluated as*

$$\begin{aligned} C^{float}(S_0,T) = {} & 1_{\{\frac{2r}{\sigma^2} > -1\}} \left[S_0 - \frac{S_0}{rT} \left(1 - \frac{(\frac{2}{\sigma^2 T})^{\frac{2r}{\sigma^2}}}{\Gamma(2+\frac{2r}{\sigma^2})} \phi\left(\frac{2r}{\sigma^2}, \frac{2r}{\sigma^2} + 2, -\frac{2}{\sigma^2 T} \right) \right) \right] \\ & + 1_{\{\frac{2r}{\sigma^2} > 1\}} \frac{S_0}{rT} \left[1 - \frac{(\frac{2}{\sigma^2 T})^{\frac{2r}{\sigma^2}-1}}{\Gamma(\frac{2r}{\sigma^2})} \phi\left(\frac{2r}{\sigma^2} - 1, \frac{2r}{\sigma^2}, -\frac{2}{\sigma^2 T} \right) \right] e^{-rT} \\ & + 1_{\{\frac{2r}{\sigma^2} > 3\}} S_0 \left(\frac{2}{\sigma^2 T} \right)^{1+\frac{2r}{\sigma^2}} e^{-\frac{2}{\sigma^2 T}} \sum_{m=0}^{[\frac{\frac{2r}{\sigma^2}-3}{2}]} \xi_m e^{-\frac{\sigma^2 T}{2}(m+2)(\frac{2r}{\sigma^2}-1-m)} \\ & + S_0 \left(\frac{2}{\sigma^2 T} \right)^{\frac{1}{2}+\frac{r}{\sigma^2}} e^{-\frac{2}{\sigma^2 T}} \int_0^\infty \xi_\mu e^{-\frac{\sigma^2 T}{2}(\delta+\mu^2)} d\mu \end{aligned} \tag{2.55}$$

where

$$\xi_m = \frac{\left(\frac{\sigma^2 T}{2}\right)^{m+2} (-1)^m L_m^{\frac{2r}{\sigma^2}-3-2m}\left(\frac{2}{\sigma^2 T}\right)}{(m+1)(m+2)\Gamma(\frac{2r}{\sigma^2}-3-2m)(\frac{2r}{\sigma^2}-2-2m)_{m+2}} \tag{2.56}$$

and

$$\xi_u = \frac{1}{2\pi}\left|\frac{\Gamma(-\frac{r}{\sigma^2}-\frac{1}{2}+i\mu)}{\Gamma(2i\mu)}\right|^2\left(\frac{2}{\sigma^2 T}\right)^{i\mu}\psi\left(\frac{3}{2}-\frac{r}{\sigma^2}+i\mu, 1+2i\mu, \frac{2}{\sigma^2 T}\right) \tag{2.57}$$

and finally $\delta = \frac{(\frac{2r}{\sigma^2}+1)^2}{4}$.

Proof. Using the asset-numeraire defined by (2.17), this price could be computed through the relation

$$e^{-rT}E^Q\left(S_T - \frac{Y_T}{T}\right)^+ = S_0 E^{Q^S}\left(1 - \frac{Y_T}{TS_T}\right)^+ = S_0 E^{Q^S}\left(1 - \frac{X_T}{T}\right)^+ \tag{2.58}$$

The Itô lemma gives $X_t = \frac{Y_t}{S_t}$ as the strong solution of

$$dX_t = (1 - rX_t)dt + \sigma X_t dW_t^{Q^S}$$

with $X_0 = 0$.

But, the alternative formulation of Theorem 2.2.2 shows that X_T has the same distribution as $\int_0^T e^{(-r-\frac{\sigma^2}{2})t+\sigma W_t}dt$. In other terms, the starting floating-strike call option is worth $S_0 e^{-rT}$ units of the starting fixed-strike Asian put option of strike 1 on an asset of initial value 1 and interest rate[14] $-r$. This result, though interesting as it brings together two apparently unrelated options, depends on the lognormal assumption for the asset distribution

Alternatively, the expression (2.55) can also be obtained as the price of a vanilla call option with null strike on a stock paying a dividend $\frac{1}{T}$ and starting at S_0, with the interest rate r. Indeed, from Theorem 2.2.2, if X_t satisfies

$$dX_t = \left(-\frac{S_0}{T} + rX_t\right)dt + \sigma X_t dW_t^Q$$

[14] It is straightforward to generalise the results of this chapter to an asset model which includes proportional dividends or comparable features which can turn the asset drift negative. Typically, in the foreign exchange market, negative drifts are sometimes encountered. Considering a short rate $-r$ is thus not shocking in financial terms, since there is no discounting at a negative rate in this formula. This relation between fixed and floating strike options is therefore not merely a pure mathematical object, it has a financial sense.

then X_T has the same distribution as $S_T - \frac{Y_T}{T}$. □

iv. Integral formula for the seasoned floating-strike call

Theorem 2.2.8. *Noting $a = \frac{\int_0^t S_u du}{T}$, the seasoned floating strike call option price of payoff $\left(S_T - \frac{\int_0^T Y_t dt}{T}\right)^+$ is*

$$C^{float}(S_t,t,T,a) = 1_{\{\frac{2r}{\sigma^2}>-1\}}\left[S_t - \frac{S_t}{rT}\left(1 - \frac{(\frac{2}{\sigma^2 T})^{\frac{2r}{\sigma^2}}}{\Gamma(\frac{2r}{\sigma^2}+2)}\phi\left(\frac{2r}{\sigma^2},\frac{2r}{\sigma^2}+2,-\frac{2}{\sigma^2 T}\right)\right)\right]$$

$$+1_{\{\frac{2r}{\sigma^2}>1\}}\left(\frac{S_t}{rT} - a\right)\left[1 - \frac{(\frac{2}{\sigma^2 T})^{\frac{2r}{\sigma^2}-1}}{\Gamma(\frac{2r}{\sigma^2})}\phi(\frac{2r}{\sigma^2}-1,\frac{2r}{\sigma^2},\frac{-2}{\sigma^2 T})\right]e^{-rT}$$

$$+1_{\{\frac{2r}{\sigma^2}>3\}}\frac{2S_t}{\sigma^2 T}\left(\frac{2}{\sigma^2 T}\right)^{\frac{2r}{\sigma^2}} e^{-\frac{2}{\sigma^2 T}} \sum_{m=0}^{[\frac{\frac{2r}{\sigma^2}-3}{2}]} \xi_m(S_t,a) e^{-\frac{\sigma^2 T}{2}(m+2)(\frac{2r}{\sigma^2}-1-m)}$$

$$+S_t^{\frac{1-\frac{2r}{\sigma^2}}{2}} a^{\frac{1+\frac{2r}{\sigma^2}}{2}} e^{-\frac{2}{\sigma^2 T}} \int_0^\infty \xi_\mu(S_t,a) e^{-\frac{\sigma^2}{2}T(\delta+\mu^2)} d\mu \tag{2.59}$$

with

$$\xi_m(S_t,a) = \frac{\left(\frac{a\sigma^4 T^2}{4S_t}\right)^{m+2} L_{m+2}^{\frac{2r}{\sigma^2}-3-2m}\left(\frac{2S_t}{\sigma^2 Ta}\right) m! L_m^{\frac{2r}{\sigma^2}-3-2m}\left(\frac{2}{\sigma^2 T}\right)}{(\frac{2r}{\sigma^2}-2-2m)_{m+2}\Gamma(\frac{2r}{\sigma^2}-3-2m)} \tag{2.60}$$

and

$$\xi_u(S_t,a) = \frac{1}{2\pi}\left|\frac{\Gamma(-\frac{r}{\sigma^2}-\frac{1}{2}+i\mu)}{\Gamma(2i\mu)}\right|^2 \left(\frac{4S_t}{a\sigma^4 T^2}\right)^{i\mu}$$

$$\times\psi\left(-\frac{1}{2}-\frac{r}{\sigma^2}+i\mu, 1+2i\mu, \frac{2S_t}{\sigma^2 Ta}\right)\psi\left(\frac{3}{2}-\frac{r}{\sigma^2}+i\mu,1+2i\mu,\frac{2}{\sigma^2 T}\right) \tag{2.61}$$

where $\delta = \frac{(\frac{2r}{\sigma^2}+1)^2}{4}$.

Proof. This option can be seen as a vanilla call option with strike a on a stock paying a dividend $\frac{S_t}{T}$ and starting at S_t with the instantaneous interest rate r. This representation follows from Theorem 2.2.2. □

Remark. In this section and the following one, our choice to evaluate either one of the call or put option price is determined by their mathematical properties. For example, the payoff of a fixed-strike Asian put is bounded, which makes it easier to price than the corresponding call. It is yet possible to retrieve one from the other with the classical call-put parity relations.
For fixed-strike options,

$$C^{fix}(S_0, K, T) - P^{fix}(S_0, K, T) = \frac{1 - e^{-rT}}{rT} S_0 - e^{-rT} K \tag{2.62}$$

which comes from the fact that[15]

$$E\left(\int_0^T S_u du\right) = \frac{e^{rT} - 1}{r} S_0 \tag{2.63}$$

For floating-strike options,

$$C^{float}(S_0, T) - P^{float}(S_0, T) = \frac{rT - 1 + e^{-rT}}{rT} S_0 \tag{2.64}$$

2.2.6 Eigenfunction expansion technique: series form.

We start by observing that the fixed-strike Asian option is a barrier option, since the underlying is a strictly increasing process. This would naturally appeal to the application of countable eigenfunction expansion methods for barriers options like the one used in Lo and al. [52]. However, among the two Markovian one-factor representations we can use for this valuation, only the second one (second formulation of section 2.2.1) would allow us to work with a bounded state-space. But, this representation is valid only at maturity, given that the distributional equivalence is not valid at the process level: path-properties and hence barrier features cannot be exploited. It is still possible to consider a barrier option and exploit this eigenfunction

[15]Straightforward application of Fubini theorem.

representation as an approximation to the Asian option in the limit that the barrier H, well above the strike level K, recesses to infinity. The approximate option can then be priced, following the same method as in Lo and al. [52], the main intuition being that, for a regular Sturm-Liouville problem, the associated Green function will only have discrete negative simple poles and can hence be (relatively) simply inverted. A brief account of the theory of eigenfunction expansions can be found in Zauderer [78].

i. Series approximation for the fixed-strike put.

Theorem 2.2.9. *For any $H \geq KT$ sufficiently large, the fixed strike Asian put price can be approximated by*

$$P(S_0, K, T) = e^{-rT} \sum_{n=0}^{\infty} c_n \|\xi(\lambda_n, \cdot)\|^{-2} e^{\lambda_n T} \tag{2.65}$$

with λ_n the (negative) zeroes of the function $\xi(H, \lambda_n)$, where

$$\xi(y, \lambda_n) = \left(\frac{2S_0}{y\sigma^2}\right)^{\mu^+(\lambda_n)} \psi\left(\mu^+(\lambda_n), 2\mu^+(\lambda_n) - \frac{2r}{\sigma^2} + 2, \frac{2S_0}{\sigma^2 y}\right) \tag{2.66}$$

and

$$\mu_-^+(\lambda) = \frac{-(\frac{\sigma^2}{2} - r) \pm \sqrt{(\frac{\sigma^2}{2} - r)^2 + 2\sigma^2\lambda}}{\sigma^2} \tag{2.67}$$

c_n is defined as (with $\mu_n = \mu(\lambda_n)$)

$$c_n = \frac{2}{\sigma^2 T}\left(\frac{\sigma^2}{2S_0}\right)^{-\frac{2r}{\sigma^2}} \left(\frac{2S_0}{\sigma^2 KT}\right)^{\mu_n - \frac{2r}{\sigma^2}} e^{-\frac{2S_0}{\sigma^2 KT}} \psi\left(\mu_n + 2, 2\mu_n - \frac{2r}{\sigma^2} + 2, \frac{2S_0}{\sigma^2 KT}\right) \tag{2.68}$$

And $\|\xi(\lambda_n, \cdot)\|^2$ is given by the equivalent expressions

$$\|\xi(\lambda_n, \cdot)\|^2 = \int_0^H \frac{2\xi^2(y, \lambda_n) y^{\frac{2r}{\sigma^2} - 2} e^{-\frac{2S_0}{\sigma^2 y}}}{\sigma^2} dy = -\frac{\xi(y, \lambda_n)}{\eta(y, \lambda_n)} \frac{\frac{\partial \eta}{\partial x}(H, \lambda) \frac{\partial \xi}{\partial \lambda}(H, \lambda)}{e^{-\frac{2S_0}{\sigma^2 H}} H^{\frac{2r}{\sigma^2}}} \tag{2.69}$$

for any y and

$$
\begin{aligned}
\eta(x,\lambda) &= \varphi^+(x) - \tfrac{\varphi^+(H)}{\varphi^-(H)}\varphi^-(x) \\
\varphi^{\pm}_{-}(x) &= x^{-\mu^{\pm}_{-}}\phi\big(\mu^{\pm}_{-}(\lambda), 2\mu^{\pm}_{-}(\lambda) - \tfrac{2r}{\sigma^2} + 2, \tfrac{2S_0}{\sigma^2 x}\big) \\
\tfrac{\partial\eta}{\partial x}(x,\lambda) &= \tfrac{\partial\varphi^+}{\partial x}(x) - \tfrac{\varphi^+(H)}{\varphi^-(H)}\tfrac{\partial\varphi^-}{\partial x}(x) \\
\tfrac{\partial\varphi^{\pm}_{-}}{\partial x}(x) &= (\mu^{\pm}_{-}(\lambda))x^{-\mu^{\pm}_{-}-1}\phi\big(\mu^{\pm}_{-}(\lambda)+1, 2\mu^{\pm}_{-}(\lambda) - \tfrac{2r}{\sigma^2} + 2, \tfrac{2S_0}{\sigma^2 x}\big)
\end{aligned}
\tag{2.70}
$$

Proof. To avoid the payoff integrability problems encountered in Lewis [49] and Appendix 2.5.1 and to work on a positive state space, we will use the second formulation of Section 2.2.1, i.e. we consider X_t following

$$dX_t = (S_0 + rX_t)dt + \sigma X_t dW_t^Q, \ X_0 = 0$$

which, at a given date t, has the same distribution as the point Y_t.

We note $P(S_0, K, T)$ the price of the put option of strike K, maturity T and time to maturity $\tau = T - t$. Then, from the results of section 2.2.1, the non-discounted put price $\bar{P}(X_0, \tau) = e^{rT}P(S_0, K, T)$ is the solution of the second-order partial differential equation

$$\frac{\partial \bar{P}(x,\tau)}{\partial \tau} = \frac{\sigma^2 x^2}{2}\frac{\partial^2 \bar{P}(x,\tau)}{\partial x^2} + (S_0 + rx)\frac{\partial \bar{P}(x,\tau)}{\partial x} \tag{2.71}$$

with the initial condition

$$\bar{P}(x,0) = \left(K - \frac{x}{T}\right)^+$$

and the boundary conditions $\forall \tau \leq T$, $\bar{P}(H,\tau) = 0$.

We define $\mathcal{G}(x,\lambda) = \int_0^\infty e^{-\lambda\tau}\bar{P}(x,\tau)d\tau$ the Laplace transform of $\bar{P}$ with respect to τ. Since $\bar{P}$ is bounded by K, this Laplace transform exists at least for any $\lambda > 0$. Then, for any λ for which the transform exists, $\mathcal{G}(x,\lambda)$ follows the ODE

$$\lambda\mathcal{G}(x,\lambda) - \bar{P}(x,0) = \frac{\sigma^2 x^2}{2}\frac{\partial^2 \mathcal{G}(x,\lambda)}{\partial x^2} + (S_0 + rx)\frac{\partial \mathcal{G}(x,\lambda)}{\partial x} \tag{2.72}$$

with the boundary condition $\mathcal{G}(H,\lambda) = 0$.

We follow the eigenfunction expansion methodology described in Appendix C (and use the same notations). Indeed, even if there is no absorption condition at 0, the expansion is still valid with appropriate changes since there is a solution of the associated homogeneous equation tending to a limit at 0 (see Zauderer [78]). We construct the Green function from the solutions to this homogeneous equation, i.e.

$$\frac{\partial^2 \mathcal{G}_h(x,\lambda)}{\partial x^2} + \frac{2(S_0+rx)}{\sigma^2 x^2}\frac{\partial \mathcal{G}_h(x,\lambda)}{\partial x} = \frac{2\lambda}{\sigma^2 x^2}\mathcal{G}_h(x,\lambda) \tag{2.73}$$

It can be put in a self-adjoint form with

$$s(x) = e^{\int^x \frac{2(S_0+rz)}{\sigma^2 z^2}dz} = e^{-\frac{2S_0}{\sigma^2 x}} x^{\frac{2r}{\sigma^2}}$$

and

$$w(x) = \frac{2}{\sigma^2} x^{\frac{2r}{\sigma^2}-2} e^{-\frac{2S_0}{\sigma^2 x}}$$

This ODE is the same as the equation (2.24) in section 2.2.2 with $\alpha = S_0$ and $\beta = -r$. Hence (the form of the solutions does not depend on the sign of the parameters), two independent solutions of (2.73) are, with the same notations as in 2.2.2, φ^+ and φ^- defined in (2.70). Solutions of (2.72) can then be formally constructed (for $x > 0$)[16] as

$$-\mathcal{G}(x,\lambda) = \eta(x,\lambda)\int_0^x \frac{2\xi(y,\lambda)\bar{P}(y,0)}{\sigma^2 y^2 W_{(\xi,\eta)}(y)}dy + \xi(x,\lambda)\int_x^H \frac{2\eta(y,\lambda)\bar{P}(y,0)}{\sigma^2 y^2 W_{(\xi,\eta)}(y)}dy \tag{2.74}$$

where $\eta(x,\lambda)$ and $\xi(x,\lambda)$ are solutions of (2.73) satisfying some conditions discussed below and $W_{(\xi,\eta)}(y)$ is the Wronskian of these two solutions. As mentioned in the discussion in Lewis [49], this Green function solution exists if some sufficient integrability conditions are verified. More precisely, noting $\rho(y) = \frac{2}{\sigma^2 y^2 W_{(\xi,\eta)}(y)}$, the integrals in (2.74) exist if $\xi \in \mathcal{L}_{2,\rho}(0,x)$, $\eta \in \mathcal{L}_{2,\rho}(x,H)$ and $\bar{P}(.,0) \in \mathcal{L}_{2,\rho}(0,H)$. These conditions are sufficient to ensure the existence of the considered integrals but not always

[16] The function is then prolonged by continuity at 0.

necessary. More specifically, Lewis [49] provides a counter-example in which the last integrability condition on $\bar{P}(.,0)$ is not satisfied but the integrals are finite. In our setting, though, this condition on the final payoff is readily verified by the put payoff, although the diffusion considered here is very similar[17] to the one in Lewis [49]. From these observations, the first step to choose ξ and η is to determine the form of $\rho(y)$. As $W_{(\xi,\eta)}(y) = \frac{C(\lambda)}{s(y)}$, we have $\rho(y) = \frac{w(y)}{C(\lambda)}$. Now, considering the asymptotics of the Kummer function ϕ (See Erdelyi [28]) as $h \to \infty$

$$\phi(a,c,h) = \frac{\Gamma(c)}{\Gamma(a)} h^{a-c} e^h [(1 + \mathcal{O}(|h|^{-1})] \tag{2.75}$$

it is clear there is only one linear combination (up to a multiplicative factor) of φ^+ and φ^- which satisfies the ρ-square integrability condition near 0. We choose the multiplicative constant that enables us to express it directly in terms of the Kummer confluent hypergeometric function of the second kind ψ. This leads us to the definition of $\xi(y,\lambda)$ given in (2.66), where

$$\psi\left(\mu^+, 2\mu^+ - \frac{2r}{\sigma^2} + 2, \frac{2S_0}{\sigma^2 y}\right) = \frac{\Gamma(-2\mu^+ + \frac{2r}{\sigma^2} - 1)}{\Gamma(-\mu^+ + \frac{2r}{\sigma^2} - 1)} \phi\left(\mu^+, 2\mu^+ - \frac{2r}{\sigma^2} + 2, \frac{2S_0}{\sigma^2 y}\right)$$
$$+ \frac{\Gamma(2\mu^+ - \frac{2r}{\sigma^2} + 1)}{\Gamma(\mu^+)} \phi\left(\mu^-, 2\mu^- - \frac{2r}{\sigma^2} + 2, \frac{2S_0}{\sigma^2 y}\right) \left(\frac{2S_0}{y\sigma^2}\right)^{\mu^- - \mu^+}$$

The Kummer function of the second kind ψ has the known asymptotics (See Erdelyi [28]) when $|h| \to \infty$, $-\frac{3}{2}\pi < \arg(h) < \frac{3}{2}\pi$ and $N = 0, 1, 2, \ldots$

$$\psi(a,c,h) = \sum_0^N (-1)^n \frac{(a)_n (a-c+1)_n}{n!} h^{-a-n} + \mathcal{O}(|h|^{-a-N-1}) \tag{2.76}$$

which confirms that $\eta(x,\lambda)$ is ρ-square integrable. It is actually the only bounded solution in $[0, H]$.

[17] This difference is mainly due to the sign of α which decides whether $C(y)$ vanishes or explodes near 0. α, positive in our case, is negative in the case presented in Lewis [49], hence limiting there the range of C-square integrable payoffs.

Now, coming to $\eta(y,\lambda)$, the integrability condition on $[x,H]$ is automatically verified by any solution, but for the condition $\mathcal{G}(H,\lambda)=0$ to be satisfied for any λ for which $\mathcal{G}$ is defined, the supplementary condition $\eta(H,\lambda)=0$ is needed. The natural choice is therefore the one given in (2.70).
The Green function is then equal to

$$\mathcal{G}_{-\lambda}(x,y)=w(y)\frac{\xi(\lambda,\min(x,y))\eta(\lambda,\max(x,y))}{C(\lambda)}$$

As outlined in Appendix C, we can exploit this expression to determine the eigenvalues and the normalised eigenfunctions. The eigenvalues λ_n, $n=0\ldots\infty$ are the discrete negative (real) values for which $\xi(H,\lambda_n)=0$. The corresponding eigenfunctions are multiples of $\eta(\lambda_n,x)$ and $\xi(\lambda_n,x)$. Their norms can be computed by using the definition of the scalar product (C.16) or the relation (C.17) with $C(\lambda)$. For the latter, we notice that

$$C(\lambda)=W_{(\xi,\eta)}(H)p(H)=\xi(H,\lambda)\frac{\partial\eta}{\partial x}(H,\lambda)p(H)$$

From the relation (see Erdelyi [28])

$$\frac{d^n}{dx^n}[x^{a+n-1}\phi(a,c,x)]=(a)_n x^a\phi(a+n,c,x)$$

the derivative of η is

$$\frac{\partial\eta}{\partial x}(H,\lambda)=\mu^+\left(\frac{2S_0}{\sigma^2H}\right)^{\mu^++1}\phi\left(\mu^++1,2\mu^+-\frac{2r}{\sigma^2}+2,\frac{2S_0}{\sigma^2H}\right)$$

$$-\frac{\varphi^+(H)}{\varphi^-(H)}\mu^-\left(\frac{2S_0}{\sigma^2H}\right)^{\mu^-+1}\phi\left(\mu^-+1,2\mu^--\frac{2r}{\sigma^2}+2,\frac{2S_0}{\sigma^2H}\right)$$

Finally,

$$\|\xi(\lambda_n,\cdot)\|^2=-\frac{\xi(y,\lambda_n)}{\eta(y,\lambda_n)}\frac{\partial\eta}{\partial x}(H,\lambda)\frac{\partial\xi}{\partial\lambda}(H,\lambda)p(H)$$

We still have to calculate the expansion coefficients

$$c_n = \int_0^H \xi(y, \lambda_n)\bar{P}(y, 0)w(y)dy$$

which, divided by the norms, are the projections of $\bar{P}(., 0)$ on the orthonormal complete basis of eigenfunctions

$$\forall y \in (0, H),\ \bar{P}(y, 0) = \sum_{n=0}^{\infty} c_n \tilde{\xi}(y, \lambda_n) \tag{2.77}$$

After making the change of variable $h = \frac{2S_0}{\sigma^2 x}$ and using the relations (see Erdelyi [28])

$$\frac{d^n}{dx^n}[e^{-x}x^{c-a+n-1}\psi(a, c, x)] = (-1)^n e^{-x}x^{c-a-1}\psi(a-n, c, x)$$

we can compute explicitly the coefficients, which leads to the expression (2.68).

□

Remark. Considering the solutions of the homogeneous equation (2.73), a first idea would have been to use the relation between the confluent hypergeometric function and the Laguerre polynomials and decompose the final payoff in terms of Laguerre polynomials. But, this would lead to a divergent series, since the implied λ_n would be positive. Yet, as seen earlier, using eigenfunction expansion without limiting the state space (i.e. without using the option barrier property) leads to a formula counting the first Laguerre terms and replacing the series with positive λ_n by an integral of a Kummer function and other terms.

ii. Series approximation for the unseasoned floating strike call.

Theorem 2.2.10. *For any $H \geq T$ sufficiently large, the floating strike Asian put price can be approximated by*

$$C^{float}(S_0, T) = S_0 \sum_{n=0}^{\infty} c_n \|\xi(\lambda_n, \cdot)\|^{-2} e^{\lambda_n T} \tag{2.78}$$

with λ_n *the (negative) zeroes of the function* $\xi(H, \lambda_n)$ *, where*

$$\xi(y, \lambda_n) = \left(\frac{2}{y\sigma^2}\right)^{\mu^+(\lambda_n)} \psi\left(\mu^+(\lambda_n), 2\mu^+(\lambda_n) + \frac{2r}{\sigma^2} + 2, \frac{2}{\sigma^2 y}\right) \tag{2.79}$$

and

$$\mu^+(\lambda) = \frac{-\left(\frac{\sigma^2}{2} + r\right) + \sqrt{\left(\frac{\sigma^2}{2} + r\right)^2 + 2\sigma^2\lambda}}{\sigma^2} \tag{2.80}$$

c_n *is defined as (with* $\mu_n = \mu^+(\lambda_n)$*)*

$$c_n = \frac{2}{\sigma^2 T}\left(\frac{\sigma^2}{2}\right)^{\frac{2r}{\sigma^2}} \left(\frac{2}{\sigma^2 T}\right)^{\mu_n + \frac{2r}{\sigma^2}} e^{-\frac{2}{\sigma^2 T}} \psi\left(\mu_n + 2, 2\mu_n + \frac{2r}{\sigma^2} + 2, \frac{2}{\sigma^2 T}\right) \tag{2.81}$$

The other terms are defined in the same way as in Theorem 2.2.9 except for

$$\begin{array}{lcl} \|\xi(\lambda_n, \cdot)\|^2 & = & \int_0^H \frac{2\xi^2(y,\lambda_n) y^{\frac{2r}{\sigma^2}-2} e^{-\frac{2}{\sigma^2 y}}}{\sigma^2} dy = -\frac{\xi(y,\lambda_n)}{\eta(y,\lambda_n)} \frac{\frac{\partial \eta}{\partial x}(H,\lambda) \frac{\partial \xi}{\partial \lambda}(H,\lambda)}{e^{-\frac{2}{\sigma^2 H}} H^{\frac{2r}{\sigma^2}}} \\ \varphi_-^+(x) & = & x^{-\mu_-^+} \phi(\mu_-^+(\lambda), 2\mu_-^+(\lambda) - \frac{2r}{\sigma^2} + 2, \frac{2}{\sigma^2 x}) \\ \frac{\partial \varphi_-^+}{\partial x}(x) & = & (\mu_-^+(\lambda)) x^{-\mu_-^+ - 1} \phi(\mu_-^+(\lambda) + 1, 2\mu_-^+(\lambda) - \frac{2r}{\sigma^2} + 2, \frac{2}{\sigma^2 x}) \end{array} \tag{2.82}$$

Proof. The floating strike options call price can be derived, as previously, from the standard formulation of section 2.2.1

$$e^{-rT} E^Q \left(S_T - \frac{Y_T}{T}\right)^+ = S_0 E^{Q^S}\left(1 - \frac{Y_T}{T S_T}\right)^+ = S_0 E^{Q^S}\left(1 - \frac{X_T}{T}\right)^+$$

with $X_0 = 0$ and

$$dX_t = (1 - rX_t)dt + \sigma dW_t^{Q^S} \tag{2.83}$$

This expectation is the price of a fixed-strike Asian option with $S_0 = 1$, $K = 1$ and interest rate $-r$. Given that the Theorem 2.2.9 is valid for any real r, this leads to the result. □

iii. Series approximation for the seasoned floating strike call.

The seasoned option price can be deduced in the same way as the unseasoned one, by using the asset-numeraire and a process satifying the SDE (2.83). The starting value of the process is not 0 anymore but $a/S0$, where $a = \frac{\int_0^t S_u du}{T}$. To obtain the price, it is sufficient to multiply each term of the series (2.78) by $\xi(\frac{a}{S0}, \lambda_n)$.

2.2.7 Numerical applications

In their reference paper, Geman and Eydeland [35] studied a group of cases that have been generally used afterwards as a benchmark to evaluate the performance of other numerical methods. We will thus check our formulae against these cases.

Case	r	σ	T	K	S_0	GY	GE-MC	Integral
1	0.05	0.5	1	2	1.9	0.195	0.191	0.193174
2	0.05	0.5	1	2	2.1	0.308	0.306	0.30622
3	0.02	0.1	1	2	2	0.058	0.056	0.055986
4	0.18	0.3	1	2	2	0.227	0.217	0.218387
5	0.0125	0.25	2	2	2	0.1722	0.1711	0.172269
6	0.05	0.5	2	2	2	0.351	0.347	0.350095

Table 2.1: Fixed-strike Asian options, explicit integral form .

Table 2.1 compares the results by Geman and Eydeland on the numerical inversion of the Geman-Yor Laplace transform (column GY) and a Monte-Carlo estimation (column GE-MC) with the numerical evaluation of formula (2.49) adding put-call parity (column Integral). We are able to obtain a good accuracy with our numerical integration completed through the built-in *NIntegrate* routine of Mathematica.

Case	r	σ	T	S_0	Integral
1	0.05	0.5	1	1.9	0.191499
2	0.05	0.5	1	2.1	0.211657
3	0.02	0.1	1	2	0.265788
4	0.18	0.3	1	2	0.062566
5	0.0125	0.25	2	2	0.148986
6	0.05	0.5	2	2	0.265788

Table 2.2: Floating-strike Asian options, explicit integral form .

Table 2.2 shows the corresponding results for the floating strike options.

It should be noted that the numerical evaluation of our formula does not present much difficulties, except in case 3. As $\sigma\sqrt{T}$ decreases, the formula requires a finer and more time-consuming integration.

In Table 2.3, we consider exactly the same reference cases described in Table 2.1 with the approximate series method presented in Section 2.2.6.

Case	Value	B	N
1	0.193174	1.6	5
2	0.30622	1.6	5
4	0.218387	1.3	6
5	0.172269	1.5	6
6	0.350095	2.2	4

Table 2.3: Fixed-strike Asian options, approximate series form .

The column Value contains the value obtained trough the series method, B the minimal value such that placing the absorbing point at $H = B * K * T$ leads to a sum less than 10^{-4} away from the correct price. The maximal level this B reaches in these numerical application is 2.2. N represents the number of terms needed for the partial sum to be less than 10^{-4} away from the limit in relative value. As can be seen, for these levels for the barrier, the series converge in only 5 to 6 terms. This series is therefore extremely fast-converging.

However, as for all the other methods to evaluate these options, the troubles start when $\sigma\sqrt{T}$ decreases too much. For the case 3, we could not produce a result for this reason.

The behaviour of the series depends on the barrier level and, hence, on B. As this parameter decreases, the basis function η becomes more and more oscillatory and the frequency or number of zeroes in an interval increases while the speed of convergence of the series decreases at the same time. This is illustrated by Table 2.4 and Figure 2.2.

B	1	2	4	5	8	10	15
	1.76	.68	0.28	0.21	0.12	0.10	0.06
	4.14	2.01	1.08	.90	0.63	0.54	0.42
	6.87	3.65	2.12	1.81	1.33	1.17	0.92
	9.92	5.54	3.37	2.92	2.20	1.94	1.56
	13.25	7.67	4.80	4.19	3.20	2.84	2.32
	16.84	10.00	6.39	5.61	4.34	3.87	3.19
	20.67	12.53	8.14	7.18	5.60	5.02	4.15
	24.73	15.24	10.04	8.88	6.98	6.27	5.22
	29.02	18.12	12.07	10.72	8.47	7.63	6.37
	33.51	21.17	14.24	12.67	10.07	9.09	7.62
	38.21	24.39	16.53	14.75	11.77	10.64	8.95

Table 2.4: Evolution of the location of the zeroes with respect to B .

