Dynamic Models of Exchange Rates

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Abstract of Thesis

The thesis starts by describing those aspects of economic theory which are relevant to the construction of dynamic models of exchange rates. The deterministic flexible price monetary model, and the deterministic price inertia (Dornbusch) model are then derived for two countries of similar size. The implied restrictions on the economic parameters of the models due to no-arbitrage considerations when three countries are connected are also investigated. A variation of the price inertia model in which the relevant economic variables and the price adjustment procedure, are in current value terms rather than traditional volumes, is then investigated. The resulting systems of ordinary differential equations are solved for the nominal and real exchange rate, and the implied limiting values for these exchange rates are considered.

The thesis subsequently describes the extension of the flexible price monetary model and the price inertia model, to formulations containing a stochastic variable. The differential equations describing the models are derived using stochastic calculus, and quantitative and qualitative solutions are obtained for the two models, respectively. The thesis then shows how the stochastic models can be used to describe the movement of the exchange rate in a target zone.

Finally, an extension of the stochastic flexible price monetary model is considered for a small network of more than two countries. The situation in which some countries in the network have a target zone imposed on their exchange rate while others have freely floating rates is considered. In particular, the solution describing the movement of the exchange rate between two countries, one of which is participating in a target zone arrangement with a third country, is derived. These models are then extended to cover the situation of a small network of countries in which one of them has a central position compared with the others.

Contents

Introduction

The aim of this thesis is to provide a critical exposition and assessment of the literature relevant to the formulation of the two main mechanisms for describing exchange rate movements, the monetary and price inertia models. After discussing the basic economic framework in Chapter 1, we will continue in Chapter 2 to present a coherent derivation of the deterministic versions of the flexible price monetary model and of the price inertia model. Both models will be formulated in their most general terms for two countries of similar size, rather than one domain being considered as 'the rest of the world' which is a simplifying approach adopted in most standard texts. Explicit solutions for how the exchange rate moves with time will be obtained under certain conditions, and the important overshooting phenomenon of the price inertia model will be discussed. In addition, the existing results will be extended by examining what, if any, restrictions are implied for the parameters of the models, if networks of more than two countries are considered.

In most standard texts it is assumed that in the derivation of the price inertia model the macroeconomic parameters are the same for the two countries considered. However, in Chapter 3, it will be shown that tractability of the analysis is little changed if different macroeconomic parameters are assumed. In addition, the Chapter shows that with an alternative formulation of the basic equations in nominal, rather than in the more usual real terms, a range of limiting behaviour of the exchange rate with time can be deduced. These new results indicate that by incorporating such changes into the formulation of the model it is possible to obtain a mechanism for explaining why observed exchange rates have a greater volatility than the fundamental variables, such as money supply, upon which they are assumed to depend. As such, the mechanism offers an alternative to the overshooting effect of the price inertia model for describing this phenomenon.

Chapter 4 contains an exposition of the relatively new literature concerned with the stochastic versions of the monetary and price inertia models. For the monetary model it is shown how, with the aid of stochastic calculus, an explicit solution can be obtained for the movement of the exchange rate in relation to the fundamental variables of the model. In the case of the price inertia model, although an explicit solution cannot in general be obtained, the qualitative nature of the solution paths can be described. In the following Chapter we apply both stochastic models to the description of how the exchange rate moves in a target zone, as this was one of the main motivating reasons for the development of stochastic models by researchers such as Krugman. Chapter 5 also considers the relevant boundary conditions needed to obtain solutions, a topic which has become an important subject in the literature because of its relationship to areas of research in finance.

In Chapter 6, the approach of Jørgensen and Mikkelsen to applying stochastic target zone models to networks having a third country not involved in the target zone is discussed. This approach is then extended to cover the new situation of networks containing multiple target zones, and it is shown how explicit solutions for exchange rate movements between countries in different zones can be obtained. Finally, the previously unconsidered situation of a network with one country having a central role is discussed. The analysis indicates that it is possible to have target zones between each non-central country and the country with a central role, without potential conflict, provided all interventions are performed by the non-central countries. The result is of interest because it gives some justification for the non-intervention policy of Germany in the 'black Wednesday' crisis when the UK came out of the ERM.

1 Economic framework

1.1 The exchange rate

This chapter aims to define a number of the most important variables and their relationships, which are referred to in later sections of the thesis. Many of the descriptions and derivations are given in fairly 'non-technical' terms, since the economic theory and justification behind those relationships is not the primary aim of this thesis. However, some rationale of the 'building blocks' which go in to the formulation of exchange rate models is important in order to understand the nature of the models and the interpretation of results stemming from their use. As a first step, we need to consider precisely what we mean by an exchange rate. The exchange rate between two countries can be defined in various ways but the conventional means of doing so is as follows. The **exchange rate**, denoted by S_{12}, is the price of a unit of country 2's currency, measured in units of country 1's currency. For example, if country 1 is the United Kingdom and country 2 is the United States of America, then S_{12} would be the number of pounds Sterling per dollar. Of course, the exchange rate between the two countries may vary over time unless efforts are made to keep it fixed, and even then the rate will inevitably change at some point, as it has done in all 'fixed' exchange rate regimes. Therefore, the exchange rate can be considered as a function of time t, that is, more formally the exchange rate should be written as $S_{12}(t)$. In particular, and unless otherwise stated, $S_{12}(t)$ will be the rate for exchanging money at the moment of time t, that is the so called spot exchange rate. Other interpretations of an exchange rate are possible, such as the rate for exchanging money in say, time t plus a year, which would be a forward exchange rate. In general, unless we need to emphasise the dependency on time, we shall simply use the notation S_{12} for the rate. The dependency of the exchange rate on time is not purely arbitrary, since many variables could affect its movement, and in various ways. In Chapter 2 we shall discuss two of the most important models which attempt to describe their relationship.

1.2 Purchasing power parity

Suppose that we consider two goods which we shall label A and B, that are traded *within* a certain country, but produced in possibly different locations. Let $P^A(t)$ and $P^B(t)$ be the price at time t, for which they are traded in the locations in which they are produced. Again, unless we need to emphasise the dependency on time, t will be suppressed in what follows. Let us consider the situation when these two goods are perfect, or at least close, substitutes. If product A was available for sale at, say, £5 less than product B, then there would be an incentive for some entrepreneur to purchase as many items of product A as they could afford and then resell them at (or slightly below) the price at which product B was selling. Since the two products are close or perfect substitutes, consumers would have no hesitation in preferring to purchase product A over product B. This would have the effect of driving up the price of product A, by increasing its demand; driving down the price of product B, and, in the process, make a profit for the entrepreneur. The incentive to continue the process will cease when the two prices are equal, as there will then be no profit. This result is known as **the law of one price**, and is usually stated as follows; if two goods are identical, they must sell for the same price. In addition, the process of exploiting the price differential is known as arbitrage. It may be defined as follows. **Arbitrage** is the process of buying or selling something in order to exploit a price differential so as to make a riskless profit.

In the present context, the 'something' which is being bought and sold is a good or service, however, it may also be assumed to apply to earnings from securities. The implications of this latter assumption will be discussed when we consider Uncovered Interest Rate Parity, later in this chapter. Returning for the present to our example of the law of one price; in the derivation of this result it was implicitly assumed that there would be no other costs influencing our entrepreneur. However, in practice there may well be extra expenses, such as transport costs associated with obtaining and moving product A to the current purchasers of product B. These extra costs are what we shall refer to as **transaction costs,** and may be defined as all the costs associated with a transaction, over and above the cost of the good which actually changes hands. If we consider our example of two products again but now with the possibility of transaction costs being involved, it is not difficult to see that arbitrage will cease provided that

$$P^B - P^A \leq c$$

where c is the total transaction costs of supplying a unit of product A in place of product B. The maximum selling price for product B can therefore be considered to be

$$P^B = P^A + c. \tag{1}$$

Suppose now that the two products are produced in different countries but that trading between these two countries is allowed. Let us assume that product A is produced in country 2 and sells domestically at a price P_2, while product B is produced in country 1 and sells in that country at a price P_1. Let us further assume, for the present at least, that the transaction costs in supplying product A in place of product B are negligible. Then applying the law of one price to trade in these substitutes, we would obtain

$$P_1 = S_{12}P_2 \tag{2}$$

where S_{12} is the exchange rate between the two countries, defined as in the earlier section, so $S_{12}P_2$ would represent the selling price in country 2 in terms of country 1's currency. As the two goods are assumed perfect or close substitutes we have suppressed the labels for A and B. As the same consideration holds for all tradeable items between the two countries a similar equation could be justified for all of them. Hence, if we further assume that the weights used in compiling each country's domestic price indices are identical, then the above equation would continue to hold, where P_1 and P_2 now represent the respective price indices in the two countries. This result amounts to the following proposition; the general level of prices, when converted to a common currency, will be the same in every country. This assertion is usually referred to as the doctrine of purchasing power parity (PPP), in its absolute form. It was assumed in the above derivation that transaction costs were negligible. This is justified in the economic literature in a number of ways. Firstly, for many individual products it may well be that transport and other transaction costs are small enough, relative to selling prices, to be ignored for practical purposes. Secondly, even if transaction costs are not negligible, it may be argued that they vary, at least approximately, in proportion to the value of the goods in question. For instance, tariffs, which are charges imposed by one country on the imports of another, are often charged *ad valorem.* The transportation costs of some products are frequently related to the value of goods shipped, and items like insurance premiums will almost invariably bear some relation to the unit selling price.

If it is assumed that all such costs are proportional to price, then the application of the law of one price to trade in our two products A and B, would result in the following modified relationship

$$P_1 = kS_{12}P_2$$

where k is a positive constant. If this proportional increase for transaction costs is similar for all products, then this equation would continue to apply when we aggregate over all tradeable goods. Under these circumstances, transaction costs have the same effect as a once-over rescaling of prices. The functional form of the relationship between prices, as predicted by PPP, is therefore preserved. Finally, if neither of the above assumptions is acceptable, it may be postulated that absolute PPP, or some variation of it, holds only in the long term, with deviations allowed in the short term. This is one of the assumptions in the formulation of the price inertia model which will be discussed in a later chapter.

Reliance upon the international application of the law of one price is not however the only justification for the doctrine of PPP. Copeland (1994), gives an alternative, macroeconomic approach. Suppose, instead of the prices of individual goods and services being equated internationally, it is the general price levels themselves which are considered. Let us assume that each national price level is determined by that country's macroeconomic policy, in much the same way it would be if the country was not engaged in international trade. Then, given these independently determined price levels, the exchange rate is postulated to move in such a way as to satisfy the PPP relationship. One reason why this might occur, could be that the pressures from international trade might encourage similar costs of production between countries, at least in some overall sense. Taking the general level of prices as a measure of the costs of production in the country, we could compare the costs in another country with our own, by adjusting their price index by the exchange rate. If those costs of production are the same then it follows that the PPP relationship holds.

If the above argument is correct, then a number of consequences follow. Firstly, the argument amounts to an assertion that the exchange rate is determined by the following equation

$$P_1(t) = S_{12}(t)P_2(t), \tag{3}$$

where P_1 and P_2 are indices of the general level of prices in the two countries. This, as will be described in later chapters, is one of the main assumptions behind the so called monetary model.

Secondly, it follows that this scenario presupposes a floating exchange rate. In periods when the exchange rate is fixed, either PPP fails to hold, or at least the mechanism inducing the relationship between the variables is assumed to function in a different way.

Thirdly, in principle at least, this approach does not rely on the law of one price. However, it might still apply in a sense, even if there were wide divergences between countries in the price of individual goods and services. For instance, there may well be forces generating a broad equality between the cost of living (as measured by domestic price indices) in different countries. With this interpretation, we would not require equally weighted price indices, but it would now require that if there was a good which was overpriced (relative to that expected from PPP) which contributed say 1% to the weight of the domestic price index, then there would be another good, or goods, which were underpriced, and also contributed 1% to the index. Such requirements may seem unlikely, and, not surprisingly, the evidence for PPP holding on a continuous basis is not very convincing. Indeed, testing PPP in its absolute form is not easy. In particular, all published price indices have shortcomings, some theoretical because of their method of construction, and others relating to how well in practise, a particular index represents the overall level of prices as required in this context. It may be therefore, that the PPP relationship is valid but only with the 'true' price level variables which are *unobservable,* and only distantly related to what we actually observe using the published price indices.

As we have seen, if PPP could be relied upon to obtain at all times, competitiveness as measured in this way, would not only be constant, it would in some sense be equalised across different countries. No country would have a price advantage over another, at least in terms of the broad spread of goods and services represented by the general price level. As possible deviations from PPP play an important role in developing exchange rate models it will be useful to have a means of measuring such movements. The usual way of doing this, is through the concept of the so called real exchange rate, defined as follows. The **real exchange rate,** Q, is the price of foreign relative to domestic goods and services. It is calculated as

$$Q(t) = \frac{S_{12}(t)P_2(t)}{P_1(t)}, \tag{4}$$

where P_1 and P_2 are indices of the general level of prices in the two countries. If PPP held, then as noted above, the countries involved can be considered to be equally competitive. Deviations of the real exchange rate from unity are therefore measures of relative international competitiveness between the countries. For instance, values of Q less than 1, imply that country 2 has a competitive edge over country 1, at least in terms of the price of its goods, taken as a whole. Similarly, if Q is greater than 1, then country 1 can be considered to have the price advantage when selling goods.

1.3 Uncovered interest rate parity

The uncovered interest rate parity condition plays a major role in many models of exchange rate movements. Some detailed consideration of its derivation is therefore well justified. We shall suppose that an investor wishes to purchase a standard security for a fixed period, at the end of which they will receive interest on that investment. For convenience, we shall take the period of investment to be a year although the argument is readily adapted to shorter periods with the same results. We shall be concerned with the alternatives open to an investor, with a stock of wealth in liquid form available for such investments. In particular, the choices we shall be concerned with, will be between a deposit in a domestic bank, or similar institution, and deposits in a foreign bank. It shall be assumed that there is a perfect flow of capital between countries, that is, no barriers to international transfers. This in fact is fairly realistic as far as today's currency markets are concerned, at least for the developed world. We shall also assume that our investor is exempt from all taxes that might affect the choice of where to make a deposit.

Suppose that an investor has the opportunity to deposit 1 unit of domestic currency in a bank within his own country. At the end of a year he receives interest equal to the domestic going rate, denoted by r_1(assumed constant over the period) So at the end of the period, the investment will have 'grown' to

$$1 + r_1.$$

Alternatively, the investor has the opportunity to invest his 1 unit of currency in a foreign country (country 2). If the current exchange rate between the two countries is S_{12}, as defined in Section 1.1., then 1 unit of domestic currency will be worth $1/S_{12}$ units of currency in the foreign country. That is, if we assume that it is not possible to make a profit by converting money into another currency and back again, no-arbitrage considerations ensure that

$$S_{21} = \frac{1}{S_{12}}. \tag{5}$$

Hence, if the same amount of domestic currency is invested in the foreign country at an interest rate of r_2 (also assumed constant over the period), then at the end of the period, the investment will have 'grown' to

$$\frac{1 + r_2}{S_{12}}$$

in terms of country 2's currency. In order to compare the two investments, this return must be converted back in to units of domestic currency. At the beginning of the investment period it is not known what the exchange rate will be at the end of the period. So in order to enable a comparison to be made let us assume that our investor *anticipates* that by the end of the period the exchange rate will be S_{12}^a (the forecast being made at the start of the period of course). Therefore, the anticipated return on this investment, in domestic currency will be

$$\frac{(1 + r_2)S_{12}^a}{S_{12}}.$$

Now before comparing these two investment opportunities, we need to consider the risk involved with both investments. The investment in the domestic bank has no financial risk involved. It requires no forecasting of the exchange rate, or possible changes to the policy of a free movement in capital etc. which may change before the end of the period. The investment in the foreign country does involve these extra unknowns however, and so is far from riskless. In order to consider the possible effect of this on the investment strategy, we shall introduce the following concept. The **risk premium** is the reward, usually in the form of an anticipated excess return, that an economic agent (investor) gets in order to persuade him to bear risk.

Why do agents usually require an incentive in order to undertake risky projects? Economic theory assumes that for the most part, individuals prefer riskless to risky investments, other things being equal. This feature of preferences is what explains why risky investments usually yield a higher return than riskless ones. Not that this is always true. There are instances when agents seem content to run risks for no reward, or even in some cases to pay for the privilege of bearing risk. Therefore, when considering the returns from investments in differing countries it will be important to make clear the assumptions being made about the attitude of agents. As such we make the following definition. **Risk averters** are economic agents who require a (positive) risk premium in order to persuade them to hold risky assets. By contrast, **risk lovers** are willing to pay a premium for the privilege of bearing risk, while **risk neutral** agents are willing to do so in return for a zero risk premium.

Let us assume that economic agents are risk neutral. In general, then, for equilibrium (that is a situation where there is, on balance, no tendency for funds to move into or out of the domestic economy) we require that the return from depositing 1 unit of currency in a domestic bank, be just equal to the return from depositing the same amount (also in domestic currency) in the foreign bank. Hence we have

$$(1+r_1) = \frac{(1+r_2)S^a_{12}}{S_{12}}. \tag{6}$$

Suppose we let ΔS^a_{12} represent the *anticipated* rate of depreciation, defined as follows.

$$\Delta S^a_{12} = \frac{S^a_{12} - S_{12}}{S_{12}},$$

where the time interval in this case is one year. Rearranging this equation gives

$$\frac{S^a_{12}}{S_{12}} = 1 + \Delta S^a_{12}. \tag{7}$$

Using this relationship in equation (6) for the returns in the two countries, gives

$$(1 + r_1) = 1 + r_2 + \Delta S_{12}^a + r_2 \Delta S_{12}^a.$$

Now the cross-product term $r_2 \Delta S_{12}^a$, is made up of two rates; the rate of interest and the anticipated rate of depreciation of the currency. Provided the latter is not exorbitant (for example hyperinflation) this cross-product term will be of the second order, and can therefore be ignored. On so doing, we would have

$$\Delta S_{12}^a = r_1 - r_2. \tag{8}$$

This result is known as the **Uncovered Interest Rate Parity condition (UIP)**. In words it may be stated as; the domestic interest rate must be higher (lower) than the foreign interest rate by an amount equal to the anticipated depreciation (appreciation) of the domestic currency.

Although the above relationship was derived on the assumption that the investment period was for a year, the same argument can be applied to any period of time. In particular, as the period of time $\Delta t \to 0$, we obtain

$$\frac{S_{12}^a(t)'}{S_{12}(t)} = r_1(t) - r_2(t) \tag{9}$$

where the left hand side of the above equation now represents the anticipated instantaneous rate of change in the exchange rate. If s_{12} represents the natural log of the exchange rate, then using the standard result for the differential of the log of a variable, the above relationship can be written as

$$\frac{d(s_{12}^a)}{dt} = r_1 - r_2 \tag{10}$$

which is the form in which it is often encountered in standard texts. It is also worth noting that it is not difficult to show that the foregoing relationship holds exactly, rather than approximately, if interest is assumed to be compounded continuously.

In many standard texts, what we have referred to as the anticipated rate of change, is often called the expected rate of change. This implies that the markets anticipated change in the exchange rate is equal to the expected value of the change in a random variable, usually taken to be the actual exchange rate However, this step does require further assumptions, and at this stage is not required for a basic statement of UIP. Although the assumption of UIP is quite plausible the evidence for it holding is not overwhelming. This is partly due to the difficulty of testing any hypothesis about UIP independently from that of how the markets anticipated change in the exchange rate is formed. However, McCallum (1994) claims that the current evidence can be interpreted as being at least not inconsistent with UIP. Whatever the evidence, UIP remains a central assumption in many exchange rate models.

1.4 The goods sector in the open economy

In an open economy (that is, one which engages in international trade) the planned aggregate demand for goods and services in a given period is equal to those goods and services which are used within the country plus net exports to other countries. This is usually expressed as follows

$$\textit{planned demand} = C + I + G + B \qquad (11)$$

where C and I are the expenditure on consumption and investment, respectively, G is net government purchases of goods and services, and B is the excess of the country's exports over its imports. Here, all the components are measured in volume, that is real money terms. We may ask what factors will influence these components of aggregate demand. As far as consumption is concerned, an individual household's consumption is likely to increase as its own income rises. Aggregating across all households therefore, it seems reasonable to assume that other things being equal, total private sector consumption will depend positively on the volume of national income in the period, Y. Conversely, the higher the level of domestic interest rates, the more the incentive is to save rather than to consume. Household consumption therefore, can be assumed to be negatively related to the level of interest rates, r.

Investment spending, is assumed to be primarily that undertaken by the corporate sector, although similar considerations would apply to investment decisions in other sectors of the economy. Investment spending by the corporate sector will depend on a comparison of the cost of funds for the purchase of equipment and similar items, with the profits to be expected from the investment. Although this will involve many factors and uncertainties, based upon elementary economic considerations at least one conclusion seems firm. That is, other things being equal, the higher the interest rate in the economy, the greater the cost of capital, and hence the less likely it is that any given prospective investment will appear profitable to the decision maker. In general therefore, aggregate investment will also be negatively related to interest rates.

Government spending will be taken as exogenously given. That is, its level is considered to be determined outside the model by political and other factors. Its level is not necessarily constant, but can be taken as given, from a mathematical viewpoint, at any point in time.

Finally, one of the major factors the level of net exports for a country will depend upon is its relative competitiveness compared to other countries. Now as noted in the Section 1.2, relative competitiveness can be measured by Q, the real exchange rate. The higher the level of Q, the greater the competitive edge the domestic country has in terms of prices. In other words, the higher Q, the greater will be the domestic country's exports and smaller its imports. Hence, the level of net exports can be considered to be positively related to the real exchange rate. Of course, the real exchange rate is generally changing all the time, so either we must consider that it is the average real exchange rate (or some similar measure) over the period which is relevant, or that the period of time under consideration is sufficiently small to be of the same magnitude as the periods in which changes to Q are measured. The net level of exports may depend upon other factors in addition to the real exchange rate. In particular, exports are likely to depend upon the level of national output in the foreign countries. The higher their national output, the greater is likely to be their propensity to import goods, all other things remaining constant. However, for the time being we shall assume that either these factors are relatively unimportant compared with the real exchange rate, or at least their movements are likely to be relatively small over the periods we are considering.

Lastly, as in the standard texts on the topic, we may assume for simplicity that the dependency of aggregate demand on all components is linear, or more precisely, that the relationship can be adequately represented by the first term of its Taylor expansion. Therefore, taking all the above, postulated dependencies in to account, implies the following relationship for aggregate demand in the domestic country

$$\textit{planned demand} = \delta_1 Y - \gamma_1 r + G_1 + \frac{\eta_{12} S_{12} P_2}{P_1} \tag{12}$$

where δ_1, γ_1 and η_{12} are positive parameters. In addition, $\delta_1 < 1$, since total personal sector consumption of domestic goods and services cannot exceed their production, as measured by national income. In the above derivation, international trade has been assumed to take place with just one other country. An alternative interpretation could be that this represents net trade with the rest of the world in which case S_{12} and P_2 would represent some form of average exchange rate and price index. In theory, there are many possible candidates for what should be used to measure the level of domestic interest rates r; bond yields, returns on shares etc. However, as Copeland(1994) points out, in practise the returns on different assets are highly correlated, so it is enough to choose one interest rate, such as the yield on Treasury Bills, and use that as *the* interest rate for these discussions.

Now equilibrium in the goods market is defined to mean equality between planned aggregate demand and the aggregate supply of goods and services in the economy, over the period of time. Aggregate supply can be taken as synonymous with the national income generated by the production and sale of those goods and services, that is actual demand Y. Equilibrium in the goods market is therefore equivalent to the condition

$$\textit{actual demand} = \textit{planned demand}.$$

Therefore, substituting for the relationship for planned demand derived in equation (12) we obtain the following

$$Y = \delta_1 Y - \gamma_1 r + G_1 + \frac{\eta_{12} S_{12} P_2}{P_1}. \tag{13}$$

Rearranging equation (13), we obtain

$$\beta_1 Y + \gamma_1 r - \frac{\eta_{12} S_{12} P_2}{P_1} = G_1 \tag{14}$$

where $\beta_1 = 1 - \delta_1$. This relationship is usually referred to as the IS equation in standard economic texts. In most exchange rate models utilising the IS relationship or the theory of aggregate demand from which it is derived, the equations are given in log-linear form. Hence, the IS relationship is usually expressed as follows

$$\beta_1 y + \gamma_1 r - \eta_{12}(s_{12} + p_2 - p_1) = g_1 \tag{15}$$

where all variables, except the interest rate, are now in natural logs. The theoretical justification for this form does not seem much stronger than that for a straightforward linear relationship. However, it does have the advantage of being simpler to handle mathematically in many exchange rate models, which no doubt explains its widespread use.

1.5 The money sector in the open economy

By holding a proportion of their wealth in the form of money, people are able to enjoy considerable economic advantages. These advantages include near-universal acceptability, easy conversion in to other assets and low transaction costs with its use. It follows that the greater their money balances, the more benefit from these advantages they enjoy. The less they hold, the more frequently they have to realize other assets such as stocks or bonds, so as to pay for transactions. The holding of wealth in the form of money is therefore always a desirable property, other things being equal. However, other forms of assets while not enjoying the same advantages as money, or at least to the same degree, do have the counter-attraction of a return which may take the form of interest on saving deposits, yields on bonds, dividends on shares etc. Hence, the relationship governing the demand for money is usually explained along the lines of the following arguments.

It is assumed that, other things being equal, the more transactions there are in a country in a given period of time, the greater will be peoples' demand for money in order to carry out those transactions. For these purposes, transactions can be considered as any activity where money changes hands for goods and services. Suppose that the structure of the economy is fairly stable over some period. Then it might be assumed that the volume of transactions bears some stable relationship to the level of economic activity in the economy, which in turn might be measured by aggregate national income.

In addition, it may be assumed in such a period, that the relationship between the demand for money and the number of transactions is also constant. Therefore, if we ignore the effect that returns from holding other forms of assets will have for the present, the demand for money, M^d, may be assumed to be related to national income in money terms, W, as follows

$$M^d = lW, \qquad l > 0. \tag{16}$$

In this relationship, both money demand and national income are in nominal, that is money terms. Now the volume of a variable such as national income, is defined to mean the value of that variable in money terms, deflated by a relevant price index. In this case therefore, since we have already denoted the volume of national output by Y, and the index of overall prices in an economy by P, (neglecting subscripts, as we are dealing with a single economy at present) we must have

$$W = PY. \tag{17}$$

Substituting for this in the equation for money demand and rearranging, we therefore obtain

$$\frac{M^d}{P} = lY.$$

The quantity M^d/P is referred to as **the demand for real money balances**, and the relationship itself as the Cambridge quantity equation. However, as already indicated, the demand for money is likely to depend not only on the level of transactions for goods but also on the proportion of wealth held in assets other than money. In particular, the higher the return from assets such as bonds and shares, the greater will be the demand to hold wealth in these types of assets rather than as money. Hence, assuming a linear relationship, the equation for real money balances can be modified as follows

$$\begin{aligned} \frac{M^d}{P} &= lY - kr \\ l, k &> 0. \end{aligned} \tag{18}$$

where as in Section 1.4, r is a typical measure of domestic interest rates.

In line with most standard economic texts, the above derivation assumes that it is real money balances which is a linear function of domestic interest rates. However, the justification for this is not overpowering and in Chapter 3 we explore the effect of formulating this relationship rather differently.

Let M^s denote the supply of money in nominal terms. Then equilibrium exists in the money market when demand is equal to supply. When that happens we must have

$$\frac{M^s}{P} = lY - kr. \tag{19}$$

This result is known as the LM equation, and effectively defines combinations of national income and interest rates consistent with equilibrium in money markets for given levels of money supply and prices. As with the IS equation discussed earlier, in most applications of the LM relationship to the modelling of exchange rates, the equation is usually reformulated as a log-linear equation. The relationship then takes the form

$$m - p = ly - kr \tag{20}$$

where lower case letters denote the natural logs of variables (except interest rates) and l and k are positive parameters as before. The superscript on m is usually dropped as in equilibrium, money demand equals money supply. As in the formulation of the IS equation, the use of a log-linear relationship usually simplifies its incorporation in to an exchange rate model.

One final point of interest is that M, P and r are all variables which can be measured at a given point in time t. However, national income Y, is by definition, a quantity which only has meaning over an *interval* of time. Therefore, it may be better to think of Y as the annual national income, if the current, instantaneous, level of output was maintained. The same observation does of course apply to the consideration of the IS equation in Section 1.4.

