The London School of Economics and Political Science

Equilibria of Dynamic Mutual Choice Mating Games

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A thesis submitted to the Operational Research Group under the
Department of Management of the London School of Economics for
the degree of Doctor of Philosophy.

London, February 2009

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Abstract

We examine dynamic mutual choice mating games: members of two populations (males and females) are randomly matched in successive periods and form couples only if they mutually accept each other. Players are heterogeneous and their "types" are distributed in an interval. The utility that a player obtains from a mating depends on both his type and the type of his partner.

We consider three type of preferences: (i) homotypic (preference for similar types), (ii) common (preference for high types) and (iii) age dependent preferences.

In case (i), we explore the equilibrium behaviour when the sex ratio r is $1:1$. We extend the results of Alpern and Reyniers (1999) two period continuous type game. Next, we develop an algorithm, for reducing the potential equilibrium strategies in the two period discrete type model. Using this algorithm, we are able then to determine the equilibria in some discrete type models; we find multiple equilibria in some cases. Even when we do not assume the sexes adopt identical strategies, we find that this always occurs at equilibrium.

We also explore the equilibrium behaviour and the mating patterns when players have mixed preferences (combination of mixed and common preferences) with the help of a discrete type model.

In case (ii), we extend the Alpern and Reyniers (2005) common preferences model to the case of a sex ratio $r > 1$. Males remaining unmated after the end of the game have negative utility $-c$. We analyse how the equilibria of this mating game are formed, depending on the parameters r and c. It is proved that males are not always choosy at equilibrium and for some (r, c) there are multiple equilibria. In a region of (r, c) space with multiple equilibria, we compare these, and analyse their "efficiency" in several respects (stability and welfare).

Finally in (iii), based on an idea of Alpern and Reyniers (1999) and Alpern (2008) we analyse the equilibrium strategies in a steady state model where individuals have age dependent preferences and they seek partners who provide to them the longest possible common fertile life. We determine the equilibrium strategies as the sex ratio of the incoming population changes and comment of the efficiency of equilibria.

Statement of Originality

All the work is mine alone, except sections 3.2 and 3.4. Section 3.2 is a slightly modified version of the LSE-CDAM-2007-32 research report "Equilibria of two - sided matching games", presenting joined work with Steve Alpern on common preferences. Section 3.4 also presents joint work with Steve Alpern and it is going to be the subject of a joint paper with him.

Acknowledgement 1 *During the last four years I learned a great deal about game theory and research. I would like to express my gratitude to all those who gave me the possibility to complete this thesis.*

I would first like to thank my supervisor Professor Steve Alpern for his constant advice and support, his enthusiasm and his patience during all the years I was a research student. Without his encouragement, inspiration, guidance, stimulating suggestions and his great efforts to explain things in a clear and simple way, I would not have been able to complete this thesis. I am very grateful I wrote this thesis under his supervision.

I am also grateful to my second supervisor Dr John Howard for all his help, especially during the last stages of the thesis. I am indebted to him for his useful and sound advice and comments and his kind assistance with writing the thesis.

I would also like to thank my parents and my sister Maria who stood to my side all the years I was a research student, encouraged me, supported me and showed great patience, love and understanding during all the stages of my PhD.

May thanks as well to my friends, who showed understanding and support during all the years I have been in LSE and remained my friends. They helped me relax and put things into perspective. Special thanks to my little friend and flatmate Amrit for his love and support and for reminding me that things are not too serious, especially during the last stages of the PhD.

I am also grateful to my fellow research students and the staff of both the Department of Mathematics and the Operational Research Group in LSE, for providing a pleasant environment where I could work. I would also like to thank my students in Operational Research and in Mathematics, who made my days in LSE to be fun.

Finally, I would like to acknowledge the financial and academic support of London School of Economics and its staff, particularly in the awards of a Maths Doctoral Training Grant and a Postgraduate Research Studentship in Operational Research.

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1 Introduction

The forming of couples and partnerships is a common occurrence in everyday life. Individuals form couples for multiple reasons and in a variety of contexts: animals and people search for mates to reproduce; people need partners to exchange goods or information, to play games, to socialise or to share experiences; companies or organisations seek individuals or other organisations to collaborate, to create partnerships, to make social or commercial transactions. Mating of individuals is a continuously recurring process and it has a strong impact on the decisions and relationships of both humans and animals. Therefore it does not come as a surprise that many disciplines try to explore the dynamics of mate selection. Mating has been analysed extensively by biologists, social scientists and economists, all trying to understand its underlying mechanisms. Its analysis permits us to explore, influence or even improve the products of the mating process. Furthermore, it allows us to make forecasts and analyse the effects of the mating behaviour of individuals on themselves, their partners and their opponents.

Depending on the availability of data and the complexity of the situation, different techniques have been used in order to study mate selection. Observation, laboratory and field experiments and simulation are extremely common, especially in the analysis of sexual selection (Kalick and Hamilton (1986), Breeching and Hopp (1999), Kodric-Brown and Nicoletto (2001)). In addition, especially lately, a plethora of game theoretical models has been developed, analysing mating behaviour in a variety of contexts. It is interesting to note that Johnstone (1997) proposes the use of game theoretic analysis when both sexes are selective because it can model the inter-dependency of the strategies of the two sexes at equilibrium.

We focus on the game theoretical study of the mating procedure, analysing mating as a dynamic situation. Game theory investigates mate selection as a game played between two populations. For simplicity, we call the two populations males and females. The aim of the game is the formation of couples and the game outcome is a couple pattern (mating structure). We use the term "pairing" or "matching" to denote the random meeting of two unmated individuals and the term "mating" to denote the permanent coupling of two individuals.

Players are characterised by "types". Types are a measure of the quality of players. They may represent quality, size, colour, hue, age etc.

The utility of a mating (x, y) between a male of type x and a female of type y is determined by the types of both partners and may be different for male and female. Hence the payoff that players expect to receive depends on the probability of finding a partner by the end of the game, on their type and on the type of their potential partner.

Game theory studies the strategies that players use in order to maximise their payoffs. It investigates the effect of competition and population distribution on players' decisions and examines the formation of choices and strategies under different circumstances. Additionally, it provides a better understanding of the effect of preferences, types and utilities on the mating behaviour of individuals. Therefore, it has proved to be a very useful and flexible tool in the examination of mating process and it is extensively used by different disciplines.

1.1 Main Background: The Alpern and Reyniers "Homotypic" and "Common" Preferences Models (1999, 2005)

Kalick and Hamilton's (1986) simulation motivated Alpern and Reyniers (1999, 2005) to produce two papers focusing on the use of game theoretical analysis for the prediction of behaviour of humans/animals under homotypic or common preferences. Under homotypic preferences individuals prefer partners similar to themselves, while under common preferences individuals prefer partners of a high rank or of a specific characteristic. Alpern and Reyniers analysis of both type of preferences has applications in biology and ecology, as well as in social sciences and in some economic problems. Their work has provided us with the theoretical framework to develop our own analysis.

Hence, we regard it as necessary to present briefly Alpern and Reyniers' results, before illustrating our own models. This will help the reader to understand the ideas on which our models were built. (A description of the Kalick and Hamilton (1986) model can be found in chapter 5.)

1.1.1 Alpern and Reyniers (1999, 2005)

Alpern and Reyniers constructed a decentralised dynamic game, where cohorts of males and females meet randomly in sequential periods and form couples only when they both accept each other. Following the assumptions of Kalick and Hamilton (1986), they assumed that the sex ratio in the beginning of the game is $1:1$. They did not permit remarrying or divorcing. Therefore mated individuals in each period leave the game, without being given the opportunity to enter the game again; otherwise they remain in the game and continue in the next period unmated. All players are better off by being mated than by remaining unmated after the end of the game.

Alpern and Reyniers used a notion of equilibrium similar to a Subgame Perfect Nash equilibrium in a continuum of players. At equilibrium, given players' utility functions, players accept those types which provide them with at least as high utility as their expected utility in the next period.

They constructed two basic models; in the first (1999), they assumed that players seek partners similar to themselves (homotypic or similarity preferences) and in the second (2005), that players search for high quality partners (maximising or common preferences).

- *Alpern and Reyniers (1999)*

Alpern and Reyniers analysed a model with no replacement played over n periods. In the beginning of the game, types are uniformly distributed in a continuous interval $[-1,1]$ or a discrete interval $\{\frac{-m}{2m+1}, \frac{-m+1}{2m+1}, ..., \frac{1}{2m+1}, 0, \frac{1}{2m+1}, ..., \frac{m-1}{2m+1}, \frac{m}{2m+1}\}$. Two kinds of symmetry are assumed: symmetrical behaviour of males and females and of negative and positive types. The cost of a mating (x,y) is the same for both partners and it is the absolute distance $|x-y|$ between the two types x, y. Players are minimisers, seeking partners close to their own type.

Alpern and Reyniers found equilibrium conditions for the continuous type game for any initial type distribution. Specifically in the continuous type game where types are initially uniformly distributed, they partly determined the equilibrium strategies. Furthermore, they calculated numerically the equilibrium for a discrete type game.

In the same paper, they also constructed a similar model with replacement (allowing a fixed number of individuals to enter the game in each period). They proposed, without analysing it, that it would be interesting to analyse another model with replacement where "preferences over mates include the number of periods since mate has entered the model". Alpern (2008) proposed an age preferences model based on this idea, assuming that the sex ratio is $1:1$.

- *Alpern and Reyniers (2005)*

Alpern and Reyniers (2005) examined the effect of common preferences on the equilibrium behaviour. They studied a model with no replacement played over n periods. Types are non negative and they are uniformly distributed in a continuous interval $[0,1]$ or a discrete interval $\{0, \frac{1}{b}, ..., \frac{b-1}{b}, 1\}$. The symmetry around the central type 0 is no longer important for the analysis. The utility of a mating (x,y) for a type x is the type y of his partner (respectively for y it is type x). Players are maximisers, seeking high type partners.

Alpern and Reyniers showed that in the n period model there always exists an equilibrium. Furthermore they proved that strategies are decreasing sequences (of lower bounds of acceptable partners) over time.

They focused on the two and three period model, where types are uniformly distributed in $[0,1]$. They showed that when when the sex ratio is $1:1$, there exists a unique symmetric equilibrium.

Finally, they analysed the effect of learning on the equilibrium behaviour.

1.2 Brief Presentation of our Models

Our work extends and in some cases generalises Alpern and Reyniers' models (1999, 2005) and Alpern's model (2008). We present three basic models and some extensions of them. Our interest is focused on the existence and differences of multiple equilibria, as well as on the effect of a skewed population sex ratio on the equilibrium behaviour.

1.2.1 Extension of Alpern and Reyniers (1999)

In chapter 2 we extend the analysis of the homotypic preferences models of Alpern and Reyniers (1999) and offer some additional computational results on a discrete type game with mixed preferences (combination of homotypic and common preferences). The majority of the results concerning homotypic preferences appearing in this chapter are presented in LSE-CDAM-2006-20 research report "Multiple equilibria in a Dynamic Mating Game with Discrete Types and Similarity Preferences".

Alpern and Reyniers analysed only the equilibrium behaviour of a limited range of types of the two period model where types are uniformly distributed in $[-1, 1]$. We extend their results, exploring the behaviour of the remaining types. This analysis is presented in Alpern, Katrantzi, Reyniers (2005).

Next we examine the equilibrium behaviour in the discrete type two period game. We find only symmetric equilibria, even in the cases where we relax the assumptions of symmetry of Alpern and Reyniers (1999) and permit to players to behave independently. In the case where symmetry is assumed between negative (types in $[-1, 0)$) and positive types (types in $(0, 1]$), and between males and females, we develop an algorithm which reduces the number of strategies needed to be checked in order to find all existing equilibria. Using this algorithm, we are lead to the most important finding of this chapter, which is that multiple equilibria exist in some discrete type games. The latter result has not yet been encountered in any of the models presented in the relevant literature, where it is implicitly assumed that equilibrium is unique. In all the discrete type games examined, we compare the intra-couple correlation and the marital stability of the equilibria found. (We say that a pair of couples (x_1, y_1) and (x_2, y_2) is marital unstable if a male from one couple and a female from the other both prefer each other to their current partner.) Furthermore, we calculate the expected payoff that each equilibrium offers to different types of players, drawing conclusions on the effect of the number of types on the equilibrium strategies.

In chapter 3 we also study how strategies and mating patterns would change in a model where players have a combination of homotypic and common preferences, as the impact of either preference becomes stronger. We present our computational results on a discrete type game played over 2 and 3 periods. Our conclusions for both two and three period models are similar. Higher types find a mate more quickly when common preferences are stronger. On the other hand, when players have strictly homotypic preferences, extremely high and low types have slightly lower probabilities that the rest of the types to find a partner before the last period. In general, a random player has higher chances to be mated before the final period of the game when homotypic preferences are stronger. Focusing especially on the three period model, it becomes obvious that under all combinations of preferences, the probability that two random types mutually accept each other is higher in the second period than in the first period. An exception is the case where players have purely homotypic preferences, since in this case mating takes place between the same types in both periods.

1.2.2 Extension of Alpern and Reyniers (2005)

In chapter 3, we investigate in detail the effect of common preferences on equilibrium behaviour when the sex ratio is not 1 : 1. The second section of the chapter is a modified version of the LSE-CDAM-2007-32 research report "Equilibria of two-sided matching games", presenting our joint work with Steve Alpern on common preferences. A shorter version of this report is accepted by the European Journal of Operational Research (Alpern and Katrantzi (2008). In the same chapter, we also explore the effect of the equilibrium strategies on social and individual welfare. Finally we comment on how the model could be amended so that it is more applied in biological problems.

In the second section of chapter 3, we extend the mutual choice model with common preferences of Alpern and Reyniers (2005) by assuming a male biased sex ratio r and a penalty c for players (males) who leave the game unmated. We offer analytical results for the 2 and partially for the three period model. We prove that under specific conditions males are better off by being unchoosy, even when they are given the chance to be selective, offering an answer to the question of "when do we have mutual and when do we have one sided choice?". However, the most significant result of this chapter is the existence of multiple equilibria for specific combinations of sex ratio r and non-mating penalty c. In the two period game, we divide the (r, c) space into three regions, I, II and III. We prove that in the first two regions, a unique equilibrium exists. However, in region III there are always three equilibria: two mutual choice and one female choice equilibrium. At mutual choice equilibria, both sexes are choosy, while at the female choice equilibrium, males are unchoosy and accept all female types.

We prove that at equilibrium choosiness moves in the same direction for both sexes. Hence in area III, we call "choosy" the mutual choice equilibrium where both males and females have high acceptance standards and "easy" the mutual choice equilibrium where both sexes have lower acceptance standards. The existence of multiple equilibria seems to be a quite robust result, since the exhaustive analysis of the three period discrete type problem and the non exhaustive analysis of the 4 period discrete type problem revealed the existence of multiple equilibria under specific combinations of (r, c).

We are particularly interested in the comparison of the "efficiency"of the equilibria in region III, studying their dynamical stability and their marital stability. We suggest that both choosy and female choice equilibria are dynamically stable, while easy equilibrium is dynamically unstable. Our initial results on basins of attraction in area III indicate that the choosy equilibrium has a significantly larger attracting area and hence it might be more likely to appear than female choice equilibrium. However, this observation is based on specific cases and it should be examined further. Furthermore, we note that the choosy equilibrium has the highest marital stability, while the female choice equilibrium has in all cases a marital stability of 0.5, which is the lowest possible. So, it is expected that when players are choosier, they have higher probabilities to be content with their partner.

In section 3.3, we complete the analysis of the "efficiency" of the equilibria in area III, by examining the social and individual welfare at equilibrium. Males are proved to be equally content at all equilibria, while females are better off at the female choice and worst off at the choosy equilibrium. The payoff of different female and male types varies at each equilibrium. As a result, specific types may prefer that different equilibria appear given their expected payoffs. High male, low male and high female types are better off at the choosy equilibrium, whilst medium male, medium female and low female types prefer the female or the easy equilibrium. This result, in combination with our results on stability, provides important information to researchers for making inferences on which equilibrium would appear under different circumstances.

The biological implications of our results on common preferences are multiple. The last section of chapter 3 focuses on the biological interpretation of our analysis. We amend our model by maintaining a biased sex ratio, but removing the non mating penalty and assuming that males and females types are uniformly distributed in $[h-\gamma, h+\gamma]$ and in $[l-\kappa, l+\kappa]$. Our goal is to examine how female variation in combination with a biased sex ratio affects the equilibrium behaviour. A complete analysis of the model is going to be the subject of a joint paper with Steve Alpern.

1.2.3 Age Preferences Model

In chapter 4, we focus on the role of age in the mating procedure. We analyse the equilibrium strategies in a steady state model where individuals have age dependent preferences. We assume that the age of an individual represents the periods he has been fertile and that there is a maximum age limit for male and females. In other words, individuals can be fertile for a given number of periods. Individuals seek partners so that to maximise the number of periods that both them and their partners are able to produce offspring.

We analyse the behaviour at equilibrium, given that it depends purely on age and on the incoming population sex ratio R. The incoming population sex ratio R is defined as the ratio of males over females entering in the game in every period.

Age dependent preferences is a subject that is not extensively studied, probably due to its complexity. It is often difficult to isolate the effect of age on mating behaviour, since age is connected to multiple factors. Steve Alpern (2008) built a model on age preferences, assuming a sex ratio $r = 1$, but he has not analysed the equilibria of this model. We generalise Alpern's model (2008), assuming that it is possible to have a biased population sex ratio r and we investigate the equilibrium strategies for different incoming population sex ratios R.

Under age preferences, it is expected that younger individuals have a greater incentive than older to find a young partner, since they have more years left to be mated or to search for a potential partner in the game. Hence, at equilibrium, a type is always at least as choosy as any higher type (of an older age). Types in the last two years of their life are always unchoosy, accepting any partner, while types in the first year of their life are accepted by all players.

We focus on the cases that females live for two and three years and males live for at least as many years as females. We investigate the equilibrium behaviour when the incoming population sex ratio R is male or female biased. In the cases we examine, it is shown that when the incoming population of males is the same as the incoming population of females in each period, the equilibrium is trivial and all players are mated in the first period of their life, unless they are not paired with a partner. Nevertheless, when R is biased, more interesting equilibrium type distributions appear. In all the models we examine, there is a unique equilibrium for any R; the exception is the case where males live for 4 periods and females live for 3 periods, where there is no equilibrium for some male biased R. In the model where males live for 4 periods and females live for 3 periods, it is interesting to note there is a pair of equilibria where the population sex ratios r are the same.

In the last part of chapter 4, we examine the population productivity; it is proved that it is the maximum when the incoming population sex ratio R is $1 : 1$, while it tends to decrease as the incoming population sex ratio becomes biased. Moreover, we analyse the average age at marriage of males and females. It is shown that it is the minimum when the incoming population sex ratio R

is 1 : 1, while it increases as the incoming population sex ratio becomes biased.

After the reader becomes familiar with our analysis on homotypic, common and age dependent preferences, we consider that it is easier to make connections with other models, related to ours. In chapter 5 we briefly present the area of mate selection, focusing on game theoretical analysis. It is important to note that most models are based on very different assumptions that ours and comparisons are difficult to be made. Finally in chapter 6, we offer a summary of our results and comment on possible extensions of our models.

1.3 Summary

In all the models we examine, we are particularly concerned about the role of preferences and outside constraints (such as the sex ratio) on the equilibrium behaviour. Our goal is to contribute to the exploration of the area of dynamic mate choice and to assist the analysis of real life problems. We are especially interested in the implications of our results for operational research problems and in biology. As part of our effort to offer a complete description of the mating process and its products, in all the cases we examine, we investigate the stability and effectiveness of the equilibria as well as the mating pattern of the couples created. We hope our results help to lead to a better understanding of mate choice and encourage the advance of research into how mating decisions are formed.

The dissertation is organised as follows. In the next chapter, we present our results on homotypic preferences and on combination preferences, before illustrating our analysis on an asymmetric game with common preferences in chapter 3. Next, in chapter 4, we present a steady state model, where players have age preferences. We examine the role of age and of asymmetry in the incoming population sex ratio on the formation of equilibrium strategies and distribution. In chapter 5, we offer a discussion on related approaches in the literature. This chapter can be considered as a presentation of the general area of mating games or as an examination of the connections of other models with ours. Finally, in the last chapter, a summary of our conclusions and comments is given, and proposals for further research are made.

2 Homotypic Preferences

2.1 Introduction

In this chapter we focus on games with homotypic preferences and we present some additional results on a game with mixed preferences (a combination of homotypic and common preferences). The majority of the results concerning homotypic preferences were first presented in LSE-CDAM-2006-20 research report "Multiple equilibria in a Dynamic Mating Game with Discrete Types and Similarity Preferences". The report appears in the website of the LSE Mathematics department[1].

Under homotypic or else similarity preferences, individuals prefer partners similar to themselves. Homotypic preferences are the least examined in literature, even though they appear frequently in real life (Alpern and Reyniers (1999), Belgstrom and Real (2000)). Their analysis has particular interest since it has multiple applications in the social, economic and natural worlds.

In biology and in ecology there is evidence that a variety of animal species, especially monogamous, illustrate homotypic preferences (Cooke and Davies (1983), Amundsen (2000), Arnqvist et al (1996), Barlow and Rogers (1978)), searching for partners having the same traits, such as colour and age, as themselves or their parents. In psychology, there are also findings suggesting that humans tend to define their criteria of beauty according to their own traits (Yela and Sangrador (2002)) and they often prefer partners who have common characteristics to themselves (Russel and Bartrip (1989), Alvarez and Jaffe (2004)). Furthermore, in economy and in social sciences, similarity preferences appear in different contexts. People search for competitors in sports and arts who have the same abilities as themselves; flatmates need to have the same expectations for the flat they share; partners who work together in a firm often need to have the same goals.

Alpern and Reyniers (1999) were the first to built a game theoretic model on homotypic preferences; we base our analysis on the dynamic decentralised non atomic mating games Γ_n and $\Gamma_n(m)$ initially presented by them. In the Alpern and Reyniers (A&R) game, two populations X (males) and Y (females) are randomly paired over n successive periods. Players have one dimensional types which are uniformly distributed over a continuous or a discrete interval (in Γ_n or in $\Gamma_n(m)$ respectively), which includes the origin 0 and is symmetric around it; hence for every positive type, there exist an equivalent negative type. Symmetric positive and negative types are taken to have symmetric preferences and behaviour. No new players can enter the game in any period, hence the population density of each type changes each time new couples are formed.

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[1] http://www.cdam.lse.ac.uk/Reports/Files/cdam-2006-20.pdf

In each period after pairing takes place, each party of a matched pair (i, j) can either accept or reject the other. If both accept, then they form a mated couple and leave the game, with both paying a cost of $|i - j|$. Otherwise, they both proceed unmated into the next period. At the end of the game, all players prefer to be mated than to remain unmated. Even though every pairing has the same cost $|i - j|$ for both the matched male i and female j, it is not always true that when a female (male) accepts a male (female), the male (female) will also accept the female (male). The mating decisions of a player in a period depends on his/her expected cost in the following periods. The expected cost differs for each type, since the type distribution changes in every period. As a result, it is possible that a type i is choosier in a period than an other type j and not willing to accept the latter, even though j is willing to be mated with i. In general, central types are choosier, since they have more chances to meet someone close to their own type.

In Alpern and Reyniers (1999) and in the relevant literature, it seems to be implicitly assumed that there exists a unique equilibrium, since in the examples examined only one equilibrium is presented. Hence the search for multiple equilibria comes as a natural question. The basic contribution of this chapter is to prove the existence of multiple equilibria in some discrete type games. Specifically, it is found that when there are $m = 3$ or $m = 5$ positive types (hence 7 or 11 types in total respectively), there exist 3 equilibrium strategies in the two period game. The existence of multiple equilibria, seems to be a fairly robust result; computer search, which was not exhaustive, shows that the existence of multiple equilibria is possible also when the number of periods is larger. For instance, it was found that at least two equilibria exist in the four period model when there are $m = 8$ positive types (hence 17 types in total).

In the discrete type game, the total number of strategies is always finite. Nevertheless, as the number of types becomes higher, it is too costly to check all possible strategies to pick out the equilibria. We present a method for reducing the number of strategies needed to be checked in order to find the equilibrium profiles in the discrete type game $\Gamma_2(m)$. Furthermore, we provide an overview of the equilibrium strategies used in Γ_2, expanding the analysis of the continuous type Γ_2 game of Alpern and Reyniers.

Unlike the original Alpern and Reyniers paper, we examine as well the equilibrium behaviour without assuming that females and males and negative and positive types behave in a symmetric way in the two period discrete type game. However, even when negative and positive types and males and females have the opportunity to use variant strategies, we observe only symmetric equilibria. Hence only equilibria symmetric over the origin 0 and symmetric for males and females exist even when the symmetry assumption is not embedded in the model.

In the next chapter we will analyse the equilibrium behaviour when players have common preferences. Before that, it is interesting to explore how the equilibrium behaviour is affected when players have a combination of common and homotypic preferences. The final section of this chapter is devoted to the analysis of the equilibrium behaviour when individuals have a cost when choosing mates who are not close to them, but simultaneously search for partners of a high type. This model could describe the case, when a tennis player needs a competitor close to his ability, but at the same time he is interested in a competitor of a higher ability in order to have the opportunity to exercise and improve his skills. Another example could be a setting where a buyer of a car needs a car that has a price covered by his budget but on the same time he is fascinated by an expensive car of a higher quality. Similar preferences may be displayed by humans who seek partners close to their type, but at they same time they believe they are better off with partners who have the best possible fitness.

We analyse a discrete type game, played over two or three periods. Our goal is to examine how the equilibrium strategies "evolves" and also how the couple pattern and the acceptance range of each type change, as preferences vary. It seems that at equilibrium players are the most choosy when they have homotypic preferences and the least choosy when they have mixed preferences, giving the same weight to maximise the type of their partner and to minimise the cost of a mating. The number of couples formed before the last period is the highest when there is a combination of common and homotypic preferences, but the importance of homotypic preferences is significantly higher.

This chapter is organised as follows. In the first section we illustrate the symmetric continuous type game Γ_n of Alpern and Reyniers and describe the equilibrium strategy for all types in Γ_2. Next, we focus on the corresponding discrete type game $\Gamma_n(m)$, where we assume having m positive types, m negative types and a type 0. We describe a method for finding all the equilibrium strategies in $\Gamma_2(m)$, and then we present our numerical results for the two period game when there are $m = 1$ to $m = 9$ positive types, analysing the equilibrium strategies, their effectiveness (in reducing mean intra-couple type difference) and their stability. Finally, we discuss the existence of multiple equilibria in games of n periods (for $n > 2$), and give an example of the existence of multiple equilibria in the 4 period game $\Gamma_4(m)$. In the last section we explore the relationship between similarity and common preferences and seek to see how the equilibrium changes as the importance of the latter increases.

2.2 Description of the Non Atomic Mating Alpern and Reyniers Continuous Type Mutual Choice Game Γ_n

In the Γ_n mutual choice game, two identical populations of individuals, of known one dimensional types, are uniformly distributed in an interval $[-1, 1]$. Individuals have similarity preferences as described before (preferring potential partners having types close to their own) and the distance between the individuals' types $|x - y|$ is a measure of the cost of a mating of an individual of type x with an individual of type y. We assume that there exist a continuum of players.