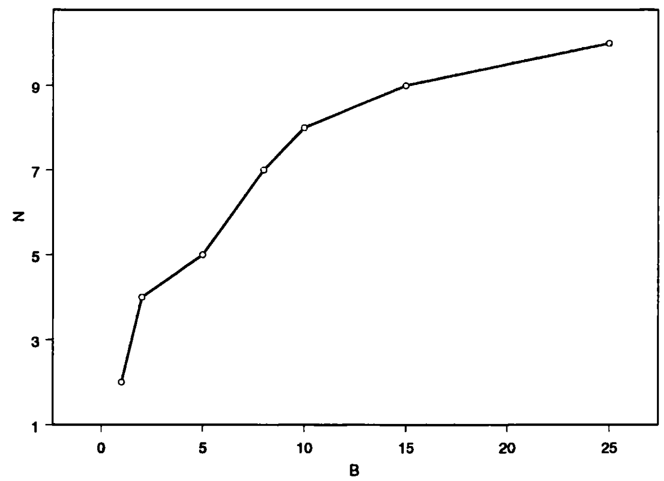


Figure 2.2: *Evolution of the speed of convergence with respect to B*

2.2.8 An excursion to jump-diffusions

The two reformulations to a one-dimensional markovian problem studied in Part 2.2.2 and the alternative derivation of the Geman-Yor Laplace transform provided in Part 2.2.3 simplify considerably the framework in which the Asian option pricing problem is worked out. The foundation for this simplification lies on the nature of the strong solution of the SDE (2.12) and on the S_t-numeraire measure whose properties turn out to remain valid under more general assumptions, deviations from the lognormal Black-Scholes model, as for example non-flat interest rate, stochastic volatility, etc (see Vecer and Xu [74]). However, it may become difficult to obtain a close-form solution with such extensions of the model. In this part of the chapter, we attempt a generalisation to a multiplicative jump model and show that in some cases, we can still solve for the Asian option price in close-form. This problem has for example been considered by Andreasen [2].

We consider an equity defined by

$$S_t = S_0 e^{\left(r-\lambda(m-1)-\frac{\sigma^2}{2}\right)t+\sigma W_t}\left(\prod_{j=1}^{N_t} Z_j\right) \tag{2.84}$$

where N_t is a Poisson process of intensity λ, $Z_j = (1+U_j)$ independent and identically distributed random variables on $]-1,\infty[$, of mean $m = E(Z_1)$. N_t and the Z_j are assumed independent of the Brownian motion and the density of the variable $\ln Z_1$ is denoted $f^L(\cdot)$. As previously, we denote $Y_T = \int_0^T S_u du$.

Andreasen [2] showed how the diffusion process gets modified under the asset-numeraire for gaussian log-jumps. The following proposition generalises this result.

Proposition 2.2.1. *Under the asset measure Q^S given as before by $\frac{dQ^S}{dQ} = \frac{S_T}{E^Q(S_T)}$, the Poisson process remains one but with the intensity λm and the distribution of the* *log-jumps becomes the Esscher transformed distribution, i.e. under* Q^S*, the* $\ln Z_j$ *have the density* $\frac{f^L(v)e^v}{\int f^L(v)e^v dv}$.

Proof. Defining $\Lambda(t) = e^{-\lambda(m-1)t+\sum_{j=1}^{N_t}\ln Z_j}$, Girsanov's theorem for Poisson processes implies that N is a Poisson process with intensity λm under the measure given from the risk-neutral measure by the Radon-Nykodym derivative $\Lambda(T)$.

$$E_s\left(\prod_{j=N_s+1}^{N_t} Z_j^u \frac{\Lambda(t)}{\Lambda(s)}\right) = e^{\lambda m(t-s)\left(\frac{g(u+1)}{m}-1\right)}$$

$g(-u) = E(Z_1^{-u})$ standing for the MGF of $\ln Z_1$. But, the random variable of density $\frac{f^L(v)e^v}{\int f^L(v)e^v dv}$ has for MGF $\frac{g(u+1)}{m}$. $\square$

As in the Black and Scholes case, we can work with the payoff process $X_t = \frac{K}{S_t} - \frac{Y_t}{TS_t}$ since $E\left(\frac{Y_T}{T} - K\right)^+ = E^{Q^S}(X_T)^-$. This process satisfies a modified linear SDE with jumps under Q^S, SDE whose parameters can be deduced from the results of Proposition 2.2.1. But, as in the Black-Scholes case, we need only its diffusion under Q to work out the Laplace transform of the option

$$dX_t = \left(-\frac{1}{T} - (r-\sigma^2-\lambda m)X_t\right)dt - \sigma dW_t^Q - \frac{Z-1}{Z}dN_t^Q \tag{2.85}$$

As in Section 2.2.3, we deduce the Laplace transform from the MGF of the hitting time $\tau_x = \inf\{t \geq 0,\ X_t < x\}$ for $x = 0$. To this effect, we study

$$\gamma f(x) = (\alpha - \beta x)\frac{\partial f}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} + \lambda\left(\int_0^\infty f(x(1+y)\nu(y)dy - f(x)\right) \tag{2.86}$$

with $\alpha = -\frac{1}{T}$, $\beta = r - \sigma^2 - \lambda m$ and $\nu(\cdot)$ the density of $\tilde{Z} = \frac{1-Z}{Z}$

Using the change of variable $z = \frac{1}{x}$ as in the sheer Black and Scholes setting,

$$\frac{\sigma^2 z^2}{2}g''(z) + z(\sigma^2+\beta z-\alpha)g'(z) - \gamma g(z) + \lambda\left(\int_0^\infty g\left(\frac{z}{1+y}\right)\nu(y)dy - g(z)\right) = 0 \tag{2.87}$$

Assuming the form $g(z) = \sum_{n=0}^{\infty} a_n z^{n+\mu}$, we try the analytic coefficient method and infer an equation for μ from the lower-order terms

$$\frac{\sigma^2}{2}\mu^2 + \left(\frac{\sigma^2}{2} + \beta\right)\mu - (\lambda + \gamma) = -\lambda M_\mu$$

where $M_\mu = \int_0^\infty \left(\frac{1}{1+y}\right)^\mu \nu(y)dy$ represents the generalised moment $E(Z^\mu)$. With $D(\mu) = \frac{\sigma^2}{2}\mu^2 + \left(\frac{\sigma^2}{2} + \beta\right)\mu - (\lambda + \gamma) + \lambda M_\mu$, we notice that $D(0) = -\gamma < 0$ and $D(-\infty) = D(+\infty) = +\infty$ and therefore conclude that there exist two roots for the equation $D(\mu) = 0$, one positive and one negative. We shall call those roots μ^+ and μ^- hereafter. Since we want a solution bounded on $[0, \infty[$, only the solution with μ^+ is left. We redefine $g(z) = \sum_{n=0}^{\infty} a_n z^{n+\mu^+}$ with the coefficients $a_0 = 1$ and

$$a_n = -\frac{\alpha(n + \mu^+ - 1)}{n\mu\sigma^2 + n^2\frac{\sigma^2}{2} + (\beta + \frac{\sigma^2}{2})n + \lambda(M_{n+\mu} - M_\mu)} a_{n-1}$$

To complete the calculations, we would have to find the asymptotics of this solution $g_\infty = \lim_{z \longrightarrow \infty} g(z)$ as when $\mu^+ - 1 > 0$

$$\int_0^\infty e^{-\lambda T} C(S_0, K, T)dT = \frac{1}{g_\infty \lambda} \sum_{n=0}^{\infty} \frac{a_n}{\mu^+ + n - 1} \left(\frac{S_0}{K}\right)^{\mu^+ + n - 1} \tag{2.88}$$

which depend on the jump distribution chosen.

For this whole approach to work, we need to assume smoothness conditions on the jump density and that the MGF solved for is analytic with a very fast-convergent power series representation since the distribution should have moments of any order and the integral and the summation in $\int_0^\infty g\left(\frac{z}{1+y}\right)\nu(y)dy$ should be interchangeable. These conditions should all be checked. We present below an application with a specific jump distribution.

In the case $\tilde{Z}$ is exponentially distributed with intensity ρ, we have to consider the following equation:

$$\gamma f(x) = (\alpha x - \beta)f'(x) + \frac{\sigma^2 x^2}{2} f''(x) + \lambda\left(\int_0^\infty f(xe^v)\rho e^{-\rho v}dv - f(x)\right) = 0$$

This equation can be solved with the methodology described above or with the change of variable $xe^v = u$. Defining $A_0 = \rho\gamma$, $B_1 = \gamma + (\alpha - \sigma^2)(\rho+1) + \lambda$, $B_2 = -\beta(\rho+2)$, $C_2 = \alpha - \left(2 + \frac{\rho}{2}\right)\sigma^2$, $C_3 = C_1 = -\beta$ and $D_3 = -\frac{\sigma^2}{2}$, μ^+ is the positive root of

$$A_0 + B_1\mu + C_2\mu(\mu+1) + D_3\mu(\mu-1)(\mu-2) = 0$$

Although this equation has actually three roots, there is no contradiction with the previously shown result as, since $D_3 < 0$, there is a zero of this equation below $-\rho$ which does not count since moments are not defined for this value. Denoting $-\zeta_1$ and $-\zeta_2$ the two - possibly complex - solutions of the second-order equation

$$B_1 + C_2(\zeta + (2\mu^+ - 1)) + D_3(\zeta^2 + (3\mu^+ - 3)\zeta + 3(\mu^+)^2 - 6\mu^+ + 2) = 0 \qquad (2.89)$$

we obtain

$$\frac{a_n}{a_{n-1}} = -\frac{C_3(n+\mu^+-1)(n+\mu^+-2+\frac{B_2}{C_3})}{nD_3(n+\zeta_1)(n+\zeta_2)}$$

which means that

$$g(z) = z^{\mu^+}\ {}_2F_2\left(\mu^+, \mu^+ - 1 + \frac{B_2}{C_3}; \zeta_1 + 1, \zeta_2 + 1; -\frac{C_3}{D_3}z\right) \qquad (2.90)$$

As g has the representation

$$g(z) = \frac{\Gamma(\mu^+)\int_0^1 y^{-2+\mu^+ + \frac{B_2}{C_3}}(1-y)^{1-\mu^+ - \frac{B_2}{C_3} + \zeta_2} z^{\mu^+}\phi(\mu^+, \zeta_1 + 1, -\frac{C_3}{D_3}zy)dy}{\Gamma(2-\mu^+ - \frac{B_2}{B_3} + \zeta_2)\Gamma(\zeta_1 + 1)}$$

its limit at infinity is

$$\lim_{z\longrightarrow\infty} g(z) = \frac{\Gamma(\mu^+)\left(\frac{C_3}{D_3}\right)^{-\mu^+} B\left(\frac{B_2}{C_3} - 1, \zeta_2\frac{B_2}{C_3}\mu^+\right)}{\Gamma(2-\mu^+ - \frac{B_2}{C_3} + \zeta_2)\Gamma(\zeta_1 + 1 - \mu^+)} \qquad (2.91)$$

This little excursion concludes our study of Asian options on geometric Brownian motion underlyings.

2.3 The square-root process case

In this part, we show that it is possible to obtain simpler or rather easier to compute formulae for the Asian option prices when the underlying asset follows the Cox-Ross [16] square-root process instead of the Black and Scholes geometric Brownian motion. Vanilla options (Cox and Ross [16]) and other exotic options (barrier options in Lo and al. [52]) have been studied and given in explicit form for the specific tractable square-root CEV equity model. But, to the best of our knowledge, no such attempt has been made for Asian options.

In this section, S_t will refer to the square-root equity-spot

$$dS_t = rS_t dt + \sigma\sqrt{S_t}dW_t$$

and, as previously, $Y_t = \int_0^t S_u du$.

After some needed preliminary results, we will analyse the distribution of Y_t by deriving its density and probability distribution function, thus extending the distributional results provided in Section 1.1.3, Chapter 1. We then tackle the valuation of the Asian option in this model.

2.3.1 Preliminary results

The quantity basically needed for valuing the Asian option is the expectation[18] $E(e^{-\lambda Y_T}1_{\{Y_T>K\}})$. This expectation can be determined by double integration of $e^{-\lambda y}1_{\{y>K\}}$ against the joint-density of (S_T, Y_T) or alternatively, by inversion of the corresponding Laplace transform, this last method being the one adopted here.

The first step is then to find the Laplace transform of this expectation.

[18]Indeed, taking $\lambda = 0$ gives the probability of being in-the-money and differentiating with respect to λ at 0 gives $E(Y_T 1_{\{Y_T>K\}})$, the other term appearing in the option valuation.

Theorem 2.3.1. *The Laplace transform of* $E(e^{-\lambda Y_T}1_{\{Y_T>K\}})$ *for* $\lambda \geq 0$ *is*

$$\int_0^\infty e^{-\mu K} E(e^{-\lambda Y_T}1_{\{Y_T>K\}})dK = \frac{E(e^{-\lambda Y_T})}{\mu} - \frac{E(e^{-(\lambda+\mu)Y_T})}{\mu} \tag{2.92}$$

Proof. This can be seen by inverting expectation and integration with respect to K, using a dominated convergence argument. □

Now, given that the first term on the right-part of (2.92) can straightforwardly be inverted into $E(e^{-\lambda Y_T})$, only the second term is left to be inverted. To this effect, we first set some notations and objects.

Definitions and notations. First defining the scaling quantity $\alpha = \frac{\sigma^2}{8}$ and the parameters $D_\xi = -\frac{r}{\sigma^2} - \frac{\xi}{2}$ and $\tilde{C} = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2K\alpha}}e^{\frac{(S_0+(n+1)\sigma^2 T)\xi}{2}}$, we build a multiply-indexed series M from another series $\tilde{O}$ with the following procedure.

- Induction rules
  - For $M_{p,q,n}$,

$$M_{p,q,n}(\xi, K) = \sqrt{\frac{2}{\pi}}\frac{e^{-\xi^2 K\alpha}}{\sqrt{2K\alpha}^{p+q+3}}\tilde{O}_{p,q-1,n}(K) - \xi M_{p,q-1,n}(\xi, K) \tag{2.93}$$

  - For $\tilde{O}_{p,q,n}$,

$$\tilde{O}_{p,q,n}(K) = \left\{ 1_{\{p=0\}}\left[He_{q+2}\left(\frac{S_0 + (n+1)\sigma^2 T}{2\sqrt{2K\alpha}}\right)e^{-\frac{(S_0+(n+1)\sigma^2 T)^2}{16K\alpha}}\right] \right.$$

$$\left. + p\tilde{O}_{p-1,q,n}(K) + \frac{r}{\sigma^2}\tilde{O}_{p,q-1,n}(K) \right\} 2\sqrt{2K\alpha} \tag{2.94}$$

- Initialisations
  - For $M_{p,-p-3,n}$,

$*$ *If* $\xi \neq -\frac{2r}{\sigma^2}$,

$$\begin{aligned} M_{p,-p-3,n}(\xi,K) &= \tfrac{p}{D_\xi}M_{p-1,-p-2,n}(\xi,K) - \tfrac{\tilde{C}}{2D_\xi}\check{O}_{p,n}(D_\xi,K) \\ M_{0,-3,n}(\xi,K) &= \tfrac{\tilde{C}}{2D_\xi}\big(\check{O}_{0,n}(0,K) - \check{O}_{0,n}(D_\xi,K)\big) \end{aligned} \tag{2.95}$$

$*$ *If* $\xi = -\frac{2r}{\sigma^2}$,

$$M_{p,-p-3,n}\bigg(-\frac{2r}{\sigma^2},K\bigg) = \frac{\tilde{C}}{2}\frac{1}{p+1}\check{O}_{p+1,n}(0,K) \tag{2.96}$$

∘ For $\tilde{O}_{p,-p-3,n}$,

$$\begin{aligned} \tilde{O}_{p,-p-3,n}(K) &= -\bigg(S_0 + (n+1)\sigma^2 T - 8K\alpha\frac{r}{\sigma^2}\bigg)\tilde{O}_{p-1,-p-2,n}(K) \\ &+8K\alpha\bigg\{1_{\{p=1\}}e^{-\frac{(S_0+(n+1)\sigma^2T)^2}{16K\alpha}} + (p-1)\tilde{O}_{p-2,-p-1,n}(K)\bigg\} \end{aligned} \tag{2.97}$$

and

$$\tilde{O}_{0,-3,n}(K) = (2\sqrt{\pi K\alpha})\mathrm{erfc}\bigg(\frac{S_0 + (n+1)\sigma^2 T - 8K\alpha\frac{r}{\sigma^2}}{4\sqrt{K\alpha}}\bigg) \tag{2.98}$$

∘ For the additional initialising terms $\check{O}_{p,n}(D_\xi,K)$,

$$\begin{aligned} \check{O}_{p,n}(D_\xi,K) &= -\big(S_0 + (n+1)\sigma^2 T + 4K\alpha(\xi + 2D_\xi)\big)\check{O}_{p-1,n}(D_\xi,K) \\ &+8K\alpha\bigg\{1_{\{p=1\}}e^{-\frac{(S_0+(n+1)\sigma^2T+4K\alpha\xi)^2}{16K\alpha}} + (p-1)\check{O}_{p-2,n}(D_\xi,K)\bigg\} \end{aligned} \tag{2.99}$$

and

$$\check{O}_{0,n}(D_\xi,K) = (2\sqrt{\pi K\alpha})\mathrm{erfc}\bigg(\frac{S_0 + (n+1)\sigma^2 T + 4K\alpha(\xi + 2D_\xi)}{4\sqrt{K\alpha}}\bigg) \tag{2.100}$$

Theorem 2.3.2. *For* $\lambda \in \mathbb{R}^+$, *the inverse Laplace transform of the modified MGF* $\frac{E(e^{-(\lambda+\mu)Y_T})}{\mu}$ *with respect to* μ *is*[19] $\mathrm{MMI}(K,\lambda) + \mathrm{MMI}^{\mathrm{a}}(K,\lambda)$, *where the component due to absorption has been separated in* $\mathrm{MMI}^{\mathrm{a}}(K,\lambda)$. *The non-absorbed part is*

$$\mathrm{MMI}(K,\lambda) = S_0\frac{e^{-\frac{r(2S_0+rK)}{2\sigma^2}}}{2\beta(\lambda)}\sum_{n=0}^{\infty}\frac{m_n(K,\lambda)}{n+1} \tag{2.101}$$

[19] *MMI standing for Modified MGF Inverse*

where $m_n(K, \lambda)$ *is given by*

$$\sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!} \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!} (-S_0)^q \big(M_{p,q,n}(-\beta(\lambda), K) - M_{p,q,n}(\beta(\lambda), K)\big) \tag{2.102}$$

with $\beta(\lambda) = \sqrt{\frac{4r^2}{\sigma^4} + \frac{8\lambda}{\sigma^2}}$.

Proof. See Appendix 2.5.2 to this chapter. □

Now, coming to the case where the spot is null at maturity, i.e. has been absorbed

Theorem 2.3.3. *$\beta(\lambda)$ being defined as in Theorem 2.3.2, the component due to absorption is*

$$\mathrm{MMI}^{a}(K, \lambda) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{\binom{n}{p}}{p!} \frac{(-S_0)^p}{2\beta(\lambda)} \big(G_{p,n}(-\beta, K) - G_{p,n}(\beta, K)$$

$$-G_{p,n+1}(-\beta, K) + G_{p,n+1}(\beta, K)\big) \tag{2.103}$$

with

$$G_{-1,n}(\xi, K) = \left(e^{\frac{(S_0+nT\sigma^2)\xi}{2}}\right) \mathrm{erfc}\left(\frac{S_0 + nT\sigma^2 + 4K\alpha\xi}{4\sqrt{K\alpha}}\right) \tag{2.104}$$

and

$$G_{p,n}(\xi, K) = \sqrt{\frac{2}{\pi}} \frac{e^{-\xi^2 K\alpha}}{\sqrt{2K\alpha}^{p+1}} He_p\left(\frac{S_0 + nT\sigma^2}{2\sqrt{2K\alpha}}\right) e^{-\frac{(S_0+nT\sigma^2)^2}{16K\alpha}}$$

$$-\xi G_{p-1,n}(\xi, K) \tag{2.105}$$

Proof. See Appendix 2.5.3. □

2.3.2 The distribution of Y_T

The calculations done in the previous section give the basis for finding the marginal distribution of Y_T.

The density of Y_T

The marginal density of Y_T can be obtained by integration with respect to s of the joint density given in Theorem 1.1.7, Section 1.1.3, Chapter 1.

Theorem 2.3.4. *The density of Y_T is*

$$f^Y(y) = f_0^Y(y) + \frac{S_0\alpha}{2\sqrt{2\pi}}\frac{1}{(y\alpha)^2}e^{-\frac{r^2 Y_t}{2\sigma^2}-\frac{rS_0}{\sigma^2}}\sum_{n=0}^{\infty}\frac{O_n^y(y)}{n+1} \tag{2.106}$$

with

$$O_n^Y(y) = \sum_{p=0}^{n}\frac{\binom{n+1}{n-p}}{p!}\left(\frac{-1}{\sqrt{2y\alpha}}\right)^p\sum_{q=0}^{n}\frac{\binom{n+1}{n-q}}{q!}\left(\frac{-S_0}{\sqrt{2y\alpha}}\right)^q\tilde{O}_{p,q,n}(y) \tag{2.107}$$

where the $\tilde{O}_{p,q,n}(y)$ are the ones defined in the Theorem 2.3.2 and $f_0^Y(y)$ was defined in Theorem 1.1.8, Section 1.1.3, Chapter 1.

Proof. This comes from the definition of $\tilde{O}_{p,q,n}(y)$ given in Appendix 2.5.2. □

The distribution function of Y_T

The distribution function of Y_T can be straightforwardly deduced from Theorem 2.3.2.

Theorem 2.3.5. *The distribution function of Y_T is given by*

$$P(Y_T \le K) = \mathrm{MMI}(K,0) + \mathrm{MMI}^{\mathrm{a}}(K,0) \tag{2.108}$$

Proof. This comes from Theorem 2.3.2 and Theorem 2.3.1. □

2.3.3 Fixed strike Asian option

Definitions and notations. We need to define two elements $C^{\mathrm{na}}_{\mathrm{sqr}}$ and $C^{\mathrm{a}}_{\mathrm{sqr}}$.

- The non-absorption component $C^{\mathrm{na}}_{\mathrm{sqr}}$

This component can be written as

$$C^{\mathrm{na}}_{\mathrm{sqr}}(S_0, K, T) = \mu_1^{\mathrm{na}} - KP^{\mathrm{na}} + S_0 \frac{e^{-\frac{r(2S_0+rK)}{2\sigma^2}}}{2\beta(\lambda)} \sum_{n=0}^{\infty} \frac{c_n^{\mathrm{na}}(S_0, K, T)}{n+1} \tag{2.109}$$

where

· The probability of non-absorption P^{na} is worth

$$P^{\mathrm{na}} = P(S_T > 0) = 1 - e^{-\frac{S_0 r(1+e^{-rT})}{\sigma^2(1-e^{-rT})}} \tag{2.110}$$

· The truncated moment μ_1^{na} is

$$\mu_1^{\mathrm{na}} = E(Y_T 1_{\{S_T>0\}}) = S_0 \frac{e^{rT}-1}{r} - e^{-\frac{S_0 r(1+e^{-rT})}{\sigma^2(1-e^{-rT})}} \Big(- \frac{S_0}{r} \frac{1+e^{-rT}}{1-e^{-rT}} + T\frac{S_0 e^{-rT}}{1-e^{-rT}} + \frac{S_0 T e^{-rT}(1+e^{-rT})}{(1-e^{-rT})^2} \Big) \tag{2.111}$$

· The terms $c_n^{\mathrm{na}}(S_0, K, T)$ are given by

$$\sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!} \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!} (-S_0)^q \Big(K.M_{p,q,n}\Big(-\frac{2r}{\sigma^2}, K\Big) - K.M_{p,q,n}\Big(\frac{2r}{\sigma^2}, K\Big) + DM_{p,q,n}\Big(-\frac{2r}{\sigma^2}, K\Big) + DM_{p,q,n}\Big(\frac{2r}{\sigma^2}, K\Big)\Big) \tag{2.112}$$

where the $M_{p,q,n}(\xi, K)$ and other notations have been defined in Theorem 2.3.2 and the $DM_{p,q,n}(\xi, K)$ are given by the recursions

$$\begin{cases} DM_{p,q,n}(\xi,K) &= -\frac{4K\alpha}{r} M_{p,q+1,n}(\xi,K) - \frac{2(p+q+3)}{r} M_{p,q-1,n}(\xi,K) \\ & \quad + \big(\frac{S_0+(n+1)\sigma^2 T - 4K\alpha\xi}{r}\big) M_{p,q,n}(\xi,K) + \frac{1}{r} M_{p+1,q-1,n}(\xi,K) \\ DM_{p,-p-3,n}(\xi,K) &= \frac{S_0+(n+1)\sigma^2 T}{r} M_{p,-p-3,n}(\xi,K) \\ & \quad + \frac{1}{r} M_{p+1,-p-4,n}(\xi,K) - \frac{4K\alpha}{r} \tilde{C}\check{O}_{p,n}(D_\xi,K) \end{cases} \tag{2.113}$$

- The absorption component $C^{\mathrm{a}}_{\mathrm{sqr}}$

The absorption component has a similar structure

$$C^{\mathrm{a}}_{\mathrm{sqr}}(S_0, K, T) = \mu^{\mathrm{a}}_1 - KP^{\mathrm{a}} + \frac{1}{2\beta(\lambda)} \sum_{n=0}^{\infty} c^{\mathrm{a}}_n(S_0, K, T) \tag{2.114}$$

with

· The probability of absorption

$$P^{\mathrm{a}} = P(S_T = 0) = e^{-\frac{S_0 r(1+e^{-rT})}{\sigma^2(1-e^{-rT})}} \tag{2.115}$$

· The truncated moment

$$\mu^{\mathrm{a}}_1 = E(Y_T 1_{\{S_T=0\}}) = e^{-\frac{S_0 r(1+e^{-rT})}{\sigma^2(1-e^{-rT})}} \Bigg(-\frac{S_0}{r}\frac{1+e^{-rT}}{1-e^{-rT}}$$

$$+T\frac{S_0 e^{-rT}}{1-e^{-rT}} + \frac{S_0 T e^{-rT}(1+e^{-rT})}{(1-e^{-rT})^2}\Bigg) \tag{2.116}$$

· The series terms

$$c^{\mathrm{a}}_n(S_0, K, T) = \sum_{p=0}^{n} \frac{\binom{n}{p}}{p!}(-S_0)^p \Bigg(G_{p,n}\Big(-\frac{2r}{\sigma^2}, K\Big) - G_{p,n}\Big(\frac{2r}{\sigma^2}, K\Big)$$

$$-G_{p,n+1}\Big(-\frac{2r}{\sigma^2}, K\Big) + G_{p,n+1}\Big(\frac{2r}{\sigma^2}, K\Big) + DG_{p,n}\Big(-\frac{2r}{\sigma^2}, K\Big)$$

$$+DG_{p,n}\Big(\frac{2r}{\sigma^2}, K\Big) - DG_{p,n+1}\Big(-\frac{2r}{\sigma^2}, K\Big) - DG_{p,n+1}\Big(\frac{2r}{\sigma^2}, K\Big)\Bigg) \tag{2.117}$$

where the $DG_{p,n}(\xi, K)$ satisfy the recursions

$$\left\{ \begin{array}{lcl} DG_{p,n}(\xi, K) & = & -G_{p-1,n}(\xi, y) - \xi . DG_{p-1,n}(\xi, K) \\ & & -2K\xi e^{-\xi^2 K\alpha}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2K\alpha}^{p+1}} He_p\big(\frac{S_0+nT\sigma^2}{2\sqrt{2K\alpha}}\big) e^{-\frac{(S_0+nT\sigma^2)^2}{16K\alpha}} \\ DG_{-1,n}(\xi, K) & = & \sqrt{\frac{2}{\pi}} e^{\frac{S_0+nT\sigma^2}{2}\xi} e^{-\frac{(S_0+nT\sigma^2+4K\alpha\xi)^2}{16K\alpha}} \\ & & +\frac{S_0+nT\sigma^2}{2}\big(e^{\frac{S_0+nT\sigma^2}{2}\xi}\big)\mathrm{erfc}\big(\frac{S_0+nT\sigma^2+4K\alpha\xi}{4\sqrt{K\alpha}}\big) \end{array} \right. \tag{2.118}$$

Theorem 2.3.6. *The value of the fixed-strike Asian option $C_{\text{sqr}}(S_0, K, T) = C^{\text{na}}_{\text{sqr}}(S_0, K, T) + C^{\text{a}}_{\text{sqr}}(S_0, K, T)$ for the square-root process is the inverse Laplace transform of*

$$\mathcal{L}C^{(S_0,K,T)}_{\text{sqr}}(\mu) = E\Big(\frac{Y_T}{\mu T} + \frac{1}{\mu^2}(e^{-\mu\frac{Y_T}{T}} - 1)\Big)$$

with respect to μ.

Proof. The Laplace transform comes from

$$\int_0^\infty e^{-\mu K} E\Big(\frac{Y_T}{T} - K\Big)^+ dK = E\Big(\int_0^{\frac{Y_T}{T}} e^{-\mu T}\Big(\frac{Y_T}{T} - K\Big)dK\Big)$$

The proof of the inversion is gathered in Appendix 2.5.4. □

2.3.4 Numerical illustrations

We once again come back to the reference cases of Table 2.1 and adapt the diffusion parameters of the geometric Brownian motion to obtain similar levels of asset, local variance and mean for the square-root process.

Case	r	σ	T	K	S_0	Moment	Intrinsic	Option	N
1	0.05	0.69	1	2	1.9	1.8533	0	0.1902	4
2	0.05	0.72	1	2	2.1	2.0484	0.1459	0.3098	5
3	0.02	0.14	1	2	2	1.9801	0.0197	0.0197	0
4	0.18	0.42	1	2	2	1.8303	0.1598	0.2189	11
5	0.01	0.35	2	2	2	1.9752	0.0246	0.1725	7
6	0.05	0.71	2	2	2	1.9033	0.0936	0.3339	3

Table 2.5: Fixed-strike Asian options.

Table 2.5 collects the moments of $E(\frac{Y_T}{T})$, the intrinsic values and the prices of the options. N represents the number of terms needed for the error to be inferior to 10^{-4}. Two points spring to the eyes from this table. Firstly, the series converges very rapidly for a broad range of cases. Secondly, the prices obtained for the square-root process are quite close to the Black and Scholes Asian ones. The differences between the models are to be computed in basis points rather than in percents, except in cases 3 and 6. The greater differences appearing in these two cases could be accounted for by the relatively greater importance of the tails there as the synthetic parameter $\sigma\sqrt{T}$ becomes very small for case 1 and quite large for case 6.

As in the mean-reverting interest rate case for the marginal distribution (Section 1.1.4 of Chapter 1 and Section 3.1.8 of Chapter 3), the series converges more and more rapidly as T or σ increase. This can be seen from Table 2.6 and Table 2.7. Here, as well, the results were cross-tested against a numerical Laplace inversion. We once again observe that these algorithms get strained and even become unreliable as the volatility or the maturity increase.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.71	0.1	2	2	1.9950	0.0050	0.0751	30
0.05	0.71	0.5	2	2	1.9752	0.0246	0.1725	7
0.05	0.71	1	2	2	1.9508	0.0484	0.2468	5
0.05	0.71	2	2	2	1.9033	0.0936	0.3339	3
0.05	0.71	5	2	2	1.7696	0.2120	0.3733	2

Table 2.6: Evolution with the maturity.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.1	1	2	2	1.9508	0.0484	0.0484	0
0.05	0.3	1	2	2	1.9508	0.0484	0.1207	18
0.05	0.5	1	2	2	1.9508	0.0484	0.1827	7
0.05	0.7	1	2	2	1.9508	0.0484	0.2446	5

Table 2.7: Evolution with the volatility.

Table 2.5 highlighted a strange or rather unexpected behaviour of N with respect to the volatility. The analysis of Table 2.7 enables us to explain it. The series converges slowly for small σ. But for too tiny σ, as nothing can really happen for such values, the put option is almost worthless and the series converges in one term.

We finally analyse the evolution with respect to the strike K. Table 2.8 shows results in agreement with our observations concerning the series representing the density in Section 1.1.3, Chapter 1. In fact, all the results obtained here corroborate the observations and comments made in that section concerning the effect of the different parameters on the speed of convergence. This is due to the fact that the leading term remains the same.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.71	1	1	2	1.9508	0.9996	1.0017	2
0.05	0.71	1	1.5	2	1.9508	0.5240	0.5644	4
0.05	0.71	1	2	2	1.9508	0.0484	0.2468	5
0.05	0.71	1	2.5	2	1.9508	-0.4273	0.0822	6
0.05	0.71	1	3	2	1.9508	-0.9029	0.0210	8

Table 2.8: Evolution with the strike.

2.4 Conclusion

In this chapter, we studied Asian derivatives on geometric Brownian motion and square-root process underlying assets. For the geometric Brownian motion, after having presented the different possibilities to modelise the valuation as one-dimensional Markov problem, we derive explicit exact integral expressions and approximate series form for fixed and floating strike seasoned and unseasoned options. We tested the numerical implementation of these formulae against benchmark results in the literature.

We also considered the alternative model constituted by the square-root process and applied the methods and results proposed in the previous chapter to construct explicit series solutions for Asian options in this model. These formulae turn out to be simpler than for the Black and Scholes model. We also analysed the performance of the numerical evaluation of this series and found it very rapid in general but also more specially for large volatilities and maturities. This hence provides a model, interesting on its own but also as a benchmark to test numerics in the Black and Scholes model.

2.5 Appendices to this chapter

2.5.1 Call option on an absolute-dividend paying stock: Proof of Theorem 2.2.5

Using no-arbitrage argument, the call price function follows the second-order partial differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (rS - D)\frac{\partial C}{\partial S} - rC = \frac{\partial C}{\partial \tau} \tag{2.119}$$

where $\tau = T - t$ represents the time to maturity.