1.6 The supply side

In Section 1.4, the factors which influence the demand for goods and services were considered. When the demand for these items was in equilibrium with supply, the so called IS relationship was defined. In this Section we look at the factors and assumptions which affect the supply of those goods and services. As a starting point, we shall assume that the quantity of equipment, plant and other non-labour factors of production in a country are fixed, or at least are changing slowly over the timescales we are considering. In other words, it is assumed that output depends only on the manpower employed.

In the first scenario, we shall assume that prices, including wage rates, in an economy, are perfectly flexible. That is, they respond instantly to any factors which might influence them, especially the willingness and ability of employers to vary wage rates. The effect of a price rise in this situation is described in detail by Copeland (1994), of which the following are the essentials. Suppose the supply side of the economy is in equilibrium, in the sense that at a given level of prices, there will be a wage level at which the supply of labour is just equal to that demanded by employers. At this level, there will be a certain number of people employed producing a level of output related to this number. If now prices suddenly change by a given percentage, due to changes in demand for products for instance, then all other things being equal, employers will be willing to pay wages which are higher by the same percentage, at any particular level of output. In order for employees to maintain the same relationship between consumption and leisure time, they would also need wages to be increased by the same percentage at any given level of employment. The upshot of all this is that the demand and the supply of labour will be in equilibrium again if wages rise by the same percentage as prices. What is more, the number of people employed, and the related level of output at the new equilibrium point, will be exactly the same as before. The only thing that has changed is that both prices and wages have increased or decreased by the same percentage. This result maybe summarised as follows. With the assumption of flexible prices, the aggregate supply curve is vertical at the long-run capacity output level of the economy, often referred to as the **natural level of output**. This is because, as the price level fluctuates up or down, the money wage adjusts to keep the real wage constant. Hence, employment and output never vary, and the price level itself is determined by aggregate demand. The supply curve under these assumptions is referred to as **Classical**. These results will be important in the discussion of the flexible price monetary model in Section 2.1.

In the next scenario, we shall assume that money wages can not change. This might arise because money wages are fixed, at least in the medium time horizon, by employment contracts. In many cases the contract might be negotiated between employers and trade unions, in which case the mechanism for adjusting money wages could be quite time consuming. In this situation, faced with an increase in prices, due to increased demand, employers will wish to take on more labour in order to meet the extra demand at this higher price. It is assumed that workers will willingly supply more labour at the current rate, either because it is uncertain how temporary the extra demand will be, or because of the inertia in the wage negotiating mechanism already mentioned. The effect will be to increase both the level of employment and the associated level of output: while real wages fall. In other words, an increase in prices results in an increase in output, that is, a so called upward-sloping supply curve. As part of this scenario, it was assumed that prices had risen because of increased demand. In practise, if wages formed the biggest part of employers' costs, or if non-labour costs were constant, then faced with an increase in demand, rather than increase its prices, a competitive firm would prefer to maintain its prices (in order not to lose market share) and simply increase production at the fixed money wage. In the extreme limit of this scenario, therefore, all prices, whether wages or selling prices, are constant, and supply is determined by demand. This is known as a **Keynesian supply curve.**

It may be argued that the flexible wage assumption is an implausible description of the labour market in the short term, while the fixed wage alternative is equally unrealistic as a description of the longer term behaviour of an economy. Consider therefore a compromise between the two extremes. According to this scenario, an increase in the demand for goods and services will result, initially, in a rise in employment and output, in order to meet that extra demand. Workers would be willing to supply more labour at the initial wage rate because of existing contracts or the other factors mentioned earlier. The selling prices of goods are also unaffected in this phase. As time passes, contracts are renegotiated, and money wages are progressively bid up in response to the extra demand for labour. In turn, firms try to recoup the rise in labour costs by raising selling prices, which itself gives further impetus to the increase in wage rates. More importantly, the increased selling prices will tend to reduce demand again, so output now starts to fall; at each stage short term supply meeting demand. The process continues until output is back at its natural level, but with selling prices higher than before and with wage rates increased by the same percentage as prices. The final position therefore being that predicted by the classical supply assumptions. This scenario, or some variation of it, is that assumed by the price inertia models of exchange rate dynamics.

1.7 Summary

In this Chapter the basic macroeconomic framework for countries operating in an international trading environment has been formulated. The goods and money sectors of the economy were discussed and, in particular, the circumstances when equilibrium exists in these markets. In addition, the two important concepts of purchasing power parity and uncovered interest rate parity were introduced. With this framework in place, we are now in a position to formulate the two major dynamic models of exchange rate movements. Section 2.1 considers the deterministic monetary model, and Section 2.2 formulates the deterministic price inertia model in one of its most general terms. The Chapter then goes on to consider the important overshooting effect of the price inertia model, and for the situation of a fixed short term supply of goods in a country, obtains an explicit relationship for how the exchange rate moves with time. The Chapter concludes by considering for the same model, the restrictions on macroeconomic parameters that are implied if exchange rates are to fullfil no-arbitrage conditions in networks consisting of more than two countries.

2 The monetary and price inertia models

2.1 The monetary model

2.1.1 Two countries

The flexible price monetary model was developed by Frenkel (1976), Mussa (1976) and Bilson (1978).The monetary model rests essentially on the following assumptions. Firstly, the aggregate supply curve is assumed vertical. As noted in Section 1.6, this does not imply that output is fixed in the long term, simply that it can only vary as the result of a change in the productivity of the economy. That is, through technical progress, the accumulation of capital, growth in the labour force or its educational level, and changes in related factors. Again as noted in Section 1.6, a vertical supply curve presupposes perfect price flexibility in all markets.

Secondly, money supply and demand are assumed to be equal when there is equilibrium in the money market. In addition, the demand for real money balances is assumed to depend upon domestic real income and interest rates, in a conventional manner. Hence, the money demand function for country i, in log- linear form can, as indicated by equation (20) in Section 1.5, be written as follows

$$m_i - p_i = l_i y_i - k_i r_i \tag{21}$$

where m_i is money supply, p_i is the price level, y_i is domestic real output, and r_i is the interest rate. All variables except the interest rate are measured in logs.

Thirdly, purchasing power parity in absolute form, PPP, is assumed to hold at all times. As in Section 1.2, this means that in log form, the exchange rate between countries 1 and 2, is related to the domestic price level in those countries as follows

$$s_{12} = p_1 - p_2 \tag{22}$$

where s_{12} is the logarithm of the exchange rate between the two countries, and p_i is the logarithm of the price level in country i.

Substituting for p_1 and p_2 in the PPP relationship from the respective money demand relationships given in equation (21), gives

$$s_{12} = m_1 - m_2 + l_1 y_1 - l_2 y_2 + k_1 r_1 - k_2 r_2. \tag{23}$$

In most formulations of this model, the macroeconomic parameters in the two countries are assumed identical, particularly the semi-elasticity on interest rates, k_i. This enables further assumptions about the components of the interest rate, and the differential in the rates, to be incorporated in to the model. For instance, Pilbeam(1992), Chapter 7.8, assumes that the interest rate is equal to an underlying 'real' rate of interest, plus the market's anticipated rate of price inflation.

In this monetary model, a sudden increase in a variable such as the money supply in country 1, would generate an immediate excess demand for goods in that country. As real output cannot exceed y_1, until productivity levels change, this in turn leads to inflation. The price level therefore rises immediately in order to keep money supply and demand in equilibrium, thereby choking-off the excess demand for goods. At the same time, the exchange rate depreciates, and in so doing maintains purchasing power parity.

Of course, in the above description, nothing has been said about the effects, if any, on interest rates. In practice, interest rate differentials cannot be arbitrary. In particular, if we assume that the uncovered interest rate parity (UIP) condition holds in asset markets, then the interest rate differential must be equal to the market's anticipated rate of change in the exchange rate. The value of the exchange rate is therefore subject to a form of feedback mechanism, and will be dependent upon the exact form of that mechanism. In the following section, some possible alternatives for how the market's anticipations are formed will be discussed. If in addition, **rational expectations** are assumed to hold in the market, (see Section 2.2.1 for more details) then the market's anticipated value of the exchange rate will be, on average, be equal to the actual exchange rate. We shall make use of this approach, when discussing the stochastic version of the monetary model, in Section 4.1.

2.1.2 More than two countries

As we have already seen from equation (5), in order that there can be no arbitrage by converting one currency in to another and then back again, for any pair of countries we must have

$$S_{ij}(t) = \frac{1}{S_{ji}(t)}. \tag{24}$$

This may be written in the following, alternative form

$$S_{ij}(t)S_{ji}(t) = 1. \tag{25}$$

Suppose that we have a number of countries at least some of which are linked through international trade and so there exists an exchange rate between them. We can consider the countries as the vertices of a graph. A pair ij of countries is an edge of the graph if the exchange rate $S_{ij}(t)$ is defined. If every pair of countries is linked by an exchange rate, then the graph is **complete.** In general, we do not assume that the graph is complete, although we do assume that it is **connected:** that is, any given country can be reached from any other country by a sequence of links (edges of the graph). Consider any cycle in the graph, comprising the countries arbitrarily labelled, $i, j, \cdots, l$. It will be assumed that it is not possible to make a profit or loss by a sequence of exchanges around the cycle. This can be viewed as an extension of the previous no-arbitrage relationship, and is usually referred to as the **cyclic no-arbitrage condition.** The restriction on the exchange rates becomes

$$S_{ij}(t)S_{jk}(t)\cdots S_{li}(t) = 1. \tag{26}$$

Taking natural logarithms leads to the corresponding result

$$s_{ij}(t) + s_{jk}(t) + \cdots + s_{li}(t) = 0. \tag{27}$$

The justification for this assumption is that currency exchanges operate so rapidly and with so little cost (for the major players) that any deviation cannot persist for more than a few moments. In the past, this was not always true.

Suppose for simplicity that we have a network of just three countries, all of which are connected by exchange rates. Assuming that exchange rate are determined by the flexible price monetary model, then the rate between any pair of countries i, j will be given as follows

$$s_{ij} = m_i - m_j + l_i y_i - l_j y_j + k_i r_i - k_j r_j. \tag{28}$$

The cyclic no-arbitrage condition would require that

$$s_{12} + s_{23} + s_{31} = 0. \tag{29}$$

It is straightforward to check that with the foregoing model, the cyclic no-arbitrage condition will be met for all time t. In addition, it can be seen that the argument is readily extended to cycles consisting of any number of countries within a network.

2.2 The price inertia model

2.2.1 Two countries

The assumption of flexible prices in the monetary model has, as noted earlier, some important consequences. In particular, short term supply and long term supply, are constant at the full employment level of the economy. Any excess demand pressures are translated immediately in to price rises which remove the excess demand. In addition, as purchasing power parity (PPP) is assumed to hold continuously, the exchange rate between the two countries is determined by the relative price levels at any point in time. An influential study by Meese and Rogoff (1983), and other empirical evidence, indicated that the observed variability of actual exchange rates was much greater than might be expected from that predicted by the monetary model after recorded changes in the variables upon which it depends, that is the so called fundamentals of the model. In addition, as noted by Copeland (1994, p76) the assumption of PPP holding continuously does not seem to agree with the observed facts, at least in the industrialised countries.

In order to explain some of these observations, Dornbusch (1976) developed a model which emphasised the *stickiness* of prices in product markets. This model, in its short-run features, fits in to the Keynesian tradition, with prices having considerable inertia to movement. On the other hand, it displays some of the long-run characteristics of the flexible price monetary model. In particular, the long-run value of the real exchange rate remains unchanged, provided national outputs and government expenditures remain unaltered. In addition, as prices can no longer adjust immediately, any changes to variables such as money supply result in changes to interest rates in order to maintain equilibrium in money markets. As the UIP condition is still assumed to hold in asset markets, there will be a corresponding movement by the exchange rate to a new equilibrium level. A consequence of the model is that the exchange rate jumps immediately on a change in money supply, but further than the new equilibrium level (at least for certain combinations of parameters), the so called *over-shooting* effect. A number of variations of the basic model discussed by Dornbusch have been proposed, but all of these variations will be referred to under the general heading of price inertia models.

As already noted, there are many descriptions of the price inertia model, but the variation discussed below is probably one of the most general formulations, and is based on the principles outlined by Gärtner(1993). However, Gärtner formulates the model in terms of one country verses the rest of the world. The approach here, considers the interaction between two countries of comparable economic size, and therefore includes aspects of the formulation by Baillie and McMahon(1994). As in the monetary model, we assume that money supply in country i is equal to money demand; both of which will be represented by m_i. In addition, money demand is assumed to be given by the standard LM equation in log-linear form, as shown below

$$m_i - p_i = l_i y_i - k_i r_i, \tag{30}$$

where m_i, p_i, y_i, and r_i are functions of time, defined as in Section 1.5.

In the goods sector, equilibrium will be assumed to be determined by a standard IS equation in log-linear form. Assuming that trade is possible between two countries, then as in Chapter 1, s_{12} will represent the natural log of the (spot) exchange rate between them, with country 1 being taken as the domestic country. Then as shown in equation (15) the IS equation for country 1 can be written as follows

$$\beta_1 y_1 + \gamma_1 r_1 = g_1 + \eta_{12}(s_{12} - p_1 + p_2), \tag{31}$$

where the quantities involved are as defined in Chapter 1. The IS equation for country 2 can be written similarly

$$\beta_2 y_2 + \gamma_2 r_2 = g_2 + \eta_{21}(s_{21} - p_2 + p_1). \tag{32}$$

As noted in Chapter 1, if the goods market is assumed to be in continuous equilibrium as represented by the IS equation, then this implies that the demand for goods is in equilibrium with short term supply, although not necessarily with the long term supply level; see Section 1.6.

In Section 1.3, equation (5), arbitrage considerations required that $S_{21} = 1/S_{12}$, which in terms of the logarithm of the exchange rate, is equivalent to $s_{21} = -s_{12}$. Therefore, the IS equation for country 2 can be written as follows

$$\beta_2 y_2 + \gamma_2 r_2 = g_2 - \eta_{21}(s_{12} - p_1 + p_2). \tag{33}$$

Prices are assumed to change in proportion to (in log-linear form) the difference between the demand for goods, y^d, and the long term, natural level of supply, when prices are assumed once again to be stable. That is

$$\frac{dp_i}{dt} = \phi_i(y_i^d - \overline{y}_i) \tag{34}$$

where $\overline{y}_i$ denotes the natural level of output in country i, y_i^d the demand for goods and services at time t, and p_i the price level in country i at time t. All variables being measured in natural logs. The parameter ϕ_i governs the rate at which prices adjust. For the present, we shall assume short term equilibrium exists in the goods market, so demand equates with national output. That is $y_i^d = y_i$.

Proposition 1 *With the above assumptions, the rate of change of the price differential satisfies the differential equation*

$$\begin{aligned}
\frac{dp_{12}}{dt} &= \left(\frac{\phi_1}{\beta_1 k_1 + l_1\gamma_1} + \frac{\phi_2}{\beta_2 k_2 + l_2\gamma_2}\right)(s_{12} - p_{12}) - \phi_1\overline{y}_1 + \phi_2\overline{y}_2 \\
&\quad + \left(\frac{\phi_1}{\beta_1 k_1 + l_1\gamma_1}\right)(k_1 g_1 + \gamma_1 m_1 - \gamma_1 p_1) \\
&\quad - \left(\frac{\phi_2}{\beta_2 k_2 + l_2\gamma_2}\right)(k_2 g_2 - \gamma_2 m_2 - \gamma_2 p_2).
\end{aligned}$$

where $p_{12} = p_1 - p_2$.

Proof

Solving the LM and IS equations for the level of output and interest rates which maintain equilibrium in both the money and goods sectors, gives for country 1,

$$y_1 = \frac{1}{\beta_1 k_1 + l_1\gamma_1}\left(k_1 g_1 + k_1\eta_{12}\left(s_{12} - p_1 + p_2\right) + \gamma_1 m_1 - \gamma_1 p_1\right) \tag{35}$$

$$r_1 = \frac{1}{\beta_1 k_1 + l_1\gamma_1}\left(g_1 l_1 + l_1\eta_{12}\left(s_{12} - p_1 + p_2\right) - \beta_1 m_1 + \beta_1 p_1\right).$$

Similarly, the levels of output and interest rates which maintain equilibrium in the money and goods sectors in country 2 are

$$y_2 = \frac{1}{\beta_2 k_2 + l_2\gamma_2}\left(g_2 l_2 - k_2\eta_{21}\left(s_{12} - p_1 + p_2\right) + \gamma_2 m_2 - \gamma_2 p_2\right) \tag{36}$$

$$r_2 = \frac{1}{\beta_2 k_2 + l_2\gamma_2}\left(g_2 l_2 - l_2\eta_{21}\left(s_{12} - p_1 + p_2\right) - \beta_2 m_2 + \beta_2 p_2\right).$$

Suppose we define P_{12} as the ratio of the price indices in countries 1 and 2, so

$$P_{12} = \frac{P_1}{P_2}. \tag{37}$$

Then upon taking logs we have

$$p_{12} = p_1 - p_2.$$

Differentiating p_{12} with respect to time, and substituting for the rates of the individual price adjustments, we have

$$\frac{dp_{12}}{dt} = \phi_1(y_1^d - \overline{y}_1) - \phi_2(y_2^d - \overline{y}_2). \tag{38}$$

As we are assuming that short term output is equal to demand, we can substitute for y_1 and y_2, from equations (35) and (36), which describe equilibrium in the money and goods sectors in the respective countries.

Upon doing so and rearranging, we obtain

$$\begin{aligned}\frac{dp_{12}}{dt} &= \left(\frac{\phi_1}{\beta_1 k_1 + l_1\gamma_1} + \frac{\phi_2}{\beta_2 k_2 + l_2\gamma_2}\right)(s_{12} - p_{12}) - \phi_1\overline{y}_1 + \phi_2\overline{y}_2 \\ &+ \left(\frac{\phi_1}{\beta_1 k_1 + l_1\gamma_1}\right)(k_1 g_1 + \gamma_1 m_1 - \gamma_1 p_1) \\ &- \left(\frac{\phi_2}{\beta_2 k_2 + l_2\gamma_2}\right)(k_2 g_2 - \gamma_2 m_2 - \gamma_2 p_2).\end{aligned} \tag{39}$$

□

The other relevant consideration which affects the dynamics of the system is the relative interest rates in the two countries. In Section 1.3, it was shown that if the uncovered interest rate parity condition (UIP) is assumed to hold then the markets anticipated rate of change in the exchange rate is approximately equal to the interest rate differential, that is

$$\frac{d(s_{12}^a)}{dt} = r_1 - r_2.$$

As noted in Section 1.3, there are a range of possible assumptions for the precise way in which the market forms its view of the anticipated rate of change in the exchange rate and in particular how this relates to the exchange rate itself. A common assumption is that the market forms its view of the anticipated rate of change on the basis of the difference between the actual exchange rate at time t and an assumed long term equilibrium level of the rate. At this long term equilibrium, the exchange rate is usually presumed to be at rest, as are prices. For the time being however, all that is required is to assume that the anticipated rate of change is proportional, in log linear terms, to the difference between the exchange rate at time t, and some particular level, denoted by $\overline{s}_{12}$. The assumption then is

$$\frac{d(s_{12}^a)}{dt} = -\alpha(s_{12} - \overline{s}_{12}) \qquad \alpha > 0. \tag{40}$$

On this assumption, the anticipated rate of change is greater the further the exchange rate is from $\overline{s}_{12}$, and in the direction towards this level. This process is usually referred to as one of adaptive expectations.

Another variation, is to assume that however the market comes to its view on the anticipated rate of change, its view is invariably correct. That is, the market's anticipated rate of change at time t is equal to the actual rate of change at time t. This is referred to as perfect foresight in, for example Gärtner(1993), and Pilbeam(1992). The assumption is therefore

$$\frac{d(s_{12}^a)}{dt} = \frac{d(s_{12})}{dt}. \tag{41}$$

The assumption of perfect foresight asserts that there is complete certainty as to the movement in the exchange rate; it follows the market's anticipated movement, however that may be formed. A possibly more realistic assumption is that the market's anticipated change is correct, *on average.* This requires the concept of rational expectations, which may be defined as follows. The **rational expectations hypothesis** states that the market's anticipated value, made at time t, is equal to the mathematical expected value of the variable in question, conditional on the set of all available information at time t. Hence, considering future values of the exchange rate to be random variables, the rational expectations hypothesis would maintain that the market's anticipated value for the rate at any time in the future, would be equal to the conditional expectation of the exchange rate for the same point in time. If investors are rational in forming their anticipations, they will often be wrong, in fact they may be wrong all of the time, but however large their errors, on average they will be correct. In real life it would be irrational to employ any forecasting method which could be improved. Since any forecasting process that generates systematic errors can be improved upon by a method which exploits the pattern in the errors, the assumption that the rational expectations hypothesis applies to the market's estimates of future values of the exchange rate is not without some merit. If this assumption holds then the market's anticipated rate of change, made at time t, can be written as follows

$$\frac{d(s_{12}^a)}{dt} = \frac{E(ds_{12})_t}{dt} \tag{42}$$

where the subscript t, indicates that all information up to and including time t, is assumed known. Most importantly, this implies that $s_{12}(t)$ is known. The use of rational expectations will be relevant to the discussion of stochastic models in Chapter 4, where the approach relies on the taking of expectations in order to derive ordinary differential equations which describe the system. Many standard texts, for example Pilbeam (1992), also assume the rational expectations hypothesis applies in this non-stochastic situation, or at least they use the term 'market expectations' in a very general, undefined sense. However, in most cases all that is required is the assumption that market anticipations are formed in a particular way, without the need to introduce the further assumption of rational expectations.

Conventionally, the system is assumed to start evolving following an unexpected change in one of the variables such as the money supply in one of the countries. It is then assumed that prices and the exchange rate continue to adjust until output is once again at its long term natural level in the two countries. At this point, prices and the exchange rate are also assumed to cease evolving. Assuming they exist, we may enquire what these limiting, or equilibrium values as they are conventionally referred to, for prices and the exchange rate would be. In order to obtain explicit answers to this question we shall need to make some simplifying assumptions. In particular, we shall assume that the macroeconomic parameters in the two countries are identical. In this case, the following results are obtained.

Proposition 2 *If the uncovered interest rate parity condition is assumed, the equilibrium price differential and exchange rate is given as follows*

$$\overline{p}_{12} = m_{12} - l\overline{y}_{12}$$

$$\overline{s}_{12} = m_{12} + \frac{(\beta - 2\eta l)\overline{y}_{12}}{2\eta} - \frac{g_{12}}{2\eta},$$

where the subscripts on the macro-economic parameters have been dropped, as these are now assumed identical in both countries, and $g_{12} = g_1 - g_2$, with similar definitions for $\overline{y}_{12}$ and m_{12}.

Proof.

With the assumption of identical macroeconomic parameters in the two countries, the equation for the rate of change in the price differential simplifies to the following

$$\frac{dp_{12}}{dt} = \frac{\phi}{\beta k + l\gamma}\left(kg_{12} + 2k\eta\left(s_{12} - p_{12}\right) + \gamma m_{12} - \gamma p_{12}\right) - \phi\overline{y}_{12}. \tag{43}$$

When the system is in equilibrium, by definition prices cease to change. Hence, putting $dp_{12}/dt = 0$, and rearranging, we obtain

$$2k\eta s_{12} - (2k\eta + \gamma)p_{12} = (\beta k + l\gamma)\overline{y}_{12} - kg_{12} - \gamma m_{12}. \tag{44}$$

This equation effectively constrains the levels of the exchange rate and relative prices, in order that there is no inflation.

Taking the difference in the respective LM equations for the two countries gives

$$m_{12} - p_{12} = ly_{12} - k(r_1 - r_2). \tag{45}$$

Assuming that the UIP condition holds continuously, therefore implies

$$m_{12} - p_{12} = ly_{12} - k\frac{d(s_{12}^a)}{dt}. \tag{46}$$

At equilibrium, the exchange rate ceases to change. Therefore, assuming a method of formation of market anticipations, such as adaptive expectations or perfect foresight, the market's anticipated rate of change at equilibrium will also be zero. Alternatively, we could assume this to be the situation directly, as part of the definition of system equilibrium. In either case, at equilibrium it is implied that the two interest rates are equal. Hence at equilibrium we must have

$$\overline{p}_{12} = m_{12} - l\overline{y}_{12} \tag{47}$$

since it is assumed that supply is back at its long term natural level at equilibrium. This can be viewed as the new equilibrium price level. Substituting this for the equilibrium relative price in equation (44), we obtain

$$2k\eta\overline{s}_{12} - (2k\eta + \gamma)(m_{12} - l\overline{y}_{12}) = (\beta k + l\gamma)\overline{y}_{12} - kg_{12} - \gamma m_{12}. \tag{48}$$

After simplifying, the following expression for the equilibrium exchange rate is obtained

$$\bar{s}_{12} = m_{12} + \frac{(\beta - 2\eta l)\bar{y}_{12}}{2\eta} - \frac{g_{12}}{2\eta}. \tag{49}$$

□

The precise path of the exchange rate, and in particular how it approaches the equilibrium value, depends upon the assumption of how market anticipations are formed. In the original formulation by Dornbusch, adaptive expectations are assumed, and this is still the case in the majority of texts. With this assumption, and with plausible values for the economic parameters, the price inertia model exhibits the so called *overshooting* effect. The conditions imposed on the macroeconomic parameters for overshooting to occur are as follows.

Proposition 3 *The 'jump' in the exchange rate immediately following a change in relative money supply will be greater than that required to reach the new equilibrium level provided*

$$\beta - 2\eta l > 0.$$

Proof.

To see this, first of all recall equation (46) for the relationship for the difference in the two countries LM equations assuming that UIP holds;

$$m_{12} - p_{12} = ly_{12} - k\frac{d(s^a_{12})}{dt}.$$

With the assumption of adaptive expectations, described in equation (40), we would then have

$$m_{12} - p_{12} = ly_{12} - k\alpha(\bar{s}_{12} - s_{12}). \tag{50}$$

After substituting for the previously derived expressions for y_1 and y_2 in equations (35) and (36), we obtain

$$\begin{aligned} m_{12} - p_{12} &= \frac{l}{\beta k + l\gamma}(kg_{12} + 2k\eta(s_{12} - p_{12}) + m_{12}\gamma - \gamma p_{12}) \\ &\quad + kas_{12} - k\alpha\bar{s}_{12}. \end{aligned} \tag{51}$$

Rearranging this equation and simplifying, gives

$$p_{12}(\beta - 2\eta l) = \frac{m_{12}\beta}{\beta k + l\gamma} - \frac{s_{12}(2k\eta + \beta k + l\gamma)}{\beta k + l\gamma} + \alpha\bar{s}_{12} - \frac{g_{12}l}{\beta k + l\gamma}. \tag{52}$$

An inspection of this equation indicates that the partial derivative of the price differential with respect to the exchange rate will be negative provided that

$$\beta - 2\eta l > 0. \tag{53}$$

□

Gärtner (1993) shows that this result can also be obtained by solving the above equation for s_{12} and taking the partial derivative with respect to money supply, while keeping prices constant at the moment of change in money levels. This derivative is greater than 1 (the change needed to bring the exchange rate to the new equilibrium level) under the same conditions on the economic parameters. In such circumstances, following an unexpected increase in a variable such as relative money supply, m_{12}, the equilibrium exchange rate will rise proportionately, as will the equilibrium level of the price differential. Then as the current price differential starts to rise, the current exchange rate starts to fall. Assuming the system was in equilibrium before the change in money supply, this implies that the current exchange rate must have increased at the same instant as the money supply change, but by a greater amount than the new equilibrium level. This is the so called overshooting effect of the original Dornbusch formulation of the model, and is a means of explaining why, in practice, the variability in exchange rates seems more than what would be implied if exchange rates moved to new equilibrium levels in some 'smooth' way. Of course, this condition on the economic parameters may not hold, in which case both prices and the exchange rate increase steadily following a disturbance, towards their new equilibrium levels. However, as Gärtner points out, theoretical considerations make this latter possibility much less plausible. In order to consider the possibility of whether or not the exchange rate overshoots in more detail, it will be helpful to rewrite the inequality for overshooting, (53), as below

$$\frac{2\eta l}{\beta} < 1. \tag{54}$$

The general sequence of events following a change in money supply can be summarised as follows. Assume initially, the exchange rate is at equilibrium with no movement anticipated by the market. Therefore, according to the UIP condition the interest rates in the two countries must be the same. Suppose that there is a sudden, unexpected increase in money supply in country 1, so the exchange rate immediately rises, that is country 1's currency depreciates. This depreciation results in an increased demand for goods via higher exports, and hence increased supply, the amount depending upon the elasticity of net exports to the real exchange rate, η and the coefficient β. An increase in money supply must also result in an equal increase in money demand, as these are assumed in equilibrium at all times. Hence the demand for real money balances in equation (30) also increases. Part of this increase will be met by the extra supply of goods resulting from the higher exports, the effect being governed by the value of the income elasticity of money demand, l. The remaining increase in demand for real money balances is met by a fall in interest rates within the country. However, the change in the exchange rate will also result in a reduction in net exports from the second country, equal to the increase from country 1. In order for the relationship for real money balances in country 2 to continue to hold, interest rates must fall there also in order to offset the reduction in supply resulting from the lower exports. If relationship (54) is an equality rather than an inequality, that is if

$$\frac{2\eta l}{\beta} = 1, \tag{55}$$

then it transpires that the reduction in interest rates required in the two countries is identical and so they remain equal in level also. As the UIP condition, equation (10), is assumed to hold, the identical interest rates would imply that the market does not anticipate any further change in the exchange rate. In other words, following the change in money supply the exchange rate moves directly to its new equilibrium level.

