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Structural and decomposition results for binet matrices, bidirected graphs and signed-graphic matroids

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

In this thesis we deal with binet matrices and the class of signed-graphic matroids which is the class of matroids represented over $\mathbb{R}$ by binet matrices. The thesis is divided in three parts. In the first part, we provide the vast majority of the notions used throughout the thesis and some results regarding the class of binet matrices. In this part, we focus on the class of linear and integer programming problems in which the constraint matrix is binet and provide methods and algorithms which solve these problems efficiently. Results of the part regarding the optimization with binet matrices are joint work with G. Appa, B. Kotnyek and L. Pitsoulis and have been published in [5]. The main new result is that the existing combinatorial methods can not solve the $\{0, \frac{1}{2}\}$-separation problem (special case of the well known separation problem) with integral binet matrices.

The main new results of the whole thesis are provided in the next two parts. In the second part, we present a polynomial time algorithm to construct a bidirected graph for any totally unimodular matrix B by finding node-edge incidence matrices Q and S such that $QB = S$. Seymour's famous decomposition theorem for regular matroids states that any totally unimodular matrix can be constructed through a series of composition operations called k-sums starting from network matrices and their transposes and two compact representation matrices B_1 and B_2 of a certain ten element matroid. Given that B_1 and B_2 are binet matrices, we examine the k-sums of network and binet matrices ($k = 1, 2, 3$). It is shown that the k-sum of a network and a binet matrix is a binet matrix, but binet matrices are not closed under this operation for $k = 2, 3$. A new class of matrices is introduced, the so-called tour matrices, which generalises network and totally unimodular matrices. For any such matrix there exists a bidirected graph such that the columns represent a collection of closed tours in the graph. It is shown that tour matrices are closed under 1-, 2- and $\oplus_3$-sum as well as under elementary operations on their rows and columns. Given the constructive proofs of the above results regarding the k-sum operations and existing recognition algorithms for network and binet matrices, an algorithm is presented which constructs a bidirected graph for any totally unimodular matrix. I should note here that many results of this part are joint work with G. Appa, B. Kotnyek and L. Pitsoulis; these results can be found in a joint journal article [61].

In the third part of this thesis we deal with the frame matroid of a signed graph, or simply the signed-graphic matroid. Several new results are provided in this last part of the thesis. Specifically, given a signed graph, we provide methods to find representation matrices of the associated signed-graphic matroid over $GF(2)$, $GF(3)$ and $\mathbb{R}$. Furthermore, two new matroid recognition algorithms are presented in this last part. The first one determines whether a binary matroid is signed-graphic or not and the second one determines whether a (general) matroid is binary signed-graphic or not. Finally, one of the most important new results of this thesis is the decomposition theory for the class of binary signed-graphic matroids which is provided in the last chapter. In order to achieve this result, we employed Tutte's theory of bridges. The proposed decomposition differs from previous decomposition results on matroids that have appeared in the literature in the sense that it is not based on k-sums, but rather on the operation of deletion of a cocircuit. Specifically, it is shown that certain minors resulting from the deletion of a cocircuit of a binary matroid will be graphic matroids except for one that will be signed-graphic if and only if the matroid is signed-graphic. The decomposition theory for binary signed-graphic matroids is a joint work with G. Appa and L. Pitsoulis.

Dedicated to my parents: Athanassios Papalamprou and Eleni Mparpagianni
to my brother: Marios Papalamprou
and to Vasiliki Samartzi

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Contents

List of Figures

Part I

Introduction

Chapter 1

Basic definitions and notation

In this chapter we define basic notions and notation used throughout the thesis. Furthermore, we provide some well-known results which are presented in books of graph theory, matroid theory and combinatorial optimization. Our basic references for graph theory are the books of Diestel [20], and Bondy and Murty [11] and for matroid theory are the books of Oxley [56] and Truemper [78]. The concepts and results of this chapter related to combinatorial optimization can be found in books written by Schrijver [64, 65] and in the book of Nemhauser and Wolsey [55]. Finally, we assume that the reader is familiar with the basic elements of linear algebra and linear programming. For that reason we do not define notions such as: a set, a matrix, transpose or inverse of a matrix, linear independence, determinant, rank, linear program etc.

1.1 Sets and fields

The (distinct) objects of a set are usually called the *elements* or *members* of the set. The notation $a \in A$ means that a is an element of the set A while the notation $b \notin A$ means that b is not an element of A. For instance, a set A with elements $a_1, \ldots, a_{n-1}$ and a_n is denoted by $A = \{a_1, \ldots, a_{n-1}, a_n\}$. A set may have no members in which case it is called *empty* and is denoted by $\emptyset$. Given a set A, we denote its collection of subsets and cardinality by 2^A and $|A|$, respectively . We also denote by $[A]^k$ the family of all subsets of A having cardinality k. We assume that the reader is familiar with the basic operations on a pair of sets A and B, namely the *union* denoted by $A \cup B$ and the intersection denoted by $A \cap B$. The *difference* between two sets A and B, denoted by $A \backslash B$ or by $A - B$, is the set $\{x \in A \mid x \notin B\}$. Another operation between two sets A and B is the *symmetric difference*, denoted by $A \triangle B$, which is defined as follows: $A \triangle B = (A \backslash B) \cup (B \backslash A)$. Note that often we abbreviate $A \cup \{e\}$, $A \backslash \{e\}$ and $A - \{e\}$ to $A \cup e$, $A \backslash e$ and $A - e$, respectively. For sets A and B, the notation $B \subseteq A$ means that B is a subset of A while the notation $B \subset A$ means that $B \subseteq A$ and $B \neq A$. Moreover, when $B \subseteq A$ we usually say that B is *contained* in A. If $B \subset A$ and $B \neq \emptyset$ then B is called a *proper subset* of A. Two sets A and B such that $A \cap B = \emptyset$ are called *disjoint*.

A set consists of distinct elements. If we relax this condition, i.e if any element can appear multiple times, then we get what is called a *multiset*, which generalizes the notion of a set. The order in which the elements appear in a multiset has no significance, for example, the multisets $\{a, b, a, b, c\}$ and

$\{c, a, a, b, b\}$ are equal. An ordered list C of objects $c_1, c_2, \ldots, c_m$ is called a *sequence* and is denoted by $C = (c_1, c_2, \ldots, c_n)$. Like a set, a sequence contains objects which are called *members* or *elements* of the sequence but, unlike a set, the order in which the members appear in a sequence has significance and any element may appear more than once at different positions in a sequence. A *collection* is synonymous with a set which is particularly used for a set whose members are sets. A *class* is another synonymous with a set which is particularly used for a set whose members are specific structures, e.g. graphs, matrices.

The *cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all possible ordered n-tuples whose first element is a member of A_1, whose second element is a member of A_2 and so on, that is, $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \quad 1 \leq i \leq n\}$. If $A_1 = A_2 = \ldots = A_n = A$ we abbreviate $A_1 \times A_2 \times \ldots \times A_n$ to A^n.

Given a collection $\mathcal{A}$ of subsets of A, we say that $B \in \mathcal{A}$ is a *minimal* member of $\mathcal{A}$ if there is no $C \in \mathcal{A}$ such that $C \subset B$. Similarly, we say that an $E \in \mathcal{A}$ is a *maximal* member of $\mathcal{A}$ if there is no $F \in \mathcal{A}$ such that $E \subset F$. Furthermore, we denote the maximal member and the minimal member of a collection $\mathcal{A}$ by maximal$\mathcal{A}$ and minimal$\mathcal{A}$, respectively. A collection $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of subsets of a set A is called a *partition* of A if $A_i \cap A_j = \emptyset$ for all $1 \leq i \neq j \leq n$ and $A_1 \cup A_2 \cup \ldots \cup A_n = A$. A function f from a set A to a set B is denoted by $f : A \rightarrow B$.

For some number a, we denote by $|a|$ its absolute value ; by $\lfloor a \rfloor$ the highest integer which is less than or equal to a; and by $\lceil a \rceil$ the smallest integer not less than a. With $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ we denote the sets of *real*, *integer*, *natural* and *rational* numbers, respectively. The script '+' restricts a set to the positive numbers, so for example $\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x > 0\}$. Given a finite set A consisting of real numbers we denote the largest number and the smallest number in A by $\max A$ and $\min A$, respectively.

Finite fields (or Galois fields) are used quite often in this thesis. The finite fields which appear in this work are: the *binary* field denoted by $GF(2)$, the *ternary* field denoted by $GF(3)$ and the *quaternary* field denoted by $GF(4)$. Proofs regarding the existence and uniqueness as well as other basic concepts about finite fields can be found in many graduate algebra textbooks. A detailed presentation of the finite fields is outside the scope of this thesis and for that reason we only provide the necessary and basic background. For q prime the finite fields $GF(q)$ coincide with $\mathbb{Z}_q$, the ring of integers modulo q. Thus, the fields $GF(2)$ and $GF(3)$ have elements in $\{0, 1\}$ and $\{0, 1, -1\}$, respectively, and the addition and multiplication of these elements are performed modulo q. The field $GF(4)$ can be viewed as having four elements, viz. $0, 1, \alpha, \alpha + 1$, and the addition and multiplication for this field are given in the following tables:

$+$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

$\times$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

For a finite field F the *characteristic* of F is defined to be the smallest number of times needed to add 1 to itself in order to get 0; F is said to have characteristic 0 if this repeated sum never becomes 0. For the finite fields discussed above, the characteristic of both $GF(2)$ and $GF(4)$ is 2 while that of $GF(3)$ is 3.

1.2 Matrices and polyhedra

Let A be a matrix with row set R and column set S. Matrix A is also denoted by $[a_{ij}]$ while by a_{ij} or A_{ij} is denoted the element in row $i \in R$ and column $j \in S$ of A. For any $i \in R$ and any $j \in S$, we denote by $A_{i\bullet}$ and by $A_{\bullet j}$ the i^{th} row and j^{th} column of A, respectively. Furthermore, in this thesis whenever we say simply that a is a vector we mean that a is a column vector and by a_i we denote the element of a in row i. Matrices or vectors whose elements are integers are called *integral*. That is, n-dimensional integral vectors are those in $\mathbb{Z}^n$ and $m \times n$ integral matrices are in $\mathbb{Z}^{m \times n}$. Similarly, a matrix or vector is called *rational* if all of its elements are members of $\mathbb{Q}$. An $m \times n$ matrix whose elements are integer multiples of some rational number r is called r-integral, for instance, if $r = 1/2$ then the term *half-integral* matrix is used. If B' is the matrix obtained by reducing each element of an integral matrix B modulo k then we write $B' = B \bmod k$. Furthermore, a matrix B' which is obtained from a matrix B by replacing its non-zeros by 1s is called the *binary support* of matrix B.

A matrix of size $m \times n$ that contains only zeros is called an *all-zeros* matrix and is denoted by $\mathbf{0}_{m \times n}$. The identity matrix of size $m \times m$ is denoted by I_m. Whenever the size of an all-zeros matrix (identity matrix) is clear from the context we simplify the notation by using the symbol $\mathbf{0}$ (I). In matrix notation, $[A\ B]$ or $[A|B]$ denotes a matrix with submatrices A and B in which submatrix A is on the left of B. We also denote by $\begin{bmatrix} C \\ D \end{bmatrix}$ or $\left[\begin{array}{c} C \\ \hline D \end{array}\right]$ a matrix with submatrices C and D in which the submatrix C is above D. Sometimes in order to simplify our notation we leave blank the parts of a matrix that contain 0 elements. So, for example, $\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$ is the matrix that has submatrices B and C in the upper right and bottom left corner, respectively, while it has an all-zero submatrix in the bottom right corner and a submatrix A in the upper left corner.

A matrix of the form

$$\left[\begin{array}{c|c|c|c} A_1 & & & \\ \hline & A_2 & & \\ \hline \cdots & \cdots & \ddots & \cdots \\ \hline & & & A_n \end{array}\right]$$

is called *decomposable* if $n \geq 2$ and its structure is its *decomposition* into matrices $A_1, A_2, \ldots, A_n$. Moreover, any matrix which cannot become decomposable by a series of row and column permutations is called *connected* . If $A_1, A_2, \ldots, A_n$ are square then they are *blocks* of A in which case A is also called *block diagonal.*

Sometimes we index the rows and the columns of a matrix. The row indices are written to the left of a given matrix while the column indices are written above the matrix. For example, we may write:

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

The transpose of a matrix A is denoted by A^T; furthermore, if A is square then the determinant and the inverse of A are denoted by $det(A)$ and A^{-1}, respectively. A matrix is of *full row rank* if its rank, denoted by $r(A)$, is equal to the number of its rows, that is, the rows of A are linearly independent. A *basis* of a full row rank matrix A is a square submatrix B of A such that $r(B) = r(A)$. Two or more vectors with elements in some field $\mathbb{F}$ are called *$\mathbb{F}$-independent* if they are linearly independent in $\mathbb{F}$.

A matrix A is *projectively equivalent* to a matrix B, denoted by $A \sim B$, if there are nonsingular matrices π and Δ such that $B = \pi A \Delta$ and Δ is diagonal. In other words, we have that $A \sim B$ whenever matrix A can be converted to B by a sequence of elementary row operations and column scaling. An important operation on matrices is that of pivoting. *Pivoting* on a non-zero element ϵ of a matrix A over some field is defined as the replacement of

$$A = \left[\begin{array}{c|c} \epsilon & y \\ \hline & \\ x & D \end{array}\right] \text{ by } B = \left[\begin{array}{c|c} \frac{1}{\epsilon} & \frac{y}{\epsilon} \\ \hline & \\ -\frac{x}{\epsilon} & D - \frac{1}{\epsilon}xy \end{array}\right],$$

where x and y is a column and a row vector, respectively, and D is a submatrix of A of appropriate size. Equivalently, pivoting can be viewed as a series of row operations executed on $[A\ I]$; specifically, it is the same as premultiplying $[A\ I]$ by the inverse of a basis.

Given a set $J \subseteq \mathbb{R}^n$, an $x \in \mathbb{R}^n$ is a *convex combination* of members of J if there exist $x_1, \ldots, x_k \in J$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ with $\lambda_1 + \ldots + \lambda_k = 1$ and $x = \lambda_1 x_1 + \ldots + \lambda_k x_k$. The *convex hull* of J, denoted by $conv(J)$ is the set of all vectors which are convex combinations of vectors in J.

A set of vectors P of $\mathbb{R}^n$ is called a *polyhedron* if $P = \{x \mid Ax \leq b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Furthermore, any inequality of the form $c^T x \leq d$ is called a *valid inequality* for P if it holds for each $x \in P$. A bounded polyhedron is called a *polytope*. Equivalently, a set of vectors P of $\mathbb{R}^n$ is called a polytope if it is the convex hull of a finite set L of vectors in $\mathbb{R}^n$, i.e. $P = conv(L)$. We say that P is a *rational polyhedron* if A is a rational matrix and b is a rational vector. A vector of a polyhedron P is also called a *point* of P. A point of P is an *extreme point* if it cannot be written as a convex combination of two points of the polyhedron. If a polyhedron P is the convex hull of the integer vectors contained in P then it is called an *integral polyhedron*. Therefore, a polytope P is an *integral polytope* if all the extreme points of P are integral.

Given a rational polyhedron P, the convex hull of the integer points of P is called the *integer hull* of P and is denoted by P_I, i.e. $P_I = conv\{P \cap \mathbb{Z}^n\}$. Clearly $P_I \subseteq P$ while there are several cases and problems in which $P \neq P_I$. In order to tackle these cases and find solutions to integer programming problems different methods have been developed. One of the most well known techniques is based on the cutting-plane method [34]. An important notion of this technique is the Chvátal-Gomory cut defined as follows. Given an integral polyhedron $P_I = conv\{x \in \mathbb{Z}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, a *Chvátal-Gomory cut* is a valid inequality for P_I of the type: $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in \mathbb{R}^m_+$ is such that $\lambda^T A \in \mathbb{Z}^n$. Notice that only when $\lambda \in [0,1)^m$ we obtain undominated cuts, since otherwise we can replace λ by $\lambda - \lfloor \lambda \rfloor$ and get a stronger cut. The *rank-1 closure* of a polyhedron P, denoted by P_1, is the intersection of P with the half-spaces defined by all possible undominated Chvátal-Gomory cuts, i.e. $P_1 = \{x \in P \mid \lambda^T Ax \leq \lfloor \lambda^T b \rfloor \text{ for } \lambda \in [0,1)^m \text{ such that } \lambda^T A \in \mathbb{Z}^n\}$. Given a polyhedron $P = \{x \mid Ax \leq b\}$, if $P_I = P_1$ for all integral vectors b then the matrix A is said to have *Chvátal rank* 1. An integral $m \times n$ matrix A has strong Chvátal rank 1 if $P'_1 = P'_I = conv\{P' \cap \mathbb{Z}^n\}$ for polyhedron

$P' = \{x \mid l \leq x \leq u, a \leq Ax \leq b\}$ and all integral vectors l, u, a, b.

1.3 Graphs

Basic Notions

A graph G is a pair (V, E) consisting of a finite set V of *vertices* and a multiset E of *edges* such that each $e \in E$ is a multiset consisting of at most two vertices. Therefore, for some $u, v \in V$, we have four types of edges: an edge $e = \{u, v\}$ is a *link*, an edge $e = \{v, v\}$ is a *loop*, an edge $e = \{v\}$ is a *half-edge*, while an edge $e = \emptyset$ is a *loose edge*. We often denote the set of vertices and the set of edges of a graph G by $V(G)$ and $E(G)$, respectively. If $e = \{u, v\}$ is an edge of a graph G then we say that *e joins* u and v. We say that a vertex v of a graph G is *incident with* an edge e of G and that e is *incident with* v if $v \in e$. The vertices incident with an edge are called its *end-vertices*. Edges with the same end-vertices are called *parallel*. We say that two vertices u and v of G are *adjacent* or that u is *adjacent to* v if $\{u, v\}$ is an edge of G. Furthemore, two edges are called *adjacent* if they have an end-vertex in common. The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v, where each loop counts as two edges.

Observe that except for half-edges and loose edges all the definitions above can be found in textbooks of graph theory (e.g. [11, 20]). For that reason a graph without half-edges and loose edges is called an *ordinary graph*. An ordinary graph with no loops or parallel edges is called a *simple graph*. If G is a simple graph such that for each pair $\{u, v\} \subseteq V(G)$, u and v are adjacent then G is called a *complete* graph. A complete graph with n vertices is denoted by K_n. A simple graph G is called *bipartite* if $V(G)$ can be partitioned into two sets V_1 and V_2 such that $\{u, v\} \in E(G)$ only if $u \in V_1$ and $v \in V_2$; furthermore, if every vertex in V_1 is adjacent to every vertex in V_2 then G is called a *complete bipartite graph* and is denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$.

The above structures are called graphs mainly because they can be represented graphically. Specifically, each vertex is indicated by a point (small circle) and each edge by a line joining its end-vertices; loose edges are not depicted in a graphical representation of a graph. In Figure 1.1, we provide a graphical representation of a graph G whose vertex set is $\{v_1, v_2, v_3, v_4\}$ and whose edge set is $\{e_1, e_2, e_3, e_4, e_5\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$ and $e_3 = \{v_3, v_4\}$ are links, $e_4 = \{v_1, v_1\}$ is a loop, and $e_5 = \{v_4\}$ is a half-edge.

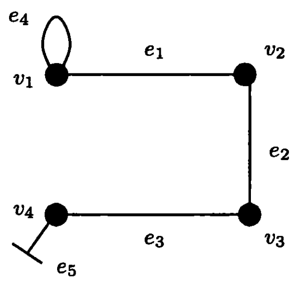


Figure 1.1: A graph.

We say that G' is a *subgraph* of G, denoted by $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For some $V' \subseteq V(G)$, if G' is the subgraph of G whose vertex set is V' and whose edge set consists of all edges of G which have both end-vertices in V' then G' is called the subgraph of G *induced* by V' and is denoted by $G[V']$. Similarly we define the subgraphs of G induced by subsets of $E(G)$. For some $E' \subseteq E(G)$, if G' is the subgraph of G whose edge set is E' and vertex set consists of all end-vertices of edges of E' then G' is called the subgraph of G *induced* by E' and is denoted by $G[E']$.

For any graph G, any $v \in V(G)$ and any edge $e \in E(G)$, we denote by $G\backslash e$ and $G\backslash v$ the subgraphs of G defined as $G\backslash e := (V(G), E(G)\backslash e)$ and $G\backslash v := G[V(G)\backslash v]$, respectively. We further say that $G\backslash e$ and $G\backslash v$ arise from G by *deleting edge e* and by *deleting vertex v*, respectively.

Identification of two vertices u and v in a graph G is the operation where the vertices u and v are replaced by a new single vertex v' and any edge incident with u or v in G is replaced by a new edge as follows:
(i) any half-edge at u or v is replaced by a half-edge at v',
(ii) any loop at v or w or any link incident with v and w is replaced by a loop at v', and
(iii) any link incident with either u or v and some other vertex $x \in V(G) - \{u, v\}$ is replaced by a link incident with v' and x.

For any graph G and any edge $e \in E(G)$, we denote by G/e the following subgraph of G: (i) if $e = \{u, v\}$ with $u \neq v$, i.e. if e is a link, then G/e is the graph obtained from $G\backslash e$ by identifying u and v; (ii) if $e = \{v\}$ is a half-edge then G/e is the graph obtained from $G\backslash e$ by replacing all links incident with v by half-edges incident with their other end-vertex and replacing all loops and half-edges incident with v by loose edges and then deleting v from the graph so-obtained; (iii) if $e = \emptyset$ or $e = \{u, u\}$, i.e. if e is a loose edge or a loop, then $G/e = G\backslash e$. We further say that G/e arises from G by *contracting edge e*.

A graph G' is called a *minor* of G if it is obtained from G by a sequence of deletions and contractions of edges and deletions of vertices. A series of deletions of elements in some $X \subseteq E(G)$ from G is denoted by $G\backslash X$ and a series of contractions of elements in some $Y \subseteq E(G)$ from G is denoted by G/Y. For disjoint subsets X, Y of $E(G)$, the minor of G obtained from G by deleting the elements of X and contracting the elements of Y is denoted by $G\backslash X/Y$. Note here that, although our definition of graphs is more general than the usual one (in e.g. [11, 20]), the above operations of deletion and contraction in graphs commute both with each other and with themselves (see [94]); in other words, the order in which we delete and contract edges from a graph G in order to obtain the minor $G\backslash X/Y$ does not matter. Clearly, a subgraph of a graph is also its minor. If G has a minor isomorphic to a graph H then we will often say that G has an *H-minor* or G has *H as a minor*. For some $S \subseteq E(G)$, we say that the subgraph H of G is the *deletion* of *G to S*, denoted by $H = G\backslash .S$, if $E(H) = S$ and $V(H)$ is the set of end-vertices of all edges in S. Clearly, for $S \subseteq E(G)$, $G\backslash .S$ is the graph obtained from $G\backslash(E(G) - S)$ by deleting the isolated vertices (if any); note also that $G\backslash .S = G[S]$. Moreover, for $T \subseteq E(G)$, a subgraph K of G is the *contraction* of *G to K*, denoted $K = G/.T$, if K is the graph obtained from $G/(E(G) - S)$ by deleting the isolated vertices (if any). A *subdivision* of a link or a loop e of G is the operation of deleting edge e, adding a new vertex x and joining x to the end-vertices of e (when e is a link this amounts to replacing e by a path of length 2). Any graph obtained from a graph G via a sequence of subdivisions is called a *subdivision* of G.

Two graphs G_1 and G_2 are called *isomorphic*, denoted by $G_1 \cong G_2$, if there exists a bijection p :

$V(G_1) \to V(G_2)$ such that $\{u, v\} \in E(G_1)$ if and only if $\{p(u), p(v)\} \in E(G_2)$. We also say that two graphs are *disjoint* if they have no vertex in common. Given two graphs G_1 and G_2 we define their *union*, denoted by $G_1 \cup G_2$, as the graph $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Furthermore, we define the operation of *reversing*, which is also known as *twisting*, as follows (see [56]). Let G_1 and G_2 be two disjoint graphs where each of G_1 and G_2 has at least two vertices $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively, and let G be the graph obtained from G_1 and G_2 by identifying u_1 of G_1 and u_2 of G_2 as the vertex u of G and identifying v_1 of G_1 and v_2 of G_2 as the vertex v of G. In a *reversing* G' of G *about* (u, v) *with parts* (G_1, G_2) we identify, instead, u_1 of G_1 with v_2 of G_2 and v_1 of G_1 with u_2 of G_2 (see Figure 1.2). The subgraphs G_1 and G_2 of G are called the *reversing parts* of the reversing.

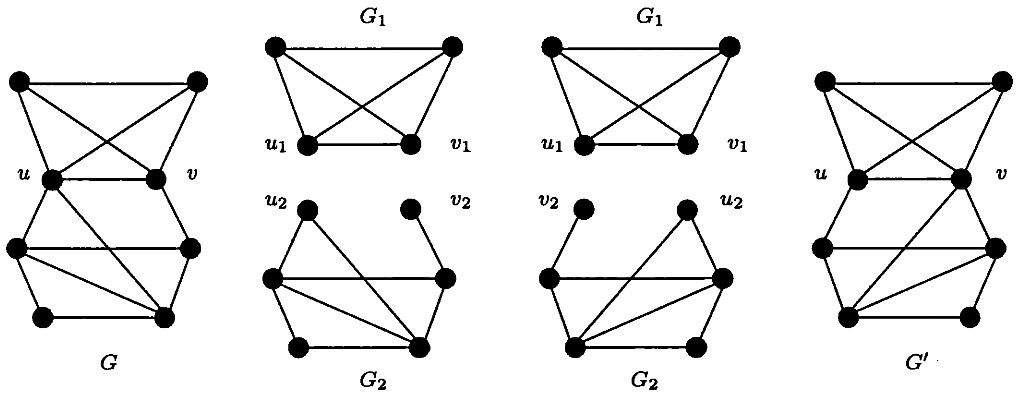


Figure 1.2: Reversing of G about (u, v) with parts (G_1, G_2).

A *walk* in a graph G is a sequence $P = (v_1, e_1, v_2, e_2, \ldots, e_{t-1}, v_t)$, where v_i and v_{i+1} are end-vertices of edge e_i $(i = 1, \ldots, t-1)$ including the case where $v_i = v_{i+1}$ and e_i is a half-edge. The vertices v_1 and v_t are called *outer-vertices* of P while all the other vertices are called *inner vertices* of P. The number of edges in P (i.e. the number $t-1$) is called the *length* of P. We say that P is a *closed walk* if $v_1 = v_t$. If P consists only of links and $v_1, v_2, \ldots, v_t$ are all distinct then we call P a *path.* If P is a closed walk such that all $v_1, v_2, \ldots, v_{t-1}$ are different then P is called a cycle. If all the edges of P are different then P is called a *tour*. Notice that if P is a path then it is a tour as well, but a tour is not necessarily a path. A *closed tour* is a tour in which the first and last vertices coincide. Furthermore, if G is an ordinary graph and there exists a closed tour in G which contains all its edges then G is called *Eulerian.* A walk with no half-edges is called an *ordinary walk.* Two or more walks are called *internally disjoint* if they have no inner vertex in common. At this point we should note that, if no confusion may arise, we sometimes identify the walk P with the subgraph $(V(P), E(P))$ of G, where $V(P)$ is the set of vertices in P and $E(P)$ is the set of edges in P. If P is a cycle then the edge set $E(P)$ is called a *circle* of G. Notice also that if e is a half-edge or a loop of a graph G then the subgraph of G induced by e is a cycle.

If H and K are two subgraphs of G, each having at least one edge, such that $E(H) \cup E(K) = E(G)$ and $V(H) \cap V(K) = \{v\}$ then v is called a *cut-vertex* of G . Any partition $\{T, U\}$ of $V(G)$ into two nonempty sets T and U leads to a *cut* of G, denoted by $Q(T, U)$, defined as the set of links incident with a vertex in T and a vertex in U. A cut of the form $Q(v, V(G) - v)$ is called the *star* of v and is denoted by $st_G(v)$. A minimal nonempty cut of a graph G is called a *bond* of G. The following well-known result of graph theory (see e.g. [11, 20]) shows the exact relationship between cuts and bonds.

Theorem 1.1. *A set $S \subseteq E(G)$ is a cut if and only if S is a disjoint union of bonds of G.*

A *theta graph* is the union of three internally disjoint paths with common outer vertices, including the case in which one or more of these paths has exactly one edge (an example theta graph T is given in Figure 1.3). Observe that a loop or a half-edge cannot be contained in a theta graph. An ordinary graph with n vertices is called a *wheel graph*, denoted by W_n, if it consists of a cycle of length n along with a vertex which is adjacent to every vertex of the cycle. In Figure 1.3 we depict the wheel graph W_4.

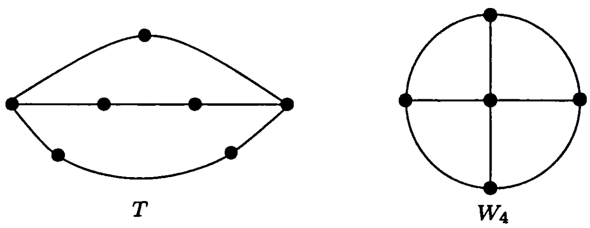


Figure 1.3: A theta graph and the wheel graph W_4.

We also introduce the important class of planar graphs. A graph is called *planar* if it can be drawn on the plane such that edges meet only at points corresponding to their common end-vertices. Theorem 1.2 provides two well-known and important characterizations of planar graphs due to Kuratowski [47] and Wagner [87].

Theorem 1.2. *The following statements are equivalent for a graph G:*

(i) G is planar

(ii) G contains neither K_5 nor $K_{3,3}$ as a minor (Wagner)

(iii) G has no subgraph that is a subdivision of K_5 or $K_{3,3}$ (Kuratowski).

Finally, with any graph G we can associate a matrix called the *node-edge incidence matrix* or simply the *incidence matrix*. The *incidence matrix* of a graph G with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{e_1, \dots, e_m\}$ is the $n \times m$ matrix $A = [a_{v_i e_j}]$ (where $i = 1, \dots, n$ and $j = 1, \dots, m$) defined by:

$$a_{v_i e_j} := \begin{cases} 1 & \text{if } v_i \in e_j \text{ and } e_j \text{ is not a loop,} \\ 2 & \text{if } v_i \in e_j \text{ and } e_j \text{ is a loop,} \\ 0 & \text{otherwise} \end{cases}$$

Connectivity

A graph G is called *connected* if for every pair of its vertices there is a walk of G containing both vertices; if G is not connected then it is called a *disconnected graph*. Also, we say that a vertex v is *connected to* a vertex u if there is a walk containing both u and v. It is easily seen that this is an equivalence relation on the set of vertices, therefore it partitions $V(G)$ into equivalence classes, whose induced subgraphs are called the *connected components* or simply the *components* of G. Thus, a connected graph consists of exactly one component. In addition, it can easily be shown that a graph G is connected if and only if its incidence matrix is connected.

A graph without cycles is a *forest*. A *tree* is a forest with one component. A connected graph containing exactly one cycle is called an *1-tree*. A *spanning tree* T of a connected graph G is a subgraph $T = (V(G), E')$ of G which is a tree (where $E' \subseteq E(G)$); furthermore, notice that, for any edge $e \in E(G) - E'$, the subgraph $(V(G), E' \cup e)$ of G contains a unique cycle. These cycles are called the *fundamental cycles* of G *with respect to the spanning tree* T.

There are several notions of higher connectivity in graphs that have appeared in the literature. Each of which has its own advantages and disadvantages. Here we will define two; namely, Tutte connectivity and vertical connectivity. The concepts related to Tutte connectivity for graphs are due to Tutte [85] and their main feature is that they are in agreement with the corresponding concepts defined for matroids.

Let k be a positive integer. Then, for a connected graph G, a partition (A, B) of $E(G)$ is a *Tutte k-separation* if:

(T1) $\min\{|A|, |B|\} \geq k$, and

(T2) $|V(G[A]) \cap V(G[B])| \leq k$.

A connected graph G for which there exists a partition (A, B) of $E(G)$ such that $(T1)$ and $(T2)$ are satisfied is called *Tutte k-separable*. For n positive integer, we say that G is *Tutte n-connected* if for all l, $1 \leq l < n$, G has no Tutte l-separation. If G is Tutte k-separable, for some positive integer k, then we define the *Tutte connectivity* $\lambda(G)$ of G as the minimum integer j such that G is j-separable; otherwise, we say that the connectivity $\lambda(G)$ is infinite. For example, for the graphs G_1 and G_2 of Figure 1.4 we have that $\lambda(G_1) = \infty$ and $\lambda(G_2) = 2$, respectively. Note that we often abbreviate the terms Tutte k-separation, Tutte k-separable and Tutte n-connected to k-separation, k-separable and n-connected, respectively. Furthermore, it can be shown that the graphs with infinite connectivity are precisely the 2-connected graphs of at most three edges (Theorem IV.6 in [84]).

Let G be a connected graph and k be a positive integer such that $1 \leq k \leq (|V(G)| - 1)$. Then a partition (A, B) of $E(G)$ is a *vertical k-separation* if:

(V1) $\min\{|V(G[A])|, |V(G[B])|\} > k$, and

(V2) $|V(G[A]) \cap V(G[B])| \leq k$.

A connected graph G for which there exists a partition (A, B) of $E(G)$ such that $(V1)$ and $(V2)$ are satisfied is called *vertically k-separable*. If G is vertically k-separable for some positive integer k we define the vertical connectivity $k(G)$ of G as the minimum positive integer j such that G is vertically j-separable; if G is not vertically k-separable for any $1 \leq k \leq (|V(G)| - 1)$ then we take $k(G) = |V(G)| - 1$. For example, for the graphs G_1 and G_2 of Figure 1.4 we have that $k(G_1) = 2$ and $k(G_2) = 3$, respectively. Furthermore, for n positive integer exceeding one, we say that G is *vertically n-connected* if $n \leq k(G)$.

Vertical n-connectivity is the most common type of connectivity appearing in the graph theory literature and a well-known equivalent definition for vertically k-connected graphs, where k is a positive integer

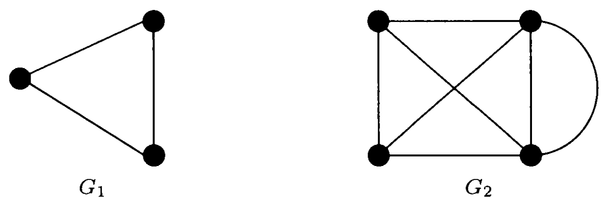


Figure 1.4: Example graphs illustrating the notions of connectivity and vertical connectivity.

exceeding one, goes as follows: a connected graph G is called *vertically k-connected* if $|V(G)| > k$ and the graph $G[V(G) - X]$ is connected for every $X \subseteq V(G)$ with $|X| < k$. Note also that often in the graph theory literature (e.g. in [11, 20]) vertical k-connectivity is defined as k-connectivity. However, for the purposes of this work, we call Tutte n-connectivity just n-connectivity in contrast to the normal usage in the literature, where vertical n-connectivity is called n-connectivity, mainly because the connectivity of a graph and that of the corresponding graphic matroid, as we will see later, coincide under this definition. The following result of [19] provides a useful connection between vertical connectivity and (Tutte) connectivity of a graph:

Proposition 1.3. *Let G be a connected graph having at least three vertices and suppose that $G \not\cong K_3$. Then G is k-connected if and only if G is vertically k-connected and has no cycles of length less than k.*

A *block* of a graph G is defined as a maximal 2-connected subgraph of G. Evidently, if G is 2-connected then it contains exactly one block, namely G itself. A loop or a half-edge of G along with its incident vertex always constitute a block of G, since it forms a 2-connected subgraph of G (actually it is infinitely connected) which cannot be part of a larger 2-connected subgraph of G. Let $\{B_1, \dots, B_k\}$ be the set of blocks of a graph G and let $\{c_1, \dots, c_t\}$ be the set of cut-vertices of G. We construct the graph $C(G)$ with vertex set $\{B_1, \dots, B_k, c_1, \dots, c_t\}$ where two vertices c_i and B_j are joined by an edge if and only if c_i is a vertex of the block B_j of G for $i = 1, \dots, t$ and $j = 1, \dots, k$, called the *block graph* of G . Then, it can be shown that $C(G)$ is a tree (forest) if G is connected (disconnected) (see e.g. [11]). For example, in Figure 1.5, we depict a connected graph G containing five blocks (denoted by B_1, B_2, B_3, B_4 and B_5) and the associated block graph, where the cut-vertices of G (denoted by c_1 and c_2) are indicated by using white colour instead of black.

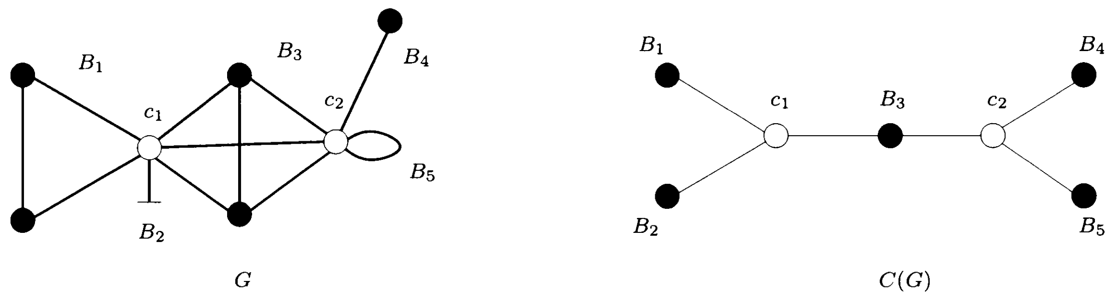


Figure 1.5: A graph G and the associated block graph $C(G)$.

1.4 Digraphs and network matrices

Digraphs are ordinary graphs except that there is a clear distinction between the two end-vertices of each edge. Formally, a *directed graph* or *digraph* $D = (V, A)$ is defined as a finite set of vertices V and a set of edges $A \subseteq V^2$ where identical elements are allowed. Thus, by the ordered pair $a = (u, v)$ is denoted an edge a of D with end-vertices u and v, where the vertex u and v is called the *tail* and the *head* of a, respectively. Thus, with any digraph D we can associate an (ordinary) graph on the same vertex set simply by ignoring the distinction on the end-vertices of the edges of D. This ordinary graph is called the *underlying* graph of D and is denoted by $G(D)$. The opposite process in which we start from an ordinary graph G and obtain a digraph D by making the aforementioned (head-tail) distinction to the end-vertices of each edge of G is called an *orientation* of G; for that reason we say that the term *undirected graph* is synonymous with ordinary graph. Moreover, a walk $P = (v_1, e_1, v_2, e_2, \ldots, e_{t-1}, v_t)$ of the underlying graph of a digraph D is called a *directed walk* if, for any $e_i \in P(i = 1 \ldots, t-1)$ v_i is a tail and v_{i+1} is a head of e_i. For the purposes of this work, whenever a concept defined for graphs is used for a digraph, it refers to its underlying graph. For instance, a digraph D is said to be k-connected if $G(D)$ is k-connected, a path of D is a path of $G(D)$, and so on. An edge $a = (u, v)$ of a path P of D is called a *forward* edge of P if u appears before v in P while in the opposite case a is called a *backward* edge of P.

The graphical representation of a digraph consists of the graphical representation of its underlying graph along with a set of arrows on the edges; in particular, there is exactly one arrow on each edge and each arrow points towards the head of the corresponding edge. The *node-edge incidence matrix* or *incidence matrix* of a digraph D with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $A = \{a_1, \ldots, a_m\}$ is the $n \times m$ matrix $B = [b_{v_i a_j}]$ (where $i = 1, \ldots, n$ and $j = 1, \ldots, m$) defined by:

$$b_{v_i a_j} := \begin{cases} -1 & \text{if } v_i \text{ is a tail of } a_j, \\ 1 & \text{if } v_i \text{ is a head of } a_j, \\ 0 & \text{otherwise} \end{cases}$$

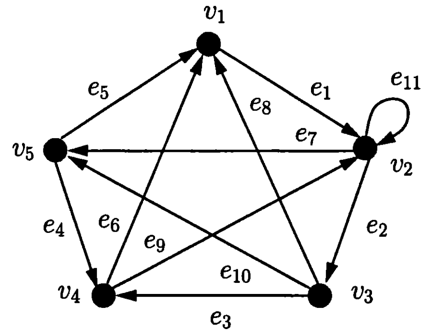
for any $v_i \in V$ and any nonloop edge $a_j \in A$. If a_j is a loop, we set $b_{v_i a_j} := 0$ for each vertex $v_i \in V$. An example digraph and the associated incidence matrix is provided in Figure 1.6.

With any digraph we can associate another important class of matrices, known as network matrices. We initially provide the graphical definition of network matrices which directly relates a network matrix to its associated digraph and then we use the incidence matrix of a digraph to provide an equivalent algebraic definition of network matrices. The graphical definition goes as follows. Given is a connected digraph $D = (V, A)$ and a spanning tree $T = (V, A')$. Let N be the $|A'| \times |A - A'|$ matrix defined as follows. For any $a' \in A'$ and $a = (u, v) \in (A - A')$, let P_{uv} be the unique path from u to v in T. Define

$$N_{a'a} := \begin{cases} 1 & \text{if } a' \text{ is a forward edge of } P_{uv}, \\ -1 & \text{if } a' \text{ is a backward edge of } P_{uv}, \\ 0 & \text{if } a' \text{ does not occur in } P_{uv}. \end{cases}$$

Then the matrix $N = [N_{a'a}]$ is the *network matrix* of D with respect to T.

For the algebraic definition, let E be the incidence matrix of a connected digraph D. It can be shown that the matrix E' obtained by deleting one arbitrarily selected row of E is of full row rank. Let B be a



$$
\begin{array}{c c}
 & \begin{array}{ccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} &
\left[\begin{array}{ccccccccccc}
-1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

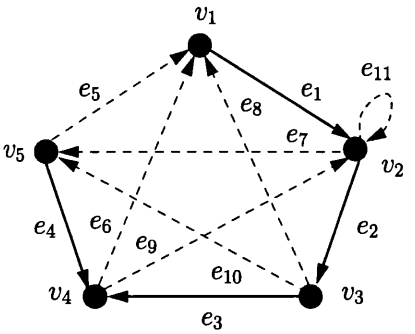
Figure 1.6: A digraph and the associated incidence matrix.

basis of E' and suppose that $E' = [B\ S]$. The matrix $N = B^{-1}S$ is called a *network matrix.* It can be easily shown that the edges in D corresponding to the columns of B are the edges of a spanning tree T of D and furthermore, that N is equal to the network matrix of D with respect to the tree T as determined by the graphical definition. The edges in D corresponding to the columns of B and S are called *tree* edges and *non-tree* edges, respectively. Notice that there is one-to-one correspondence between the tree edges of D and the rows of N as well as between the non-tree edges of D and the columns of N. For each non-tree edge e the cycle in D which contains e and some tree edges of T is unique and is called the *fundamental cycle* of e; the set of these cycles is the set of *fundamental cycles* of D with respect to T. The digraph in which the tree and the non-tree edges are clearly indicated in D is called a *network representation* of N. Usually there exist more than one network representations for a given network matrix N. For example, in Figure 1.7 we present the network matrix N_1 of the digraph of Figure 1.6 with respect to the spanning tree $\{e_1, e_2, e_3, e_4\}$. In the same figure a network representation of N_1 is also provided, where the tree and the non-tree edges are indicated by bold and dashed lines, respectively.

1.5 Totally unimodular matrices

A matrix is *totally unimodular* (TU) if all of its square submatrices have determinant equal to $0, +1$, or -1. There are numerous other characterizations for the class of TU matrices (see [55, 64]). In Theorem 1.4 we provide a famous characterization given by Ghouila-Houri [32].

Theorem 1.4. *A $\{0, \pm 1\}$ matrix is totally unimodular if and only if for each collection of columns or rows, there exists a scaling of the selected columns or rows by ± 1 such that the sum of the scaled columns*



$$N_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Figure 1.7: A network matrix and an associated network representation.

or rows is a vector of elements in $\{0, \pm 1\}$.

The class of TU matrices is of great importance in integer programming and combinatorial optimization. Their importance stems mainly from the following characterization of Hoffman and Kruskal [41]:

Theorem 1.5. *Let* A *be an integral matrix. Then* A *is totally unimodular if and only if for each integral vector* b *the polyhedron* $P = \{x \mid x \geq 0; Ax \leq b\}$ *is integral.*

Therefore, by Theorem 1.5, the integer programming problem $\max\{cx \mid x \in P, x \text{ integer}\}$ can be solved as a linear programming problem in polynomial time. It can also be shown that TU matrices are closed under some well known matrix operations.

Proposition 1.6. *Totally unimodular matrices are closed under the following operations:*

(i) taking the transpose,

(ii) permuting rows or columns,

(iii) multiplying a row or a column by -1,

(iv) pivoting,

(v) repeating a row or a column,

(vi) adding an all-zero row or column, or adding a row or a column with exactly one non-zero element being ± 1.

One of the most important subclasses of TU matrices is that of network matrices (and their transposes). There exist TU matrices that are neither network matrices nor the transposes of such matrices as can be shown by the two well-known matrices B_1 and B_2 of (1.1) below (see e.g. [55, 64]).

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

However, by a deep decomposition result of Seymour [66], we know that the network matrices and their transposes and the matrices B_1 and B_2 are the building blocks for TU matrices. Specifically, there exist operations such that any TU matrix can be composed from these building blocks by a sequence of such operations. These operations are those of Proposition 1.6 and the operations of *matrix k-sums* ($k = 1, 2, 3$) defined as follows [78]:

Definition 1.7. *If A, B are matrices, a, d are column vectors and b, c are row vectors of appropriate size in $\mathbb{R}$ then*

1-sum: $A \oplus_1 B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

2-sum: $\begin{bmatrix} A & a \end{bmatrix} \oplus_2 \begin{bmatrix} b \\ B \end{bmatrix} := \begin{bmatrix} A & ab \\ 0 & B \end{bmatrix}$

3-sum: $\begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \oplus_3 \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} := \begin{bmatrix} A & ab \\ dc & B \end{bmatrix}$ *or*

$$\begin{bmatrix} A & 0 \\ b & 1 \\ c & 1 \end{bmatrix} \oplus^3 \begin{bmatrix} 1 & 1 & 0 \\ a & d & B \end{bmatrix} := \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$$

where, in the $\oplus^3$, b and c are $\mathbb{R}$-independent row vectors and a and d are $\mathbb{R}$-independent column vectors such that $[\frac{b}{c}] = [D_1 | \bar{D}]$, $[a|d] = [\frac{\bar{D}}{D_2}]$ and $\bar{D}$ is a square non-singular matrix. Then, $D = [a|d]\bar{D}^{-1}[\frac{b}{c}]$.

Note that there are two alternative definitions for 3-sum, distinguished by $\oplus_3$ and $\oplus^3$. The indices of the isolated columns and rows in the 2-sum and 3-sum operations are called *connecting elements* of the 2-sum and the 3-sum, respectively.

It is well known that total unimodularity is preserved under the k-sum compositions ($k = 1, 2, 3$) [55, 64].

Theorem 1.8. *Totally unimodular matrices are closed under the k-sum operations* ($k = 1, 2, 3$).

The definition of the k-sums may seem complicated at first glance, but these operations essentially provide a way to decompose a TU matrix into smaller TU matrices provided that the matrix admits such a decomposition. Specifically suppose that we have a TU matrix N which under row and column permutations can take the form

$$N = \begin{array}{c c} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X_1 \\ X_2 \end{array} & \left[\begin{array}{c|c} A & D_1 \\ \hline D_2 & \end{array}\right] \end{array}, \tag{1.2}$$

where X_1 and X_2 are the row sets of the submatrices A and B of N, respectively, and Y_1 and Y_2 are the column sets of A and B, respectively. Suppose also that the following two conditions are satisfied for some $k = 1, 2, 3$:

(i) $\min\{|X_1 \cup Y_1|, |X_2 \cup Y_2|\} > k$ and ,

(ii) $r(D_1) + r(D_2) = k - 1$.

Then the matrix N of (1.2) can be decomposed under a k-sum operation into two matrices of smaller size which are submatrices of N, preserving total unimodularity. In the case of 3-sum we note from the definition that there are two alternative operations, reflecting the fact that condition (ii) above can be satisfied in two different ways (i.e. $r(D_1) = r(D_2) = 1$ or $r(D_1) = 0, r(D_2) = 2$). However, it can be shown that in the case of TU matrices both definitions of 3-sum are equivalent under pivoting in either $GF(2)$ or $\mathbb{R}$. Moreover, whenever we have to make a distinction between the two alternative 3-sum operations, we shall use the terms $\oplus_3$-sum and $\oplus^3$-sum in order to refer to the matrix operation associated with $\oplus_3$ and $\oplus^3$, respectively.

We conclude by providing the composition theorem for TU matrices due to Seymour [66, 68] which plays a central role in this work. We should also note here that Seymour's approach yielded the first polynomial-time algorithm to test a $\{0, 1, -1\}$ matrix for total unimodularity.