Defining x as $\frac{\gamma}{S}$ and $\tilde{t}$ as $\frac{\sigma^2\tau}{2}$ and using otherwise the same notations as in Theorem 2.2.5 , the function $c(x, \tilde{t}) = C(S, \tau)$ has a Laplace transform $\tilde{c}(s)$ with respect to time which satisfies the second-order ordinary differential equation

$$x^2 \frac{\partial^2 \tilde{c}}{\partial x^2} + (x^2 + (2-\beta)x)\frac{\partial \tilde{c}}{\partial x} - \beta\tilde{c} = s\tilde{c} - c_0$$

where $c_0(x) = c(x, 0)$. As also explained in (2.2.9), the solution of this inhomogeneous equation can be formed as a Green function from η and ξ two solutions of the corresponding homogeneous equation

$$\tilde{c}(x, \lambda) = -\eta(x, \lambda) \int_0^x \rho(y)\xi(y, \lambda)c_0(y)dy - \xi(x, \lambda) \int_x^\infty \rho(y)\eta(y, \lambda)c_0(y)dy \tag{2.120}$$

with $\rho(y) = \frac{1}{y^2 W_{\eta,\xi}(y)}$ where $W_{\eta,\xi}(y)$ is the Wronskian of the two solutions and the new variable $\lambda = -(s+\frac{(\beta+1)^2}{4}) = -(s+\delta)$ has been introduced for convenience of notation. Defining the square-root as the branch with positive imaginary part, Lewis shows that the functions verifying integrability conditions are $\xi(x,\lambda) = e^{(i\pi(1-c)-x)}x^{1+a-c}\phi(1-a, 2-c, x)$ and $\eta(x,\lambda) = e^{-x}x^a\psi(c-a,c,x)$, where $a = \frac{\beta-1}{2} + i\sqrt{\lambda}$ and $c = 1+2i\sqrt{\lambda}$ and that their Wronskian is $W_{\eta,\xi}(y) = \frac{\Gamma(2-c)}{\Gamma(1-a)}x^{2a-c}e^{(-i\pi c-x)}$, leading to (noting $I_1(x,\lambda)$ the first integral appearing on the right side of (2.120) and $I_2(x,\lambda)$ the second)

$$I_1(x,\lambda) = \omega(\lambda)\Bigg\{ -1_{\{k\le x\}}\left(\frac{k^{-a-1}}{a(a+1)}\phi(-a-1,2-c,k)\right)$$
$$-1_{\{k>x\}}\left(\frac{x^{-a-1}}{a(a+1)}\phi(-a-1,2-c,x) - \frac{(x^{-a-1}-x^{-a}k^{-1})}{a}\phi(-a,2-c,x)\right)\Bigg\}$$

and

$$I_2(x,\lambda) = 1_{\{k>x\}}\omega(\lambda)e^{i\pi c}\Bigg\{\frac{(x^{c-a-2}-x^{c-a-1}k^{-1})}{a(c-a-1)}\psi(c-a-1,c,x)$$
$$+\frac{k^{c-a-2}\psi(c-a-2,c,k) - x^{c-a-2}\psi(c-a-2,c,x)}{a(a+1)(c-a-1)(c-a-2)}\Bigg\}$$

where $\omega(\lambda) = \gamma\frac{\Gamma(1-a)}{\Gamma(2-c)}$ and $k = \frac{\gamma}{K}$.

As explained by Lewis, the final payoff is not ρ-square-integrable, which leads to the transform having a few more simple poles than what the classical eigenfunction expansion theory would imply. However, this function is still having only a finite number of simple poles and a branch cut line along the positive half-axis, coming from the branch choice made for the square-root function. The following residues and branch cut integrals computed by Lewis are still valid in our case

<u>Residue at $\lambda = -\frac{(\beta-1)^2}{4}$</u>: $1_{\{\beta>1\}}\left(\frac{D}{r}-K\right)\left[1-\frac{(\frac{\gamma}{S_0})^{\beta-1}}{\Gamma(\beta)}\phi(\beta-1,\beta,\frac{-\gamma}{S_0})\right]e^{-rT}$

<u>Branch cut integral</u>: $S_0^{\frac{1-\beta}{2}}K^{\frac{1+\beta}{2}}e^{-\frac{\gamma}{S_0}}\int_0^\infty \xi_\mu(S_0,K)e^{-\frac{\sigma^2}{2}T(\delta+\mu^2)}d\mu$

The following residues are changed compared to Lewis's work:

Residue at $\lambda = -\frac{(\beta+1)^2}{4}$:

When $\beta \geq -1$, then $a = -1$ and $c = -\beta$. This is a pole of both I_1 and I_2. I_2 contributes to nothing since $\psi(c-a-2, c, z) = \psi(-\beta-1, -\beta, z) = z^{\beta+1}$. After calculation and reexpressing ψ function in terms of ϕ, the residue is then $\gamma\left\{\frac{1}{x} - \frac{1}{\beta} + \frac{x^\beta}{\beta\Gamma(2+\beta)}\phi(\beta, \beta+2, -x)\right\}$.

When $\beta < -1$, then $a = \beta$ and $c = 2+\beta$. This is only a pole of I_2. But, once more, given that $\psi(c-a-2, c, z) = \psi(0, 2+\beta, z) = 1$, the contribution of I_2 is nil.

Therefore, the overall residue is $1_{\{\beta\geq -1\}}\gamma\left\{\frac{1}{x} - \frac{1}{\beta} + \frac{x^\beta}{\beta\Gamma(2+\beta)}\phi(\beta, \beta+2, -x)\right\}$.

Residue at the poles of $\Gamma(1-a)$:

They appear when $1-a$ is a negative integer $-m$, i.e. when $i\sqrt{\lambda} = n + \frac{3-\beta}{2}$ for $0 \leq m \leq \frac{\beta-3}{2}$. Then, using the relation $\psi(\alpha, \gamma, z) = z^{1-c}\psi(\alpha-\gamma+1, 2-\gamma, z)$, expressing ψ in terms of ϕ gives for the residue of $\tilde{c}$ (after some manipulations)

$$\frac{(-1)^m}{(m+2)!}\frac{\gamma e^{-x}x^\beta}{\Gamma(c_m-1)}(xk)^{-n-2}(-c_m)_m\phi(-n, c_m, x)\phi(-n-2, c_m, k)$$

Using the relation with Laguerre polynomials, $\phi(-m, \alpha, z)(\alpha)_m = m!L_m^{\alpha-1}(z)$, the contributions of these poles (noticing these poles exist only if $\beta > 3$) is

$$1_{\{\beta\geq 3\}}\gamma\left(\frac{\gamma}{S_0}\right)^\beta e^{-\frac{\gamma}{S_0}}\sum_{m=0}^{[\frac{\beta-3}{2}]}\xi_m(S_0, K)e^{-\frac{\sigma^2}{2}T(\delta+\lambda_m)}$$

2.5.2 Square-root Asian option: proof of the preliminary Theorem 2.3.2

The following conventions will be used for this appendix and all the appendices linked to Section 2.3: all the Laplace transforms will have μ as argument and all inverse transforms will have y (for the value of Y_T) as argument.

Coming to the Theorem 2.3.2 now, we need to invert $\frac{E(e^{-(\lambda+\mu)Y_T})}{\mu}$. From the result of Section 1.1.3, Chapter 1, for the non-absorbed part, this amounts to invert

$$e^{-\frac{r^2 y}{2\sigma^2}-\frac{r(S_0-s)}{\sigma^2}} \frac{S_0}{\mu} \frac{4\gamma^2 e^{-\gamma T}}{\sigma^4} e^{-(s+S_0)\frac{\gamma}{\sigma^2}} \sum_{n=0}^{\infty} \frac{L_n^1\left(\frac{2s\gamma}{\sigma^2}\right) L_n^1\left(\frac{2S_0\gamma}{\sigma^2}\right)}{n+1} e^{-\gamma n t} \tag{2.121}$$

with

$$\gamma = \sqrt{r^2 + 2\sigma^2(\lambda+\mu)}$$

and to integrate this resulting inverse with respect to s on $\mathbb{R}^+$.

Denoting as in (1.40), Section 1.1.3, Chapter 1, $\alpha_n = \frac{s+S_0+(n+1)\sigma^2 T}{2}$ and starting with the observation that

$$\left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{\mu} = \frac{8}{\sigma^2}\left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{\left(\frac{2\gamma}{\sigma^2}-\beta(\lambda)\right)\left(\frac{2\gamma}{\sigma^2}+\beta(\lambda)\right)}$$

$$= \frac{8}{\sigma^2}\left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{2\beta(\lambda)} \left(\frac{1}{\frac{2\gamma}{\sigma^2}-\beta(\lambda)} - \frac{1}{\frac{2\gamma}{\sigma^2}+\beta(\lambda)}\right)$$

it appears that only inverse Laplace transforms of terms of the type

$$\left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{\frac{2\gamma}{\sigma^2}+\xi} \tag{2.122}$$

are needed for the inversion of (2.121).

Specialising to the case $\xi^2 = (\pm\beta(\lambda))^2 = \frac{4r^2}{\sigma^4} + \frac{8\lambda}{\sigma^2}$ and given that the inverse of $(\sqrt{\mu})^n e^{-\nu\sqrt{\mu}}$ is $\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2y}^{n+2}} He_{n+1}(\frac{\nu}{\sqrt{2y}}) e^{-\frac{\nu^2}{4y}}$, then it follows that the inverse of (2.122) is

$$\int_0^{\infty} e^{-\nu\xi} \sqrt{\frac{2}{\pi}} \frac{\alpha}{(\sqrt{2y\alpha})^{p+q+4}} e^{-\xi^2 y\alpha} He_{p+q+3}\left(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\right) e^{-\frac{(\alpha_n+\nu)^2}{4y\alpha}} d\nu \tag{2.123}$$

This result follows from the scaling and shifting properties of Laplace transform[20] $\mathcal{L}^{-1}_{[\sqrt{u+v\mu}]}(y) = \frac{e^{-\frac{uy}{v}}}{v} \mathcal{L}^{-1}_{[\sqrt{\mu}]}\left(\frac{y}{v}\right)$ and from the Fubini-Tonelli theorem allowing to change

[20]Here, the round brackets contain the argument of the inverse transform $\mathcal{L}^{-1}$ while the square-brackets in the indices contain the argument of the Laplace transform $\mathcal{L}$

the order of integration after taking the Laplace transform in (2.123) and noting that $\int_0^\infty \left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} e^{-\alpha_n \frac{2\gamma}{\sigma^2}} e^{-\nu\left(\frac{2\gamma}{\sigma^2}+\xi\right)} d\nu = \left(\frac{2\gamma}{\sigma^2}\right)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{\frac{2\gamma}{\sigma^2}+\xi}$

The expression (2.123) will subsequently be named $G_{p,q,n}(\xi, y, s)$. Integrating it by part leads to

$$G_{p,q,n}(\xi,y,s) = e^{-\xi^2 y\alpha}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2y\alpha}^{p+q+3}}\left\{He_{p+q+2}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\alpha_n^2}{4y\alpha}}\right.$$
$$\left.-\int_0^\infty He_{p+q+2}\left(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\right)e^{-\frac{(\alpha_n+\nu)^2}{4y\alpha}}\xi e^{-\nu\xi}d\nu\right\} \tag{2.124}$$

The inverse can hence be written

$$G_{p,q,n}(\xi,y,s) = e^{-\xi^2 y\alpha}\sqrt{\frac{2}{\pi}}\alpha\left\{\sum_{r=0}^{p+q+2}\frac{(-1)^r\xi^r}{\sqrt{2y\alpha}^{p+q+3-r}}He_{p+q+2-r}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\alpha_n^2}{4y\alpha}}\right.$$
$$\left.+(-1)^{p+q+3}\int_0^\infty\frac{e^{-\frac{(\alpha_n+\nu)^2}{4y\alpha}}}{\sqrt{2y\alpha}}\xi^{p+q+3}e^{-\nu\xi}d\nu\right\}$$

where the last integral is simply an exponential multiplied by a complementary error function erfc.

Recalling now that this inverse has to be integrated with respect to s, the following notation is introduced

$$M'_{p,q,n}(\xi,y) = \int_0^\infty s^p G_{p,q,n}(\xi,y,s)e^{-\chi s}dx \tag{2.125}$$

where $\chi = -\frac{r}{\sigma^2}$.

The relation (2.124) implies

$$M'_{p,q,n}(\xi,y) = \sqrt{\frac{2}{\pi}}\frac{\alpha e^{-\xi^2 y\alpha}}{\sqrt{2y\alpha}^{p+q+3}}\int_0^\infty s^p He_{p+q+2}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\alpha_n^2}{4y\alpha}-\chi s}dx$$
$$-\xi M'_{p,q-1,n}(\xi,y) \tag{2.126}$$

Noting $\tilde{O}_{p,q,n}(y) = \int_0^\infty s^p He_{p+q+3}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\alpha_n^2}{4y\alpha}-\chi s}dx$, integration by part leads to

$$\tilde{O}_{p,q,n}(y) = -\left[s^p e^{-\chi s}2\sqrt{2y\alpha}He_{p+q+2}\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\alpha_n^2}{4y\alpha}}\right]_0^\infty$$

$$+\int_0^\infty e^{-\chi s}(px^{p-1}-\chi s^p)2\sqrt{2y\alpha}He_{p+q+2}\Big(\frac{\alpha_n}{\sqrt{2y\alpha}}\Big)e^{-\frac{\alpha_n^2}{4y\alpha}}dx$$

which implies

$$\tilde{O}_{p,q,n}(y)=\Big\{1_{\{p=0\}}He_{q+2}\Big(\frac{\tilde{\alpha}_n}{\sqrt{2y\alpha}}\Big)e^{-\frac{\tilde{\alpha}_n^2}{4y\alpha}}+p\tilde{O}_{p-1,q,n}(y)-\chi\tilde{O}_{p,q-1,n}(y)\Big\}2\sqrt{2y\alpha}$$

where $\tilde{\alpha}_n=\frac{S_0+(n+1)\sigma^2T}{2}$.

Therefore, the $\tilde{O}$ can be recursively computed once the $\tilde{O}_{p,-p-3,n}(y)$ are known. But, $\tilde{O}_{p,-p-3,n}(y)$ belongs to a wider class that proves convenient to be defined here

$$O_k(A,B)=\int_0^\infty s^k e^{-\frac{(Ax+B)^2}{4y\alpha}}dx$$

where A is a strictly positive real and B a real. This class satisfies the following recursive relation

$$O_k(A,B)=-\frac{B}{A}O_{k-1}(A,B)+\Big(\frac{2y\alpha}{A^2}\Big)\Big\{1_{\{k=1\}}e^{-\frac{B^2}{4y\alpha}}$$
$$+(k-1)O_{k-2}(A,B)\Big\}$$

which comes from

$$s^k e^{-\frac{(Ax+B)^2}{4y\alpha}}=\Big(-\frac{2y\alpha}{A^2}\Big)\Big[-s^{k-1}\frac{A}{2y\alpha}(Ax+B)e^{-\frac{(Ax+B)^2}{4y\alpha}}\Big]-\frac{B}{A}s^{k-1}e^{-\frac{(Ax+B)^2}{4y\alpha}}$$

and integration by part of the first resulting integral. The Os can be therefore computed through recursion from the first term

$$O_0(A,B)=\int_0^\infty e^{-\frac{(Ax+B)^2}{4y\alpha}}dx=\frac{\sqrt{\pi y\alpha}}{A}\text{erfc}\Big(\frac{B}{2\sqrt{y\alpha}}\Big)$$

Coming back to $\tilde{O}$, they are related to O through

$$\tilde{O}_{p,-p-3,n}(y)=e^{4y\alpha\chi^2+2\tilde{\alpha}_n\chi}O_p\Big(\frac{1}{2},\tilde{\alpha}_n+4y\alpha\chi\Big)$$

Now the $\tilde{O}$ are explicitly given, the initialising $M_{p,-p-3,n}(\xi, y)$ are still left to be computed in order to obtain $M_{p,q,n}(\xi, y)$ from (2.93)

$$M'_{p,-p-3,n}(\xi, y) = \int_0^\infty s^p e^{-\chi s} \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-\nu\xi} \frac{\alpha}{\sqrt{2y\alpha}} e^{-\xi^2 y\alpha} e^{-\frac{(\nu+\alpha_n)^2}{4y\alpha}} d\nu dx$$
$$= \tilde{C'} \int_0^\infty s^p e^{-xD_\xi} \int_{\frac{s}{2}+\tilde{\alpha}_n+2y\alpha\xi}^\infty e^{-\frac{h^2}{4y\alpha}} dh dx \tag{2.127}$$

where $\tilde{C'} = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\sqrt{2y\alpha}} e^{\xi\tilde{\alpha}_n}$ and $D_\xi = \chi - \frac{\xi}{2}$. Since $\int_g^\infty e^{-\frac{h^2}{4y\alpha}} dh = \mathbf{o}\Big(e^{-\frac{g^2}{4y\alpha}}\Big)$, if $D_\xi \neq 0$ and $p > 0$,

$$M'_{p,-p-3,n}(\xi, y) = \tilde{C'} \Bigg\{ \frac{p!}{D_\xi^{p+1}} \int_{\tilde{\alpha}_n+2y\alpha\xi}^\infty e^{-\frac{h^2}{4y\alpha}} dh$$
$$-\frac{1}{2} \int_0^\infty \left(\sum_{k=0}^{p} \frac{p!}{(p-k)!} \frac{s^{p-k}}{D_\xi^{k+1}} \right) e^{-D_\xi s - \frac{(\alpha_n+2y\alpha\xi)^2}{4y\alpha}} dx \Bigg\} \tag{2.128}$$

Noting $\check{O}_{k,n}(D_\xi, y) = \int_0^\infty s^k e^{-D_\xi s - \frac{(\alpha_n+2y\alpha\xi)^2}{4y\alpha}} dx$,

$$M'_{p,-p-3,n}(\xi, y) = \tilde{C'} \Bigg\{ \frac{p}{D_\xi} \frac{(p-1)!}{D_\xi^p} \int_{\tilde{\alpha}_n+2y\alpha\xi}^\infty e^{-\frac{h^2}{4y\alpha}} dh$$
$$-\frac{p}{2D} \sum_{k=-1}^{p-1} \frac{(p-1)!}{((p-1-k)!)} \frac{1}{D_\xi^{k+1}} \check{O}_{p-1-k,n}(D_\xi, y) \Bigg\}$$

which leads to

$$M'_{p,-p-3,n}(\xi, y) = \frac{p}{D_\xi} M'_{p-1,-p-2,n}(\xi, y) - \frac{\tilde{C'}}{2D_\xi} \check{O}_{p,n}(D_\xi, y)$$

For the case $p = 0$, integration by part gives

$$M'_{0,-3,n}(\xi, y) = \frac{\tilde{C'}}{2D_\xi} (\check{O}_{0,n}(0, y) - \check{O}_{0,n}(D_\xi, y))$$

Now, if $D_\xi = 0$, i.e $\chi = \frac{\xi}{2}$, by integration by part

$$M'_{p,-p-3,n}(\xi, y) = \frac{\tilde{C'}}{2} \frac{1}{p+1} \check{O}_{p+1,n}(0, y)$$

The $\check{O}$ can be computed through

$$\check{O}_{k,n}(D_\xi, y) = e^{4y\alpha D_\xi^2 + 2D_\xi(\tilde{\alpha}_n+2y\alpha\xi)} O\Big(k, \frac{1}{2}, \tilde{\alpha}_n + 2y\alpha\xi + 4y\alpha D_\xi\Big)$$

which completes the calculation.

2.5.3 Square-root Asian option: proof of the preliminary Theorem 2.3.3 for absorption

The absorption case is treated in the same way as the non-absorption. From (1.45), Section 1.1.3, Chapter 1, the result is the inverse of

$$\frac{1}{\mu}e^{-S_0\frac{\gamma}{\sigma^2}}(1-e^{-\gamma t})\sum_{n=0}^{\infty}L_n(\frac{2S_0\gamma}{\sigma^2})e^{-\gamma nt}$$

Denoting $\beta_n=\frac{S_0+nt\sigma^2}{2}$, the result is built, as previously, from the inverse of

$$\left(\frac{2\gamma}{\sigma^2}\right)^p\frac{e^{-\beta_n\frac{2\gamma}{\sigma^2}}}{\frac{2\gamma}{\sigma^2}+\xi}$$

which is

$$G'_{p,n}(\xi,y)=\int_0^\infty e^{-\nu\xi}\sqrt{\frac{2}{\pi}}\frac{\alpha e^{-\xi^2 y\alpha}}{(\sqrt{2y\alpha})^{p+2}}He_{p+1}\left(\frac{\beta_n+\nu}{\sqrt{2y\alpha}}\right)e^{-\frac{(\beta_n+\nu)^2}{4y\alpha}}d\nu$$

Similarly to (2.124), the $G'_{p,n}(\xi,y)$ satisfy

$$G'_{p,n}(\xi,y)=e^{-\xi^2y\alpha}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2y\alpha}^{p+1}}\left\{He_p\left(\frac{\beta_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\beta_n^2}{4y\alpha}}\right.$$
$$\left.-\int_0^\infty He_p\left(\frac{\beta_n+\nu}{\sqrt{2y\alpha}}\right)e^{-\frac{(\beta_n+\nu)^2}{4y\alpha}}\xi e^{-\nu\xi}d\nu\right\} \quad (2.129)$$

The initial value is

$$G'_{-1,n}(\xi,y)=\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{\alpha e^{-\xi^2y\alpha}}{\sqrt{2y\alpha}}e^{-\frac{(\beta_n+\nu)^2}{4y\alpha}}e^{-\nu\xi}d\nu$$
$$=(\alpha e^{\beta_n\xi})\mathrm{erfc}\left(\frac{\beta_n+2y\alpha\xi}{2\sqrt{y\alpha}}\right) \quad (2.130)$$

Hence the result.

2.5.4 Fixed strike Asian option: proof of Theorem 2.3.6

Given that we have already solved for the probability to be in the money at maturity, we only need to compute $E(Y_T 1_{\{Y_T>K\}})$. Therefore, we need to differentiate the expressions given in Theorem 2.3.2 and Theorem 2.3.3 with respect to λ at 0, which means differentiating with respect to ξ at $\pm\frac{2r}{\sigma^2}$, then multiplying the result by $\frac{d\beta(\lambda)}{d\lambda}|_{\lambda=0} = -\frac{2}{r}$.

Recalling from (2.125) the following expression for $M_{p,q,n}(\xi, y)$

$$\int_0^\infty s^p e^{-\chi s} \int_0^\infty e^{-\nu\xi} \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{(\alpha_n+\nu)^2}{4y\alpha}}}{(\sqrt{2y\alpha})^{p+q+4}} e^{-\xi^2 y\alpha} He_{p+q+3}\Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big) d\nu dx$$

where we use the same notations as in Appendix 2.5.2, we can differentiate under the integral sign, which gives

$$\int_0^\infty \int_0^\infty \frac{s^p e^{-\chi s-\nu\xi-\frac{(\alpha_n+\nu)^2}{4y\alpha}-\xi^2 y\alpha} He_{p+q+3}\big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\big)}{\sqrt{\pi y\alpha}(\sqrt{2y\alpha})^{p+q+3}}(-2y\alpha\xi-\nu)d\nu dx \qquad (2.131)$$

For $p+q+3>0$, using

$$He_{p+q+4}\Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big) = \Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big) He_{p+q+3}\Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big)$$
$$-(p+q+3)He_{p+q+2}\Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big)$$

we are lead to

$$\frac{\partial M_\xi}{\partial\xi}(p,q,n,y) = -2y\alpha\xi M_{p,q,n}(\xi,y) - 2y\alpha M_{p,q+1,n}(\xi,y)$$
$$+\frac{1}{2}M_{p+1,q-1,n}(\xi,y) + \tilde{\alpha}_n M_{p,q,n}(\xi,y) - (p+q+3)M_{p,q-1,n}(\xi,y)$$

recalling $\tilde{\alpha}_n = \frac{S_0+(n+1)\sigma^2 T}{2}$.

Now, for $p=-q-3$,

$$\frac{\partial M_{p,-p-3,n}}{\partial\xi}(\xi,y) = \tilde{\alpha}_n M_{p,-p-3,n}(\xi,y) + \frac{1}{2}M_{p+1,-p-4,n}(\xi,y)$$

$$-2y\alpha\tilde{C}(\check{O}_{p,n}(D))$$

which can be seen from manipulating (2.131) or differentiating (2.127).

These recursions enable us to compute the derivative for the non-absorption case. For the absorption part, we obtain by differentiation of (2.129) and (2.130)

$$\frac{\partial G'_{p,n}}{\partial \xi}(\xi, y) = -2y\alpha\xi e^{-\xi^2 y\alpha}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2y\alpha}^{p+1}}He_p\left(\frac{\beta_n}{\sqrt{2y\alpha}}\right)e^{-\frac{\beta_n^2}{4y\alpha}}$$

$$-G'_{p,n-1}(\xi, y) - \xi\frac{\partial G'_{p,n-1}}{\partial \xi}(\xi, y)$$

and

$$\frac{\partial G'_{-1,n}}{\partial \xi}(\xi, y) = \beta_n(\alpha e^{\beta_n \xi})\text{erfc}\left(\frac{\beta_n + 2y\alpha\xi}{2\sqrt{y\alpha}}\right) + \sqrt{\frac{2}{\pi}}\alpha e^{\beta_n\xi}e^{-\frac{(\beta_n+2y\alpha\xi)^2}{4y\alpha}}$$

Chapter 3

Interest rate derivatives

The concept of interest rate belongs to our every-day life. That lending money must be rewarded and that a certain amount of money possessed today will not be worth the same tomorrow is common knowledge and wisdom although an abstract formulation of this can be quite complex. Fixed-income theory thus constitutes a major area of mathematical finance. Moreover, the properties of positivity and mean-reversion of the CEV processes render them natural driving diffusions for interest rates and hence a valuable application field for us.

The square-root process of Cox Ingersoll and Ross [15] has played a considerable role in fixed-income theory and practice for its tractability and positivity make it one of the most popular one-factor model. A number of important quantities and instruments are though not known explicitly, but only in closed-form. This chapter mainly focuses on the use of the distributional results derived and methodologies proposed in Section 1.1, Chapter 1 to obtain explicit solutions for some such derivatives. The very last part of the chapter deals with the general mean-reverting CEV instantaneous rate model for which we propose a closed-form for the zero-coupon bonds under some parameter restrictions.

3.1 The Cox Ingersoll Ross model

3.1.1 Introduction

The theory of interest rate modelling was originally based on the construction of a one-dimensional diffusion for the instantaneous short rate process. All the pertaining quantities could then be derived by no-arbitrage arguments as the expectation of functionals of this short rate under the risk-neutral measure. This approach has been pioneered by Vasicek [72], who proposed a mean-reverting gaussian Ornstein-Uhlenbeck for the short rate.

Cox, Ingersoll and Ross [15] later developed a general equilibrium approach, modelling the short rate as a square-root process. Belonging to the class of time-homogeneous endogenous processes initially employed to represent the short rate, the CIR model has been a benchmark for many years because of its allying both strictly positive interest rates - unlike the Vasicek short model - and a relative analytical tractability, unlike many other positive rate models.

i. The model

Under the risk-neutral measure, the short rate is assumed to follow

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t \tag{3.1}$$

Proposition 3.1.1. *(See Cox Ingersoll and Ross [15]). Under this model, the price at time t of a zero-coupon bond maturing at time T is*

$$P(t,T) = \left(\frac{2\gamma e^{\frac{(b-\gamma)(T-t)}{2}}}{(\gamma+b)+e^{-\gamma(T-t)}(\gamma-b)}\right)^{\frac{2a}{\sigma^2}} e^{-r_t\left(\frac{2\mu(1-e^{-\gamma(T-t)})}{(\gamma+b)+e^{-\gamma(T-t)}(\gamma-b)}\right)} \tag{3.2}$$

Proof. This comes from setting $\lambda = 0$ and $\mu = 1$ in the joint moment generating function given in Proposition 1.1.4, Section 1.1.1, Chapter 1. □

ii. Forward measure and options on zero-coupon bonds

Thanks to the use of the forward measure, it is possible to obtain an analytical expression for the price of options on zero-coupon bonds.

Proposition 3.1.2. *(See Lamberton and Lapeyre [46]). Under the forward measure Q^T defined by*

$$\frac{dQ^T}{dQ} = \frac{e^{-\int_0^T r_t dt}}{P(0,T)} \tag{3.3}$$

the short rate retains a square-root structure, but with modified parameters

$$dr_t = (a - (b + \sigma^2 \Upsilon(T)) r_t) dt + \sigma \sqrt{r_t} dW_t^T \tag{3.4}$$

where the function $\Upsilon(t)$ is the one defined in Proposition 1.1.4, Section 1.1.1, Chapter 1

$$\Upsilon(t) = \frac{\lambda((\gamma - b) + e^{-\gamma t}(\gamma + b)) + 2\mu(1 - e^{-\gamma t})}{\sigma^2 \lambda (1 - e^{-\gamma t}) + (\gamma + b) + e^{-\gamma t}(\gamma - b)} \tag{3.5}$$

Proof. See, for example, Lamberton and Lapeyre [46]. □

This forward measure, along with the distributional results concerning r_t - see Chapter 1 -, enables us to formulate explicitly the price of any derivative whose payoff depends solely on r_T. Options on zero-coupon bonds are an important example of such derivatives.

Proposition 3.1.3. *(See Lamberton and Lapeyre [46]) The price of a call maturing at θ on a zero-coupon bond, $P(\theta, T)$ maturing at T, is*

$$E\left(e^{-Y_t}(P(\theta,T) - K)^+\right) = P(0,T) F_{\frac{4a}{\sigma^2},\zeta_1}\left(\frac{r^*}{L_1}\right) - K P(0,\theta) F_{\frac{4a}{\sigma^2},\zeta_2}\left(\frac{r^*}{L_2}\right) \tag{3.6}$$

where

$$\begin{aligned}\zeta_1 &= \frac{8r_0\gamma^* e^{\gamma^*\theta}}{\sigma^2(e^{\gamma^*\theta}-1)(\gamma^*(e^{\gamma^*\theta}+1)+(\sigma^2\Upsilon(T-\theta)+b)(e^{\gamma^*\theta}-1))} \\ \zeta_2 &= \frac{8r_0{\gamma^*}^2 e^{\gamma^*\theta}}{\sigma^2(e^{\gamma^*\theta}-1)(\gamma^*(e^{\gamma^*\theta}+1)+b(e^{\gamma^*\theta}-1)}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}L_1 &= \frac{\sigma^2}{2}\frac{e^{\gamma^*\theta}-1}{\gamma^*(e^{\gamma^*\theta}+1)+(\sigma^2\Upsilon(T-\theta)+b)(e^{\gamma^*\theta}-1)} \\ L_2 &= \frac{\sigma^2}{2}\frac{e^{\gamma^*\theta}-1}{\gamma^*(e^{\gamma^*\theta}+1)+b(e^{\gamma^*\theta}-1)}\end{aligned} \tag{3.8}$$

with the notations

$$\begin{aligned}r^* &= -\frac{a\Theta(T-\theta)+\ln K}{\Upsilon(T-\theta)} \\ \gamma^* &= \sqrt{b^2+2\sigma^2}\end{aligned} \tag{3.9}$$

Remark. F represents here the non-central chi-square probability distribution function, which can be computed, for example by integration of the power series density given in Chapter 1. This distribution has actually been acutely studied and a few algorithms are available to compute it.

iii. A simple extension

As mentioned above, the CIR model belongs to the important classical but early interest rate models. One of its major drawbacks is that it is endogenous, i.e. the term-structure is an output rather than an input and we can only hope to get as close as possible to a specific term-structure rather than fitting it perfectly. A possible simple extension which improves this point is the following model

$$\begin{cases} dX_t &= (a-bX_t)dt+\sigma\sqrt{X_t}dW_t \\ r_t &= X_t+h(t)\end{cases} \tag{3.10}$$

The addition of the deterministic function $h(t)$ gives us more degrees of freedom to perfectly fit a yield-curve while the main structure is preserved. A discussion of this model, the so-called *CIR++*, can be found in Brigo and Mercurio [9]. Although the results of this chapter deal with the basic CIR, they can be easily generalised to this framework.

3.1.2 Preliminary results

We proceed in the same way as in Section 2.3.1 of Chapter 2 and for the same reasons - i.e. to compute the same type of truncated moments and probabilities - we first try to invert the Laplace transform $\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}$ with respect to μ. We also derive as a preliminary result its derivative with respect to λ for simplicity of presentation.

i. First result

Before stating the actual result, we start by defining some sequences of functions on which the inverse Laplace transform will depend.

Definitions and notations. Using the parameters $\beta = \frac{1}{2\sigma^2}$ and

$$\vartheta = \sqrt{b^2 + 2\lambda\sigma^2} \tag{3.11}$$

we define three sequences $\tilde{I}^3_{p,q}(y,\lambda,\varpi)$, $\tilde{I}^1_{p,q}(y,\lambda,\varpi)$ and $\tilde{I}^2_{p,q}(y,\lambda,\varpi)$ recursively constructed in the following way for $q \geq 1$ and $p \geq 0$.