In every period, individuals are randomly paired and they form a couple if they both accept each other; otherwise they move on to the next period, where a random matching takes place again. The process continues until period n, where all matchings are mutually accepted and form mated couples.

A strategy is defined as an acceptance rule indicating the maximum distance $s_k(x)$ acceptable in period k to a type x individual. That is, a type x male will accept a type y female in period k if

$$s_k(x) \geq |x - y| \tag{1}$$

Both sexes are assumed to use the same strategies in this symmetric continuous type model. Hence a type x male is going to use the same strategy $s_k(x)$ as a type x female in every period k.

When the population is uniformly distributed between $[-1, 1]$, types x and $-x$ are also assumed to share the same strategy.

$$s_k(x) = s_k(-x) \tag{2}$$

It follows from (1) that in period k, given that type x uses strategy $s_k(x)$ and type y uses strategy $s_k(y)$, a mating occurs if and only if

$$|x - y| \leq \min\{s_k(x), s_k(y)\} \tag{3}$$

The expected cost for an individual of type x to enter period k unmated is denoted by $C_k(x)$; this depends on the strategy s, so $C_k(x) = C_k(s, x)$.

At equilibrium, no player can decrease his expected cost by using other than the equilibrium strategy $\hat{s}$, given that the rest of the population is using strategy $\hat{s}$. Hence at equilibrium

$$\hat{s}_k(x) = C_{k+1}(\hat{s}, x), \text{ for } 1 \leq k \leq n-1 \text{ and } -1 \leq x \leq 1 \tag{4}$$

If F is the normalised cumulative probability function of types y in the last period, the expected cost for a type x entering in the last period is

$$C_n(x) = \int_{-1}^{1} |x - y| \, dF(y)$$

Before presenting the $\Gamma_n(m)$ game and illustrating our results, it is important to mention two theorems of Alpern and Reyniers, which are important for our analysis. In these theorems it is assumed that F can take any form, hence the theorems can be applied also in the case that F is a discrete function as in $\Gamma_n(m)$.

Theorem 2.1 (of Alpern and Reyniers (1999)) *If F denotes the final period cumulative probability distribution, then the final period cost function C_n is a symmetric convex function, minimised at 0, with the following properties:*

$$\begin{aligned} C_n(-1) &= C_n(1) = 1 && (5)\\ C_n'(-1) &= -1,\ C_n'(1) = 1 \text{ and } C_n'(0) = 0 && (6)\\ C_n'(x) &= 2F(x) - 1 \end{aligned}$$

Theorem 2.2 (of Alpern and Reyniers (1999)) *In a two period model, at equilibrium, if $x_0 < x_1 < x_2$, and x_0 accepts x_2, then x_1 also accepts x_2.*

Since C_n is increasing in $|x|$ by Theorem 2.1, the choosier individuals tend to be nearer to the centre. As we move away from the centre, types become less choosy, with the extreme types accepting every type between themselves and the middle types. Combining Theorem 2.1 with Theorem 2.2, we can show that in the two period model, central types tend to do better than extreme types, as they have more chances to be accepted in the first period. This fact is verified in the analysis of the discrete two period game in section 2.7.

2.3 Extended Alpern and Reyniers Two Period Model

Alpern and Reyniers focus on the two period mutual choice matching game, where the populations are uniformly distributed in an interval $[-1, 1]$. In the two period model, a mating occurs in the first period if and only if there is mutual acceptance between the paired individuals; thus an individual x is mated in the first period if and only if he is paired with an individual y in the set A_x described below.

$$A_x = \{y : |x - y| \leq \min(s(x), s(y))\} \qquad (7)$$

For simplicity reasons, since a strategy is defined only in the first period and a expected cost function is defined only in the second period, $s(x) = s_1(x)$ will denote the strategy of a type x in the first period and $C(x) = C_2(x)$ will denote the cost that a type x expects to get for entering in the last period unmated.

Let $p(x)$ be the probability that an individual of type x is mated in the first period; then

$$p(x) = \mu(A_x) = \mu(\{y : |x - y| \leq \min(s(x), s(y))\}) \tag{8}$$

where μ is the normalised Lebesgue measure on $[-1, 1]$. The pre-normalised population density at the beginning of period 2 is $(1 - p(x))$ and given $p(-x) = p(x)$ the total population Q is

$$Q = \int_{-1}^{1} (1 - p(x))d\mu(x) = \int_{0}^{1} (1 - p(x))dx$$

Therefore the normalised population density function at the beginning of period 2 is

$$F'(x) = \frac{1 - p(x)}{2Q}. \tag{9}$$

In order to define the equilibrium acceptance strategies, A_x and $p(x)$ have to be evaluated.

Let

$$x_U = \min\{x : x \textit{ accepts } 1\} = \min\{x : x + s(x) = 1\} \tag{10}$$

(x_U is the unique solution)

and

$$x_L = \max\{x : -x \textit{ accepts } x\} = \max\{x : -x + s(-x) = x\} \tag{11}$$

$$= \max\{x : -x + s(x) = x\}$$

(x_L is the unique solution)

We define

$$I_1 = [0, x_L], \quad I_2 = [x_L, x_U] \quad \text{and} \quad I_3 = [x_U, 1]$$

and

$$R(x) = R_s(x) = x + s(x)$$
$$L(x) = L_s(x) = x - s(x)$$

where $R(x)$ and $L(x)$ are the rightmost and leftmost types that x accepts under strategy s. Hence $R^{-1}(x)$ is the individual that is furthest to the left of individual x and accepts x.

Since everything is symmetric around 0, it is sufficient to analyse the behaviour of the non negative types; negative types are assumed to behave like positive types.

We know that for $x \epsilon [0, x_L]$, a type $x - s(x)$ always accepts x (as proved in Lemma $A.1$ in Appendix $A.1$)

Hence for $x \epsilon [0,1]$ we have that

$$A_x = \begin{cases} [L(x), R(x)] = [x - s(x), x + s(x)] & for\ x \in I_1 \\ [R^{-}1(x), R(x)], & for\ x \in I_2 \\ [R^{-}1(x), 1], & for\ x \in I_3 \end{cases}$$

and

$$p(x) = \begin{cases} s(x), & for\ x \in I_1 \\ \frac{x+s(x)-R^{-1}(x)}{2}, & for\ x \in I_2 \\ \frac{1-R^{-1}(x)}{2}, & for\ x \in I_3 \end{cases}$$

Types in I_3 are universal acceptors; if a type x accepts a type y in I_3, then y always accepts x as well. Therefore in order to determine the equilibrium behaviour, it is only needed to define the equilibrium strategy in I_1 and I_2.

At equilibrium,

$$\begin{aligned} s(x) &= C(x) \text{ according to (4)} \\ \Rightarrow s''(x) &= 2F'(x) \\ \Rightarrow s''(x) &= \frac{1}{Q}(1 - p(x)) \end{aligned} \tag{12}$$

Therefore,

(i) for $x \in I_1$ (Alpern and Reyniers(1999)),

$$s''(x) = \frac{1}{Q}(1 - s(x)) \tag{13}$$

where s is symmetric around 0 and $s(x) = s(-x)$, $s'(0) = 0$.

It is important to note that Q is underdetermined and it depends on the equilibrium acceptance strategy s.

Hence,

$$\begin{aligned} s(x, C) &= 1 - a\cos\left(\sqrt{\frac{1}{Q}}x\right) \\ &= 1 - (1 - s(0))\cos\left(\sqrt{\frac{1}{Q}}x\right). \end{aligned} \tag{14}$$

(ii) for $x \in I_2$,

$$s''(x) = \frac{1}{Q}\left(1 - \frac{x + s(x) - R_1^{-1}(x)}{2}\right) \tag{15}$$

where $R^{-1}(x)$ belongs in I_1or in I_{-1}, since

(i) $(x + s(x))$ is an increasing function

(ii) $-x_L + s(-x_L) = x_L$ and

(iii) we assume that $x_U \leq x_L + 2x_L$

Hence the acceptance strategy of $R^{-1}(x)$ is already known from (14). Moreover, $s(x_L)$ and $s\prime(x_L)$ are known from (14).

(iii) for $x \in I_3$,

$$\begin{aligned} s''(x) &= \frac{1}{Q}\left(1 - \frac{1 - R^{-1}(x)}{2}\right) \\ &= \frac{R^{-1}(x) - 1}{2Q} \end{aligned} \tag{16}$$

where $R^{-1}(x)$ belongs either to I_{-1} or to I_1 or to I_2, since

(i) $x + s(x)$ is an increasing function,

(ii) $-x_L + s(-x_L) = x_L$,

(iii) we assume that $x_U \leq x_L + 2x_L$,

(iv) $x_L + s(x_L) = 3x_L$ and

(v) $x_U + s(x_U) = 1$

Hence its acceptance strategy is already known from (14) or from (15). Furthermore it is known that $s(1) = 1$ and $s'(1) = 1$ from (5), since at equilibrium $s(1) = C(1)$.

The solution of the previous differential equations can be quite complicated. Alpern and Reyniers tried to approximate a possible equilibrium of the game by using a discrete approximation of the continuous model. They assumed we have $2m + 1$ continuous types. In order to keep the symmetry of the continuous model, each type i is taken to belong in the set I_m, where

$$I_m = \{-m, -m+1, -m+2, \ldots, -2, -1, 0, 1, 2, \ldots, m-2, m-1, m\}$$

Hence a type i corresponds to a type $\frac{i}{m}$ in the continuous model. They found the equilibrium strategy for $m = 40$ and they used it to calculate an approximated value for $s(0)$ and Q ($s(0) = 0.447$ and $Q = 0.51$). Using these results, they showed that the equilibrium strategy in the interval I_1 of the continuous type game is a good approximation of the equilibrium strategy of all types in the discrete case, therefore a fairly good approximation for the equilibrium strategy in all the intervals of the continuous type game.

If we take $s(x) = 1 - 0.447\cos(\sqrt{\frac{1}{0.51}}x)$, we have $x_L = 0.295$ and $x_U = 0.384$, as it is illustrated in Figure 2.1.

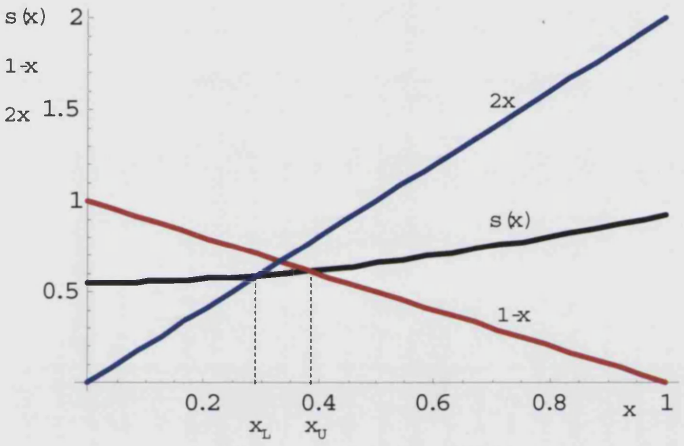


Figure 2.1: Plots of $1 - 0.447\cos(\sqrt{\frac{1}{0.51}}x)$, $1 - x$ and $2x$.

2.4 Discrete Type two-period Model $\Gamma_2(m)$

In the discrete type game $\Gamma_n(m)$, as in the continuous game Γ_n, cohorts of males and females are randomly paired for n periods. Nevertheless, as we saw in the previous sections, there are $2m+1$ types belonging in the set

$$I_m = \{-m, -m+1, -m+2, \ldots, -2, -1, 0, 1, 2, \ldots, m-2, m-1, m\}$$

The types are uniformly distributed so that at the beginning of the game the fraction of the population being of type i is $\frac{1}{2m+1}$. The requirement of mutual acceptance is maintained and the definition of a strategy s remains unchanged as in the continuous type model; $s_k(i)$ denotes the maximum distance that any type can have from i in order to be accepted by the latter in period k. It is important to note though that in the discrete type model, a strategy $s_k(i)$ is always an integer.

In the two period game $\Gamma_2(m)$ since a strategy is defined only in the first period and a expected cost function is defined only in the second period, $s(i) = s_1(i)$ denotes the strategy of a type i in the first period and $C_d(i) = C_{d_2}(i)$ denotes the cost that a type i expects to get for entering in the last period unmated, as the C in the continuous type two period game.

At equilibrium, the strategy of a type i in $\Gamma_2(m)$ is equal to the floor function of the expected cost for i in the last period. Hence

$$s(i) = \lfloor C_d(i,s) \rfloor \text{ and } C_d(i,s) - 1 < s(i) \leq C_d(i,s) \tag{17}$$

where $C_d(i,s)$ is the cost that i expects to get if it enters in the second period unmated, when s strategy is used.

As a result, even though two types may not have the same expected cost for entering unmated in the last period, they can have the same strategy. For instance, if $C_d(i,s) = 2.1$ and $C_d(j,s) = 2.9$, then $s(i) = s(j) = 2$.

It is necessary to note that Alpern and Reyniers consider that males and females use the same strategies and that positive and negative types behave in the same way as well; we maintain this assumption in the next two sections. Nonetheless, we also examined how the strategies change if we removed this assumption, allowing the players to act independently, and calculated the equilibria in the discrete type game for m up to 5; our results remained the same as the one found in section 2.7. The exhaustive research showed no difference in the equilibria; only symmetric equilibria were found.

2.5 Properties of Equilibrium Strategies in $\Gamma_2(m)$

Alpern and Reyniers found an equilibrium strategy in $\Gamma_2(m)$ by seeking fixed points of a best response function on the strategy space. Starting with any strategy s, they used the iterative method in order to find a new strategy $\tilde{s}$, whose value function has the same floor for all types as $\tilde{s}$. This iterative procedure in general may miss some equilibria - especially the ones corresponding to repelling fixed points. To ensure that we find all the equilibria, we must adopt a more thorough method; hence for m positive types, in order to be certain we find all the equilibria, it is sufficient to check all $\frac{(2m+1)!}{(m-1)!}$ possible strategies, examining which ones satisfy the equilibrium condition (17). However, as the number m of types increases, this method becomes extremely time consuming and the need to rule out some of the strategies becomes apparent. Theorem 2.1 can be used to reduce the number of eligible equilibrium strategies.

Before enumerating the properties that an equilibrium strategy has according to Theorem 2.1, it is important to stress that even if the expected cost function

C_d is strictly convex, it does not follow that its floor approximation strategy s is also strictly convex. This is due to the fact that a strategy $s(i)$ of a type i is always an integer, but the corresponding cost $C_d(i)$ does not have to be an integer and it can be larger than $s(i)$; $s(i) \leq C_d(i) < s(i) + 1$. For example, a cost function $C_d = (1.99, 2.1, 2.4, 2.9)$ is strictly convex, while its corresponding strategy $s = (1, 2, 2, 2)$ is not. However, we will show, that s must be "almost convex", in senses that we make precise in Theorem 2.3.

Lemma 2.1 *If $s(i) - s(j) \geq 2$, then $C_d(i) - C_d(j) > 1$.*

Proof. By (17), $C_d(i) \geq s(i)$ and $C_d(j) < s(j) + 1$.

Hence $C_d(i) - C_d(j) > s(i) - s(j) - 1 \geq 1$. ■

Lemma 2.2 *If $C_d(i) - C_d(j) \geq 1$, then $s(i) - s(j) \geq 1$.*

Proof. By (17), $C_d(i) < s(i) + 1$ and $C_d(j) \geq s(j)$.

Hence $s(i) - s(j) > C_d(i) - 1 - C_d(j) \geq 0$.

Since $s(i)$ and $s(j)$ are both integers, it follows that $s(i) - s(j) \geq 1$. ■

Given Lemmas 2.1 and 2.2, we will determine the properties of the floor function of a convex function.

Theorem 2.3 *Let $s = \lfloor C_d \rfloor$ be an equilibrium strategy of $\Gamma_2(m)$, with corresponding cost function C_d. Then s is of the form $s = (s(0), s(1), s(2), ..., s(m))$, where $s(i)$, for $i \geq 0$, denotes the strategy that a type i uses at equilibrium. Then,*

(i) $s(m) = m$

The extreme types m and $-m$ are universal acceptors, accepting all non negative and all non positive types respectively.

(ii) For all i, $s(i+1) \geq s(i)$.

A strategy s has to be a non decreasing list.

(iii) For all $i \geq m-2$, if $s(i+1) - s(i) \geq 2$ then $s(i+2) - s(i+1) \geq s(i+1) - s(i) - 1$.

(iv) For all $i \geq m-4$, if $s(i+2) \geq s(i) + 2$ then $s(i+4) > s(i+2)$.

Proof. (i) This is an immediate consequence of Theorem 2.1, since we have $s(m) = \lfloor v(m) \rfloor = m$ according to (5). Same way, for $-m$.

(ii) From (5) and (17), s is the non decreasing floor function of a non decreasing function C_d, hence non decreasing.

(iii) Suppose that s is an equilibrium strategy, violating this condition. Then for some $l \geq 2$, we have $s(i) = a$, $s(i+1) = a + l$ and $s(i+2) = a + 2l - 2$. For strategy s to be an equilibrium strategy then from Theorem 1, it has to be true that

$$C_d(i+1) - C_d(i) \leq C_d(i+2) - C_d(i+1)$$

By (17), we have

$$\begin{aligned} &C_d(i) \in [a, a+1) \\ &C_d(i+1) \in [a+l, a+l+1) \\ &C_d(i+2) \in [a+2l-2, a+2l-1) \end{aligned}$$

Consequently

$$l-1 < C_d(i+1) - C_d(i) < l+1$$

and

$$l-3 < C_d(i+2) - C_d(i+1) < l-1$$

Hence $C_d(i+1) - C_d(i) > C_d(i+2) - C_d(i+1)$, which contradicts our initial claim.

(iv) By Theorem 2.1, $C_d(i+4) - C_d(i+2) \geq C_d(i+2) - C_d(i)$. But by Lemma 2.1, if $s(i+2) \geq s(i)+2$ then $C_d(i+2) - C_d(i) > 1$. Hence $C_d(i+4) - C_d(i+2) > 1$.

It follows by Lemma 2.2 that $s(i+4) - s(i+2) \geq 1$ and therefore we have $s(i+4) > s(i+2)$. ■

Having determined the properties of the equilibrium strategies, it is then easy to identify the potential equilibrium strategies and check which ones satisfy the equilibrium condition (17). In order to do so, given a potential equilibrium strategy s, we first have to find the set of potential types j with which every type i can be mated. Then, given the initial type distribution, we can calculate the probability of each type i remaining unmated and moving to the last period. This probability helps us next to determine the new type distribution in the last period and consequently the expected cost $C_d(i)$ for each type i entering the second period unmated; if $s(i) = \lfloor C_d(i) \rfloor$ for every type i, then s is proved to be an equilibrium strategy. The algorithm for identifying equilibrium strategies is illustrated in Figure 2.2.

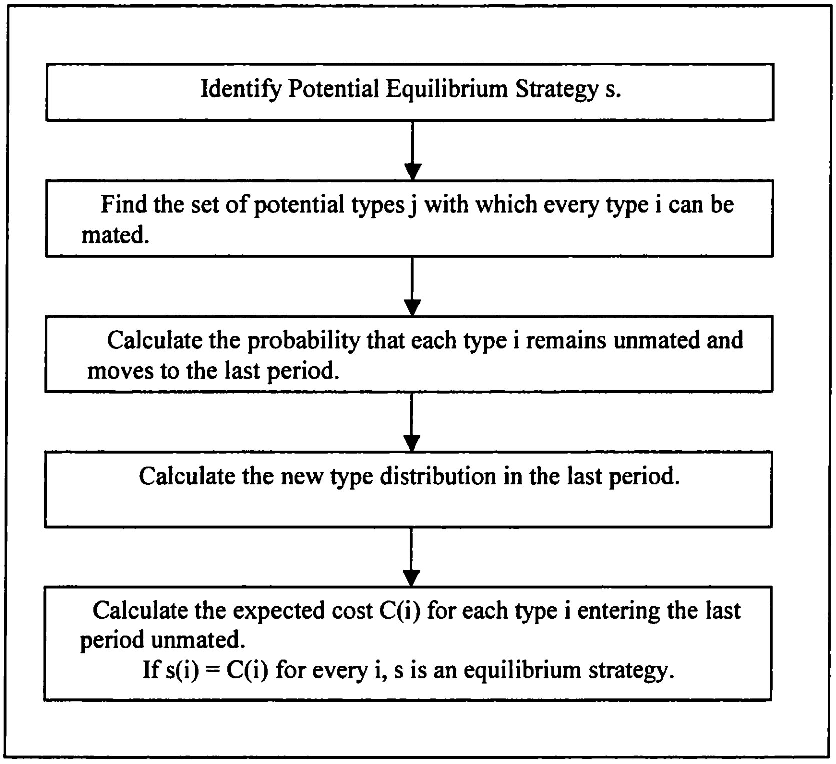


Figure 2.2: Algorithm for identifying equilibrium strategies

2.6 Analysis of the Two Period Mutual Choice Discrete Type Game $\Gamma_2(m)$ with $m = 0$ to $m = 9$ Positive Types

We applied the algorithm described in Figure 2.2 to identify all the equilibrium strategies in $\Gamma_2(m)$ for $m = 1$ to $m = 9$ positive types. Next, we calculated the probability to be mated in the first period if the equilibrium strategies are used, and the expected cost for any type i in each period and in total; finally we computed the intra-couple correlation and the inter-couple stability (marital stability) in each case.

The most important result of our analysis is the existence of 3 equilibrium strategies when there are $m = 3$ and $m = 5$ positive types, since multiple equilibria had not been noted in $\Gamma_2(m)$ till now. Nevertheless, the equilibrium analysis in the models where there is a unique equilibrium is also interesting since it helps us to get a better insight into the game itself. Connections between the strategy, the probability of being mated and the mating cost are made in the following sections. Furthermore the role of the number of types is studied. We find that the correlation is not affected by the number of types in the game, while the inter-couple (marital) stability seems to be inversely correlated with the number of types.

2.6.1 Equilibrium Strategies

In all the examined games there exists an odd number of equilibria, ranging from 1 to 3, a fact that is in accordance with general theory. In the majority of the games (for $m = 1, 2, 4, 6, 7, 8, 9$) there exists a unique equilibrium strategy. Nevertheless, when there are $m = 3$ and $m = 5$ positive types, there exist 3 equilibrium strategies. The equilibrium strategies for all the games are listed in Figure 2.6. A strategy denotes the maximum distance that each type i accepts in the first period. For example, when $m = 4$, at equilibrium we have $s(1) = 2$, hence type 1 accepts types i, such that $-1 \leq i \leq 3$ and type -1 accepts types j such that $-3 \leq j \leq 1$.

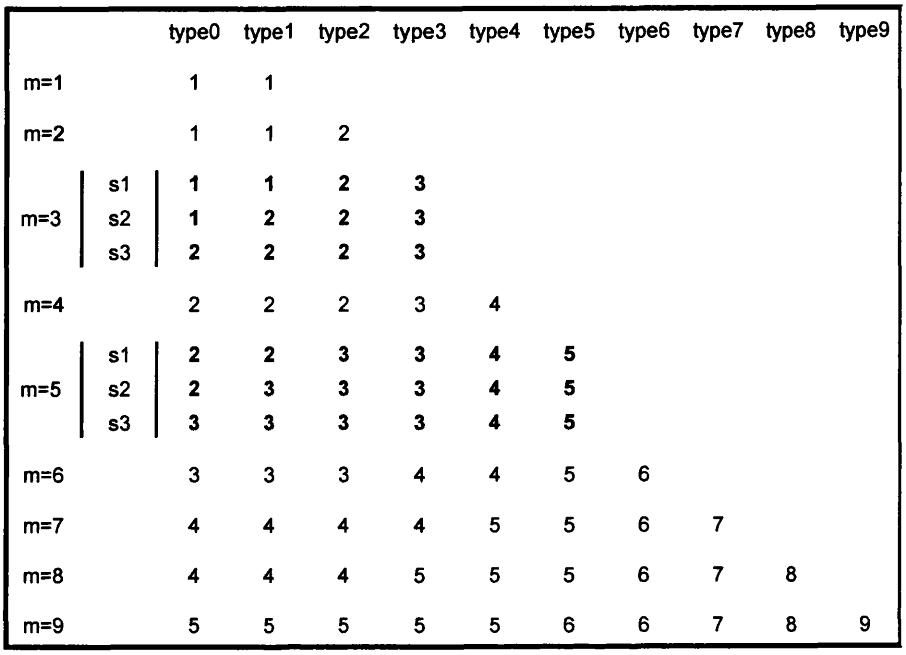
We also examined the expected costs connected with each strategy. Figure 2.4 illustrates the expected cost $C_d(i)$ that each type i will pay if he enters in the last period unmated when the equilibrium strategy is used, for all examined m. The floor functions of C_d are the equilibrium strategies. However, it is interesting to examine the costs C_d, since they may give some extra information in the case where there are multiple equilibria. The plotted expected costs C_d are divided by m, so that $\frac{C_d(m)}{m} = 1$; by normalising C_d, it is easier to make comparisons when m varies.

By looking at Figure 2.3 and at Figure 2.4, it becomes obvious that in every game the middle type (type 0) is the choosiest type. As we move from the middle type (type 0) to the extreme types (m and $-m$), types become less choosy. The extreme types (m and $-m$) are the least choosy, being universal acceptors. Moreover, it is evident that as the number m of types increases, the graphs of the strategies seem to approach each other, a fact that suggests that for very large m we may have a unique equilibrium. This assumption needs to be checked and it is based mostly on intuition.

Figures 2.5 and 2.6 show the 3 equilibrium strategies when $m = 3$ and when $m = 5$ respectively, named $s1$, $s2$ and $s3$ in Figure 2.3. The equilibrium strategies in each case differ only on the strategies used by the middle types (close to 0), while the strategies used by the more extreme types (closer to m and $-m$) are the same. Nevertheless, even when the strategies are the same for some types j, where $j \neq m, -m$ (Figure 2.3) the expected cost for entering in the last period unmated differs (Figures 2.4, 2.5 and 2.6).

It is important to note that we checked whether there are any changes in the equilibria if we remove the assumptions of sex and around the 0 type symmetry in the equilibrium strategies. We calculated the equilibria in the games where

m takes values between 1 and 5 by using exhaustive research and we did not find any additional equilibria. Players behave symmetrically even if they have the opportunity not to do so.



		type0	type1	type2	type3	type4	type5	type6	type7	type8	type9
m=1		1	1								
m=2		1	1	2							
m=3	s1	**1**	**1**	**2**	**3**						
	s2	**1**	**2**	**2**	**3**						
	s3	**2**	**2**	**2**	**3**						
m=4		2	2	2	3	4					
m=5	s1	**2**	**2**	**3**	**3**	**4**	**5**				
	s2	**2**	**3**	**3**	**3**	**4**	**5**				
	s3	**3**	**3**	**3**	**3**	**4**	**5**				
m=6		3	3	3	4	4	5	6			
m=7		4	4	4	4	5	5	6	7		
m=8		4	4	4	5	5	5	6	7	8	
m=9		5	5	5	5	5	6	6	7	8	9

Figure 2.3: Equilibrium strategies in the first period of the 2 period discrete type game.

