Pricing and Hedging in an Incomplete Interest Rate Market: Applications of the Laplace Transform

Christopher Solon Strom

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Department of Statistics,
The London School of Economics and Political Science,
The University of London

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Declaration

The work presented in this thesis and to the examiners' committee is my own, except where I specifically refer to other publications.

Christopher S. Strom

Abstract

This thesis explores pricing models for interest rate markets. The model used to describe the short rate is based on the discontinuous shot noise process. As a consequence the market is incomplete, meaning that not all securities contingent on the short rate can be replicated perfectly with a dynamically adjusted portfolio of a bond and cash. This framework is still consistent with the absence of arbitrage as evidenced by the existence of an equivalent martingale measure. This measure is not unique, however, due to the incompleteness of the market.

Two approaches to pricing contingent claims are pursued. The first, risk-neutral pricing, evaluates the expected value of the pay-off at expiration under an equivalent martingale measure. A parameterized class of martingales, based on the Esscher transform, allows for the definition of a flexible set of equivalent martingale measures and results in a formula for the conditional joint Laplace transform of the short rate and its time-integral. The pricing formula for a discount bond follows trivially from these results. A method for pricing a European call option is also proposed, requiring numerical inversion of the aforementioned Laplace transform.

The second approach, mean-variance hedging, addresses the incompleteness of the market. A contingent claim is priced by forming a portfolio of a bond and cash. The portfolio is dynamically updated to mimic the pay-off of the claim at expiration. The replicating portfolio is restricted to be self-financing and predictable. This approach leads to a closed-form pricing formula for a discount bond and formulæ for European call and put options, requiring the numerical Laplace inversion methods mentioned above. All this is in the context of a discrete-time model that includes as a special case a discrete-time version of the shot noise process.

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Chapter 1

Introduction

1.1 Martingales and Arbitrage

Martingale theory has provided a suitable framework for pricing securities, particularly contingent claims. A key element of this framework is the exclusion of arbitrage opportunities, which started with the work on option pricing by Black and Scholes [6].

Harrison and Kreps [26] relate the concept of arbitrage to the valuation of contingent claims in a general setting for the discrete-time case using results from martingale theory, while Harrison and Pliska [27] expand these results to the continuous-time case. See Harrison and Pliska [27] and Baxter and Rennie [5] for specific derivations of the Black-Scholes option pricing formula that are based more explicitly on the relation between martingale theory and the absence of arbitrage than that of Black and Scholes.

This section summarizes some results of the work mentioned above relevant to this thesis. The notation used in Harrison and Pliska [27] will be adopted in what follows. We begin by calling into existence a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$, satisfying the usual conditions (see Harrison and Pliska [27] or Protter [35]). In a frictionless market with unrestricted borrowing and short selling let the price processes (which are right continuous with left-hand limits or RCLL) of $K+1$ securities be given by positive components $S^0, S^1, \dots, S^K$ which are adapted and collected in a vector S. The market is assumed to evolve from time 0 to time T. Security zero plays the role of a cash bond, the price of which, if it is absolutely continuous, may be written as $e^{\int_0^t \lambda_s ds}$ where λ_t is interpreted to be the instantaneous short rate at time t. It will be convenient to normalize the price processes by dividing by the price of security zero. Thus the discounted price processes $Z^0, Z^1, \ldots, Z^K$ are obtained, where Z^0 is identical to 1 for $0 \leq t \leq T$.

A trading strategy $\varphi = \{\varphi_t : 0 < t \leq T\}$ is defined to be a predictable vector process with components $\varphi^0, \varphi^1, \ldots, \varphi^K$. The vector process φ is predictable if its components φ^i, $i = 0, \ldots, K$ are measurable with respect to the σ-algebra generated by the adapted processes with left-continuous paths. A strategy φ defines a portfolio with value process $V(\varphi) = \sum_{i=0}^K \varphi^i S^i$. The gains process of portfolio φ is defined to be the stochastic integral $G(\varphi) = \int \varphi dS \equiv \sum_{i=0}^K \int \varphi^i dS^i$ or $G_t(\varphi) = \int_0^t \varphi_u dS_u$. A strategy φ is said to be self-financing if $V(\varphi) = V_0(\varphi) + G(\varphi)$, where $V_0(\varphi)$ is the initial value (investment) of the portfolio.

The portfolio may also be viewed in terms of the discounted securities. The discounted value process is defined as $V^*(\varphi) = \sum_{i=0}^K \varphi^i Z^i = \varphi^0 + \sum_{i=1}^K \varphi^i Z^i$ and the discounted gains process as $G^*(\varphi) = \int \varphi dZ \equiv \sum_{i=1}^K \int \varphi^i dZ^i$. Harrison and Pliska [27] prove that φ is self-financing if and only if $V^*(\varphi) = V_0^*(\varphi) + G^*(\varphi)$.

Next the existence is assumed of at least one measure P^* which satisfies the following: 1) P^* is equivalent to P and 2) the discounted price processes $Z^1, \ldots, Z^K$ are P^*-martingales. It will be necessary to determine the existence of P^* in specific applications. If such a P^* exists then for predictable φ^i the stochastic integrals $\int \varphi^i dZ^i$ are local P^*-martingales. Moreover, because the Z^i are positive, $\int \varphi^i dZ^i$ are supermartingales under P^*.

An arbitrage opportunity is defined to be a self-financing strategy φ such that $V_0(\varphi) = 0$, $V_0(\varphi) \geq 0$ for $0 < t \leq T$, but $V_T(\varphi) > 0$ with positive probability. There can be no arbitrage opportunities if an equivalent martingale measure exists. This is because, as mentioned above, under any equivalent martingale measure P^*, $V^*(\varphi)$ is known to be a positive supermartingale and thus must remain zero if it starts there. Under certain conditions the reverse implication also holds true yielding the Fundamental Theorem of Asset Pricing, which states that the existence of an equivalent martingale measure is equivalent to absence of arbitrage. See, for example, Schachermayer [37] for a proof of this theorem when time is finite discrete. Other treatments of the theorem can be found in Harrison and Kreps [26], Harrison and Pliska [27] and Delbaen and Schachermayer [15, 16].

A contingent claim is defined to be a positive, integrable random variable X (by convention $\mathcal{F} = \mathcal{F}_T$, thus X is $\mathcal{F}_T$-measurable). Such a claim is attainable if there exists a self-financing strategy such that $V_T^*(\varphi) = (S_T^0)^{-1} X$, then φ is called a replicating strategy and $\pi = V_0^*(\varphi)$ the price associated with X. Traditionally the trick has been to limit the range of trading strategies φ to predictable processes for which $V^*(\varphi)$ is a P^*-martingale, in which case $\pi = E^*[V_T^*(\varphi)] = E^*\left[(S_T^0)^{-1} X\right]$. Attainability is then defined with regards to this more limited set of strategies. A market is complete if all contingent claims are attainable. If a market is complete then the equivalent martingale measure is unique, see Harrison and Pliska [27].

The search for an equivalent martingale measure is thus justified in two ways. Firstly, finding at least one equivalent martingale measure implies the model does not allow arbitrage opportunities, providing an economic rationale to support the theory. Secondly, if the securities market is complete the martingale measure is unique and yields a unique price for any contingent claim.

Completeness is a fairly specialist property, which even minor modifications of a complete model may not possess. Examples are diffusion processes with a stochastic volatility component and stochastic processes with jumps. This thesis explores an incomplete model of the latter type. When the market is no longer complete not all contingent claims are attainable. The absence of arbitrage is still implied by the existence of an equivalent martingale measure, but as this measure is no longer necessarily unique, neither is the price of a non-attainable contingent claim.

Since no replicating strategies exist for an non-attainable contingent claim, one possible approach is to choose a strategy that is as close as possible to a replicating strategy according to a some (subjective) criterion. Föllmer and Sondermann [22] drop the condition that a replicating strategy be self-financing. In their approach the strategy is certain to have a terminal value equal to the pay-out of the claim at expiration, but may require cash in- and out-flows during its lifetime. The strategy is chosen by minimizing a risk function, which is defined as the expected value at present of the square of the cash flows to the strategy incurred over the remaining time to expiration. Föllmer and Sondermann [22] only consider the case where $K = 1$ and the discounted price process is already a martingale under the base measure. It turns out that the risk minimizing strategy is mean-self-financing, that is the cost process, defined to be the accumulated cash flows to the strategy, is a martingale. Föllmer and Schweizer [23] extend this result to the general case where the discounted price process is a semimartingale and the strategy is locally risk-minimizing. Schweizer [39] provides an overview of the above as well as the alternative approach, mean-variance hedging, where the replicating strategy is required to be self-financing and where the expected value of the squared difference between the value of the replicating portfolio at expiration and the pay-off of the contingent claim is minimized.

1.2 Interest Rate Models

The main topic of this thesis is the pricing of securities that are contingent on interest rates. As with all securities, martingales and arbitrage play a key role in pricing those that depend on interest rates. An extensive literature is devoted to this topic, but the paper by Heath, Jarrow and Morton [28] is central. In this paper the authors build on the work of Harrison and Pliska [27] to construct a unifying theory for valuing contingent claims under a stochastic term structure of interest rates. Their model, based on diffusion-type processes, describes a complete market with a unique equivalent martingale measure and unambiguous prices for contingent claims. The earlier companion papers by Cox, Ingersoll and Ross [8, 9], also describe a term-structure theory based on diffusion-type processes, but do not use the martingale theory-based argument of no-arbitrage to value securities. Instead, their approach is based on economic equilibrium. Baxter and Rennie [5] provide an overview of the various term-structure models available in the literature that are special cases of the very general Heath-Jarrow-Morton (HJM) model.

Some terminology standard in the term-structure literature follows. Analogous to section 1.1, a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\{\mathcal{F}_t; 0 \leq t \leq \tau\}$ describe the uncertainty in the economy and evolution of information, respectively. Default-free discount (zero-coupon) bonds trade with various maturities $T \in [0, \tau]$. The price at time t of the T maturity bond is denoted by $P(t,T)$. It is required that $P(T,T) = 1$, $P(t,T) > 0$ for $t \in [0,T]$ and $\partial \log P(t,T) / \partial T$ exists.

The instantaneous forward rate at time t for date $T > t$ is then defined by

$$f(t,T) = -\partial \log P(t,T) / \partial T \quad \text{for} \quad T \in [0,\tau], \quad t \in [0,T]$$

This rate represents the forward price at time t of instantaneous risk-free borrowing at a later time T. It follows that

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\, ds\right) \quad \text{for} \quad T \in [0,\tau], \quad t \in [0,T]$$

The short rate at time t is the instantaneous forward rate at time t for date t and is given by

$$\lambda_t = f(t,t) \quad \text{for} \quad t \in [0,\tau]$$

The value of a continuously compounding cash bond or deposit is then given by $B_t = e^{\int_0^t \lambda_s ds}$, the same as security 0 from section 1.1 above. This security will have the special role of being a numeraire for other securities. Thus the discounted zero coupon bond price is $Z(t,T) = B_t^{-1} P(t,T)$. Under the assumption of no arbitrage an equivalent martingale measure P^* exists that allows a given interest rate contingent claim to be priced according to $B_t E^*\left(B_T^{-1} X \middle| \mathcal{F}_t\right)$, where X is the pay-off at expiration. A direct consequence is that the discount bond price is given by $P(t,T) = B_t E^*\left(B_T^{-1} \middle| \mathcal{F}_t\right)$.

In the HJM model the forward rate curve evolves over time according to a (possibly multi dimensional) stochastic process. Though many popular interest rate models are defined in terms of a short rate process, they can be written as a forward rate model along the lines of HJM. The interest rate model examined in this thesis is a short rate model. Instead of a diffusion-type process, however, the short rate model is based on the discontinuous shot-noise process. Unlike HJM, this model describes an incomplete market. The approach in this thesis follows that of Dassios [10], Dassios and Jang [11, 12] and Jang [31], where the model has been explored extensively in an insurance mathematics context. The shot noise process is introduced in the next chapter.

1.3 Overview

This dissertation is organized as follows: Chapter 2 presents an overview of the shot-noise process. An important martingale is introduced which leads to an expression for the joint Laplace transform of the interest rate and its time integral. The martingale is also used to define an equivalent martingale measure based on the Esscher transform. These results will be used in subsequent chapters to derive results for pricing interest rate contingent claims. Finally, a short overview is given of the important affine jump-diffusion model, of which the shot-noise process is a special case.

Chapter 3 develops some results in a continuous-time risk-neutral pricing context. An interest rate model based on the shot-noise process is introduced. The price of a zero-coupon bond is derived by taking the expectation of the discounted pay-off under an equivalent martingale measure. The derivation of this result relies on the Laplace transform from Chapter 2. A method is presented to invert the Laplace transform when the jumps of the joint noise process are assumed to be exponentially distributed. This yields the joint probability density function of the interest rate and its time integral, which is used to compute the price of a call option on a bond, by taking the expectation of the pay-off under an equivalent martingale measure. The Laplace inversion method used here is well-known in the telecommunications literature [1, 2] as it is well suited for problems in queuing-theory, but we are unaware of its application heretofore in the field of mathematical finance.

In Chapter 4 a discrete-time interest rate model is introduced, which has as a special case a discrete-time version of the shot-noise process. This model is more general, however, and allows for non-deterministic paths between jumps. The discrete nature of this model makes it easier to price a non-attainable security by tracking the value of a replicating portfolio that is designed to have a value as close as possible (in a mean-square sense) to the pay-off of the target security at expiration. The replicating strategy is derived by solving a recursive optimization problem. Using this methodology, a general contingent claim, which has zero-coupon bonds and European call and put options as special cases, is priced by forming replicating portfolios of a longer-dated zero-coupon bond and a cash account. The derived pricing formula is closed-form up to an expectation operator. The expected value is evaluated using Laplace inversion techniques in the context of a discount bond as well as a European call and put option. The expectation result is closed-form in the case of the bond while numerical Laplace inversion techniques are required for the options. These techniques lead to numerical evaluations that have not been before and the option pricing result is central to this thesis. The equivalent martingale measure implied by the mean-variance replicating bond price is examined and an explicit representation for the Rado-Nikodym derivative is derived. This is also a new result.

The model in chapter 4 is parameterized so that the length of the time-increments can be made arbitralily small. A limit argument then ties much of the discrete-time theory developed in 4 in with the continuous-time model presented in chapter 3. The framework developed for pricing bonds and European call and put options on bonds allows for a rich set of short rate processes that are discrete-time approximations to processes which can include multiple sources of jumps and diffusion-type behavior between jumps. These results are another novel contribution of this thesis to the mathematical finance literature.

Chapter 5 concludes with numerical implementations of the results developed in the previous chapters.

Chapter 2

Overview of the Shot Noise Process

2.1 Introduction

This chapter presents an overview of the shot-noise process. Much of the theory is based on piece-wise deterministic Markov processes introduced by Davis [13, 14]. We give an overview of the generator and the extended generator of a Markov process. Next, the shot noise process is defined along with its generator. An important martingale is introduced which leads to an expression for the joint Laplace transform of the shot noise process and its time integral. The martingale is also used to define an equivalent martingale measure based on the Esscher transform. These results are from Dassios [10], Jang [31] and Dassios and Jang [11] and will be used in subsequent chapters to derive results for pricing interest rate contingent claims.

Finally, a short overview is given of the important affine jump-diffusion model, of which the shot noise process is a special case. This is based on Duffie, Pan and Singleton [19].

2.2 Strong Generator

In this and the following section notation similar to that in Davis [14] will be used and the reader is referred to this publication for further details. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), (x_t), (P_x, x \in E))$ be a Markov family on a state space E. The initial distribution is δ_x, i.e. $P_x[x_0 = x] = 1$, while the transition function p is related to the measure P_x by

$$P_x[x_t \in A] = p(t, x, A)$$

and satisfies the Chapman-Kolmogorov equation

$$p(t+s, x, A) = \int_E p(s, y, A)\, p(t, x, dy)$$

The expectation operator associated with the above Markov family is then defined as

$$E_x[f(x_t)] = \int_E f(y)\, p(t, x, dy)$$

If we let $B(E)$ denote the set of bounded measurable functions on E we may define an operator $P_t : B(E) \rightarrow B(E)$ by

$$P_t f(x) = E_x[f(x_t)] \tag{2.1}$$

Note that $P_t f(x)$ is again a function of x, the initial value of the process x_t. The Chapman-Kolmogorov equation is equivalent to the following semi-group property of P_t for all $s, t \geq 0$,

$$P_t P_s = P_{t+s}$$

The strong generator of a (vector) Markov process x_t acting on a function $f(x)$, denoted by $\hat{A}f(x)$, is defined as

$$\hat{A}f(x) \equiv \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t} \tag{2.2}$$

for any function $f(x)$ in $B(E)$, for which this limit exists. The strong generator can be regarded as a generalization of the derivative of $E_x[f(x_t)]$ with respect to t evaluated at 0. It can be proved (see Davis [14] or Öksendal [33]) that

$$E_x[f(x_t)] = f(x) + E_x\left[\int_0^t \hat{A}f(x_u)\,du\right] \tag{2.3}$$

for any stopping time $t \geq 0$. This is known as Dynkin's formula. The set of functions for which the limit in (2.2) exists is called the domain of $\hat{A}$ and denoted by $D\left(\hat{A}\right)$.

2.3 Extended Generator

Consider the Markov process x_t from the previous section. Let $D(A)$ denote the set of functions f in $B(E)$ for which the following property holds: for every $f \in D(A)$ there exists a measurable function $h_f : E \to R$ (which may vary with f) such that the function $t \to h_f(x_t)$ is integrable P_x-a.s. and the process

$$C_t^f = f(x_t) - f(x) - \int_0^t h_f(x_u)\,du \tag{2.4}$$

is a martingale, where $t \geq 0$. We can then write $Af(x) \equiv h_f(x)$ and call A the extended generator of the process x_t. We write

$$C_t^f = f(x_t) - f(x) - \int_0^t Af(x_u)\,du$$

Taking expectations on both sides gives us

$$E_x\left[C_t^f\right] = E_x[f(x_t)] - E_x[f(x)] - E_x\left[\int_0^t Af(x_u)\,du\right] = 0$$

As C_t^f is a martingale, $E_x\left\{C_t^f\right\} = C_0^f = 0$ for any stopping time $t \geq 0$, and thus

$$P_t f(x) = f(x) + \int_0^t P_u A f(x)\,du \tag{2.5}$$

which is essentially the same as (2.3), Dynkin's formula. It is often much easier to find easily checked sufficient conditions for a function's membership of $D(A)$ than of $D\left(\hat{A}\right)$.

2.4 The Shot Noise Process

The shot noise process consists of a series of random jumps occurring at poisson times. In between jumps, process decays deterministically at an exponential rate. The following definition is from Dassios and Jang [11]. At time t the process is represented as

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}$$

where:

λ_0	initial value of λ
y_i	size of jump i with $y_i > 0$ and $E(y_i) < \infty$
s_i	time at which jump i occurs, where $s_i < t < \infty$
δ	exponential decay
ρ	the rate of jump arrival.

The jump-size distribution is denoted $G(y)$. The aggregated process x_t is defined as the time-integral of λ_t:

$$x_t = \int_0^t \lambda_s ds$$

Following Jang [31] and Dassios and Jang [11], a slightly more general type of shot noise process will be considered, where the parameters and jump-size distribution depend on time:

$$\delta(t), \rho(t), G(y;t) \tag{2.6}$$

This is a special case of the so-called Piecewise Deterministic Markov Processes

(PDPs), a general class of non-diffusion stochastic processes, developed by Davis [13, 14].

Using the fact that the triplet (x_t, λ_t, t) is jointly Markov, the notation in (2.1) can be adapted to suit the particular case of the shot noise process

$$P_t f(x, \lambda, 0) = E\left[f(x_t, \lambda_t, t) \,|\, (x_0, \lambda_0, 0) = (x, \lambda, 0)\right] \tag{2.7}$$

When conditioning on a starting point different from $(x, \lambda, 0)$, (2.7) implies that

$$P_t f(x_s, \lambda_s, s) = E\left[f(x_{t+s}, \lambda_{t+s}, t+s) \,|\, (x_s, \lambda_s, s)\right] \tag{2.8}$$

Also, the generator of the generalized shot-noise process described above, acting on a function $f(x, \lambda, t)$ is given by

$$\begin{aligned} Af(x, \lambda, t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} - \delta(t) \lambda \frac{\partial f}{\partial \lambda} + \\ &\quad \rho(t) \left[\int_{\mathbb{R}^+} f(x, \lambda + y, t)\, dG(y; t) - f(x, \lambda, t) \right] \end{aligned} \tag{2.9}$$

where the domain of the generator, $D(A)$, is the set of functions f such that for all $i = 1, 2, \ldots,$

- f is absolutely continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times [s_{i-1}, s_i)$, that is, between jumps.
- $E\left[\left| f(x_{s_i}, \lambda_{s_i}, s_i) - f\left(x_{s_i}, \lambda_{s_i^-}, s_i\right) \right|\right] < \infty$

This follows from the results presented in Davis [13, 14] and is stated directly in Jang [31].

The next result is from Jang [31] and can also be found in Dassios and Jang [11].

Proposition 2.4.1 (Theorem 2.1.11 by Jang) *For constants $k, \nu \geq 0$ and the aggregated process x_t associated with a generalized shot noise process λ_t as defined above, the function defined by*

$$\begin{aligned} &e^{-\nu x_t} \exp\left(-\left(ke^{\Delta(t)} - \nu e^{\Delta(t)} \int_0^t e^{-\Delta(r)} dr\right)\lambda_t\right) \times \\ &\exp\left(\int_0^t \rho(s)\left[1 - \hat{g}\left(ke^{\Delta(s)} - \nu e^{\Delta(s)} \int_0^s e^{-\Delta(r)} dr; s\right)\right] ds\right) \end{aligned} \tag{2.10}$$

is a martingale where $\Delta(t) = \int_0^t \delta(s)\,ds$ *and* $\hat{g}$ *is the Laplace transform of the jump-size distribution* G.

Proposition 2.4.1 leads directly to the next useful result:

Corollary 2.4.2 *The joint Laplace transform of* λ_{t_2} *and* x_{t_2}*, given* $\mathcal{F}_{t_1}$*, is*

$$E\left[e^{-\nu_1 x_{t_2}} e^{-\nu_2 \lambda_{t_2}} \middle| \mathcal{F}_{t_1}\right] =$$
$$e^{-\nu_1 x_{t_1}} \exp\left(-\left(\nu_2 e^{-(\Delta(t_2)-\Delta(t_1))} + \nu_1 e^{\Delta(t_1)} \int_{t_1}^{t_2} e^{-\Delta(r)} dr\right)\lambda_{t_1}\right) \times$$
$$\exp\left(-\int_{t_1}^{t_2} \rho(s)\left[1-\hat{g}\left(\nu_2 e^{-(\Delta(t_2)-\Delta(s))} + \nu_1 e^{\Delta(s)} \int_s^{t_2} e^{-\Delta(r)} dr; s\right)\right] ds\right) \qquad (2.11)$$

Proof: Because (2.10) is a martingale

$$E\left[e^{-\nu x_{t_2}} \exp\left(-\left(k e^{\Delta(t_2)} - \nu e^{\Delta(t_2)} \int_0^{t_2} e^{-\Delta(r)} dr\right)\lambda_{t_2}\right) \middle| \mathcal{F}_{t_1}\right] \times$$
$$\exp\left(\int_0^{t_2} \rho(s)\left[1-\hat{g}\left(k e^{\Delta(s)} - \nu e^{\Delta(s)} \int_0^s e^{-\Delta(r)} dr; s\right)\right] ds\right) =$$
$$e^{-\nu x_{t_1}} \exp\left(-\left(k e^{\Delta(t_1)} - \nu e^{\Delta(t_1)} \int_0^{t_1} e^{-\Delta(r)} dr\right)\lambda_{t_1}\right) \times$$
$$\exp\left(\int_0^{t_1} \rho(s)\left[1-\hat{g}\left(k e^{\Delta(s)} - \nu e^{\Delta(s)} \int_0^s e^{-\Delta(r)} dr; s\right)\right] ds\right)$$

Substituting $\nu = \nu_1$ and $k = \nu_2 e^{-\Delta(t_2)} + \nu_1 \int_0^{t_2} e^{-\Delta(r)} dr$ into the above produces the desired result in (2.11). Note that from (2.11) and the fact that λ_t is a Markov process, it follows that

$$e^{\nu_1 x_{t_1}} E\left[e^{-\nu_1 x_{t_2}} e^{-\nu_2 \lambda_{t_2}} \middle| \mathcal{F}_{t_1}\right] = E\left[e^{-\nu_1 \left(x_{t_2} - x_{t_1}\right)} e^{-\nu_2 \lambda_{t_2}} \middle| \lambda_{t_1}\right]$$

Also note that x_t denotes a specific scalar-valued process in this section and is distinct from x_t in the previous section where it denotes any (possibly vector-valued) Markov process. In fact, (x_t, λ_t, t) in this section taken together as a single vector-valued Markov process is analogous to x_t from the previous section. For the special case $\delta(t) = \delta$, the martingale in (2.10) reduces to

$$e^{-\nu x_t} e^{-\left(k e^{\delta t} - \frac{\nu}{\delta}\left(e^{\delta t}-1\right)\right)\lambda_t} \exp\left(\int_0^t \rho(s)\left[1-\hat{g}\left(k e^{\delta s} - \frac{\nu}{\delta}\left(e^{\delta s}-1\right); s\right)\right] ds\right) \qquad (2.12)$$

When none of the parameters depend on time, the martingale further reduces to

$$e^{-\nu x_t} e^{-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta t}-1\right)\right)\lambda_t} \exp\left(\rho \int_0^t \left[1-\hat{g}\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right)\right)\right] ds\right) \tag{2.13}$$

Assume the latter case, where no parameters depend on time. Let λ_t describe the short-rate process. Then the value of a cash bond (or deposit) at time t is e^{x_t}. The extended generator of (x_t, λ_t, t) acting on a function $f(x, \lambda, t)$ is given by

$$\begin{aligned} Af(x,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} - \delta\lambda \frac{\partial f}{\partial \lambda} + \\ &\quad \rho \left[\int_{\mathbb{R}^+} f(x, \lambda + y, t)\, dG(y) - f(x,\lambda,t) \right] \end{aligned} \tag{2.14}$$

a result found in Dassios [10] and a special case of (2.9).