Theorem 1.9. *Any totally unimodular matrix is up to row and column permutations and scaling by ± 1 factors a network matrix, or the transpose of such a matrix, or the matrix B_1 or B_2 of (1.1), or may be constructed recursively from these matrices using matrix 1-, 2- and 3-sums.*

1.6 Relevant matroid theory

1.6.1 Definitions and important classes of matroids

Definition 1.10. *A matroid M is an ordered pair $(E, \mathcal{I})$ of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:*

(I1) $\emptyset \in \mathcal{I}$

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$

(I3) If U and V are members of $\mathcal{I}$ with $|U| < |V|$ then there exists $x \in V - U$ such that $U \cup x \in \mathcal{I}$.

Given a matroid $M = (E, \mathcal{I})$, the set E is called the *ground set* of M and the members of $\mathcal{I}$ are the *independent sets* of M; furthermore, any subset of E not in $\mathcal{I}$ is called a *dependent set* of M. A maximal independent set of M is called a *basis* or *base* of M while a minimal dependent set is called a *circuit* of M. By condition (I3), it is evident that all bases have the same cardinality. Whenever several matroids are considered, we shall often write $E(M)$ and $\mathcal{I}(M)$ for the ground set and the collection of independent sets of a matroid M, respectively. Furthermore, the set of circuits and the set of bases of a matroid M are usually denoted by $\mathcal{C}(M)$ and $\mathcal{B}(M)$, respectively. The *rank function* $r_M : 2^E \rightarrow \mathbb{Z}_+$ of a matroid M is a function defined by: $r_M(A) = max(|X| : X \subseteq A, X \in \mathcal{I})$, where $A \subseteq E$. The *rank* of a matroid M, denoted by $r(M)$, is equal to $r_M(E)$ and thus $r(M) = |B|$, where $B \in \mathcal{B}(M)$. The axiomatic Definition 1.10 for a matroid on a given ground set uses its independent sets. However, there are several equivalent ways to define a matroid which can be found in [56]. For example, a matroid M on a given ground set E can be defined through its rank function, through its set of bases or through its set of circuits; in these cases M is given as a pair (E, r_M), $(E, \mathcal{B}(M))$ or $(E, \mathcal{C}(M))$, respectively. Specifically, the definition of a matroid M through its rank function goes as follows:

Definition 1.11. *A matroid M is an ordered pair (E, r_M) of a finite set E and a rank function $r_M : 2^E \longrightarrow \mathbb{Z}_+$ satisfying the following three conditions:*

(R1) $0 \le r_M(X) \le |X|$, *for all* $X \subseteq E$

(R2) $r_M(X) \le r_M(Y)$, *for all* $X \subseteq Y \subseteq E$

(R3) $r_M(X) + r_M(Y) \ge r_M(X \cap Y) + r_M(X \cup Y)$, *for all* $X, Y \subseteq E$.

Two matroids M_1 and M_2 are called *isomorphic* if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that $X \in \mathcal{I}(M_1)$ if and only if $\psi(X) \in \mathcal{I}(M_2)$. We denote that M_1 and M_2 are isomorphic by $M_1 \cong M_2$. A standard example of matroids is that of *uniform* matroids. Let E be a ground set of cardinality n and k be a positive integer such that for $I \subseteq E$ we have $I \in \mathcal{I}$ if $|I| \le k$; then it can be proved that $M = (E, \mathcal{I})$ is a matroid, called the *uniform* matroid of rank k and denoted by $U_{k,n}$.

Let E be a finite set of vectors from a vectorspace over some field F and let $\mathcal{I}$ be the collection of linearly independent subsets of E; then it can be proved that $M = (E, \mathcal{I})$ is a matroid called *vector matroid.* Furthermore, any matroid isomorphic to M is called a *representable matroid* over F. Matroids representable over $GF(2)$ are called *binary* and matroids representable over $GF(3)$ are called *ternary.*

Let A be a matrix whose columns are the vectors of the ground set of a vector matroid M. It is evident that there is one-to-one correspondence between the linearly independent columns of A and the independent sets of M, so the matroid M can be fully characterized by matrix A. Matrix A is called a *representation matrix* of M and we denote the vector matroid with representation matrix A by $M[A]$. For example, the following matrix A_{F_7} over $GF(2)$ represents a special binary matroid, the so-called *Fano matroid* denoted by F_7, which appears frequently in the relevant literature.

$$A_{F_7} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Suppose now that we delete from A all the linearly dependent rows and from the matrix A' so-obtained we choose a basis B. Clearly, linear F-independence of columns is not affected by such a deletion of rows. By pivoting on non-zero elements of B we can transform A' to matrix $[I \ B']$. Pivotings do not affect linear F-independence of a matrix and thus, $M = M[I \ B']$. The matrix B' is called a *compact representation matrix* of M, and $M(B')$ will denote a matroid with compact representation matrix B'. Furthermore, a matroid M is called *uniquely representable* over some field $\mathbb{F}$ if and only if any two representation matrices of M (over $\mathbb{F}$) are projectively equivalent. Note that a binary matroid is uniquely representable over every field in which it can be represented, while ternary matroids are uniquely representable over $GF(3)$ (for more see Chapter 10 in [56]).

Let G be an ordinary graph and let $\mathcal{C}$ be the collection of edge sets of cycles of G. Then it can be shown that the pair $(E(G), \mathcal{C})$ is a matroid called the *cycle matroid* of G and is denoted by $M(G)$. A matroid M such that $M \cong M(G)$, for some ordinary graph G, is called *graphic.* If A is the incidence matrix of an orientation of G then it can be shown that $M(G) \cong M[A]$. Thus, for any network matrix N with respect to some tree of G we have that $M(G) \cong M(N)$, since the way we obtain N from A is also the way we can obtain from A a compact representation matrix of $M[A]$. Recall that our definition for a graph is more general than the usual one. Thus, in order to be consistent with the literature the following important convention has been agreed. We do not define a matroid of a graph $G = (V, E)$ which contains half-edges or loose edges; thus, whenever we refer to a graphic matroid $M(G)$ we assume that the graph G is an ordinary graph. Up to here we have defined the notion of isomorphism between two graphs and between two matroids. Clearly, for two isomorphic graphs G and H we have that $M(G) \cong M(H)$. However, the interesting part is the opposite that is, what someone could say for two graphs G and H whose cycle matroids are isomorphic (i.e. $M(G) \cong M(H)$)? Theorems 1.12 and 1.13, which are due to Whitney [89], provide the relationship between two graphs with isomorphic matroids. [1]

Theorem 1.12. *If G and H are 3-connected matroids then $M(G) \cong M(H)$ if and only if $G \cong H$.*

Theorem 1.13. *If G and H are 2-connected graphs then $M(G) \cong M(H)$ if and only if H can be obtained from G via a sequence of reversings.*

A matroid representable over every field is *regular*. Furthermore, there is a clear connection between regular matroids and TU matrices. Specifically, any TU matrix is the representation matrix of some regular matroid and any regular matroid has a TU representation matrix (in $\mathbb{R}$). Tutte proved that representability over $GF(2)$ and $GF(3)$ is sufficient for a matroid M to be regular [82]. These results of Tutte with a different proof can also be found in [78] (Theorem 9.2.9). Finally, since regular matroids may be viewed as binary matroids representable over every field we have that regular matroids are uniquely representable over every field.

Theorem 1.14. *For a matroid M the following are equivalent:*

(i) M is regular.

(ii) M has a TU compact representation matrix (in $\mathbb{R}$).

(iii) M is representable over every field.

(iv) M is representable over $GF(2)$ and $GF(3)$.

(v) M is representable over $GF(3)$ by a matrix A which when viewed over $\mathbb{R}$ is TU.

1.6.2 Duality and minors

Given is a matroid $M = (E, \mathcal{I})$. The ordered pair $(E, \{E - S : S \notin \mathcal{I}\})$ is a matroid called the *dual matroid* of M and is denoted by M^*. It is clear that $(M^*)^* = M$. The set $\mathcal{B}(M^*)$ of bases of M^* is the set of *cobases* of M, the set $\mathcal{C}(M^*)$ of circuits of M^* is the set of *cocircuits* of M and so on; in other words, the prefix 'co' is used to dualize a term. We also denote the set of cobases and cocircuits of M by $\mathcal{B}^*(M)$ and $\mathcal{C}^*(M)$, respectively. A useful result, which can be found in [56], is the following:

Proposition 1.15. *Let M be a matroid with ground set E and let D be a non-empty subset of E. Then D is a circuit of M if and only if $|D \cap C^*| \neq 1$ for every cocircuit C^* of M.*

[1] Clearly, Theorem 1.12 can be obtained from Theorem 1.13, since the reversing operation is defined on 2-connected and not 3-connected graphs. However, since we mainly use Theorem 1.12 in the following chapters, we have included it here as a separate result.

We should note here that not all the classes of matroids are closed under duality. For instance, uniform and representable matroids are closed under duality, but the class of *cographic* matroids, which consists of the duals of graphic matroids, is not. Furthermore, for a representable matroid $M(A)$ with compact representation matrix A we have that $M^*[I\ A] = M[I\ A^T]$.

Deletion and *contraction* are two fundamental matroid operations. Formally, given a matroid $M = (E, \mathcal{C})$ on a ground set E defined by its collection of circuits $\mathcal{C}$ the *deletion of* some $e \in E$ from M is the matroid $M\backslash e$ on $E\backslash e$ with collection of circuits

$$\mathcal{C}(M\backslash e) := \{C \in \mathcal{C}(M) | e \notin C\}. \tag{1.3}$$

The *contraction of* some $e \in E$ is the matroid M/e on $E\backslash e$ with collection of circuits

$$\mathcal{C}(M/e) := \text{minimal}\{C\backslash e | C \in \mathcal{C}(M)\}. \tag{1.4}$$

A series of deletions of elements in some $X \subseteq E$ from M is denoted by $M\backslash X$ and a series of contractions of elements in some $X \subseteq E$ from M is denoted by M/X. Any matroid which can be obtained from M by a series of deletions and contractions is called a *minor* of M. For disjoint subsets X, Y of E, the minor of M obtained from M by deleting the elements of X and contracting the elements of Y is denoted by $M\backslash X/Y$. A minor $M\backslash X/Y$ of M such that X and Y are not both empty is called a *proper minor* of M. Note also that the operations of deletion and contraction commute both with each other and with themselves (see [56]) and, as a consequence, $M\backslash X/Y = M/Y\backslash X$. If M has a minor isomorphic to a matroid N then we will often say that M has an *N-minor* or M has N *as a minor*. A matroid N is called an *excluded minor* for a class of matroids $\mathcal{M}$ if $N \notin \mathcal{M}$ but every proper minor of N is in $\mathcal{M}$.

Furthermore, deletion and contraction may be viewed as dual operations in the sense that the deletion or contraction of a set $T \subseteq E(M)$ from M is translated as the contraction or deletion of T from M^*, respectively. In a symbolic way this is expressed as follows:

$$M\backslash T = (M^*/T)^* \text{ and } M/T = (M^*\backslash T)^* \tag{1.5}$$

For a matter of convenience in the chapters that will follow, we also employ the complement notions of deletion and contraction. Specifically, the *deletion* of M *to* a set $X \subseteq E(M)$, denoted by $M\backslash.X$, is defined as

$$M\backslash.X := M\backslash(E(M) - X),$$

while the *contraction* of M *to* a set $X \subseteq E(M)$, denoted by $M/.X$, is defined as

$$M/.X := M/(E(M) - X).$$

Important classes of matroids have been characterized in terms of excluded minors. Three of the most important and well-known such characterizations are due to Tutte which can be found in [79, 80, 82]:

Theorem 1.16. *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

Theorem 1.17. *A binary matroid is graphic if and only if it has no minor isomorphic to* F_7, F_7^*, $M^*(K_{3,3})$ *or* $M^*(K_5)$.

Theorem 1.18. *A regular matroid is graphic if and only if it has no minor isomorphic to* $M^*(K_{3,3})$ *or* $M^*(K_5)$.

Finally, we should note here that another way to characterize a class of matroids is through the development of a decomposition theory for that class. The development of a decomposition theory for the class of regular matroids is one of the most important results of matroid theory and was accomplished by Seymour in [66]. Specifically, Seymour proved Theorem 1.19 in [66], which states that any regular matroid can be decomposed by means of matroid 1-, 2- and 3-sum operations into graphic matroids, cographic matroids and copies of a specific regular matroid called R_{10}. We do not define the matroid 1-, 2-, and 3-sum operations here; however, the interested reader can find these definitions in [66, 78]. The matroid R_{10}, which first appeared in [8], is isomorphic to the vector matroid of $[I \ B_{R_{10}}]$ viewed over $GF(2)$, where $B_{R_{10}}$ is the matrix of (1.6). The converse of Theorem 1.19 (namely that the 1-sum, 2-sum or 3-sum of two regular matroids is a regular matroid), which completes the characterization of regular matroids, is easy and is mainly due to results of Brylawski [12].

$$B_{R_{10}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{1.6}$$

Theorem 1.19. *Every regular matroid can be decomposed into graphic and cographic matroids and matroids isomorphic to* R_{10} *by repeated* $1-$, $2-$, *and* $3-$*sums.*

1.6.3 Connectivity

Consider a matroid M defined by a rank function $r_M : E(M) \to \mathbb{Z}$. A *separator* of M, is any $S \subseteq E(M)$ which satisfies $r_M(S) + r_M(E(M) - S) = r(M)$. Furthermore, if S is minimal with respect to that property, then S is called an *elementary separator* of M. Generally for some positive integer k, a partition (X, Y) of $E(M)$ is called a *k-separation* of M if the following two conditions are satisfied:

(M1) $\min\{|X|, |Y|\} \geq k$, and

(M2) $r_M(X) + r_M(Y) - r(M) \leq k - 1$.

When equality occurs in condition (M2), then (X, Y) is an *exact k-separation*. Therefore, a separator defines 1-separation and vice versa. If M has a k-separation then M is called *k-separated* or *k-separable*. In the case that M has 1-separation then M is usually called *disconnected*. We say that M is *k-connected* when it does not have an l-separation for $1 \leq l \leq k - 1$. The following useful Proposition 1.20 regarding connected matroids can be found in [56].

Proposition 1.20. *A matroid* M *is connected if and only if, for every pair of distinct elements of* $E(M)$, *there is a circuit containing both.*

If a matroid M is k-separated for some integer k then the *connectivity of a matroid* of M is the smallest integer j for which M is j-separated; otherwise, we take the connectivity of M to be infinite. In the relevant literature this connectivity is also known as Tutte connectivity of M. Note also that there exist other types of connectivity for matroids; for instance, the vertical connectivity and the cyclic connectivity of a matroid, which generalize corresponding types of graph connectivity, are discussed in [19, 56].

One of the most important properties of the connectivity of a matroid M is that M is k-connected if and only if M^* is k-connected. Furthermore, with regard to the graphic matroids we have the following two theorems which relate the two types of connectivity defined for a graph to the connectivity of the corresponding graphic matroid. The first one is a result of Tutte in [83] while the second is a well-known result which is implied by Proposition 1.3 and Theorem 1.21.

Theorem 1.21. *Let G be a connected graph. Then G is k-connected if and only if $M(G)$ is k-connected.*

Theorem 1.22. *Let G be a connected graph with $|V(G)| \geq 3$ and suppose that $G \ncong K_3$. Then $M(G)$ is k-connected if and only if G is vertically k-connected and has no cycles of length less than k.*

Chapter 2

Bidirected graphs and binet matrices

There exist more general graphs than those described in Chapter 1. In this chapter such a generalisation of graphs and digraphs is provided by describing the class of bidirected graphs. Bidirected graphs have appeared several times in the relevant literature from the time they were introduced by Edmonds [21]. The reason we describe these graphs is because they serve as the background for describing binet matrices, which were introduced by Appa and Kotnyek in [3]. Binet matrices are of central importance in this thesis and are defined on bidirected graphs in a similar way to how network matrices are defined on digraphs.

Bidirected graphs are discussed in section 2.1 in which basic concepts and useful properties are provided. Section 2.2 deals with the class of binet matrices, where a formal definition of binet matrices is given along with fundamental properties of these matrices. In the same section we present an algorithm which determines a binet matrix using its bidirected graph representation and we discuss two related classes of matrices, namely the 2-regular and dyadic matrices. We should note that, apart from some propositions in section 2.2, the results contained in these two sections can be found in works of others and the relevant references are provided within these two sections. The new results of our work are contained in section 2.3 which deals with the linear and integer programming problems with binet constraint matrices.

2.1 Bidirected graphs

The class of bidirected graphs constitutes a generalisation of both graphs and digraphs defined in the previous chapter. In the case of a bidirected graph the edges can be oriented in more ways than they can be oriented in the case of a digraph. Specifically, in bidirected graphs, apart from a directed orientation, we may have the case in which both end-vertices of an edge are either heads or tails. As a consequence, a graph may be viewed as a bidirected graph in which the end-vertices of every edge are heads. We discuss bidirected graphs mainly because they are the basis on which binet matrices are defined in the second part of this chapter.

Bidirected graphs have appeared several times in the relevant literature since they were introduced by Edmonds in [21]. For example, Edmonds and Johnson have shown that important combinatorial optimization problems can be expressed by bidirected network flow models and can be solved efficiently by techniques used to solve the matching problem (see [22]). Furthermore, they also showed that the incidence matrix of a bidirected graph has strong Chvátal rank 0 or 1 [22, 23]. Finally, Zaslavsky studied extensively the closely related concept of signed graphs and also tried to establish a common terminology for bidirected graphs and the related notions (see [101]). In the following sections concerning bidirected graphs we use Zaslavsky's terminology and we largely adopt definitions and concepts appearing in works of Appa and Kotnyek ([3, 45]).

2.1.1 Basic notions

A bidirected graph Γ is defined as $\Gamma = (G, s)$, where G is a graph called the *underlying graph* of Γ and where s assigns to each $e \in E(G)$ and $v \in e$ a sign $s_e(v) \in \{+1, -1\}$. If $e = \{v, v\}$, i.e. e is a loop, then we may assign different signs on the two occurrences of v. The set of vertices $V(G)$ and the set of edges $E(G)$ of Γ are also denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. Moreover, if $s_e(v) = +1$ then v is an *in-vertex* or *head* of e, otherwise it is an *out-vertex* or *tail* of e. This terminology derives from the graphical representation of bidirected graphs appearing in the relevant literature. Specifically, the graphical representation of a bidirected graph consists of the graphical representation of its underlying graph with arrows on the end-vertices of each edge such that incoming and outgoing arrows on an edge represent positive and negative labels, respectively. Furthermore, there is a sign $\sigma_e \in \{+1, -1\}$ assigned to each edge $e \in E(\Gamma)$ defined as follows:

$$\begin{array}{ll}
\text{if } e \text{ is a link } e = \{u, v\}: & \sigma(e) = -s_e(u)s_e(v) \\
\text{if } e \text{ is a loop } e = \{u, u\}: & \sigma(e) = -s_e(u)s_e(u) \\
\text{if } e \text{ is a half-edge } e = \{u\}: & \sigma(e) = -1 \\
\text{if } e \text{ is a loose edge } e = \emptyset: & \sigma(e) = +1
\end{array}$$

If $\sigma(e) = +1$ then the edge e is called *positive*, otherwise e is a *negative edge*. Observe that in the above definition we make the convention that half-edges are always negative while loose edges are always positive. A bidirected graph without loose edges and half-edges is called an *ordinary bidirected graph*. We also say that a positive link or a positive loop is a *directed* edge while all the other edges apart from loose edges are called *bidirected*. For simplicity, if e is a directed edge of a bidirected graph Γ then in the graphical representation of Γ we depict only the head of e and thus, there is one arrow on e instead of two.

In Figure 2.1 we depict an example bidirected graph, where the edges e_1 and e_5 are negative links with $s_{e_1}(v_2) = s_{e_1}(v_3) = +1$ and $s_{e_5}(v_1) = s_{e_5}(v_3) = -1$, respectively; e_3 is a positive link with $s_{e_3}(v_3) = -s_{e_3}(v_2) = +1$; e_2 and e_8 are negative loops with $s_{e_2}(v_2) = +1$ and $s_{e_8}(v_3) = -1$, respectively; e_4 and e_7 are half-edges and hence, negative edges, with $s_{e_4}(v_3) = -1$ and $s_{e_7}(v_1) = +1$, respectively; $e_6 = \{v_1, v_1\}$ is a positive loop with $s_{e_6}(v_1) = +1$ for the one occurrence of v_1 and $s_{e_6}(v_1) = -1$ for the other occurrence of v_1.

By the definition of a bidirected graph, every concept defined for graphs automatically applies to bidirected graphs. Thus, a walk in a bidirected graph is a walk in its underlying graph; a bidirected graph is connected if and only if its underlying graph is connected and so on. Therefore, a *cycle of a bidirected graph* is a closed path in its underlying graph and thus it can be a loose edge, a loop, a half-edge or a closed path consisting only of links. The sign of a cycle of a bidirected graph is equal to the product of the signs

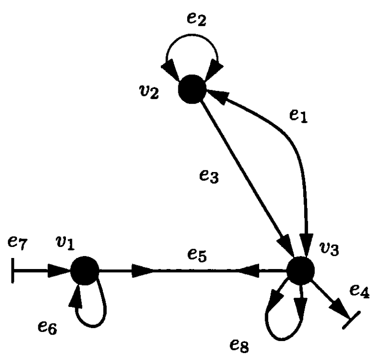


Figure 2.1: Graphical representation of a bidirected graph.

of its edges. So a cycle is called *positive* if it contains an even number of bidirected edges; otherwise, it is called *negative.* Thus, a half-edge or a negative loop is always a negative cycle while a positive loop is always a positive one. Furthermore, a bidirected graph containing at least one negative cycle is a *negative* bidirected graph; otherwise, it is called *positive.* We also say that in a walk $(v_0, e_1, v_1, \ldots, e_l, v_l)$ of $\Gamma = (G, s)$, an inner vertex v_i is *consistent* if $s_{e_i}(v_i) = -s_{e_{i+1}}(v_i)$; otherwise v_i is *inconsistent.*

As in the case of graphs and digraphs, with any bidirected graph $\Gamma = (G, s)$ we associate a matrix called the node-edge incidence matrix or simply the incidence matrix of Γ whose rows and columns are identified with the vertices and edges of Γ, respectively. Formally, the *incidence matrix* of Γ with vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$ and edge set $E(\Gamma) = \{e_1, \ldots, e_m\}$ is the $n \times m$ matrix $A = [a_{v_ie_j}]$ defined by:

$$a_{v_ie_j} := \begin{cases} s_{e_j}(v_i) & \text{if } v_i \in e_j \text{ and } e_j \text{ is a link or a half-edge,} \\ 2s_{e_j}(v_i) & \text{if } v_i \in e_j \text{ and } e_j \text{ is a negative loop,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, the incidence matrix of the bidirected graph of Figure 2.1 is the following:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 & 0 & -2 \end{bmatrix} \end{array}$$

Furthermore, given any integral $m \times n$ matrix $A = [a_{ij}]$ such that

$$\sum_{i=1}^{m} |a_{ij}| \leq 2 \text{ for } j = 1, \ldots, n \tag{2.1}$$

we can find a bidirected graph Γ such that A is the incidence matrix of Γ. Therefore, (2.1) characterizes the incidence matrix of a bidirected graph, i.e. $A = [a_{ij}]$ is the incidence matrix of a bidirected graph if and only if A satisfies (2.1). We denote by $\Gamma(A)$ the bidirected graph whose incidence matrix is A. Clearly, any submatrix of A also satisfies (2.1) and thus it has an associated bidirected graph. Finally, as in the case of graphs, note that a bidirected graph is connected if and only if its incidence matrix is not decomposable.

2.1.2 Some operations and the circuits of bidirected graphs

As we have seen, a bidirected graph is fully determined by its incidence matrix and there is a bidirected graph associated with any matrix $A = [a_{ij}]$ for which (2.1) is satisfied. In this section we briefly translate some well-known operations on the incidence matrix A of a bidirected graph to operations on $\Gamma(A) = \Gamma$. The interested reader is referred to [3, 45] for a more detailed discussion.

For the rest of this section let $\Gamma = (G, \sigma)$. Deletion of a column e is equivalent to deleting edge e from the graph. Deletion of a row r corresponds to the removal of the related vertex r together with the edge-ends incident to the vertex. That is, links incident to this vertex become half-edges while loops and half-edges located at the deleted vertex r become loose edges. All other edges remain unchanged. If we multiply a column e of A by -1 then we change the signs on the end-vertices of e in Γ, i.e. in-vertices of e become out-vertices and vice versa. This operation is called *edge reversing* and leaves the sign of e unaltered. If a row v of A is multiplied by -1 then, for any edge $e \in E(G)$ with $v \in e$, $s_e(v)$ becomes $-s_e(v)$. In other words, if v was an in-vertex of some edge e of Γ then it becomes an out-vertex of e and vice versa. This operation is called *switching at* a vertex v. There are two useful propositions regarding switching which can be found in [4, 94].

Proposition 2.1. *Switching at vertices of a bidirected graph does not alter the sign of a cycle.*

Proposition 2.2. *If a bidirected graph Γ has no negative cycles then it can be transformed to a graph with only directed edges and loose edges by a sequence of switchings at the vertices of Γ.*

The final operation on Γ that we will define here corresponds to the following operation on A (this operation appears in [31] as well):

$$\text{Replacing } A = \begin{bmatrix} a & c \\ b & D \end{bmatrix} \text{ by } A' = D - abc \tag{2.2}$$

where a is a non-zero element, b is a column vector, c is a row vector and D is a submatrix of A of appropriate size. It can be easily shown that this operation when applied to the incidence matrix of a bidirected graph maintains property (2.1). Moreover, the graph $\Gamma(A')$ will have one less vertex and one less edge than the graph Γ. Let u be the vertex and e be the edge corresponding to the first row and the first column of A, respectively. Then the operation (2.2) on A is translated to an operation on Γ which is called *contracting* edge e with vertex u . For the different kind of edges the operations must be defined carefully.

If e is a half-edge or a negative loop then u and e are removed from the graph, the incident loops and half-edges at u other than e become loose edges while the incident links lose one end-vertex and become half-edges. For example, in Figure 2.2(i) we depict the bidirected graph obtained from the graph of Figure 2.1 by contracting e_4 or e_8 with v_3.

If e is a positive link, then we get an operation similar to the ordinary graph contraction. The vertex u and the edge e are removed, the edges connected to u in Γ will be connected to v in $\Gamma(A')$ while positive links or negative links parallel to e will become positive loops or negative loops at v, respectively.

For example, in Figure 2.2(ii) we depict the bidirected graph obtained from the graph of Figure 2.1 by contracting e_3 with v_2. Note that the same graph is obtained if we contract a positive link with either of its end-vertices.

If e is negative link then contracting e with u means switching at u and then contracting the now positive link e. For example, in Figure 2.2(iii) we depict the bidirected graph obtained from the graph of Figure 2.1 by contracting e_5 with v_1.

Finally, in the literature the contraction of a positive loop and the contraction of a loose edge is also defined; specifically, contracting a positive loop or a loose edge e means deleting e from the bidirected graph. For example, in Figure 2.2(iv) we depict the bidirected graph obtained from the graph of Figure 2.1 by contracting e_6.

Remark 1. *We should note here that contracting a negative link $e = \{u, v\}$ with u and contracting with v result in different bidirected graphs. But the two graphs are switching equivalent and specifically, one graph can be obtained from the other by a single switching at the remaining vertex (u or v). When it does not cause problems, i.e. when the signs of the edges incident to u are not relevant, we shall use the shorter "contracting edge e" instead of the full "contracting e with u" and we shall denote the graph obtained from Γ by contracting edge e by Γ/e. Therefore, contraction of a negative link is well-defined only up to switching. However, in the relevant literature (see e.g. [94]), it is regarded as common practice to define contraction of a negative link as above since it has been shown that switching equivalent graphs have many common characteristics e.g. equal collections of negative cycles, equal associated matroids etc.*

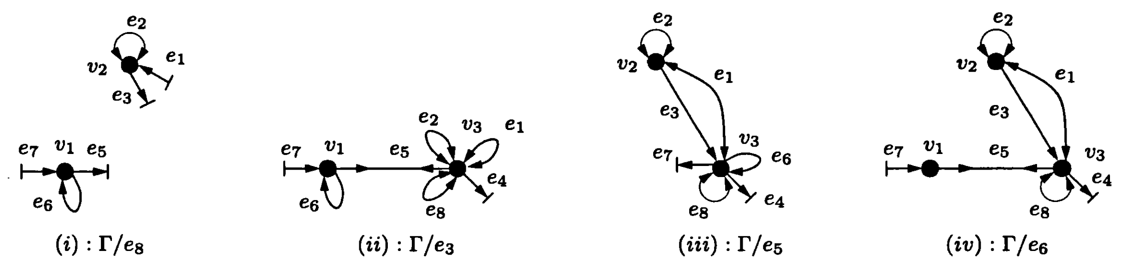


Figure 2.2: Contracting an edge from the bidirected graph of Figure 2.1.

We are also interested in the graphical characterization of the set of edges of Γ which corresponds to a maximal set of linearly independent columns in A. Such a result is given in Corollary 2.4 which is based on the following Theorem 2.3 of Zaslavsky appearing in [94]. Zaslavsky provides Theorem 2.3 using matroid terminology while Appa and Kotnyek proved this theorem and also gave Corollary 2.4 in [4] without recourse to matroids.

Theorem 2.3. *Let R be a square submatrix of the incidence matrix of a bidirected graph Γ. Then, R is non-singular if and only if each connected component of $\Gamma(R)$ is a negative 1-tree.*

If C is a minimal set of linearly dependent columns of A then $\Gamma(C)$ is called a *circuit* of Γ. We close this section with the following two consequences of Theorem 2.3 which can be also found in [4, 45].

Corollary 2.4. *Let A be a full row rank incidence matrix of a bidirected graph Γ, R be a set of columns of A and T be a set of linearly independent columns of A. Then each connected component of $\Gamma(T)$ either* *forms a tree or a negative* 1*-tree. Conversely, if every component of a subgraph* $\Gamma(R)$ *forms a tree or a negative* 1*-tree, then the columns of* R *are linearly independent.*

Corollary 2.5. *Let* C *be a subgraph of a bidirected graph* Γ*. Then,* C *is a circuit of* Γ *if and only if* C *falls in one of the following categories:*

(i) it is a subgraph of Γ *consisting of a positive cycle, or*

(ii) it is a subgraph of Γ *consisting of a pair of negative cycles with exactly one common vertex, or*

(iii) it is a subgraph of Γ *consisting of a pair of vertex disjoint negative cycles connected by a minimal path which has no common vertex with the cycles except its end-points*

A circuit of a bidirected graph which falls in category (i) of Corollary 2.5 is called a *positive circuit*, while the circuits of categories (ii) and (iii) are called *handcuffs*. In particular, a circuit falling in category (ii) is called a *handcuff of Type I* while a circuit falling in category (iii) is called a *handcuff of Type II*. In Figure 2.3, we provide examples of positive circuits and handcuffs of Type I and Type II.

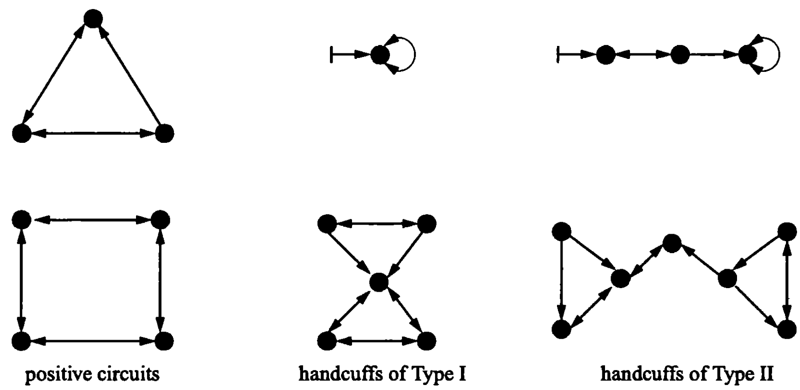


Figure 2.3: Examples of bidirected circuits.

2.2 Binet matrices and related classes of matrices

Binet matrices were introduced by Appa and Kotnyek [3, 45] and furnish a direct generalisation of network matrices. They arise from bidirected graphs in much the same way network matrices arise from digraphs. In section 2.2.1 we provide an algebraic definition of binet matrices and also present an algorithm which determines a binet matrix using its bidirected graph representation. We should note that, largely, definitions and basic concepts regarding binet matrices are taken from [4, 45].

In addition, binet matrices possess interesting properties from an optimization viewpoint. This stems mainly from the fact that a binet matrix is also a 2-regular matrix, i.e. the inverse of every non-singular square submatrix of a binet matrix is half-integral, which implies that if a binet matrix is the constraint matrix of a linear program with an integral right hand-side then all the basic optimal solutions are half-integral. Another generalisation of binet matrices discussed is that of dyadic matrices (or totally

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

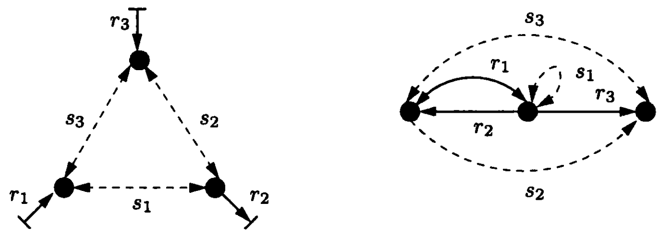


Figure 2.4: A binet matrix and two associated binet graphs.

2-modular), i.e. matrices for which every non-singular square submatrix has determinant equal to an integral power of 2. Both classes are discussed in section 2.2.2. Main references for 2-regularity are the works of Appa and Kotnyek [2, 3]. Dyadic matrices appear more frequently in the relevant literature (see e.g. [49, 91, 94]) mainly because they are representation matrices for the class of dyadic matroids.

2.2.1 Binet matrices

In what follows we denote by $A = [R\ S]$ the full row rank incidence matrix of a bidirected graph Γ, where R is a basis of A. The algebraic definition of a binet matrix goes as follows.

Definition 2.6. *Let $A = [R\ S]$ be a full row rank incidence matrix of a bidirected graph Γ, where R is a basis of A. The matrix $B = R^{-1}S$ is called a* binet matrix.

The subgraph $\Gamma(R)$ is called a *basis* of the bidirected graph; the edges in $\Gamma(R)$ are called *basic* while the remaining edges are called *non-basic*. By Corollary 2.4, the bidirected graph $\Gamma(R)$ is a collection of negative 1-trees. The unique cycle in a component of $\Gamma(R)$ is called a *basic cycle*. Furthermore, there is one-to-one correspondence between the basic edges of a bidirected graph Γ and the rows of the associated binet matrix B as well as between the non-basic edges of Γ and the columns of B. Specifically, if B is an $m \times n$ matrix, then the i^{th} row of B is indexed by the index of the i^{th} column of R and the j^{th} column of B is indexed by the index of the j^{th} column of S ($i = 1, \ldots, m$ and $j = 1, \ldots, n$). For each non-basic edge s the circuit in the bidirected graph which contains s and some basic edges is unique and is called the *fundamental circuit* of s; furthermore, the set of these circuits is the set of *fundamental circuits* of the bidirected graph with respect to a basis.

When in a bidirected graph Γ representing a binet matrix B the set of basic edges J is clearly indicated then we call it a *binet graph* or a *binet representation* of B and we denote it by (Γ, J). We also establish the following convention when we draw a binet graph: basic edges are indicated by solid lines while non-basic edges are indicated by dashed lines. Usually, there exist more than one bidirected graphs giving rise to the same binet matrix as the example of Figure 2.4 illustrates, where $J = \{r_1, r_2, r_3\}$ is the set of basic edges and $S = \{s_1, s_2, s_3\}$ is the set of non-basic ones.

We now provide the BINET MATRIX ALGORITHM which computes any column of a binet matrix from its binet graph. Clearly, by repeated application of this algorithm the binet matrix associated with a given binet graph is also computed. This algorithm, which appears in [102], helps us to prove properties of binet matrices and also makes handling of binet matrices easier. A similar algorithm can also be found in [4]. We define here some of the notions used within the algorithm. A *minimal covering walk* of a circuit C in a bidirected graph Γ, denoted by $w(C)$, is a closed walk of minimal length containing all the edges of C. By Corollary 2.5, the walk $w(C)$ covers each edge of C at most twice. More specifically, only when C is a handcuff of Type II there are edges covered twice by $w(C)$; it is not difficult to see that these are the edges contained in the path connecting the two vertex disjoint cycles of the handcuff. A minimal covering walk is called *consistent* if every inner vertex in it is consistent. It can be shown that a minimal covering walk of a circuit in a bidirected graph consisting of n edges can become consistent by a sequence of at most $n-1$ edge reversings (a detailed discussion regarding the bidirected orientation of graphs can be found in [4, 99]).

BINET MATRIX ALGORITHM

Input: A binet graph (Γ, J) and a non-basic edge s.
Output: The entries of column s of the binet matrix $B = [B_{ij}]$ associated with (Γ, J).

Step 1. For the fundamental circuit C of s, find the minimal covering walk $w(C)$ of C.

Step 2. Reverse edges in $E(C) - s$ so that $w(C)$ becomes consistent.

Step 3. Assign the weight -1 to each single covered edge which is not a half-edge; the value -2 to any half-edge and any other edge covered twice; the value 0 to all other edges in J.

Step 4. Negate the weight of the reversed edges.

Step 5. Divide by $+2$ and/or negate the weights of all the edges in $w(C)$, if necessary, to ensure that the weight of s is -1.

Step 6. Output the weights of edges in J. The entry B_{rs} in columns s of B equals the weight of the corresponding $r \in J$.

Binet matrices have entries in $\{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$; a fact which is also implied by the BINET MATRIX ALGORITHM. Moreover, in the following straightforward proposition which stems easily from the same algorithm we present the possible entries of a column of a binet matrix with respect to the number of half-edges in the corresponding fundamental circuit in the binet graph.

Proposition 2.7. *Let C be the fundamental circuit of a non-basic edge s in a binet graph (Γ, J) and let B_s be the corresponding column of the associated binet matrix. Then the following statements are valid:*

(i) *If C is a positive circuit then the entries of B_s are in $\{0, +1, -1\}$.*

(ii) *Suppose that C is a Type I handcuff. The entries of B_s are in :*$\{0, +1, -1\}$ *if C contains* 0 *or* 2 *half-edges;* $\{0, +1/2, -1/2\}$ *if the non-basic edge s is the unique half-edge in C;* $\{0, +1, -1, +2, -2\}$ *if some basic edge r is the unique half-edge in C.*

(iii) *Suppose that C is a Type II handcuff. The entries of B_s are in:* $\{0, 1, -1, +2, -2\}$ *if C does not contain half-edges and s is not contained in the path connecting the cycles of C;* $\{0, +1/2, -1/2, +1, -1\}$ *if C does not contain half-edges and s is contained in the path connecting the cycles of C or if C contains exactly one half-edge;*$\{0, 1, -1\}$ *if C contains* 2 *half-edges.*

We obtain the following corollary from Proposition 2.7.

Corollary 2.8. *If B is a binet matrix then it has no column containing both a ± 2 and a $\pm 1/2$ entry.*

Furthemore, we prove the following new Proposition 2.9 using the algebraic definition of binet matrices. However, this result can also derived easily by using the BINET MATRIX algorithm.

Proposition 2.9. *Let (Γ', J') be the binet graph obtained from a binet graph (Γ, J) by replacing a basic half-edge $r = \{v\}$ by a basic negative loop $r' = \{v, v\}$. Then the binet matrix B' associated with (Γ', J') is obtained from the binet matrix B associated with (Γ, J) by dividing all the elements of row r of B with either a $+2$ or a -2. Specifically, we divide by $+2$ if v is either a head or a tail in both r and r'; otherwise, we divide by -2.*

Proof: Let $A = [R\ S]$ and $A' = [R'\ S']$ be the incidence matrices of (Γ, J) and (Γ', J'), respectively; where by R and R' we denote a basis of A and A', respectively. By the definition of binet matrices, $RB = S$ and $R'B' = S'$. By hypothesis of the proposition $S = S'$ and R' is obtained from R by changing the single ± 2 entry in column r of A to a ± 1 in column r' of A'. By the rules of matrix multiplication the result follows easily. □

Similarly it can be shown that the following proposition is true.

Proposition 2.10. *Let (Γ', J') be the binet graph obtained from a binet graph (Γ, J) by replacing a non-basic half-edge $s = \{v\}$ by a non-basic negative loop $s' = \{v, v\}$. Then the binet matrix B' of (Γ', J') is obtained from the binet matrix B of (Γ, J) by multiplying all the elements of column s of B with either a $+2$ or a -2. Specifically, we multiply by $+2$ if v is either a head or a tail in both r and r'; otherwise, we multiply by -2.*

In [4, 45] we find the following two results regarding the operations on binet matrices.

Lemma 2.11. *Binet matrices are closed under the following operations:*

(a) Multiplying a row or column with -1

(b) Deleting a row or a column

(c) Pivoting (in $\mathbb{R}$) on a non-zero element

(d) Repeating a row or a column

(e) Adding a unit row or a unit column

Lemma 2.12. *Switching at a vertex of a binet graph does not alter the binet matrix.*

In [45] we find that a non-integral binet matrix can be transformed to an integral one by a finite number of pivot operations. More specifically:

Theorem 2.13. *Let B be an $m \times n$ non-integral binet matrix. Then, by at most $2m$ pivot operations B can be transformed to an integral binet matrix.*

We close this section with some important examples of binet matrices.

Example 1. *Incidence matrices of bidirected graphs.*

If A is the incidence matrix of a bidirected graph then the matrix $[I\ A]$ is also an incidence matrix of a bidirected graph since (2.1) is still satisfied. If we take $R = I$ and $S = A$ then we have that $R^{-1}S = A$ and, thus, A is a binet matrix.

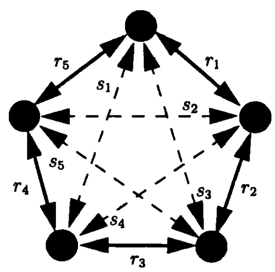
Example 2. *Network matrices.*

If N is a network matrix then there exists an incidence matrix $[R\ S]$ of a digraph such that $N = R'^{-1}S'$, where R' and S' are submatrices of R and S, respectively, obtained from $[R\ S]$ by deleting a row. Clearly, $[R'\ S']$ satisfies (2.1) and, therefore, it is the incidence matrix of a bidirected graph. Since we also have that $N = R'^{-1}S'$, it follows that N is a binet matrix.

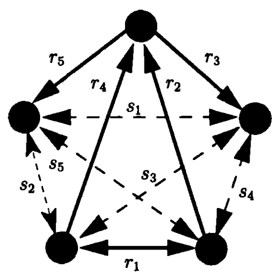
Example 3. *The matrices B_1 and B_2 of* (1.1).

Matrices B_1 and B_2 play an important role in the decomposition of totally unimodular matrices. Neither B_1 nor B_2 is a network matrix or the transpose of a network matrix. However, it has been shown in [4, 45] that they are binet matrices. Specifically, a binet graph for each is depicted below:

$$B_1 = \begin{array}{cc} & \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{array} \tag{2.3}$$



$$B_2 = \begin{array}{cc} & \begin{array}{ccccc} s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{2.4}$$



2.2.2 Dyadic and 2-regular matrices

Theorem 1.5 is a cornerstone in integer programming and exhibits the importance of totally unimodular matrices. Specifically, given a polyhedron $P = \{x \mid Ax \leq b, x \geq 0\}$, where A is an integral matrix,

$P = P_I$ for any integral vector b if and only if A is totally unimodular. However, notice that A is assumed to be integral in this important result. Therefore, a natural research question could be stated as follows: what happens if A is a non-integral matrix or, more specifically, under which conditions a specific class of rational matrices A ensures that $P = P_I$. One could also ask what we can say for matrices A which ensure that $P = P_I$ for special right hand-side vectors b, e.g. if b is a 2-integral vector. It is not difficult to see that these two questions are related to each other, since there always exists a positive integer k such that the matrix $B = kA$ is integral while each element of the vector $d = kb$ is an integer multiple of k. In other words, the polyhedron $P = \{x \mid Ax \leq b, x \geq 0\}$ is equal to $P' = \{x \mid Bx \leq d, x \geq 0\}$, where $B = kA$ is integral and $d = kb$ is a k-integral vector. Therefore, we seek a characterisation of integral matrices B for which the polyhedron $P' = \{x \mid Bx \leq d, x \geq 0\}$ is integral for all k-integral vectors d.

This question was answered in [3] for the more general case in which B is a rational matrix. It was shown there that these matrices are precisely the *k-regular* matrices defined as follows:

Definition 2.14. *Let A be a rational matrix and k be a positive integer. Then, A is called* k-regular *if for each of its non-singular square submatrices R,* kR^{-1} *is integral.*

The important theorem of [3] which answers the above question goes as follows:

Theorem 2.15. *Let A be a rational matrix and k be a positive integer. The polyhedron* $P = \{x \mid Ax \leq kb, x \geq 0\}$ *is integral for each integral vector b if and only if A is k-regular.*

Clearly, by Theorems 1.5 and 2.15, the class of k-regular matrices may be viewed as a generalisation of totally unimodular matrices. Moreover, in [3], it is shown that for $k = 2$, the class of 2-regular matrices contains as a subclass that of binet matrices.

Theorem 2.16. *Every binet matrix is 2-regular.*

A corollary of Theorem 2.15 is that the 2-regularity of a matrix A implies that for every vertex x of the polyhedron $\{x \mid Ax \leq b, x \geq 0\}$, $2x$ is integral. In other words, if the linear program: $\max\{cx \mid Ax \leq b, x \geq 0\}$ (b integral) has a 2-regular constraint matrix A then it has half-integral optimal solutions for any rational vector c. However, a stronger version of this result holds for binet matrices (Theorem 2.17) which also appears in [3].

Theorem 2.17. *If B is a binet matrix and* l, u, a, b *are integral vectors of appropriate size, then the basic solutions of the optimization problem*

$$\max\{cx \mid l \leq x \leq u,\ a \leq Bx \leq b\} \tag{2.5}$$

are all half-integral.

Furthemore, in [3] the authors proved that if A is an integral 2-regular matrix of size $m \times n$, then for polyhedron $Q = \{x \mid Ax \leq b, x \geq 0\}$ with integral b, the rank-1 closure Q_1 can be achieved by only half-integral cuts, i.e., valid inequalities of the form $\lambda Ax \leq \lfloor \lambda b \rfloor$ where $\lambda \in \{0, \frac{1}{2}\}^m$ and λA is integral. Specifically, Theorem 2.18 can be found in [3], where by $Q_{\frac{1}{2}}$ we denote the intersection of Q with the half-spaces induced by all the possible half-integral cuts.

Theorem 2.18. *Let A be an $m \times n$ integral matrix and $Q = \{x \mid Ax \leq b, x \geq 0\}$. Then, A is 2-regular if and only if $Q_1 = Q_{\frac{1}{2}}$ for all $b \in \mathbb{Z}^m$.*

Another interesting generalisation of totally unimodular matrices is the class of *dyadic* (or *totally 2-modular*) matrices. The definition of dyadic matrices goes as follows [3]:

Definition 2.19. *A matrix is called* dyadic *if for all of its square submatrices R, $det(R) \in \{0, \pm 2^k \mid k \in \mathbb{Z}\}$.*

We should note here that in [3] the more general class of totally k-modular matrices is defined; however, for the purposes of this work we restrict ourselves to the $k = 2$ case. Since integral binet matrices have elements in $\{0, \pm 1, \pm 2\}$ and, by Theorem 2.16, are also 2-regular, the proof of Theorem 2.20 is straightforward.

Theorem 2.20. *Integral binet matrices are dyadic.*

In contrast to 2-regular matrices the class of dyadic matrices does not possess any important property from an optimization point of view. However, the matroids which can be represented by a dyadic matrix over $\mathbb{Q}$, the so-called *dyadic matroids* appear frequently in the matroid representation theory literature. From a seminal result of Tutte we know that a matroid is representable over $GF(2)$ and the reals if and only if it has a totally unimodular matrix representation. A first natural extension of this result would be a similar characterization of the matroids representable over the $GF(3)$ and the reals. For this case, Whittle proved in [90] that a matroid is representable over $GF(3)$ and the reals if and only if it has a dyadic matrix representation. We would like also to stress here that this is only one of the many important results of this work of Whittle and we strongly recommend the interested reader in matroid representation theory to view also the results in [91, 92].

2.3 Optimization with binet matrices

Probably the first natural question arising from Theorem 2.17 is how to find efficiently a solution to the maximisation problem of (2.5). The answer to this question is discussed in [5] and [45] and in the rest of this section we present the main results of these works and pose an interesting question.

We start with the case in which we are looking for a solution to the linear programming problem defined by (2.5) and then we discuss the case in which the variables are restricted to the integral domain. Note that if the matrix B of (2.5) is a network matrix and l, u, a, b are integral vectors then the optimal solution vector is always integral. This stems from the fact that the class of network matrices is a subclass of totally unimodular matrices and therefore Theorem 1.5 applies. However, this is not the case for binet matrices and therefore we have to consider the two cases separately. Therefore, in section 2.3.1 we present known algorithms that solve the linear programming problems with binet constraint matrices while in section 2.3.2 we show that there exists a polynomial-time algorithm for the integer programming problems whose constraint matrices are binet. In section 2.3.3 the $\{0, \frac{1}{2}\}$-separation problem (a special case of the so-called separation problem in combinatorial optimization) is defined and it is shown that for integral binet matrices this problem is solvable in polynomial time through the ellipsoid method. Finally, we pose the interesting question of devising a combinatorial algorithm for this particular case and provide some related results.