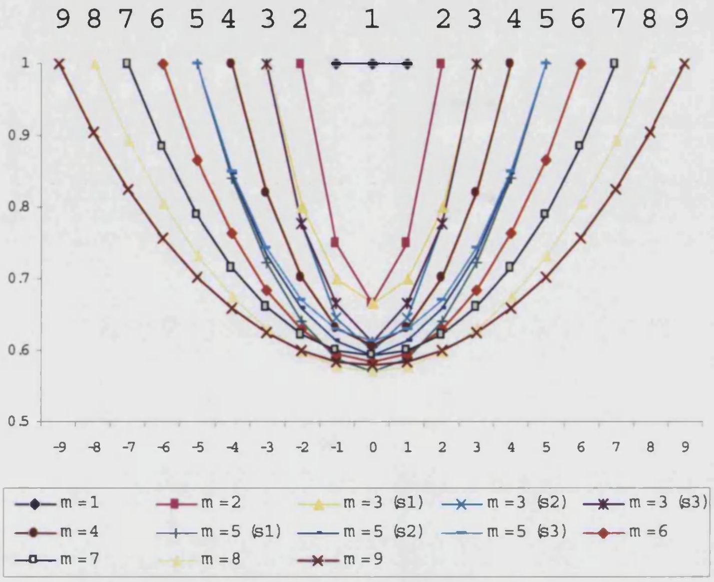


Figure 2.4: Normalised equilibrium expected costs for entering in the second period unmated in the 2 period discrete type game.

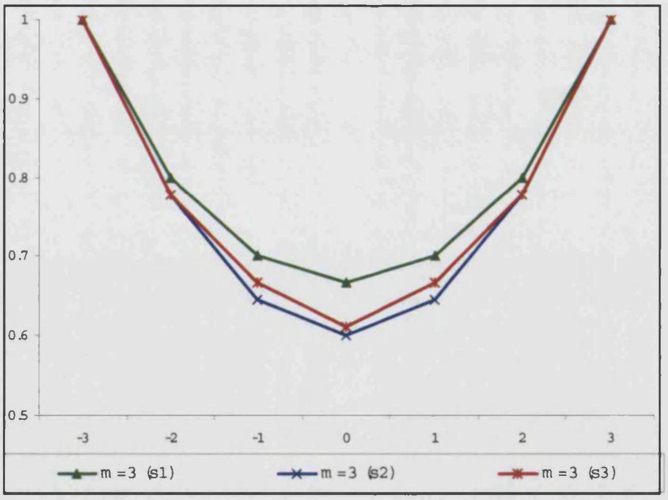


Figure 2.5: Normalised expected cost for entering in the second period unmated for $m = 3$

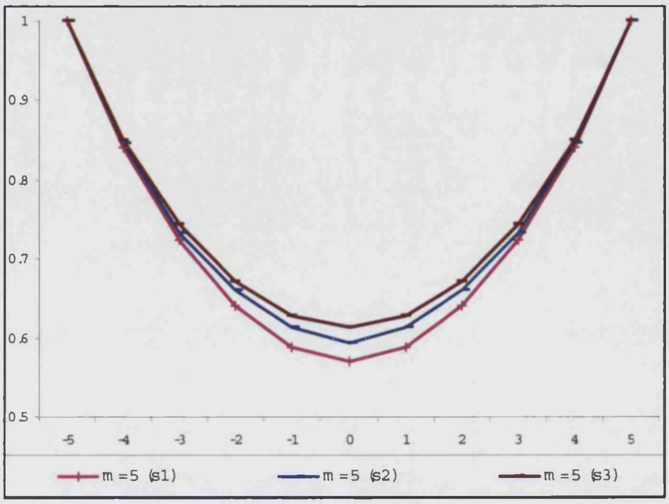


Figure 2.6: Normalised equilibrium expected costs for entering in the second period unmated for $m = 5$.

2.6.2 Probability of Getting Mated in the First Period

We explored how the probability of getting mated in the first period changes, depending on the type and on the number of positive types m. In that way, we can estimate which types we expect to find a partner more quickly. Figure 2.7 illustrates the probability for each type, in each game, to be mated in the first period. The probability is described as a percentage. A more accurate figure of the probability is given in Appendix A.2. It is obvious that the types closer to the middle (type 0) have higher probability of being mated in the first period (higher than 0.5 in most cases except for $m = 5$ and $m = 3$), while the more extreme types (close to m and $-m$) have lower chances to find a mate in the initial period (lower than 0.5 in every case, except when $m = 1$). Additionally, it is apparent that in the cases where there are multiple equilibria, when the middle types are less choosy it is more probable they are mated in the first period, while whether the extreme types have a smaller chance of being mated, depends not only on the behaviour of the middle types (type 0), but also on the behaviour of the types between the extreme (m and $-m$) and the middle (type 0).

		type0	type1	type2	type3	type4	type5	type6	type7	type8	type9
m=1		100	67 0								
m=2		60	60	40							
m=3	s1	43	43	43	29						
	s2	43	71	43	43						
	s3	71	71	57	43						
m=4		56	56	56	44	33					
m=5	s1	45	45	55	45	36	36				
	s2	45	64	64	45	45	36				
	s3	64	64	64	55	45	36				
m=6		54	54	54	54	46	39	31			
m=7		60	60	60	60	53	47	40	47		
m=8		53	53	53	59	53	47	41	35	35	
m=9		58	58	58	58	58	53	47	42	37	32

Figure 2.7: Probability (as percentage) of getting mated in the first period at equilibrium.

2.6.3 Total Expected Cost

Depending on the strategy used, individuals of each type i have an expected total cost $TC(i) = TC(i,s)$ for entering the game, that is the expected distance $|i-j|$ from their eventual mate j. The total expected cost $TC(i)$ for a type i, depends on the probability $p_1(i)$ of i being mated in the first period, the expected cost $d_1(i)$ of i when mated in first period and the expected cost $d_2(i)$ of i when mated in the second period. Hence

$$TC(i) = p_1(i)d_1(i) + (1 - p_1(i))d_2(i)$$

It is expected that types having a greater probability of being mated in the first period will have a bigger cost in the first period, but probably a smaller total cost. This is confirmed by examining the middle types cost as illustrated in Figure 2.8. In Figure 2.8, we have normalised the cost, by multiplying it with 100, in order to make it easier to make comparisons. A more accurate figure of the cost is given in Appendix $A.2$. For each game (hence for each

m), the first row in the table in Figure 2.8 illustrates the normalised average cost $100p_1(i)d_1(i)$ that each type i receives when mated in the first period. The second row of the table illustrates the normalised average cost $100(1-p_1(i))d_2(i)$ that each type i receives when mated in the second period. The third row illustrates the normalised average total cost $100TC(i)$ that a type i expects to receive when he enters the game. Since the probability of getting mated in the first period is higher for types around 0, they have the lowest cost of mating in total, but a large cost in the first period. The total mating cost tends to increase as we move from the middle types (type 0) to the extreme types (m and $-m$), which is probably explained by the fact that the mating probability in the first period tends to decrease as we move away from type 0. Furthermore, the difference between the costs in the two periods increases as we move from the middle types (type 0), to the extreme types (m and $-m$) and as we add more types in the game.

In the case of multiple equilibria, Figure 2.8 shows which strategy is preferred by each type, since it illustrates which strategy has the smallest expected total cost for each type. Thus, for $m = 3$, it becomes obvious that $s1 = (1, 1, 2, 3)$ is better for middle types $|i| \leq 1$, $s2 = (1, 2, 2, 3)$ is better for types $|j| = 2$ and both $s2 = (1, 2, 2, 3)$ and $s3 = (2, 2, 2, 3)$ are equally good for extreme types $|z| = 3$. For $m = 5$, $s1 = (2, 2, 3, 3, 4, 5)$ is preferred by $|i'| \leq 2$, $s2 = (2, 3, 3, 3, 4, 5)$ is preferred by $|j'| = 4$ and $s1 = (3, 3, 3, 3, 4, 5)$ is favoured by $|z| = 3$, while the extreme types $|z'| = 5$ are indifferent between all equilibrium strategies.

			type0	type1	type2	type3	type4	type5	type6	type7	type8	type9	average
m=1		1rst period	67	33									44
		2nd period	0	33									22
		total	67	67									67
m=2		1rst period	40	40	20								32
		2nd period	53	60	120								83
		total	93	100	140								115
m=3	s1	1rst period	29	29	29	14							25
		2nd period	103	111	133	214							146
		total	131	139	162	229							170
	s2	1rst period	29	86	29	43							49
		2nd period	105	57	133	171							118
		total	133	143	162	214							167
	s3	1rst period	86	86	57	43							65
		2nd period	57	60	103	171							104
		total	143	146	160	214							169
m=4		1rst period	67	67	67	44	33	55					54
		2nd period	108	112	125	183	267	318					165
		total	175	179	192	227	300	373					219
m=5	s1	1rst period	55	55	82	55	36	55					56
		2nd period	156	160	146	197	268	318					212
		total	210	215	228	252	304	373					268
	s2	1rst period	55	109	109	55	64	55					76
		2nd period	162	112	120	200	231	318					193
		total	216	221	229	255	295	373					269
	s3	1rst period	109	109	109	82	64	55					86
		2nd period	112	114	122	169	232	318					184
		total	221	223	231	251	296	373					270
m=6		1rst period	92	92	92	92	69	54	46				76
		2nd period	162	165	174	190	247	320	415				245
		total	254	257	267	282	316	374	462				321
m=7		1rst period	133	133	133	133	107	87	73	67			107
		2nd period	166	168	175	186	234	295	371	467			264
		total	299	301	308	319	341	382	444	533			370
m=8		1rst period	118	118	118	147	118	94	77	65	88		104
		2nd period	215	217	225	208	254	310	379	462	518		315
		total	333	335	342	355	372	404	455	527	606		419
m=9		1rst period	158	158	158	158	158	132	111	95	84	79	127
		2nd period	220	222	227	236	249	299	359	429	514	616	343
		total	378	380	385	394	407	431	469	524	599	695	471

Figure 2.8: Total expected cost at equilibrium.

2.6.4 Intra-Couple Correlation

The intra-couple correlation measures how alike mated couples are and it can be used as an indicator of the pleasure the players derive at the equilibrium. In order to find the intra-couple correlation coefficient $\check{r}$, we have to find first the fraction of couples (i,j) in periods 1 and 2.

In the first period, the fraction of individuals of type i is $\frac{1}{2m+1}$ and the fraction of individuals of type j is $\frac{1}{2m+1}$. Hence $\frac{1}{2m+1}$ of the individuals of type i are paired with an individual of type j. The fraction of couples (i,j) created in the first period is therefore $\frac{1}{2m+1}\frac{1}{2m+1}$.

Given that strategy s is used in the first period, we define as $\hat{q}(i,s)$ the probability that an individual of type i remains unmated in the first period and enters in the second period. Hence in the second period, there are $\frac{1}{2m+1}\hat{q}(i,s)$ individuals of type i and $\frac{1}{2m+1}\hat{q}(j,s)$ individuals of type j. Every individual accepts anyone he is paired with in the last period. So the probability of a type i being mated with a type j is $\frac{\hat{q}(j,s)}{\sum_{k=-m}^{k=m}\hat{q}(k,s)}$. Thus, the proportion of couples (i,j) in the second period when strategy s is used is

$$\frac{1}{2m+1}\hat{q}(i,s)\frac{\hat{q}(j,s)}{\sum_{k=-m}^{k=m}\hat{q}(k,s)}$$

We define the function

$$\begin{aligned}\hat{a}(s,i,j) &= \begin{cases} 0, & \text{if } i \text{ and } j \text{ are not mated in the first period} \\ 1, & \text{if } i \text{ and } j \text{ are mated in the first period} \end{cases} \\ &= \begin{cases} 0, & \text{if } |i-j| > s(i) \text{ or } |i-j| > s(j) \\ 1, & \text{if } |i-j| \le s(i) \text{ and } |i-j| \le s(j) \end{cases}\end{aligned}$$

The total fraction $\hat{N}(s,i,j)$ of couples (i,j) in the first and second period when strategy s is used is

$$\hat{N}(s,i,j) = \hat{a}(s,i,j)\left(\frac{1}{2m+1}\right)^2 + \frac{1}{2m+1}\hat{q}(i,s)\frac{\hat{q}(j,s)}{\sum_{k=-m}^{k=m}\hat{q}(k,s)} \tag{18}$$

Knowing the fraction of couples (i,j) in every period, it is easy then to calculate the correlation coefficient $\check{r}$.

The intra-couple correlation $\check{r}$ seems to be quite small in all the games analysed, taking values between 0.306 and 0.36, probably due to the fact that all types are willing to accept a range of types including theirs. There are no significant differences between the different games or in the same game between different equilibria. We would expect that correlation $\check{r}$ would probably be higher if the number of periods n becomes larger.

	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
s1	0.36	0.306	0.337	0.34	0.33	0.34	0.34	0.336
s2		0.357		0.347				
s3		0.357		0.347				

Figure 2.9: Intra-couple correlation coefficient $\check{r}$ at equilibrium.

2.6.5 Instability of Created Couples

It is important to know how stable are the couples created given a specific strategy used. We need to know how willing the players would be to change partners after the end of the game. Instability can also be used as a measure of the pleasure that players get from the game. Assuming as before that in every couple (i, j) created, i represents a male and j a female, we choose two couples (i, j) and (k, l) at random, where i and k are males and j and l are females. We define instability as the probability that either i prefers l better than j and l prefers i better than k, or that j prefers k better than i and k prefers j better than l.

We define a function $\hat{w}$ such as

$$\begin{aligned}
\hat{w}(s,(i,j),(k,l)) &= \begin{cases} 1, & \begin{cases} \text{if } i \text{ prefers } l \text{ than } j \text{ and } l \text{ prefers } i \text{ than } k \\ \qquad\qquad\qquad \textit{or} \\ \text{if } j \text{ prefers } k \text{ than } i \text{ and } k \text{ prefers } j \text{ than } l \end{cases} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \begin{cases} \text{if } |i-j| > |i-l| \text{ and } |l-k| > |i-l| \\ \qquad\qquad\qquad \textit{or} \\ \text{if } |i-j| > |j-k| \text{ and } |k-l| > |k-j| \end{cases} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Given that initially we have fixed cohorts of males and females, the probability of choosing 2 random couples (i, j) and (k, l) is $\hat{N}(s, i, j)\hat{N}(s, k, l)$ from (18). Hence

$$instability(s)=\sum_{i=-m}^{i=m}\sum_{j=-m}^{j=m}\sum_{k=-m}^{k=m}\sum_{l=-m}^{l=m}\hat{w}(s,(i,j),(k,l))\hat{N}(s,i,j)\hat{N}(s,k,l)$$

It becomes apparent from Figure 2.10, that as we add more types to the game, the couples created become more unstable. Hence in a separate game where types were permitted to divorce, we would expect that the probability of taking a divorce would increase as the number of types becomes larger. Focusing on the games with multiple equilibria, we can observe that there are no important differences between the different strategies. In any case, when $m=3$, $s_1=(1,1,2,3)$ seems to be the most stable strategy and $s_3=(2,2,2,3)$ the most unstable and when $m=5$, $s_1=(2,2,3,3,4,5)$ seems to be the most stable strategy and $s_3=(3,3,3,3,4,5)$ the most unstable. As expected, the choosier on average the players are, the more stable are their matings; this is also observed in the common preferences model.

	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
s1	0.181	0.206	0.245	0.257	0.275	0.293	0.296	0.307
s2		0.223		0.267				
s3		0.238		0.272				

Figure 2.10: Instability index at equilibrium.

2.7 Multiple Equilibria in an n-Period Game $\Gamma_n(m)$, where $n>2$

In the two period discrete games analysed, we were able to find all the equilibrium strategies and comment on their efficiency. Nonetheless, as we add more periods in the game, it becomes complicated to find all the existing equilibrium strategies, since even the method described before and used in the two period games becomes time-consuming. The 3 period game and the 4 period game, where there exist $m=8$ positive types (and 8 symmetric negative types and a 0 type) were analysed by using computer search methods, and in the case of the 4 periods, there were found at least two equilibrium strategies, namely

$s_1=((2,2,2,2,2,3,3,4,4),(3,3,3,3,3,4,4,5,5),(4,4,4,5,5,5,6,7,8))$ and

$s_2=((2,2,2,2,3,3,3,3,4),(3,3,3,3,3,4,4,5,5),(4,4,5,5,5,6,6,7,8))$.

This fact illustrates that the equilibrium in the discrete game is not unique even in some discrete type games of more than 2 periods. Thus, it becomes apparent that it is possible that there exist multiple equilibria in games of any number of periods n, even though this is not investigated thoroughly at this point and further research needs to be done.

2.8 Mixed Preferences - Combination of Similarity and Common Preferences

In our analysis so far, we assumed that players have homotypic preferences. However, in real life it is not always true that preferences are purely homotypic or purely maximising. In many cases, individuals have a combination of homotypic and maximising preferences. They want to find partners similar to them, in order to minimise the cost of mating but simultaneously they search for a partner that is of a high type, since they value greatly the utility they get from his type. When the interests that individuals have are conflicting, they make their choice based on the importance they assign to their needs.

In our model, we define the utility of a mating (i, j) for type i as a weighted average between its cost and the utility j gives to i. Hence when a type i is mated with a type j, the utility u_{ij} of i and the utility u_{ji} of j are

$$\begin{array}{l} u_{ij} = -\omega \left| i - j \right| + (1 - \omega) j \\ u_{ji} = -\omega \left| j - i \right| + (1 - \omega) i \end{array} \text{, where } \omega \in [0, 1] \tag{19}$$

As individuals become less interested in mating with a high type, but value the mating with a type close to their own greatly, the weight ω becomes larger, while as the need to find a high type partner becomes stronger ω becomes smaller. When $\omega = 1$, players have homotypic preferences and when $\omega = 0$, players have common preferences.

We assume that males and females have the same preferences, weighting in the same way the cost of mating (weight ω) and the value of a mating (weight $(1 - \omega)$). This permits us to maintain the sex symmetry displayed in the pure homotypic preferences model.

Apart from the player's preferences, the rest of the characteristics of the game remain mostly the same as in the previous sections. Players have one dimensional types which are uniformly distributed over an interval and are randomly paired over n periods. We focus on the discrete type decentralised non atomic two and three period game, where types belong in an interval $\{0, 1, ..., m-1, m\}$. The assumption of symmetric behaviour of types around the origin 0 has no longer any meaning therefore it is removed. However, the game remains a mutual choice game, where no new players are permitted to enter after the beginning of the game and in the last period all players accept any available partner, exactly as in the games described in the previous sections.

A strategy $s_k(i)$ indicates the minimum utility that i accepts in period k. Hence at equilibrium, mating (i, j) takes place in period k when the utility u_{ij} and u_{ji} that it offers to i and j respectively is higher than the utility they would expect to receive if they entered in the next period unmated.

$$u_{ij} \geq \hat{s}_k(i) \text{ and } u_{ji} \geq \hat{s}_k(j)$$

Therefore a type i accepts a type j when

$$\begin{aligned}
-\omega\,|i-j| + (1-\omega)j \geq \hat{s}_k(i) \ &\Leftrightarrow \\
-\omega\,|i-j| \geq \hat{s}_k(i) - (1-\omega)j &\Leftrightarrow \\
|i-j| \leq \frac{-\hat{s}_k(i) + (1-\omega)j}{\omega} &\Leftrightarrow \\
\frac{\hat{s}_k(i) - (1-\omega)j}{\omega} \leq i - j \leq \frac{-\hat{s}_k(i) + (1-\omega)j}{\omega} &\Leftrightarrow \\
-\hat{s}_k(i) + (1-\omega)j + \omega i \geq \omega j \geq \hat{s}_k(i) - (1-\omega)j + \omega i &\Leftrightarrow \\
-\hat{s}_k(i) + \omega i \geq (2\omega - 1)j \text{ and } j \geq \hat{s}_k(i) + \omega i &
\end{aligned}$$

Thus

$$\begin{aligned}
&\text{If } \omega > \frac{1}{2}, \text{ then } i \text{ accepts } j \text{ if } \frac{-\hat{s}_k(i) + \omega i}{2\omega - 1} \geq j \geq \hat{s}_k(i) + \omega i \\
&\text{If } \omega < \frac{1}{2}, \text{ then } i \text{ accepts } j \text{ if } j \geq \max\{\hat{s}_k(i) + \omega i, \frac{-\hat{s}_k(i) + \omega i}{2\omega - 1}\} \\
&\text{If } \omega = \frac{1}{2} \text{ then } i \text{ accepts } j \text{ if } \hat{s}_k(i) \leq \frac{i}{2} \text{ and } j \geq \hat{s}_k(i) + \frac{i}{2}
\end{aligned}$$

Hence a mating between i and j takes place when

$$\begin{aligned}
&\text{If } j \leq i, \left.\begin{cases} s_k(i) \leq -\omega\,(i-j) + (1-\omega)j \\ s_k(j) \leq -\omega\,(i-j) + (1-\omega)i \end{cases}\right| \Leftrightarrow \left.\begin{cases} s_k(i) \leq j - \omega i \\ s_k(j) \leq j\omega + (-2\omega + 1)i \end{cases}\right| \\
&\text{If } i \leq j, \left.\begin{cases} s_k(i) \leq \omega\,(i-j) + (1-\omega)j \\ s_k(j) \leq \omega\,(i-j) + (1-\omega)i \end{cases}\right| \Leftrightarrow \left.\begin{cases} s_k(i) \leq \omega i - 2\omega j + j \\ s_k(j) \leq -j\omega + i \end{cases}\right|
\end{aligned}$$

Given that the expected utility for an individual of type i to enter period k unmated is denoted by $V_k(s, i)$, at equilibrium

$$\hat{s}_k(i) = V_{k+1}(\hat{s}, i) \text{ for } 1 \leq k \leq n-1 \tag{20}$$

It is expected that as ω decreases, the total utility that players receive decreases as well. We expect that for all ω, the highest types have the smallest expected utility for entering in the last period unmated, a fact confirmed by the numerical results presented in the next sections.

We examine how the equilibrium behaviour changes as ω increases from 0 to 1 with steps of 0.1 when there are 10 discrete types, uniformly distributed between 0 and 9. We did not assume that there is sex or type symmetry at equilibrium, nevertheless we found that the equilibrium strategies of males and females are symmetric. We present our results for the two and three period games.

2.8.1 Two Period Game

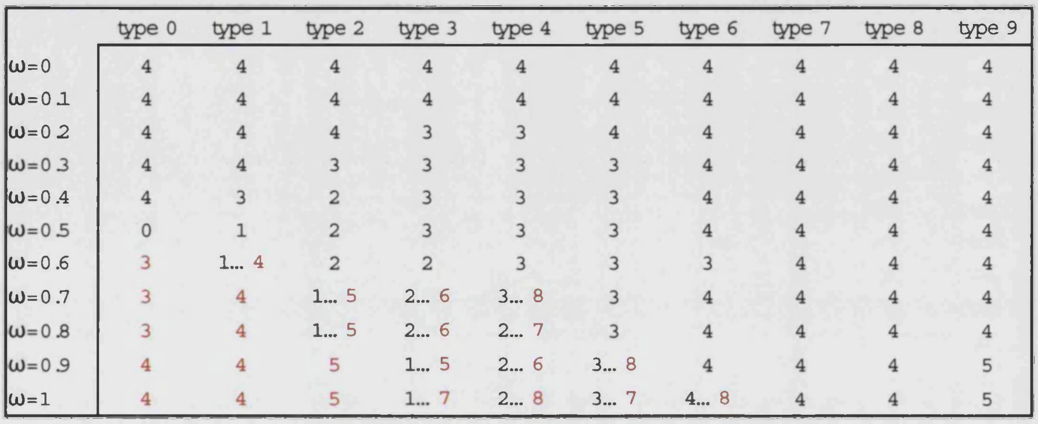
The analysis of the two period game offers us the opportunity to examine the differences in the behaviour of different types as their preferences change. We note that the majority of types are the choosiest when they have homotypic preferences and are the least choosy when they regard as equally important finding a high type mate and a mate close to them. We note that we measure the choosiness of a type according to the number of types he is willing to accept. In general, extreme types (close to 0 and 9) are always choosier than the middle types, accepting higher types when $\omega < 0.5$, and types closer to their own otherwise. The effect of the weight ω on the acceptance range of different types varies and it is shown that the acceptance range of low and middle types is influenced by ω in a greater extent than that of the high types. As a result the pattern and the number of couples created before the last period varies significantly for different ω. In most games, especially for $\omega < 0.5$, the majority of couples formed in the first period are created between high and middle types. However, as ω increases low types participate as well in the mating process in the first period. The number of couples created in the first period is the minimum when $\omega = 0$ and the maximum when $\omega = 0.9$.

Figures 2.11 show the equilibrium strategies found via iteration for each ω and Figure 2.12 illustrates the range of types that each type is willing to accept at equilibrium. In Appendix A.3 we plot the equilibrium strategies in Figure A.3. It is worth reminding to the reader that a strategy that a type i uses in a period denotes the minimum utility that type i is willing to accept from a pairing in that period; the strategy of i is equal with the utility that type i expects to receive if he enters in the next period unmated. It is expected for types to have different expected utilities in the last period, but the same acceptance range. Since males and females use the same strategy and have the same behaviour at equilibrium, Figures 2.11 and 2.12 illustrate both male and female strategies and range of acceptance. In Figure 2.12, we do not list all the types that a type accepts, since there is never discontinuity at the range of acceptance of any type. If a type i accepts a type j and a type z, it always accepts all types between j and z. Thus, in order to define the range of acceptance of any type i, we just need to know the highest and the lowest type accepted by i. When type i accepts all types between j and 9 (including j and 9), only the lowest possible type j that a type i accepts is given in non bold writing. When type i accepts all types between 0 and $\mathbf{z}$ (including 0 and $\mathbf{z}$), only the highest possible type $\mathbf{z}$ that a type i accepts is given in bold writing and in red colour. When i accepts any types between j and $\mathbf{z}$ (including j and $\mathbf{z}$), where $j > 0$ and $\mathbf{z} < 9$, both the lowest and the highest types $j ... \mathbf{z}$ that i accepts are given.