2.5 The Esscher Transform and Equivalent Martingale Measures

The Esscher transform is a tool borrowed from actuarial science, as noted in Gerber and Shiu [25], which has also proved useful in finance. The transform induces an equivalent martingale probability measure on certain stochastic processes which can be useful in valuing derivative securities. Gerber and Shiu [24, 25] apply Esscher transforms to option pricing for several types of stochastic processes. Bühlmann, Delbaen, Embrechts and Shiryaev [7] discuss Esscher transforms as they relate to the no-arbitrage argument in mathematical finance. Jang [31] applies the Esscher transform method to the pricing of insurance derivatives.

Let P^* denote a change of measure, equivalent to P. Then the expectation with respect to P^* is defined as

$$E^* \left[X \frac{P^*}{P} \right] \tag{2.15}$$

for a random variable X, which is measurable wrt $\mathcal{F}$. The expectation of a process X_t, adapted to $\mathcal{F}_t$, conditional on $\mathcal{F}_s$ can then be defined using the martingale $M_t = E\left[\frac{dP^*}{dP}\middle|\mathcal{F}_t\right]$ as

$$E^*\left[X_t \frac{M_t}{M_s}\middle|\mathcal{F}_s\right] \tag{2.16}$$

The Esscher transform allows us to reverse-engineer P^* by defining M_t as follows: Let $R(x,\lambda,t)$ be a measurable function such that $M_t = e^{R(x_t,\lambda_t,t)}$ is a martingale. Then we will use M_t to define the Radon-Nikodym derivative as in (2.16). This Radon-Nikodym derivative is a special case of the Esscher transform. For any measurable function f the expectation of the random variable $f(x_t,\lambda_t,t)$ with respect to the newly defined probability measure is then given by

$$E^*\left[f(x_t,\lambda_t,t)|\,\mathcal{F}_s\right] = \frac{E\left[f(x_t,\lambda_t,t)\,e^{R(x_t,\lambda_t,t)}\middle|\,\mathcal{F}_s\right]}{E\left[e^{R(x_t,\lambda_t,t)}\middle|\,\mathcal{F}_s\right]} \tag{2.17}$$

The following result is due to Jang [31], where it is proved in the slightly narrower context of the strong generator.

Theorem 2.5.1 *Let x_t and λ_t be as defined above. Let A denote their extended generator given by (2.14) and let A^* denote the extended generator of (x_t,λ_t,t) under the new measure given by (2.17). Also, let R be a function as defined above and furthermore satisfy the same restrictions as f given below (2.9). Then A^* acting on a function $f(x_t,\lambda_t,t)$ is given by*

$$A^* f(x,\lambda,t) = \frac{A\left\{f(x,\lambda,t)\,e^{R(x,\lambda,t)}\right\}}{e^{R(x,\lambda,t)}} \quad a.e. \tag{2.18}$$

Proof: Let P denote the base measure with respect to which the process (x_t,λ_t,t) is originally defined and let P^* denote the change of measure implied by using $e^{R(x_t,\lambda_t,t)}$ to define the Radon-Nikodym derivative. Recall from the previous section that $Af(x_t,\lambda_t,t)$ is defined to be the process $h_f(x_t,\lambda_t,t)$, where $h_f(x,\lambda,t)$ is a measurable function of (x,λ,t), such that the process

$$C_t^f = f(x_t,\lambda_t,t) - f(0,\lambda_0,0) - \int_0^t h_f(x_u,\lambda_u,u)\,du$$

is a P-martingale. Similarly, for (2.18) to hold, we need the process

$$C_t^{*f} = f(x_t, \lambda_t, t) - f(0, \lambda_0, 0) - \int_0^t \frac{A\left\{f(x_u, \lambda_u, u)\, e^{R(x_u, \lambda_u, u)}\right\}}{e^{R(x_u, \lambda_u, u)}} du \quad (2.19)$$

to be a P^*-martingale. As C_t^{*f} is adapted to $\mathcal{F}_t$, for $t \geq s$ we have

$$\begin{aligned} E^*\left[C_t^{*f} - C_s^{*f} \middle| \mathcal{F}_s\right] &= E^*\left[f(x_t, \lambda_t, t) \middle| \mathcal{F}_s\right] - f(x_s, \lambda_s, s) - \\ & E^*\left[\int_s^t \frac{A\left[f(x_u, \lambda_u, u)\, e^{R(x_u, \lambda_u, u)}\right]}{e^{R(x_u, \lambda_u, u)}} du \middle| \mathcal{F}_s\right] \end{aligned} \quad (2.20)$$

We will examine the last term more closely:

$$\begin{aligned} & E^*\left\{\int_s^t \frac{A\left[f(x_u, \lambda_u, u)\, e^{R(x_u, \lambda_u, u)}\right]}{e^{R(x_u, \lambda_u, u)}} du \middle| \mathcal{F}_s\right\} = \\ & \int_s^t E^*\left\{\frac{A\left[f(x_u, \lambda_u, u)\, e^{R(x_u, \lambda_u, u)}\right]}{e^{R(x_u, \lambda_u, u)}} \middle| \mathcal{F}_s\right\} du = \\ & \int_s^t P_{u-s}^*\left\{\frac{A\left[f(x_s, \lambda_s, s)\, e^{R(x_s, \lambda_s, s)}\right]}{e^{R(x_s, \lambda_s, s)}}\right\} du = \\ & \int_0^{t-s} P_r^*\left\{\frac{A\left[f(x_s, \lambda_s, s)\, e^{R(x_s, \lambda_s, s)}\right]}{e^{R(x_s, \lambda_s, s)}}\right\} dr \end{aligned} \quad (2.21)$$

In the first and second step above we made use of the Fubini theorem and Markov property of (x_t, λ_t, t), respectively. The operator P_t^* is defined analogous to (2.7) and (2.8). Using (2.17) yields

$$\begin{aligned} & P_t^* f(x_s, \lambda_s, s) = E^*\left[f(x_{t+s}, \lambda_{t+s}, t+s) \middle| (x_s, \lambda_s, s)\right] = \\ & E\left[f(x_{t+s}, \lambda_{t+s}, t+s) \frac{\exp(R(x_{t+s}, \lambda_{t+s}, t+s))}{\exp(R(x_s, \lambda_s, s))} \middle| (x_s, \lambda_s, s)\right] = \\ & \frac{P_t\left[f(x_s, \lambda_s, s) \exp(R(x_s, \lambda_s, s))\right]}{\exp(R(x_s, \lambda_s, s))} \end{aligned} \quad (2.22)$$

Substituting (2.22) into (2.21) results in

$$\frac{\int_0^{t-s} P_r \left\{ \frac{A[f(x_s,\lambda_s,s)\exp(R(x_s,\lambda_s,s))]}{\exp(R(x_s,\lambda_s,s))} \exp\left(R\left(x_s, \lambda_s, s\right)\right) \right\} dr}{\exp\left(R\left(x_s, \lambda_s, s\right)\right)} =$$

$$\frac{P_{t-s}\left[f\left(x_s, \lambda_s, s\right) \exp\left(R\left(x_s, \lambda_s, s\right)\right)\right]}{\exp\left(R\left(x_s, \lambda_s, s\right)\right)} - \frac{f\left(x_s, \lambda_s, s\right) \exp\left(R\left(x_s, \lambda_s, s\right)\right)}{\exp\left(R\left(x_s, \lambda_s, s\right)\right)} =$$

$$P^*_{t-s}\left[f\left(x_s, \lambda_s, s\right)\right] - f\left(x_s, \lambda_s, s\right) \tag{2.23}$$

The RHS of the first equality in (2.23) follows from (2.5). In order to use (2.5) the function $f(x, \lambda, t) \exp(R(x, \lambda, t))$ must lie in the domain of A, which it does by the restrictions imposed on R in the hypothesis. Noting that

$$E^*\left[f\left(x_t, \lambda_t, t\right) | \mathcal{F}_s\right] = P^*_{t-s}\left[f\left(x_s, \lambda_s, s\right)\right]$$

then shows that the last term in (2.20) cancels out the first two. We conclude that C_t^{*f} is a martingale and thus that the generator of (x_t, λ_t, t) is characterized by (2.18).

We now turn to a specific form of $e^{R(x_t,\lambda_t,t)}$, namely a slightly rewritten version of the martingale given in (2.13):

$$e^{R(x_t,\lambda_t,t)} = e^{-\kappa_1 \delta x_t} e^{-\left(\kappa_1 + \kappa_2 e^{\delta t}\right)\lambda_t} e^{\rho \int_0^t \left[1 - \hat{g}\left(\kappa_1 + \kappa_2 e^{\delta s}\right)\right] ds}$$

Here κ_1 and κ_2 are constants such that $\kappa_1 \geq 0$ and $\kappa_2 \geq -\kappa_1 e^{-\delta t^*}$ and it is assumed that the process evolves up to time t^*(see Dassios [10] for details). Thus $R(x, \lambda, t)$ has the following form:

$$R(x, \lambda, t) \equiv -\kappa_1 \delta x - \left(\kappa_1 + \kappa_2 e^{\delta t}\right) \lambda + \rho \int_0^t \left[1 - \hat{g}\left(\kappa_1 + \kappa_2 e^{\delta s}\right)\right] ds$$

The Radon-Nykodim derivative process may then be defined as

$$M_t \equiv e^{R(x_t,\lambda_t,t)} \equiv e^{-\kappa_1 \delta x_t} e^{-\left(\kappa_1 + \kappa_2 e^{\delta t}\right)\lambda_t} e^{\rho \int_0^t \left[1 - \hat{g}\left(\kappa_1 + \kappa_2 e^{\delta s}\right)\right] ds} \tag{2.24}$$

Let A^* denote the generator under the equivalent measure defined using (2.24).

In order to find $A^* f(x,\lambda,t)$ we will need $A\left\{f(x,\lambda,t)\, e^{R(x,\lambda,t)}\right\}$. Define

$$\varphi(x,\lambda,t) = f(x,\lambda,t)\, e^{R(x,\lambda,t)}$$

From (2.14) we know that

$$\begin{aligned} A\varphi(x,\lambda,t) &= \frac{\partial\varphi}{\partial t} + \lambda\frac{\partial\varphi}{\partial x} - \delta\lambda\frac{\partial\varphi}{\partial\lambda} + \\ &\quad \rho\left[\int_{\mathbf{R}^+} \varphi(x,\lambda+y,t)\, dG(y) - \varphi(x,\lambda,t)\right] \end{aligned} \tag{2.25}$$

Furthermore, it is easily verified that

$$\frac{\partial\varphi}{\partial t} = e^{R(x,\lambda,t)}\left(\frac{\partial f}{\partial t} + f\frac{\partial R}{\partial t}\right)$$

$$\frac{\partial\varphi}{\partial x} = e^{R(x,\lambda,t)}\left(\frac{\partial f}{\partial x} + f\frac{\partial R}{\partial x}\right)$$

and

$$\frac{\partial\varphi}{\partial\lambda} = e^{R(x,\lambda,t)}\left(\frac{\partial f}{\partial\lambda} + f\frac{\partial R}{\partial\lambda}\right)$$

where

$$\frac{\partial R(x,\lambda,t)}{\partial t} = -\delta\kappa_2 e^{\delta t}\lambda + \rho\left[1 - \hat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right)\right]$$

$$\frac{\partial R(x,\lambda,t)}{\partial\lambda} = -\left(\kappa_1 + \kappa_2 e^{\delta t}\right)$$

and

$$\frac{\partial R(x,\lambda,t)}{\partial x} = -\kappa_1\delta$$

Additionally,

$$\varphi(x,\lambda+y,t) = f(x,\lambda+y,t)\, e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)y} e^{R(x,\lambda,t)}$$

Substituting the above equations into (2.25) yields

$$A\varphi(x,\lambda,t) =$$
$$e^{R(x,\lambda,t)}\left\{\frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} - \delta\lambda\frac{\partial f}{\partial\lambda} + f\rho\left[1 - \hat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right)\right]\right\} +$$

$$e^{R(x,\lambda,t)}\rho\left[\int_{\mathbf{R}+} f(x,\lambda+y,t)\, e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)y} dG(y) - f(x,\lambda,t)\right] \quad (2.26)$$

And thus

$$\begin{aligned} A^* f(x,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} - \delta\lambda\frac{\partial f}{\partial \lambda} + \\ &\quad \rho\int_{\mathbf{R}+} f(x,\lambda+y,t)\, e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)y} dG(y) - \\ &\quad \rho f(x,\lambda,t)\, \hat{g}\left(\kappa_1+\kappa_2 e^{\delta t}\right) \\ &= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\int_{\mathbf{R}+} f(x,\lambda+y,t)\, dG^*(y;t) \\ &\quad - \rho^*(t) f(x,\lambda,t) \end{aligned} \quad (2.27)$$

where

$$\rho^*(t) = \rho\hat{g}\left(\kappa_1+\kappa_2 e^{\delta t}\right) \quad (2.28)$$

and

$$dG^*(y;t) = \frac{e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)y} dG(y)}{\hat{g}\left(\kappa_1+\kappa_2 e^{\delta t}\right)} \quad (2.29)$$

This result is due to Jang [31] (Theorem 3.2.5) where it is noted that (2.27) is a special case of (2.9) and thus, under the new equivalent measure P^*, λ_t follows a generalized shot noise process with time-dependent parameter $\rho^*(t)$ and time-dependent jump-size distribution function $G^*(y;t)$.

We will now obtain an expression for the joint conditional Laplace transform (LT) of (x_t, λ_t, t) under the equivalent martingale measure P^*. This result can be obtained in two ways: Firstly, by using the generator A^* in (2.27). The second way is by using the Laplace transform under the base measure P, given in (2.11).

We start by using the result in (2.27) combined with (2.12). Since

$$e^{-\nu x_t}\exp\left(-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta t}-1\right)\right)\lambda_t\right)\exp\left(\int_0^t \rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds\right) \quad (2.30)$$

is a P^*-martingale we have

$$E^*\left[\exp\left(-\nu x_{t_2}-\left(ke^{\delta t_2}-\frac{\nu}{\delta}\left(e^{\delta t_2}-1\right)\right)\lambda_{t_2}+\int_0^{t_2}\rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds\right)\middle|\mathcal{F}_{t_1}\right]=$$
$$e^{-\nu x_{t_1}}\exp\left(-\left(ke^{\delta t_1}-\frac{\nu}{\delta}\left(e^{\delta t_1}-1\right)\right)\lambda_{t_1}\right)\exp\left(\int_0^{t_1}\rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds\right)$$

Take

$$\nu_1=\nu$$

and

$$\nu_2=ke^{\delta t_2}-\frac{\nu}{\delta}\left(e^{\delta t_2}-1\right)$$

which implies

$$\nu_2 e^{-\delta t_2}+\frac{\nu_1}{\delta}\left(1-e^{-\delta t_2}\right)=k$$

and thus

$$ke^{\delta t_1}-\frac{\nu}{\delta}\left(e^{\delta t_1}-1\right)=\nu_2 e^{-\delta(t_2-t_1)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)$$

This leads to

$$\begin{aligned}
&E^*\left[e^{-\nu_1 x_{t_2}}e^{-\nu_2\lambda_{t_2}}\middle|\mathcal{F}_{t_1}\right]=\\
&e^{-\nu_1 x_{t_1}}\exp\left(-\left(\nu_2 e^{-\delta(t_2-t_1)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)\right)\lambda_{t_1}\right)\times\\
&\exp\left(-\int_{t_1}^{t_2}\rho^*(s)\left[1-\hat{g}^*\left(\nu_2 e^{-\delta(t_2-s)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-s)}\right);s\right)\right]ds\right)=\\
&e^{-\nu_1 x_{t_1}}\exp\left(-\left(\nu_2 e^{-\delta(t_2-t_1)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)\right)\lambda_{t_1}\right)\times\\
&\exp\left(-\rho\int_{t_1}^{t_2}\left[\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)-\hat{g}\left(\nu_2 e^{-\delta(t_2-s)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-s)}\right)+\kappa_1+\kappa_2 e^{\delta s}\right)\right]ds\right)
\end{aligned}\quad(2.31)$$

The second method allows us to prove the result in (2.31) without the transformed generator. Using the joint Laplace transform of x_{t_2} and λ_{t_2}, given λ_{t_1}, in (2.11) leads to

$$E^*\left[e^{-\nu_1 x_{t_2}}e^{-\nu_2\lambda_{t_2}}\middle|\mathcal{F}_{t_1}\right]=$$

$$E\left[\left.\frac{e^{-(\nu_1+\kappa_1\delta)x_{t_2}}e^{-\left(\nu_2+\kappa_1+\kappa_2e^{\delta t_2}\right)\lambda_{t2}}\exp\left(\rho\int_0^{t_2}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)}{E\left(e^{-\kappa_1\delta x_{t_2}}e^{-\left(\kappa_1+\kappa_2e^{\delta t_2}\right)\lambda_{t_2}}\exp\left(\rho\int_0^{t_2}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)\middle|\mathcal{F}_{t_1}\right)}\right|\mathcal{F}_{t_1}\right]=$$

$$e^{-(\nu_1+\kappa_1\delta)x_{t_1}}\exp\left(-\left(\left(\nu_2+\kappa_1+\kappa_2e^{\delta t_2}\right)e^{-\delta(t_2-t_1)}+\frac{(\nu_1+\kappa_1\delta)}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)\right)\lambda_{t_1}\right)\times$$

$$\exp\left(\rho\int_0^{t_2}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)\times$$

$$\frac{\exp\left(-\rho\int_{t_1}^{t_2}\left[1-\hat{g}\left(\left(\nu_2+\kappa_1+\kappa_2e^{\delta t_2}\right)e^{-\delta(t_2-s)}+\frac{(\nu_1+\kappa_1\delta)}{\delta}\left(1-e^{-\delta(t_2-s)}\right)\right)\right]ds\right)}{e^{-\kappa_1\delta x_{t_1}}e^{-\left(\kappa_1+\kappa_2e^{\delta t_1}\right)\lambda_{t_1}}\exp\left(\rho\int_0^{t_1}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)}=$$

$$e^{-(\nu_1+\kappa_1\delta)x_{t_1}}\exp\left(-\left(\left(\nu_2e^{-\delta(t_2-t_1)}+\kappa_2e^{\delta t_1}\right)+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)+\kappa_1\right)\lambda_{t_1}\right)\times$$

$$\exp\left(\rho\int_0^{t_2}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)\times$$

$$\frac{\exp\left(-\rho\int_{t_1}^{t_2}\left[1-\hat{g}\left(\left(\nu_2+\kappa_1+\kappa_2e^{\delta t_2}\right)e^{-\delta(t_2-s)}+\frac{(\nu_1+\kappa_1\delta)}{\delta}\left(1-e^{-\delta(t_2-s)}\right)\right)\right]ds\right)}{e^{-\kappa_1\delta x_{t_1}}e^{-\left(\kappa_1+\kappa_2e^{\delta t_1}\right)\lambda_{t_1}}\exp\left(\rho\int_0^{t_1}\left[1-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)}=$$

$$e^{-\nu_1x_{t_1}}\exp\left(-\left(\nu_2e^{-\delta(t_2-t_1)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)\right)\lambda_{t_1}\right)\times$$

$$\exp\left(\rho\int_{t_1}^{t_2}\left[\hat{g}\left(\frac{\nu_1}{\delta}-\left(\frac{\nu_1}{\delta}-\nu_2\right)e^{-\delta(t_2-s)}+\kappa_1+\kappa_2e^{\delta s}\right)-\hat{g}\left(\kappa_1+\kappa_2e^{\delta s}\right)\right]ds\right)$$

which is the same result as (2.31), yielded by using the generator.

2.6 Affine Jump-Diffusion Models

The shot noise process described above is a special (and simple) case of a more general class of processes called affine jump-diffusion (AJD) processes. AJD models nevertheless have sufficient structure to yield closed- or nearly closed-form expressions for securities prices and many models of the term structure are special cases. For an introduction to AJD term-structure models, see Duffie and Kan [18]. Duffie, Pan and Singleton [19] develop an option pricing methodology within the AJD framework. Duffie et. al. [17] provide a rigorous definition and characterization of regular affine processes.

Following Duffie, Pan and Singleton [19], but modifying the notation to conform to the previous section, the AJD model is defined as follows: Let a probability space $(\Omega, \mathcal{F}, P)$ and information filtration $\{\mathcal{F}_t; 0 \leq t \leq \tau\}$ be given. Let ζ be a Markov process in some state space $D \subset \mathbb{R}^n$, solving the stochastic differential equation

$$d\zeta_t = \mu(\zeta_t)\,dt + \sigma(\zeta_t)\,dB_t + dJ_t \tag{2.32}$$

where B is an $(\mathcal{F}_t)$-standard Brownian motion in $\mathbb{R}^n$; $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^{n\times n}$, and J is a pure jump process whose jumps have a fixed probability distribution G on $\mathbb{R}^n$ and arrive with intensity $\{\rho(\zeta_t) : t \geq 0\}$, for some $\rho : D \to [0, \infty)$. The generator of the process in (2.32) acting on a function $f(\zeta)$ is then given by

$$\begin{aligned} Af(\zeta) &= \mu(\zeta)\frac{\partial f}{\partial \zeta} + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 f}{\partial \zeta \partial \zeta^T}\sigma(\zeta)\sigma(\zeta)^T\right] + \\ &\quad \rho(\zeta)\int_{\mathbb{R}^n}[f(\zeta+y) - f(\zeta)]\,dG(y) \end{aligned} \tag{2.33}$$

An affine structure is imposed on the functions μ, $\sigma\sigma^T$ and ρ. Moreover, the short rate is also assumed to be affine in its dependence on the state variable ζ. The affine structure has made the model in (2.32) analytically tractable compared to its general form.

Comparing (2.33) to (2.14) it is immediately clear that for $n = 1$ and trivially affine transformations $\mu(x) = -\delta x$, $\sigma(x) = 0$ and $\rho(x) = \rho$, the SDE in (2.32) defines a shot noise process.

The framework for the AJD models allows for an extension of the shot-noise process, described in section 2.4, by adding a Brownian perturbation. This process will be referred to as a diffusion shot noise process and is defined by the SDE

$$d\lambda_t = -\delta\lambda_t dt + \sigma dB_t + dJ_t \tag{2.34}$$

Let its aggregated process x be defined by

$$x_t = \int_0^t \lambda_s ds \tag{2.35}$$

From (2.14) and (2.33) the generator of the process (x_t, λ_t, t) acting on a function $f(x, \lambda, t)$ is given by

$$\begin{aligned} Af(x, \lambda, t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} - \delta \lambda \frac{\partial f}{\partial \lambda} + \frac{1}{2} \frac{\partial^2 f}{\partial \lambda^2} \sigma^2 + \\ & \rho \left[\int_{\mathbf{R}^+} f(x, \lambda + y, t) \, dG(y) - f(x, \lambda, t) \right] \end{aligned} \tag{2.36}$$

With (2.36) it is possible to extend the result in (2.13).

Theorem 2.6.1 *Let λ and x be as defined in (2.34) and (2.35), respectively, evolving up to a fixed time t^*. Also, let κ_1 and κ_2 be constants such that $\kappa_1 \geq 0$ and $\kappa_2 \geq -\kappa_1 e^{-\delta t^*}$. Then the process given by*

$$e^{-\kappa_1 \delta x_t} e^{-\left(\kappa_1 + \kappa_2 e^{\delta t}\right) \lambda_t} \exp \left\{ \rho \int_0^t \left[1 - \widehat{g}\left(\kappa_1 + \kappa_2 e^{\delta s}\right)\right] ds - \frac{1}{2} \sigma^2 \int_0^t \left(\kappa_1 + \kappa_2 e^{\delta s}\right)^2 ds \right\} \tag{2.37}$$

is a martingale.

Proof: Following Dassios [10], define

$$w_t = \delta x_t + \lambda_t$$

and

$$z_t = \lambda_t e^{\delta t}$$

We will try a function of the form

$$f^*(w, z, t) = e^{-\kappa_1 w} e^{-\kappa_2 z} h(t) = e^{-\kappa_1 (\delta x + \lambda)} e^{-\kappa_2 \lambda e^{\delta t}} h(t)$$

or

$$f(x, \lambda, t) = f^*\left(\delta x + \lambda, \lambda e^{\delta t}, t\right) = e^{-\kappa_1 \delta x} e^{-\left(\kappa_1 + \kappa_2 e^{\delta t}\right) \lambda} h(t) \tag{2.38}$$

Substituting (2.38) into (2.36) yields

$$\begin{aligned} & Af(x, \lambda, t) = \\ & e^{-\kappa_1 \delta x} e^{-\left(\kappa_1 + \kappa_2 e^{\delta t}\right) \lambda} \left(h'(t) + \frac{1}{2} \sigma^2 \left(\kappa_1 + \kappa_2 e^{\delta t}\right)^2 h(t) \right) + \end{aligned}$$

$$e^{-\kappa_1 \delta x} e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)\lambda} h(t)\, \rho \left[\widehat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right) - 1\right]$$

Setting this to zero yields

$$h'(t) = h(t)\left[\rho\left(1 - \widehat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right)\right) - \frac{1}{2}\sigma^2\left(\kappa_1 + \kappa_2 e^{\delta t}\right)^2\right] \tag{2.39}$$

Solving for $h(t)$ results in

$$h(t) = K \exp\left\{\rho \int_0^t \left[1-\widehat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\right] ds - \frac{1}{2}\sigma^2 \int_0^t \left(\kappa_1+\kappa_2 e^{\delta s}\right)^2 ds\right\} \tag{2.40}$$

where K is an arbitrary constant. This completes the proof.

Theorem 2.6.1 leads directly to a generalization of the result in 2.11.