2.3.1 Binet linear programming problems

We first deal with the binet linear programming problem, defined as:

$$\max\{cx \mid l \leq x \leq u,\ a \leq Bx \leq b\} \tag{2.6}$$

in which B is a binet matrix. Note that in (2.6) l, u, a, b are vectors whose elements are not necessarily integers. Clearly we can find an optimal solution to (2.6) by applying general-purpose methods, where the obvious choice would be the well-known simplex algorithm which is not a polynomial-time method in the worst case. Polynomial-time methods for solving linear programming problems do exist. The best known such algorithms are variants of the so-called *ellipsoid method* of Khachiyan, which was based on earlier work on nonlinear optimization problems, and the algorithm of Karmarkar which falls in the category of interior point methods for linear programming. Both algorithms, along with interesting references, can be found in books on linear programming and optimization such as [37, 64]. In both algorithms the size of the input numbers influence the number of elementary arithmetic operations (viz. addition, subtractions, multiplications, divisions and comparisons) to be performed during the algorithm. Thus, an input with larger numbers causes not only more complicated arithmetic operations but also increases the number of these operations. A class of algorithms which does not have this drawback is that of strongly polynomial algorithms. Formally, an algorithm consisting of elementary arithmetic operations is called *strongly polynomial* if: (i) the number of arithmetic operations needed is polynomially bounded in the dimension of the input, where the *dimension of the input* is the number of data items in the input (each number is considered to add one in the dimension of the input), and (ii) when it is applied to rational input, the numbers occurring in the algorithm are of size polynomially bounded in the dimension of the input and the size of the input numbers [37, 76].

An important result of Tardos in [76] states that there exists a strongly polynomial algorithm which solves any linear programming problem $\max\{cx \mid x \geq 0, Ax \leq b\}$ in which the elements of A are bounded; specifically, the number of arithmetic operations in this algorithm depends only on the size of the elements of A. The elements of a binet matrix are small rationals, viz. rationals between -2 and $+2$, and therefore the algorithm of Tardos has a strongly polynomial worst-case running time for linear programming problems with a binet constraint matrix.

Theorem 2.21. *There exists a strongly polynomial time algorithm for the binet linear programming problem.*

In general, although Tardos' algorithm has a very "good" computational complexity, it should be mainly considered a theoretical contribution as she herself mentions in [76]. In addition, most of recent implementations of the simplex algorithm outperform this strongly polynomial method in practice since Tardos' algorithm relies on Khachiyan's method [5, 45]. For that reason, we describe a method which, although not polynomial in the worst case, works well in practice since it utilizes the underlying combinatorial structure of binet matrices.

As shown in [5, 45] an alternative and efficient method of solving (2.6) is the *generalized network simplex algorithm* which is a known adaptation of the simplex method. A *generalized network* consists of a connected digraph G together with a real non-zero multiplier p_e associated with each edge $e = (i, j)$

of G where we assume that if a unit flow leaves the tail i of e, then p_e units arrive at j. Furthermore, it is assumed that the multiplier of a loop cannot be $+1$ since such a loop would be redundant. If all multipliers are equal to $+1$ then we have the well-known *network*. A generalized network on a digraph $G = (V, E)$ is determined by its incidence matrix. The *incidence matrix of a generalized network* is the $|V| \times |E|$ matrix whose rows and columns are indexed by the vertices and edges of G, respectively, where: the column $e \in E$ corresponding to a non-loop edge $e = (u, v)$ contains a -1 in row u, a p_e in row v and 0 elsewhere; the column $f = (y, y)$ corresponding to a loop $f \in E$ contains a $p_f - 1$ in row y and 0 elsewhere.

The generalized network simplex method works on linear programming problems whose constraint matrix is projectively equivalent to the incidence matrix of a generalized network. The main point in this algorithm is that the main steps of the simplex method are performed directly on the generalized network associated with the constraint matrix of (2.6) and thus the special structure of the constraint matrix is utilized. The reader may also recall that the same idea is behind the well-known network simplex algorithm. An introduction to generalized networks through a number of applications can be found in [33] while the reader is referred to [1, 53] for a more thorough analysis of the generalized networks and the generalized network simplex algorithm.

Specifically, in [5, 45], it is shown that, with particular column transformations, (2.6) can be converted to an equivalent linear programming problem whose constraint matrix is projectively equivalent to the incidence matrix of a generalized network and thus, an optimal solution can be found by the generalized network simplex method. However, as argued in [5, 45], the generalized network simplex method presented in [1, 53] can be adapted directly to include the case in which the constraint matrix is the incidence matrix of a bidirected graph. This in turn implies that the aforementioned column transformations are not really necessary. Finally, as expected, in the worst-case the generalized network simplex method is not polynomial. However, we should note that in the majority of cases it outperforms substantially the classic simplex method as well as the strongly polynomial method of Tardos (see Reference Notes in Chapter 15 of [1]).

2.3.2 Binet integer programming problems

We turn now to the more interesting *binet integer programming problem*:

$$\max\{cx \mid l \le x \le u,\ a \le Bx \le b,\ x \text{ integral}\} \tag{2.7}$$

where B is an integral binet matrix and l, u, a, b are integral vectors. Let $A = [R\ S]$ be the incidence matrix of a bidirected graph of full row rank, where R is a basis of A, such that $B = R^{-1}S$. Starting from (2.7) we present the following sequence of transformations. Initially, we introduce new variables $z = -Bx$ to get:

$$\max\{cx + \mathbf{0}z \mid l \le x \le u,\ -b \le z \le -a,\ Bx + z = \mathbf{0},\ x, z \text{ integral}\} \tag{2.8}$$

and then we multiply the equality in (2.8) by R to get:

$$\max\{cx + \mathbf{0}z \mid l \le x \le u,\ -b \le z \le -a,\ Sx + Rz = \mathbf{0},\ x, z \text{ integral}\} \tag{2.9}$$

In these transformations, firstly notice that the integrality of B implies the integrality of z and secondly, it is clear that the integer programming problems (2.7) and (2.9) are equivalent. Moreover, if we substitute $x - l$ and $z + b$ by x and z, respectively, (2.9) can be brought to the following form:

$$\max\{wy \mid 0 \leq y \leq h,\ Ay = d,\ y \text{ integral}\} \tag{2.10}$$

As argued in [5, 45], (2.10) can be viewed as a bidirected network flow problem in which the objective is to maximize the weighted integer flow in a bidirected network under some supply (or demand) and capacity constraints. More specifically, (2.10) is a bidirected network flow problem where y stands for the flow on the edges in a bidirected graph with incidence matrix A, h is the capacity vector and d is the vector containing the supply or demand at the vertices. The bidirected network flow problems were introduced by Edmonds in [21]. More importantly, Edmonds has also shown that this class of problems is part of a bigger class of problems, the so-called class of *general matching problems* (see also [48]). Thus, a polynomial time algorithm for the general matching problem would mean that there exists a polynomial time algorithm for (2.7). For that reason, in what follows we provide the necessary definitions regarding the general matching problems, for which our main sources are the works of Gerards [28, 30].

The *general matching problem* is defined on an ordinary graph $G = (V, E)$, where we denote by $\gamma(v)$ and $\delta(v)$ the set of loops and links incident with vertex $v \in V(G)$, respectively. Moreover, we associate with every edge $e \in E$ an edge-weight $w_e \in \mathbb{R}$ and a capacity $c_e \in \mathbb{R} \cup \{\infty\}$. Also, by $a_v \in \mathbb{R} \cup \{\infty\}$ and $b_v \in \mathbb{R} \cup \{\infty\}$ we denote the lower and the upper bound, respectively, of the degree of a vertex $v \in V$. The objective of the general matching problem is to find a minimum or maximum weight integral vector $x = [x_e] \in \mathbb{R}^E$, satisfying:

$$\begin{array}{ll} \alpha_v \leq 2\sum_{e\in\gamma(v)} x_e + \sum_{e\in\delta(v)} x_e \leq b_v & \text{for all } v \in V \\ 0 \leq x_e \leq c_e & \text{for all } e \in E \end{array} \tag{2.11}$$

An integral vector x satisfying (2.11) is called a *general matching*. Clearly, if $\alpha_v = 0$ and $b_v = 1$ for all $v \in V$ then we get the well-known *matching* problem. Moreover, if $\alpha_v = b_v = 1$ for all $v \in V$ then we get the classical *perfect matching problem*. The matching problem and the perfect matching problem are treated in many combinatorial optimization books, e.g. [16, 58, 65]. If $\alpha_v = 0$ and b_v is arbitrary for all $v \in V$ and $c_e = \infty$ for all $e \in E$ then (2.11) is called a *b-matching problem*. The *perfect b-matching* problem is the problem described by (2.11) in which $\alpha_v = b_v$ for all $v \in V$ and $c_e = \infty$ for all $e \in E$.

Maybe the most exciting aspect of matching theory is that not only the classical matching problem is a special case of the general matching problem but also the fact that the general matching problem can be reduced to the classical matching problem. Furthermore, according to these reductions which are described in [28, 30], a polynomial time algorithm for the b-matching problem implies a polynomial time algorithm for the general matching problem.

Edmonds showed that any bidirected network flow problem is equivalent to a b-matching problem (see [48]). Thus, a polynomial time algorithm for the b-matching problem implies that the problem (2.10) and, equivalently, the binet integer programming problem can be solved in polynomial time. Edmonds also proved that a b-matching problem can be solved by solving exactly one general matching problem of polynomial size in a bipartite graph and exactly one perfect matching problem of polynomial size (see [28, 30]). The general matching problem in a bipartite graph has been shown to be equivalent to a minimum cost flow problem, the so-called minimum cost circulation problem. The first polynomial time algorithm for the minimum cost flow problem was devised by Edmonds and Karp in [24]. In the same paper they posed the development of a strongly polynomial algorithm for this problem as an open question. Tardos was the first to settle this question in [75]. Thus, there is a strongly polynomial algorithm for the general matching problem which in turn implies the following result of [5, 45]:

Theorem 2.22. *There exists a strongly polynomial time algorithm for the binet integer programming problem.*

2.3.3 Strong Chvátal rank and the $\{0, \frac{1}{2}\}$-separation problem

From an optimization viewpoint, an important characteristic of binet matrices is that they have strong Chvátal rank 1 [5]. We should note here that not many known non-trivial classes of matrices have this property. Specifically, we are aware of three other such classes of matrices. The first and oldest one is due to Edmonds and Johnson who showed that the incidence matrices of bidirected graphs have strong Chvátal rank 1 [22, 23] (for that reason matrices with strong Chvátal rank 1 are also known as matrices having the *Edmonds-Johnson property*). The second class is due to Gerards and Schrijver who characterized the class of transposes of incidence matrices of bidirected graphs having the Edmonds-Johnson property[31]. Finally, very recently Del Pia and Zambelli in [59, 60] extended the result of Edmonds and Johnson by providing a more general class of matrices and furthermore, they showed that any matrix obtained from a totally unimodular matrix with two non-zero elements per row by multiplying some of its columns by 2 has the Edmonds-Johnson property. In [5] we can find Theorem 2.23 showing that binet matrices belong to the class of matrices having the Edmonds-Johnson property.

Theorem 2.23. *If B is an integral binet matrix, then it has strong Chvátal rank 1.*

As binet matrices are 2-regular (Theorem 2.16), we immediately get the following corollary from Theorems 2.18 and 2.23.

Corollary 2.24. *If B is an integral binet matrix and b is an integral vector, then the integer hull of $Q = \{x \mid Bx \leq b, x \geq 0\}$ can be achieved by half-integral cuts, i.e., $Q_I = Q_{\frac{1}{2}}$.*

This result has an interesting consequence in separation. The $\{0, \frac{1}{2}\}$*-separation problem*, as defined in [15], is the following:

> Given $x \in Q = \{x \mid Ax \leq b\}$, decide if x is in $Q_{\frac{1}{2}}$ or not, and if it is not, find a half-integral cut that separates it, i.e., a $\lambda \in \{0, \frac{1}{2}\}^m$ such that $\lambda A \in \mathbb{Z}^n$ and $\lambda A x > \lfloor \lambda b \rfloor$.

It is well known (see e.g. [65]), that the general separation problem can be shown to be polynomially equivalent to the optimization problem through the ellipsoid method. In the special case of $\{0, \frac{1}{2}\}$-separation, it means that if we can optimize linear functions over $Q_{\frac{1}{2}}$ in polynomial time, then we can decide the separation question in polynomial time. As shown in section 2.3.2, the integer optimization (i.e., optimizing linear functions over Q_I) with integral binet constraint matrices is polynomially solvable, since it is equivalent to a matching problem. As a result we have the following consequence of Corollary 2.24:

Corollary 2.25. *If A is an integral binet matrix, then the $\{0, \frac{1}{2}\}$-separation problem can be solved in polynomial time.*

If A or its transpose is a network matrix, then the $\{0, \frac{1}{2}\}$-separation is trivial, as $Q_{\frac{1}{2}} = Q$. This is because for totally unimodular matrices $Q_I = Q$, and for any polyhedron $Q_I \subseteq Q_{\frac{1}{2}} \subseteq Q$. Corollary 2.25 extends this result to integral binet matrices.

However, the result of Corollary 2.25 is based on the ellipsoid method which is polynomial (in the worst case) in theory, but slow in practice. It is therefore of interest to find a combinatorial algorithm for the $\{0, \frac{1}{2}\}$-separation problem with integral binet matrices. Although we leave this as an open problem, in the following discussion we shall show that integral binet matrices do not fall into the known classes of special cases for which $\{0, \frac{1}{2}\}$-separation can be solved polynomially by a combinatorial algorithm.

The first polynomially solvable cases of $\{0, \frac{1}{2}\}$-separation were obtained by Caprara and Fischetti as follows [15]:

Theorem 2.26. *Let $Q = \{x \mid Ax \leq b\}$. The $\{0, \frac{1}{2}\}$-separation is polynomially solvable if the matroid with compact representation matrix A mod 2 is either graphic or the dual of a graphic matroid.*

These results were further generalized by Letchford in [50] (see Theorem 2.27) who showed that the binary matroid M with compact representation matrix A mod 2 does not necessarily have to be graphic or cographic. Specifically, it suffices for M not to have the matroid F_7 or the matroid F_7^* as a minor. The relevant result goes as follows:

Theorem 2.27. *Let $Q = \{x \mid Ax \leq b\}$. The $\{0, \frac{1}{2}\}$-separation is polynomially solvable if the binary matroid with compact representation matrix A mod 2 does not contain F_7 or F_7^* as a minor.*

The separation problem in Theorem 2.27 is transformed into a minimum weight odd cycle problem on binary matroids, where given a weight function $w : E \to \mathbb{R}_+$ and a parity on the elements of E, a minimum weight cycle of odd parity is required. This problem is then solved by a combinatorial algorithm given in [38].

We now show that Theorem 2.27 does not cover the case in which A is a binet matrix. A compact representation matrix B_{F_7} of the matroid F_7 is the following:

$$B_{F_7} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Thus, $B_{F_7}^T$ is a compact representation matrix of F_7^*. Both B_{F_7} and $B_{F_7}^T$ are integral binet matrices, since we have below incidence matrices of bidirected graphs, namely $[R_{B_{F_7}} S_{B_{F_7}}]$ and $[R_{B_{F_7}}^T S_{B_{F_7}}^T]$, such that $B_{F_7} = R_{B_{F_7}}^{-1} S_{B_{F_7}}$ and $B_{F_7}^T = (R_{B_{F_7}}^T)^{-1} S_{B_{F_7}}^T$.

$$R_{B_{F_7}} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, S_{B_{F_7}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$R_{B^T_{F_7}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } S_{B^T_{F_7}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Therefore, the matrix $B = \begin{bmatrix} B_{F_7} & \mathbf{0} \\ \mathbf{0} & B^T_{F_7} \end{bmatrix}$ is an integral binet matrix, while the matroid with compact representation matrix B mod 2 contains both F_7 and F_7^* as minors.

Part II

k-Sums of matrices

Chapter 3

k-sums of network and binet matrices

Seymour's decomposition theorem for totally unimodular matrices states that all totally unimodular matrices can be constructed recursively by applying k-sum operations ($k = 1, 2, 3$) on network matrices, transposes of network matrices and two special totally unimodular matrices, namely B_1 and B_2 of (1.1), which are neither network matrices nor transposes of network matrices. In [4], Appa and Kotnyek have shown that B_1 and B_2 are binet matrices. Thus, all the building blocks of totally unimodular matrices or their transposes are binet. It would be of interest to see whether the totally unimodular matrices produced by the k-sum operations between network and binet matrices are either network or binet since very efficient ways for solving the linear and integer programming problems with network and binet constraint matrices exist. An investigation of this question has led to several new results. Specifically, the known result that the k-sum of two network matrices is a network matrix is shown here in a constructive way along with the new result stating that the k-sum of a network and a binet matrix is a binet matrix. Moreover, given the network or binet representations of network or binet matrices respectively, we provide graphical methods to construct representations of the k-sums of these matrices. However, for $k = 2, 3$ we show that the k-sum of two binet matrices is not necessarily a binet matrix. Based on this we can state that not all totally unimodular matrices are binet.

This chapter is organized as follows. In section 3.1 we examine the operation of k-sums between network matrices and in section 3.2 we examine the k-sum operations between a network and a binet matrix. In both sections the most general case for $k = 3$ is treated and a graphical construction of the k-sum operations is presented. The negative result stating that binet matrices are not closed under k-sums is provided in section 3.3. Finally, an interesting real-life application with a totally unimodular matrix being neither network nor the transpose of a network matrix is discussed in section 3.4.

3.1 The k-sum operations between network matrices

In this section we shall prove that network matrices are closed under the k-sum operations. Furthermore, since network matrices are compact representation matrices of graphic matroids, a direct consequence of our results is the well known fact (see [56]) that graphic matroids are closed under matroidal k-sum operations. The most general case of 3-sum will be examined since, as it will be shown in Theorem 3.2, the other sum operations follow.

Lemma 3.1. *If N_1 and N_2 are network matrices such that*

$$N_1 = \begin{array}{c} \\ \\ e_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} & e_1 & e_2 \end{array} \\ \begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \end{array}, \quad N_2 = \begin{array}{c} \\ f_3 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} f_1 & f_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} \end{array},$$

then $N = N_1 \oplus_3 N_2$ is a network matrix.

Proof: Because of the definition of the 3-sum operation we have that in a possible graphical representation of N_1 the fundamental cycle of e_1 consists of the edges that correspond to non-zero elements in a. The fundamental cycle of e_2 has all these edges and e_3. This means that e_1, e_2 and e_3 should form a triangle. Similarly, f_1, f_2 and f_3 form a triangle in any network representation of N_2. Now let $[R_1|S_1]$ and $[R_2|S_2]$ be the incidence matrices associated with N_1 and N_2, respectively. Up to permutations of rows and columns and/or multiplications of rows with ± 1, we have that:

$$[R_1|S_1] = \begin{array}{c} \begin{array}{cccccc} & e_3 & & e_1 & e_2 \end{array} \\ \left[\begin{array}{cc|ccc} r_1 & -1 & s_1 & 0 & -1 \\ r_1' & 1 & s_1' & -1 & 0 \\ r_1'' & 0 & s_1'' & 1 & 1 \\ R_1' & \mathbf{0} & S_1' & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}, \quad [R_2|S_2] = \begin{array}{c} \begin{array}{ccccc} f_3 & & f_1 & f_2 & \end{array} \\ \left[\begin{array}{cc|ccc} 0 & r_2 & -1 & -1 & s_2 \\ -1 & r_2' & 0 & 1 & s_2' \\ 1 & r_2'' & 1 & 0 & s_2'' \\ \mathbf{0} & R_2' & \mathbf{0} & \mathbf{0} & S_2' \end{array}\right] \end{array} \tag{3.1}$$

where $\mathbf{0}$ is a vector or matrix of zeros of appropriate size, $r_i, r_i', r_i'', s_i, s_i'$ and s_i'' are row vectors and R_i', S_i' are matrices of appropriate size ($i = 1, 2$). By the definition of network matrices the following two equations hold:

$$R_1 N_1 = S_1, \qquad R_2 N_2 = S_2 \tag{3.2}$$

For N_1 using (3.1) and (3.2) we have that:

$$\left[\begin{array}{c|c} r_1 & -1 \\ r_1' & 1 \\ r_1'' & 0 \\ \hline R_1' & \mathbf{0} \end{array}\right] \left[\begin{array}{c|c|c} A & a & a \\ \hline c & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c|c} s_1 & 0 & -1 \\ s_1' & -1 & 0 \\ s_1'' & 1 & 1 \\ \hline S_1' & \mathbf{0} & \mathbf{0} \end{array}\right]$$

where upon decomposing the block matrix multiplications we derive the following equations:

$$\begin{bmatrix} r_1 \\ r_1' \\ r_1'' \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s_1' \\ s_1'' \end{bmatrix}; \quad \begin{bmatrix} r_1 \\ r_1' \\ r_1'' \end{bmatrix} a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \tag{3.3}$$

$$\begin{bmatrix} r_1 \\ r_1' \\ r_1'' \end{bmatrix} a + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad R_1'A = S_1'; \quad R_1'a = \mathbf{0}.$$

Similarly, for N_2 using (3.1) and (3.2) we have that:

$$\left[\begin{array}{c|c} 0 & r_2 \\ -1 & r_2' \\ 1 & r_2'' \\ \hline \mathbf{0} & R_2' \end{array}\right] \left[\begin{array}{c|c|c} 1 & 0 & b \\ \hline d & d & B \end{array}\right] = \left[\begin{array}{c|c|c} -1 & -1 & s_2 \\ 0 & 1 & s_2' \\ 1 & 0 & s_2'' \\ \hline \mathbf{0} & \mathbf{0} & S_2' \end{array}\right]$$

where upon decomposing the block matrix multiplications we derive the following equations:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} r_2 \\ r_2' \\ r_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} r_2 \\ r_2' \\ r_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \tag{3.4}$$
$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} b + \begin{bmatrix} r_2 \\ r_2' \\ r_2'' \end{bmatrix} B = \begin{bmatrix} s_2 \\ s_2' \\ s_2'' \end{bmatrix}; \quad R_2'd = \mathbf{0}; \quad R_2'B = S_2'.$$

Using block matrix multiplication and equations in (3.3) and (3.4), it is easy to show that the following equality holds:

$$\underbrace{\left[\begin{array}{c|c} r_1 & r_2 \\ r_1' & r_2' \\ r_1'' & r_2'' \\ \hline R_1' & \mathbf{0} \\ \hline \mathbf{0} & R_2' \end{array}\right]}_{R'} \underbrace{\left[\begin{array}{c|c} A & ab \\ \hline dc & B \end{array}\right]}_{N} = \underbrace{\left[\begin{array}{c|c} s_1 & s_2 \\ s_1' & s_2' \\ s_1'' & s_2'' \\ \hline S_1' & \mathbf{0} \\ \hline \mathbf{0} & S_2' \end{array}\right]}_{S'} \tag{3.5}$$

The matrix $[R'|S']$ is the incidence matrix of a directed graph since each column contains a $+1$ and a -1. It remains to be shown that the matrix $\hat{R}$ obtained by deleting one row of R' is non-singular. If we delete the first row of R' we have that:

$$\hat{R} = \left[\begin{array}{c|c} r_1' & r_2' \\ r_1'' & r_2'' \\ R_1' & \mathbf{0} \\ \hline \mathbf{0} & R_2' \end{array}\right]$$

If we delete the first row from R_1 then we obtain the matrix $\begin{bmatrix} r_1' & 1 \\ r_1'' & 0 \\ R_1' & \mathbf{0} \end{bmatrix}$ which is non-singular. Expanding now the determinant of this matrix along the last column we can see that the matrix $\begin{bmatrix} r_1'' \\ R_1' \end{bmatrix}$ is also non-singular. Therefore, within the submatrix $\begin{bmatrix} r_1' \\ r_1'' \\ R_1' \end{bmatrix}$ of $\hat{R}$, r_1' can be written as a linear combination of the other rows:

$$r_1' + u\, r_1'' + qR_1{}' = 0 \tag{3.6}$$

where u is a scalar, and q is a column vector of appropriate size with elements in $\mathbb{R}$. Also, we have that $u \neq 0$ since if we delete e_3 in R_1 then the matrix obtained corresponds to a forest in which the vertices which correspond to rows r_1' and r_1'' belong to the same tree of that forest. Using (3.6) we get:

$$det(\hat{R}) = det\left(\left[\begin{array}{c|c} r_1' & r_2' \\ r_1'' & r_2'' \\ R_1{}' & \mathbf{0} \\ \hline \mathbf{0} & R_2{}' \end{array}\right]\right) = det\left(\left[\begin{array}{c|c} 0 & r_2' + u\, r_2'' \\ r_1'' & r_2'' \\ R_1{}' & \mathbf{0} \\ \hline \mathbf{0} & R_2{}' \end{array}\right]\right) = det\left(\left[\begin{array}{c|c} r_1'' & r_2'' \\ R_1{}' & \mathbf{0} \\ \hline 0 & r_2' + u\, r_2'' \\ 0 & R_2{}' \end{array}\right]\right) \tag{3.7}$$

So, matrix $\hat{R}$ is block diagonal and its blocks are square. Thus:

$$\begin{aligned} det\left(\left[\hat{R}\right]\right) &= det\left(\left[\begin{array}{c} r_1'' \\ R_1{}' \end{array}\right]\right) \cdot det\left(\left[\begin{array}{c} r_2' + u\, r_2'' \\ R_2{}' \end{array}\right]\right) \\ &= det\left(\left[\begin{array}{c} r_1'' \\ R_1{}' \end{array}\right]\right) \cdot \left(det\left(\left[\begin{array}{c} r_2' \\ R_2{}' \end{array}\right]\right) + u \cdot det\left(\left[\begin{array}{c} r_2'' \\ R_2{}' \end{array}\right]\right)\right) \end{aligned} \tag{3.8}$$

If we delete from R_2 its first row then the matrix so-obtained is non-singular and, since it is a submatrix of a TU matrix, it has to be TU as well, i.e. its determinant should be equal to ± 1. Expanding the determinant of that matrix along its first column we take:

$$det\left[\begin{array}{c} r_2' \\ R_2{}' \end{array}\right] + det\left[\begin{array}{c} r_2'' \\ R_2{}' \end{array}\right] = \pm 1 \tag{3.9}$$

Furthermore $det\left[\begin{array}{c} r_2' \\ R_2{}' \end{array}\right], det\left[\begin{array}{c} r_2'' \\ R_2{}' \end{array}\right] \in \{0, \pm 1\}$ since the corresponding matrices are TU. From (3.9) we see that exactly one of these matrices has a non-zero determinant. Combining this with (3.8) and the fact that $u \neq 0$ we have that $\hat{R}$ is non-singular.
Finally, it is obvious that the matrix $[R'|S']$ contains a -1 and a $+1$ in each column since its columns are columns of $[R_1|S_1]$ and $[R_2|S_2]$. We can conclude that the 3-sum of two network matrices is a network matrix with incidence matrix $[R'|S']$.

□

Theorem 3.2. *Network matrices are closed under k-sums* ($k = 1, 2, 3$).

Proof: For $k = 1$ it is straightforward. For $k = 2$ it is enough to observe that if $N_1 = \begin{bmatrix} A & a \end{bmatrix}$ and $N_2 = \begin{bmatrix} b \\ B \end{bmatrix}$ are network matrices, then the matrices $\bar{N}_1 = \begin{bmatrix} A & a & a \\ 0 & 0 & 1 \end{bmatrix}$ and $\bar{N}_2 = \begin{bmatrix} 1 & 0 & b \\ 0 & 0 & B \end{bmatrix}$ are network matrices too, since we have only duplicated columns and added unitary rows and columns. But then $N_1 \oplus_2 N_2 = \bar{N}_1 \oplus_3 \bar{N}_2$ which we know from Lemma 3.1 to be network. For the alternative 3-sum operation, since network matrices are closed under pivoting the result follows. □

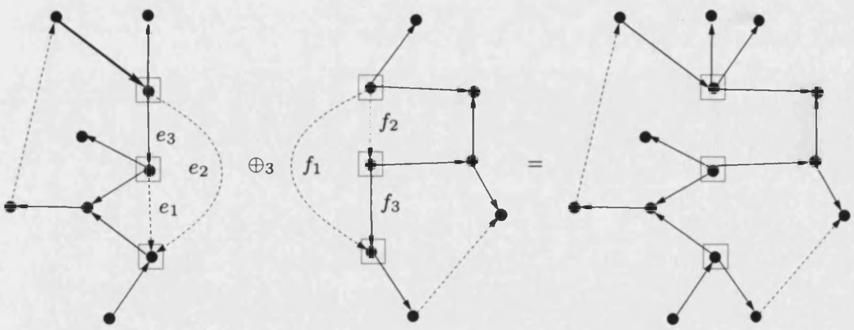


Figure 3.1: The network representation of the 3-sum of two network matrices.

Graphical Representation of Network $\oplus_k$ Network:
Using $[R'|S']$ from (3.5), we can draw a network representation of $N_1 \oplus_3 N_2$ using the network representations associated with N_1 and N_2. Glueing together the triangles (e_1, e_2, e_3) and (f_1, f_2, f_3) such that e_3 meets f_2, e_1 meets f_3 and e_2 meets f_1 is the procedure that gives rise to $[R'|S']$ which is described in (3.5). An illustrative example is given in Figure 3.1, where the solid edges correspond to basic edges and the boxed vertices are the vertices along which the gluing takes place.

We have already mentioned that the 2-sum operation is a special case of the 3-sum operation; to see this, observe that for two network matrices $N_1 = \begin{array}{c} \quad\; e \\ \begin{bmatrix} A & a \end{bmatrix} \end{array}$ and $N_2 = \begin{array}{cc} f & \begin{bmatrix} b \\ B \end{bmatrix} \end{array}$ we have that $N_1 \oplus_2 N_2 = \bar{N}_1 \oplus_3 \bar{N}_2$, where $\bar{N}_1 = \begin{bmatrix} A & a & a \\ 0 & 0 & 1 \end{bmatrix}$ and $\bar{N}_2 = \begin{bmatrix} 1 & 0 & b \\ 0 & 0 & B \end{bmatrix}$ can be shown to be network matrices (see proof of Theorem 3.2). Thus, using $[R'|S']$ from (3.5), it is evident that a graphical representation of $N_1 \oplus_2 N_2$ can be obtained by glueing together the network representations of N_1 and N_2 along e and f and then deleting the edges e and f from the unified graph. Finally, it is straightforward to see that a network representation of $N_1 \oplus_1 N_2$ can be obtained from the identification of the network representations of N_1 and N_2 along some vertex.

3.2 The k-sum operations between a network and a binet matrix

In this section we examine the k-sum operations between network and binet matrices. As shown in section 2.2.1 (Example 2), network matrices are binet matrices. First, in section 3.2.1, we provide two binet representations of a network matrix. These binet representations are of central importance in section 3.2.2 in which we prove that the k-sum of a network and a binet matrix is a binet matrix.

3.2.1 Two binet representations of network matrices

Let N be a network matrix and $[R|S]$ be the incidence matrix of an associated digraph such that $RN = S$. The first binet representation of a network matrix is straightforward. We know that by deleting any arbitrary row from $[R|S]$ we obtain a full-row rank matrix $[R'|S']$ for which we have that $R'N = S'$ and

R' is non-singular. By deleting this row some columns of $[R'|S']$ contain only one non-zero element and, therefore, these columns represent half-edges in the binet representation of N. In Figure 3.2 (ii) a binet representation of the network matrix associated with the network of Figure 3.2 (i) is depicted. Vertex v is deleted and edges incident with it in Figure 3.2 (i) are replaced by half-edges in Figure 3.2 (ii). The fundamental circuits of the non-tree and non-basic edges s_i $(i \in \{1,2,3\})$ for the network and binet representation of N, respectively, are also depicted. In addition, in [4], it has been shown that if a binet matrix has a representation in which all negative edges are half-edges, then it is a network matrix .

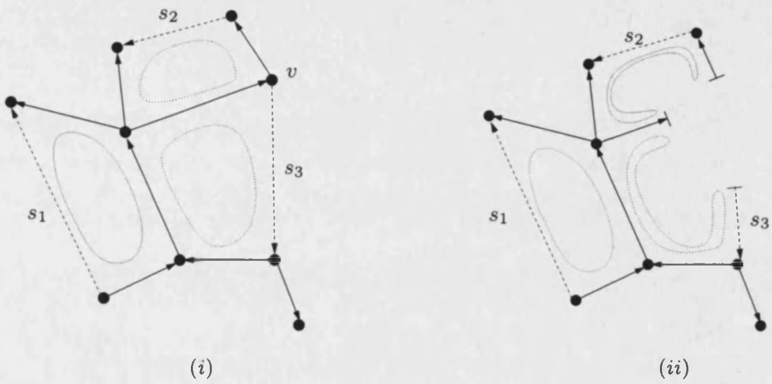


Figure 3.2: The fundamental circuits of different non-tree and non-basic edges in the directed and the bidirected representation graphs, respectively.

For the second possible binet representation of a network matrix N let again $[R|S]$ be an incidence matrix associated with N such that $RN = S$ and let e be a column of S. We shall show that there exists a binet representation in which edge e is a negative loop at any one of its end-vertices. In order to do this, we add a negative link f parallel to e and as a result we have that the binet matrix B' associated with this graph is equal to the original network matrix N plus an all-zero row. Thus, if we delete this all-zero row then we get the original matrix N. The equivalent graphical operation is contraction of edge f. Contraction of f involves switching at one end-vertex of f (say at v), since f is a negative link. This way e becomes a negative loop (see Figure 3.3).

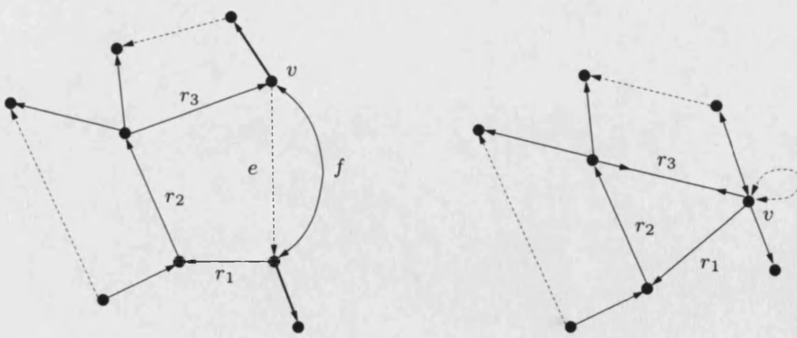


Figure 3.3: Inserting a negative edge f, and then contracting it by switching at v.

In matrix terms, we have that starting from $[R|S] = \begin{array}{cc} & \begin{array}{ccc} & e & \end{array} \\ v & \left[\begin{array}{c|cc} & -1 & \\ R & 1 & \bar{S} \\ & 0 & \end{array}\right] \end{array}$ and following the aforementioned procedure we obtain $[R'|S'] = \begin{array}{cc} & \begin{array}{ccc} & e & \end{array} \\ v & \left[\begin{array}{c|cc} & 2 & \\ \hat{R} & 0 & \hat{S} \\ & 0 & \end{array}\right] \end{array}$, such that $R'N = S'$ and $[R'|S']$ is an incidence matrix associated with a binet representation of N.

3.2.2 Network $\oplus_k$ Binet

Now we investigate what would happen to Lemma 3.1 if N_2 is a binet matrix. Then in a possible representation of N_2, its edges could be not only links but also loops and half-edges. Most importantly, because of the structure of matrix N_2 we have that the edges f_1, f_2 and f_3 should be of a specific type (loop, link, or half-edge) in order to form a binet representation of N_2. We examine below all the possible cases.

If f_3 is a link in the basic cycle (in which case we can assume that it is a positive link due to switching), then (by the BINET MATRIX ALGORITHM of section 2.2.1) f_1 and f_2 cannot be half-edges, because the fundamental circuit of a half-edge uses all the cycle edges, and the values on the cycle edges determined by the fundamental circuit are halves, so there can be neither 0 nor 1 in the row f_3 and columns f_1 and f_2 of N_2. Furthermore, f_2 cannot be a loop, because the fundamental circuit of any loop uses all edges of the basic cycle, despite the 0 in the corresponding position of the matrix. So either both f_1 and f_2 are links, or f_1 is a loop and f_2 is a link. If they are both links, then they are both positive or both negative. Otherwise the fundamental circuit of one of them would use the negative edge in the cycle, the other would not, and they use the same edges except for the positive f_3. Moreover, f_1, f_2 and f_3 must form a triangle, so by a switching at a vertex we can make both f_1 and f_2 positive.

If f_3 is a (negative) loop, then f_1 cannot be a half-edge, because then the entry in row f_3 and column f_1 of N_2 would be a half. If f_1 is a negative loop, then vector d of N_2 contains ± 2 entries, but this is impossible because then f_2 would be an edge whose fundamental circuit uses non-cycle edges twice but does not use the basic cycle (whose edge set is $\{f_3\}$). So f_1 must be a link, which implies that f_2 is also a link, and f_1 is negative and f_2 is positive, because the fundamental circuit of f_1 uses the basic cycle, while that of f_2 does not.

If f_3 is a half-edge, then f_2 must be a positive link, as its fundamental circuit does not use the basic cycle formed by f_3. This also implies that f_1 is a half-edge.

If f_3 is a non-cycle link, then f_1 cannot be a loop, as then it would have ± 2 on f_3 in the fundamental circuit. So either f_1 is a link and then f_2 is a link or a loop; or f_1 is a half-edge in which case f_2 is also a half-edge.

Therefore the cases that may appear are the following six:

case (a): f_3 is a positive link in the (basic) cycle and f_1, f_2 are positive links;

case (b): f_3 is a positive link in the (basic) cycle, f_1 is a negative loop and f_2 is a negative link;

case (c): f_3 is a negative loop, f_1 is a negative link and f_2 is a positive link;

case (d): f_1, f_3 are half-edges and f_2 is a positive link;

case (e): f_3 is a non-cycle link, f_1 is a link and f_2 is a link or a negative loop; and

case (f): f_3 is a non-cycle link and f_1, f_2 are half-edges.

Lemma 3.3. *If N_1 is a network matrix and N_2 is a binet matrix such that*

$$N_1 = \begin{array}{c} \\ \\ e_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \quad & e_1 & e_2 \end{array} \\ \left[\begin{array}{ccc} A & a & a \\ c & 0 & 1 \end{array}\right] \end{array}, \quad N_2 = \begin{array}{c} \\ f_3 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} f_1 & f_2 & \quad \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & b \\ d & d & B \end{array}\right] \end{array},$$

then $N = N_1 \oplus_3 N_2$ is a binet matrix.

Proof: Since N_1 is a network matrix we have that e_1, e_2 and e_3 should form a triangle. Therefore, w.l.o.g. we can assume for all the cases that the incidence matrix associated with the network matrix N_1 is the following one:

$$[R_1|S_1] = \begin{array}{c} \begin{array}{cccccc} \quad & e_3 & \quad & e_1 & e_2 \end{array} \\ \left[\begin{array}{cc|ccc} r_1 & -1 & s_1 & 0 & -1 \\ r_1' & 1 & s_1' & -1 & 0 \\ r_1'' & 0 & s_1'' & 1 & 1 \\ R_1' & \mathbf{0} & S_1' & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}, \tag{3.10}$$

where $\mathbf{0}$ is a zero matrix, $r_i, r_i', r_i'', s_i, s_i'$ and s_i'' are vectors and R_i' and S_i' are matrices of appropriate size $(i = 1, 2)$.

Case (a): For case (a) we have that the incidence matrix associated with the binet matrix N_2 can have the following form:

$$[R_2|S_2] = \begin{array}{c} \begin{array}{ccccc} f_3 & \quad & f_1 & f_2 & \quad \end{array} \\ \left[\begin{array}{cc|ccc} 0 & r_2 & -1 & -1 & s_2 \\ -1 & r_2' & 0 & 1 & s_2' \\ 1 & r_2'' & 1 & 0 & s_2'' \\ \mathbf{0} & R_2' & \mathbf{0} & \mathbf{0} & S_2' \end{array}\right] \end{array}$$

The proof for this case is very similar to the one regarding the 3-sum of two network matrices in Lemma 3.1. Because of the structure of matrix N_2, we have that f_1, f_2, and f_3 should form a triangle in any binet representation of N_2. Although we omit the full proof for this case because of its similarity to the one of Lemma 3.1, we provide the incidence matrix $[R'|S']$ of the binet graph associated with the binet matrix N produced by the 3-sum:

$$[R'|S'] = \left[\begin{array}{cc|cc} r_1 & r_2 & s_1 & s_2 \\ r_1' & r_2' & s_1' & s_2' \\ r_1'' & r_2'' & s_1'' & s_2'' \\ R_1' & \mathbf{0} & S_1' & \mathbf{0} \\ \mathbf{0} & R_2' & \mathbf{0} & S_2' \end{array}\right] \tag{3.11}$$

Case (b): For this case we have that the incidence matrix associated with the binet matrix N_2 can have the following form:

$$
\begin{array}{cc|ccc}
f_3 & & f_1 & f_2 & \\
\end{array}
$$

$$
[R_2|S_2] = \left[\begin{array}{cc|ccc}
-1 & r_2 & -2 & -1 & s_2 \\
1 & r_2' & 0 & -1 & s_2' \\
\mathbf{0} & R_2' & \mathbf{0} & \mathbf{0} & S_2'
\end{array}\right] \tag{3.12}
$$

Initially, we convert the network representation $[R_1|S_1]$ of N_1 to a binet representation in which e_2 is a negative loop. This can be done by introducing an artificial link parallel to e_2 and then contracting it, as shown in section 3.2.1. Thus, e_1 becomes a negative link, as contraction involves switching at the vertex with which e_1 and e_2 are incident. Graphically this case is illustrated in Figure 3.5 which shows such an alternative binet representation of the matrix represented by the directed graph in Figure 3.4. Therefore, the incidence matrix $[R_1|S_1]$ of the binet graph associated with N_1 can have the following form:

$$
\begin{array}{cc|ccc}
 & e_3 & & e_1 & e_2 \\
\end{array}
$$

$$
[R_1|S_1] = \left[\begin{array}{cc|ccc}
r_1 & -1 & s_1 & -1 & -2 \\
r_1' & 1 & s_1' & -1 & 0 \\
R_1' & \mathbf{0} & S_1' & \mathbf{0} & \mathbf{0}
\end{array}\right] \tag{3.13}
$$

We have that the following equations hold:

$$
R_1 N_1 = S_1, \qquad R_2 N_2 = S_2 \tag{3.14}
$$

From (3.13) and (3.14) we have that:

$$
\left[\begin{array}{c|c}
r_1 & -1 \\
r_1' & 1 \\ \hline
R_1' & \mathbf{0}
\end{array}\right]
\left[\begin{array}{c|c|c}
A & a & a \\ \hline
c & 0 & 1
\end{array}\right]
=
\left[\begin{array}{c|c|c}
s_1 & -1 & -2 \\
s_1' & -1 & 0 \\ \hline
S_1' & \mathbf{0} & \mathbf{0}
\end{array}\right],
$$

where upon decomposing the block matrix multiplications we derive the following equations.

$$
\begin{bmatrix} r_1 \\ r_1' \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s_1' \end{bmatrix}; \quad \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} a = \begin{bmatrix} -1 \\ -1 \end{bmatrix};
$$

$$
\begin{bmatrix} r_1 \\ r_1' \end{bmatrix} a + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}; \quad R_1'A = S_1'; \quad R_1'a = \mathbf{0}. \tag{3.15}
$$

From (3.12) and (3.14) we have that:

$$
\left[\begin{array}{c|c}
-2 & r_2 \\
1 & r_2' \\ \hline
\mathbf{0} & R_2'
\end{array}\right]
\left[\begin{array}{c|c|c}
1 & 0 & b \\ \hline
d & d & B
\end{array}\right]
=
\left[\begin{array}{c|c|c}
-2 & -1 & s_2 \\
0 & -1 & s_2' \\ \hline
\mathbf{0} & \mathbf{0} & S_2'
\end{array}\right]
$$

and, thus:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} d = \begin{bmatrix} -2 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} d = \begin{bmatrix} -1 \\ -1 \end{bmatrix};$$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} b + \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} B = \begin{bmatrix} s_2 \\ s_2' \end{bmatrix}; \quad R_2{}'d = \mathbf{0}; \quad R_2{}'B = S_2{}'. \tag{3.16}$$

Using block matrix multiplication and the equations in (18) and (19), the following equality holds:

$$\underbrace{\left[\begin{array}{c|c} r_1 & r_2 \\ r_1' & r_2' \\ \hline R_1{}' & \mathbf{0} \\ \hline \mathbf{0} & R_2{}' \end{array}\right]}_{R'} \underbrace{\left[\begin{array}{c|c} A & ab \\ \hline dc & B \end{array}\right]}_{N} = \underbrace{\left[\begin{array}{c|c} s_1 & s_2 \\ s_1' & s_2' \\ \hline S_1{}' & \mathbf{0} \\ \hline \mathbf{0} & S_2{}' \end{array}\right]}_{S'}$$

and $[R'|S']$ is the incidence matrix associated with N.

Case (c): This case is very similar to case (b). Here we have again to find an alternative binet representation of N_1. This can be obtained if we take the binet representation of N_1 in which e_1 is a negative loop. In this case the incidence matrix associated with a binet representation of N_1 can be:

$$[R_1|S_1] = \begin{array}{c} \begin{array}{cccccc} & e_3 & & e_1 & e_2 \end{array} \\ \left[\begin{array}{cc|ccc} r_1 & 1 & s_1 & 0 & 1 \\ r_1' & -1 & s_1' & 2 & 1 \\ R_1{}' & \mathbf{0} & S_1{}' & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}$$

and w.l.o.g. we can also assume that the incidence matrix associated with the binet matrix N_2 is:

$$[R_2|S_2] = \begin{array}{c} \begin{array}{ccccc} f_3 & & f_1 & f_2 & \end{array} \\ \left[\begin{array}{cc|ccc} 0 & r_2 & 1 & 1 & s_2 \\ 2 & r_2' & 1 & -1 & s_2' \\ \mathbf{0} & R_2{}' & \mathbf{0} & \mathbf{0} & S_2{}' \end{array}\right] \end{array}$$

Using the same methodology as we did in cases (a) and (b) it can be shown that for case (c) an incidence matrix $[R'|S']$ associated with matrix N, i.e. such that $R'N = S'$, is:

$$[R'|S'] = \left[\begin{array}{cc|cc} r_1 & r_2 & s_1 & s_2 \\ r_1' & r_2' & s_1' & s_2' \\ R_1{}' & \mathbf{0} & S_1{}' & \mathbf{0} \\ \mathbf{0} & R_2{}' & \mathbf{0} & S_2{}' \end{array}\right]$$

Case (d): Similarly, the incidence matrix associated with N_2 can be:

$$[R_2|S_2] = \begin{array}{c} \begin{array}{cccccc} f_3 & & f_1 & f_2 & \end{array} \\ \left[\begin{array}{cc|ccc} -1 & r_2 & -1 & 0 & s_2 \\ 0 & r_2' & 1 & -1 & s_2' \\ \mathbf{0} & R_2{}' & \mathbf{0} & \mathbf{0} & S_2{}' \end{array}\right] \end{array}$$

As shown in section 3.2.1, we can delete the third row from matrix $[R_1|S_1]$ of (3.10) in order to get a binet representation of N_1. Therefore, we can assume that in this case the incidence matrix associated with N_1 is:

$$[R_1|S_1] = \begin{array}{c} \begin{array}{ccccc} & e_3 & & e_1 & e_2 \end{array} \\ \left[\begin{array}{cc|ccc} r_1 & -1 & s_1 & 0 & -1 \\ r_1' & 1 & s_1' & -1 & 0 \\ \hat{R}_1 & \mathbf{0} & \hat{S}_1 & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}$$

Using the same methodology as we did in all the previous cases it is easy to show that an incidence matrix associated with N is:

$$[R'|S'] = \left[\begin{array}{cc|cc} r_1 & r_2 & s_1 & s_2 \\ r_1' & r_2' & s_1' & s_2' \\ R_1{}' & \mathbf{0} & S_1{}' & \mathbf{0} \\ \mathbf{0} & R_2{}' & \mathbf{0} & S_2{}' \end{array}\right] \tag{3.17}$$

Case (e) is directly analogous to the case (a) and (b) where f_2 is a link and f_2 is a loop, respectively. Case (f) is directly analogous to the case (d). For this reason we omit the proof for these cases.

For each of the aforementioned cases it is obvious that $[R'|S']$ is the incidence matrix of a bidirected graph, since the set of columns of this matrix is a combination of columns of $[R_1|S_1]$ and $[R_2|S_2]$. The rows/columns of R' in each case are linearly independent, something that can be proved in much the same way as we did for the R' in Lemma 3.1. Alternatively, the non-singularity of R' stems also from the graphical explanation we give in the following section. Specifically, since there is one-to-one correspondence between the matrix R' and the corresponding bidirected graph, we show that the graph induced by the edges corresponding to the columns of R' is a negative 1-tree in the unique bidirected graph associated with $[R'|S']$ found in each case. □

In the next theorem we show that for a network matrix N_1 and a binet matrix N_2, the matrix $N_1 \oplus_k N_2$ is a binet matrix for any $k \in \{1, 2, 3\}$.

