Models for Investment Capacity Expansion

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Abstract

The objective of this thesis is to develop and analyse two stochastic control problems arising in the context of investment capacity expansion. In both problems the underlying market fluctuations are modelled by a geometric Brownian motion. The decision maker's aim is to determine admissible capacity expansion strategies that maximise appropriate expected present-value performance criteria.

In the first model, capacity expansion has price/demand impact and involves proportional costs. The resulting optimisation problem takes the form of a singular stochastic control problem. In the second model, capacity expansion has no impact on price/demand but is associated with fixed as well as proportional costs, thus resulting in an impulse control problem.

Both problems are completely solved and the optimal strategies are fully characterised. In particular, the value functions are constructed explicitly as suitable classical solutions to the associated Hamilton-Jacobi-Bellman equations.

Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Contents

Chapter 1

Introduction

A standard capacity expansion model, which is a special case of the model studied by Kobila (1993), can be described as follows. We model market uncertainty by means of the geometric Brownian motion given by

$$dX_t^0 = bX_t^0\,dt + \sqrt{2}\sigma X_t^0\,dW_t, \quad X_0^0 = x > 0, \tag{1.1}$$

for some constants b and $\sigma \neq 0$, where W is a standard one-dimensional Brownian motion. The random variable X_t^0 can represent an economic indicator such as the price of or the demand for one unit of a given investment project's output at time t. The firm behind the project can invest additional capital at proportional costs at any time, but cannot disinvest from the project. We denote by y the project's initial capital at time 0 and by ζ_t the total additional capital invested by time t. We assume that there is no capital depreciation, so the total capital invested at time t is

$$Y_t = y + \zeta_t, \quad Y_0 = y \geq 0. \tag{1.2}$$

The investor's objective is to maximise the total expected discounted payoff resulting from the project's management, which is given by the performance index

$$J_{x,y}^0(\zeta) = \mathbb{E}\left[\int_0^\infty e^{-rt}h(X_t^0, Y_t)\,dt - K\int_{[0,\infty[} e^{-rt}\,d\zeta_t\right], \tag{1.3}$$

over all capacity expansion strategies ζ. The discounting rate $r > 0$ and the cost of each additional unit of capital $K > 0$ are constants, while h is an appropriate running payoff function.

Under suitable assumptions on the problem data, the solution to this stochastic control problem is characterised by a threshold given by a strictly increasing free-boundary function $G^0 : \mathbb{R}_+ \to \mathbb{R}_+$. In the special case that arises when $h(x, y) = x^\alpha y^\beta$, for some $\alpha > 0$ and $\beta \in \,]0, 1[$, namely, when h is a so-called Cobb-Douglas production function,

$$G^0(y) = \left(\frac{rK(\alpha - m)}{-m\beta}\right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}} \quad \text{for } y \geq 0,$$

where $m < 0$ is an appropriate constant. If the initial condition (x, y) is strictly below the graph of the function G^0 in the x-y plane, then it is optimal to invest so that the joint process (X^0, Y) has a jump at time 0 that positions it in the graph of G^0. Otherwise, it is optimal to take minimal action so that the process (X^0, Y) does not fall below the graph of G^0, which amounts to reflecting it in G^0 in the positive y-direction.

Irreversible capacity expansion models have attracted considerable interest and can be traced back to Manne (1961) (see Mieghem (2003) for a survey). More relevant to this thesis models have been studied by several authors in the economics literature: see Dixit and Pindyck (1994, Chapter 11) and references therein. Related models that have been studied in the mathematics literature include Davis et al. (1987), Davis (1993), Øksendal (2000), Wang (2003), Chiarolla and Haussmann (2005), Bank (2005), Alvarez (2006, 2010), Løkka and Zervos (2011b) and references therein. Furthermore, capacity expansion models with costly reversibility were introduced by Abel and Eberly (1996), and were further studied by Guo and Pham (2005), Merhi and Zervos (2007), Guo and Tomecek (2008b,a), Guo et al. (2011) and Løkka and Zervos (2011a).

In the model that we have briefly discussed above, additional investment does not influence the underlying economic indicator, which is unrealistic if one considers supply and demand issues. The nature of the optimal strategy is such that, if $b < \sigma^2$, then $\lim_{t\to\infty} X_t^0 = 0$ and the investment's maximal optimal capacity level remains finite for realistic choices of the problem data. On the other hand, if $b \geq \sigma^2$, then $\limsup_{t\to\infty} X_t^0 = \infty$ and the optimal capacity level typically converges to ∞ as $t \to \infty$.

The model that we study in Chapter 2 assumes that additional investment has a strictly negative effect on the value of the underlying economic indicator. In particular, we model market uncertainty by the solution to the SDE

$$dX_t = bX_t\,dt - X_t \circ d\zeta_t + \sqrt{2}\sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{1.4}$$

where

$$\int_0^t X_s \circ d\zeta_s = c\int_0^t X_s\,d\zeta_s^{\mathrm{c}} + \sum_{0\le s<t} X_s\big(1 - e^{-c\Delta\zeta_s}\big),$$

for some constant $c > 0$, in which expression, ζ^{c} denotes the continuous part of the increasing process ζ. The objective is to maximise over all admissible capacity expansion strategies ζ the performance criterion

$$J_{x,y}(\zeta) = \mathbb{E}\left[\int_0^\infty e^{-rt}h(X_t, Y_t)\,dt - K\int_{[0,\infty[} e^{-rt}\,d\zeta_t\right], \tag{1.5}$$

where $r, K > 0$ are constants and the running payoff function h satisfies Assumption 2.1.1 in Chapter 2.

The solution to this problem is again characterised by a threshold defined by a strictly increasing free-boundary function G. Informally, the optimal strategy can be described as the one in the problem defined by (1.1)–(1.3). However, reflection in the free-boundary G is oblong rather than in the positive y-direction (see Figures 2.5.1–2.5.3). Furthermore, the negative effect that additional investment has on the underlying economic indicator X results in a maximal optimal capacity level that is bounded in cases of special interest, such as the ones arising, e.g., when the running payoff function h is a Codd-Douglass production function (see Example 2.2.1).

From a stochastic control theoretic perspective, the problem that we solve in Chapter 2 has the features of singular stochastic control, which was introduced by Bather and Chernoff (1967) who considered a simplified model of spaceship control. In their seminal paper, Beneš et al. (1980) were the first to solve rigorously an example of a finite-fuel singular control problem. Since then, the area has attracted considerable interest in the literature. Apart from references that we have discussed in the context of capacity expansion models, Bahlali et al. (2009) Chiarolla and Haussmann (1994), Chow et al. (1985), Davis and Zervos (1998), Fleming and Soner (1993, Chapter VIII), Haussmann and Suo (1995a,b), Harrison and Taksar (1983), Jack et al. (2008), Jacka (1983, 2002), Karatzas (1983), Ma (1992), Menaldi and Robin (1983), Øksendal (2000), Shreve et al. (1984), Soner and Shreve (1989), Sun (1987) and Zhu (1992), provide an alphabetically ordered list of further contributions.

In the references discussed above, the controlled process affects the state dynamics in a purely additive way: the change of the state process due to control action does not depend on the state process itself. Singular stochastic control models in which changes of the state process due to control action may depend on the state process were introduced and studied by Dufour and Miller (2002) and Motta and Sartori (2007). To the best of our knowledge, problems with state dynamics such as the ones given by (1.4) have not been considered in the literature before. Furthermore, the problem that we solve is the very first one that involves control action that does not affect the state dynamics in a purely additive way and admits an explicit solution.

The model that we study in Chapter 3, takes a different perspective. In this case, we assume that additional investment does not affect the value of the underlying economic indicator. In particular, we model market uncertainty by means of the geometric Brownian motion given by (1.1). The objective of the stochastic control problem that we study is to maximise over all admissible capacity expansion strategies ζ the performance criterion

$$J_{x,y}(Z) = \mathbb{E}\left[\int_0^\infty e^{-rt}[(X_t^0)^\alpha Y_t^\beta - K_1 Y_t]\, dt - \sum_{0\leq t} e^{-rt}(K_2\Delta Z_t + c)1_{\{\Delta\zeta_t>0\}}\right] \qquad (1.6)$$

where $r, \alpha > 0$ and $\beta \in]0,1[$, $K_1 \geq 0$ and $K_2, c > 0$ are constants.

The solution to this problem is now characterized by two free-boundary functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G_1(y) < G_0(y)$ for all $y \geq 0$. If the initial condition (x, y) is below the graph of G_0 in the x-y plane, then it is optimal to invest so that the joint process (X^0, Y) has a jump at time 0 that positions it in the graph of G_1. After time 0, it is optimal to invest each time (X^0, Y) hits the graph G_0 so that the process (X^0, Y) has a jump in the vertical direction of the x-y plane that positions it inside the graph of G_1.

The model that we study in Chapter 3 was introduced by Merhi (2006, Chapter 3) who considered a general running payoff function $(x, y) \mapsto h(x, y)$ rather than the Cobb-Douglas running payoff function $(x, y) \mapsto x^\alpha y^\beta$ that we consider here. The analysis of Merhi (2006, Chapter 3) failed to determine the free-boundary functions G_0 and G_1 in a satisfactory way. The possibility of solving completely an important special case of the more general problem was the motivation for the study we present in Chapter 3. Unfortunately, the complete solution to the problem still remains elusive (see the assumptions of Lemma 3.3.1). Plainly, the problem formulation and the Hamilton-Jacobi-Bellman equation, which takes the form of a quasi-variational inequalities, that we present in Chapter 3 follow very closely the corresponding parts of Merhi (2006, Chapter 3). However, the rest of the analysis is different.

From a stochastic control theoretic perspective, the problem that we analyse in Chapter 3 has the features of a genuinely two-dimensional impulse control problem. Stochastic impulse control problems have been studied in the context of various fields, including mathematical finance, economic and operations research. The study of impulse control problems by means of quasi-variational inequalities was introduced by Bensoussan and Lions (1973). The corresponding theory is developed extensively in the book by Bensoussan and Lions (1984). Recent expositions of the general theory of stochastic impulse control can be found in the books by Øksendal and Sulem (2007) and Pham (2009).

The impulse control of one-dimensional diffusions has attracted considered interest in the literature. Notable contributions include Richard (1977), Harrison et al. (1983), Jeanblanc-Picqué and Shiryaev (1995), Mundaca and Øksendal (1998), Cadenillas and Zapatero (1999), Korn (1999), Bar-Ilan et al. (2002), Alvarez (2004), Bar-Ilan et al. (2004), Ohnishi and Tsujimura (2006), Alvarez and Koskela (2007), Cadenillas et al. (2010), Djehiche et al. (2010) and Feng and Muthuraman (2010). To the best of our knowledge, the problem studied by Merhi (2006) and by this thesis is the first genuinely two-dimensional one that has been analysed with mathematical rigour at the depth we present here with a view to an explicit solution.

Chapter 2

Irreversible capital accumulation with economic impact

2.1 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by $\mathbb{P}$-negligible sets, and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{Z}$ the family of all increasing càglàd $(\mathcal{F}_t)$-adapted processes ζ such that $\zeta_0 = 0$.

We consider an investment project that produces a given commodity and we assume that the project's capacity, namely, its rate of output, can be increased at any given time and by any finite amount up to a maximum level $\bar{y} \in \,]0, \infty]$. We denote by Y_t the project's capacity at time t and we model cumulative capacity increases by a process $\zeta \in \mathcal{Z}$. In particular, given times $0 \leq s \leq t$, $\zeta_{t+} - \zeta_s$ is the total capacity increase incurred by the project management's decisions during the time interval $[s, t]$. The project's capacity process Y is therefore given by

$$Y_t = y + \zeta_t, \quad Y_0 = y \geq 0, \tag{2.1}$$

where $y \geq 0$ is the project's initial capacity.

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t\,dt - X_t \circ d\zeta_t + \sqrt{2}\sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{2.2}$$

for some constants b and $\sigma \neq 0$, where

$$\int_0^t X_s \circ d\zeta_s = c\int_0^t X_s\,d\zeta_s^{\mathrm{c}} + \sum_{0\leq s<t} X_s\bigl(1 - e^{-c\Delta\zeta_s}\bigr), \tag{2.3}$$

for some constant $c > 0$, in which expression, ζ^{c} denotes the continuous part of the increasing process ζ. In practice, X_t can be an economic indicator reflecting, e.g., the value of one unit of the output commodity or the output commodity's demand or both, at time t. Using Itô's formula, we can check that

$$X_t = X_t^0 e^{-c\zeta_t^{\mathrm{c}}} \prod_{0\leq s<t} e^{-c\Delta\zeta_s} = X_t^0 e^{-c\zeta_t}, \tag{2.4}$$

where X^0 is the geometric Brownian motion defined by

$$dX_t^0 = bX_t^0\,dt + \sqrt{2}\sigma X_t^0\,dW_t, \quad X_0^0 = x > 0. \tag{2.5}$$

To simplify the notation, we denote by $\mathcal{S}$ the problem's state space, so that

$$\mathcal{S} = \bigl\{(x,y) \in \mathbb{R}^2 \mid\ x > 0 \text{ and } 0 \leq y \leq \bar{y}\bigr\}.$$

With each decision policy ζ we associate the performance criterion

$$J_{x,y}(\zeta) = \mathbb{E}\left[\int_0^\infty e^{-rt}h(X_t, Y_t)\,dt - K\int_{[0,\infty[} e^{-rt}\,d\zeta_t\right], \tag{2.6}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function and $K, r > 0$ are constants. Here, h models the running payoff resulting from the project's operation, while K models the costs associated with increasing the project's capacity level.

Definition 2.1.1 The set $\mathcal{A}$ of all admissible strategies is the family of all processes $\zeta \in \mathcal{Z}$ such that

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-rt}\,d\zeta_t\right] < \infty \tag{2.7}$$

and $Y_t \in [0,\bar{y}] \cap \mathbb{R}_+$ for all $t \geq 0$. □

The objective of the control problem is to maximise the performance index $J_{x,y}$ over all admissible strategies $\zeta \in \mathcal{A}$. Accordingly, we define the problem's value function v by

$$v(x,y) = \sup_{\zeta \in \mathcal{A}} J_{x,y}(\zeta), \quad \text{for } (x,y) \in \mathcal{S}. \tag{2.8}$$

For the stochastic control problem to be well-defined, we make the following assumption.

Assumption 2.1.1 $K > 0$, the function h is C^3,

$$h(\cdot, y) \text{ is increasing for all } y \in [0, \bar{y}] \cap \mathbb{R}_+, \tag{2.9}$$

$$\int_0^x s^{-m-1} |h(s,y)| \, ds + \int_x^\infty s^{-n-1} |h(s,y)| \, ds < \infty \quad \text{for all } x > 0 \text{ and } y \in [0, \bar{y}] \cap \mathbb{R}_+, \tag{2.10}$$

where the constants $m < 0 < n$ are defined by (2.75) in Appendix II (see also (2.78)–(2.79) in Appendix II). If we define

$$H(x,y) = h_y(x,y) - cxh_x(x,y) - rK, \quad \text{for } x > 0 \text{ and } y \in \,]0, \bar{y}[, \tag{2.11}$$

then there exists a point $x_0 \geq 0$ and a continuous strictly increasing function $y^\dagger : \,]x_0, \infty[\to \mathbb{R}_+$ such that

$$0 \leq y_0 := \lim_{x \downarrow x_0} y^\dagger(x) < \lim_{x \to \infty} y^\dagger(x) =: y_\infty \leq \bar{y}, \quad y_0 = 0 \text{ if } x_0 > 0, \tag{2.12}$$

$$H(x,y) \begin{cases} < 0, & \text{if } (x,y) \in \mathcal{H}_-, \\ = 0, & \text{if } (x,y) \in \mathcal{S} \setminus (\mathcal{H}_- \cup \mathcal{H}_+), \\ > 0, & \text{if } (x,y) \in \mathcal{H}_+, \end{cases} \tag{2.13}$$

$$\liminf_{x \to \infty} H(x,y) > 0 \quad \text{for all } y \in \,]y_0, y_\infty[, \tag{2.14}$$

$$\text{the function } H(x, \cdot) \text{ is strictly decreasing for all } y \in \,]y_0, y_\infty[, \tag{2.15}$$

where

$$\mathcal{H}_- = \{(x,y) \in \mathcal{S} \mid \; x \leq x_0 \text{ or } x > x_0 \text{ and } y > y^\dagger(x)\},$$
$$\mathcal{H}_+ = \{(x,y) \in \mathcal{S} \mid \; x > x_0 \text{ and } y < y^\dagger(x)\},$$

and the function $x^\dagger$ is defined by

$$x^\dagger(y) = \begin{cases} 0, & \text{if } 0 \le y < y_0, \\ (y^\dagger)^{-1}(x), & \text{if } y_0 \le y < y_\infty, \\ \infty, & \text{if } y_\infty \le y < \bar{y}. \end{cases} \tag{2.16}$$