Corollary 2.6.2 *The joint Laplace transform of* λ_{t_2} *and* x_{t_2}*, given* λ_{t_1} *is*

$$\begin{aligned}
&E\left[e^{-\nu_1 x_{t_2}} e^{-\nu_2 \lambda_{t_2}} \middle| \mathcal{F}_{t_1}\right] = \\
&e^{-\nu_1 x_{t_1}} e^{-\left(\nu_1 \frac{1-e^{-\delta(t_2-t_1)}}{\delta} + \nu_2 e^{-\delta(t_2-t_1)}\right)\lambda_{t_1}} \times \\
&\exp\left[-\rho \int_{t_1}^{t_2} \left(1 - \widehat{g}\left(\nu_1 \frac{1-e^{-\delta(t_2-s)}}{\delta} + \nu_2 e^{-\delta(t_2-s)}\right)\right) ds\right] \times \\
&\exp\left[\frac{1}{2}\sigma^2 \int_{t_1}^{t_2} \left(\nu_1 \frac{1-e^{-\delta(t_2-s)}}{\delta} + \nu_2 e^{-\delta(t_2-s)}\right)^2 ds\right]
\end{aligned} \tag{2.41}$$

Proof: Because (2.37) is a martingale,

$$\begin{aligned}
&E\left\{ e^{-\kappa_1 \delta x_{t_2} - \left(\kappa_1+\kappa_2 e^{\delta t_2}\right)\lambda_{t_2}} \exp\left[\int_0^{t_2}\left[\rho\left(1-\widehat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\right) - \frac{\sigma^2}{2}\left(\kappa_1+\kappa_2 e^{\delta s}\right)^2\right] ds\right] \middle| \mathcal{F}_{t_1} \right\} = \\
&e^{-\kappa_1 \delta x_{t_1}} e^{-\left(\kappa_1+\kappa_2 e^{\delta t_1}\right)\lambda_{t_1}} \exp\left[\rho \int_0^{t_1}\left(1-\widehat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\right) ds - \frac{1}{2}\sigma^2 \int_0^{t_1}\left(\kappa_1+\kappa_2 e^{\delta s}\right)^2 ds\right]
\end{aligned} \tag{2.42}$$

Dividing both sides of (2.42) by

$$\exp\left[\int_0^{t_2}\left[\rho\left(1-\widehat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\right) - \frac{\sigma^2}{2}\left(\kappa_1+\kappa_2 e^{\delta s}\right)^2\right] ds\right]$$

and substituting the values $\kappa_1 = \frac{\nu_1}{\delta}$ and $\kappa_2 = \left(\nu_2 - \frac{\nu_1}{\delta}\right) e^{-\delta t_2}$ yields the desired result.

Chapter 3

A Continuous-Time Interest Rate Model

3.1 Introduction

This chapter introduces an interest rate model based on the shot-noise process described in chapter 2. The price of a zero-coupon bond is derived by taking the expectation of the discounted pay-off under an equivalent martingale measure. The derivation of this result relies on the Laplace transform results from Chapter 2.

Weeks' method to invert Laplace transforms is introduced and applied in the context of a shot-noise process with exponentially distributed jumps. This yields the joint probability density function of the interest rate and its time integral, which is used to compute the price of a call option on a bond, by taking the expectation of the pay-off under an equivalent martingale measure. This equivalent martingale measure is often interpreted as a risk-neutral measure and this approach to option pricing will be referred to as risk-neutral valuation.

Though Weeks' method is well-known in the telecommunications literature, as it is well suited for problems in queuing-theory, its current application to option pricing is new.

3.2 Definition of the Model

We will use the shot noise process λ_t with parameters ρ and δ and jump size distribution $G(y)$ (all independent of time), as defined in section 2.4, to model the instantaneous short rate. Also, let x_t denote the aggregate process $\int_0^t \lambda_s ds$. The shot noise process is a suitable model model for an interest rate because it does not assume negative values as long as the domain of the jump-size probablility density g is $\mathbb{R}^+$. Clearly, a model that produces occasional negative short rates is unrealistic because interest rates are rarely below zero. People are usually not paid to borrow money. That said, there are examples of interest rate models where the short rate is allowed to go negative. The Ho and Lee model is an example (see Baxter and Rennie [5] for an overview).

The generator of the Markov process (x_t, λ_t, t) acting on a function $f(x, \lambda, t)$ is given by (2.14). The value of a continuously compounding cash bond or deposit is now given by $B_t = e^{x_t}$. Let $P(t, T)$ denote the price at time t of a zero-coupon or discount bond maturing at time T and let $Z(t, T)$ denote the corresponding discounted bond price process $B_t^{-1} P(t, T)$. We will use P to denote the original measure with respect to which (λ_t, x_t, t) is defined and P^* to denote an equivalent martingale measure, suitably chosen with the aid of an Esscher transform as described in section 2.5.

As discussed in sections 1.1 and 1.2, a bond maturing at time T may be viewed as a contingent claim with payoff 1 at time T and thus may be priced within a risk-neutral framework as $P(t, T) = B_t E^* \left(B_T^{-1} \middle| \mathcal{F}_t \right)$, where E^* is the expectation with respect to P^*, an equivalent martingale measure. As it turns out, the equivalent martingale measure is not unique, even when it is defined using the Esscher transform M_t, defined in (2.24). Recall that according to (2.27), under this new measure, the process is a generalized shot-noise process, the parameters of which are given right below (2.27). We will use (2.12) to find an expression for the discounted bond price process $Z(t, T) = E^* \left(B_T^{-1} \middle| \mathcal{F}_t \right)$. In fact,

$$e^{-\nu x_t} e^{-\left(ke^{\delta t} - \frac{\nu}{\delta}\left(e^{\delta t} - 1\right)\right)\lambda_t} e^{\int_0^t \rho^*(s)\left[1 - \hat{g}^*\left(ke^{\delta s} - \frac{\nu}{\delta}\left(e^{\delta s} - 1\right); s\right)\right] ds}$$

is a P^*-martingale and thus we have

$$E^*\left(e^{-\nu x_T}e^{-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta T}-1\right)\right)\lambda_T}e^{\int_0^T \rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds}\middle|\mathcal{F}_t\right)=$$
$$e^{-\nu x_t}e^{-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta t}-1\right)\right)\lambda_t}e^{\int_0^t \rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds}$$

or

$$E^*\left(e^{-\nu x_T}e^{-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta T}-1\right)\right)\lambda_T}\middle|\mathcal{F}_t\right)=$$
$$e^{-\nu x_t}e^{-\left(ke^{\delta t}-\frac{\nu}{\delta}\left(e^{\delta t}-1\right)\right)\lambda_t}e^{-\int_t^T \rho^*(s)\left[1-\hat{g}^*\left(ke^{\delta s}-\frac{\nu}{\delta}\left(e^{\delta s}-1\right);s\right)\right]ds}$$

We will choose $\nu=1$ and $k=\frac{1}{\delta}\left(1-e^{-\delta T}\right)$. This yields

$$E^*\left(e^{-x_T}\middle|\mathcal{F}_t\right)=$$
$$e^{-x_t}e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t}e^{-\int_t^T \rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds}$$

Recall from (2.27), (2.28) and (2.29) that the generalized shot-noise process in question is defined by

$$\rho^*(t)=\rho\hat{g}\left(\kappa_1+\kappa_2e^{\delta t}\right)$$

and

$$dG^*(y;t)=\frac{e^{-\left(\kappa_1+\kappa_2e^{\delta t}\right)y}dG(y)}{\hat{g}\left(\kappa_1+\kappa_2e^{\delta t}\right)}$$

It is easily established that

$$\begin{aligned}\hat{g}^*(\xi;s) &= \int_{\mathbf{R}+}e^{-y\xi}dG^*(y;t)=\int_{\mathbf{R}+}e^{-y\xi}\frac{e^{-\left(\kappa_1+\kappa_2e^{\delta t}\right)y}dG(y)}{\hat{g}\left(\kappa_1+\kappa_2e^{\delta t}\right)}= \\ & \frac{\hat{g}\left(\xi+\kappa_1+\kappa_2e^{\delta t}\right)}{\hat{g}\left(\kappa_1+\kappa_2e^{\delta t}\right)}\end{aligned}$$

and thus

$$\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]=$$

$$\rho\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\left[1-\frac{\hat{g}\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right)+\kappa_1+\kappa_2 e^{\delta s}\right)}{\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)}\right]=$$
$$\rho\left[\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)-\hat{g}\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right)+\kappa_1+\kappa_2 e^{\delta s}\right)\right]$$

We finally obtain the desired expression for the discounted bond price process

$$Z(t,T)=E^*\left(e^{-x_T}\middle|\mathcal{F}_t\right)=$$
$$e^{-x_t}e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t}e^{\rho\int_t^T\left[\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}+\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right)\right)-\hat{g}\left(\kappa_1+\kappa_2 e^{\delta s}\right)\right]ds} \tag{3.1}$$

$Z(t,T)$ is a P^*-martingale, because

$$E^*\left[Z(t,T)\middle|\mathcal{F}_s\right]=E^*\left\{E^*\left[e^{-x_T}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right\}=E^*\left[e^{-x_T}\middle|\mathcal{F}_s\right]=Z(s,T)$$

We will not worry about how κ_1 and κ_2 should be determined but assume they have been chosen suitably in some sense.

If κ_1 and κ_2 are zero, i.e. assuming no change of measure, the price process for a discount bond maturing at time T is

$$P(t,T)=e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t}e^{\rho\int_t^T\left[\hat{g}\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right)\right)-1\right]ds}$$

As was pointed out in section 2.6, the interest rate model defined in this section is a special case of the affine jump-diffusion process presented in Duffie, Pan and Singleton [19]. It is not surprising that the change of measure they propose is equivalent to the Esscher transform used above when their model is limited to the special case of the shot noise process. In fact, since the shot noise process becomes a generalized shot noise process with deterministic, but time-dependent parameters under the change of measure, the affine structure of the model is preserved, just as in the more general case of the AJD model. This allows the machinery developed for affine processes to be used for derivatives pricing and it is easily verified that the AJD model leads to the same bond price as in (3.1). Beyond preserving the affine structure of the model, the issue of choosing an appropriate equivalent martingale measure is largely ignored.

The remainder of this chapter is in line with this philosophy and assumes that a suitable change of measure has already been decided on. Chapter 4 will address the choice of equivalent martingale measure in detail, albeit indirectly.

3.3 Inverse Laplace Transform

3.3.1 Introduction

The Fundamental Theorem of Asset Pricing justifies pricing a contingent claim by computing the expected value of the pay-off at expiration under an equivalent martingale measure. The pricing formula for a discount bond in (3.1) is an example of this approach, which also illustrates the key role that Laplace transforms play, especially when the discounted pay-off at expiration can be written as an exponential function of an affine combination of x_T and λ_T, where T is the expiration date.

Duffie, Pan and Singleton [19] pursue a similar approach for pricing claims by using a type of extended Fourier transform. Specifically, using the notation in section 2.6, the extended transform of ζ_T, given the information available at time t, is

$$E\left[\exp\left(-\int_t^T R(\zeta_s, s)\, ds\right)(v_0 + v_1 \cdot \zeta_T)\, e^{u\cdot\zeta_T}\middle|\, \mathcal{F}_t\right] \tag{3.2}$$

where the short rate R is an affine function of ζ_T, possibly time dependent, and v_0, v_1 and u may be complex-valued. This extended transform differs from the (conditional) characteristic function (Fourier transform) because the discounting at rate $R(\zeta_t, t)$ and the term $v_0 + v_1 \cdot \zeta_T$. Though a closed-form expression for (3.2) will automatically produce a closed-form expression for the price of a discount bond, pricing more complicated claims, such as options, is more involved. As an example, Duffie, Pan and Singleton [19] consider the price at time 0 of an option that pays off $\left(e^{d\cdot\zeta_T} - c\right)^+$ at time T, for given $d \in \mathbb{R}^n$ and strike c. For any real y and a and b in $\mathbb{R}^n$, let $G_{a,b}(y)$ denote the price of a security that pays $e^{a\cdot\zeta_T}$ at time T in the event that $b \cdot \zeta_T \leq y$. The price of the call option at time 0 is then

$$p = G_{d,-d}\left(-\log c\right) - cG_{0,-d}\left(-\log c\right)$$

The function $G_{a,b}(z)$ is given by (3.2) for the complex coefficient vector $u = a + izb$, with $v_0 = 1$ and $v_1 = 0$. Because of the affine structure of the model, a closed-form expression exists for the Fourier transform of the function $G_{a,b}(\cdot)$ defined by

$$\mathcal{G}_{a,b}(z) = \int_{-\infty}^{\infty} e^{izy} dG_{a,b}(y)$$

The closed-form solution is

$$\mathcal{G}_{a,b}(z) = e^{\alpha(0)+\beta(0)\cdot\zeta_0}$$

where α and β solve known, complex-valued ordinary differential equations with boundary conditions at T determined by z. In some cases these ODEs have explicit solutions, in others they need to be solved numerically. The function $G_{a,b}(\cdot)$ can then be be obtained by inverting the Fourier transform $\mathcal{G}_{a,b}(\cdot)$. This example in the beginning of Duffie, Pan and Singleton [19] is fairly typical of their approach toward pricing various types of options.

The approach pursued here in the context of the shot-noise process is somewhat different. In (2.31) an expression was presented for the joint conditional Laplace transform of x_t and λ_t, given and λ_{t_1} for $t_1 \leq t$, under the equivalent martingale measure P^*. The simple nature of the shot-noise process, when jump-sizes are assumed to be exponentially distributed, allows for an explicit evaluation of the Laplace transform in (2.31). The inverse Laplace transform is equivalent to the joint conditional probability distribution of x_t and λ_t. Given a few regularity conditions, with the inverse transform the conditional expected value of many types of functions of x_T and λ_T can be computed, allowing for the evaluation of a general set of contingent claims. The remainder of this section will focus on the implementation of Weeks' method of Laplace transform inversion. Though implementing this method requires numerical integration, it produces a closed-form expression of the inverse Laplace transform, which can be manipulated analytically to compute integrals of functions of x_T and

λ_T. In the next section the inverse Laplace transform will be used to compute the price of a European call option by taking the conditional expectation of the pay-off at expiration under the equivalent martingale measure P^*.

3.3.2 Laplace Transform

Abusing our notation slightly, we will denote the conditional joint probability density of x_t and λ_t, given $\mathcal{F}_{t_1}$, by $\Pi(x_t, \lambda_t)$, ignoring any reference to $\mathcal{F}_{t_1}$.The next theorem presents the Laplace transform of Π for the specific case when the jump sizes follow an exponential distribution.

Theorem 3.3.1 *Let λ and x be as defined in section 2.4, where δ and ρ are time-invariant and the jump-size distribution is exponential, that is*

$$g(y) = \alpha e^{-\alpha y} \tag{3.3}$$

and its Laplace transform is

$$\hat{g}(\nu) = \frac{\alpha}{\alpha + \nu} \tag{3.4}$$

Then the joint LT of x_{t_2} and λ_{t_2}, given λ_{t_1} under the measure P^*, as given in (2.31), takes on the specific form

$$\begin{aligned}
&\widehat{\Pi}(\nu_1, \nu_2) = E^*\left\{ e^{-\nu_1 x_{t_2}} e^{-\nu_2 \lambda_{t_2}} \middle| \mathcal{F}_{t_1} \right\} = \\
&e^{-\nu_1 x_{t_1}} \exp\left(-\left(\tfrac{\nu_1}{\delta} + \left(\nu_2 - \tfrac{\nu_1}{\delta}\right) e^{-\delta(t_2 - t_1)}\right) \lambda_{t_1} \right) \exp\left(\rho\alpha(t_2 - t_1) \left(\tfrac{1}{\alpha + \kappa_1 + \frac{\nu_1}{\delta}} - \tfrac{1}{\alpha + \kappa_1} \right) \right) \times \\
&\left(\frac{\alpha + \kappa_1 + \kappa_2 e^{\delta t_1} + \nu_2 e^{-\delta(t_2 - t_1)} + \frac{\nu_1}{\delta}\left(1 - e^{-\delta(t_2 - t_1)}\right)}{\alpha + \kappa_1 + \kappa_2 e^{\delta t_2} + \nu_2} \right)^{\frac{\rho\alpha}{\delta(\alpha + \kappa_1) + \nu_1}} \times \\
&\left(\frac{\alpha + \kappa_1 + \kappa_2 e^{\delta t_1}}{\alpha + \kappa_1 + \kappa_2 e^{\delta t_2}} \right)^{\frac{\rho\alpha}{\delta(\alpha + \kappa_1)}}
\end{aligned} \tag{3.5}$$

Proof: Recall from (2.31) that

$$\begin{aligned}
&E^*\left\{ e^{-\nu_1 x_{t_2}} e^{-\nu_2 \lambda_{t_2}} \middle| \mathcal{F}_{t_1} \right\} = e^{-\nu_1 x_{t_1}} e^{-\left(\frac{\nu_1}{\delta} + \left(\nu_2 - \frac{\nu_1}{\delta}\right) e^{-\delta(t_2 - t_1)}\right) \lambda_{t_1}} \times \\
&\exp\left(-\int_{t_1}^{t_2} \rho^*(s) \left[1 - \hat{g}^*\left(\tfrac{\nu_1}{\delta} + \left(\nu_2 - \tfrac{\nu_1}{\delta}\right) e^{-\delta(t_2 - s)}; s \right) \right] ds \right)
\end{aligned}$$

Also, recall from (2.28) and (2.29) that

$$\rho^*(t) = \rho\hat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right)$$

and

$$dG^*(y;t) = \frac{e^{-\left(\kappa_1+\kappa_2 e^{\delta t}\right)y} dG(y)}{\hat{g}\left(\kappa_1 + \kappa_2 e^{\delta t}\right)}$$

Using (3.4) yields

$$\rho^*(t) = \frac{\rho\alpha}{\alpha + \kappa_1 + \kappa_2 e^{\delta t}}$$

$$g^*(y;t) = \left(\alpha + \kappa_1 + \kappa_2 e^{\delta t}\right) e^{-\left(\alpha+\kappa_1+\kappa_2 e^{\delta t}\right)y}$$

and

$$\hat{g}^*(\nu;t) = \frac{\alpha + \kappa_1 + \kappa_2 e^{\delta t}}{\alpha + \kappa_1 + \kappa_2 e^{\delta t} + \nu}$$

Thus

$$\hat{g}^*\left(\tfrac{\nu_1}{\delta}+\left(\nu_2-\tfrac{\nu_1}{\delta}\right)e^{-\delta(t_2-s)};t\right) = \frac{\alpha+\kappa_1+\kappa_2 e^{\delta t}}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta t}}$$

and

$$\rho^*(t)\,\hat{g}^*\left(\tfrac{\nu_1}{\delta}+\left(\nu_2-\tfrac{\nu_1}{\delta}\right)e^{-\delta(t_2-s)};t\right) = \frac{\rho\alpha}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta t}}$$

From (2.31) it is clear that we need to evaluate

$$\int_{t_1}^{t_2} \rho^*(s)\left[1 - \hat{g}^*\left(\tfrac{\nu_1}{\delta}+\left(\nu_2-\tfrac{\nu_1}{\delta}\right)e^{-\delta(t_2-s)};s\right)\right] ds =$$
$$\int_{t_1}^{t_2} \left[\frac{\rho\alpha}{\alpha+\kappa_1+\kappa_2 e^{\delta s}} - \frac{\rho\alpha}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta s}}\right] ds \tag{3.6}$$

Both terms in the integral on the RHS of (3.6) are of the form $\frac{1}{A+Be^{\delta s}}$. These integrals may be evaluated as follows:

$$\int_{t_1}^{t_2} \frac{ds}{A+Be^{\delta s}} = \frac{1}{\delta}\int_{e^{\delta t_1}}^{e^{\delta t_2}} \frac{du}{(A+Bu)\,u} =$$
$$\frac{1}{\delta}\int_{e^{\delta t_1}}^{e^{\delta t_2}} \left[\frac{1}{Au} - \frac{B}{A(A+Bu)}\right] du \tag{3.7}$$

We have

$$\int_{e^{\delta t_1}}^{e^{\delta t_2}} \frac{du}{Au} = \frac{1}{A} \left[\log u\right]_{e^{\delta t_1}}^{e^{\delta t_2}} = \frac{\delta\left(t_2 - t_1\right)}{A} \tag{3.8}$$

and

$$\int_{e^{\delta t_1}}^{e^{\delta t_2}} \frac{du}{A + Bu} = \frac{1}{B} \log\left(\frac{A + Be^{\delta t_2}}{A + Be^{\delta t_1}}\right) \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) yields

$$\frac{1}{\delta} \int_{e^{\delta t_1}}^{e^{\delta t_2}} \left[\frac{1}{Au} - \frac{B}{A\left(A + Bu\right)}\right] du = \frac{1}{\delta A} \left[\delta\left(t_2 - t_1\right) - \log\left(\tfrac{A+Be^{\delta t_2}}{A+Be^{\delta t_1}}\right)\right] \tag{3.10}$$

Applying (3.10) to the two terms on the RHS of (3.6) yields

$$\int_{t_1}^{t_2} \frac{\rho\alpha ds}{\alpha + \kappa_1 + \kappa_2 e^{\delta s}} = \frac{\rho\alpha}{\delta\left(\alpha + \kappa_1\right)} \left[\delta\left(t_2 - t_1\right) - \log\left(\tfrac{\alpha+\kappa_1+\kappa_2 e^{\delta t_2}}{\alpha+\kappa_1+\kappa_2 e^{\delta t_1}}\right)\right] \tag{3.11}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{\rho\alpha ds}{\alpha + \kappa_1 + \frac{\nu_1}{\delta} + \left(\kappa_2 + \left(\nu_2 - \frac{\nu_1}{\delta}\right) e^{-\delta t_2}\right) e^{\delta s}} = \\
&\frac{\rho\alpha}{\delta\left(\alpha + \kappa_1 + \frac{\nu_1}{\delta}\right)} \left[\delta\left(t_2 - t_1\right) - \log\left(\tfrac{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta t_2}}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta t_1}}\right)\right] = \\
&\frac{\rho\alpha}{\delta\left(\alpha + \kappa_1 + \frac{\nu_1}{\delta}\right)} \left[\delta\left(t_2 - t_1\right) - \log\left(\tfrac{\alpha+\kappa_1+\kappa_2 e^{\delta t_2}+\nu_2}{\alpha+\kappa_1+\kappa_2 e^{\delta t_1}+\frac{\nu_1}{\delta}+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta(t_2-t_1)}}\right)\right]
\end{aligned} \tag{3.12}$$

Thus combining (3.6), (3.11) and (3.12) yields

$$\begin{aligned}
&\exp\left\{-\int_{t_1}^{t_2} \rho^*(s)\left[1-\hat{g}^*\left(\tfrac{\nu_1}{\delta}+\left(\nu_2-\tfrac{\nu_1}{\delta}\right)e^{-\delta(t_2-s)};s\right)\right]ds\right\} = \\
&\exp\left\{-\int_{t_1}^{t_2}\left[\tfrac{\rho\alpha}{\alpha+\kappa_1+\kappa_2 e^{\delta s}}-\tfrac{\rho\alpha}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta s}}\right]ds\right\} = \\
&\exp\left\{\rho\alpha\left[\int_{t_1}^{t_2}\tfrac{ds}{\alpha+\kappa_1+\frac{\nu_1}{\delta}+\left(\kappa_2+\left(\nu_2-\frac{\nu_1}{\delta}\right)e^{-\delta t_2}\right)e^{\delta s}}-\int_{t_1}^{t_2}\tfrac{ds}{\alpha+\kappa_1+\kappa_2 e^{\delta s}}\right]\right\} =
\end{aligned}$$

$$\left(\frac{\alpha+\kappa_1+\kappa_2 e^{\delta t_1}+\nu_2 e^{-\delta(t_2-t_1)}+\frac{\nu_1}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\alpha+\kappa_1+\kappa_2 e^{\delta t_2}+\nu_2}\right)^{\frac{\rho\alpha}{\delta(\alpha+\kappa_1)+\nu_1}} \times$$

$$\exp\left(\rho\alpha(t_2-t_1)\left(\frac{1}{\alpha+\kappa_1+\frac{\nu_1}{\delta}}-\frac{1}{\alpha+\kappa_1}\right)\right)\left(\frac{\alpha+\kappa_1+\kappa_2 e^{\delta t_1}}{\alpha+\kappa_1+\kappa_2 e^{\delta t_2}}\right)^{\frac{\rho\alpha}{\delta(\alpha+\kappa_1)}} \tag{3.13}$$

The desired result follows from substituting (3.13) into (2.31).

3.3.3 Weeks' Method

For the purpose of pricing a European call option we will need to invert (3.5) w.r.t. ν_2. It will become clear at the end of section 3.4 why (3.5) does not need to be inverted w.r.t. ν_1. In a nutshell, the option pricing formula derived will turn out to have the form of a Laplace transform wrt x_{t_2}. Though this inversion problem is not known to be analytically tractable, techniques exist to approximate $\widehat{\Pi}(\nu_1, \nu_2)$ with a linear combination of functions (of ν_2) whose inverse Laplace transforms are known.

The inversion technique applied here is based on Laguerre polynomials and known as Weeks' method [40]. Though this method was one of the first successful implementations, the idea of using Laguerre functions to invert Laplace transforms goes as far back as Widder [41]. He proved that, under certain regularity conditions, the inverse transform may be represented as an infinite weighted sum of Laguerre functions. Many transform inversion methods based on that of Weeks have been proposed since. An example is that presented in the companion papers by Abate et al. [1, 2], which describe a method for the univariate and multivariate cases, respectively. The paper by Kano et al. [32] provides a description of Weeks' method and an extension to the case of matrix functions.

The description of Weeks' method by Kano et al. [32] is presented next, adapted to the present context. Weeks' method returns an analytical formula for the inverse transform function. It assumes that a function $\Pi(x)$ can be represented as an expansion in terms of Laguerre polynomials $L_n(x)$

$$\Pi(x) = e^{\sigma x}\sum_{n=0}^{\infty} c_n e^{-bx} L_n(2bx) \tag{3.14}$$

The Laguerre polynomials $L_n(x)$ form an orthogonal basis in $L_2(\mathbb{R})$ and are commonly used to approximate functions. The two scaling parameters σ and b need to be chosen so that $b > 0$ and $\sigma > \sigma_0$, where σ_0 is the abscissa of convergence. If the infinite Laguerre expansion converges uniformly then $\Pi(x)$ may be approximated for numerical purposes by

$$\Pi(x) \approx e^{\sigma x} \sum_{n=0}^{N} c_n e^{-bx} L_n(2bx) \tag{3.15}$$

where N is chosen so that the error of the approximation is below the desired tolerance.