Theorem 3.4. *The k-sum of a network matrix and a binet matrix is binet* $((k = 1, 2, 3))$.

Proof: The proof is similar to that of Theorem 3.2 since binet matrices are also closed under duplication of columns and rows, addition of unitary rows and pivoting. □

Graphical Representation of Network $\oplus_k$ Binet:

We begin by providing a graphical representation of the (binet) matrix $N_1 \oplus_3 N_2$, where N_1 is a network matrix and N_2 is a binet matrix. An illustration regarding case (a) is depicted in Figure 3.4, where the triangles (e_1, e_2, e_3) and (f_1, f_2, f_3) are glued together and their edges are deleted from the unified graph. Note that, as in the case of network $\oplus_k$ network, we put a box around each vertex taking part in this gluing. In this way, we obtain a bidirected graph whose associated incidence matrix is the one given by (3.11). In case (b), we convert the network representation of N_1 to a binet representation in which

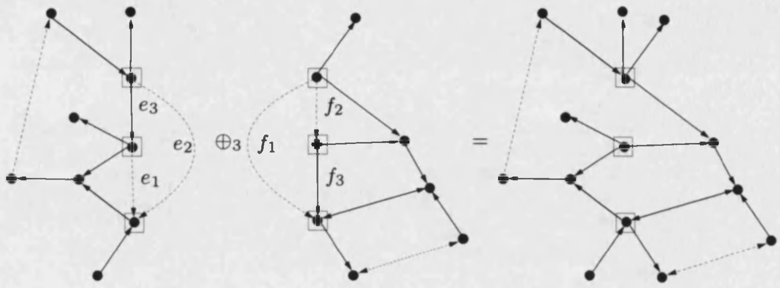


Figure 3.4: The binet representation of the 3-sum of a network and a binet matrix. The case when f_1, f_2, f_3 are links.

e_2 is a loop. As described in the proof of Lemma 3.3, this can be done by introducing an artificial link parallel to e_2 and then contracting it. In this way e_1 becomes a negative link, since contraction involves switching at the vertex with which e_1 and e_2 are incident, but this does not affect the gluing of e_1 and f_2, since f_2 is also a negative link because its fundamental circuit uses the negative link of the basic cycle. This case is illustrated in Figure 3.5. That figure shows the alternative binet representation of the matrix represented by the directed graph in Figure 3.4. For case (c), see Figure 3.6 for an illustration. To make

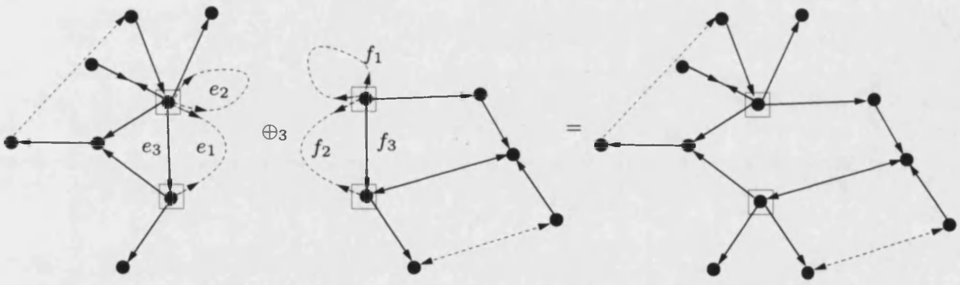


Figure 3.5: The binet representation of the 3-sum of a network and a binet matrix. The case when f_1 is a loop, f_2 is a negative link, f_3 is a positive link.

a similar representation for N_1, we can convert e_1 to a negative loop with a contraction. The binet graph representing N_1 in Figure 3.6 is an alternative representation to the directed graph in Figure 3.4. In case (d) the three edges f_1,f_2 and f_3 are positioned as in Figure 3.7. We can have a similar position of edges e_1, e_2, e_3 if we delete a vertex that is incident with e_1 and e_2. The leftmost graph in Figure 3.7 shows such a binet representation of the network matrix represented by the directed graph in Figure 3.4. Cases (e) and (f) can be handled with the techniques described previously. If an edge among f_1, f_2, f_3 is a negative loop, then contract an artificial edge in the directed graph representation of N_1 to make the corresponding edge a negative loop. If two edges among f_1, f_2, f_3 are half-edges, then delete an appropriate vertex from the directed graph.

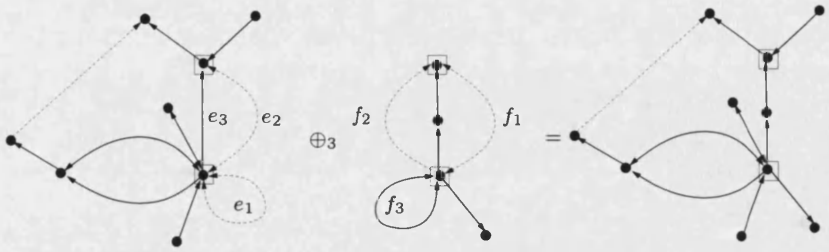


Figure 3.6: The binet representation of the 3-sum of a network and a binet matrix. The case when f_3 is a loop.

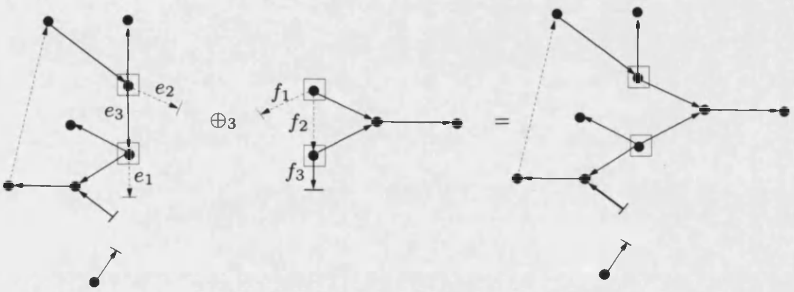


Figure 3.7: The binet representation of the 3-sum of a network and a binet matrix. The case when f_3 is a half-edge.

In order to obtain a binet representation of $N_1 \oplus_2 N_2$, where $N_1 = \begin{array}{c} \quad\quad e \\ \begin{bmatrix} A & a \end{bmatrix} \end{array}$ is a network matrix and $N_2 = \begin{array}{c} f \\ \\ \end{array}\begin{bmatrix} b \\ B \end{bmatrix}$ is a binet matrix we have to do a similar analysis to the one we did for the graphical representation of the 2-sum of two network matrices in section 3.1. Specifically by such an analysis it can be easily shown that we can obtain a binet representation of $N_1 \oplus_2 N_2$ as follows: initially, we take a binet representation of N_1 such that e and f are edges of the same type in the binet representations of N_1 and N_2 (we know that such a representation of N_1 exists because of the results in section 3.2.1), then we glue together the two binet representation of N_1 and N_2 along e and f such that any tail (head) of e is glued with a tail (head) of f (note that the operation of switching guarantees that such a glueing is possible) and then, we delete e and f from the unified graph. Finally, it is easy to see that a binet representation of $N_1 \oplus_1 N_2$ can be obtained from the identification of a network representation of N_1 and a binet representation of N_2 along some vertex.

3.2.3 Binet $\oplus_k$ Network

A very similar analysis of the cases can be done here. The role of e_1, e_2 and e_3 is analogous to f_1, f_2 and f_3 as in the previous section. Specifically, if N_1 is a binet matrix and N_2 is a network matrix then all the cases can be handled in much the same way, by finding a suitable alternative representation of N_2 as we did for N_1 in the proof of Lemma 3.3.

Lemma 3.5. *If N_1 is a binet matrix and N_2 is a network matrix such that*

$$N_1 = \begin{array}{cc} & \begin{array}{ccc} & e_1 & e_2 \end{array} \\ \begin{array}{c} \\ e_3 \end{array} & \left[\begin{array}{ccc} A & a & a \\ c & 0 & 1 \end{array}\right] \end{array}, \quad N_2 = \begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & \end{array} \\ \begin{array}{c} f_3 \\ \\ \end{array} & \left[\begin{array}{ccc} 1 & 0 & b \\ d & d & B \end{array}\right] \end{array},$$

then $N = N_1 \oplus_3 N_2$ is a binet matrix.

Theorem 3.6. *The k-sum of a binet matrix and a network matrix is binet ($k = 1, 2, 3$).*

Proof: The proof is similar to that of Theorem 3.2 since binet matrices are also closed under duplication of columns and rows, addition of unitary rows and pivoting. □

3.3 The k-sum operations between binet matrices

It is straightforward to prove that the 1-sum of two binet matrices B_1 and B_2 is a binet matrix. Specifically, a binet graph of $B_1 \oplus_1 B_2$ may be obtained from the union of the binet graphs associated with B_1 and B_2. In this section we shall prove that the k-sum, for $k = 2, 3$, of two binet matrices is not necessarily a binet matrix. Using a counterexample, we show that the 2-sum of the two well-known binet, non-network and totally unimodular matrices B_1 and B_2 (see (1.1)), which are the two unique compact representation matrices for the R_{10} matroid, is not a binet matrix. The column of B_1 as well as the row of B_2 used in our 2-sum counterexample are indicated below.

$$B_1 = \left[\begin{array}{c|c} A & a \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right], B_2 = \left[\begin{array}{c} b \\ \hline B \end{array}\right] = \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Let M be the 2-sum of B_1 and B_2 which according to the 2-sum definition is:

$$M = \left[\begin{array}{c|c} A & ab \\ \hline \mathbf{0} & B \end{array}\right] = \begin{array}{cc} & \begin{array}{ccccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \end{array} & \left[\begin{array}{cccc|ccccc} 0 & 0 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Note that the rows and columns of M are labelled by r_i and s_i ($i = 1, \ldots, 9$), respectively. If we assume that M is a binet matrix, then r_i and s_i label the basic and non-basic edges, respectively, in a binet representation of M. Furthermore, matrix M is integral and since it is also binet then any possible binet representation of M up to switchings should be one of the following two types (see Lemmas 5.10 and 5.12 of [45]):

type I: Every basic cycle is a half-edge, and all other basic edges are directed.

type II: There are no half-edges in the binet graph, the basis is connected and there is only one bidirected edge in the basis.

We shall show that M has neither of the above two representations, thereby it can not be binet. We make use of the following lemma in [45]:

Lemma 3.7. *Let us suppose that a binet matrix B is totally unimodular. Then it is a network matrix if and only if it has a binet representation in which each basic cycle is a half-edge.*

Lemma 3.8. *Matrix $M = B_1 \oplus_2 B_2$ does not have a binet representation of type I or type II.*

Proof: Suppose that M has a binet representation of type I. Combining the fact that M is totally unimodular with Lemma 3.7 we have that M is a network matrix. It is well-known that any submatrix of a network matrix is a network matrix itself (e.g. see [55]). B_1 is a submatrix of M which is known to be non-network. Thus, M can not have a binet representation of type I.

Assume that M has a binet representation Σ of type II. Let Σ_R be the subgraph of Σ induced by the edges in $R = \{r_1, \dots, r_9\}$, i.e. Σ_R is the basis of Σ. Let also C be the set of edges that constitute the unique cycle in Σ_R, i.e. C is the edge set of the basic cycle of the binet graph Σ. Because of column s_5 of M the subgraph of Σ_R induced by the basic edges in $S = \{r_1, r_2, r_3, r_6, r_7, r_8, r_9\}$ is connected. Our first claim is that $C \subseteq S$. If we assume the contrary, i.e. that $C \nsubseteq S$, then the edges in S should form a path in Σ_R. Moreover, observe that each non-basic edge of the set $\{s_6, s_7, s_8, s_9\}$ is using edges of S in order to create the associated fundamental circuit in Σ. Combining this with the fact that the edges in S induce a path of Σ, we have that $\begin{bmatrix} ab \\ B \end{bmatrix}$ must be a network matrix. But this can't happen since this matrix contains B_2 as a submatrix which is not a network matrix and thus, our claim is true, i.e. $C \subseteq S$. Furthermore, since there is only one cycle in Σ_R, we have that $\{r_4, r_5\} \notin C$.

Let $D = \{r_1, r_2, r_3\}$ and $E = S - C = \{r_6, r_7, r_8, r_9\}$; our second claim is that $C \nsubseteq D$. If we assume the contrary, i.e. that $C \subseteq D$ then because of column s_5 of M we have that the corresponding fundamental circuit in Σ should be either a handcuff of Type I or a handcuff of Type II. However, it can not be a handcuff of Type II since then a ± 2 would appear in M (see BINET MATRIX ALGORITHM in section 2.2.1). Therefore, it is a handcuff of Type I and thus the basic edges in $(D - C) \cup E$ induce a path in the basis graph. Thus, the edges in E and one or more edges of D are the parts of this path in the basis graph. Moreover, from the fundamental circuits of Σ described by the columns of $\begin{bmatrix} A \\ \mathbf{0} \end{bmatrix}$ part of M we have that the subgraph Σ_T of Σ_R induced by the set of edges in $T = \{r_1, r_2, r_3, r_4, r_5\}$ is connected. Observe now that the edges in D appear in all the fundamental circuits of Σ corresponding to the columns of $\begin{bmatrix} ab \\ B \end{bmatrix}$. Therefore, because of the structure of these fundamental circuits and the fact that Σ_T is connected, we have that in Σ_R the following conditions must be satisfied: (i) r_6 and r_9 are adjacent, (ii) r_6 and r_7 are adjacent, (iii) r_7 and r_8 are adjacent, and (iv) r_8 and r_9 are adjacent. We show now that this can not happen. Assume, w.l.o.g., that r_9 is on the right side of r_6 then because of (ii) r_7 should be put on the left side of r_6. Moreover, because of (iii) r_8 should be on the left side of r_7. But now condition (iv) can not be satisfied. Thus, our assumption that $C \subseteq D$ is not correct and this completes the proof of our second claim.

Since we have shown that $\{r_4, r_5\} \notin C$ and that $C \nsubseteq D$ we have that Σ_T is a tree in Σ_R. We show now that any two edges in D do not share a common end-vertex. Note that the following procedure can be used in much the same way for any pair of edges in D. Specifically, suppose that r_1 and r_2 share an end-vertex and without loss of generality suppose that r_2 stands on the right side of r_1. Consider the fundamental circuits of Σ determined by the columns of the $\begin{bmatrix} A \\ \mathbf{0} \end{bmatrix}$ part of M. Because of the columns s_3 and s_4 we have that r_5 stands on the left side of r_1. Moreover, because of the columns s_1 and s_3 we have that r_4 has a common end-vertex with r_1 and r_2. But now, we can not satisfy the fundamental circuit defined by s_2 because edge r_1 is in the middle of r_4 and r_5. Thus, we can conclude that any two edges of D do not share a common end-vertex. However, we have that Σ_T (which contains r_4 and r_5) is a tree and that the edges in S (which does not contain r_4 and r_5) induce a connected subgraph in Σ_R containing a basic cycle. This can only happen if Σ_R contains at least two cycles. In other words, in order to satisfy the fundamental circuits described by the columns of M we have that Σ_R should contain at least two cycles. This is in contradiction with the fact that connected binet graphs contain at most one basic cycle in the basis graph. Therefore, M does not have a binet representation of type II. □

Based on a recent result, appearing in [62], which gave the complete list of the regular excluded minors for the class of signed-graphic matroids, we provide here an alternative proof of Lemma 3.8. There are 31 minors in this list, 29 of which are cographic. Moreover, as we will see in section 6.1 (Theorem 6.1), binet matrices represent over $\mathbb{R}$ the class of signed-graphic matroids. Let R be the regular matroid whose representation matrix over $\mathbb{R}$ is $M = B_1 \oplus_2 B_2$. Clearly, if we could show that R contains as a minor one of the regular excluded minors for the class of signed-graphic matroids then R could not be signed-graphic. In order to check this, we used the MACEK software ([40]) which can compute, for relatively small matroids, if a given matroid contains another matroid as a minor. As input to MACEK we gave the matrix M mod 2, which represents R over $GF(2)$, and a $GF(2)$-representation matrix for each of the 31 regular excluded minors for signed-graphic matroids. The software identified that the matroid R (whose representation matrix is M mod 2) contains as a minor one of these 31 excluded minors, namely the cographic matroid of the graph H depicted in Figure 3.8 with the following compact representation matrix over $GF(2)$ (which was part of the input to the software):

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

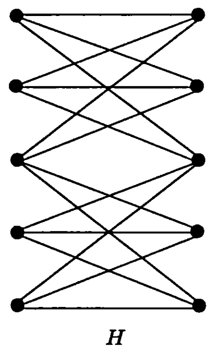


Figure 3.8: The graph H.

Thus, since binet matrices represent signed-graphic matroids we have that M is not binet.

Therefore, we can state the following theorem.

Theorem 3.9. *Totally unimodular binet matrices are not closed under k-sums for* $k = 2, 3$.

Proof: For $k = 2$ the Lemma 3.8 provides a counterexample. For $k = 3$ it is enough to observe that for $c = 0$ in the Definition 1.7, the 3-sum of two matrices reduces to the 2-sum of some submatrices obtained by the deletion of columns and rows. Since binet and TU matrices are closed under row and column deletions, the result follows. □

3.4 An example

In an unpublished manuscript [46], a realistic problem in which the constraint matrix is totally unimodular but it is neither a network matrix nor the transpose of a network matrix is provided. This matrix, which we call A, is given in (3.18). We reproduce this realistic problem in the following lines since it shows that there are interesting applications of totally unimodular matrices in which these matrices must not necessarily be network matrices or transposes of network matrices. Most importantly, we would like to stress that this problem/application shows that the optimization methods presented in section 2.3 can be applied to a wider set of real-life problems with totally unimodular constraint matrices which are not solvable by the network simplex method.

$$A = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \tag{3.18}$$

This problem goes as follows. Suppose that two factories F_1 and F_2 produce a product P which can be sold in two markets, S_1 and S_2. Product P may be transported directly from either factory to either market and it can also be transported from market S_1 to market S_2. We assume that the demand at the markets is stochastic described by two scenarios, scenario 1 and scenario 2, having probability p_1 and p_2, respectively. We denote by d_{jk} the demand at market j in scenario k. Let also w_1 and w_2 be the capacities of F_1 and F_2, respectively. We denote by x_{ij} the variables ($i = 1, 2$ and $j = 1, 2$) which represent the amount of product P transported from F_i to S_j. Furthermore, by y_1 and y_2 we denote the amount of P transported from S_1 to S_2 in scenarios 1 and 2, respectively. At the factories the following capacity constraints must be satisfied:

$$x_{11} + x_{12} \leq w_1$$
$$x_{21} + x_{22} \leq w_2$$

At the markets the following demand constraints must be satisfied under scenario 1:

$$x_{11} + x_{21} - y_1 \leq d_{11}$$
$$x_{12} + x_{22} + y_1 \leq d_{21}$$

At the markets the following demand constraints must be satisfied under scenario 2:

$$x_{11} + x_{21} - y_2 \leq d_{12}$$
$$x_{12} + x_{22} + y_2 \leq d_{22}$$

Clearly, the constraint matrix given by these constraints is A. At this point, we should mention that A is a submatrix of the constraint matrix in a stochastic ground holding problem in air traffic control [63] and it was this problem that inspired Kotnyek to think of this "realistic" problem in [46]. Specifically, as mentioned in [46], although A is not the constraint matrix of a ground holding problem, the model discussed in [63] was the basis for getting A.

The objective of this problem is to maximize the expected profit. As defined in [46], the profit of one unit of P transported from F_1 to S_1, denoted by a_{11}, is the difference between the selling price at S_1 and the cost (cost consists of the cost of production at F_1 and the cost of transportation). Similarly, we can calculate the profit a_{12}, a_{21} and a_{22} on one unit of x_{12}, x_{21} and x_{22}, respectively. The (possibly negative) profit h of a unit of P sent from S_1 to S_2 is the difference in price decreased by the cost of transportation. The objective function of the problem is given in (3.19). Note that in the objective function the price for each unit of P sent to S_1 is counted; if it is not sold but resent to S_2, then we correct this in h [46].

$$max \sum_{i,j=1}^{2} a_{ij}x_{ij} + h(p_1y_1 + p_2y_2). \tag{3.19}$$

Matrix A is neither B_1 nor B_2 and thus, according to Seymour's decomposition theory for totally unimodular matrices, A must be decomposable. We note here that Kotnyek [46] provides a long proof in order to show that A is totally unimodular and that neither A nor its transpose is a network matrix. However, one could observe directly that A is a totally unimodular representation of the well-known special matroid R_{12} which appears frequently in the matroid theory literature (see e.g. [56, 66, 78]). This matroid has the special characteristic that it contains exactly two circuits of cardinality three and these circuits are disjoint. It is well-known that R_{12} is neither graphic nor cographic which in turn implies that A is neither network nor the transpose of a network matrix. Furthermore, R_{12} is of central importance in Seymour's decomposition result in [66] mainly due to the fact that a matroid having an R_{12}-minor has a 3-separation. This implies that matrix A can be written as a 3-sum of two other matrices. In the following lines we show that one of these matrices is network while the other is binet and, therefore, A is a binet matrix due to the results of section 3.2. Moreover, we use the earlier results of this chapter to provide a binet representation of A.

First, we pivot in the non-zero element $A_{r_3 s_1} = 1$ of A, getting the following A':

$$A' = \begin{array}{c} \\ r_1 \\ r_2 \\ s_1 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{cccccc} r_3 & s_2 & s_3 & s_4 & s_5 & s_6 \end{array} \\ \begin{bmatrix} -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

We obtain the following matrix by rearranging rows and columns of A' and multiplying some of them (namely, r_3, r_5 and s_1) by -1:

$$A'' = \begin{array}{c} \\ r_2 \\ r_1 \\ -s_1 \\ r_4 \\ r_6 \\ -r_5 \end{array} \begin{array}{c} \begin{array}{ccc|ccc} s_3 & s_2 & s_4 & -r_3 & s_5 & s_6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right] \end{array}$$

Matrix A'' is the 3-sum of the following two matrices:

$$A_1 = \begin{array}{c} \\ r_2 \\ r_1 \\ -s_1 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc|cc} s_3 & s_2 & s_4 & e_1 & e_2 \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array} \qquad A_2 = \begin{array}{c} \\ f_3 \\ r_4 \\ r_6 \\ -r_5 \end{array} \begin{array}{c} \begin{array}{cc|ccc} f_1 & f_2 & -r_3 & s_5 & s_5 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right] \end{array}$$

The matrix A_1 is a network matrix as shown by its network representation in Figure 3.9(i) while the matrix A_2 is a binet matrix as shown by its binet representation in Figure 3.9(ii).

By Lemma 3.3, matrix A'' is a binet matrix since it the 3-sum of a network and a binet matrix. As explained in section 3.2.2, a binet representation of A' can be obtained from the two graphs of Figure 3.9; specifically, in order to get a binet representation of A'' we identify e_3 with f_2, e_1 with f_3, and e_2 with f_1 and then we delete these edges from the unified graph. This procedure is depicted in Figure 3.10,

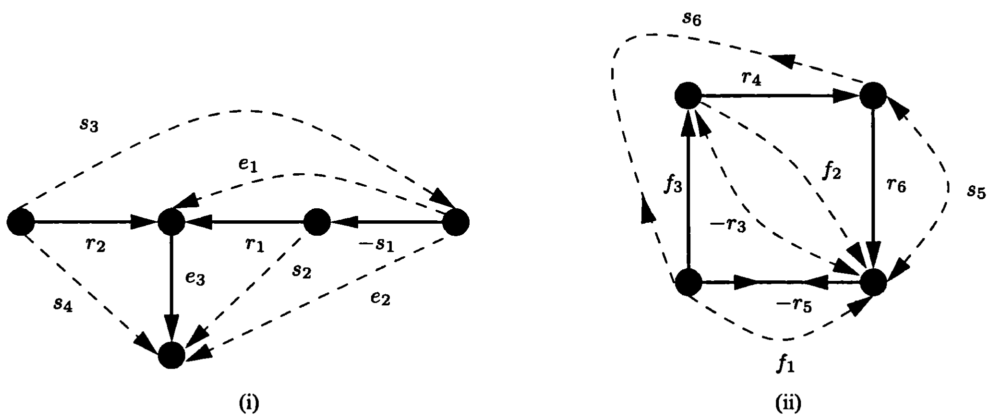


Figure 3.9: The network representation and the binet representation of A_1 and A_2, respectively.

where the network representation of A_1 and the binet representation of A_2 have been redrawn in order to facilitate the identification.

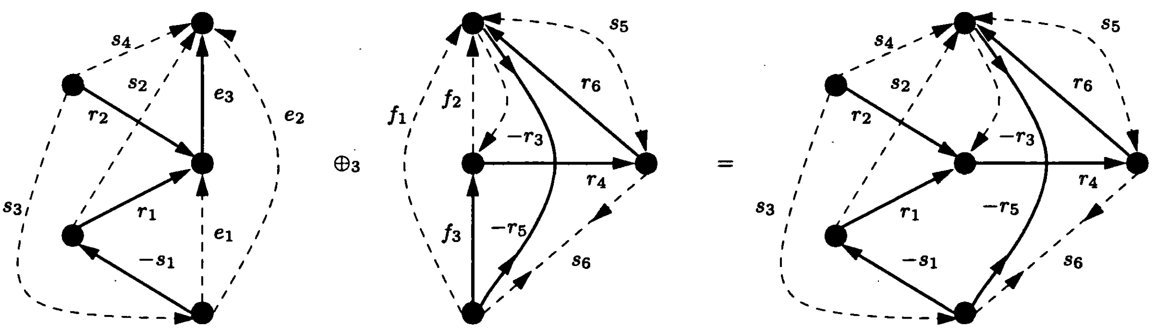


Figure 3.10: The binet representation of A''.

Since A'' is binet then, by Lemma 2.11, A is a binet matrix. In order to get a binet representation of A we have to change the direction of edges r_3, r_5 and s_1 and exchange the labels of r_3 and s_1 to express pivoting. This binet representation of A is depicted in Figure 3.11 .

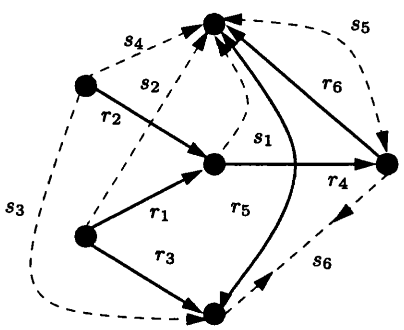


Figure 3.11: The binet representation of A.

Chapter 4

Representing totally unimodular matrices on bidirected graphs

As we have mentioned, totally unimodular matrices are of great importance in combinatorial optimization due to the integrality property of the associated polyhedron. The most important subclass of totally unimodular matrices is formed by the class of network matrices. It is well-known that there exist efficient methods, such as the network simplex algorithm, which solve integer programming problems whose constraint matrix is network. Moreover, it has been reported in the literature [33] that these methods can be up to 200 times faster than general-purpose linear programming codes (e.g. simplex method). In Chapter 2 we have shown that the generalized network simplex method is very efficient in solving integer programming problems whose constraint matrix is binet. In both cases (network and binet) the underlying digraph or bidirected graph was utilized. Given the fact that Seymour's decomposition result states that the building blocks for totally unimodular matrices are binet matrices (B_1 and B_2), network matrices and their transposes, one could ask whether the graphical representability of the building blocks of totally unimodular matrices could mean that a graphical representation may exist for all totally unimodular matrices. More importantly, one could ask if the combinatorial characteristics of this representation allow us to devise a very efficient algorithm for solving integer programming problems whose constraint matrix is totally unimodular. Clearly, such an algorithm must utilize the combinatorial characteristics of the underlying graphical representation like the network simplex method and the generalized network simplex method do. In this chapter we build the combinatorial structure (graph) on which this method may be applied. Specifically, we show that each column of a totally unimodular matrix represents closed tours on a bidirected graph. This case resembles the case of network matrices where each column represents a path on a digraph. Therefore, apart from its structural and theoretical importance, we believe that this result provides the means of devising an algorithm which could be practically very efficient in solving the class of integer programming problems with totally unimodular constraint matrices.

To pursue graphical representability of totally unimodular matrices a new class of $\{0, \pm 1\}$-matrices is introduced, the so-called *tour matrices* where each column represents closed tours in bidirected graphs. In contrast to binet matrices, it is shown that tour matrices are closed under 2- and $\oplus_3$-sums. These results and the fact that totally unimodular matrices are closed under pivoting are utilized together with

Theorem 1.9 to provide an algorithm which delivers a bidirected graph representation for any totally unimodular matrix.

The rest of this chapter is organised as follows. Tour matrices are defined in section 4.1 and in the same section it is shown that tour matrices are closed under k-sums ($k = 1, 2, \oplus_3$). In section 4.2, we gather the results presented so far in order to provide the main result of this chapter, that is, an algorithm for constructing a non-trivial bidirected graph for any totally unimodular matrix. We close this chapter with some remarks regarding possible directions for future research in section 4.3.

4.1 Tour matrices

4.1.1 Definition and properties

The definition of tour matrices goes as follows:

Definition 4.1. *Let A be the incidence matrix of an ordinary bidirected graph Σ without negative loops, Q be a column submatrix of A such that $\Sigma(Q)$ consists only of links and S be the remaining column submatrix of A (i.e. up to column permutations, $A = [Q|S]$). Then a $\{0, \pm 1\}$ matrix T such that $QT = S$ is called a tour matrix.*

The edges in $\Sigma(Q)$ are called *prime* and the edges in $\Sigma(S)$ are called *non-prime* while we also say that $\Sigma(Q)$ ($\Sigma(S)$) is the primal (non-primal) subgraph of Σ. When in a bidirected graph representing a tour matrix T the prime and non-prime edges are clearly indicated, we call it a *tour representation* or a *tour graph* of T. It is also clear, by Definition 4.1, that the prime edges of a tour graph are all links while a non-prime edge is either link or positive loop. Note that we identify the rows and columns of a tour matrix T with prime and non-prime edges, respectively, a technique we will use throughout the thesis. Specifically, if T is an $m \times n$ tour matrix then row i of T is indexed by the index of the i^{th} column of Q while column j of T is indexed by the index of the j^{th} column of S ($i = 1, \dots, m$ and $j = 1, \dots, n$). We also establish the following convention when we draw a tour graph: prime edges are indicated by solid lines while non-prime edges are indicated by dashed lines. Note that two or more tour matrices may have the same tour representation. For example, the following tour matrices T_1 and T_2 have the tour representation depicted in Figure 4.1.

$$T_1 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \qquad T_2 = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Lemma 4.2. *Let Σ be a tour representation of an $m \times n$ tour matrix T. For any column s_i of T (which corresponds to the non-prime edge s_i in Σ), let $Q(s_i)$ be the set consisting of prime edges corresponding to the rows of T which contain a non-zero element in column s_i ($i = 1, \dots, n$). Then, the subgraph of Σ induced by the edges in $Q(s_i) \cup s_i$ is a collection of closed tours.*

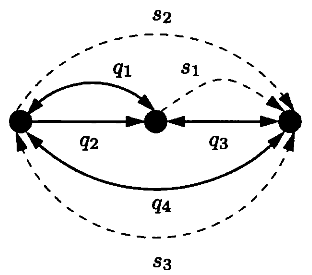


Figure 4.1: A tour representation of T_1 and T_2.

Proof: Since T is a tour matrix, the incidence matrix of Σ can be written in the form $[Q|S]$ such that $QT = S$. Let Q' be the column submatrix of Q consisting of the columns which correspond to the prime edges of $Q(s_i)$. Then $A' = [Q'|S_{\bullet s_i}]$ is the incidence matrix of the subgraph Σ' of Σ induced by $Q(s_i) \cup s_i$. Since $QT_{\bullet s_i} = S_{\bullet s_i}$ for all $i \in \{1, \ldots, n\}$, there exists, for any row a' of A', a column vector $t_{a'}$ with elements in $\{+1, -1\}$ such that $a't_{a'} = \mathbf{0}$. This implies that the sum of the absolute values of the elements of a' is an even number. But this sum is equal to the degree of the vertex of Σ' corresponding to the row a'. Therefore, the degree of every vertex in Σ' is an even number and thus, by a well known result of graph theory [11, 20], the connected components of Σ' are Eulerian. This in turn implies that the subgraph Σ' of Σ is a collection of closed tours. □

As an illustration to Lemma 4.2, one can easily see that the set of edges indexed by the non-zero entries in a column s_i of T_1 (or T_2) correspond to edges which, along with s_i, induce a subgraph of the graph of Figure 4.1 which is a closed tour. For example, the union of the edges $\{q_1, q_3\}$, indexed by the non-zeros in column s_2 of T_2, and $\{s_2\}$ induce a cycle and thereby, a closed tour of the graph of Figure 4.1.

In the following lemmas we provide some elementary operations, which if applied to a tour matrix result in a tour matrix.

Lemma 4.3. *Let Σ be a tour representation of a tour matrix T and $Q \subseteq E(\Sigma)$ be the set of its prime edges. Then, if Σ' is a bidirected graph obtained from Σ by switching at a vertex then Σ' with set of prime edges Q is also a tour representation of T.*

Proof: Let $D = [Q|S]$ be the incidence matrix of Σ where $QT = S$. Switching at a vertex v in (the bidirected graph) Σ is interpreted as multiplying by -1 the row of the incidence matrix which corresponds to vertex v. Let Q' and S' be the matrices obtained after multiplying by -1 the aforementioned row of D. Since $QT = S$, from matrix multiplication we also have that $Q'T = S'$. □

Lemma 4.4. *Tour matrices are closed under the following operations:*

(a) Permuting rows or columns.

(b) Multiplying a row or a column by -1.

(c) Duplicating a row or a column.

(d) Deleting a column or a unit row.

(e) Adding a unit row or a unit column.

Proof: If T is a tour matrix then by definition there exists a bidirected graph Σ with incidence matrix $D = [Q|S]$ such that $QT = S$. Let T' be the matrix obtained by applying one of the above operations on T. We show in each case that T' is a tour matrix by providing the incidence matrix $D' = [Q'|S']$ of the associated tour representation.

(a) When permutation of columns takes place let $Q' = Q$ and S' be the matrix obtained from S by permuting the columns of S in the same way that columns of T were permuted. When permutation of rows takes place let $S' = S$ and Q' be the matrix obtained from Q by permuting its columns in the same way that rows of T were permuted. From matrix multiplication rules we have that $Q'T' = S'$ and thus, $D' = [Q'|S']$ is the incidence matrix of a bidirected graph associated with T'.

(b) If row e of T is multiplied by -1 then let Q' be Q with column e multiplied by -1 and $S' = S$. If we multiply a column f of T by -1 then let $Q' = Q$ and S' be S with column f multiplied by -1. Obviously, in both cases T' is a tour matrix since from matrix multiplication rules we have that $Q'T' = S'$.

(c) If we duplicate a column f in T, let $Q' = Q$ and S' be S with column f duplicated. It is easy to check then that T' satisfies the conditions of a tour matrix.

Row duplication is a bit more involved. Let f be the row of T to be duplicated which corresponds to a prime link $f = \{u, s\}$ (see Figure 4.2). First duplicate row s in $[Q|S]$ to create a new row t,make all the elements of row s zero except the element in position f and then in row t make the element in position f zero. Finally, add a zero column f' and let Q' be the column submatrix of the matrix so-obtained defined by the indices of the columns of Q and f where we set

$$Q'_{tf'} = -Q'_{sf'} = Q_{sf}.$$

and let S' be its remaining column submatrix The matrix $[Q'|S']$ is the incidence matrix of a bidirected graph by construction and it can be checked easily that $Q'T' = S'$. In Figure 4.2 the graphical equivalent of the above procedure is shown where it is clear that $[Q'|S']$ is the incidence matrix of a bidirected graph Σ' without half-edges and negative loops and $\Sigma'(Q)$ contains only links.

(d) Deletion of a column in a tour matrix is simply the deletion of the corresponding non-prime edge in the corresponding tour graph. Deletion of a unit row f is a bit more involved. Suppose that it contains a nonzero element in column s of T. Then, observe that we can assume that f is adjacent to s in a tour representation Σ of T and that there exists no prime link parallel to f. It can be checked easily that the graph obtained by contracting the edge f in Σ is a tour representation of T'.

(e) Let us suppose that we add a unit column in T with an 1 in row e and suppose that this row is the ith row of T'. Then let $Q' = Q$ and S' be the matrix obtained from S by adding a copy of the column labelled by e in Q such that this new column is the ith column of S'. Then, it can be easily seen that $Q'T' = S'$ and therefore, T' is a tour matrix.

Let us suppose now that we add a unit row in T. Furthermore, suppose that the added unit row of T' is the ith row of T' and that this row contains an 1 in column f. We can apply the same operation as in part (c). So let us subdivide edge f with a new node s into edges f and f' as shown in Figure 4.2. The new edge f' will be a prime edge corresponding to the additional row and f remains non-prime. Let Q' be the incidence matrix of the prime subgraph of the new graph which can be obtained from Q by adding the column associated with f such that this column is the ith column of Q'. It can be easily seen that Q' is full-row rank and that $Q'T = S$ and thus, T is a tour matrix. $\square$

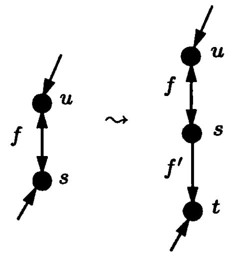


Figure 4.2: The graphical equivalent of duplicating row f

We should note here that multiplying a row (column) by -1 in a tour matrix, is graphically equivalent to reversing the direction of the corresponding prime (respectively non-prime) edge in the associated bidirected graph. On the other hand, duplicating a column amounts to creating a parallel non-prime edge in the tour graph.

Finally, we close this section by providing the following small $\{0, \pm 1\}$ matrix which can be easily checked as not being a tour matrix.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

4.1.2 The k-sum operations between tour matrices

In what follows we present results on the k-sums of tour matrices. As with the k-sum of network and binet matrices in Chapter 3 we shall only examine the case of $\oplus_3$-sum since the 1-sum and the 2-sum operations could be reduced to it by the addition of unitary rows and duplication of columns.

Lemma 4.5. *If K and L are tour matrices such that*

$$K = \begin{array}{c c} & \begin{array}{ccc} & e_1 & e_2 \end{array} \\ e_3 & \begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \end{array}, \quad L = \begin{array}{c c} & \begin{array}{ccc} f_1 & f_2 & \end{array} \\ f_3 & \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} \end{array},$$

then $M = K \oplus_3 L$ is a tour matrix.

Proof: We call $\Sigma(D_1)$ and $\Sigma(D_2)$ the tour graphs associated with K and L. Let $D_1 = [Q_1|S_1]$ and $D_2 = [Q_2|S_2]$ be incidence matrices of $\Sigma(D_1)$ and $\Sigma(D_2)$, respectively, where the columns of Q_1 and Q_2 correspond to the prime edges in $\Sigma(D_1)$ and $\Sigma(D_2)$, respectively. Due to the form of the tour matrices K and L, we have that $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ form triangles in any tour representation of K and L, respectively. Furthermore, by Lemma 4.3, we can also assume that the connecting elements e_1, e_2, e_3 and f_1, f_2, f_3 are all positive links in the corresponding tour graphs.

By Lemma 4.4, the incidence matrices D_1 and D_2 of the tour representations of K and L can take the following form:

$$[Q_1|S_1] = \begin{array}{c} \begin{matrix} & e_3 & & e_1 & e_2 \end{matrix} \\ \left[\begin{array}{cc|ccc} q_1 & -1 & s_1 & 0 & -1 \\ q_1' & 1 & s_1' & -1 & 0 \\ q_1'' & 0 & s_1'' & 1 & 1 \\ {Q_1}' & \mathbf{0} & {S_1}' & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{matrix} u \\ v \\ y \\ \end{matrix} \end{array}, \quad [Q_2|S_2] = \begin{array}{c} \begin{matrix} & f_3 & & f_1 & f_2 & \end{matrix} \\ \begin{matrix} u' \\ v' \\ y' \\ \end{matrix} \left[\begin{array}{cc|ccc} 0 & q_2 & -1 & -1 & s_2 \\ -1 & q_2' & 0 & 1 & s_2' \\ 1 & q_2'' & 1 & 0 & s_2'' \\ \mathbf{0} & {Q_2}' & \mathbf{0} & \mathbf{0} & {S_2}' \end{array}\right] \end{array} \tag{4.1}$$

where $\mathbf{0}$ is a vector or matrix of zeros of appropriate size; $q_i, q_i', q_i'', s_i, si'$ and s_i'' are row vectors and ${Q_i}', {S_i}'$ are matrices of appropriate size; and ${Q_i}'$ is the incidence matrix of the primal subgraph of the tour representation $(i = 1, 2)$. Also, u, v and y label the first three rows of D_1 and consequently the corresponding vertices of $\Sigma(D_1)$. Similarly, u', v', y' label the first three rows of D_2 and the corresponding vertices of $\Sigma(D_2)$. We have that the following equations hold:

$$Q_1 K = S_1, \qquad Q_2 L = S_2 \tag{4.2}$$

For K using (4.1) and (4.2) we have that:

$$\left[\begin{array}{c|c} q_1 & -1 \\ q_1' & 1 \\ q_1'' & 0 \\ \hline {Q_1}' & \mathbf{0} \end{array}\right] \left[\begin{array}{c|c|c} A & a & a \\ \hline c & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c|c} s_1 & 0 & -1 \\ s_1' & -1 & 0 \\ s_1'' & 1 & 1 \\ \hline {S_1}' & \mathbf{0} & \mathbf{0} \end{array}\right]$$

From the above equation we take the following equations:

$$\begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} A + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c = \begin{bmatrix} s_1 \\ s_1' \\ s_1'' \end{bmatrix}; \quad \begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} a = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};$$
$$\begin{bmatrix} q_1 \\ q_1' \\ q_1'' \end{bmatrix} a + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad {Q_1}'A = {S_1}'; \quad {Q_1}'a = \mathbf{0}. \tag{4.3}$$

Similarly, for L using (4.1) and (4.2) we have that:

$$\left[\begin{array}{c|c} 0 & q_2 \\ -1 & q_2' \\ 1 & q_2'' \\ \hline \mathbf{0} & {Q_2}' \end{array}\right] \left[\begin{array}{c|c|c} 1 & 0 & b \\ \hline d & d & B \end{array}\right] = \left[\begin{array}{c|c|c} -1 & -1 & s_2 \\ 0 & 1 & s_2' \\ 1 & 0 & s_2'' \\ \hline \mathbf{0} & \mathbf{0} & {S_2}' \end{array}\right]$$

From the above equation we take the following equations:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} d = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} b + \begin{bmatrix} q_2 \\ q_2' \\ q_2'' \end{bmatrix} B = \begin{bmatrix} s_2 \\ s_2' \\ s_2'' \end{bmatrix}; \quad Q_2'd = \mathbf{0}; \quad Q_2'B = S_2'. \tag{4.4}$$

Using block matrix multiplication and equations in (4.3) and (4.4), it is easy to show that the following equation holds:

$$\underbrace{\left[\begin{array}{c|c} q_1 & q_2 \\ q_1' & q_2' \\ q_1'' & q_2'' \\ \hline Q_1' & \mathbf{0} \\ \hline \mathbf{0} & Q_2' \end{array}\right]}_{Q'} \underbrace{\left[\begin{array}{c|c} A & ab \\ \hline dc & B \end{array}\right]}_{M} = \underbrace{\left[\begin{array}{c|c} s_1 & s_2 \\ s_1' & s_2' \\ s_2'' & s_2'' \\ \hline S_1' & \mathbf{0} \\ \hline \mathbf{0} & S_2' \end{array}\right]}_{S'} \tag{4.5}$$

Clearly, $D' = [Q'|S']$ is the incidence matrix of a bidirected graph Σ' without positive loops and half-edges and $\Sigma(Q')$ consists only of links. □

Let us examine the structure of the bidirected graph $\Sigma(D')$ so-obtained, from the $\oplus_3$-sum operation on tour matrices. From (4.5) we have that $\Sigma(D')$ is obtained by gluing $\Sigma(D_1)$ and $\Sigma(D_2)$ such that u and u', v and v', y and y' become single vertices u, v and y, respectively, and edges e_1, e_2, e_3, f_1, f_2 and f_3 are deleted from the unified graph. In other words, this can also be seen as gluing together the graphs $\Sigma(D_1)$ and $\Sigma(D_2)$ along the triangles (e_1, e_2, e_3) and (f_1, f_2, f_3) so that e_1 meets f_3, e_2 meets f_1 and e_3 meets f_2 and deleting the glued triangle from the unified graph. Obviously, we can say that in $\Sigma(D')$ the edge e_3 which was deleted is substituted by the tour associated with f_2 in $\Sigma(D_2)$ and that the edge f_3 which was deleted is substituted by the tour associated with e_1 in $\Sigma(D_1)$. Therefore, now any tour that used e_3 will instead go through the tour associated with f_2 giving rise to the non-zero part of dc in $K \oplus_3 L$. The tours that went through f_3 use the tour of e_1 in the unified graph, as determined by the ab part of $K \oplus_3 L$. All other tours remain unchanged, as expressed by the fact that if c or b had a zero element then dc or ab has an all-zero column in the same position. From Lemma 4.5 and the fact that 1-, and 2-sum operations are special cases of the $\oplus_3$-sum operation we obtain the following theorem:

Theorem 4.6. *Tour matrices are closed under k-sums for* $k = 1, 2, \oplus_3$.

Proof: The proof is similar to that of Theorem 3.2 since tour matrices are also closed under duplication of columns and rows and addition of unitary rows. □

Example 4. *In Figure 4.3 we provide a tour representation of the following matrix M which is the 2-sum of the matrices B_1 and B_2 of* (1.1). *Recall also that it has been shown in section 3.3 that M is not a binet* *matrix. However, as shown here M is a tour matrix and thereby, representable on a bidirected graph.*

$$M = \begin{array}{cc} & \begin{array}{ccccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{array} & \left[\begin{array}{cccc|ccccc} 0 & 0 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

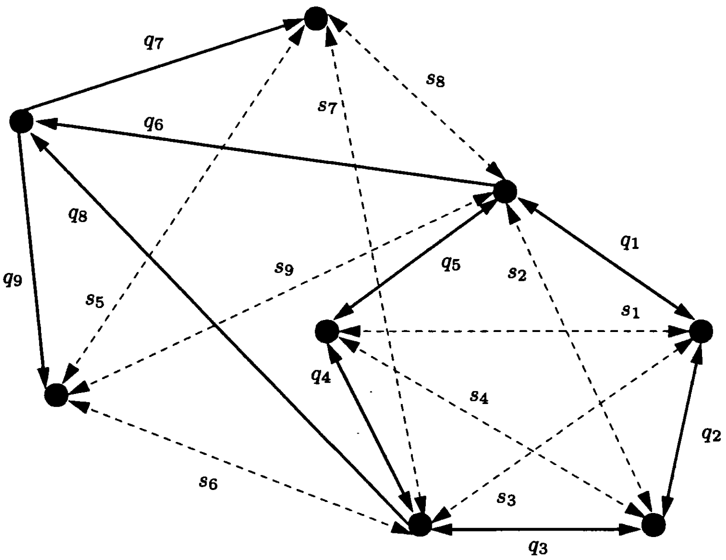


Figure 4.3: A tour representation of matrix M.

4.2 Bidirected graph representation of totally unimodular matrices

4.2.1 Totally unimodular matrices are tour matrices

In this section we shall demonstrate that all totally unimodular matrices have a bidirected graph representation by showing that they are a subclass of tour matrices. A simple but uninformative way of showing this is to provide a method to create a "trivial" tour representation for any given totally unimodular matrix. Such a case is illustrated in the following theorem.

Theorem 4.7. *All totally unimodular matrices are tour matrices.*

Proof: Let T be an $n \times m$ totally unimodular matrix. By Theorem 1.4 (Ghouila-Houri characterization of totally unimodular matrices [32]), we have that there exists a vector $x^T \in \{\pm 1\}^n$ such that $x^T T = y^T \in \{0, \pm 1\}^n$; that is multiplying the rows by ± 1 the resulting matrix has columns which sum up to $\{0, \pm 1\}$. Therefore, we can have $\underbrace{\begin{bmatrix} x^T \\ x^T \end{bmatrix}}_{Q} T = \underbrace{\begin{bmatrix} y^T \\ y^T \end{bmatrix}}_{S}$ and $[Q|S] = \left[\begin{array}{c|c} x^T & y^T \\ x^T & y^T \end{array}\right]$ is the incidence matrix of a bidirected graph Σ since the sum of the absolute values of the elements in each column is less or equal to 2. Furthermore, Σ consists only of links and $\Sigma(Q)$ does not contain loops and thus, T is a tour matrix. $\square$

However, a tour matrix may have multiple bidirected graph representations, and in the proof of Theorem 4.7 the bidirected graph so constructed does not have enough structural information with respect to the linear independence of the columns of the associated matrix. Moreover, the network or binet building blocks for totally unimodular matrices have directed or bidirected graph representations which do not have positive loops in contrast with the graph provided in the proof of Theorem 4.7. In addition, as we show in Propositions 4.8 and 4.9, large subclasses of totally unimodular matrices have richer and more informative tour representations than that described in the proof of Theorem 4.7. Therefore, there must exist a way to obtain a bidirected graph representation with a richer structure for a given totally unimodular matrix. This is accomplished by an algorithm provided in the next section for which the following two propositions are of importance.