Also, there exist a decreasing function $\Psi :]y_0, y_\infty[\to]0, \infty[$ such that $\lim_{y \downarrow 0} \Psi(y) < \infty$ if $x_0 > 0$ as well as constants $C_0 > 0$ and $\vartheta \in]0, n[$ such that

$$-C_0(1+y) \le h(x,y) \le C_0(1+y)\big(1 + x^{n-\vartheta}\big) \quad \text{for all } (x,y) \in \mathcal{S}, \tag{2.17}$$

$$H(x,y) \le \Psi(y)\big(1 + x^{n-\vartheta}\big) \quad \text{for all } x > 0 \text{ and } y \in]0, \bar{y}[, \tag{2.18}$$

where $n > 0$ is given by (2.75) in Appendix II. □

Example 2.1.1 Suppose that $\bar{y} = \infty$ and h is a so-called Cobb-Douglas function, given by

$$h(x,y) = x^\alpha y^\beta, \quad \text{for } (x,y) \in \mathcal{S}, \tag{2.19}$$

where $\alpha \in]0, n[$ and $\beta \in]0, 1]$ are constants. In this case, we can check that

$$H(x,y) = \big(\beta y^{-1} - c\alpha\big) x^\alpha y^\beta - rK.$$

If we define

$$y_0 = 0, \quad y_\infty = \frac{\beta}{c\alpha} \quad \text{and} \quad x_0 = \begin{cases} (rK)^{1/\alpha}, & \text{if } \beta = 1, \\ 0, & \text{if } \beta \in]0, 1[, \end{cases}$$

then we can see that the calculations

$$\frac{\partial H(x,y)}{\partial x} = \alpha\big(\beta y^{-1} - c\alpha\big) x^{\alpha-1} y^\beta \begin{cases} > 0 & \text{for all } y \in]y_0, y_\infty[, \\ < 0 & \text{for all } y > y_\infty, \end{cases} \tag{2.20}$$

$$\lim_{x \downarrow 0} H(x,y) = -rK < 0 \text{ for all } y > 0, \quad \text{and}$$

$$\lim_{x \to \infty} H(x,y) = \begin{cases} \infty, & \text{for all } y \in]y_0, y_\infty[, \\ -\infty, & \text{for all } y > y_\infty, \end{cases} \tag{2.21}$$

imply that there exists a unique function $y^\dagger :]x_0, \infty[\to \mathbb{R}_+$ such that (2.12)–(2.13) hold true. Furthermore, differentiating the identity $H\big(x, y^\dagger(x)\big) = 0$ with respect to x, we can see that the derivative $\dot{y}^\dagger$ of $y^\dagger$ satisfies

$$\dot{y}^\dagger(x) = \frac{\alpha y(\beta - c\alpha y)}{\beta x\big[(1-\beta) + c\alpha y\big]} > 0 \quad \text{for all } y \in \,]y_0, y_\infty[,$$

so $y^\dagger$ is indeed strictly increasing. Also, it is straightforward to check that (2.14)–(2.15) and (2.17)–(2.18) are all satisfied. Indeed, (2.14) (resp., (2.15)) follows immediately from (2.21) (resp., (2.20)), while (2.17)–(2.18) follow from (2.19) and (2.20) for the choices $\vartheta = n - \alpha$ and

$$\Psi(y) = \begin{cases} 1, & \text{if } \beta = 1, \\ y^{-(1-\beta)}, & \text{if } \beta \in \,]0, 1[. \end{cases}$$

□

2.2 The solution to the control problem

We solve the stochastic control problem that we consider by constructing an appropriate classical solution $w : \mathcal{S} \to \mathbb{R}$ to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\big\{\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y), \\ w_y(x,y) - cxw_x(x,y) - K\big\} = 0, \quad (x,y) \in \mathcal{S}, \tag{2.22}$$

where $w_y(x,0) = \lim_{y\downarrow 0} w_y(x,y)$. To obtain qualitative understanding of this equation, we consider the following heuristic arguments. At time 0, the project's management has two options. The first one is to wait for a short time Δt and then continue optimally. Bellman's principle of optimality implies that this option, which is not necessarily optimal, is associated with the inequality

$$v(x,y) \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-rt} h(X_t^0, y)\, dt + e^{-r\Delta t} v\big(X_{\Delta t}^0, y\big)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$\sigma^2 x^2 v_{xx}(x,y) + bxv_x(x,y) - rv(x,y) + h(x,y) \leq 0. \tag{2.23}$$

The second option is to increase capacity by $\varepsilon > 0$, and then continue optimally. This action is associated with the inequality

$$v(x,y) \geq v(x - cx\varepsilon, y + \varepsilon) - K\varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$v_y(x,y) - cxv_x(x,y) - K \leq 0. \tag{2.24}$$

Furthermore, the Markovian character of the problem implies that one of these options should be optimal and one of (2.23), (2.24) should hold with equality at any point in the state space $\mathcal{S}$. It follows that the problem's value function v should identify with an appropriate solution w of the HJB equation (2.22).

To construct the solution w to (2.22) that identifies with the value function v, we first consider the existence of a strictly increasing function $G : \,]y_0, y_\infty[\, \to \,]0, \infty[$ that partitions the state space $\mathcal{S}$ into two regions, the "waiting" region $\mathcal{W}$ and the "investment" region $\mathcal{I}$, defined by

$$\begin{aligned}
\mathcal{W} = &\big\{(x,0) \mid \ 0 < x \leq x_0 \text{ if } x_0 > 0\big\} \\
&\cup \big\{(x,y) \mid \ y \in \,]y_0, y_\infty[\text{ and } 0 < x \leq G(y)\big\} \\
&\cup \big\{(x,y) \mid \ x > 0 \text{ and } y \in [y_\infty, \bar{y}] \cap \mathbb{R}\big\}, \\
\mathcal{I} = &\big\{(x,0) \mid \ x > x_0 \text{ if } x_0 > 0\big\} \\
&\cup \big\{(x,y) \mid \ x > 0 \text{ and } y \in [0, y_0] \text{ if } y_0 > 0\big\} \\
&\cup \big\{(x,y) \mid \ y \in \,]y_0, y_\infty[\text{ and } x > G(y)\big\}.
\end{aligned}$$

(see Figures 2.5.1–2.5.3 in Appendix III). Inside the region $\mathcal{W}$, the heuristic arguments that we have briefly discussed above suggest that w should satisfy the differential equation

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) = 0. \tag{2.25}$$

In light of the theory that we review in Appendix II and the intuitive idea that the value function should remain bounded as $x \downarrow 0$, every relevant solution to this ODE is given by

$$w(x,y) = A(y)x^n + R(x,y), \tag{2.26}$$

for some function A, where n is given by (2.75) and $R(\cdot, y)$ is defined by (2.80) for $k = h(\cdot, y)$, i.e.,

$$R(x,y) = \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m-1} h(s,y)\, ds + x^n \int_x^\infty s^{-n-1} h(s,y)\, ds \right]. \tag{2.27}$$

On the other hand, w should satisfy

$$w_y(x,y) - cxw_x(x,y) = K, \quad \text{for } (x,y) \in \mathcal{I}, \tag{2.28}$$

which implies that

$$w_{yx}(x,y) - cxw_{xx}(x,y) - cw_x(x,y) = 0, \quad \text{for } (x,y) \in \mathcal{I}. \tag{2.29}$$

Remark 2.2.1 At this point, it is worth making a comment on the qualitative dependence of the optimal strategy arising from the considerations above and depicted by Figure 2.5.1 on the parameters c and K. The constant $c > 0$ determines the magnitude of the effect that investment has on the state process X (see (2.2)-(2.3)). Therefore, as $c \downarrow 0$, we expect that the curved arrows in Figure 2.5.1 become vertical because additional investment has less and less effect on the state dynamics. On the other hand, as $c \to \infty$, we expect that the curved arrows in Figure 2.5.1 bend more and more towards the horizontal axis because additional investment has increasing effect on the dynamics of X. The constant $K > 0$ determines the degree at which additional investment is penalised by the performance criterion defined by (2.6). As $K \downarrow 0$, we expect that the free-boundary G moves higher and higher in the x-y plane and the investment region $\mathcal{I}$ spreads to cover the entire $\mathbb{R}_+^2$ because additional investment is penalized less and less. On the other hand, as $K \to \infty$, we expect that G moves lower and lower in the x-y plane and the investment region $\mathcal{I}$ shrinks because additional investment is increasingly penalized.

To determine A and G, we postulate that w is $C^{2,1}$, in particular, along the free-boundary G. Such a requirement and (2.26)–(2.29) yield the system of equations

$$\big[\dot{A}(y) - ncA(y)\big] G^n(y) = -\Big[R_y\big(G(y), y\big) - cG(y) R_x\big(G(y), y\big) - K\Big], \tag{2.30}$$

$$\big[\dot{A}(y) - ncA(y)\big] G^n(y) = -\frac{G(y)}{n} \Big[R_{yx}\big(G(y), y\big) - cG(y) R_{xx}\big(G(y), y\big) - cR_x\big(G(y), y\big)\Big]. \tag{2.31}$$

In view of the definition (2.27) of R, the associated expression (2.85) for the function $x \mapsto xR_x(x, y)$ and (2.84), we can see that this system is equivalent to

$$q\big(G(y), y\big) = 0, \tag{2.32}$$

$$\dot{A}(y) = ncA(y) - \frac{1}{\sigma^2(n-m)} \int_{G(y)}^{\infty} s^{-n-1} H(s, y)\, ds, \tag{2.33}$$

where H is defined by (2.11) and

$$q(x, y) = \int_0^x s^{-m-1} H(s, y)\, ds. \tag{2.34}$$

We can also check that the solution to (2.33) is given by

$$A(y) = \frac{e^{cny}}{\sigma^2(n-m)} \int_y^{y_\infty} e^{-cnu} \int_{G(u)}^{\infty} s^{-n-1} H(s, u)\, ds\, du, \quad \text{for } y_0 < y < y_\infty, \tag{2.35}$$

if the integrals converge.

The following result, the proof of which we develop in Appendix I, is concerned with the solution to the system of equations (2.32)–(2.33).

Lemma 2.2.1 Suppose that Assumption 2.1.1 holds true. The equation $q(x, y) = 0$ for $x > 0$ defines uniquely a strictly increasing C^1 function $G :]y_0, y_\infty[\to]0, \infty[$, which satisfies

$$x^\dagger(y) < G(y) \text{ for all } y \in]y_0, y_\infty[, \quad \lim_{y \downarrow y_0} G(y) = 0, \text{ if } y_0 > 0, \quad \text{and} \quad \lim_{y \uparrow y_\infty} G(y) = \infty, \tag{2.36}$$

where $x^\dagger$ is defined by (2.16). Furthermore, the function A given by (2.35) is well-defined and real-valued, and there exists a constant $C_1 > 0$ such that

$$0 < A(y)G^n(y) \le C_1 \Psi(y) \left[1 + G^{n-\vartheta}(y)\right] \quad \text{for all } y \in]y_0, y_\infty[, \tag{2.37}$$

where the decreasing function Ψ and the constant $\vartheta > 0$ are as in (2.18), and

$$g^{-1}(x) + \left[1 + g^{-1}(x)\right] G^{n-\vartheta}\big(g^{-1}(x)\big) \le C_1\left[1 + x^{n-\vartheta}\right] \quad \text{for all } x > x_0, \tag{2.38}$$

where g^{-1} is the inverse of the strictly increasing function g that is defined by

$$g(y) = e^{cy} G(y), \quad \text{for } y \in]y_0, y_\infty[. \tag{2.39}$$

Example 2.2.1 Suppose that h is a Cobb-Douglas function given by (2.19) in Example 2.1.1. In this case, we can check that

$$G(y) = \left[\frac{rK(\alpha - m)}{-m} \frac{y^{1-\beta}}{\beta - \alpha c y} \right]^{1/\alpha}, \quad \text{for } y \in \,]y_0, y_\infty[\,\equiv\,]0, \beta/c\alpha[. \tag{2.40}$$

Figures 2.5.2 and 2.5.3 illustrate this example. □

To complete the construction of the solution w to the HJB equation (2.22) that identifies with the problem's value function v, we note that there exists a mapping $z : \mathcal{I} \to \mathbb{R}_+$ such that

$$z(x, y) \in \,](y_0 - y)^+, y_\infty - y[\quad \text{and} \quad x e^{-cz(x,y)} = G\big(y + z(x, y)\big) \quad \text{for all } (x, y) \in \mathcal{I}. \tag{2.41}$$

Indeed, this claim follows immediately from the calculations

$$\lim_{z \uparrow y_\infty - y} \Big[x e^{-cz} - G(y + z) \Big] = -\infty,$$

$$\frac{\partial}{\partial z} \Big[x e^{-cz} - G(y + z) \Big] = -c x e^{-cz} - G'(y + z) < 0, \quad \text{for } z \in \,](y_0 - y)^+, y_\infty - y[,$$

$$\lim_{z \downarrow (y_0 - y)^+} \Big[x e^{-cz} - G(y + z) \Big] = \left\{ \begin{matrix} x e^{-c(y_0 - y)} - \lim_{u \downarrow y_0} G(u), & \text{if } y \le y_0, \\ x - G(y), & \text{if } y > y_0 \end{matrix} \right\} > 0,$$

in which, we have used (2.36) and the fact that G is increasing. We prove the following result in Appendix I.

Lemma 2.2.2 Suppose that Assumption 2.1.1 holds true. The function w defined by

$$w(x, y) = \begin{cases} R(x, y), & \text{if } (x, y) \in \mathcal{W} \cap \big(\mathbb{R}_+ \times [y_\infty, \bar{y}]\big), \\ A(y) x^n + R(x, y), & \text{if } (x, y) \in \mathcal{W} \cap \big(\mathbb{R}_+ \times [y_0, y_\infty[\big), \\ w\big(x e^{-cz(x,y)}, y + z(x, y)\big) - K z(x, y), & \text{if } (x, y) \in \mathcal{I}, \end{cases} \tag{2.42}$$

where A is defined by (2.35) and z is given by (2.41), is a $C^{2,1}$ solution to the HJB equation (2.22). Furthermore, the function $w(\cdot, y)$ is increasing and there exists a constant $C_2 > 0$ such that

$$-C_2(1 + y) \le w(x, y) \quad \text{for all } (x, y) \in \mathcal{S}, \tag{2.43}$$

$$w\big(G(y), y\big) \le C_2[\Psi(y) + y]\big[1 + G^{n-\vartheta}(y)\big] \quad \text{for all } y \in \,]y_0, y_\infty[, \tag{2.44}$$

where the decreasing function Ψ is as in (2.17)–(2.18).

We can now establish the main result of the paper.

Theorem 2.2.1 Suppose that Assumption 2.1.1 holds true. The value function v of the control problem formulated in Section 2.1 identifies with the solution w to the HJB equation (2.22) given by (2.42) in Lemma 2.2.2 and the optimal capacity expansion strategy ζ^{o} is given by

$$\zeta_t^{\mathrm{o}} = \begin{cases} 0, & \text{if } y > y_0 \text{ and } e^{cy} \sup_{0\le s\le t} X_s^0 \le \overline{g}(y), \\ g^{-1}\left(e^{cy} \sup_{0\le s\le t} X_s^0\right), & \text{if } y < y_\infty \text{ and } e^{cy} \sup_{0\le s\le t} X_s^0 > \overline{g}(y), \end{cases} \quad \text{for } t > 0, \tag{2.45}$$

where

$$\overline{g}(y) = \begin{cases} 0, & \text{if } y_0 > 0 \text{ and } y \le y_0, \\ g(y), & \text{if } y \in \,]y_0, y_\infty[, \\ \infty, & \text{if } y \in [y_\infty, \bar{y}] \cap \mathbb{R}_+, \end{cases} \tag{2.46}$$

g is defined by (2.39), and X^0 is the geometric Brownian motion given by (2.5).