The principal challenge is to compute the coefficients c_n, which is described next. Given a Laplace transform $\widehat{\Pi}(\nu)$, its inverse is defined as a contour integral in the complex plane

$$\Pi(x) = \frac{1}{2\pi i} \oint_{\Gamma} e^{\nu x} \widehat{\Pi}(\nu)\, d\nu$$

where $i = \sqrt{-1}$ and Γ is the Bromwhich contour $\Gamma(\nu) = \sigma + iy$, with $\sigma > \sigma_0$, $y \in R$. Then

$$\Pi(x) = \frac{e^{\sigma x}}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \widehat{\Pi}(\sigma + iy)\, dy \tag{3.16}$$

Equating (3.14) and (3.16) yields

$$\sum_{n=0}^{\infty} c_n e^{-bx} L_n(2bx) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \widehat{\Pi}(\sigma + iy)\, dy \tag{3.17}$$

The weighted Laguerre functions have the Fourier representation

$$e^{-bx} L_n(2bx) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyx} \frac{(iy - b)^n}{(iy + b)^{n+1}} dy \tag{3.18}$$

Substituting (3.18) into (3.17) and interchanging sum and integral, which is valid if (3.14) converges uniformly, leads to

$$\sum_{n=0}^{\infty} c_n \frac{(iy - b)^n}{(iy + b)^{n+1}} = \widehat{\Pi}(\sigma + iy) \tag{3.19}$$

Next, a Mobius transformation is applied to map ν to a new complex variable w

$$w = \frac{\nu - \sigma - b}{\nu - \sigma + b}$$

Using $\nu = \sigma + iy$ leads to

$$w = \frac{iy - b}{iy + b}$$

and the inverse is

$$y = \frac{ib\,(w+1)}{w-1}$$

Equation (3.19) becomes

$$\sum_{n=0}^{\infty} c_n w^n = (iy + b)\,\widehat{\Pi}\,(\sigma + iy) = \frac{2b}{1-w}\widehat{\Pi}\left(\sigma - b\frac{w+1}{w-1}\right)$$

The coefficients c_n may be computed using Cauchy's integral formula by integrating along the unit circle $w = e^{i\theta}$

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_{|w|=1} \frac{1}{w^{n+1}}\frac{2b}{1-w}\widehat{\Pi}\left(\sigma - b\frac{w+1}{w-1}\right) dw = \\ &\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\frac{2b}{1-e^{i\theta}}\widehat{\Pi}\left(\sigma - b\frac{e^{i\theta}+1}{e^{i\theta}-1}\right) d\theta \end{aligned}$$

The resulting inversion formula is then

$$\Pi\,(x) = e^{\sigma x}\sum_{n=0}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\frac{2b}{1-e^{i\theta}}\widehat{\Pi}\left(\sigma - b\frac{\left(e^{i\theta}+1\right)}{e^{i\theta}-1}\right) d\theta\right) e^{-bx} L_n\,(2bx)$$

For numerical purposes, the coefficients c_n need to be evaluated via numerical integration. Using the midpoint rule leads to the approximation

$$c_n \approx \frac{e^{-\frac{in\pi}{(2M)}}}{2M}\sum_{m=-M}^{M-1} e^{-in\theta_m}\frac{2b}{1-e^{i\theta_{m+1/2}}}\widehat{\Pi}\left(\sigma - b\frac{e^{i\theta_{m+1/2}}+1}{e^{i\theta_{m+1/2}}-1}\right) \qquad (3.20)$$

where $\theta_m = \frac{m\pi}{M}$, $m = -M, \ldots, M-1$ and $n = 0, \ldots, N$.

Kano et al. [32] also provide error bounds for the approximation due to numerical rounding error and truncation error. They ignore the error induced by numerical integration, though numerical integration error estimates are readily available in e.g. [4].

The Laguerre polynomials $L_n(x)$ can be defined via the generating function

$$(1-t)^{-1} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) t^n \tag{3.21}$$

Laguerre polynomials satisfy the well-known triple recursion relation

$$L_n(x) = \frac{2n-1-x}{n} L_{n-1}(x) - \frac{n-1}{n} L_{n-2}(x) \tag{3.22}$$

which may be proved with (3.21) or by using their orthogonality property [4]. The recurrence relation in (3.22) can be useful for computing the values of the Laguerre polynomials to evaluate the approximation in (3.15). It is possible, however, that the Laguerre polynomials become large as their order increases, which can lead to a prohibitive loss in precision. In this case Kano et al. [32] recommend evaluating the sum in (3.15) by using the backward Clenshaw algorithm as described in [34]. In the applications in chapter 5 there is no apparent need for using the backward Clenshaw algorithm and it has not been implemented here.

We will need to evaluate integrals of the form

$$I_n(x, \nu) = \int_0^x e^{-\nu y} L_n(y)\, dy \tag{3.23}$$

The generating function (3.21) may be used to develop a recursion that is useful for computing (3.23). Evaluating the integral of the left-hand side of (3.21) yields

$$(1-t)^{-1} \int_0^x e^{-\nu y} e^{-\frac{yt}{1-t}} dy = \frac{1 - e^{-x\nu} e^{-x\frac{t}{1-t}}}{(1-t)\nu + t} = \frac{1 - e^{-x\nu}(1-t)\sum_{n=0}^{\infty} L_n(x) t^n}{(1-t)\nu + t}$$

Equating this to the same integral of the right-hand side of (3.21) yields

$$1 - e^{-x\nu}(1-t)\sum_{n=0}^{\infty} L_n(x) t^n = ((1-t)\nu + t)\sum_{n=0}^{\infty} t^n I_n(x, \nu)$$

or

$$1 - e^{-x\nu} \sum_{n=0}^{\infty} L_n(x)\, t^n + e^{-x\nu} \sum_{n=0}^{\infty} L_n(x)\, t^{n+1} =$$
$$\nu \sum_{n=0}^{\infty} t^n I_n(x,\nu) + (1-\nu) \sum_{n=0}^{\infty} t^{n+1} I_n(x,\nu)$$

Equating powers of t leads to

$$I_0(x,\nu) = \frac{1 - e^{-x\nu} L_0(x)}{\nu}$$

and

$$I_n(x,\nu) = \frac{e^{-x\nu}}{\nu}\left(L_{n-1}(x) - L_n(x)\right) - \frac{1-\nu}{\nu} I_{n-1}(x,\nu)$$

When $x \to \infty$ the integral in (3.23) becomes the Laplace transform of $L_n(y)$. It is well-known, see e.g. [1], that the Laplace transform of $L_n(y)$ is

$$\widehat{L}_n(\nu) = \lim_{x\to\infty} I_n(x,\nu) = \frac{(\nu-1)^n}{\nu^{n+1}} \tag{3.24}$$

Since $\widehat{\Pi}(\nu_1,\nu_2)$ is assumed to be known from the outset, it can be used to assess the accuracy of the approximation in the Laplace transform domain.

Note 3.3.2 *Weeks' method implicitly assumes that the inverse Laplace transform has support on* $\mathbb{R}^+$*. It is clear from (3.16), however, that the support could just as well be* $\mathbb{R}$*, because this inversion formula does not distinguish between a Laplace transform and a bilateral Laplace transform of the form*

$$\widehat{h}(\nu) = \int_{-\infty}^{+\infty} e^{-\nu x} h(x)\, dx \tag{3.25}$$

On the other hand, because the inner product with respect to which Laguerre polynomials are orthogonal only has support on $\mathbb{R}^+$*, the Laguerre approximation in (3.15) is only accurate on* $\mathbb{R}^+$*. Moreover, while the exponential term in (3.14) will produce appropriate behavior for the right tail of a probability density function (i.e. approach zero at a suitable rate), it will explode in the left tail as its argument tends to minus infinity.*

This problem is easily fixed by splitting the support into two components,

$\mathbb{R}^-$ *and* $\mathbb{R}^+$. *If* B *denotes the bilateral Laplace transform operator it is easy to see that*

$$B^{-1}\left\{\widehat{h}\left(\nu\right)\right\}\left(-y\right) = B^{-1}\left\{\widehat{h}\left(-\nu\right)\right\}\left(y\right) \tag{3.26}$$

Thus to evaluate $h\left(y\right)$ *for negative values of* y *we can invert* $\widehat{h}\left(-\nu\right)$, *in addition to using the inverse of* $\widehat{h}\left(\nu\right)$ *to evaluate* $h\left(y\right)$ *for positive* y.

3.4 A Call Option Pricing Formula

In subsection 3.2 brief mention was made of pricing contingent claims. Let $P\left(t,T\right) = B_t E^*\left(B_T^{-1}\middle|\mathcal{F}_t\right)$ denote the price at time t of a zero-coupon or discount bond maturing at time T and let $Z\left(t,T\right)$ denote the corresponding discounted bond price process $B_t^{-1}P\left(t,T\right)$ with $B_t = e^{\int_0^t \lambda_s ds} = e^{x_t}$. Then

$$\begin{array}{l} E^*\left(e^{-x_T}\middle|\mathcal{F}_t\right) = \\ e^{-x_t}e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t}\exp\left(-\int_t^T \rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right) \end{array}$$

and

$$\begin{aligned} P\left(t,T\right) &= e^{x_t}E^*\left(e^{-x_T}\middle|\mathcal{F}_t\right) = \\ & e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)\lambda_t}\exp\left(-\int_t^T \rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right) \end{aligned}$$

We follow the same logic here as in section 3.2, namely that a discount bond may be viewed as a contingent claim with a deterministic payoff of 1 at time T. Thus its value at time $t < T$ is the conditional expected value under a suitably chosen measure P^*. Having established the price process of the discount bond, one may extend the above logic to determine the value of a claim contingent on that bond. We will price a European call option at time t_1 to purchase a discount bond (maturing at time T) at time t_2 at strike price k, where $T > t_2 > t_1$. At the expiration date t_2, the call option's pay-off is

$\max(P(t_2,T)-k,0)$. Thus the option value at time t_1 is

$$e^{x_{t_1}}E^*\left(e^{-x_{t_2}}\max(P(t_2,T)-k,0)\middle|\mathcal{F}_{t_1}\right)$$

where

$$\max(P(t_2,T)-k,0)=P(t_2,T)-k,\ \text{if}\ P(t_2,T)>k,\ 0\ \text{otherwise}$$

Thus

$$\begin{aligned}
&P(t_2,T)>k\Leftrightarrow\\
&e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t_2)}\right)\lambda_{t_2}}\exp\left(-\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right)>k\Leftrightarrow\\
&-\frac{1}{\delta}\left(1-e^{-\delta(T-t_2)}\right)\lambda_{t_2}-\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds>\log k\Leftrightarrow\\
&\lambda_{t_2}<-\frac{\delta}{1-e^{-\delta(T-t_2)}}\left(\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds+\log k\right)\equiv c^* \qquad (3.27)
\end{aligned}$$

Recall that

$$\begin{aligned}
&\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds=\\
&\frac{\rho\alpha}{\delta(\alpha+\kappa_1)}\left[\delta(T-t_2)-\log\left(\frac{\alpha+\kappa_1+\kappa_2e^{T}}{\alpha+\kappa_1+\kappa_2e^{t_2}}\right)\right]-\\
&\frac{\rho\alpha}{\delta\left(\alpha+\kappa_1+\frac{1}{\delta}\right)}\left[\delta(T-t_2)-\log\left(\frac{\alpha+\kappa_1+\kappa_2e^{\delta T}}{\alpha+\kappa_1+\kappa_2e^{\delta t_2}+\frac{1}{\delta}\left(1-e^{-\delta(T-t_2)}\right)}\right)\right]
\end{aligned}$$

Continuing, the option price may be written as

$$\begin{aligned}
&e^{x_{t_1}}E^*\left(e^{-x_{t_2}}\max(P(t_2,T)-k,0)\middle|\mathcal{F}_{t_1}\right)=\\
&e^{x_{t_1}}E^*\left\{\max\left[e^{-x_{t_2}}P(t_2,T)-e^{-x_{t_2}}k,0\right]\middle|\mathcal{F}_{t_1}\right\}=\\
&e^{x_{t_1}}\int_0^{c^*}\int_0^{\infty}\Pi(x_{t_2},\lambda_{t_2})e^{-x_{t_2}}e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t_2)}\right)\lambda_{t_2}}dx_{t_2}d\lambda_{t_2}\times\\
&\exp\left(-\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\frac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right)-\\
&e^{x_{t_1}}\int_0^{c^*}\int_0^{\infty}\Pi(x_{t_2},\lambda_{t_2})e^{-x_{t_2}}k\,dx_{t_2}d\lambda_{t_2}=\\
&e^{x_{t_1}}\int_0^{c^*}L_x\{\Pi\}(1,\lambda_{t_2})e^{-\frac{1}{\delta}\left(1-e^{-\delta(T-t_2)}\right)\lambda_{t_2}}d\lambda_{t_2}\times
\end{aligned}$$

$$
\begin{aligned}
&\exp\left(-\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\tfrac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right) - \\
&e^{x_{t_1}}\int_0^{c^*} L_x\{\Pi\}(1,\lambda_{t_2})\,d\lambda_{t_2}k \approx \\
&e^{x_{t_1}}\sum_{n=0}^{N}\frac{c_n}{2b}I_n\left(2bc^*,\frac{\frac{1-e^{-\delta(T-t_2)}}{\delta}+b-\sigma}{2b}\right)\times \\
&\exp\left(-\int_{t_2}^{T}\rho^*(s)\left[1-\hat{g}^*\left(\tfrac{1}{\delta}\left(1-e^{-\delta(T-s)}\right);s\right)\right]ds\right) - \\
&e^{x_{t_1}}k\sum_{n=0}^{N}\frac{c_n}{2b}I_n\left(2bc^*,\frac{b-\sigma}{2b}\right)
\end{aligned}
\tag{3.28}
$$

where $L_x\{\Pi\}(\nu,\lambda)$ denotes the LT of $\Pi(x,\lambda)$ w.r.t. x, where

$$
I_n\left(2bc^*,\frac{\frac{1-e^{-\delta(T-t_2)}}{\delta}+b-\sigma}{2b}\right)
$$

and

$$
I_n\left(2bc^*,\frac{b-\sigma}{2b}\right)
$$

are as defined in (3.23) and where c^* is as defined in (3.27). From (3.28) it is clear that we only need to invert a Laplace transform w.r.t. λ, as was alluded to in subsection 3.3.3.

Numerical experiments using the option pricing formula in (3.28) are presented in section 5.1.

Chapter 4

Pricing in Discrete Time

4.1 Motivation

This chapter presents one of the main contributions of this thesis, namely a pricing formula for a claim contingent on a discrete-time short rate. Special cases are discount bonds and European call or put options on those bonds. Explicit pricing formulæ are presented for these claims.

In chapter 3 a continuous-time interest rate model was defined, which is based on the shot noise process. Within this framework pricing formulæ were developed for a discount bond as well as a European call option on such a bond. The pricing was based on evaluating the expected value of the pay-off at expiration under an equivalent martingale measure. This risk-neutral valuation approach, justified by the Fundamental Theorem of Asset Pricing, was originally used for models of complete markets, starting with Black and Scholes [6] and made rigorous by Harrison and Kreps [26] and Harrison and Pliska [27]. Heath, Jarrow and Morton [28] apply this approach in an interest rate context.

Risk-neutral pricing has also been used in the context of models of incomplete markets such as those in Gerber and Shiu [24, 25] and Duffie, Pan and Singleton [19]. As noted in section 2.6, the shot noise process is a special case of the AJD proposed in the latter publication and it is not unreasonable to expect the pricing results from chapter 3 to be equivalent to those in [19].

As mentioned in chapter 1, this approach is not entirely satisfactory. The equivalent martingale measure for pricing a given claim is not unique and the approach offers no guidance for making an appropriate choice. Moreover, this approach ignores the risk incurred when selling a contingent claim and hedging the resulting exposure with the underlying asset and cash. This risk results from the fact that claim is unattainable; the discontinuities (jumps) in the short-rate process, which cannot be anticipated beforehand, make it impossible to construct a portfolio of cash and the underlying security that perfectly replicates the pay-off of the claim at expiration.

The remainder of this thesis is devoted to finding a partial remedy for this problem in the context of an interest rate market. Though the ambiguity of the martingale measure will not be removed entirely, it will be limited to a single security, namely the discount bond with the longest maturity that is priced in a market. Once an equivalent martingale measure has been decided on for the long bond, the price for any shorter dated bond or other claim can be derived by replicating it with a dynamically adjusted portfolio of the long bond and cash. The replicating portfolio is constructed so that the expected value at present of the squared difference between the discounted portfolio value and the discounted pay-off of the claim at expiration is minimized. The squared difference is an arbitrary but not unreasonable criterion. Once this criterion and the equivalent martingale measure of the long bond have been decided on, all other securities that are contingent on the interest rate can, in principle, be priced.

Two types of quadratic hedging strategies that specifically deal with pricing in incomplete markets have been covered in the literature. The first, local risk minimization, forces the replicating strategy to have the same value as the claim at expiration and minimizes the expected squared cash in- and out-flows (Föllmer and Sondermann [22], Föllmer and Schweizer [23], Runggaldier and Schweizer [36] and Schweizer [38]). As the name suggests, this strategy hedges only locally without taking future evolutions of the price or short rate into account when minimizing risk. The results in these publications are based on the Kunita-Watanabe decomposition and its generalization by Föllmer and Schweizer, which essentially allows the value of a claim to be represented as the sum of a scalar and two orthogonal martingales, one of which is a stochastic integral with respect to the (martingale) price process of an underlying asset. The stochastic integral can be interpreted as the value process of the replicating strategy. The optimality of the strategy, in a mean-variance sense, is based on the orthogonality of the two component martingales.

The second quadratic approach, mean-variance hedging, forces the replicating strategy to be self-financing and minimizes the expected squared difference between its value and that of the claim at expiration. The minimization problem solved in this approach is global. Duffie and Richardson [20] explore mean-variance hedging in the context of futures markets (without jumps). Heath et al. [29, 30] compare local risk minimization and mean-variance hedging in the context of option pricing with several stochastic volatility models (again without jumps). Schweizer [39] provides a more general and theoretical overview of both approaches. Though the theory in these papers is fairly general in nature, specific option pricing results for non-trivial price/short rate processes with jumps have remained elusive.

The approach explored in this thesis clearly belongs to the mean-variance category. In order to solve the minimization problem of the replicating portfolio, the decision is made to develop the framework in terms of a discrete-time interest rate model. This short rate model, though specific, is quite flexible and allows for incremental movements with both a diffusion nature as well as a jump nature.

A central result in this thesis is a pricing formula for a claim contingent on any discount bond with a maturity shorter than or equal to that of the long bond. This claim has a "European" flavor in the sense that it can only be exercised on the expiration date. Special cases are discount bonds and European call or put options on those bonds. The option pricing results in particular are new as no explicit pricing formulas based on mean-variance hedging in an incomplete market with jumps have been presented in the literature so far.

This model is parameterized in such a way that the length of the time steps can be reduced arbitrarily. Some limit results are presented which demonstrate the equivalence of the discrete-time model in this chapter to the continuous-time model in chapter 2. In principle, continuous-time counterparts of the pricing results developed in this chapter can be obtained in the limit as well. This is a lengthy and messy affair, however, which was not found to add any further insights or yield more effecient algorithms for computing prices. Hence these limit results have been omitted from the dissertation.

Sections 4.2 and 4.3 below develop some useful results for the discrete-time approximation of the shot noise process. In section 4.4 the optimization problem is solved for the case where a contingent claim with a fairly general pay-off at maturity is priced by hedging it with cash and a longer dated discount bond. Section 4.5 applies the general pricing results to the special case of a discount bond. Section 4.6 develops pricing formulæ for both European call and put options.

4.2 A Discrete-Time Interest Rate Model

The discrete-time short-rate model we consider will be restricted to the time interval $[0, t^*]$. Here t^* is chosen to be suitably large, but finite. An integer n will parameterize the granularity of the time-steps by dividing $[0, t^*]$ into n sub-intervals $[i\Delta^n, (i+1)\Delta^n)$ for $i = 0, \ldots, n-1$ and $\Delta^n = \frac{t^*}{n}$. Then t^n denotes a discrete map of $t \in [0, t^*]$ defined by

$$t^n = i\Delta^n \tag{4.1}$$

whenever

$$i\Delta^n \le t < (i+1)\Delta^n \tag{4.2}$$

for some $i = 0, \ldots, n-1$ and $t^n = t^*$ when $t = t^*$. Equations (4.1) and (4.2) imply

$$t - \Delta^n < t^n \le t \tag{4.3}$$

for $t \le t^*$. And, since $\lim_{n\to\infty} \Delta^n = 0$, we have

$$\lim_{n\to\infty} t^n = t \tag{4.4}$$

Next, consider the discrete-time process λ_t^n, which represents the short rate and evolves as follows:

$$\lambda_t^n = \lambda_{t-\Delta^n}^n (1 - \alpha\Delta^n) + X_t^n \tag{4.5}$$

Here $0 < \alpha < 1$ and λ_t^n and X_t^n are discrete in the sense that $\lambda_t^n = \lambda_{i\Delta^n}^n$ and $X_t^n = X_{i\Delta^n}^n$ for some $i = 0, \ldots, n-1$, analogous to (4.1) and (4.2). The increments $X_{i\Delta^n}^n$ are i.i.d. with d.f. $h^n(y)$, which is thus allowed to depend on n. Clearly, λ_t^n is a Markov process. The filtration generated by λ_t^n will be denoted $\mathcal{F}_t^n$.

It will be useful to define

$$N_n(t) = \frac{t^n}{\Delta^n} \tag{4.6}$$

Thus, when $[0, t^*]$ is divided into n sub-intervals, $N_n(u)$ returns the number of whole sub-intervals on $[0, u]$, for $u \in [0, t^*]$. Clearly, $N_n(t^*) = \frac{t^*}{\Delta^n} = n$.

The aggregated process x_t^n (not to be confused with X_t^n) can now be defined as

$$x_t^n = \sum_{j=1}^{N_n(t)} \lambda_{j\Delta^n}^n \Delta^n \tag{4.7}$$

The following result expresses λ_t^n and x_t^n as functions of $X_{j\Delta^n}^n$, $j \leq N_n(t)$ in addition to some boundary conditions:

Lemma 4.2.1 *The discrete-time shot noise process λ_t^n may be written as*

$$\lambda_t^n = \lambda_{t-\tau^n}^n (1 - \alpha\Delta^n)^{N_n(\tau)} + \sum_{j=N_n(t)-N_n(\tau)+1}^{N_n(t)} X_{j\Delta^n}^n (1 - \alpha\Delta^n)^{N_n(t)-j} \tag{4.8}$$

and the aggregated process may be written as

$$x_t^n - x_{t-\tau^n}^n = \lambda_{t-\tau^n}^n \frac{1 - \alpha\Delta^n - (1 - \alpha\Delta^n)^{N_n(\tau)+1}}{\alpha} +$$

$$\sum_{j=N_n(t)-N_n(\tau)+1}^{N_n(t)} X^n_{j\Delta^n} \frac{1-(1-\alpha\Delta^n)^{N_n(t)-j+1}}{\alpha} \tag{4.9}$$

Proof: The result in (4.8) follows easily from (4.5) by carrying out repeated substitutions. To prove the result in (4.9), note that by (4.7) the aggregate process x^n_t can be written as

$$x^n_t - x^n_{t-\tau^n} = \sum_{j=N_n(t)-N_n(\tau)+1}^{N_n(t)} \lambda^n_{j\Delta^n}\Delta^n \tag{4.10}$$

Also note that

$$\sum_{i=N_n(t)-N_n(\tau)+1}^{N_n(t)} (1-\alpha\Delta^n)^i =$$
$$(1-\alpha\Delta^n)^{N_n(t)-N_n(\tau)+1} \sum_{i=0}^{N_n(\tau)-1} (1-\alpha\Delta^n)^i =$$
$$(1-\alpha\Delta^n)^{N_n(t)-N_n(\tau)+1} \frac{(1-\alpha\Delta^n)-(1-\alpha\Delta^n)^{N_n(\tau)}}{\alpha\Delta^n}$$

The result then follows by repeatedly substituting (4.8) into (4.10). This concludes the proof of lemma 4.2.1.