On the other hand, if inequality (54) holds, then the initial depreciation of the exchange rate results in a larger fall in the interest rate in the first country compared to the second country. According to the UIP condition, this must be reflected in the markets anticipation of an appreciation to the exchange rate. So the exchange rate must have overshot the new equilibrium level initially. Similarly, if inequality (54) is reversed, the initial depreciation results in so much extra exports and hence supply of goods in country 1, that it more than offsets the increase needed in real money balances. In this case interest rates may actually have to increase in country 1 in order to maintain equality in real money balances. Consideration of the UIP condition therefore indicates in this case that following the initial jump, the exchange rate continues to depreciate towards its new equilibrium level.

Hence, if overshooting does not occur, this may be due to an usually high elasticity for net exports η, resulting in more than the necessary increase in supply needed to keep real money balances in equilibrium. Alternatively, the same effect would be produced by an unusually large income elasticity of money demand l, so even small increases in supply result in relatively big increases in the demand for real money balances. The final alternative of a very small value for the coefficient β, would by the derivation of equation (14) imply that the fixed proportion of output which goes in domestic personal sector consumption of goods and services is very high. Any increase in the output in country 1 needed to meet a modest rise in exports would be accompanied by a large increase for domestic personal consumption; the latter resulting from the earnings from the extra exports. The two contributions together are once again more than is needed to maintain the relationship for real money balances, and so the exchange rate once again undershoots its new equilibrium level.

The possibility of the exchange rate overshooting is an important property of the price inertia model but of equal interest is consideration of the actual path the exchange rate takes in reaching its equilibrium level. This we shall now turn to. In order to examine how prices and the exchange rate change explicitly with time, we first return to the expression for the rate of change in relative prices previously derived in Proposition 2,

$$\frac{dp_{12}}{dt} = \frac{\phi}{\beta k + l\gamma}\left(kg_{12} + 2k\eta\left(s_{12} - p_{12}\right) + \gamma m_{12} - \gamma p_{12}\right) - \phi\overline{y}_{12}.$$

This can be rearranged to give the following

$$\frac{dp_{12}}{dt} = \frac{\phi}{\beta k + \gamma l}\left(kg_{12} + 2k\eta\left(s_{12} - p_{12}\right) + \gamma m_{12} - \gamma p_{12} - \beta k\overline{y}_{12} - \gamma l\overline{y}_{12}\right). \tag{56}$$

Eliminating money supply from equations (47) and (49) gives the following relationship between the equilibrium levels of the relative prices and exchange rate,

$$2\eta(\overline{s}_{12} - \overline{p}_{12}) = -g_{12} + \beta\overline{y}_{12}. \tag{57}$$

Then, after rearranging, the rate of change can be written as follows

$$\frac{dp_{12}}{dt} = \frac{\phi}{\beta k + \gamma l}\left(2k\eta\left(s_{12} - \overline{s}_{12}\right) - 2k\eta\left(p_{12} - \overline{p}_{12}\right) + \gamma\left(m_{12} - p_{12} - l\overline{y}_{12}\right)\right).$$

This can be simplified further to give

$$\frac{dp_{12}}{dt} = \frac{\phi}{\beta k + \gamma l}\left(2k\eta\left(s_{12} - \overline{s}_{12}\right) - \left(2k\eta + \gamma\right)\left(p_{12} - \overline{p}_{12}\right)\right). \tag{58}$$

In order to obtain an explicit solution to this differential equation, we shall reformulate it with an additional, simplifying assumption. It will be assumed that the short term supply of goods is fixed at the long term natural level in both countries. That is, $y_i(t) = \overline{y}_i$, for all times t. In these circumstances, the short term supply and demand for goods will not be in equilibrium, unless of course the latter happens to equal the natural level of output in an economy.

Proposition 4 *If the short term supply of goods is fixed, the movement of the exchange rate is given by the following equation*

$$s_{12}(t) = \bar{s}_{12} + (s_{12}(0) - \bar{s}_{12})e^{-v_{12}t}$$

where $\bar{s}_{12}$ *is the equilibrium level of the exchange rate, and* $s_{12}(0)$ *the level at time* $t = 0$, *immediately following a change in the fundamental variables.*

Proof

With the assumption of fixed supply in both countries, the planned demand for goods and services in country 1 will be given by the equivalent expression to equation (12) in Section 1.4 , but in log-linear form and with national output equal to $\bar{y}_1$, that is

$$y_1^d = \delta_1\bar{y}_1 - \gamma_1 r_1 + g_1 + \eta(s_{12} + p_2 - p_1). \tag{59}$$

The corresponding equation for the demand for goods and services in country 2 is obtained similarly. As noted previously, the rate of price inflation in the two countries is given by

$$\frac{dp_i}{dt} = \phi(y_i^d - \bar{y}_i).$$

Hence, after substituting for y_i^d, and as before, assuming identical macroeconomic parameters in the two countries, the rate of change in relative prices can be written as

$$\frac{dp_{12}}{dt} = \phi\left(2\eta\left(s_{12} - p_{12}\right) + \left(\delta - 1\right)\bar{y}_{12} + g_{12} - \gamma\left(r_1 - r_2\right)\right). \tag{60}$$

As derived in equation (57), the relationship between equilibrium levels of relative prices and the exchange rate, can be shown to be

$$2\eta(\bar{s}_{12} - \bar{p}_{12}) = -g_{12} + \beta\bar{y}_{12}$$

where, using the same notation as in Section 1.4 , we have put $\beta = 1 - \delta$.

Hence, substituting in the rate of change equation and rearranging, we obtain

$$\frac{dp_{12}}{dt} = \phi\left(2\eta\left(s_{12} - \overline{s}_{12}\right) - 2\eta\left(p_{12} - \overline{p}_{12}\right)\right). \tag{61}$$

Taking the difference in the LM equations for the two countries gives in this case

$$m_{12} - p_{12} = l\overline{y}_{12} - k(r_1 - r_2), \tag{62}$$

where the aggregate supply of goods is now constrained to equal the natural level in each country. Once again, assuming adaptive exchange rate anticipations, implies the above equation can be written as follows

$$m_{12} - p_{12} = l\overline{y}_{12} - k\alpha(\overline{s}_{12} - s_{12}). \tag{63}$$

As before, at equilibrium when the exchange rate stops evolving, we have that

$$\overline{p}_{12} = m_{12} - l\overline{y}_{12}. \tag{64}$$

The difference in the LM relationships, equation (63), can therefore be written as

$$\overline{p}_{12} - p_{12} = k\alpha(s_{12} - \overline{s}_{12}). \tag{65}$$

Substituting this in to equation (61) for the rate of change, gives

$$\frac{dp_{12}}{dt} = \left(\frac{2\eta\left(\overline{p}_{12} - p_{12}\right)}{k\alpha} - 2\eta\left(p_{12} - \overline{p}_{12}\right)\right). \tag{66}$$

This can be simplified to give

$$\frac{dp_{12}}{dt} = -2\eta\phi\left(1 + \frac{1}{k\alpha}\right)\left(p_{12} - \overline{p}_{12}\right). \tag{67}$$

This linear first-order differential equation with constant coefficients has solution

$$p_{12}(t) = \overline{p}_{12} + (p_{12}(0) - \overline{p}_{12})e^{-v_{12}t} \tag{68}$$

where $v_{12} = 2\eta\phi\left(1 + \frac{1}{k\alpha}\right)$, and $p_{12}(0)$ is the initial relative price level when relative money supply is changed. Because $v_{12} > 0$, the relative price level converges to its long term equilibrium level. The change in prices is linked to the change in the exchange rate by the following, previously derived relationship in equation (65),

$$\overline{p}_{12} - p_{12}(t) = k\alpha(s_{12}(t) - \overline{s}_{12}).$$

As this holds for all times, including at time $t = 0$, immediately following a change in the fundamental variables, we obtain

$$\overline{p}_{12} - p_{12}(0) = k\alpha(s_{12}(0) - \overline{s}_{12}). \tag{69}$$

Therefore, substituting for price differences in terms of the exchange rate in equation (68) gives

$$s_{12}(t) = \overline{s}_{12} + (s_{12}(0) - \overline{s}_{12})e^{-v_{12}t} \tag{70}$$

where $s_{12}(0)$ is the value of the exchange rate immediately following the change in money supply.

□

Hence, following a change in a variable such as money supply, the exchange rate also moves exponentially towards its equilibrium value.

It can be seen that the speed at which the exchange rate moves towards its equilibrium value depends upon v_{12} and hence upon the four parameters α, η, ϕ and k.

A number of variations are possible in the formulation of the price inertia model. As noted above, short term output can either be assumed to be fixed or in equilibrium with the short term demand for goods. On the other hand, the demand for goods itself is sometimes assumed, for reasons of simplicity, not to be related to interest rates at all; or more realistically, related to interest rates adjusted for the anticipated rate of price inflation in the corresponding country. For example, the latter assumption is made by Frankal (1979) and in Miller and Weller (1990) and followed in the description of the stochastic version of the price inertia model in Section 4.2.

2.2.2 More than two countries

Consider a small network consisting of three countries. Then, as in Section 2.1.2, the cyclic no-arbitrage condition would require that at any point in time

$$s_{12} + s_{23} + s_{31} = 0. \tag{71}$$

Suppose that prices and the exchange rate adjust according to the equations obtained in Section 2.2.1 where short term supply was assumed constant in each country. Then, with the same notation as in Section 2.2.1, equation (70) would give the corresponding solution for each pair of countries. Cyclic no-arbitrage would then require

$$\begin{aligned}\bar{s}_{12} + (s_{12}(0) - \bar{s}_{12})e^{-v_{12}t} + \bar{s}_{23} + (s_{23}(0) - \bar{s}_{23})e^{-v_{23}t} + \bar{s}_{31}\\ + (s_{31}(0) - \bar{s}_{31})e^{-v_{31}t} = 0,\end{aligned}$$

where v_{23} and v_{31} are defined as the corresponding combinations of macro economic parameters as v_{12}. Now, as in the previous Section, the new equilibrium level for the exchange rate between countries 1 and 2 can be shown to be

$$\bar{s}_{12} = m_{12} + \frac{(\beta - 2\eta l)\bar{y}_{12}}{2\eta} - \frac{g_{12}}{2\eta}. \tag{72}$$

Assuming that the macroeconomic parameters are the same in the three countries, and recalling that the no-arbitrage condition requires that $\bar{s}_{31} = -\bar{s}_{13}$, it is readily verified that

$$\bar{s}_{12} + \bar{s}_{23} + \bar{s}_{31} = 0.$$

If in addition, the parameters relating to rates of adjustment, α and ϕ are also identical, then the difference between the initial and the equilibrium exchange rates for a pair of countries given in equation (69), can be written as follows

$$(s_{ij}(0) - \overline{s}_{ij}) = \frac{(\overline{p}_{ij} - p_{ij}(0))}{k\alpha}. \tag{73}$$

It is then readily verified that the above cyclic no-arbitrage condition,

$$s_{12} + s_{23} + s_{31} = 0,$$

is met.

It is also worth noting that unless the macroeconomic and rate of adjustment parameters are assumed to be identical as we have just done, it would not be possible for the cyclic no-arbitrage condition to be met in practice, at least for all time t. For instance, no-arbitrage at all times would certainly require that $v_{12} = v_{23} = v_{31}$. Hence, even if the macroeconomic parameters k, η and ϕ are assumed identical in the three countries, which may be plausible, at least approximately, for countries of similar levels of economic development, the definition of the v_{ij} would still require that

$$\alpha_{12} = \alpha_{23} = \alpha_{31}. \tag{74}$$

That is, the rate of adjustment is the same for the exchange rate between each country. Thus, the rates of adjustment are not independent but are constrained by the cyclic no-arbitrage condition, which is invariably observed in practice.

2.3 Summary

In Section 2.1 the monetary model for two countries was formulated. The assumptions of PPP holding continuously and that prices are completely flexible, result in a relatively simple model for exchange rate movements. However, the dynamics of the system are not necessarily as straightforward as they may appear from an inspection of the model. In particular, the basic formulation says nothing about how interest rates behave. In reality these cannot be arbitrary. For instance, if UIP is assumed to hold then the interest differential must be equal to the market's anticipated rate of change in the exchange rate. This in turn may be assumed to be formulated in many different ways; perfect foresight, adaptive expectations, etc. However, because of the PPP assumption, the model will always result in the cyclic no-arbitrage condition being met for networks of more than two countries, whatever mechanism we may postulate for interest rate movements.

The price inertia model for two countries of comparable size was derived in Section 2.2. With plausible assumptions for the economic parameters the model was shown to exhibit the so called overshooting effect for the exchange rate. This effect is important as it is a mechanism for explaining the observed phenomenon that the variability in exchange rate movements is much larger than might be expected from the known changes in economic variables such as money supply. As a means of explaining the observed large variation in exchange rate movements, the price inertia model has certainly had a considerable influence since its initial formulation by Dornbusch. The final result of this Section is of some interest although little mentioned in the literature. Because PPP does not hold continuously, if the cyclic no-arbitrage condition is assumed to hold for a network of countries (which it is invariably observed to do in practice) the macroeconomic and other parameters of the various countries cannot be independent. Although this result was derived for a rather simplified model in which corresponding macroeconomic parameters in each country are assumed identical and the supply of goods fixed in each country, a less restrictive model is more rather than less likely to exhibit similar constraints. It is a mute point whether this has more to say about the assumptions behind the model, or about unknown economic mechanisms which might constrain parameters in some way.

In the following Chapter, an extension to the standard analysis of the price inertia model is considered. Most analyses in the literature assume for simplicity that the macroeconomic parameters are identical in each country. In Chapter 3, the tractability and implications for the model of making asymmetric assumptions about macroeconomic parameters for countries, together with a formulation in nominal money terms, is examined, and the limiting behaviour of the exchange rate discussed.

3 A price inertia model in nominal, linear form

3.1 Motivation

In Section 2.2 the basic price inertia model for the exchange rate between two countries was derived in traditional form. However, the results derived in the literature from the model invariably rely on the assumption that the corresponding macroeconomic parameters are the same for both countries. Usually the justification for this approach is that it simplifies the analysis while preserving the essential nature of the results. In this Chapter the tractability of the analysis is explored when different macroeconomic parameters are allowed in each country. In addition, while the equations employed in the consideration of the goods and monetary sectors usually have many of the variables in real terms, that is after deflation by a relevant price index, the economic arguments can also be formulated in nominal money terms, that is in current prices. The current Chapter therefore explores the nature of the exchange rate dynamics with such a modification to the formulation. Further, although many of the basic relationships are usually, as in Section 2.2, derived in log-linear form, it will be shown that a linear formulation of the equations provides an useful framework for enabling an explicit relationship between the exchange rate and time to be produced. For the model considered, it will be shown that following a disturbance, there are a range of possible limiting values for both the nominal and real exchange rates.

3.2 Derivation of the system of equations

In the usual derivation of the IS equation, the value of the relevant variables are given in real, that is in deflated or volume terms. However, as the fundamental identity between the demand and supply of goods, from which the IS equation is derived, is usually stated in nominal terms, it seems more logical to obtain the IS equation in these terms also. In addition, exogenous variables such as government spending, are more readily interpreted when measured in nominal money units, rather than as money volumes. As in the derivation of the IS equation in Section 2.2.1, savings are assumed to increase with rising income and interest rates, while investment falls with increasing interest rates. Therefore, if we also assume a linear relationship with these variables, we may write the IS equation for the first country in nominal terms as follows

$$\beta_1 W_1 + \gamma_1 r_1 = G_1^n + \text{value of net trade}, \tag{75}$$

where G_1^n is government spending in nominal terms, which is taken to be exogenous to the model, and W_1 is national output in country 1, also measured in nominal money terms.

Although the value of net trade on the current account may depend upon many variables, it is reasonable to assume that the first domain's goods will become more competitive as the real exchange rate $S_{12}P_2/P_1$, increases and thereby influence the level of trade between the two domains. As in Section 2.2.1, we shall assume therefore that the value of net trade is a function of the real exchange rate. Although this functional dependence may be quite general, for convenience we shall assume that relationship is linear, or more precisely that the relationship can be adequately represented by the first term of its Taylor expansion. As a first approximation we may therefore write the IS equation as follows

$$\beta_1 W_1 + \gamma_1 r_2 = G_1^n + \frac{\eta_{12} S_{12} P_2}{P_1} \tag{76}$$

where η_{12} is a constant. The IS equation for the second country will take a similar form but noting that the real exchange rate for the second country will be given by $S_{21}P_1/P_2$. Hence, we have

$$\beta_2 W_2 + \gamma_2 r_2 = G_2^n + \frac{\eta_{21} S_{21} P_1}{P_2}. \tag{77}$$

For the money sector, the demand for money can be taken to be positively related to the level of national output, in order to facilitate transactions in goods and services. However it can also be assumed to decline with the rising opportunity cost of holding money, as measured by interest rates. The modified LM equation for the first country can therefore be written as

$$M_1 = l_1 W_1 - k_1 r_1 \quad (l_1, k_1 > 0), \tag{78}$$

where M_1 is the national money supply in nominal terms, and is assumed to be equal to money demand at all times. Similarly, and with corresponding notation, we have for the second country

$$M_2 = l_2 W_2 - k_2 r_2. \tag{79}$$

For each country, equilibrium in the money and goods sectors occurs when the national output and interest rate are at levels which simultaneously fulfil the IS and LM equations. As in Section 2.2.1, it is implicit in this approach that the short term supply of goods is able to adjust to meet demand at all times. In addition, it will be assumed that as in Section 2.2.1, it is the movement of demand for goods away from the long term natural level of supply, which drives the change in prices within each country. With the foregoing derivations of the IS and LM equations, the values of national output and interest rates which simultaneously solve the IS and LM equations in country 1 are

$$W_1 = \frac{1}{\beta_1 k_1 + \gamma_1 l_1}\left(k_1 G_1^n + \frac{k_1 \eta_{12} S_{12} P_2}{P_1} + M_1 \gamma_1\right) \tag{80}$$

$$r_1 = \frac{l_1}{\beta_1 k_1 + \gamma_1 l_1}\left(G_1^n + \frac{\eta_{12} S_{12} P_2}{P_1} - \frac{\beta_1 M_1}{l_1}\right). \tag{81}$$

The corresponding solutions in country 2 being

$$W_2 = \frac{1}{\beta_2 k_2 + \gamma_2 l_2}\left(k_2 G_2^n + \frac{k_2 \eta_{21} S_{21} P_1}{P_2} + M_2 \gamma_2\right) \tag{82}$$

$$r_2 = \frac{l_2}{\beta_2 k_2 + \gamma_2 l_2}\left(G_2^n + \frac{\eta_{21} S_{21} P_1}{P_2} - \frac{\beta_2 M_2}{l_2}\right). \tag{83}$$

Unlike in Section 2.2.1 however, we shall not assume that the macroeconomic parameters are identical in the two countries, as this will allow the formulation to be as general as possible. As noted for equation (5) in Section 1.3, the no-arbitrage condition requires that $S_{21} = 1/S_{12}$ So, the solutions for the second country can be written as follows

$$W_2 = \frac{1}{\beta_2 k_2 + \gamma_2 l_2}\left(k_2 G_2^n + \frac{k_2 \eta_{21} P_1}{S_{12} P_2} + M_2 \gamma_2\right) \tag{84}$$

$$r_2 = \frac{l_2}{\beta_2 k_2 + \gamma_2 l_2}\left(G_2^n + \frac{\eta_{21} P_1}{S_{12} P_2} - \frac{\beta_2 M_2}{l_2}\right). \tag{85}$$

We shall show that the above assumptions, coupled with that of perfect foresight, leads to a dynamical system which can be described by the following differential equations.

Proposition 5 *The system of differential equations describing the evolution of the model with perfect foresight is as follows*

$$\frac{dS_{12}}{dt} = c_1 S_{12} + \frac{c_2 S_{12}^2}{P_{12}} + c_3 P_{12}$$

$$\frac{dP_{12}}{dt} = c_4 P_{12} + c_5 S_{12} + \frac{c_6 P_{12}^2}{S_{12}},$$

where P_{12} is the ratio of prices in the two countries, and the coefficients c_i, are combinations of the macroeconomic parameters for the countries.

Proof.

As noted in Section 1.3, the assumption of perfect substitutability of assets with no restrictions on capital movements, implies that uncovered interest rate parity holds continuously. That is, the market's anticipated rate of change in the exchange rate is approximately equal to the interest rate differential, as below

$$\frac{S_{12}^{a\,\prime}}{S_{12}} = r_1 - r_2.$$

If, in addition, the market is assumed to have perfect foresight so that the anticipated rate of change in the exchange rate is equal to the actual rate of change, then the uncovered interest rate parity condition can be written as

$$\frac{S'_{12}}{S_{12}} = r_1 - r_2. \tag{86}$$

Upon substituting the previously derived values for r_1 and r_2 from equations (83) and (85), in to this relationship, we obtain

$$\begin{aligned} S'_{12} &= \frac{S_{12}l_1}{\beta_1 k_1 + \gamma_1 l_1}\left(G_1^n - \frac{\beta_1 M_1}{l_1} + \frac{\eta_{12}S_{12}P_2}{P_1}\right) \\ &\quad - \frac{S_{12}l_2}{\beta_2 k_2 + \gamma_2 l_2}\left(G_2^n - \frac{\beta_2 M_2}{l_2} + \frac{\eta_{21}P_1}{S_{12}P_2}\right). \end{aligned} \tag{87}$$

In Section 2.2.1, it was noted that in the price inertia model the rate of change of prices is assumed to be a function of the excess demand for goods in each country. We shall therefore assume that within each country there exists a level, in nominal terms, such that when output is at this level there is no price inflation. We shall represent this level by $\overline{W}_i$. In the simplest situation, the function will be taken to be a linear relationship, with a coefficient of proportionality ϕ_i for the ith country. That is

$$\frac{P'_i}{P_i} = \phi_i(W_i^d - \overline{W}_i), \tag{88}$$

where W_i^d represents the demand for goods and services in money terms. The assumption that short term supply is equal to the demand for goods implies that $W_i = W_i^d$. Therefore, substituting from equations (80) and (82), the rate of price adjustment in the two countries is

$$P'_1 = \frac{\phi_1 P_1}{\beta_1 k_1 + \gamma_1 l_1}\left(k_1 G_1^n + \frac{k_1 \eta_{12} S_{12} P_2}{P_1} + M_1 \gamma_1\right) - \phi_1 P_1 \overline{W}_1 \tag{89}$$

and

$$P'_2 = \frac{\phi_2 P_2}{\beta_2 k_2 + \gamma_2 l_2}\left(k_2 G_2^n + \frac{k_2 \eta_{21} P_1}{s_{12} P_2} + M_2 \gamma_2\right) - \phi_2 P_2 \overline{W}_2. \tag{90}$$

The system can be rewritten by consideration of the two endogenous variables S_{12}, and the ratio of relative prices $P_{12} = P_1/P_2$. After some manipulation, the rates of change of the two exogenous variables can be obtained as follows

$$\frac{dS_{12}}{dt} = c_1 S_{12} + \frac{c_2 S_{12}^2}{P_{12}} + c_3 P_{12} \tag{91}$$

and

$$\frac{dP_{12}}{dt} = c_4 P_{12} + c_5 S_{12} + \frac{c_6 P_{12}^2}{S_{12}}, \tag{92}$$

where

$$c_1 = \frac{l_1}{\beta_1 k_1 + \gamma_1 l_1}\left(G_1^n - \frac{\beta_1 M_1}{l_1}\right) - \frac{l_2}{\beta_2 k_2 + \gamma_2 l_2}\left(G_2^n - \frac{\beta_2 M_2}{l_2}\right)$$

$$c_2 = \frac{l_1 \eta_{12}}{\beta_1 k_1 + \gamma_1 l_1}$$

$$c_3 = \frac{-l_2 \eta_{21}}{\beta_2 k_2 + \gamma_2 l_2}$$

$$c_4 = \phi_1\left(\frac{1}{\beta_1 k_1 + \gamma_1 l_1}(k_1 G_1^n + M_1\gamma_1) - \overline{W}_1\right) - \phi_2\left(\frac{1}{\beta_2 k_2 + \gamma_2 l_2}(k_2 G_2^n + M_2\gamma_2) - \overline{W}_2\right)$$

$$c_5 = \frac{\phi_1 k_1 \eta_{12}}{\beta_1 k_1 + \gamma_1 l_1}$$

$$c_6 = \frac{-\phi_2 k_2 \eta_{21}}{\beta_2 k_2 + \gamma_2 l_2}.$$

□

The above pair of equations can be solved in order to determine how the nominal exchange rate varies with time. However, before doing so it will be helpful to examine the evolution of the real exchange rate over time.

3.3 Evolution of the real exchange rate

Suppose, as in Section 1.2, we denote the real exchange rate by Q. That is

$$Q = \frac{S_{12}P_2}{P_1}. \tag{93}$$

Then we shall show the following result.

Proposition 6 *The real exchange rate fulfils the following relationship*

$$\frac{Q-\alpha}{Q-\delta} = \widetilde{C}e^{t\alpha(\alpha-\delta)},$$

where α and δ are the roots of a quadratic equation involving the macroeconomic parameters, and $\widetilde{C}$ is a constant of integration.

Proof.

On taking logarithms of the equation defining Q and differentiating with respect to time, we obtain

$$\frac{Q'}{Q} = \frac{S'_{12}}{S_{12}} - \frac{P'_1}{P_1} + \frac{P'_2}{P_2}. \tag{94}$$

In the previous Section it was assumed that UIP holds continuously and the market has perfect foresight. This led to the following relationship

$$\frac{S'_{12}}{S_{12}} = r_1 - r_2.$$

It was further assumed that the rate of price inflation in the ith country is determined by the following equation

$$\frac{P'_i}{P_i} = \phi_i(W_i^d - \overline{W}_i).$$

Hence, after inserting these relationships into equation (94) for the real exchange rate and substituting the previously derived values for r_i and W_i^d which simultaneously solve the IS and LM equations, we obtain

$$\frac{Q'}{Q} = aQ + b + \frac{c}{Q}, \tag{95}$$

where

$$a = \frac{\eta_{12}}{\beta_1 k_1 + \gamma_1 l_1}(l_1 - \phi_1 k_1),$$

$$\begin{aligned} b &= \frac{l_1}{\beta_1 k_1 + \gamma_1 l_1}\left(G_1^n - \frac{\beta_1 M_1}{l_1} - \frac{\phi_1 k_1 G_1^n}{l_1} + \frac{\phi_1 M_1 \gamma_1}{l_1}\right) \\ &\quad - \frac{l_2}{\beta_2 k_2 + \gamma_2 l_2}\left(G_2^n - \frac{\beta_2 M_2}{l_2} - \frac{\phi_2 k_2 G_2^n}{l_2} + \frac{\phi_2 M_2 \gamma_2}{l_2}\right), \end{aligned}$$

$$c = \frac{\eta_{21}}{\beta_2 k_2 + \gamma_2 l_2}(\phi_2 k_2 - l_2).$$

Hence. on multiplying through by Q and separating the variables we obtain

$$\frac{dQ}{aQ^2 + bQ + c} = dt. \tag{96}$$

That is

$$\frac{dQ}{a(Q^2 + \frac{bQ}{a} + \frac{c}{a})} = dt. \tag{97}$$

If we assume the quadratic in Q has roots α and δ, which may be real or complex, then on using partial fractions we obtain the relation

$$\frac{dQ}{a(\alpha - \delta)(Q - \alpha)} - \frac{dQ}{a(\alpha - \delta)(Q - \delta)} = dt. \tag{98}$$

The result of integrating this relation will depend upon the signs of $Q - \alpha$ and $Q - \delta$. Therefore, for the present, assume that the roots α and δ are real and that both $Q-\alpha$ and $Q-\delta$ are positive. Then integrating both sides yields

$$\log\left(\frac{Q-\alpha}{Q-\delta}\right)^{\frac{1}{a(\alpha-\delta)}} = t + C, \tag{99}$$

where C is a constant of integration. Taking antilogs gives a solution of the form

$$\frac{Q-\alpha}{Q-\delta} = \widetilde{C}e^{ta(\alpha-\delta)}. \tag{100}$$

$\widetilde{C}$ can be taken as constant provided a, b and c remain unchanged.