Proposition 4.8. *Let A be the incidence matrix of a network representation of a network matrix N and Q be the column submatrix of A corresponding to the tree edges. Then a tour representation of N is the bidirected graph Σ with incident matrix A and prime subgraph $\Sigma(Q)$.*

Proof: Let S be the column submatrix of A corresponding to the non-tree edges of the network representation, then, since N is a network matrix, $QN = S$. Furthermore, the bidirected graph Σ with incident matrix A consists only of links and $\Sigma(Q)$ does not contain loops and thus, the result follows. $\square$

Proposition 4.9. *Let A be the incidence matrix of a binet representation of a totally unimodular, binet and non-network matrix N and Q be the column submatrix of A corresponding to the basic edges. Then a tour representation of N is the bidirected graph Σ with incident matrix A and prime subgraph $\Sigma(Q)$.*

Proof: Let S be the column submatrix of A corresponding to the non-basic edges of the binet representation, then, since N is binet, $QN = S$. It remains to show that the bidirected graph Σ with incidence matrix A does not have negative loops or half-edges and that $\Sigma(Q)$ consists only of links. Since N is totally unimodular and non-network, the regular matroid $M(N)$ is not graphic; moreover, it is known that binet matrices are compact representation matrices for signed-graphic matroids and thereby, $M(N)$ is signed-graphic. Slilaty [72] has shown that if the signed-graphic matroid $M(N)$ is regular and non-graphic then any binet representation Σ of N has no two vertex disjoint negative cycles and no balancing vertex, that is, no vertex whose deletion leaves a balanced bidirected graph. Therefore, Σ contains no negative loops or half-edges since, if we suppose the contrary, Σ would have a balancing vertex. Finally, $\Sigma(Q)$ contains no positive loops since otherwise, a positive loop would be a basic edge in the binet representation of N, a contradiction. $\square$

4.2.2 A bidirected graph for any totally unimodular matrix

According to Theorem 1.9, any totally unimodular matrix can be composed by applying k-sum operations on specific building blocks each of which is either a network matrix or the transpose of a network matrix or the matrices B_1 or B_2. Therefore, it is easy to see that any building block of a totally unimodular matrix is either: (i) a network matrix, (ii) a binet and non-network matrix, or (iii) a non-binet matrix. By Propositions 4.8 and 4.9, any building block falling in category (i) or (ii) has an informative graphical representation while for a non-binet matrix the trivial representation given in the proof of Theorem 4.7 can be used. In the following lines we first describe methods to obtain a tour representation of the $\oplus_1$, $\oplus_2$, $\oplus_3$ and $\oplus^3$ of two totally unimodular matrices and then, based on these results, we provide a polynomial time algorithm which given a totally unimodular matrix as input will provide a tour representation of this matrix as output.

Tour representation of the $\oplus_1$, $\oplus_2$ and $\oplus_3$ of two tour matrices

In the proof of Theorem 4.5, given the tour representations of two tour matrices, we provide a tour representation of the matrix obtained by taking the $\oplus_3$-sum of these tour matrices. Furthermore, in Theorem 4.6, we show that the 1-sum and the 2-sum of two tour matrices are special cases of the $\oplus_3$-sum and thereby, a tour representation of a matrix obtained by taking the 1-sum or the 2-sum of two tour matrices is also provided.

Tour representation of the $\oplus^3$ of two totally unimodular matrices

Tour matrices are not closed under pivoting and therefore, the fact that tour matrices are closed under $\oplus_3$-sum does not imply that they are closed under $\oplus^3$-sum. However, if two tour matrices A and B are also totally unimodular then, by Theorems 1.8 and 4.7, $A \oplus^3 B$ is a tour matrix. In the following lines we show how a tour representation of $A \oplus^3 B$ can be obtained from the tour representations of A and B.

Suppose that $M = A \oplus^3 B$ then by applying a specific pivoting on M we can obtain a matrix M' such that $M' = A' \oplus_3 B'$, where A' and B' are pivoted versions of A and B (for more see Chapter 11 in [78]). Therefore, due to the proof of Lemma 4.5 we can find a tour representation of M'. Therefore, it remains to show how the pivoting on M' which gives rise to M is translated in the associated tour representation.

Since tour matrices are closed under permutation of rows and columns, let's assume that $M' = \begin{bmatrix} 1 & c \\ b & D \end{bmatrix}$ with a tour representation Σ with incidence matrix $[f\ Q \mid e\ S]$ such that

$$[f\ Q]M' = [e\ S], \tag{4.6}$$

where $[f\ Q]$ is the part of the incidence matrix corresponding to the prime edges of Σ. We shall show that the matrix $M = \begin{bmatrix} -1 & c \\ b & D - bc \end{bmatrix}$ which is obtained from M' by pivoting on the M'_{11} element has a tour representation with incidence matrix $[e\ Q| - f\ S]$, where $[e\ Q]$ is the part corresponding to the prime edges of this tour representation of M.

Clearly, the columns f and e correspond to a prime edge f and a non-prime edge e of the tour graph Σ of M', respectively. Consider the bidirected graph Σ' with incidence matrix $[e\ Q \mid -f\ S]$, that is Σ with edge f having its end-vertices reversed in sign. We shall show that Σ' with prime edges corresponding to the columns of $[e\ Q]$ and non-prime edges corresponding to the columns of $[-f\ S]$ is a tour representation of M. Thus, we shall show that

$$[e\ Q]M = [-f\ S] \tag{4.7}$$

We know from (4.6) that $f + \sum_i b_i Q_{\bullet i} = e$. Therefore,

$$-f = -e + \sum_i b_i Q_{\bullet i}, \tag{4.8}$$

which shows that the first column of M is a collection of tours in Σ'. Take any other column j of M. If $c_j = 0$ the relationship (4.7) follows. If $c_j = +1$ then we know from (4.6) that

$$f + \sum_i D_{ij} Q_{\bullet i} = S_{\bullet j},$$

and the corresponding product in (4.7) will be

$$e + \sum_i (D_{ij} - b_i) Q_{\bullet i}.$$

Partition the indices of the differences in the above summation into three sets: I_1 which corresponds to indices where both $D_{ij}, b_i \neq 0$, I_2 where $D_{ij} \neq 0$ and $b_i = 0$ and I_3 where $D_{ij} = 0$ and $b_i \neq 0$. Replacing e by (4.8) we have

$$\begin{aligned} e + \sum_i (D_{ij} - b_i) Q_{\bullet i} &= f + \sum_i b_i Q_{\bullet i} + \sum_{i \in I_1} (D_{ij} - b_i) Q_{\bullet i} + \sum_{i \in I_2} D_{ij} Q_{\bullet i} - \sum_{i \in I_3} b_i Q_{\bullet i} \\ &= f + \sum_i D_{ij} Q_{\bullet i} = S_{\bullet j} \end{aligned}$$

Similarly for the case where $c_j = -1$ (or alternatively use (b) of Lemma 4.4).

Consequently, in order to find a tour representation of $M = A \oplus^3 B$ we do the following. We pivot at a non-zero element of M, say at the element in row indexed by r and column indexed by s, so that the matrix M' so-obtained is the $\oplus_3$ of two matrices A' and B' (see [78] for the existence of such a pivoting), where A' and B' can be obtained from A and B, respectively, by a specific pivoting. Moreover, if any of the matrices A and B is network or binet, then a network or binet representation of A' and B' can be found by exchanging a tree with a non-tree edge or a basic with a non-basic edge, respectively. Therefore, using Propositions 4.8 and 4.9 and the trivial representation given in the proof of Theorem 4.7, a tour representation of A' and B' can be found and therefore, by Lemma 4.5, a tour representation of M' can be constructed. Then, a tour representation of M can be obtained from M' by reversing the prime edge r and then exchanging the reversed r with the non-prime edge s (i.e. by making the reversed r non-prime and s prime).

The algorithm

Based on the analysis we did in this section, we shall now present an algorithm which, given a totally unimodular matrix T, will construct a bidirected graph Σ or equivalently an incidence matrix, where each column of T represents a collection of closed tours in Σ.

TU REPRESENTATION ALGORITHM

Input: A totally unimodular matrix T.
Output: A tour representation of T.

Step 1. Decompose T via k-sums into matrices $T_1, \ldots, T_n$ each of which, by Seymour's decomposition theorem (Theorem 1.9), is a network matrix, or the transpose of such a matrix or the matrix B_1 or B_2 of (1.1). A separation algorithm for finding k-sum decompositions can be found in Truemper's book [78].

Step 2. For each matrix T_i one of the following cases will be true:

- **2.1.** Check whether T_i is a network matrix; this can be done by the Tutte's recognition algorithm [10, 81] (resulting from his decomposition theory for graphic matroids). If T_i is network then the algorithm provides a network representation of T_i. Using this network representation, create a tour representation D_{Σ_i} of T_i as described in Proposition 4.8.
- **2.2.** Check whether T_i is a binet matrix; this can be done by using the algorithm given in [54]. If T_i is binet then the algorithm provides a binet representation of T_i. Using this binet representation, create a tour representation D_{Σ_i} of T_i as described in Proposition 4.9.
- **2.3** If neither of the above cases is true, then T_i is the transpose of a network matrix which is not binet. If so construct a tour representation D_{Σ_i} of T_i as described in the proof of Theorem 4.7.

Step 3. Starting from the k-sum decompositions indicated in step 1 and the tour representations D_{Σ_i} resulting from step 2, compose a tour representation of T as described in the "Tour representation of the $\oplus_1$, $\oplus_2$ and $\oplus_3$ of two tour matrices" and "Tour representation of the $\oplus^3$ of two totally unimodular matrices" parts of this section.

All of the above steps can be performed in polynomial time with respect to the size of the matrix T, since all the routines used in this algorithm have been shown to run in polynomial time in the worst case.

The fact that case 2.3 in the above algorithm is possible, that is the existence of a transpose of a network matrix which is not binet, is verified by a recent work of Slilaty [69] in which he identifies a set of 29 cographic excluded minors for the class of signed-graphic matroids. Examination of these 29 excluded minors, reveals that the representation matrices of all the k-sum decomposable ones are a 2- or 3-sum of two binet matrices without positive loops, therefore by Lemma 4.5, tour matrices with a bidirected graph representation without positive loops. However, we were unable to generalize this to an arbitrary non-binet transpose of a network matrix, therefore we use the trivial bidirected graph representation given in the proof of Theorem 4.7.

4.3 Concluding remarks

In the following lines we shall present three directions for further research:
1. Totally unimodular matrices characterize a class of well solved integer programming problems, due to the integrality property of the associated polyhedron. In this chapter we exploited the decomposition theorem of Seymour for totally unimodular matrices, and provided a graphical representation for every such matrix on a bidirected graph, such that the structural information of the decomposition building blocks is mostly retained. In order to do this, we defined the class of tour matrices and we showed that tour matrices are closed under the k-sum operations for $k = 1, 2, \oplus_3$. However, there must exists a tour representation without positive loops for any totally unimodular matrix (provided that it has no all-zero columns). Such a statement would be correct if the transpose N^T of every network matrix N (of a graph G) were a network matrix. However, it has been proved that N^T is a network matrix if and only if G is planar [80, 88].

Furthermore, we could say that all totally unimodular matrices are tour matrices if the transposes of all network matrices were binet. As we have mentioned earlier, due to very recent work of Slilaty [69] where he identifies a set of excluded minors for a cographic matroid to be signed-graphic, we know that the transpose of a network matrix is not necessarily binet. Therefore, one direction in proving that every totally unimodular matrix has a bidirected graph representation without positive loops is to prove that the transposes of network matrices are tour matrices having tour representations without positive loops. We provide some possible directions towards such a graphical representation of totally unimodular matrices.

Theorem 4.10. *The transposes of network matrices associated with graphs K_5 and $K_{3,3}$ are binet.*

Proof: Let N_i^T denote the transpose of a network matrix of a graph $i \in \{K_{3,3}, K_5\}$. Then it can be easily shown that such N_i^T can be:

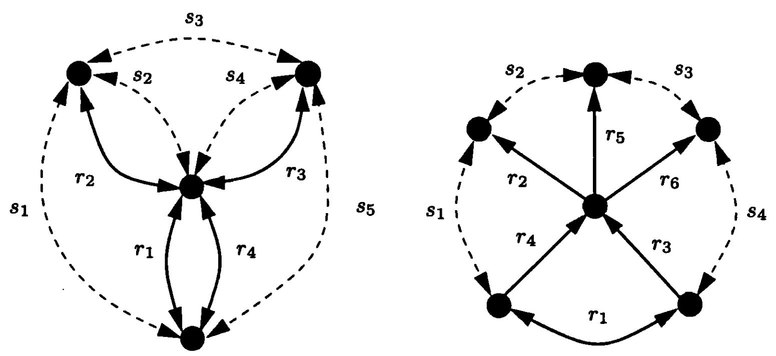
$$N_{K_{3,3}}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad N_{K_5}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We first show that these matrices are binet by providing the associated incidence matrix $[R_i|S_i]$ of the bidirected graph for each case. Note that for every $i \in \{K_{3,3}, K_5\}$ we have that R_i is non-singular.

$$[R_{K_{3,3}}|S_{K_{3,3}}] = \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$[R_{K_5}|S_{K_5}] = \left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

From the bidirected graphs corresponding to the aforementioned incidence matrices we can conclude that all N_i^T matrices ($i \in \{K_{3,3}, K_5)$ are binet (see Figure 4.4). Finally, any other network matrix associated with graph i is, up to row and column permutation and scaling of rows and columns by -1, a pivoted version of $N_{K_{3,3}}^T$ or $N_{K_5}^T$. By Lemma 2.11, binet matrices are closed under row and column permutations, scalings of rows and columns by -1 and pivotings and, therefore, the result follows. □



$N_{K_{3,3}}^T$ $N_{K_5}^T$

Figure 4.4: Binet representations of $N_{K_{3,3}}^T$ and $N_{K_5}^T$

In view of Theorem 4.6 and Kuratowski's characterization of planar graphs (see Theorem 1.2), an approach to show that the transpose of any network matrix is a tour matrix having a representation without positive loops would be to show that subdivisions and additions of edges in K_5 and $K_{3,3}$ give rise to graphs whose associated network matrices are the transposes of tour matrices having representations without positive loops. This is true for subdivisions as shown in the following Theorem 4.11. Note that when we say that a matrix N' is a subdivision of a network matrix N, we mean that N' is the network matrix of the graph G' which has been produced by subdividing an edge from the graph G associated with N.

Theorem 4.11. *If N is a network matrix, N^T is binet and N' is a subdivision of N, then N'^T is a binet matrix.*

Proof: If e is a basic edge in G and, therefore, associated with a row r_e of N then the network matrix N' will be the matrix N plus one extra row r'_e which will be identical to row r_e. Since, by Lemma 2.11, binet matrices are closed under duplication of columns we have that N'^T is binet.

If e is a non-basic edge in G then it will be associated with a column s_e of N. Therefore, it can be easily seen that N' will be N plus an extra column s'_e with exactly one non-zero element. Since, by Lemma 2.11, binet matrices are closed under addition of a unitary row we have that N'^T will be binet.

□

However, it seems difficult to prove that when the transpose of a network matrix N associated with a graph G is binet, the addition of an edge in G results in a network matrix N' whose transpose is a tour matrix with a representation without positive loops. Therefore, we state the following conjecture, an affirmative answer to which would imply that all totally unimodular matrices with no all-zero columns have a tour representation without positive loops.

Conjecture 4.12. *If N is a network matrix with no all-zero columns then N^T has a tour representation without positive loops.*

2. One of the key motivations for representing totally unimodular matrices on bidirected graphs is the potential to devise an efficient combinatorial algorithm which will solve the associated integer programming problem. Therefore, an open research question is how can we utilize the tour representation of a totally unimodular matrix in order to devise an algorithm for solving the associated integer programming problem. From our point of view, there are two possible approaches to this problem. According to the first approach, the tour representation of the original totally unimodular matrix is constructed and the combinatorial attributes of the tour graph are used in order to provide an efficient algorithm for the associated integer programming problem. More specifically, due to the similarities of the graph representations of network and binet matrices with the tour representation of totally unimodular matrices, i.e. in all cases the non-zeros in a column represent walks in a directed or a bidirected graph, we expect that such an algorithm must be based on the ideas behind the network simplex and the generalized network simplex methods.

As concerns the second approach, this is a "bottom-up approach" [1]. Specifically, the main idea in the second approach is to decompose the integer programming problem into smaller problems whose constraint matrices are building blocks of the original matrix, solve these problems by using fast available methods (network simplex, generalized network simplex etc.) and then combine the solutions of all these smaller problems into a solution for the original problem. Obviously, the difficult part in this approach is to devise a method which, given the solutions of the smaller problems, provides a solution to the original problem. In other words, given the solution of two smaller problems with A_1 and A_2 constraint matrices, the question is how we could obtain a solution to the problem whose constraint matrix is a k-sum of A_1 and A_2. From our point of view, the way that the graphical representation of A is obtained from the graphical representations of A_1 and A_2 may provide the insight which could make such a method possible. We should also note that a "bottom-up approach" is used in [77], where decomposition results for specific classes of graph (namely, perfect graphs and even-hole-free graphs) are used effectively in order to construct polynomial time combinatorial optimization algorithms. Therefore, ideas and directions of this recent work may prove useful in answering the open question of devising an algorithm of combinatorial nature for the class of integer programming problems with totally unimodular constraint matrices.

3. Further research could be conducted in the area of optimization. Possible research question could be the following: What could we say for linear or integer programming problems whose constraint matrix is a tour matrix? Can we devise efficient algorithms for such problems (e.g. similar to those for the case of binet matrices in section 2.3)? The class of tour matrices is a large subclass of $\{0, \pm 1\}$-matrices and, therefore, efficient methods for solving the associated linear and integer programming problems would

[1] The term "bottom-up" approach was taken from [77].

be of great importance. At the moment it seems very difficult to devise such methods for the entire class of tour matrices. However, devising combinatorial methods for linear and integer programming problems whose constraint matrix belongs to a subclass of tour matrices (such as TU matrices) could be the first step towards answering the above questions.

Part III

Signed-graphic matroids

Chapter 5

Signed graphs and the three associated matroids

In this chapter we discuss mainly two new classes of graphs, signed graphs and biased graphs, which are closely related to the class of bidirected graphs. Loosely speaking, given a bidirected graph, if we ignore the signs at the end-vertices of its edges and instead focus only on whether an edge is positive or negative, then a signed graph is obtained from that bidirected graph. It is easy to see that more than one bidirected graph may give rise to the same signed graph, while there is only one signed graph associated with a given bidirected graph. Furthermore, it will be shown that the class of signed graphs can be regarded as a special case of the most general class of graphs discussed in this work which is the class of biased graphs.

Harary first introduced signed graphs in [39]. Both biased graphs and signed graphs have been extensively studied by Zaslavsky [96, 98]. We should also note here that Zaslavsky is also the creator of the glossary of signed and gain graphs [101] and of the annotated bibliography of signed and gain graphs [100], which is updated regularly. Both these works are essential tools for someone interested in biased graphs. Among others, Gerards has also studied signed graphs and mainly a binary matroid associated with the class of signed graphs, which in Zaslavsky's terminology, is the complete lift matroid of a signed graph. We mention the work of Gerards for two reasons; firstly, because decomposition theorems for signed graphs were provided (in Chapter 3 of [27]) and secondly, because using signed graphs Gerards et.al. in [25, 29] were able to re-prove important theorems of matroid theory in an elegant way. Apart from the complete lift matroid of a biased graph, two other matroids discussed in the following sections are the lift matroid and the frame matroid of a biased graph.

We should note that central in this thesis is the notion of the signed graph and the notion of the frame matroid of a signed graph, which is known as signed-graphic matroid. For that reason, most of this chapter is devoted to results concerning signed graphs and signed-graphic matroids. This chapter is organised as follows. In section 5.1, we provide some basic notions regarding signed graphs and biased graphs, where the clear connection between signed graphs and bidirected graphs is also discussed. In the same section, the definitions of the three aforementioned matroids associated with biased graphs are also provided. The terminology, definitions and notation provided in this section was mainly taken from works of Zaslavsky [94, 98, 101]. In the following sections we focus on the matroids associated with signed-graphs. In section 5.2 we provide results regarding the lift and the complete lift matroid of a signed graph and show that the extended even cycle matroid appearing in the relevant literature is the complete lift matroid. Later sections are devoted to signed-graphic matroids which is of central importance in this work. Specifically, in section 5.3 we provide some interesting preliminary results for this class while in 5.4 we examine an important subclass of signed graphs, the so-called tangled signed graphs, and, among other results, we show the exact relationship between the class of tangled signed graphs and the class of binary signed-graphic matroids.

5.1 Signed graphs and biased graphs

5.1.1 Signed graphs

A *signed graph* is defined as $\Sigma := (G, \sigma)$ where G is a graph called the *underlying graph* of Σ and σ is a sign function $\sigma : E(G) \rightarrow \{\pm 1\}$; by definition $\sigma(e) = -1$ if e is a half-edge and $\sigma(e) = +1$ if e is a loose edge while links and loops can be positive or negative. Thus, a signed graph is a graph where the edges are labelled as positive or negative. We denote by $V(\Sigma)$ and $E(\Sigma)$ the vertex set and the edge set of a signed graph Σ, respectively. The graphical representation of a signed graph consists of the graphical representation of its underlying graph with a $+$ or a $-$ on each edge stating that the corresponding edge is positive or negative, respectively. In Figure 5.1 we depict an example signed graph.

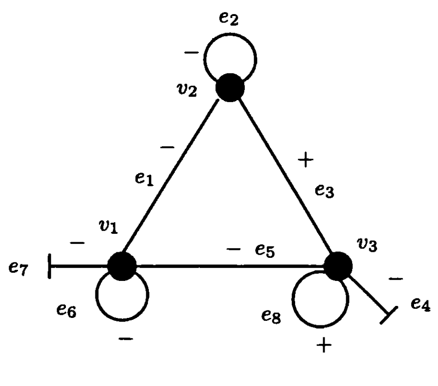


Figure 5.1: Graphical representation of a signed graph.

We can define a series of similar operations on signed graphs as we did for graphs. All operations on signed graphs are defined through a corresponding operation on the underlying graph and the sign function. In the following definitions assume that we have a signed graph $\Sigma = (G, \sigma)$. *Deletion* of a vertex v is defined as $\Sigma \backslash v := (G \backslash v, \sigma)$. The operation of *switching* at a vertex v results in a new signed graph $(G, \bar{\sigma})$ where $\bar{\sigma}(e) := -\sigma(e)$ for each link e adjacent to v, while $\bar{\sigma}(e) := \sigma(e)$ for all other edges. *Deletion* of an edge e is defined as $\Sigma \backslash e := (G \backslash e, \sigma)$. The *contraction* of an edge e is more complicated and consists of three cases:

1. if e is a positive loop, a loose edge or a positive link, then $\Sigma / e := (G/e, \sigma)$
2. if $e = \{v, v\}$ is a negative loop or $e = \{v\}$ is a half-edge, then $\Sigma / e := (G/e, \sigma')$ where, for any

$f \in E(\Sigma/e)$, $\sigma'(f) = \sigma(f)$ if f was not incident with v in Σ; while, by definition, any half-edge or any loose edge created by contracting e has a -1 or a $+1$ sign, respectively

3. if $e = \{u, v\}$ is a negative link, then $\Sigma/e := (G/e, \bar{\sigma})$ where $\bar{\sigma}$ is a result of switching at either one of the end vertices of e. [1]

A signed graph Σ' obtained from a signed graph Σ by a sequence (possibly empty) of contractions of edges and deletions of edges and/or vertices of Σ is called a *minor of* Σ. The *sign of a cycle* is the product of the signs of its edges, so we have a *positive cycle* if the number of negative edges in the cycle is even, otherwise the cycle is *negative.* Both negative loops and half-edges are by definition negative cycles. A signed graph is called *balanced* if it contains no negative cycles; otherwise, it is called *unbalanced.* Therefore, a positive (negative) cycle may also be called a *balanced (unbalanced) cycle.* We also call a *circle* of Σ *positive (negative)* if the corresponding cycle of Σ is positive(negative). A vertex $v \in V(\Sigma)$ is called a *balancing vertex* if $\Sigma\backslash v$ is balanced. Furthermore, we define the *b-star* of a vertex v of a signed graph Σ as the set of edges having v as an end-vertex and are not positive loops. All remaining notions used for a signed graph are as defined for graphs in Chapter 1 (as applied to its underlying graph). For example, for some $S \subseteq E(\Sigma)$ we have that $\Sigma[S] = (G[S], \sigma)$, Σ is k-connected if and only if G is k-connected etc.

Obviously, a bidirected graph is also a signed graph since bidirected graphs have also a sign assigned to each edge. The sign of an edge of a bidirected graph is determined by the signs at the end-vertices of the edge. Therefore, we may view a bidirected graph Γ as an oriented version of some signed graph Σ, i.e. we can orient the edges of Σ in order to obtain Γ with same signs on the corresponding edges. In order to do this we allocate arbitrary signs on the ends of every edge of the signed graph so that positive edges become directed and negative edges become bidirected; this procedure is called *orientation of a signed graph.* More specifically, if $e = \{u, v\}$ is a link or a loop of a signed graph Σ then the sign $s_e(u)$ of e at u and the sign $s_e(v)$ of e at v in an orientation of Σ are determined by $s_e(v) = -\sigma(e)s_e(u)$. In this case, we usually say that Σ is the *underlying signed graph* of Γ. Clearly, more than one bidirected graph may have the same underlying signed graph. Finally, based on this relation, we can say that every result provided for bidirected graphs can be easily converted to signed graphs.

5.1.2 Biased graphs and the associated matroids

A *biased graph* $\Omega = (G, \mathcal{B})$ consists of a graph G which is called the *underlying graph* of Ω and a linear subclass $\mathcal{B}$ of the ordinary circles of G, where a class $\mathcal{B}$ of ordinary circles is a *linear subclass* if it has the property: if C_1 and C_2 belong to $\mathcal{B}$ and $C_1 \cup C_2$ is the edge set of a theta graph then the third ordinary circle $C_1 \triangle C_2$ also belongs to $\mathcal{B}$. The circles in $\mathcal{B}$ are called *balanced circles.* Any circle (ordinary or not) which is not in $\mathcal{B}$ is called *unbalanced.* Therefore, a biased graph is a graph together with a distinguished class of ordinary circles, which are called balanced, such that no theta subgraph contains exactly two balanced

[1] Note that contraction of a negative link is well-defined only up to switching. Specifically, contracting a negative link $e = \{u, v\}$ after switching at u and contracting after switching at v result in different but switching equivalent signed graphs. In the rest of this work the sign of the cycles of a signed graph is of importance and not the sign of the edges incident with a vertex u; for that reason we shall use the shorter "contracting edge e" instead of saying "contracting e after switching at u". This situation is also discussed in [94] and it is regarded as standard practice to contract a negative link in a signed graph as defined above.

circles. Furthermore, a subgraph or edge set of G is called *balanced* if every circle in it is balanced and has no half-edges; otherwise, it is unbalanced.

Another property of a subclass of the ordinary circles of a graph, which is stronger than linearity, is *additivity*. Specifically, a subclass of the ordinary circles of a graph is called *additive*, if in any theta subgraph an odd number of circles belong to the subclass. A biased graph $\Omega = (G, \mathcal{B})$ such that $\mathcal{B}$ is an additive subclass of circles in G is called an *additively biased graph*. Given now a signed graph Σ with underlying graph G let us consider the pair $(G, \mathcal{C}_+(\Sigma))$, where $\mathcal{C}_+(\Sigma) = \{\text{positive ordinary circles in } \Sigma\}$. The following proposition, which is the Example 6.4 in [96], shows that a signed graph may also be viewed as an additively biased graph.

Proposition 5.1. *If $\Sigma = (G, \sigma)$ is a signed graph then the pair $(G, \mathcal{C}_+(\Sigma))$ is an additively biased graph.*

The following theorem, which appears in [93], shows that the converse of Proposition 5.1 is also correct.

Theorem 5.2. *$\Omega = (G, \mathcal{B})$ is an additively biased graph if and only if there exists a signed graph $\Sigma = (G, \sigma)$ such that $\Omega = (G, \mathcal{C}_+(\Sigma))$.*

Another important class of graphs which may be viewed as biased graphs is that of gain graphs (see [96, 98]). Formally, a *gain graph* $\Phi = (G, \phi)$ consists of an underlying graph G together with a mapping $\phi : E_* \to \mathcal{G}$, where E_* is the set of ordinary edges of G and $\mathcal{G}$ is a group of elements. We think of the edges in E_* as directed in an arbitrary but fixed way, so that if e is an edge in one direction, then e^{-1} is the same edge in the opposite direction. The gain of an edge e whose orientation has been reversed is defined as: $\phi(e^{-1}) = [\phi(e)]^{-1}$. Let us suppose that $P = \{e_1, e_2, \ldots, e_n\}$ is the edge set of a cycle C. Reverse the orientation of the edges so that all of them are oriented in the same direction, i.e. C becomes a directed cycle. The gain $\phi(P)$ of circle P is the product of the direction-adjusted gains of the cycle-edges, i.e. $\phi(P) = \prod_{i=1}^{n} [\phi(e_i)]^{k_i}$, where $k_i = -1$ if e_i was reversed and $+1$ otherwise. Obviously, the gain of a circle depends on the chosen starting point and direction unless the gain is $+1$. A circle whose gain is $+1$ is called *balanced*; the class of balanced circles of Φ will be denoted by $\mathcal{B}(\Phi)$. It has been shown in [96] that the pair $(G, \mathcal{B}(\Phi))$ is a biased graph and in that sense we can say that any gain graph may be viewed as a biased graph. Observe also that if $\mathcal{G}$ is a group of two elements then a gain graph is a signed graph (with only minor differences in terminology). Thus, the class of gain graphs stands between that of signed graphs and that of biased graphs. Apart from that, another important reason for which we mention gain graphs is that the generalized networks discussed in section 2.3.1 are clearly based on gain graphs with $\mathcal{G} = R_+$.

Zaslavsky defined three matroids associated with a biased graph in [98]. We shall present these important definitions in the rest of this section. Let $\Omega = (G, \mathcal{B})$ be a biased graph; the definition of the *frame matroid* of Ω, denoted by $B(\Omega)$, with respect to its bases goes as follows:

Definition 5.3. *The ground set of the frame matroid $B(\Omega)$ of a biased graph $\Omega = (G, \mathcal{B})$ is the set of edges of G. A basis of $B(\Omega)$ consists of the edge set of a spanning tree in each balanced component of Ω and the edge set of an unbalanced 1-tree in each other component.*

Equivalently, one can define the frame matroid of a biased graph with respect to its circuits. A circuit of $B(\Omega)$ is the edge set of: (i) a balanced cycle of Ω, (ii) a pair of unbalanced cycles of Ω that meet in exactly one vertex (iii) a pair of vertex-disjoint unbalanced cycles with a minimal path which connects these cycles, or (iv) a theta subgraph with no balanced cycles (see Figure 5.2 (a), (b) and (c) for examples of (ii), (iii) and (iv), respectively, where all depicted cycles are unbalanced).

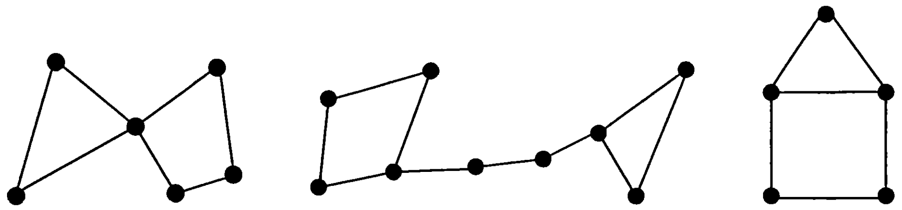


Figure 5.2: Example subgraphs of a biased graph Ω whose edge sets are circuits of $B(\Omega)$ (all cycles depicted are unbalanced).

Associated with a biased graph $\Omega = (G, \mathcal{B})$, there are two other matroids which are discussed in depth in [98]. These matroids are the *lift matroid* $L(\Omega)$ and the *complete lift matroid* $L_0(\Omega)$ of Ω defined as follows:

Definition 5.4. *The ground set of the lift matroid $L(\Omega)$ of the biased graph $\Omega = (G, \mathcal{B})$ is the set $E(G)$. A circuit of $L(\Omega)$ is the edge set of: (i) a balanced cycle of Ω, (ii) two unbalanced cycles of Ω with at most one vertex in common, or (iii) a theta subgraph of Ω with no balanced cycles.*

It follows from Definition 5.4 that the edge-set of the subgraph (of Ω) depicted in Figure 5.2(b) is not a lift circuit of $L(\Omega)$ while those of Figure 5.2 (a) and (c) are.

Definition 5.5. *The ground set of the complete lift matroid $L_0(\Omega)$ of the biased graph $\Omega = (G, \mathcal{B})$ is the set $E(G) \cup \{e_0\}$, where e_0 is an extra element. A circuit of $L_0(\Omega)$ is the edge set of: (i) a balanced cycle of Ω, (ii) two unbalanced cycles of Ω with at most one vertex in common, (iii) a theta subgraph of Ω with no balanced cycles, or (iv) an unbalanced cycle of Ω union with $\{e_0\}$.*

By Definitions 5.4 and 5.5, it is not difficult to see that we can define the lift matroid via the complete lift matroid and vice versa (see also e.g. [71, 98]).

Definition 5.6. *The lift matroid $L(\Omega)$ of a biased graph Ω is $L_0(\Omega)\backslash e_0$.*

Definition 5.7. *The complete lift matroid $L_0(\Omega)$ of a biased graph Ω is $L(\Omega_0)$, where Ω_0 consists of Ω along with an unbalanced loop e_0 attached to a new vertex.*

Also, by Definitions 5.4 and 5.5, we can easily derive the following lemma[2].

Lemma 5.8. *Let Ω' be a biased graph obtained from a biased graph Ω by replacing any number of half-edges by unbalanced loops and vice versa, then $L(\Omega') = L(\Omega)$ and $L_0(\Omega') = L_0(\Omega)$.*

For a signed graph $\Sigma = (G, \sigma)$, the frame matroid of the biased graph $\Omega_\Sigma = \{G, \mathcal{C}_+(\Sigma)\}$ is called the *frame matroid of* Σ or the *signed-graphic matroid of* Σ and is denoted by $M(\Sigma)$. Similarly, the lift

[2]The straightforward proof of Lemma 5.8 is similar to that of Lemma 5.23 which is presented later in this chapter.

matroid and the complete lift matroid of Ω_Σ are called the *lift matroid of* Σ and the *complete lift matroid of* Σ, respectively. Using the standard notation, the lift matroid of Σ is denoted by $L(\Sigma)$ while the complete lift matroid of Σ is denoted by $L_0(\Sigma)$. Finally, we note that in this work we are mostly interested in the signed-graphic matroid of a signed graph. For that reason in the next section we review some important results regarding the lift matroid and the complete lift matroid of a signed graph while we devote the rest of this chapter to the signed-graphic matroid which is one of the central notions of the next chapters.

5.2 The lift and the complete lift matroid of a signed graph

A work considering the matroids of a signed graph would not be complete without referring to the two lift matroids associated with a signed graph. This is mainly due to the fact that many important results have been produced with the assistance of the complete lift matroid of a signed graph. However, in the literature this matroid has been studied by different viewpoints and was given different names. For these reasons, although the lift matroids of signed graphs do not play a central role in this work, we shall present some of these results using one common terminology, viz. Zaslavsky's terminology, in order to stress the importance of these matroids.

Since a signed graph Σ is also an additively biased graph, no three unbalanced circles may be contained in a theta subgraph of Σ. Thus, by Definition 5.5, we obtain the following Definition 5.9 for the complete lift matroid of Σ. Similarly, one can also obtain the Definition 5.10 for the lift matroid of a signed graph from Definition 5.4.

Definition 5.9. *The ground set of the complete lift matroid $L_0(\Sigma)$ of the signed graph $\Sigma = (G, \sigma)$ is the set $E(G) \cup \{e_0\}$, where e_0 is an extra element. A circuit of $L_0(\Sigma)$ is the edge set of: (i) a balanced cycle of Σ, (ii) two unbalanced cycles of Σ with at most one vertex in common, or (iii) an unbalanced cycle of Σ union with $\{e_0\}$.*

Definition 5.10. *The ground set of the lift matroid $L(\Sigma)$ of the signed graph $\Omega = (G, \sigma)$ is the set $E(G)$. A circuit of $L(\Sigma)$ is the edge set of: (i) a balanced cycle of Σ, or (ii) two unbalanced cycles of Σ with at most one vertex in common.*

There is a "natural" compact representation matrix N_Σ of $L_0(\Sigma)$ over $GF(2)$. The construction of N_Σ goes as follows. By Lemma 5.8 we may suppose that $\Sigma = (G, \sigma)$ contains no half-edges. Let M_G be the node-edge incidence matrix of G. Adjoin an additional row x_G to M_G recording if an edge is positive or negative. Specifically, row x_G has a $+1$ in column e if $\sigma(e) = -1$ and has a 0 if $\sigma(e) = +1$. Finally, we add a new column corresponding to e_0 having a 1 in the additional row and 0 elsewhere. Thus, N_Σ has the following form:

$$N_\Sigma = \begin{bmatrix} 1 & x_G \\ 0 & M_G \end{bmatrix}$$

Lemma 5.11. *The matrix N_Σ is a compact representation matrix for $L_0(\Sigma)$ over $GF(2)$.*

Proof: By Definition 5.7, we have that $L_0(\Sigma) = L(\Sigma_0)$, where Σ_0 consists of Σ along with a negative loop e_0 attached to a new vertex. Let T be a subset of the columns of N_Σ which constitutes the column submatrix N_T of N_Σ. By Definition 5.10 of the lift matroid of a signed graph, it suffices to show that T

is a minimal set of linearly dependent columns of N_Σ if and only if the set of edges corresponding to the columns in T induce either a positive cycle or two negative cycles with at most one common vertex in Σ_0.

The "if" part of the above statement stems easily from the structure of N_Σ. For the "only if" part, we first observe that every row of N_T must have an even number of 1s. Therefore, the subgraph of Σ_0 induced by the edges corresponding to the columns in T is Eulerian and thus, it is a union of, say k ($k \in \mathbb{Z}_+$), edge-disjoint cycles. If $k = 1$ then there exist an even number of 1s in the first row of T which means that the cycle induced by T is a positive cycle and, thus, the result follows. If $k = 2$, and if one of the cycles was positive then there would exist a proper subset T' of T consisting of linearly dependent columns and thus, T would not be minimal. Therefore, both cycles have to be negative and, thus, the result follows. Finally, if $k \geq 3$ then at least two cycles are positive or negative. In both cases, the columns corresponding to those two cycles constitute a proper subset T' of T which is a minimal set of dependent columns of N_Σ. This means that T is not minimal and, thus, we cannot have $k \geq 3$. □

In [95], Zaslavsky proves the following interesting Theorem 5.12.

Theorem 5.12. *The complete lift matroid $L_0(\Omega)$ of a biased graph Ω is binary if and only if Ω is a signed graph.*

Gerards also defined a binary matroid represented over $GF(2)$ by the matrix N_Σ. In some of his works (e.g. [29]) this matroid is called the extended even cycle matroid of a signed graph. By Lemma 5.11 we have that the extended even cycle matroid of a signed graph Σ is precisely the complete lift matroid of Σ. Gerards also defined his terminology associated with this matroid; however, for a matter of simplicity, when we present results of his works (e.g. Theorem 5.15) we shall express them using Zaslavsky's terminology.

An interesting lemma which can be found in [42] is Lemma 5.13 which provides sufficient conditions for a complete lift matroid to be graphic. Moreover, this lemma is used in [42] in order to provide an alternative proof of Theorem 1.12.

Lemma 5.13. *Let $L_0(\Sigma)$ be the complete lift matroid of a signed graph Σ. If Σ is vertically 3-connected and $L_0(\Sigma)$ is 3-connected then $L_0(\Sigma)$ is graphic.*

Many of the results of Gerards which are associated with the complete lift matroid of a signed graph appear in [27]. In [31], Gerards and Schrijver characterized signed graphs with incident matrices whose transpose has strong Chvátal rank 1. Furthermore, as discussed also in section 2.3.3, this class of matrices along with integral binet matrices are among the few known classes of matrices having this property. A central role in this characterization (Theorem 5.14) is played by the odd-K_4 signed graphs. An *odd-K_4* signed graph is a signed subdivision of K_4 such that each of the four cycles in it having exactly three vertices of degree three, is negative.

Theorem 5.14. *The transpose of the incidence matrix of any orientation of a signed graph Σ has strong Chvátal rank 1 if and only if Σ does not contain an odd-K_4 as a subgraph.*

The following Theorem 5.15 of [27] shows how the result of Theorem 5.14 is related to the complete lift matroid of Σ.

Theorem 5.15. *A signed graph Σ contains no odd-K_4 as a subgraph if and only if $L_0(\Sigma)$ has no F_7^* minor using e_0.*

Finally, using the complete lift matroid of a signed graph Gerards et.al. were able to re-prove in an elegant way important theorems of matroid theory. Specifically, in [29], we find such a proof for Theorem 1.17, while in [25] a key theorem for the proof of the regular matroid decomposition theorem of Seymour, which is one of the deepest theorems in matroid theory, is given via signed graphs and the associated complete lift matroid.

5.3 The signed-graphic matroid

5.3.1 Basic notions

We begin this section with a characterization of signed-graphic matroids, based on the collection of its circuits. By doing this we have a direct correspondence between the circuits of the matroid and the circuits of the associated signed graph. This correspondence is further extended to other concepts known in matroids, such as cocircuits and elementary separators, and the associated sets of edges are characterized in the signed graph. It is noted that although many concepts such as circuits, cocircuits and connectivity in matroids were generalized from graphs, here we instead specialize the aforementioned concepts to signed graphs.

Characterizations

Two equivalent characterizations for the *signed-graphic matroid* of a signed graph Σ are provided, both of which were introduced in [94]. In the first one (Theorem 5.16) a rank function on the edge set of Σ is defined such that the axioms of Definition 1.11 are satisfied and the resulting matroid is the signed-graphic matroid of Σ. In the second one (Theorem 5.17), the sets of edges in Σ which correspond to the circuits of the associated signed-graphic matroid $M(\Sigma)$ are identified. Note also here that Theorem 5.17 may also easily be derived by the circuit characterization of the frame matroid of a biased graph and the fact that signed graphs are additively biased graphs (see Proposition 5.1).

Theorem 5.16. *Given a signed graph Σ define a function $r : 2^{E(\Sigma)} \to \mathbb{Z}^+$ as $r(S) = |V(S)| - b(S)$ for $S \subseteq E(\Sigma)$, where $V(S)$ is the set of vertices incident with the edges of S and $b(S)$ is the number of balanced components of $\Sigma[S]$. Then $M(\Sigma) = (E(\Sigma), r)$ is the signed-graphic matroid of Σ with rank function r.*

Theorem 5.17. *Given a signed graph Σ let $C \subseteq 2^{E(\Sigma)}$ be the collection of minimal edge sets inducing a subgraph in Σ which is either:*

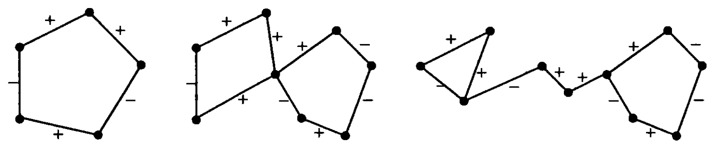
(i) a positive cycle, or

(ii) two negative cycles which have exactly one common vertex, or

(iii) two vertex-disjoint negative cycles connected by a path which has no common vertex with the cycles apart from its end-vertices.

Then $M(\Sigma) = (E(\Sigma), \mathcal{C})$ is the signed-graphic matroid of Σ with collection of circuits $\mathcal{C}$.

The subgraphs of Σ induced by the edges corresponding to a circuit of $M(\Sigma)$ are called *circuits* of Σ. Therefore, a circuit of Σ can be one of three types. The circuits of Σ described by (ii) and (iii) of Theorem 5.17 are also called *handcuffs* of Type I and Type II, respectively. Examples of the different types of circuits of a signed graph are depicted in Figure 5.3.1.



(a) positive cycle (b) Type I handcuff (c) Type II handcuff

Figure 5.3: Different types of circuits in a signed graph.

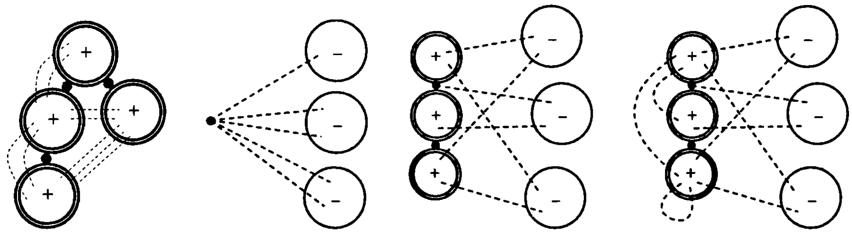
Therefore, for each signed graph Σ with edge set $E(\Sigma)$, there is an associated signed-graphic matroid $M(\Sigma)$ on the set of elements $E(\Sigma)$. However, for a given signed-graphic matroid M there may exist several signed graphs Σ_i such that $M = M(\Sigma_i)$ where, $i \geq 1$ $(i \in \mathbb{Z}^+)$. Therefore, signed-graphic matroids can be viewed as the abstract constructs, while their corresponding signed graphs their representations in a graphical context. Later we will also mention analogous representations using matrices, that is in an algebraic context.

Duality and Representability

We know that for any matroid M there exists a dual matroid M^* on the same ground set. With the following Theorem 5.18 appearing in [94], we can characterize the sets of edges in a signed graph Σ which correspond to circuits of $M^*(\Sigma)$.

Theorem 5.18. *Given a signed graph Σ and its corresponding matroid $M(\Sigma)$, $Y \subseteq E(\Sigma)$ is a cocircuit of $M(\Sigma)$ if and only if Y is a minimal set of edges whose deletion increases the number of balanced components of Σ.*

The sets of edges defined in Theorem 5.18 (i.e. the minimal sets of edges whose removal increases the number of balanced components of Σ) are called *bonds* of Σ. In analogy with the different types of circuits a signed-graphic matroid has, bonds can also be classified into different types according to the signed graph obtained upon their deletion. Specifically, for a given connected and unbalanced signed graph Σ, the deletion of a bond Y results in a signed graph $\Sigma \backslash Y$ with exactly one balanced component due to the minimality of Y. Note here that in contrast with ordinary graphs, although the minimality of Y restricts $\Sigma \backslash Y$ to have exactly one balanced component, the number of unbalanced components is unlimited. Thus, $\Sigma \backslash Y$ may be a balanced connected graph in which case we call Y a *balanced bond* or it may consist of one balanced component and some unbalanced components. In the latter case, if the balanced component is a vertex, i.e. the balanced component is empty of edges, then we say that Y is a *star bond*, while in the case that the balanced component is not empty of edges Y can be either an *unbalanced bond* or a *double bond*. Specifically, if the balanced component is not empty of edges and there is no edge in Y such that both of its end-vertices are vertices of the balanced component, then Y is an unbalanced bond. On the other hand, if there exists at least one edge of Y whose both end-vertices are vertices of the balanced component then Y is a double bond. Therefore, double bond can be viewed as a combination of a balanced and an unbalanced bond, where Y also contains edges between vertices of the balanced component. In Figure 5.3.1, we illustrate the four aforementioned types of bonds by providing an example signed graph for each case. The dashed lines represent the edges of a bond, a circle depicts a connected graph and two homocentric circles depict a block, where in each case a positive (negative) sign is used to indicate whether the connected or 2-connected component is balanced (unbalanced).



(a) balanced bond (b) star bond (c) unbalanced bond (d) double bond

Figure 5.4: The four types of bonds in signed graphs. Dashed lines represent the edges of the bond in each case.

A further classification of bonds is based on whether the matroid $M(\Sigma)\backslash Y$ is connected or not for some $Y \in C^*(M(\Sigma))$. In the case that $M(\Sigma)\backslash Y$ is disconnected we call Y a *separating bond* of Σ, otherwise we say that Y is a *non-separating bond.*

In [94], it is proved that signed-graphic matroids are ternary, while the following Theorem 5.19 of [57] provides necessary and sufficient conditions on a signed graph Σ for $M(\Sigma)$ to be binary.

Theorem 5.19. *Let Σ' be the signed graph obtained from a connected signed graph Σ by contracting all its balanced blocks. Then $M(\Sigma)$ is binary if and only if Σ' contains no pair of vertex disjoint negative cycles.*

Moreover by Theorem 1.14, it is evident that the class of binary signed-graphic matroids is a subclass of regular matroids. In Figure 5.5 the relationship between the class of signed-graphic matroids and other well-known classes of representable matroids is depicted.