Theorem 4.2.2 *The joint Laplace Transform of $\lambda^n_{\tau_2}$ and $x^n_{\tau_2}$, given $\mathcal{F}_{\tau_1}$, is*

$$E\left[\exp\left(-\nu_1\left(x^n_{\tau_2}-x^n_{\tau_1}\right)-\nu_2\lambda^n_{\tau_2}\right)\Big|\,\mathcal{F}^n_{\tau_1}\right] =$$
$$\exp\left(-\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}\right)\lambda^n_{\tau_1}\right)\times$$
$$\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(\nu_1\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i}\right) \tag{4.11}$$

Proof: Using (4.8) and (4.9) leads to

$$\nu_1\left(x^n_{\tau_2}-x^n_{\tau_1}\right)+\nu_2\lambda^n_{\tau_2} =$$
$$\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}\right)\lambda^n_{\tau_1}+$$
$$\sum_{j=N_n(\tau_1)+1}^{N_n(\tau_2)} \left(\nu_1\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-j}\right)X^n_{j\Delta^n} \tag{4.12}$$

Note that, because $u^n = N_n(u)\Delta^n$, we can write

$$N_n(\tau_2) - N_n(\tau_1) = \frac{1}{\Delta^n}\left(N_n(\tau_2)\Delta^n - N_n(\tau_1)\Delta^n\right) = \frac{1}{\Delta^n}\left(\tau_2^n - \tau_1^n\right)$$

Moreover, $E\left[e^{-\nu X_{j\Delta^n}^n}\middle|\mathcal{F}_{i\Delta^n}^n\right] = E\left[e^{-\nu X_{j\Delta^n}^n}\right] = \hat{h}^n(\nu)$ when $j > i$ and thus

$$E\left[\exp\left(-\left(\nu_1\tfrac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-j}\right)X_{j\Delta^n}^n\right)\right] = \hat{h}^n\left(\nu_1\tfrac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-j}\right)$$

Because the $X_{j\Delta^n}^n$ are i.i.d., we have

$$E\left[\exp\left(-\sum_{j=N_n(\tau_1)+1}^{N_n(\tau_2)}\left(\nu_1\tfrac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-j}\right)X_{j\Delta^n}^n\right)\right] = \prod_{j=N_n(\tau_1)+1}^{N_n(\tau_2)}\hat{h}^n\left(\nu_1\tfrac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-j}\right) \quad (4.13)$$

where $\hat{h}^n$ is the Laplace transform of h^n. Combining (4.13) with (4.12) leads to the desired result. This concludes the proof. Note that from (4.11) and the Markov property of λ_t^n it is clear that

$$E\left[\exp\left(-\nu_1\left(x_{\tau_2}^n - x_{\tau_1}^n\right) - \nu_2\lambda_{\tau_2}^n\right)\middle|\mathcal{F}_{\tau_1}^n\right] = E\left[\exp\left(-\nu_1\left(x_{\tau_2}^n - x_{\tau_1}^n\right) - \nu_2\lambda_{\tau_2}^n\right)\middle|\lambda_{\tau_1}^n\right]$$

If we are willing to settle for the base-measure P as the risk-neutral measure, theorem 4.2.2 can be used to evaluate $P_t^{T,n}$, the price at time $t^n \in [0, t^*]$ of a zero-coupon bond maturing at time $T^n \in [t^n, t^*]$, as discussed in sections 1.1 and 1.2. This is achieved by setting $\nu_1 = 1$ and $\nu_2 = 0$ in (4.11). Since the pay-off of a zero-coupon bond at maturity is unity, its discounted price at time t^n, $e^{-x_t^n}P_t^{T,n}$, is the conditional expectation of the discounted pay-off:

$$\begin{aligned} e^{-x_t^n}P_t^{T,n} &= E\left[e^{-x_T^n}\middle|\mathcal{F}_t^n\right] = \\ &\quad e^{-x_t^n}\exp\left(-\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n\right)\times \end{aligned}$$

$$\prod_{j=N_n(t)+1}^{N_n(T)} \hat{h}^n \left(\frac{1-(1-\alpha\Delta^n)^{N_n(T)-j+1}}{\alpha}\right) \tag{4.14}$$

Clearly, the discounted bond price process $e^{-x_t^n} P_t^{T,n}$ is a martingale since $E\left[E\left[e^{-x_T^n}\middle|\mathcal{F}_t^n\right]\middle|\mathcal{F}_s^n\right] = E\left[e^{-x_T^n}\middle|\mathcal{F}_s^n\right]$ for $s \leq t$.

4.3 Change of Measure

We would like to go a bit further than (4.14) and use a measure different from the base-measure as an equivalent martingale measure. This section introduces a martingale that will allow us to define a change of measure analogous to the continuous-time case described in section 2.5, and specifically given in (2.24). This martingale is presented in the following

Theorem 4.3.1 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\hat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n. Then the following process is a martingale:*

$$\frac{\exp\left(-\nu x_t^n - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(t)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_t^n\right)}{\prod_{i=1}^{N_n(t)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i} + \frac{\nu}{\alpha}\right)} \tag{4.15}$$

Proof: In (4.11) we see that on the LHS $\lambda_{\tau_2}^n$ is multiplied by ν_2, while on the RHS $\lambda_{\tau_1}^n$ is multiplied by

$$\nu_1 \frac{1-\alpha\Delta^n - (1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha} + \nu_2 (1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)} \tag{4.16}$$

which contains ν_2 again. Hence there seems to be an element of recursion in this equation. To obtain a martingale we wish to obtain an expression for ν_2 which is internally consistent in the following sense: Define $\gamma(\tau_2) = \nu_2$, and $\nu = \nu_1$ such that

$$\nu_1 \frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha} + \nu_2 (1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)} =$$

$$\nu \, \frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha} + \gamma(\tau_2)\,(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)} \;=\; \gamma(\tau_1)$$

It is easily verified that

$$\gamma(\tau_1) = \nu \, \frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha} + \gamma(\tau_2)\,(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}$$

is a solution. It follows immediately that

$$\nu\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha} + \gamma(\tau_2)\,(1-\alpha\Delta^n)^{N_n(\tau_2)-j} = \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^j} + \frac{\nu}{\alpha}$$

To make the non-stochastic product term on the RHS consistent as well, note that

$$E\left[\exp\left(-\nu x^n_{\tau_2} - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + (1-\alpha\Delta^n)\frac{\nu}{\alpha}\right)\lambda^n_{\tau_2}\right)\middle|\mathcal{F}^n_{\tau_1}\right] = \exp\left(-\nu x^n_{\tau_1} - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_1)}} + (1-\alpha\Delta^n)\frac{\nu}{\alpha}\right)\lambda^n_{\tau_1}\right) \times \prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)$$

Thus

$$E\left[\frac{\exp\left(-\nu x^n_{\tau_2} - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda^n_{\tau_2}\right)}{\prod_{i=1}^{N_n(\tau_2)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\middle|\mathcal{F}^n_{\tau_1}\right] = \frac{\exp\left(-\nu x^n_{\tau_1} - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_1)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda^n_{\tau_1}\right)}{\prod_{i=1}^{N_n(\tau_1)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}$$

And thus the process in (4.15) is a martingale, which concludes the proof.

The martingale in (4.15) may be used to construct a Radon-Nykodim derivative as follows

$$\frac{\dfrac{\exp\left(-\nu x_t^n-\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(t)}}+\frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_t^n\right)}{\prod_{i=1}^{N_n(t)}\hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}}{E\left[\dfrac{\exp\left(-\nu x_t^n-\left(\frac{c-\frac{\nu}{\alpha}}{\left(1-\alpha\frac{t^*}{n}\right)^{N_n(t)}}+\frac{\nu}{\alpha}\left(1-\alpha\frac{t^*}{n}\right)\right)\lambda_t^n\right)}{\prod_{i=1}^{N_n(t)}\hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\right]} =$$

$$\frac{\exp\left(-\nu x_t^n-\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(t)}}+\frac{\nu}{\alpha}\left(1-\alpha\Delta^n\right)\right)\lambda_t^n\right)}{e^{-(c-\nu\Delta^n)\lambda_0^n}\prod_{i=1}^{N_n(t)}\hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)} \tag{4.17}$$

This Radon-Nykodim derivative can be used to define a change of measure. As before let P denote the base measure and P^* the transformed measure using (4.17). Define the expectations E and E^* similarly. We will use the convention $\prod_{j=k}^{i}\equiv 1$ when $k>i$. The P^*-analogue of (4.11) is presented in the following

Theorem 4.3.2 *The joint Laplace transform of $\lambda_{\tau_2}^n$ and $x_{\tau_2}^n$ conditional on $\mathcal{F}_{\tau_1}^n$ under the changed measure P^*, as defined above, is*

$$E^*\left[e^{-\nu_1 x_{\tau_2}^n-\nu_2\lambda_{\tau_2}^n}\middle|\mathcal{F}_{\tau_1}^n\right]=$$

$$\frac{E\left[\dfrac{\exp\left(-(\nu+\nu_1)x_{\tau_2}^n-\left(\nu_2+\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}}+\frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)}{\prod_{i=1}^{N_n(\tau_2)}\hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\middle|\mathcal{F}_{\tau_1}^n\right]}{E\left[\dfrac{\exp\left(-\nu x_{\tau_2}^n-\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}}+\frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)}{\prod_{i=1}^{N_n(\tau_2)}\hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\middle|\mathcal{F}_{\tau_1}^n\right]}=$$

$$e^{-\nu_1 x_{\tau_1}^n}\times$$

$$\exp\left(-\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}\right)\mathcal{F}_{\tau_1}^n\right)\times$$

$$\frac{\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\hat{h}^n\left(\nu_1 a_n^{\tau_2}(i\Delta^n)+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i}+b_n(i\Delta^n)\right)}{\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\hat{h}^n\left(b_n\left(i\Delta^n\right)\right)} \tag{4.18}$$

where the step functions a_n^τ and b_n are defined as

$$a_n^\tau(t) = \frac{1-(1-\alpha\Delta^n)^{\frac{\tau^n - t^n + \Delta^n}{\Delta^n}}}{\alpha} \tag{4.19}$$

and

$$b_n(t) = \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{\frac{t^n}{\Delta^n}}} + \frac{\nu}{\alpha} \tag{4.20}$$

Proof: From the definition of E^* in (4.17), the LHS of (4.18) is

$$E^*\left[e^{-\nu_1 x_{\tau_2}^n - \nu_2\lambda_{\tau_2}^n}\middle|\mathcal{F}_{\tau_1}^n\right] = \frac{E\left[\frac{\exp\left(-(\nu+\nu_1)x_{\tau_2}^n - \left(\nu_2 + \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)}{\prod_{i=1}^{N_n(\tau_2)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\middle|\mathcal{F}_{\tau_1}^n\right]}{E\left[\frac{\exp\left(-\nu x_{\tau_2}^n - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)}{\prod_{i=1}^{N_n(\tau_2)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right)}\middle|\mathcal{F}_{\tau_1}^n\right]} \tag{4.21}$$

Using theorem 4.3.1, it follows that the denominator of (4.21) is

$$\begin{aligned} &E\left[\exp\left(-\nu x_{\tau_2}^n - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)\middle|\mathcal{F}_{\tau_1}^n\right] = \\ &\exp\left(-\nu x_{\tau_1}^n - \left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_1)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_1}^n\right) \times \\ &\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i}+\frac{\nu}{\alpha}\right) \end{aligned} \tag{4.22}$$

while the numerator is evaluated as

$$\begin{aligned} &E\left[\exp\left(-(\nu+\nu_1)x_{\tau_2}^n - \left(\nu_2 + \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_2)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_2}^n\right)\middle|\mathcal{F}_{\tau_1}^n\right] = \\ &e^{-(\nu+\nu_1)x_{\tau_1}^n} \times \\ &\exp\left(-\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha} + \nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}\right)\lambda_{\tau_1}^n\right) \times \\ &\exp\left(-\left(\frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^{N_n(\tau_1)}} + \frac{\nu}{\alpha}(1-\alpha\Delta^n)\right)\lambda_{\tau_1}^n\right) \times \end{aligned}$$

$$\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(\nu_1 \frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha} + \nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i} + \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i} + \frac{\nu}{\alpha}\right) \quad (4.23)$$

Note that, from (4.19),

$$a_n^{\tau_2}(i\Delta^n) = \frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha}$$

and from (4.20),

$$b_n(i\Delta^n) = \frac{c-\frac{\nu}{\alpha}}{(1-\alpha\Delta^n)^i} + \frac{\nu}{\alpha}$$

Substituting (4.22) and (4.23) into (4.21) concludes the proof.

The change of measure defined using (4.17) allows for a richer set of bond price processes. The price at time t of a zero-coupon bond maturing at time T can now be defined as $P_t^{T,n} = e^{x_t^n} E^*\left[e^{-x_T^n}\middle|\mathcal{F}_t^n\right]$ and is evaluated in the following

Lemma 4.3.3 *The expression for the discount bond price, defined as $P_t^{T,n} = e^{x_t^n} E^*\left[e^{-x_T^n}\middle|\mathcal{F}_t^n\right]$, is*

$$P_t^{T,n} = \exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n\right)\Lambda_{N_n(t)}^{N_n(T)} \quad (4.24)$$

where

$$\Lambda_{N_n(t)}^{N_n(T)} = \prod_{i=N_n(t)+1}^{N_n(T)} \frac{\hat{h}^n\left(a_n^T(i\Delta^n)+b_n(i\Delta^n)\right)}{\hat{h}^n\left(b_n(i\Delta^n)\right)} \quad (4.25)$$

where $a_n^T(i\Delta^n)$ and $b_n(i\Delta^n)$ are as defined in (4.19) and (4.20), respectively.

Proof: This follows directly from (4.18).

Also, using virtually the same argument as that below (4.14), it follows that the discounted bond price process $e^{-x_t^n}P_t^{T,n}$, with $P_t^{T,n}$ given in (4.24), is a P^*-martingale. There is, of course, no longer any reason for $e^{-x_t^n}P_t^{T,n}$ to be a P-martingale.

4.4 Mean-Variance Hedging

Let $Z_t^{T,n}$ denote the discounted version of the bond price $P_t^{T,n}$ defined in (4.24):

$$Z_t^{T,n} = e^{-x_t^n} \exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n\right) \Lambda_{N_n(t)}^{N_n(T)} \tag{4.26}$$

with $\Lambda_{N_n(t)}^{N_n(T)}$ as defined in (4.25).

Consider a contingent claim which expires at time t^n. The discounted pay-off ϕ_t^n of this contract is a random variable measurable with respect to $\mathcal{F}_t^n$. As an example, the discounted pay-off of a European call option to purchase on date t^n a discount bond with maturity $T^n \geq t^n$ at strike price k is $\phi_t^n = \left(Z_t^{T,n} - e^{-x_t^n}k\right)^+$.

To determine the value of this claim, this problem may be viewed in the context of a multi-period hedging problem. We will try to replicate the discounted pay-off at expiration with a self-financing portfolio of cash and the underlying bond. The portfolio is rebalanced at discrete time points using only information available before that time. The discounted value of the replicating portfolio at expiration of the claim can be written as

$$V_t^n = V_0^n + \pi_0^n \Delta Z_{\Delta^n}^{T,n} + \ldots + \pi_{t-\Delta^n}^n \Delta Z_t^{T,n} \tag{4.27}$$

or, equivalently, as the recursion

$$V_t^n = V_{t-\Delta^n}^n + \pi_{t-\Delta^n}^n \Delta Z_t^{T,n} \tag{4.28}$$

The hedging error is

$$\varepsilon_t^n = V_t^n - \phi_t^n \tag{4.29}$$

The values of $V_0^n, \pi_0^n, \ldots, \pi_{t-\Delta^n}^n$ will be determined by minimizing the expected squared hedging error $E\left[(\varepsilon_t^n)^2 \middle| \mathcal{F}_0^n\right]$. If we define

$$W_t^n = \left[V_0^n, \pi_0^n, \ldots, \pi_{t-\Delta^n}^n\right] \tag{4.30}$$

with

$$W_0^n = V_0^n \tag{4.31}$$

then using (4.29) this minimization problem can be restated as

$$\min_{W_t^n} E\left[(V_t^n)^2 - 2V_t^n \phi_t^n \middle| \mathcal{F}_0^n \right] \tag{4.32}$$

The solution to this minimization problem will yield V_0^n, the price at time 0 of the contingent claim with pay-off ϕ_t^n at time t. This is the main result of this section and is presented in theorem 4.4.1 below. In order to prove this theorem we solve this minimization problem iteratively, for one value of $\pi_{t-i\Delta}$ at a time, $i = 1, \ldots, N_n(t)$.

The toolt needed to solve these one-step minimizations are provided by lemma 4.4.2 and corollary 4.4.3 below. These intermediate results further depend on a number of supporting lemmas which are listed below the main results of this section in order of progression, the proof each lemma depending on results in prior lemmas.

First, we present the main result:

Theorem 4.4.1 *The price at time* 0 *of a contingent claim with pay-off* ϕ_t^n *at time* t^n*, obtained by solving the minimization problem in (4.32) is*

$$V_0^n = E\left[\phi_t^n \frac{M_t^n}{M_0^n} \middle| \mathcal{F}_0^n \right] \tag{4.33}$$

where M_t^n *is a martingale defined by*

$$M_t^n = \prod_{j=1}^{N_n(t)} \left(1 + \frac{\left[\widehat{h}^n\left(a_n^T(t_j)\right) - \widehat{h}_{t_j}^n\left(a_n^T(t_j)\right) \right] \left[\widehat{h}^n\left(a_n^T(t_j)\right) - \frac{Z_{t_j}^{T,n}}{Z_{t_{j-1}}^{T,n}} \widehat{h}_{t_j}^n\left(a_n^T(t_j)\right) \right]}{\widehat{h}^n\left(2a_n^T(t_j)\right) - \widehat{h}^n\left(a_n^T(t_j)\right)^2} \right) \tag{4.34}$$

with $M_0^n = 1$,

$$t_j = j\Delta^n \quad \textit{for} \quad j = 0, \ldots, n \tag{4.35}$$

and

$$\widehat{h}_t^n(\nu) = \frac{\widehat{h}^n(\nu + b_n(t))}{\widehat{h}^n(b_n(t))} \tag{4.36}$$

Also, $a_n^T(t_j)$ *and* $b_n(t_j)$ *are defined in (4.19) and (4.20), respectively.*

As mentioned above, the proof of theorem 4.4.1 is carried out by solving the minimization problem iteratively. This is accomplished by casting an intermediate minimization problem as a recursion:

Lemma 4.4.2 *Consider the minimization problem*

$$\min_{W_{t-\tau^n}^n} E\left[\left(V_{t-\tau^n}^n\right)^2 - 2V_{t-\tau^n}^n \phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\middle| \mathcal{F}_0^n\right] \tag{4.37}$$

where $\Delta^n \leq \tau^n \leq t^n$, ϕ_t^n *denotes the pay-off of a contingent claim at expiration described at the start of this section,* $V_{t-\tau^n}^n$ *is the value of the replicating portfolio at time* $t^n - \tau^n$ *given in (4.27),* M_t^n *is the martingale given in (4.34) and, similar to (4.30),* $W_{t-\tau^n}^n$ *is given by*

$$W_{t-\tau^n}^n = \left[V_0^n, \pi_0^n, \ldots, \pi_{t-\tau^n-\Delta^n}^n\right]$$

Then by solving for $\pi_{t-\tau^n-\Delta^n}^n$ *the minimization problem in (4.37) can be cast as the recursion*

$$\min_{W_{t-\tau^n}^n} E\left[\left(V_{t-\tau^n}^n\right)^2 - 2V_{t-\tau^n}^n \phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\middle| \mathcal{F}_0^n\right] =$$
$$-E\left[\frac{E\left(\phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n} \Delta Z_{t-\tau^n}^{n,N} \middle| \mathcal{F}_{t-\tau^n-\Delta^n}^n\right)^2}{E\left(\left(\Delta Z_{t-\tau^n}^{n,N}\right)^2 \middle| \mathcal{F}_{t-\tau^n-\Delta^n}^n\right)}\middle| \mathcal{F}_0^n\right] +$$

$$\left(1-\frac{E\left[\Delta Z^{T,n}_{t-\tau^n}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]^2}{E\left[\left(\Delta Z^{T,n}_{t-\tau^n}\right)^2\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]}\right)\times$$
$$\min_{W^n_{t-\tau^n-\Delta^n}} E\left[\left(V^n_{t-\tau^n-\Delta^n}\right)^2-2V^n_{t-\tau^n-\Delta^n}\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n-\Delta^n}}\middle|\mathcal{F}^n_0\right] \tag{4.38}$$

Proof: The minimization problem in (4.37) can be written as

$$\min_{W^n_{t-\tau^n}} E\left[\left(V^n_{t-\tau^n}\right)^2-2V^n_{t-\tau^n}\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\middle|\mathcal{F}^n_0\right]=$$
$$\min_{W^n_{t-\tau^n-\Delta^n}} E\left\{\min_{\pi^n_{t-\tau^n-\Delta^n}} E\left[\left(V^n_{t-\tau^n}\right)^2-2V^n_{t-\tau^n}\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]\middle|\mathcal{F}^n_0\right\} \tag{4.39}$$

Using (4.28) the term inside the outer expectation (conditional on $\mathcal{F}^n_0$) can be expanded as

$$\left(V^n_{t-\tau^n-\Delta^n}\right)^2-2V^n_{t-\tau^n-\Delta^n}E\left[\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]+$$
$$\min_{\pi^n_{t-\tau^n-\Delta^n}} 2\pi^n_{t-\tau^n-\Delta^n}E\left[\left(V^n_{t-\tau^n-\Delta^n}-\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\right)\Delta Z^{n,N}_{t-\tau^n}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]+$$
$$\left(\pi^n_{t-\tau^n-\Delta^n}\right)^2E\left[\left(\Delta Z^{n,N}_{t-\tau^n}\right)^2\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right] \tag{4.40}$$

Solving the minimization problem for $\pi^n_{t-\tau^n-\Delta^n}$ in (4.40) yields

$$\pi^n_{t-\tau^n-\Delta^n}=-\frac{E\left[\left(V^n_{t-\tau^n-\Delta^n}-\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\right)\Delta Z^{n,N}_{t-\tau^n}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]}{E\left[\left(\Delta Z^{n,N}_{t-\tau^n}\right)^2\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]} \tag{4.41}$$

Substituting $\pi^n_{t-\Delta^n}$ back into (4.40) results in

$$\left(V^n_{t-\tau^n-\Delta^n}\right)^2-2V^n_{t-\tau^n-\Delta^n}E\left[\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]-$$
$$\frac{E\left[\left(V^n_{t-\tau^n-\Delta^n}-\phi^n_t\frac{M^n_t}{M^n_{t-\tau^n}}\right)\Delta Z^{n,N}_{t-\tau^n}\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]^2}{E\left[\left(\Delta Z^{n,N}_{t-\tau^n}\right)^2\middle|\mathcal{F}^n_{t-\tau^n-\Delta^n}\right]}=$$

$$-\frac{E\left[\left(\phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\Delta Z_{t-\tau^n}^{n,N}\right)\middle| \mathcal{F}_{t-\tau^n-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_{t-\tau^n}^{n,N}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]}+$$

$$\left(V_{t-\tau^n-\Delta^n}^n\right)^2\left(1-\frac{E\left[\Delta Z_{t-\tau^n}^{n,N}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_{t-\tau^n}^{n,N}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]}\right)-2V_{t-\tau^n-\Delta^n}^n\times$$

$$\left[E\left(\phi_t^n\frac{M_t^n}{M_{t-\tau^n}^n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)-\frac{E\left(\phi_t^n\frac{M_t^n}{M_{t-\tau^n}^n}\Delta Z_{t-\tau^n}^{n,N}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)E\left(\Delta Z_{t-\tau^n}^{n,N}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)}{E\left[\left(\Delta Z_{t-\tau^n}^{n,N}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]}\right] \tag{4.42}$$

The term multiplied with $V_{t-\tau^n-\Delta^n}^n$ in (4.42) can be simplified by applying lemma 4.4.14. Substituting the result into (4.39) completes the proof of lemma 4.4.2.

The first step in the minimization problem in (4.32) follows directly from lemma 4.4.2 and is stated as

Corollary 4.4.3 *By solving for $\pi_{t-\Delta^n}^n$, the minimization problem in (4.32) can be rewritten as*

$$\min_{W_t^n} E\left[(V_t^n)^2 - 2V_t^n\phi_t^n\middle|\mathcal{F}_0^n\right] =$$

$$\min_{W_{t-\Delta^n}^n} E\left[\left(V_{t-\Delta^n}^n\right)^2 - 2V_{t-\Delta^n}^n\phi_t^n\frac{M_t^n}{M_{t-\Delta^n}^n}\middle|\mathcal{F}_0^n\right]\left(1-\frac{E\left(\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right)^2}{E\left(\left(\Delta Z_t^{T,n}\right)^2\middle|\mathcal{F}_{t-\Delta^n}^n\right)}\right)-$$

$$E\left[\frac{E\left(\phi_t^n\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right)^2}{E\left(\left(\Delta Z_t^{T,n}\right)^2\middle|\mathcal{F}_{t-\Delta^n}^n\right)}\middle|\mathcal{F}_0^n\right] \tag{4.43}$$

where M_t^n is defined in (4.34).

Proof of theorem 4.4.1: Repeated applications of lemma 4.4.2 to (4.32), starting with the result in corollary 4.4.3, allows us to solve for $\pi_0^n, \ldots, \pi_{t-\Delta^n}^n$ yielding

$$\min_{W_t^n} E\left[(V_t^n)^2 - 2V_t^n\phi_t^n\middle|\mathcal{F}_0^n\right] =$$

$$-\sum_{j=1}^{N_n(t)} \prod_{i=N_n(t-t_j)+2}^{N_n(t)} \left(1 - \frac{E\left[\Delta Z_{t_i}^{T,n} \middle| \mathcal{F}_{t_{i-1}}^n\right]^2}{E\left[\left(\Delta Z_{t_i}^{T,n}\right)^2 \middle| \mathcal{F}_{t_{i-1}}^n\right]}\right) \times$$

$$E\left(\frac{E\left[\left(\phi_t^n \frac{M_t^n}{M_{t-t_j+\Delta^n}^n} \Delta Z_{t-t_j+\Delta^n}^{n,N}\right) \middle| \mathcal{F}_{t_j}^n\right]^2}{E\left[\left(\Delta Z_{t-t_j+\Delta^n}^{n,N}\right)^2 \middle| \mathcal{F}_{t_j}^n\right]} \middle| \mathcal{F}_0^n \right) +$$

$$\prod_{j=1}^{N_n(t)} \left(1 - \frac{E\left[\Delta Z_{t_j}^{T,n} \middle| \mathcal{F}_{t_{j-1}}^n\right]^2}{E\left[\left(\Delta Z_{t_j}^{T,n}\right)^2 \middle| \mathcal{F}_{t_{j-1}}^n\right]}\right) \min_{V_0^n} E\left[(V_0^n)^2 - 2V_0^n \phi_t^n \frac{M_t^n}{M_0^n}\right] \quad (4.44)$$

Note that from (4.31), $W_0^n = V_0^n$. We can now easily solve for V_0^n to yield the result in (4.33).