□

The roots α and δ will of course be real provided $b^2 - 4ac > 0$, and complex otherwise. When the roots are real and a is negative, the solution given in equation (100) can be shown to be equivalent to the following

$$\frac{Q+\frac{b}{2a}}{\xi} = \coth(\widetilde{C} - ta\xi), \tag{101}$$

where

$$\xi = \frac{\sqrt{b^2-4ac}}{2a}.$$

If a is positive, the coth function is replaced by tanh.

For real roots but differing assumptions about the sign of $Q-\alpha$ or $Q-\delta$ the solutions can be obtained similarly while ensuring that the arguments of the appropriate log function are positive.

When the roots are complex and a is positive, the solution given in equation (100) can be shown to be equivalent to

$$\frac{Q+\frac{b}{2a}}{\varsigma} = \tan(\widetilde{C} + ta\varsigma), \tag{102}$$

where

$$\varsigma = \frac{\sqrt{4ac - b^2}}{2a}.$$

If a is negative then tan is replaced by cot.

3.4 Implications for the real exchange rate

If we examine the solution for the evolution of the real exchange rate with time, some deductions can readily be made concerning its limiting behaviour. For example, the solution for real roots, and initial value of Q following a disturbance such that $Q - \alpha$ and $Q - \delta$ being both positive, was earlier found to be

$$\frac{Q - \alpha}{Q - \delta} = \widetilde{C} e^{ta(\alpha - \delta)}.$$

It can be observed that if α is negative, then since $\alpha - \delta$ is positive, as $t \to \infty$ the right hand side of this equation approaches zero. This implies that $Q \to \alpha$ in order that $\frac{Q-\alpha}{Q-\delta}$ should also approach zero. This then is the long term limiting value for the real exchange rate under these assumptions. In general, for values of b which ensure at least one positive real root, the limiting value of Q will depend upon the signs of the coefficients a and c. The limits of Q for the possible combinations of a and c are listed below

1. If $a < 0, c > 0$, then for any initial value of $Q > 0$, $Q \to \alpha$ as $t \to \infty$.

2. If $a < 0, c < 0$, then for an initial value of $Q > \delta$, $Q \to \alpha$ as $t \to \infty$. However, if initially, $0 < Q < \delta$, then $Q \to 0$ in finite time.

3. If $a > 0, c > 0$, then for an initial values, $0 < Q < \alpha$, $Q \to \delta$ as $t \to \infty$. If the initial value is $Q > \alpha$, then $Q \to \infty$, in finite time.

4. If $a > 0, c < 0$, then for initial values, $0 < Q < \alpha$, $Q \to 0$ in finite time. If the initial value is $Q > \alpha$, then $Q \to \infty$, in finite time.

Except in the unlikely event that the combined magnitudes of the various exogenous variables and macroeconomic parameters are such that either α

or δ is equal to 1, purchasing power parity in absolute terms ($Q = 1$) will not hold, even in the long term, whatever the initial conditions might be.

Having examined the possible limits for the real exchange rate we may inquire as to the limiting behaviour of the nominal exchange rate itself. In order to do so, we shall first obtain the relationship between the nominal exchange rate and the ratio of relative prices. This we can do by returning to the system of equations in Section 3.2.

3.5 The nominal exchange rate

In Section 3.2 the system of equations for the rate of change of the nominal exchange rate and the relative prices was shown in equations (91) and (92) to be as follows

$$\frac{dS}{dt} = c_1 S + \frac{c_2 S^2}{P} + c_3 P \tag{103}$$

$$\frac{dP}{dt} = c_4 P + c_5 S + \frac{c_6 P^2}{S}, \tag{104}$$

where, for convenience, we have dropped the subscripts on S and P. This system of equations is not of course defined along the lines $P = 0$ and $S = 0$. However, as the system of equations is homogenous in S and P, an explicit solution to the system can be found. We first show that the system can be reduced to a single differential equation

Proposition 7 *The system of equations can be reduced to the following*

$$\begin{aligned}\frac{dP}{P} &= \frac{K_1 dQ}{Q} + \frac{K_2 \left(2 \left(c_2 - c_5\right) Q + \left(c_1 - c_4\right)\right) dQ}{\left(\left(c_2 - c_5\right) Q^2 + \left(c_1 - c_4\right) Q + \left(c_3 - c_6\right)\right)} \\ &\quad + \frac{K_3 dQ}{Q^2 + \frac{(c_1 - c_4)}{(c_2 - c_5)} Q + \frac{(c_3 - c_6)}{(c_2 - c_5)}},\end{aligned}$$

where Q is the real exchange rate, and the coefficients K_j are functions of the c_i.

Proof.

Providing that $S \neq 0$ and $P \neq 0$, combining the two equations (103) and (104) gives the following

$$\frac{dS}{dP} = \frac{S(c_1 SP + c_2 S^2 + c_3 P^2)}{P(c_4 SP + c_5 S^2 + c_6 P^2)}. \tag{105}$$

From the definition of the real exchange rate Q, we can write $S = QP$. Hence, upon substituting for S in the above equation and simplifying, we obtain

$$\frac{dS}{dP} = \frac{Q(c_1 Q + c_2 Q^2 + c_3)}{c_4 Q + c_5 Q^2 + c_6}. \tag{106}$$

As dS/dP can be written as

$$\frac{dS}{dP} = Q + \frac{dQ}{dP}P,$$

from equation (106), we obtain the following relationship

$$Q + \frac{dQ}{dP}P = \frac{Q(c_1 Q + c_2 Q^2 + c_3)}{c_4 Q + c_5 Q^2 + c_6}. \tag{107}$$

On separating the variables and using partial fractions, we obtain

$$\begin{aligned} \frac{dP}{P} &= \frac{K_1 dQ}{Q} + \frac{K_2 \left(2\left(c_2 - c_5\right)Q + \left(c_1 - c_4\right)\right)dQ}{\left(\left(c_2 - c_5\right)Q^2 + \left(c_1 - c_4\right)Q + \left(c_3 - c_6\right)\right)} \\ &\quad + \frac{K_3 dQ}{Q^2 + \frac{(c_1 - c_4)}{(c_2 - c_5)}Q + \frac{(c_3 - c_6)}{(c_2 - c_5)}}, \end{aligned} \tag{108}$$

provided the relevant terms in the denominators do not vanish, and where

$$K_1 = \frac{c_6}{c_3 - c_6},$$

$$K_2 = \frac{(c_3 c_5 - c_2 c_6)}{2\left(c_3 - c_6\right)\left(c_2 - c_5\right)},$$

$$K_3 = \frac{2\left(c_3c_4 - c_1c_6\right)\left(c_2 - c_5\right) - \left(c_3c_5 - c_2c_6\right)}{2\left(c_2 - c_5\right)^2\left(c_3 - c_6\right)}.$$

□

As with the evolution of the real exchange rate, the solution of the above equation will depend upon the signs of the various component terms. For example, let the quadratic in Q in the last term of the equation have roots ω and τ. Further, let us assume that these roots are real and that $Q - \omega$ and $Q - \tau$ are positive. Then provided the quadratic in Q in the second term of the integrand is also positive, integrating both sides gives the following

$$\begin{aligned} C + \log P &= K_1 \log Q + K_2 \log\left(\left(c_2 - c_5\right)Q^2 + \left(c_1 - c_4\right)Q + \left(c_3 - c_6\right)\right) \\ &\quad + \frac{K_3}{(\omega - \tau)} \log\left(\frac{Q - \omega}{Q - \tau}\right), \end{aligned} \tag{109}$$

where C is a constant of integration. On writing $C = \log \widetilde{C}$ and substituting S/P for Q in equation (109), we obtain the result

$$\widetilde{C}P = \left(\frac{S}{P}\right)^{K_1}\left(\left(c_2 - c_5\right)\left(\frac{S}{P}\right)^2 + \left(c_1 - c_4\right)\left(\frac{S}{P}\right) + \left(c_3 - c_6\right)\right)^{K_2}\left(\frac{\frac{S}{P} - \omega}{\frac{S}{P} - \tau}\right)^{\frac{K_3}{\omega - \tau}}$$

The roots ω and τ are of course given by the following

$$\begin{aligned} \omega &= -\frac{\left(c_1 - c_4\right)}{2\left(c_2 - c_5\right)} + \sqrt{\frac{\left(c_1 - c_4\right)^2}{4\left(c_2 - c_5\right)} - \frac{\left(c_3 - c_6\right)}{\left(c_2 - c_5\right)}} \\ \tau &= -\frac{\left(c_1 - c_4\right)}{2\left(c_2 - c_5\right)} - \sqrt{\frac{\left(c_1 - c_4\right)^2}{4\left(c_2 - c_5\right)} - \frac{\left(c_3 - c_6\right)}{\left(c_2 - c_5\right)}}. \end{aligned}$$

After some manipulation, it can be seen that the roots ω and τ are identical with the roots of the quadratic in the solution of the real exchange rate equation, that is, $\omega = \alpha$ and $\tau = \delta$.

In Section 3.3, the limiting behaviour of the real exchange rate when the coefficients where such that, $a < 0$, $c > 0$, and the initial value of Q was greater than α, with real roots, was shown to be that $Q \to \alpha$ as $t \to \infty$. Therefore. using this limiting value of Q in the above equation indicates that in these circumstances $P \to 0$ as $t \to \infty$. As the limiting value of S/P is finite, the limiting value of S must also be 0. The limiting values of S and P for other initial values of Q and different assumptions about the coefficients, can in principle be obtained similarly.

3.6 Summary

The foregoing results indicate that the analysis of the model is certainly tractable even with asymmetric assumptions concerning the macroeconomic parameters between countries. However, the simplifying assumption of identical macroeconomic parameters would not, in general, affect the qualitative nature of solutions, since the parameters are absorbed into the coefficients of the equations describing the system, for example equations (91) and (92). This non-reliance upon symmetry includes the semi-elasticities for interest rates at least for the perfect foresight formulation of market anticipations used in this formulation.

In general, the solutions to the model are typified by a large range of possible limiting behaviour, both for the real exchange rate and for its component parts; the nominal exchange rate and relative prices. After some disturbance, such as a change in one country's money supply, the system is typified by a continuous adjustment of both S and P. In practise, this limiting behaviour is almost certain to be interrupted by further disturbances and consequential adjustment of the rates. As the type of limiting behaviour may be quite different after each disturbance, the movements in both the nominal and the real exchange rates may be much more volatile than what might have been anticipated from a relatively minor change in one exogenous variable. This alternative framework of working with nominal variables and linear equations appears both tractable and fruitful in terms of describing the dynamics of the exchange rate. However, one possible criticism is that in order for the analysis to be tractable, the price adjustment mechanism, equation (88), needed to be formulated using the difference in current supply and its long term level, both measured in nominal terms. It might be more reasonable to argue that the rate of change in prices should still, as in Section 2.2, be related to differences measured in volume, that is deflated terms. Possibly, further consideration of the dimensionality of the variables involved would enable one or more of the basic equations to be reformulated so that a system incorporating a more realistic price adjustment mechanism would then be tractable.

In both Chapters 2 and 3 the relationships between the variables, assumed as part of the models, were purely deterministic. However, it might be considered unlikely that these relationships, such as the one postulated for money demand, will contain all the relevant factors affecting the dynamics of the system. Therefore, in the next Chapter, in order to allow for this uncertainty random variables are incorporated in to some of the basic model relationships, giving rise to stochastic models. In subsequent Chapters it is demonstrated how these models can be used to describe the movements of exchange rates when limits are placed on their range of movements by central banks.

4 Extension to stochastic versions

4.1 The monetary model

4.1.1 Derivation of the differential equation

In previous Chapters, the models of the exchange rate were completely deterministic. However, since the relatively simple basic relationships postulated for variables such as money demand, are unlikely to capture all the relevant factors which might conceivably affect the exchange rate, a more realistic proposal is that some or all of the relationships contain a random or stochastic element, in order to take into account the effect of factors not incorporated in to the model explicitly. Alternatively, such elements might be considered to represent the assumption that the 'true' relationship is in any event stochastic in the first place and writing the relationships in deterministic form is merely a convenient simplification. Then, by making further assumptions concerning the properties of these random elements the dynamics of the exchange rate can be investigated. As a first step, we shall assume that the exchange rate between two countries of similar size is determined by the flexible price monetary model. This is relatively simple compared with other models, but is interesting because a closed form solution exists for this variation. The approach, at least in its current form, was first described in Krugman (1988) and more widely in Krugman (1991) and Krugman (1992). However, Williamson and Miller (1987), had previously discussed a similar but more elaborate model and Smith (1987) had discussed some related problems in stochastic process switching.

In the derivation of the deterministic monetary model the LM relationship in log-linear form for country i ($i = 1, 2$) was shown in equation (21) of Section 2.1.1, to be as follows

$$m_i - p_i = l_i y_i - k_i r_i.$$

As before, m_i is the money supply for country i, p_i is the price level, y_i is real output, and r_i is the interest rate. All variables except interest rates, are measured in logs. In the short term, if money supply, real output and interest rates are unchanged then domestic prices are constant. However, it is an assumption of the flexible price monetary model that if other variables do change, then prices can adjust instantaneously in order that the above relationship continues to hold.

It will now be assumed that the demand for real money balances depends not only on real output and interest rates but possibly also on other factors. The cumulative effect of these other factors will be represented by a stochastic variable. The LM equation for country i may now therefore be written as follows

$$m_i - p_i = l_i y_i - k_i r_i - e_i, \tag{110}$$

where e_i is (the log of) the assumed stochastic demand variable, whose properties we shall return to shortly. As before, prices are assumed suitably flexible so that the above equation holds at all times. As noted in Section 2.1.1, a further assumption of the flexible price monetary model is that purchasing power parity (PPP) holds in its absolute form, that is

$$s_{12} = p_1 - p_2,$$

where s_{12} is the log of the exchange rate between the two countries. Substituting for p_1 and p_2 from the respective money demand relationships summarised in equation (110) gives

$$s_{12} = m_1 - m_2 + l_2 y_2 - l_1 y_1 + k_1 r_1 - k_2 r_2 - e_1 + e_2. \tag{111}$$

Further, if it is assumed that UIP holds with no restrictions on the flow of capital, then the market's anticipated rate of change in the exchange rate equals the difference in interest rates. However, if rational expectations are assumed to hold for the market, then the market's anticipated value of the future exchange rate is, on average, equal to the actual exchange rate at that point in time. Hence, by equation (41), viewing future values of the (log of) the exchange rate as a random variable, the market's anticipated rate of change in the exchange rate, given all the information available at time t, can be written as

$$\frac{\mathrm{E}(ds_{12})_t}{dt},$$

where E is the mathematical expectations operator. The assumption of UIP therefore results in the relationship

$$\frac{\mathrm{E}(ds_{12})_t}{dt} = r_1 - r_2. \tag{112}$$

If it is further assumed that the interest semi-elasticity for the demand for money, k, is identical in the two countries, then substituting for the interest rate differential in equation (111) gives

$$s_{12} = m_1 - m_2 + l_2 y_2 - l_1 y_1 - e_1 + e_2 + k\frac{\mathrm{E}(ds_{12})_t}{dt}. \tag{113}$$

Suppose we define the fundamental measure for a country, f_i, as

$$f_i = m_i - \nu_i \tag{114}$$

where $\nu_i = e_i - l_i y_i$. If y_i is assumed constant, at least in the short term, then the dynamics of ν_i and e_i will be identical.

The fundamental measure summarises the level of the underlying variables upon which the strength of a country's currency depends. If we let $f_{ij} = f_i - f_j$ be an index of the difference in fundamental determinants for the two currencies then equation (113) can be written as follows

$$s_{12} = f_{12} + k\frac{\mathrm{E}(ds_{12})_t}{dt}. \tag{115}$$

Before obtaining the solution for the exchange rate, it is worth observing that although this model was derived on the assumption that money demand depended upon a stochastic variable, it could also be formulated with a random element to total money supply or in the PPP relationship. As indicated by Delgado and Dumas (1990, 1992), in this case, ν_i would represent a composite random variable, being the sum of the random elements in money supply, demand and the PPP relation.

4.1.2 The stochastic demand process

The dynamics of the exchange rate will depend of course on the nature of the process governing the variables ν_i. One of the simplest assumptions (from the point of view of being able to obtain an explicit solution for the exchange rate) is that the stochastic demand variables e_i, and hence ν_i, follow a Brownian motion. This being the continuous time equivalent of a random walk. With this assumption, the change in ν_i can be written as follows

$$d\nu_i = \mu_i dt + \sigma_i dz_i, \tag{116}$$

where μ_i and σ_i are constant and ≥ 0, and dz_i is the increment of a standard Wiener process. In the original description of the model by Krugman the so called drift parameter μ_i was assumed to be zero, but later authors, for instance Svenson (1991), and Delgado and Dumas (1992) show that the non-zero situation can be readily incorporated in to the model.

A standard Wiener process, $z(t)$, has the property that the change in $z(t)$ over a short time period, dz, is a random variable drawn from a normal distribution. In addition, the mean of the change over the short time period is zero, and the variance of the change is equal to the period dt. One way of writing this is as follows

$$dz = \phi\sqrt{dt}, \tag{117}$$

where ϕ is a standard normal deviate. The choice of $\sqrt{dt}$ as the scaling factor is important since other values would result in either trivial or explosive results as $dt \longrightarrow 0$. The term σdz therefore represents a Wiener process whose variance is $\sigma^2 dt$. It is worth noting that the sample paths of a Wiener process are continuous but nowhere differentiable functions of time.

In the above equation for $d\nu_i$, μ_i represents a deterministic trend in ν_i, and σ^2 is referred to as the variance per unit time period for the process. Suppose we let $\nu_{12} = \nu_1 - \nu_2$. Now the first term in the expressions for the incremental change in ν_1 and ν_2 are purely deterministic, and therefore the first term in the expression for the incremental change in ν_{12} will represent the deterministic trend $\mu_1 - \mu_2$. However, the second term in the incremental change to ν_i is proportional to an increment of a standard Wiener process and therefore, as noted above, may be written, at least symbolically, as

$$dz_i = \phi_i\sqrt{dt},$$

where ϕ_i is a standard normal deviate.

As the difference in two normal variables is also a normal variable, the second term in the expression for the incremental change in ν_{12} also behaves as a Wiener process with variance per unit time period of $\sigma_1^2 + \sigma_2^2$, and can be written as

$$(\sigma_1^2 + \sigma_2^2)^{\frac{1}{2}} \phi_{12} \sqrt{dt},$$

where ϕ_{12} is also a standard normal variable. From equation (116), the incremental change in ν_{12} may therefore be written as

$$d\nu_{12} = \mu_{12} dt + \sigma_{12} dz_{12}, \tag{118}$$

where $\mu_{12} = \mu_1 - \mu_2$, $\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2$ and dz_{12} is the increment of a standard Wiener process.

The fundamental index, upon which the exchange rate depends, is a function not only of the stochastic variable ν_{12} but also of relative money supply between the two countries. By adjusting relative money supply countries can therefore attempt to influence the exchange rate. However, in order to obtain an explicit solution for the dynamics of the exchange rate we shall firstly consider the situation of when there is no change in money supply by either country, that is without intervention. As we shall see later, the dynamics of the system with a target zone imposed on the exchange rate can be obtained as a special case of the non-intervention solution with appropriate boundary conditions. The approach relies on results stemming from Harrison (1985) in which he showed how to derive solutions for a variety of problems in controlled Brownian motion. Harrison had previously described some of these ideas, in Harrison and Taksar (1983), and Harrison, Sellke and Taylor (1983).

4.1.3 Dynamics of the exchange rate without intervention

If countries do not attempt to influence the exchange rate by altering relative money supply, then the latter can be considered constant. Under these conditions the incremental change for the fundamental index defined in relationship (114), will be given as follows

$$df_{12} = d\nu_{12} = \mu_{12}dt + \sigma_{12}dz_{12}. \tag{119}$$

As dz_{12} is a random variable rather than deterministic, the above is a form of stochastic differential equation, and the fundamental index will be a stochastic variable. If the fundamental index had been deterministic, Taylor's theorem could have been used to obtain a relationship between a small change in the exchange rate resulting from a small change in the fundamental index. However, as the latter is a stochastic variable to achieve the equivalent relationship we need to appeal to Ito's lemma. With the assumption that s_{12} is a twice differentiable function of f_{12}, and using the relationship for df_{12} obtained in equation (119), application of Ito's lemma indicates that the incremental change to s_{12} can be written as follows

$$ds_{12} = \frac{ds_{12}}{df_{12}}\sigma_{12}dz_{12} + \left(\mu_{12}\frac{ds_{12}}{df_{12}} + \frac{\sigma_{12}^2 d^2 s_{12}}{2df_{12}^2}\right)dt \tag{120}$$

Taking expectations conditional on the information set at time t, and noting that the expectation of the standard Wiener process dz_{12} is zero, gives

$$\frac{\mathrm{E}(ds_{12})_t}{dt} = \mu_{12}\frac{ds_{12}}{df_{12}} + \frac{\sigma_{12}^2 d^2 s_{12}}{2df_{12}^2}. \tag{121}$$

Hence, substituting for the conditional expectation in the relationship between the exchange rate and the fundamental index, represented by equation (115), gives the following

$$s_{12} = f_{12} + k\mu_{12}\frac{ds_{12}}{df_{12}} + \frac{k\sigma_{12}^2 d^2 s_{12}}{2df_{12}^2}. \tag{122}$$

This of course is an ordinary differential equation in the variable f_{12} and has general solution

$$s_{12} = f_{12} + k\mu_{12} + A_1 e^{\rho_1 f_{12}} + A_2 e^{\rho_2 f_{12}}, \tag{123}$$

where $\rho_1 > 0$ and $\rho_2 < 0$, are the roots of the equation

$$\frac{\sigma_{12}^2\rho^2}{2} + \mu_{12}k\rho - 1 = 0,$$

and A_1 and A_2 are constants of integration.

If we impose the further assumption that the exchange rate should not grow explosively as $f_{12} \rightarrow \pm\infty$ (which is possible since there is no restriction on the index in the non-intervention case), then this implies both constants of integration must be zero. The relationship between the exchange rate and the fundamental index, given in equation (123), would then reduce to

$$s_{12} = f_{12} + k\mu_{12}. \tag{124}$$

This is usually referred to as the free float solution for the exchange rate. In the simplest situation of no deterministic trend, the (log of) the exchange rate is just equal to the fundamental index.

4.2 The price inertia model

4.2.1 Derivation of the fundamental differential equation

In the previous Section, it was assumed that the LM equation for each country contained a stochastic variable in order to represent the cumulative effect of those factors excluding real output and interest rates which might affect the demand for real money balances. However, in the price inertia model it is customary to assume that the stochastic variable representing uncertainty, forms part of the price adjustment equation. For convenience, we shall adopt the same approach but it is worth noting that the derivation is substantively unaltered if the stochastic variable appears in, say, the LM equation or even the IS equation, either in place of, or in addition to, a stochastic term in the price equation. In the latter case the variable we finally work with can be considered to be a composite of the individual stochastic variables. The approach adopted below, is based upon that given in Miller and Weller (1990) and Miller and Weller (1991). Following the same notation used in equation (20) of Section 1.5 the LM relationship for country i may be written as

$$m_i - p_i = ly_i - kr_i.$$

Miller and Weller assume an IS equation for country 1 of the following form

$$y_1 = -\gamma(r_1 - \pi_1) + \eta(s_{12} - p_1 + p_2). \tag{125}$$

This is essentially of the same form as that discussed in Section 2.2.1 but with the following modifications. Firstly, the consumption of goods, and the level of investment, are assumed to depend on real interest rates, that is, domestic interest rates adjusted for the anticipated rate of domestic price inflation, π_1.This is usually accepted as being more realistic than assuming dependency on interest rates alone. Secondly, if net government expenditure is assumed constant over the time scale covered by the model, the dynamics of the system will not be affected qualitatively by its precise value. For convenience therefore, it has been taken as zero. Lastly, to simplify the presentation but without effecting the dynamics of the model, the coefficient of national real income has been taken as unity. This is equivalent to dividing through by the original coefficient which is arbitrary except for being positive.

Similarly, using the same assumptions and notation, the IS equation for the second country will be as follows

$$y_2 = -\gamma(r_2 - \pi_2) - \eta(s_{12} - p_1 + p_2). \tag{126}$$

As before, for convenience the macroeconomic parameters in the two countries have been assumed to be identical.

In Section 2.2.1 it was assumed that the rate of price adjustment in the price inertia model was proportional to the excess of the short term demand for goods and services over the long term equilibrium level of supply, $\overline{y}$. In this Section we shall make the same assumption but, in addition, we shall assume that the change in price also depends upon the change in a stochastic variable e. Upon making the same assumption of equality between short term demand and national output as was done in Section 2.2.1, the price adjustment equation for country i can therefore be written as

$$dp_i = \phi(y_i - \overline{y}_i)dt + de_i. \tag{127}$$

If rational expectations are assumed to hold, then the market's anticipated inflation rate can be equated with the expected value of the actual price change. Hence, using the same notation for the expected value of a random variable made at time t, with all information up to that point assumed known, we can write

$$\pi_i = \frac{\mathrm{E}(dp_i)_t}{dt}. \tag{128}$$

Finally, we shall assume that the UIP condition holds continuously. As noted in Section 2.2.1, if rational expectations apply in the market, this is equivalent to requiring that the expected rate of change of the exchange rate is equal to the interest rate differential. That is

$$\frac{\mathrm{E}(ds_{12})_t}{dt} = r_1 - r_2. \tag{129}$$

Taking the difference in the price adjustment relationship for the two countries, given by equation (127), gives

$$dp_1 - dp_2 = \phi(y_1 - y_2 - \overline{y}_1 + \overline{y}_2)dt + de_1 - de_2. \tag{130}$$

Upon using the respective LM and IS equations to eliminate y_1 and y_2, in equation (130), we obtain

$$dp_{12} = \frac{(2\phi k\eta s_{12} - p_{12}(\gamma\phi + 2k\eta\phi) + \phi\gamma m_{12} - \phi(l\gamma + k)\overline{y}_{12})dt}{l\gamma + k - \phi\gamma k} + de_1 - de_2 \tag{131}$$

where, $p_{12} = p_1 - p_2$, $m_{12} = m_1 - m_2$ and $\overline{y}_{12} = \overline{y}_1 - \overline{y}_2$. Similarly, after substituting for the equilibrium interest rates from the respective LM and IS equations, in to the UIP condition given by equation (129), we obtain

$$\mathrm{E}(ds_{12})_t = \frac{(2l\eta s_{12} + p_{12}(1 - \gamma\phi - 2l\eta) - \phi\gamma l\overline{y}_{12} - (1 - \gamma\phi)m_{12})dt}{l\gamma + k - \phi\gamma k} \tag{132}$$

If we assume that each of the stochastic variables in the price adjustment equations follow a Brownian motion, with no deterministic trend, then as discussed in Section 4.1.1 their rate of change can be written as

$$de_i = \sigma_i dz_i \tag{133}$$

where σ_i^2 is the variance per unit time period for the variable and dz_i is the increment to a standard Wiener process. Again, as noted in Section 4.1.1, the difference in the rates of change is also a Brownian motion with variance per unit time period $\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2$ and with an increment to a standard Wiener process of dz_{12}. Then the following result can be obtained.