Proof. Fix any initial condition $(x, y) \in \mathcal{S}$ and any admissible strategy $\zeta \in \mathcal{A}$. In view of Itô-Tanaka-Meyer's formula and the left-continuity of the processes X, Y, we can see that

$$\begin{aligned} e^{-rT} w(X_{T+}, Y_{T+}) = w(x, y) &+ \int_0^T e^{-rt} \big[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t)\big]\, dt \\ &+ \int_{[0,T]} \big[w_y(X_t, Y_t) - cX_t w_x(X_t, Y_t)\big]\, d\zeta_t^{\mathrm{c}} + M_T \\ &+ \sum_{0\le t\le T} e^{-rt} \big[w(X_{t+}, Y_{t+}) - w(X_t, Y_t)\big], \end{aligned}$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t. \tag{2.47}$$

Combining this calculation with the observation that

$$\begin{aligned} w(X_{t+}, Y_{t+}) - w(X_t, Y_t) &\overset{(2.4)}{=} \int_0^{\Delta\zeta_t} \frac{dw\big(X_t e^{-cs}, Y_t + s\big)}{ds}\, ds, \\ &= \int_0^{\Delta\zeta_t} \big[w_y\big(X_t e^{-cs}, Y_t + s\big) - cX_t e^{-cs} w_x\big(X_t e^{-cs}, Y_t + s\big)\big]\, ds, \end{aligned}$$

we obtain

$$\int_0^T e^{-rt}h(X_t, Y_t)\,dt - K\int_{[0,T]} e^{-rt}\,d\zeta_t + e^{-rT}w(X_{T+}, Y_{T+})$$
$$= w(x,y) + \int_0^T e^{-rt}\big[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t) + h(X_t, Y_t)\big]\,dt$$
$$+ \int_{[0,T]} \big[w_y(X_t, Y_t) - cX_t w_x(X_t, Y_t) - K\big]\,d\zeta_t^c + M_T$$
$$+ \sum_{0\le t\le T} e^{-rt}\int_0^{\Delta\zeta_t}\big[w_y\big(X_t e^{-cs}, Y_t + s\big) - cX_t e^{-cs} w_x\big(X_t e^{-cs}, Y_t + s\big) - K\big]\,ds. \tag{2.48}$$

Since w satisfies the HJB equation (2.22), it follows that

$$\int_0^T e^{-rt}h(X_t, Y_t)\,dt - K\int_{[0,T]} e^{-rt}\,d\zeta_t + e^{-rT}w(X_{T+}, Y_{T+}) \le w(x,y) + M_T. \tag{2.49}$$

In view of the integration by parts formula and (2.1), we can see that

$$e^{-rT}Y_{T_+} - y = -r\int_0^T e^{-rt}Y_t\,dt + \int_{[0,T]} e^{-rt}\,d\zeta_t. \tag{2.50}$$

This identity, the admissibility condition (2.7) in Definition 2.1.1 and the monotone convergence theorem imply that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-rt}Y_t\,dt\right] &= \lim_{T\to\infty}\mathbb{E}\left[\int_0^T e^{-rt}Y_t\,dt\right]\\ &\le \lim_{T\to\infty}\left(\frac{y}{r} + \frac{1}{r}\mathbb{E}\left[\int_{[0,T]} e^{-rt}\,d\zeta_t\right]\right)\\ &= \frac{y}{r} + \frac{1}{r}\mathbb{E}\left[\int_{[0,\infty[} e^{-rt}\,d\zeta_t\right]\\ &< \infty,\end{aligned} \tag{2.51}$$

which implies that

$$\liminf_{T\to\infty}\mathbb{E}\left[e^{-rT}Y_{T+}\right] = 0. \tag{2.52}$$

The lower bound in (2.17), the estimate (2.43) and (2.50) imply that

$$\begin{aligned}\int_0^T e^{-rt} h(X_t, Y_t)\, dt - K \int_{[0,T]} e^{-rt}\, d\zeta_t + e^{-rT} w(X_{T+}, Y_{T+}) \\ \geq -C_0 \int_0^T e^{-rt}(1+Y_t)\, dt - K \int_{[0,T]} e^{-rt}\, d\zeta_t - C_2 e^{-rT}(1+Y_{T+}) \\ \geq -C_0 \int_0^T e^{-rt}(1+Y_t)\, dt - (K+C_2) \int_{[0,T]} e^{-rt}\, d\zeta_t - C_2(1+y) \\ \geq -\left(\frac{C_0}{r} + C_2 + C_2 y\right) - C_0 \int_0^\infty e^{-rt} Y_t\, dt - (K+C_2) \int_{[0,\infty[} e^{-rt}\, d\zeta_t.\end{aligned}$$

The admissibility condition (2.7) and (2.51) imply that the random variable on the right-hand side of these inequalities has finite expectation. Combining this observation with (2.49), we can see that $\mathbb{E}\left[\inf_{T\geq 0} M_T\right] > -\infty$. Therefore, the stochastic integral M is a supermartingale and $\mathbb{E}\left[M_T\right] \leq 0$ for all $T > 0$. Furthermore, Fatou's lemma implies that

$$J_{x,y}(\zeta) \leq \liminf_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-rt} h(X_t, Y_t)\, dt - K \int_{[0,T]} e^{-rt}\, d\zeta_t\right].$$

Taking expectations in (2.49) and passing to the limit, we obtain

$$J_{x,y}(\zeta) \leq w(x,y) + \liminf_{T\to\infty} e^{-rT} \mathbb{E}\left[-w(X_{T+}, Y_{T+})\right].$$

The inequality $J_{x,y}(\zeta) \leq w(x,y)$ now follows because the estimate (2.43) implies that

$$\liminf_{T\to\infty} e^{-rT} E\big[-w(X_{T+}, Y_{T+})\big] \leq \lim_{T\to\infty} C_2 e^{-rT} + C_2 \liminf_{T\to\infty} e^{-rT} E\left[Y_{T+}\right] \overset{(2.52)}{=} 0.$$

Thus, we have proved that $v(x,y) \leq w(x,y)$.

To prove the reverse inequality and establish the optimality of the process ζ° given by (2.45), we first consider the possibility that $[y_\infty, \bar{y}] \cap \mathbb{R}_+ \neq \emptyset$ and $y \in [y_\infty, \bar{y}]$. In this case, $\zeta_t^{\circ} = 0$ for all $t \geq 0$, and

$$J_{x,y}(\zeta^{\circ}) = \mathbb{E}\left[\int_0^\infty e^{-rt} h(X_t^0, y)\, dt\right] \overset{(2.27),(2.82)}{=} R(x,y) \overset{(2.42)}{=} w(x,y),$$

which establish the required claims.

In the rest of the proof, we assume that $y < y_\infty$. In this case,

$$Y_t^{\circ} = \begin{cases} y, & \text{if } y \in \,]y_0, y_\infty[\text{ and } e^{cy} \sup_{0\leq s\leq t} X_s^0 \leq \overline{g}(y), \\ g^{-1}\left(e^{cy} \sup_{0\leq s\leq t} X_s^0\right), & \text{if } e^{cy} \sup_{0\leq s\leq t} X_s^0 > \overline{g}(y), \end{cases} \tag{2.53}$$

for all $t > 0$, and, apart from a possible initial jump of size $(g^{-1}(e^{cy}x) - y)^+$ at time 0, the process $(e^{cy}X^0, Y^{\circ})$ is reflecting in the free-boundary g in the positive direction. In particular,

$$Y_t^{\circ} \in [y_0, y_\infty[, \quad e^{cy}X_t^0 \le g(Y_t^{\circ}) \quad \text{and} \quad \zeta_t^{\circ} - \zeta_0^{\circ} = \int_{]0,t[} \mathbf{1}_{\{e^{cy}X_s^0 = g(Y_s^{\circ})\}} \, d\zeta_s^{\circ} \quad \text{for all } t > 0.$$

In view of (2.4) and the definition (2.39) of g, we can see that

$$e^{cy}X_t^0 \le g(Y_t^{\circ}) \Leftrightarrow X_t^{\circ} \le G(Y_t^{\circ}) \quad \text{and} \quad \{e^{cy}X_t^0 = g(Y_t^{\circ})\} = \{X_t^{\circ} = G(Y_t^{\circ})\},$$

where X° is the solution of (2.2) given by (2.4). It follows that the process (X°, Y°) satisfies

$$Y_t^{\circ} \in [y_0, y_\infty[, \quad X_t^{\circ} \le G(Y_t^{\circ}) \quad \text{and} \quad \zeta_t^{\circ} - \zeta_0^{\circ} = \int_{]0,t[} \mathbf{1}_{\{X_s^{\circ} = G(Y_s^{\circ})\}} \, d\zeta_s^{\circ} \quad \text{for all } t > 0. \tag{2.54}$$

Since the function g is strictly increasing, $\zeta_0^{\circ} > 0$ if and only if $xe^{cy} > g(y) \overset{(2.39)}{=} e^{cy}G(y)$. Therefore,

$$\zeta_0^{\circ} = \left(g^{-1}(e^{cy}x) - y\right)^+ > 0 \text{ if and only if } (x, y) \in \mathcal{I}. \tag{2.55}$$

Furthermore, given any $(x, y) \in \mathcal{I}$, we note that

$$z = g^{-1}(xe^{cy}) - y \Leftrightarrow xe^{cy} = e^{c(y+z)}G(y + z) \Leftrightarrow xe^{-cz} = G(y + z),$$

which implies that $\zeta_0^{\circ} = z(x, y)$, where the function z is given by (2.41). It follows that

$$w(X_{0+}^{\circ}, Y_{0+}^{\circ}) - w(x, y) = w\big(xe^{-cz(x,y)}, y + z(x, y)\big) - w(x, y) \overset{(2.42)}{=} Kz(x, y). \tag{2.56}$$

In light of (2.54)–(2.56) and the construction of the solution w of the HJB equation (2.22), we can see that (2.48) implies that

$$\int_0^T e^{-rt}h\big(X_t^{\circ}, Y_t^{\circ}\big)\, dt - K \int_{[0,T]} e^{-rt}\, d\zeta_t^{\circ} + e^{-rT}w\big(X_T^{\circ}, Y_T^{\circ}\big) = w(x, y) + M_T^{\circ} \tag{2.57}$$

for all $T > 0$, where the local martingale M° is defined as in (3.45).

To show that ζ° is indeed admissible, we use (2.38) and (2.53) to calculate

$$Y_t^{\circ} = y\mathbf{1}_{\{Y_t^{\circ} = y\}} + g^{-1}\left(e^{cy} \sup_{0 \le s \le t} X_s^0\right) \mathbf{1}_{\{Y_t^{\circ} > y\}} \le y + C_1 + C_1 e^{c(n-\vartheta)y} \left(\sup_{0 \le s \le t} X_s^0\right)^{n-\vartheta}.$$

Combining these inequalities with the first estimate in (2.77), we can see that

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-rT}Y_T^{\circ}\right] = 0 \quad \text{and} \quad \mathbb{E}\left[\int_0^\infty e^{-rt}Y_t^{\circ}\,dt\right] < \infty.$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\int_{[0,\infty[} e^{-rt}\,d\zeta_t^{\circ}\right] &= \lim_{T\to\infty} \mathbb{E}\left[\int_{[0,T]} e^{-rt}\,d\zeta_t^{\circ}\right] \\ &\overset{(2.50)}{=} \lim_{T\to\infty}\left(\mathbb{E}\left[e^{-rT}Y_T^{\circ}\right] + r\mathbb{E}\left[\int_0^T e^{-rt}Y_t^{\circ}\,dt\right] - y\right) \\ &< \infty, \end{aligned} \tag{2.58}$$

which proves that $\zeta^{\circ} \in \mathcal{A}$.

To proceed further, we note that the inequality in (2.54), the fact that $w(\cdot, y)$ is increasing and the bound given by (2.44) imply that, given any $t > 0$,

$$\begin{aligned} w(X_t^{\circ}, Y_t^{\circ}) &\leq w\big(G(Y_t^{\circ}), Y_t^{\circ}\big) \\ &\leq C_2\big[\Psi(Y_t^{\circ}) + Y_t^{\circ}\big]\big[1 + G^{n-\vartheta}(Y_t^{\circ})\big] \leq C_2\big[\Psi(Y_{0+}) + Y_t^{\circ}\big]\big[1 + G^{n-\vartheta}(Y_t^{\circ})\big], \end{aligned}$$

the last inequality following because Ψ is decreasing. Also, (2.17) and (2.54) imply that

$$h(X_t^{\circ}, Y_t^{\circ}) \leq C_0(1 + Y_t^{\circ})(1 + X_t^{\circ n-\vartheta}) \leq C_0(1 + Y_t^{\circ})\big[1 + G^{n-\vartheta}(Y_t^{\circ})\big].$$

The estimate (2.38) and (2.53) imply that

$$\begin{aligned} (1 + Y_t^{\circ})G^{n-\vartheta}(Y_t^{\circ}) &= (1 + y)G^{n-\vartheta}(y)\mathbf{1}_{\{Y_t^{\circ}=y\}} \\ &\quad + \left[1 + g^{-1}\left(e^{cy}\sup_{0\leq s\leq t} X_s^0\right)\right] G^{n-\vartheta}\left(g^{-1}\left(e^{cy}\sup_{0\leq s\leq t} X_s^0\right)\right)\mathbf{1}_{\{Y_t^{\circ}>y\}} \\ &\leq (1 + y)G^{n-\vartheta}(y)\mathbf{1}_{\{y>y_0\}} + C_1 + C_1 e^{c(n-\vartheta)y}\left(\sup_{0\leq s\leq t} X_s^0\right)^{n-\vartheta}. \end{aligned}$$

It follows that there exists a constant $C_3 = C_3(y)$ such that

$$w(X_t^{\circ}, Y_t^{\circ}) \leq C_3\left[1 + \left(\sup_{0\leq s\leq t} X_s^0\right)^{n-\vartheta}\right] \quad \text{and} \quad h(X_t^{\circ}, Y_t^{\circ}) \leq C_3\left[1 + \left(\sup_{0\leq s\leq t} X_s^0\right)^{n-\vartheta}\right]$$

for all $t > 0$. These inequalities and the estimates (2.77) imply that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T>0}\left(\int_0^T e^{-rt}h\big(X_t^{\mathrm{o}},Y_t^{\mathrm{o}}\big)\,dt + e^{-rT}w\big(X_T^{\mathrm{o}},Y_T^{\mathrm{o}}\big)\right)\right] \\
&\le C_3\left(\frac{(1+r)}{r} + \int_0^\infty \mathbb{E}\left[e^{-rt}\left(\sup_{0\le s\le t} X_s^0\right)^{n-\vartheta}\right]dt + \mathbb{E}\left[\sup_{T>0} e^{-rT}\left(\sup_{0\le s\le T} X_s^0\right)^{n-\vartheta}\right]\right) \\
&< \infty,
\end{aligned} \tag{2.59}$$

and

$$\liminf_{T\to\infty} e^{-rT}\mathbb{E}\big[-w(X_T^{\mathrm{o}},Y_T^{\mathrm{o}})\big] \ge -C_3 \lim_{T\to\infty} e^{-rT}\left(1+\mathbb{E}\left[\left(\sup_{0\le s\le T} X_s^0\right)^{n-\vartheta}\right]\right) = 0. \tag{2.60}$$

In view of (2.57) and (2.59), we can see that $\mathbb{E}\left[\sup_{T>0} M_T^{\mathrm{o}}\right] < \infty$. Therefore, the stochastic integral M^{o} is a submartingale and $\mathbb{E}\left[M_T^{\mathrm{o}}\right] \ge 0$ for all $T > 0$. Furthermore, Fatou's lemma implies that

$$J_{x,y}(\zeta^{\mathrm{o}}) \ge \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-rt}h(X_t^{\mathrm{o}},Y_t^{\mathrm{o}})\,dt - K\int_{[0,T]} e^{-rt}\,d\zeta_t^{\mathrm{o}}\right].$$

In view of these observations and (2.60), we can take expectations in (2.57) and pass to the limit to obtain

$$J_{x,y}(\zeta^{\mathrm{o}}) \ge w(x,y) + \limsup_{T\to\infty} e^{-rT}\mathbb{E}\left[-w(X_T^{\mathrm{o}},Y_T^{\mathrm{o}})\right] \ge w(x,y).$$

This result and the inequality $v(x,y) \le w(x,y)$ that we have proved above, imply that $v(x,y) = w(x,y)$ and that ζ^{o} is optimal. □

2.3 Appendix I: proof of Lemmas 2.2.1 and 2.2.2

Proof of Lemma 2.2.1. Given any $y \in \,]y_0, y_\infty[$, we observe that

$$\frac{\partial}{\partial x}q(x,y) = x^{-m-1}H(x,y)\begin{cases} < 0, & \text{for all } x \in \,]0, x^\dagger(y)[, \\ = 0, & \text{for all } x = x^\dagger(y), \\ > 0, & \text{for all } x > x^\dagger(y), \end{cases}$$

where $x^\dagger$ is defined by (2.16) in Assumption 2.1.1. Also, we note that (2.13) and (2.14) in Assumption 2.1.1 imply that there exist constants $\varepsilon_1 = \varepsilon_1(y)$ and $x_1 = x_1(y) > x^\dagger(y)$

such that $H(x,y) \geq \varepsilon_1$ for all $x \geq x_1$. Given such a choice of constants, we calculate

$$\begin{aligned}\lim_{x\to\infty} q(x,y) &= \lim_{x\to\infty}\left[q(x_1,y) + \int_{x_1}^{x} s^{-m-1}H(s,y)\,ds\right]\\ &\geq \lim_{x\to\infty}\left[q(x_1,y) + \frac{\varepsilon_1}{m}x_1^{-m} - \frac{\varepsilon_1}{m}x^{-m}\right]\\ &= \infty,\end{aligned}$$

because $m < 0$. Combining these observations with the fact that $q(0,y) = 0$, we can see that the equation $q(x,y) = 0$ for $x > 0$ has a unique solution $G(y) > x^{\dagger}(y)$ for all $y \in \,]y_0, y_\infty[$, and that G satisfies (2.36).

To see that the function $G : \,]y_0, y_\infty[\, \to \,]0,\infty[$ is C^1 and strictly increasing, we differentiate the identity $q\big(G(y),y\big) = 0$ with respect to y to obtain

$$\dot{G}(y) = -G^{m+1}(y)H^{-1}\big(G(y),y\big)\int_0^{G(y)} s^{-m-1}H_y(s,y)\,ds > 0, \tag{2.61}$$

the inequality following from (2.15) in Assumption 2.1.1.