To prove that M_t^n in (4.34) is a martingale, note that from the definition of $Z_t^{T,n}$ in (4.26)

$$\frac{Z_t^{T,n}}{Z_{t-\Delta^n}^{T,n}} = \frac{\exp\left(-a_n^T(t)\, X_t^n\right)}{\hat{h}_t^n\left(a_n^T(t)\right)} \quad (4.45)$$

Then, using the definition of M_t^n in (4.34),

$$\frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n} = 1 + \frac{\left[\widehat{h}^n\left(a_n^T(t_j)\right) - \widehat{h}_{t_j}^n\left(a_n^T(t_j)\right)\right]\left[\widehat{h}^n\left(a_n^T(t_j)\right) - \exp\left(-a_n^T(t_j) X_{t_j}^n\right)\right]}{\widehat{h}^n\left(2a_n^T(t_j)\right) - \widehat{h}^n\left(a_n^T(t_j)\right)^2} \quad (4.46)$$

for $j = 1, \ldots, N_n(t)$. It is clear from (4.46) that

$$E\left[\frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n} \middle| \mathcal{F}_\tau^n\right] = 1 \quad (4.47)$$

for $\tau \leq t_j - \Delta^n$. Also,

$$M_t^n = \prod_{j=1}^{N_n(t)} \frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n} \quad (4.48)$$

Hence, using the fact that the $\frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n}$ are independent (because the $X_{t_j}^n$ are i.i.d.),

$$E\left[M_t^n \middle| I_\tau^n\right] = \prod_{j=1}^{N_n(\tau)} \frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n} \prod_{j=N_n(\tau)+1}^{N_n(t)} E\left[\frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n}\middle|\mathcal{F}_\tau^n\right] = M_\tau^n \quad (4.49)$$

This completes the proof of theorem 4.4.1.

Note that because $M_0^n = 1$ we could have used the notation M_t^n instead of $\frac{M_t^n}{M_0^n}$ in (4.33). We opted for the latter to emphasize the similarity with the notation in (2.16), which indicates a change of martingale measure.

In the remainder of this section we list lemmas 4.4.4 through 4.4.14, which are needed to prove lemma 4.4.2 and ultimately theorem 4.4.1:

Lemma 4.4.4 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then*

$$Z_{t-\Delta^n}^{T,n} = E^*\left[e^{-x_T^n}\middle|\mathcal{F}_{t-\Delta^n}^n\right] =$$
$$e^{-x_{t-\Delta^n}^n} \exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right) \prod_{i=N_n(t)}^{N_n(T)} \widehat{h}_{t_i}^n\left(a_n^T(t_i)\right) \quad (4.50)$$

Proof: This follows directly from (4.26) and the definitions in (4.35) and (4.36).

Lemma 4.4.5 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then*

$$E\left[Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right] =$$
$$e^{-x_{t-\Delta^n}^n} \exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right) \times$$
$$\widehat{h}^n\left(a_n^T(t)\right) \prod_{i=N_n(t)+1}^{N_n(T)} \widehat{h}_{t_i}^n\left(a_n^T(t_i)\right) \quad (4.51)$$

Proof: This follows from (4.26) and theorem 4.3.2.

Lemma 4.4.6 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then the expected value of the first difference of the discounted bond price is*

$$E\left[\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right]=E\left[Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right]-Z_{t-\Delta^n}^{T,n}=$$
$$e^{-x_{t-\Delta^n}^n}\exp\left(-\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right)\times$$
$$\left[\widehat{h}^n\left(a_n^T(t)\right)-\widehat{h}_t^n\left(a_n^T(t)\right)\right]\prod_{i=N_n(t)+1}^{N_n(T)}\widehat{h}_{t_i}^n\left(a_n^T(t_i)\right)\tag{4.52}$$

Proof: This follows from lemmas 4.4.4 and 4.4.5. Lemma 4.4.6 leads directly to the following result:

Lemma 4.4.7 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then*

$$E\left[\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right]^2=e^{-2x_{t-\Delta^n}^n}\exp\left(-2\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right)\times$$
$$\left[\widehat{h}^n\left(a_n^T(t)\right)-\widehat{h}_t^n\left(a_n^T(t)\right)\right]^2\prod_{i=N_n(t)+1}^{N_n(T)}\widehat{h}_{t_i}^n\left(a_n^T(t_i)\right)^2\tag{4.53}$$

The following two lemmas present conditional expected values useful in evaluating $E\left[\left(\Delta Z_t^{T,n}\right)^2\middle|\mathcal{F}_{t-\Delta^n}^n\right]$.

Lemma 4.4.8 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ be* *as defined in (4.19). Then*

$$E\left[e^{-2x_t^n}\exp\left(-2\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n\right)\Big|\,\mathcal{F}_{t-\Delta^n}^n\right] =$$
$$e^{-2x_{t-\Delta^n}^n}\exp\left(-2\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right)\widehat{h}^n\left(2a_n^T(t)\right) \quad (4.54)$$

Lemma 4.4.9 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ be as defined in (4.19). Then*

$$E\left[e^{-x_t^n}\exp\left(-\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n\right)\Big|\,\mathcal{F}_{t-\Delta^n}^n\right] =$$
$$e^{-x_{t-\Delta^n}^n}\exp\left(-\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right)\widehat{h}^n\left(a_n^T(t)\right) \quad (4.55)$$

Both lemmas 4.4.8 and 4.4.9 follow directly from theorem 4.3.2.

Lemma 4.4.10 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then the conditional expected value of the squared difference of the discounted bond price at time t^n, given $\mathcal{F}_{t-\Delta^n}^n$, is*

$$E\left[\left(\Delta Z_t^{T,n}\right)^2\Big|\,\mathcal{F}_{t-\Delta^n}^n\right] = e^{-2x_{t-\Delta^n}^n}\exp\left(-2\tfrac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n\right)\times$$
$$\left[\widehat{h}^n(2a_n^T(t))+\widehat{h}_t^n(a_n^T(t))^2-2\widehat{h}_t^n(a_n^T(t))\widehat{h}^n(a_n^T(t))\right]\prod_{i=N_n(t)+1}^{N_n(T)}\widehat{h}_{t_i}^n\left(a_n^T(t_i)\right)^2 \quad (4.56)$$

Proof: Using (4.25), (4.26), (4.36) and (4.50) it is easily verified that

$$\Delta Z_t^{T,n} = e^{-x_t^n-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n}\Lambda_{N_n(t)}^{N_n(T)} -$$
$$e^{-x_{t-\Delta^n}^n-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n}\widehat{h}_t^n\left(a_n^T(t)\right)\Lambda_{N_n(t)}^{N_n(T)} \quad (4.57)$$

Taking the square of (4.57) results in

$$\begin{aligned}\left(\Delta Z_t^{T,n}\right)^2 = \; & e^{-2x_t^n - 2\frac{1-\alpha\Delta^n - \left(1-\alpha\Delta^n\right)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n} \left(\Lambda_{N_n(t)}^{N_n(T)}\right)^2 - \\ & e^{-2x_{t-\Delta^n}^n - 2\frac{1-\alpha\Delta^n - \left(1-\alpha\Delta^n\right)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n} \widehat{h}_t^n\left(a_n^T(t)\right)^2 \left(\Lambda_{N_n(t)}^{N_n(T)}\right)^2 - \\ & 2e^{-x_t^n - \frac{1-\alpha\Delta^n - \left(1-\alpha\Delta^n\right)^{N_n(T)-N_n(t)+1}}{\alpha}\lambda_t^n} \times \\ & \quad e^{-x_{t-\Delta^n}^n - \frac{1-\alpha\Delta^n - \left(1-\alpha\Delta^n\right)^{N_n(T)-N_n(t)+2}}{\alpha}\lambda_{t-\Delta^n}^n} \widehat{h}_t^n\left(a_n^T(t)\right) \left(\Lambda_{N_n(t)}^{N_n(T)}\right)^2 \end{aligned} \tag{4.58}$$

The expected values of those terms in (4.58) that are stochastic conditional on $\mathcal{F}_{t-\Delta^n}^n$, that is, the terms involving λ_t^n and x_t^n, were evaluated in (4.54) and (4.55). Substituting these results into (4.58) leads directly to the desired expression for $E\left[\left(\Delta Z_t^{T,n}\right)^2 \middle| \mathcal{F}_{t-\Delta^n}^n\right]$ in (4.56), proving the lemma.

Lemma 4.4.11 *Let λ_t^n be the short rate process defined in (4.5) and let x_t^n be its aggregated process defined in (4.7). Furthermore, let $\widehat{h}^n$ be the Laplace transform of h^n, the density function of the increments of λ_t^n, and let $a_n^T(t)$ and $b_n(t)$ be as defined in (4.19) and (4.20), respectively. Finally, let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then the ratio $\frac{E\left[\Delta Z_t^{T,n} \middle| \mathcal{F}_{t-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_t^{T,n}\right)^2 \middle| \mathcal{F}_{t-\Delta^n}^n\right]}$ is non-stochastic and evaluates to*

$$\frac{E\left[\Delta Z_t^{T,n} \middle| \mathcal{F}_{t-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_t^{T,n}\right)^2 \middle| \mathcal{F}_{t-\Delta^n}^n\right]} = \frac{\left[\widehat{h}^n\left(a_n^T(t)\right) - \widehat{h}_t^n\left(a_n^T(t)\right)\right]^2}{\widehat{h}^n\left(2a_n^T(t)\right) + \widehat{h}_t^n\left(a_n^T(t)\right)^2 - 2\widehat{h}_t^n\left(a_n^T(t)\right)\widehat{h}^n\left(a_n^T(t)\right)} \tag{4.59}$$

Proof: This follows from (4.53) and (4.56).

Lemma 4.4.12 *Let λ_t^n be the short rate process defined in (4.5) and let $Z_t^{T,n}$ be the discounted bond price process given in (4.26). Then*

$$\frac{E\left[\Delta Z_t^{T,n} \middle| \mathcal{F}_{t-\Delta^n}^n\right]}{E\left[\left(\Delta Z_t^{T,n}\right)^2 \middle| \mathcal{F}_{t-\Delta^n}^n\right]} = -\frac{e_t^{T,n}}{Z_{t-\Delta^n}^{T,n}} \tag{4.60}$$

where $e_t^{T,n}$ is defined as

$$e_t^{T,n} = \frac{\hat{h}_t^n\left(a_n^T(t)\right)\left[\hat{h}_t^n\left(a_n^T(t)\right) - \hat{h}^n\left(a_n^T(t)\right)\right]}{\hat{h}^n\left(2a_n^T(t)\right) + \hat{h}_t^n\left(a_n^T(t)\right)^2 - 2\hat{h}_t^n\left(a_n^T(t)\right)\hat{h}^n\left(a_n^T(t)\right)} \tag{4.61}$$

Proof: This follows from (4.26) and lemmas 4.4.6 and 4.4.10.

Lemma 4.4.13 *Let $Z_t^{T,n}$ be the discounted bond price process given in (4.26) and let $e_t^{T,n}$ be as defined in (4.61). Furthermore, let ϕ_t^n denote the pay-off of a contingent claim at expiration as described at the start of this section. Then*

$$\frac{E\left(\phi_t^n \Delta Z_{t-\tau^n}^{T,n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)E\left(\Delta Z_{t-\tau^n}^{T,n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)}{E\left[\left(\Delta Z_{t-\tau^n}^{T,n}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]} =$$

$$-\left[E\left(\phi_t^n \frac{Z_{t-\tau^n}^{T,n}}{Z_{t-\tau^n-\Delta^n}^{T,n}}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right) - E\left(\phi_t^n\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)\right]e_{t-\tau^n}^{T,n}$$

Proof: This follows from lemma 4.4.12 and the definition of $e_t^{T,n}$ in (4.61).

Lemma 4.4.14 *Let $Z_t^{T,n}$ be the discounted bond price process given in (4.26) and let M_t^n be the martingale defined in (4.34). Furthermore, let ϕ_t^n denote the pay-off at expiration described at the start of this section. Then*

$$E\left(\phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right) - \frac{E\left(\phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\Delta Z_{t-\tau^n}^{T,n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)E\left(\Delta Z_{t-\tau^n}^{T,n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right)}{E\left[\left(\Delta Z_{t-\tau^n}^{T,n}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]} =$$

$$\left(1 - \frac{E\left[\Delta Z_{t-\tau^n}^{T,n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_{t-\tau^n}^{T,n}\right)^2\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right]}\right)E\left(\phi_t^n \frac{M_t^n}{M_{t-\tau^n-\Delta^n}^n}\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right) \tag{4.62}$$

Proof: From (4.60) the LHS of (4.62) can be written as

$$E\left[\phi_t^n \frac{M_t^n}{M_{t-\tau^n}^n}\left(1 - e_{t-\tau^n}^{T,n} + \frac{Z_{t-\tau^n}^{T,n}}{Z_{t-\tau^n-\Delta^n}^{T,n}}e_{t-\tau^n}^{T,n}\right)\middle|\mathcal{F}_{t-\tau^n-\Delta^n}^n\right] \tag{4.63}$$

Using (4.59) it follows that

$$1-\frac{E\left[\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_t^{T,n}\right)^2\middle|\mathcal{F}_{t-\Delta^n}^n\right]}=\frac{\widehat{h}^n\left(2a_n^T(t)\right)-\widehat{h}^n\left(a_n^T(t)\right)^2}{\widehat{h}^n\left(2a_n^T(t)\right)+\widehat{h}_t^n\left(a_n^T(t)\right)^2-2\widehat{h}_t^n\left(a_n^T(t)\right)\widehat{h}^n\left(a_n^T(t)\right)} \tag{4.64}$$

and the definition of M_t^n in (4.34) yields

$$\frac{M_t^n}{M_{t-\Delta^n}^n}=1+\frac{\left[\widehat{h}^n\left(a_n^T(t)\right)-\widehat{h}_t^n\left(a_n^T(t)\right)\right]\left[\widehat{h}^n\left(a_n^T(t)\right)-\frac{Z_t^{T,n}}{Z_{t-\Delta^n}^{T,n}}\widehat{h}_t^n\left(a_n^T(t)\right)\right]}{\widehat{h}\left(2a_n^T(t)\right)-\widehat{h}\left(a_n^T(t)\right)^2} \tag{4.65}$$

Using (4.64), (4.65) as well as the definition $e_t^{T,n}$ in (4.61) yields

$$1-e_t^{T,n}+\frac{Z_t^{T,n}}{Z_{t-\Delta^n}^{T,n}}e_t^{T,n}=\left[1-\frac{E\left[\Delta Z_t^{T,n}\middle|\mathcal{F}_{t-\Delta^n}^n\right]^2}{E\left[\left(\Delta Z_t^{T,n}\right)^2\middle|\mathcal{F}_{t-\Delta^n}^n\right]}\right]\frac{M_t^n}{M_{t-\Delta^n}^n} \tag{4.66}$$

Substituting (4.66) into (4.63) produces the desired result in (4.62).

4.5 Pricing a Discount Bond

We will now consider a special case of the contingent claim priced in the previous section, namely a discount bond with unit pay-off at expiration. The discounted pay-off in this case is $\phi_t^n = e^{-x_t^n}$. The price is given in the following

Theorem 4.5.1 *Let $\phi_t^n = e^{-x_t^n}$ be the discounted pay-off function of a discount bond with maturity t^n. The value of V_0^n that solves the minimization problem in (4.32) for this special case is*

$$\begin{aligned} V_0^n &= E\left[e^{-x_t^n}\middle|\mathcal{F}_0\right]\times \\ &\prod_{j=1}^{N_n(t)}\left[1+\frac{\left[\widehat{h}^n\left(a_n^T\left(t_j\right)\right)-\frac{\hat{h}^n\left(a_n^t\left(t_j\right)+a_n^T\left(t_j\right)\right)}{\hat{h}^n\left(a_n^t\left(t_j\right)\right)}\right]\left[\widehat{h}^n\left(a_n^T\left(t_j\right)\right)-\widehat{h}_{t_j}^n\left(a_n^T\left(t_j\right)\right)\right]}{\widehat{h}^n\left(2a_n^T\left(t_j\right)\right)-\widehat{h}^n\left(a_n^T\left(t_j\right)\right)^2}\right] \end{aligned} \tag{4.67}$$

where $\widehat{h}^n$ is the Laplace transform of h^n, the density function of the increments of λ_t^n, and $a_n^T(t)$ and $b_n(t)$ are as defined in (4.19) and (4.20), respectively.

Proof: Using theorem 4.2.2 and the definitions of λ_t^n and x_t^n in (4.5) and (4.7), respectively, yields

$$\begin{aligned}
E\left(e^{-x_t^n}\middle|\mathcal{F}_{t_j}^n\right) &= e^{-x_{t_j}^n}\exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(t)-N_n(t_j)+1}}{\alpha}\lambda_{t_j}^n\right)\times \\
&\quad \prod_{i=N_n(t_j)+1}^{N_n(t)}\hat{h}^n\left(a_n^t(t_i)\right) \\
&= e^{-x_{t_j-\Delta^n}^n}\exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(t)-N_n(t_j)+2}}{\alpha}\lambda_{t_j-\Delta^n}^n\right)\times \\
&\quad \exp\left(-a_n^t(t_j)X_{t_j}^n\right)\prod_{i=N_n(t_j)+1}^{N_n(t)}\hat{h}^n\left(a_n^t(t_i)\right)
\end{aligned} \tag{4.68}$$

Another application of theorem 4.2.2 combined with (4.68) then yields

$$\begin{aligned}
&E\left(e^{x_t^n}\exp\left(-a_n^T(t_j)X_{t_j}^n\right)\middle|\mathcal{F}_{t_j-\Delta^n}^n\right) = \\
&E\left(E\left(e^{x_t^n}\middle|\mathcal{F}_{t_j}^n\right)\exp\left(-a_n^T(t_j)X_{t_j}^n\right)\middle|\mathcal{F}_{t_j-\Delta^n}^n\right) = \\
&e^{-x_{t_j-\Delta^n}^n}\exp\left(-\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(t)-N_n(t_j)+2}}{\alpha}\lambda_{t_j-\Delta^n}^n\right)\times \\
&\hat{h}^n\left(a_n^t(t_j)+a_n^T(t_j)\right)\prod_{i=N_n(t_j)+1}^{N_n(t)}\hat{h}^n\left(a_n^t(t_i)\right) = \\
&E\left(e^{-x_t^n}\middle|\mathcal{F}_{t_j-\Delta^n}^n\right)\frac{\hat{h}^n\left(a_n^t(t_j)+a_n^T(t_j)\right)}{\hat{h}^n\left(a_n^t(t_j)\right)}
\end{aligned} \tag{4.69}$$

From (4.69) and the definition of M_t^n in (4.34)

$$\begin{aligned}
&E\left(e^{x_t^n}\frac{M_{t_j}^n}{M_{t_j-\Delta^n}^n}\middle|\mathcal{F}_{t_j-\Delta^n}^n\right) = \\
&E\left(e^{x_t^n}\middle|\mathcal{F}_{t_j-\Delta^n}^n\right)\left[1+\frac{\left[\widehat{h}^n\left(a_n^T(t_j)\right)-\frac{\hat{h}^n\left(a_n^t(t_j)+a_n^T(t_j)\right)}{\hat{h}^n\left(a_n^t(t_j)\right)}\right]\left[\widehat{h}^n\left(a_n^T(t_j)\right)-\widehat{h}_{t_j}^n\left(a_n^T(t_j)\right)\right]}{\widehat{h}^n\left(2a_n^T(t_j)\right)-\widehat{h}^n\left(a_n^T(t_j)\right)^2}\right]
\end{aligned} \tag{4.70}$$

Repeated applications of (4.70) then allows us to evaluate

$$E\left(e^{x_t^n}\frac{M_t^n}{M_0^n}\middle|\mathcal{F}_0^n\right) =$$
$$E\left(E\left(e^{x_t^n}\frac{M_t^n}{M_{t-\Delta^n}^n}\middle|\mathcal{F}_{t-\Delta^n}^n\right)\frac{M_{t-\Delta^n}^n}{M_0^n}\middle|\mathcal{F}_0^n\right) =$$
$$E\left(\ldots E\left(E\left(e^{x_t^n}\frac{M_t^n}{M_{t-\Delta^n}^n}\middle|\mathcal{F}_{t-\Delta^n}^n\right)\frac{M_{t-\Delta^n}^n}{M_{t-2\Delta^n}^n}\middle|\mathcal{F}_{t-2\Delta^n}^n\right)\ldots\frac{M_{\Delta^n}^n}{M_0^n}\middle|\mathcal{F}_0^n\right)$$

completing the proof.

4.6 Option Pricing

This section explores the special case of pricing European call and put options. In both cases the security underlying the derivative is a T-maturity discount bond. The derivative is a contract to buy (call) or sell (put) the underlying at time t, obviously with $0 < t < T$, at strike price k. The object is to determine the price of these derivative contracts at time 0.

The option contracts are considered to be be European in the sense that the holder can exercise the right to call or put on the expiration date. This is distinct from American options where the holder can exercise at any time prior to expiration as well.

For the special case of a European call option to buy a T-discount bond at time t and strike price k the discounted pay-off is $\phi_t^n = \left(Z_t^{T,n} - ke^{-x_t^n}\right)^+$. Using (4.12) in theorem 4.2.2 and the definition of $Z_t^{T,n}$ in (4.26), this pay-off can be written as

$$\left(Z_t^{T,n} - e^{-x_t^n}k\right)^+ =$$
$$\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^T(\Delta^n)\right)\Lambda_{N_n(t)}^{N_n(T)}\exp\left(-\sum_{j=1}^{N_n(t)}a_n^T(j\Delta^n)X_{j\Delta^n}\right)\mathbf{1}\left(C^n\right) -$$
$$\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^t(\Delta^n)\right)k\exp\left(-\sum_{j=1}^{N_n(t)}a_n^t(j\Delta^n)X_{j\Delta^n}\right)\mathbf{1}\left(C^n\right) \qquad (4.71)$$

where $\mathbf{1}(\cdot)$ is the indicator function which equals 1 when the condition in its argument is met and 0 otherwise and $\mathcal{C}^n$ is the event that the call option is in-the-money:

$$\mathcal{C}^n = \left\{ Z_t^{T,n} - e^{-x_t^n} k \geq 0 \right\} =$$
$$\left\{ \sum_{j=1}^{N_n(t)} \left(a_n^T(j\Delta^n) - a_n^t(j\Delta^n) \right) X_{j\Delta^n} \leq \log \frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n \left(a_n^T(0) - a_n^t(0) \right) \right\} \quad (4.72)$$

Likewise, the pay-off of a put option can be written as

$$\left(e^{-x_t^n} k - Z_t^{T,n} \right)^+ =$$
$$\exp\left(-\lambda_0^n (1-\alpha\Delta^n) a_n^t(\Delta^n)\right) k \exp\left(- \sum_{j=1}^{N_n(t)} a_n^t(j\Delta^n) X_{j\Delta^n} \right) \mathbf{1}\left(\mathcal{P}^n\right) -$$
$$\exp\left(-\lambda_0^n (1-\alpha\Delta^n) a_n^T(\Delta^n)\right) \Lambda_{N_n(t)}^{N_n(T)} \exp\left(- \sum_{j=1}^{N_n(t)} a_n^T(j\Delta^n) X_{j\Delta^n} \right) \mathbf{1}\left(\mathcal{P}^n\right) \quad (4.73)$$

where $\mathcal{P}^n$ is the event that the put option is in-the-money:

$$\mathcal{P}^n = \left\{ e^{-x_t^n} k - Z_t^{T,n} \geq 0 \right\} =$$
$$\left\{ \sum_{j=1}^{N_n(t)} \left(a_n^T(j\Delta^n) - a_n^t(j\Delta^n) \right) X_{j\Delta^n} \geq \log \frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n \left(a_n^T(0) - a_n^t(0) \right) \right\} \quad (4.74)$$

In what follows we base our notation on Duffie, Pan and Singleton [19], which was touched on in subsection 3.3.1 above. For any real y and step functions $a_n, b_n : [0, t^*] \to \mathbb{R}$, let $G(y; a_n, b_n, t)$ denote the price of a security that pays

$$\exp\left(- \sum_{j=1}^{N_n(t)} X_{j\Delta^n} a_n(j\Delta^n) \right)$$

at time $t \in [0, t^*]$ in the event that

$$\sum_{j=1}^{N_n(t)} X_{j\Delta^n} b_n(j\Delta^n) \leq y$$

Then, using (4.71) and (4.72), the price at time 0 of a European call option to buy a T-discount bond at time t and strike price k is

$$
\begin{aligned}
&C_0^n(k,t) = \\
&\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^T(\Delta^n)\right)\Lambda_{N_n(t)}^{N_n(T)} G\left(\log\frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n\left(a_n^T(0)-a_n^t(0)\right); a_n^T, a_n^T - a_n^t, t\right) - \\
&\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^t(\Delta^n)\right) kG\left(\log\frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n\left(a_n^T(0)-a_n^t(0)\right); a_n^t, a_n^T - a_n^t, t\right)
\end{aligned}
\tag{4.75}
$$

Similarly, using (4.73) and (4.74), the price at time 0 of a European put option to sell a T-discount bond at time t and strike price k is

$$
\begin{aligned}
&P_0^n(k,t) = \\
&\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^t(\Delta^n)\right) kG\left(-\left(\log\frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n\left(a_n^T(0)-a_n^t(0)\right)\right); a_n^t, -\left(a_n^T - a_n^t\right), t\right) - \\
&\exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^T(\Delta^n)\right)\Lambda_{N_n(t)}^{N_n(T)} G\left(-\left(\log\frac{\Lambda_{N_n(t)}^{N_n(T)}}{k} - \lambda_0^n\left(a_n^T(0)-a_n^t(0)\right)\right); a_n^T, -\left(a_n^T - a_n^t\right), t\right)
\end{aligned}
\tag{4.76}
$$

In order to evaluate the option prices in (4.75) and (4.76) we will need to obtain an expression for $G(y; a_n, b_n, t)$. From theorem 4.4.1 we have

$$
G(y; a_n, b_n, t) = E\left[\exp\left(-\sum_{j=1}^{N_n(t)} X_{j\Delta^n} a_n(j\Delta^n)\right) \mathbf{1}\left(\sum_{j=1}^{N_n(t)} X_{j\Delta^n} b_n(j\Delta^n) \leq y\right) \frac{M_t^n}{M_0^n} \middle| \mathcal{F}_0^n\right]
\tag{4.77}
$$

The main result of this section is a formula for the expectation in (4.77), presented in theorem 4.6.1. In order to evaluate the integral in (4.77) we rely heavily on an important insight similar in spirit to that in Duffie, Pan and Singleton (see [19], equation (2.10) and following). This insight is captured by lemma 4.6.3, which states that integrals of a particular form are easily evaluated in the Laplace transform domain. It is, of course, necessary to invert the results back to the real world, which in our case involves resorting to numerical Laplace transform inversion methods such as those discussed in section 3.3.