Proposition 8 *Suppose $s_{12}-m_{12} = f(p_{12}-m_{12})$ where f is a deterministic, twice differentiable function, then*

$$\frac{\sigma_{12}^2 f''}{2} + (a_{11}(p_{12} - m_{12}) + a_{12}f)\, f' - \left(a_{21}(p_{12} - m_{12}) + a_{22}f\right) = 0$$

where the coefficients a_{ij} are the ij th elements of a matrix A.

Proof

Firstly, with the foregoing notation, equation (131) for the rate of change in relative prices can be written as

$$dp_{12} = \frac{(2\phi k\eta s_{12} - p_{12}(\gamma\phi + 2k\eta\phi) + \phi\gamma m_{12} - \phi(l\gamma + k)\overline{y}_{12})dt}{l\gamma + k - \phi\gamma k} + \sigma_{12}dz_{12} \tag{134}$$

The system of equations representing the dynamics of the process can then be represented in more succinct form by the following matrix relationship

$$\begin{bmatrix} dp_{12} \\ \mathrm{E}(ds_{12})_t \end{bmatrix} = A \begin{bmatrix} (p_{12} - m_{12})dt \\ (s_{12} - m_{12})dt \end{bmatrix} + B \begin{bmatrix} 0 \\ \overline{y}_{12}dt \end{bmatrix} + \begin{bmatrix} \sigma_{12}dz_{12} \\ 0 \end{bmatrix} \tag{135}$$

where

$$A = \frac{1}{\Delta} \begin{bmatrix} -\phi(\gamma + 2k\eta) & 2\phi k\eta \\ 1 - \gamma\phi - 2l\eta & 2l\eta \end{bmatrix}$$

$$B = \frac{1}{\Delta} \begin{bmatrix} 0 & -\phi(l\gamma + k) \\ 0 & -\phi\gamma l \end{bmatrix}$$

and $\Delta = l\gamma + k - \phi\gamma k$.

In order to simplify notation, redefine the variables $p_{12} - m_{12}$ and $s_{12} - m_{12}$ as deviations from the long term equilibrium values of the deterministic model. The system of equations may then be written as

$$\begin{bmatrix} dp_{12} \\ \mathrm{E}(ds_{12})_t \end{bmatrix} = A \begin{bmatrix} (p_{12} - m_{12})dt \\ (s_{12} - m_{12})dt \end{bmatrix} + \begin{bmatrix} \sigma_{12} dz_{12} \\ 0 \end{bmatrix}. \tag{136}$$

Suppose $s_{12} - m_{12}$ is a deterministic, twice differentiable function of $(p_{12} - m_{12})$ which we shall denote by f. Then, by Ito's lemma, the incremental change to s_{12} can be written as follows

$$ds_{12} = f' dp_{12} + \left(\frac{\sigma_{12}^2 f''}{2} \right) dt \tag{137}$$

from which it follows that

$$\mathrm{E}(ds_{12})_t = f' \mathrm{E}(dp_{12})_t + \left(\frac{\sigma_{12}^2 f''}{2} \right) dt. \tag{138}$$

Now from the system of equations (136), we obtain $\mathrm{E}(ds_{12})_t = a_{21} p_{12} dt + a_{22} s_{12} dt$ and $\mathrm{E}(dp_{12})_t = a_{11} p_{12} dt + a_{12} s_{12} dt$, where a_{ij} denotes the appropriate element of the matrix A. Hence, substituting for $\mathrm{E}(ds_{12})_t$ and $\mathrm{E}(dp_{12})_t$ in relationship (138), obtained by Ito's lemma, we obtain

$$\frac{\sigma_{12}^2 f''}{2} + \left(a_{11}(p_{12} - m_{12}) + a_{12} f\right) f' - \left(a_{21}(p_{12} - m_{12}) + a_{22} f\right) = 0 \tag{139}$$

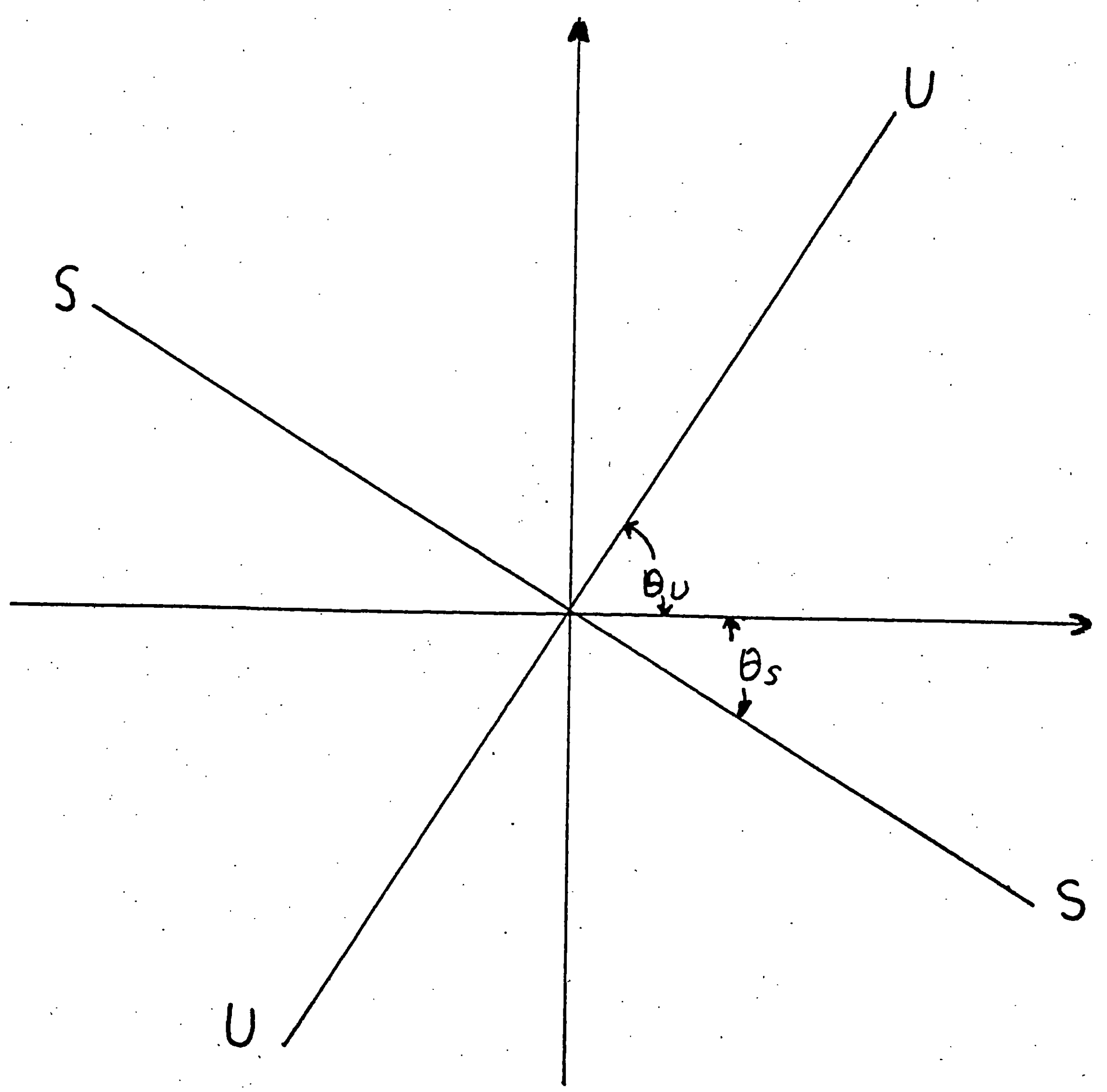
which is the required result. We shall refer to this as the fundamental differential equation.

□

The above second-order, nonlinear differential equation has no closed form solutions in general, but it is possible to characterize the qualitative features of the solutions under certain boundary conditions.

In order to characterize the solution paths, we shall make use of the eigenvectors of matrix A, which are denoted by SS and UU in figure 4.1. In fact, these are themselves solutions of the fundamental differential equation corresponding to the situation $\sigma = 0$ and therefore represent the solution paths for the equivalent deterministic system. The slopes of these eigenvectors are obtained as the roots of a quadratic equation in the elements of A. As this equation plays a role in describing the solutions to the stochastic model it is worth obtaining it explicitly at this point.

Figure 4.1 The eigenvectors of the system.



Let an eigenvector of A, normalized on its first element, be written as follows

$$\begin{bmatrix} 1 \\ \hline \theta \end{bmatrix}.$$

It must, by definition, satisfy the condition that

$$A \begin{bmatrix} 1 \\ \hline \theta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \hline \theta \end{bmatrix}$$

or, in detail,

$$\begin{aligned} a_{11} + a_{12}\theta &= \lambda & (140) \\ a_{21} + a_{22}\theta &= \lambda\theta & (141) \end{aligned}$$

where λ is a root of the characteristic equation of A. Eliminating λ from equations (140) and (141), gives the following requirement

$$p(\theta) = a_{21} + (a_{22} - a_{11})\theta - a_{12}\theta^2 = 0. \qquad (142)$$

So $p(\theta)$ represents the required quadratic expression whose zeros, θ_S and θ_U, give the slopes of the eigenvectors of A.

Next, consider all solutions to the differential equation satisfying the boundary condition $f(0) = 0$. Since a second-order differential equation requires two boundary conditions to tie down a unique solution, there will be an infinite family of trajectories passing through the origin. As noted in Section 2.2.1, deterministic price inertia models have qualitatively different exchange rate paths for their solutions depending upon whether the combination of economic parameters leads to an overshooting effect of the exchange rate or not. Although figure 4.1 illustrates the eigenvector positions corresponding to the usual overshooting case in which money supply remains unchanged after some initial disturbance, the following approach for qualitatively describing the solutions can readily be used with combinations of macroeconomic parameters which lead to eigenvectors with differing patterns of slopes.

An inspection of the fundamental differential equation indicates that if $f(p_{12}-m_{12})$ is a solution satisfying $f(0)=0$, then $f(p_{12}-m_{12}) = -f(-(p_{12}-m_{12}))$. Therefore, all solutions are antisymmetric so we need only consider in detail the two quadrants in which $p_{12}-m_{12} > 0$. In addition, it is worth noting that because of the requirement that $f(0)=0$ the fundamental differential equation implies that $f''(0) = 0$ also. In order to analyse the trajectories, we shall firstly examine the properties of the solutions close to the equilibrium point, and then look at the more global properties of the solutions. The approach is that outlined in Miller and Weller (1989).

4.2.2 Curvature around equilibrium

The curvature of paths in the neighbourhood of the origin, here the equilibrium point, can be determined by approximating the value taken by the second differential of any solution in that vicinity. Suppose, for ease of notation, we let $x = p_{12} - m_{12}, y = s_{12} - m_{12}$ and we drop the subscripts on σ^2. Then if $f(x)$ is a solution with the condition $f(0) = 0$, to a first approximation we have the following result.

Proposition 9 *In the vicinity of the origin*

$$\frac{\sigma^2}{2} f''(x) \simeq p(\theta)x$$

where $p(\theta)$ is the quadratic equation whose zeros give the slope of the eigenvectors of matrix A.

Proof.

Suppose that the second derivative of $f(x)$ meets the requirements for it to have a Maclaurin's expansion as far as the first order around the origin. Then, to a first order approximation we have

$$\frac{\sigma^2}{2} f''(x) \simeq \frac{\sigma^2}{2}(f''(0) + f'''(0)x). \tag{143}$$

Since the second derivative is zero at the origin, this implies

$$\frac{\sigma^2}{2} f''(x) \simeq \frac{\sigma^2}{2} f'''(0)x. \tag{144}$$

Recall that the fundamental differential equation for a solution is given by

$$\frac{\sigma^2}{2} f''(x) + (a_{11}x + a_{12}f(x))\, f'(x) - (a_{21}x + a_{22}f(x)) = 0.$$

So

$$\frac{\sigma^2}{2} f''(x) = -\left(a_{11}x + a_{12}f(x)\right) f'(x) + \left(a_{21}x + a_{22}f(x)\right). \tag{145}$$

Differentiating both sides, evaluating the result at $x = 0$, and then substituting for $f''(0)$ in approximation (144), gives

$$\frac{\sigma^2}{2} f''(x) \simeq \left(-f'(0)\left(a_{11} + a_{12}f'(0)\right) + a_{21} + a_{22}f'(0)\right) x. \tag{146}$$

Letting $\theta = f'(0)$ leads to the following

$$\frac{\sigma^2}{2} f''(x) \simeq p(\theta)x \tag{147}$$

which is the required result.

□

An inspection of the elements of matrix A indicates that for the values of the macroeconomic parameters postulated, a_{11} will be negative while a_{12} and a_{22} will be positive. In addition, in order for the eigenvectors to represent the overshooting case illustrated in figure 4.1, a_{21} also needs to be positive. It therefore follows that $p(0) = a_{21} > 0$ and that $p(\theta)$ reaches a maximum between θ_S and θ_U. A sketch of the function would therefore look as in figure 4.2. This now enables us to calculate the sign of f'' in the neighbourhood of the origin. As the angle $\tan^{-1}(\theta)$ goes from 0 to $\pm$ 90° the sign of f'' changes with each crossing of an eigenvector. The results for the other two quadrants can be readily obtained on recalling that solutions must be antisymmetric. Figure 4.3 summarises these results. To complete the qualitative description of solutions, we now need to examine the trajectory paths away from the origin.

Figure 4.2 The value of $p(\theta)$ in the half plane $x > 0$.

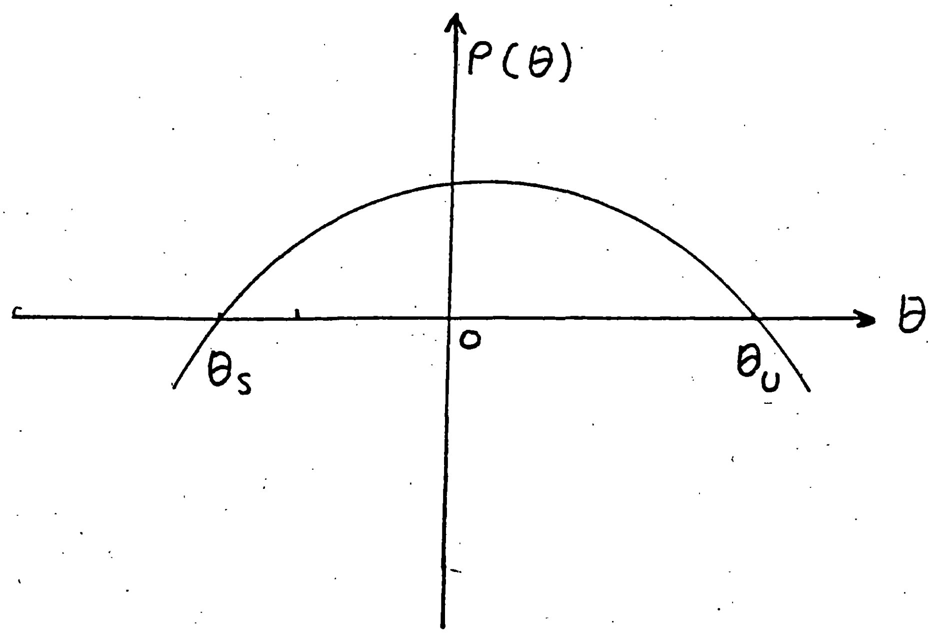
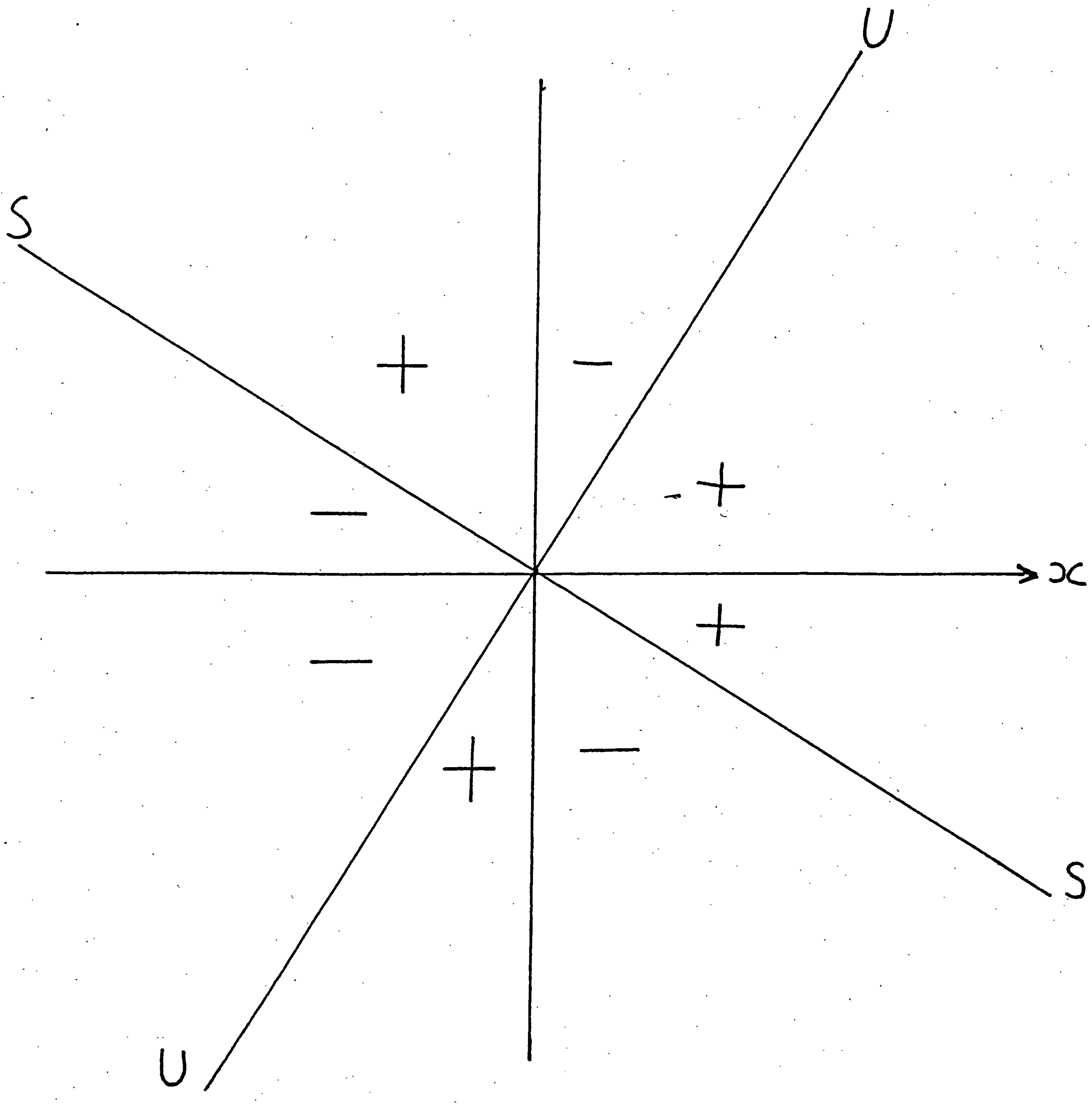


Figure 4.3 The sign of f'' in the neighbourhood of the origin.



Firstly, we shall obtain the line of expected stationarity for x. Using the same notation as in the previous section, we have from the system of equations (136)

$$dx = a_{11}xdt + a_{12}ydt + \sigma dz. \tag{148}$$

Taking expectations gives

$$\mathrm{E}(dx)_t = a_{11}xdt + a_{12}ydt. \tag{149}$$

Therefore, along the line of expected stationarity for x we have

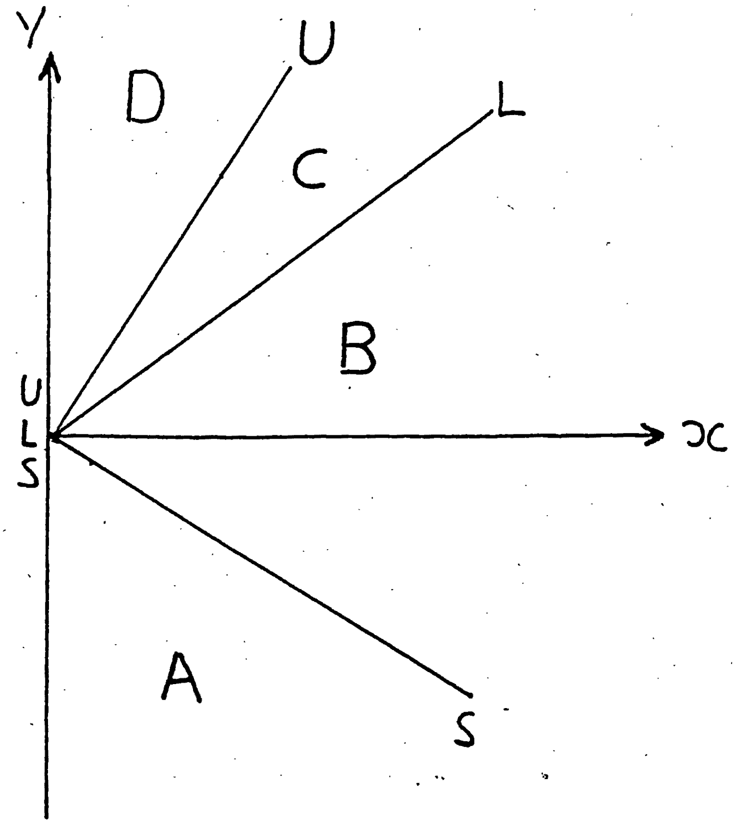
$$\frac{\mathrm{E}(dx)_t}{dt} = a_{11}x + a_{12}y = 0.$$

Hence, for the line of expected stationarity

$$slope = -\frac{a_{11}}{a_{12}}. \tag{150}$$

Because of the symmetry of solutions, it is only necessary to consider the behaviour of trajectories in one half of the plane. Therefore, taking the right hand half-plane, where $x \geq 0$, we divide this into four regions as shown in figure 4.4. Regions A and B lie on either side of the eigenvector SS and below the line of expected stationarity in x, labelled as LL. Regions C and D lie on either side of the second eigenvector, UU and above line LL. We shall consider the regions in pairs. We first observe that by a standard theorem on differential equations, such as in Birkhoff and Rota (1969), p152, for example, the two initial conditions $f(0) = 0$ and $f'(0) = 0$, are sufficient to determine a unique solution to the fundamental differential equation in any compact, convex region of the (x,y) plane: so the solution trajectories do not intersect other than at the origin. We now turn to the first pair of regions.

Figure 4.4 The four regions in the global analysis of solution paths.



4.2.3 Trajectories in regions A and B

Rather than working directly with the function $f(x)$, it will be convenient to use the function $g(x)$ which measures the vertical distance between the function $f(x)$ and the corresponding point on the eigenvector SS. Hence

$$g(x) = f(x) - \theta_S x. \tag{151}$$

It follows that

$$g' = f' - \theta_S$$

and

$$g'' = f''.$$

Therefore, while the slope of g differs from that of f by a constant, the convexity/concavity of the two functions is the same for any value of x. As a first step in describing the trajectories we shall obtain the differential equation that must be satisfied by the function $g(x)$.

Proposition 10 *The function $g(x)$ must satisfy the following differential equation*

$$\frac{\sigma^2}{2} g'' = -\left(\left(a_{11} + a_{12}\theta_S\right) x + a_{12} g\right) g' + \left(a_{22} - \theta_S a_{12}\right) g.$$

Proof.
First, recall the fundamental differential equation for f

$$\frac{\sigma^2}{2} f''(x) + \left(a_{11} x + a_{12} f(x)\right) f'(x) - \left(a_{21} x + a_{22} f(x)\right) = 0.$$

Upon substituting for f in terms of g we obtain

$$\frac{\sigma^2}{2} g'' = -\left(g' + \theta_S\right)\left(a_{11} x + a_{12}\left(g + \theta_S x\right)\right) + a_{21} x + a_{22}\left(g + \theta_S x\right). \tag{152}$$

This can be rearranged in order to give

$$\frac{\sigma^2}{2} g'' = -\left(\left(a_{11} + a_{12}\theta_S\right) x + a_{12} g\right) g' + p\left(\theta_S\right) x + \left(a_{22} - \theta_S a_{12}\right) g. \qquad (153)$$

Upon noting that $p(\theta_S) = 0$, we obtain the result.

□

In order to characterize paths in these regions it is necessary first to demonstrate that the terms multiplying g and g' in the equation (153) are both positive. The former coefficient

$$a_{22} - \theta_S a_{12},$$

is certainly positive because by assumption a_{22}, $a_{12} > 0$, and $\theta_S < 0$. For the latter coefficient

$$-\left(\left(a_{11} + a_{12}\theta_S\right) x + a_{12} g\right),$$

we note that in regions A and B f lies below the LL locus, which has slope $-a_{11}/a_{12}$. Therefore it follows that

$$g = f - \theta_S x < \frac{-a_{11} x}{a_{12}} - \theta_S x.$$

From which we can deduce that

$$a_{12} g < -\left(a_{11} + a_{12}\theta_S\right) x.$$

Finally, we obtain the required result that

$$0 < -\left(\left(a_{11} + a_{12}\theta_S\right) x + a_{12} g\right). \qquad (154)$$

We shall examine first the solution paths in region A. In this region, by assumption $f'(0) < \theta_S$ so by construction $g'(0) < 0$. In region A the following result holds.

Proposition 11 *For $x>0$, $g'(x)<0$.*

Proof.
Suppose that this is not the case and that x_1 is the smallest strictly positive value of x such that $g''(x_1) = 0$. By construction, g and g' will be negative in a neighbourhood of the origin, and from the result of Proposition 9, f'' and hence g'' will also be negative in such a neighbourhood. Recall that the differential equation that must be satisfied by g is given as follows

$$\frac{\sigma^2}{2}g'' = -\left(\left(a_{11} + a_{12}\theta_S\right)x + a_{12}g\right)g' + \left(a_{22} - \theta_S a_{12}\right)g.$$

It follows that since both the coefficients of g and g' are positive, that either $g(x_1) \geq 0$ or $g'(x_1) \geq 0$. Assume firstly the case $g'(x_1) \geq 0$. Then since g' is negative in the neighbourhood of the origin it follows that between the neighbourhood and x_1, g'' must be positive for some values of x. As $g'' < 0$ in a neighbourhood of the origin, it follows that since g is continuous, that there exists a value x_2, where $x_2 < x_1$ such that $g'' = 0$, contrary to hypothesis. Similarly, if $g(x_1) \geq 0$, then because g is negative in a neighbourhood of the origin, there must exist values of $x < x_1$ for which $g'(x) > 0$. Then by the foregoing argument, there must the be at least one value of $x < x_1$ at which $g'' = 0$, contrary to hypothesis again.

□

An analogous argument establishes that $g'' > 0$ for $x > 0$ in region B, that is for all solution paths satisfying the condition

$$0 < g'(0) < \frac{-a_{11}}{a_{12}} - \theta_S.$$

Thus, the curvature found in a neighbourhood of the origin applies throughout each of these regions. On the boundary between the two regions, where $g' = \theta_S$, $g'' = 0$ for all $x > 0$.

4.2.4 Trajectories in regions C and D

Here, it will be convenient to define a function $h(x)$ to represent the 'distance' of f from the second eigenvector UU. That is

$$h(x) = f(x) - \theta_U x.$$

The first and second derivatives of h are therefore as follows

$$\begin{aligned} h' &= f' - \theta_U \\ h'' &= f''. \end{aligned} \tag{155}$$

As in the analysis in regions A and B, we start by substituting for f in the fundamental differential equation.(139). After simplification, this gives the following

$$\frac{\sigma^2}{2} h'' = -\left(\left(a_{11} + a_{12}\theta_U \right) x + a_{12} h \right) h' + \left(a_{22} - \theta_U a_{12} \right) h. \tag{156}$$

We shall show that the coefficients of both h and h' in the above equation will be negative in regions C and D. As for the former term, $a_{22} - \theta_U a_{12}$, we note that

$$p(\theta_U) = -a_{12}\theta_U^2 + (a_{22} - a_{11})\theta_U + a_{21} = 0. \tag{157}$$

So

$$a_{22} - \theta_U a_{12} = \frac{-a_{21}}{\theta_U} + a_{11}. \tag{158}$$

The result follows after noting that $a_{21} > 0$, and $\theta_U,\ a_{11} < 0$.