To establish (2.38), we note that

$$\lim_{y\downarrow y_0} G^{n-\vartheta}(y) = e^{-c(n-\vartheta)y_0}\lim_{y\downarrow y_0} g^{n-\vartheta}(y) \leq \lim_{y\downarrow y_0} g^{n-\vartheta}(y)$$

and

$$0 \leq \lim_{y\uparrow y_\infty}(1+y)g^{-n+\vartheta}(y) \leq \lim_{y\uparrow y_\infty}(1+y)G^{n-\vartheta}(y)g^{-n+\vartheta}(y) = \lim_{y\uparrow y_\infty}(1+y)e^{-c(n-\vartheta)y} < \infty,$$

where we have used (2.36) and the facts that G is increasing and $n-\vartheta > 0$. Combining these inequalities with the fact that G and g are continuous increasing functions with the same domain $]y_0, y_\infty[$, we can see that there exists a constant $C_1 > 0$ such that

$$1 + y + (1+y)G^{n-\vartheta}(y) \leq C_1\big[1 + g^{n-\vartheta}(y)\big] \quad \text{for all } y \in \,]y_0, y_\infty[.$$

For $x > x_0$ and $y = g^{-1}(x)$, this inequality implies the estimate in (2.38).

In view of (2.18) and the fact that the functions G, $-\Psi$ are increasing, we can see that, given any $y \in \,]y_0, y_\infty[$,

$$\begin{aligned}A(y)G^n(y) &\leq \frac{e^{cny}}{\sigma^2(n-m)}G^n(y)\int_y^{y_\infty} e^{-cnu}\Psi(u)\left[\frac{1}{n}G^{-n}(u) + \frac{1}{\vartheta}G^{-\vartheta}(u)\right]du\\ &\leq \frac{e^{cny}}{\sigma^2(n-m)}\left[\frac{1}{n}\int_y^{y_\infty} e^{-cnu}\Psi(u)\,du + \frac{1}{\vartheta}G^{n-\vartheta}(y)\int_y^{y_\infty} e^{-cnu}\Psi(u)\,du\right]\\ &\leq \frac{1}{\sigma^2(n-m)}\Psi(y)\left[\frac{1}{cn^2} + \frac{1}{cn\vartheta}G^{n-\vartheta}(y)\right],\end{aligned}$$

which implies (2.37). Finally, the strict positivity of A follows from (2.13) and the inequality in (2.36). □

Proof of Lemma 2.2.2. In view of its construction, we will prove that w is $C^{2,1}$ if we show that w_y, w_x and w_{xx} are continuous along the free-boundary G. To this end, we consider any $(x,y) \in \mathcal{I}$, we recall the definition (2.42) of w and the definition (2.41) of z, and we use (2.28)–(2.29) to calculate

$$\begin{aligned}
w_y(x,y) &= \frac{\partial}{\partial y}\Big[w\big(xe^{-cz(x,y)}, y+z(x,y)\big) - Kz(x,y)\Big] \\
&= w_y\big(xe^{-cz(x,y)}, y+z(x,y)\big) \\
&\quad + \Big[w_y\big(xe^{-cz(x,y)}, y+z(x,y)\big) - cxe^{-cz(x,y)}w_x\big(xe^{-cz(x,y)}, y+z(x,y)\big) - K\Big] z_y(x,y) \\
&= w_y\big(xe^{-cz(x,y)}, y+z(x,y)\big), \qquad (2.62) \\
w_x(x,y) &= \frac{\partial}{\partial x}\Big[w\big(xe^{-cz(x,y)}, y+z(x,y)\big) - Kz(x,y)\Big] \\
&= w_x\big(xe^{-cz(x,y)}, y+z(x,y)\big)e^{-cz(x,y)} \\
&\quad + \Big[w_y\big(xe^{-cz(x,y)}, y+z(x,y)\big) - cxe^{-cz(x,y)}w_x\big(xe^{-cz(x,y)}, y+z(x,y)\big) - K\Big] z_x(x,y) \\
&= w_x\big(xe^{-cz(x,y)}, y+z(x,y)\big)e^{-cz(x,y)} \qquad (2.63)
\end{aligned}$$

and

$$\begin{aligned}
w_{xx}(x,y) &= \frac{\partial}{\partial x}\Big[w_x\big(xe^{-cz(x,y)}, y+z(x,y)\big)e^{-cz(x,y)}\Big] \\
&= w_{xx}\big(xe^{-cz(x,y)}, y+z(x,y)\big)e^{-2cz(x,y)} \\
&\quad + \Big[w_{xy}\big(xe^{-cz(x,y)}, y+z(x,y)\big) - cxe^{-cz(x,y)}w_{xx}\big(xe^{-cz(x,y)}, y+z(x,y)\big) \\
&\qquad - cw_x\big(xe^{-cz(x,y)}, y+z(x,y)\big)\Big] e^{-cz(x,y)} z_x(x,y) \\
&= w_{xx}\big(xe^{-cz(x,y)}, y+z(x,y)\big)e^{-2cz(x,y)} \qquad (2.64)
\end{aligned}$$

These calculations imply the required continuity results because $\lim_{n\to\infty} z(x_n, y_n) = 0$ for every convergent sequence (x_n, y_n) in $\mathcal{I}$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} G(y_n)$.

To prove (2.43)–(2.44), we note that the bounds of h in (2.17), the definition (2.27) of R and the identity $\sigma^2 mn = -r$ imply that

$$-\frac{C_0}{r}(1+y) \le R(x,y) \le C_0(1+y)\left[\frac{1}{r} + \frac{1}{\sigma^2(n-m-\vartheta)\vartheta}x^{n-\vartheta}\right]. \qquad (2.65)$$

The lower of these bounds and the positivity of A (see (2.37)) imply that

$$-\frac{C_0}{r}(1+y) \leq A(y)x^n + R(x,y) = w(x,y) \quad \text{for all } (x,y) \in \mathcal{W}. \tag{2.66}$$

In light of (2.9) and (2.83) in Appendix II, we can see that $R(\cdot, y)$ is increasing for all $y \in [0, \bar{y}] \cap \mathbb{R}$. Combining this observation with the inequalities $A > 0$ and $n > 0$, we deduce that $w_x(x,y) \geq 0$ for all $(x,y) \in \mathcal{W}$. This result, (2.41) and (2.63) imply that $w(\cdot, y)$ is increasing for all $y \in [0, \bar{y}] \cap \mathbb{R}$, which, combined with (2.66), implies (2.43). Also, (2.44) follows immediately from (2.37) and the upper bound in (2.65).

It remains to show that w satisfies the HJB equation (2.22). By the construction and the $C^{2,1}$ continuity of w, we will achieve this if we show that

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \leq 0 \quad \text{for all } (x,y) \in \mathcal{I}, \tag{2.67}$$

$$w_y(x,y) - cxw_x(x,y) - K \leq 0 \quad \text{for all } (x,y) \in \mathcal{W} \cap \big(\mathbb{R}_+ \times]y_0, \bar{y}[\big). \tag{2.68}$$

To see (2.67), we consider any $(x,y) \in \mathcal{I}$ and we use (2.42), (2.63)–(2.64) and the fact that w satisfies the ODE (2.25) inside $\mathcal{W}$ to calculate

$$\begin{aligned}
&\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \\
&\quad = \sigma^2 \big[xe^{-cz(x,y)}\big]^2 w_{xx}\big(xe^{-cz(x,y)}, y + z(x,y)\big) + b\big[xe^{-cz(x,y)}\big] w_x\big(xe^{-cz(x,y)}, y + z(x,y)\big) \\
&\qquad - rw\big(xe^{-cz(x,y)}, y + z(x,y)\big) + rKz(x,y) + h(x,y) \\
&\quad = -h\big(xe^{-cz(x,y)}, y + z(x,y)\big) + h(x,y) + rKz(x,y) \\
&\quad = -\int_0^{z(x,y)} \left[\frac{\partial h\big(xe^{-cu}, y+u\big)}{\partial u} - rK\right] du \\
&\quad \overset{(2.11)}{=} -\int_0^{z(x,y)} H\big(xe^{-cu}, y+u\big)\, du.
\end{aligned}$$

These calculations, (2.13), (2.36), (2.41) and the continuity of z imply (2.67).

To prove (2.68), we first consider the possibility that $y_\infty < \bar{y}$. In this case, we use the fact that $w = R$ inside $\mathcal{W} \cap \big(\mathbb{R}_+ \times [y_\infty, \bar{y}]\big)$, the definition (2.27) of R, the associated expression (2.85) for the function $x \mapsto xR_x(x,y)$ and (2.84) to calculate

$$\begin{aligned}
w_y(x,y) - cxw_x(x,y) - K &= R_y(x,y) - cxR_x(x,y) - K \\
&= \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m-1} H(s,y)\, ds + x^n \int_x^\infty s^{-n-1} H(s,y)\, ds\right] \\
&\leq 0 \qquad \text{for all } (x,y) \in \mathcal{W} \cap \big(\mathbb{R}_+ \times [y_\infty, \bar{y}[\big),
\end{aligned} \tag{2.69}$$

the inequality following thanks to (2.13) in Assumption 2.1.1.

To proceed further, we note that, inside $\mathcal{W} \cap \big(\mathbb{R}_+ \times \,]y_0, y_\infty[\big)$, the definition (2.42) of w, (2.30), (2.32), calculations similar to the ones in (2.69) and the definition (2.11) of H imply that

$$\begin{aligned} \varrho(x,y) &:= w_y(x,y) - cxw_x(x,y) - K \\ &= \frac{1}{\sigma^2(n-m)} \left[-x^m \int_x^{G(y)} s^{-m-1} H(s,y)\, ds + x^n \int_x^{G(y)} s^{-n-1} H(s,y)\, ds \right]. \end{aligned} \tag{2.70}$$

In light of (2.13), (2.36) and the fact that $m < 0 < n$, we can see that

$$\begin{aligned} \varrho_x(x,y) &= \frac{1}{\sigma^2(n-m)} \left[-mx^{m-1} \int_x^{G(y)} s^{-m-1} H(s,y)\, ds + nx^{n-1} \int_x^{G(y)} s^{-n-1} H(s,y)\, ds \right] \\ &\geq 0 \qquad \text{for all } x \in [x^\dagger(y), G(y)], \end{aligned}$$

which, combined with the identity $\varrho\big(G(y), y\big) = 0$, implies that

$$\varrho(x,y) \leq 0 \quad \text{for all } x \in [x^\dagger(y), G(y)]. \tag{2.71}$$

Also, we can use the inequality

$$\int_x^{G(y)} s^{-m-1} H(s,y)\, ds > 0 \quad \text{for all } x \in \,]0, G(y)[,$$

which follows from (2.13) in Assumption 2.1.1 and (2.32), to calculate

$$\begin{aligned} \lim_{x \downarrow 0} \varrho(x,y) &\leq \frac{1}{\sigma^2(n-m)} \lim_{x \downarrow 0} x^n \int_x^{G(y)} s^{-n-1} H(s,y)\, ds \\ &= \frac{1}{\sigma^2(n-m)} \lim_{x \downarrow 0} x^n \int_x^{x^\dagger(y)} s^{-n-1} H(s,y)\, ds \\ &\leq 0, \end{aligned} \tag{2.72}$$

the inequality following from (2.13) and the fact that $n > 0$.

Finally, we can use the fact that m, n are the solutions of the quadratic equation (2.74) and straightforward calculations to obtain

$$\sigma^2 x^2 \varrho_{xx}(x,y) + bx\varrho_x(x,y) - r\varrho(x,y) = -H(x,y) > 0 \quad \text{for all } x \in \,]0, x^\dagger(y)[.$$

This inequality and the maximum principle imply that the function ϱ has no positive maximum inside $]0, x^\dagger(y)[$, which, combined with (2.71)–(2.72), implies that $\varrho(x,y) \leq 0$ for all $y \in \,]y_0, y_\infty[$ and $x \in \,]0, G(y)]$, and (2.68) follows. □

2.4 Appendix II: a second order linear ODE

In this section, we review certain results regarding the solvability of a second order linear ODE on which our analysis has been based. All of the claims that we do not prove here are standard and can be found in several references (e.g., with the exception of (2.77), which is proved in Merhi and Zervos (2007, Lemma 1), all results can be found in Knudsen et al. (1998)).

Every solution of the homogeneous ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) = 0 \tag{2.73}$$

is given by

$$u(x) = Ax^n + Bx^m,$$

for some $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\sigma^2\lambda^2 + (b - \sigma^2)\lambda - r = 0, \tag{2.74}$$

given by

$$m, n = \frac{-(b-\sigma^2) \pm \sqrt{(b-\sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{2.75}$$

It follows that, if λ is a constant, then

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(X_t^0\right)^\lambda dt\right] &= x^\lambda \int_0^\infty e^{\left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right]t}\mathbb{E}\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt \\ &= \begin{cases} \infty, & \text{if } \lambda \le m \text{ or } \lambda \ge n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in \,]m, n[, \end{cases} \end{aligned} \tag{2.76}$$

where X^0 is the geometric Brownian motion given by (2.5). Furthermore, for all $\lambda \in \,]0, n[$, there exist constants $\varepsilon, C > 0$ such that

$$e^{-rT}\mathbb{E}\left[\left(\sup_{0\le t\le T} X_t^0\right)^\lambda\right] \le Cx^\lambda e^{-\varepsilon T} \quad \text{and} \quad \mathbb{E}\left[\sup_{T\ge 0} e^{-rT}\left(\sup_{0\le t\le T} X_t^0\right)^\lambda\right] \le Cx^\lambda \tag{2.77}$$

for all $x > 0$.

A Borel measurable function $k : \,]0, \infty[\, \to \mathbb{R}$ satisfies

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\left|k(X_t^0)\right| dt\right] < \infty \quad \text{for all } x > 0, \tag{2.78}$$

if and only if

$$\int_0^x s^{-m-1} |k(s)| \, ds + \int_x^\infty s^{-n-1} |k(s)| \, ds < \infty \quad \text{for all } x > 0. \tag{2.79}$$

In the presence of these equivalent integrability conditions, the function R defined by

$$R(x) = \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m-1} k(s) \, ds + x^n \int_x^\infty s^{-n-1} k(s) \, ds \right], \quad \text{for } x > 0, \tag{2.80}$$

is a special solution to the non-homogeneous ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) + k(x) = 0 \tag{2.81}$$

that admits the probabilistic expression

$$R(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} k(X_t^0) \, dt \right]. \tag{2.82}$$

Furthermore,

$$\text{if } k \text{ is increasing, then } R \text{ is increasing,} \tag{2.83}$$

$$\text{and, if } k \text{ is constant, then } rR(x) = k \text{ for all } x > 0. \tag{2.84}$$

In our analysis we have used the following result.

Lemma 2.4.1 Consider any C^1 function $k : \,]0, \infty[\, \to \mathbb{R}$ satisfying the equivalent integrability conditions (2.78)–(2.79) and suppose that there exists $\varepsilon > 0$ such that

$$\forall x < \varepsilon, \text{ either } k'(x) \geq 0 \text{ or } k'(x) \leq 0 \quad \text{and} \quad \forall x > \varepsilon^{-1}, \text{ either } k'(x) \geq 0 \text{ or } k'(x) \leq 0.$$

Then

$$xR'(x) = \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m} k'(s) \, ds + x^n \int_x^\infty s^{-n} k'(s) \, ds \right], \quad \text{for all } x > 0. \tag{2.85}$$

Proof. We first note that the integrability condition (2.79) implies that the limits

$$\lim_{z \downarrow 0} \int_z^x s^{-m-1} k(s) \, ds \quad \text{and} \quad \lim_{z \to \infty} \int_x^z s^{-n-1} k(s) \, ds$$

exist and that

$$\liminf_{z \downarrow 0} z^{-m} |k(z)| = 0 \quad \text{and} \quad \liminf_{z \to \infty} z^{-n} |k(z)| = 0. \tag{2.86}$$

To see the latter claim, suppose that $\liminf_{z\downarrow 0} z^{-m}|k(z)| > 0$. In such a case, there exist constants $\varepsilon, z_1 > 0$ such that $z^{-m}|k(z)| \geq \varepsilon$ for all $z \leq z_1$. Therefore,

$$\int_0^{z_1} s^{-m-1}|k(s)|\, ds \geq \varepsilon \int_0^{z_1} s^{-1}\, ds = \infty,$$

which contradicts (2.79). We can argue similarly by contradiction to prove the second limit in (2.86).