Given Figure 2.12, it is easy to construct Figure 2.13. The latter presents the pattern of couples expected to be formed in the first period when the equilibrium strategies are used. For each particular ω, we construct a 10×10 table. In each table, rows and columns range from 0 to 9. Each row i represents a male type i and each column j a female type j. A cell that belongs in row i and column j is blue when types i and j accept each other and form a couple when they meet in period 1 or it is white otherwise.

	type 0	type 1	type 2	type 3	type 4	type 5	type 6	type 7	type 8	type 9
ω=0	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***	***3.375***
ω=0.1	2.7	2.769	2.806	**2.812**	2.788	2.75	2.7	2.638	2.563	*2.475*
ω=0.2	2.026	2.16	2.23	**2.233**	2.184	2.115	2.02	1.898	1.751	*1.577*
ω=0.3	1.342	1.541	**1.637**	1.632	1.556	1.449	1.312	1.134	0.915	*0.656*
ω=0.4	0.676	0.938	**1.062**	**1.062**	0.966	0.828	0.648	0.414	0.124	*-0.22*
ω=0.5	0	0.339	0.518	**0.536**	0.429	0.268	0.054	-0.232	-0.589	*-1.018*
ω=0.6	-0.696	-0.304	**-0.119**	**-0.119**	-0.235	-0.419	-0.673	-0.996	-1.412	*-1.919*
ω=0.7	-1.53	-1.046	-0.75	**-0.642**	-0.696	-0.858	-1.1	-1.45	-1.908	*-2.473*
ω=0.8	-2.365	-1.812	-1.473	**-1.319**	-1.35	-1.504	-1.75	-2.119	-2.612	*-3.258*
ω=0.9	-3.446	-2.788	-2.338	-2.062	**-1.958**	-2.027	-2.235	-2.581	-3.1	*-3.792*
ω=1	*-4.5*	-3.75	-3.215	-2.857	**-2.679**	**-2.679**	-2.857	-3.214	-3.75	*-4.5*

Figure 2.11: Equilibrium strategies for $m = 9$. At equilibrium, a strategy of a type i in period 1 is equal with the utility i expects to receive in period 2.



	type 0	type 1	type 2	type 3	type 4	type 5	type 6	type 7	type 8	type 9
ω=0	4	4	4	4	4	4	4	4	4	4
ω=0.1	4	4	4	4	4	4	4	4	4	4
ω=0.2	4	4	4	3	3	4	4	4	4	4
ω=0.3	4	4	3	3	3	3	4	4	4	4
ω=0.4	4	3	2	3	3	3	4	4	4	4
ω=0.5	0	1	2	3	3	3	4	4	4	4
ω=0.6	3	1... 4	2	2	3	3	3	4	4	4
ω=0.7	3	4	1... 5	2... 6	3... 8	3	4	4	4	4
ω=0.8	3	4	1... 5	2... 6	2... 7	3	4	4	4	4
ω=0.9	4	4	5	1... 5	2... 6	3... 8	4	4	4	5
ω=1	4	4	5	1... 7	2... 8	3... 7	4... 8	4	4	5

Figure 2.12: Acceptance range at equilibrium.

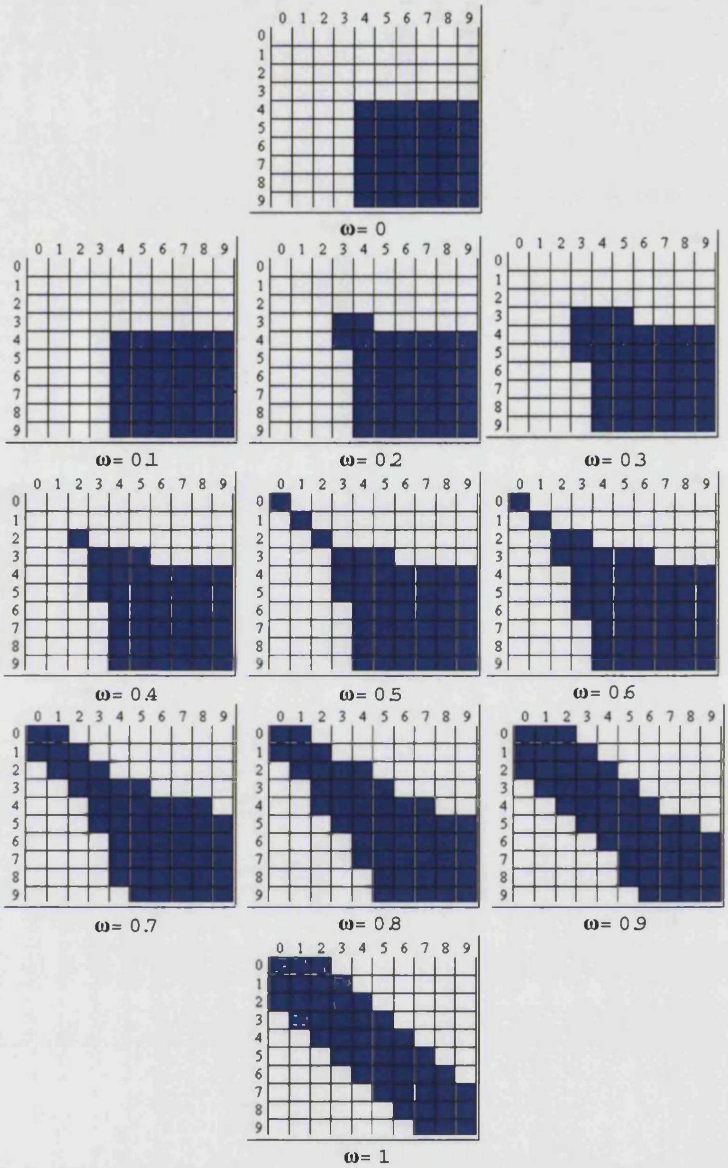
It becomes apparent from Figures 2.11 and 2.12 that when players have common preferences ($\omega = 0$), the equilibrium strategy is a cutoff point, common for all types; all types are willing to accept any type at least as high as 4. Furthermore, for $\omega = 0$, the expected utility $C_2(\hat{s}, i)$ that a type i expects to get when entering in the second period is much higher that the utility i expects to receive in any other game examined. As a result, compared with all games where $\omega \leq 0.5$, most types are choosier when $\omega = 0$. However, for $\omega > 0.5$, most types tend to be even choosier than for $\omega = 0$.

If we look at Figure 2.11, we observe that for a given type i, the expected utility $C_2(\hat{s}, i)$ for entering in the second period unmated decreases as ω increases. However the decrease of expected utility does not imply necessarily a decrease of types' choosiness. If we look at Figure 2.12, we note that most types become less choosy as ω increases from 0 to 0.5, while the opposite is true as ω increases from 0.5 to 1. Furthermore, while for $\omega \leq 0.5$ all types are willing to accept all types higher that 4, for $\omega > 0.5$ they often reject high types. For $\omega > 0.5$, low types (types $0, 1, 2$) never prefer types higher than 5, low middle types (types $3, 4$) never accept extreme types 0 and 9, high middle types (types $5, 6$) do not always accept type 9 (evident as ω approaches 1) and high types (types 7, 8 and 9) accept all high and middle high types but not lower types (they even reject type 4 as ω approaches 1). Looking at Figure 2.12, we observe that on average players are the choosiest when they have homotypic preferences ($\omega = 1$) and are the least choosy when $\omega = 0.5$ and therefore they value the same the cost and the return of a mating.

Focusing on each particular game, we see that the extreme types (close to

0 and 9) are most of the times choosier than the middle types. Exceptions are $\omega = 0$, $\omega = 0.1$ and $\omega = 0.5$. When $\omega = 0$ and $\omega = 0.1$ all types are equally choosy, while when $\omega = 0.5$ type 0 is a universal acceptor and choosiness of types increases as we move from type 0 to type 9. For $\omega < 0.5$, extreme (0 and 9) and high types are the choosiest, while middle types are the least choosy. (It has to be noted that for $\omega < 0.5$ the choosiness and the strategy of type 0 are irrelevant for the rest of the players, since he is rejected by all types and he always enters in the last period unmated.) For $0.5 < \omega < 0.8$, the importance of the cost of a mating increases. As a result, low types are the choosiest and as we move to higher types closer to type 9 the range of acceptance increases. For $\omega > 0.8$ the range of acceptance does not have great differences between the types, since most types are willing to accept as many as 5 types of the opposite sex.

For a given ω, we also examine the changes of the expected utility of type i entering in the next period unmated as we move from $i = 0$ to $i = 9$. We observe that in all games, the strategy of type $i = 0$ is quite high, it increases as we move to higher types, until it reaches its maximum for type $i = 2$, 3 or 4 depending on the value of ω, and then it starts decreasing again as i increases, until it reaches its minimum for type $i = 9$. When players have homotypic preferences ($\omega = 1$), it is worth mentioning that the strategies of types are symmetric around middles types 4 and 5; $s(4 - \xi) = s(5 - \xi)$ for any integer ξ. Hence we see that the symmetry observed in the previous sections around the middle type 0, remains in this game around the middle types 4 and 5.



ω= 0

ω= 0.1

ω= 0.2

ω= 0.3

ω= 0.4

ω= 0.5

ω= 0.6

ω= 0.7

ω= 0.8

ω= 0.9

ω= 1

Figure 2.13: Couple formation at equilibrium, $n = 2$.

Figure 2.13 gives an image of the couples that are formed in the first period in each game. If we take the diagonal passing from cells (i, i), it is clear that the pattern of couples formed is always symmetric around it, as a consequence of the symmetric behaviour of females and male types. The number of couples formed in the first period is the minimum when $\omega = 0$ and maximum when $\omega = 0.9$. As ω increases, the number of couples created increases, reaches its maximum and then it decreases slightly for $\omega = 1$. Hence, under homotypic preferences ($\omega = 1$) more couples than under common preferences ($\omega = 0$) are formed in the first period. If we choose a male i and a female j at random in the first period, there exist a 0.36 probability they accept each other when they have common preferences and a 0.44 probability to mutually accept to form a couple when they have homotypic preferences. Looking at all games, the higher probability that a random male of type i and a random female of type j accept each other is 0.5 (for $\omega = 0.9$).

With the exception of $\omega = 1$, couples are concentrated in the right bottom corner of each table, hence in the majority of couples only middle and high types are involved. When ω is close to 0 couples are created only between types which are of a middle or of a higher type. Nevertheless, as ω increases lower types start to participate in the formation of couples as well. For $\omega < 0.5$, lower middle types form couples with types next to them and for $\omega \geq 0.5$ even low types participate in the couples created, forming couples with types close to them. As we move towards $\omega = 1$, especially after $\omega = 0.7$, high and middle types tend to form couples only with types very close to them. As a result, the number of blue squares in the bottom right of the Figure 2.13 decreases as ω approaches 1. At $\omega = 1$, all types participate in the formation of couples in the first period, but each type only mates with types which are close to him (maximum distance between mates is 4).

It becomes clear that when populations have a combination of homotypic and common preferences, but the significance of homotypic preferences is much higher, it is more probable for a type to be mated in the first period. On the other hand, as populations become more focused on searching for a high type mate, the probability that a type is mated in the first period is significantly lower. Under homotypic preferences a significantly higher number of couples is formed in the first period than under common preferences. High types under any mixture of preferences have a high chance to find a partner in the first period. However, the probability for high types not to enter the second period is higher when common preferences are stronger. The opposite is true for low types; when players have common preferences, lower types have no chances to be mated in the first period. Middle types have the highest probability to find a mate when common and homotypic preferences are equally strong.

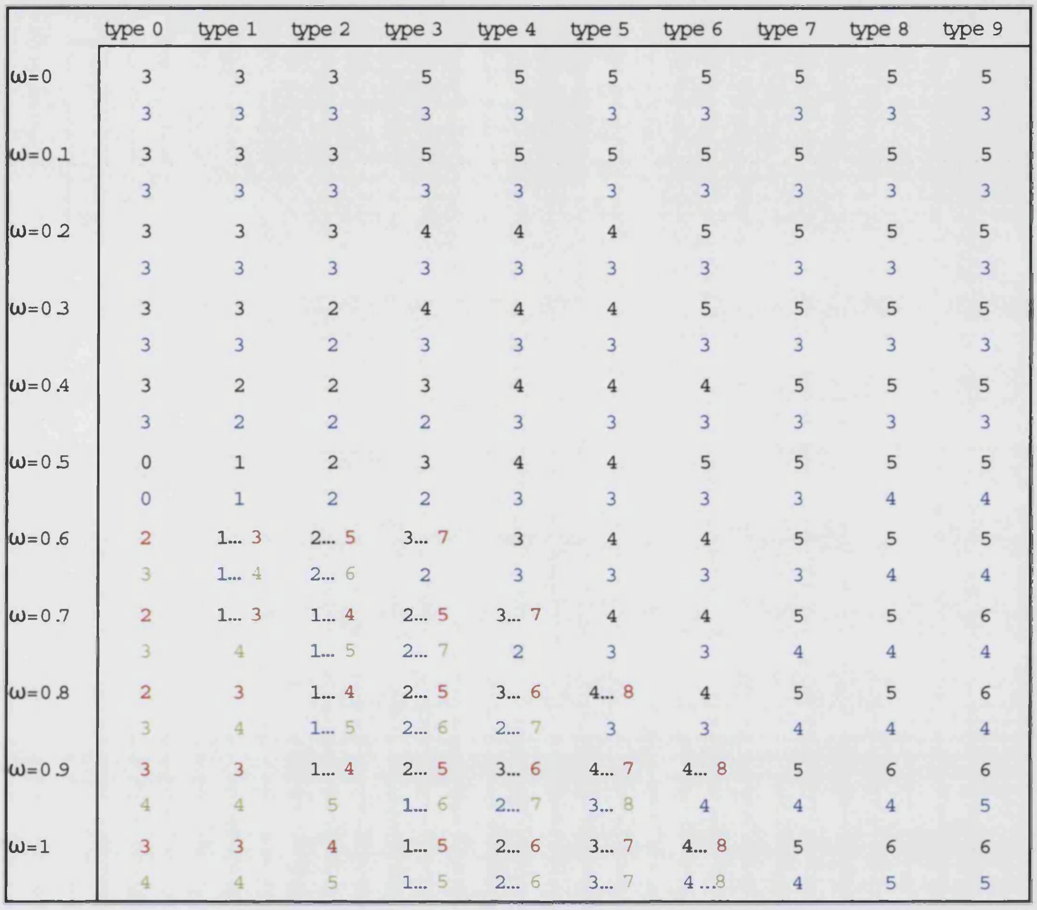
2.8.2 Three Period Game

In the three period game, the equilibrium behaviour presents great similarities with the equilibrium behaviour in the two period game. Any differences are the product of the additional opportunity that players have to find a partner of their choice in the second period. As in the two period game, players are on average the choosiest when they have homotypic preferences and are the least choosy when they value the same the cost of a mating and the value of their partner. Players are choosier in the first period or they are equally choosy in both periods. The range of acceptance of each particular type includes different types as ω changes. High types are the choosiest and low types are the least choosy when $\omega \leq 0.5$, but when $\omega > 0.5$, all types become choosier, not accepting more than 5 types in any case. This affects the number of couples created before the last period. More couples are formed in the first two periods when players tend to weight more the cost of mating. The number of couples created in the first two periods is the highest for $\omega = 0.7$ and the lowest for $\omega = 0$.

Figure 2.14 illustrates the equilibrium strategies found via iteration; for each ω, the first line (written in black) describes the equilibrium strategy of each type in the first period and the second line (written in blue) describes the equilibrium strategy of each type in the second period. In Appendix A.3 we plot the equilibrium strategies in Figure A.4. Figure 2.15 presents the range of acceptance of each type at equilibrium. For each ω, the first line (written in black and red) refers to the range of acceptance of each type in period 1 and the second line (written in blue and green) in period 2. The range of acceptance of all types remains continuous in the three period game (as in the two period game), hence it is illustrated in Figure 2.15 in the same way as in Figure 2.12 of the previous section. Only the lower limit is given when types accept type 9, only the higher limit is given (in bold writing and in red for the first period or in green for the second period) when player accept type 0, or both limits are given when neither 0 or 9 are accepted. In Figure 2.16 we draw the couples expected to be formed in the first and second period. We construct Figure 2.16 in the say way as Figure 2.13 in the 2 period game. The only difference is that the couples formed in the first period are described in red colour and any additional couples created in the second period are represented in green colour. All combinations of types forming couples in the first period, also form couples in the second period.

	type 0	type 1	type 2	type 3	type 4	type 5	type 6	type 7	type 8	type 9
ω=0	2.667	2.667	2.667	4.333	4.333	4.333	4.333	4.333	4.333	4.333
	2.667	2.667	2.667	2.667	2.667	2.667	2.667	2.667	2.667	2.667
ω=0.1	2.133	2.192	2.208	3.667	3.683	3.667	3.633	3.583	3.517	3.433
	2.133	2.192	2.208	2.183	2.142	2.0833	2.017	1.942	1.858	1.767
ω=0.2	1.6	1.716	1.747	2.99	3.014	2.982	2.923	2.828	2.698	2.533
	1.6	1.716	1.747	1.695	1.607	1.491	1.361	1.214	1.049	0.867
ω=0.3	1.071	1.237	1.287	2.329	2.371	2.327	2.241	2.101	1.907	1.661
	1.071	1.237	1.271	1.993	1.074	0.904	0.713	0.495	0.249	-0.235
ω=0.4	0.538	0.747	0.908	1.633	1.671	1.621	1.51	1.339	1.091	0.767
	0.538	0.747	0.767	0.661	0.504	0.291	0.045	-0.234	-0.552	-0.912
ω=0.5	0	0.33	0.569	0.897	1.156	1.117	1.003	0.796	0.499	0.108
	0	0.304	0.414	0.358	0.209	-0.02	-0.288	-0.606	-0.974	-1.407
ω=0.6	-0.534	-0.166	0.089	0.298	0.407	0.38	0.256	0.036	-0.303	-0.763
	-0.622	-0.259	-0.134	-0.213	-0.398	-0.662	-0.977	-1.344	-1.776	-2.292
ω=0.7	-1.08	-0.688	-0.419	-0.28	-0.216	-0.168	-0.256	-0.44	-0.772	-1.271
	-1.417	-1.002	-0.8	-0.765	-0.893	-1.093	-1.377	-1.728	-2.189	-2.758
ω=0.8	-1.768	-1.243	-0.935	-0.814	-0.804	-0.836	-0.917	-1.098	-1.493	-2.033
	-2.263	-1.758	-1.469	-1.369	-1.447	-1.652	-1.947	-2.314	-2.799	-3.428
ω=0.9	-2.479	-1.852	-1.553	-1.423	-1.392	-1.425	-1.508	-1.656	-2.002	-2.656
	-3.377	-2.761	-2.352	-2.115	-2.061	-2.149	-2.383	-2.741	-3.221	-3.878
ω=1	-3.339	-2.628	-2.277	-2.161	-2.084	-2.084	-2.161	-2.277	-2.628	-3.339
	-4.5	-3.793	-3.303	-2.968	-2.797	-2.797	-2.968	-3.303	-3.793	-4.5

Figure 2.14: Equilibrium strategies when $n = 3$.



	type 0	type 1	type 2	type 3	type 4	type 5	type 6	type 7	type 8	type 9
ω=0	3	3	3	5	5	5	5	5	5	5
	3	3	3	3	3	3	3	3	3	3
ω=0.1	3	3	3	5	5	5	5	5	5	5
	3	3	3	3	3	3	3	3	3	3
ω=0.2	3	3	3	4	4	4	5	5	5	5
	3	3	3	3	3	3	3	3	3	3
ω=0.3	3	3	2	4	4	4	5	5	5	5
	3	3	2	3	3	3	3	3	3	3
ω=0.4	3	2	2	3	4	4	4	5	5	5
	3	2	2	2	3	3	3	3	3	3
ω=0.5	0	1	2	3	4	4	5	5	5	5
	0	1	2	2	3	3	3	3	4	4
ω=0.6	2	1... 3	2... 5	3... 7	3	4	4	5	5	5
	3	1... 4	2... 6	2	3	3	3	3	4	4
ω=0.7	2	1... 3	1... 4	2... 5	3... 7	4	4	5	5	6
	3	4	1... 5	2... 7	2	3	3	4	4	4
ω=0.8	2	3	1... 4	2... 5	3... 6	4... 8	4	5	5	6
	3	4	1... 5	2... 6	2... 7	3	3	4	4	4
ω=0.9	3	3	1... 4	2... 5	3... 6	4... 7	4... 8	5	6	6
	4	4	5	1... 6	2... 7	3... 8	4	4	4	5
ω=1	3	3	4	1... 5	2... 6	3... 7	4... 8	5	6	6
	4	4	5	1... 5	2... 6	3... 7	4 ...8	4	5	5

Figure 2.15: Acceptance range at equilibrium when $n = 3$.

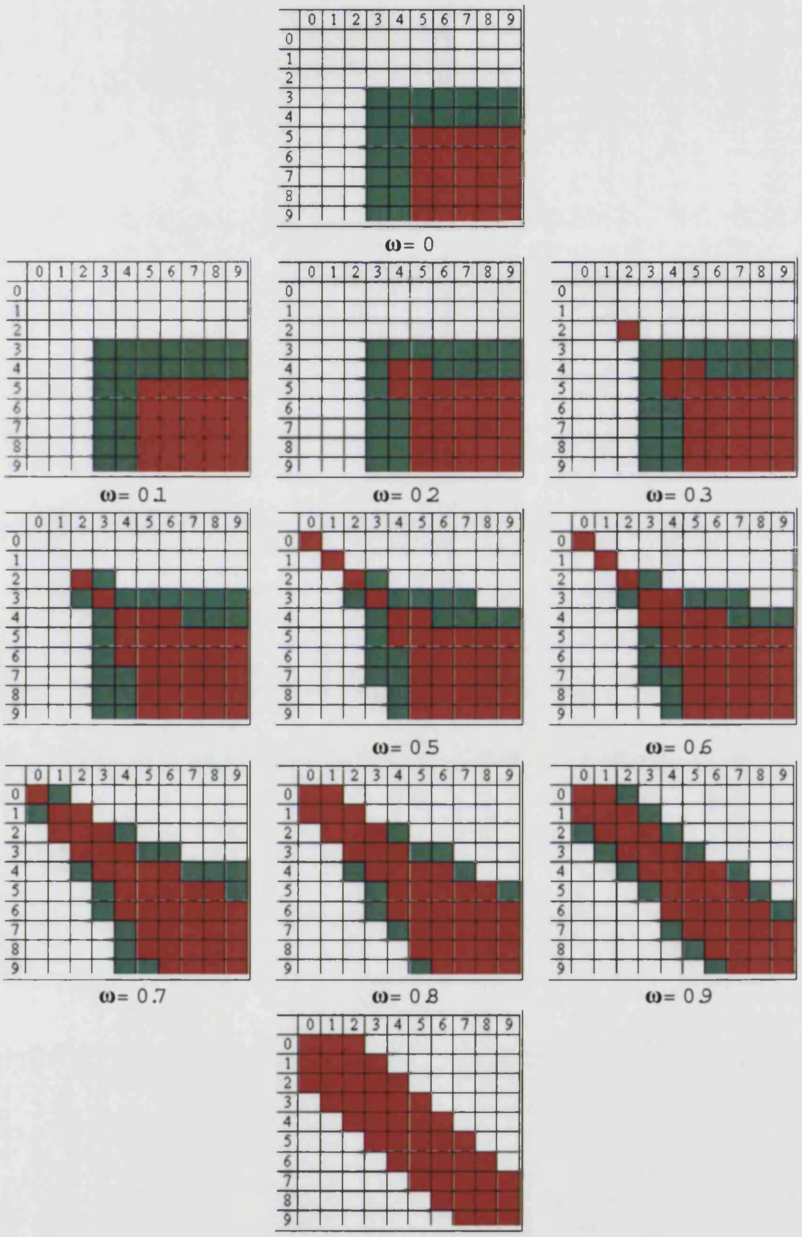
From the definition of the equilibrium, as it is given in equation (20), it is expected that the strategy of a type in period 2 will never be higher that his strategy in period 1. Hence, from Figure 2.14, we note that the strategy in period 1 is higher than in period 2, apart from the equilibrium strategy of type 0 and of low types close to 0 for $\omega < 0.5$. For $\omega < 0.5$ the types close to 0 use the same strategy in both periods. The explanation of the latter lies on the fact that types around 0 are never accepted in the first two periods and therefore expect that they will enter the last period unmated.

If we look at each column of the table in Figure 2.14, it is clear that the utility that a type i expects to receive in the second period decreases as ω increases; the same is true for the utility a type i expects to receive in the last period. For any $\omega > 0$, comparing the strategy that types have in each period, it has the same pattern as in the two period game; it increases as types become higher, until it reaches its maximum and then it starts decreasing again. Furthermore, for any ω, the type having the maximum strategy in period 1 is the same or higher than the type having the maximum strategy in period 2. The minimum strategy in period 1 is always used by type 1 when $\omega \geq 0.5$ and by type 9 otherwise, while the minimum strategy in period 2 is always used by type 9. When players have homotypic preferences the strategies of types are symmetric around middle types 4 and 5 in both periods, as in the two period game.

When players have common preferences ($\omega = 0$) only the strategy of the high type players is relevant at equilibrium, since the low types will never be accepted before the last period. As we will show in the next chapter, under common preferences the strategy in the two period game is a cutoff point, the same for all players. In the three period game, all players use the same strategy in the second period, while in the first period high and low types have different strategies. The high types are choosier, since they expect to be accepted in the second period. The low types, who remain unmated till the last period, are less choosy; they have the same strategy in both the first and the second period. In Figure 2.15 it is noted that players of type $0, 1$ and 2 do not accept anyone below 3 in both the first and the second period, while all higher types do not accept anyone below 5 in the first period. The strategy of types 0, 1 and 2 does not affect the couples created, since these types are rejected by all types in the first and second period. All strategies where types 0, 1 and 2 accept any type higher than i, where $i \geq 3$, in the first period, are strategies that would result in the same pattern of couples. However, only the strategy where types 0, 1 and 2 accept any type higher than 3 in the first period is an equilibrium, according to the equilibrium definition given in Chapter 1.

The strategies of all types are at least as high in the first period as in the second period; no type is choosier in the second period. Comparing the choosiness of types for given ω, for any $\omega \leq 0.2$ and for $\omega = 0.5$ types are choosier the higher they are. For $\omega = 0.3$ and $\omega = 0.4$ types very close to 0 tend to be choosier than the low middle types next to them, while the rest of the types become choosier the closer they are to 9. (However, for $\omega = 0.3$ the strategy of types very close to 0 is irrelevant for the rest of the players, since these types always enter the last period unmated.) For all $\omega < 0.5$, the range of all types includes type 9 but not type 0. This is not always true for $\omega > 0.5$. For $\omega > 0.5$, the acceptance range tends to be smaller on average than for $\omega \leq 0.5$ and it does not always include high types. No type i accept a type j such that $|i - j| > 5$.

For a given ω and a given type i, the range of acceptance of i in the first period includes the same or at most 2 types less that his range of acceptance in the second period. The range of acceptance in the second period is very similar for all types when $\omega < 0.5$. However, as the cost of mating becomes more important, significant differences arise between the acceptance ranges of different types.


$\omega= 0$

$\omega= 0.1$ $\omega= 0.2$ $\omega= 0.3$

$\omega= 0.5$ $\omega= 0.6$

$\omega= 0.7$ $\omega= 0.8$ $\omega= 0.9$

$\omega= 1$

Figure 2.16: Couple formation at equilibrium, $n = 3$.