Connectivity and Minors

By Theorem 1.21, the connectivity of a graphic matroid is equal to that of the associated graph. This is not the case for signed-graphic matroids, since the connectivity of a signed graph is solely defined on the underlying graph and does not take into account the sign function. Therefore for a graph, an elementary separator of the associated graphic matroid is the edge set of a block in the graph. In [94, 98], the edge sets of a signed graph which correspond to elementary separators in the associated signed-graphic matroid are determined. Before we present this result in Theorem 5.20, appearing in [98], we have to provide some necessary definitions. An *inner block* of Σ is a block that is unbalanced or lies on the path between two unbalanced blocks in the block graph of Σ. Any other block is called *outer*. The *core* of Σ is the union of

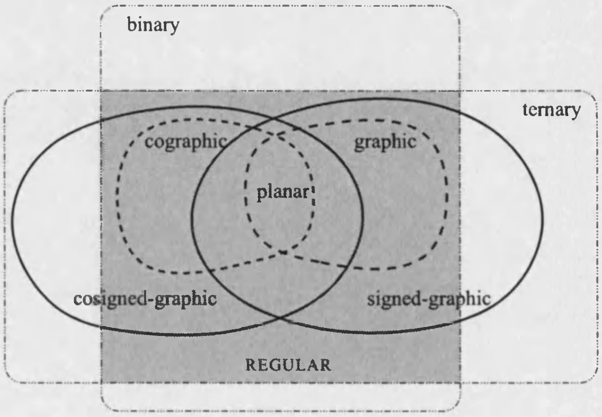


Figure 5.5: Representable matroids

all inner blocks . A *necklace* is a special type of 2-connected unbalanced signed graph, which is composed of maximal 2-connected balanced subgraphs Σ_i, which are called *blocks of the necklace*, joined in a cyclic fashion as illustrated in Figure 5.6. Observe that any negative cycle in a necklace has to contain at least one edge from each Σ_i.

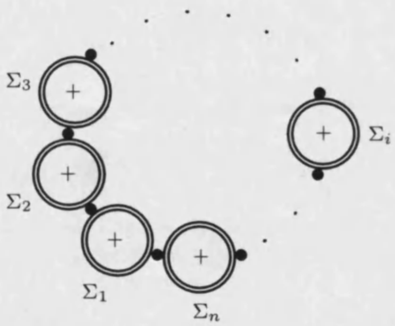


Figure 5.6: The structure of a necklace.

Theorem 5.20. *Let Σ be a connected signed graph. The elementary separators of $M(\Sigma)$ are the edge sets of the outer blocks and the core, except when the core is a necklace where in that case each block of the necklace is individually an elementary separator.*

If B is an elementary separator of $M(\Sigma)$ then the subgraph $\Sigma\backslash.B$ is called a *separate* of Σ. There is an equivalence of the deletion and contraction operations on a signed-graphic matroid, with respect to the associated signed graphic operations of deletion and contraction defined in Section 5.1.1, as indicated by Theorem 5.21 appearing in [94].

Theorem 5.21. *Let Σ be a signed graph and $S \subseteq E(\Sigma)$. Then $M(\Sigma\backslash S) = M(\Sigma)\backslash S$ and $M(\Sigma/S) = M(\Sigma)/S$.*

Based on Theorem 5.21, we can relate the two different classifications of bonds defined in the "Duality

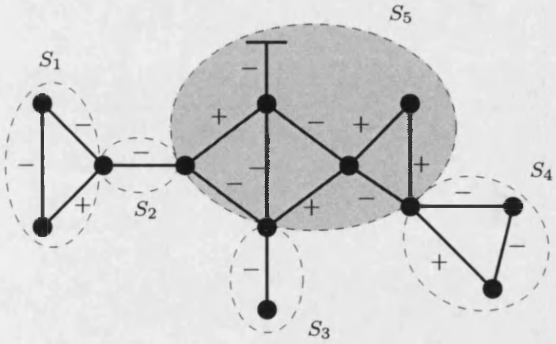


Figure 5.7: The separates of a signed graph.

and Representability" subsection of this section. More specifically, we show that an unbalanced or double bond is always separating, while a star bond and a balanced bond may be non-separating.

Proposition 5.22. *Let $M(\Sigma)$ be a connected signed-graphic matroid and $Y \in \mathcal{C}(M^*(\Sigma))$. If $M(\Sigma)\backslash Y$ is connected then Y is either a star bond or a balanced bond of Σ.*

Proof: By way of contradiction, suppose that $M(\Sigma)\backslash Y$ is connected and that Y is neither a star bond nor a balanced bond. Thus, Y is an unbalanced bond or a double bond and, therefore, $\Sigma\backslash Y$ consists of at least two components each of which is non-empty of edges. If e and f are edges which belong to different components of $\Sigma\backslash Y$ then obviously there is no circuit in $\Sigma\backslash Y$ containing both e and f. Thus, the matroid $M(\Sigma\backslash Y)$ does not have a circuit which contains $\{e, f\}$. By Theorem 5.21, the matroid $M(\Sigma)\backslash Y = M(\Sigma\backslash Y)$ has no circuit containing both e and f. By Proposition 1.20, the matroid $M(\Sigma)\backslash Y$ is disconnected which is in contradiction with our initial assumption. □

5.3.2 Some invariant operations

In this section we essentially provide three operations which when applied to a signed graph Σ produce a new signed graph Σ' with the same matroid, i.e. $M(\Sigma') = M(\Sigma)$. The following Lemma 5.23, which appears as a basic statement in works of others (see [71, 98]), is proved here.

Lemma 5.23. *Let Σ' be a signed graph obtained from Σ by replacing any number of negative loops by half-edges and vice versa, then $M(\Sigma') = M(\Sigma)$.*

Proof: Let $\mathcal{C}$ and $\mathcal{C}'$ be the collections of circuits of $M(\Sigma)$ and $M(\Sigma')$, respectively. We known that $\mathcal{C}$ and $\mathcal{C}'$ are also the circuit families of Σ and Σ', respectively. If $C_0 \in \mathcal{C}$ is the edge set of a positive cycle of Σ or the edge set of handcuff of Type I or II containing no half-edges or positive loops, then clearly $C_0 \in \mathcal{C}'$, since no alteration is made to the edges of C_0 by any possible sequence of replacements. For the remaining case, any handcuff of Type I or II of Σ (Σ') that contains a half-edge or a negative loop which is replaced by a negative loop or a half-edge remains a handcuff of the same type in Σ' (Σ). Therefore, $\mathcal{C} = \mathcal{C}'$ and furthermore, since $E(M(\Sigma)) = E(M(\Sigma'))$, $M(\Sigma') = M(\Sigma)$. □

The following Lemma 5.24, which appears in [94], is an immediate application of the fact that the signs of the cycles of a signed graph do not change by the switching operation and thereby, the set of circuits of the associated matroid remain also unaltered.

Lemma 5.24. *Let Σ and Σ' be two signed graphs on the same underlying graph. Then, Σ' can be obtained from Σ by a sequence of switchings if and only if $M(\Sigma') = M(\Sigma)$.*

The following lemma is a generalization of the reversing operation in graphs, which can be proved easily.

Lemma 5.25. *Let Σ' be the reversing graph of a signed graph Σ about (u, v) and let Σ_1 and Σ_2 be the reversing parts of Σ. If Σ_1 is balanced or all of its negative cycles contain u and v then $M(\Sigma') = M(\Sigma)$.*

5.3.3 Signed graphs with graphic matroids

With any signed graph $\Sigma = (G, \sigma)$ we can associate the signed-graphic matroid of Σ and the graphic matroid of its underlying graph G. The results of this section state necessary conditions upon which $M(\Sigma) = M(G)$. The following Proposition 5.26 appears in [94] and here we provide our own proof.

Proposition 5.26. *If $\Sigma = (G, \sigma)$ consists only of positive edges then $M(\Sigma) = M(G)$.*

Proof: Since $\Sigma = (G, \sigma)$, $M(G)$ and $M(\Sigma)$ are matroids on the same ground set. Moreover, by the hypothesis that Σ contains no negative edges, any cycle of Σ is positive and therefore a circuit of $M(\Sigma)$. This means that the circuits of $M(\Sigma)$ is the collection of the edge sets of cycles in the underlying graph G. But this collection is also the set of circuits of $M(G)$. Thus, $M(\Sigma)$ and $M(G)$ are matroids on the same ground set and $\mathcal{C}(M(\Sigma)) = \mathcal{C}(M(G))$ which implies that $M(\Sigma) = M(G)$. □

Clearly, Proposition 5.26 implies the following Corollary 5.27 and also provides a way to obtain a signed graph Σ from a graph G such that $M(\Sigma) = M(G)$.

Corollary 5.27. *If M is a graphic matroid then M is signed-graphic.*

Given any signed graph $\Sigma = (G, \sigma)$ where G is a tree consisting only of links, there exists a procedure to make all the links positive by a series of switchings. Specifically, choose any vertex $r \in V(\Sigma)$ and create the rooted tree based on the distance from r. Starting from r traverse all the non-leaf vertices of the tree in a breadth-first manner, where at each vertex v_i in the sequence apply the switching operation at its child vertex v_j if and only if the link (v_i, v_j) is negative. As it will be shown in the next result, which also can be easily derived by results in [94], this can be further generalized for balanced signed graphs.

Proposition 5.28. *If $\Sigma = (G, \sigma)$ is balanced then $M(\Sigma) = M(G)$. Specifically, there exist a series of switchings such that Σ can be transformed into a signed graph with only positive links.*

Proof: Since $\Sigma = (G, \sigma)$ is balanced it does not contain negative cycles, half-edges or negative loops. If we take any spanning forest F of G then all the edges contained in F can become positive by a series of switchings at some vertices. Let $\Sigma' = (G, \bar{\sigma})$ where $\bar{\sigma}$ is the sign function so obtained from σ by these switchings at the vertices of F. Note that for any edge e in F we have $\bar{\sigma}(e) = +1$. For any other edge e of Σ' not in F we have that $G[F \cup e]$ contains a unique cycle C and $e \in C$. Since Σ is balanced and the signs of the cycles remain the same under switchings, C should be positive in Σ', therefore $\bar{\sigma}(e) = +1$. By Lemma 5.24, $M(\Sigma) = M(\Sigma')$ and by Proposition 5.26, $M(\Sigma') = M(G)$. Thus, $M(\Sigma) = M(G)$. □

The following interesting Proposition 5.29 appears in [71]. We provide here an alternative proof.

Proposition 5.29. *If the collection of negative cycles of $\Sigma = (G, \sigma)$ consists only of negative loops and half-edges then $M(\Sigma)$ is graphic. Moreover, a graph G' such that $M(\Sigma) = M(G')$ can be obtained from G by adding a new vertex v and replacing any negative loop and half-edge by a link joining its end-vertex with v.*

Proof: Clearly the matroids $M(G')$ and $M(\Sigma)$ have the same ground set $E = E(G)$. We have to show that $\mathcal{C}(M(\Sigma)) = \mathcal{C}(M(G'))$. If N is the edge set containing all the negative loops and half-edges of Σ, then the subgraph G'' of G' induced by the edges in $E - N$ is the underlying graph of the subgraph Σ' of Σ induced by the edges in $E - N$. Since Σ' is balanced, by Proposition 5.28, we have that a set $C \subseteq (E - N)$ being a circuit of $M(\Sigma)$ is also a circuit of $M(G')$ and vice versa. Any circuit D of $M(G')$ for which $D \cap N \neq \{\emptyset\}$ consists of a pair of elements $P \subseteq N$ and those elements of $E - N$ corresponding to a path of G'' which along with the edges in P create a cycle in G'. The elements of any such circuit of $M(G')$ constitute a circuit in $M(\Sigma)$ as well since they correspond to either a Type I or a Type II handcuff in Σ. Similarly, it can be shown that any circuit D' of $M(\Sigma)$ for which $D' \cap N \neq \{\emptyset\}$ is also a circuit in $M(G')$ and, therefore, the result follows. □

The following two propositions show that a signed graph with a balancing vertex has some interesting properties. Proposition 5.31 was taken from [71] where the corresponding proof is also provided.

Proposition 5.30. *If Σ has a balancing vertex v then a signed graph Σ' can be obtained from Σ via a sequence of switchings such that all the negative edges of Σ' are incident with v.*

Proof: The graph $\Sigma_1 = \Sigma \backslash v$ is balanced and thus, by Proposition 5.28, there exists a sequence of switchings at the vertices of Σ_1 such that all its edges become positive. Since $V(\Sigma_1) \subseteq V(\Sigma)$, we can apply the same sequence of switchings to Σ. By doing so we have that in the graph Σ' so-obtained any edge being negative must have one end-vertex v. □

Proposition 5.31. *If Σ is a signed graph having a balancing vertex v then $M(\Sigma)$ is graphic.*

A straightforward result implied by Proposition 5.31 is the following corollary regarding the case in which Σ is a necklace. Let Σ' be the signed graph obtained from a necklace signed graph Σ by deleting some vertex $v \in V(\Sigma)$ which is common to two of its blocks. Then, there is no cycle in Σ' whose edges intersect the edge set of more than one block of Σ' which means that the collection of cycles of Σ' consists of the cycles of its balanced blocks. This in turn implies that Σ' is balanced and thus v is a balancing vertex.

Corollary 5.32. *If Σ is a necklace then $M(\Sigma)$ is graphic.*

5.4 Tangled signed graphs

5.4.1 Preliminary results for tangled signed graphs

Tangled signed graphs form an important class of signed graphs which plays an important role in this work.

Definition 5.33. *A connected signed graph is called tangled if it has no balancing vertex and no two vertex disjoint negative cycles.*

For our purposes, the importance of tangled signed graphs stems mainly from Theorem 5.42, according to which, if a binary matroid is signed-graphic but not graphic then it has a tangled graphical representation. For a tangled signed graph Σ and a bond Y, the main results of this section are three theorems regarding the structure of the signed graph $\Sigma \backslash Y$. We initially provide a proof for the following Proposition 5.34 appearing in [71].

Proposition 5.34. *If Σ is a tangled signed graph then it contains exactly one unbalanced block.*

Proof: Suppose that Σ contains two unbalanced blocks U and V. For these blocks we have that one of the following holds: (i) U and V are vertex-disjoint, or (ii) U and V have a vertex v in common. If U and V are vertex-disjoint, then Σ can not be tangled since there are two vertex disjoint negative cycles in Σ. For case (ii), we can say that all negative cycles of U and V must have v as vertex, since otherwise there would exist two vertex disjoint negative cycles in Σ. But then Σ has a balancing vertex and, therefore, it is not tangled. □

The following Proposition 5.35 can be also viewed as a special case of a result of Slilaty (Lemma 4 in [70]).

Proposition 5.35. *If a signed graph Σ is 2-connected and tangled then any star of Σ is a bond.*

Proof: Let J be the star of a vertex v of Σ. Since Σ is 2-connected, $\Sigma \backslash J$ consists of a vertex v and a connected signed graph Σ'. By definition tangled signed graphs have no balancing vertex and therefore Σ' is unbalanced. Thus, J is a set of edges whose deletion from Σ results in a signed graph with one balanced component. The minimality of J stems from the fact that for any edge $e \in J$, the signed graph $\Sigma \backslash (J - e)$ is an unbalanced connected signed graph. □

The following Proposition 5.36 stems easily from Propositions 5.34 and 5.35.

Proposition 5.36. *If a signed graph Σ is tangled then any b-star of Σ is a disjoint union of bonds.*

Moreover, a consequence of Theorem 5.19 is the following theorem which restricts the types of bonds of a tangled signed graph.

Theorem 5.37. *If a signed graph Σ is tangled then Σ has no double bond.*

Proof: By way of contradiction, suppose that a tangled signed graph Σ has a double bond Y. Then $Y = Y_1 \cup Y_2$ with $Y_1, Y_2 \neq \emptyset$, where the removal of Y_1 cuts Σ into Σ_1 and Σ_2 and Y_2 is a minimal set whose removal makes one of Σ_1 or Σ_2 balanced. Obviously both Σ_1 and Σ_2 are not balanced since otherwise Y_2 must be empty because of the minimality of a bond. This means that in Σ there are two vertex disjoint subgraphs (Σ_1 and Σ_2) with negative cycles. This in turn implies that Σ has two vertex disjoint negative cycles and thus, Σ is not tangled which is in contradiction with our assumption. □

As a consequence of Theorem 5.37, if Y is a bond of a tangled signed graph Σ then Y is a star-bond, a balanced bond or an unbalanced bond of Σ. Clearly, if Y is a balanced bond then the graph $\Sigma \backslash Y$ consists of one component. The next theorem shows that if Y is a bond other than balanced then $\Sigma \backslash Y$ consists of exactly two components.

Theorem 5.38. *Let Y be a bond of a tangled signed graph Σ. If Y is a star bond or an unbalanced bond then $\Sigma\backslash Y$ consists of exactly two components.*

Proof: By the definitions of a star bond and an unbalanced bond, we have that, for any signed graph Σ, $\Sigma\backslash Y$ will have at least two components. By way of contradiction, suppose that there exists a tangled signed graph Σ' and a bond Y' of Σ' such that $\Sigma'\backslash Y'$ consists of n components (where n is a positive integer greater than two). Then, $n-1$ of the n components of $\Sigma'\backslash Y'$ are unbalanced since otherwise Y' would not be minimal. However, in that case there are negative cycles in different components of $\Sigma'\backslash Y'$ which implies that Σ' contains vertex disjoint negative cycles. This is in contradiction with our hypothesis that Σ is tangled and thus, the result follows. □

By definition, if Y is a balanced bond of Σ then the blocks of $\Sigma\backslash Y$ are all balanced. As shown in the following theorem, for a tangled signed graph Σ and a star bond or an unbalanced bond Y of Σ, the blocks of $\Sigma\backslash Y$ are all balanced except for one.

Theorem 5.39. *If Σ is a tangled signed graph and Y is a star bond or an unbalanced bond of Σ then all the blocks of $\Sigma\backslash Y$ are balanced except for one which is unbalanced.*

Proof: By Theorem 5.38, $\Sigma\backslash Y$ consists of two components which we shall call Σ_1 and Σ_2. Exactly one of Σ_1 or Σ_2 has to be balanced; without loss of generality, let's assume that Σ_2 is the balanced component. Clearly, all the blocks of Σ_2 are balanced. By way of contradiction, suppose that Σ_1 has more than one unbalanced block. Then, all these unbalanced blocks must share exactly one common vertex v, since otherwise the connected Σ would contain two vertex disjoint negative cycles which is in contradiction with the fact that Σ is tangled. Moreover, it is easy to see that all the negative cycles of Σ containing edges of Y must also contain v, since otherwise Σ would contain two vertex disjoint negative cycles which is in contradiction with the fact that Σ is tangled. This means that all the negative cycles of Σ contain v. Therefore, v is a balancing vertex of Σ which contradicts the fact that Σ is tangled and the result follows. □

5.4.2 Binary signed-graphic matroids and tangled signed graphs

The following result provides a connection between the connectivity of a tangled signed graph Σ and the connectivity of the associated signed-graphic matroid $M(\Sigma)$.

Theorem 5.40. *Let Σ be a tangled signed graph. Then Σ is 2-connected if and only if $M(\Sigma)$ is connected.*

Proof: For the "only if" part, by way of contradiction, suppose that there exists a 2-connected tangled signed graph Σ such that the matroid $M(\Sigma)$ is disconnected. Since $M(\Sigma)$ is disconnected and Σ is 2-connected, by Theorem 5.20, we have that Σ is a necklace and thus, Σ contains a balancing vertex. Therefore, Σ can not a tangled signed graph which is in contradiction with our assumption.

For the "if" part, by way of contradiction, suppose that $M(\Sigma)$ is connected and it does have a tangled representation Σ which has an 1-separation. Therefore, Σ is connected and contains at least two blocks. By Proposition 5.34, Σ contains exactly one unbalanced block B_N and, thus, Σ must also contain at least one balanced block B_P. According to Theorem 5.20, the edge-sets of B_N and B_P are elementary separators of $M(\Sigma)$ and thus, $M(\Sigma)$ has more than one elementary separators which is in contradiction with our hypothesis. □

We prove the following new important structural result which provides the means of a more direct and shorter proof of Theorem 5.42 than that appearing in [72].

Lemma 5.41. *Let $M(\Sigma)$ be a binary signed-graphic matroid and let Σ' be the signed graph obtained from Σ by contracting the edges of all balanced blocks . Then, $M(\Sigma')$ is graphic if and only if $M(\Sigma)$ is graphic.*

Proof: The "if" part is straightforward. Let S be the set consisting of the edges of all the balanced blocks of Σ. By Theorem 5.21, $M(\Sigma') = M(\Sigma/S) = M(\Sigma)/S$ which implies that $M(\Sigma')$ is a minor of the graphic matroid $M(\Sigma)$. It is well known that graphic matroids are closed under minor taking (see [56]) and therefore, $M(\Sigma')$ is graphic.

For the proof of the "only if" part of the lemma we proceed as follows. If Σ contains no unbalanced blocks then, by Proposition 5.28, $M(\Sigma)$ is graphic. If Σ contains exactly one unbalanced block then this block is the core, say K, of Σ. By Theorem 5.20, K is either a necklace or a separate of Σ. If Σ is a necklace then, by Corollary 5.32, all the elementary separators of $M(\Sigma)$ are graphic; while, if K is a separate of Σ then again, by the hypothesis of the lemma, all the elementary separators of $M(\Sigma)$ are graphic. Therefore, in both cases, $M(\Sigma)$ is graphic since it is well-known that if the elementary separators of a matroid M are graphic then M itself is graphic (see Proposition 4.2.15 in [56]). Thus, we can assume that Σ contains more than one unbalanced blocks. Moreover, we can make two further assumptions for Σ. The first one is that Σ contains no half-edges, since by Lemma 5.23 we can replace the half-edges of a signed graph by negative loops with no effect on the associated matroid. The second assumption is that Σ contains no outer blocks, since if the signed-graphic matroid of the core of a signed graph T is graphic then all the elementary separators of $M(T)$ are graphic and therefore, by Proposition 4.2.15 in [56], $M(T)$ is graphic. Thus, in the rest of the proof we assume that Σ: (i) has no half-edges, (ii) has no outer blocks, and (iii) contains at least two unbalanced blocks.

All the blocks of Σ' (which is the signed graph obtained from Σ by contracting the edges of all the balanced blocks) must have some vertex v in common since otherwise Σ' will have two vertex disjoint negative cycles which, by Theorem 5.19, would imply that $M(\Sigma)$ is not binary, a contradiction. Moreover, if all the negative cycles of each block of Σ' do not have v as a vertex then Σ' will have two vertex disjoint negative cycles and thus, $M(\Sigma')$ would be not binary, a contradiction. Thus, there exists a set of cut-vertices $U \subseteq V(\Sigma)$ in Σ such that every negative cycle of Σ contains a $u_i \in U$ and moreover, every u_i is replaced by v after the contraction of the edges of all the balanced blocks of Σ which gives rise to Σ'.

For any $u_i \in U$, let $C(u_i)$ be the union of the unbalanced blocks of Σ containing u_i. We can assume that in Σ all the negative edges of each $C(u_i)$ are incident with the corresponding u_i. To see this, consider some unbalanced block $B \subseteq C(u_i)$; then due to the fact that all the negative cycles of B contain u_i we have that u_i is a balancing vertex. Therefore, by Proposition 5.28, there exists a sequence of switchings at the vertices of $V(B) - u_i$ such that all the edges of the graph induced by $V(B) - u_i$ are positive. Since switchings do not alter the matroid of the associated signed graph, we can assume that all the negative edges of B are incident with u_i. We apply such a sequence of switching to every block of every $C(u_i)$ and, since at most one switching takes places at each vertex, we can assume in what follows that all the negative edges of each $C(u_i)$ are incident with the corresponding u_i in Σ.

We shall now construct a graph G' and show that $M(G') = M(\Sigma)$. Let Σ'' be the signed graph obtained from Σ by adding one new vertex w and by replacing every negative link $\{u, u_i\}$ and every negative loop $\{u_i, u_i\}$ of Σ by a link $\{u, w\}$ and a link $\{u_i, w\}$, respectively, for any $u \in (V(\Sigma) - U)$ and $u_i \in U$. We call G' the underlying graph of Σ''. Obviously the matroids $M(\Sigma)$ and $M(G')$ are defined on the same ground set and thus, it remains to be shown that $\mathcal{C}(M(\Sigma)) = \mathcal{C}(M(G'))$. In other words, in order to have $M(G') = M(\Sigma)$, we have to show that every circuit of $M(\Sigma)$ is a circuit of $M(G')$ and vice versa.

A positive cycle in Σ which does not contain some $u_i \in U$ must be a cycle in G' by construction. Any other positive cycle of Σ contains two edges having some $u_i \in U$ as an end-vertex. These edges must be of the same sign and therefore, the edge set of such a cycle induces a cycle in G'. Furthermore, every negative cycle of Σ contains exactly one negative edge which is incident with some $u_i \in U$. Therefore, the edge set of any Type I or Type II handcuff, which we call R, in Σ will contain exactly two negative edges; these edges will be incident with w in G' and thereby, R will be the edge set of a cycle in G'. Thus, by Theorem 5.17, we can conclude that every circuit of $M(\Sigma)$ is a circuit of $M(G')$.

We now show that every circuit of $M(G')$ is a circuit of $M(\Sigma)$. Let F be a circuit of $M(G')$ which is the edge set of a cycle J of G'. If J does not contain w then, by construction, F is the edge set of a positive cycle in Σ. Suppose now that J contains w and thus, it contains two edges e_1 and e_2 being incident with w. Then, by construction, all the edges of F in Σ are positive apart from e_1 and e_2 which, furthermore, belong to different cycles in Σ; thus, F induces a handcuff of Type I or Type II in Σ. Finally, by Theorem 5.17, we have that every circuit of $M(G')$ is a circuit of $M(\Sigma)$. □

It has already been mentioned that the tangled signed graphs are of great importance for our work in the next chapters regarding the class of binary signed-graphic matroids. The importance stems mainly from the following Theorem 5.42 which appears in [72]. Based on Lemma 5.41 we provide an alternative shorter proof of that theorem.

Theorem 5.42. *Let Σ be a connected signed graph. Then, $M(\Sigma)$ is binary if and only if*

(i) Σ is tangled, or

(ii) $M(\Sigma)$ is graphic.

Proof: The "if" part is straightforward. Specifically, the class of graphic matroids is a subclass of binary matroids and thus, if $M(\Sigma)$ is graphic then it is also binary. Moreover, by Definition 5.33, a tangled signed graph Σ does not contain two vertex disjoint negative cycles and therefore, by Theorem 5.19, $M(\Sigma)$ is binary.

For the "only if" part, by Propositions 5.28 and 5.31, if Σ is balanced or has a balancing vertex then the binary matroid $M(\Sigma)$ is graphic. Therefore, by Definition 5.33, it suffices to prove that if Σ is unbalanced, has no balancing vertex and has two vertex disjoint negative cycles then the binary matroid $M(\Sigma)$ is graphic. In order to prove this statement we proceed as follows. If the two vertex disjoint negative cycles of Σ are contained in the same block then, by Theorem 5.19, $M(\Sigma)$ will be nonbinary. Therefore, the two vertex disjoint negative cycles of Σ are in different blocks of Σ. Let Σ' be the graph obtained from Σ by contracting the edges of all balanced blocks. Suppose, by way of contradiction, that there is no vertex u of Σ' that is common to all negative cycles. Then, Σ' does have two vertex disjoint negative cycles and therefore, by Theorem 5.19, $M(\Sigma')$ is not binary. Moreover, by Theorem 5.21, $M(\Sigma')$ is a minor of $M(\Sigma)$ and, since representable matroids are closed under taking of minors, $M(\Sigma)$ is not binary, which is a contradiction. Thus, all the negative cycles of Σ' have some vertex u as a common vertex. This in turn implies that u is a balancing vertex of Σ' and therefore, by Proposition 5.31, $M(\Sigma')$ is graphic. Thus, by Lemma 5.41, $M(\Sigma)$ is graphic. □

Chapter 6

Representability and characterizations

In [94], it is shown that signed-graphic matroids are representable over any field of characteristic not 2. Therefore, it would be of interest to determine when a signed-graphic matroid is representable over fields of characteristic 2. In a paper of Whittle ([91]) we find, among other interesting results, the following:

- a matroid M is representable over $GF(3)$, $\mathbb{Q}$ and a field of characteristic 2 if and only if M is representable over all fields except possibly $GF(2)$, and
- a matroid representable over $GF(3)$ and $\mathbb{Q}$ is not representable over $GF(4)$ if and only if M is representable over all fields whose characteristic is not 2.

By these results and Theorem 1.14, it follows that a signed-graphic matroid M falls in one of the following categories [57]:

(i) If M is binary, then it is regular and therefore, representable over all fields.

(ii) If M is representable over $GF(4)$ but not binary, then it is representable over all fields except $GF(2)$.

(iii) If M is not representable over $GF(4)$, then it is representable over all fields of characteristic other than 2.

In [57], Pagano goes further by characterizing the classes of signed graphs whose associated signed-graphic matroid is binary or quaternary. For the binary case, we have already provided the characterization in section 5.3.1 (see Theorem 5.19).

The rest of this chapter is organised as follows. In section 6.1, given a signed graph Σ, we provide possible representations of $M(\Sigma)$ over $GF(2)$, $GF(3)$ and $\mathbb{R}$. In section 6.2, we re-prove Theorem 5.19 and provide the BINARY RECOGNITION ALGORITHM which decides whether a binary matroid is signed-graphic or not. In section 6.3 we provide a characterization of signed-graphic matroids inspired mainly by a similar result of Seymour for graphic matroids in [67]. Based on this characterization and the BINARY RECOGNITION ALGORITHM, our contribution is to provide the GENERAL RECOGNITION ALGORITHM which decides whether a matroid is binary signed-graphic or not in the last section of this chapter.

6.1 Representation matrices of signed-graphic matroids

As already mentioned, signed-graphic matroids are representable over $GF(3)$, and not all signed-graphic matroids are representable over $GF(2)$. Here given a signed graph Σ, we show how to obtain representation matrices of the associated signed-graphic matroid $M(\Sigma)$ over $\mathbb{R}$, $GF(2)$ (provided that $M(\Sigma)$ is binary) and $GF(3)$. Note also here that, in the rest of this chapter, given a signed graph Σ, if B is a binet matrix that can be obtained from the incidence matrix of an orientation of Σ (see Definition 2.6) then we shall say that *B is a binet matrix associated with* Σ. We begin by showing that the incidence matrix of any orientation of Σ is a representation matrix of $M(\Sigma)$ over $\mathbb{R}$ and that any binet matrix associated with Σ is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$.

Theorem 6.1. *Let A_Σ be the incidence matrix of any orientation of a signed graph Σ. Then A_Σ is a representation matrix of $M(\Sigma)$ over $\mathbb{R}$ and any binet matrix associated with Σ is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$.*

Proof: As defined in section 5.1.1, A_Σ is the incidence matrix of a bidirected graph Γ achieved by an orientation of Σ. By Corollary 2.5, a set of columns of A_Σ is a minimal set of linearly dependent columns if and only if the subgraph induced by the corresponding edges in Γ is an oriented version of one of the subgraphs listed in Theorem 5.17. Hence, $M[A_\Sigma] = M(\Sigma)$. Furthemore, by Definition 2.6, the way we obtain a binet matrix B from A_Σ is also the way we can obtain from A_Σ a compact representation matrix of $M[A_\Sigma]$ and thus, B is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$. □

We turn now to the $GF(3)$ matrix representations of $M(\Sigma)$. Signed-graphic matroids are known to be ternary [94]. This is proved in [45, 94] by taking the excluded minors for ternary matroids, viz. $F_7, F_7^*, U_{2,5}$ and $U_{3,5}$, and showing that they do not belong to the class of signed-graphic matroids. In Theorem 6.2 we provide a different proof for the $GF(3)$ matrix representation which is constructive.

Theorem 6.2. *Signed-graphic matroids are ternary. Moreover, a ternary representation of a signed-graphic matroid $M(\Sigma)$ is obtained by reducing every element of the incidence matrix of any orientation of Σ modulo* 3.

Proof: Let $D(\Sigma)$ be an arbitrary orientation of Σ and $A_{D(\Sigma)}$ be the incidence matrix of $D(\Sigma)$. We denote by $A'_{D(\Sigma)}$ the matrix obtained from $A_{D(\Sigma)}$ by reducing each entry of $A_{D(\Sigma)}$ modulo 3. Thus, $A'_{D(\Sigma)}$ is obtained from $A_{D(\Sigma)}$ by replacing each -2 by a $+1$ and each $+2$ by a -1, whereas all the other entries remain unchanged. We call $D(\Sigma')$ the graph obtained by replacing any ingoing (resp. outgoing) negative loop at a vertex of $D(\Sigma)$ by an outgoing (resp. ingoing) half-edge at the same vertex. By Lemma 5.23, $M(\Sigma) = M(\Sigma')$ and furthermore, if $A'_{D(\Sigma')}$ is the matrix obtained from the incidence matrix of $D(\Sigma')$ by reducing each entry modulo 3, then $A'_{D(\Sigma)} = A'_{D(\Sigma')}$. Thus, it suffices to show that $A'_{D(\Sigma')}$ represents $M(\Sigma')$ over $GF(3)$.

Let C be a circuit of $M(\Sigma')$. Then, C is also a circuit of $D(\Sigma')$. If C is a positive loop or a loose edge, then the corresponding column of $A'_{D(\Sigma')}$ is zero, hence C is a circuit of $M[A'_{D(\Sigma')}]$. Otherwise, let $w(C) = (v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_k, v_1)$ be a minimal covering walk of C (i.e. a closed walk of minimal length containing all the edges of C), where $v_i \in V(\Sigma')$ and $e_j \in E(\Sigma')$ $(i, j \in \mathbb{Z}_+)$, and let also $|w(C) = n|$. We label the first edge appearing in $w(C)$ (i.e. e_1) by b_1, the second edge of $w(C)$

by b_2 and so on. Let also $v(b_i)$ be a column which is equal to the column of $A'_{D(\Sigma')}$ that corresponds to the edge of C labelled by b_i ($i = 1, \dots, n$). As mentioned in section 2.2.1 we can apply the appropriate reversings of the edges of C such that every inner vertex in $w(C)$ becomes a consistent vertex. Now let α_i ($i = 1, \dots, n$) be -1 or $+1$ according to whether the edge labelled by b_i has been reversed or not, respectively. Therefore, since $A'_{D(\Sigma')} = A_{D(\Sigma')}$, for these a_is we have that $\sum_{i=1}^{k} \alpha_i v(b_i) = 0$, which implies that C is dependent in $M[A'_{D(\Sigma')}]$.

Now suppose that $\{f_1, f_2, \dots, f_m\}$ is a circuit of $M[A'_{D(\Sigma')}]$. If $m = 1$ then clearly f_1 is a positive loop or a loose edge in $D(\Sigma')$, so $\{f_1\}$ is a circuit of $M(\Sigma')$. Assume, then, that $m > 1$ and let $v(f_1), v(f_2), \dots, v(f_m)$ be the columns of $A'_{D(\Sigma')}$ corresponding to $f_1, f_2, \dots, f_m$, respectively. Evidently, for some non-zero members $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ of $GF(3)$, $\sum_{i=1}^{m} \epsilon_i v(f_i) = 0$. Thus, every row of the matrix $[v(f_1)v(f_2)\cdots v(f_m)]$ containing at least one non-zero, contains at least two non-zero entries. But this matrix is a column submatrix of $A'_{D(\Sigma')}$ and therefore, its rows correspond to vertices of $D(\Sigma')$. Hence, in the subgraph $D(\Sigma'_1)$ of $D(\Sigma')$ induced by $\{f_1, f_2, \dots, f_m\}$ every vertex has degree at least two, which implies that $D(\Sigma'_1)$ must contain a cycle. Furthermore, since the column submatrix of $A'_{D(\Sigma')}$ consisting of $v(f_1), v(f_2), \dots, v(f_m)$ has to be connected due to the minimality of the circuit $\{f_1, f_2, \dots, f_m\}$, the subgraph $D(\Sigma'_1)$ is connected as well. We consider two cases. In the first case, we assume that every vertex of $D(\Sigma'_1)$ has degree exactly two, i.e. $D(\Sigma'_1)$ is a cycle. This means that every row of $[v(f_1)v(f_2)\cdots v(f_m)]$ contains exactly two non-zero entries and, thus, there exists a scaling with $\epsilon_1, \epsilon_2, \dots, \epsilon_m \in \{\pm 1\}$ of the corresponding columns of this matrix such that each row of the matrix $\bar{A}$ so obtained by these scalings contains a $+1$ and a -1. Thus, the bidirected graph with incidence matrix $\bar{A}$ is a positive cycle. Since scaling of a column in the incidence matrix does not alter the sign of an edge in the associated bidirected graph we have that $D(\Sigma'_1)$ is also a positive cycle. Hence the edge set of $D(\Sigma'_1)$ is a circuit of $D(\Sigma')$ which implies that the set $\{f_1, f_2, \dots, f_m\}$ is a circuit of $M(\Sigma')$. In the second case, we have that the sum of the degrees of the vertices in $D(\Sigma'_1)$ is at least $2m + 1$. This implies that the connected subgraph $D(\Sigma'_1)$ does contain at least two cycles (positive or negative). By Corollary 2.5 and due to the fact that any theta subgraph of a signed graph contains at least one positive cycle (see Proposition 5.1), it is easy to see that the edge-set of $D(\Sigma'_1)$ contains a circuit of $D(\Sigma')$ which implies that the set $\{f_1, f_2, \dots, f_m\}$ contains a circuit of $M(\Sigma')$.

We have shown that every circuit of $M(\Sigma')$ is dependent in $M[A'_{D(\Sigma')}]$ and that every circuit of $M[A'_{D(\Sigma')}]$ is dependent in $M(\Sigma')$, therefore, by a well known of matroid theory (see Lemma 2.1.19 in [56]), it follows that $M(\Sigma') = M[A'_{D(\Sigma')}]$. □

By Theorem 6.1, any binet matrix B associated with a signed graph Σ is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$. In the following lines, given a binet matrix B, we show how to obtain a $GF(3)$ representation matrix of $M(\Sigma)$ from B . Specifically, in Theorem 6.4 it is shown that by reducing the elements of B modulo 3 we obtain a ternary compact representation matrix of $M(\Sigma)$. We note here that for the proof of Theorem 6.4 the main idea was taken from a paper of Lee (see Proposition 3.1 in [49]) and that we also make use of Proposition 6.3 which can be found in [56] (Proposition 6.4.5 in [56]).

Proposition 6.3. *Let $[I_r|D_1]$ and $[I_r|D_2]$ be $r \times n$ matrices over the fields F and F', respectively, with the columns of each matrix being labelled, in order, by $e_1, e_2, \dots, e_n$. Then the identity map on $\{e_1, e_2, \dots, e_n\}$ is an isomorphism from $M[I_r|D_1]$ to $M[I_r|D_2]$ if and only if, whenever D'_1 and D'_2 are* *corresponding square submatrices of D_1 and D_2 respectively, $det(D_1') = 0$ exactly when $det(D_2') = 0$.*

Theorem 6.4. *Let B be an integral binet matrix and $M(\Sigma)$ be the signed-graphic matroid represented by B over $\mathbb{R}$ (i.e. $M(\Sigma) \cong M(B)$). Then, the matrix $B' = B \bmod 3$ is a compact representation matrix of $M(\Sigma)$ over $GF(3)$.*

Proof: Let $D = [d_{ij}]$ and $D' = [d'_{ij}]$ be corresponding $m \times m$ submatrices of B and B', where by hypothesis $d'_{ij} = d_{ij} \bmod 3$, for all i and j $(i, j = 1, \dots, m)$. By Proposition 6.3 enough to show that $det(D) = 0$ if and only if $det(D') = 0$. Using the permutation definition of the determinant (see e.g. [73]) we have

$$det(D) = \sum_{p \in S_m} sign(p) \prod_{i=1}^{m} d_{ip(i)}, \quad (in\ \ \mathbb{Z})$$

$$det(D') = \sum_{p \in S_m} sign(p) \prod_{i=1}^{m} d'_{ip(i)}, \quad (in\ \ GF(3))$$

where S_m is the set of permutations of $\{1, \dots, m\}$ and $sign(p) = \pm 1$ is the sign of the permutation p. Relating the two determinants we have

$$\begin{array}{rcll} det(D') & = & \sum_{p \in S_m} sign(p) \prod_{i=1}^{m} d'_{ip(i)}, & (\text{in } GF(3)) \\ & = & \left(\sum_{p \in S_m} sign(p) \ \bmod 3 \prod_{i=1}^{m} d_{ip(i)} \ \bmod 3\right) \ \bmod 3\,, & (\text{in } \mathbb{Z}) \\ & = & \left(\sum_{p \in S_m} sign(p) \prod_{i=1}^{m} d_{ip(i)}\right) \ \bmod 3\,, & (\text{in } \mathbb{Z}) \\ & = & det(D) \ \bmod 3. & \end{array}$$

Clearly if $det(D) = 0$ then we have that $det(D') = 0$. If $det(D') = 0$ then we have that $det(D)$ mod $3 = 0$, which implies that $|det(D)| = 3k$, for some $k \in \mathbb{Z}$. According to Theorem 2.20, if $k \neq 0$ we will have

$$2^r = 3k, \quad \text{for some } \ k \in \mathbb{Z}^+, r \in \mathbb{N}.$$

This in turn implies that there exists an integer with two different prime factorizations which is in contradiction with the fundamental theorem of arithmetic (that is, every positive integer has a unique prime factorization). Therefore, $det(D) = 0$ and the result follows. □

Observe that in Theorem 6.4 we make the assumption that B is an integral matrix, i.e. B has elements in $\{0, \pm 1, \pm 2\}$. However, if B is a non-integral matrix then, by Theorem 2.13, we can obtain from B an integral binet matrix with at most $2m$ pivots, where m is the number of rows of B. Thus, given any binet matrix we are in a position to find a $GF(3)$ compact representation matrix of the associated signed-graphic matroid.

Although not all signed-graphic matroids are binary, it would be desirable to obtain a binary compact representation matrix for a binary signed-graphic matroid $M(\Sigma)$. We prove in Theorem 6.5 that the binary support of an integral binet matrix is actually one such representation. Moreover, using Theorem 2.13 we can obtain a binary compact representation matrix of $M(\Sigma)$ from a non-integral binet matrix B.

Theorem 6.5. *Let B be an integral binet matrix and $M(\Sigma)$ be the binary signed-graphic matroid represented by B over $\mathbb{R}$ (i.e. $M(\Sigma) \cong M(B)$). Then, the binary support of B is a compact representation matrix of $M(\Sigma)$ over $GF(2)$.*

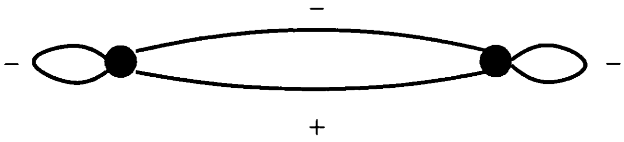


Figure 6.1: A signed graph whose signed-graphic matroid is isomorphic to $U_{2,4}$.

Proof: By Lemma 6.2, the matrix $B' = B \bmod 3$ is a ternary compact representation matrix of $M(\Sigma)$. Binary signed-graphic matroids are regular [57] and thus, by Theorem 1.14, if we view matrix B' over $\mathbb{R}$ then it is totally unimodular. Moreover, the binary support B'' of B' is a $GF(2)$ representation of $M(\Sigma)$ due to a well-known result of matroid theory (see Lemma 9.2.8 in [78]). Finally, it is easy to see that the matrix B'' is equal to the binary support of B. □

6.2 Binary signed-graphic matroids and binet matrices

In this section, by $M(\Sigma)$ is denoted a binary signed-graphic matroid and by B is denoted a binet matrix associated with a signed graph Σ. By Proposition 6.1, we know that $M(\Sigma) = M(B)$. Moreover, by Theorem 1.16, a matroid is binary if and only if it does not have the $U_{2,4}$ uniform matroid as a minor. However, a signed-graphic matroid can have a $U_{2,4}$-minor. Specifically, the signed-graphic matroid of the signed graph depicted in Figure 6.1 may be easily checked to be isomorphic to $U_{2,4}$. Any graph which can be obtained from this graph by a sequence of switchings and replacements of negative loops by half-edges and vice versa will be called a Γ-*graph*. By Theorem 6.6 we have that any signed graphical representation of $U_{2,4}$ is a Γ-graph.

Theorem 6.6. *The signed-graphic matroid $M(\Sigma)$ of a signed graph Σ is binary if and only if Σ does not contain a Γ-graph as a minor.*

Proof: For the "only if" part, by way of contradiction, suppose that Σ contains a Γ-graph Γ_1 as a minor and $M(\Sigma)$ is binary. Then there exist disjoint subsets X, Y of $E(\Sigma)$ such that, up to deletion of isolated vertices, $\Sigma \backslash X/Y \cong \Gamma_1$. Then, by Theorem 5.21, $M(\Sigma)\backslash X/Y = M(\Sigma\backslash X/Y) \cong M(\Gamma_1)$. But $M(\Gamma_1) \cong U_{2,4}$ (to see this, observe that any Γ-graph consists of four edges and in any Γ-graph every pair of edges does not create a circuit while any three edges form a circuit). This in turn implies that $M(\Sigma)$ has a $U_{2,4}$-minor. Thus, by Theorem 1.16, the matroid $M(\Sigma)$ is not binary.

For the "if" part, we prove the equivalent statement: if $M(\Sigma)$ is not binary then Σ contains a Γ-graph as a minor. Let $M(\Sigma)\backslash X/Y$ be a minor of $M(\Sigma)$ such that $M(\Sigma)\backslash X/Y \cong U_{2,4}$. By Theorem 5.21, $M(\Sigma)\backslash X/Y = M(\Sigma\backslash X/Y) \cong U_{2,4}$. Since $U_{2,4}$ is a connected matroid we have that $\Sigma\backslash X/Y$ must be connected. Furthermore, if we assume that $\Sigma\backslash X/Y$ is balanced then, by Proposition 5.28, the matroid $M(\Sigma\backslash X/Y) \cong U_{2,4}$ is a graphic matroid, which is in contradiction with the fact that $U_{2,4}$ is not binary and therefore, not graphic. Since $\Sigma\backslash X/Y$ is a connected unbalanced signed graph and the rank of $U_{2,4}$ is equal to 2, by Theorem 5.16 we have that $\Sigma\backslash X/Y$ is a signed graph on two vertices consisting of four edges. Furthermore, since $\Sigma\backslash X/Y$ can not have circuits with two or less edges, it can have at most one negative loop or half-edge incident with each vertex and two links of different sign. Thus, $\Sigma\backslash X/Y$ is a Γ-graph which in turn implies that Σ has a Γ-graph as a minor. □

The following Lemma 6.7 is important for the proof of Theorem 6.8.

Lemma 6.7. *A signed-graphic matroid $M(\Sigma)$ is non-binary if and only if there exists a compact representation matrix of $M(\Sigma)$ which is binet and contains the matrix* $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ *as a submatrix.*

Proof: For the "if" part, observe that $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is a compact representation matrix of $U_{2,4}$ over $\mathbb{R}$. This means that $M(\Sigma)$ has a minor isomorphic to $U_{2,4}$ and thus, by Theorem 1.16, $M(\Sigma)$ is a non-binary matroid.

For the "only if" part, let Σ' be the signed graph obtained from Σ by replacing every half-edge by a negative loop incident at the same vertex. By Lemma 5.23 and Theorem 6.6, $M(\Sigma') = M(\Sigma)$ and Σ' contains a Γ-graph as a minor. Since Σ' does not contain half-edges, there exists a sequence of deletions of vertices and edges and contractions of edges of Σ' which results in a Γ-graph Γ_1 without half-edges. Thus, up to switchings, Γ_1 is the Γ-graph depicted in Figure 6.1. It is not difficult to check that any binet matrix associated with Γ_1 can be obtained from matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ by a sequence (possibly empty) of row and column permutations, pivotings and row and column scalings by ± 1 factors. Let B be any binet matrix associated with Σ'. Then by Theorem 6.1, B is a compact representation of $M(\Sigma)$. Furthermore, by Theorem 5.21, $M(\Gamma_1)$ is a minor of $M(\Sigma')$. Thus, a matrix B' which contains a compact representation matrix D of $M(\Gamma_1)$ as a submatrix can be obtained from B by a series of pivotings. By Lemma 2.11, B' and D are binet matrices and, as shown above, D can be obtained from matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ by a sequence (possibly empty) of row and column permutations, pivotings and row and column scalings by ± 1 factors. This implies that there exist a series of these operations on B' which result in a matrix B'' containing the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ as a submatrix. By Lemma 2.11, B'' is a binet matrix. Finally, B'' is projectively equivalent with B and thus, it is a compact representation matrix of $M(\Sigma)$. □

By providing Theorem 6.8, Pagano [57] characterised the class of signed graphs whose signed-graphic matroid is binary. We provide a simpler proof of this theorem using results of this section as well as the BINET MATRIX ALGORITHM appearing in section 2.2.1.

Theorem 6.8. *Let Σ be a connected signed graph. The signed-graphic matroid $M(\Sigma)$ is binary if and only if the signed graph obtained by contracting all the balanced blocks of Σ does not contain two vertex-disjoint negative cycles.*

Proof: If Σ contains half-edges then let Σ' be the signed graph obtained from Σ by replacing every half-edge by a negative loop incident at the same vertex. Also let T and T' be the signed graphs obtained by contracting all the balanced blocks of Σ and Σ', respectively. By Lemma 5.23, $M(\Sigma') = M(\Sigma)$ and furthermore, T does not contain two vertex-disjoint negative cycles if and only if T' does not contain two vertex-disjoint negative cycles. Thus, it suffices to prove the theorem for a signed graph Σ which contains no half-edges and in the rest of the proof we assume that Σ is such a signed graph.