Using the integration by parts formula, we calculate

$$x^{-m}k(x) - z^{-m}k(z) = -m\int_z^x s^{-m-1}k(s)\, ds + \int_z^x s^{-m}k'(s)\, ds \quad \text{for all } 0 < z < x. \tag{2.87}$$

The assumptions that we have made on k' and the monotone convergence theorem imply that the limit $\lim_{z\downarrow 0}\int_z^x s^{-m}k'(s)\, ds$ exists. Therefore, we can pass to the limit as $z \downarrow 0$ in (2.87) to obtain

$$x^{-m}k(x) = -m\int_0^x s^{-m-1}k(s)\, ds + \int_0^x s^{-m}k'(s)\, ds \quad \text{for all } x > 0.$$

Similarly, we can see that

$$-x^{-n}k(x) = -n\int_x^\infty s^{-n-1}k(s)\, ds + \int_x^\infty s^{-n}k'(s)\, ds \quad \text{for all } x > 0.$$

The required result follows immediately from these calculations and the expression

$$xR'(x) = \frac{1}{\sigma^2(n-m)}\left[mx^m\int_0^x s^{-m-1}k(s)\, ds + nx^n\int_x^\infty s^{-n-1}k(s)\, ds\right].$$

□

2.5 Appendix III: Illustration of the free boundary function

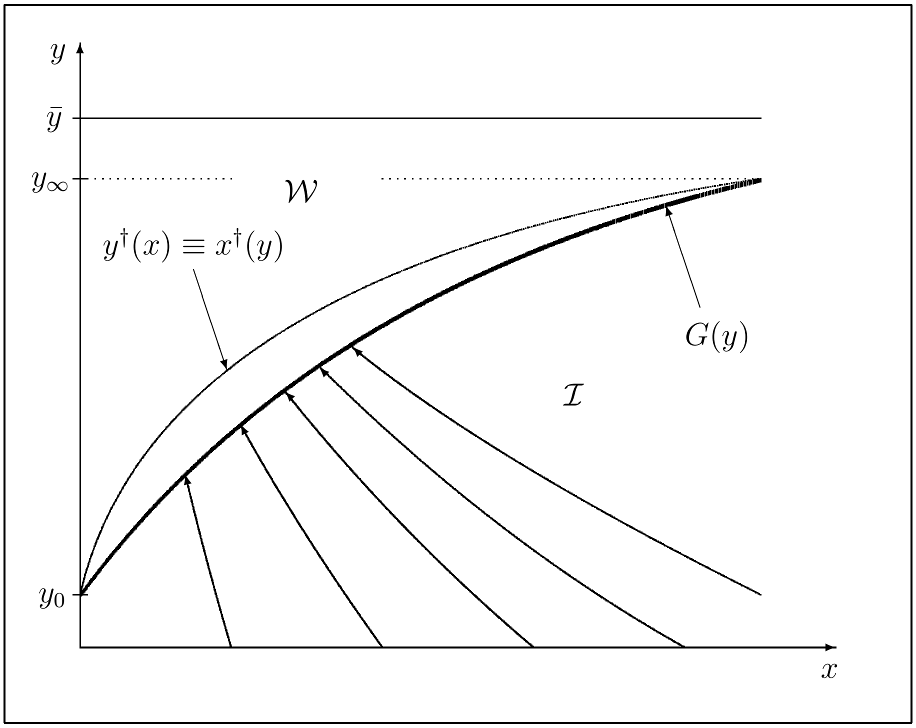

Figure 2.5.1 Graph of the free-boundary function G in the general context. If the initial condition (x, y) is inside the "investment" region $\mathcal{I}$, then it is optimal to invest so that the joint process (X°, Y°) has a jump at time 0 that positions it in the graph of G along the curved arrows. It is optimal to take no action, i.e., wait, as long as the process (X°, Y°) takes values in the interior of the waiting region $\mathcal{W}$. Otherwise, it is optimal to take minimal action so that the process (X°, Y°) does not fall below the graph of G, which amounts to reflecting it in G in the direction indicated by the curved arrows.

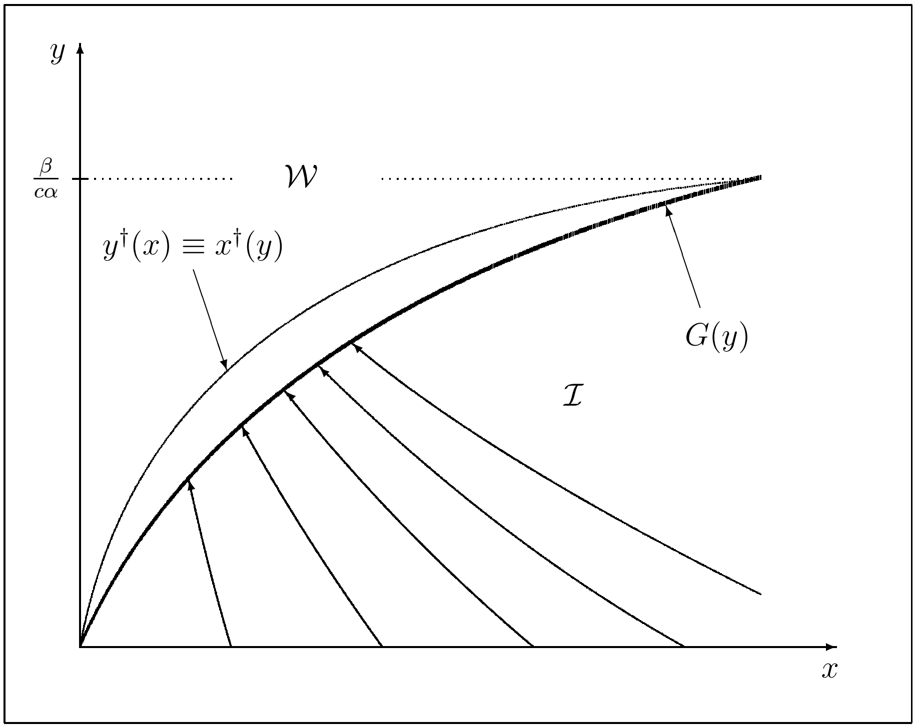

Figure 2.5.2 Graph of G when h is a Cobb-Douglass function with $\beta \in]0,1[$. The qualitative nature of the optimal strategy can be described in the same way as in Figure 2.5.1. In this case, $y_0 = 0$, $\bar{y} = y_\infty = \infty$ and G is given by (2.40) in Example 2.1.1.

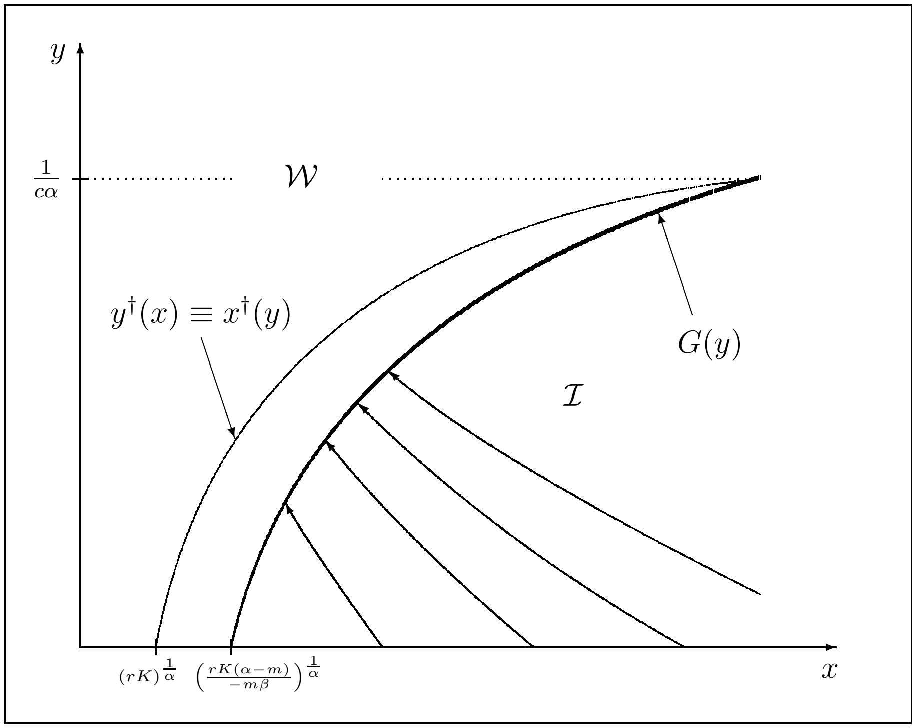

Figure 2.5.3 Graph of G when h is a Cobb-Douglass function with $\beta = 1$. The qualitative nature of the optimal strategy can be described in the same way as in the previous figures. In this case, $y_0 = 0$, $\bar{y} = y_\infty = \frac{1}{c\alpha}$ and G is given by (2.40) in Example 2.1.1. Comparing with Figures 2.5.2, it is worth noting that the free-boundary function does not intersect the y-axis in this example.

Chapter 3

Impulsive irreversible capacity expansion

3.1 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by $\mathbb{P}$-negligible sets, and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{Z}$ the family of all càglàd $(\mathcal{F}_t)$-adapted increasing and piecewise constant processes Z such that $Z_0 = 0$.

We consider an investment project that produces a given commodity, and we assume that the project's capacity, namely its rate of output, can be increased at any given time and by any amount. We denote by Y_t the project's capacity at time t, and we model capacity increases by the jumps of an impulse control process $Z \in \mathcal{Z}$. The capacity process Y is therefore given by

$$Y_t = y + Z_t, \quad Y_0 = y \geq 0, \tag{3.1}$$

where $y \geq 0$ is the project's initial capacity. Every process $Z \in \mathcal{Z}$ is characterised by the collection $(\tau_1, \tau_2, ..., \tau_n, ...; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, ..., \Delta Z_{\tau_n}, ...)$ where τ_n is the $(\mathcal{F}_t)$-stopping time at which the n-th jump of Z occurs, while ΔZ_{τ_n} is the associated jump size. If the project's management adopts the capacity expansion strategy modelled by Z, then the project's capacity is increased at the times τ_n by an amount $\Delta Y_{\tau_n} = \Delta Z_{\tau_n} > 0$, for

$n \geq 1$.

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t\,dt + \sqrt{2}\sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{3.2}$$

for some constants b and σ. In practice, X_t can be an economic indicator reflecting, e.g., the value of one unit of the output commodity or the output commodity's demand or both, at time t.

To simplify the notation, we define

$$\mathcal{S} = \left\{(x, y) \in \mathbb{R}^2 \mid \ x > 0,\ y \geq 0\right\},$$

so that $\mathcal{S}$ is the set of all possible initial conditions. With each decision policy Z we associate the performance criterion

$$J_{x,y}(Z) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(X_t^\alpha Y_t^\beta - K_1 Y_t\right)dt - \sum_{0\leq t} e^{-rt}\big(K_2\Delta Z_t + c\big)\mathbf{1}_{\{\Delta Z_t > 0\}}\right], \tag{3.3}$$

where $r, \alpha > 0$, $\beta \in \,]0, 1[$, $K_1 \geq 0$ and $K_2, c > 0$ are given constants. In particular, K_2 and c provide a proportional and a fixed cost incurred each time that the project's capacity level is changed.

Definition 3.1.1 An investment strategy $Z \in \mathcal{Z}$ is admissible if

$$\mathbb{E}\left[\sum_{0\leq t} e^{-rt}\big(\Delta Z_t + 1\big)\mathbf{1}_{\{\Delta Z_t > 0\}}\right] < \infty. \tag{3.4}$$

We denote by $\mathcal{A}$ the family of all admissible decision policies. □

The objective is to maximise this performance index over all admissible capacity expansion strategies $Z \in \mathcal{A}$. The value function of the resulting optimisation problem is defined by

$$v(x, y) = \sup_{Z\in\mathcal{A}} J_{x,y}(Z). \tag{3.5}$$

For the control problem to be well-posed, we make the following assumption on the problem data.

Assumption 3.1.1 $r, \alpha > 0$, $\beta \in]0,1[$, $K_1 \geq 0$, $K_2, c > 0$ and

$$\frac{\alpha}{1-\beta} \in]0,n[\quad \Leftrightarrow \quad \frac{n\beta}{n-\alpha} \in]0,1[, \tag{3.6}$$

where $n > 0$ is given by (3.11) in the next section. □

3.2 Well-posedness of the control problem

It is well-known that every solution to the Euler ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) + x^\alpha = 0 \tag{3.7}$$

is given by

$$u(x) = Ax^n + Bx^m + \Gamma x^\alpha, \tag{3.8}$$

for some constants $A, B \in \mathbb{R}$, where

$$\Gamma = \frac{1}{\sigma^2(\alpha - m)(n-\alpha)} > 0, \tag{3.9}$$

and the constants $m < 0 < n$ are the solutions to the quadratic equation

$$\sigma^2 k^2 + (b - \sigma^2)k - r = 0, \tag{3.10}$$

given by

$$m, n = \frac{-(b-\sigma^2) \pm \sqrt{(b-\sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{3.11}$$

Also, given any constant $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-rt} X_t^\lambda \, dt\right] &= x^\lambda \int_0^\infty e^{\left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right]t} \mathbb{E}\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt \\ &= \begin{cases} \infty, & \text{if } \lambda \leq m \text{ or } \lambda \geq n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in]m,n[, \end{cases} \end{aligned} \tag{3.12}$$

and, if $\lambda \in [0, n[$, then there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\mathbb{E}\big[e^{-rt}\bar{X}_t^\lambda\big] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda e^{-\varepsilon_1 t} \quad \text{and} \quad \mathbb{E}\left[\sup_{t \geq 0} e^{-rt}\bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda, \tag{3.13}$$

where $\bar{X}_t = \sup_{s \leq t} X_s$ (for the latter claim, see Lemma 1 in Merhi and Zervos (2007)).

The following result is concerned with the well-posedness of the control problem as well as with its reformulation to a simpler one.

Lemma 3.2.1 Consider the stochastic control problem formulated in Section 3.1. The capacity process Y that is associated with any admissible investment strategy $Z \in \mathcal{A}$ is such that

$$\liminf_{T\to\infty} \mathbb{E}\left[e^{-rT} Y_T\right] = 0. \tag{3.14}$$

Given any initial condition $(x, y) \in \mathcal{S}$,

$$0 \le v(x,y) = \tilde{v}(x,y) - \frac{K_1}{r} y < \infty, \tag{3.15}$$

where $\tilde{v}$ is the value function defined by

$$\tilde{v}(x,y) = \sup_{Z\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt - \sum_{0\le t} e^{-rt}\big(K\Delta Z_t + c\big)\mathbf{1}_{\{\Delta Z_t > 0\}}\right],$$

with

$$K = \frac{K_1}{r} + K_2 > 0. \tag{3.16}$$

Proof. Throughout the proof, we fix any initial condition $(x, y) \in \mathcal{S}$. Also, we note that (3.12) and (3.6) imply that

$$\mathbb{E}\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt\right] < \infty. \tag{3.17}$$

Given any admissible investment strategy $Z \in \mathcal{A}$, we can use the integration by parts formula and (3.1) to calculate

$$\begin{aligned} \sum_{0\le t\le T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} &= \int_{[0,T]} e^{-rt} \, dZ_t \\ &= e^{-rT} Z_{T+} + r\int_{[0,T]} e^{-rt} Z_t \, dt \\ &= r\int_{[0,T]} e^{-rt} Y_t \, dt + e^{-rT} Y_{T+} - y. \end{aligned} \tag{3.18}$$

Combining this result with (3.4) and the monotone convergence theorem, we can see that

$$\mathbb{E}\left[\int_0^\infty e^{-rt} Y_t \, dt\right] \le \frac{y}{r} + \frac{1}{r}\mathbb{E}\left[\sum_{0\le t} e^{-rt}\Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}}\right] < \infty, \tag{3.19}$$

which implies (3.14). Furthermore, we can see that (3.18), (3.14) and the monotone convergence theorem imply that

$$\mathbb{E}\left[\sum_{0\le t} e^{-rt}\Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}}\right] = r\mathbb{E}\left[\int_0^\infty e^{-rt} Y_t \, dt\right] - y.$$

It follows that the prerformance index defined by (3.3) admits the expression

$$J_{x,y}(Z) = \mathbb{E}\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt - \sum_{0\le t} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t>0\}}\right] + \frac{K_1}{r} y$$

$$= \mathbb{E}\left[\int_0^\infty e^{-rt} \left(X_t^\alpha Y_t^\beta - rKY_t\right) dt - \sum_{0\le t} e^{-rt} c \mathbf{1}_{\{\Delta Z_t>0\}}\right] + K_2 y, \qquad (3.20)$$

where K is defined by (3.16), and the identity in (3.15) has been established.

Given any constant $Q > 0$, we can verify that

$$Qz^\beta - rKz \le \frac{rK(1-\beta)}{\beta} \left(\frac{\beta}{rK}\right)^{1/(1-\beta)} Q^{1/(1-\beta)} \quad \text{for all } z \ge 0.$$

Combining this inequality with (3.19), (3.20) and Hölder's inequality, we obtain

$$\begin{aligned} J_{x,y}(Z) &\le \mathbb{E}\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt\right] - rK\mathbb{E}\left[\int_0^\infty e^{-rt} Y_t \, dt\right] + K_2 y \\ &\le \left(\mathbb{E}\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt\right]\right)^{1-\beta} \left(\mathbb{E}\left[\int_0^\infty e^{-rt} Y_t \, dt\right]\right)^\beta \\ &\quad - rK\mathbb{E}\left[\int_0^\infty e^{-rt} Y_t \, dt\right] + K_2 y \\ &\le \frac{rK(1-\beta)}{\beta} \left(\frac{\beta}{rK}\right)^{1/(1-\beta)} \mathbb{E}\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt\right] + K_2 y, \end{aligned}$$

which proves that $v(x, y) < \infty$ because the right-hand side of these inequalities is finite and independent of Z. Finally, the positivity of $v(x, y)$ follows immediately from the observation that the strategy $Z \equiv 0$, which involves no capacity changes, has positive payoff. □

We conclude this section by showing that (3.6) in Assumption 3.1.1 is essential for the value function of our optimisation problem to be finite.