As ω increases, the changes in the range of acceptance result in changes in the pattern of mated couples. Figure 2.16 gives an image of the couples created before the last period in each game. The same types form couples in both periods under homotypic preferences ($\omega = 1$); hence in this case, the probability of finding a partner is higher in the first period than in the second period. The number of couples created in the first period is minimum when $\omega = 0$ (common preferences) and maximum when $\omega = 0.8$. For $\omega = 1$ (homotypic preferences) more couples than for $\omega = 0$ (common preferences) are formed in the first period. However, their number is lower than the maximum. As ω increases, the number of couples created in the first period increases, since lower types start participating in the mating process, but after it reaches its maximum (for $\omega = 0.8$), it decreases again.

The number of new types which are added in the mating process in period 2 is not increasing as ω increases. As a result, for ω close to 1, the probability that a random type has to find a partner is not always higher in the second period than in the first period. The opposite is true for ω close to 0. On average, a player entering the game, has the highest probability of getting mated in the first two periods when $\omega = 0.7$. The probability of a player entering unmated in the last period is the highest when $\omega = 0$, it then decreases as ω increases and after it reaches its minimum (for $\omega = 0.7$) it starts increasing again up to $\omega = 1$ when it slightly decreases again.

Examining each type separately, high types have higher chances to be mated in the first two periods for ω close to 0.5 and lower chances for ω close to 0 and to 1, while low types have higher chances to be mated before period 3 for ω close to 1. Middle types have little differences in the probability to be mated before the last period as ω varies.

2.9 Conclusions

We extended the solution given by Alpern and Reyniers (1999) for the two period continuous type game and defined the necessary equilibrium conditions for all types in $[-1, 1]$. Furthermore, we developed an algorithm, which allowed to us to reduce the number of calculations needed in order to find all possible equilibria in a discrete type game. With the help of this algorithm, we derived the equilibrium profiles for the discrete type two period game where the number m of positive types is between 1 and 9. We found multiple equilibria when $m = 3$ and $m = 5$ in the two period game $\Gamma_2(m)$. Non exhaustive search showed that multiple equilibria may also appear when the game is played for longer than 2 periods. This finding may have several implications. In some real life situations, players may not want or may not be able to distinguish between continuous types. For example, if a type represents age, colour or size, animals and humans tend to divide types into groups, where the members of each group are considered to belong in the same category or in other words in the same discrete type. In that way, even though types may be continuous, they may be discretised. For instance, often humans are categorised according to their height to be tall, of medium height or short, although humans who are considered tall may not all have the same height.

In the two period discrete type games which we studied, we focused on the "effectiveness" of the equilibria into reducing the mean intra-couple type difference and their stability. The type correlation in the couples formed at equilibrium is in general quite low and it does not present important differences between models. However, this is not a surprising result in a game played for a small number of periods. When the number of types is very small, although the probability of two individuals forming a couple in the first period is very high, players are relatively less choosy than in games with a larger number of types m; therefore it is more probable that players find a partner of a type not so close to themselves. On the other hand, when the number of types increases, players become relatively choosier, so they are mated with types closer to their own type in the first round, whilst the probability of entering unmated in the second round and being obliged to accept any type increases significantly. In contrast with correlation, the marital stability of the equilibria is quite high and it is significantly higher in all models than the stability under random mating (which is 0.5).

In the two period discrete type models, we also studied the welfare (payoff) of different types. In the case of multiple equilibria, we found that different types are better off in different equilibria. Given that types have an incentive to support the appearance of a particular equilibrium, it would be interesting to observe which equilibrium would appear in real life and which factors would affect its appearance.

We also analysed and compared the two and three period models under homotypic or common (or a combination of both) types of preferences. This analysis, allowed us to gain a better insight into how different preferences affect the equilibrium strategies and therefore the correlation and the stability of the couples created. Kalick and Hamilton (1986) showed in their simulation model (their work is described in chapter 5) that both similarity and maximising preferences lead to assortative mating. Looking at the pattern of the couples created at equilibrium before the last period(Figures 2.16 and 2.19), it becomes obvious that in general it is true that high types tend to mate with high types and low types tend to mate with low types under any combination of preferences; nevertheless, how quickly players find a partner differs greatly and it depends on their preferences. High types tend to mate more quickly under common preferences. On the other hand, under homotypic preferences most types have the same probabilities to get mated before the last period; only the extremely high and extremely low types have slightly lower chances than the rest of the types to find a partner before the final period. It is expected that when the impact of common preferences is higher, there will be a higher density of low types in the last period. On the contrary, the population will be more uniform in the last period when the effect of homotypic preferences is stronger.

In the three period game, independently of the preferences of the players, all types are at least as choosy in the first period as in the second period. As a result, there is a wider range of types mutually accepting each other in the second period; exception is the game where players have purely homotypic preferences. Under homotypic preferences the same pairings lead to matings in both periods. In this case, even though extreme types become less choosy in the second period, middle types remains as choosy as in the first period. As a result, the pattern of couples formed in the first period, does not change in the second period.

The decrease of choosiness over time was noted in the common preferences model by Alpern and Reyniers (2005). It is an immediate result of the definition of the equilibrium. Players accept those type which provide them with a utility at least as high as the utility they expect to receive in the next period. Hence if players expect to receive a better utility in the future, they will be choosier in the current period, waiting for a more profitable mating in a future period.

3 Common preferences

3.1 Introduction

This chapter extends Alpern and Reyniers(2005). It is divided into three sections. Sections 3.2 and 3.4 focus on work done with Steve Alpern, while section 3.3 is personal work.

Section 3.2 is a slightly modified version of the LSE-CDAM-2007-32 research report "Equilibria of two-sided matching games", presenting joined work with Steve Alpern on common preferences; the research report appears in the site of the LSE Mathematics department[2]. A shorter version of this report is accepted by the European Journal of Operational Research. It is necessary to note that section 3.2 may include introductory material that the reader may find repetitive.

Section 3.3 compliments the analysis of the efficiency of equilibria done in section 3.2.

The final section, briefly presents the results of an equivalent model to the one presented in section 3.2 that is more applied in biology. A more analytical study of these results is going to be the subject of a joint paper with Steve Alpern.

3.2 Extended Version of the European Journal of Operational Research Article

The problem of pairwise matching of individuals from distinct sets (or sexes) X and Y occurs in many guises: buyers and sellers, employers and employees, medical schools and interns, males and females. We shall use the terminology of the last case, calling the larger group X the males. We assume that individuals of each group have common preferences over whom they would like to be matched with in the other group.

The so-called 'stable marriage' problem proposed by Gale and Shapley (1962) seeks a matching among equal sized finite sets X and Y such that for any two matched pairs (x_1, y_1) and (x_2, y_2), in neither unmatched couple (x_1, y_2) or (x_2, y_1) would each member prefer (with an arbitrary preference relation) their new partner to the one in the original matching. To analyze such questions one must look at complete matchings without considering how they might arise in practice. This 'centralized' problem has received much study (see Roth and Sotomayor (1990)).

More recently, the processes by which complete matchings may arise over time have been analyzed as dynamic games played by the individuals in the two

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[2] http://www.cdam.lse.ac.uk/Reports/Files/cdam-2007-32.pdf

groups. The utilities of these players are often modeled (and will be so here) as 'common preferences' by all members of one sex over individuals of the other. For this reason we can give each individual a 'type' (called x for males, y for females) such that when a couple (x, y) is formed, the male x gets utility y, and the female y gets utility x. We assume that the 'mating season' is short with respect to the time the couple will be together, so that we may ignore the utility consequences of the time (period) in which the couple is formed – there are no search costs in our model. By assuming that an individual's utility is the relative rank of their partner within his or her group, we can normalize these types to the unit interval $[0, 1]$. A male who is unmated at the end of the n(final) period gets a utility $-c$, where c is a known parameter representing the cost of failure to mate. In the 'mutual choice', or 'two-sided', models we shall extend in this section, individuals are randomly paired in each period (that is, the smaller group of females is randomly paired with an equally large randomly chosen set of males - the remaining males are not paired in that period). Then if each member of a matched pair chooses to accept the other rather than go into the next period unmated, they form a couple and are permanently mated. In the final period, players always accept. We call this game $\Gamma_n(r, c)$, where $r \geq 1$ (the 'sex ratio') is the initial number of males divided by the initial number of females. This game has been analyzed by Alpern and Reyniers (2005) in the symmetric case $r = 1$. Johnstone (1997) considered a similar dynamic game model and Kalick and Hamilton (1986) simulated a social psychology version. Related games have been studied by Ramsey (2008) and Eriksson, Sjöstrand and Strimling (2008a).

A strategy for a player in $\Gamma_n(r, c)$ is a rule specifying which potential matches to accept in each period, by determining the least valuable acceptable mate. A strategy profile is called an *equilibrium* if prospective mates are accepted if and only if their type (utility) exceeds the expected utility of the chooser of going into the next period unmated - this is essentially a Subgame perfect Nash equilibrium defined in the same way as in chapter 2. In the symmetric case $(r = 1)$ studied by Alpern and Reyniers (2005), only a single equilibrium was found. In this generalization to $r \geq 1$, we find a region of (r, c) space with multiple equilibria. For example, when $n = 2$ we find three equilibria: a *choosy* equilibrium, where both groups have high acceptance standards; an *easy* equilibrium, where both groups have low but positive acceptance standards; and a *one-sided* (female choice) equilibrium, where males accept anyone. Much of this section is devoted to analyzing and comparing these in terms of dynamical stability and marital stability (a randomized version of Gale's stability condition based on that of Eriksson and Strimling (2004) and Eriksson and Häggström (2008)). For $n = 2$ (and numerically, for higher n) we find that choosiness at equilibrium goes in the same direction for males and females; equilibria with choosy males have choosy females). We find that the choosy and one-sided equilibria are dynamically stable (attracting fixed points of a dynamical system); but the easy equilibrium is dynamically unstable. The equilibrium where both sexes are choosy has the highest marital stability; the equilibrium where only females choose has the lowest. We note that the *existence* of an equilibrium follows from a simple application of Brouwer's Fixed Point Theorem in the same way as established for $r = 1$ by Alpern and Reyniers (2005). As shown there, equilibria are fully determined by a pair of non-increasing $n-1$ tuples of threshold values $(u_1, u_2, \ldots, u_{n-1})$ and $(v_1, v_2, \ldots, v_{n-1})$, where u_i is the lowest type female that a top male ($x = 1$) will accept in period i (similarly for v_i for female choice). At equilibrium, a pairing (x, y) in period i will mutually accept and form a couple if and only if $x \geq v_i$ and $y \geq u_i$. The v_i will always be positive. If all the u_i are 0, we call it a 'one-sided' (or female choice) equilibrium; otherwise we call it a 'two-sided' (or mutual choice) equilibrium.

From the point of view of a single player, a sort of 'secretary problem' (see Ferguson (1989)) is being played out over time, in that he is being presented with a random succession of secretaries. As in the original secretary problem, he may not go back and accept someone he has rejected. However there are many differences: The distribution in each period depends on previous choices of other players; a secretary may reject him; the objective is expected rank. The closest version of the secretary problem is that of Eriksson, Sjöstrand and Strimling (2007).

In contrast to two-sided search models such as the well known one of McNamara and Collins (1990), our model is not steady-state. Each period is different: the sex ratio increases and the distribution of types changes according to the strategies employed. The cohorts are initially uniformly distributed but not in any future period. At all equilibria, individuals become less choosy over time, as suggested in the Pennebaker et al (1979) social science analysis of the country and western song "Don't the girls get prettier at closing time". A good analysis of the effects of changing and uncertain distributions of male quality on female choice has been given in by Collins, McNamara and Ramsey (2006).

Two-sided matching models have been used in various aspects of economic theory, principally by Burdett and Coles (1997,1999), Bloch and Ryder (2000), Eeckhout (1999) and Eriksson and Häggström (2007). In biology and psychology, they have been used to describe and analyze mating behavior in animals (Alpern and Reyniers (1999), Alpern, Katrantzi and Reyniers (2005), Bergstrom and Real (2000)), and in humans (Kalick and Hamilton (1986)). Connections with two-sided spatial matching ('rendezvous search') will be discussed in the Conclusions section.

Some notes on terminology. As our model involves two matching processes, the random pairing of unmated individuals at the start of each period and the permanent coupling of pairs who accept each other, we distinguish these by calling the former process *matching* and the latter *mating.* Some results are obtained numerically, and these will be denoted as *Propositions,* covering the region $1 \leq r \leq 2.5$, $0 \leq c \leq 2.5$.

The section is organized as follows. Section 3.2.2 gives a complete treatment of the two period problem. We find formulae for the three equilibria: e^1 (one-sided), e^2 (easy), e^3 (choosy). We determine the regions of (r, c) space where they exist (Theorem 3.1). We show that male and female choosiness vary in the same way at equilibria (Monotonicity Lemma 3.3). We show that only e^1 and e^3 are dynamically stable (Proposition 3.1); We show that e^3 is the most maritally stable whereas e^1 is the least (Proposition 3.2). In Section 3.2.3 we use both analytical and numerical methods to establish that these properties of equilibria for $n = 2$ periods tend to hold for models with $n > 2$ periods.

We wish to thank an anonymous referee of Alpern and Reyniers (2005) for suggesting that an extension of that paper with a nontrivial sex ratio might yield new phenomena – which it has. The addition of the sex ratio has required new techniques to deal with multiple equilibria, as Alpern and Reyniers (2005) established uniqueness for the trivial (unit) sex ratio case. In addition, they dealt only symmetric equilibria, whereas a large part of the story of this chapter is about the asymmetries of equilibrium strategies resulting from a skewed sex ratio.

3.2.1 The Two Period Game $\Gamma_2(r, c)$

We begin with populations of females and males, with types (quality) uniformly distributed on $[0, 1]$. The females have unit density (and unit population), while the males have density (and population) r (the sex ratio) which is at least 1. Let u and v be the male and female first period threshold strategies; females accept a male x iff $x \geq v$ while males accept female y iff $y \geq u$. A matched male-female pair with types (x, y) will be mated by mutual acceptance if both $x \geq u$ and $y \geq v$ and with random matching the number (understood as a proportion of the female population) of such couples will be

$$k = (1 - u)(1 - v). \tag{21}$$

as shown in the unshaded regions of both the female and male populations of Figure 3.1. In the left square, females are located according to their type (horizontal y axis) and the type of the male they are matched with (vertical x axis). Those in the left rectangle are rejected by their partner and those in the bottom right rectangle reject their partner. The rectangle on the right similarly plots all males, with the additional lower rectangle of unmatched males.

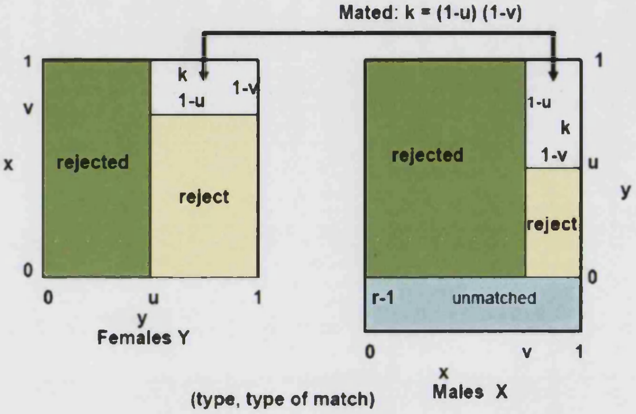


Figure 3.1: Couple formation

The mean value μ_x of the $r-k$ males x that enter the final period unmated (those not in upper right unshaded rectangle) is calculated by dividing them into those with $x<v$ (of average type $v/2$) and those with $x \geq v$ (of average type $(1+v)/2$). The first group of males have population (area) rv, while the second have population $(1-v)(r-1+u)$. Hence

$$\mu_x = \phi(u,v) = \frac{[rv](v/2) + [(1-v)(r-1+u)](1+v)/2}{r-k}. \tag{22}$$

The value (expected payoff) of any female who enters the final period unmated is simply the mean type of the second period male population, that is, μ_x. So in period 1 she should accept a male x iff $x \geq \mu_x$. Hence the female equilibrium condition (f.e.c.) is simply

$$v = \phi(u,v). \tag{23}$$

For v in the range of ϕ, we can solve uniquely for u, giving the f.e.c. (23) as

$$u = f(v) = \frac{-r-2v+2rv+v^2+1}{-2v+v^2+1}. \tag{24}$$

To calculate the corresponding male equilibrium equation (m.e.c.) we first need to obtain the mean female type μ_y in the final period. By symmetry when $r=1$, we obtain this by interchanging u and v in (22):

$$\mu_y = \frac{u(u/2) + (1-u)(v)(1+u)/2}{1-(1-u)(1-v)}. \tag{25}$$

Unlike the simpler case for females, a male entering the final period will not obtain an expected payoff of μ_y - he will only get this if he is lucky enough to be matched. Otherwise he will have the cost c (utility $-c$). The probability p of a male being matched in the second period is the inverse sex ratio

$$p = \frac{1-k}{r-k}. \tag{26}$$

Hence a male entering the final period has an expected payoff, which we will call $\tilde{\mu}_y$, given by

$$\begin{aligned} \tilde{\mu}_y &= p\,\mu_y + (1-p)(-c)\,, \text{ or} && (27)\\ \tilde{\mu}_y &= \psi(u,v) = \frac{1}{2}\frac{2c+v-u^2v-2cr+u^2}{r+u+v-uv-1} && (28) \end{aligned}$$

The justification for the notation $\tilde{\mu}_y$ is that we may add to the final period female population a number $r-1$ of imaginary females with type $-c$, and in this case $\tilde{\mu}_y$ would indeed be the mean type of such a population. Hence the male equilibrium equation (m.e.c.) is

$$u = (\psi(u,v))^+\,, \text{ where } (a)^+ = \max(a,0)\,, \tag{29}$$

since a threshold value of 0 is equivalent to a negative one. Solving the m.e.c. (29) for v as a function of u, for u in the range of ψ^+, gives

$$v = g(u) \equiv \frac{u^2 + (2r-2)u + 2c(r-1)}{(u-1)^2} \tag{30}$$

Definition: A pair (u,v), $0 \leq u,v \leq 1$, satisfying (23) and (29) is called an *equilibrium* of the game $\Gamma_2(c,r)$. If $u=0$, the equilibrium (u,v) is called a *female-choice equilibrium (f.e.c.)* (or, one-sided choice equilibrium) and if $u>0$ it is called a *mutual-choice equilibrium* $(m.e.c.)$ (or two-sided equilibrium). Let E_1 denote the set of all one-sided equilibria, E_2 the two-sided ones, and $E = E_1 \cup E_2$ the set of all equilibria.

Note that at *any* equilibrium we have $v>0$, since the mean of the final period males is always positive.

3.2.2 Equilibrium Theorem

In the symmetric case $r=1$ studied in Alpern and Reyniers (2005) (where c is irrelevant, as all males end up mated), the unique equilibrium was shown to be the mutual-choice equilibrium $u = v = \frac{3-\sqrt{5}}{2} \approx 0.381\,97$. For general r and c the situation is more complicated, though indeed for r sufficiently close to 1 (depending on c) there is still a unique equilibrium which is of mutual-choice type. More generally, we show in Theorem 3.1 that equilibrium behavior partitions (c, r) space by two curves: $r = r_F(c)$, called the 'F' curve; and $r = r_M(c)$, called the 'M' curve. Female choice equilibria exist only on or above the F curve, while mutual choice equilibria exist only on or below the (higher) M curve. These curves are defined as

$$r = r_F(c) = \frac{(c+1/2)^2}{c(c+1)} \text{ and} \tag{31}$$

$$r = r_M(c) = \begin{cases} \dfrac{27(1+c) - \sqrt{27(-5-10c+27c^2)}}{32}, & c \geq 1 \\ r_F(c), & c < 1 \end{cases} \tag{32}$$

For $c \geq 1$, we have $r_M(c) \geq r_F(c)$ with equality only at $c = 1$ (where they are both $\frac{9}{8}$, and tangent to each other). Figure 3.2 shows how the two curves F ($r = r_F(c)$) and M ($r = r_M(c)$) divide c, r space into three open regions defined by

$$I = \{(r,c) : r > r_M(c) \text{ and } c \leq 1, \text{ or } r > r_F(c) \text{ and } c > 1\}, \tag{33}$$

$$II = \{(r,c) : r < r_F(c)\}, \text{ and} \tag{34}$$

$$III = \{\{(r,c) : r_F(c) < r < r_M(c) \text{ and } c < 1\}\}. \tag{35}$$

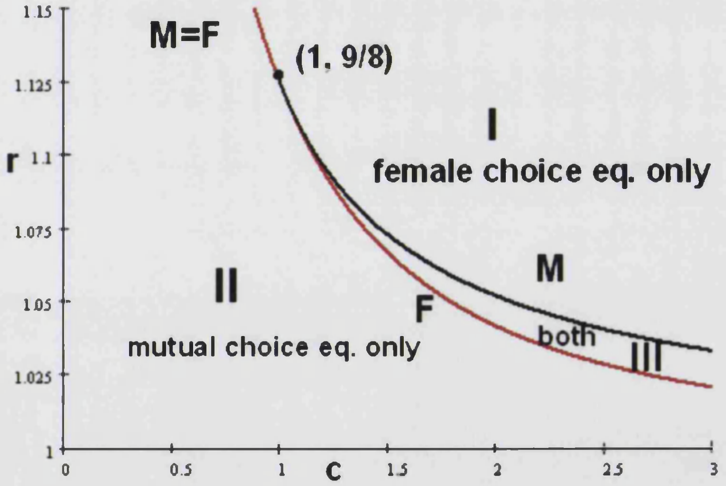


Figure 3.2: Illustration of theorem 3.1

We use this numbering of the regions because: region I has a *one*-sided (female-choice) equilibrium, region II has a *two*-sided (mutual-choice) equilibrium, and region III has *three* equilibria (one female-choice and two mutual choice). The following is our main result for the two period game, and will be proved in the next section.

Theorem 3.1 (common preferences) *Consider the two period game* $\Gamma_2(r,c)$, *for* $r \geq 1$ *and* $c \geq 0$, *and let the regions* I, II, *and* III *be defined as in (33-35).*

1. *If* $(r,c) \in I$, *then there is a unique equilibrium and it is a female-choice equilibrium*
2. *If* $(r,c) \in II$, *then there is a unique equilibrium and it is a mutual-choice equilibrium*
3. *If* $(r,c) \in III$, *then there are three equilibria: one of them is a female-choice equilibrium, and the other two are mutual-choice equilibria.*

Figure 3.3 illustrates equilibria in regions I, II, III, given as the intersection of the female equilibrium condition (23) drawn in purple (thin) and the male equilibrium condition (29) drawn in green (thick).

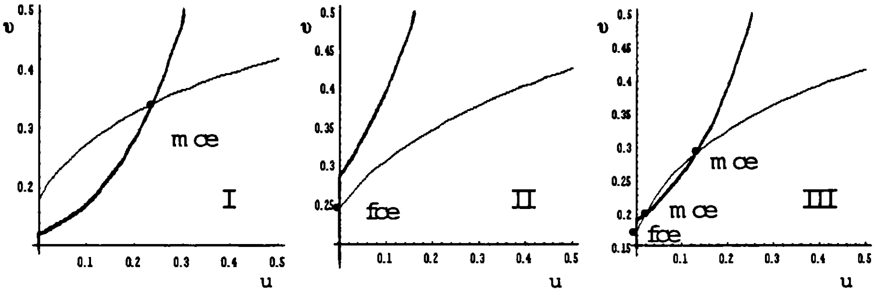


Figure 3.3: Male (green) and female (red) equilibrium conditions

The equilibria in the intersections of regions I, II and III are illustrated in Appendix B.1.

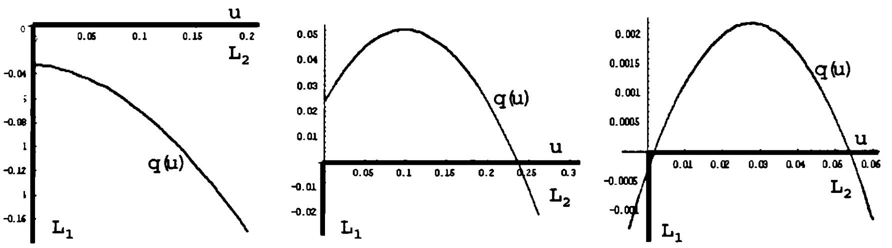
3.2.3 Proof of Theorem 3.1

In Figure 3.3 we illustrated Theorem 3.1 by exhibiting the equilibria E symmetrically with respect to male and female strategies u and v, as the intersection of the male and female equilibrium conditions. However to prove Theorem 3.1, we now take an asymmetric approach in which we determine only the male threshold strategies u which form half of an equilibrium pair (u,v). Of course if we know u, then v is uniquely determined by the female equilibrium condition (23). So for the time being we forget about v and concentrate only on u.

To establish Theorem 3.1, we show that the set $E = E_1$(one sided, f.c.e.'s) $\cup$ E_2 (two-sided, mce's) of equilibria is determined by the intersections of a certain cubic polynomial $q(u)$ with the disjoint union

$$\Gamma = L_1 \cup L_2, \tag{36}$$

where L_1 is the negative $y-$axis $\{(0,y), -\infty < y \leq 0\}$ and L_2 is the open interval $\{(u,0) : 0 < u < 1\}$. Intersections with L_1 give fce's and those with L_2 give mce's. This is illustrated in Figure 3.4 for the three regions discussed in Theorem 3.1: For region I, q intersects only L_1; for region II, q only intersects L_2; for region III, q intersects L_1 and then intersects L_2 twice, once before and once after the local maximum of q. Figure 3.4 should be compared with the earlier Figure 3.3, noting that the earlier one indicated both coordinates (u and v) of each equilibrium, while this figure indicates only the u coordinate.



I: r=1.12 c=1.2 II: r=1.05 c=1.2 III: r=1.095 c=1.2

Figure 3.4: Intersection of q with $L_1 \cup L_2$, regions I,II,III

The following result establishes that the intersections shown in Figure 3.4 are indeed equilibria.

Lemma 3.1 (common preferences) *Fix any parameters* $r \geq 1$ *and* $c \geq 0$. *Then*

$$(u, g(u)) \in E \textit{ if and only if } (u, q(u)) \in \Gamma. \tag{37}$$

Furthermore

$$(u, g(u)) \in E_i \textit{ if and only if } (u, q(u)) \in L_i. \tag{38}$$

Proof. Recall that $v = g(u)$ (30) is a form of the male equilibrium equation. In this analysis u is always in $[0,1]$. If we combine the two (female and male) equilibrium conditions in the form of (24) and (30), we can obtain all mutual-choice equilibria by seeking solutions $u \in (0,1)$ to the fixed point equation

$$\begin{aligned} u &= f(g(u)), \text{ or equivalently, solving} && (39)\\ h(u) &= u - f(g(u)) = 0. && (40) \end{aligned}$$

We can factor the rational function $h(u)$ in the form

$$h(u) = \frac{-(1-u)\ \ q(u)}{(1+2c-2cr-2ru)^2}, \text{ where } q = q(u) \text{ is the cubic} \tag{41}$$

$$q = ru^3 - 3ru^2 + \left(5r + 4cr - 4r^2 - 4cr^2\right)u - 4c^2r^2 - 4cr^2 + 8c^2r + 8cr + r - 4c^2 - 4c - 1 \tag{42}$$

For $0 < u < 1$, $q(u)$ is 0 if and only if $h(u)$ is 0, which is equivalent to (38) for $i = 2$. The condition $f(g(0)) < 0$ is equivalent to $(0, g(0))$ being a female-choice equilibrium, because $(0, g(0))$ always satisfies the male equilibrium condition (30) and $(f(g(0)), g(0))$ satisfies the female equilibrium condition (24). But a negative threshold strategy $f(g(0))$ for the males is strategically equivalent to $u = 0$ (as there as no females of negative type y). Hence $(0, g(0))$ is an equilibrium. But the condition $f(g(0)) < 0$ is equivalent to $0 - f(g(0)) > 0$, or $q(0) < 0$. Thus (38) holds for $i = 1$ as well, and hence the main condition (37) also holds. ■

Lemma 3.1 reduces the proof of Theorem 3.1 to the determination of the intersections of the cubic curve $q(u)$ with the set Γ, for different values of the parameters r and c. The analysis of the cubic q is given in the following Lemma. For Theorem 3.1 we will need information about the location α of the local maximum and its height $q(\alpha)$.