First, we present the main result of this section:

Theorem 4.6.1 *Assume that the increments $X_{j\Delta^n}$ in the short rate model*

(4.5) for $j = 1, 2, \ldots, n$ are independently distributed with identical density function $h^n(y)$ with support on $\mathbb{R}$, so that its (possibly bilateral) Laplace transform is $\hat{h}^n(\nu)$. Also, let $G(y; a_n, b_n, t)$ be as defined above. Then for $y \geq 0$

$$G(y; a_n, b_n, t) = L_\nu^{-1}\left\{\frac{1}{\nu}\prod_{j=1}^{N_n(t)}\left[A(t_j)\widehat{h}^n(a_n(t_j)+\nu b_n(t_j))+B(t_j)\widehat{h}^n\left(a_n(t_j)+a_n^T(t_j)+\nu b_n(t_j)\right)\right]\right\}(y) \tag{4.78}$$

$$G(-y; a_n, -b_n, t) = G(\infty; a_n, b_n, t) - G(y; a_n, b_n, t) \tag{4.79}$$

and

$$G(\infty; a_n, b_n, t) = \prod_{j=1}^{N_n(t)}\left[A(t_j)\widehat{h}^n(a_n(t_j)) + B(t_j)\widehat{h}^n\left(a_n(t_j) + a_n^T(t_j)\right)\right] \tag{4.80}$$

where $A(t)$ and $B(t)$ are defined as

$$A(t) = \frac{\widehat{h}^n\left(2a_n^T(t)\right) - \widehat{h}_t^n\left(a_n^T(t)\right)\widehat{h}^n\left(a_n^T(t)\right)}{\widehat{h}^n(2a_n^T(t)) - \widehat{h}^n(a_n^T(t))^2} \tag{4.81}$$

and

$$B(t) = \frac{\widehat{h}^n\left(a_n^T(t)\right) - \hat{h}_t^n\left(a_n^T(t)\right)}{\widehat{h}^n(2a_n^T(t)) - \widehat{h}^n(a_n^T(t))^2} \tag{4.82}$$

Proof: First, note that from the definition of $Z_t^{T,n}$ in (4.26)

$$\frac{Z_t^{T,n}}{Z_{t-\Delta^n}^{T,n}} = \frac{\exp\left(-a_n^T(t)X_t^n\right)}{\hat{h}_t^n(a_n^T(t))} \tag{4.83}$$

From the definition of the martingale M_t^n in (4.34) it is then easy to see that

$$M_t^n = \prod_{j=1}^{N_n(t)} \left[A(t_j) - e^{-a_n^T(t_j)X_{t_j}^n} B(t_j)\right] \tag{4.84}$$

with $A(t_j)$ and $B(t_j)$ given in (4.81) and (4.82), respectively. We can then evaluate the expectation in (4.77) as

$$\begin{aligned} &E\left[\exp\left(-\sum_{j=1}^{N_n(t)} X_{j\Delta^n a_n(j\Delta^n)}\right) \mathbf{1}\left(\sum_{j=1}^{N_n(t)} X_{t_j}^n b_n(t_j) \le y\right) \frac{M_t^n}{M_0^n}\middle| \mathcal{F}_0^n\right] = \\ &E\left[\mathbf{1}\left(\sum_{j=1}^{N_n(t)} X_{t_j}^n b_n(t_j) \le y\right) \prod_{j=1}^{N_n(t)} \left[e^{-a_n(t_j)X_{t_j}^n} A(t_j) - e^{-\left(a_n(t_j)+a_n^T(t_j)\right)X_{t_j}^n} B(t_j)\right]\middle| \mathcal{F}_0^n\right] \end{aligned} \tag{4.85}$$

An application of lemma 4.6.3 below then leads to the result in (4.78).

To prove (4.79), note that

$$\begin{aligned} &G(-y; a_n, -b_n, t) = \\ &E\left[\left(1 - \mathbf{1}\left(\sum_{j=1}^{N_n(t)} X_{t_j}^n b_n(t_j) \le y\right)\right) \exp\left(-\sum_{j=1}^{N_n(t)} X_{t_j}^n a_n(t_j)\right) \frac{M_t^n}{M_0^n}\middle| \mathcal{F}_0^n\right] = \\ &E\left[\exp\left(-\sum_{j=1}^{N_n(t)} X_{t_j}^n a_n(t_j)\right) \frac{M_t^n}{M_0^n}\middle| \mathcal{F}_0^n\right] - G(y; a_n, b_n, t) \end{aligned} \tag{4.86}$$

The first term on the RHS of the second equality in (4.86) may be defined as $G(\infty; a_n, b_n, t)$, since setting $y = \infty$ is equivalent to removing the restriction imposed by the indicator function. This yields (4.79). The expression for $G(\infty; a_n, b_n, t)$ in (4.80) now follows trivially. This completes the proof of theorem 4.6.1.

The formula in (4.79) allows us to rewrite the formula for the price of a put option in (4.76) as the price of a call option minus the price of a forward, a contract that obliges the purchase of the underlying security on a certain date, at a certain strike price. This result is known as put-call parity (see Baxter and Rennie [5]) and is presented in

Corollary 4.6.2 (Put-call parity) *The put option pricing formula in (4.76)* *is equivalent to*

$$P_0^n(k,t) = C_0^n(k,t) - F_0^n(k,t) \tag{4.87}$$

where $F_0^n(k,t)$ denotes the price of a forward contract, the obligation to buy a T-discount bond at time t and strike price k and is given by

$$\begin{aligned} F_0^n(k,t) &= \exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^T(\Delta^n)\right)\Lambda_{N_n(t)}^{N_n(T)}G\left(\infty;a_n^T,a_n^T-a_n^t,t\right) - \\ &\quad \exp\left(-\lambda_0^n(1-\alpha\Delta^n)a_n^t(\Delta^n)\right)kG\left(\infty;a_n^t,a_n^T-a_n^t,t\right) \end{aligned} \tag{4.88}$$

It is easy to verify (see, for example, (4.71)) that the pay-off of the forward contract at expiration is $\phi_t^n = Z_t^{T,n} - ke^{-x_t^n}$.

In the proof of theorem 4.6.1 we made use of the following

Lemma 4.6.3 *Let real $y, d_j, A_j, B_j, \alpha_j, \beta_j \geq 0$ and let X_j be independent and identically distributed random variables with density function h with support on $\mathbb{R}$ and with Laplace transform $\widehat{h}$, for $j = 1, \ldots, N$. Then*

$$\begin{aligned} &\int \prod_{j=1}^{N} \left(A_j e^{-\alpha_j X_j} + B_j e^{-\beta_j X_j}\right) \mathbf{1}\left(\sum_{j=1}^{N} d_j X_j < y\right) \prod_{j=1}^{N} h(X_j)\, dX_j = \\ &L_\nu^{-1}\left\{\frac{1}{\nu}\prod_{j=1}^{N}\left[A_j\widehat{h}(\alpha_j + \nu d_j) + B_j\widehat{h}(\beta_j + \nu d_j)\right]\right\}(y) \end{aligned} \tag{4.89}$$

where L_ν^{-1} is the inverse (possibly bilateral) Laplace transform operator with respect to ν.

Proof: The indicator function on the LHS of (4.89) can be written as

$$\mathbf{1}\left(\sum_{j=1}^{N} d_j X_j < y\right) = U\left(y - \sum_{j=1}^{N} d_j X_j\right) \tag{4.90}$$

where U is Heavyside's unit function. The Laplace transform of (4.90), as a function of y, is

$$L_y\left\{U\left(y - \sum_{j=1}^{N} d_j X_j\right)\right\}(\nu) = \frac{1}{\nu}\prod_{j=1}^{N} e^{-\nu d_j X_j}$$

The Laplace transform of the entire integral term on the LHS of (4.89), with respect to y, is

$$\frac{1}{\nu}\int\prod_{j=1}^{N}\left[A_j e^{-(\alpha_j+\nu d_j)X_j}+B_j e^{-(\beta_j+\nu d_j)X_j}\right]h(X_j)\,dX_j=$$

$$\frac{1}{\nu}\prod_{j=1}^{N}\left[A_j\widehat{h}(\alpha_j+\nu d_j)+B_j\widehat{h}(\beta_j+\nu d_j)\right] \tag{4.91}$$

Inverting (4.91) yields the desired result.

Note 4.6.4 *To apply Weeks' method for numerical Laplace transform inversion to the transform in (4.89) it is advisable to invert*

$$L_\nu^{-1}\left\{\prod_{j=1}^{N}\left[A_j\widehat{h}(\alpha_j+\nu d_j)+B_j\widehat{h}(\beta_j+\nu d_j)\right]\right\}(y) \tag{4.92}$$

instead. Then (4.89) may be computed as

$$\int_0^y L_\nu^{-1}\left\{\prod_{j=1}^{N}\left[A_j\widehat{h}(\alpha_j+\nu d_j)+B_j\widehat{h}(\beta_j+\nu d_j)\right]\right\}(u)\,du \tag{4.93}$$

Weeks' method may be used to compute the inverse Laplace transforms in (4.93). The integrals can then be computed using integrals of the Laguerre functions as described in (3.23) and below.

4.7 Limit Results

In this section we motivate the choice of model introduced in section 4.2. It will be shown that a special case of λ^n in (4.5) converges in law to the shot-noise process λ presented in section 2.4 for $n \to \infty$. Also, a slightly more general version is shown to converge in law to the diffusion shot noise process in section 2.6. Both these results are proved by evaluating the limit of the Laplace transforms of the conditional density functions of these processes. The model presented in this chapter is much more general than either of those special cases, however. Resorting to a discrete-time framework is the compromise struck in part to achieve this generality.

We will see that other results from the continuous-time theory also have parallels in the discrete-time framework in this chapter. The bond price derived in (4.26) is a discrete-time approximation to the one derived in (3.1). On the other hand, the pricing results developed in the context of the mean-variance hedging framework, especially the option prices, have no known parallel derivation in continuous time. These pricing results were the main motivation for abandoning the continuous-time framework. It is, of course, possible to evaluate the limit of these pricing results for $n \to \infty$, but this exercise is lengthy and was not found to offer any additional insight and hence has been omitted.

The first result in this section unveils the form of the Laplace transform in (4.11) when $n \to \infty$. A restriction on $\hat{h}^n$ is required to guarantee decorum in the limit.

Theorem 4.7.1 *Let λ_t^n and x_t^n be as defined in (4.5) and (4.7) above. Furthermore, let $\hat{h}^n$ be of the form*

$$\widehat{h}^n(\nu) = 1 + \Delta^n \widehat{\gamma}^n(\nu) + o(\Delta^n) \tag{4.94}$$

where $\widehat{\gamma}^n$ is a Laplace transform. Then the limit of the Laplace transform in (4.11) for $n \to \infty$ is

$$\begin{aligned} &\lim_{n\to\infty} E\left[\exp\left(-\nu_1\left(x_{\tau_2}^n - x_{\tau_1}^n\right) - \nu_2\lambda_{\tau_2}^n\right)\middle| \mathcal{F}_{\tau_1}^n\right] = \\ &\exp\left(-\left(\nu_1 \frac{1-e^{-\alpha(\tau_2-\tau_1)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2-\tau_1)}\right)\lambda_{\tau_1}\right) \times \\ &\exp\left[\int_{\tau_1}^{\tau_2} \widehat{\gamma}\left(\nu_1 \frac{1-e^{-\alpha(\tau_2-s)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2-s)}\right) ds\right] \end{aligned} \tag{4.95}$$

Proof: Define the function

$$\eta^n(u; \nu_1, \nu_2, \tau_2) = \nu_1 \frac{1-(1-\alpha\Delta^n)^{\frac{\tau_2^n - u + \Delta^n}{\Delta^n}}}{\alpha} + \nu_2 (1-\alpha\Delta^n)^{\frac{\tau_2^n - u}{\Delta^n}} \tag{4.96}$$

so that

$$\eta^n(j\Delta^n; \nu_1, \nu_2, \tau_2) = \nu_1 \frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-j+1}}{\alpha} + \nu_2 (1-\alpha\Delta^n)^{N_n(\tau_2)-j} \tag{4.97}$$

Note that

$$\lim_{n\to\infty}(1-\alpha\Delta^n)^{N_n(t)-N_n(\tau)+1} = \lim_{n\to\infty}(1-\alpha\Delta^n)^{\frac{t^n-\tau^n+\Delta^n}{\Delta^n}} = e^{-\alpha(t-\tau)} \tag{4.98}$$

It then follows that

$$\lim_{n\to\infty}\eta^n(u;\nu_1,\nu_2,\tau_2) = \eta(u;\nu_1,\nu_2,\tau_2) = \nu_1\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha}+\nu_2 e^{-\alpha(\tau_2-u)} \tag{4.99}$$

It also follows that the limit of first term on the RHS of (4.11) is

$$\lim_{n\to\infty}\exp\left(-\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n\left(\tau_2-\tau_1^n\right)+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n\left(\tau_2-\tau_1^n\right)}\right)\lambda^n_{\tau_1}\right) = \exp\left(-\left(\nu_1\frac{1-e^{-\alpha(\tau_2-\tau_1)}}{\alpha}+\nu_2 e^{-\alpha(\tau_2-\tau_1)}\right)\lambda_{\tau_1}\right) \tag{4.100}$$

Using (4.94) and (4.97), the second term on the RHS of (4.11) can be written as

$$\begin{aligned}&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\hat{h}^n\left(\eta^n\left(i\Delta^n;\nu_1,\nu_2,\tau_2\right)\right) = \\ &\exp\left[\sum_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\log\left(1+\Delta^n\widehat{\gamma}^n(\eta^n(i\Delta^n;\nu_1,\nu_2,\tau_2))+o(\Delta^n)\right)\right] = \\ &\exp\left[\Delta^n\sum_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\widehat{\gamma}^n(\eta^n(i\Delta^n;\nu_1,\nu_2,\tau_2))+\sum_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}o(\Delta^n)\right] = \\ &\exp\left[\int_{\tau_1+\Delta^n}^{\tau_2}\widehat{\gamma}^n(\eta^n(u;\nu_1,\nu_2,\tau_2))du+O\left(\tfrac{1}{n}\right)\left\{\tau_2^n-\tau_1^n-\Delta^n\right\}\right] \to \\ &\exp\left[\int_{\tau_1}^{\tau_2}\widehat{\gamma}\left(\eta\left(u;\nu_1,\nu_2,\tau_2\right)\right)du\right]\end{aligned} \tag{4.101}$$

Combining (4.100) with (4.101) leads to the desired result in (4.95).

The next two results follow readily from theorem 4.7.1. They show that two special cases of the discrete-time model presented in this chapter converge in law their continuous-time counterparts discussed in chapter 2.

Corollary 4.7.2 *Let λ_t^n and x_t^n be as defined in (4.5) and (4.7) above. Fur-*

thermore, let h^n be of the form

$$h^n(y) = p\Delta^n g(y) + (1 - p\Delta^n)\,\delta(y) \tag{4.102}$$

where g is a probability density function with Laplace transform $\hat{g}$. Then, as $n \to \infty$, the finite dimensional distributions of (λ_t^n, x_t^n) converge to those of (λ_t, x_t), the continuous-time shot noise process with constant decay and its aggregated process, introduced in section 2.4.

Proof: The Laplace transform of h^n in (4.102) is of the same form as (4.94) with

$$\widehat{\gamma}^n(\nu) = p\,(\widehat{g}^n(\nu) - 1) \tag{4.103}$$

From theorem 4.7.1 it then follows that

$$\begin{aligned} &E\left[\exp\left(-\nu_1\left(x_{\tau_2}^n - x_{\tau_1}^n\right) - \nu_2\lambda_{\tau_2}^n\right)\Big|\,\mathcal{F}_{\tau_1}^n\right] \to \\ &\exp\left(-\left(\nu_1\frac{1-e^{-\alpha(\tau_2-\tau_1)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2-\tau_1)}\right)\lambda_{\tau_1}\right) \times \\ &\exp\left\{p\int_{\tau_1}^{\tau_2}\left[\hat{g}\left(\nu_1\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2-u)}\right) - 1\right]du\right\} \end{aligned} \tag{4.104}$$

Clearly (4.104) has the same form as (2.13), the Laplace transform of the conditional joint density of the continuous-time shot noise process and its aggregated process as proved in Jang [31] and Dassios and Jang [11].

Let $\Pi^n\left(\lambda_{\tau_2}^n, x_{\tau_2}^n\middle|\,\lambda_{\tau_1}^n, x_{\tau_1}^n\right)$ denote the joint conditional distribution of $\lambda_{\tau_2}^n$ and $x_{\tau_2}^n$, given $\lambda_{\tau_1}^n$ and $x_{\tau_1}^n$. Similarly, let $\Pi\left(\lambda_{\tau_2}, x_{\tau_2}\middle|\,\lambda_{\tau_1}, x_{\tau_1}\right)$ denote the joint conditional distribution of λ_{τ_2} and x_{τ_2}, given λ_{τ_1} and x_{τ_1}. Then from (4.104) it follows that

$$\lim_{n\to\infty}\Pi^n\left(\lambda_{\tau_2}^n, x_{\tau_2}^n\middle|\,\lambda_{\tau_1}^n, x_{\tau_1}^n\right) = \Pi\left(\lambda_{\tau_2}, x_{\tau_2}\middle|\,\lambda_{\tau_1}, x_{\tau_1}\right)$$

where λ_t^n is the discrete-time process with characteristic $(p, \alpha, g(\cdot))$ and λ_t is the continuous-time process with characteristic $(\rho, \delta, g(\cdot))$, with $p = \rho$ and $\alpha = \delta$.

Let $\Pi^n\left(\lambda^n_{\tau_1}, x^n_{\tau_1}, \ldots, \lambda^n_{\tau_N}, x^n_{\tau_N}\right)$ denote the joint distribution of $\lambda^n_{\tau_1}, x^n_{\tau_1}, \ldots,$ $\lambda^n_{\tau_N}, x^n_{\tau_N}$ for any $\tau_1 < \ldots < \tau_N \leq t^*$ and $N > 0$. Define $\Pi\left(\lambda_{\tau_1}, x_{\tau_1}, \ldots, \lambda_{\tau_N}, x_{\tau_N}\right)$ similarly. Then by the joint Markov property of (λ^n_t, x^n_t) and (λ_t, x_t) we have

$$\prod_{i=1}^{N} \Pi^n\left(\lambda^n_{\tau_i}, x^n_{\tau_i} \middle| \lambda^n_{\tau_{i-1}}, x^n_{\tau_{i-1}}\right) = \Pi^n\left(\lambda^n_{\tau_1}, x^n_{\tau_1}, \ldots, \lambda^n_{\tau_N}, x^n_{\tau_N}\right)$$

and

$$\prod_{i=1}^{N} \Pi\left(\lambda_{\tau_i}, x_{\tau_i} \middle| \lambda_{\tau_{i-1}}, x_{\tau_{i-1}}\right) = \Pi\left(\lambda_{\tau_1}, x_{\tau_1}, \ldots, \lambda_{\tau_N}, x_{\tau_N}\right)$$

where we define $\tau_0 \equiv 0$. Finally, it follows that

$$\lim_{n\to\infty} \Pi^n\left(\lambda^n_{\tau_1}, x^n_{\tau_1}, \ldots, \lambda^n_{\tau_N}, x^n_{\tau_N}\right) = \Pi\left(\lambda_{\tau_1}, x_{\tau_1}, \ldots, \lambda_{\tau_N}, x_{\tau_N}\right)$$

This concludes the proof.

The next result demonstrates equivalence with the diffusion shot noise process in section 2.6:

Corollary 4.7.3 *Let λ^n_t and x^n_t be as defined in (4.5) and (4.7) above. Furthermore, let h^n be of the form*

$$h^n(y) = \left(h^n_1 * h^n_2\right)(y) \tag{4.105}$$

where $*$ denotes the convolution operator, let

$$h^n_1(y) = p\Delta^n g(y) + (1 - p\Delta^n)\,\delta(y) \tag{4.106}$$

with the Laplace transform of g continuous and bounded and let

$$h^n_2(y) = \frac{1}{\sqrt{2\pi\Delta^n}\sigma} \exp\left(-\frac{y^2}{2\Delta^n\sigma^2}\right) \tag{4.107}$$

Then, as $n \to \infty$, the finite dimensional distributions of (λ^n_t, x^n_t) converge to those of (λ_t, x_t), the continuous-time shot noise process with constant decay, perturbed by Brownian motion and its aggregated process as described in section 2.6.

Proof: We will prove that the limit of the joint Laplace transform in (4.11) is of the form (2.41) as $n \to \infty$, when h^n is of the form (4.105). First note that the LT of h^n is

$$\hat{h}^n(\nu) = \hat{h}_1^n(\nu)\,\hat{h}_2^n(\nu) \tag{4.108}$$

with

$$\hat{h}_1^n(\nu) = p\Delta^n \hat{g}(\nu) + (1 - p\Delta^n) \tag{4.109}$$

and

$$\hat{h}_2^n(\nu) = \exp\left(-\frac{\Delta^n \sigma^2 \nu^2}{2}\right) \tag{4.110}$$

From (4.108) it is clear that the joint Laplace Transform of $\lambda_{\tau_2}^n$ and $x_{\tau_2}^n$, given $\mathcal{F}_{\tau_1}^n$ in (4.11) takes on the form

$$\begin{aligned}
&E\left[\exp\left(-\nu_1\left(x_{\tau_2}^n - x_{\tau_1}^n\right) - \nu_2\lambda_{\tau_2}^n\right)\Big|\,\mathcal{F}_{\tau_1}^n\right] = \\
&\exp\left(-\left(\nu_1\frac{1-\alpha\Delta^n-(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-N_n(\tau_1)}\right)\lambda_{\tau_1}^n\right) \times \\
&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}_1^n\left(\nu_1\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i}\right) \times \\
&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}_2^n\left(\nu_1\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i}\right)
\end{aligned} \tag{4.111}$$

The limits of the first two terms on the RHS of (4.111), as $n \to \infty$ were already determined in corollary 4.7.2. The limit of the third term is

$$\begin{aligned}
&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \exp\left(-\tfrac{\Delta^n\sigma^2}{2}\left(\nu_1\frac{1-(1-\alpha\Delta^n)^{N_n(\tau_2)-i+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i}\right)^2\right) = \\
&\exp\left(-\tfrac{\sigma^2\Delta^n}{2}\sum_{i=N_n(\tau_1)+1}^{N_n(\tau_2)}\left(\nu_1\frac{1-(1-\alpha\Delta^n)^{\frac{\tau_2^n-i\Delta^n}{\Delta^n}+1}}{\alpha}+\nu_2(1-\alpha\Delta^n)^{\frac{\tau_2^n-i\Delta^n}{\Delta^n}}\right)^2\right) \to \\
&\exp\left(-\tfrac{\sigma^2}{2}\int_{\tau_1}^{\tau_2}\left(\nu_1\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha}+\nu_2 e^{-\alpha(\tau_2-u)}\right)^2 du\right)
\end{aligned} \tag{4.112}$$

Combining the result in (4.112) with that in (4.104) yields

$$
\begin{aligned}
&E\left[\exp\left(-\nu_1\left(x^n_{\tau_2}-x^n_{\tau_1}\right)-\nu_2\lambda^n_{\tau_2}\right)\Big|\mathcal{F}^n_{\tau_1}\right]\to\\
&\exp\left(-\left(\nu_1\frac{1-e^{-\alpha(\tau_2-\tau_1)}}{\alpha}+\nu_2e^{-\alpha(\tau_2-\tau_1)}\right)\lambda_{\tau_1}\right)\times\\
&\exp\left\{p\int_{\tau_1}^{\tau_2}\left[\hat{g}\left(\nu_1\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha}+\nu_2e^{-\alpha(\tau_2-u)}\right)-1\right]du\right\}\times\\
&\exp\left[-\frac{\sigma^2}{2}\int_{\tau_1}^{\tau_2}\left(\nu_1\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha}+\nu_2e^{-\alpha(\tau_2-u)}\right)^2du\right]
\end{aligned}
\tag{4.113}
$$

Clearly (4.113) has the same form as (2.37).

A similar argument to that used in the proof of corollary 4.7.2 then allows us to conclude that the finite dimensional distributions of the discrete-time model with characteristic $(p,\alpha,g(\cdot),\sigma)$ converge to those of the continuous-time diffusion shot noise process with characteristic $(\rho,\delta,g(\cdot),\sigma)$, where $p=\rho$ and $\alpha=\delta$, as the sampling frequency increases. This concludes the proof.

The next theorem is the equivalent martingale measure version of theorem 4.7.1.