- For $\tilde{I}^3_{p,q}(y,\lambda,\varpi)$

  - Recursion

$$\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) = -\frac{\varpi}{q}\tilde{I}^3_{p,q}(y,\lambda,\varpi) + 1_{\{p>0\}}\left\{\frac{p}{q}\tilde{I}^3_{p-1,q}(y,\lambda,\varpi)\right\}$$

$$+1_{\{q>1\}}\left\{\frac{2y\beta}{q}\tilde{I}^3_{p,q-1}(y,\lambda,\varpi)\right\} + 1_{\{q=1\}}\left\{\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}\beta\sqrt{2}}{\sqrt{\pi(2y\beta)^{p+1}}}He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right)\right\} \tag{3.12}$$

- Initial value recursion

$$\tilde{I}^3_{p+1,1}(y,\lambda,\varpi) = -b\tilde{I}^3_{p,1}(y,\lambda,\varpi) + \frac{\beta\sqrt{2}e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{\sqrt{\pi(2y\beta)^{p+3}}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) \tag{3.13}$$

- Starting point

$$\tilde{I}^3_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2 y\beta}}{2y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{3.14}$$

- <u>For $\tilde{I}^1_{p,q}(y,\lambda,\varpi)$</u>

  - Recursion

$$\tilde{I}^1_{p,q+1}(y,\lambda,\varpi) = \frac{1}{\vartheta}\tilde{I}^1_{p,q}(y,\lambda,\varpi) - \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) \tag{3.15}$$

  - Initial value recursion

$$\tilde{I}^1_{p+1,1}(y,\lambda,\varpi) = (\vartheta - b)\tilde{I}^1_{p,1}(y,\lambda,\varpi) + \frac{1}{2\vartheta}\tilde{I}^3_{p,1}(y,\lambda,\varpi) \tag{3.16}$$

  - Starting point

$$\tilde{I}^1_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta\varpi}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi - 2y\beta\vartheta}{2\sqrt{y\beta}}\right) - \frac{e^{-\vartheta^2 y\beta}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{3.17}$$

- For $\tilde{I}^2_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\tilde{I}^2_{p,q+1}(y,\lambda,\varpi) = -\frac{1}{\vartheta}\tilde{I}^1_{p,q}(y,\lambda,\varpi) + \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) \tag{3.18}$$

  - Initial value recursion

$$\tilde{I}^2_{p+1,1}(y,\lambda,\varpi) = -(b+\vartheta)\tilde{I}^2_{p,1}(y,\lambda,\varpi) + \frac{1}{2\vartheta}\tilde{I}^3_{p,1}(y,\lambda,\varpi) \tag{3.19}$$

  - Starting point

$$\tilde{I}^2_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2 y\beta}}{4\vartheta^2 y}\text{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) - \frac{e^{\vartheta\varpi}}{4\vartheta^2 y}\text{erfc}\left(\frac{\varpi + 2y\beta\vartheta}{2\sqrt{y\beta}}\right) \tag{3.20}$$

We can now formulate the main result.

Theorem 3.1.1. *For $\lambda \geq -\mu$, the inverse Laplace transform of $\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}$ with respect to $\mu > 0$ is*

$$\mathcal{G}_{a,b,\sigma}(y,\lambda) = e^{b\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m}\tilde{I}_{k,k-n}(y,\lambda,\varpi_m) \tag{3.21}$$

or equivalently

$$\mathcal{G}_{a,b,\sigma}(y,\lambda) = \lim_{K\to\infty} e^{b\frac{at+x_0}{\sigma^2}} \sum_{m=0}^{K}\sum_{k=m}^{K}\sum_{n=k-m}^{k} u_{k,n,m}\tilde{I}_{k,n}(y,\lambda,\varpi_m) \tag{3.22}$$

where $u_{k,n,m} = (-1)^m \frac{\left(-\frac{x_0}{\sigma^2}\right)^{k-n}}{2^n(k-n)!}\binom{k+\frac{2a}{\sigma^2}-1}{n}\binom{n}{m+n-k}$ and $\varpi_m = \frac{at+x_0}{\sigma^2} + mt$ are the same as in Part 1.1.4 *of* Chapter 1.

For $p \geq 0$, $q \geq 0$ and $\varpi > 0$, the main term $\tilde{I}_{p,q}(y,\lambda,\varpi)$ is given by

$$\begin{aligned}
\tilde{I}_{p,q}(y,\lambda,\varpi) &= 1_{\{q=0\}}\left\{2y\beta\Big(\vartheta^2(\tilde{I}^1_{p,1}(y,\lambda,\varpi) - \tilde{I}^2_{p,1}(y,\lambda,\varpi)) + \tilde{I}^3_{p,1}(y,\lambda,\varpi)\Big)\right\} \\
&+ 1_{\{q=1\}}\left\{2y\beta\vartheta\Big(\tilde{I}^1_{p,1}(y,\lambda,\varpi) + \tilde{I}^2_{p,1}(y,\lambda,\varpi)\Big)\right\} \\
&+ 1_{\{q>1\}}\left\{2y\beta\Big(\tilde{I}^1_{p,q-1}(y,\lambda,\varpi) - \tilde{I}^2_{p,q-1}(y,\lambda,\varpi)\Big)\right\}
\end{aligned} \tag{3.23}$$

Proof. See Appendix 3.4.1. □

Remark. The subscript a, b, σ appearing in the notation $\mathcal{G}_{a,b,\sigma}$, is meant to point out the dependence of the inverse Laplace transform $\mathcal{G}$ on the parameters of the diffusion. It ought to be added to all functions involved in this section, yet has been dropped for most of them to lighten the notations. We retained this notation only for a couple of main functions.

The formulation of Theorem 3.1.1 given above is helpful to understand the structure of the inverse Laplace transform. Yet, it is possible to reexpress the main element $\tilde{I}_{p,q}(y, \lambda, \varpi)$ in terms of a different set of sequences, which reduces the actual amount of calculations needed to compute $\tilde{I}_{p,q}(y, \lambda, \varpi)$.

Definitions and notations. Keeping the same notations, we build two sequences $\tilde{H}_{p,q}(y, \lambda, \varpi)$ and $\tilde{G}_p(y, \lambda, \varpi)$ from $\tilde{I}^3_{p,q}(y, \lambda, \varpi)$ defined above in (3.12), (3.13) and (3.14)

- Recursion

$$\tilde{H}_{p,q+1}(y, \lambda, \varpi) = \frac{1}{\vartheta^2}(\tilde{H}_{p,q-1}(y, \lambda, \varpi) - \tilde{I}^3_{p,q+1}(y, \lambda, \varpi)) \tag{3.24}$$

- Initial value recursions

$$\begin{cases} \tilde{H}_{p,2}(y, \lambda, \varpi) &= \vartheta \tilde{G}_p(y, \lambda, \varpi) - b\tilde{H}_{p,1}(y, \lambda, \varpi) \\ \tilde{H}_{p+1,1}(y, \lambda, \varpi) &= \vartheta \tilde{G}_p(y, \lambda, \varpi) - b\tilde{H}_{p,1}(y, \lambda, \varpi) \\ \tilde{G}_{p+1}(y, \lambda, \varpi) &= \vartheta \tilde{H}_{p,1}(y, \lambda, \varpi) - b\tilde{G}_p(y, \lambda, \varpi) + \frac{1}{\vartheta}\tilde{I}^3_{p,q}(y, \lambda, \varpi) \end{cases} \tag{3.25}$$

- Starting points

$$\begin{cases} \tilde{G}_0(y, \lambda, \varpi) &= \frac{e^{-\vartheta\varpi}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi - 2y\beta\vartheta}{2\sqrt{y\beta}}\right) - \frac{e^{\vartheta\varpi}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi + 2y\beta\vartheta}{2\sqrt{y\beta}}\right) \\ \tilde{H}_{0,1}(y, \lambda, \varpi) &= \frac{e^{-\vartheta\varpi}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi - 2y\beta\vartheta}{2\sqrt{y\beta}}\right) + \frac{e^{\vartheta\varpi}}{4\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi + 2y\beta\vartheta}{2\sqrt{y\beta}}\right) \\ & \quad - \frac{e^{-\vartheta^2 y\beta}}{2\vartheta^2 y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \end{cases} \tag{3.26}$$

Theorem 3.1.2. *The terms $\tilde{I}_{p,q}(y,\lambda,\varpi)$ appearing in (3.21) and (3.22) in Theorem 3.1.1 can be computed with*

$$\begin{aligned}\tilde{I}_{p,q}(y,\lambda,\varpi) &= 1_{\{q=0\}}\left\{2y\beta\left(\vartheta^2\tilde{H}_{p,1}(y,\lambda,\varpi)+\tilde{I}^3_{p,1}(y,\lambda,\varpi)\right)\right\}\\ &+ 1_{\{q=1\}}\left\{2y\beta\vartheta\tilde{G}_p(y,\lambda,\varpi)\right\}\\ &+ 1_{\{q>1\}}\left\{2y\beta\tilde{H}_{p,q-1}(y,\lambda,\varpi)\right\}\end{aligned}\tag{3.27}$$

Proof. This result comes from setting

$$\begin{aligned}\tilde{H}_{p,q}(y,\lambda,\varpi) &= \tilde{I}^1_{p,q}(y,\lambda,\varpi)-\tilde{I}^2_{p,q}(y,\lambda,\varpi)\\ \tilde{G}_p(y,\lambda,\varpi) &= \tilde{I}^1_{p,1}(y,\lambda,\varpi)+\tilde{I}^2_{p,1}(y,\lambda,\varpi)\end{aligned}\tag{3.28}$$

(3.15) and (3.18) then give the general recursion formula (3.24). (3.16) and (3.19) lead to the recursion formula for the initial values in q (3.25). And, finally, the starting values (3.26) - initial values for $p=0$ and $q=1$ - can be devised from (3.17) and (3.20). □

ii. Second result

As mentioned in the introduction of this section, the second intermediate result we need is the inverse Laplace transform of $\frac{E(Y_t e^{-(\lambda+\mu)Y_t})}{\mu}$ with respect to μ.

Definitions and notations. we define three new sequences $\hat{I}^3_{p,q}(y,\lambda,\varpi)$, $\hat{I}^1_{p,q}(y,\lambda,\varpi)$ and $\hat{I}^2_{p,q}(y,\lambda,\varpi)$ constructed recursively for $q\geq 1$ and $p\geq 0$ from $\tilde{I}^3_{p,q}(y,\lambda,\varpi)$, $\tilde{I}^1_{p,q}(y,\lambda,\varpi)$ and $\tilde{I}^2_{p,q}(y,\lambda,\varpi)$.

- For $\hat{I}^3_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\hat{I}^3_{p,q+1}(y,\lambda,\varpi) = -\frac{\varpi}{q}\hat{I}^3_{p,q}(y,\lambda,\varpi)+1_{\{p>0\}}\left\{\frac{p}{q}\hat{I}^3_{p-1,q}(y,\lambda,\varpi)\right\}$$

$$+1_{\{q>1\}}\left\{\frac{2y\beta}{q}\hat{I}^3_{p,q-1}(y,\lambda,\varpi)\right\}+1_{\{q=1\}}\left\{\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{\sqrt{2\pi(2y\beta)^{p-1}}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)\right\} \tag{3.29}$$

o Initial value recursion

$$\hat{I}^3_{p+1,1}(y,\lambda,\varpi)=-b\hat{I}^3_{p,1}(y,\lambda,\varpi)+\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{\sqrt{2\pi(2y\beta)^{p+1}}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right) \tag{3.30}$$

o Starting point

$$\hat{I}^3_{0,1}(y,\lambda,\varpi)=\frac{e^{-\vartheta^2 y\beta}}{2}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{3.31}$$

- <u>For $\hat{I}^1_{p,q}(y,\lambda,\varpi)$</u>

o Recursion

$$\hat{I}^1_{p,q+1}(y,\lambda,\varpi)=\frac{\hat{I}^1_{p,q}(y,\lambda,\varpi)}{\vartheta}+\frac{\sigma^2\tilde{I}^1_{p,q}(y,\lambda,\varpi)}{\vartheta^3}-\frac{\frac{\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{2}+\sigma^2\tilde{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^2} \tag{3.32}$$

o Initial value recursion

$$\hat{I}^1_{p+1,1}(y,\lambda,\varpi)=(\vartheta-b)\hat{I}^1_{p,1}(y,\lambda,\varpi)-\frac{\sigma^2}{\vartheta}\tilde{I}^1_{p,1}(y,\lambda,\varpi)+\frac{\hat{I}^3_{p,1}(y,\lambda,\varpi)+\frac{\sigma^2\tilde{I}^3_{p,1}(y,\lambda,\varpi)}{\vartheta}}{2\vartheta} \tag{3.33}$$

o Starting point

$$\hat{I}^1_{0,1}(y,\lambda,\varpi)=\frac{\sigma^2 e^{-\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi-2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}+\frac{1}{\vartheta}\right]$$
$$-\sqrt{\frac{\beta}{\pi y}}\frac{\sigma^2 e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{2\vartheta^3}-\frac{e^{-\vartheta^2 y\beta}}{2\vartheta^2}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right)\left[\frac{1}{2}+\frac{\sigma^2}{\vartheta^2 y}\right] \tag{3.34}$$

- <u>For $\hat{I}^2_{p,q}(y,\lambda,\varpi)$</u>

o Recursion

$$\hat{I}^2_{p,q+1}(y,\lambda,\varpi)=-\frac{\hat{I}^1_{p,q}(y,\lambda,\varpi)}{\vartheta}-\frac{\sigma^2\tilde{I}^1_{p,q}(y,\varpi)}{\vartheta^3}+\frac{\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{2\vartheta^2}+\frac{\sigma^2\tilde{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^4} \tag{3.35}$$

∘ Initial value recursion

$$\hat{I}^2_{p+1,1}(y,\lambda,\varpi) = \frac{\sigma^2}{\vartheta}\tilde{I}^2_{p,1}(y,\lambda,\varpi) - (b+\vartheta)\hat{I}^2_{p,1}(y,\lambda,\varpi) + \frac{\hat{I}^3_{p,1}(y,\lambda,\varpi) + \frac{\sigma^2\tilde{I}^3_{p,1}(y,\lambda,\varpi)}{\vartheta}}{2\vartheta} \tag{3.36}$$

∘ Starting point

$$\hat{I}^2_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2 y\beta}}{2\vartheta^2}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right)\left[\frac{1}{2}+\frac{\sigma^2}{\vartheta^2 y}\right]$$

$$+\frac{\sigma^2 e^{\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi+2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}-\frac{1}{\vartheta}\right]+\sqrt{\frac{\beta}{y\pi}}\frac{\sigma^2 e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{2\vartheta^3} \tag{3.37}$$

Theorem 3.1.3. *For $\lambda > -\mu$, the inverse Laplace transform of $\frac{E(Y_t e^{-(\lambda+\mu)Y_t})}{\mu}$ with respect to $\mu > 0$ is*

$$\hat{\mathcal{G}}_{a,b,\sigma}(y,\lambda) = e^{b\frac{at+x_0}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m}\hat{I}_{k,k-n}(y,\lambda,\varpi_m) \tag{3.38}$$

or equivalently

$$\hat{\mathcal{G}}_{a,b,\sigma}(y,\lambda) = \lim_{K\to\infty} e^{b\frac{at+x_0}{\sigma^2}}\sum_{m=0}^{K}\sum_{k=m}^{K}\sum_{n=k-m}^{k} u_{k,n,m}\hat{I}_{k,n}(y,\lambda,\varpi_m) \tag{3.39}$$

where, for $p \geq 0$, $q \geq 0$ and $\varpi > 0$, $\hat{I}_{p,q}(y,\lambda,\varpi)$ is given by

$$\begin{aligned}
\hat{I}_{p,q}(y,\lambda,\varpi) &= 1_{\{q=0\}}\Big\{2y\beta\Big(\vartheta^2(\hat{I}^1_{p,1}(y,\lambda,\varpi)-\hat{I}^2_{p,1}(y,\lambda,\varpi))+\hat{I}^3_{p,1}(y,\lambda,\varpi)\Big)\\
&\quad -2y(\tilde{I}^1_{p,1}(y,\lambda,\varpi)-\tilde{I}^2_{p,1}(y,\lambda,\varpi))\Big\}\\
&+ 1_{\{q=1\}}\Big\{2y\beta\vartheta\Big(\hat{I}^1_{p,1}(y,\lambda,\varpi)+\hat{I}^2_{p,1}(y,\lambda,\varpi)\Big)\\
&\quad -\tfrac{y}{\vartheta}\Big(\tilde{I}^1_{p,1}(y,\lambda,\varpi)+\tilde{I}^2_{p,1}(y,\lambda,\varpi)\Big)\Big\}\\
&+ 1_{\{q>1\}}\Big\{2y\beta\Big(\hat{I}^1_{p,q-1}(y,\lambda,\varpi)-\hat{I}^2_{p,q-1}(y,\lambda,\varpi)\Big)\Big\}
\end{aligned} \tag{3.40}$$

Proof. Since, from Theorem 3.1.1, $\mathcal{G}(y,\lambda)$ is the inverse Laplace transform of $\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}$, we have for any $\mu > 0$

$$\int_0^\infty e^{-\mu y}\mathcal{G}(y,\lambda)dy = \frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}$$

Thus, for $\lambda_\epsilon \in]\lambda - \frac{\mu}{2}, \lambda + \frac{\mu}{2}[$,

$$\int_0^\infty e^{-\mu y}\frac{\partial \mathcal{G}(y,\lambda_\epsilon)}{\partial \lambda_\epsilon}dy = \frac{E(Y_t e^{-(\lambda_\epsilon+\mu)Y_t})}{\mu}$$

That it is possible to interchange the order of the differentiation and the integration operators is straightforward to prove for the right-hand side of the equation but more difficult to show for the left-hand side. Instead, for the left-hand side, it is easier to first consider each atom $\mathcal{G}_k(y,\lambda_\epsilon)$ separately

$$\int_0^\infty e^{-\mu y}\frac{\partial \mathcal{G}_k(y,\lambda_\epsilon)}{\partial \lambda_\epsilon}dy = \frac{1}{\mu}\frac{\partial \mathcal{L}_k^Y(\mu+\lambda_\epsilon)}{\partial \lambda_\epsilon} \tag{3.41}$$

where

$$\mathcal{G}_k(y,\lambda_\epsilon) = e^{b\frac{at+x_0}{\sigma^2}}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m}\tilde{I}_{k,k-n}(y,\lambda_\epsilon,\varpi_m)$$

and, with $\gamma = \sqrt{b^2 + 2\sigma^2(\lambda+\mu)}$,

$$\mathcal{L}_k^Y(\mu) = \sum_{n=0}^{k}\frac{1}{n!}\frac{x_0^n}{\sigma^{2n}}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\frac{(\gamma-b)^k(1-e^{-\gamma t})^{k-n}e^{-\gamma n t}}{(2\gamma)^{k-n}}$$

Indeed, $\frac{\partial \mathcal{G}_k(y,\lambda_\epsilon)}{\partial \lambda_\epsilon}$ is integrable for $y \in \mathbb{R}^+$, allowing the differentiation under the integral sign leading to (3.41). It is then left to prove that the sum of the inverses $\mathcal{G}_k(y,\lambda)$ converges to the inverse of the sum $\sum_0^\infty \mathcal{L}_k^Y(\mu+\lambda)$, which can be done by following the same steps as the convergence proof given in Appendix 1.5.1 of Chapter 1.

Setting $\hat{I}_{p,q}(y,\lambda,\varpi) = -\frac{\partial \tilde{I}_{p,q}(y,\lambda,\varpi)}{\partial \lambda}$, $\hat{I}^1_{p,q}(y,\lambda,\varpi) = -\frac{\partial \tilde{I}^1_{p,q}(y,\lambda,\varpi)}{\partial \lambda}$, $\hat{I}^2_{p,q}(y,\lambda,\varpi) = -\frac{\partial \tilde{I}^2_{p,q}(y,\lambda,\varpi)}{\partial \lambda}$ and $\hat{I}^3_{p,q}(y,\lambda,\varpi) = -\frac{\partial \tilde{I}^3_{p,q}(y,\lambda,\varpi)}{\partial \lambda}$, the manipulation of the equations used in Theorem 3.1.1 results in the formulation of Theorem 3.1.3. □

Just like $\tilde{I}_{p,q}(y,\lambda,\varpi)$ in the first result, it is possible to reexpress $\hat{I}_{p,q}(y,\lambda,\varpi)$ in a more efficient way, computation-wise.

Definitions and notations. Similarly, we build two derivative sequences $\hat{H}_{p,q}(y,\lambda,\varpi)$ and $\hat{G}_p(y,\lambda,\varpi)$ based on $\tilde{H}_{p,q}(y,\lambda,\varpi)$ and $\tilde{G}_p(y,\lambda,\varpi)$.

∘ Recursion

$$\hat{H}_{p,q+1}(y,\lambda,\varpi) = \frac{\hat{H}_{p,q-1}(y,\lambda,\varpi) - \hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^2} + \sigma^2 \frac{\tilde{H}_{p,q-1}(y,\lambda,\varpi) - \tilde{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^3} \tag{3.42}$$

∘ Initial value recursions

$$\begin{cases} \hat{H}_{p,2}(y,\lambda,\varpi) &= \vartheta\hat{G}_p(y,\lambda,\varpi) - \frac{\sigma^2}{\vartheta}\tilde{G}_p(y,\lambda,\varpi) - b\hat{H}_{p,1}(y,\lambda,\varpi) \\ \hat{H}_{p+1,1}(y,\lambda,\varpi) &= \vartheta\hat{G}_p(y,\lambda,\varpi) - \frac{\sigma^2}{\vartheta}\tilde{G}_p(y,\lambda,\varpi) - b\hat{H}_{p,1}(y,\lambda,\varpi) \\ \hat{G}_{p+1}(y,\lambda,\varpi) &= \vartheta\hat{H}_{p,1}(y,\lambda,\varpi) - \frac{\sigma^2}{\vartheta}\tilde{H}_{p,1}(y,\lambda,\varpi) - b\hat{G}_p(y,\lambda,\varpi) \\ & \quad +\frac{1}{\vartheta}\hat{I}^3_{p,q}(y,\lambda,\varpi) + \frac{\sigma^2}{\vartheta^3}\tilde{I}^3_{p,q}(y,\lambda,\varpi) \end{cases} \tag{3.43}$$

∘ Starting points

$$\begin{cases} \hat{G}_0(y,\lambda,\varpi) &= \frac{\sigma^2 e^{-\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi-2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}+\frac{1}{\vartheta}\right] \\ & \quad +\frac{\sigma^2 e^{\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi+2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}-\frac{1}{\vartheta}\right] \\ \hat{H}_{0,1}(y,\lambda,\varpi) &= \frac{\sigma^2 e^{-\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi-2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}+\frac{1}{\vartheta}\right] - \sqrt{\frac{\beta}{\pi y}}\frac{\sigma^2 e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{\vartheta^3} \\ & \quad -\frac{\sigma^2 e^{\vartheta\varpi}}{2\vartheta^3 y}\mathrm{erfc}\left(\frac{\varpi+2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}+\frac{1}{\vartheta}\right] \\ & \quad -\frac{e^{-\vartheta^2 y\beta}}{\vartheta^2}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right)\left[\frac{1}{2}+\frac{\sigma^2}{\vartheta^2 y}\right] \end{cases} \tag{3.44}$$

Given these derivative sequences, it is possible to rewrite the last expression (3.40) of Theorem 3.1.3.

Theorem 3.1.4. *The terms $\hat{I}_{p,q}(y,\lambda,\varpi)$ appearing in (3.38) and (3.39) in Theorem*

3.1.3 can be computed with

$$\begin{aligned}\hat{I}_{p,q}(y,\lambda,\varpi) &= 1_{\{q=0\}}\Big\{2y\beta\Big(\vartheta^2\hat{H}_{p,1}(y,\lambda,\varpi)+\hat{I}^3_{p,1}(y,\lambda,\varpi)\Big)-2y\tilde{H}_{p,1}(y,\lambda,\varpi)\Big\}\\ &+ 1_{\{q=1\}}\Big\{2y\beta\vartheta\hat{G}_p(y,\lambda,\varpi)-\tfrac{y}{\vartheta}\tilde{G}_p(y,\lambda,\varpi)\Big\}\\ &+ 1_{\{q>1\}}\Big\{2y\beta\hat{H}_{p,q-1}(y,\lambda,\varpi)\Big\}\end{aligned} \tag{3.45}$$

Proof. Setting

$$\begin{aligned}\hat{H}_{p,q}(y,\lambda,\varpi) &= -\tfrac{\partial\tilde{H}_{p,q}(y,\lambda,\varpi}{\partial\lambda} = \hat{I}^1_{p,q}(y,\lambda,\varpi)-\hat{I}^2_{p,q}(y,\lambda,\varpi)\\ \hat{G}_p(y,\lambda,\varpi) &= -\tfrac{\partial\tilde{G}_p(y,\lambda,\varpi)}{\partial\lambda} = \hat{I}^1_{p,q}(y,\lambda,\varpi)+\hat{I}^2_{p,q}(y,\lambda,\varpi)\end{aligned} \tag{3.46}$$

and manipulating the equations defining $\hat{I}^1_{p,q}(y,\lambda,\varpi)$, $\hat{I}^2_{p,q}(y,\lambda,\varpi)$ and $\hat{I}_{p,q}(y,\lambda,\varpi)$ lead to this formulation. □

3.1.3 Probability distribution function of the cumulative interest rate

Theorem 3.1.5. *For $y>0$, the probability distribution function of Y_t is given by*

$$P(Y_t\le y)=\mathcal{G}_{a,b,\sigma}(y,0) \tag{3.47}$$

Proof. For $\mu>0$, the Laplace transform of the probability distribution function is

$$\int_0^\infty e^{-\mu y}E\big(1_{\{Y_t\le y\}}\big)dy=E\bigg(\int_{Y_t}^\infty e^{-\mu y}dy\bigg)=E\bigg(\frac{e^{-\mu Y_t}}{\mu}\bigg)$$

which, combined with Theorem 3.1.1, implies the result. □

3.1.4 Truncated expectation of the cumulative interest rate

Theorem 3.1.6. *For $y>0$, the truncated expectation of Y_t is given by*

$$E\big(Y_t1_{\{Y_t\le y\}}\big)=\hat{\mathcal{G}}_{a,b,\sigma}(y,0) \tag{3.48}$$

Proof. For $\mu > 0$, the Laplace transform of the probability distribution function is

$$\int_0^\infty e^{-\mu y} E\big(Y_t 1_{\{Y_t \le y\}}\big) dy = E\Big(Y_t \int_{Y_t}^\infty e^{-\mu y} dy\Big) = E\Big(Y_t \frac{e^{-\mu Y_t}}{\mu}\Big)$$

From Theorem 3.1.3, it leads to the result. □

3.1.5 Guaranteed endowment option

This option pays out the shortfall between 1 and the amount accumulated in a standard savings account $(1 - Ke^{Y_T})^+$. It is simply used to guarantee a minimal accrual on an initial amount of cash, a common insurance feature.

Theorem 3.1.7. *For $1 \ge K > 0$, the guaranteed endowment put option is given by the expectation of the discounted expectation of its payoff*

$$GEO^{\mathrm{p}}(K,T) = E(e^{-Y_T} - K)^+ = \mathcal{G}_{a,b,\sigma}(-\ln K, 1) - K\mathcal{G}_{a,b,\sigma}(-\ln K, 0) \qquad (3.49)$$

Proof. The option price can be split in two parts

$$\mathrm{GEO}^{\mathrm{p}}(K,T) = E(e^{-Y_T} 1_{\{Y_T \le -\ln K\}}) - KP(Y_T \le -\ln K)$$

We denote $k = -\ln K$. Since $1 \ge K > 0$, k is strictly positive and we therefore have, for $\mu > 0$,

$$\int_0^\infty e^{-\mu k} E\big(e^{-Y_T} 1_{\{Y_T \le k\}}\big) dk = E\Big(e^{-Y_T} \int_{Y_T}^\infty e^{-\mu k} dk\Big) = E\Big(\frac{e^{-(\mu+1)Y_T}}{\mu}\Big)$$

Therefore, Theorem 3.1.1 allows us to explicit the first part. Theorem 3.1.5 gives the second part. □

Remark. The assumption $K < 1$ is due to the fact that $e^{-Y_T} < 1$. For $K > 1$, the option is simply worthless.

Theorem 3.1.8. *The guaranteed endowment call option can be deduced with the parity relation*

$$GEO^c(K,T) - GEO^p(K,T) = P(t,T) - K \tag{3.50}$$

Proof. This results follows from

$$(e^{-Y_T} - K)^+ - (K - e^{-Y_T})^+ = e^{-Y_T} - K$$

□

3.1.6 Binary Asian options

In general, binary - or digital - options are classified in two classes: the ones paying in cash units and the one paying in asset units, i.e. paying an interest rate here. In the context of Asian interest rate derivatives, their respective payoff is $E(e^{-Y_T}1_{\{Y_T \geq K\}})$ and $E(Y_T e^{-Y_T}1_{\{Y_T \geq K\}})$ for cap - or call - options. For floor - put - options, those payoffs are $E(e^{-Y_T}1_{\{Y_T \leq K\}})$ and $E(Y_T e^{-Y_T}1_{\{Y_T \leq K\}})$.

Theorem 3.1.9. *For $K > 0$, the cash binary Asian floor is worth*

$$CBA^f(K,T) = E(e^{-Y_T}1_{\{Y_T \leq K\}}) = \mathcal{G}_{a,b,\sigma}(K,1) \tag{3.51}$$

Proof. We have , for $\mu > 0$,

$$\int_0^\infty e^{-\mu K} E(e^{-Y_T}1_{\{Y_T \leq K\}})dK = E\Big(e^{-Y_T}\int_{Y_T}^\infty e^{-\mu K}dK\Big) = E\Big(\frac{e^{-(\mu+1)Y_T}}{\mu}\Big)$$

Hence the result. □

Theorem 3.1.10. *For $K > 0$, the rate binary Asian floor is worth*

$$RBA^f(K,T) = E(Y_T e^{-Y_T}1_{\{Y_T \leq K\}}) = \hat{\mathcal{G}}_{a,b,\sigma}(K,1) \tag{3.52}$$

Proof. In the same way,

$$\int_0^{\infty} e^{-\mu K} E\big(Y_T e^{-Y_T} 1_{\{Y_T \le K\}}\big) dK = E\Big(\frac{Y_T e^{-(\mu+1)Y_T}}{\mu}\Big)$$

□

We end this subsection with the call-put parity relation which allows us to deduce cap option prices from the floor option prices given above.

Theorem 3.1.11. *The options are linked in the following way*

$$\begin{aligned} CBA^c(K,T) + CBA^f(K,T) &= P(0,T) \\ RBA^c(K,T) + RBA^f(K,T) &= P(0,T)E^T(Y_T) \end{aligned} \tag{3.53}$$

Proof. Indeed,

$$\begin{aligned} E(e^{-Y_T} 1_{\{Y_T \le K\}}) + E(e^{-Y_T} 1_{\{Y_T > K\}}) &= E(e^{-Y_T}) \\ E(Y_T e^{-Y_T} 1_{\{Y_T \le K\}}) + E(Y_T e^{-Y_T} 1_{\{Y_T \le K\}}) &= E(Y_T e^{-Y_T}) \end{aligned}$$

□

3.1.7 Regular Asian options

Interest rates Asian options have been developed to cover needs similar to those having created equity Asian. Those instruments have been noticeably studied by Leblanc and Scaillet [47] and Chacko and Das [12].

Theorem 3.1.12. *The regular Asian option can be computed as the inverse Laplace transform with respect to μ of*

$$\frac{E(Y_T e^{-Y_T})}{\mu} + \frac{E(e^{-(\mu+1)Y_T} - e^{-\mu Y_T})}{\mu^2} \tag{3.54}$$

Proof. Straightforward application of the previous results. □

Remarks.

1. The first term of the expression (3.54) can easily be obtained explicitly as the derivative of the MGF $\frac{\partial}{\partial \xi}\mathcal{L}^{X,Y}(0,\xi)$ at $\xi = 1$. We do not present the actual formula here as the expression turns out to be quite heavy.

2. To compare our Laplace transform result to the existing solutions to this problem, the Laplace transform approach proposed in Theorem 3.1.12 is simpler. Indeed, Leblanc and Scaillet [47] proposed to first compute the density of Y_T through numerical Laplace transform inversion of its MGF and then integrate it against the discounted payoff of the option, which is a quite heavy numerical procedure. Chacko and Das [12] expressed the option as a sum of Asian binary calls at ascending strikes and used Fourier inversion. Our expression is more immediate to calculate.

We can yet also propose a completely explicit solution for this option.

Theorem 3.1.13. *For $K > 0$, the Asian floor is worth*

$$AO^f(K,T) = E((K - Y_T)^+ e^{-Y_T}) = K\mathcal{G}_{a,b,\sigma}(K,1) - \hat{\mathcal{G}}_{a,b,\sigma}(K,1) \tag{3.55}$$

Proof. The regular Asian option is the difference between two binary, one paying in cash unit and the other in rate unit

$$\mathrm{AO}^f(K,T) = K.\mathrm{CBA}(K,T) - \mathrm{RBA}(K,T)$$

□

Remark. For the actual analytical inversion of the Laplace transform (3.54), we decompose the option in binary options. The computation of these options being made at the same time, there are some recurrent terms used in the computations of both binary derivatives and some simplifications in the algorithm.

As previously, we present a call-put parity result:

Theorem 3.1.14. *For $K > 0$, we have*

$$AO^c(K,T) - AO^f(K,T) = E(Y_T e^{-Y_T}) - KP(0,T) \tag{3.56}$$

3.1.8 Numerical results

The works of Chacko and Das [12] on one hand and Leblanc and Scaillet [47] on the other provide us with material for comparison. We consider the prices of regular Asian caps, $\mathrm{AO}^c(K,T) = E\big((\frac{Y_T}{T} - K)^+ e^{-Y_T}\big)$, cash binary caps, $\mathrm{CBA^C}(K,T) = E\big(e^{-Y_T} 1_{\{Y_T > KT\}}\big)$, and additionally the valuation of rate binary caps, $\mathrm{RBA^C}(K,T) = E\big(Y_T e^{-Y_T} 1_{\{Y_T > KT\}}\big)$, truncated moments defined as $\mathrm{TM^C}(K,T) = E\big(Y_T 1_{\{Y_T > KT\}}\big)$ and probabilities, $P(Y_T > KT)$.