The "if" part can be restated as follows: if $M(\Sigma)$ is a non-binary matroid then the signed graph T obtained from Σ by contracting all the balanced blocks contains two vertex-disjoint negative cycles. Since

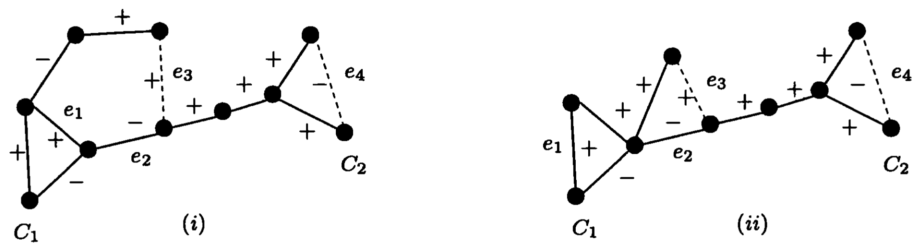


Figure 6.2: Examples signed graphs for cases (i) and (ii)

$M(\Sigma)$ is not binary, by Lemma 6.7, there is a compact representation matrix of $M(\Sigma)$ which is binet and contains the matrix:

$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

as a submatrix, where the labels e_1, e_2, e_3 and e_4 label also the corresponding edges of Σ. Since the column e_4 of A contains a 2 and Σ does not contain half-edges, then, by the BINET MATRIX ALGORITHM, we have that in an orientation of Σ the fundamental circuit of e_4 is a handcuff of Type II. Moreover, the edge e_2 is a link in the path connecting the two vertex-disjoint negative cycles C_1 and C_2 of this handcuff and e_1 is an edge of either C_1 or C_2; without loss of generality, we shall assume that $e_1 \in C_1$.

We consider two cases. In the first case, we assume that C_1 and C_2 belong to the same block of Σ. Then T contains two vertex disjoint negative cycles and the result follows immediately. In the second case, we assume that C_1 and C_2 belong to different blocks of Σ. By way of contradiction, suppose that C_1 and C_2 are not vertex disjoint in T. This implies that e_2 belongs to a balanced block of Σ. From the BINET MATRIX ALGORITHM, we have that the fundamental circuit C_3 of e_3 is either: (i) a positive cycle containing both e_1 and e_2, or (ii) a handcuff of Type I containing both e_1 and e_2 (see Figure 6.2). In case (i), e_1 and e_2 belong to the same block of Σ which is unbalanced since this block contains C_1, and in case (ii) e_2 belongs to some unbalanced block since e_2 is an edge of a negative cycle. Thus, in both cases e_2 belongs to an unbalanced block of Σ which is in contradiction with our assumption that C_1 and C_2 are not vertex disjoint in T (which, as mentioned above, implies that e_2 belongs to a balanced block of Σ).

For the "only if" part, we prove the following: if T contains two vertex-disjoint negative cycles, then $M(\Sigma)$ is not binary. We consider two cases.

Case (a): There is no block of T which contains two vertex disjoint negative cycles. Let C_1 and C_2 be two vertex disjoint negative cycles of T which belong to two different blocks T_1 and T_2, respectively. Since T consists only of unbalanced blocks, without loss of generality, we can assume that T_1 and T_2 have a common vertex v. Since C_1 and C_2 are vertex disjoint and T_1 and T_2 have a common vertex, at most one of C_1 and C_2 can be a negative loop. Therefore, without loss of generality, let us assume that C_1 is a cycle consisting of links. If u_1 and w_1 are two vertices of C_1, then, since T_1 is 2-connected, there exist two paths P_1 and P_2 with end-vertices $\{u_1, v\}$ and $\{w_1, v\}$, respectively, which are internally disjoint (see Figure 6.3 for an illustrative example). It is evident that the graph $C_1 \cup P_1 \cup P_2$ is a theta graph. Moreover, since signed graph are additively biased graphs (i.e. in any theta subgraph, either 1 or 3 cycles are positive), at least one of the cycles of this theta graph is positive. Since C_1 is a negative cycle

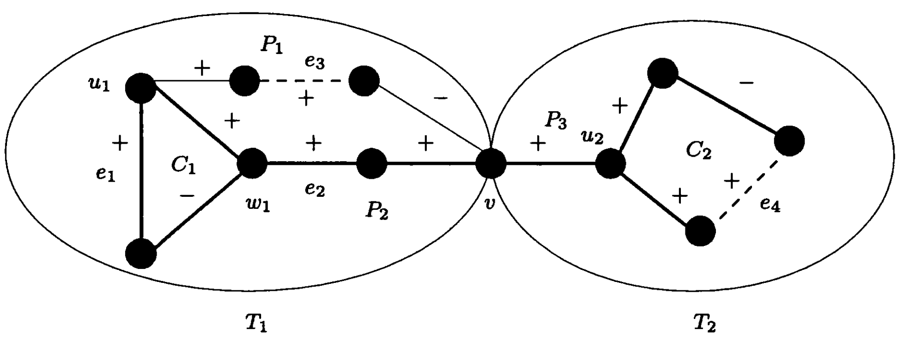


Figure 6.3: The subgraph $C_1 \cup C_2 \cup P_1 \cup P_2 \cup P_3$ of an example signed graph in Case (a).

we have that one of the two other cycles is positive. Let e_1 be an edge of this positive cycle which is also an edge of C_1. Let us call e_2 an edge of P_2, e_3 an edge of P_1, e_4 an edge of C_2, and let P_3 be a path in T_2 from v to a vertex u_2 of C_2. Then, in any orientation $\overrightarrow{T}$ of T there must exist a basis R such that $C_1 \cup (P_1 \backslash e_3) \cup P_2 \cup P_3 \cup (C_2 \backslash e_4) \subseteq R$. We apply the BINET MATRIX ALGORITHM two times: the first time with input the binet graph $(\overrightarrow{T}, R)$ and the non-basic edge e_3 and the second time with input the binet graph $(\overrightarrow{T}, R)$ and the non-basic edge e_4, in order to get the columns e_3 and e_4 of the binet matrix associated with $(\overrightarrow{T}, R)$. Then it is not difficult to see that the binet matrix associated with $(\overrightarrow{T}, R)$ will have, up to scalings by ± 1 factors the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} e_3 & e_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

as a submatrix. Thus, by Theorem 6.1 and Lemma 6.7, the result follows easily.

Case (b): There is a block T' of T which contains two vertex disjoint negative cycles. Let C_1' and C_2' be the two vertex disjoint negative cycles of T'. Since T' is 2-connected, neither C_1' nor C_2' can be a negative loop and moreover, any two edges of T' belong to some cycle. Therefore, there exists two vertex disjoint paths P_1' and P_2' with end-vertices $\{u_1', u_2'\}$ and$\{w_1', w_2'\}$, respectively, where u_i' and w_i' are vertices of C_i' $(i = 1, 2)$. Without loss of generality, we can assume that P_1' and P_2' are of minimal length and thus, they have no common vertices with C_1' and C_2' other than their end-vertices (see Figure 6.4 for an illustrative example). Let us call P_3' and $\bar{P}_3'$ the two paths of C_2' with first vertex u_2' and last vertex w_2'. Then the graph $C_1' \cup P_1' \cup P_2' \cup P_3'$ is a theta graph. Since signed graphs are additively biased graphs, we can assume that one of the cycles in this theta graph (other than the negative C_1') is positive. Let e_1' be an edge of this positive cycle which is also an edge of C_1'. Let us call e_2' an edge of P_2', e_3' an edge of P_1', and e_4' an edge of $\bar{P}_3'$. Then, in any orientation $\overrightarrow{T}$ of T there must exist a basis R' such that $C_1' \cup (P_1' \backslash e_3') \cup P_2' \cup (C_2' \backslash e_4') \subseteq R'$. If we apply the BINET MATRIX ALGORITHM two times: the first time with input the binet graph $(\overrightarrow{T}, R')$ and the nonbasic edge e_3' and the second time with input the binet graph $(\overrightarrow{T}, R')$ and the nonbasic edge e_4', in order to get the columns e_3' and e_4' of the binet matrix associated with $(\overrightarrow{T}, R')$, then it is not difficult to see that the binet matrix associated with $(\overrightarrow{T}, R')$ will

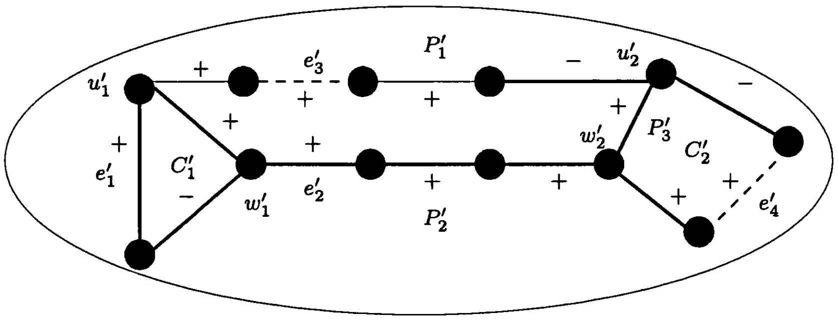


Figure 6.4: The subgraph $C_1' \cup C_2' \cup P_1' \cup P_2'$ of an example signed graph in Case (b).

have, up to scalings by ±1 factors, the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} e_3' & e_4' \end{array} \\ \begin{array}{c} e_1' \\ e_2' \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

as a submatrix. Therefore, by Theorem 6.1 and Lemma 6.7, the result follows easily.

□

If Σ is a tangled signed graph then, by definition, it contains no two vertex disjoint negative cycles and furthermore, by Proposition 5.34, Σ contains exactly one unbalanced block. Using the BINET MATRIX ALGORITHM appearing in section 2.2.1, it can be easily shown that any binet matrix associated with Σ contains no ±2s' or $\pm\frac{1}{2}$s'(Proposition 6.9).

Proposition 6.9. *If a signed graph Σ is tangled then any binet matrix associated with Σ has elements in* $\{0, \pm1\}$.

Furthermore, by Theorem 5.42 and Proposition 6.9 we get the following corollary.

Corollary 6.10. *If a signed-graphic matroid $M(\Sigma)$ is binary and a binet matrix associated with Σ has an element other than $\{0, \pm1\}$ then $M(\Sigma)$ is graphic.*

We also prove that any $\{0, \pm1\}$ compact representation matrix of a binary signed-graphic matroid is binet and totally unimodular.

Theorem 6.11. *Let A be a real compact representation matrix of a signed-graphic matroid $M(\Sigma)$ with elements in $\{0, \pm1\}$. Then, $M(\Sigma)$ is binary if and only if A is totally unimodular and binet.*

Proof: The "if" part is clear, since any totally unimodular matrix is a representation matrix of some binary matroid (see Theorem 1.14 (ii) and (iv)). For the "only if" part, we first show that there exists a compact representation matrix of $M(\Sigma)$ which is binet and totally unimodular. By Theorem 5.42, one of the following cases apply: (i) $M(\Sigma)$ is graphic, or (ii) Σ is a tangled signed graph. In case (i), there exists a network matrix which is the compact representation of $M(\Sigma)$ over $\mathbb{R}$ (see Chapter 11 in [78]). By Example 2 in section 2.2.1, this matrix is binet as well. In case (ii), by Theorem 6.1 and Proposition 6.9, any binet matrix associated with Σ is a real compact representation matrix of $M(\Sigma)$ and has elements in $\{0, \pm 1\}$. Thus, in all cases there exists a binet matrix A' with elements in $\{0, \pm 1\}$ which is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$. By Theorem 6.4, a compact representation of $M(\Sigma)$ over $GF(3)$ is matrix A' itself. By Theorem 1.14 (parts (i) and (v)), A' is totally unimodular as well.

Using matrix A', we shall now show that any compact representation matrix of $M(\Sigma)$ with elements in $\{0, \pm 1\}$ is totally unimodular and binet. By Theorem 1.14 and due to the fact that signed-graphic matroids are ternary, $M(\Sigma)$ is a regular matroid. Regular matroids are uniquely representable (see Corollary 10.1.4 in [56]), which implies that a matrix with elements in $\{0, \pm 1\}$ is a compact representation matrix of $M(\Sigma)$ if and only if it can be obtained from any $\{0, \pm 1\}$ compact representation matrix of $M(\Sigma)$ by a (possibly empty) sequence of pivotings, scalings of rows and columns by ± 1 factors and permutations of columns and rows (see Proposition 6.3.13 in [56]). Therefore, A can be obtained from A' by a sequence of pivotings, scalings of rows and columns by ± 1 factors and permutations of columns and rows. Binet matrices and totally unimodular matrices are closed under pivotings, scalings of rows and columns by ± 1 factors and permutations of columns and rows and thus, A is binet and totally unimodular.

□

Theorem 6.11 has an important implication. We know by Camion's algorithm, which is a direct consequence of the results in [13, 14], that there exists a unique signing (i.e. replacement of the non-zero entries of a matrix by $+1$ or -1) of a binary representation matrix A of a regular matroid into a totally unimodular matrix A', up to multiplying rows or columns by -1. Therefore, given any binary compact representation matrix of a binary signed-graphic matroid $M(\Sigma)$ we can find a binet matrix associated with Σ. Based on this, we partially answer an open question appearing in [54] which asks whether the binet recognition algorithm presented in that work may be used to determine if a matroid is signed-graphic or not. The following algorithm provides a partial answer.

BINARY RECOGNITION ALGORITHM

Input: A binary matrix A.

Output: The matroid $M = M(A)$ is identified as signed-graphic or not. Moreover, a signed graph Σ such that $M = M(\Sigma)$ is provided.

Step 1. Test whether M is regular using the test given in [66] (see also [78]). If M is not regular then M is not signed-graphic.

Step 2. Apply Camion's algorithm [14] (see also [17]) in order to sign A into a totally unimodular matrix A'.

Step 3. Test whether A' is binet using the test given in [54]. If so, then M is signed-graphic and, moreover, $M = M(\Sigma)$, where Σ is the underlying signed graph of the bidirected graph provided by this test; otherwise, M is not signed-graphic.

Regularity of a binary matroid can be checked in polynomial time (see e.g. [78]) and we can decide whether a real matrix is binet or not in polynomial time [54]. Furthermore, Camion's algorithm has also been shown to be polynomial (see e.g. [17]). Therefore, all the procedures used in the above algorithm run in polynomial time which in turn implies that the above algorithm has a polynomial running time. Finally, the proof of correctness of this algorithm is straightforward and is omitted.

6.3 Characterizing signed-graphic matroids

In this section we provide necessary and sufficient conditions for a matroid to be the signed-graphic matroid of a given signed graph. This result was inspired by an interesting paper of Seymour ([67]) in which he gave necessary and sufficient conditions for a matroid to be the graphic matroid of a given graph. We note here that for the proof of this result (Theorem 6.12), we are adopting techniques used to prove a similar theorem for the class of bicircular matroids (see Theorem 3.1 in [18]), where a bicircular matroid is the frame matroid of a biased graph with no balanced circles. Finally, while working towards the results of this section, it has come to our attention that a similar characterization for the more general class of biased graphs was given in [36].

Theorem 6.12. *Let M be a matroid with ground set E and let $M(\Sigma)$ be the signed-graphic matroid of a connected signed graph Σ with edge set $E(\Sigma) = E$, where at least one edge of Σ is not a positive loop. Then $M = M(\Sigma)$ if and only if*

(i) the b-star of every vertex of Σ is a union of cocircuits of M,

(ii) the edge set of a vertex-disjoint union of negative cycles of Σ is independent in M,

(iii) the edge set of every positive cycle of Σ is dependent in M, and

(iv) $r(M) \leq r(M(\Sigma))$

Proof: For the "only if" part, we have that (ii) and (iii) follow from Theorem 5.17 while (iv) holds trivially. For (i), let $st_\Sigma(v)$ be the b-star of a vertex v of Σ. Since Σ is connected and has at least one edge which is not a positive loop, we have that $st_\Sigma(v)$ is non-empty. Therefore, the signed graph $\Sigma \setminus st_\Sigma(v)$ has at least one more balanced component than Σ and thus, by Theorem 5.18, $st_\Sigma(v)$ contains a cocircuit D_1 of $M(\Sigma)$. If D_1=$st_\Sigma(v)$, then the result follows. In the remaining case, consider the signed graph $\Sigma' = \Sigma \setminus D_1$. By the same argument, $st_{\Sigma'}(v)$ contains a cocircuit D_2 of $M(\Sigma') = M(\Sigma \setminus D_1) = M(\Sigma) \setminus D_1$. Therefore, by the definition of the matroid contraction operation (see (1.4) in section 1.6.2), $D_2 \cup S$ is a cocircuit of $M(\Sigma)$, where $S \subseteq D_1 \subseteq st_\Sigma(v)$. If $D_1 \cup D_2 = st_\Sigma(v)$ then, as before, the result follows. In the remaining case, if $D_1 \cup D_2 \subseteq st_\Sigma(v)$, then let $\Sigma'' = \Sigma' \setminus D_2$. Continuing this process, provides the result.

For the "if" part we first prove the following claim.

Claim. *Let H be a subgraph of Σ. If each component of H is either a tree or a negative 1-tree, then $E(H)$ is an independent set of M.*

Proof: Assume the contrary and let H be a counterexample with $|E(H)|$ minimum. If H is a union of negative cycles then, by condition (ii), $E(H)$ is an independent set of M, a contradiction. Therefore, H has a degree-one vertex v. Let e be the edge of H being incident with v. By condition (i), e is an element of a cocircuit D of M such that $D \cap E(H) = \{e\}$. Thus, if C is a circuit of M such that $C \subseteq E(H)$ then $e \notin C$ due to Proposition 1.15. Since each component of $H \setminus e$ is a tree or a negative 1-tree and $H \setminus \{e\}$ has one less edge than H, $E(H) \setminus e$ is an independent set of M due to the minimality of $E(H)$. This is in contradiction with our assumption stating that there is a circuit of M contained in $E(H)$ and the fact that this circuit can not contain e. □

By this claim, we can conclude that a basis B of $M(\Sigma)$ (which as implied by Theorem 5.17 corresponds to a subgraph of Σ whose components are trees or negative 1-trees) will be an independent set of M. Therefore, $r(M(\Sigma)) \leq r(M)$, which combined with condition (iv) gives $r(M) = r(M(\Sigma))$. Thus, B is a basis of M. Therefore, in order to prove that $M = M(\Sigma)$, it remains to show that every basis of M is a basis of $M(\Sigma)$. If every basis of M is independent in $M(\Sigma)$ then, since $r(M) = r(M(\Sigma))$, the result follows. Therefore, by way of contradiction, let B be a basis of M and suppose that B is dependent in $M(\Sigma)$. In this case, $\Sigma[B]$ contains a circuit K of Σ. If K is a Type II handcuff, then let e be an edge of one of the cycles of K; otherwise, let e be any edge of K. Since $E(K)\backslash e$ is independent in $M(\Sigma)$ and Σ is connected, $E(K)\backslash e$ can be extended to a basis B' of $M(\Sigma)$ such that $\Sigma[B']$ is connected. Since $r(M) = r(M(\Sigma))$, B' is a basis of M. Thus, $B' \cup e$ contains a unique circuit C of M. By condition (i) and Proposition 1.15, $\Sigma[C]$ has no vertices with degree one. We now consider two cases depending on the type of circuit K.

Case (i): Suppose that K is a handcuff of Type I or Type II. Then, by condition (ii), $C = E(K)$. This is a contradiction, since $C = E(K) \subseteq B$.

Case (ii): Suppose that K is a positive cycle. If $\Sigma[B']$ has a cycle then we call D this cycle. Suppose that $E(K) \cap E(D) = \emptyset$. Then, by condition (ii), $E(K) \subseteq C$, and by condition (iii), $E(K) = C$, a contradiction. Suppose now that $E(K) \cap E(D) \neq \emptyset$. Then $K \cup D$ is a theta graph. Thus, $G[C]$ is either the entire theta graph or one of the cycles contained in this theta graph. The theta graph contains a positive cycle and thus, by condition (iii), $G[C] \neq K \cup D$. Furthermore, since signed graphs are additively biased graphs, K is the only positive cycle contained in this theta graph. Therefore, by conditions (ii) and (iii), $C = E(K)$, a contradiction. □

We turn now our attention to the special case in which Σ is tangled. By the definition of tangled signed graphs and Proposition 5.36, we easily obtain the following corollary of Theorem 6.12.

Theorem 6.13. *Let M be a matroid with ground set E and let $M(\Sigma)$ be the signed-graphic matroid of a connected tangled signed graph Σ with edge set $E(\Sigma) = E$. Then, $M = M(\Sigma)$ if and only if*

(i) for each $v \in V(\Sigma)$, every bond of Σ included in the b-star of v is a cocircuit of M,

(ii) the edge set of every negative cycle of Σ is independent in M,

(iii) the edge set of every positive cycle of Σ is dependent in M, and

(iv) $r(M) \leq r(M(\Sigma))$

In section 6.2 we presented the BINARY RECOGNITION ALGORITHM which decides whether a given binary matroid N is signed-graphic. Furthermore, if N is found to be signed-graphic, this algorithm provides a signed graph Σ such that $N \cong M(\Sigma)$. Following the steps of the algorithm for recognizing graphic matroids in [67], we can use Theorem 6.13 in order to provide an algorithm which determines whether a general matroid M (not necessarily binary) belongs to the class of binary signed-graphic matroids. As usual (see e.g. [67]), we shall assume that M is given by means of an independence testing oracle, that is we can decide whether a subset of $E(M)$ is independent or not in unit time.

GENERAL RECOGNITION ALGORITHM

Input: A matroid M given by its independence oracle.
Output: M is identified as binary signed-graphic or not.

Step 1. Test whether M is graphic using the test given in [67]. If so, then M is binary signed-graphic.
Step 2. Pick a basis B of M and for each element of $x \in E(M)\backslash B$ find the unique circuit C_x in $B \cup x$. Construct the $B \times (E(M) - B)$ matrix A as follows: for any $e \in B$ and any $x \in (E(M) - B)$, let $A_{ex} = 1$ if $e \in C_x$, and 0 otherwise.
Step 3. Test whether $M' = M(A)$ is binary signed-graphic using the BINARY RECOGNITION ALGORITHM of section 6.2. If M' is not binary signed-graphic then M is not a signed-graphic matroid; otherwise, let Σ be the signed graph provided by the BINARY RECOGNITION ALGORITHM such that $M' = M(\Sigma)$.
Step 4. Check whether Σ satisfies the conditions (i), (ii) and (iii) of Theorem 6.13. If yes, then M is a binary signed-graphic matroid; otherwise, M is not binary signed-graphic.

The proof that the aforementioned algorithm is correct goes as follows. Clearly, if M is binary then $M = M'$. Thus, if M' is not a binary signed-graphic matroid, then either $M = M'$ and M is not signed-graphic, or $M \neq M'$ and thus, M is not binary and hence not binary signed-graphic. Therefore, if M' is not binary signed-graphic then neither is M. By Theorem 5.42, if M' is binary signed-graphic then one of the following two cases may happen: (i) M' is graphic or, (ii) any signed graph H such that $M' \cong M(H)$ is tangled. We shall examine the two cases separately. For case (i), we apply the algorithm of [67] and decide if M is graphic or not. If M is graphic then M is also signed-graphic. If M is not graphic then M is not binary (since if M is binary then $M = M'$) and therefore M is not binary signed-graphic. If case (ii) applies, then there exists a tangled signed graph Σ such that $M' = M(\Sigma)$. Moreover, $M = M(\Sigma)$ if and only if M is binary signed-graphic. In order to see this, first notice that the "only if" part is trivial while for the "if" part we have that M is binary and thus $M = M' = M(\Sigma)$. Thus, it remains to test if $M = M(\Sigma)$ which can be done by using Theorem 6.13. Finally, observe that condition (iv) of Theorem 6.13 is satisfied because of the way M' is constructed and for that reason, it does not have to be checked at the last step of the algorithm.

Chapter 7

Decomposition of Binary Signed-Graphic Matroids

A class of representable matroids can be characterized either by providing a set of excluded minors, or through a decomposition result which is usually much harder to obtain. For the latter case, a celebrated example is the decomposition theorem for regular matroids by Seymour [66] which led to a polynomial algorithm to recognize whether a $\{0, \pm 1\}$-matrix is totally unimodular (see the book of Truemper [78] for details). In [25], using signed graphs Gerards et.al. were able to re-prove in an elegant and shorter way a key result used by Seymour in his proof of the regular matroid decomposition theorem; specifically, they proved that each 3-connected regular matroid that is neither graphic nor cographic contains an R_{10}- or an R_{12}-minor. To do this, at the first step, they proved a structural theorem for signed graphs and translated this result to the associated complete lift matroid while at the second step, they combined this matroidal result with the Splitter Theorem of Seymour (see [66, 56]). Moreover, as we have already mentioned in previous chapters, the class of signed-graphic matroids has attracted the attention of many researchers over the past years(see [57, 69, 70, 72, 97, 98] among others), while recently it has also been conjectured by Whittle et.al. that they may be the building blocks of a k-sum decomposition of dyadic and near-regular matroids [51, 92]. Therefore, signed graphs and their associated matroids may be of great importance in obtaining decomposition results and recognition algorithms for important classes of matroids and thus, further knowledge regarding the structure and the building blocks of these matroids is desirable.

In this chapter we employ Tutte's theory of bridges to derive a decomposition theorem for the class of binary signed-graphic matroids. The proposed decomposition differs from previous decomposition results on matroids that have appeared in the literature in the sense that it is not based on k-sums, but rather on the operation of deletion of a cocircuit. Specifically, it is shown that all minors resulting from the deletion of a cocircuit of a binary matroid will be graphic matroids except for exactly one that will be signed-graphic if and only if the matroid is signed-graphic.

The theory of bridges was developed by Tutte in [80] in order to answer fundamental questions regarding graphs and their matroids. Two well known results which are a consequence of the theory of bridges, is Tutte's recognition algorithm for graphic matroids in [81] and Bixby and Cunningham's efficient algorithm for testing whether a matroid is 3-connected or not in [9]. Moreover, in his last book [86]

Tutte expressed the belief that this theory is rich enough to enjoy more theoretical applications. In this work we use the theory of bridges to derive a decomposition result for binary signed-graphic matroids.

An overview of previous decomposition results regarding signed-graphic matroids and signed graphs can be found in [72]. However, the majority of the results presented in that work are mainly decomposition results for signed graphs rather than for signed-graphic matroids. Specifically, based on previous results of Pagano [57] and Gerards [27], the authors of [72] provide two main decomposition theorems for a signed graph Σ; one theorem concerning the case in which the associated signed-graphic matroid $M(\Sigma)$ is binary and one theorem concerning the case in which $M(\Sigma)$ is quaternary. The notion of k-sums of signed graphs is introduced by Pagano in [57] while Gerards introduces the similar notion of k-splits ($k = 1, 2, 3$) in order to provide decomposition results for signed graphs whose complete lift matroids are regular. In [72], these notions are slightly altered and extended so that the signed-graphic matroid of the k-sum of two signed graphs Σ_1 and Σ_2 to be equal to the matroidal k-sum of the signed-graphic matroids $M(\Sigma_1)$ and $M(\Sigma_2)$. By using the k-sum operations as defined for signed graphs and the results of [27, 57], the above mentioned decomposition theorems regarding the class of signed graphs with binary or quaternary signed-graphic matroids are proved in [72].

The rest of this chapter is structured in the following way. In section 7.1 we present some definitions and preliminary results regarding the theory of bridges, which will be needed in the next sections. In section 7.2 the cocircuits of binary signed-graphic matroids are further classified into graphic and non-graphic, depending on whether their deletion produces a graphic matroid or not. Section 7.3 is the main section of this chapter, in which the necessary structural theorems providing the connection between a tangled signed graph and its corresponding signed-graphic matroid are presented. An excluded minor characterization for signed-graphic matroids with all graphic cocircuits is given in section 7.3.1, while the decomposition based on non-graphic cocircuits is presented in section 7.3.2. In section 7.4, we shall present the underlying direction of our research as well as we will mention a recent unpublished work of Gerards et.al. regarding matroid minors and its impact on our work. Finally, we should note that the majority of the results in this chapter have to do with the structure of tangled signed graphs, and the relationship between cocircuits in a binary signed-graphic matroid and bonds in the corresponding signed graph representation .

7.1 Bridges

Let Y be a cocircuit of a binary matroid M. We define the *bridges* of Y in M to be the elementary separators of $M\backslash Y$. If $M\backslash Y$ has more than one bridge then we say that Y is a *separating* cocircuit; otherwise it is *non-separating*. Let B be a bridge of Y in M; the matroid $M/.(B \cup Y)$ is called an Y-*component* of M. From [80], we know that if M is connected then each Y-component of M is connected. Furthermore, for any bridge B of Y in M, we denote by $\pi(M, B, Y)$ the family of all minimal non-null subsets of Y which are intersections of cocircuits of $M/.(B \cup Y)$. The following theorem and its corollary, which are known results (see [74, 81]), relate $\pi(M, B, Y)$ for binary matroids to the family of cocircuits of a given minor.

Theorem 7.1. *Let Y be a cocircuit of a binary matroid M. Two elements a and b of Y belong to the same members of $\pi(M, B, Y)$ if and only if they belong to the same cocircuits of $(M/.(B \cup Y))\backslash.Y$.*

In [82], Tutte proved that if M is binary the members of $\pi(M, B, Y)$ are disjoint and their union is Y. We usually refer to $\pi(M, B, Y)$ as the *partition of Y determined by B*. Moreover, in [81] we find the following useful theorem:

Theorem 7.2. *Let Y be a cocircuit of a matroid M. If M is regular then $\pi(M, B, Y) = C^*((M/.(B \cup Y))\backslash.Y)$.*

Let B_1 and B_2 be two bridges of Y in M. The bridges B_1 and B_2 are said to *avoid* each other if there exists $S \in \pi(M, B_1, Y)$ and $T \in \pi(M, B_2, Y)$ such that $S \cup T = Y$; otherwise we say that B_1 and B_2 *overlap* one another. A cocircuit Y is called *bridge-separable* if its bridges can be classified into two classes U and V such that no two members of the same class overlap. Tutte has shown in [82] that all cocircuits of graphic matroids are bridge-separable while if a matroid has a cocircuit which is not bridge-separable, then it will contain a minor isomorphic to $M^*(K_5)$, $M^*(K_{3,3})$ or F_7^*. Recall that by definition there is one-to-one correspondence between the collection of edge-sets of the separates of $\Sigma\backslash Y$ and the collection of bridges of Y in $M(\Sigma)$. Suppose now that B is a bridge of Y in $M(\Sigma)$ and let Σ_i be the component of $\Sigma\backslash Y$ such that $\Sigma\backslash.B \subseteq \Sigma_i$. Then, if v is a vertex of $V(\Sigma\backslash.B)$, we denote by $C(B, v)$ the component of $\Sigma_i\backslash B$ having v as a vertex. Finally, we denote by $Y(B, v)$ the set of all $y \in Y$ such that one end-vertex of y in Σ is a vertex of $C(B, v)$.

7.2 Cocircuits and Bonds

In this section we deal with cocircuits of connected binary signed-graphic matroids and their graph representation in the corresponding tangled signed graphs. Let Y be a cocircuit of a connected binary signed-graphic and non-graphic matroid $M(\Sigma)$. Clearly, Y is a bond of Σ and by Theorems 5.40 and 5.42 we have that Σ is 2-connected and tangled. By the classification of bonds based on the nature of $\Sigma\backslash Y$ presented in section 5.3.1 (see Figure 5.3.1) and Theorem 5.37 we know that Y can be one of the following types of bonds in Σ (see Figure 7.1):

(a) balanced bond

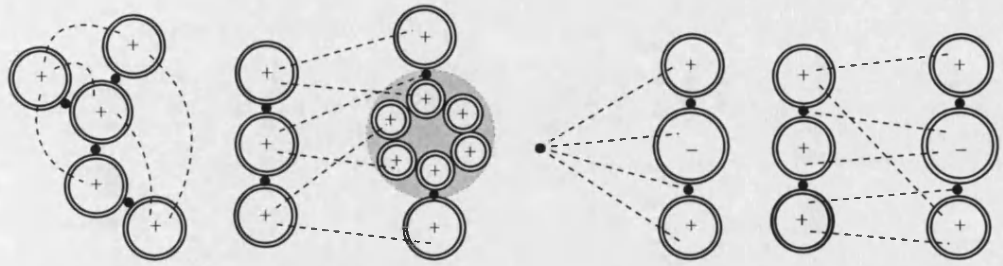
(b) star or unbalanced bond such that the core of $\Sigma\backslash Y$ is a necklace

(c) star bond such that the core of $\Sigma\backslash Y$ is not a necklace

(d) unbalanced bond such that the core of $\Sigma\backslash Y$ is not a necklace

By Theorem 5.20, any bridge B of Y in $M(\Sigma)$ will correspond to a 2-connected subgraph $\Sigma\backslash.B$ in Σ, and by Theorem 5.39, $\Sigma\backslash Y$ will contain at most one unbalanced block. Note also that the only case in which a block of $\Sigma\backslash Y$ does not correspond to a separator of $M(\Sigma)\backslash Y$, is when the block is unbalanced and a necklace (i.e. see (b) in Figure 7.1). In this case the edge-sets of the blocks of the necklace are separators of the matroid.

We observe that if Y is either of type (a) or of type (b), then $M(\Sigma)\backslash Y$ is graphic since all of its separators have a balanced signed-graph representation (see Proposition 5.28). We shall say that a cocircuit Y of a binary matroid M is *graphic* if $M\backslash Y$ is a graphic matroid; otherwise Y will be called *non-graphic*. Therefore, if M is signed-graphic and Y is a non-graphic cocircuit then, in any signed graph Σ such



(a) balanced bond (b) bond with a necklace (c) star bond (d) unbalanced bond

Figure 7.1: Bonds in tangled signed graphs

that $M = M(\Sigma)$, Y will be a bond of type (c) or type (d) only. As it turns out non-graphic cocircuits have similar structural characteristics to cocircuits of graphic matroids, and as it will be demonstrated in section 7.3.2 they provide a means of decomposing binary signed-graphic matroids.

7.3 Decomposition

In this section we will present a decomposition theory for binary signed-graphic matroids which utilizes the theory of bridges by Tutte [80, 82]. The main result is that of Theorem 7.15, which states that deletion of a non-graphic cocircuit naturally decomposes a binary signed-graphic matroid into minors which are all graphic except for one which is signed-graphic. This decomposition follows the same path as an analogous result for graphic matroids by Tutte in [80, 82]. However it differs in many ways mainly due to the more complex nature of cocircuits in signed-graphic matroids with respect to cocircuits of graphic matroids. In section 7.3.1 we provide an excluded minor characterization for the class of signed-graphic matroids without non-graphic cocircuits while in section 7.3.2 our decomposition result is provided.

7.3.1 Graphic Cocircuits

We have already mentioned in Corollary 5.27 that the class of graphic matroids is a subclass of signed-graphic matroids. In contrast neither the cographic matroids nor the regular matroids are a subclass of signed-graphic matroids. Two important theorems which associate signed-graphic matroids with cographic matroids and regular matroids in terms of excluded minors have been shown by Slilaty et.al. in [62, 69]. Specifically, of the 35 forbidden minors for projective planar graphs 29 are not 1-separable; these 29 graphs, which we call $G_1, G_2, \ldots, G_{29}$, can be found in [6, 52]. Slilaty has shown in [69] that the collection of the cographic matroids of these 29 graphs $\mathcal{M} = \{M^*(G_1), M^*(G_2), \ldots, M^*(G_{29})\}$ forms the complete list of the cographic excluded minors for the class of signed-graphic matroids. Clearly, since cographic matroids are a subclass of regular matroids we expect the list of regular excluded minors for signed-graphic matroids to contain the matroids in $\mathcal{M}$ and some other matroids. Those other matroids are the R_{15} and R_{16} whose binary compact representation matrices are the following:

$$A_{R_{15}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}, A_{R_{16}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, in [69] we find Theorem 7.3 and in [62] we find Theorem 7.4.

Theorem 7.3. *A cographic matroid M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_1), \ldots, M^*(G_{29})$.*

Theorem 7.4. *A regular matroid M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_1), \ldots, M^*(G_{29}), R_{15}$ or R_{16}.*

The following two lemmas are essential for the proof of the main result of this section which characterizes the regular matroids with graphic cocircuits. Central in this characterization are the cographic matroids of graphs G_{17} and G_{19}. The graph G_{17} is isomorphic to the graph $K_{3,5}$ while the graph G_{19} is isomorphic to $K_{4,4} \backslash e$, where e is any edge of the graph $K_{4,4}$.

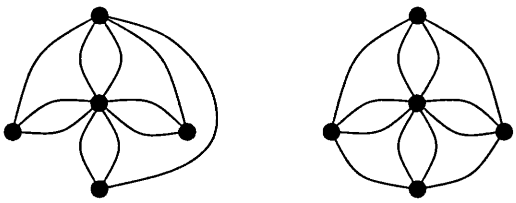
Lemma 7.5. *If a matroid M is isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$ then any cocircuit of M is graphic.*

Proof: By (1.5), we can equivalently show that, for any circuit Y of M^*, the matroid M^*/Y is cographic. The matroid M^* is graphic and thus, regular. Therefore, by Theorem 1.18, we have to show that for any circuit Y of $M^* \in \{M(G_{17}), M(G_{19}))\}$ the matroid M^*/Y has no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. We know that G_{17} is isomorphic to the graph $K_{3,5}$ and G_{19} is isomorphic to $K_{4,4} \backslash e$, where e is any edge of $K_{4,4}$. Since $M(G_{19})$ is a graphic matroid we have that $M(G_{19}) \cong M(K_{4,4} \backslash e) = M(K_{4,4}) \backslash e$. Therefore, $M(K_{4,4})$ has a minor isomorphic to $M(G_{19})$ and by (1.3), any circuit of $M(G_{19})$ is a circuit of $M(K_{4,4})$. Thus, in order to prove the theorem, it suffices to prove that for any circuit $Y_1 \in \mathcal{C}(M(K_{3,5}))$ and $Y_2 \in \mathcal{C}(M(K_{4,4}))$ the matroids $M(K_{3,5})/Y_1 = M(K_{3,5}/Y_1)$ and $M(K_{4,4})/Y_2 = M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$.

Since $K_{3,5}$ and $K_{4,4}$ are 3-connected, by Theorem 1.12, we get that Y_1 and Y_2 correspond to circles of $K_{3,5}$ and $K_{4,4}$, respectively. The 3-connected graphs $K_{3,5}$ and $K_{4,4}$ are also bipartite and therefore, they have no circle of odd cardinality and moreover, they have no parallel edges. This means that $K_{3,5}/Y_1$ and $K_{4,4}/Y_2$ have at most five vertices each. Therefore, the matroids $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have rank at most 4 which is less than the rank of $M(K_{3,3})$. Thus, $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ can not have a minor isomorphic to $M(K_{3,3})$.

It remains to be shown that $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_5)$. Let us suppose that Y_1 and Y_2 are circuits of cardinality four. Then, since $K_{3,5}$ and $K_{4,4}$ are 3-connected we have (by Theorem 1.12) that Y_1 and Y_2 are circles of $K_{3,5}$ and $K_{4,4}$, respectively, with cardinality four. Observe now that for any Y_1 and Y_2, the graphs $K_{3,5}/Y_1$ and $K_{4,4}/Y_2$ are isomorphic to the graphs $\bar{G}$

and $\hat{G}$ of Figure 7.2, respectively. Furthermore, parallel edges of a graph correspond to parallel elements in the associated graphic matroid. Therefore, any simple minor of $M(\bar{G})$ or $M(\hat{G})$ has at most seven or eight elements, respectively. The matroid $M(K_5)$ is simple and has ten elements. Therefore, $M(K_5)$ can not be a minor of $M(\bar{G}) \cong M(K_{3,5}/Y_1)$ or $M(\hat{G}) \cong M(K_{4,4}/Y_2)$. For the remaining case, that is, if Y_1 or Y_2 has more than four elements, one can use a similar argument to the one followed in order to prove that $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_{3,3})$ and for that reason is omitted. □



$\bar{G}$ $\hat{G}$

Figure 7.2: The graphs $\bar{G}$ and $\hat{G}$.

Lemma 7.6. *If N is a minor of a matroid M then for any cocircuit C_N of N there exists a cocircuit C_M of M such that $N\backslash C_N$ is a minor of $M\backslash C_M$.*

Proof: We have that $N = M\backslash X/Y$, for some disjoint $X, Y \subseteq E(M)$. By the definitions of deletion and contraction given in (1.3) and (1.4), we have that for any cocircuit C_N of N there exists a cocircuit C_M of M such that:

(i) $C_N \subseteq C_M$, and

(ii) $E(N) \cap C_M = C_N$,

which in turn imply that $C_M - C_N \subseteq X$. Therefore, $M\backslash X$ is a minor of $M\backslash\{C_M - C_N\}$ and since N is a minor of $M\backslash X$ we obtain that N is a minor of $M\backslash\{C_M - C_N\}$. By

$$M\backslash C_M = M\backslash\{C_M - C_N\}\backslash C_N$$

and the fact that N is a minor of $M\backslash\{C_M - C_N\}$ we have that $N\backslash C_N$ is a minor of $M\backslash C_M$. □

We are now ready to prove the main result of this section.

Theorem 7.7. *Let M be a regular matroid such that all its cocircuits are graphic. Then, M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$.*

Proof: The "only if" part is clear because of Theorem 7.3. For the "if" part, by way of contradiction, assume that M is not signed-graphic. By Theorem 7.3, M must contain a minor N which is isomorphic to some matroid in the set

$$\mathcal{M} = \{M^*(G_1), \ldots, M^*(G_{16}), M^*(G_{18}), M^*(G_{20}), \ldots, M^*(G_{29}), R_{15}^*, R_{16}^*\}.$$

By case analysis, verified also by the MACEK software [40], it can be shown that for each matroid $M' \in \mathcal{M}$ there exists a cocircuit $Y' \in \mathcal{C}(M'^*)$ such that the matroid $M'\backslash Y'$ does contain an $M^*(K_{3,3})$ or an $M^*(K_5)$ as a minor. Thus, by Theorem 1.18, there exists a cocircuit $Y_N \in \mathcal{C}(N^*)$ such that $N\backslash Y_N$ is not graphic. Therefore, by Lemma 7.6, there is a cocircuit $Y_M \in \mathcal{C}(M^*)$ such that $N\backslash Y_N$ is a minor of $M\backslash Y_M$. Thus, $M\backslash Y_M$ is not graphic which is in contradiction with our assumption that M has graphic cocircuits. □

Based on Theorem 7.7, we can provide some sufficient conditions for a regular matroid to be signed-graphic.

Theorem 7.8. *Let M be a regular matroid. If*

(i) all the cocircuits of M are graphic, and

(ii) for any circuit $X \in \mathcal{C}(M)$, the matroid M/X is graphic

then M is signed-graphic.

Proof: By Theorem 7.7, enough to show that if condition (ii) is true then M does not have a minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$. Assume by contradiction that M contains a minor $N \in \{M^*(G_{17}), M^*(G_{19})\}$ and conditions (i) and (ii) hold for M. It can be easily shown that N has a circuit $C_N \in \mathcal{C}(N)$ such that N/C_N is not graphic. Using the dual version of Lemma 7.6, there exists a circuit $C_M \in \mathcal{C}(M)$ such that M/C_M is not graphic, which is a contradiction. □

By simple duality arguments and the fact that regular matroids are closed under duality, we obtain the following corollary from Theorem 7.8, which gives sufficient conditions for a regular matroid to be both signed-graphic and cosigned-graphic.

Corollary 7.9. *If M is a regular matroid such that*

(i) for any cocircuit $Y \in \mathcal{C}(M^*)$, the matroid $M\backslash Y$ is graphic and cographic, and

(ii) for any circuit $X \in \mathcal{C}(M)$, the matroid M/X is graphic and cographic

then M and M^* are signed-graphic.

7.3.2 Non-Graphic Cocircuits

The following technical lemma is necessary for the proof of Theorem 7.11.

Lemma 7.10. *Let Y be an unbalanced bond or a star bond of a tangled signed graph Σ such that the core of $\Sigma\backslash Y$ is not a necklace and let $\Sigma\backslash.B$ be an unbalanced separate of $\Sigma\backslash Y$. Then:*

(i) $M(\Sigma)$ is graphic, or

(ii) there exists a series of switchings at the vertices of Σ such that all the edges of the separates of $\Sigma\backslash Y$ other than $\Sigma\backslash.B$ become positive and for any $v_i \in V(\Sigma\backslash.B)$ such that $Y(B, v_i) \neq \emptyset$, the edges of $Y(B, v_i)$ have the same sign.

Proof: Let $\bar{V}_B := \{v_i \in V(\Sigma\backslash.B) \mid Y(B,v_i) \neq \emptyset\}$. By Theorems 5.38 and 5.39, we have that $\Sigma\backslash Y$ consists of two components Σ_1 and Σ_2 and that there is exactly one unbalanced block in $\Sigma\backslash Y$, which without loss of generality we assume that it is contained in Σ_1. Since this unbalanced block is not a necklace, its edge-set is a bridge B of Y in $M(\Sigma)$ and therefore, $\Sigma\backslash.B$ is a separate of $\Sigma\backslash Y$. By Proposition 5.28, there exists a series of switchings at the vertices of the balanced subgraphs $C(B,v_i)$ (for all $v_i \in V(\Sigma\backslash.B)$) and at the vertices Σ_2 such that all the edges in $\Sigma_1\backslash B$ and Σ_2 become positive. We call Σ', Σ_1' and Σ_2' the signed graphs so-obtained from Σ, Σ_1 and Σ_2, respectively, by applying these switchings.

For each $v_i \in V(\Sigma'\backslash.B)$ let $Y^+(B,v_i)$ and $Y^-(B,v_i)$ be the positive and negative edges of $Y(B,v_i)$, respectively, and

$$V_B := \{v_i \in \bar{V}_B \mid Y^+(B,v_i), Y^-(B,v_i) \neq \emptyset\}.$$

Suppose that $|V_B| \geq 2$, and let $V_B = \{v_1,\ldots,v_k\}$ for some positive integer $k \geq 2$. Since $\Sigma_1'\backslash B$ and Σ_2' consists of only positive edges, the signed graph $\Sigma'/.(B\cup Y)$ is obtained from Σ' by contracting every component $C(B,v_i)$ of $\Sigma_1'\backslash B$ to v_i and by contracting Σ_2' to a single vertex, which we shall call u. For example, Figure 7.3 depicts the graph $\Sigma'/.(B\cup Y)$ obtained from Σ' where the dashed lines indicate the edges of Y. Therefore, in $(\Sigma'/.(B\cup Y))\backslash.Y$, each $Y(B,v_i)$ with $v_i \in V_B$ will become a class of parallel edges and all the edges in Y will have u as an end-vertex (see Figure 7.3). Thus, the following collection $\mathcal{L}$ is a collection of bonds of $(\Sigma'/.(B\cup Y))\backslash.Y$:

$$\mathcal{L} = \{\{Y^-(B,v_1),\ldots,Y^-(B,v_k)\},\{Y^+(B,v_1),\ldots,Y^+(B,v_k)\},\{Y(B,v_1)\},\ldots,\{Y(B,v_k)\}\}.$$

Therefore, by Theorems 5.18 and 5.21 and Lemma 5.24, $\mathcal{L}$ is a set of cocircuits of $M((\Sigma'/.(B \cup$

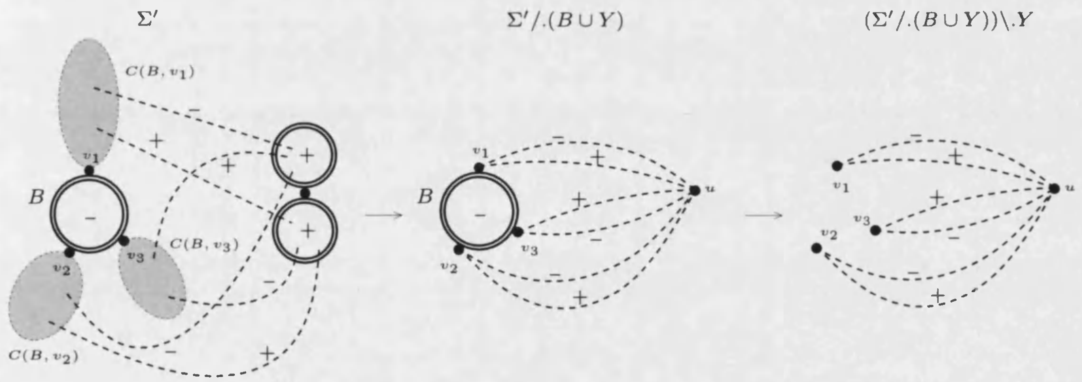


Figure 7.3: The signed graphs Σ', $\Sigma'/.(B\cup Y)$ and $(\Sigma'/.(B\cup Y))\backslash.Y$.

$Y))\backslash.Y) = (M(\Sigma')/.(B\cup Y))\backslash.Y = (M(\Sigma)/.(B\cup Y))\backslash.Y$. Recall that binary signed-graphic matroids are regular and therefore, by Corollary 7.2, we have that:

$$\mathcal{C}^*((M(\Sigma)/.(B\cup Y))\backslash.Y) = \pi(M(\Sigma),B,Y).$$

But we know by (7.3) in [80], that the members of $\pi(M(\Sigma),B,Y)$ should be disjoint which is a contradiction. Therefore, $|V_B| < 2$.