Lemma 3.2 (analysis of cubic q) *The cubic $q(u)$ increases from $-\infty$ to it's local maximum $q(\alpha)$ at α, then decreases until its local minimum at β, from which point it increases to infinity. The numbers α and β (the two solutions of the quadratic equation $q'(u) = 0$) are given by*

$$\begin{aligned} \alpha &= 1 - \sqrt{2/3}\sqrt{D} < 1, \text{ and } \beta = 1 + \sqrt{2/3}\sqrt{D} > 1, \text{ where} && (43)\\ D &\equiv 2r - 1 + 2c(r-1) > 1. && (44) \end{aligned}$$

For all parameter values, we have $q(1) < 0$, but the values of $q(0)$ and $q(\alpha)$ and

α depend on the parameters r and c in that

$$
\begin{aligned}
sign\left(q\left(0\right)\right) &= sign\left(r_F\left(c\right)-r\right), & (45)\\
if\, r &< r_M\left(c\right),\ c\geq 1,\ then\ q\left(\alpha\right)>0 & (46)\\
if\, r &> r_M\left(c\right),\ c\geq 1, then\ either\ q\left(\alpha\right)<0\ or\ \alpha<0 & (47)\\
sign\left(\alpha\right) &= sign\ \left(r_3\left(c\right)-r\right),\ where\ r_3\left(c\right)=\frac{4c+5}{4c+4}. & (48)
\end{aligned}
$$

We can now use our two Lemmas to give a simple proof of Theorem 3.1 which involves breaking up region I into two regions I_a (with $c \leq 1$) and I_b (with $c > 1$), as shown below in Figure 3.5.

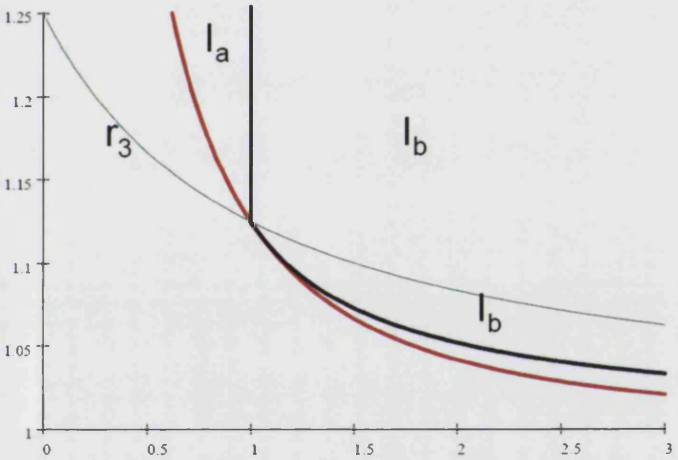


Figure 3.5: Partition of I into I_a and I_b

Proof of Theorem 3.1: The portions of the Theorem concerning female choice equilibria follow immediately from (38) for $i = 1$ and (45). That is, there is a female-choice equilibrium if $q(0) \leq 0$, which is equivalent to $r > r_F(c)$. So we need to consider only mutual-choice equilibria. Recall that u corresponds to a mutual-choice equilibrium if $0 < u < 1$ and $q(u) = 0$. We have shown in the previous Lemma (3.2) that $q(1)$ is always negative. We know that q is decreasing from α to $\beta > 1$. We now prove the cases in turn.

1a If $(c, r) \in I_a$, then $r \geq r_F(c)$ and $r \geq r_3(c)$. It follows from (45) that $q(0) \leq 0$ and from (48) that $\alpha \leq 0$. Consequently q is decreasing between 0 and 1, and can have no root in that open interval. Hence there is no mutual-choice equilibrium.

1b If $(c,r) \in I_b$, $r > r_M(c)$ and hence also $r > r_F(c)$. The latter condition ensures by (45) that $q(0)$ is negative. Since $r > r_M(c)$ we have by (47) that $q(\alpha) < 0$ or $\alpha \leq 0$. If $\alpha \leq 0$, then as in the previous part, q is decreasing between 0 and 1. If $\alpha > 0$, q will increase until α, but $q(\alpha) < 0$, so q has no root less than α. From α to 1 it is decreasing and the result follows.

2 If $(c,r) \in II$, we have $r < r_F(c)$ and so by (45) we have $q(0) > 0$. Since $q(1) < 0$ for all parameters, the Intermediate Value Theorem guarantees at least one root of q between 0 and 1, hence at least one mutual choice equilibrium. If q had two roots between 0 and 1, then it would have a local minimum between them. But $q(u)$ has only one local minimum, at $\beta > 1$. Hence in this case there is exactly one mutual-choice equilibrium.

3 If $(c,r) \in III$, then $r > r_F(c)$, $r < r_M(c)$, and hence $r < r_3(c)$. So by (45) we have $q(0) < 0$, by (46) we have $q(\alpha) > 0$ and by (48) we have $\alpha > 0$. Hence by the Intermediate Value Theorem, q has a root between 0 and α and another root between α and 1. We have already explained above why q cannot have more than two roots between 0 and 1. Hence there are two mutual-choice equilibria.

3.2.4 Analysis of Equilibria

In Theorem 3.1 (common preferences) we determined the number and type of equilibria, as a discrete function (regions I, II, III) of the parameter values r and c. Here we obtain explicit formulae for these equilibria and analyze how they depend continuously on the parameters r and c. Our first observation is that when comparing equilibria, the level of choosiness (acceptance level) goes in the same direction for both males and females, the monotonicity Lemma. In other words, one of the equilibria is choosier than the other (for both sexes). To see this, recall that any equilibrium pair (u,v) satisfies the female equilibrium equation (24) $u = f(v)$, so $f'(v) = 2vr/(1-v)^3 > 0$ implies that u is increasing in v, giving the following.

Lemma 3.3 (common preferences) (Monotonicity) *Given any two equilibria* (u,v) *and* (u',v'), *we have* $u \neq u'$ *and*

$$(u'-u)(v'-v) > 0 \tag{49}$$

As an application of this Lemma, the three potential equilibria can be ordered in terms of *choosiness* as $e^i = (\bar{u}_i, \bar{v}_i)$, $i = 1,2,3$, where for $i < j$ we have both $\bar{u}_i < \bar{u}_j$ and $\bar{v}_i < \bar{v}_j$.We name these (where the latter two are mutual choice equilibria) as:

e^1, the *female choice equilibrium*, which exists on and above the F curve,

e^2, the *easy equilibrium*, which exists between the M and F curves, and

e^3, the *choosy equilibrium*, which exists on and below the M curve.

For the female choice equilibrium e^1, we obviously have $\bar{u}_1 = 0$, and can obtain v_1 directly from the female equilibrium condition (24) $0 = f(v)$, or $0 = -r - 2v + 2rv + v^2 + 1$, with unique positive solution

$$\bar{v}_1 = 1 - r + \sqrt{r^2 - r}. \tag{50}$$

For the mutual choice equilibria $e^i = (\bar{u}_i, \bar{v}_i)$, $i = 2, 3$, we obtain the formula for $\bar{u}_i$ by explicitly solving the cubic equation $q(u) = 0$ for $u = \bar{u}_i$, getting the corresponding $\bar{v}_i$ from the formula $\bar{v}_i = g(\bar{u}_i)$ (30).

$$\bar{u}_2 = 2\sqrt{\gamma}\cos\left(\frac{t + 2\pi}{3}\right) + 1, \quad \bar{u}_3 = 2\sqrt{\gamma}\cos(\frac{t + 4\pi}{3}) + 1, \text{ where} \tag{51}$$

$$\begin{aligned} t &= \arccos\left((2 - a_1 - a_0)/2\sqrt{\gamma^3}\right), \gamma = (3 - a_1)/3, \\ a_1 &= 5 - 4c(r-1) - 4r, a_0 = \left(-1 + 4c(r-1) + 4c^2(r-1)\right)(1-r)/r \end{aligned}$$

To see what the equilibria look like in u, v space, for a 4×5 grid of c, r parameters taken to cover points in all three regions I, II, and III (see Figure 3.6), we draw the equilibria in a similarly arranged array of boxes in Figure 3.7. In each box (with r and c fixed), we plot any female choice equilibria with a red triangle and any mutual choice equilibria with blue diamonds.

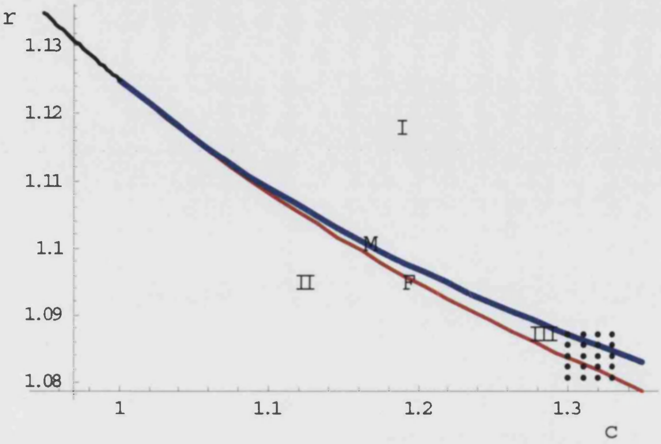


Figure 3.6: The 20 grid points for equilibrium analysis

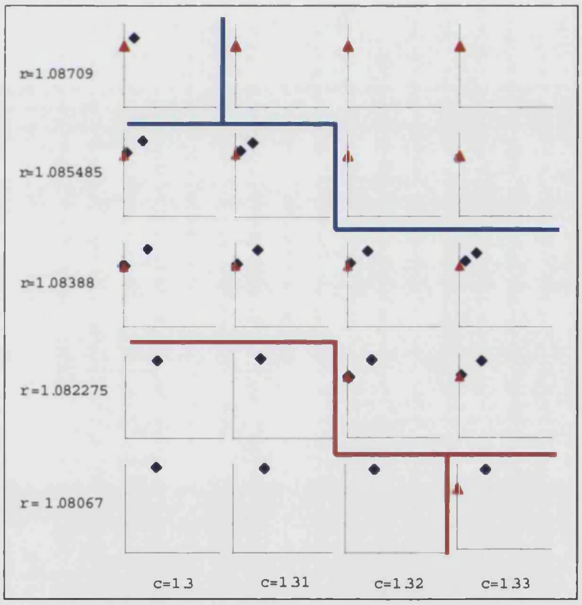


Figure 3.7: Equilibria at the 20 c, r grid points

We have drawn the M line in blue and the F line in red. Note that the grid points were chosen so that the former goes through the upper left box and the latter goes through the lower right box. For that reason we have drawn the lines as splitting around these boxes. Observe that in all cases in Figure 3.7, a line between two equilibria is always upwards sloping, as follows from the Monotonicity Lemma. Note that as we go up (increasing r) the column of boxes corresponding to $c = 1.32$, we start with one mutual choice equilibrium (which is the choosy one e^3), then get all three, and finally get only the female choice equilibrium e^1. A better way of seeing these transitions is to consider the bifurcation diagram drawn in Figure 3.8 with the sex ratio r increasing to the right, and the male and female acceptance levels drawn in the vertical axis. The lower (black) curves describe the male equilibrium acceptance levels u, while the top (red) lines describe the female levels v. In region III, the equilibrium values for each sex appear, from top to bottom, in the order e^3, e^2, e^1.

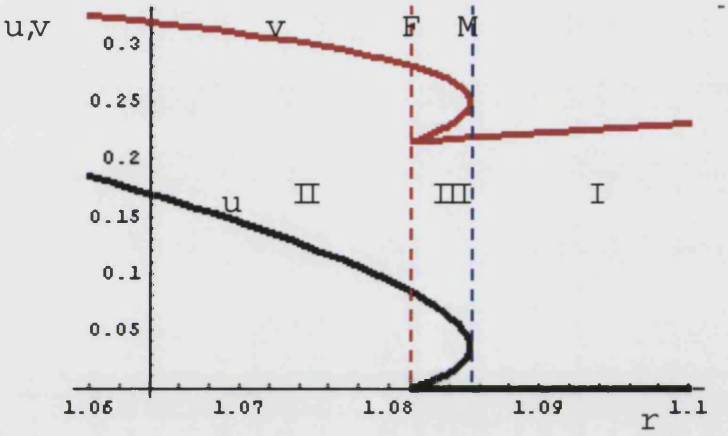


Figure 3.8: Bifurcation in r of equilibria for $c = 1.32$

3.2.5 Dynamical Stability of Equilibria

The equilibria e^i are solutions to both the male and female equilibrium conditions (29,23), or equivalently are fixed points of the mapping T given by

$$T(u,v) = \left(\psi^+(u,v), \phi(u,v)\right). \tag{52}$$

In this section we determine the dynamical stability of the equilibria e^i as fixed points of the mapping T. That is, a fixed point is stable if iterations of T applied to nearby points converge back to it. To do this, we must determine the matrix norm

$$N^j(u,v) = \left\| \begin{pmatrix} \frac{\partial\psi^+}{\partial u} & \frac{\partial\psi^+}{\partial v} \\ \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v} \end{pmatrix}^j \right\|. \tag{53}$$

A fixed point (u, v) will be dynamically stable if for some j, $N^j(u, v) < 1$. Since we found the formulae for the equilibria $e^i(r, c)$ in the previous section, we can evaluate the Jacobian matrix at these u, v values. We find that $N^2\left(e^1(r, c)\right) < 1$ where e^1 exists (on and above the F curve) and $N^2\left(e^3(r, c)\right) < 1$ where e^3 exists (on and below the M curve). Furthermore $N^1\left(e^1(r, c)\right) < 1$ in the bigger part of area II and $N^1\left(e^3(r, c)\right) < 1$ in the bigger part of area I. On the other hand both eigenvalues of the Jacobian of $\left(\psi^+, \phi\right)$ at the easy equilibrium e^2 have absolute values larger than 1. Summarizing these numerical results, we have the following.

Proposition 3.1 *Let* e^1, e^2 *and* e^3 *be the female, easy and choosy equilibria. The equilibria* e^1 *and* e^3 *are dynamically stable and the equilibrium* e^2 *is unstable.*

Our numerical results supporting the above proposition and some examples on the basins of attraction of each equilibrium type are illustrated in Appendix B.2.4.

3.2.6 Marital Stability σ of Equilibria

Suppose we look at the distribution of couples over the (x, y) square that arises at the end of the play of our game, or indeed that arises in any way. We ignore the unmated males in this analysis. For the moment, suppose that agent preferences are arbitrary. We say that a pair of couples (x_1, y_1) and (x_2, y_2) is *unstable* if a male from one couple and a female from the other both prefer each other to their current partner. In our common preference model, where type equals utility to the opposite sex, this means that the better (higher type) male and the better female belong to distinct couples, or that $(x_2 - x_1)(y_2 - y_1) < 0$. If a pair of couples is not unstable, we say it is *stable.* We define the *Stability* σ of a given distribution to be the probability that a randomly and independently chosen pair of couples is stable. This definition is similar in spirit to that proposed by K. Eriksson and P. Strimling in (2004). In our two period game, every strategy pair (u, v) (not only equilibrium pairs) leads to a couple distribution that is uniform (with some constant density) on each of the four sub-rectangles rectangles R_k of the unit square in x, y space determined by the lines $x = v$ and

$y = u$ drawn in Figure 3.9.

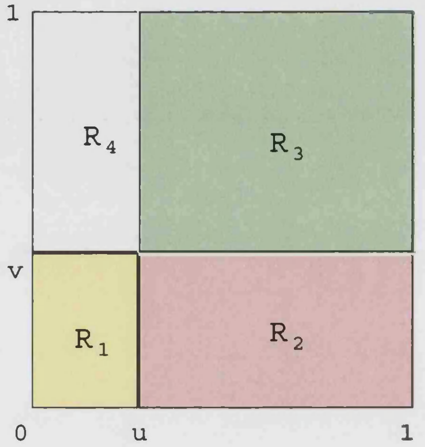


Figure 3.9: Couple distribution on the type square.

Let π_k denote the probability that a couple belongs to R_k. Note that $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ and that $\pi_1 + \pi_2 = 1 - u$, while $\pi_1 + \pi_3$ depends on r. Define a symmetric 4×4 matrix S so that $s_{i,j}$ is the probability that a pair of couples is stable, given that the couples belong to R_i and R_j. It is easy to see that two couples belonging to R_2 and R_4 form an unstable pair, while a pair belonging to R_1 and R_3 form a stable pair. Otherwise, the couples belong to the same rectangle or two rectangles whose union is a rectangle R, and which are each preserved under a symmetry transformation θ of R. Observe that θ transposes pairs of such couples in such a way that if one is stable then the other is unstable. Hence for all these cases, $s_{ij} = 1/2$. For example, Figure 3.10 illustrates how $\theta(y) = u - y$ transposes stable with unstable couple pairs R_1 and R_2.

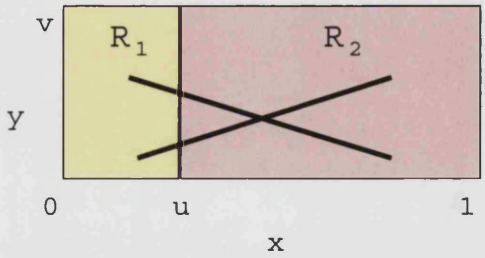


Figure 3.10: $s_{3,4} = 1/2$

Thus

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{54}$$

So if the distribution over the rectangles is $\pi = \pi(u, v) = (\pi_1, \pi_2, \pi_3, \pi_4)$, the stability σ is given by

$$\sigma = \sigma(u, v) = \pi S \pi = \frac{1}{2}(1 - 2\pi_2\pi_4 + 2\pi_1\pi_3). \tag{55}$$

Note that at any female choice equilibrium we have $u = 0$, hence $\pi_1 = \pi_4 = 0$, so by (55) we have $\sigma = 1/2$. More generally, we calculated σ at the three equilibria in III, observing that

Proposition 3.2 *For any r and c, in region III, the choosy equilibrium e^3 is the most stable one and the female choice equilibrium e^1 is the most unstable one. That is,*

$$0.5 = \sigma(e^1(r, c)) \leq \sigma(e^2(r, c)) \leq \sigma(e^3(r, c)).$$

Furthermore, $\sigma(e^2(r, c)) \leq 0.54$ and $\sigma(e^3(r, c)) < 0.59$.

The distribution π is shown in Appendix B.3 and the numerical results supporting proposition 3.2 are illustrated in Appendix B.4.

Figure 3.11 plots the marital stability σ of the three equilibria as a function of r, for fixed $c = 1.32$. The red line is $\sigma(e^1)$, the black line $\sigma(e^2)$ and the green line $\sigma(e^3)$.

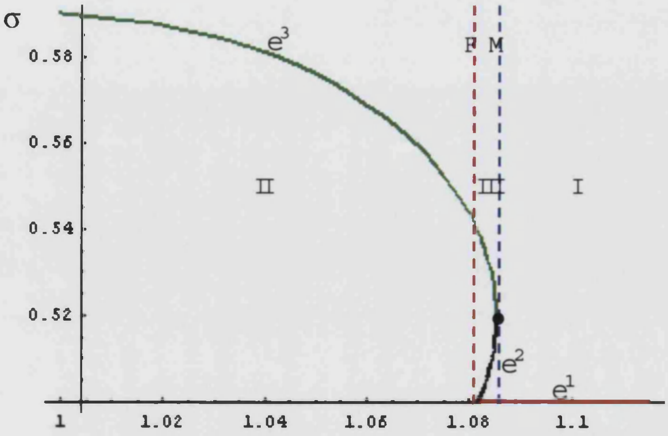


Figure 3.11: Marital Stability σ of the e^i.

3.2.7 n-Period Model, $n > 2$

In the previous section, we were able to obtain a complete analytical description of the equilibria for the two period model, with explicit formulae. Due to the complexities involved, this will no longer be possible for models with $n > 2$ periods. However, we are still able to obtain some analytical solutions for the continuous (uniform distribution) model and fairly complete numerical solutions for models where both sexes come in m discrete types. We have extensive results for $n = 3$ and very partial results for $n = 4$.

Recall that in the two period game all (say) females who enter the last period unmated do equally well, they get the mean male of the final period. Consequently at equilibrium all female types have the same threshold strategy in the first period (namely this male mean). This observation remains true in the penultimate period (last strategic period) in the n-period game – but not in earlier periods. Consider the three period game, and an equilibrium in which the males are choosy in period 2, accepting females equal or above some type a. Hence in period 1, females of low types $[0, a)$ do worse if they enter period 2 unmated than those females of high types $[a, 1]$. So the low female types accept males of types above some v_L and the high types accept those above v_H, with $v_H > v_L$. However since the low female types will not be accepted by males in period 2, they also will not be accepted in period 1 (where males are more choosy). So v_L is not relevant to any important equilibrium property and can be ignored in the analysis. This fact, the irrelevance of low type strategies, was already observed in Alpern and Reyniers (2005). So the only relevant numbers are the equilibrium strategies in each period of the top male ($x = 1$) and female ($y = 1$) types. Once the equilibrium strategies of the high types are determined, those of the lower types can if desired by calculated as well, but they don't affect which couples form at equilibrium. It is worth noting in this multi period context that the threshold strategies for top types are decreasing in time (period). This is an observation obtained earlier in Alpern and Reyniers (2005).

3.2.8 Female-Choice Equilibrium for $n = 3$

This section presents our only analytical result for $n > 2$, the determination of the unique female choice equilibrium for $n = 3$. We assume that any male accepts any female in any period. For females, we denote by v^1 and v^2, $v^2 < v^1$, the lowest male type that a top female ($y = 1$) will accept in periods 1 and 2. Let $p_i = 1/r_i$ be the inverse sex ratio, the ratio of females to males at the beginning of period i. This is the probability that a male entering period i will be matched in that period. Let $q_i = 1 - p_i$ be the complementary probability of not being matched in period i, so that

$$q_1 = 1 - 1/r, \quad q_2 = 1 - v_1/\left(v_1 + r - 1\right).$$

There are three type-classes of male: High, of type in $H = [v_1, 1]$, with initial probability $P_H = 1 - v^1$ and mean $\mu_H = \left(1 + v^1\right)/2$; Medium, with types in $M = \left[v^2, v^1\right]$ with initial probability $P_M = v^1 - v^2$ and mean $\mu_M = \left(v^1 + v^2\right)/2$; Low, with types in $L = \left[0, v^2\right]$ with initial probability $P_L = v^2$ and mean $\mu_L = v^2/2$. The probabilities that males of these types reach the final period unmated are given by

$$\bar{P}_L = 1, \bar{P}_M = q_2, \bar{P}_H = q_1 q_2.$$

The expected payoff e_3 to a female entering the final period is simply the mean type of the final period male distribution, and hence given by

$$e_3 = \frac{P_L \bar{P}_L \mu_L + P_M \bar{P}_M \mu_M + P_H \bar{P}_H \mu_H}{P_L \bar{P}_L + P_M \bar{P}_M + P_H \bar{P}_H}. \tag{56}$$

Thus

$$e_3 = \frac{r^2 - v_1^2 - 2r + v_1^2 r + v_1 v_2^2 r + 1}{2v_1 r - 4r - 2v_1 + 2v_1 v_2 r + 2r^2 + 2}. \tag{57}$$

The expected utility e_2 for a female entering period 2 unmated is calculated as follows: If she meets a Low male, she goes into final period and gets e_3; if she meets a Middle male she accepts and gets on average $\left(v^1 + v^2\right)/2$; if she meets a High male, she accepts and gets on average $\left(1 + v^1\right)/2$. Hence her expected payoff is given by

$$e_2 = \frac{v^2 r}{\kappa} e_3 + \left(\left(v^1 - v^2\right) r/\kappa\right) \left(\frac{v^1 + v^2}{2}\right) + \frac{\left(1 - v^1\right)(r - 1)}{\kappa} \left(\frac{1 + v^2}{2}\right), \tag{58}$$

where $\kappa = r - \left(1 - v^1\right)$ is the male population in period 2. Hence the female equilibrium condition is given by the two equations,

$$v^2 = e_3 \text{ and } v^1 = e_2 \tag{59}$$

The solution $v^1(r)$ and $v^2(r)$ to the female equilibrium equations is drawn (using *Mathematica*) in the following figure. Of course, these will be equilibria only if c is sufficiently large so that males will always accept, that is, if the male equilibrium equations are also satisfied.

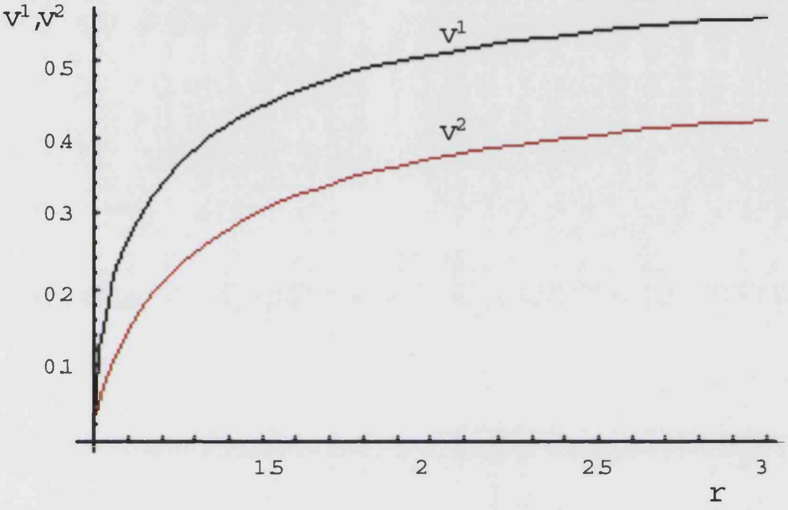


Figure 3.12: Female equilibrium strategies in the one sided choice 3 period game.

To determine when v^1, v^2 forms an equilibrium with males always accepting $(u^1 = u^2 = 0)$ we must find when a male of type 1 will accept a female of type 0 in period 1. Clearly he will do this only if his expected payoff w_2, if he goes into period 2 unmated, is not positive. A type 1 male will be accepted if matched, and will on average be matched with a type $\frac{1}{2}$ female. If he is not matched in either period, he gets $-c$. Thus

$$w_2 = w_2(r,c) = (p_2)\,\frac{1}{2} + (q_2\;p_3)\,\frac{1}{2} + (q_2\;q_3)\,(-c)\,. \tag{60}$$

Solving the equation $0 = w_2(r,c)$ for r as a function of c gives the line (which we again call the F line) above which we have one female choice equilibrium and below which we have none. This line is drawn as $F = F^3$ (again using Mathematica) in Figure 3.13, alongside the F and M lines of the two period model of Figure 3.2.