Following Chacko and Das [12], we first analyse the evolution with respect to the maturity T and strike K.

Maturity	Type	$K=0.08$	$K=0.09$	$K=0.10$	$K=0.11$	$K=0.12$
$T=0.1$	$P(Y_T > KT)$	0.1061	0.2936	0.5337	0.7427	0.8790
	$\mathrm{CBA^C}$	0.9631	0.8104	0.4812	0.1763	0.0387
	$\mathrm{TM^C}$	0.0098	0.0085	0.0053	0.0021	0.0006
	$\mathrm{RBA^C}$	0.0097	0.0084	0.0052	0.0021	0.0005
	$\mathrm{AO^c}$	0.0199	0.0109	0.0043	0.0011	0.0002
$T=0.5$	$P(Y_T > KT)$	0.1549	0.3261	0.5280	0.7107	0.8441
	$\mathrm{CBA^C}$	0.8018	0.6377	0.4452	0.2718	0.1459
	$\mathrm{TM^C}$	0.0444	0.0285	0.0275	0.0180	0.0103
	$\mathrm{RBA^C}$	0.0421	0.0351	0.0260	0.0169	0.0097
	$\mathrm{AO^c}$	0.0201	0.0128	0.0074	0.0039	0.0018
$T=1$	$P(Y_T > KT)$	0.1878	0.3535	0.5354	0.6979	0.8209
	$\mathrm{CBA^C}$	0.7301	0.5779	0.4125	0.2662	0.1565
	$\mathrm{TM^C}$	0.0867	0.0726	0.0553	0.0383	0.0242
	$\mathrm{RBA^C}$	0.0777	0.0647	0.0490	0.0337	0.0211
	$\mathrm{AO^c}$	0.0193	0.0127	0.0078	0.0044	0.0023
$T=2$	$P(Y_T > KT)$	0.1789	0.3509	0.5395	0.7050	0.8276
	$\mathrm{CBA^C}$	0.6644	0.5193	0.3633	0.2291	0.1317
	$\mathrm{TM^C}$	0.1744	0.1451	0.1093	0.0746	0.0465
	$\mathrm{RBA^C}$	0.1402	0.1155	0.0859	0.0578	0.0354
	$\mathrm{AO^c}$	0.0170	0.0110	0.0066	0.0037	0.0019

Table 3.1: Evolution with T, Chacko and Das parameters.

The diffusion parameters of the instantaneous rate are a=0.15, b=1.5, σ = 0.2

and $r_0 = 0.1$. The results we obtained in Table 3.1 do not actually exactly tally with the ones presented by Chacko and Das [12]. Yet, the Abate and Whitt algorithm confirm our results.

To double-check the validity of our method, we also consider Asian options on yields in the setting of Leblanc and Scaillet [47]. The yield $\mathcal{Y}(T, T+\tau)$ of maturity τ at time T is defined as $-\frac{\ln(B(T,T+\tau))}{\tau} = \frac{a\Theta(\tau)+r_T\Upsilon(\tau)}{\tau}$, with the functions defined in Theorem 1.1.4, Chapter 1 for $\lambda = 0$ and $\mu = 1$. Asian call options on yields are then given by

$$C^{\mathcal{Y}} = E\left[\left(\frac{1}{T}\int_0^T \mathcal{Y}(u, u+\tau)du - K\right)^+ e^{-Y_T}\right] \tag{3.57}$$

They are related to the Asian options on the instantaneous rate trough

$$C^{\mathcal{Y}} = \frac{\Upsilon(\tau)}{\tau}\mathrm{AO^c}\left(\frac{\tau K - a\Theta(\tau)}{\Upsilon(\tau)}, T\right)$$

Case	a	b	σ^2	T	τ	K	LS	Fusai	Series
1	0.02	0.2	0.02	1	10	0.1	0.000949	0.00094927	0.000949272
1	0.02	0.2	0.02	0.25	10	0.1	0.00012	0.00012019	0.00012019
1	0.02	0.2	0.02	1	0.25	0.1	0.008131	0.00813132	0.00813132
1	0.02	0.2	0.02	0.25	0.25	0.1	0.008131	0.00477464	0.00477464

Table 3.2: Asian options on yield.

Table 3.2 confirms that the values computed with our series are correct. Indeed, the column LS and Fusai collect the prices respectively produced by Leblanc and Scaillet [47] and Fusai [33].

This cross-checking done, we come back to the parameters proposed by Chacko and Das [12]. We first observe how higher maturities, common in fixed-income, affect the results in Table 3.3.

Maturity	Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 5$	$P(Y_T > KT)$	0.1061	0.2936	0.5337	0.7427	0.8790
	CBA^c	0.5354	0.4130	0.2637	0.1399	0.0631
	TM^c	0.4607	0.3806	0.2665	0.1571	0.0791
	RBA^c	0.2730	0.2208	0.1499	0.0851	0.0410
	AO^c	0.0118	0.0070	0.0036	0.0016	0.0006
$T = 10$	$P(Y_T > KT)$	0.0436	0.2201	0.5255	0.7936	0.9343
	CBA^c	0.3504	0.2756	0.1576	0.0634	0.0185
	TM^c	0.9668	0.8154	0.5248	0.2442	0.0834
	RBA^c	0.3495	0.2853	0.1732	0.0747	0.0234
	AO^c	0.0069	0.0037	0.0016	0.0005	0.0001

Table 3.3: Higher maturities, Chacko and Das parameters.

The volatility of $\frac{Y_T}{T}$ should start from a low level, increase with T and then decrease again for high maturities, the mean-reversion pulling the rate to its long-term level $\frac{a}{b}$. The call prices and the probabilities corroborate this intuition. The monotonous evolutions with respect to K are as expected.

Type	T	0.08	0.09	0.10	0.11	0.12	T	0.08	0.09	0.10	0.11	0.12
$\mathcal{G}_{a,b,\sigma}(KT,0)$	0.1	50	51	54	57	59	0.5	16	19	19	21	21
$\mathcal{G}_{a,b,\sigma}(KT,1)$		49	51	53	56	53		16	19	19	21	21
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,0)$		49	51	53	57	53		16	19	19	21	21
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,1)$		50	51	54	57	59		16	20	19	21	21
$\mathcal{G}_{a,b,\sigma}(KT,0)$	1	15	14	12	14	14	2	11	10	11	10	9
$\mathcal{G}_{a,b,\sigma}(KT,1)$		15	14	12	14	14		11	10	11	10	9
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,0)$		15	14	12	15	14		11	10	11	10	9
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,1)$		15	14	12	15	14		11	10	11	10	9
$\mathcal{G}_{a,b,\sigma}(KT,0)$	5	9	7	8	7	8	10	8	6	6	6	6
$\mathcal{G}_{a,b,\sigma}(KT,1)$		9	3	8	7	6		8	6	6	6	5
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,0)$		9	7	8	7	8		8	6	6	6	6
$\hat{\mathcal{G}}_{a,b,\sigma}(KT,1)$		9	7	8	7	6		8	6	6	6	6

Table 3.4: Evolution of the speed of convergence with T.

The series performance, i.e speed of convergence, indicated in Table 3.4 are in agreement with our observation concerning the behaviour of the density series $f^Y(y)$ in Section 1.1.4 of Chapter 1. The figures of Table 3.4 represent the minimal number of terms needed to ensure the relative error is inferior to 10^{-4}. As for the density, the four series converge more and more quickly as T increase. N remains roughly of the same order throughout the strike curve.

Once again, all the results produced in this section have been cross-tested again numerical Laplace transform inversion and numerical integration of (B.1). For maturities superior than five years, the Abate-Whitt starts becoming unstable and for ten years maturities, the numerical integration cannot be performed successfully with Mathematica.

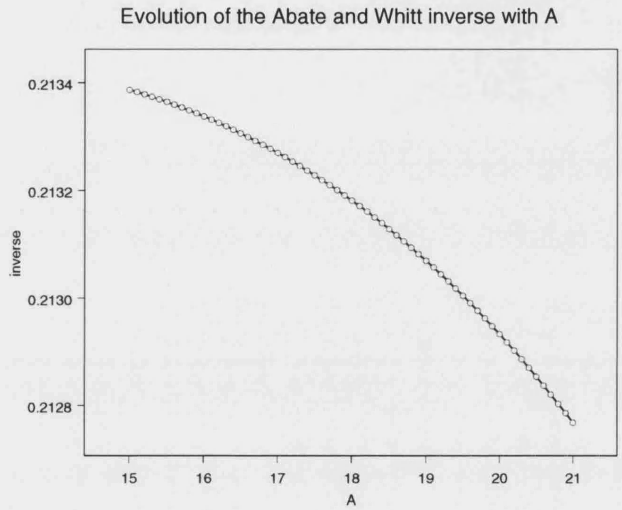


Figure 3.1: $r_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ *and* $T = 10$

Figure 3.1 shows no hint can obtained as to the location of the real inverse for $T = 10$ with the Abate and Whitt algorithm. As mentioned earlier, long-dated instruments are not rare in fixed-income markets. Our series brings a quick and effective solution for those problematic high maturities regions.

The same is true for high volatilities as expected.

Table 3.5 contains the results for the higher volatility case considered in Chacko and Das [12], $\sigma = 0.3$, all other base parameters remaining the same. Not surprisingly,

K	N				Value				
	$\mathcal{G}(KT,0)$	$\mathcal{G}(KT,1)$	$\check{\mathcal{G}}(KT,0)$	$\check{\mathcal{G}}(KT,1)$	P	CBA^c	TM^c	RBA^c	AO^c
0.08	8	8	11	11	0.3040	0.6204	0.0803	0.0711	0.0215
0.09	11	10	12	11	0.4308	0.5039	0.0695	0.0612	0.0158
0.1	11	11	11	11	0.5534	0.3924	0.0579	0.0506	0.0114
0.11	10	10	10	10	0.6625	0.2942	0.0464	0.0403	0.0079
0.12	10	10	10	10	0.7533	0.2133	0.0360	0.0310	0.0054

Table 3.5: $\sigma = 0.3$, $T = 1$, Chacko and Das parameters.

our series converge faster as the volatility increases whereas the numerical inversions routines start having difficulties.

K	N				Value				
	$\mathcal{G}(KT,0)$	$\mathcal{G}(KT,1)$	$\check{\mathcal{G}}(KT,0)$	$\check{\mathcal{G}}(KT,1)$	P	CBA^c	TM^c	RBA^c	AO^c
0.08	11	11	11	11	0.6173	0.3475	0.0370	0.0335	0.0057
0.09	10	10	9	9	0.7724	0.2050	0.0238	0.0214	0.0030
0.1	11	11	12	12	0.8771	0.1097	0.0139	0.0124	0.0014
0.11	11	11	11	11	0.9391	0.0539	0.0074	0.0066	0.0007
0.12	10	10	10	11	0.9720	0.0246	0.0037	0.0032	0.0003

Table 3.6: $\frac{a}{b} = 0.05$, $T = 1$, Chacko and Das parameters.

Table 3.6 presents the last group of cases analysed in Chacko and Das [12], a shift in the long-term mean level. We observe that our series converge slightly faster.

This numerical analysis of the performance of the series we derived concludes our study of the CIR model. We will now consider the general CEV model.

3.2 The CEV model for the instantaneous rate

As a short-term interest model, the CEV process has been introduced by Chan, Karolyi, Longstaff and Sanders [13] in its mean-reverting form

$$dr_t = (a - br_t)dt + \sigma r_t^\alpha dW_t \tag{3.58}$$

with $a > 0$ and $b > 0$. This model proved a better statistical fit to historical data than other one-factor models.

As for the equity model studied in Chapter 1, we will suppose $0 < \alpha < 1$ and $\alpha = \frac{p}{q}$.

Theorem 3.2.1. *Any function belonging to $C^{1,2}(\mathbb{R}^+, \mathbb{R}^+)$, bounded with bounded derivatives for its second variable, bounded on any compact interval for its first variable and satisfying the partial differential equation*

$$\frac{\partial P}{\partial t} = \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 P}{\partial x^2} + (a - bx)\frac{\partial P}{\partial x} - xP \tag{3.59}$$

with the initial condition

$$\forall x_0 \geq 0, \quad P(0, x_0) = 1 \tag{3.60}$$

is the zero-coupon bond price for this short-rate model.

Proof. The proof follows the same line as the demonstration of Theorem 1.3.2 in Chapter 1. □

3.2.1 A Laplace transform approach

Trying to solve the differential equation (3.59) with a Laplace transform with respect to time comes across quite naturally.

Theorem 3.2.2. *The Laplace transform of the zero-coupon bond price in this model $\hat{P}(\lambda, x) = \int_0^\infty e^{-\lambda T} E(e^{-\int_0^T r_t dt} | X_0 = x) dT$ - clearly defined for $\lambda > 0$ - satisfies the ODE*

$$\hat{P}(\lambda, x) - 1 = \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 \hat{P}(\lambda, x)}{\partial x^2} + (a - bx)\frac{\partial \hat{P}(\lambda, x)}{\partial x} - x\hat{P}(\lambda, x) \tag{3.61}$$

Proof. This simply results from taking the transform of the equation (3.59), taking into account the condition (3.60). □

Theorem 3.2.3. *For $q < 2p$, i.e. $\alpha > \frac{1}{2}$, no analytic solution to the equation (3.61) exists. For $q > 2p$, i.e. $\alpha < \frac{1}{2}$, both the vector space of solutions to the homogeneous equation and the affine space of solutions to the non-homogeneous equation are composed of analytic functions.*

Proof. As in the equity case in Chapter 1, we use the change of variables $y = x^{\frac{1}{q}}$, $h(\lambda, y) = g(\lambda, x)$ which, with $l = 2(q-p)$, gives

$$0 = \frac{\sigma^2}{2q^2} y^2 \frac{\partial^2 h}{\partial y^2} + \left[\frac{\sigma^2}{2} \frac{1-q}{q^2} y - \frac{b}{q} y^{l+1} + \frac{a}{q} y^{1-q+l} \right] y \frac{\partial h}{\partial y} - \mu y^{q+l} h - \lambda y^l h + y^l$$

for the non-homogeneous ODE. When applying Frobenius method (see Theorem 2.2.3, Chapter 2 for another application of this analytic coefficient method) and trying to find a solution in the form $\sum_{n=0}^{\infty} u_n y^{n+\beta}$, different cases arise. If $q > 2p$, there are two possibilities: $\beta = 0$ and $\beta = q(1 - \frac{2a}{\sigma^2})$. When $q < 2p$, there is only one possibility: $\beta = 0$. And it turns out that, even in that case, the recursions lead to a divergent series for $q < 2p$. The same analysis can be carried out for the homogeneous equation. □

We could not find a way to solve this ODE in explicit form for the case $q < 2p$ and no such solution seems available in the mathematical literature.

We will thus focus here on the case $q > 2p$, for which it is possible to obtain an explicit formulation for the Laplace transform of the zero-coupon bond prices.

Definitions and notations. Given $l = 2(q-p)$, $A = \frac{\sigma^2}{2}\frac{1-q}{q^2}$, $B = \frac{-b}{q}$, $C = \frac{\sigma^2}{2q^2}$ and $D = \frac{a}{q}$, $h^\lambda(\nu_0, \nu_1, x)$ refers, in this section, to the power series

$$h^\lambda(\nu_0, \nu_1, x) = \sum_{k=0}^{\infty} u_k^\lambda(\nu_0, \nu_1) x^{\frac{k}{q}}$$

$$\begin{cases} u_0^\lambda(\nu_0,\nu_1) &= \nu_0 & \text{for } k=0 \\ u_k^\lambda(\nu_0,\nu_1) &= 0 & \text{for } 0<k<l-q \\ u_k^\lambda(\nu_0,\nu_1) &= -\frac{D(k+q-l)}{k[A+C(k-1)]}u_{k+q-l}^\lambda(\nu_0,\nu_1) & \text{for } l-q<k<l,\ k\neq q \\ u_q^\lambda(\nu_0,\nu_1) &= \nu_1 & \text{for } k=q \\ u_k^\lambda(\nu_0,\nu_1) &= \frac{[\lambda-B(k-l)]u_{k-l}^\lambda(\nu_0,\nu_1)-D(k+q-l)u_{k+q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \text{for } l\leq k<q+l \\ u_k^\lambda(\nu_0,\nu_1) &= \frac{[\lambda-B(k-l)]u_{k-l}^\lambda(\nu_0,\nu_1)+u_{k-q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \text{for } q+l\leq k \\ & \quad -\frac{[D(k+q-l)]u_{k+q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \end{cases} \tag{3.62}$$

Theorem 3.2.4. *When $q > 2p$, the solutions to the homogeneous equation*

$$\lambda h(x) = \frac{\sigma^2}{2}x^{2\alpha}\frac{\partial^2 h(x)}{\partial x^2} + (a-bx)\frac{\partial h(x)}{\partial x} - xh(x)$$

are the functions $h^\lambda(\nu_0,\nu_1,x)$.

Proof. This follows from assuming the form $h(\lambda,x) = \sum_{n=0}^{\infty} u_n x^{\frac{k}{q}}$ and deducing the recursion formulae from the homogeneous differential equation. □

We need to set more notations for the inhomogeneous equation.

Definitions and notations. With the same notations, $i^\lambda(\nu_0,\nu_1,x)$ refers, in this section, to the power series

$$i^\lambda(\nu_0,\nu_1,x) = \sum_{k=0}^{\infty} v_k^\lambda(\nu_0,\nu_1)x^{\frac{k}{q}}$$

$$\begin{cases} v_0^\lambda(\nu_0,\nu_1) &= \nu_0 & \text{for } k=0 \\ v_k^\lambda(\nu_0,\nu_1) &= 0 & \text{for } 0<k<l-q \\ v_k^\lambda(\nu_0,\nu_1) &= -\frac{D(k+q-l)}{k[A+C(k-1)]}v_{k+q-l}^\lambda(\nu_0,\nu_1) & \text{for } l-q<k<l,\ k\neq q \\ v_q^\lambda(\nu_0,\nu_1) &= \nu_1 & \text{for } k=q \\ v_l^\lambda(\nu_0,\nu_1) &= \frac{\lambda\nu_0-Dq\nu_1-1}{l[A+C(l-1)]} & \text{for } k=l \\ v_k^\lambda(\nu_0,\nu_1) &= \frac{[\lambda-B(k-l)]v_{k-l}^\lambda(\nu_0,\nu_1)-D(k+q-l)v_{k+q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \text{for } l<k<q+l \\ v_k^\lambda(\nu_0,\nu_1) &= \frac{[\lambda-B(k-l)]v_{k-l}^\lambda(\nu_0,\nu_1)+v_{k-q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \text{for } q+l\leq k \\ & \quad -\frac{[D(k+q-l)]v_{k+q-l}^\lambda(\nu_0,\nu_1)}{k[A+C(k-1)]} & \end{cases} \tag{3.63}$$

Theorem 3.2.5. *When $q>2p$, the solutions to the inhomogeneous ODE (3.61) are the $i^\lambda(\nu_0,\nu_1,x)$.*

Proof. This follows from assuming the form $i(\lambda,x)=\sum_{n=0}^{\infty} v_n x^{\frac{k}{q}}$ and computing the resulting analytic coefficients. □

As in the equity case, we proceed to approximate the model by adding an absorption at the high level B since we do not know the asymptotics of the $h^\lambda(\nu_0,\nu_1,x)$.

Theorem 3.2.6. *For $q>2p$ with an absorbing condition at the origin 0, the Laplace transform of the zero-coupon bond price can be approximated to any arbitrary level of closeness by the function*

$$\hat{P}_B(\lambda)=\left[i^\lambda\left(\frac{1}{\lambda},0,x\right)-\frac{i^\lambda(\frac{1}{\lambda},0,B)}{i^\lambda(\frac{1}{\lambda},1,B)}i^\lambda(\frac{1}{\lambda},1,x)\right]\left(1-\frac{i^\lambda(\frac{1}{\lambda},0,B)}{i^\lambda(0,1,B)}\right)^{-1} \tag{3.64}$$

Proof. This proof can be conducted in the same way as the proof of Theorem 1.3.7 in Chapter 1. □

3.2.2 Considering an eigenfunction expansion approach

The zero-coupon bonds can, in theory, be expanded in eigenfunctions for $q > 2p$. To discuss this possibility, we follow the steps of the methodology presented in the general Appendix C.

The homogeneous equation can be transformed into a self-adjoint one with

$$s(x) = e^{\int^x \frac{2(a-by)}{\sigma^2 y^{\frac{2p}{q}}} dy} = e^{\frac{2}{\sigma^2}\left(\frac{aq}{q-2p} x^{1-\frac{2p}{q}} - \frac{bq}{l} x^{2-\frac{2p}{q}}\right)} \tag{3.65}$$

and

$$w(x) = \frac{2e^{\frac{2}{\sigma^2}\left(\frac{aq}{q-2p} x^{1-\frac{2p}{q}} - \frac{bq}{l} x^{2-\frac{2p}{q}}\right)}}{\sigma^2 x^{\frac{2p}{q}}} \tag{3.66}$$

The solutions to the homogeneous equation so that $\eta(\lambda, 0) = 0$ and $\xi(\lambda, B) = 0$ are

$$\eta(\lambda, 0) = h^\lambda(0, 1, x) \tag{3.67}$$

and

$$\xi(\lambda, 0) = h^\lambda(1, 0, x) - \frac{h^\lambda(1, 0, B)}{h^\lambda(0, 1, B)} h^\lambda(0, 1, x) \tag{3.68}$$

The eigenvalues are the zeroes of the function $\lambda \mapsto \eta(\lambda, B)$ in λ.

The coefficients of this expansion are

$$\begin{aligned} \alpha_n &= \int_0^B h^{\lambda_n}(0, 1, x) \frac{2e^{\frac{2}{\sigma^2}\left(\frac{aq}{q-2p} x^{1-\frac{2p}{q}} - \frac{bq}{l} x^{2-\frac{2p}{q}}\right)}}{\sigma^2 x^{\frac{2p}{q}}} dx \\ &= \sum_{k=0}^{\infty} u_k^\lambda(0, 1) \int_0^B x^{\frac{k}{q}} \frac{2e^{\frac{2}{\sigma^2}\left(\frac{aq}{q-2p} x^{1-\frac{2p}{q}} - \frac{bq}{l} x^{2-\frac{2p}{q}}\right)}}{\sigma^2 x^{\frac{2p}{q}}} dx \end{aligned} \tag{3.69}$$

and the norms

$$\| h^{\lambda_n}(0, 1, x) \|^2 = \int_0^B (h^{\lambda_n}(0, 1, x))^2 \frac{2e^{\frac{2}{\sigma^2}\left(\frac{aq}{q-2p} x^{1-\frac{2p}{q}} - \frac{bq}{l} x^{2-\frac{2p}{q}}\right)}}{\sigma^2 x^{\frac{2p}{q}}} dx \tag{3.70}$$

Though expansions and other tricks could be considered to deal with these integrals, these calculations seem quite involved. Yet, the eigenfunction expansion could be very useful for this problem, as it would allow a fast computation of the whole term-structure of zero-coupon prices, once the main quantities relating to the calculation, λ_n, α_n and $\| h^{\lambda_n}(0,1,x) \|^2$ have been evaluated once. Practically, the question implied is: what amount of initial calculation would be acceptable or can this amount be reduced to an acceptable level? This issue is here left to be explored in a future research work, as it should be the object of a separate study. We though briefly wanted to present how the methodology would work, since the idea remains valuable.

3.2.3 Numerical example

As in the equity case, we present one example in details. We place ourselves in the case: $b = 0.2$, $\frac{a}{b} := 0.1$, $\sigma = 0.2$, $p = 1$ and $q = 4$.

For[1] $\lambda = 7.5$, we study the behaviour of the solution (3.64) for different r_0.

Table 3.7 draws the basis functions $i^\lambda(\frac{1}{\lambda}, 0, x)$ denoted $f1$ and $i^\lambda(\frac{1}{\lambda}, 1, x)$ denoted $f2$. $n1$ and $n2$ represent the terms needed in these respective summations for the error to be inferior to 10^{-4}. Both functions recess to infinity.

x	f1	n1	f2	n2
0.1	0.133333	0	0.133432	0
0.15	0.133333	0	0.133825	1
0.2	0.133333	0	0.134861	4
0.3	0.133344	0	0.141089	11
0.4	0.133533	10	0.164404	22
0.5	0.135931	25	0.305733	34
0.6	0.174035	45	2.138934	56
0.7	1.538270	64	65.998457	86
0.8	179.310360	120	8376.717839	134
0.9	131727.134	190	6158053.022	204
1	951344667.3	287	44474027808	302
1.25	1.77061E+25	752	8.27734E+26	770
1.5	1.30359E+58	1804	6.09409E+59	1824

Table 3.7: Evolution of the basis functions with respect to r_0 .

[1] This choice would come from the choice $A = 15$ in the Abate and Whitt algorithm for example.

From Table 3.8, we can see that the approximate Laplace transform is quite stable in the regions of interest and $B = 1$ is sufficient. "const" in this table denotes the adjustment ratio $\frac{i^{\lambda}(\frac{1}{\lambda},0,B)}{i^{\lambda}(\frac{1}{\lambda},1,B)}$.

B	0.7	1	1.25	1.5
const	0.023308	0.021391	0.021391	0.021391
x		$\hat{P}_B$		
0.1	0.133331	0.133331	0.133331	0.133331
0.15	0.133322	0.133323	0.133323	0.133323
0.2	0.133297	0.133300	0.133300	0.133300
0.3	0.133159	0.133174	0.133174	0.133174
0.4	0.132796	0.132858	0.132858	0.132858
0.5	0.131879	0.132220	0.132220	0.132220
0.6	0.127145	0.131086	0.131086	0.131086
0.7		0.129261	0.129261	0.129261
0.8		0.126557	0.126557	0.126557
0.9		0.122799	0.122815	0.122815
1			0.117943	0.117943
1.25				0.101069

Table 3.8: Evolution of the approximate Laplace transform for different B .

We also draw the real part of the Laplace transform for complex λ in Figure 3.2. Although the corresponding figures are not presented here for simplicity, we observed that the number of terms needed to reach convergence in the basis functions increase as the imaginary part of λ increases.

Globally, the series converge fast for the r_0 we would expect. Computing the basis functions at the boundary B point can yet take many more terms. But, the advantage of this method resides in the possibility of applying powerful techniques like fast fourier inversion.

3.3 Conclusion

In this chapter, we focus on the applications of the results derived in Chapter 1 to the world of fixed-income derivatives. We analysed one of the most fundamental

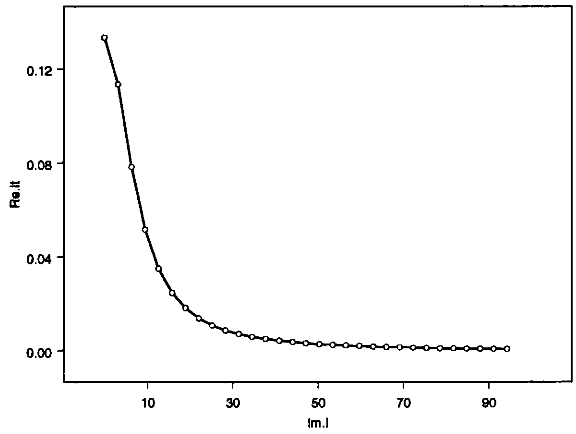


Figure 3.2: *Real part of the Laplace transform along a complex vertical line of abscissa 7.5.*

models in interest rate theory, the CIR model and produced explicit series formulae for the probability distribution function of the average instantaneous rate as well as other Asian or average-rate claims options. As previously, we observed that our series is a theoretical tool but also an efficient method to evaluate those quantities numerically specially when the volatility or the maturity is large.

We also considered the general CEV instantaneous rate model, adapting the results on equity CEV processes.

3.4 Appendices to this chapter

3.4.1 Proof of Theorem 3.1.1

From Appendix 1.5.1 of Chapter 1, the expectation that we are trying to Laplace-invert can be decomposed as

$$\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu} = e^{-(\gamma-b)\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m} \frac{(\gamma-b)^k e^{-m\gamma t}}{\gamma^{k-n}\mu}$$

where

$$\gamma = \sqrt{b^2 + 2\sigma^2(\lambda+\mu)}$$

Given that $\mu = \frac{(\gamma-\vartheta)(\gamma+\vartheta)}{2\sigma^2}$, ϑ being defined in (3.11), we are mainly interested in the inverse Laplace transforms of terms of the type $\frac{(\gamma-b)^p}{(\gamma-\vartheta)\gamma^q(\gamma+\vartheta)}e^{-\gamma\varpi}$, $\varpi \in \mathbb{R}^+$. The inverse of this expression, named $\tilde{I}_{p,q}(y,\varpi)$ hereafter, can be written in an integro-differential way, with the same type of arguments[2] as those used in Appendix 1.5.1 of Chapter 1. The case $q > 0$ and $q = 0$ give rise to different expressions and need hence to be treated separately.

<u>CASE $q > 0$</u>

The inverse Laplace transform in this case can be expressed in the form

$$\tilde{I}_{p,q}(y,\varpi) = (-1)^p \frac{\partial^p}{\partial\zeta^p}\left[\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \tilde{J}_q^0(y,\varpi,\zeta,u,h^+,h^-)dh^+dh^-du\right] \tag{3.71}$$

where $\tilde{J}_q^0(y,\varpi,\zeta,u,h^+,h^-)$ is given by

$$\beta\frac{e^{-\frac{(\varpi+\zeta+u+h^++h^-)^2}{4y\beta}+b\zeta+\vartheta(h^--h^+)-\vartheta^2 y\beta}(\varpi+\zeta+u+h^++h^-)u^{q-1}}{2\sqrt{\pi(y\beta)^3}(q-1)!}$$

[2]Justifications for interchanging integrals or integrals and differentiation can be worked out in the same way.

However, it is easier to study the simpler expression

$$\tilde{K}_{p,q}(y,\varpi) = (-1)^p \frac{\partial^p}{\partial\zeta^p}\left[\int_0^\infty\int_0^\infty\int_0^\infty \tilde{J}^1_q(y,\varpi,\zeta,u,h^+,h^-)dh^+dh^-du\right] \tag{3.72}$$

where:

$$\tilde{J}^1_q(y,\varpi,\zeta,u,h^+,h^-) = \beta\frac{e^{-\frac{(\varpi+\zeta+u+h^++h^-)^2}{4y\beta}+b\zeta+\vartheta(h^--h^+)-\vartheta^2y\beta}u^{q-1}}{2\sqrt{\pi(y\beta)^3}(q-1)!}$$

To link this with the original expression (3.71), we integrate by part

$$\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+u+h^++h^-)^2}{4y\beta}+b\zeta+\vartheta(h^--h^+)-\vartheta^2y\beta}(\varpi+\zeta+u+h^++h^-)u^{q-1}}{2\sqrt{\pi(y\beta)^3}(q-1)!}du$$

$$= 2y\beta\left(1_{\{q=1\}}\tilde{J}^1_1(y,\varpi,\zeta,0,h^+,h^-) + 1_{\{q>1\}}\int_0^\infty \tilde{J}^1_{q-1}(y,\varpi,\zeta,u,h^+,h^-)du\right)$$

which implies that, for $q > 0$,

$$\tilde{I}_{p,q}(y,\varpi) = 1_{\{q>1\}}\{2y\beta\tilde{K}_{p,q-1}(y,\varpi)\} + 1_{\{q=1\}}\left\{(-1)^p(2y\beta)\frac{\partial^p}{\partial\zeta^p}\tilde{L}(y,\varpi)\right\} \tag{3.73}$$

where

$$\tilde{L}(y,\varpi) = \int_0^\infty\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+h^++h^-)^2}{4y\beta}+b\zeta+\vartheta(h^--h^+)-\vartheta^2y\beta}}{2\sqrt{\pi(y\beta)^3}}dh^+dh^-$$

The inverse $\tilde{I}_{p,q}(y,\varpi)$ can thus be recovered from the study of $\tilde{K}_{p,q-1}(y,\varpi)$ and $\tilde{L}_p(y,\varpi) = (-1)^p\frac{\partial^p}{\partial\zeta^p}\tilde{L}(y,\varpi)$.