Lemma 3.2.2 Consider the stochastic control problem formulated in Section 3.1, and suppose that $n < \frac{\alpha}{1-\beta}$. Then $v(x, y) = \infty$ for every initial condition $(x, y) \in \mathcal{S}$.

Proof. Throughout the proof, we fix any initial condition $(x, y) \in \mathcal{S}$. If $x < 1$, then we we define $i_x = 0$, otherwise, we denote by i_x the unique integer such that

$$2^{i_x - 1} \le x < 2^{i_x}. \qquad (3.21)$$

If $n \leq \alpha$, then we can see that the strategy $Z \equiv 0$, which involves no capacity changes, has payoff

$$J_{x,y}(Z) = y^{\beta}\mathbb{E}\left[\int_0^{\infty} e^{-rt} X_t^{\alpha}\, dt\right] + \frac{K_1}{r} y = \infty,$$

the second identity following from (3.12). We therefore assume that $\alpha < n < \frac{\alpha}{1-\beta}$ in what follows, we define $\lambda = \frac{n-\alpha}{\beta} > 0$ and we note that $\lambda < n$. We consider the capacity expansion strategy given by

$$Z_{t+} = \mathbf{1}_{\{\bar{X}_t < 1\}} + \sum_{j=1}^{\infty} 2^{\lambda j} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}, 2^j[\}}, \quad \text{for } t \geq 0,$$

where $\bar{X}_t = \sup_{s \leq t} X_s$ and $Z_{t+} = \lim_{s \downarrow t} Z_s$. The associated capacity level process satisfies

$$Y_{t+}^{\beta} \mathbf{1}_{\{\bar{X}_t < 1\}} = (y+1)^{\beta} \mathbf{1}_{\{\bar{X}_t < 1\}} \geq X_t^{n-\alpha} \mathbf{1}_{\{\bar{X}_t < 1\}}$$

and

$$\begin{aligned} Y_{t+}^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}, 2^j[\}} &= (y + 2^{\lambda j})^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}, 2^j[\}} \\ &\geq \left(y + \bar{X}_t^{\lambda}\right)^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}, 2^j[\}} \\ &\geq X_t^{n-\alpha} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}, 2^j[\}}. \end{aligned}$$

Combining these inequalities with (3.12), we can see that

$$\mathbb{E}\left[\int_0^{\infty} e^{-rt} X_t^{\alpha} Y_t^{\beta}\, dt\right] \geq \mathbb{E}\left[\int_0^{\infty} e^{-rt} X_t^{n}\, dt\right] = \infty. \tag{3.22}$$

Next, we define the sequence of stopping times

$$\begin{aligned} \tau_j &= \inf\left\{t \geq 0 \mid \ X_t \geq 2^j\right\} \\ &= \inf\left\{t \geq 0 \,\middle|\, \frac{b-\sigma^2}{\sqrt{2}|\sigma|} t + \frac{\sigma}{|\sigma|} W_t \geq \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right)\right\}, \quad \text{for } j = i_x, i_x+1, \ldots. \end{aligned}$$

Since the process $\frac{\sigma}{|\sigma|}W$ is a standard Brownian motion, we can use Exercise 3.5.10 in Karatzas and Shreve (1991) and the definition (3.11) of $n > 0$ to calculate

$$\begin{aligned} \mathbb{E}\left[e^{-r\tau_j}\right] &= \exp\left(\frac{b-\sigma^2}{2\sigma^2} \ln\left(\frac{2^j}{x}\right) - \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right) \sqrt{\frac{(b-\sigma^2)^2}{2\sigma^2} + 2r}\right) \\ &= \left(\frac{x}{2^j}\right)^{n}. \end{aligned}$$

In view of this calculation and (3.21), we can see that

$$\begin{aligned}\mathbb{E}\left[\sum_{0\le t} e^{-rt}\big(K\Delta Z_t + c\big)\mathbf{1}_{\{\Delta Z_t>0\}}\right] &= K2^{\lambda i_x} + c + \mathbb{E}\left[\sum_{j=i_x}^{\infty} e^{-r\tau_j}\left[K\left(2^{(j+1)\lambda} - 2^{j\lambda}\right) + c\right]\right] \\ &= K2^{\lambda i_x} + c + \sum_{j=i_x}^{\infty}\left[K(2^{\lambda}-1)2^{\lambda j} + c\right]\mathbb{E}\left[e^{-r\tau_j}\right] \\ &= K2^{\lambda i_x} + c + K(2^{\lambda}-1)x^n \sum_{j=i_x}^{\infty}\left(\frac{1}{2^{n-\lambda}}\right)^j + cx^n \sum_{j=i_x}^{\infty}\left(\frac{1}{2^n}\right)^j \\ &< \infty,\end{aligned}$$

the inequality being true because $n-\lambda>0$. Combining this result with (3.22), we can see that $J_{x,y}(Z)=\infty$. □

3.3 The solution to the control problem

In view of Lemma 3.2.1, we may assume that $K_1=0$ and $K_2=K>0$ in what follows. We solve the resulting control problem by constructing a solution to its Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}\max\big\{&\sigma^2x^2w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + x^{\alpha}y^{\beta}, \\ &- w(x,y) - c + \sup_{z>0}\big[w(x,y+z) - Kz\big]\big\} = 0, \quad (x,y)\in\mathcal{S}. \quad (3.23)\end{aligned}$$

To get a qualitative feeling about the origins of this equation, observe that, at time 0, the project's management has two options. The first one is to wait for a short time Δt and then continue optimally. In view of Bellman's principle of optimality, this option, which is not necessarily optimal, is associated with the inequality

$$v(x,y) \ge \mathbb{E}\left[\int_0^{\Delta t} e^{-rt}X_t^{\alpha}y^{\beta}\,dt + e^{-r\Delta t}v(X_{\Delta t},y)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t\downarrow 0$, we obtain

$$\sigma^2x^2v_{xx}(x,y) + bxv_x(x,y) - rv(x,y) + x^{\alpha}y^{\beta} \le 0. \quad (3.24)$$

The second option is to increase capacity by $\Delta Z_0 = z > 0$, and then continue optimally. Since such a capacity increase is not necessarily optimal, this action is associated with the inequality

$$v(x, y) \geq v(x, y + z) - Kz - c,$$

which implies that

$$\sup_{z>0} \bigl[v(x, y + z) - Kz\bigr] - v(x, y) - c \leq 0, \tag{3.25}$$

because $z > 0$ has been arbitrary. Since these two are the only options available, we expect that, given any initial condition $(x, y) \in \mathcal{S}$, one of them should be optimal, so that one of the inequalities (3.24)–(3.25) should hold with equality. This observation and (3.24)–(3.25) suggest that the value function v should identify with a solution w to the HJB equation (3.23).

We postulate that the optimal strategy is characterised by two strictly increasing C^∞ functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G_1(y) < G_0(y)$ for all $y \geq 0$. The function G_0 separates the state space $\mathcal{S}$ into two regions, the waiting region $\mathcal{W}$ and the investment region $\mathcal{I}$, while the function G_1 provides the capacity level that should be reached whenever it is optimal to increase the project's capacity (see Figure 3.5.1 in Appendix V). We denote by $\mathcal{G}_0$, $\mathcal{G}_1$ the inverses of the functions G_0, G_1, so that

$$G_i\bigl(\mathcal{G}_i(x)\bigr) = x \text{ for all } x \geq G_i(0) \quad \text{and} \quad \mathcal{G}_i\bigl(G_i(y)\bigr) = y \text{ for all } y \geq 0. \tag{3.26}$$

Remark 3.3.1 It is worth making a comment on the qualitative dependence of the optimal strategy that we have considered above, which is depicted by Figure 3.5.1, on the parameter c. The constant $c > 0$ provides the fixed cost that each additional investment incurs. Therefore, as $c \downarrow 0$, we expect that the free-boundary functions G_0, G_1 move closer and closer together until they confound because the fixed costs become negligible relative to the proportional costs. On the other hand, as c takes larger and larger values, we expect that G_0 and G_1 move further and further apart because increasing fixed costs discourage frequent investment.

In light of the heuristic arguments discussed above, we look for a solution to the

HJB equation (3.23) that satisfies

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + x^\alpha y^\beta = 0 \tag{3.27}$$

in the interior of $\mathcal{W}$ and is given by

$$w(x,y) = w\big(x, \mathcal{G}_1(x)\big) - K\big[\mathcal{G}_1(x) - y\big] - c, \quad \text{for } (x,y) \in \mathcal{I}. \tag{3.28}$$

Every solution to (3.27) that remains bounded as $x \downarrow 0$ is given by

$$w(x,y) = A(y)x^n + \Gamma x^\alpha y^\beta, \tag{3.29}$$

for some function A, where the constants $\Gamma, n > 0$ are given by (3.9), (3.11). To determine the functions A, $\mathcal{G}_0$ and $\mathcal{G}_1$, we first note that (3.28) for $y = \mathcal{G}_0(x)$ and the inequality

$$w\big(x, \mathcal{G}_0(x) + z\big) - w\big(x, \mathcal{G}_0(x)\big) - Kz - c \le 0 \quad \text{for all } z > 0,$$

which is associated with the HJB equation (3.23), imply that the function

$$z \mapsto w\big(x, \mathcal{G}_0(x) + z\big) - w\big(x, \mathcal{G}_0(x)\big) - Kz - c$$

has a local maximum at $z = \mathcal{G}_1(x) - \mathcal{G}_0(x)$. Therefore,

$$w_y\big(x, \mathcal{G}_1(x)\big) \equiv \dot{A}\big(\mathcal{G}_1(x)\big)x^n + \beta\Gamma x^\alpha \mathcal{G}_1^{\beta-1}(x) = K. \tag{3.30}$$

Next, we postulate that w is $C^{1,1}$ at the free-boundary $\mathcal{G}_0$. The requirement that w_y should be continuous yields

$$\lim_{u\downarrow\mathcal{G}_0(x)} w_y(x,u) \equiv \dot{A}\big(\mathcal{G}_0(x)\big)x^n + \beta\Gamma x^\alpha \mathcal{G}_0^{\beta-1}(x) = K \equiv \lim_{u\uparrow\mathcal{G}_0(x)} w_y(x,u), \tag{3.31}$$

while the requirement that w_x should be continuous gives rise to the identities

$$\begin{aligned}
\lim_{u\to\mathcal{G}_0(x)} w_x\big(x,u\big) &\equiv nA\big(\mathcal{G}_0(x)\big)x^{n-1} + \alpha\Gamma x^{\alpha-1}\mathcal{G}_0^\beta(x) \\
&= \lim_{\varepsilon\downarrow 0} \frac{w\big(x+\varepsilon, \mathcal{G}_0(x)\big) - w\big(x, \mathcal{G}_0(x)\big)}{\varepsilon} \\
&= \lim_{\varepsilon\downarrow 0} \frac{w\big(x+\varepsilon, \mathcal{G}_1(x+\varepsilon)\big) - w\big(x, \mathcal{G}_1(x)\big) - K\big[\mathcal{G}_1(x+\varepsilon) - \mathcal{G}_1(x)\big]}{\varepsilon} \\
&\overset{(3.30)}{=} w_x\big(x, \mathcal{G}_1(x)\big) \\
&= nA\big(\mathcal{G}_1(x)\big)x^{n-1} + \alpha\Gamma x^{\alpha-1}\mathcal{G}_1^\beta(x),
\end{aligned}$$

which imply that

$$A\big(\mathcal{G}_0(x)\big)x^n + \frac{\alpha\Gamma}{n}x^\alpha \mathcal{G}_0^\beta(x) = A\big(\mathcal{G}_1(x)\big)x^n + \frac{\alpha\Gamma}{n}x^\alpha \mathcal{G}_1^\beta(x). \tag{3.32}$$

Recalling the notation introduced by (3.26), we can see that (3.30) and (3.31) are equivalent to

$$\dot{A}(y)G_1^n(y) + \beta\Gamma y^{\beta-1}G_1^\alpha(y) = K, \quad \text{for } y \geq \mathcal{G}_1\big(G_0(0)\big) \tag{3.33}$$

and

$$\dot{A}(y)G_0^n(y) + \beta\Gamma y^{\beta-1}G_0^\alpha(y) = K, \quad \text{for } y \geq 0. \tag{3.34}$$

These identities imply that G_0 and G_1 should satisfy

$$F\left(y^{-\frac{1-\beta}{\alpha}}G_1(y), y^{-\frac{1-\beta}{\alpha}}G_0(y)\right) = 0 \quad \text{for all } y \geq \mathcal{G}_1\big(G_0(0)\big), \tag{3.35}$$

where

$$F(z_1, z_0) = z_0^{-n}\left(\beta\Gamma z_0^\alpha - K\right) - z_1^{-n}\left(\beta\Gamma z_1^\alpha - K\right). \tag{3.36}$$

On the other hand, combining (3.28) for $y = \mathcal{G}_0(x)$ with (3.32), we can see that $\mathcal{G}_0$ and $\mathcal{G}_1$ should satisfy

$$\Phi\big(x, x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_0(x), x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_1(x)\big) = 0 \quad \text{for all } x \geq G_0(0), \tag{3.37}$$

where

$$\Phi(x, p_0, p_1) = p_1^\beta - p_0^\beta - \frac{nK}{(n-\alpha)\Gamma}(p_1 - p_0) - \frac{nc}{(n-\alpha)\Gamma}x^{-\frac{\alpha}{1-\beta}}. \tag{3.38}$$

To summarise the heuristic discussion above, suppose that there exist strictly increasing functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (3.35) and (3.37). Both of G_0 and G_1 are C^∞ because F and Φ are C^∞. If we choose

$$\begin{aligned} A(y) &= \beta\Gamma \int_y^\infty u^{-(1-\beta)}G_0^{-(n-\alpha)}(u)\,du - K\int_y^\infty G_0^{-n}(u)\,du \\ &= \beta\Gamma \int_y^\infty G_0^{-n}(u)\left[\left(u^{-\frac{1-\beta}{\alpha}}G_0(u)\right)^\alpha - \frac{K}{\beta\Gamma}\right]du, \end{aligned} \tag{3.39}$$

then, assuming the integrals are well-defined and finite, (3.30)–(3.32) and the function w, defined by (3.28) if $(x,y) \in \mathcal{I}$ and by (3.29) if $(x,y) \in \mathcal{W}$, is $C^{1,1}$ along the free-boundary $\mathcal{G}_0$ and satisfies the HJB equation (3.23).

The next result, which we prove in Appendix IV, is concerned with this construction.

Lemma 3.3.1 Suppose that Assumption 3.1.1 holds, $K_1 = 0$ and $K_2 = K > 0$. Also, assume that there exist strictly increasing functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the system of equations (3.35) and (3.37). Such functions G_0, G_1 are C^∞,

$$\left(\frac{K}{\beta\Gamma}\right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}} < G_1(y) < \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}} \quad \text{for all } y \geq 0, \tag{3.40}$$

or, equivalently,

$$\left(\frac{(n-\alpha)\beta\Gamma}{nK}\right)^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}} < \mathcal{G}_1(x) < \left(\frac{\beta\Gamma}{K}\right)^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}} \quad \text{for all } x > 0, \tag{3.41}$$

and there exist strictly positive constants $\underline{C} < \overline{C}$ such that

$$\underline{C}\left(1 \vee y^{\frac{1-\beta}{\alpha}}\right) < G_0(y) < \overline{C}\left(1 \vee y^{\frac{1-\beta}{\alpha}}\right) \quad \text{for all } y \geq 0. \tag{3.42}$$

The function $w : \mathcal{S} \to \mathbb{R}$ defined by

$$w(x, y) = \begin{cases} A(y)x^n + \Gamma x^\alpha y^\beta, & \text{if } x < G_0(y), \\ w\big(x, \mathcal{G}_1(x)\big) - K\big[\mathcal{G}_1(x) - y\big] - c, & \text{if } x \geq G_0(y), \end{cases} \tag{3.43}$$

where the constants $\Gamma, n > 0$ are given by (3.9), (3.11) and $A > 0$ is given by (3.39), is $C^{1,1}$ and $C^{\infty,\infty}$ outside the graph of G_0. Also, w is a classical solution to the HJB equation (3.23) such that

$$0 < w(x, y) \leq C\left(1 + y + x^\alpha y^\beta + x^{\frac{\alpha}{1-\beta}}\right) \quad \text{for all } (x, y) \in \mathcal{S}, \tag{3.44}$$

for some constant $C > 0$, and the function $w(\cdot, y)$ is strictly increasing for all $y \geq 0$.

We can now prove the main result of the paper.