For the second coefficient, $-\left(\left(a_{11} + a_{12}\theta_U \right) x + a_{12} h \right)$, the requirement that

$$f > \frac{-a_{11}}{a_{12}} x$$

implies that

$$h = f - \theta_U x > \frac{-a_{11}}{a_{12}} x - \theta_U x = -\left(\frac{a_{11}}{a_{12}} + \theta_U \right) x. \tag{159}$$

So

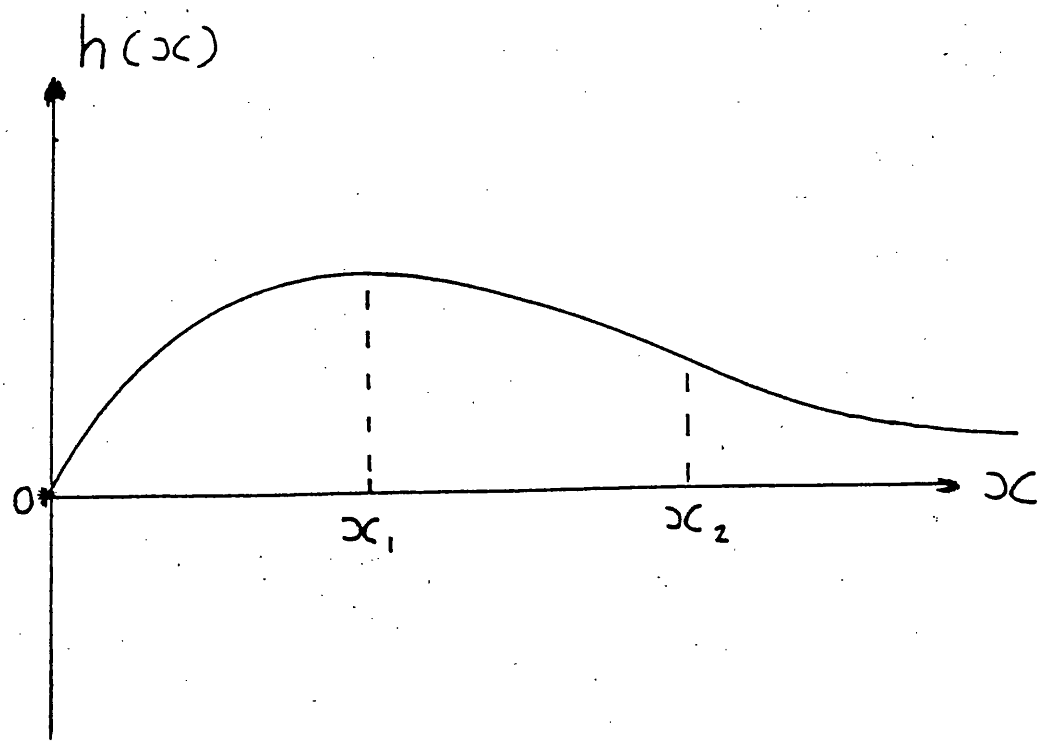
$$a_{12}h > -\left(a_{11} + a_{12}\theta_U\right)x.$$

Hence we obtain the required result that

$$0 > -\left(\left(a_{11} + a_{12}\theta_U\right)x + a_{12}h\right). \tag{160}$$

To examine the curvature of solutions near eigenvector UU, we begin in region D where all the trajectories are connected to the origin. By construction, this will also be the curvature exhibited by function h. We intend to show that trajectories have the qualitative characteristics shown in figure 4.5. That is, it has a positive slope but is concave near the origin, reaches a maximum, and passes through a unique point of inflection before approaching a limiting value as x increases without limit.

Figure 4.5 The function $h(x)$ in region D.



From proposition 9 it follows that $f'' < 0$ in the neighbourhood of the origin for region D,and so by the definition of $h(x)$, establishes that $h''(x) < 0$ for paths starting from the origin with a positive slope. The general characteristics of a solution path, as x moves away from the origin, are established as follows.

Proposition 12 *There exists a value x_1 satisfying $h'(x_1)=0$*

Proof

Suppose the proposition is not true. Since $h'(0) > 0$ in region D, then *ex hypothesi,* $h' > 0$ for all $x > 0$ and h is bounded away from zero as x increases, and is always positive. However, inspection of the differential equation for h

$$\frac{\sigma^2}{2}h'' = -\left(\left(a_{11} + a_{12}\theta_U\right)x + a_{12}h\right)h' + \left(a_{22} - \theta_U a_{12}\right)h$$

implies that $h'' < 0$ for all $x \geq 0$ and is bounded away from zero as x increases. Hence, h' is strictly decreasing and so for some value of x, x_1 say, $h'(x_1) = 0$, contrary to hypothesis.

□

Proposition 13 *$h(x)$ is strictly concave over the range $0 \leq x \leq x_1$ and reaches a maximum at x_1*

Proof

As $h(0) = 0$ and $h'(0) > 0$, it follows from the fact that the coefficients of both h and h' are negative in the differential equation (156), that $h'' < 0$ over the range $0 \leq x \leq x_1$. Since $h'(x_1) = 0$ and $h''(x_1) < 0$, it follows that $h(x_1)$ is a maximum.

□

Proposition 14 *For all $x > x_1$, $h'(x) < 0$*

Proof
Since the line $h = 0$ is itself a solution of the differential equation for h, it follows from the uniqueness theorem in Birkhoff and Rota (1969), that any other solution with $h(0) = 0$ and $h'(0) > 0$, must remain above the $h = 0$ line. Hence, $h > 0$ for all $x > 0$.

It follows from the fact that the coefficients for both h and h' are negative in the differential equation (156) for h, that at any point where $h' = 0$, $h'' < 0$. So if there existed another turning point following x_1, this would be a local maxima, and hence x_1 would have to be a local minima, which can not be so. So there can be no such points.

□

The characterization of the trajectories is completed by the following two results.

Proposition 15 *There exists a unique point of inflection for $h(x)$ at $x_2 > x_1$.*

Proof
Since h is bounded from below by zero, and $h'(x_1) = 0$, $h''(x_1) < 0$, it follows from the differential equation for h, that there must exist some value of $x > x_1$, at which h'' passes through zero from below. That is

$$\begin{aligned} h''(x) &= 0, \\ h'''(x) &> 0. \end{aligned}$$

Let x_2 be the smallest such value. Then, differentiating both sides of differential equation (156) for h, and using the fact that $h''(x_2) = 0$, gives

$$\frac{\sigma^2}{2} h'''(x_2) = -\left(a_{11} + a_{12}\theta_U\right) h'(x_2) - a_{12} h'(x_2)^2 + \left(a_{22} - \theta_U a_{12}\right) h'(x_2) > 0.$$

It follows from the above relationship that

$$h'(x_2) > \frac{a_{22} - a_{11}}{a_{12}} - 2\theta_U.$$

Suppose there exists another point of inflection at x_3, such that $x_3 > x_2$. Then by assumption, $h''(x_3) = 0$ and $h'' > 0$ for $x_2 < x < x_3$. Therefore h'' must pass through zero from above at x_3. That is $h'''(x_3) < 0$.

Now since $h'' > 0$, for $x_2 < x < x_3$, it must follow that

$$h'(x_3) > \frac{a_{22} - a_{11}}{a_{12}} - 2\theta_U.$$

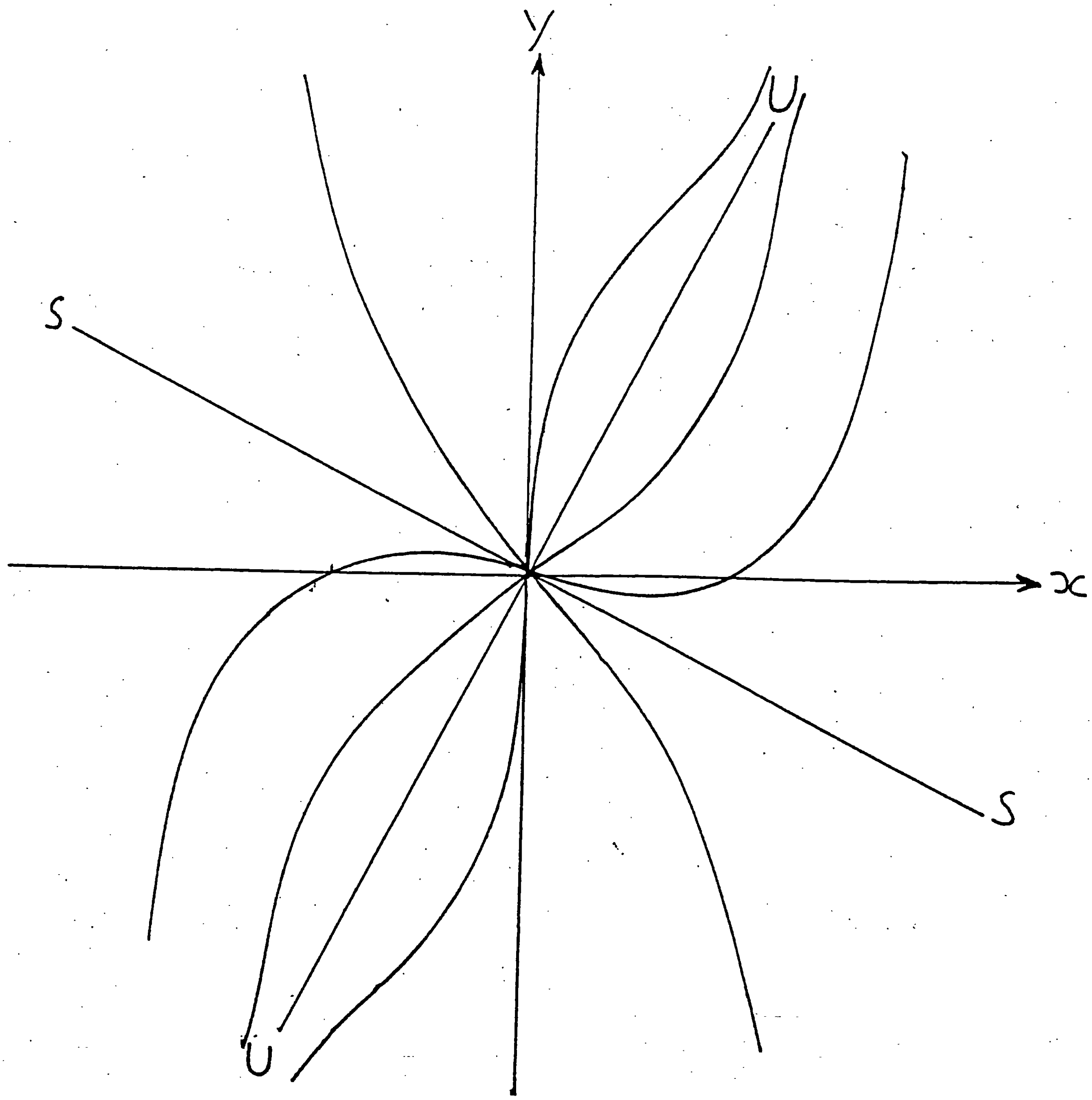
But, it is also a consequence of the earlier relationship that such a value for h' would imply that $h'''(x_3) > 0$, which is a contradiction. Therefore the point of inflection at x_2 is unique.

$\square$

Finally, we note that proposition 14, together with the fact that h is bounded below by zero, ensures that h approaches a well defined limit as $x \to \infty$.

The arguments needed to characterise paths in region C are very similar to those above. The only point of note is that in region C some paths enter from region B, and it may be true that $h' > 0$ when they enter. Therefore, the analogue of proposition 12 is a demonstration that there exists x_1 satisfying $h'(x_1) \geq 0$, where we pick $x_1$to be the smallest value of x satisfying the inequality. Subsequent proofs then follow similarly to before, but with suitable changes of sign. The foregoing results for the regions A to D, together with the requirement that solution paths are antisymmetric, implies that the nature of solutions is as shown in figure 4.6.

Figure 4.6 Stochastic solution paths.



Two special cases are worth noting for the solution paths. First, if $a_{12} = 0$ then the eigenvector UU coincides with the vertical axis, and the regions C and D disappear. Solution paths lying in the half-plane where $x \geq 0$ are globally concave or convex and all points of inflection, other than those at the origin, vanish. Second, if $a_{12} = a_{22} = 0$, then the two eigenvectors, SS and UU, correspond to the horizontal and vertical axes respectively. In this case, all turning points vanish.

4.3 Summary

In this Chapter it has been demonstrated how stochastic versions of the monetary and price inertia models can be justified and formulated. If the random elements in these models are assumed to follow a Wiener process then an explicit relationship, up to some constants of integration, between the exchange rate and the assumed fundamental variables upon which it depends, can be obtained for the monetary model. In general, the exchange rate paths will be non-linear except in the situation of a free float when no interventions in order to control the exchange rate are assumed. Although an explicit solution for the price inertia model is not in general possible, investigation of the qualitative nature of those solution paths indicates that they will also be highly non-linear. In both models the potential range of solution paths seems extremely large even if their general characteristics can be described, and seem to emphasise the complexity and uncertainty in trying to obtain a particular solution which adequately describes a practical situation. However, if further constraints can be incorporated in to the models, for instance rules concerning when changes to money supply can take place, the potential solution paths can be narrowed down considerably or even made unique. In the next Chapter we shall impose just such constraints in the context of modelling the movement of the exchange rate within an allowable band, a so called target zone.

5 Target zones

5.1 The monetary model

In equation (123) of Section 4.1 the stochastic monetary model for two countries was shown to have the following general solution

$$s_{12}(f_{12}) = f_{12} + k\mu_{12} + A_1 e^{\rho_1 f_{12}} + A_2 e^{\rho_2 f_{12}} \tag{161}$$

where A_1 and A_2 are constants of integration. Suppose now that a target zone is imposed on the range of the fundamental index. As we shall see later, this is equivalent to a target zone on the exchange rate itself. In order to obtain an explicit solution for the exchange rate as a function of the fundamental index, the following assumptions will be made.

1. Intervention in the foreign exchange market by central banks occurs at the boundaries of the target zone. This intervention takes the form of an adjustment to relative money supply in order to prevent the fundamental index from exceeding the target zone boundary. The interventions are assumed to be instantaneous and infinitesimal. More complicated intervention rules are possible, for instance discrete interventions, or an intra marginal intervention of some form. However, the current assumption allows the method of approach to be demonstrated most clearly.

2. There is full co-operation between the central banks to render the target zone completely credible. This includes full knowledge by the market of the target zone and the method of intervention. The assumption of credible co-operation between central banks is an important aspect of many discussions of exchange rate regimes whether or not involving formal target zones. For example Avesani (1990), Delgado and Dumas (1993), Lockwood, Miller and Zhang (1996), and Walsh (1995), discuss the role of central banks and their reputation.

3. The exchange rate, s_{12}, is assumed to be a continuous function of f_{12} over the entire target zone interval. This precludes 'jumps' in the exchange rate which would otherwise lead to anticipated excess profit.

The actual method of intervention can be described in more detail as follows. Suppose that when the fundamental index reaches the upper level of a target zone, defined for the present on f_{12} and denoted by $\overline{f}_{12}$, relative money supply is adjusted in order to keep the index within the zone. This can be done either by country 1 decreasing its money supply or by country 2 increasing its supply. Relative money supply is adjusted in a similar fashion when the index reaches the lower boundary of the target zone, denoted by $\underline{f}_{12}$.Therefore, from the relationship (114), the incremental change in the fundamental index will now take the form

$$df_{12} = dm_{12} + d\nu_{12}, \tag{162}$$

where $m_{12} = m_1 - m_2$.

Inside the band there are no interventions, i.e. $dm_{12} = 0$, so as in equation (119) the fundamental index follows

$$df_{12} = d\nu_{12} = \mu_{12}dt + \sigma_{12}dz_{12}. \tag{163}$$

At the edges of the band, there are infinitesimal interventions to prevent the fundamental index from moving outside the band. These interventions can be represented by an lower and upper 'regulator', L and U, such that

$$dm_{12} = dL - dU, \tag{164}$$

where dL and dU are non-negative. The change dL, represents increases in money supply and is positive only if $f_{12} = \underline{f}_{12}$, and dU represents reductions in money supply and is positive only if $f_{12} = \overline{f}_{12}$. Once the fundamental index moves inside the band, the intervention ceases. The process governing the movement of f_{12} is therefore a form of *regulated* Brownian motion. Therefore, in these circumstances the incremental change in the fundamental index can be represented as follows

$$df_{12} = \mu_{12}dt + \sigma_{12}dz_{12} + dL - dU. \tag{165}$$

However, since

$$df_{12} = \mu_{12}dt + \sigma_{12}dz \text{ for } f_{12} \in \left(\underline{f}_{12}, \overline{f}_{12}\right),$$

some member of the general solution under a free float must also characterise the movement in the exchange rate when f_{12} is in the interior of $\left[\underline{f}_{12}, \overline{f}_{12}\right]$. However, since s_{12} is assumed to be continuous over the whole interval, including at the boundaries, it cannot coincide with a function of a particular form in the interior of the interval unless it coincides with the same function at the boundaries. Therefore, the dynamics of s_{12} under regulated Brownian motion will take the same form as under a free float with appropriate boundary conditions.

Proposition 16 *The appropriate boundary conditions are that*

$$\frac{ds_{12}}{df_{12}} = 0$$

when $f_{12} = \underline{f}_{12}$ *and when* $f_{12} = \overline{f}_{12}$.

Proof

In order to obtain these conditions, we firstly recall that under a free float the relationship for the conditional expected rate of change was shown in equation (121) to be as follows

$$\frac{E(ds_{12})_t}{dt} = \mu_{12}\frac{ds_{12}}{df_{12}} + \frac{\sigma_{12}^2 d^2 s_{12}}{2df_{12}^2}.$$

As the exchange rate is continuous at the boundaries, the above expression for the conditional expected rate of change must also hold at the boundaries as well as in the interior of the interval. In particular, the expression holds at the upper boundary $\overline{f}_{12}$. Hence

$$E(ds_{12}(\overline{f}_{12}))_t = \left(\mu_{12}\frac{ds_{12}(\overline{f}_{12})}{df_{12}} + \frac{\sigma_{12}^2 d^2 s_{12}(\overline{f}_{12})}{2df_{12}^2}\right) dt. \tag{166}$$

However, $(\overline{f}_{12}, s_{12}(\overline{f}_{12}))$ is also an equilibrium point for a target zone even though the evolution of the fundamental index now follows the process

$$df_{12} = \mu_{12}dt + \sigma_{12}dz_{12} + dL - dU.$$

If Ito's lemma is applied with this definition of the incremental change for f_{12} , then at the upper boundary we obtain

$$E(ds_{12}(\overline{f}_{12}))_t = \left(\mu_{12}\frac{ds_{12}(\overline{f}_{12})}{df_{12}} + \frac{\sigma_{12}^2 d^2 s_{12}(\overline{f}_{12})}{2df_{12}^2}\right) dt - \frac{ds_{12}(\overline{f}_{12})}{df_{12}}dU, \quad (167)$$

using the fact that $dL = 0$ at $f_{12} = \overline{f}_{12}$. However, comparing the two equations for the conditional expectations, they would be mutually contradictory unless

$$\frac{ds_{12}(\overline{f}_{12})}{df_{12}} = 0. \quad (168)$$

Similarly, the point $(\underline{f}_{12}, s_{12}(\underline{f}_{12}))$ can lie on a solution for a free float and for a target zone, only if

$$\frac{ds_{12}(\underline{f}_{12})}{df_{12}} = 0. \quad (169)$$

□

Therefore, these are the relevant boundary conditions for determining the constants of integration. This approach to determining the boundary condition is that adopted by Froot and Obstfeld (1991 a and b). Alternative, more heuristic justifications for these boundary conditions have been given in some sources, for instance Krugman (1991), and a comprehensive discussion of the general topic of smooth pasting conditions is given in Dixit (1993).

Differentiating equation (161), and equating the result to zero at the boundaries gives

$$0 = 1 + \rho_1 A_1 e^{\rho_1 \overline{f}_{12}} + \rho_2 A_2 e^{\rho_2 \overline{f}_{12}} \quad (170)$$

$$0 = 1 + \rho_1 A_1 e^{\rho_1 \underline{f}_{12}} + \rho_2 A_2 e^{\rho_2 \underline{f}_{12}}. \tag{171}$$

Solving these two equations for A_1 and A_2 produces

$$A_1 = \frac{e^{\rho_2 \underline{f}_{12}} - e^{\rho_2 \overline{f}_{12}}}{\rho_1 \left(e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}}\right)} \tag{172}$$

and

$$A_2 = \frac{e^{\rho_1 \overline{f}_{12}} - e^{\rho_1 \underline{f}_{12}}}{\rho_2 \left(e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}}\right)}. \tag{173}$$

The solution for the value of the exchange rate under the target zone is therefore obtained by substituting these values for the coefficients in to equation (161), giving

$$\begin{aligned} s_{12} &= f_{12} + k\mu_{12} + \frac{\left(e^{\rho_2 \underline{f}_{12}} - e^{\rho_2 \overline{f}_{12}}\right) e^{\rho_1 f_{12}}}{\rho_1 \left(e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}}\right)} \\ &\quad + \frac{\left(e^{\rho_1 \overline{f}_{12}} - e^{\rho_1 \underline{f}_{12}}\right) e^{\rho_2 f_{12}}}{\rho_2 \left(e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}}\right)}. \end{aligned} \tag{174}$$

As this function is monotonically increasing over its domain of definition, confining the fundamentals to the zone $\left[\underline{f}_{12}, \overline{f}_{12}\right]$ is equivalent to restricting the exchange rate to the target zone $\left[s_{12}(\underline{f}_{12}), s_{12}(\overline{f}_{12})\right]$. It is also worth noting that if we let the lower barrier, $\underline{f}_{12}$, go to minus infinity, then the solution for the exchange rate simplifies to

$$s_{12} = f_{12} + k\mu_{12} - \rho_1^{-1} e^{\rho_1 (f_{12} - \overline{f}_{12})}. \tag{175}$$

If in addition we let the upper barrier, $\overline{f}_{12}$, go to plus infinity, we obtain the previously derived linear solution under a free float

$$s_{12} = f_{12} + k\mu_{12}.$$

Although the majority of the literature regarding the stochastic monetary model assumes that interventions at the boundary of the target zone are infinitesimal, some consideration, for instance in Flood and Garber (1989) and Miller and Weller (1991), has been given to the effect of discrete interventions in such models. After the initial development of the stochastic monetary model and its solution by Krugman, the model was quickly applied to the subject of speculative attacks on target zones. A collection of papers on this topic subsequently appeared in Krugman and Miller (1992). More recently, Miller and Zhang (1996), use the model to obtain optimal target zone bands given proportional costs of intervention, while other authors have applied the basic model to the situation of a managed floating currency but without a formal target zone. Svensson (1994) for instance, discusses a managed float where monetary policy is chosen to deliver the desired combination of exchange rate stability and domestic monetary independence, while Papi (1993), Laskar (1994), and Miller and Papi (1997), discuss optimal policies with quadratic intervention costs.

5.2 The price inertia model

5.2.1 A target zone on the real exchange rate

In advocating target zones for exchange rates, Williamson(1985) proposed that the monetary authorities in the major industrial countries be prepared to adjust the stance of monetary policy so as to keep their real exchange rates within broad bands. It was intended that adjustments be made only at the edges of these bands, however, so that monetary control could be aimed at domestic anti-inflationary objectives within these target zones. This Section uses the price inertia model to discuss some of the properties of an exchange rate operating in such a zone, and follows the approach described in Miller and Weller (1990) and (1991).

Proposition 17 *Suppose we define (the log of) relative real money balances as $b = m_{12} - p_{12}$. In addition, let $q = s_{12} - p_{12}$ denote (the log of) the real exchange rate. Then the dynamics of the system can be represented by the following matrix equation*

$$\begin{bmatrix} db \\ E(dq)_t \end{bmatrix} = C \begin{bmatrix} bdt \\ qdt \end{bmatrix} + \begin{bmatrix} \sigma_{12} dz_{12} \\ 0 \end{bmatrix}$$

where

$$C = \frac{1}{\Delta}\left[\begin{array}{cc} -k\phi & -2\phi k\eta \\ -1 & 2\eta(l-\phi k) \end{array}\right]$$

and Δ is as defined in Section 4.2.

Proof

In equation (134) of Section 4.2, the rate of change of the relative prices was shown to be

$$dp_{12} = \left(\frac{2k\varphi\eta s_{12} - p_{12}(\gamma\varphi + 2k\eta) + \varphi\gamma m_{12} - \varphi(l\gamma + k)\overline{y}_{12}}{l\gamma + k - \varphi\gamma k}\right) dt + \sigma_{12}dz_{12}.$$

Then using the previously defined notation for the real money balances and exchange rate, the above equation can be written as follows

$$dp_{12} = \frac{\gamma\phi bdt + 2\phi k\eta qdt - \phi(l\gamma + k)\overline{y}_{12}dt}{l\gamma + k - \phi\gamma k} + \sigma_{12}dz_{12}. \tag{176}$$

As $db = dm_{12} - dp_{12}$, if there is no intervention it follows that since m_{12} does not change

$$db = -dp_{12} = \frac{-\gamma\phi bdt - 2\phi k\eta qdt + \phi(l\gamma + k)\overline{y}_{12}dt}{l\gamma + k - \phi\gamma k} - \sigma_{12}dz_{12}. \tag{177}$$

Similarly, the expected rate of change in the exchange rate was shown in equation (132) to be as follows

$$\mathrm{E}(ds_{12})_t = \left(\frac{2l\eta s_{12} + p_{12}(1 - \gamma\varphi - 2l\eta) - \varphi\gamma l\overline{y}_{12} - (1-\gamma\varphi)m_{12}}{l\gamma + k - \varphi\gamma k}\right) dt$$

It follows from the definition of the real exchange rate, q, that

$$\mathrm{E}(dq)_t = \mathrm{E}(ds_{12})_t - \mathrm{E}(dp_{12})_t. \tag{178}$$

Since, from equation (176), the expected rate of change in relative prices can be written as follows

$$\mathrm{E}(dp_{12})_t = \frac{\gamma\phi bdt + 2\phi k\eta qdt - \phi(l\gamma + k)\overline{y}_{12}dt}{l\gamma + k - \phi\gamma k} \tag{179}$$

then substituting for the respective terms in equation (178 we obtain

$$\mathrm{E}(dq)_t = \frac{2l\eta qdt - (1-\gamma\phi)bdt - \gamma\phi l\overline{y}_{12} - \gamma\phi bdt - 2\phi k\eta qdt + \phi(l\gamma + k)\overline{y}_{12}dt}{l\gamma + k - \phi\gamma k}.$$

Upon simplifying, we obtain the expression

$$\mathrm{E}(dq)_t = \frac{-bdt + 2\eta(l-\phi k)cdt + \phi k\overline{y}_{12}dt}{l\gamma + k - \phi\gamma k}. \tag{180}$$

Therefore, writing the two expressions for db and $\mathrm{E}(dq)_t$ in matrix form, and using the previous definition of $\triangle$ we obtain

$$\begin{bmatrix} db \\ \mathrm{E}(dq)_t \end{bmatrix} = C \begin{bmatrix} bdt \\ qdt \end{bmatrix} + \begin{bmatrix} \sigma_{12}dz_{12} \\ 0 \end{bmatrix} \tag{181}$$

where

$$C = \frac{1}{\triangle}\begin{bmatrix} -k\phi & -2\phi k\eta \\ -1 & 2\eta(l-\phi k) \end{bmatrix}$$

and as in Section 4.2 to simplify the notation, we have redefined s_{12} and p_{12} as deviations from the long-run equilibrium of the deterministic system.