To start with the calculation of $\tilde{K}_{p,q}(y,\varpi)$, the double integral

$$\tilde{K}^a_q(y,\varpi) = \int_0^\infty\int_0^\infty \tilde{J}^1_q(y,\varpi,\zeta,u,h^+,h^-)dh^+dh^-$$

can be simplified using the fact that its inner integral can be reexpressed as

$$\int_0^\infty \tilde{J}^1_q(y,\varpi,\zeta,u,h^+,h^-)dh^+ = \int_{\frac{\varpi+\zeta+u+h^-+2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \tilde{J}^2_q(y,\varpi,\zeta,u,h^-,g)dg$$

$$\tilde{J}^2_q(y,\varpi,\zeta,u,h^-,g) = \frac{e^{-\frac{g^2}{2}+\vartheta(\varpi+\zeta+u+2h^-)+b\zeta}u^{q-1}}{y\sqrt{2\pi}(q-1)!}$$

Integrating by part $\int_0^\infty \int_{\frac{\varpi+\zeta+u+h^- +2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \tilde{J}_q^2(y,\varpi,\zeta,u,h^-,g)dgdh^-$ then gives the following expression

$$\tilde{K}_q^a(y,\varpi) = \beta \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u+h^- +2y\beta\vartheta)^2}{4y\beta}+b\zeta+\vartheta(\varpi+\zeta+u)}}{4\vartheta\sqrt{\pi(y\beta)^3}(q-1)!}(e^{2\vartheta h^-}-1)u^{q-1}dh^- \tag{3.74}$$

Denoting

$$\tilde{J}_q^3(y,\varpi,\zeta,u,h^-) = \beta\frac{e^{-\frac{(\varpi+\zeta+u+h^- +2y\beta\vartheta)^2}{4y\beta}+b\zeta+\vartheta(\varpi+\zeta+u+2h^-)}}{4\vartheta\sqrt{\pi(y\beta)^3}(q-1)!}u^{q-1}$$

we split $\tilde{K}_q^a(y,\varpi)$ in two and are next first interested in the quantity

$$\tilde{K}_q^{b,1}(y,\varpi) = \int_0^\infty \int_0^\infty \tilde{J}_q^3(y,\varpi,\zeta,u,h^-)dh^-du$$

After transforming the inner integral as previously,

$$\int_0^\infty \tilde{J}_q^3(y,\varpi,\zeta,u,h^-)dh^- = \int_{\frac{\varpi+\zeta+u-2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta-\vartheta(\varpi+\zeta+u)}}{2\vartheta y\sqrt{2\pi}(q-1)!}u^{q-1}dg$$

integration by part of $\tilde{K}_q^{b,1}(y,\varpi)$ leads to:

$$\tilde{K}_q^{b,1}(y,\varpi) = \beta\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u-2y\beta\vartheta)^2}{4y\beta}+b\zeta-\vartheta(\varpi+\zeta+u)}}{4\vartheta\sqrt{\pi(y\beta)^3}q!}u^q du$$
$$+\int_0^\infty \int_{\frac{\varpi+\zeta+u-2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta-\vartheta(\varpi+\zeta+u)}}{2y\sqrt{2\pi}q!}u^q dgdu \tag{3.75}$$

Defining

$$\tilde{K}_q^c(y,\varpi) = \beta\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u-2y\beta\vartheta)^2}{4y\beta}+b\zeta-\vartheta(\varpi+\zeta+u)}}{4\vartheta\sqrt{\pi(y\beta)^3}(q-1)!}u^{q-1}du \tag{3.76}$$

(3.75) becomes

$$\tilde{K}_q^{b,1}(y,\varpi) = \vartheta\tilde{K}_{q+1}^{b,1}(y,\varpi) + \tilde{K}_{q+1}^c(y,\varpi) \tag{3.77}$$

The second expression we need to study to get (3.74) is

$$\tilde{K}_q^{b,2}(y,\varpi) = \beta\int_0^\infty \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u+h^- +2y\beta\vartheta)^2}{4y\beta}+b\zeta+\vartheta(\varpi+\zeta+u)}}{4\vartheta\sqrt{\pi(y\beta)^3}(q-1)!}u^{q-1}dh^-du$$

As previously, integration by part gives

$$\tilde{K}_q^{b,2}(y,\varpi) = \beta \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u+2y\beta\vartheta)^2}{4y\beta}+b\zeta+\vartheta(\varpi+\zeta+u)}}{4\vartheta\sqrt{\pi(y\beta)^3}q!}u^q du$$

$$- \int_0^\infty \int_{\frac{\varpi+\zeta+u+2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta+\vartheta(\varpi+\zeta+u)}}{2y\sqrt{2\pi}q!}u^q dg du \tag{3.78}$$

We notice the the first integral on the right-side of (3.78) is exactly the same as the one in (3.75). The counterpart to (3.77) is thus

$$\tilde{K}_{p,q}^{b,2}(y,\varpi) = -\vartheta \tilde{K}_{p,q+1}^{b,2}(y,\varpi) + \tilde{K}_{p,q+1}^{c}(y,\varpi) \tag{3.79}$$

Computing $\tilde{K}_q^{b,1}(y,\varpi)$ and $\tilde{K}_q^{b,2}(y,\varpi)$ then amounts to compute their initial values and as well as the integrals $\tilde{K}_q^{c,2}(y,\varpi)$ and $\tilde{K}_q^{c,2}(y,\varpi)$.

The initial values can actually also be computed through integration by parts but expressing the integrand as the product of a couple of functions different from the one chosen in (3.78):

$$\tilde{K}_1^{b,1}(y,\varpi) = \int_0^\infty \int_{\frac{\varpi+\zeta+u-2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta-\vartheta(\varpi+\zeta+u)}}{2\vartheta y\sqrt{2\pi}} dg du$$

$$= \int_{\frac{\varpi+\zeta-2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta-\vartheta(\varpi+\zeta)}}{2\vartheta^2 y\sqrt{2\pi}} dg - \beta \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u-2y\beta\vartheta)^2}{4y\beta}+b\zeta-\vartheta(\varpi+\zeta+u)}}{4\vartheta^2\sqrt{\pi(y\beta)^3}} du$$

Therefore,

$$\tilde{K}_1^{b,1}(y,\varpi) = \beta \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}+b\zeta-\vartheta^2 y\beta}}{4\vartheta^2\sqrt{\pi(y\beta)^3}}(e^{\vartheta u}-1)du$$

In the same way,

$$\tilde{K}_2^{b,1}(y,\varpi) = \int_0^\infty \int_{\frac{\varpi+\zeta+u+2y\beta\vartheta}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta+\vartheta(\varpi+\zeta+u)}}{2\vartheta y\sqrt{2\pi}} dg du$$

$$= \beta \int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}+b\zeta-\vartheta^2 y\beta}}{4\vartheta^2\sqrt{\pi(y\beta)^3}}(1-e^{-\vartheta u})du$$

To obtain $\tilde{K}_{p,q}(y,\varpi)$, we need now to differentiate p times these different expressions with respect to ζ.

Denoting

$$\begin{aligned}\tilde{I}^1_{p,q}(y,\varpi) &= (-1)^p\tfrac{\partial^p}{\partial\zeta^p}\tilde{K}^{b,1}_q(y,\varpi)\\ \tilde{I}^2_{p,q}(y,\varpi) &= (-1)^p\tfrac{\partial^p}{\partial\zeta^p}\tilde{K}^{b,2}_q(y,\varpi)\\ \tilde{I}^3_{p,q}(y,\varpi) &= 2\vartheta(-1)^p\tfrac{\partial^p}{\partial\zeta^p}\tilde{K}^c_{p,q}(y,\varpi)\end{aligned}$$

(3.77) and (3.79) are respectively transformed into

$$\tilde{I}^1_{p,q+1}(y,\varpi) = \frac{1}{\vartheta}\tilde{I}^1_{p,q}(y,\varpi) - \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\varpi)$$

and

$$\tilde{I}^2_{p,q+1}(y,\varpi) = -\frac{1}{\vartheta}\tilde{I}^2_{p,q}(y,\varpi) + \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\varpi)$$

The term

$$\tilde{I}^3_{p,q}(y,\varpi) = (-1)^p\frac{\partial^p}{\partial\zeta^p}\left[\int_0^\infty \frac{\beta e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}+b\zeta-\vartheta^2 y\beta}}{4\vartheta\sqrt{\pi(y\beta)^3}(q-1)!}u^{q-1}du\right]$$

is very similar to the term

$$\tilde{J}_{p,q}(y,\varpi) = (-1)^p\beta\frac{\partial^p}{\partial\zeta^p}\int_0^\infty e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-b^2y\beta}\frac{e^{b\zeta}}{2\sqrt{\pi(y\beta)^3}}\frac{u^{q-1}}{(q-1)!}du$$

studied in Appendix 1.5.2, Chapter 1. The analysis of $\tilde{I}^3_{p,q}(y,\varpi)$ can be carried out in the same way with only minor changes in the resulting formulae. These changes are basically only the replacement of $b^2y\beta$ with $\vartheta^2y\beta$ in the exponentials. Hence, (1.108) becomes:

$$\begin{aligned}\tilde{I}^3_{p+1,q}(y,\varpi) &= -b\tilde{I}^3_{p,q}(y,\varpi) + 1_{\{q>1\}}\tilde{I}^3_{p,q-1}(y,\varpi)\\ &+1_{\{q=1\}}\left\{\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2y\beta}}{(\sqrt{2y\beta})^p}\frac{\beta}{2\sqrt{\pi(y\beta)^3}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)\right\}\end{aligned} \tag{3.80}$$

Similarly, we obtain (3.12), (3.13) and (3.14) respectively.

For the initial values

$$\tilde{I}^1_{p+1,1}(y,\varpi) = (-1)^{p+1}\frac{\partial^{p+1}}{\partial\zeta^{p+1}}\left[\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-\vartheta^2 y\beta+b\zeta}(e^{\vartheta u}-1)}{4\vartheta^2\sqrt{\pi(y\beta)^3}}du\right] \tag{3.81}$$

$$= -b\tilde{I}^1_{p,1}(y,\varpi) - (-1)^p\frac{\partial^p}{\partial\zeta^p}\left\{\left[\beta\frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-\vartheta^2 y\beta+b\zeta}}{4\vartheta^2\sqrt{\pi(y\beta)^3}}(e^{\vartheta u}-1)\right]_0^\infty - \int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-\vartheta^2 y\beta+b\zeta}}{4\vartheta\sqrt{\pi(y\beta)^3}}e^{\vartheta u}du\right\}$$

which gives (3.16).

Setting $p=-1$ in (3.81) leads to the expression (3.17).

In the same way,

$$\tilde{I}^2_{p+1,1}(y,\varpi) = (-1)^{p+1}\frac{\partial^{p+1}}{\partial\zeta^{p+1}}\left[\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-\vartheta^2 y\beta+b\zeta}(1-e^{-\vartheta u})}{4\vartheta^2\sqrt{\pi(y\beta)^3}}du\right] \tag{3.82}$$

which, similarly, lead to (3.19) and (3.20).

Now $\tilde{I}^1_{p,q}(y,\varpi)$ and $\tilde{I}^2_{p,q}(y,\varpi)$ are fully determined, it is possible to compute the expression (3.72) we wanted:

$$\tilde{K}_{p,q}(y,\varpi) = \tilde{I}^1_{p,q}(y,\varpi) - \tilde{I}^2_{p,q}(y,\varpi) \tag{3.83}$$

which arises from (3.74).

The term $\tilde{L}_p(y,\varpi) = (-1)^p\frac{\partial^p}{\partial\zeta^p}\tilde{L}(y,\varpi)$ in the equation (3.73) still remains to be analysed:

$$\tilde{L}(y,\varpi) = \int_0^\infty\int_{\frac{(\varpi+\zeta+h^-+2y\beta)}{\sqrt{2y\beta}}}^\infty \frac{e^{-\frac{g^2}{2}+b\zeta+\vartheta(\varpi+\zeta+2h^-)}}{y\sqrt{2\pi}}dh^+dh^-$$

$$= \frac{\beta}{2\vartheta}\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}+b\zeta-\vartheta^2 y\beta}}{2\sqrt{\pi(y\beta)^3}}(e^{\vartheta u}-e^{-\vartheta u})du \tag{3.84}$$

This, in turn, leads to

$$\begin{aligned}\tilde{L}_p(y,\varpi) &= (-1)^p \frac{\beta}{2\vartheta}\frac{\partial^p}{\partial\zeta^p}\left[\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}+b\zeta-\vartheta^2 y\beta}}{2\sqrt{\pi(y\beta)^3}}(e^{\vartheta u}-e^{-\vartheta u})du\right] \\ &= \vartheta(\tilde{I}^1_{p,1}(y,\varpi)+\tilde{I}^2_{p,1}(y,\varpi))\end{aligned}$$

which completes the inversion for $q>0$

CASE $q=0$

In this case, the inverse Laplace transform of $\frac{(\gamma-b)^p e^{-\gamma\varpi}}{(\gamma+\vartheta)(\gamma-\vartheta)}$ is given by

$$\tilde{I}_{p,q}(y,\varpi) = (-1)^p \frac{\partial^p}{\partial\zeta^p}\left[\int_0^\infty\int_0^\infty \tilde{J}^4(y,\varpi,\zeta,h^+,h^-)dh^+dh^-\right]$$

with

$$\tilde{J}^4(y,\varpi,\zeta,h^+,h^-) = \frac{\beta e^{-\frac{(\varpi+\zeta+h^++h^-)^2}{4y\beta}+b\zeta-\vartheta(\vartheta y\beta-h^-+h^+)}(\varpi+\zeta+h^++h^-)}{2\sqrt{\pi(y\beta)^3}}$$

once again following the kind same of derivation as the one used in the Appendix 1.5.1 of Chapter 1.

Manipulating the inner integral as previously and integrating by parts

$$\tilde{I}_{p,0}(y,\varpi) = (-1)^p\frac{\partial^p}{\partial\zeta^p}\left[(2y\beta)\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+h^-)^2}{4y\beta}+b\zeta-\vartheta(\vartheta y\beta-h^-)}}{2\sqrt{\pi(y\beta)^3}}dh^-\right.$$

$$\left.-(2y\beta)\vartheta\int_0^\infty\int_0^\infty \beta\frac{e^{-\frac{(\varpi+\zeta+h^-+h^+)^2}{4y\beta}+b\zeta-\vartheta(\vartheta y\beta+h^+-h^-)}}{2\sqrt{\pi(y\beta)^3}}dh^+dh^-\right]$$

$$= 2y\beta\Big(2\vartheta^2\tilde{I}^1_{p,1}(y,\varpi)+\tilde{I}^3_{p,1}(y,\varpi)\Big) - 2y\beta\vartheta^2\Big(\tilde{I}^1_{p,1}(y,\varpi)+\tilde{I}^2_{p,1}(y,\varpi)\Big)$$

This identity completes the study and proves the first line in (3.23).

Chapter 4

Stochastic volatility models

The hypothesis of constant volatility assumed by Black, Scholes and Merton has been questioned early. In 1976, Fisher Black writes: "Suppose we use the standard deviation ... of possible future returns on a stock ... as a measure of its volatility. Is it reasonable to take that volatility as constant over time? I think not"(See Lewis [50]). For most financial equities and indexes, statistical tests strongly reject the the idea they could have been generated by a constant volatility. Derivatives markets practitioners also observe that call and put options prices emerging from the the economic law of supply and demand produce a pattern for the implied volatilities - the smile - that invalidates Black-Scholes assumptions. This smile, happily named after the shape of the implied volatility surface, started to be particularly noticeable after the crash of 1987.

Indeed, the implied volatility, σ^{imp}, is defined as the positive real value, which fed into the Black-Scholes analytic formula, BS^C for call options of right maturity T and strike K gives the actual market price of the call $C(S, K, T, r)$

$$\text{BS}^C(S, K, T, \sigma^{\text{imp}}, r, \delta) = C(S, K, T, r, \delta) \tag{4.1}$$

δ denoting the continuous proportional dividend rate[1].

Under the Black-Scholes assumptions, this value should remain constant across maturity and strikes. In practice, the implied volatility is rather a local parabola (smile) or can even exhibit others shapes (smirks, etc.). Stochastic volatility models are a natural extension of the Black-Scholes model which allow the implied volatility patterns to appear (See Das and Sundaram [19] for example). Although many other alternatives can be considered (jumps, local volatlity approach, etc.) to obtain an implied volatility surface, stochastic volatility models are relatively simple and capture most of the important features of the smile.

Stochastic volatility models constitute for us both an interesting and important area of application. Indeed, the main models developed are based on constant elasticity of variance processes: the Geometric Brownian motion for Hull and White [41], the Ornstein-Ulenbech for Stein and Stein [67] and the square-root process for Heston [40]. The temporal integral of the variance related CEV process is involved in all of these model and generally in a complex way. We therefore devote a whole chapter to this class of models.

After having recalled the main results concerning these reference models, we will first provide more insight in their structure, by highlighting the role of two conditioning variables. We will explain how some results on these reference models can be retrieved in a very simple and intuitive way through the use of moment generating functions and characteristic functions. We will also extend and apply this method to the Hull and White model so as to obtain a close-form solution by incorporating results gained in the not so independent problem of Asian options pricing. We

[1]Until now, we have not introduced proportional dividend rate in our equity derivatives model, as our framework and results can be straightforwardly generalised in this case. In this chapter though, it is quite convenient to introduce it because as we will see later, stochastic volatility brings a stochastic dividend rate into play.

then focus on approximating the call option price through expansion, more precisely moments-based expansion. After providing new expressions for the joint moments of the conditioning variables for the reference models, we will compare and extend different asymptotic expansions proposed in the literature. We will finally present two convergent expansions, which are of theorical interest for their convergence and of practical utility when it comes to compute a whole surface prices for different strikes and maturity.

We evaluate either one of the call or put options as the model independent call-put parity allows us to deduce one from the other.

4.1 Reference models

4.1.1 The Hull-White model

This model is of historical importance, since the geometric Brownian motion was the first diffusion proposed to model the stochastic volatility of the index, otherwise conditionally lognormal itself.

$$\begin{cases} \frac{dS_t^{\mathrm{HW}}}{S_t^{\mathrm{HW}}} & = & \mu dt + \sigma_t^{\mathrm{HW}} d\hat{W}_t^1 \\ \frac{d\sigma_t^{\mathrm{HW}}}{\sigma_t^{\mathrm{HW}}} & = & \mu_\sigma dt + \xi d\hat{W}_t^2 \\ < d\hat{W}_t^1, d\hat{W}_t^2 > & = & \rho dt \end{cases} \tag{4.2}$$

The market premium for volatility risk is then taken to be null, meaning that, under the resulting risk-neutral measure, we have:

$$\begin{cases} \frac{dS_t^{\mathrm{HW}}}{S_t^{\mathrm{HW}}} & = & (r-\delta)dt + \sigma_t^{\mathrm{HW}} dW_t^1 \\ \frac{d\sigma_t^{\mathrm{HW}}}{\sigma_t^{\mathrm{HW}}} & = & \mu_\sigma dt + \xi dW_t^2 \\ < d\hat{W}_t^1, d\hat{W}_t^2 > & = & \rho dt \end{cases} \tag{4.3}$$

Hull and White then produce an approximate solution series in the case the two Brownian motions are independent. The underlying process is then:

$$S_t^{\mathrm{HW}} = S_0 e^{rT - \frac{1}{2}\int_0^T (\sigma_s^{\mathrm{HW}})^2 ds + \int_0^T \sigma_s^{\mathrm{HW}} dW_s} \tag{4.4}$$

This expression is actually valid for any stochastic volatility diffusion as long as the asset S_t follows the same stochastic differential equation and the Brownian motions leading the asset and the volatility are independent. The argument of the exponential is a normal of mean $(r-\delta)T - \frac{1}{2}\int_0^T (\sigma_s^{\mathrm{HW}})^2 ds$ and variance $\int_0^T (\sigma_s^{\mathrm{HW}})^2 ds$ conditionally on the accumulated variance $\int_0^T (\sigma_s^{\mathrm{HW}})^2 ds$. Hence, the price of the a call option in this framework is an average of Black-Scholes call option prices:

$$C(S_0, \sigma_0, K, T) = \int_0^\infty \mathrm{BS}^C(S_0, K, T, \sqrt{v}, r, \delta) f_T^{HW}(v|\sigma_0) dv \tag{4.5}$$

where $f_T^{HW}(v|\sigma_0)$ represents the density of the random variable $\frac{\int_0^T (\sigma_s^{\mathrm{HW}})^2 ds}{T}$ given the initial value σ_0.

Proposition 4.1.1. *(See Hull and White [41]). When $\mu_\sigma = 0$, Hull and White give the approximation to the call option price*

$$\begin{aligned} C^{\mathrm{HW}}(S_0, \sigma_0, K, T) &\simeq \mathrm{BS}^C(S_0, K, T, \sigma_0, r, \delta) \\ &+ \frac{S_0 \sigma_0 \sqrt{T} \mathcal{N}'(d_1)(d_1 d_2 - 1)}{8} \left[\frac{2(e^k - k - 1)}{k^2} - 1 \right] \\ &+ \frac{S_0 \sqrt{T} \mathcal{N}'(d_1)[(d_1 d_2 - 3)(d_1 d_2 - 1) - (d_1^2 + d_2^2)]}{48 \sigma_0^5} \\ &+ \sigma_0^6 \frac{e^{3k} - (9 + 18k)e^k + (8 + 24k + 18k^2 + 6k^3)}{3k^3} + \dots \end{aligned} \tag{4.6}$$

where $\mathcal{N}$ is the probability distribution function of the standardised normal and d_1, d_2 are the functions used inside the Black-Scholes formula. More precisely,

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma_0^2}{2}\right)T}{\sigma_0 \sqrt{T}} \\ d_2 &= d_1 - \sigma_0 \sqrt{T} \end{aligned}$$

A similar formula is also proposed for $\mu \neq 0$, with one order less, though, in the Taylor development.

Hull and White also provide a Monte-Carlo analysis for the correlated case ($\rho \neq 0$) and present the evolution of the the prices with the strike for different correlations.

4.1.2 The Stein and Stein model

The previous model had two main drawbacks. Firstly, the volatility does not have a stationary limit. In the case $\mu_\sigma \neq 0$, the volatility either increases to infinity or decreases to 0 with time. This behaviour is not desirable in a volatility model. The second main defect of the Hull-White model is its lack of tractability.

The Stein and Stein model brings both tractability and a convergence in time to a stable economy by choosing an Ornstein-Uhlenbeck volatility process.

$$\left\{ \begin{array}{rcl} \frac{dS_t^{\mathrm{SS}}}{S_t^{\mathrm{SS}}} & = & \mu dt + \sigma_t^{\mathrm{SS}} d\hat{W}_t^1 \\ d\sigma_t^{\mathrm{SS}} & = & (\hat{a} - b\sigma_t^{\mathrm{SS}})dt + \xi d\hat{W}_t^2 \end{array} \right. \tag{4.7}$$

where the two Brownian motions $\hat{W}_t^1$ and $\hat{W}_t^2$ are independent.

Assuming a constant market price of volatility risk λ, the process diffusion under the risk-neutral measure becomes:

$$\left\{ \begin{array}{rcl} \frac{dS_t^{\mathrm{SS}}}{S_t^{\mathrm{SS}}} & = & (r - \delta)dt + \sigma_t^{\mathrm{SS}} dW_t^1 \\ d\sigma_t^{\mathrm{SS}} & = & (a - b\sigma_t^{\mathrm{SS}})dt + \xi dW_t^2 \end{array} \right. \tag{4.8}$$

where $a = \hat{a} - \lambda\xi$

The independence of the Brownian motions allows us to write the underlying process under the form (4.4). Stein and Stein solve the call option pricing problem in the following way:

Proposition 4.1.2. *(See Stein and Stein [67]). The call price can be evaluated by integrating the payoff $(S_T - K)^+$ against the density $f_T^{SS}(s)$ of S_T^{SS} in the risk-neutral* *measure.*

$$C^{\mathrm{SS}}(S_0, \sigma_0, K, T) = e^{-rT} \int_K^{\infty} (s-K) f_T^{SS}(s) ds \tag{4.9}$$

This density can be numerically calculated as an inverse Fourier transform

$$f_T^{SS}(s) = \frac{1}{S_0} \frac{e^{-rT}}{2\pi} \left(\sqrt{\frac{S_0}{s}} \right)^3 \int_{-\infty}^{\infty} I^{\mathrm{SS}} \left(\left(\eta^2 + \frac{1}{4} \right) \frac{T}{2} \right) e^{i\eta \ln \frac{s}{S_0}} d\eta \tag{4.10}$$

where the function I^{SS} is given in Appendix 4.5.1.

This method is implying first a numerical Fourier inversion and then another numerical integration, hence a double integral. It is actually possible to solve the problem with a less burdensome method involving one and only one Fourier transform inversion, a real one-dimensional integral. This will be investigated in Section 4.1.4.

This section on the Stein and Stein model cannot be ended without pointing out the important fact that, although this model does not possess the two major drawbacks of the Hull-White lognormal stochastic volatility setting, it still has one major defect: the volatility can go negative. A lot of controversy arised from this. Stein and Stein justified their model by emphasizing the fact that the volatility enters the process only as in squared way (see (4.4)). These authors conclude from this that the sign of σ_t^{SS} does not matter and their model can be interpreted as equivalent to reflecting σ_t^{SS} at 0. Ball and Roma [5] contradicted this interpretation by proving that reflecting the Ornstein-Uhlenbech at 0 would completely modify the density of the process, which is not what happens here. Though right in this, they wrongly conclude that Stein and Stein are taking the absolute value of σ_t^{SS}. It is true that σ_t^{SS} enters the model only in a squared-fashion. Yet, the value and hence the sign of σ_t^{SS} condition the subsequent behaviour of the volatility for time above t. This can easily be intuitively seen for a mean level $\frac{a}{b} > 0$. Suppose that $|\sigma_0^{\mathrm{SS}}|$ is slightly inferior to the mean-level; we will then expect the volatility to raise due to the attraction towards the mean level. When the initial volatility is negative, the process will then to decrease first in absolute value as the volatility increases in real value. Hence, the average $\frac{\int_0^T (\sigma_t^{\mathrm{SS}})^2 ds}{T}$ is likely to be inferior in the case $\sigma_0^{\mathrm{SS}} < 0$ compared to the case $\sigma_0^{\mathrm{SS}} > 0$. This shows that the sign of the volatility has an indirect influence on the value of $\frac{\int_0^T (\sigma_t^{\mathrm{SS}})^2 ds}{T}$ and therefore on the process behaviour. In the case, the two Brownian motions W_t^1 and W_t^2 are correlated, it becomes even more obvious that the sign of the volatility has an effect on the process, since σ_t^{SS} enters the formula in a non-squared fashion as well. Extending the Stein and Stein model by adding correlation and deducing the corresponding closed-form solution has been recently done by Masoliver and Perello [53] as well as Schobel and Zhu [64].

4.1.3 The Heston model

i. The original model

This model is labelled as typical by Lewis [50], in the sense that it displays the qualitative properties we expect in general from time-homogeneous cases. Drägulescu and V. M. Yakovenko [23] also show that probability distribution functions resulting from the Heston model fit quite well market data and possess a number of common properties. Under this model, the equity follows the diffusion:

$$\begin{cases} \frac{dS_t}{S_t} & = \mu dt + \sqrt{v_t} d\hat{W}_t^1 \\ dv_t & = (a - \hat{b} v_t)dt + \xi \sqrt{v_t} d\hat{W}_t^2 \\ < d\hat{W}_t^1, d\hat{W}_t^2 > & = \rho dt \end{cases} \tag{4.11}$$

This model groups the three advantages of both the previous reference models: the volatility cannot go negative, is mean-reverting and still leads to tractable closed-form expressions for call option prices.

Heston [40] assumes a proportional form for the market price of volatility risk, following economical consumption-based models. He proves that, under the risk-neutral measure arising from this assumption, the asset is driven by:

$$\begin{cases} \frac{dS_t}{S_t} & = & (r-\delta)dt + \sqrt{v_t}dW_t^1 \\ dv_t & = & (a - bv_t)dt + \xi\sqrt{v_t}dW_t^2 \\ < dW_t^1, dW_t^2 > & = & \rho dt \end{cases} \tag{4.12}$$

Basically, with this choice of volatility risk premium, the process retains the same form, which is very practical. Only the reversion strength, b, is different from its historical value.

Heston then solves for the price of a call option in the form

$$C^{\mathrm{H}}(S_0, v_0, K, T) = S_0 P_1(S_0, v_0, K, T) - KB(0,T)P_2(S_0, v_0, K, T) \tag{4.13}$$

where $B(0,T)$ is the price of the zero-coupon bond with the same maturity, T, as the option and $P_j, j = 1,2$ are the functions to be computed. More precisely, these functions are given by $P_1(S_0, v_0, K, T) = E\Big(\frac{S_T}{S_0}1_{\{S_T \geq K\}}\Big)$ and $P_2(S_0, v_0, K, T) = E\big(1_{\{S_T \geq K\}}\big)$ and therefore are both probabilities. Indeed, $P_1(S_0, v_0, K, T)$ is the probability of $\{S_T > K\}$ under S_T-numeraire measure given by the Radon-Nykodim derivative $\frac{dQ^x}{dQ} = \frac{S_T}{S_0}$. Heston characterises these probabilities by their Fourier transform.

Proposition 4.1.3. *(See Heston [40]). For $j \in \{1,2\}$, the $P_j(S_0, v_0, K, T)$ can be computed by numerical integration*

$$P_j(S_0, v_0, K, T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(\ln S_0, v_0, T, \phi)}{i\phi}\right] d\phi \tag{4.14}$$

where the transforms are given by:

$$f_j(x, v, T, \phi) = e^{C(T,\phi) + D(T,\phi)v + i\phi \ln S_0} \tag{4.15}$$

with

$$\begin{cases} C(T,\phi) = (r-\delta)\phi iT + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)T - 2\ln\left[\frac{1-ge^{dT}}{1-g}\right]\right\} \\ D(T,\phi) = \frac{b_j-\rho\sigma\phi i+d}{\sigma^2}\left[\frac{1-e^{dT}}{1-ge^{dT}}\right] \\ g = \frac{b_j-\rho\sigma\phi i+d}{b_j-\rho\sigma\phi i-d} \\ d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)} \end{cases}$$

and the parameters

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad b_1 = b - \rho\sigma, \quad b_2 = b$$

Numerous papers (Bakshi and al [4], Duffie and al. [24], Pan [59], Tompkins [69]) compared option pricing derived from this model and its extensions with empirical data on option pricing and found that the Heston model describes the empirical options prices, much better than the Black-Scholes theory and that its extension even further the agreement. Drägulescu and Yakovenko [23] also showed that the Dow Jones actual distribution itself is close to the theorical distribution resulting from the Heston model and analyse some of its important asymptotic properties, which turn out to be empirically verified by the index time-series.

ii. The Roma and Ball approach

Roma and Ball [5] considered in their paper the special case in which the correlation ρ is null, i.e. the Brownian motions W_t^1 and W_t^2 are independent and showed that the methodology followed by Stein and Stein in [67] can be replicated here.

Proposition 4.1.4. *The call option price can be computed as in (4.9) by integrating the payoff against the density function of the asset in the risk neutral measure. This density is given by:*

$$f^{\mathrm{H}}(s) = e^{(r-\delta)\frac{T}{2}}\frac{\sqrt{S_0}}{\sqrt{s^3}}\int_{-\infty}^{\infty}\frac{I^{\mathrm{H}}\left[\left(\eta^2+\frac{1}{4}\right)\frac{T}{2}\right]\cos\left[\left(\ln\frac{s}{S_0} - rT\right)\eta\right]}{2\pi}d\eta \tag{4.16}$$

with

$$I^{\mathrm{H}}(z) = \mathcal{L}^{X,Y}\left(0, \frac{z}{T}\right) \tag{4.17}$$

where $\mathcal{L}^{X,Y}(\lambda, \mu)$ is the joint MGF given in Proposition 1.1.4, Chapter 1.

The authors proceed in exactly the same way as Stein and Stein [67], with only the function $I(z)$ being changed. This method has therefore the same weakness; it involves a double numerical integration, one of which is a real Fourier inversion.

4.1.4 Moment generating\characteristic functions approaches

For the rest of this chapter, we will consider, except otherwise specified, the following general diffusion which encapsulates the previous models.

$$\begin{cases} dS_t & = & (r-\delta)S_t dt + \sigma_t S_t dW_t^1 \\ d\sigma_t & = & d(t,\sigma_t)dt + vv(t,\sigma_t)\xi dW_t^2 \\ < dW_t^1, dW_t^2 > & = & \rho dt \end{cases} \tag{4.18}$$

i. Zero correlation

We consider here the case the driving Brownian motions dW_t^1 and dW_t^2 are independent, i.e. $\rho = 0$. Both Stein and Stein [67] results and Ball and Roma [5] analysis are examples of such diffusions. We here point out that the methodology these authors follow can be improved. Indeed, standard vanilla option price can be computed with only one real numerical integration, instead of a double one.

In this zero-correlation framework, it is theorically possible to recover put option prices - and call prices by parity - in the following way.

Theorem 4.1.1.

$$C(S_0, \sigma_0, K, T) = S_0 - \frac{M}{2\pi}\int_{-\infty}^{\infty} \frac{I\left[\left(\eta^2+\frac{1}{4}\right)\frac{T}{2}\right]\cos\left[\eta\left((r-\delta)T + \ln\frac{S_0}{K}\right)\right]}{\eta^2+\frac{1}{4}} d\eta \tag{4.19}$$

where $I(z)$ is the MGF of the average variance $\frac{\int_0^T \sigma_s^2 ds}{T}$ for $z \in \mathbb{R}^+$ and $M = \sqrt{KS_0}e^{-(r+\delta)\frac{T}{2}}$ is a geometric mean.