Theorem 3.3.1 Consider the capacity control problem formulated in Section 3.1 and suppose, without loss of generality, that $K_1 = 0$ and $K_2 = K > 0$. Also assume that there exist strictly increasing functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the system of equations (3.35) and (3.37). The value function v identifies with the solution to the HJB equation (3.23) given by (3.43) in Lemma 3.3.1. Apart from an initial jump of size $\big[\mathcal{G}_1(x) - y\big]^+$ at time 0, the optimal capacity level process Y° has jumps of sizes provided by the function $\mathcal{G}_1 - \mathcal{G}_0$ that occur at the $(\mathcal{F}_t)$-stopping times when the process (X, Y°) hits the graph of $\mathcal{G}_0$, and is given by (3.48)–(3.50) in the proof below.

Proof. Fix any initial condition $(x, y) \in \mathcal{S}$ and any admissible strategy $Z \in \mathcal{A}$. Since Y is piecewise constant and $w(\cdot, y)$ is C^1 along the free-boundary $\mathcal{G}_0$ and C^2 outside the graph of $\mathcal{G}_0$, for all $y \geq 0$, we can use the Itô-Tanaka-Meyer formula and the fact that X has continuous sample paths to obtain

$$e^{-rT} w(X_T, Y_{T+}) = w(x, y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t)\right] dt \\ + M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_{t+}) - w(X_t, Y_t)\right],$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t. \tag{3.45}$$

This implies that

$$\begin{aligned} &\int_0^T e^{-rt} X_t^\alpha Y_t^\beta \, dt - \sum_{0 \leq t \leq T} e^{-rt} \big(K \Delta Z_t + c\big) \mathbf{1}_{\{\Delta Z_t > 0\}} + e^{-rT} w(X_T, Y_{T+}) \\ &\quad = w(x, y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t) + X_t^\alpha Y_t^\beta\right] dt \\ &\quad + M_T + \sum_{0 \leq t \leq T} e^{-rt} \big[w(X_t, Y_t + \Delta Z_t) - w(X_t, Y_t) - K\Delta Z_t - c\big] \mathbf{1}_{\{\Delta Z_t > 0\}}. \end{aligned} \tag{3.46}$$

Since w is positive and satisfies the HJB equation (3.23), we can see that

$$\int_0^T e^{-rt} X_t^\alpha Y_t^\beta \, dt - \sum_{0 \leq t \leq T} e^{-rt} \big(K \Delta Z_t + c\big) \mathbf{1}_{\{\Delta Z_t > 0\}} \leq w(x, y) + M_T, \tag{3.47}$$

which implies that

$$\inf_{T \geq 0} M_T \geq -w(x, y) - \sum_{0 \leq t \leq T} e^{-rt} \big(K \Delta Z_t + c\big) \mathbf{1}_{\{\Delta Z_t > 0\}}.$$

The random variable on the right hand side of this inequality has finite expectation thanks to (3.4). It follows that the stochastic integral M is a supermartingale, and therefore, $\mathbb{E}\left[M_T\right] \leq 0$ for all $T > 0$. Taking expectations in (3.47) and passing to the limit using the monotone convergence theorem, we obtain

$$J_{x,y}(Z) \leq w(x, y),$$

and the inequality $v(x, y) \leq w(x, y)$ follows.

To establish the reverse inequality, we let

$$\tau_0 = 0 \quad \text{and} \quad Z_t^0 = \big[\mathcal{G}_1(x) - y\big] \mathbf{1}_{\{y < \mathcal{G}_0(x)\}} \mathbf{1}_{\{0 < t\}}, \tag{3.48}$$

and we define iteratively the $(\mathcal{F}_t)$-stopping times τ_ℓ and the processes Z^ℓ by

$$\tau_{\ell+1} = \inf \Big\{ t \geq \tau_\ell \mid \ X_t \geq G_0\big(y + Z_t^{(\ell)}\big) \Big\}, \quad \text{for } \ell = 0, 1, \ldots, \tag{3.49}$$

$$Z_t^{(\ell+1)} = Z_t^{(\ell)} + \big[\mathcal{G}_1(X_{\tau_{\ell+1}}) - \mathcal{G}_0(X_{\tau_{\ell+1}})\big] \mathbf{1}_{\{t > \tau_{\ell+1}\}}, \quad \text{for } \ell = 0, 1, \ldots. \tag{3.50}$$

Observing that $\lim_{\ell \to \infty} \tau_\ell = \infty$, $\mathbb{P}$-a.s., we define the capacity expansion process Z^{o} by $Z_t^{\mathrm{o}} = Z_t^{(\ell)}$ for $t < \tau_\ell$, and we note that the associated capacity process Y^{o} satisfies

$$Y_t^{\mathrm{o}} \leq y \mathbf{1}_{\{\bar{X}_t \leq G_0(y)\}} + \mathcal{G}_1(\bar{X}_t) \mathbf{1}_{\{\bar{X}_t > G_0(y)\}},$$

where $\bar{X}_t = \sup_{s \leq t} X_s$. This inequality and (3.41) imply that

$$Y_t^{\mathrm{o}} \leq y + \left(\frac{\beta \Gamma}{K}\right)^{\frac{1}{1-\beta}} \bar{X}_t^{\frac{\alpha}{1-\beta}}$$

and

$$X_t^\alpha \big(Y_t^{\mathrm{o}}\big)^\beta \leq \bar{X}_t^\alpha \big(Y_t^{\mathrm{o}}\big)^\beta \leq y^\beta \bar{X}_t^\alpha + \bar{X}_t^\alpha \mathcal{G}_1^\beta(\bar{X}_t) \leq y^\beta \bar{X}_t^\alpha + \left(\frac{\beta \Gamma}{K}\right)^{\frac{\beta}{1-\beta}} \bar{X}_t^{\frac{\alpha}{1-\beta}}.$$

In view of the estimate (3.44) of w, we can therefore see that there exists a constant $\tilde{C} > 0$ such that

$$0 < w(X_T, Y_{T+}^{\mathrm{o}}) \leq \tilde{C} \left(1 + \bar{X}_t^\alpha + \bar{X}_T^{\frac{\alpha}{1-\beta}}\right). \tag{3.51}$$

Recalling Assumption 3.1.1, we can combine these inequalities with (3.13) to obtain

$$\mathbb{E}\left[\int_0^\infty e^{-rt} X_t^\alpha \big(Y_t^{\mathrm{o}}\big)^\beta \, dt + \sup_{T > 0} e^{-rT} w(X_T, Y_{T+}^{\mathrm{o}})\right] < \infty. \tag{3.52}$$

In light of the construction of Y^{o}, we can see that (3.46) implies

$$\int_0^T e^{-rt} X_t^\alpha, \big(Y_t^{\mathrm{o}}\big)^\beta \, dt - \sum_{0 \leq t \leq T} e^{-rt} \big(K \Delta Z_t^{\mathrm{o}} + c\big) \mathbf{1}_{\{\Delta Z_t^{\mathrm{o}} > 0\}}$$

$$+ e^{-rT} w(X_T, Y_{T+}^{\mathrm{o}}) = w(x, y) + M_T^{\mathrm{o}}, \tag{3.53}$$

where M° is defined by (3.45) for $Y = Y^{\circ}$. This identity and (3.52) imply that $\mathbb{E}\left[\sup_{T>0} M^{\circ}_T\right] < \infty$, so the stochastic integral M° is a submartingale. In view of this observation, we can take expectations in (3.53) to obtain

$$\mathbb{E}\left[\int_0^T e^{-rt} X_t^{\alpha}, \left(Y_t^{\circ}\right)^{\beta} dt\right] - \mathbb{E}\left[\sum_{0\leq t\leq T} e^{-rt}\left(K\Delta Z_t^{\circ} + c\right)\mathbf{1}_{\{\Delta Z_t^{\circ}>0\}}\right] + \mathbb{E}\left[e^{-rT} w(X_T, Y_{T+}^{\circ})\right] \geq w(x,y). \quad (3.54)$$

The first inequality in (3.13) and (3.51) imply that

$$\lim_{T\to\infty} e^{-rT}\mathbb{E}\left[w(X_T, Y_{T+}^{\circ})\right] = 0.$$

Combining this result with (3.52), (3.54) and the monotone convergence theorem, we can see that

$$\mathbb{E}\left[\sum_{0\leq t} e^{-rt}\left(K\Delta Z_t^{\circ} + c\right)\mathbf{1}_{\{\Delta Z_t^{\circ}>0\}}\right] < \infty,$$

which implies that $Z^{\circ} \in \mathcal{A}$ (see Definition 3.1.1). In particular, we can pass to the limit in (3.54) using the monotone convergence theorem to obtain

$$J_{x,y}(Z^{\circ}) \geq w(x,y).$$

Recalling the inequality $v(x,y) \leq w(x,y)$ that we proved above, we deduce that $v(x,y) = w(x,y)$ and that Z° is optimal. □

3.4 Appendix IV: Proof of Lemma 3.3.1

To establish Lemma 3.3.1, we first need to consider a pair of preliminary results. The first one is concerned with a study of the function F defined by (3.36).

Lemma 3.4.1 Suppose that Assumption 3.1.1 holds. Given $z_1 > 0$ fixed, equation (3.36) has a solution $z_0 > z_1$ if and only if $z_1 \in \mathcal{D}_F$, where

$$\mathcal{D}_F = \left]\left(\frac{K}{\beta\Gamma}\right)^{\frac{1}{\alpha}}, \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}\right[.$$

In particular, there exists a strictly decreasing C^∞ function $L_F : \mathcal{D}_F \to \mathbb{R}_+$ such that

$$\left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}} < L_F(z_1) \quad \text{and} \quad F\big(z_1, L_F(z_1)\big) = 0 \quad \text{for all } z_1 \in \mathcal{D}_F, \tag{3.55}$$

$$\lim_{z_1 \downarrow \left(\frac{K}{\beta\Gamma}\right)^{\frac{1}{\alpha}}} L_F(z_1) = \infty, \qquad \lim_{z_1 \uparrow \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}} L_F(z_1) = \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}} \tag{3.56}$$

and

$$F\big(u, L_F(z_1)\big) = \begin{Bmatrix} > 0, & \text{if } u < z_1 \\ < 0, & \text{if } u > z_1 \end{Bmatrix} \quad \text{for all } z_1 \in \mathcal{D}_F. \tag{3.57}$$

Proof. In view of the definition (3.36) of F and the assumption that $\alpha < n$, we calculate

$$F(z_1, z_1) = 0$$

and

$$\lim_{z_0 \to \infty} F(z_1, z_0) = -\beta\Gamma z_1^{-n}\left(z_1^\alpha - \frac{K}{\beta\Gamma}\right) \begin{cases} > 0, & \text{if } z_1 < \left(\frac{K}{\beta\Gamma}\right)^{\frac{1}{\alpha}}, \\ < 0, & \text{if } z_1 > \left(\frac{K}{\beta\Gamma}\right)^{\frac{1}{\alpha}}. \end{cases}$$

Combining these observations with the calculation

$$\frac{\partial}{\partial z_0} F(z_1, z_0) = -(n-\alpha)\beta\Gamma z_0^{-n-1}\left(z_0^\alpha - \frac{nK}{(n-\alpha)\beta\Gamma}\right) \begin{cases} < 0, & \text{if } z_0 < \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}, \\ > 0, & \text{if } z_0 > \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}, \end{cases}$$

we can see that the equation $F(z_1, z_0) = 0$ has a solution $z_0 > z_1$ if and only if $z_1 \in \mathcal{D}_F$, and that there exists a unique function $L_F : \mathcal{D}_1 \to \mathbb{R}_+$ satisfying (3.55) as well as (3.56). Differentiating $F\big(z_1, L_F(z_1)\big) = 0$ with respect to z_1, we obtain

$$\dot{L}_F(z_1) = \frac{z_1^{-n-1}\left[z_1^\alpha - \frac{nK}{(n-\alpha)\beta\Gamma}\right]}{L_F^{-n-1}(z_1)\left[L_F^\alpha(z_1) - \frac{nK}{(n-\alpha)\beta\Gamma}\right]} < 0 \quad \text{for all } z_1 \in \mathcal{D}_F,$$

which proves that L_F is strictly decreasing. Finally, (3.57) follows from the calculation

$$\frac{\partial}{\partial u} F\big(u, L_F(z_1)\big) = (n-\alpha)\beta\Gamma u^{-n-1}\left(u^\alpha - \frac{nK}{(n-\alpha)\beta\Gamma}\right) \begin{cases} < 0, & \text{if } u < \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}, \\ > 0, & \text{if } u > \left(\frac{nK}{(n-\alpha)\beta\Gamma}\right)^{\frac{1}{\alpha}}, \end{cases}$$

and the identities

$$F\big(z_1, L_F(z_1)\big) = F\big(L_F(z_1), L_F(z_1)\big) = 0.$$

□

The next result is concerned with a study of the function Φ defined by (3.38).

Lemma 3.4.2 Suppose that Assumption 3.1.1 holds, consider the points

$$\underline{\underline{p}}_1 < \underline{p}_1 < \overline{p}_1 < \overline{\overline{p}}_1,$$

given by

$$\underline{\underline{p}}_1 = 0, \quad \underline{p}_1 = \left(\frac{(n-\alpha)\beta\Gamma}{nK}\right)^{\frac{1}{1-\beta}}, \quad \overline{p}_1 = \left(\frac{\beta\Gamma}{K}\right)^{\frac{1}{1-\beta}} \quad \text{and} \quad \overline{\overline{p}}_1 = \left(\frac{(n-\alpha)\Gamma}{nK}\right)^{\frac{1}{1-\beta}}, \tag{3.58}$$

and define

$$\mathcal{D}_\Phi = \left\{(x, y) \in \mathcal{S} \mid \underline{\underline{p}}_1 < p_1 < \overline{\overline{p}}_1 \text{ and } x \geq \mathbb{X}(p_1)\right\},$$

where

$$\mathbb{X}(p_1) = \left[\frac{(n-\alpha)\Gamma}{nc} p_1^\beta \left(1 - \frac{nK}{(n-\alpha)\Gamma} p_1^{1-\beta}\right)\right]^{-\frac{1-\beta}{\alpha}}, \quad \text{for } p_1 \in \left]\underline{\underline{p}}_1, \overline{\overline{p}}_1\right[.$$

Given $(x, p_1) \in \mathcal{S}$, equation (3.38) has a solution $p_0 < p_1$ if and only if $(x, p_1) \in \mathcal{D}_\Phi$. In particular, there exists a C^∞ function $L_\Phi : \mathcal{D}_\Phi \to \mathbb{R}_+$ such that

$$L_\Phi(x, p_1) < p_1 \wedge \underline{p}_1 \quad \text{and} \quad \Phi\big(x, L_\Phi(x, p_1), p_1\big) = 0 \quad \text{for all } (x, p_1) \in \mathcal{D}_\Phi. \tag{3.59}$$

Furthermore,

$$\lim_{x \to \infty} L_\Phi\left(x, \underline{p}_1\right) = \underline{p}_1, \tag{3.60}$$

$$\frac{\partial}{\partial x} L_\Phi(x, p_1) > 0 \text{ for all } p_1 \in \left]\underline{\underline{p}}_1, \overline{\overline{p}}_1\right[\quad \text{and} \quad \frac{\partial}{\partial p_1} L_\Phi(x, p_1) < 0 \text{ for all } p_1 \in \left]\underline{p}_1, \overline{\overline{p}}_1\right[. \tag{3.61}$$

Proof. Recalling the definition (3.38) of Φ and Assumption 3.1.1, we calculate

$$\begin{aligned}
\Phi(x, p_1, p_1) &= -\frac{nc}{(n-\alpha)\Gamma} x^{-\frac{\alpha}{1-\beta}} < 0, \\
\frac{\partial}{\partial p_0} \Phi(x, p_0, p_1) &= -\beta p_0^{-(1-\beta)} \left(1 - \frac{nK}{(n-\alpha)\beta\Gamma} p_0^{1-\beta}\right) \\
&= -\beta p_0^{-(1-\beta)} \left(1 - \left(\frac{p_0}{\underline{p}_1}\right)^{1-\beta}\right) \begin{cases} < 0, & \text{if } p_0 < \underline{p}_1, \\ > 0, & \text{if } p_0 > \underline{p}_1, \end{cases}
\end{aligned}$$

and

$$\Phi(x,0,p_1) = p_1^{\beta}\left(1 - \frac{nK}{(n-\alpha)\Gamma} p_1^{1-\beta}\right) - \frac{nc}{(n-\alpha)\Gamma} x^{-\frac{\alpha}{1-\beta}}.$$

Noting that

$$\Phi(x,0,p_1) = -\frac{nc}{(n-\alpha)\Gamma} x^{-\frac{\alpha}{1-\beta}} \left(1 - \left(\frac{x}{\mathbb{X}(p_1)}\right)^{\frac{\alpha}{1-\beta}}\right), \quad \text{if } p_1 \in \left]\underline{\underline{p}}_1, \overline{\overline{p}}_1\right[,$$

we can see that these results imply that

if $p_1 \geq \overline{\overline{p}}_1$, then there exist no $x > 0$ and $p_0 \in [0, p_1]$ such that $\Phi(x, p_0, p_1) = 0$,

if $p_1 < \overline{\overline{p}}_1$, then there exists a unique $p_0 \in]0, p_1[$ such that $\Phi(x, p_0, p_1) = 0$

if and only if $x > \mathbb{X}(p_1)$,

and, in the latter case, the solution p_0 of $\Phi(x, p_0, p_1) = 0$ is strictly less than $\underline{p}_1$. It follows that equation (3.38) defines uniquely a function $L_\Phi : \mathcal{D}_\Phi \to \mathbb{R}_+$ such that (3.59) as well as (3.60) hold true. This function is C^∞ because Φ is.