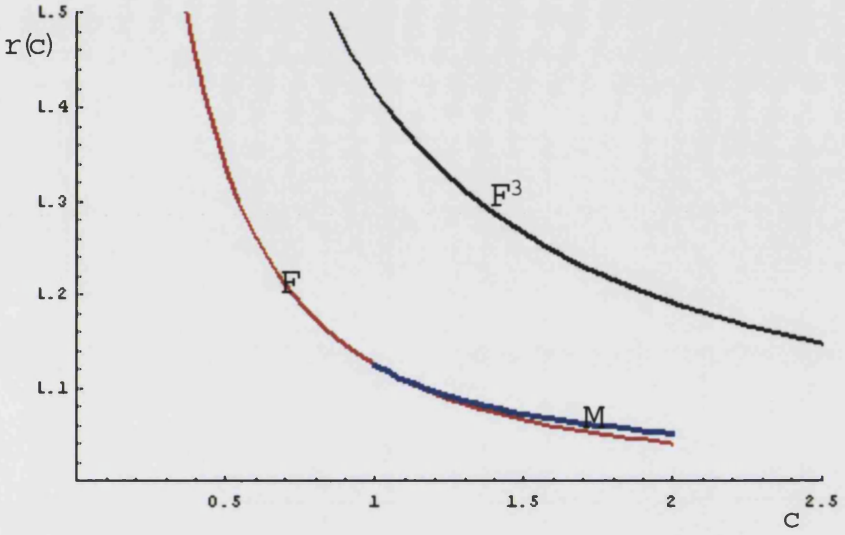
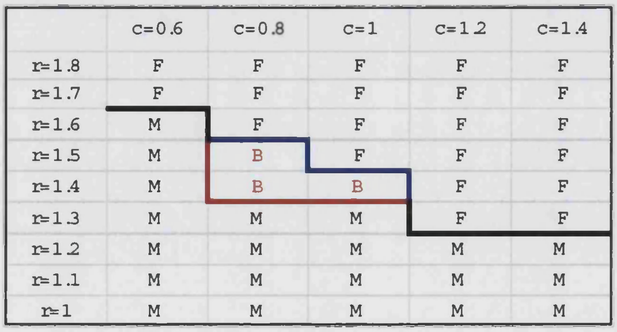


Figure 3.13: The F line for $n = 3$.

3.2.9 Discrete Type Model for $n = 3$

Although it was possible in the previous section to obtain the female choice equilibrium for $n = 3$ and the uniform distribution by analytical means, the mutual choice equilibria cannot be obtained in this way. For this reason we now turn to the game $\Gamma_{n,m}(r, c)$ in which both sexes are initially distributed with equal amounts of types $0, 1, 2, \ldots, m-1$. To align our results with the continuous model where types belong to the interval $[0, 1]$, the cost to a male of not mating will be given as $c\ m$. For $n = 3$ and small m we can obtain *all* the equilibria by a modified exhaustive search technique illustrated in Appendix B.5. Figure 3.14 charts, for $m = 8$, the qualitative aspects of this search, for r and c in the grid. Here, F represents just a female choice equilibrium, M just a male choice equilibrium, and B the presence of both types. The 'F' line is drawn in red, the 'M' line in blue, and the portion M=F is drawn in black. One can easily detect the same qualitative partitioning of c, r space into the regions I, II, and III of the table in Figure 3.14. Note that, compared with the two period problem, region III (B's) is smaller and the 'M' line is lower.



	c=0.6	c=0.8	c=1	c=1.2	c=1.4
r=1.8	F	F	F	F	F
r=1.7	F	F	F	F	F
r=1.6	M	F	F	F	F
r=1.5	M	B	F	F	F
r=1.4	M	B	B	F	F
r=1.3	M	M	M	F	F
r=1.2	M	M	M	M	M
r=1.1	M	M	M	M	M
r=1	M	M	M	M	M

Figure 3.14:. Regions I, II, III for $n = 3$, $m = 8$

To obtain a more quantitative analysis of the equilibria, as functions of r and c, we describe in Figure 3.16 the equilibria corresponding to a grid of r and c values. This is analogous to Figure 3.7 for $n = 2$, except that for $n = 2$ an equilibrium could be represented by a single point (u, v), whereas for $n = 3$ we represent each equilibrium by a line segment between the lower male equilibrium values (u_1, u_2) and the higher female values (v_1, v_2). The grid lines correspond to $u, v \in \{0, 1, 2, 3, 4\}$. Female choice equilibria, which have lower point $(u_1, u_2) = (0, 0)$, are drawn in red and mutual choice equilibria are drawn in black.

In order to make it easier for the reader to interpret Figure 3.16, we will focus on an example. We examine the case where $c = 1.4$ and $r = 0.6$; we have one mutual choice equilibrium where $(u_1, u_2) = (1, 0)$ and $(v_1, v_2) = (4, 2)$. The equilibrium is represented by two dots. The coordinates of the lower dot indicate the male strategy (u_1, u_2) in periods 1 and 2 and the coordinates of the higher dot indicate the female strategy (v_1, v_2) in periods 1 and 2. Figure 3.15 illustrates the equilibrium strategies for $c = 1.4$ and $r = 0.6$.

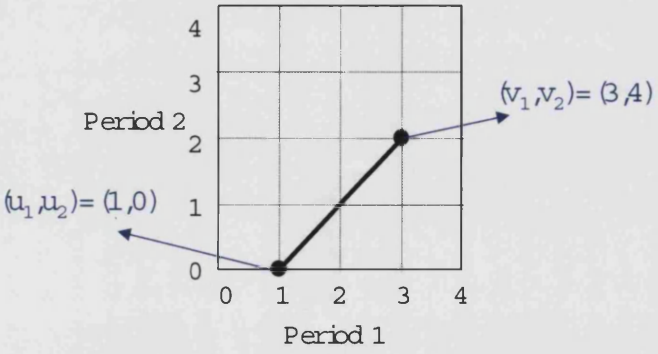


Figure 3.15: Equilibrium for discrete uniform distribution, $m = 8$, when $r = 0.6$ and $c = 1.4$.

Note in Figure 3.16 that most of the mutual choice equilibria start at the bottom ($u_2 = 0$), so in these the males are only choosy in the first period. In fact, only in the two boxes corresponding to $r = 1.1$ and $c = .6$ and $.8$ is there a mutual choice equilibrium where the males are choosy in both periods.

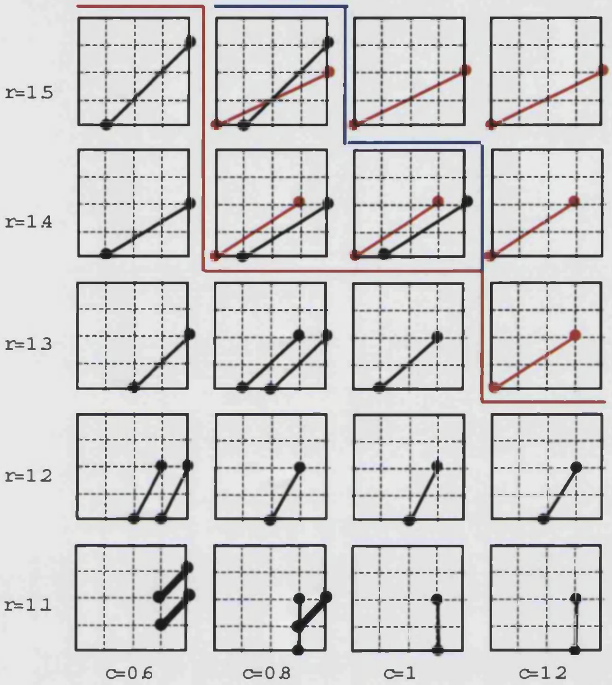


Figure 3.16: Equilibria for discrete uniform distribution, $m = 8$.

The red line between the boxes is the F line and the blue line is the M line.

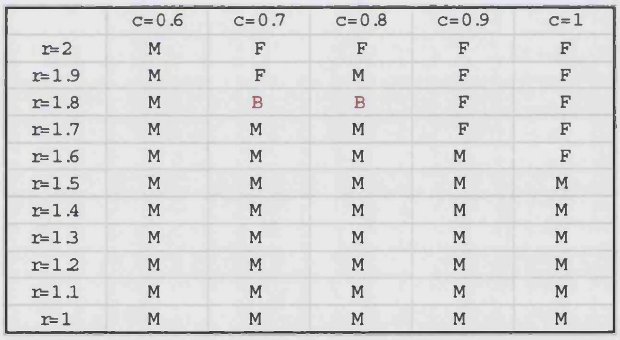
We also analysed a discrete 8 type model played over 3 periods, where types are not uniformly distributed in the first period. We assumed a truncated normal distribution of the population, so that middle types are more common than extreme types. Our results are presented in Appendix B.6. We found that there are multiple equilibria even when the distribution is not uniform and that it is possible that either female choice or mutual choice may appear for the same pair (r, c).

3.2.10 Equilibria for $n = 4$

When there are $n = 4$ periods, it is extremely time consuming to do an exhaustive search for all equilibria, even for small numbers of types. So we adopt the iterative methods employed by Alpern and Reyniers (1999) in which an analog of the T mapping of (52) is iterated to find attracting fixed points. Of course in this case T acts on the 6-dimensional space,

$$T\left(\left(u^1, u^2, u^3\right), \left(v^1, v^2, v^3\right)\right) = \left(\left(w_2, w_3, w_4\right), \left(e, e_3, e_4\right)\right),$$

where w_j (resp. e_j) is the expected payoff for a male (resp. female) entering period j unmated, given the strategies u^i, v^i. For each pair of r and c, we start the iteration at a number of different points, and note the fixed points (all orbits of T appear to converge). In the cases where only one type of equilibrium is observed (female or mutual choice), we indicate this by an F or M; if both appear we write down a B. Of course any F or M might become a B if we added the right additional starting point.



	c=0.6	c=0.7	c=0.8	c=0.9	c=1
r=2	M	F	F	F	F
r=1.9	M	F	M	F	F
r=1.8	M	B	B	F	F
r=1.7	M	M	M	F	F
r=1.6	M	M	M	M	F
r=1.5	M	M	M	M	M
r=1.4	M	M	M	M	M
r=1.3	M	M	M	M	M
r=1.2	M	M	M	M	M
r=1.1	M	M	M	M	M
r=1	M	M	M	M	M

Figure 3.17: Equilibria for $n = 4$, $m = 8$

The pattern is similar for the case $n = 3$ shown in Figure 3.14, except for the M between an F and a B at the top of column $c = .08$. Possibly the M is really a B.

3.2.11 Conclusions

This section generalized the earlier matching model of Alpern and Reyniers (2005) by considering unequal sized groups to be matched. Calling the larger group 'males', and letting $r \geq 1$ denote the 'sex ratio' of males to females, we observed that a fraction $\frac{1}{r}$ of the males will end the game unmated. We set the utility of this eventuality to an unmated male as a cost (negative utility)

$c \geq 0$. We then analyzed the equilibria of the resulting n-period game $\Gamma_n(r, c)$. We analytically determined the equilibria in terms of the parameters r and c for $n = 2$. We found two regions with unique equilibria (one-sided and two-sided) and a more interesting region with three simultaneous equilibria: a choosy equilibrium (both groups with high acceptance standards), an easy equilibrium (both groups have low standards) and a one-side equilibrium (males accept anyone). It is an interesting question as to which equilibrium one would expect to find in practice. If the process is one that is repeated each season (hiring season, mating season), one might expect that the equilibrium is determined in an evolutionary manner, in which case we would expect either the choosy or one-sided equilibria, which are dynamically stable (and the choosy one has a larger basin of attraction). If equilibria are chosen by society to be stable with respect to deviations after the couple formation (e.g. divorce), then we would also expect to see the choosy equilibrium, because it has the highest marital stability index σ. Our preliminary investigations (to be carried further in a subsequent article) indicate that different quality individuals (bands of types) fare unequally in the three equilibria, and so we might expect that the power of these groups (expressed through their numbers or otherwise) might be a determinant of the equilibrium that occurs.

The section takes the two parameters of the game, r and c, as given. Stepping back a level to the design problem, it can be seen that the organizer of the matching process may have some control over these values. The organizer of the process will usually have some direct control over r, typically by limiting the entries to the process of the larger group (called males), and so making it closer to 1. This is in fact often done by 'capping' in 'singles' events, and indirectly done by setting application dates or entry fees (e.g. ladies free nights) in other matching processes. In captive breeding populations (in zoos or elsewhere), the sex ratio is chosen to optimize population growth, for a given set of resources. In order to optimize the choice of r, it is of course necessary to know the result (or results, if multiple equilibria) of the game with a given value of r. Our model shows that in some cases shifting r a small amount will convert an ambiguous situation with multiple equilibria into a more predictable unique situation. Control over the c (non mating) cost parameter is more problematical. At the governmental level, it is reflected in tax benefits to married couples and in tourism by the use of 'singles supplements'. Some local councils give a tax break to single occupancy apartments.

In this section we have taken the usual route of not explicitly modeling the process that pairs unmated individuals at the start of each period. Presumably, in order to be matched, the pairs must come into spatial contiguity by some process. A good candidate for this process, assuming individuals *want* to be matched, is what is known in the literature as *rendezvous search*, which temporally optimizes the search for a partner. Originally posed by the first author in 1976, this problem has been extensively studied (mostly, but not exclusively, for two searchers). See, for example, Alpern and Gal (2003), Alpern (2002),Gal (1999), and Howard (1999). With more work in this area for multiple searchers, we might have to modify our assumption of *random* pairing in each period. For example in the housing market some real estate agents may cater mostly for expensive houses and rich buyers (a rendezvous focal point), giving some degree of assortative matching even before choice is taken into account. Similarly, in the biological setting, Cronin (1991) has suggested that assortative pairing may arise due to non-random arrival times at the breeding ground, another non-choice factor.

3.3 Welfare

We study the average payoff (welfare) that females and males expect to receive at each equilibrium in the area III of (r, c) space (presented in Figure 3.2). This welfare analysis complements the analysis of the equilibria in the area III of (r, c) space. Welfare analysis in combination with dynamical and marital stability analysis offers a complete study of the "efficiency" of the equilibria of the two period continuous type game. Its conclusions can be proven to be useful to explain or predict the equilibrium behaviour under different circumstances.

The study of the social and individual welfare allows us to examine any conflicts of interest between male and female populations and between the members of the same sex. Hence welfare analysis can be useful to understand how the equilibrium behaviour is formed under different conditions.

Welfare analysis may also be relevant to equilibrium selection. If a particular group has a dominant role, it may have the power to direct the rest of the population to follow an equilibrium that is more profitable for it. There are cases where male or female behaviour can be controlled or influenced by a specific group, or by a leader of a particular type. For instance, biological experiments show that in fish, younger females often copy the behaviour of older females (Gibson and Hoglund (1992), Dugatkin and Godin (1993)).

We note that in this section, we use the same notation as in section 3.2. Hence we name e^1 the female choice equilibrium, e^2 the easy equilibrium and e^3 the choosy equilibrium and we assume that at an equilibrium e^i, the male equilibrium strategy is the u_i and the female equilibrium strategy is the v_i.

3.3.1 Social Welfare

Given (r, c), the mean male welfare remains the same at all equilibria, as we will prove next. However, this is not true for mean female welfare which is affected by the strategies used. We prove that females are better off at the female choice equilibrium and worst off at the choosy equilibrium.

3.3.1.1. *Definition of Mean Male and Female Welfare*

The mean male welfare is independent of the strategies used by both males and females. Given (r, c), only $\frac{1}{r}$ males are mated after the end of the game and $\frac{r-1}{r}$ males remain unmated, having a cost c. Hence the mean male welfare $\overline{W}_M(e^i)$ depends only on the sex ratio r and on the expected cost c and it is the following

$$\overline{W}_M(e^i) = \frac{1}{r}\frac{1}{2} + \frac{r-1}{r}c \tag{61}$$

Hence from (61) follows Theorem 3.2.

Theorem 3.2 (male social welfare) *For given (r, c) in region III, the average male payoff $\overline{W}_M(e^i)$ is not affected by the strategies used by males or females.*

The mean female welfare coincides with the average male type which is mated in the game; hence it is affected by the strategies used at equilibrium. At an equilibrium $e^i(r, c) = (u_i, v_i)$ only $k_i = (1 - u_i)(1 - v_i)$ (as defined in (21)) females are expected to be mated in the first period, receiving a utility higher than v_i, and the rest of the females are mated in the second period, receiving an average utility of v_i. Thus, the mean female welfare $\overline{W}_F(e^i)$ is the following

$$\begin{aligned} \overline{W}_F(e^i) &= k_i \frac{1+v_i}{2} + (1 - k_i)\, v_i \\ &= \frac{1-u_i}{2}(1 - v_i)^2 + v_i \end{aligned} \tag{62}$$

Obviously, at the female choice equilibrium e^1, males are non choosy ($u_1 = 0$) and the average female payoff is of the form

$$\overline{W}_F(e^1) = \frac{1+v_1^2}{2} \tag{63}$$

3.3.1.2. *Theorem 3.3 on Female Social Welfare*

Theorem 3.3 (female social welfare) *For given (r, c) in region III, the average female payoff $\overline{W}_F(e^i)$ is the highest at the female choice equilibrium e^1 and the lowest at the choosy equilibrium e^3.*

In order to prove Theorem 3.3, we will use the following Lemma.

Lemma 3.4 *Given* (r,c) *let* (u,v) *and* (u',v') *be equilibria where* $u, u' \neq 0$. *If* $v < v'$ *then the mean female welfare* $\bar{W}_F(e^i)$ *is higher at the* (u,v).

Proof. We rewrite $\bar{W}_F(e^i)$, replacing u with the equivalent function

$$f(v) = \frac{-r - 2v_i + 2rv_i + v_i^2 + 1}{-2v_i + v_i^2 + 1}$$

(from (24)) for $i = 2, 3$. Hence we have

$$\begin{aligned}
\overline{W}_F(e^i) &= \overline{W}_F(u_i, v_i) = \overline{W}_F(f(v_i), v_i) \\
&= \frac{1 - f(v_i)}{2}(1 - v_i)^2 + v_i \\
&= \frac{1 - \left(\dfrac{-r - 2v_i + 2rv_i + v_i^2 + 1}{-2v_i + v_i^2 + 1}\right)}{2}(1 - v_i)^2 + v_i \\
&= \frac{r}{2} - (r - 1)\, v_i \qquad (64)
\end{aligned}$$

Hence $\overline{W}_F(e^i)$ decreases in v_i. ■

With the help of Lemma 3.4, we can now prove Theorem 3.3.

Proof of theorem 3.3 (female social welfare). We are first going to prove that females are better off at the female choice equilibrium e^1 than at the easy equilibrium e^2.

From (50), we know that $v_1 = 1 - r + \sqrt{-r + r^2}$ at the female choice equilibrium. Hence from (63), and (64) the difference between the mean female welfare at the female choice equilibrium e^1 and at the easy equilibrium e^2 is the following

$$\begin{aligned}
\bar{W}_F(e^1) - \bar{W}_F(e^2) &= \frac{1 + v_1^2}{2} - \left(\frac{r}{2} - (r - 1)\, v_2\right) \\
&= \frac{1 + (1 - r + \sqrt{-r + r^2})^2}{2} - \frac{r}{2} + (r - 1)\, v_2 \\
&= \sqrt{r\,(r - 1)} - r\sqrt{r\,(r - 1)} - 2r + r^2 + 1 + (r - 1)\, v_2 \\
&= (r - 1)\left(v_2 - \left(1 - r + \sqrt{r\,(r - 1)}\right)\right) \\
&= (r - 1)\,(v_2 - v_1) \geq 0
\end{aligned}$$

The fact that females are better off at the easy equilibrium than at the choosy equilibrium, is an immediate consequence of Lemmas 3.3 and 3.4. ■

The following figure shows the mean female welfare $\bar{W}_F(e^i)$ at equilibrium e^i when $r = 1.05$ and $c = 2$.

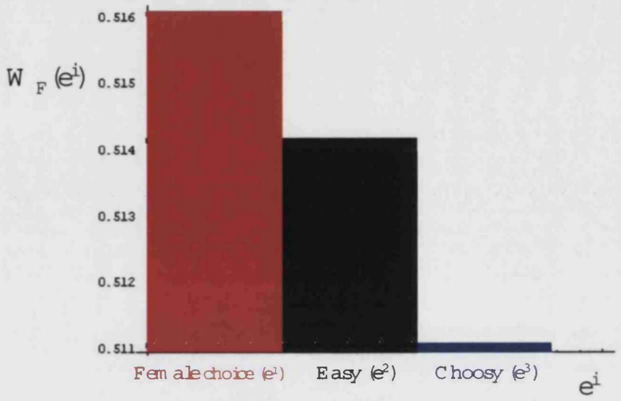


Figure 3.18: Average female payoff $\bar{W}_F(e^i)$ at equilibrium e^i when $r = 1.05$, and $c = 2$.

3.3.2 Individual Welfare

In the previous section, we analysed the effect of the equilibrium strategies on the mean male and female welfare. In this section, we study the impact of the equilibrium strategies on the expected payoff of specific groups of female and male types. As not all types are accepted in the first period, the welfare of different types of males and females differs greatly as strategies change. Types of the same sex can have conflicting individual interests and it is probable that the social and individual interests cannot be satisfied simultaneously. According to Theorem 3.2, the sum of the expected payoffs of all males is constant and equal to $\frac{1}{2} - (r-1)c$. Hence when some types of males are better off at a particular equilibrium, inevitably other male types are worst off. Total female welfare does not remain constant, but it varies as the strategies change. Nevertheless, it becomes clear from our analysis that different types of females can have different payoffs in each equilibrium and as a result different preferences for different equilibria to appear.

3.3.2.1. *Definition of Expected Payoff of Male Type x and Female Type y at a Mutual Choice Equilibrium $e^i = (u_i, v_i)$*

At an equilibrium, types accepted in the first period expect to enjoy a higher expected utility than the rest of the types. At a mutual choice equilibrium $e^i = (u_i, v_i)$, a male of a type higher than v_i expects to get $\frac{1+u_i}{2}$ if he is mated in the first period (probability $(1-u_i)\frac{1}{r}$) and u_i if he enters in the second period unmated (probability $1 - (1-u_i)\frac{1}{r}$). However, a male of a type lower than v_i

expects to get a utility of u_i in the game, since he can only be mated in the second period.

Hence at a mutual choice equilibrium e^i, the expected utility $W_M(x, e^i)$ that a male of type x expects to have when he enters the game is the following

$$W_M(x, e^i) = \begin{cases} u_i & \text{for } x \leq v_i \\ \frac{u_i+1}{2}\left(1-u_i\right)\frac{1}{r} + \left(1-\left(1-u_i\right)\frac{1}{r}\right)u_i & \text{for } x \geq v_i \end{cases} \tag{65}$$

Similarly, at a mutual equilibrium $e^i = (u_i, v_i)$ a female of a type higher than u_i expects to receive $\frac{v_i+1}{2}$ if she is mated in the first period (probability$(1 - v_i)$) and v_i if she enters the second period unmated (probability v_i). A female of a type lower than u_i expects to receive v_i.

Hence, the expected utility $W_F(y, e^i)$ that a female of type y expects to have when she enters the game is the following

$$W_F(z_f, e^i) = \begin{cases} v_i & \text{for } y \leq u_i \\ \left(1-v_i\right)\frac{v_i+1}{2} + v_i^2 & \text{for } y \geq u_i \end{cases} \tag{66}$$

3.3.2.2. *Definition of Expected Payoff of Male Type x and Female Type y at a Female Choice Equilibrium $e^1 = (0, v_1)$*

At a female choice equilibrium not all males expect to have the same payoff. Males of a type higher than v_1 have a higher expected utility, since they have a chance of $\frac{1}{r}$ to be mated in the first period. The rest of the types never expect to find a partner in the first period. In the second period, players of all types have the same chance of finding a partner. (The probability of being mated in the second period is the ratio of unmated females over unmated males, hence it is $\frac{v_i}{r-k_1}$). The average payoff that a male expects to receive if he enters unmated in the second period is $\tilde{\mu}_y$, as it is given in equation (27).

Hence the payoff $W_M(x, e^1)$ that a male of type x expects to receive when he enters the game is

$$\begin{aligned} W_M(x, e^1) &= \begin{cases} \tilde{\mu}_y & \text{for } x \leq v_1 \\ \frac{1}{2}\frac{1}{r} + \frac{r-1}{r}\tilde{\mu}_y & \text{for } x \geq v_1 \end{cases} \\ &= \begin{cases} \frac{v_1}{r-1+v_1}\frac{1}{2} - \left(1 - \frac{v_1}{r-1+v_1}\right)c & \text{for } x \leq v_1 \\ \frac{1}{2}\frac{1}{r} + \frac{r-1}{r}\left(\frac{v_1}{r-1+v_1}\frac{1}{2} - \left(1 - \frac{v_1}{r-1+v_1}\right)c\right) & \text{for } x \geq v_1 \end{cases} \end{aligned} \tag{67}$$

It is important to note that at a female choice equilibrium, a male who remains unmated after the first period expects that he is going to receive a maximum payoff of 0 in the second period. Hence males of a type lower than

v_1 expect to receive a non positive payoff, while the rest of the males expect to receive a payoff that is not higher than $\frac{1}{2}\frac{1}{r}$.

$$W_M(x, e^1) \leq \begin{cases} 0 & \text{for } x \leq v_1 \\ \frac{1}{2}\frac{1}{r} & \text{for } x \geq v_1 \end{cases} \tag{68}$$

At a female choice equilibrium e^1, females of all types have the same expected payoff when they enter the game; since no male rejects them, they all have the same probability to be mated in the first or the second period. The expected payoff $W_F(y, e^1)$ that any female type y expects to receive is the same as the average male type $\bar{W}_F(e^1)$ in the couples created. Hence for all $y \in [0, 1]$, we have

$$W_F(y, e^1) = \bar{W}_F(e^1) \tag{69}$$

3.3.2.3. *Categories of Male and Female Types*

For any given (r, c), from Lemma **3.3** it is known that in area III, the strategy u_i of males and the strategy v_i of females are the lowest at the female choice equilibrium $e^1(r, c)$ and the highest at the choosy equilibrium $e^3(r, c)$.

$$u_3 \geq u_2 \geq u_1 \text{ and } v_3 \geq v_2 \geq v_1$$

Given (r, c), we divide male types in the following four categories

$$\begin{aligned} LL_m &= [0, v_1] \\ L_m &= [v_1, v_2) \\ M_m &= [v_2, v_3) \\ H_m &= [v_3, 1] \end{aligned}$$

We call LL_m the male types rejected in the first period at all equilibria, L_m the male types rejected in the first period at the easy and the choosy equilibrium but accepted at the female choice equilibrium, M_m the male types rejected in the first period at the choosy equilibrium but accepted at the easy and the female choice equilibrium and H_m the male types accepted always, at all equilibria.