□

For this system, assume that there exists a deterministic functional relationship between real balances and the real exchange rate, which we shall denote by $l(b)$, such that $q = l(b)$. Then, upon using Ito's lemma, the fundamental differential equation again takes the form

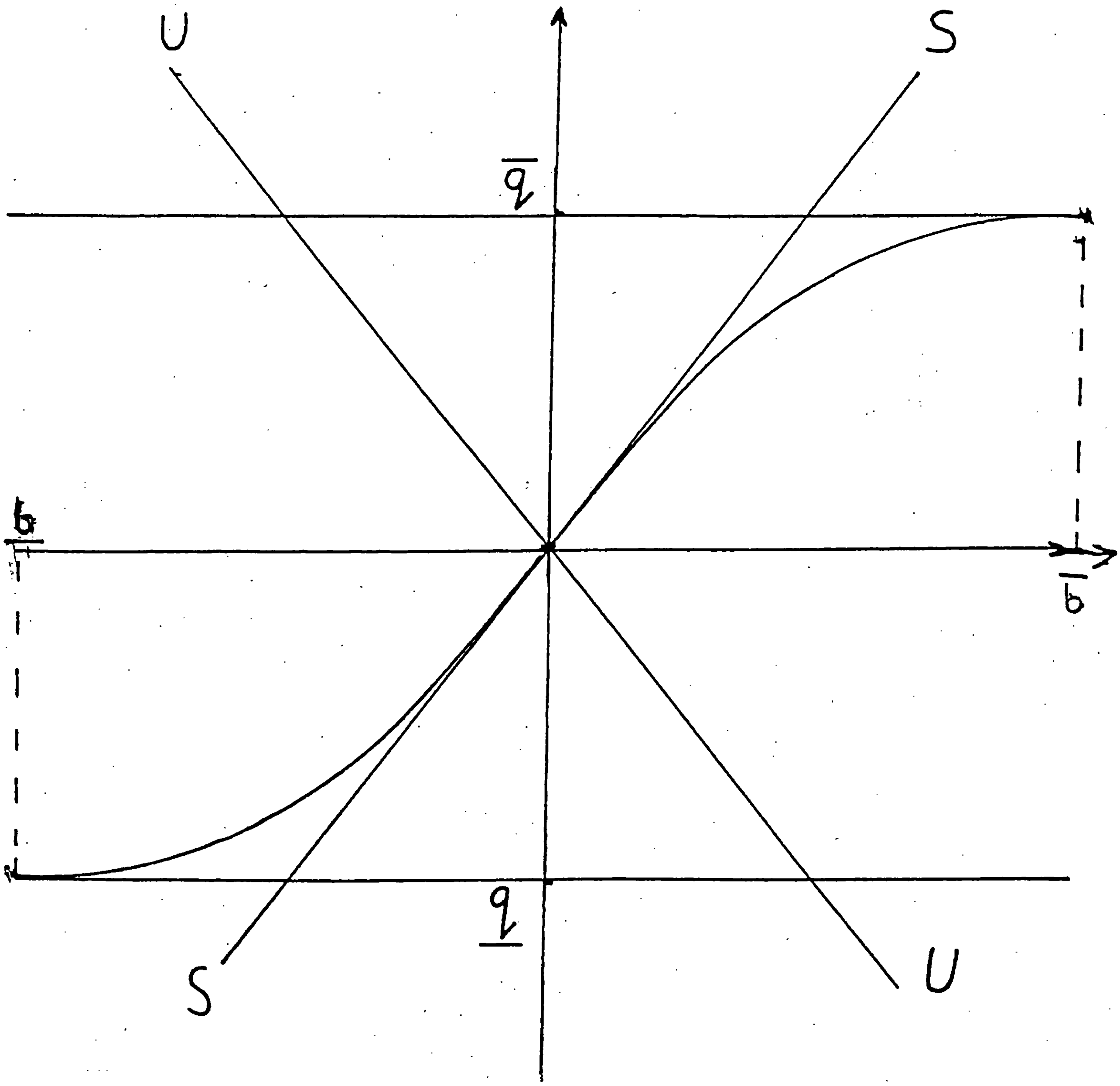
$$\frac{\sigma_{12}^2}{2}\frac{d^2l}{db^2} + (c_{11}b + c_{12}l)\frac{dl}{db} - (c_{21}b + c_{22}l) = 0. \tag{182}$$

where c_{ij} are the elements of C.

The solutions to the above equation which satisfy the condition $l(0) = 0$, are qualitatively similar to those discussed in the previous Chapter. However, for the usual overshooting case, the signs of the relevant coefficients will mean that the slopes of the two eigenvectors (of C) will be the reverse of that shown in figure 4.1 .

Suppose now that a target zone is imposed on real money balances b. Hence, central banks intervene to prevent b from exceeding the upper boundary $\bar{b}$, and from falling below $\underline{b}$. As in the discussion of the monetary model, the central banks make this intervention by infinitesimal adjustments to relative money supply. Provided the market is aware of this rule and believes it, then this arrangement is equivalent to imposing a target zone on the real exchange rate q. That is, as in Section 5.1, control of b between $\underline{b}$ and $\bar{b}$, simultaneously confines q to a range $\underline{q}$ to $\bar{q}$, provided there exists a solution to the above differential equation satisfying the boundary conditions, which provides a 1 to 1 relationship between b and q. As with the monetary model discussed in Section 5.1, the appropriate boundary condition to be applied is that the path of the real exchange rate is tangent to the edges of the band (Miller and Weller (1989) and (1990)). The path of the real exchange rate will therefore be as illustrated in figure 5.1.

Figure 5.1 Smooth pasting solution in a real exchange rate band.



5.2.2 A target zone on the nominal exchange rate

Of course, most practical implementations of currency bands involve a zone imposed on the nominal exchange rate, with the possibility of realignment of the central rate. It would therefore be useful to examine the dynamics of the exchange rate within such a zone, subject to price inertia.

In such an arrangement, we shall assume that the target zone imposed on the nominal exchange rate between the two countries is denoted by $[\underline{s}_{12}, \overline{s}_{12}]$. As before, we assume money stocks are held fixed while the exchange rate is in the interior of the band and there is infinitesimal monetary interventions required to keep the exchange rate within the zone. Under such an arrangement, it was shown in the previous Chapter that assuming there exits a deterministic functional relationship between the nominal exchange rate and relative prices, that relationship is given by the solution of equation (139), that is

$$\frac{\sigma_{12}^2 f''}{2} + \left(a_{11}(p_{12} - m_{12}) + a_{12} f\right) f' - \left(a_{21}(p_{12} - m_{12}) + a_{22} f\right) = 0$$

where a_{ij} are the appropriate elements of the matrix A as defined in Section 4.2. Once again, the appropriate boundary condition will be that the solution is tangential to the edges of the band. The qualitative properties of the solutions for a nominal exchange rate band can be seen by considering the situation where the equilibrium point for the exchange rate is initially in the middle of the band and the smooth pasting points also lie symmetrically on either side of the equilibrium at $\underline{p}$ and $\overline{p}$. Suppose that the exchange rate is at the lower end of its band at $\underline{s}_{12}$ so that relative prices are at $\overline{p}$. If relative prices increase further due to a random shock generated by the Wiener process, the exchange rate would be affected unless there was a change in money supply. In the case of a real exchange rate band, the change in money supply had to be equal to the change in price level in order to keep $m_{12} - p_{12}$ constant. However, with a nominal band, because of the form of the functional relationship between the exchange rate and relative prices,viz.

$$s_{12} - m_{12} = f(p_{12} - m_{12}) \tag{183}$$

the adjustment to money supply need not exactly offset the change in prices in order to keep the exchange rate at $\underline{s}_{12}$.

5.3 Summary

If rules such as those listed in Section 5.1, are assumed for how and when central banks make interventions then it has been shown that a unique solution path exists for both the monetary and the price inertia stochastic models, and in the case of the monetary model an explicit relationship between the exchange rate and the fundamental variables was obtained. This result indicated that although no monetary interventions take place within a target zone, the solution path for the exchange rate is far from the linear relationship obtained for the pure free float situation described at the end of Section 4.1.3. For the price inertia model, the solution paths are also non-linear whether the real or nominal exchange rate is being considered. In both the monetary model and the price inertia model the solution paths generally have an elongated S shaped appearance. This similarity is not too surprising since identical intervention rules are assumed for both. In particular, the assumption that the exchange rate is a continuous function of the relevant fundamental variables results in the requirement that the function has a turning point at the zone boundaries for both models.

One interesting feature of the stochastic approach is that exchange rates are allowed to move, because of the Wiener process involved, even though fundamental variables such as money supply may not change. The models therefore provide an additional mechanism for explaining the observation previously noted in Chapter 2, that the variation in exchange rate movements appears much larger than the changes in variables such as money supply would normally lead us to expect. On the other hand, as emphasised in both this and the previous Chapter, the tractability of the models results largely from the assumption that the random element in the model follows a Wiener process, and the special properties these processes have. While such an assumption is not unreasonable, it is nevertheless just one out of many potential processes that might describe a random variable in these circumstances. In addition, the derivations and hence results make considerable use of Ito's stochastic calculus. This is of course the most widely applied version of stochastic calculus but other variations exist with differing rules and properties. Both the properties of the Wiener process itself and the properties of Ito's calculus might therefore be supposed to influence the nature of the results obtained for both models.

The stochastic models discussed in this Chapter were aimed at describing the exchange rate dynamics for two countries operating a target zone. In the next Chapter we shall extend the results for the monetary model to the situation in which both countries involved in the target zone exist in a network with a number of other countries. The dynamics of the exchange rate between each pair of countries will be examined, firstly when only one target zone exists in the network and then when two zones are assumed to be in operation. Lastly, the possibility of a network in which a large number of target zones are operating, but with one country taking a central role, is considered.

6 Stochastic monetary models for more than two countries

6.1 A small network of countries

The majority of the literature on modelling the dynamics of the exchange rate has so far been concerned with representing the movement between a single pair of countries. However, in the real world, bilateral exchange rates exist between a group of countries, and because of practical restrictions such as the cyclic no-arbitrage considerations noted in Section 2.1.2, they cannot be independent of each other. In this Section we shall develop a model, using the stochastic monetary approach, to represent the dynamics of a small network of countries. In particular, we shall examine some of the properties of the exchange rate between two countries, one of which is involved in a target zone relationship with a third country, and the other country whose currency is free floating.

In equation (115) of Section 4.1 it was shown that the equation for the (log of) the exchange rate between two countries, assuming identical macroeconomic parameters in the flexible price monetary model, could be written as follows

$$s_{12} = f_{12} + k\frac{\mathrm{E}(ds_{12})_t}{dt}.$$

If a target zone is imposed on the exchange rate between these two countries then, with the assumptions of Section 5.1, this had the general solution given in equation (123), that is

$$s_{12}(f_{12}) = f_{12} + k\mu_{12} + A_1 e^{\rho_1 f_{12}} + A_2 e^{\rho_2 f_{12}}$$

where

$$A_1 = \frac{e^{\rho_2 \underline{f}_{12}} - e^{\rho_2 \overline{f}_{12}}}{\rho_1 (e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}})}$$

and

$$A_2 = \frac{e^{\rho_1 \overline{f}_{12}} - e^{\rho_1 \underline{f}_{12}}}{\rho_2 (e^{\rho_1 \underline{f}_{12} + \rho_2 \overline{f}_{12}} - e^{\rho_1 \overline{f}_{12} + \rho_2 \underline{f}_{12}})}.$$

As before, $\underline{f}_{12}$ and $\overline{f}_{12}$ are the lower and upper boundaries on the fundamental index corresponding to the exchange rate target zone, and ρ_1 and ρ_2 are solutions of the equation

$$\frac{\sigma_{12}^2 \rho^2}{2} + \mu_{12} k \rho - 1 = 0. \tag{184}$$

If we assume that there is no deterministic trend in the exchange rate movement (that is $\mu_{12} = 0$), then, as can be seen from equation (184)

$$\rho_1 = -\rho_2 = \sqrt{\frac{2}{k\sigma_{12}^2}}, \tag{185}$$

where σ_{12}^2 is the sum of the variance per unit time period for each of the two Wiener processes involved. On writing $\rho = \rho_1 = -\rho_2$, the two coefficients can be simplified to the following

$$A_1 = \frac{e^{-2\rho \underline{f}_{12} - \rho \overline{f}_{12}} - e^{-\rho \underline{f}_{12} - 2\rho \overline{f}_{12}}}{\rho (e^{-2\rho \overline{f}_{12}} - e^{-2\rho \underline{f}_{12}})} \tag{186}$$

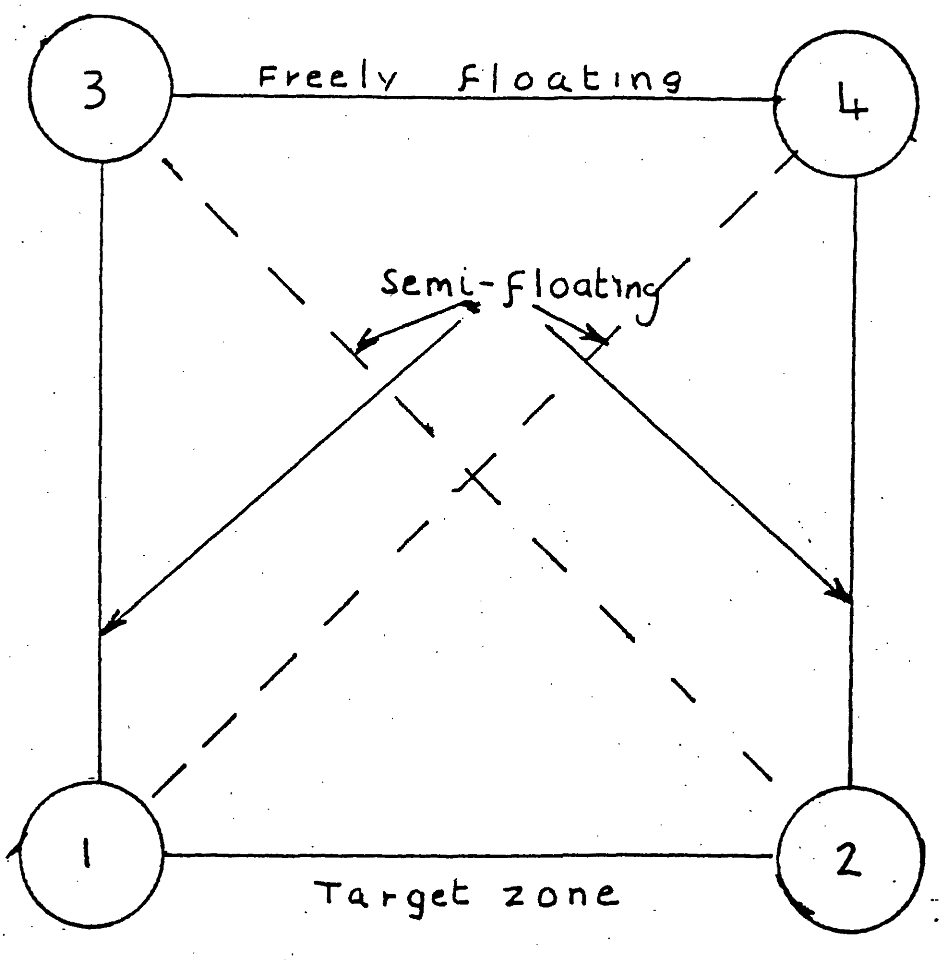
and

$$A_2 = \frac{e^{-\rho \overline{f}_{12}} - e^{-\rho \underline{f}_{12}}}{\rho (e^{-2\rho \overline{f}_{12}} - e^{-2\rho \underline{f}_{12}})}. \tag{187}$$

It should be noted that $A_1 \leq 0 \leq A_2$.

Suppose that in addition to countries 1 and 2, which have a credible target zone imposed on their common bilateral exchange rate, there exits two further countries, 3 and 4, which let their currencies float freely with no interventions to their money supply levels, as illustrated in figure 6.1.

Figure 6.1 A four country model.



The bilateral exchange rate between countries 3 and 4 will therefore be a fully free floating exchange rate. However, assuming all four countries have bilateral exchange rates against each other, any exchange rate between country 1 or 2, and either of the other two countries, 3 or 4, will be neither free floating, nor a fully targeted rate. Such rates will therefore be referred to as semi-floating rates. As before, let s_{ij} denote the log of the exchange rate between countries i and j measured in units of currency of country i per unit of currency of country j.

Countries 3 and 4 practise identical policies (non-intervention) and therefore the qualitative behaviour of their exchange rates vis-a-vis a given third country should also be identical. Hence s_{13} and s_{14} should share the same qualitative properties as should s_{23} and s_{24}.

Before attempting to derive solutions for the semi-floating exchange rates, it should be noted that in a two country target zone model there is no need to specify how intervention in the target zone is allocated between the two countries. This can be observed from the definition of the index of differences in fundamental determinants in Section 4.1, i.e.

$$f_{ij} = f_i - f_j = m_i - m_j - v_{ij}.$$

Provided f_{ij} is kept within its bounds, it does not matter whether this is achieved by one country increasing its money supply or the other country decreasing its own money supply. However, since a semi-floating exchange rate is only influenced by the intervention of only one of the target zone countries, an exact specification of intervention policy is now needed.

6.2 Intervention policies

Intervention by increasing or decreasing the level of money supply occurs when the fundamental index, or equivalently the exchange rate, reaches a boundary of the relevant zone. At this point we need to specify the proportion of the change which is achieved by one country increasing its money supply compared with a decrease by the second country. In order to do so, Jørgensen and Mikkelsen (1997) first define m_i^c and m_i^w to be the 'integral' of all money created and the 'integral' of all money withdrawn in a given country i. They claim that the usual money supply variable is simply the difference in these two quantities, that is

$$m_i = m_i^c - m_i^w, \tag{188}$$

and that the fundamental measure, as defined in Section 4.1 can therefore be written as

$$f_i = m_i^c - m_i^w - \nu_i. \tag{189}$$

However, this relationship is not quite as obvious as it may seem. When deriving the stochastic version of the monetary model in Section 4.1, the variable m_i, was defined as the *logarithm* of the money supply. It does not therefore seem obvious that it should equal either the difference in money created and withdrawn, or the difference in their logarithms. However, we shall show that the assumption made by Jørgensen and Mikkelsen is valid, at least to a first approximation.

Proposition 18 *To a first approximation,* $m_i = m_i^c - m_i^w$

Proof.

Suppose we let the money supply for country i (before taking logs) be represented by M_i. Then at any point in time, money supply will equal the initial money supply at time zero, m_i^0, plus the change in supply up time t, denoted by $M_i^{'}$. The log of the money supply can therefore be written as follows

$$m_i = \ln(M_i) = \ln(m_i^0 + M_i^{'}) = \ln(m_i^0) + \ln\left(1 + \frac{M_i^{'}}{m_i^0}\right). \tag{190}$$

To a first approximation, this can be represented as

$$m_i = \ln(m_i^0) + \frac{M_i^{'}}{m_i^0} = \ln(m_i^0) + \frac{m_i^c}{m_i^0} - \frac{m_i^w}{m_i^0}, \tag{191}$$

since the change in money supply will simply be the difference between m_i^c and m_i^w. As the units for measuring money supply are arbitrary within a given country, let us choose the initial money supply to be one unit. In which case, we obtain (at least to a first approximation),

$$m_i = m_i^c - m_i^w,$$

as used by Jørgensen and Mikkelsen.

□

Using this approximation, the index of the difference in fundamental determinants between countries 1 and 2, can be written as follows

$$f_{12} = f_1 - f_2 = (m_1^c + m_2^w) - (m_1^w + m_2^c) - \nu_{12}. \tag{192}$$

That is, if for example currency 1 is too strong relative to currency 2, whether country 1 increases or country 2 decreases money supply does not matter for the target zone exchange rate since both have the same effect on the first term from the right hand side of the above equation. Also, if currency 2 is too strong the monetary authorities have two possible solutions. The first is to let country 1 withdraw some of its money and the second is to let country 2 create more of its own. Alternatively, interventions may be split between the countries in some systematic fashion according to some rules. Before looking at a possible formulation of such a set of rules, it is worth noting that $(m_1^c + m_2^w)$ and $(m_1^w + m_2^c)$ correspond to the variables L and U respectively, as introduced in Section 5.1, when discussing the solution of the monetary model with a target zone.

Let us suppose that at the upper boundary of the target zone country 1 withdraws a certain amount of its currency while country 2 creates money, and that the amounts withdrawn and created are in a fixed proportion. Similarly, at the lower boundary, we shall suppose that the amount of money withdrawn and created by countries 2 and 1 respectively, are also in a constant proportion. These rules governing the amount of intervention at the edges of the target zone can be summarised as follows

$$\begin{aligned} dm_1^w &= \gamma_1 dm_2^c & (193)\\ dm_2^w &= \gamma_2 dm_1^c & (194)\end{aligned}$$

where γ_1 and γ_2 are non-negative constants. Since a country defends its currency when it is relatively weak by withdrawing some of its own money from the market, γ_i can be interpreted as an index of the degree of "self-defence" when country i's currency is weak. If $\gamma_i > 1$ ($\gamma_i < 1$) country i intervenes more (less) relative to the other country when its currency is weak. It will be convenient in what follows to define the following relative weights

$$\lambda_i = \frac{\gamma_i}{1+\gamma_i}, \qquad i = 1, 2 \tag{195}$$

where $0 \leq \lambda_i \leq 1$ by construction and $\lambda_i > 0.5$ ($\lambda_i < 0.5$) corresponds to $\gamma_i > 1$ ($\gamma_i < 1$).

Proposition 19 *With the foregoing notation, the exchange rate between country 1 and country j (j=3,4), is as follows*

$$s_{1j} = f_{1j} + \lambda_1 A_1 e^{\rho f_{12}} + (1 - \lambda_2) A_2 e^{-\rho f_{12}}.$$

Proof.

Consider the two components $A_1 e^{\rho f_{12}}$ and $A_2 e^{-\rho f_{12}}$ which arise in the solution of

$$s_{12} = f_{12} + k \frac{\mathrm{E}(ds_{12})_t}{dt}.$$

They are due to the expectations term in the above equation. As noted earlier, $A_1 < 0$, so it follows that $A_1 e^{\rho f_{12}}$ is the term which results from interventions at the upper boundary of the target zone, that is, as currency 1 weakens. (It was noted in Section 4.1 that if there was no upper boundary to the zone, the constant of integration A_1, must be zero in order to avoid explosive growth). This term arises as a result of investors' expectations to the publicly announced credible intervention that will take place should the rate reach its upper margin. Since everybody knows the distribution of intervention among the two countries represented by the weight λ_1, the term can be divided into country specific components. As the proportion of the intervention that will be performed by country 1 is simply λ_1, the term arising from the expectation of the intervention that will be performed by country 1 when the exchange rate reaches its upper boundary is $\lambda_1 A_1 e^{\rho f_{12}}$

and the corresponding term for country 2 is $(1-\lambda_1)A_1e^{\rho f_{12}}$. Similarly, since $A_2 > 0$, $(1-\lambda_2)A_2e^{-\rho f_{12}}$ arises from the expected intervention that country 1 performs if the exchange rate reaches its lower boundary (currency 2 weakens) and $\lambda_2 A_2 e^{-\rho f_{12}}$ is a consequence of the promised intervention by country 2 in this case. Given the nature of the monetary model, provided all relevant macroeconomic and financial parameters are assumed to be identical in the countries concerned, intervention by a given country influences all exchange rates between that country and other countries, equally. Hence, an exchange rate between a country participating in a target zone and a third country, will be fully affected by the expectation of intervention by the given country from the target zone. Thus, the implied solution for the exchange rate between country 1 and country j $(j = 3, 4)$ will take the form

$$s_{1j} = f_{1j} + \lambda_1 A_1 e^{\rho f_{12}} + (1-\lambda_2)A_2e^{-\rho f_{12}}. \tag{196}$$

□

There is no term arising from the expectation of interventions by country j in the above solution since this country's known policy is not to alter its money supply (to affect its exchange rate at least). Similarly, the implied semi-floating solutions involving country 2 are

$$s_{2j} = f_{2j} - (1-\lambda_1)A_1e^{\rho f_{12}} - \lambda_2 A_2 e^{-\rho f_{12}}. \tag{197}$$

The minus sign in front of the two exponential terms arises because whenever country 1 intervenes by withdrawing money (for example at the upper boundary of the target zone), country 2 intervenes by creating money. This would of course have the opposite effect on the exchange rate between country 1 and a third country, and between country 2 and a third country. The same argument applies at the lower boundary of the zone.

The foregoing solutions for the semi-floating rates provide intuitively acceptable results at the extremities of the intervention policy. For instance, suppose $\lambda_1 = 1$ and $\lambda_2 = 0$. This would correspond to country 1 making all the interventions at the extremes of the target zone. This policy would amount to a unilaterally enforced target zone. In such an arrangement, the exchange rates between country 1 and third countries should be fully affected by the expectations of those interventions, while those between country 2 and third countries should not be affected at all and are effectively free floating. An inspection of the above semi-floating solutions indicates that this is precisely the result which follows upon putting $\lambda_1 = 1$ and $\lambda_2 = 0$. The equations for the exchange rates being as follows

$$\begin{aligned} s_{1j} &= f_{1j} + A_1 e^{\rho f_{12}} + A_2 e^{-\rho f_{12}} \\ s_{2j} &= f_{2j}. \end{aligned}$$

Conversely, if in the above solutions we put $\lambda_1 = 0$ and $\lambda_2 = 1$, only exchange rates involving country 2 are affected by the expected interventions, in line with what might be anticipated intuitively.

In addition to agreeing with what should be predicted from theory, the semi-floating solutions must also obey the cyclic no-arbitrage conditions given in equation (27) of Section 2.1.2. For instance, for a cycle of three countries involving countries 1 and 2 and either country 3 or 4, when dealing with the logarithms of exchange rates we must have

$$s_{12} + s_{2j} + s_{j1} = 0 \quad (j = 3, 4). \tag{198}$$

Since, from Section 2.1.2, $s_{j1} = -s_{1j}$, the above no-arbitrage condition is equivalent to

$$s_{12} + s_{2j} = s_{1j}. \tag{199}$$

Substitution from the relevant formulae for the targeted and semi-floating exchange rates, indicates that the above no arbitrage condition is indeed met. Similarly, it is not difficult to verify that the no-arbitrage condition is fulfilled for all other cycles in this small network.

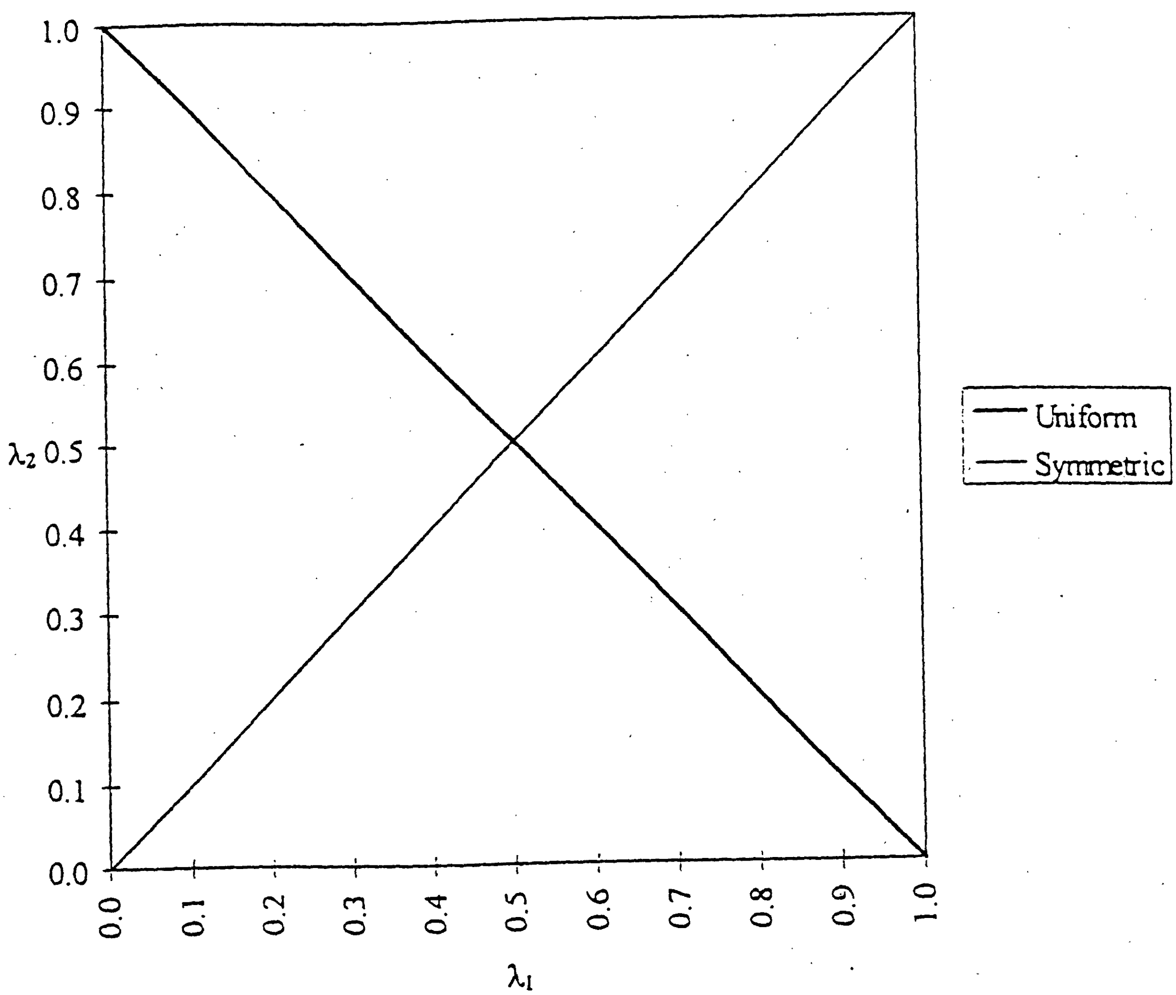
Although the foregoing derivation of the solution for semi-floating exchange rates is valid for quite general values for λ_1 and λ_2, Jørgensen and Mikkelsen describe two particular intervention policies in which λ_1 and λ_2 are related. These are as follows.