Suppose now that $V_B = \{v\}$. If we assume that there exist $y_1 \in Y^-(B,v)$ and $y_2 \in Y^+(B,v)$

such that both y_1 and y_2 do not have v as an end-vertex then the negative cycle N formed by these two edges and the positive paths between their end-vertices in $C(B, v)$ and Σ'_2, respectively, is vertex disjoint with some negative cycle in the unbalanced separate $\Sigma' \backslash .B$. This is in contradiction with the fact that Σ' is tangled and, therefore, all edges of $Y^-(B, v)$ or all edges of $Y^+(B, v)$ have v as an end-vertex in Σ'. Furthermore, since N has no vertex in $\Sigma \backslash .B$ other than v we have that any negative cycle in $\Sigma \backslash .B$ must be adjacent to v; otherwise two vertex disjoint negative cycles are contained in Σ' which is in contradiction with the fact that Σ' is tangled.

Assume that C is a negative cycle in Σ' not adjacent to v. Since C is not contained in $\Sigma \backslash .B$ it contains edges from Y and, furthermore, since Y is an unbalanced bond the number of these edges has to be an even number. Say that these edges are those contained in $Y_C = \{y_1, y_2, \ldots, y_{2n}\}$. Arrange the edges in Y_C, each of which has one end-vertex in Σ'_1 and one in Σ'_2, as shown in Figure 7.4, where the dashed lines are the n paths between the end-vertices of the edges of Y_C in Σ'_1 and Σ'_2. Since C is negative there exist vertices w_i and w_{i+1} of C in Σ'_2 such that the path $(w_i, y_i, u_i, \ldots, u_{i+1}, y_{i+1}, w_{i+1})$ is negative. Assume

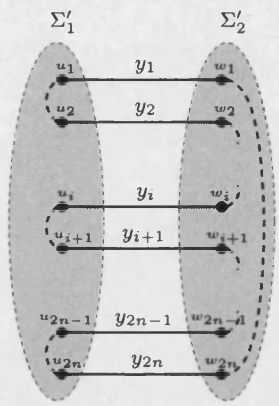


Figure 7.4: An arrangement of the edges of Y_C.

that $y_i, y_{i+1} \in Y(B, v_l)$ for some $v_l \in \bar{V}_B$. Then $v_l \neq v$, since C is not adjacent to v. If $v_l \in \bar{V}_B - \{v\}$, then the path between u_i and u_{i+1} in $C(B, v_l)$ is positive or empty, which implies that y_i and y_{i+1} are of different sign. This in turn implies that $|V_B| > 1$. Therefore, there exist vertices v_1 and v_2 in $\bar{V}_B$ such that $v_1 \neq v_2 \neq v$ and $y_i \in Y(B, v_1)$ and $y_{i+1} \in Y(B, v_2)$ Contract the edges of $C(B, v_1)$ and $C(B, v_2)$ and switch at v_1 or v_2 if y_i and y_{i+1} have different signs, such that the path from v_1 to v_2 of C contained in $\Sigma' \backslash .B$ is positive and y_i and y_{i+1} have different sign. Moreover since this path, which we shall call $v_1 - v_2$ path, is positive we can perform switching at its vertices other than v_1 and v_2 to make all of its edges positive. Contract now the positive edges of the $v_1 - v_2$ path into a new vertex w and call Σ'' the graph so-obtained.

The effect of contracting the $v_1 - v_2$ path in $\Sigma' \backslash .B$ can be seen inductively if we consider the contraction of a single edge e from the path. Consider the graph $(\Sigma' \backslash .B)/e$, where $e = (w_1, w_2)$ is some positive edge from the $v_1 - v_2$ path and C' any negative cycle of $\Sigma' \backslash .B$, which we know that it is adjacent to v. If $e \in C'$, then $E(C') - e$ is the edge set of a negative cycle in $(\Sigma' \backslash .B)/e$ (recall that we have shown that C' is adjacent to v). If $e \notin C'$, then either C' is still a negative cycle in $(\Sigma' \backslash .B)/e$ or C' is not a cycle any more in the graph because it is not minimal. This is because the contraction of e creates two cycles

C_1' and C_2' in $(\Sigma'\backslash.B)/e$ whose edge-sets are contained in $E(C')$. Of the two only one is negative, and it has to be adjacent to v, since otherwise we have a negative cycle in $\Sigma'\backslash.B$ not adjacent to v. Therefore, since $w_1, w_2 \neq v$, the contraction of e into a vertex w in the 2-connected component $\Sigma'\backslash.B$ will create a 2-connected unbalanced component containing w and v and all the negative cycles of $(\Sigma'\backslash.B)/e$, and (possibly) 2-connected balanced components adjacent to w (see Figure 7.5). The 2-connected unbalanced component in $(\Sigma'\backslash.B)/e$ is not a necklace since $\Sigma'\backslash.B$ is not a necklace, i.e. the expansion of a vertex in a necklace results in a necklace.

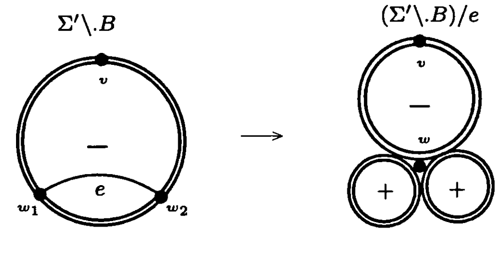


Figure 7.5: Contraction of a positive edge in an unbalanced block.

Therefore, Y is an unbalanced bond in a tangled signed graph Σ'' such that the core of $\Sigma''\backslash Y$ is not a necklace. $\Sigma''\backslash Y$ has an unbalanced block B' which contains vertices v and w, where

$$Y^+(B', v) \neq \emptyset \neq Y^-(B', v) \text{ and}$$

$$Y(B', w) = Y(B, v_1) \cup Y(B, v_2),$$

while y_i and y_{i+1} are of different sign and $y_i, y_{i+1} \in Y(B', w)$. But in this case $M(\Sigma'')$ is not binary, as shown above, which contradicts the fact that Σ'' is tangled. Therefore, our original hypothesis that there exists negative cycle C in Σ' not adjacent to v is false, which implies that v is a balancing vertex in Σ' and $M(\Sigma') = M(\Sigma)$ is graphic. □

The theorem that follows provides the graphical characterization of $\pi(M(\Sigma), B, Y)$ for a given cocircuit of a signed-graphic matroid.

Theorem 7.11. *Let $M(\Sigma)$ be a binary signed-graphic matroid and Y be a star bond or an unbalanced bond of Σ such that the core of $\Sigma\backslash Y$ is not a necklace. If $\Sigma\backslash.B$ is a separate of an end-graph Σ_i of $\Sigma\backslash Y$ then $\pi(M(\Sigma), B, Y)$ is the class of all $Y(B, v)$ such that $v \in V(\Sigma\backslash.B)$ and $Y(B, v) \neq \emptyset$.*

Proof: Let $\mathcal{L} = \{Y(B, v) \mid v \in V(\Sigma\backslash.B) \text{ and } Y(B, v) \neq \emptyset\}$. Since binary signed-graphic matroids are regular, by Corollary 7.2, we know that:

$$\pi(M(\Sigma), B, Y) = \mathcal{C}^*((M(\Sigma)/.(B \cup Y))\backslash.Y)$$

and thus, by Theorem 5.21, we have that:

$$\pi(M(\Sigma), B, Y) = \mathcal{C}^*(M((\Sigma/.(B \cup Y))\backslash.Y)).$$

Let $\mathcal{M}$ be the family of bonds of $\Sigma_b = (\Sigma/.(B \cup Y))\backslash.Y$. Since there is one-to-one correspondence between the members of $\mathcal{C}^*(M((\Sigma/.(B \cup Y))\backslash.Y)$ and the bonds of Σ_b, we shall equivalently show that, for any bridge B of Y in $M(\Sigma)$, $\mathcal{L} = \mathcal{M}$. Note that we shall show this only for the case in which Y is an unbalanced bond since the proof for the case in which Y is a star bond follows easily.

By Theorems 5.38 and 5.39, the signed graph $\Sigma\backslash Y$ will consist of two components Σ_1 and Σ_2 and will contain exactly one unbalanced block. Without loss of generality, we assume that this unbalanced block is contained in Σ_1. By Proposition 5.28, since $C(B, v)$ is balanced for any $v \in V(\Sigma\backslash.B)$ and Σ_2 is balanced, there exists a series of switchings at the vertices of $\Sigma_1\backslash B_1$ and Σ_2 such that all the edges in $\Sigma_1\backslash B_1$ and Σ_2 become positive. We call Σ', Σ'_1, Σ'_2 and Σ'_b the graphs obtained from Σ, Σ_1, Σ_2 and Σ_b, respectively, by applying these switchings. Figure 7.6 depicts an example signed graph Σ', where the dashed edges are the edges of the unbalanced bond Y.

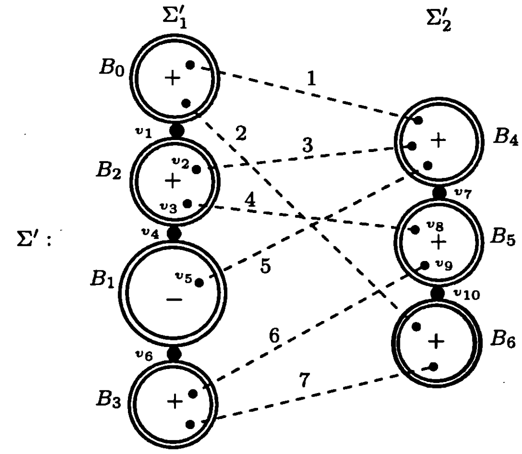


Figure 7.6: An example tangled Σ', where the dashed lines represent the edges of an unbalanced bond.

We classify the bridges of Y in $M(\Sigma') = M(\Sigma)$ in three categories based on the form of the corresponding separates in $\Sigma'\backslash Y$. Specifically, a bridge B of Y in $M(\Sigma)$ falls in:

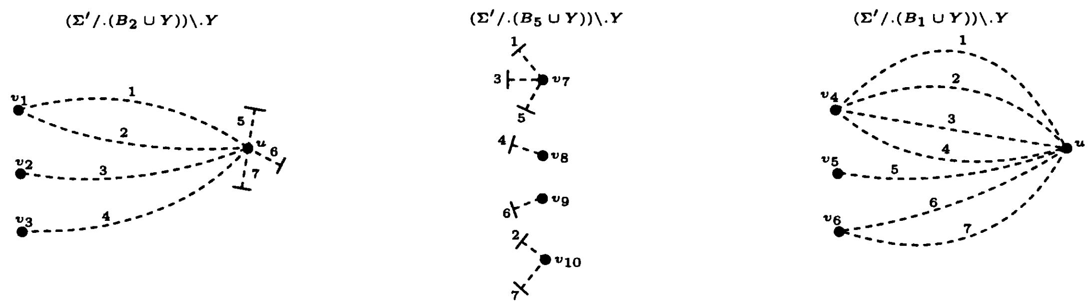
- **Category 1,** if the separate $\Sigma'\backslash.B$ of $\Sigma'\backslash Y$ is a balanced block in Σ'_1,
- **Category 2,** if the separate $\Sigma'\backslash.B$ of $\Sigma'\backslash Y$ is a balanced block in Σ'_2,
- **Category 3,** if the separate $\Sigma'\backslash.B$ of $\Sigma'\backslash Y$ is the unbalanced block in Σ'_1.

In what follows, for a bridge B of each category, we shall show that $\mathcal{L} = \mathcal{M}$. Thus, we have the following three cases.

Case 1: B is a bridge of Y in $M(\Sigma)$ which belongs to Category 1. Initially, we shall describe the effect of the series of contractions and deletions in Σ' resulting in Σ'_b. Let X be the set of common vertices of $\Sigma'\backslash.B$ and $\Sigma'_1\backslash B$. Clearly, there exists an $x_j \in X$ such that $C(B, x_j)$ contains the unbalanced block of Σ'_1. The signed graph $\bar{\Sigma}' = \Sigma'/E(\Sigma_2)$ is the graph obtained from Σ' by contracting Σ'_2 into a single vertex u and replacing the end-vertex of each edge Y in Σ'_2 by u. Furthermore, the signed graph $\Sigma'' = \bar{\Sigma}'/C(B, x_j)$, which contains Σ'_b as a minor, is obtained from $\bar{\Sigma}'$ by deleting $C(B, x_j)$ and replacing every edge in $Y(B, x_j)$ by a half-edge incident to u. In Σ'' we contract all $C(B, x_i)$ with $i \neq j$ and call Σ'_c the signed graph so-obtained. Then $\Sigma'_c = \Sigma'/.(B \cup Y)$ and every edge of $Y(B, x_i)$ with $i \neq j$ has one end-vertex being u and the other being x_i in Σ'_c. Thus, for any $v \in V(\Sigma'\backslash.B) - \{x_j\}$ such that $Y(B, v) \neq \emptyset$, $Y(B, v)$ is a set of parallel edges incident to u and v in $\Sigma'_b = \Sigma'_c\backslash B$, while all the edges in $Y(B, x_j)$ are half-edges incident with u in Σ'_b (see (a) in Figure 7.7). Furthermore, for any $v \in V(\Sigma'\backslash.B) - \{x_j\}$ such that $Y(B, v) \neq \emptyset$, the edges of $Y(B, v)$ must be of the same sign, since otherwise Σ' would have two vertex disjoint negative cycles contradicting the fact that Σ' is tangled. Thus, any $Y(B, v) \neq \emptyset$ is a bond of Σ'_b. This result and the fact that the signed graphs Σ_b and Σ'_b have equal classes of bonds imply that $\mathcal{L}$ is contained in $\mathcal{M}$. Finally, if Σ'_b had a bond which was not equal to some $Y(B, v)$ then it would have two bonds with a common element and thus, by Corollary 7.2, $M(\Sigma')$ would not be regular. By Theorem 5.42, this contradicts the fact that Σ' is tangled and thus, $\mathcal{L} = \mathcal{M}$.

Case 2: B is a bridge of Y in $M(\Sigma)$ which belongs to Category 2. Since Σ'_2 consists of positive edges and Σ'_1 contains a negative cycle, for any $v \in V(\Sigma'\backslash.B)$ such that $Y(B, v) \neq \emptyset$, $Y(B, v)$ will be a set of half-edges incident with v in $\Sigma'/.(B \cup Y)$. Thus, the edges of each $Y(B, v)$ will form a bond of Σ'_b (see (b) in Figure 7.7) which implies that $\mathcal{L}$ is contained in $\mathcal{M}$. Furthermore, Σ'_b has no other bonds, since otherwise it should have two bonds having at least one common edge. This would imply that $M(\Sigma'_b)$ would have two cocircuits which have a common element and thus, by Corollary 7.2, $M(\Sigma')$ would not be regular. By Theorem 5.42, this is in contradiction with the fact that Σ' is tangled and thus, $\mathcal{L} = \mathcal{M}$.

Case 3: B is a bridge of Y in $M(\Sigma)$ which belongs to Category 3. Since both Σ'_2 and $\Sigma'_1\backslash B$ consist



(a) Case 1 (b) Case 2 (c) Case 3

Figure 7.7: Three different cases

of positive edges, the graph $\Sigma'/.(B \cup Y)$ is obtained from Σ' by contracting Σ'_2 to a vertex u and by contracting each $C(B, v)$ (where $v \in V(\Sigma'\backslash.B)$) to v. Therefore, the edges of each $Y(B, v)$ become incident with u and v which implies that the edges of each $Y(B, v)$ are parallel edges in Σ'_b (see (c) in Figure 7.7). Furthermore, by Lemma 7.10 and since $M(\Sigma)$ is not graphic, each $Y(B, v)$ in Σ'_b consists of edges of the same sign. Thus, each $Y(B, v)$ is a bond of Σ'_b which implies that $\mathcal{L}$ is contained in $\mathcal{M}$. Finally, Σ'_b has no other bonds, since otherwise it should have two bonds having at least one common edge. This implies that $M(\Sigma')$ would have two cocircuits having a common element and thus, by Corollary 7.2, $M(\Sigma')$ would not be regular. By Theorem 5.42, this contradicts the fact that Σ' is tangled and thus, $\mathcal{L} = \mathcal{M}$ $\square$

It turns out that star bonds or unbalanced bonds whose deletion does not result in the formation of a necklace, are always bridge-separable in the corresponding signed-graphic matroid.

Theorem 7.12. *Let Y be a cocircuit of a binary signed-graphic and non-graphic matroid $M(\Sigma)$. If Y is a star bond or an unbalanced bond of Σ such that the core of $\Sigma\backslash Y$ is not a necklace then Y is a bridge-separable cocircuit of $M(\Sigma)$.*

Proof: Let Y be a star bond or an unbalanced bond of Σ such that the core of $\Sigma\backslash Y$ is not a necklace. By Theorem 5.38, $\Sigma\backslash Y$ consists of two components which we shall call Σ_1 and Σ_2. We arrange the bridges of Y in $M(\Sigma)$ in two classes T and U such that a bridge B_i is in $T(U)$ if $\Sigma\backslash.B_i$ is a separate of $\Sigma_1(\Sigma_2)$. By way of contradiction, suppose that two bridges B_1 and B_2 of T overlap. Then $\Sigma\backslash.B_1$ and $\Sigma\backslash.B_2$ are separates of Σ_1. Thus, there exist vertices v_1 of $\Sigma\backslash.B_1$ and v_2 of $\Sigma\backslash.B_2$ such that $\Sigma\backslash.B_2$ is a subgraph of $C(B_1, v_1)$ and $\Sigma\backslash.B_1$ is a subgraph of $C(B_2, v_2)$. Furthermore, every vertex in $V(\Sigma_1)$ is a vertex of $C(B_1, v_1)$ or $C(B_2, v_2)$ and therefore, we have that $Y(B_1, v_1) \cup Y(B_2, v_2) = Y$. Thus, by Theorem 7.11, we can find some $K \in \pi(M(\Sigma), B_1, Y)$ and $J \in \pi(M(\Sigma), B_2, Y)$ such that $K \cup J = Y$. This is in contradiction with our assumption that B_1 and B_2 overlap and the result follows. □

As the next result demonstrates, balanced bridges of non-graphic cocircuits result in Y-components which are graphic matroids.

Lemma 7.13. *Let Y be a non-graphic cocircuit of a binary signed-graphic matroid $M(\Sigma)$ and B be a bridge of Y in $M(\Sigma)$. If $\Sigma\backslash.B$ is balanced then $M(\Sigma)/.(B \cup Y)$ is graphic.*

Proof: Since Y is a non-graphic cocircuit, $M(\Sigma)$ is not graphic and by Theorem 5.42 Σ is a tangled signed graph. Moreover, Y will be either a star or an unbalanced bond in Σ such that the core of $\Sigma\backslash Y$ is not a necklace. It suffices to examine the case where Y is an unbalanced bond.

Let B^+ be any bridge of Y such that $\Sigma\backslash.B^+$ is balanced, while B^- be the bridge corresponding to the unique unbalanced block of $\Sigma\backslash Y$. Perform switchings at the vertices of Σ such that all the edges in the balanced blocks of $\Sigma\backslash Y$ become positive. If B^+ is in the balanced component of $\Sigma\backslash Y$, contract any other balanced block to obtain $\Sigma/.(B^+ \cup B^- \cup Y)$ (see Figure 7.8). Contracting the edges of the unbalanced block B^-, where if an edge is negative switch at one of its end-vertices, will result in one or more negative loops since this block contains negative cycles. Therefore, if we contract these negative loops according to the definition of contraction in signed graphs given in section 5.1.1, we get the signed graph $\Sigma/.(B^+ \cup Y)$ where the only negative cycles are the half-edges of Y. Therefore, by Proposition 5.29, $M(\Sigma/.(B^+ \cup Y)) = M(\Sigma)/.(B^+ \cup Y)$ is graphic.

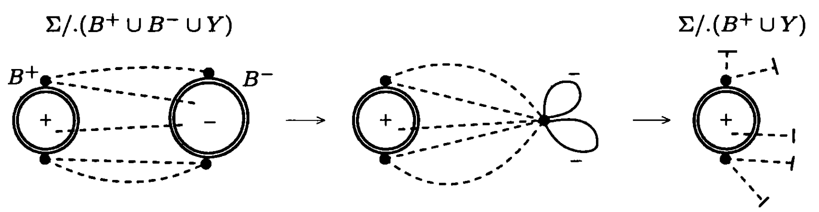


Figure 7.8: B^+ in the balanced component of $\Sigma\backslash Y$.

If B^+ is in the unbalanced component of $\Sigma\backslash Y$ the argument is similar. Contract again any other balanced block to obtain $\Sigma/.(B^+ \cup B^- \cup Y)$ (see Figure 7.9). Contraction now of the edges in the unbalanced block B^- may result in changing the sign of the edges in B^+ which are adjacent to the unique vertex of attachment v, while these edges will become half-edges upon deletion of the negative loops. Therefore, $\Sigma/.(B^+ \cup Y)$ will contain a balanced component $\bar{B}$, which is not necessarily 2-connected, and a number of half-edges from Y and B^+. If $\Sigma/.(B^+ \cup Y)$ contains a negative cycle C other than the half-edges, then C would be a negative cycle in Σ which is disjoint from v, and thereby vertex disjoint with any negative cycle in B^- implying that Σ is not tangled. Therefore, by Proposition 5.29, the matroid $M(\Sigma/.(B^+ \cup Y)) = M(\Sigma)/.(B^+ \cup Y)$ is graphic. □

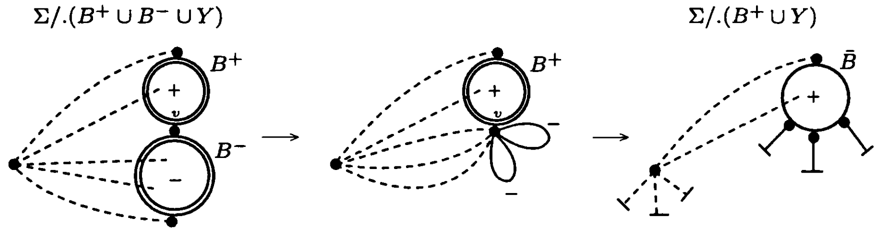


Figure 7.9: B^+ in the unbalanced component of $\Sigma\backslash Y$.

Theorem 7.14 is an extension of a result of Tutte in [80] (Theorem 8.4) regarding graphic matroids. It shows that given a signed graphic matroid M and some non-graphic cocircuit Y with no two overlapping bridges, there exists a signed graph representation of M where Y is the star of a vertex. It is an important structural result that will be used in the decomposition Theorem 7.15.

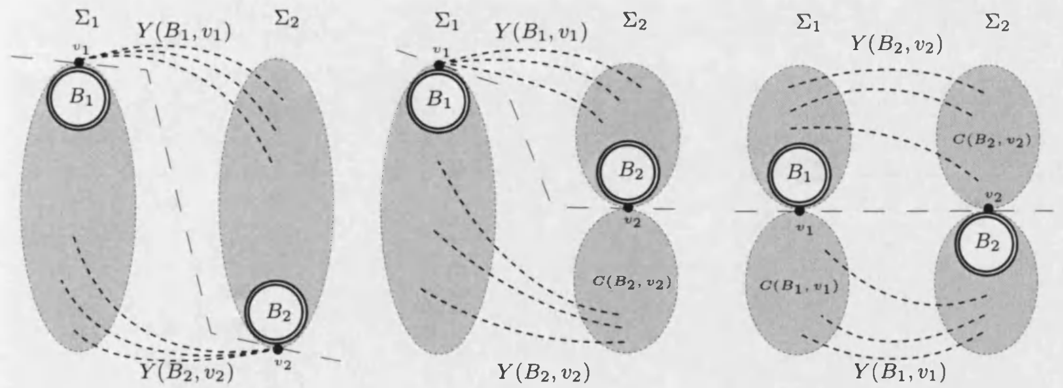
Theorem 7.14. *Let Y be a non-graphic cocircuit of a connected binary signed-graphic matroid M such that no two bridges of Y in M overlap. Then there exist a 2-connected signed graph Σ where Y is the star of a vertex $v \in V(\Sigma)$ and $M = M(\Sigma)$.*

Proof: By Theorem 5.42 there exists a tangled 2-connected signed graph Σ where $M = M(\Sigma)$ and Y is either a star bond or an unbalanced bond such that the core of $\Sigma\backslash Y$ is not a necklace. If Y is a star bond there is nothing to prove.

Let Y be an unbalanced bond of Σ, while Σ_1 and Σ_2 the two non-empty components of $\Sigma\backslash Y$, where one of them will contain the unique unbalanced block corresponding to bridge, say B^-, of Y (see Theorems 5.38 and 5.39). Furthermore, assume that we have performed switchings such that only Y and B^- may contain edges with negative sign. The main observation here, is that any two blocks in Σ_1 and Σ_2 which are bridges of Y in $M(\Sigma)$, define a 2-separation in Σ such that we can reverse on the defining vertices creating a signed graph with the same matroid. Consider any two bridges B_1 and B_2 of Y in $M(\Sigma)$, where $\Sigma\backslash.B_1$ and $\Sigma\backslash.B_2$ are blocks of Σ_1 and Σ_2 respectively. By Theorem 7.11 and the fact that B_1 and B_2 are avoiding bridges of Y, there exist some $v_1 \in V(\Sigma\backslash.B_1)$ and $v_2 \in V(\Sigma\backslash.B_2)$ such that

$$Y(B_1, v_1) \cup Y(B_2, v_2) = Y. \tag{7.1}$$

Let us call a vertex $v \in V(\Sigma\backslash.B)$ of a separate $\Sigma\backslash.B$ of $\Sigma\backslash Y$ for some bond Y a *vertex of attachment*, if the edge set of $C(B, v)$ is not empty. We can identify three cases (see Figure 7.10): (a) both v_1 and v_2 are not vertices of attachment, (b) one of v_1 and v_2 is a vertex of attachment and (c) both v_1 and



(a) No vertices of attachment (b) v_2 vertex of attachment (c) v_1 and v_2 vertices of attachment

Figure 7.10: 2-separations from non-overlapping bridges

v_2 are vertices of attachment. In all cases we can find a 2-separation in Σ as defined by v_1, v_2 and the relationship in (7.1). We will show that reversing Σ about $\{v_1, v_2\}$ produces a signed graph Σ' such that $M(\Sigma') = M(\Sigma)$.

Consider the case where both v_1 and v_2 are not vertices of attachment. The reversing parts of Σ about v_1 and v_2 are $\Sigma'_1 = \Sigma[E_1]$ and $\Sigma'_2 = \Sigma[(E(\Sigma) - E_1)]$ where $E_1 = E(\Sigma_1) \cup Y(B_2, v_2)$. Without loss of generality assume that $B^- \subseteq E(\Sigma_2)$. If Σ'_1 contains no negative cycles, then by Lemma 5.25 $M(\Sigma') = M(\Sigma)$. Let C_1 be a negative cycle in Σ'_1. Since the only negative edges of Σ'_1 appear in $Y(B_2, v_2)$ we will have that $E(C_1) \cap Y(B_2, v_2) \neq \emptyset$ which in turn implies that $v_2 \in V(C_1)$. Moreover, $B_2 = B^-$ because otherwise C_1 would be vertex disjoint with all the negative cycles contained in B^-, a contradiction since Σ is tangled. Now, Σ'_2 will contain a negative cycle C_2 such that $v_2 \notin V(C_2)$, otherwise v_2 would be a balancing vertex in Σ, contradicting the fact that $M(\Sigma)$ is a non-graphic matroid (see Proposition 5.31). Therefore C_2 is not contained in B^-, which implies that $E(C_2) \cap Y(B_1, v_1) \neq \emptyset$ and $v_1 \in C(C_1)$, otherwise C_1 and C_2 would be vertex disjoint. We can therefore conclude that any negative cycle in Σ'_1 contains the vertices v_1 and v_2, and by Lemma 5.25, $M(\Sigma') = M(\Sigma)$.

Assume now that only $v_2 \in V(\Sigma_2)$ is a vertex of attachment. The reversing parts of Σ about v_1 and v_2 are $\Sigma'_1 = \Sigma[E_1]$ and $\Sigma'_2 = \Sigma[(E(\Sigma) - E_1)]$ where $E_1 = E(\Sigma_1) \cup Y(B_2, v_2) \cup E(C(B_2, v_2))$. Let B^- be in Σ'_1, and consider any negative cycle C_1 in Σ'_2. If $B^- \subseteq E(C(B_2, v_2))$, then since $E(C_1) \cap Y(B_1, v_1) \neq \emptyset$ and $v_1 \in V(C_1)$, for C_1 not to be vertex disjoint with some negative cycle in B^- we must have that $v_2 \in V(C_1)$ also. If $B^- \subseteq E(\Sigma_1)$, then $B_1 = B^-$ and we have a similar case to (a). Alternatively, let B^- be in Σ'_2, and let C_1 be a negative cycle in Σ'_1. Then $E(C_1) \cap Y(B_2, v_2) \neq \emptyset$ and $v_2 \in V(C_1)$, while $B_2 = B^-$. In order for v_2 not to be a balancing vertex, we must have $v_1 \in V(C_1)$. Therefore in all cases C_1 contains both vertices v_1 and v_2 and by Lemma 5.25 $M(\Sigma') = M(\Sigma)$. The case where both v_1 and v_2 are vertices of attachment is similar to the previous cases.

In order to transform the underlying graph into a graph where Y is a star of a vertex, enough to observe that each component of $\Sigma \backslash Y$ is a tree of bridges, therefore we will always have at least one bridge with only one vertex of attachment. Therefore we can restrict ourselves to cases (a) and (b) only, and by a simple inductive argument it follows that by a series of the above mentioned reversings one of the components of $\Sigma \backslash Y$ can be monotonically reduced in size. $\square$

Theorem 7.15 (Decomposition). *Let M be a connected binary matroid and $Y \in \mathcal{C}^*(M)$ be a non-graphic cocircuit. Then M is signed-graphic if and only if:*

(i) Y is bridge-separable, and

ii) the Y-components of M are all graphic apart from one which is signed-graphic.

Proof: ($\Leftarrow$) Assume that M is signed-graphic. Since it is binary and not graphic, by Theorem 5.42 there exists a tangled signed graph Σ such that $M = M(\Sigma)$. Moreover, since $M\backslash Y$ is not graphic, Y cannot be a balanced bond or unbalanced bond of Σ such that $\Sigma\backslash Y$ contains a necklace. Therefore Y is either a star bond or an unbalanced bond such that $\Sigma\backslash Y$ does not contain a necklace, and by Theorem 7.12 we can conclude that Y is a bridge-separable cocircuit of M. By Theorem 5.39, $\Sigma\backslash Y$ will contain exactly one unbalanced block, say $\Sigma\backslash.B^-$ which is not a necklace, and k balanced blocks $\Sigma\backslash.B_i$, where k is a positive integer number. By Theorem 5.20, the edge-sets of these blocks are the elementary separators of $M(\Sigma\backslash Y) = M(\Sigma)\backslash Y$, and therefore the bridges of Y in $M(\Sigma)$. By Lemma 7.13, we have that $M(\Sigma)/.(B_i \cup Y)$ is graphic for each i. Now, since $M(\Sigma)/.(B^- \cup Y)$ is a minor of $M(\Sigma)$ it can be either signed-graphic or graphic. It cannot be graphic though since otherwise we would have a bridge-separable cocircuit of a connected binary matroid with all Y-components graphic, and by Theorem (8.5) of Tutte in [80], M should be graphic.

($\Rightarrow$) Assume by contradiction that there exists M and $Y \in \mathcal{C}^*(M)$ such that the "only if" part of the theorem is not true, and among these choose the one with the least $|E(M)|$.

If Y has only one bridge B, then $M = M/.(B \cup Y)$ and M is signed-graphic by assumption. Given that Y has more than one bridge and its bridge-separable, we partition the bridges of Y into two non-empty families L^- and L^+ such that no two members of the same family overlap. Furthermore, let $B^- \in L^-$, where B^- is the bridge of Y corresponding to the unique signed-graphic component $M/.(B^- \cup Y)$. Let $U^-, U^+ \subseteq E(M)$ be the unions of the members of L^- and L^+, respectively.

Let us now consider the matroids $M/.(U^- \cup Y)$ and $M/.(U^+ \cup Y)$. By Theorem (2) in [81] we know that $M/.(B \cup Y)$ is connected for any $B \in (L^+ \cup L^-)$. If S is a separator of $M/.(U^i \cup Y)$ ($i = +, -$), then there exists some $B \in L^i$, such that $S \cap (B \cup Y) \neq \emptyset$. By the definition of contraction operation, $S \cap (B \cup Y)$ would be a separator of $M/.(B \cup Y)$. We can therefore conclude that the matroids $M/.(U^- \cup Y)$ and $M/.(U^+ \cup Y)$ are connected. Moreover, by the definition of contraction, we have that Y is a cocircuit of $M/.(U^i \cup Y)$. By Theorem (8.53) in [82], we know that the bridges of Y in $M/.(U^- \cup Y)$ and $M/.(U^+ \cup Y)$ are the members of L^- and L^+, respectively, and $\pi(M/.(U^i \cup Y), B, Y) = \pi(M, B, Y)$ for all $B \in L^i$, which means that the bridges of Y are non-overlapping in both matroids. Moreover the Y-components of both $M/.(U^- \cup Y)$ and $M/.(U^+ \cup Y)$ are the Y-components of M, since $M/.(U^i \cup Y)/.(B \cup Y) = M/.(B \cup Y)$. We can therefore conclude, by Theorem (8.5) of [80], that $M/.(U^+ \cup Y)$ is a graphic matroid while $M/.(U^- \cup Y)$ is signed-graphic since it is smaller than M, and that Y is a cocircuit with non-overlapping bridges in both $M/.(U^+ \cup Y)$ and $M/.(U^- \cup Y)$.

By Theorem (8.4) in [80] there exists a 2-connected graph G^+ such that $M/.(U^+ \cup Y) = M(G^+)$ and Y is a star at a vertex say w^+. By Theorem 7.14 there exists a 2-connected tangled signed graph $\Sigma^- := (G^-, \sigma^-)$ such that $M/.(U^- \cup Y) = M(\Sigma^-)$ and Y is a star-bond at a vertex w^-. Construct now a signed graph $\Sigma := (G, \sigma)$ with $E(\Sigma) = E(M)$ and $V(\Sigma) = (V(G^+) \cup V(\Sigma^-)) - \{w^+, w^-\}$ as follows. The underlying graph G is obtained from the graphs G^+ and G^- by deleting vertices w^+ and

w^- and by adding, for any $y_i \in Y$, an edge y_i joining the end-vertex of y_i other than w^+ in G^+ with the end-vertex of y_i other than w^- in G^- (for an example see Figure 7.11). The sign function $\sigma(e)$ (where e is an edge of $E(\Sigma)$) will be:

$$\sigma(e) := \begin{cases} \sigma^-(e), & \text{if } e \in E(\Sigma^-), \\ +1, & \text{otherwise.} \end{cases}$$

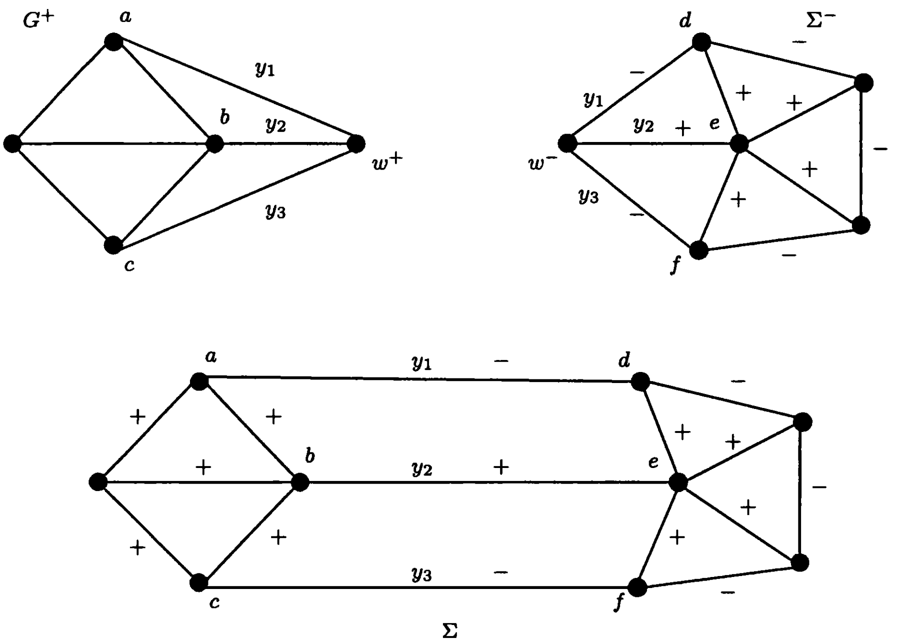


Figure 7.11: Construction of Σ from Σ^- and G^+; $Y = \{y_1, y_2, y_3\}$ is a bond of both Σ^- and G^+.

Since G^+ and G^- are 2-connected and Y is a star of a vertex in both, Y would be a minimal set of edges in Σ such that its deletion creates two components, namely $\Sigma[U^+]$ and $\Sigma[U^-]$. The component $\Sigma[U^+]$ contains only positive edges by construction, therefore it is balanced. If $\Sigma[U^-]$ did not contain a negative cycle it would imply that w^- is a balancing vertex in Σ^- which contradicts the fact that it is tangled. We therefore conclude that Y is an unbalanced bond in Σ .

Since $\Sigma[U^+]$ is balanced, we have

$$M(\Sigma)/.(U^- \cup Y) = M(\Sigma/.(U^- \cup Y)) = M(\Sigma/U^+) = M(\Sigma^-) = M/.(U^- \cup Y). \tag{7.2}$$

$\Sigma[U^-]$ contains at least one negative cycle, therefore Σ/U^- is a signed graph consisting of half-edges at the end-vertices Y contained in $V(\Sigma[U^+])$ and positive links. By Proposition 5.29 then

$$M(\Sigma)/.(U^+ \cup Y) = M(\Sigma/.(U^+ \cup Y)) = M(\Sigma/U^-) = M(G^+) = M/.(U^+ \cup Y). \tag{7.3}$$

Finally, given that both $M/(U^+ \cup Y)$ and $M/(U^- \cup Y)$ are connected, by a similar argument previously in the proof, $M(\Sigma)$ is also connected.

Now that we have established a relationship between M and $M(\Sigma)$ given by (7.2) and (7.3), by using a matroidal argument we will show that they are in fact equal. Consider the collection $\mathbb{C}$ which consists of the cocircuits of $M(\Sigma)$ being expressible as a symmetric difference of cocircuits of M (i.e. $\mathbb{C} := \{X \in \mathcal{C}^*(M(\Sigma)) \mid \exists\, X_i \in \mathcal{C}^*(M)$ such that $X = \triangle X_i\}$). Note that for $X_1, X_2 \in \mathbb{C}$ such that $X_1 \cap X_2 \neq \emptyset$, since $M(\Sigma)$ is binary we have that $X_1 \triangle X_2 \in \mathcal{C}^*(M(\Sigma))$ and $X_1 \triangle X_2 \in \mathbb{C}$.

Claim. *If $\mathcal{C}^*(M(\Sigma)) - \mathbb{C}$ is not empty then there exists an $X \in \mathcal{C}^*(M(\Sigma)) - \mathbb{C}$ such that $X - (X \cap Y)$ is a cocircuit of $M(\Sigma)\backslash Y$.*

Proof: We can assume that for all $X \in \mathcal{C}^*(M(\Sigma)) - \mathbb{C}$ we have $X \cap Y = \emptyset$, since otherwise by the deletion operation we have that $X - (X \cap Y) \in \mathcal{C}^*(M(\Sigma)\backslash Y)$. Choose such an X and assume that it is not a cocircuit of $M(\Sigma)\backslash Y$. Since $M(\Sigma)$ is connected, there exists a $T \in \mathcal{C}^*(M(\Sigma))$ such that $T \cap X \neq \emptyset$ and $T \cap Y \neq \emptyset$. Moreover, since $M(\Sigma)$ is binary $X \triangle T$ is a cocircuit of $M(\Sigma)$. If $X \triangle T$ or T does not belong to $\mathbb{C}$ the result follows. Therefore, both $X \triangle T$ and T belong to $\mathbb{C}$ and thus, the cocircuit $(X \triangle T) \triangle T = X$ belongs to $\mathbb{C}$, which is a contradiction.

By the above claim and the fact that U^- and U^+ are separators of $M(\Sigma)\backslash Y$ by construction, we can conclude that $X \subseteq (U^j \cup Y)$, where $j = +$ or $-$. Therefore since $X \in \mathcal{C}^*(M(\Sigma))$ we have that X is a cocircuit of $M(\Sigma)/.(U^j \cup Y) = M/.(U^j \cup Y)$, and a cocircuit of M. But since M is connected and binary this is a contradiction to the fact that $X \notin \mathbb{C}$. So for any cocircuit $X \in \mathcal{C}^*(M(\Sigma))$ we have

$$X = X_1 \triangle X_2 \triangle \ldots \triangle X_n$$

for $X_k \in \mathcal{C}^*(M)$ $(k = 1, \ldots, n)$. But in binary matroids the symmetric difference of cocircuits contains a cocircuit or it is empty, so we can conclude that there exists some $X' \in \mathcal{C}^*(M)$ such that $X' \subseteq X$.

Reversing the above argument we can also state that any cocircuit X' of M contains a cocircuit X of $M(\Sigma)$, and by Lemma (2.1.19) in [56] we have that $M = M(\Sigma)$ contradicting our original hypothesis. $\square$

There exist signed-graphic matroids which are not decomposable into smaller matroids as described in Theorem 7.15. Specifically, this happens when M is a binary signed-graphic matroid whose non-graphic cocircuits are all non-separating, in which case M will be called *star-based* signed-graphic. In the following theorem we provide a result regarding the structure of signed graphs whose signed-graphic matroid is star-based signed-graphic.

Theorem 7.16. *$M(\Sigma)$ is a connected star-based signed-graphic matroid if and only if Σ is a 2-connected tangled signed graph such that any non-graphic cocircuit $Y \in \mathcal{C}^*(M(\Sigma))$ is a star of Σ and $\Sigma\backslash Y$ is 2-connected, unbalanced and not a necklace.*

Proof: ($\Rightarrow$) Since $M(\Sigma)$ is binary and connected, by Theorem 5.40, Σ is tangled and 2-connected. Furthermore, any non-graphic cocircuit of $M(\Sigma)$ is non-separating and therefore, by Proposition 5.22, Y is a star bond of Σ. By Theorem 5.39, $\Sigma\backslash Y$ must have one unbalanced block. Moreover, $M(\Sigma)\backslash Y$ is connected since Y is non-separating and therefore, by Theorem 5.20, $\Sigma\backslash Y$ can not be a necklace or contain any other block except for the unbalanced one.
($\Leftarrow$) Σ is 2-connected and tangled and therefore, by Theorem 5.40, $M(\Sigma)$ is connected. Furthermore, any

non-graphic cocircuit Y is such that $\Sigma\backslash Y$ is 2-connected, unbalanced and not a necklace. Thus,by Theorem 5.20, $M(\Sigma)\backslash Y$ is connected and therefore, any non-graphic cocircuit of $M(\Sigma)$ is non-separating which implies that $M(\Sigma)$ is a connected star-based signed-graphic matroid. □

Therefore, Theorem 7.15 implies that a binary signed-graphic matroid with non-graphic cocircuits is decomposed into graphic matroids and one star-based signed-graphic matroid by a series of deletions of non-graphic cocircuits.

7.4 Concluding remarks

Future research

In this section we shall discuss the implications of the decomposition theory appearing in this chapter and also suggest some directions for future research. We believe that the decomposition theory for the class of binary signed-graphic matroids is the first major step towards the decomposition of other major classes of signed-graphic matroids. Therefore, the first *open question* is how we can use the insight gained by the decomposition of binary signed-graphic matroids in order to decompose other important classes of signed-graphic matroids (e.g. the $GF(4)$-representable signed-graphic matroids). Such decomposition results, apart from their structural and theoretical importance, will also have some important consequences.

Recognition algorithms for classes of signed-graphic matroids are expected to be the outcomes of such decomposition results, i.e. algorithms which will determine if a given matroid belongs to a specific class of signed-graphic matroids or not. We have already presented two recognition algorithms for the class of binary signed-graphic matroids in Chapter 6. Specifically, the BINARY RECOGNITION ALGORITHM in Chapter 6 determines whether a binary matroid is signed-graphic by utilizing a recognition algorithm for the class of binet matrices appearing in [54]. However, the binet recognition algorithm in [54] is very complicated and the need for a simpler and more elegant algorithm is also mentioned in that work ([54]). Therefore, a second *open question* is to find a recognition algorithm for binet matrices based on the decomposition results for signed-graphic matroids. We expect such a recognition algorithm for binet matrices to be simple and practical since it will be based on recognizing basic building blocks of binet matrices for which known fast methods are available. A well-known example which shows the power of matroid decomposition for providing recognition algorithms for the associated classes of matrices is the sole recognition algorithm for totally unimodular matrices which is based on the regular matroid decomposition theory of Seymour [66]. Another example is the recognition algorithm for network matrices appearing in [10] which is based on Tutte's decomposition results for graphic matroids. Moreover, we should note that although there were several polynomial recognition algorithms (for example, see [7, 35, 43, 44]) for network matrices, it was Tutte's decomposition results for graphic matroids which led to the first simple and practical algorithm for recognizing network matrices in [10].

Finally, it seems that the decomposition results for signed-graphic matroids will be a major step in decomposing other important classes of matroids such as the class of dyadic matroids (see section 2.2.2 for a definition) and the class of near-regular matroids which consists of the dyadic matroids representable over $GF(4)$. The importance of dyadic matroids stems mainly from a result of Whittle in [90] which states that a matroid is representable over $GF(3)$ and the rationals if and only if it is a dyadic matroid; note also that this is an analogous statement to an important result of Tutte stating that a matroid is representable over $GF(2)$ and the rationals if and only if it is regular [82]. Moreover, in [51, 92], Whittle et.al. have even conjectured that $GF(4)$-representable signed-graphic matroids may decompose near-regular matroids in a way similar to how graphic matroids decompose regular matroids in [66]. Thus, if such a decomposition for near-regular matroids and a recognition algorithm for the class of $GF(4)$-representable signed-graphic matroids do exist then a recognition algorithm for near-regular matroids would be implied. Therefore, more knowledge on the structure and decomposition of signed-graphic matroids is desirable.

Matroid Minors project and its impact on our work

During the last few years the most interesting ongoing project in matroid theory may have been the Matroid Minors project that is aimed at generalizing the results of the Graphs Minors project to matroids. The results of the Graph Minors project, published in a series of papers by Robertson and Seymour in the Journal of Combinatorial Theory Series B, are considered as groundbreaking and are probably the most profound results in graph theory until now. Clearly, analogous results for matroids are of great importance. There are two basic aims of the Matroid Minors project which are outlined in [26]; specifically, their goal is to prove that for every finite field $\mathbb{F}$: (i) any infinite set of $\mathbb{F}$-representable matroids contains two matroids, one of which is isomorphic to a minor of the other (well-quasi-ordering conjecture), and (ii) any minor closed property can be tested in polynomial time for $\mathbb{F}$-representable matroids. In early 2009, Gerards announced on his webpage that Geelen, Whittle and himself managed to prove (i) and (ii) for the finite field $GF(2)$. One of the implications of this result is that we can test in polynomial time whether a binary matroid contains another binary matroid as a minor. Although this work is not published yet, we shall try to assess its impact on our current work and future research.

An immediate consequence of this announced result on our work is that an alternative to our recognition algorithm for binary signed-graphic matroids (BINARY RECOGNITION ALGORITHM) will become available. Specifically, the binary excluded minors for signed-graphic matroids can be easily obtained by adding to the list of the 31 regular excluded minors of signed-graphic matroids (see Theorem 7.4) the binary excluded minors for regular matroids (i.e. F_7 and F_7^*), since any binary signed-graphic matroid is also regular. Thus, we can check if a binary matroid M belongs to the class of signed-graphic matroids by testing whether one of these 33 minors is a minor of M; this can be done in polynomial time due to the announced result of the Matroid Minors project. As we have mentioned in Chapter 6, not all signed-graphic matroids are representable over $GF(2)$ and therefore, this result of Matroid Minors project can not be used for the recognition of non-binary classes of signed-graphic matroids. Furthermore, although the general and powerful results of the Matroid Minors project may in the future provide decomposition results and polynomial recognition algorithms for classes of signed-graphic matroids (not necessarily binary), we think that specialized methods are needed. One main reason for which we need specialized methods is that we do not know how practical the implementetion of the algorithms derived from the Matroid Minors project is. This is also admitted in a recent work of people who are involved in the Matroid Minors project. Specifically, in [51], a conjecture regarding the decomposition of near-regular matroids into signed-graphic matroids, the duals of signed-graphic matroids and a finite set of other matroids is provided. The authors of [51] say that although the Matroid Minors project when finished should imply a decomposition of near-regular matroids, this decomposition will rely on several technicalities and for that reason they believe that specialized methods will give much more refined results.

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General Mathematical Notation

Graph Notation

Matroid Notation

Index