Theorem 4.7.4 *Let λ^n_t and x^n_t be as defined in (4.5) and (4.7) above. Furthermore, let $\hat{h}^n$ be of the form given in (4.94). Then the limit of the equivalent martingale measure Laplace transform in (4.18) for $n\to\infty$ is*

$$
\begin{aligned}
&\lim_{n\to\infty}E^*\left[e^{-\nu_1x^n_{\tau_2}-\nu_2\lambda^n_{\tau_2}}\Big|\mathcal{F}^n_{\tau_1}\right]=\\
&e^{-\nu_1x_{\tau_1}}\exp\left(-\left(\nu_1\frac{1-e^{-\alpha(\tau_2-\tau_1)}}{\alpha}+\nu_2e^{-\alpha(\tau_2-\tau_1)}\right)\lambda_{\tau_1}\right)\times\\
&\exp\left\{\int_{\tau_1}^{\tau_2}\left[\widehat{\gamma}\left(\nu_1a^{\tau_2}(u)+\nu_2e^{-\alpha(\tau_2-u)}+b(u)\right)-\widehat{\gamma}(b(u))\right]du\right\}
\end{aligned}
\tag{4.114}
$$

where

$$
a^{\tau_2}(u)=\frac{1-e^{-\alpha(\tau_2-u)}}{\alpha}\tag{4.115}
$$

and

$$b(u) = \frac{c - \frac{\nu}{\alpha}}{e^{-\alpha u}} + \frac{\nu}{\alpha} \tag{4.116}$$

Proof: It follows from (4.19) that

$$\lim_{n\to\infty} a_n^{\tau_2}(u) = a^{\tau_2}(u)$$

and from (4.20) that

$$\lim_{n\to\infty} b_n(u) = b(u)$$

Define the function

$$v_n^{\tau_2}(u) = \nu_1 a_n^{\tau_2}(u) + \nu_2 (1 - \alpha\Delta^n)^{\frac{\tau_2^n - u^n}{\Delta^n}} + b_n(u) \tag{4.117}$$

It is easy to see that

$$v_n^{\tau_2}(u) \to v^{\tau_2}(u) = \nu_1 a^{\tau_2}(u) + \nu_2 e^{-\alpha(\tau_2 - u)} + b(u) \tag{4.118}$$

Then, similar to the proof of theorem 4.7.1, we have

$$\begin{aligned}
&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(\nu_1 a_n^{\tau_2}(i\Delta^n) + \nu_2(1-\alpha\Delta^n)^{N_n(\tau_2)-i} + b_n(i\Delta^n)\right) = \\
&\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(v_n^{\tau_2}(i\Delta^n)\right) \to \exp\left[\int_{\tau_1}^{\tau_2} \hat{\gamma}\left(v^{\tau_2}(u)\right) du\right] = \\
&\exp\left[\int_{\tau_1}^{\tau_2} \hat{\gamma}\left(\nu_1 a^{\tau_2}(u) + \nu_2 e^{-\alpha(\tau_2-u)} + b(u)\right) du\right]
\end{aligned} \tag{4.119}$$

and

$$\prod_{i=N_n(\tau_1)+1}^{N_n(\tau_2)} \hat{h}^n\left(b_n(i\Delta^n)\right) \to \exp\left[\int_{\tau_1}^{\tau_2} \hat{\gamma}\left(b(u)\right) du\right] \tag{4.120}$$

Substituting (4.119) and (4.120) into (4.18) completes the proof.

Corollary 4.7.5 *Let λ_t^n and x_t^n be as defined in (4.5) and (4.7) above. Fur-*

thermore, let h^n be of the form given in (4.102). Then the limit of the equivalent martingale measure Laplace transform in (4.18) for $n \to \infty$ is

$$
\begin{aligned}
&\lim_{n\to\infty} E^* \left[e^{-\nu_1 x^n_{\tau_2} - \nu_2 \lambda^n_{\tau_2}} \middle| \mathcal{F}^n_{\tau_1} \right] = \\
&e^{-\nu_1 x_{\tau_1}} \exp\left(- \left(\nu_1 \frac{1 - e^{-\alpha(\tau_2 - \tau_1)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2 - \tau_1)} \right) \lambda_{\tau_1} \right) \times \\
&\exp\left[p \int_{\tau_1}^{\tau_2} \left(\widehat{g}\left(\nu_1 \tfrac{1 - e^{-\alpha(\tau_2 - u)}}{\alpha} + \nu_2 e^{-\alpha(\tau_2 - u)} + \tfrac{c - \frac{\nu}{\alpha}}{e^{-\alpha u}} + \tfrac{\nu}{\alpha} \right) - \widehat{g}\left(\tfrac{c - \frac{\nu}{\alpha}}{e^{-\alpha u}} + \tfrac{\nu}{\alpha} \right) \right) du \right]
\end{aligned}
\tag{4.121}
$$

Note that (4.121) is equivalent to (2.31) with

$$
\begin{aligned}
\rho &= p \\
\delta &= \alpha \\
\kappa_1 &= \frac{\nu}{\alpha} \\
\kappa_2 &= c - \frac{\nu}{\alpha}
\end{aligned}
\tag{4.122}
$$

Proof: This follows from the same argument as in (4.103), combined with the definitions of $a^{\tau_2}(u)$ and $b(u)$ given in (4.115) and (4.116), respectively.

With corollary 4.7.5 it is easy to tie the discounted bond-price process of the discrete-time model in with that based on the continuous-time shot noise process:

Corollary 4.7.6 *Let $Z_t^{T,n}$ be the discrete-time discounted bond-price process given in (4.26) with the added constraint that h^n is of the form given in (4.102). Furthermore, let $Z(t,T)$ be the continuous-time discounted bond-price process given in (3.1). Then*

$$
\lim_{n\to\infty} Z_t^{T,n} = Z(t,T) \tag{4.123}
$$

Proof: This follows from (3.1), lemma 4.3.3, (4.26) and corollary 4.7.5.

Chapter 5

Numerical Experiments

This chapter presents results from numerical implementations of the various pricing formulæ developed in the thesis. All of these experiments were implemented using S-Plus.

5.1 Continuous-Time Model

The option pricing formula using Weeks' Laplace transform inversion method from (3.28) was implemented and used to compute option prices as a numerical example.

The parameter values used are given in table 5.1. The values for N, M and b were chosen so that the error in the Laplace transform domain was small. Because $\widehat{\Pi}(\nu_1, \nu_2)$ from (3.5) does not have poles for non-negative values of ν_1 and ν_2, σ can be kept small. The computed prices of a European call option to buy a 250-day bond for various strike prices and maturities are presented in table 5.2.

5.2 Discrete-Time Model

In this section the pricing results of the refined discrete-time short rate model (4.5) from chapter 4 are explored. Several special cases of this fairly general model are examined in turn. We start with the discrete-time version of the

Table 5.1: Parameter values used to compute option prices

Parameter	Value
δ	0.02
ρ	0.35
α	125000
κ_1	500
κ_2	500
T	250
λ_0	0.0002
x_0	0
b	100000
σ	0.01
M	1000
N	1000

shot-noise process with exponentially distributed jumps.

5.2.1 Exponential Shot Noise

In this section results are presented for the short rate model where the increment size distribution h^n is defined in (4.102) with exponentially distributed jump-sizes. The parameter values used are given in table 5.3. Note that in table 5.3 the values for ν and c are specified indirectly through κ_1 and κ_2 using (4.122).

Figure 5.1 shows two realizations of the short rate using parameter values from table 5.3 evolved over 1000 time increments, which could be interpreted as 250 days with 4 updates per day.

Figure 5.2 shows two versions of the price process for the discount bond as defined in (4.24) with a maturity of 250 days. The two versions represent the price under an equivalent martingale measure ($\kappa_1 = \kappa_2 = 500$) and the base measure ($\kappa_1 = \kappa_2 = 0$).

Table 5.4 compares results for the pricing model in (4.24) with that in (4.67). Bonds with several maturities are priced. The values of κ_1 and κ_2 are as given in table 5.3. The maturity of the long bond is 250 days. The prices based on (4.67) are generated in two ways: with the pricing formula in

Table 5.2: Exponential (continuous time) shot-noise option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.030905	0.025477	0.020049	0.014621	0.009193	0.003775
96	0.033370	0.027956	0.022543	0.017129	0.011715	0.006301
114	0.035752	0.030352	0.024952	0.019552	0.014152	0.008752
132	0.038058	0.032672	0.027285	0.021899	0.016512	0.011125
150	0.040290	0.034917	0.029543	0.024170	0.018796	0.013422
168	0.042441	0.037080	0.031719	0.026358	0.020997	0.015636
186	0.044497	0.039148	0.033799	0.028450	0.023100	0.017751
204	0.046439	0.041101	0.035763	0.030425	0.025087	0.019749
222	0.048243	0.042915	0.037588	0.032260	0.026933	0.021606
240	0.049883	0.044566	0.039248	0.033930	0.028612	0.023294

Table 5.3: Parameter values used the for discrete-time short rate

Parameter	Value
Δ^n	0.25
α	0.02
p	0.35
β	125000
κ_1	500
κ_2	500
λ_0	0.0002
x_0	0

(4.67) (column "Formula (Opt)") as well as by evaluating the expected value of (4.33) through Monte Carlo simulation (10000 replications), with $\phi_t^n = e^{-x_t^n}$ (column "Simulation").

Option pricing results are displayed in tables 5.5 and 5.6. Both tables display option prices for a European call option on a 250-day discount bond for various strike prices and dates. The prices in table 5.5 are computed using formula (4.75), whereas the prices in table 5.6 are computed by evaluating the expected value of (4.33) through Monte Carlo simulation (10000 replications), with $\phi_t^n = \left(Z_t^{T,n} - ke^{-x_t^n}\right)^+$.

Figure 5.1: Two exponential shot-noise short rate paths

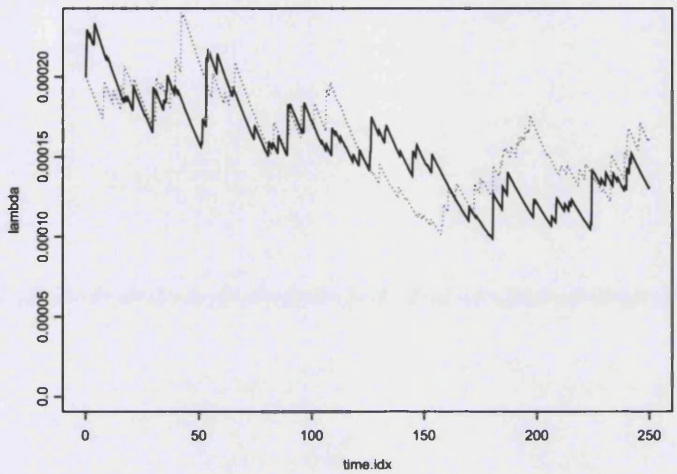

Table 5.4: Exponential shot-noise bond prices

Maturity	Formula	Formula (Opt)	Simulation
30	0.994484	0.994484	0.994428
60	0.989636	0.989636	0.989579
120	0.980987	0.980987	0.980880
200	0.970942	0.970942	0.970915

5.2.2 Gamma Shot Noise

In this section results are presented for the short rate model where the increment size distribution h^n is defined in (4.102) with gamma distributed jump-sizes, which have Laplace transform

$$\hat{g}(\nu) = \left(\frac{\beta}{\beta + \nu}\right)^r \tag{5.1}$$

The parameter values used are given in table 5.7.

Figure 5.3 shows two realizations of the short rate using parameter values from table 5.7 evolved over 1000 time increments, which could be thought of as 250 days with 4 updates per day.

Figure 5.2: Exponential shot-noise bond price process: Under the equivalent martingale measure (solid) and the base measure (dotted)

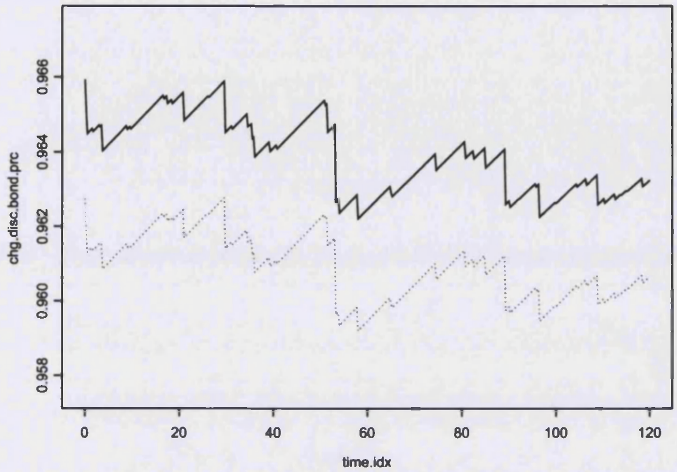

Figure 5.4 shows two versions of the price process for the discount bond as defined in (4.24) with a maturity of 250 days. The two versions represent the price under an equivalent martingale measure ($\kappa_1 = \kappa_2 = 500$) and the base measure ($\kappa_1 = \kappa_2 = 0$).

Table 5.8 compares results for the pricing model in (4.24) with that in (4.67). Bonds with several maturities are priced. The values of κ_1 and κ_2 are as given in table 5.3. The maturity of the long bond is 250 days. The prices based on (4.67) are generated in two ways: with the pricing formula in (4.67) (column "Formula (Opt)") as well as by evaluating the expected value of (4.33) through Monte Carlo simulation (10000 replications), with $\phi_t^n = e^{-x_t^n}$ (column "Simulation").

Option pricing results are displayed in tables 5.9 and 5.10. Both tables display option prices for a European call option on a 250-day discount bond for various strike prices and dates. The prices in table 5.9 are computed using formula (4.75), whereas the prices in table 5.10 are computed by evaluating the expected value of (4.33) through Monte Carlo simulation (10000 replications), with $\phi_t^n = \left(Z_t^{T,n} - ke^{-x_t^n}\right)^+$.

Table 5.5: Exponential shot-noise option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.030834	0.025406	0.019978	0.014550	0.009122	0.003703
96	0.033300	0.027886	0.022472	0.017059	0.011645	0.006231
114	0.035684	0.030284	0.024884	0.019484	0.014084	0.008684
132	0.037993	0.032607	0.027220	0.021834	0.016447	0.011061
150	0.040230	0.034856	0.029483	0.024109	0.018736	0.013362
168	0.042387	0.037026	0.031665	0.026304	0.020943	0.015582
186	0.044451	0.039101	0.033752	0.028403	0.023054	0.017705
204	0.046402	0.041064	0.035727	0.030389	0.025051	0.019713
222	0.048218	0.042890	0.037563	0.032236	0.026909	0.021582
240	0.049868	0.044551	0.039235	0.033917	0.028599	0.023287

5.2.3 Double Gamma Short Rate

The model explored in this section illustrates the flexibility of the discrete-time framework developed in chapter 4. The probability distribution of the step-sizes have the following Laplace transform

$$\hat{h^n}(\nu) = p_1\Delta^n\hat{g}_1(\nu) + p_2\hat{g_2^n}(\nu) + (1 - p_1\Delta^n - p_2) \tag{5.2}$$

with

$$\hat{g}_1(\nu) = \left(\frac{\beta_1}{\beta_1 + \nu}\right)^{r_1} \tag{5.3}$$

and

$$\hat{g_2^n}(\nu) = \left(\frac{\frac{\beta_2}{\Delta^n}}{\frac{\beta_2}{\Delta^n} + \nu}\right)^{r_2} \tag{5.4}$$

This specification of step-sizes is a mixture of two gamma distributed jumps. The first jump term in (5.2), $\hat{g}_1$, has essentially the same form as the gamma distributed jump-sizes in subsection 5.2.2, where the jump-size distribution does not vary with n, but the probability of a jump occurring decreases pro-

Table 5.6: Simulated exponential shot-noise option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.030867	0.025443	0.020019	0.014595	0.009171	0.003754
96	0.033259	0.027858	0.022458	0.017057	0.011657	0.006257
114	0.035773	0.030370	0.024967	0.019563	0.014160	0.008757
132	0.038051	0.032665	0.027279	0.021893	0.016507	0.011121
150	0.040344	0.034965	0.029585	0.024205	0.018825	0.013445
168	0.042307	0.036962	0.031617	0.026273	0.020928	0.015583
186	0.044388	0.039052	0.033717	0.028382	0.023046	0.017711
204	0.046865	0.041482	0.036098	0.030714	0.025331	0.019947
222	0.047675	0.042412	0.037149	0.031886	0.026623	0.021360
240	0.049948	0.044627	0.039307	0.033986	0.028665	0.023344

portionately with the time-increments Δ^n. The second jump term has a different form where the probability of a jump occurring in a given time step does not change with its length, whereas the jump-size distribution does. In fact, the mean of the jump-sizes defined in (5.4) is $\Delta^n r_2/\beta_2$. The parameter values chosen to implement the model defined in (5.2), (5.3) and (5.4) are given in table 5.11. Figure 5.5 shows two realizations of the short rate using parameter values from table 5.11 evolved over 1000 time increments.

Figure 5.6 shows two versions of the price process for the discount bond as defined in (4.24) with a maturity of 250 days. The two versions represent the price under an equivalent martingale measure ($\kappa_1 = \kappa_2 = 200$) and the base measure ($\kappa_1 = \kappa_2 = 0$).

Table 5.12 compares results for the pricing model in (4.24) with that in (4.67). Bonds with several maturities are priced. The values of κ_1 and κ_2 are as given in table 5.11. The maturity of the long bond is 250 days. The prices based on (4.67) are generated in two ways: with the pricing formula in (4.67) (column "Formula (Opt)") as well as by evaluating the expected value of (4.67) through Monte Carlo simulation (10000 replications), with $\phi_t^n = e^{-x_t^n}$ (column "Simulation").

Option pricing results are displayed in tables 5.13 and 5.14. Both tables

Table 5.7: Parameter values used the for discrete-time short rate (gamma)

Parameter	Value
Δ^n	0.25
α	0.02
p	0.35
r	3
β	375000
κ_1	500
κ_2	500
λ_0	0.0002
x_0	0

Table 5.8: Gamma shot-noise bond prices

Maturity	Formula	Formula (Opt)	Simulation
30	0.994478	0.994478	0.994430
60	0.989612	0.989612	0.989568
120	0.980881	0.980881	0.980819
200	0.970496	0.970497	0.970447

display option prices for a European call option on a 250-day discount bond for various strike prices and dates. The prices in table 5.13 are computed using formula (4.75), whereas the prices in table 5.14 are computed by evaluating the expected value of (4.33) through Monte Carlo simulation (10000 replications), with $\phi_t^n = \left(Z_t^{T,n} - ke^{-x_t^n}\right)^+$.

Figure 5.3: Two gamma shot-noise short rate paths

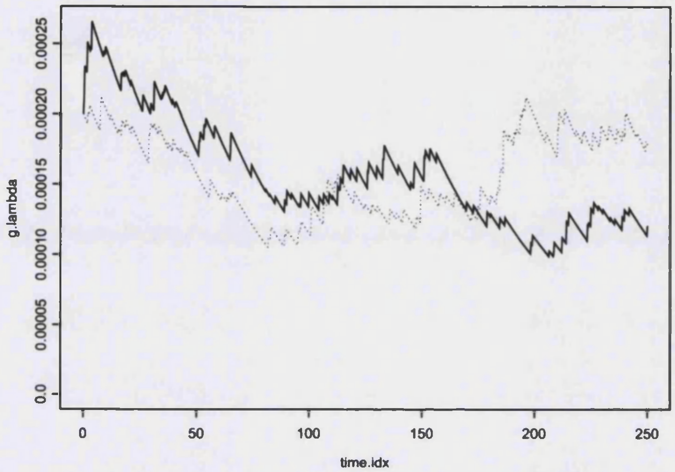

Figure 5.4: Gamma shot-noise bond price process: Under the equivalent martingale measure (solid) and the base measure (dotted)

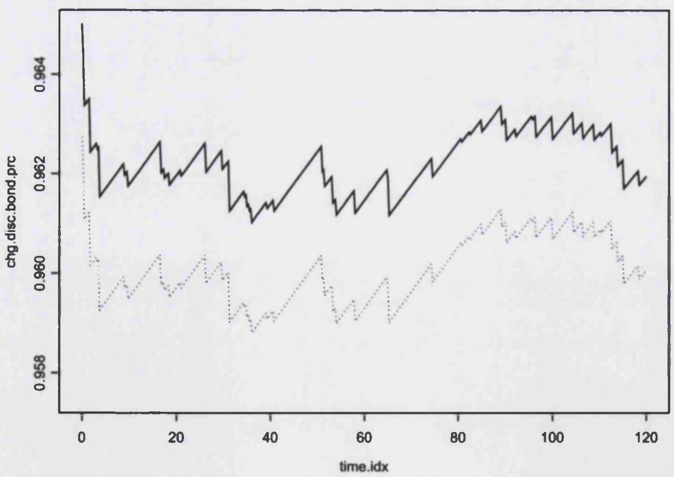

Table 5.9: Gamma shot-noise option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.027503	0.021582	0.015660	0.009739	0.003819	0.000009
96	0.029997	0.024092	0.018186	0.012281	0.006375	0.000750
114	0.032416	0.026526	0.020636	0.014745	0.008855	0.002970
132	0.034771	0.028896	0.023020	0.017145	0.011269	0.005394
150	0.037065	0.031204	0.025343	0.019482	0.013622	0.007761
168	0.039296	0.033449	0.027602	0.021756	0.015909	0.010062
186	0.041454	0.035621	0.029788	0.023955	0.018122	0.012288
204	0.043525	0.037704	0.031884	0.026064	0.020244	0.014424
222	0.045485	0.039677	0.033870	0.028062	0.022254	0.016446
240	0.047302	0.041507	0.035713	0.029921	0.024121	0.018325

Table 5.10: Simulated gamma shot-noise option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.027539	0.021621	0.015704	0.009787	0.003870	0.000011
96	0.030040	0.024138	0.018236	0.012334	0.006432	0.000790
114	0.032450	0.026564	0.020679	0.014793	0.008907	0.003026
132	0.034761	0.028897	0.023033	0.017169	0.011305	0.005441
150	0.036990	0.031148	0.025307	0.019466	0.013624	0.007783
168	0.039361	0.033513	0.027666	0.021818	0.015970	0.010123
186	0.041610	0.035764	0.029917	0.024071	0.018225	0.012378
204	0.043520	0.037709	0.031897	0.026085	0.020274	0.014462
222	0.045955	0.040094	0.034233	0.028372	0.022512	0.016651
240	0.046338	0.040667	0.034995	0.029323	0.023652	0.017980

Table 5.11: Parameter values used the for discrete-time short rate (double gamma)

Parameter	Value
Δ^n	0.25
α	0.02
p_1	0.14
r_1	3
β_1	250000
p_2	0.21
r_2	3
β_2	250000
κ_1	200
κ_2	200
λ_0	0.0002
x_0	0

Figure 5.5: Two double gamma shot-noise short rate paths

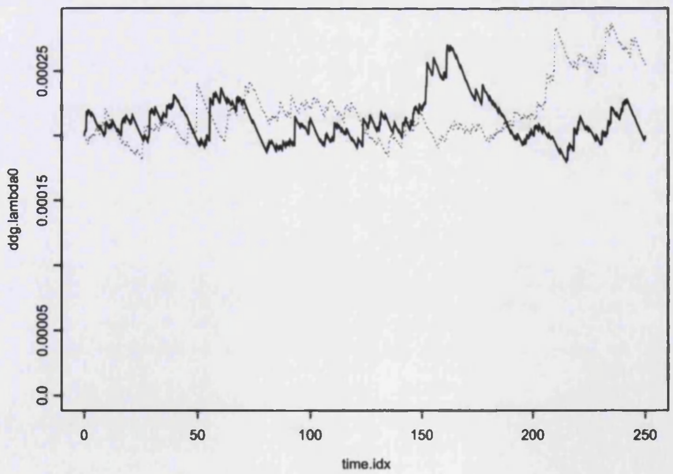

Figure 5.6: Double Gamma shot-noise bond price process: Under the equivalent martingale measure (solid) and the base measure (dotted)

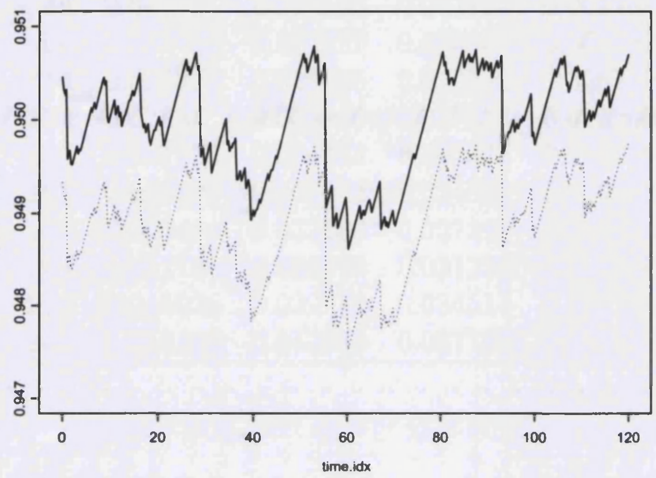

Table 5.12: Double Gamma jump bond prices

Maturity	Formula	Formula (Opt)	Simulation
30	0.993949	0.993949	0.993896
60	0.987845	0.987845	0.987777
120	0.975655	0.975655	0.975592
200	0.959794	0.959794	0.959716

Table 5.13: Double Gamma short-rate option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.017951	0.012538	0.007125	0.001781	0.000000	0.000000
96	0.021420	0.016027	0.010634	0.005242	0.000427	0.000000
114	0.024878	0.019505	0.014132	0.008759	0.003391	0.000017
132	0.028319	0.022966	0.017613	0.012260	0.006908	0.001625
150	0.031738	0.026405	0.021072	0.015739	0.010406	0.005073
168	0.035127	0.029814	0.024501	0.019188	0.013874	0.008561
186	0.038480	0.033186	0.027892	0.022598	0.017305	0.012011
204	0.041784	0.036510	0.031235	0.025960	0.020686	0.015411
222	0.045026	0.039770	0.034514	0.029259	0.024003	0.018747
240	0.048187	0.042949	0.037712	0.032474	0.027237	0.021999

Table 5.14: Simulated Double Gamma short-rate option prices

Strike price / Maturity	0.9475	0.9530	0.9585	0.9640	0.9695	0.9750
78	0.017951	0.012538	0.007125	0.001783	0.000000	0.000000
96	0.021419	0.016027	0.010634	0.005242	0.000427	0.000000
114	0.024866	0.019495	0.014124	0.008753	0.003386	0.000016
132	0.028263	0.022916	0.017570	0.012224	0.006878	0.001602
150	0.031732	0.026400	0.021068	0.015736	0.010404	0.005072
168	0.035201	0.029878	0.024555	0.019233	0.013910	0.008587
186	0.038337	0.033061	0.027785	0.022509	0.017233	0.011957
204	0.041868	0.036583	0.031298	0.026013	0.020728	0.015443
222	0.045002	0.039749	0.034496	0.029243	0.023990	0.018737
240	0.048261	0.043015	0.037770	0.032524	0.027279	0.022033

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