Proof. Recalling the result already stated in Section 4.1.1 and (4.4), the distribution of the log-asset $X_T = \ln S_T$ is conditionally normal

$$X_T \simeq \mathcal{N}\left(\ln S_0 + (r-\delta)T - \frac{1}{2}\int_0^T \sigma_s^2 ds, \int_0^T \sigma_s^2 ds \right) \Bigg| \int_0^T \sigma_s^2 ds \tag{4.20}$$

This observation enables us to put in relation the characteristic function of the log-asset and the generalised characteristic function of the cumulated variance. Using the expression for the characteristic function of the normal variable,

$$E\big(e^{i\eta X_T}\big) = E\left(E\Big(e^{i\eta X_T}\Big| \int_0^T \sigma_s^2 ds\Big)\right) = E\left(e^{-\frac{\eta}{2}(\eta+i)\int_0^T \sigma_s^2 ds}\right) e^{i\eta((r-\delta)T+\ln S_0)} \tag{4.21}$$

Now, to link this result with the call option price, we observe that the Fourier transform of the difference between e^{X_T} and the payoff function can be expressed in a nice form when the integrating variable is the logarithm of the strike, $k = \ln K$. For any positive real α and any real η ,

$$\int_{-\infty}^{\infty} e^{i\eta k}[e^{X_T} - (e^{X_T} - e^k)^+]e^{-\alpha k} dk = \frac{-e^{(1-\alpha+i\eta)X_T}}{(1-\alpha+i\eta)(i\eta-\alpha)}$$

By Fubini, we obtained an expression for the Fourier transform of a multiple of the put option price

$$\int_{-\infty}^{\infty} e^{i\eta k}(e^{-\alpha k}(S - C(S_0, \sigma_0, e^k, T)))dk = \frac{-E\big(e^{(1-\alpha+i\eta)X_T}\big)}{(1-\alpha+i\eta)(i\eta-\alpha)}e^{-rT} \tag{4.22}$$

The real part of this function is the simplest for $\alpha = \frac{1}{2}$, choice of parameter which leads to the result. □

The expression chosen for the price (choice of α) is motivated by the fact that manipulating a well-defined real function is easier. Ball and Roma [5] even preferred employing a double real integral rather than a single complex integral. It could, of course, be argued that the complex integral can always be expressed as a real integral by taking the real part of the kernel. However, one of the main difficulties in manipulating complex functions lies in the possibility of their being multivalued, property generally inherited by their real part. Some of the reference stochastic volatility models (Stein and Stein, Heston) do indeed involve multivalued characteristic functions and a naive numerical computation of the inverse Fourier integral by using the real part of the integrand, as proposed by Heston [40], may result in wrong values if the existence of different branches is not taken into account. This point has been highlighted by Schobel and Zhu [64].

ii. Non-Zero correlation

The approach followed in the previous subsection can be adapted to the case the driving Brownian motions dW_t^1 and dW_t^2 have a non-null instantaneous correlation ρ_t.

Theorem 4.1.2. *For $z = \alpha + i\phi$, α being such that $E(S_t^{\alpha+1}) < \infty$, we have*

$$C(S_0, \sigma_0, K, T) = S_0 - e^{-rT} \int_0^\infty \frac{K^{-\alpha-i\phi} I(z+1)}{2\pi(z+1)z} d\phi \tag{4.23}$$

where

$$I(z) = E\left(e^{z\left(-\frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \rho_s \sigma_s dW_s^2\right) + \frac{z^2}{2}\int_0^T (1-\rho_s^2)\sigma_s^2 ds} \right) e^{z((r-\delta)T + \ln S_0)} \tag{4.24}$$

Proof. When the Brownian motions are not independent, we have:

$$S_T = S_0 e^{(r-\delta)T - \frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \rho_s \sigma_s dW_s^2 + \int_0^T \sqrt{1-\rho_s^2}\sigma_s dW_s^{\perp 2}} \tag{4.25}$$

where $W_t^{\perp 2}$ is a Brownian motion independent of W_t^2.

The log-asset is therefore normally distributed conditionally on the couple of random variables $\Big(-\frac{1}{2}\int_0^T \sigma_s^2 ds + \int_0^T \rho_s\sigma_s dW_s^2, \int_0^T (1-\rho_s^2)\sigma_s^2 ds\Big)$ (which has also been noted by Lewis [50]) and the generalised characteristic function of the log-asset $I(z) = E\big(e^{zX_T}\big)$ is given by (4.24) for the values of z for which the right-hand side of (4.24) exists. (4.22) gives us then the Laplace transform. □

Corollary 4.1.1. *For the reference models, the generalised characteristic function $I(z)$ and therefore the Fourier transform of the put price can be directly retrieved from the joint characteristic function of*

$$\begin{array}{ll} \text{Hull} - \text{White} & \Big(\sigma_T^{\text{HW}}, \int_0^T \sigma_s^{\text{HW}} ds\Big) \\ \text{Correlated Stein and Stein} & \Big((\sigma_T^{\text{SS}})^2, \int_0^T \sigma_s^{\text{SS}} ds, \int_0^T (\sigma_s^{\text{SS}})^2 ds\Big) \\ \text{Heston} & \Big((\sigma_T^{\text{H}})^2, \int_0^T (\sigma_s^{\text{H}})^2 ds\Big) \end{array} \tag{4.26}$$

For example, for the Heston model

$$I(z) = \mathcal{L}^{X,Y}\left(\frac{\rho z}{\xi}, z\Big(-\frac{1}{2} + \frac{2\rho b}{\xi} + \frac{z}{2}(1-\rho^2)\Big)\right) e^{z\left(\left(r-\delta-\frac{\rho a}{\xi}\right)T + \ln S_0 - \frac{\rho\sigma_0^2}{\xi}\right)} \tag{4.27}$$

where $\mathcal{L}^{X,Y}(\lambda, \mu)$ is the joint MGF given in Proposition 1.1.4, Chapter 1.

Proof. Either Itô's lemma or the volatility stochastic differential equation itself give

- For Hull and White model

$$\int_0^T \sigma_s^{\text{HW}} dW_s^2 = \frac{\sigma_T^{\text{HW}} - \sigma_0}{\xi} - \frac{2\mu_\sigma}{\xi}\int_0^T \sigma_s^{\text{HW}} ds$$

- For Stein and Stein model

$$\int_0^T \sigma_s^{\text{SS}} dW_s^2 = \frac{(\sigma_T^{\text{SS}})^2 - \sigma_0^2}{2\xi} - \frac{\xi T}{2} - \frac{a}{\xi}\int_0^T \sigma_s^{\text{SS}} ds + \frac{b}{\xi}\int_0^T (\sigma_s^{\text{SS}})^2 ds$$

- For Heston model

$$\int_0^T \sigma_s^{\text{H}} dW_s^2 = \frac{(\sigma_T^{\text{H}})^2 - \sigma_0^2 - aT}{\xi} + \frac{2b}{\xi}\int_0^T (\sigma_s^{\text{H}})^2 ds$$

□

Remarks.

1. We expect the asset to have a finite expectation[2], since the discounted asset is a martingale under the risk-neutral measure. Hence, all negative values of α are valid. In order to apply fast Fourier methods, Carr and Madan [10] provide another generalised Laplace transform formulation for any case in which the generalised characteristic function of the asset is known. They also present an interesting empirical discussion for the selection of a parameter which is similar in function to α but is more constrained.

2. Theorem 4.1.2 provide an alternative formulation for the result of Heston[40]. This formulation has two advantages over Heston formulation. Firstly, it is simpler and more intuitive; indeed, we directly use in a simple way the joint MGF from Proposition 1.1.4, Chapter 1, a result that was first derived from the interest rate CIR model, whereas Heston proceeds by solving again a partial differential equation for each of the probabilities $P_j, j = 1, 2$ to get the results. Secondly, we directly provide the transform of the option price whereas Heston gives the transform of probabilities from which the price can be deduced: our formulation leads to the option price in one numerical inversion.

4.1.5 Revisiting the Hull-White model

The characteristic function approach discussed in the previous section also applies to the Hull-White model if a second Laplace transformation - with respect to time - is added.

[2]We can actually even generally expect the asset to be square-integrable.

Theorem 4.1.3. *The double Laplace transform of the put option price in maturity and log-strike $k = \ln K$ is a generalised hypergeometric function.*

$$\int_0^\infty \int_{-\infty}^\infty e^{-\alpha k - \lambda T}(S - C(S_0, \sigma_0, e^k, T))dkdT = \frac{S_0^{i\nu} W(\lambda + r + i\nu(\delta - r), \frac{\alpha\nu}{2})}{\nu\alpha} \quad (4.28)$$

with $\nu = \alpha - 1$ and $W(\beta, \gamma) = \int_0^\infty e^{-\beta T} E(e^{-\gamma V_T})dT$ is the double Laplace transform of the cumulated variance (see Fu and al. [32]).

${}_1F_2(n_1; d_1, d_2; z)$ being the generalised hypergeometric function, μ_V, ξ_V the coefficients of the $(\sigma_t^{\mathrm{HW}})^2$, which is a geometric Brownian motion as well and $y_1 = \frac{1}{2} - \frac{\mu_V}{\xi_V^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu_V}{\xi_V^2}\right)^2 + \frac{2\beta}{\xi_V^2}}$, $y_2 = \frac{1}{2} - \frac{\mu_V}{\xi_V^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu_V}{\xi_V^2}\right)^2 + \frac{2\beta}{\xi_V^2}}$

$y_1 + y_2 = 1 - \frac{2\mu_V}{\xi_V^2}$, $y_1 y_2 = -\frac{2\beta}{\xi_V^2}$

$$W(\beta, \gamma) = \frac{1}{\beta}\left[1 - \frac{\xi_V^2}{2(\mu_V - \beta)}\right]\left[1 + \frac{\gamma\sigma_0^2}{(\mu_V - \beta)}\right] + \frac{{}_1F_2(1; 1 - y_1, 1 - y_2; \frac{2\gamma}{\xi_V^2})\xi_V^2}{2\beta(\mu_V - \beta)} \quad (4.29)$$

Proof. The difference $S_0 - C(S_0, \sigma_0, e^k, T)$ will be noted $D(k, T)$ so as to emphasize its dependency on the log-strike and the time to maturity. It follows from (4.22) that

$$\int_0^\infty \int_{-\infty}^\infty e^{-\alpha k - \lambda T} D(k, T)dkdT = \int_0^\infty \frac{e^{-(\lambda + r + i\nu(\delta - r))T} E(e^{-\frac{\alpha\nu}{2}V_T})S_0^{i\nu}}{\nu\alpha}dT$$

The double transform of the temporal integral of a geometric Brownian motion is known since the work of Fu and al [32] on Asian options. We notice that this transform exist for any value of β, which completes the result. □

4.2 Moment based approximations

A number of research works have treated stochastic volatility as a perturbation (Hull and White [41], Romano and Touzi, Lewis [50], Fouque and al. [31]) and considered expansions. Moments-based expansions occur naturally in this context, all the more naturally since the distribution of the log-asset is determined by its moments. Indeed, we assume that the expectation of the asset is finite (more precisely, $E(S_T) = S_0 e^{(r-\delta)T}$). This implies that the distribution of the log-asset is subexponential and thus determined by its moment.

4.2.1 Moments for the reference models

Moments-based methods can obviously only be used if it is possible to compute the moments. For the reference models, for which the diffusion coefficients are linear, it turns out that the joint moments of the conditioning variables (δ_T, V_T) can be expressed explicitly. For the Heston model, the joint moments of $(\delta_T^{\mathrm{H}},, V_T^{\mathrm{H}})$ are given by Theorem 1.1.2, Chapter 1. The approach followed in the proof of this theorem remains fruitful when it comes to the Hull-White and Stein and Stein models.

Theorem 4.2.1. *For the Hull-White model,*

$$d(\sigma_t^{\mathrm{HW}})^2 = \mu_v (\sigma_t^{\mathrm{HW}})^2 dt + \xi_V (\sigma_t^{\mathrm{HW}})^2 dW_t^2 \tag{4.30}$$

the Laplace transform of the joint moments of $Y_T = \int_0^T (\sigma_t^{\mathrm{HW}})^2 dt$ *and* σ_T^{HW} *is given by*

$$\int_0^\infty e^{-\zeta T} E((\sigma_T^{\mathrm{HW}})^{2k} Y_T^m) dT = \frac{m!(\sigma_0^{\mathrm{HW}})^2}{\prod_{l=0}^m (\zeta - \mu_V (n-l) - \frac{\xi_V^2}{2}(n-l)(n-l-1))} \tag{4.31}$$

where $n = k + m$.

Proof. Given that the joint moments are finite, Itô's lemma applied to the product $(\sigma_T^{\mathrm{HW}})^{2k} Y_T^m$ results in

$$\frac{E((\sigma_T^{\mathrm{HW}})^{2k} Y_T^m)}{dT} = \left[\mu_V k + \frac{\xi_V^2}{2} k(k-1) \right] E((\sigma_T^{\mathrm{HW}})^{2k} Y_T^m) + m E((\sigma_T^{\mathrm{HW}})^{2k+2} Y_T^{m-1})$$

$\widehat{M}_{m,n}(\zeta)$, the Laplace transform of $M_{m,n}(T) = E(Y_T^m(\sigma_T^{\mathrm{HW}})^{2(n-m)})$ with respect to T and argument $\zeta \in \mathbb{R}^+$, is

$$\hat{M}_{m,n} = \frac{m!(\sigma_0^{\mathrm{HW}})^{2n}}{\prod_{l=0}^{m}(\zeta - \mu_V(n-l) - \frac{\xi_V^2}{2}(n-l)(n-l-1))}$$

□

It is then completely straightforward to obtain the actual moments by partial fraction decomposition. Indeed, depending on the number of different values $\mu_V(n - l) + \frac{\xi_V^2}{2}(n-l)(n-l-1)$ can attain, we have

$$\hat{M}_{m,n} = \sum_{i=i}^{I} \frac{\alpha_i}{(\zeta + \beta_i)^{\gamma^i}}$$

Therefore,

$$E(Y_T^m(\sigma_T^{\mathrm{HW}})^{2(n-m)}) = \sum_{i=i}^{I} \alpha_i \frac{T^{\gamma^i - 1} e^{-\beta_i T}}{(\gamma^i - 1)!}$$

The following corollary presents the two most common cases

Corollary 4.2.1. *When $2\mu_V$ is not a multiple of ξ_V, then*

$$E(Y_T^m(\sigma_T^{\mathrm{HW}})^{2n}) = m!(\sigma_0^{\mathrm{HW}})^{2(n+m)} \sum_{l=0}^{m} \alpha_l e^{T\lambda(n+m,l)} \tag{4.32}$$

where $\lambda(j,l) = \mu_V(j-l) + \frac{\xi_V^2}{2}(j-l)(j-l-1)$

$$\alpha_l = \left[\prod_{i=0,\ i \neq l}^{m} \Big(\lambda(n+m,l) - \lambda(n+m,i)\Big) \right]^{-1} \tag{4.33}$$

When $\mu_V = 0$ and $\mathcal{I} = \{0,..,m\} \backslash \{n-1,n\}$, then

$$E(Y_T^m(\sigma_T^{\mathrm{HW}})^{2n}) = m!(\sigma_0^{\mathrm{HW}})^{2(n+m)} \left[\sum_{l \in \mathcal{I}} \alpha_l e^{T\lambda(n+m,l)} + a_1 + a_2 T \right] \tag{4.34}$$

where

$$
\begin{aligned}
\alpha_l &= \left[\prod_{i=0,\ i\neq l}^{m}\Big(\lambda(n+m,l)-\lambda(n+m,i)\Big)\right]^{-1} \\
a_2 &= \left[\prod_{i\in\mathcal{I}}(-\lambda(n+m,i))\right]^{-1} \\
a_1 &= -\left[\prod_{i\in\mathcal{I}}(-\lambda(n+m,i))\right]^{-1}\left[\sum_{i\in\mathcal{I}}(-\lambda(n+m,i))^{-1}\right]
\end{aligned}
\tag{4.35}
$$

Remark. Geman and Yor [77] as well as Dufresne [26] provided expressions for the moments of the temporal integral of Geometric Brownian motions in the context of Asian options. Their assumption, however, is that $2\mu_V$ is not a multiple of ξ_V, which excludes the situation $\mu_V = 0$, most important case for the Hull-White model. Our result first presents the exact solution for this crucial special case and a simple general methodology to account for any case. Secondly, we also propose joint moments and not only the marginal moments of Y_T.

4.2.2 Hull-White type of approximations

i. Zero correlation

As noticed by Ball and Roma [5], the method developed by Hull-White can be applied to any volatility diffusion if $\rho = 0$.

Proposition 4.2.1. *An Hull-White type of expansion can be written for any volatility diffusion for which the average variance moments are known.*

$$
C^{\mathrm{HW}}(S_0,\sigma_0,K,T) = \mathrm{BS}^C\Big(S_0,K,T,\frac{\sqrt{E(V_T)}}{T},r,\delta\Big)
$$

$$
+\sum_{k=2}^{N}\frac{\partial^k \mathrm{BS}^C}{\partial(\sigma^2)^k}\Big(S_0,K,T,\frac{\sqrt{E(V_T)}}{T},r,\delta\Big)\frac{E(V_T-E(V_T))^k}{k!T^k} + \text{higherorderterms} \tag{4.36}
$$

where V_T represents the cumulated variance $= \int_0^T \sigma_s^2 ds$.

This formula arises from the Taylor expansion of the Black-Scholes formula around the variance. Taking the expectation of this Taylor expansion leads to (4.36). No proof of convergence has been provided by Hull and White for their original expansion. We notice that the power series of the Black-Scholes formula in variance is already divergent, because of the presence of $\frac{1}{\sqrt{\sigma^2}}$ in the formula. The moment expansion is hence expected to be divergent in most cases. Yet, this expansion still remains of practical interest, since it has been proven to be a reasonably good approximation for small volatility of volatility for the Hull and White model, though only third or fourth-order truncations have been studied in the literature.

Theorem 4.2.2. *The derivatives of the Black-Scholes call formula with respect to the variance are*

$$\frac{\partial^n \mathrm{BS}^C(S,K,T,\sqrt{v},r,\delta)}{\partial v^n} = \frac{e^{-\frac{\alpha}{2v}-\frac{v}{4}}}{\sqrt{2\pi}}(S_0 e^{-\delta T-2\alpha} - K e^{-rT+2\alpha}) \sum_{k=0}^{n} \frac{a_n^k}{v^{\frac{1}{2}+k}} \tag{4.37}$$

with $\alpha = \ln\frac{S_0}{K} + (r-\delta)T$ *and*

$$\begin{aligned} a_{2n+2}^{n+1} &= \tfrac{\alpha^2}{2} a_{2n}^n \\ a_{2n+1}^{n+1} &= -\left(2n+\tfrac{1}{2}\right) a_{2n}^n \\ a_k^{n+1} &= -\left(k-\tfrac{1}{2}\right) a_{k-1}^n + \tfrac{\alpha^2}{2} a_{k-2}^n - \tfrac{a_k^n}{4} \\ a_1^{n+1} &= -\tfrac{1}{2} a_0^n - \tfrac{a_1^n}{4} \\ a_0^{n+1} &= -\tfrac{a_0^n}{4} \end{aligned} \tag{4.38}$$

Proof. A simple recursion procedure enables us to deduce the derivatives of $\int_0^{\frac{\alpha}{\sqrt{v}}+\beta\sqrt{v}} e^{-\frac{y^2}{2}} dy.$ □

ii. Non-Zero correlation

Although the Hull and White expansion may not be convergent, practitioners still widely use it as a "quick and dirty" approximation for small volatility of volatility.

The zero-correlation constraint imposed on the method is however very restrictive. Indeed, empirical studies proved the existence of a negative correlation between log-returns of many stocks and their local volatilities. It is therefore quite useful to extend the Hull and White methodology to the case $\rho \neq 0$.

Theorem 4.2.3. *The Hull-White type of moments expansion has to be modified to*

$$C^{\mathrm{HW}}(S_0, \sigma_0, K, T) = \mathrm{BS}^C\Big(S_0, K, T, \frac{\sqrt{E(V_T)}}{T}, r, \frac{E(\delta_T)}{T}\Big) +$$

$$\sum_{n=2}^{N} \sum_{k+l=n} \frac{\partial^{k+l}\mathrm{BS}^C}{\partial(\sigma^2)^k \partial \delta^l}\Big(S_0, K, T, \frac{\sqrt{E(V_T)}}{T}, r, \frac{E(\delta_T)}{T}\Big) \frac{E\Big((V_T - E(V_T))^k (\delta_T - E(\delta_T))^l\Big)}{k!l!T^{k+l}}$$

$$+\mathrm{h.o.t} \tag{4.39}$$

when the correlation is not null. The random variables appearing in this approximation are

$$\begin{aligned} V_T &= \int_0^T (1-\rho_s^2)\sigma_s^2 ds \\ \delta_T &= \delta T + \tfrac{1}{2}\int_0^T \rho_s^2 \sigma_s^2 ds - \int_0^T \rho_s \sigma_s dW_s^2 \end{aligned} \tag{4.40}$$

Proof. The expression (4.25) establishes $\frac{\ln S_T}{\ln S_0}$ as a conditional lognormal with the modified instantaneous drift $r - \frac{\delta_T}{T}$ and volatility $\frac{\sqrt{V_T}}{T}$. The double N^{th}-order Taylor expansion of the Black-Scholes price formula around the variance at $\frac{E(V_T)}{T}$ and the compounded dividend rate at $\frac{E(\delta_T)}{T}$ thus leads to the result, when taken in expectation.

□

4.2.3 Volatility of volatility expansion

It must be pointed out that, in our previous results, we kept Hull and White notations when referring to higher order terms, which are conveniently used in the Hull and White [41] paper and can be intuitively understood, but are not rigorously defined. Hull and White expansion is an approximation for small volatility of volatility values, i.e. small ξ. An expansion in ξ might hence be mathematically more relevant. It gives more meaning to the h.o.t, which become $\mathcal{O}(\xi^n)$.

Theorem 4.2.4. *The N^{th}-order Taylor approximation of the call option price with respect to ξ is*

$$C^{\text{HW}}(S_0,\sigma_0,K,T) = \text{BS}^C\Big(S_0,K,T,\frac{\sqrt{E(V_T)}}{T},r,\frac{E(\delta_T)}{T}\Big)+$$

$$\sum_{n=2}^{N}\sum_{k+l=n}\frac{\partial^{k+l}\text{BS}^C}{\partial(\sigma^2)^k\partial\delta^l}\Big(S_0,K,T,\frac{\sqrt{E(V_T)}}{T},r,\frac{E(\delta_T)}{T}\Big)\frac{E_N(k,V_T,l,\delta_T)}{k!l!T^{k+l}}+\mathcal{O}(\xi^N) \quad (4.41)$$

where $E_N(k,V_T,l,\delta_T)$ represents the N^{th} order Taylor expansion of the corresponding joint moment $E\big((V_T-E(V_T))^k(\delta_T-E(\delta_T))^l\big)$, V_T and δ_t being the random variables defined in (4.40), Theorem 4.2.3.

Proof. To prove this theorem, it is sufficient to notice that the Taylor expansions of $V_T - E(V_T)$ and $\delta_T - E(\delta_T)$ have to start at the first order since they get neutralised when $\xi = 0$. This means that the joint moment $E\big((V_T-E(V_T))^k(\delta_T-E(\delta_T))^l\big)$ is at least of starting order $k+l$ and that the right-hand side of (4.41) contains all the terms of order inferior to N. □

Lewis [50] have also proposed a volatility of volatility series expansion. Our method is though easier to implement and complete, since our formulation can be taken to any order N for the reference models, given the expressions we formulate for their moments. Lewis approach is more complicated and he provides only the first few terms of the expansion (order ≤ 4).

4.2.4 Edgeworth expansion

The Edgeworth expansion is a common tool in statistics and has been also often used in financial mathematics, notably for the pricing of Asian options but also to take into account excess skewness and kurtosis in the stock risk-neutral distribution. This method comes quite naturally in mind when thinking about stochastic volatility, once we realise that, though the underlying process is complex, its moments are simply given in terms of the moments of the effective variance and the effective drift of the log-asset.

We first show how simple the moments of the log-return are, given the moments of the conditioning mixing variables.

Theorem 4.2.5. *The moments of X_T can be expressed in terms of the joint moments of the variables $\mu_T = \ln S_0 + (r - \delta_T)T - \frac{V_T}{2}$ and V_T, when they exist.*

$$E(X_T^n) = \sum_{s=0}^{[\frac{n}{2}]} \frac{E(\mu_T^{n-2s} V_T^s) n!}{2^s (n-2s)! s!} \tag{4.42}$$

Proof. A gaussian random variable G of mean μ and volatility σ possesses the following moments derived by differentiation of the MGF of G

$$E(G^n) = (-1)^n \frac{\partial e^{\frac{1}{2}\left(\lambda\sigma - \frac{\mu}{\sigma}\right)^2 - \frac{\mu^2}{2\sigma^2}}}{\partial \lambda} = (-\sigma)^n \widetilde{He}_n\left(-\frac{\mu}{\sigma}\right) = \sum_{s=0}^{[\frac{n}{2}]} \frac{\mu^{n-2s}\sigma^{2s} n!}{2^s (n-2s)! s!}$$

where $\widetilde{He}_n(x)$ is the polynomial previously defined in (1.55), Section 1.1.4, Chapter 1. The result then follows from $E(X^n) = E(E(X^n|(\mu_T, V_T)))$. □

Theorem 4.2.6. *Denoting $\mu_X = \ln S_0 + (r-\delta)T - \frac{\int_0^T E(\sigma_s^2)ds}{2}$, $\sigma_X^2 = \frac{\int_0^T (1-\rho_s^2)E(\sigma_s^2)ds}{2}$, κ_n the cumulants of X_T and $\hat{\kappa}_n$ the cumulants of a normal $\mathcal{N}(\mu_X, \sigma_X^2)$, the call option price $C(S_0, \sigma_0, K, T)$ can be approximated in the following way*

$$\mathrm{BS}^C\left(S_0, K, T, \frac{\sigma_X}{\sqrt{T}}, r, \frac{\mu_X}{T} + r + \delta\right) + \sum_{n=1}^{N}\left(K \sum_{p=0}^{n-2} \frac{e^{-\frac{(\ln K - \mu_X)^2}{2\sigma_X^2}} He_p\left(\frac{\ln K - \mu_X}{\sigma_X}\right)}{\sigma_X^{p+1}\sqrt{2\pi}}\right.$$

$$\left. + \frac{e^{\mu_X + \frac{\sigma_X^2}{2}}}{2} \mathrm{erfc}\left(\frac{\ln K - \mu_X - \sigma_X^2}{\sigma_X\sqrt{2}}\right)\right)\frac{T_n}{n!} e^{-rT} + \epsilon(N, K) \tag{4.43}$$

For a fixed N, the terms $T_n, n = 1..N$ are determined by the Taylor expansion

$$\exp\left\{\sum_{n=1}^{N}(\kappa_n - \hat{\kappa}_n)\frac{(it)^n}{n!}\right\} = \sum_{n=1}^{N} T_n \frac{(it)^n}{n!} + \mathcal{O}(t^{N+1})$$

$\epsilon(N,K)$ simply denotes the error term.

Proof. We apply the approximation methodology proposed by Jarrow and Rudd [44]. With $k = \ln K$, the price of a call option when expanding around a given distribution $f(x)$ is, in this framework

$$C(S_0, \sigma_0, K, T) = e^{-rT}\int_k^{\infty}(e^x - e^k)\left(f(x) + \sum_{n=1}^{N}\frac{(-1)^n T_n}{n!}\frac{\partial^n f(x)}{\partial x^n}\right)dx + \epsilon^f(N,K)$$

The approximating distribution we choose is normal as can be expected, with a mean and a variance equating those of X_T, so as to cancel two terms in the development. □

Remarks.

1. T_n is a linear combination of the difference between the theorical cumulants of X_T and the cumulants of the approximating distribution. It is hence a linear combination of the moments of X_T given previously. We expect to be needing at least a fourth order expansion since the first four cumulants are the mean $\kappa_1 = \mu_X$, the variance $\kappa_2 = \sigma_X^2$, the skewness $\kappa_3 = E(X_T - \mu_X)^3$ and a measure of kurtosis $\kappa_4 = E(X_T - \mu_X)^4 - 3\sigma_X^2$.

2. As pointed out by Jarrow and Rudd [44], there is no general analytical bound known for $\epsilon(N,K)$. These authors claim, though, that the error for the density expansion tends uniformly towards 0 as soon as the moments exist, which would mean that the Edgeworth expansion converges for any distribution which possesses all of its moments. This assertion is unfortunately false; the lognormal distribution taken around a normal one constitutes a typical counterexample. Another wrong belief would be to consider the expansion as convergent as soon as the distribution approximated is determined by its moments (see Fusai [34]). Indeed, a simple counterexample is given by the exponential distribution. When the approximating distribution is a gaussian, a sufficient condition for convergence is Cramer's, demanding "subgaussian" tails, i.e. tails decreasing at least as quickly as those of a gaussian. Although this condition is quite restrictive, Edgeworth types of expansions remain quite useful because they can produce good asymptotic approximations when they are not convergent.

4.2.5 Laguerre polynomial expansion

As discussed above, all the previous expansions have the drawback that they might not always be convergent. As a result, our aim will be here to produce convergent moment-based expansions (at the cost of more complex terms though). We explore two stategies. We first consider a polynomial expansion in the moments of the effective variance and drift. A second solution, dealt with in the next subsection, consists in continuing to use the moments of the log-returns but forcing convergence by cutting the higher tail of the volatility process.

To explain our first methodology in a simple way, we restrict ourselves to the case $\rho = 0$ and to stochastic volatility processes such that the cumulated variance is subexponential, which is the case for both the Stein and Stein and the Heston models.

Theorem 4.2.7. *Under these assumptions, the effective variance is $V_T = \int_0^T \sigma_s^2 ds$ and the call option price can be expressed in the following form*

$$C(S_0, \sigma_0, K, T) = \mathrm{BS}^C\Big(S_0, K, T, \sqrt{\frac{E(V_T)}{T}}, r, \delta\Big)$$
$$+\sum_{n=0}^{\infty} c_n \Big(\sum_{k=0}^{n} \frac{(n+p)!}{(k+p)!} \frac{E(\frac{-V_T}{\beta})^k}{k!(n-k)!} - L_n^p\Big(\frac{E(V_T)}{\beta}\Big)\Big) \tag{4.44}$$

for any $p \geq 0$ such that $E(V_T^{-p}) < \infty$ and β such that $E(e^{\frac{V_T}{2\beta}}) < \infty$.

The coefficients of this expansion are

$$c_n = \sqrt{\frac{\beta S_0 K e^{-(r+\delta)T}}{2\pi}} \sum_{m=0}^{n-1} \frac{(-1)^{m+1}\binom{n-1}{m}}{(p+1+m)!}\left(\frac{2A}{\beta}\right)^{m+p+\frac{3}{2}} K_{m+p+\frac{3}{2}}\left(\frac{|A|}{2}\right) \tag{4.45}$$

where the $K_{q+\frac{1}{2}}\left(\frac{|A|}{2}\right)$ are Bessel functions of index an integer plus one-half, special case in which they simplify to (See Gradstein and al. [37])

$$K_{q+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{l=0}^{q} \frac{(q+l)!}{l!(q-l)!(2z)^l} \tag{4.46}$$

Proof. According to the theory of Laguerre-series (for example see Dufresne [26]), any function g defined on $\mathbb{R}^+$such that $\int_0^\infty g^2(h)h^p e^{-h}dh < \infty$ has a representation $\frac{g(h^-)+g(h^+)}{2} = \sum c_n L_n^p(h)$. This series of functions is normally convergent with respect to the scalar product $< f, g >= \int_0^\infty g(h)f(h)h^p e^{-h}dh$, i.e.

$$\int_0^\infty \left(g(h) - \sum_{n=0}^{N} c_n L_n^p(h)\right)^2 h^p e^{-h}dh \longrightarrow 0$$

The Black-Scholes call option formula is bounded with respect to the volatility. It hence verifies the square-integrability condition and has a Laguerre-series representation for any positive v.

$$\mathrm{BS}^C\left(S_0, K, T, \sqrt{\frac{v\beta}{T}}, r, \delta\right) = \sum_{n=0}^{\infty} c_n L_n^p(v)$$

with

$$c_n = \frac{n!}{(n+p)!}\int_0^\infty v^p e^{-v}\mathrm{BS}^C\left(S_0, K, T, \sqrt{\frac{v\beta}{T}}, r, \delta\right)L_n^p(v)dv$$

Denoting $C_N(S_0, \sigma_0, K, T) = E\left(\sum_{n=0}^{N} c_n L_n^p\left(\frac{V_T}{\beta}\right)\right)$ and $f_{V_T}(\cdot)$ the density of the cumulated variance, the convergence in expectation follows from the Cauchy-Schwarz inequality

$$\left[C(S_0, \sigma_0, K, T) - C_N(S_0, \sigma_0, K, T)\right]^2 \leq \left[\int_0^\infty v^{-p}e^v f_{V_T}^2(v\beta)\beta dv\right]$$

$$\times\left[\int_0^\infty \left(\mathrm{BS}^C\left(S_0, K, T, \sqrt{\frac{v\beta}{T}}, r, \delta\right) - \sum_{n=0}^{N} c_n L_n^p(v)\right)^2 v^p e^{-v}\beta dv\right]$$

The first integral in the right-hand side being finite, this inequality proves the convergence of (4.44).

To compute c_n, we introduce the notations $A = \ln\frac{S_0}{K} + (r-\delta)T$, $B = \frac{A+\frac{k}{2}v\beta}{\sqrt{v\beta}}$, $\mathcal{N}(\cdot)$ the probability distribution function of the standard normal and $k = \pm 1$. We notice that

$$\int_0^\infty L_n^p(v)v^p e^{-v}\mathcal{N}(B)dv = \int_0^\infty \frac{L_{n-1}^{p+1}(v)v^{p+1}e^{-v-\frac{B^2}{2}}}{2n\sqrt{2\pi}}\left(\frac{A}{\sqrt{\beta v^3}} - \frac{k\sqrt{\beta}}{2\sqrt{v}}\right)dv$$

by integration by part, knowing that $L_n^a(x) = \frac{1}{n!}e^x x^{-a}\frac{\partial^n}{\partial x^n}(e^{-x}x^{n+a})$.

(4.45) follows then from the relation (see Gradstein and al. [37])

$$\int_0^\infty x^{\nu-1}e^{-\frac{\alpha}{x}-\gamma x}dx = 2\left(\frac{\alpha}{\gamma}\right)^{\frac{\nu}{2}}K_\nu(2\sqrt{\alpha\gamma})$$

valid for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma) > 0$.