Definition 20 *An intervention policy is* ***uniform*** *if* $\gamma_1 = \gamma_2^{-1}$ *or equivalently,* $\lambda_1 + \lambda_2 = 1$.

Definition 21 *If $\gamma_1 = \gamma_2$ or equivalently, $\lambda_1 = \lambda_2$, the intervention policy is said to be* ***symmetric.***

To illustrate these two concepts, consider figure 6.2, containing possible (λ_1, λ_2) combinations. On the uniform line, $\lambda_2 = 1 - \lambda_1$, the implication is that country 2 performs the same relative share of the intervention regardless of which currency is weak. Since the same argument holds for country 1, the level of intervention is distributed uniformly for a given country. On the line where $\lambda_1 = \lambda_2$, the intervention policy is symmetric in the sense that countries 1 and 2 perform the same fraction of the required intervention when their respective currency is weak. The two lines intersect at $\lambda_1 = \lambda_2 = \frac{1}{2}$, corresponding to a uniform and symmetric policy. Finally, it is worth reiterating the point that the derivation and solutions to these models requires that the intervention policy, including the weights λ_1 and λ_2, is announced publicly and is credible, so that investors can form expectations accordingly.

Figure 6.2 Uniform and symmetric intervention policies.



6.3 Multiple target zones

Having seen the effect that a target zone has on the semi-floating exchange rates bordering the zone, it is relevant to ask what the effect might be of more than one target zone in the network. Consider again the network of four countries illustrated in figure 6.1. In addition to the previous target zone between countries 1 and 2, suppose that there is also a target zone between countries 3 and 4. How now would the exchange rate such as that between countries 1 and 3, behave? As before, let us suppose that there is no deterministic trend in either of the two target zone exchange rates, so the two target zones between countries 1 and 2, and between countries 3 and 4 would, by reference to equation (123), have respective solutions of the form

$$s_{12} = f_{12} + A_1 e^{\rho_{12} f_{12}} + A_2 e^{-\rho_{12} f_{12}} \tag{200}$$
$$s_{34} = f_{34} + A_3 e^{\rho_{34} f_{34}} + A_4 e^{-\rho_{34} f_{34}}, \tag{201}$$

where the A_i $(i = 1, 2, 3, 4)$ are constants of integration which are evaluated as before, and subscripts on the parameter ρ indicate that this parameter need not be identical for each pair of countries.

Proposition 22 *For the above arrangement, the exchange rate between countries 1 and 3 is given as follows*

$$s_{13} = f_{13} + \lambda_1 A_1 e^{\rho_{12} f_{12}} + (1 - \lambda_2) A_2 e^{-\rho_{12} f_{12}} - \lambda_3 A_3 e^{\rho_{34} f_{34}} - (1 - \lambda_4) A_4 e^{-\rho_{34} f_{34}}.$$

Proof.

Consider the exchange rate between countries 1 and 3. As before, market expectations will be influenced by the interventions of country 1, and by the same argument as previously given, this will contribute the following term to the expected rate of change

$$\lambda_1 A_1 e^{\rho_{12} f_{12}} + (1 - \lambda_2) A_2 e^{-\rho_{12} f_{12}},$$

where λ_1 and λ_2 are defined as before. However, in addition to the interventions from country 1, the market's expected rate of change of the exchange rate s_{13}, will be influenced by interventions by country 3. By an analogous argument to the contribution from country 1, country 3 will contribute the following term to the expected rate of change

$$-\lambda_3 A_3 e^{\rho_{34} f_{34}} - (1-\lambda_4) A_4 e^{-\rho_{34} f_{34}},$$

where λ_3 and λ_4 are defined analogously to λ_1 and λ_2. The minus sign arises from two facts. Firstly, that a withdrawal of money by country 3, represented by the first component of the above term, should result in an increase to s_{13}, and secondly, that A_3 like A_1, is negative. The second component is also negative by a similar argument. The full solution for the exchange rate between countries 1 and 3 is therefore

$$\begin{aligned} s_{13} &= f_{13} + \lambda_1 A_1 e^{\rho_{12} f_{12}} + (1-\lambda_2) A_2 e^{-\rho_{12} f_{12}} \\ &\quad -\lambda_3 A_3 e^{\rho_{34} f_{34}} - (1-\lambda_4) A_4 e^{-\rho_{34} f_{34}}. \end{aligned} \tag{202}$$

□

The exchange rate between countries 1 and 3 is now driven by monetary interventions by both countries. This exchange rate is therefore at the mercy of not only the random fluctuations resulting from the Wiener process forming part of f_{13}, but also the interventions occurring whenever the exchange rates s_{12} and s_{34} reach the boundaries of their target zones. It does not therefore seem appropriate to continue to refer to such exchange rates as semi-floating, a more appropriate term might be **target-zone driven rates.**

Once again it can be shown that the exchange rate models derived in this manner obey the cyclic no-arbitrage conditions. For instance, using similar arguments to the above, the exchange rate s_{23} would be given as follows

$$\begin{aligned} s_{23} &= f_{23} - \lambda_2 A_2 e^{-\rho_{12} f_{12}} - (1-\lambda_1) A_1 e^{\rho_{12} f_{12}} \\ &\quad -\lambda_3 A_3 e^{\rho_{34} f_{34}} - (1-\lambda_4) A_4 e^{-\rho_{34} f_{34}}. \end{aligned} \tag{203}$$

Cyclic no-arbitrage conditions would require that

$$s_{12} + s_{23} = s_{13}.$$

Substitution for the three exchange rates, confirms that this condition does indeed hold. It is straightforward to check that the cyclic no arbitrage condition is fulfilled for all other possible cycles in this small network.

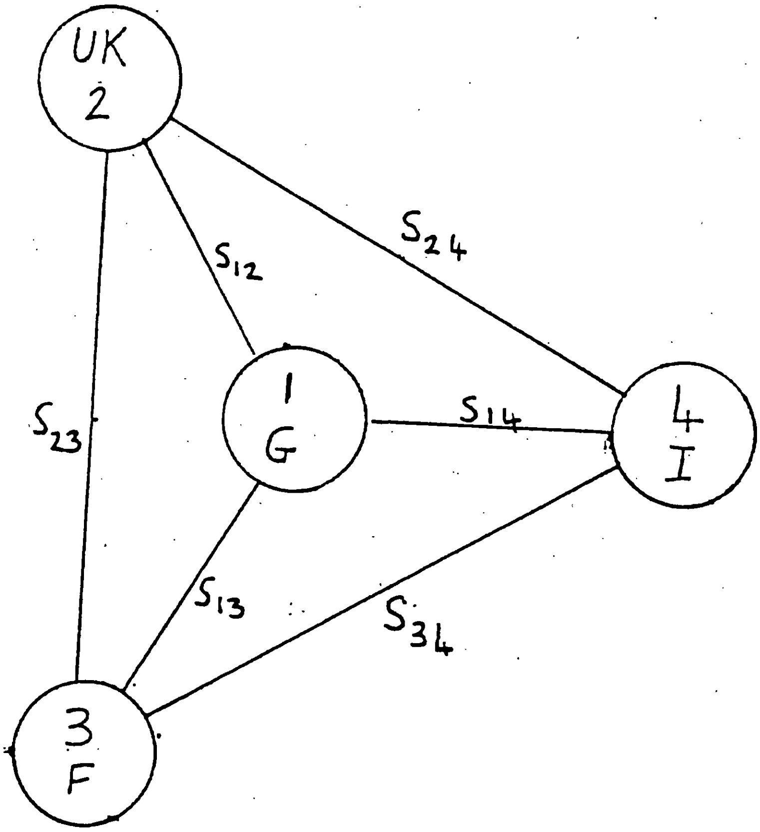
The above argument shows that it is possible in this network to incorporate more than one target zone. This is possible because in each cycle of exchange rates one of the rates is available to be driven by the others in the cycle. The cyclic no-arbitrage condition can therefore always be met. However, it is not difficult to see that including a third exchange rate with a target zone in to this network would mean that an assumption would be breached at some point. In order to demonstrate this suppose a target zone was imposed on the exchange rate between countries 1 and 3, in addition to those between countries 1 and 2, and between 3 and 4. Let us suppose s_{12} is close to its upper boundary, in which case the currency from country 1 is relatively weak. According to our earlier arguments, this would require a contraction of domestic money supply by country 1 in order to defend its currency. However, assume that at the same point in time the exchange rate s_{13} is at its lower boundary. This is quite possible since within target zone boundaries, the only mechanisms driving the exchange rates are the independent Wiener processes. Now, a withdrawal of money by country 1 at this juncture would not only strengthen its currency with respect to country 2 but with respect to country 3 also. This would therefore have the effect of making the exchange rate s_{13} smaller, thereby pushing it through its lower target zone boundary. Indeed, such an arrangement would contravene the assumption of a country only adjusting its money supply at the boundary of a particular target zone. This potential conflict seems to be inevitable, given the assumptions of this model, whenever a particular country is involved in target zone arrangements with more than one other country, as was country 1 in the above example. Other authors, for example Biggs (1993) have concluded similar results for a cycle of three countries operating within bands defined by the cost of transporting gold.

The potential conflict faced by a country involved in more than one target zone arrangement has some practical implications, as the next Section illustrates.

6.4 Multi-country exchange rate mechanisms.

Consider a network of countries in which one of them has a central role in the sense that all other countries have a target zone imposed on their exchange rate with the central country. However, non-central countries do not intervene in order to influence their exchange rate against any other non-central country. This arrangement can be represented diagrammatically as in figure 6.3. In this case, the central country has been shown as Germany, with the UK., France, and Italy as the non-central countries. This situation might for instance represent a simplified version of the arrangement under the Exchange Rate Mechanism (ERM) of the European Monetary System (EMS). In this mechanism, it was generally acknowledged that the Deutschemark held a central position or standard, compared with the other EMS currencies.

Figure 6.3 A four country network with Germany taking a central role.



In general, the formula for the exchange rate between the UK. (country 2) and Germany (country 1) would be extremely complicated, as would the formulae for the rates between the other non-central countries and Germany. This is a result of market expectations for the rate of change of the U.K./German exchange rate, being influenced not only by German interventions to preserve the £/Dm target zone but interventions by Germany to preserve other zones involving itself, also. Indeed, the frequent interventions by Germany at the edges of the various zones, would as noted in the last Section, lead to a potential conflict in maintaining all the zones simultaneously, at least in the situation of each country intervening by a known, non-zero, proportion (λ_i).

However, suppose we assume that all the target zones in the network are unilaterally imposed, as defined in Section 6.1. That is, Germany does not change its money supply at any stage, all interventions at target zone boundaries being undertaken by the corresponding non-central country. In such a situation, the exchange rate model between a non-central country, such as the UK. and Germany is, by analogy to equation (200), as follows

$$s_{12} = f_{12} + A_{12}e^{\rho_{12}f_{12}} + A_{22}e^{-\rho_{12}f_{12}}. \tag{204}$$

where the second subscript on the constants of integration indicates that these refer to the (only) target zone involving the UK., country 2. The constants of integration involved in the solution for the target zone exchange rates between Germany and the other non-central countries, can be given similar subscripts to distinguish them.

The above solution is simply the form for the exchange rate in a target zone, between any two countries with no drift parameter. Only country 2 (UK) intervenes to support the zone, and as noted before, does not intervene against any other country. The exchange rates between Germany and all other non-central countries would, because of identical arguments, take a similar form. In this situation there would be no potential conflict for the central country, since it plays a purely passive role. Although this is a relatively simple exchange rate model, the conclusion that in order to avoid potential conflict, ideally all of intervention must come from the non-central countries, at least in part justifies the perceived unwillingness by Germany to intervene during ERM crises, particularly the one involving the pound on 'black Wednesday'. Of course, in practise political considerations inevitably influence the intervention policy of the countries concerned irrespective of exchange rate models. For example, a comprehensive discussion of the actual monetary policies which affected the ERM the 1990's is given in De Grauwe (1995). These political difficulties in adopting and keeping to an intervention policy are at least some of the reasons for replacing the ERM by a single currency within the European Union, the case for which is discussed in Emerson, Gros, Italianer, Pisani-Ferry, and Reichenbach (1992). Wiliamson (1992), suggested alternative reforms to the ERM in order to overcome some of the observed difficulties in its operation.

In the network described in figure 6.3, exchange rates between any pair of non-central countries, for instance the UK. and France (country 3), will be target zone driven rates. This follows since although there is no target zone operating between the Pound and the Franc, both countries are changing their money supply in order to defend a zone against a third country. As the U.K. and France are undertaking all the interventions to defend their respective zones against Germany, the whole of these interventions will be reflected in the markets' expected rate of change for s_{23}. The exchange rate between the UK. and France would therefore, using the arguments of the previous Section, take the following form

$$s_{23} = f_{23} - A_{12}e^{\rho_{12}f_{12}} - A_{22}e^{-\rho_{12}f_{12}} + A_{13}e^{\rho_{13}f_{13}} + A_{23}e^{-\rho_{13}f_{13}}. \tag{205}$$

In determining the various effects on s_{23} in equation (205), it is worth recalling that A_{12} and A_{13} will both be negative, while A_{22} and A_{23} will both be positive. The equations for the other target zone driven exchange rates can be derived in a similar fashion.

6.5 Summary

When changes in money supply are made at the boundary of a target zone it is immaterial to the dynamics of the exchange rate whether the change results from an increase in supply by one country or a decrease in supply by the other. This is because in the monetary model only the relative monetary supply is of importance. However, the proportion of intervention undertaken by each country certainly is of relevance when the exchange rate against a third country is considered. Once the rules for allocating the proportion of intervention by each country are specified the relationship for the exchange rate between a country in the target zone and the third country can be readily obtained. These semi-floating rates consist of a combination of two terms each reflecting the expected amount of money to be created or withdrawn by the target zone participating country at each boundary.

If the network consists of two pairs of target zone participating countries then the model for the exchange rate between a pair of countries in different zones will consist of a combination of terms each reflecting the expected proportion of money creation or withdrawal at each zone boundary. These target zone driven rates are, like target zone rates themselves, non-linear even though neither of the participating countries alters its money supply in order to influence the rate between them directly. The results emphasise the fact that if a country adjusts its monetary policy as part of a target zone relationship with another country, it will inevitably have repercussions for its exchange rate movements with all other countries, intended or not. Under normal circumstances if a country is involved in more than one target zone relationship this will invariably result in a potential conflict in its monetary policy at some point in time. However, as Section 6.4 demonstrated, if all the target zone relationships are between one central country and a number of non-central countries, then no conflict need arise provided all the interventions are made by the non-central countries. The potential for such conflicts is something which groups of countries such as those which were involved in the ERM, need to bear in mind when the rules governing intervention policies are formulated.

7 Conclusion

Although some interest had been given to the movement of exchange rates prior to the 1970's, this was mainly in the context of a possible speculative attack on a fixed rate and the implications for a country's foreign currency reserves. This was at least partly the result of the Bretton Woods Agreement of 1944 whose objective was to prevent, if possible, a return to the competitive devaluations and protectionism that had characterized the period before World War II. Under the Bretton Woods system, countries undertook to preserve a fixed exchange rate until unambiguous evidence of 'fundamental disequilibrium' appeared, at which point they were expected to devalue or revalue as appropriate. That is, announce a new fixed parity against the US dollar. This arrangement continued in very much its original form until the late 1960,s, when the system came under considerable strain. The exchange rates agreed at Bretton Woods became increasingly inappropriate. The US had experienced relatively sluggish growth in its productive capacity since World War II, while the economies of continental Europe and Japan experienced rapid expansion. In addition, the US authorities were increasingly willing to print money during the 1960's in order to combat poverty at home and to finance the war in Vietnam. This not only made the dollar seem overvalued against other currencies such as the Deutsche Mark but put a strain on the fixed exchange rate of $35 per ounce of pure gold agreed at Bretton Woods. The system finally broke down in 1971 and although attempts were made to patch-up the fixed rate system through the Smithsonian Agreement this lasted barely a year, after which the modern era of floating exchange rates effectively came in to being.

The existence of floating exchange rates on a wide scale inevitably gave impetus to attempts at understanding the dynamical properties of such rates and the variables which influence them. The introduction of floating exchange rates was therefore followed fairly swiftly by the modern exposition of the deterministic monetary model by researchers such as Mussa (1976), and of the price inertia model in Dornbusch (1976). These remain the major models for describing the movement of exchange rates although modifications such as the possibility of currency substitution, have been incorporated in to the models by various authors, including Gärtner (1994). These modifications assume that domestic agents are allowed to hold foreign money as an asset in addition to domestic currency. In general, these modifications make little difference to the main properties of the models although the intensity of some phenomena, such as the overshooting effect in the price inertia model, may be affected. Currency substitution has also been given a more central role by authors such as Calvo and Rodriguez (1977), in which the demand for money and goods is related to domestic wealth rather than current income. However, giving currency substitution such a central role in the determination of exchange rate movements is only likely to be justifiable in extreme circumstances, for instance hyperinflation in the domestic economy. Therefore, it is legitimate to concentrate, as this thesis has done, on the basic monetary and price inertia models.

In Chapter 2, it was noted that both the models considered were useful in focusing attention on the variables which might affect exchange rates and in providing predictions of how rates may behave in the future. In the case of the monetary model discussed in Section 2.1, these predictions did not seem to agree very well with the data, except in the long-term, reflecting the evidence against PPP holding on a continuous basis. The price inertia models mentioned in Section 2.2 were rather more realistic in describing short-term movements in exchange rates, relying as they do on the more credible assumption of stickiness of price movements. However, as noted by Macdonald (1988), even these models and their derivatives do not track the movements of the major exchange rates particularly well. Possibly their greatest contribution has been the demonstration that mechanisms exist which explain why observed exchange rate movements can be much greater than the movement in the fundamental variables upon which they depend.

The usefulness of both the monetary and price inertia models was given a boost in the early 1990's with the development of their stochastic versions. The use of stochastic calculus in Section 4.1 allowed a closed form solution (subject to the evaluation of constants of integration) to be derived for the monetary model. This indicated that even for a relatively simplistic model of this form the solution paths are, in general, highly non-linear. Although the stochastic price inertia model discussed in Section 4.1 did not have a closed form solution, the qualitative analysis given there again illustrated the non-linear nature of potential solution paths. The use of stochastic models for modelling movements within target zones, discussed in Chapter 5, demonstrated that with the imposition of suitable boundary conditions the trajectory for the exchange rate movement can be tied down, at least in its general form. The results emphasised that these trajectories would have quite different properties to what naively might be expected from an exchange rate which experienced a free float within a target zone mechanism such as the

ERM.

Chapter 6 demonstrated how the exchange rate movement between two countries, one of which is engaged in a target zone relationship and one which has a policy of a free floating currency, could be described for an underlying monetary model. In particular, Jørgensen and Mikkelsen observed that although it does not matter which of the two countries involved in a target zone makes a monetary adjustment, when the exchange rate is at the boundary of the zone, for describing movements in that rate, it does matter from the point of view of a third country. The modelling of these so called semi-floating rates therefore required the rules for target zone interventions between participating countries to be made explicit so that market expectations for exchange rate movements could be formulated. If this is done along the lines described by Jørgensen and Mikkelsen then explicit solutions for the movement of semi-floating could readily be obtained. By using a similar approach in Sections 6.3 and 6.4, it was shown how the dynamics of exchange rates within networks containing more than one target zone or in which one country has a central role, respectively, could be described. These extensions to networks are potentially useful since real life currency arrangements such as the ERM invariably involve more than two countries often with a mixture of types of relationships between the countries. As noted earlier, the monetary model has many limitations, so it would seem potentially useful if some of the situations considered in Chapter 6 could be analysed with the price inertia model. However, many of the extra complexities of the model which make it a more realistic describer of short-term movements in exchange rates, would make analysis considerably more difficult. In particular the lack of a closed form solution for the basic price inertia model would make the formulation of a dynamical system to describe even semi-floating rates considerably more problematic.

Stochastic models have since their formulation been useful in modelling speculative attacks on currencies operating within a band and more recently in describing optimal policies for managing a floating exchange rate, for instance by Miller and Papi (1997). In some ways, the increasing number of central banks with independence of operation in recent years, for instance the Bank of England, and the European Central Bank, has if anything tended to increase the relevance of the analyses produced using these models. This follows since the assumptions behind the operation of a target zone given in Section 5.1 require that central banks co-operate in the carrying out of intervention policies and that such interventions are credible to the market.

It could be argued that both of these are more likely the greater independence central banks have of other political considerations. Walsh (1995), and Lockwood, Miller and Zhang (1996) have also discussed the role and reputation of central banks in modelling exchange rate regimes. However, although the realism of some of their proposed intervention policies may have increased over recent years, many of the fundamental assumptions behind the stochastic models, suffer from the same limitations as their deterministic counterparts, particularly those for the monetary model. Therefore, possibly their usefulness lies in the qualitative properties of their results rather than an expectation that they will track actual exchange rates any more successfully than their predecessors.

Bibliography

Avesani R. (1990), *Endogenously determined target zone and central bank optimal policy with limited reserves,* mimeo, CEPR/NBER conference on exchange rates and currency bands, University of Warwick.

Baillie R. and McMahon P. (1994), *The foreign exchange market: Theory and econometric evidence,* Cambridge: Cambridge University Press.

Biggs N.L. (1993), *Exchange rate Networks and Mechanisms,* Mathematics Pre-print series, **64**, L.S.E.

Bilson J. f. O. (1978), *The monetary approach to the exchange rate, some empirical evidence,* IMF Staff Papers, **25**, 48-75.

Birkhoff G. and Rota G. (1969), *Ordinary Differential Equations,* 2nd. Ed., Blaisdell Publishing Company.

Calvo G. and Rodriguez C. (1977), A model of exchange rate determination under currency substitution and rational expectations, *Journal of Political Economy,* **85**, 261-278.

Copeland L. S. (1994), *Exchange Rates and International Finance,* Wokingham, Addison-Wesley.

De Grauwe P (1995), Monetary policies in the EMS during the 1990's, *European Currency Crises and After,* eds C. Bordes, E. Girardin and J. Meltiz, Manchester University Press, Manchester, England.

Delgado F and Dumas B. (1990), Monetary contracting between central banks and the design of sustainable exchange rate zones, *Working Paper N3440,* NBER.

Delgado F. and Dumas B. (1992), Target Zones, Broad and Narrow, in P. Krugman and M. Miller (eds.), *Exchange Rate Targets and Currency Bands,* Cambridge: Cambridge University Press, 17-28.

Delgado F. Dumas B. (1993), Monetary contracting between central banks and the design of sustainable exchange rate zones, *Journal of International Economics,* **34**, 201-224.

Dixit A. (1993), *The art of smooth pasting,* Harwood Academic Publishers.

Dornbusch R. (1976), Expectations and exchange rate dynamics, *Journal of Political Economy,* **84**, 1161-76.

Emerson M, Gros D, Italianer A, Pisani-Ferry J, and Reichenbach H., (1992), *One Market, One Money*, Oxford, Oxford University Press.

Flood R and Garber P. (1989), The linkage between speculative attack and target zone models of the exchange rate, *Working Paper No 2918*, NBER.

Frankal J.A. (1979), On the Mark: A Theory of Floating Exchange Rates Based on Real Interest Rates Differentials, *American Economic Review*, **69**, 610-22.

Frenkel J. A. (1976), A Monetary Approach to the Exchange Rate: Doctrinal Aspects and Empirical Evidence, *Scandinavian Journal of Economics,* **78**,169-191.

Froot K.A. and Obstfeld M. (1991), Exchange-rate dynamics under stochastic regime shifts, *Journal of International Economics,* **31,** 203-29.

Gärtner M. (1993), *Macroeconomics under Flexible Exchange Rates.* L.S.E. Handbooks in Economics: Harvester Wheatsheaf.

Harrison J. Michael and Taksar M. (1983), Instantaneous control of Brownian motion, *Mathematics of Operations Research,* **8,** 439-453.

Harrison J. Michael, Sellke T. M. and Taylor A. J. (1983), Impulse control of Brownian motion, *Mathematics of Operations Research,* **8**, 454-466.

Harrison J. Michael (1985), *Brownian motion and stochastic flow systems,* New York, Wiley.

Jörgensen B. and Mikkelsen H. (1997), Semi-floating Exchange Rates, *Working Paper*, Harvard Business School.

Krugman P. R. (1988), Target zones and exchange rate dynamics, *Working Paper 2841*, NBER.

Krugman P. R. (1991), Target zones and exchange rate dynamics, *Quarterly Journal of Economics,* 669-82.

Krugman P. R. (1992), Exchange rates in a currency band: A sketch of the new approach, in P. Krugman and M Miller, eds., *Exchange rate targets and currency bands*, Cambridge University Press, Cambridge, 9-14.

Laskar D. (1994), Time inconsistency of the optimal monetary policy: a case for target zones, *CEPREMAP Working Paper No 94226.*

Lockwood B, Miller M, and Zhang L.,(1996), Central banks and reputation: some transatlantic contrasts, *Department of Economics, University of Warwick mimeo.*

McCallum B. T. (1994), A reconsideration of the uncovered interest rate parity relationship, *Journal of Monetary Economics*, **33**, 105-132.

Meese R. A. and Rogoff K. (1983), The out-of-sample failure of empirical exchange rates models: sampling error or misspecification?, in J. Frenkel ed., *International Economics*, University of Chicago Press, Chicago, 67-112.

Miller M. and Weller P. (1989), Solving Stochastic Saddlepoint Systems: A Qualitative Treatment with Economic Applications, *Centre for Economic Policy Research*, Discussion Paper Series, Number **308.**

Miller M and Weller P. (1990), Exchange Rate Bands with Price Inertia, *Centre for Economic Policy Research*, Discussion Paper Series, Number **421.**

Miller M. and Weller P. (1991), Exchange rate bands with price inertia, *The Economic Journal*, **101**, 1380-1399.

Miller M and Zhang L. (1996), Optimal target zones: How an exchange rate mechanism can improve upon discretion, *Journal of Economic Dynamics and Control*, **20**, 1641-1660.

Miller M and Papi P. (1997), The 'laissez faire' bias of managed floating, *Journal of Money and Finance*, **16**, 989-1000.

Musa M. (1976), The exchange rate, the balance of payments and monetary policy under a regime of controlled floating, *Scandinavian Journal of Economics*, **78**, 229-248.

Papi L. (1993), Essays in optimal government policy, *PhD Thesis*, University of Warwick.

Pilbeam K. (1992), *International Finance*, London, Macmillan.

Smith G.W. (1987), Stochastic process switching, *Working Paper*, Queens University.

Svensson L.E.O. (1991), Target zones and interest rate variability, *Journal of International Economics*, **31**, 27-54.

Svensson L. (1994), Why exchange rate bands? Monetary independence in spite of fixed exchange rates, *Journal of Monetary Economics*, **33**, 157-199.

Walsh C. (1995), Optimal contracts for central banks, *American Economic Review*, **85**,150-167.

Walsham and Parramore (1997), *Quantitative Methods in Finance*, London, Thomson.

Williamson J. and Miller M. (1987), Targets and Indicators: a blueprint for the international co-ordination of economic policy, *Institute for International Economics*, Washington D.C.

Williamson J. (1992), How to reform the ERM, *International Economic Insights*, **3**, 37-39.