Differentiating the identity $\Phi\big(x, L_\Phi(x, p_1), p_1\big) = 0$ with respect to x, we obtain

$$\frac{\partial}{\partial x} L_\Phi(x,p_1) = -\frac{nc\alpha x^{-\frac{\alpha}{1-\beta}-1}}{(n-\alpha)\beta(1-\beta)\Gamma L_\Phi^{-(1-\beta)}(x,p_1)\left[1 - \left(\frac{L_\Phi(x,p_1)}{\underline{p}_1}\right)^{1-\beta}\right]},$$

and the first inequality in (3.61) follows thanks to the first inequality in (3.59). Similarly, we can see that the calculation

$$\frac{\partial}{\partial p_1} L_\Phi(x,p_1) = \frac{p_1^{-(1-\beta)}\left(1 - \left(\frac{p_1}{\underline{p}_1}\right)^{1-\beta}\right)}{L_\Phi^{-(1-\beta)(x,p_1)}\left(1 - \left(\frac{L_\Phi(x,p_1)}{\underline{p}_1}\right)^{1-\beta}\right)}$$

implies the second inequality in (3.61). □

Proof of Lemma 3.3.1. The assumption that that there exists functions $G_0, G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the system of equations (3.35), (3.37) and Lemmas 3.4.1, 3.4.2 imply that

$$G_0(y) = y^{\frac{1-\beta}{\alpha}} L_F\left(y^{-\frac{1-\beta}{\alpha}} G_1(y)\right) \quad \text{for all } y \geq 0, \tag{3.62}$$

and

$$\mathcal{G}_0(x) = x^{\frac{\alpha}{1-\beta}} L_\Phi\left(x, x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_1(x)\right) \quad \text{for all } x \geq G_0(y).$$

Furthermore, Lemma 3.4.1 implies that G_1 satisfies (3.40). In view of (3.41), we can see that

$$\underline{p}_1 < x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_1(x) < \overline{p}_1,$$

where $\underline{p}_1$ and $\overline{p}_1$ are given by (3.58). Combining this observation with the fact that $p_1 \mapsto L_\Phi(x, p_1)$ is strictly decreasing in $]\underline{p}_1, \overline{p}_1[$ (see the second inequality in (3.61)), we obtain

$$L_\Phi\big(x, \overline{p}_1\big) < x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_0(x) \equiv L_\Phi\left(x, x^{-\frac{\alpha}{1-\beta}}\mathcal{G}_1(x)\right) < L_\Phi\big(x, \underline{p}_1\big) \tag{3.63}$$

textfor all $x \geq \mathbb{X}(\overline{p}_1) > G_0(y) > \mathbb{X}(\underline{p}_1)$. Since the function$x \mapsto L_\Phi(x, p_1)$ is strictly increasing (see the first inequality in (3.61)), we can see that, given any $x_* > \mathbb{X}(\overline{p}_1)$,

$$\mathcal{G}_0(x) \geq L_\Phi\big(x_*, \overline{p}_1\big)x^{\frac{\alpha}{1-\beta}} \quad \text{for all } x \geq x_*.$$

Therefore,

$$G_0(y) \leq L_\Phi^{-\frac{1-\beta}{\alpha}}\big(x_*, \overline{p}_1\big)y^{\frac{1-\beta}{\alpha}} \quad \text{for all } y \geq \mathcal{G}_0(x_*).$$

Combining this result with the inequality

$$G_0(y) \leq G_0\big(\mathcal{G}_0(x_*)\big) \quad \text{for all } y \leq \mathcal{G}_0(x_*),$$

which follows from the assumption that G_0 is increasing, we obtain the upper bound in (3.42). Similarly, we can see that the upper bound in (3.63) implies that

$$\mathcal{G}_0(x) < x^{\frac{\alpha}{1-\beta}} \lim_{x\to\infty} L_\Phi\big(x, \underline{p}_1\big) \overset{(3.60)}{=} \underline{p}_1 x^{\frac{\alpha}{1-\beta}} \quad \text{for all } x \geq G_0(0).$$

It follows that

$$\underline{p}_1^{-\frac{1-\beta}{\alpha}} y^{\frac{1-\beta}{\alpha}} < G_0(y) \quad \text{for all } y \geq 0,$$

which, combined with the inequalities $G_0(0) > \mathbb{X}(\underline{p}_1) > 0$ and the fact that G_0 is increasing, implies the lower bound in (3.42).

The lower bound in (3.40), the fact that $G_1 < G_0$ and the definition (3.39) of A imply that $A(y) > 0$ for all $y \geq 0$. In view of (3.6) in Assumption 3.1.1 and the lower bound in (3.42), we can see that

$$\begin{aligned} A(y) &< \beta\Gamma \int_y^\infty u^{-(1-\beta)} G_0^{-(n-\alpha)}(u)\, du \\ &< \beta\Gamma \underline{C}^{-(n-\alpha)} \int_y^\infty u^{-(1-\beta)} u^{-\frac{(n-\alpha)(1-\beta)}{\alpha}}\, du \\ &= \frac{\alpha\beta\Gamma\underline{C}^{-(n-\alpha)}}{n-\alpha-n\beta} y^{-\frac{n-\alpha-n\beta}{\alpha}} \quad \text{for all } y \geq 1. \end{aligned}$$

It follows that there exist a constant $\overline{C}_1 > 0$ such that

$$A(y) < \overline{C}_1 \left(1 \wedge y^{-\frac{n-\alpha-n\beta}{\alpha}}\right) \quad \text{for all } y \geq 0.$$

Combining this estimate with the upper bound in (3.42), we can see that

$$A(y)G_1^n(y) < A(y)G_0^n(y) < \overline{C}_1\overline{C}\,(1 \vee y) \quad \text{for all } y \geq 0. \tag{3.64}$$

Given any $(x, y) \in \mathcal{W}$, we can use the strict positivity of A and (3.64) to calculate

$$\begin{aligned} w(x,y) &= A(y)x^n + \Gamma x^\alpha y^\beta \\ &\leq A(y)G_0^n(y) + \Gamma x^\alpha y^\beta \\ &\leq \overline{C}_1\overline{C}(1+y) + \Gamma x^\alpha y^\beta. \end{aligned}$$

On the other hand, if $(x, y) \in \mathcal{I}$, then we can use the expression for w given by (3.43) and (3.64) to obtain

$$\begin{aligned} w(x,y) &\leq w\big(x, \mathcal{G}_1(x)\big) \\ &= A\big(\mathcal{G}_1(x)\big)x^n + \Gamma x^\alpha \mathcal{G}_1^\beta(x) \\ &< \overline{C}_1\overline{C}\big(1 + \mathcal{G}_1(x)\big) + \Gamma x^\alpha \mathcal{G}_1^\beta(x) \\ &\overset{(3.41)}{<} \overline{C}_1\overline{C}\left(1 + \left(\frac{\beta\Gamma}{K}\right)^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}}\right) + \Gamma\left(\frac{\beta\Gamma}{K}\right)^{\frac{\beta}{1-\beta}} x^{\frac{\alpha}{1-\beta}}. \end{aligned}$$

It follows that w admits an upper bound such as the one given by (3.44).

By construction, we will prove that w satisfies the HJB equation (3.23) if we show that

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + x^\alpha y^\beta \leq 0 \quad \text{for all } (x,y) \in \mathcal{I}, \tag{3.65}$$

and

$$-w(x,y) - c + \sup_{z>0}\big[w(x,y+z) - Kz\big] \leq 0 \quad \text{for all } (x,y) \in \mathcal{W}. \tag{3.66}$$

To this end, we consider any $(x,y) \in \mathcal{W}$ and we observe that (3.34) and the definition (3.36) of F imply that

$$\begin{aligned}
w_y(x,y) - K &= \dot{A}(y)x^n + \beta\Gamma x^\alpha y^{\beta-1} - K \\
&= -x^n G_0^{-n}(y)\left[\beta\Gamma\left(y^{-\frac{1-\beta}{\alpha}}G_0(y)\right)^\alpha - K\right] + \beta\Gamma\left(y^{-\frac{1-\beta}{\alpha}}x\right)^\alpha - K \\
&= -x^n y^{-\frac{n(1-\beta)}{\alpha}} F\left(y^{-\frac{1-\beta}{\alpha}}x, y^{-\frac{1-\beta}{\alpha}}G_0(y)\right) \\
&\overset{(3.62)}{=} -x^n y^{-\frac{n(1-\beta)}{\alpha}} F\left(y^{-\frac{1-\beta}{\alpha}}x, L_F\left(y^{-\frac{1-\beta}{\alpha}}G_1(y)\right)\right).
\end{aligned}$$

Combining this calculation with (3.57), we can see that

$$w_y(x,y) - K = \begin{cases} < 0, & \text{if } x < G_1(y) \ \Leftrightarrow \ y > \mathcal{G}_1(x), \\ > 0, & \text{if } x \in \left]G_1(y), G_0(y)\right[\\ & \quad \Leftrightarrow \ y > \mathcal{G}_0(x), \ \text{if } x \geq \mathcal{G}_1\big(G_0(0)\big), \ \text{and } y < \mathcal{G}_1(x), \end{cases} \tag{3.67}$$

which implies that

$$w_{yy}\big(x, \mathcal{G}_1(x)\big) \leq 0 \quad \text{for all } x \geq G_0(y). \tag{3.68}$$

In view of (3.28) and (3.67), we can see that, given any $(x,y) \in \mathcal{W}$ and $z > 0$,

$$\begin{aligned}
&-w(x,y) - c + w(x,y+z) - Kz \\
&\quad = \int_y^{y+z}\big[w_y(x,u) - K\big]\,du - \int_{\mathcal{G}_0(x)}^{\mathcal{G}_1(x)}\big[w_y(x,u) - K\big]\,du \\
&\quad = \begin{cases} -\int_{\mathcal{G}_0(x)}^{y}\big[w_y(x,u) - K\big]\,du - \int_{y+z}^{\mathcal{G}_1(x)}\big[w_y(x,u) - K\big]\,du, & \text{if } y+z \leq \mathcal{G}_1(x), \\ -\int_{\mathcal{G}_0(x)}^{y}\big[w_y(x,u) - K\big]\,du + \int_{\mathcal{G}_1(x)}^{y+z}\big[w_y(x,u) - K\big]\,du, & \text{if } y \leq \mathcal{G}_1(x) < y+z, \\ \int_y^{y+z}\big[w_y(x,u) - K\big]\,du - \int_{\mathcal{G}_0(x)}^{\mathcal{G}_1(x)}\big[w_y(x,u) - K\big]\,du, & \text{if } \mathcal{G}_1(x) < y, \end{cases} \\
&\quad \leq 0,
\end{aligned}$$

and (3.65) follows.

To establish (3.65), we first differentiate the identity $w_y\big(x, \mathcal{G}_1(x)\big) = K$ with respect to x to obtain

$$w_{xy}\big(x, \mathcal{G}_1(x)\big) = -w_{yy}\big(x, \mathcal{G}_1(x)\big)\mathcal{G}_1'(x) \geq 0,$$

the inequality following thanks to (3.68) and the fact that $\mathcal{G}_1$ is strictly increasing. Using the definition (3.43) of w and this inequality we can see that, given any $(x, y) \in \mathcal{I}$,

$$\begin{aligned} w_x(x, y) &= w_x\big(x, \mathcal{G}_1(x)\big) + \big[w_y\big(x, \mathcal{G}_1(x)\big) - K\big]\mathcal{G}_1'(x) \\ &= w_x\big(x, \mathcal{G}_1(x)\big) \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} w_{xx}(x, y) &= w_{xx}\big(x, \mathcal{G}_1(x)\big) + w_{xy}\big(x, \mathcal{G}_1(x)\big)\mathcal{G}_1'(x) \\ &\leq w_{xx}\big(x, \mathcal{G}_1(x)\big). \end{aligned}$$

In light of these inequalities, we can see that (3.65) will be established if we show that

$$\begin{aligned} \sigma^2 x^2 w_{xx}\big(x, \mathcal{G}_1(x)\big) + bxw_x\big(x, \mathcal{G}_1(x)\big) - rw\big(x, \mathcal{G}_1(x)\big) + x^\alpha \mathcal{G}_1^\beta(x) \\ + x^\alpha y^\beta - x^\alpha \mathcal{G}_1^\beta(x) + rK\big[\mathcal{G}_1(x) - y\big] + rc \leq 0 \quad \text{for all } (x, y) \in \mathcal{I}. \end{aligned}$$

Combining the fact that w satisfies (3.27) in $\mathcal{W}$ with the identities

$$\begin{aligned} rK\big[\mathcal{G}_1(x) - \mathcal{G}_0(x)\big] + rc &= \frac{r(n-\alpha)\Gamma}{n} x^\alpha \big[\mathcal{G}_1^\beta(x) - \mathcal{G}_0^\beta(x)\big] \\ &= -\frac{m}{\alpha - m} x^\alpha \big[\mathcal{G}_1^\beta(x) - \mathcal{G}_0^\beta(x)\big], \end{aligned}$$

which follow from (3.37) and the definition (3.9) of Γ, we can see that this inequality is equivalent to

$$y^\beta - \mathcal{G}_1^\beta(x) - \frac{m}{\alpha - m}\big[\mathcal{G}_1^\beta(x) - \mathcal{G}_0^\beta(x)\big] \leq 0 \quad \text{for all } (x, y) \in \mathcal{I},$$

which is indeed true because $y < \mathcal{G}_1(x)$ for all $(x, y) \in \mathcal{I}$.

Finally, we can combine the calculation

$$w_x(x, y) = nA(y)x^{n-1} + \alpha\Gamma x^{\alpha-1} y^\beta > 0 \quad \text{for all } (x, y) \in \mathcal{W},$$

with (3.69) to conclude that the function $w(\cdot, y)$ is strictly increasing for all $y \geq 0$ and that $w(x, y) > 0$ for all $(x, y) \in \mathcal{S}$. □

3.5 Appendix V: Illustration of the free boundary functions

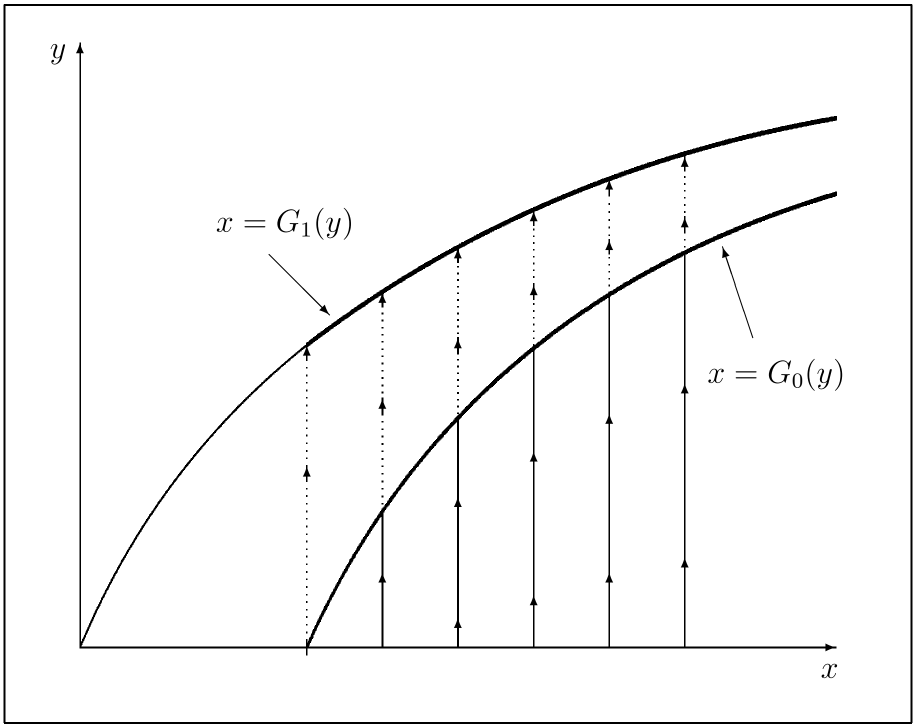

Figure 3.5.1 Graph of the free-boundary functions G_0 and G_1. If the initial condition (x, y) is below the graph of G_0, then it is optimal to invest so that the joint process (X°, Y°) has a jump at time 0 that positions it in the graph of G_1 along the arrows. After time 0, it is optimal to invest each time that (X°, Y°) hits the graph of G_0 so that the process (X°, Y°) has a jump that positions it inside the graph of G_1 along the arrows.

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