□

Remarks.

1. The methodology can still be applied for the Hull-White model by taking the reciprocal of the effective variance as the argument of the Laguerre polynomial. Indeed, V_T^{-1} is subexponential (see [26]). The coefficients c_n are then slightly modified but grossly keep the same form.

2. When $\rho \neq 0$, the result can be generalised by using multivariate Laguerre expansion. In that case, the representation becomes a double series.

4.2.6 A small parenthesis on bounded variance

We wanted to highlight that, for bounded variance, it is possible to use Hermite expansion on the moments of the log-asset rather than a Laguerre polynomial on the moments of the variance. More precisely, whenever the effective variance V_T and drift processes are bounded above, the tails of the log-asset decrease quickly enough to verify Cramer's condition and produce a convergent Edgeworth expansion.

Theorem 4.2.8. *If the effective variance V_T is bounded by some positive constant V_B and the random part of the effective drift, $\hat{\mu}_T = \mu_T - (r-\delta)T$, is also bounded, then, for $\alpha < \frac{1}{2V_B}$, $E(e^{\alpha Z_T^2}) < \infty$, where $Z_T = X_T - \ln S_0 - (r-\delta)T$.*

Proof. Conditioning on V_T and $\hat{\mu}_T$,

$$E(e^{\alpha(\ln S)^2}) = E\left(\int_0^{\infty} \frac{e^{\alpha x^2 - \frac{(x-\mu_T)^2}{2V_T}}}{\sqrt{2\pi V_T}} dx\right) = E\left(\frac{e^{\frac{\mu_T^2}{2V_T}\left[\frac{1}{1-2\alpha V_T}-1\right]}}{\sqrt{1-2\alpha V_T}}\right) < \infty$$

□

This result enables us to construct a convergent Edgeworth expansion to approximate the price under a general volatility model. Indeed, taking the bounds on V_T and $\hat{\mu_T}$ to infinity gives the price for the unbounded model. Setting these bounds to a high level should hence constitute a reasonable approximation. In this subsection, we will though take a slightly different path and propose an alternative Hermite series instead of considering a direct Edgeworth expansion. The advantage of the Hermite expansion approach is that the coefficients (as functions of the moments) are known in an analytic form for any order.

Theorem 4.2.9. *Under the same assumptions as in the previous theorem, the call price can be approximated, for $\beta > \frac{V_B}{\sqrt{2}}$, by*

$$C(S_0, \sigma_0, K, T) = \mathrm{BS}^C\Big(S_0, K, T, \frac{\beta}{\sqrt{T}}, r, \delta - \frac{\beta^2}{2T}\Big)$$

$$+\sum_{n=1}^{\infty} d_n \left(\sum_{s=0}^{\left[\frac{n}{2}\right]} (-1)^s \frac{E\left(\left(\frac{Z_T}{\beta}\right)^{n-2s}\right)}{2^s} \frac{n!}{(n-2s)!s!}\right) \tag{4.47}$$

where

$$d_n = \frac{\beta^n e^{-rT}}{n!}\left(K\sum_{p=0}^{n-2}\frac{e^{-\frac{(\ln\frac{K}{S_0}-(r-\delta)T)^2}{2\beta^2}}He_p\left(\frac{\ln\frac{K}{S_0}-(r-\delta)T}{\beta}\right)}{\sqrt{2\pi}\beta^{p+1}}\right.$$
$$\left.+\frac{e^{\frac{\beta^2}{2}}S_0e^{(r-\delta)T}}{2}\mathrm{erfc}\left(\frac{\ln\frac{K}{S0}-(r-\delta)T-\beta^2}{\beta\sqrt{2}}\right)\right) \tag{4.48}$$

Proof. We apply exactly the same kind of arguments as in the Laguerre polynomial expansion. The Laguerre weight x^pe^{-x} is changed to the Hermite weight e^{-x^2}. The function $e^{-rT}(S_0e^{\beta x+(r-\delta)T}-e^k)^+$ admits a Hermite series representation, since it is square-integrable with respect to the Hermite weight.

$$e^{-rT}(S_0e^{\sqrt{2}\beta x+(r-\delta)T}-e^k)^+ = \sum_{n=0}^{\infty} d_nH_n(x)$$

where

$$d_n = \frac{e^{-rT}}{n!\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-\frac{\sqrt{2}(y-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2}}He_n\left(\frac{y-\mu}{\sigma}\right)(S_0e^{\beta\frac{y-\mu}{\sigma\sqrt{2}}+(r-\delta)T}-e^k)^+dx$$

Applying the inequality of Cauchy-Schwarz then bounds $(C(S_0,\sigma_0,K,T)-E(\sum_0^N d_nH_n(Z_T)))^2$ by a term term tending to 0 with N, which proves convergence. □

It could be argued that this method could be used as well as an approximation for the reference model by absorbing the variance process at a high level. The moments could then typically be obtained by eigenfunction expansion methods. Indeed, for the reference models, solutions to the appropriate homogeneous differential equation are known. This method would be interesting only when we need to compute a whole smile surface, i.e call option price at numerous strike and maturities. Indeed, the major numerical work would consist in computing the eigenvalues. Once these values found, the second step for each maturity is to compute the moments, which is relatively quick given the eigenvalues. The price for all strikes at that maturity can then be done quasi-instantly. We notice that, for the Heston model, in the case $\rho = 0$, the moments can be expressed in explicit form as they can be obtained by term by term differentiation of the series

$$E(V_T^n) = (-1)^n \frac{\partial^n \mathcal{G}_{a,b,\sigma}}{\partial \lambda^n}(y, 0) \tag{4.49}$$

We deduced the first moment in the subsection 3.1.3 of Chapter 3. Repeated differentiation with the same approach lead to explicit forms for the moments of higher order. However, fast fourier methods are probably easier to implement than this type of expansion.

4.3 Numerical comparisons

The Heston [40] model remains the most popular of all and its close-form solution makes it quite practical for numerical test. For simplicity, we set $\rho = 0$ and choose to use the framework and parameters used by Roma and Ball [5].

We first extend the analysis performed by Roma and Ball [5], observing the behaviour of the Hull and White types of series for a greater number of terms. We notice that, for the Heston model, the Hull and White type of expansion and the Taylor expansion in the volatility of volatility coincide since the moments are polynomial in terms of ξ^2.

Table 4.1 presents this expansion for the cases studied in Roma and Ball [5]. The first group is based on the parameters $b = 4$, $\frac{a}{b} = 0.09$, $\xi = 0.4$, $v_0 = 0.09$ and $T = 0.5$, the second group on $b = 8$, $\frac{a}{b} = 0.1225$, $\xi = .8$, $v_0 = 0.1225$ and $T = 0.5$. The lines exact contain the value of the call options computed from numerical integration of the close-form solutions.

Chacko and Das [12] observed that adding the third term lead to a higher error.

From Table 4.1, it can be seen that the even indexed partial sum are actually generally closer to the correct value than the odd indexed ones.

	80	90	100	110	120
	First group				
Exact	21.43002	13.93501	8.35948	4.67992	2.48682
1	21.42544	13.98983	8.44700	4.74568	2.50378
2	21.42980	13.93277	8.35700	4.67738	2.48541
3	21.42587	13.93700	8.36654	4.68292	2.48330
4	21.43134	13.93458	8.35755	4.67920	2.48803
5	21.42857	13.93503	8.36162	4.68025	2.48505
6	21.43095	13.93515	8.35802	4.67984	2.48813
7	21.42922	13.93466	8.36102	4.67979	2.48540
8	21.43063	13.93554	8.35788	4.68022	2.48825
9	21.42957	13.93417	8.36144	4.67933	2.48519
10	21.43019	13.93633	8.35691	4.68093	2.48872
11	21.43035	13.93282	8.36318	4.67813	2.48450
12	21.42866	13.93878	8.35373	4.68314	2.48966
13	21.43350	13.92819	8.36905	4.67387	2.48341
14	21.42197	13.94794	8.34248	4.69176	2.49043
	Second group				
Exact	22.19201	15.09917	9.72561	5.99216	3.57592
1	22.20610	15.18579	9.84818	6.09349	3.62259
1	22.19081	15.09415	9.71944	5.98643	3.57255
3	22.18630	15.10484	9.73841	5.99958	3.57282
4	22.19497	15.09721	9.72044	5.98947	3.57794
5	22.18797	15.10046	9.73145	5.99431	3.57222
6	22.19598	15.09850	9.72023	5.99069	3.58003
7	22.18711	15.09924	9.73234	5.99325	3.57008
8	22.19819	15.09994	9.71658	5.99157	3.58431
9	22.18359	15.09669	9.73948	5.99175	3.56263
10	22.20394	15.10535	9.70221	5.99493	3.59833
11	22.17484	15.08426	9.76900	5.98338	3.53549
12	22.21584	15.13557	9.63825	6.01673	3.65320
13	22.16435	15.00729	9.91552	5.92470	3.42025
14	22.20036	15.34065	9.28263	6.18021	3.90393

Table 4.1: Volatility of volatility expansion.

We could not really find an explanation for this phenomenon. Globally, the 6^{th} terms sums seems the better, but there is no specific theoretical justification for that.

Table 4.2 shows the values of an Edgeworth expansion for the Roma and Ball reference cases. Clearly, the volatility of volatility expansion approximates the call price better the Edgeworth expansion. This probably comes from the tails of the log-asset being much fatter than a gaussian.

	80	90	100	110	120
	First group				
Exact	21.43002	13.93501	8.35948	4.67992	2.48682
1	21.42750	13.99319	8.45101	4.74948	2.50681
2	21.42750	13.99319	8.45101	4.74948	2.50681
3	21.41701	13.92592	8.46535	4.77852	2.54145
4	21.40686	13.92748	8.38425	4.73451	2.55419
5	21.40888	13.93380	8.38322	4.73145	2.55121
6	21.40694	13.93380	8.39245	4.73426	2.54592
7	21.40518	13.93177	8.39356	4.73755	2.54818
8	21.40911	13.92692	8.38501	4.73693	2.55554
	Second group				
Exact	22.19201	15.09917	9.72561	5.99216	3.57592
1	22.21072	15.19252	9.85594	6.10105	3.62908
2	22.21072	15.19252	9.85594	6.10105	3.62908
3	22.19916	15.19647	9.88459	6.15113	3.68978
4	22.16322	15.09792	9.77332	6.08417	3.69254
5	22.16806	15.10057	9.77060	6.07725	3.68500
6	22.16873	15.11551	9.78875	6.08453	3.67749
7	22.16366	15.11182	9.79173	6.09216	3.68408
8	22.16746	15.09902	9.77477	6.08888	3.69663

Table 4.2: Edgeworth expansion.

First group				Second group			
80	90	110	120	80	90	110	120
4	5	6	7	6	3	5	7

Table 4.3: Laguerre expansion.

In Table 4.3, we evaluate the performance of the Laguerre series for the same set of options. N represents the number of term needed to reach convergence within 10^{-4} relative error convergence.

4.4 Conclusion

We considered stochastic volatility as a natural application as the reference models of volatilities or variance belong to the CEV processes. We wanted first to relate the closed-form solutions derived in some of those models (Heston [40], Stein and Stein [67]) to more basic MGF results and give a more graspable structure to the models trough the use of conditioning variables. As an attempt to construct generic solutions, we then extended asymptotic expansions existing in the literature and also presented a converging Laguerre type of expansion.

A question to be worked out here is: could the moments of the integral of a general mean-reverting CEV process be computed at least in a series form so as to use these expansions?

4.5 Appendices to this chapter

4.5.1 Stein and Stein model: functions definitions

The MGF is defined as

$$I^{\mathrm{SS}}(z) = e^{L\frac{\sigma_0^2}{2}+M\sigma_0+N} \tag{4.50}$$

which involve the different functions

$$A = -\frac{b}{\xi^2}, \quad B = \frac{a}{\xi^2}, \quad -\frac{z}{\xi^2 T} \tag{4.51}$$

$$\alpha = \sqrt{A^2 - 2C}, \quad \beta = -\frac{A}{\alpha} \tag{4.52}$$

$$L = -A - \alpha \left(\frac{\sinh(\alpha\xi^2 T) + \beta \cosh(\alpha\xi^2 T)}{\cosh(\alpha\xi^2 T) + \beta \sinh(\alpha\xi^2 T)} \right) \tag{4.53}$$

$$M = B \left\{ \frac{\beta \sinh(\alpha\xi^2 T) + b^2 \cosh^2(\alpha\xi^2 T) + 1 - b^2}{\cosh(\alpha\xi^2 T) + \beta \sinh(\alpha\xi^2 T)} - 1 \right\} \tag{4.54}$$

$$\begin{aligned} N &= \tfrac{\alpha - A}{2\alpha^2}[\alpha^2 - AB^2 - B^2\alpha]\xi^2 T + \tfrac{B^2(A^2-\alpha^2)}{a^3} \left\{ \tfrac{(2A+\alpha)+(2A-\alpha)e^{2\alpha\xi^2 T}}{A+\alpha+(\alpha-A)e^{2\alpha\xi^2 T}} \right\} \\ &+ \tfrac{2AB^2[a^2-A^2]e^{2\alpha\xi^2 T}}{a^3(A+\alpha+(\alpha-A)e^{2\alpha\xi^2 T})} - \tfrac{1}{2}\ln\left\{ \tfrac{1}{2}\left(\tfrac{A}{\alpha} + 1\right) + \tfrac{1}{2}\left(1 - \tfrac{A}{\alpha}\right)e^{2\alpha\xi^2 T} \right\} \end{aligned} \tag{4.55}$$

Conclusion

Temporal integrals or averages of processes can prove delicate to study, as shown in this thesis. In many cases, their study relies upon the analysis of their moment generating function or characteristic function, i.e. the Laplace transform of their density (for positive processes). One of our aims here was to push the analytical investigation as far as we could, before giving in to numerical methods. We thus managed to provide explicit series or real integral expressions for a number of quantities which have been, up to now, mainly solved in practice by numerical Laplace or Fourier inversion algorithms and other numerical methods. Those explicit expressions help us to gain insight into the mathematical structure of the problem and the behaviour and properties of the diverse quantities studied. Moreover, they also furnish a way for systematic implementation whereas the other numerical methods generally depend on free parameters they may not be robust to and on whose location no intuition is available. The explicit expressions we derived also appear more efficient, quicker or allowing more accuracy than numerical inversion schemes for some ranges of parameters. We also obtained explicit formulae to approximate Laplace transforms when they were not available in the mathematical literature. We considered in depth two special cases of CEV processes, both of theoretical interest for their tractability and of great practical importance as they drive the most fundamental models in finance.

Of these two special cases, the square-root process stands out. Indeed, its many advantageous properties - positivity, tractability, mean-reversion - make it quite versatile and enabled us to consider a variety of applications. Moreover, it leads to simpler expressions and an easier handling than the geometric Brownian motion. The square-root process has been given special attention in interest rate and equity derivatives research for these exact reasons. In this thesis, we extended this analysis to Asian and other options depending on the integrated square-root process. To this effect, we first examined the probabilistic properties of this integrated process and arrived to a number of distributional results, among which its moments to any order, its density and probability distribution function. We then deduced Asian options and other average-rate claims prices in a series form by inverting the Laplace transform analytically. Our series perform well for medium volatilities and maturities and converge very quickly for high volatilities or maturities whereas classical numerical Laplace transform inversion algorithms become too oscillatory to be trusted for these levels of parameters. For very low volatilities or maturities, yet, our series appear slow. However, those regions of parameters are not really problematic since numerical inversions work best there and actually even some quick and dirty approximations would reasonably do. High volatilities and maturities constitute a more interesting and problematic region for which our series form a nice and quick solution. Although we considered several financial applications for the process in this thesis, a number of others can still be derived from or in the same way as the distributional results we obtained. One such example coming naturally to mind is credit derivatives, the square-root process modelling the intensity of default. Another topic for future research and development arising from our work concerns the feasibility of a generalisation of the methods and formulae we proposed to broader model assumptions, typically multifactor CIR processes.

We could not help studying the second important special case of CEV diffusions constituted by the geometric Brownian motion, the major actor on the equity stage. As indicated by the considerable amount of research devoted to Black and Scholes Asian options by financial mathematicians, the valuation of these derivatives is a difficult exercise. Our aim was once again to arrive to an explicit solution in a series form or a real integral. We analytically inverted the Laplace transform of fixed strike as well as unseasoned and seasoned floating strike options. The formulae we derived for the prices are specially useful to achieve high levels of accuracy. Along with the development of these results, we presented a synthesis of the different possible formulations of the Asian pricing problem as a one-factor Markov one and of the relation between the options. It appears that the Asian option price is linked to a number of special functions. Further research should include an attempt to interrelate the different representations of these various quantities in special functions. Another point of interest is how these methods could be extended to incorporate more elaborate model features given that we saw the one-factor Markov formulation of the Asian pricing problem is robust to a number of extensions. One such extension we considered in a small parenthesis in Chapter 2 is the addition of multiplicative jumps to the lognormal equity process, model whose analysis could be further deepened on the basis of what we derived.

Our research regarding the general CEV process is still in its initial stage as chronologically the last part of this thesis work. We built an explicit series solution for the Laplace transform of the MGF of the integral of the equity-type process as well the mean-reverting process for a subclass. The first obvious application is the valuation of zero-coupon bonds in the Chan Karolyi Longstaff and Sanders [13] model through numerical Laplace inversion. Asian options on the CEV equity models could also be obtained from these results with a double numerical Laplace inversion. Several points leave room for further development and research: could the series be represented in an integral form or in terms of special functions?

Chapter 4 devoted to stochastic volatilities models stand a bit aside from the rest of the thesis, as we wanted to develop generic results rather than formulae specific to only one model. We first gained insight in the structure of the model by showing how conditioning variables and MGF or characteristic function methods can be constructed from basic characteristic functions. We then proposed several asymptotic expansions as well as a Laguerre convergent expansion. We believe this type of expansions could be useful as in many cases, it is easier to obtain moments rather than moment generating functions. The extension of this expansion to a correlated stochastic volatility is left for future work.

Appendix A

Special functions

This brief appendix collects the definition and various results on special functions which are employed in this thesis. We advise Erdelyi [28] and Gradshteyn and Ryzhik [37] for further reference. Other specific results concerning these special are quoted inside the body of the thesis whenever needed. We also add that, for simplicity of presentation, we quoted some of the definitions and properties given in the Appendix inside the thesis itself.

A.0.2 Hermite polynomials

The Hermite polynomials are a group of orthogonal polynomials defined by

$$\frac{\partial^k e^{-\frac{x^2}{2}}}{\partial x^k} = (-1)^k He_k(x) e^{-\frac{x^2}{2}} \tag{A.1}$$

Their explicit form is given by

$$He_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!} \tag{A.2}$$

They are generated by

$$g(x,t) = e^{-\frac{t^2}{2}+tx}$$

We can deduce from this

$$He_k'(x) = nHe_{k-1}(x)$$

and

$$He_{k+1}(x) = xHe_k(x) - kHe_{k-1}(x) \tag{A.3}$$

A.0.3 Hermite polynomials of the second kind

These polynomial do not actually belong to the usual classification of special functions. We gave them this name as they are similar to Hermite polynomials. More precisely, they are defined by

$$\frac{\partial^k e^{\frac{x^2}{2}}}{\partial x^k} = He_k(x) e^{\frac{x^2}{2}} \tag{A.4}$$

which results in the explicit form

$$\widetilde{He}_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!} \tag{A.5}$$

It is useful to notice that they are generated by

$$g(x,t) = e^{\frac{t^2}{2}+tx}$$

and satisfy

$$\widetilde{He}_k'(x) = k\widetilde{He}_{k-1}(x)$$

and

$$\widetilde{He}_{k+1}(x) = x\widetilde{He}_k(x) + k\widetilde{He}_{k-1}(x) \tag{A.6}$$

A.0.4 Laguerre polynomials

Still another class of orthogonal polynomial defined by

$$L_n^a(x) = \frac{x^{-a}}{n!} e^x \frac{d}{dx} (x^{n+a} e^{-x}) \tag{A.7}$$

which leads to

$$L_n^a(x) = \sum_{m=0}^{n} (-1)^m \binom{n+a}{n-m} \frac{x^m}{m!} \tag{A.8}$$

The differential relation

$$\frac{d}{dx} L_n^a(x) = -L_{n-1}^{a+1}(x) \tag{A.9}$$

and the recursive relation

$$(n+1)L_{n+1}^a(x) - (2n+a+1-x)L_n^a(x) + (n+a)L_{n-1}^a(x) = 0 \tag{A.10}$$

should be noted.

A.0.5 Gamma functions

The Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ \mathrm{Re}(z) > 0 \tag{A.11}$$

is defined by a specific contour integral for all complex values of z except non-negative integers $-n$, which are simple poles with residues $\frac{(-1)^n}{n!}$.

The relation $\Gamma(z+1) = z\Gamma(z)$ is valid for complex values of the argument as well. Erdelyi [28] collects a good number of other relations involving this function.

A.0.6 Confluent hypergeometric functions

The confluent hypergeometric functions or Kummer functions of the first and second kind, respectively denoted $\phi(a, c, z)$ and $\psi(a, c, z)$, are two solutions to the differential equation

$$z\frac{d^2 f}{dz^2} + (c-a)\frac{df}{dz} - af = 0 \tag{A.12}$$

They are defined by

$$\phi(a,c,z) = \sum_{n=0}^{\infty} \frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)} \frac{z^n}{n!} \tag{A.13}$$

and

$$\psi(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\phi(a,c,z) + \frac{\Gamma(c-1)}{\Gamma(c)} z^{1-z}\phi(a-c+1, 2-c, z) \tag{A.14}$$

They are also called degenerate hypergeometric function, the generalised hypergeometric function being

$${}_pF_q(a_1, a_2, \dots a_p; c_1, c_2, \dots c_q, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \dots (a_p)_k}{(c_1)_k(c_2)_k \dots (c_q)_k} \frac{z^k}{k!} \tag{A.15}$$

where

$$(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$$

refers to the Pochhammer symbol.

ϕ possesses several integral representations, among which the important following one, for $0 < \mathrm{Re}(a) < \mathrm{Re}(c)$,

$$\phi(a,c,z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(a)\Gamma(c-a)} \int_0^z e^t t^{a-1}(z-t)^{c-a-1} dt \tag{A.16}$$

For ψ, we have, for $\mathrm{Re}(a)$,

$$\psi(a,c,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1}(1+t)^{c-a-1} dt \tag{A.17}$$

A number of relations and properties of this function are known, among which the Kummer relation

$$\phi(a,c,z) = e^z \phi(c-a, c, -z) \tag{A.18}$$

and the special case of negative integers for the first argument

$$\phi(-n, a+1, z) = \frac{1}{\binom{n+a}{n}} L_n^a(z) \tag{A.19}$$

The asymptotics of the Kummer functions are

$$\phi(a,c,z) = \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} e^{z}[1 + \mathcal{O}(|z|^{-1})] \tag{A.20}$$

for high positive z and

$$\phi(a,c,z) = \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}[1 + \mathcal{O}(|z|^{-1})] \tag{A.21}$$

For ψ, we have

$$\psi(a,c,z) = \sum_{n=0}^{N}(-1)^n \frac{(a)_n(a-c+1)_n}{n!} z^{-a-n} + \mathcal{O}(|z|^{-a-N-1}) \tag{A.22}$$

when $|z| \longrightarrow \infty$ and $-\frac{3}{2}\pi < \arg(z) < \frac{3}{2}\pi$.

A.0.7 Parabolic cylinder functions

This function is a linear combination of ϕ

$$D_p(z) = 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma(\frac{1-p}{2})} \phi\left(-\frac{p}{2}, \frac{1}{2}, \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma(-\frac{p}{2})} \phi\left(\frac{1-p}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \right\} \tag{A.23}$$

satisfying the recursion

$$D_{p+1}(z) - zD_p(z) + pD_{p-1}(z) = 0 \tag{A.24}$$

The parabolic cylinder function has different possible integral representations and is linked with Hermite polynomial for positive indexes

$$D_n(z) = e^{-\frac{z^2}{4}} He_n(z) \tag{A.25}$$

and with the gaussian probability distribution function for negative integers

$$\begin{aligned} D_{-1}(z) &= e^{\frac{z^2}{4}} \sqrt{\tfrac{\pi}{2}}[1 - \mathcal{N}(\tfrac{z}{\sqrt{2}})] \\ D_{-1}(z) &= e^{\frac{z^2}{4}} \sqrt{\tfrac{\pi}{2}}[\sqrt{\tfrac{2}{\pi}} e^{-\frac{z^2}{4}} - z(1 - \mathcal{N}(\tfrac{z}{\sqrt{2}}))] \end{aligned} \tag{A.26}$$

A.0.8 Bessel functions

In these thesis, we mainly use the Bessel functions and modified Bessel function

$$
\begin{aligned}
J_\nu(z) &= \tfrac{z^\nu}{2^\nu} \textstyle\sum_{k=0}^{\infty} (-1)^k \tfrac{z^{2k}}{2^{2k} k! \Gamma(\nu+k+1)} \\
I_\nu(z) &= \tfrac{z^\nu}{2^\nu} \textstyle\sum_{k=0}^{\infty} \tfrac{z^{2k}}{2^{2k} k! \Gamma(\nu+k+1)}
\end{aligned}
\tag{A.27}
$$

and K_ν which is defined in a complex way.

All these functions follow two-step recursions and simplifies ton finite sums for indexes equal to an integer plus one half.

Appendix B

Numerical Laplace transform inversion

Initially renown as a difficult and ill-conditioned, the numerical Laplace transform has become a fashionable and often employed tool. A full plethora of methods have been developed over the years to invert both Laplace and Fourier transforms. As mentioned in Craddock and al. [17], there is generally only one way to determine which of these methods is the most suitable for a specific transform: trial and comparison.

Denoting $f(\gamma) = \int_0^\infty e^{-\gamma t} F(t) dt$, the inverse can be obtained by

$$F(t) = \frac{2e^{at}}{\pi} \int_0^\infty \mathrm{Re}(f(a+iu))\cos(ut) du \tag{B.1}$$

for any a to the right of all the singularities of $f(\cdot)$.

The most popular inversion method, the Abate and Whitt algorithm, rely on a trapezoidal rule to invert this oscillatory integral. Defining $A = at$, the Abate and Whitt is the sum of the series

$$F^{\mathrm{AB}}(t) = \frac{e^{\frac{A}{2}}}{2t} \mathrm{Re}\left(f\left(\frac{A}{2t}\right)\right) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{\infty} \mathrm{Re}\left(f\left(\frac{A+2ki\pi}{2t}\right)\right) \tag{B.2}$$

The series can critically depend on the choice of A: the series may not converge to the correct values for too small A. On the other hand, it might become too difficult to evaluate numerically for large A. The correct value for the inverse can be located on a interval of A for which the sum (B.2) remains constant. This length of this interval of stability can vary greatly from one application to another.

Appendix C

Eigenfunction expansion

In some of the results developed in this thesis, we use the theory of eigenfunction expansion. Those eigenfunctions arise as the solutions of the following regular Sturm-Liouville boundary-value problem for $x \in [L, U]$

$$q(x)\frac{\partial^2 f}{\partial x^2} + p(x)\frac{\partial f}{\partial x} - (\lambda + r(x))f(x) = 0 \tag{C.1}$$

with the boundary conditions

$$f(L) = 0, \qquad f(U) = 0 \tag{C.2}$$

and the restriction $q(x) > 0,\ \forall x \in]L, U[$.

This equation should first be transformed into his self-adjoint form as the eigenfunction expansion theory is based on second-order self-adjoint differential equations. To this effect, we need to consider

$$s(x) = e^{\int_L^x \frac{p(y)}{q(y)}dy} \tag{C.3}$$

and

$$w(x) = \frac{s(x)}{q(x)} \tag{C.4}$$

Defining the operator

$$\mathbf{L}f = \frac{1}{w(x)}\frac{d}{dx}\left[s(x)\frac{df}{dx}\right] \tag{C.5}$$

the transformed differential equation $(\mathbf{L} - (\lambda + r(x)))f = 0$ is self-adjoint.

The eigenfunction expansion theory states that there is a countable infinity of couples of positive real eigenvalues and associated eigenfunctions (λ, f) satisfying the equation $(\mathbf{L} - (\lambda + r(x)))f = 0$, i.e. ODE (C.1) as well as the boundary conditions (C.2). This countable set of eigenfunctions constitute a orthogonal basis for the Hilbert space $\mathcal{L}_{2,w}(L, U)$ of functions on $[L, U]$ of finite norm with respect to the norm associated with the scalar product

$$< f, g >= \int_L^U f(x)g(x)w(x)dx \tag{C.6}$$

Normalising the eigenfunctions, we obtain a orthonormal basis for $\mathcal{L}_{2,w}(L, U)$, i.e. a sequence of couples $(\lambda_n, f_n(\cdot))$ satisfying

$$\int_L^U f_i^2(x)w(x)dx = 1 \ \ \forall i \in \mathbb{N} \tag{C.7}$$

$$\int_L^U f_i(x)f_j(x)w(x)dx = 0 \quad \forall (i,j) \in \mathbb{N}^2, \ i \neq j \tag{C.8}$$

and

$$\forall f(\cdot) : [L, U] \longrightarrow \mathbb{R}, \ \int_L^U f^2(x)w(x)dx < \infty \Longrightarrow f(\cdot) = \sum_{n=0}^{\infty} c_n f_n(\cdot) \tag{C.9}$$

where

$$c_n = \int_L^U f_n(x)f(x)w(x)dx \tag{C.10}$$

This decomposition is the core of the Sturm-Liouville theory and allows us to project any sufficiently smooth function on this eigenfunction basis. We see in this thesis that this expansion has some important applications in Finance. It has also been widely in other areas of science, typically in Physics (see Arfken [3] for some examples).

Is still left the issue of how to practically implement this expansion. The answer follows from a careful analysis of the Green function, a quantity useful in general to handle non-homogeneous equation but also of specific importance in this case as it provides a numerical - in the best case, an analytical - way of finding the eigenvalues λ_n. Representing the Dirac delta function by $\delta(\cdot)$, this Green function is defined as the bounded solution of

$$(\mathbf{L} - (\lambda + r(x)))\mathcal{G}_{-\lambda}(x,y) = \delta(x-y) \tag{C.11}$$

with the two boundary conditions $\mathcal{G}_{-\lambda}(L,y) = \mathcal{G}_{-\lambda}(U,y) = 0$.
From (C.9), we thus obtain

$$\mathcal{G}_{-\lambda}(x,y) = w(y)\sum_{n=0}^{\infty}\frac{f_n(x)f_n(y)}{\lambda_n - \lambda} \tag{C.12}$$

Denoting $\eta(\lambda,x)$ and $\xi(\lambda,x)$ two independent solutions of the homogeneous equation and $W_{\eta,\xi}(x)$ their Wronskian,

$$W_{\eta,\xi}(x) = \eta(\lambda,x)\xi'(\lambda,x) - \eta'(\lambda,x)\xi(\lambda,x) \tag{C.13}$$

it is well known that a non-homogeneous second-order differential equation of second member $F(x)$ has for solutions

$$\tilde{f}(\lambda,x) = \eta(\lambda,x)\int^x \frac{\xi(\lambda,s)F(s)}{W_{\eta,\xi}(s)}ds - \xi(x)\int^x \frac{\eta(\lambda,s)F(s)}{W_{\eta,\xi}(s)}ds \tag{C.14}$$

whenever these integrals exist.
Choosing the two solutions to the homogeneous equation in such a way that $\eta(\lambda,L) = 0$ and $\xi(\lambda,U) = 0$, we obtain the Green function

$$\mathcal{G}_{-\lambda}(x,y) = w(y)\frac{\eta(\lambda,\min(x,y))\xi(\lambda,\max(x,y))}{C(\lambda)} \tag{C.15}$$

where $C(\lambda)$ is a constant depending on λ only. More precisely, $C(\lambda) = W_{\eta,\xi}(x)s(x)$, again a classical result from the theory of second-order homogeneous equation.

Combining the representations (C.12) and (C.15) provides a complete information as to the eigenvalues and corresponding eigenfunctions. Indeed, the λ_n are the only singularities of $\mathcal{G}_{-\lambda}(x, y)$ and are thus determined by the zeroes of $C(\cdot)$, i.e the zeroes in λ of the Wronskian $W_{\eta,\xi}(x)$. The nullity of the Wronskian at the points λ_n means that for these values, the two solutions $\eta(\lambda_n, \cdot)$ and $\xi(\lambda_n, \cdot)$ are the same up to a multiplicative constant. They thus satisfy both the ODE (C.1) and the condition (C.2). They therefore are multiples of the normalised eigenfunction. The actual normalised eigenfunction can be found either by calculating the norm directly

$$\|\eta(\lambda_n, \cdot)\|^2 = \int_L^U \eta^2(\lambda_n, x) w(x) dx \tag{C.16}$$

or by computing the residual of the Green function (C.12) at the simple poles λ_n

$$\|\eta(\lambda_n, \cdot)\|^2 = \left|\frac{\zeta(\lambda_n, h)}{\eta(\lambda_n, h)}\right| \lim_{\lambda \longrightarrow \lambda_n} \frac{C(\lambda)}{\lambda_n - \lambda} = -\left|\frac{\zeta(\lambda_n, h)}{\eta(\lambda_n, h)}\right| C'(\lambda) \tag{C.17}$$

where the ratio $\frac{\zeta(\lambda_n, h)}{\eta(\lambda_n, h)}$ is independent of h.

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