In the same way, given (r, c), we divide female types in three categories

$$\begin{aligned} L_f &= [0, u_2] \\ M_f &= [u_2, u_3) \\ H_f &= [u_3, 1] \end{aligned}$$

We call L_f the female types rejected in the first period at all mutual choice equilibria but accepted at the female choice equilibrium, M_f the female types rejected at the choosy equilibrium but accepted at the easy equilibrium (and obviously at the female choice equilibrium) and H_f the female types accepted at all equilibria.

3.3.2.4. *Theorems on Individual Welfare*

We prove that each one of the categories of male and female types defined above has a list of preferences over the equilibria in area III according to the expected payoff of the types it includes. The following theorems summarise our results.

Theorem 3.4 (individual male welfare) *For given (r, c) in region III in the two period game,*

(i) Male types in LL_m get the maximum expected payoff at the choosy equilibrium e^2 and the minimum expected payoff at the female choice equilibrium e^1.

(ii) Male types in L_m get the maximum expected payoff at the female choice equilibrium e^1 and the minimum expected payoff at the easy equilibrium e^2.

(iii) Male types in M_m get the maximum expected payoff at the easy equilibrium e^2 and the minimum expected payoff at the choosy equilibrium e^3.

(iv) Male types in H_m get the maximum expected payoff at the choosy equilibrium e^3 and the minimum expected payoff at the female choice equilibrium e^1.

Proof. We are going to prove first parts (i) and (iv) and then parts (ii) and (iii).

(i) It is an immediate consequence of Lemma 3.3.

(iv) Let $x \in \mathrm{H}_m$. We are going to prove first that his expected payoff is higher at e^3 than at e^2. From (65) we have

$$\begin{aligned}
W_M(x,e^3) - W_M(x,e^2) &= \frac{u_3+1}{2}(1-u_3)\frac{1}{r} + \left(1-(1-u_3)\frac{1}{r}\right)u_3 \\
&\quad - \left(\frac{u_2+1}{2}(1-u_2)\frac{1}{r} + \left(1-(1-u_2)\frac{1}{r}\right)u_2\right) \\
&= \left(\frac{u_3+1}{2} - u_3\right)\frac{(1-u_3)}{r} + u_3 \\
&\quad - \left(\frac{u_2+1}{2} - u_2\right)\frac{(1-u_2)}{r} - u_2 \\
&= \frac{(1-u_3)^2}{2r} + u_3 - \frac{(1-u_2)^2}{2r} - u_2 \\
&= u_3 - u_2 + \frac{1}{2r}(2-u_3-u_2)(-u_3+u_2) \\
&= (u_3-u_2)\left(1-\frac{1}{2r}(2-u_3-u_2)\right) \geq 0
\end{aligned}$$

Looking at the final inequality in more detail, it becomes apparent that since $0.5 \geq u_3 \geq u_2 \geq 0$ and $2r > 2$, the difference $W_M(x,e^3) - W_M(x,e^2)$ is non negative.

Having proved that a male x in H_m is better off at e^3 that at e^2, in order to complete the proof of *(iv)* it is enough to prove that the expected payoff $W_M(x,e^1)$ that a male x gets at female choice equilibrium e^1 is at most equal to the expected payoff $W_M(x,e^2)$ he gets at the easy equilibrium e^2.

From (68), at female choice equilibrium, the expected payoff that a male expects to receive is always less than $\frac{1}{2r}$. Hence it is enough to prove that

$$W_M(l+\kappa, e^2) - \frac{1}{r}\frac{1}{2} \geq 0$$

We show this as follows

$$\begin{aligned}
W_M(l+\kappa,e^2) - \frac{1}{2}\frac{1}{r} &= \frac{u_2+1}{2}(1-u_2)\frac{1}{r} + \left(1-(1-u_2)\frac{1}{r}\right)u_2 - \frac{1}{2}\frac{1}{r} \\
&= \frac{(1-u_2)^2}{2r} + u_2 - \frac{1}{2}\frac{1}{r} \\
&= \frac{1-2u_2+u_2^2+2ru_2-1}{2r} \\
&= \frac{1}{2r}u_2(-2+u_2+2r) \geq 0.
\end{aligned}$$

■

In order to prove (ii) and (iii) we need the following lemma.

Lemma 3.5 *For given (r,c) in region III in the two periods game, a male type x which is rejected in the first period at equilibrium e^i and is accepted in the first* *period at equilibrium e^j, is always better off at equilibrium e^j.*

Proof. From (65), (67) and Lemma 3.3, for given (r, c) in region III in the 2 periods game, the highest male type 1 has the lowest expected payoff $W_M(1, e^i)$ at the female choice equilibrium, while the lowest male type 0 has the highest expected payoff $W_M(0, e^i)$ at the choosy equilibrium. A type x rejected in the first period at equilibrium e^i, but accepted in the first period at equilibrium e^j can never expect to get more than $W(0, e^3)$ at e^i and less than $W(1, e^1)$ at e^j. Hence it is sufficient to prove that the expected payoff $W(1, e^1)$ of type 1 at the female choice equilibrium is higher than the expected payoff $W(0, e^3)$ of type 0 at the choosy equilibrium.

From (65), (67) and (27) we have

$$\begin{aligned} W(1,e^1) - W(0,e^3) &= \frac{1}{2}\left(\frac{1}{r} + \frac{r-1}{r}\frac{v_1}{r-1+v_1}\right) - \frac{r-1}{r}\left(1 - \frac{v_1}{r-1+v_1}\right)c - u_3 \\ &= \frac{1}{2}\left(\frac{1}{r} + \frac{r-1}{r}\frac{1-k_1}{r-k_1}\right) - \frac{r-1}{r}\left(1 - \frac{v_1}{r-1+v_1}\right)c \\ &\quad - \left(\frac{1-k_3}{r-k_3}\mu_F(u_3,v_3) - \left(\frac{r-1+k_3}{r-k_3}\right)c\right) \\ &= \left(\frac{1}{2}\frac{1}{r} - \frac{1-k_3}{r-k_3}\mu_F(u_3,v_3)\right) + \left(\left(\frac{r-1+k_3}{r-k_3}\right)c - \frac{r-1}{r}c\right) \\ &\quad + \left(\frac{1}{2}\frac{r-1}{r}\frac{1-k_1}{r-k_1} + \frac{r-1}{r}\frac{v_1}{r-1+v_1}c\right) \geq 0 \end{aligned}$$

The explanation of the last inequality follows.

We know that $\mu_F(u_3, v_3) < \frac{1}{2}$ and $\frac{1-k_3}{r-k_3} < \frac{1}{r}$, hence the first parenthesis $\left(\frac{1}{2}\frac{1}{r} - \frac{1-k_3}{r-k_3}\mu_F(u_3, v_3)\right)$ is positive. Furthermore $\dfrac{r-1+k_3}{r-k_3} < \frac{r-1}{r}$, thus the second parenthesis $\left(\left(\dfrac{r-1+k_3}{r-k_3}\right)c - \frac{r-1}{r}c\right)$ is positive as well. The last parenthesis is positive since all terms are positive. ■

Continuation of proof of theorem 3.4 (individual male welfare).

We can now prove parts (ii) and (iii) of Theorem 3.4.

(ii) and (iii) They are immediate consequences of Lemma 3.5, Lemma 3.3 and part (i) of Theorem 3.4 (individual male welfare).

■

The following figure (3.19) shows the payoff $W_M(x, e^i(1.05, 2))$ that a male of type x expects to get at equilibrium e^i, when $r = 1.05$, and $c = 2$.

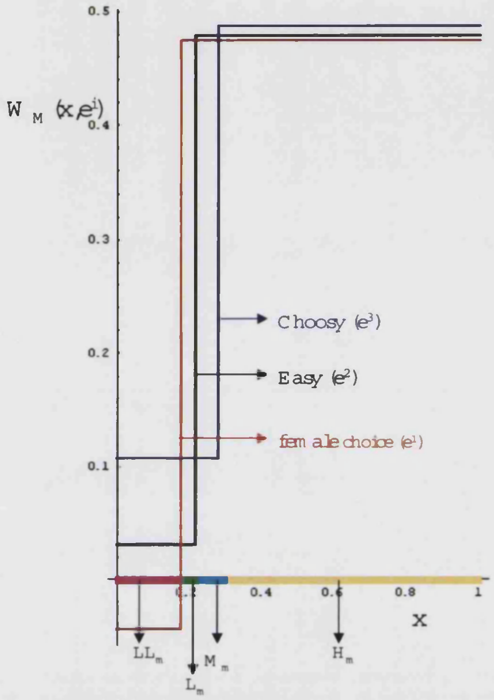


Figure 3.19: Expected payoff $W_M(x, e^i)$ of male type x at equilibrium e^i when $r = 1.05$ and $c = 2$.

Theorem 3.5 (individual female welfare) *For given (r, c) in region III in the two periods game,*

(i) Female types in L_f get the maximum expected payoff at the female choice equilibrium e^1 and the minimum expected payoff at the easy equilibrium e^2.

(ii) Female types in M_f get the maximum expected payoff at the easy equilibrium e^2 and the minimum expected payoff at the choosy equilibrium e^3.

(iii) Female types in H_f get the maximum expected payoff at the choosy choice equilibrium e^3 and the minimum expected payoff at the female choice equilibrium e^1.

Proof. We are going to prove parts (i) and (iii) and then part (ii).

(i) Let $y \in \mathrm{L}_f$. From (66) and Lemma 3.3, it follows that for all y in L_f it is true that

$$W_F(y,e^3) \geq W_F(y,e^2)$$

Hence we need to prove that the expected payoff $W_F(y,e^1)$ of a type y in L_f at the female choice equilibrium is higher than her expected payoff $W_F(y,e^3)$ at the choosy equilibrium. From (66)and (69) we have

$$W_F(y,e^1) = \frac{1+v_1^2}{2} > \frac{1}{2} > v_3 = W_F(y,e^3)$$

(iii) Let $y \in \mathrm{H}_f$. In order to prove that a type y in H_f gets her maximum payoff at the choosy equilibrium e^3, we are going to prove first that the expected payoff $W_F(y,e^i)$ of y is higher at e^3 than at e^2. From (66) we have

$$\begin{aligned} W_F(y,e^3) - W_F(y,e^2) &= (1-v_3)\frac{v_3+1}{2} + v_3^2 - \left((1-v_2)\frac{v_2+1}{2} + v_2^2\right) \\ &= \frac{v_3^2+1}{2} - \frac{v_2^2+1}{2} \\ &= \frac{1}{2}\left(v_3^2 - v_2^2\right) \geq 0 \end{aligned}$$

Having proved that $U_F(y,e^3) \geq U_M(y,e^2)$, in order to complete the proof, it is enough to prove that the payoff $W_F(y,e^i)$ that a female of type y can get at the female choice equilibrium is less that the expected payoff she can get at the easy equilibrium. Hence in the same way as before and given (66) and (69) we have

$$\begin{aligned} W_F(y,e^2) - W_F(y,e^1) &= W_F(y,e^2) - \bar{W}_F(e^1) \\ &= \frac{v_2^2+1}{2} - \frac{v_1^2+1}{2} \\ &= \frac{1}{2}\left(v_2^2 - v_1^2\right) \geq 0 \end{aligned}$$

Therefore $W_F(y,e^2) \geq W_F(y,e^1)$.

(ii) Similar to *(i)* and *(iii)*.

■

The following figure (3.20) shows the payoff $W_F(y,e^i(1.05,2))$ that a female of type y expects to get at equilibrium e^i, when $r = 1.05$ and $c = 2$.

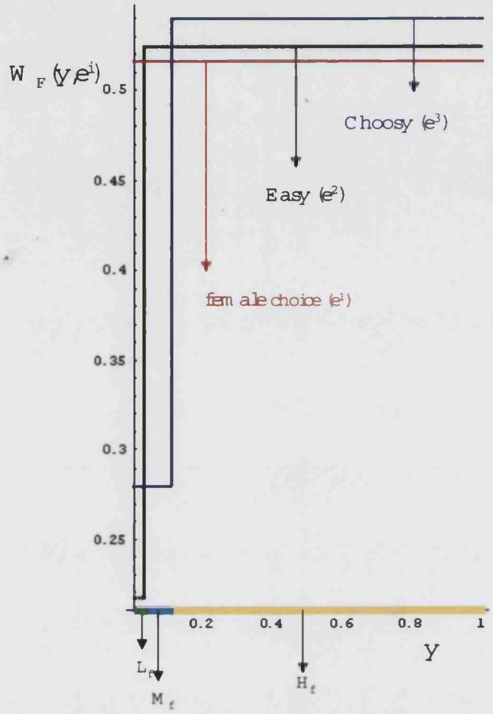


Figure 3.20: Expected payoff $W_F(y, e^i)$ of female type y at equilibrium e^i when $r = 1.05$ and $c = 2$.

3.4 Biological Implications of the Model

In this section, we briefly examine the implications of the model presented in section 3.2 in biology. We present a variation of the two period game that is more applied in biology and mate selection. The extended analysis of this model will be the subject of a joint paper with Steve Alpern. Here we summarise our main results, without providing any proofs. We try to give an answer to the question "when do we have female choice and when do we have mutual choice".

This section is added in this chapter for general information and in order to provide an insight on applications of our model.

3.4.1 Description of the Model

The same basic model as in section 3.2 is used but there are some important differences in the assumptions made concerning the types and their utility. Types represent fertility or fitness and the type space is no longer normalised in $[0, 1]$.

Instead, males are considered to be uniformly distributed in $[h-\gamma, h+\gamma]$ and females are uniformly distributed in $[l-\kappa, l+\kappa]$, where γ and κ are the maximum distances a type may have from the average male type h and the average female type l correspondingly.

Players still have maximising preferences as in section 3.2. However, we assume that their objective is to maximise the number $o(x,y)$ of offspring produced. The utility of a mating (x,y), expressed as the number of offspring $o(x,y) = Zxy$ produced, is taken to be proportional to both male type x and female type y. It is necessary to note than in section 3.2, when the lowest possible female type was mated with the lowest possible male type, the utility for both partners was zero, while in the current model, the outcome of the mating is a positive number of offspring.

Players use the type of their partners as an indicator of the number of offspring $o(x,y)$ they expect to have. Their effort to achieve the maximum number of offspring is expressed by seeking a partner of a high type.

Population is considered to be male biased (sex ratio $r > 1$), as in the original model. Consequently, not all males find a partner; however there is no cost c payable by the males remaining unmated. Males that do not manage to get mated until the end of the game do not have offspring and have a utility of 0.

It is easy to show that male variation γ and average type h do not influence female or male strategies at equilibrium, since all females are mated at the end of the game. On the other hand, female variation κ has an important role at the formation of the equilibrium behaviour. The utility of mated and unmated males at equilibrium can be significant, especially when the mean female type l is high and female variation κ is low.

3.4.2 Basic Results

We analysed the discrete type and the continuous type two period models. In the former model, we assumed we have three equiprobable male and female types.

In both models, we found that for given r and l, choosiness in both sexes increases as female variation κ increases. For κ very large, males tend to be choosy, even for r very large. When the value of female variation κ approximates the mean female type l, the utility males receive from a mating with different females varies greatly. In this case, the utility that low type females offer does not differ greatly from 0. On the other hand, high type females offer to males a significantly higher utility. As a result, males are willing to reject low type females and risk to remain unmated, in order to give to themselves more opportunities to find a partner of a higher type. For female variation κ low though, especially as l and r increase, males are not willing to take any risk and thus accept all females. When κ is low, the difference of the minimum utility $l - \kappa$ males can get by being mated and the utility of not being mated (0) becomes important, especially as l increases. If in addition to that, r is large and consequently the probability of finding a partner is low, males have one more reason not to want to be choosy.

For given l and κ, female choice is the norm when male population is significantly larger than the female, while mutual choice appears otherwise. This comes into accordance with the general theory that predicts that mutual choice appears when female variation is high and operational sex ratio is close to 1 (Trivers (1972), Kokko and Johnstone (2002), Real (1990), Kvarnemmo and Simmons (1999)).

In the discrete 3 type model, we found a unique equilibrium, while in the continuous type game we showed that it is possible the existence of multiple equilibria.

Given l, in the discrete type model there is a line F dividing the (r, κ) space in two regions. On and above F there is a unique female choice equilibrium, while below it there is a unique mutual choice equilibrium.

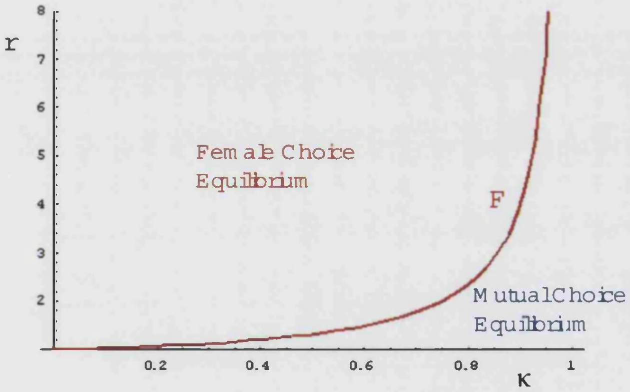


Figure 3.21: Plot of the F line for $l = 1$ (discrete type model).

However, for given l in the continuous type model, the F line separating the areas of female and mutual choice is divided into two brunches for $\kappa < \frac{l}{3}$. We call M the higher branch for $\kappa < \frac{l}{3}$. It is proved, as in section 3.2, that for $\kappa < \frac{l}{3}$, between the F and the M line, there are three equilibria, two mutual choice and one female choice equilibrium.

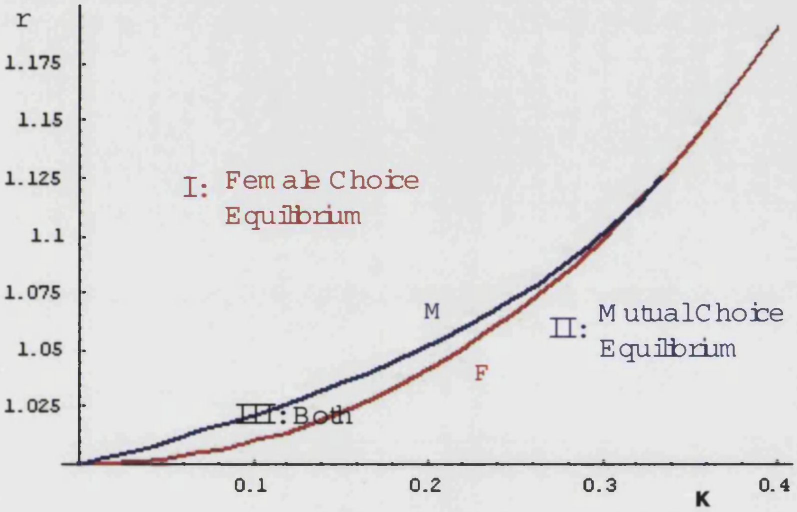


Figure 3.22: Equilibrium areas for $l = 1$ (continuous type model).

The analysis of the area between the F and the M lines is similar to the analysis of the area III in 3.2; it is easy to normalise the current model and show it is equivalent to the model presented in 3.2.

4 Age Dependent Preferences

4.1 Introduction

The connection between age and mating preferences is studied extensively by the biological, evolutionary, psychological and sociological literature, but the empirical and theoretical results often contradict each other (Brooks and Kemp (2001)). Drawing general results on the connections between age, fitness, fertility and preferences and trying to isolate each factor and analyse its relation with the others have proved to be difficult, leading to inconsistent results (Brooks and Kemp (2001)). Our research is focused on building an analytical framework which examines how age dependent preferences affect the equilibrium behaviour.

This chapter focuses on the relation between age and choice. Our analysis is different from the existing evolutionary models, examining the relation of age and time preferences (Sozou and Seymour (2003), Rogers (1994)). We do not use an evolutionary model and we examine how the age of an individual and the availability of potential partners affect the selectivity of the individual over the age of his partner, not over the time he is willing to wait in order to find a partner.

In contrast to the previous chapters, we examine a mutual choice steady state model where the types of players are determined by their age. We assume that a player of age (type) 1 is in the first year of his fertile life. Players seek partners such that they maximise the years that both they and their spouses are fertile. For instance, assuming that males are fertile for 6 years and females are fertile for 4 years, the best partner for a male of age 2 is a female of age 1, since they can both be fertile for 4 years. If a male of age 2 is mated with a female of age 3, they can produce offspring for only 2 years.

4.1.1 Related Literature on Age Preferences

In biology, most studies focus on female preferences and behaviour. It is often assumed that longevity is connected with gene quality, and therefore it is expected that females prefer to mate with old rather than young males (Anderson (1994), Trivers (1973), Manning (1985), Kokko (1998), Poesel et al (2006)). This assumption has been criticised by Hansen and Price (1995) who assert that older individuals are not always genetically better partners. Hansen and Price (1995) base their criticism on four basic facts: firstly that fitness and age are in most cases negatively correlated (Trivers (1973) and Manning (1985)); secondly that males in a younger age have usually higher breeding value for fertility; thirdly that young individuals may be better adapted to their environment as a result of their parents having been most recently exposed to the selection process; and finally that the breeding value for fitness is reduced as a result of "deleterious germ-line mutations occurring over the lifetime of a male" (Brooks and Kemp (2001)). Their arguments are supported by empirical findings, showing a negative relation between the offspring viability or reproductive quality and the age of its father (Price and Hansen (1998), Alatalo et al (1986), Beck and Powell (2000)). Furthermore, it is observed that in some species parental investment in the form of quality of ejaculates reduces with successive matings and therefore with age (Brooks and Kemp (2001), Manning (1985)). So, it is possible that females do not prefer old males. Evidence towards this direction is given by the empirical research of Jones et al (2000) and by the theoretical model of Beck and Powell (2000) who demonstrate that female mate choice based on male age is most likely to evolve in a population where females have a preference for younger and intermediate age males. "If females choose mates based on genetic quality alone, females should prefer younger rather than older males" (Beck and Powell (2000)).

On the other hand, it is proved that in many species such as insects and lizards, not only male but also female reproductive success and viability of offspring decline with age (Uetz and Norton (2007), Hercus and Hoffman, (2000), Pervez et al (2004), LeComte et al (2005)), a fact that should give an incentive to males to choose younger females as well.

In humans, it is generally accepted that during the mating process men prefer younger women and women prefer older men (Helle et al (2008), Kenrick and Keefe (1992), Kenrick, Keefe et al (1995)). Nevertheless, the preferences of women seem to change during the recent years. Einon (1997) notes that marriage statistics in UK reveal that women of breeding age choose young men. This result is supported by a study of the Office for National Statistics in UK that was published in the Daily Telegraph (2003), noting that the proportion of men marrying older women than themselves amongst men marrying for the first time has almost doubled (it is now 26%) in the last 25 years, while among women marrying for the first time, the proportion of women marrying younger men than themselves has risen from 13% per cent to 20%. This phenomenon is connected with the need of women to find mates of a high fitness. As Doosje et al (1999) argue, humans as well as other mammals aim to increase the viability of their offspring and, consequently, base their mate choices on indications of reproductive investment.

4.1.2 Summary of Our Results

Trying to examine the influence of age preferences in the mating behaviour, we built an age preference model based on an idea of Alpern and Reyniers (1999). Nevertheless, we do not assume homotypic preferences as Alpern and Reyniers

(1999), but as they propose, we explore the equilibrium behaviour under maximising preferences. In our model, two populations (males and females) meet randomly over successive periods. Players are taken to have different types, denoting their age, and the utility from a mating depends on the age of both partners. The utility of a mating is the number of years that both partners can produce offspring. Hence, the younger both partners are in a couple, the higher the utility both partners receive is. Our goal is to examine how the age of an individual and the age and availability of its potential partners affect its mating behaviour.

In contrast to the models presented so far, we use a steady state model, in which the distribution of the population remains fixed in time. Given that the age (type) of players increases over time, we consider an age limit over which players are not able to participate in the game anymore. Hence, in every period, players leave the game either because they have found a mate or because they are too old to participate any longer. In the mean time, a fixed number of new players enters the game, replacing those who left (with or without a partner). It is important to distinguish between the population sex ratio r and the incoming population sex ratio R. The population sex ratio r is the ratio of males to females in the population, while the incoming population sex ratio R, is the ratio of males of age 1 to females of age 1.

We analyse particular cases where male and female maximum age limits differ. We assume that males can never be fertile for a smaller number of periods than females. We examine how the equilibrium strategies change when the number of young males entering a period is not equal to the number of young females entering the game ($R \neq 1$). In the cases we examine, when the incoming population sex ratio R is $1:1$, the equilibria are trivial; we will show that all males of age 1 are mated with females of age 1. Nevertheless, when the incoming population sex ratio R is not taken to be $1:1$, the equilibria appearing are more interesting.

In the models examined, we do not find multiple equilibria for any given incoming population sex ratio R. However, the equilibrium in particular games often changes as R varies. The game where females are fertile for $\tau = 3$ periods and males are fertile for $\lambda = 4$ periods is of particular interest. In this game, there is no equilibrium for some values of R, while when the population is male biased ($r > 1$), there are two possible equilibria where the population sex ratio r is the same. In all the games studied, young males and females are at least as choosy as the old ones at equilibrium, a fact that accords with findings in biology (Gray (1999), Kodric - Brown and Nicoletto (2001)).

We measure the "satisfaction" of couples created according to the population productivity. We assume that both male and female populations are better off when the population productivity is high. It is shown that in all cases the population productivity is the maximum when the population sex ratio r is $1:1$. When r is $1:1$, the average age at marriage of both males and females is minimum.

We assume that on average a couple produces J offspring per period that both partners are fertile and given some special assumptions, we find when the number of entrants in each period is equal with the number of offspring produced.

The chapter is organised as follows: in the next section, the game is described in detail, before some general observations on the equilibrium behaviour are stated. Next, our results for the case where females are fertile for $\tau = 2$ periods and males are fertile for $\lambda \geq 2$ periods and the case where females are fertile for $\tau = 3$ periods and males are fertile for $\lambda \geq 3$ are presented. After the presentation of the equilibria, we focus on the population productivity and the average age at marriage as indicators of the couples satisfaction. Finally we conclude with some general remarks.

4.2 Description of the Game

We consider a steady state non atomic mating game, based an idea of Steve Alpern and Diane Reyniers (1999). Two populations, males and females, are randomly matched over successive periods. Males are fertile for λ periods and females for τ periods, where $\lambda \geq \tau$. There exists a continuum of players, who have one dimensional types denoting the number of periods they have already been fertile (including the period they are). Hence the type of an individual reveals in which period of his fertile life he is. Without loss of generality, we can assume that players are fertile from year 1. Hence in our analysis, we can assume that types simply denote the age of players.

A constant number of players (females and males) of age (type) 1 enters the game in the beginning of every period, before a random matching takes place; each party of a matched pair (i, j) can either accept or reject the other. If both accept, then they form a mated couple and leave the game, with both having a utility of $u_{i,j} = min[\lambda + 1 - i, \tau + 1 - j]$; $u_{i,j}$ can be thought as the number of periods a couple (i, j) is expected to be both fertile, or as an indicator of the number or the viability of offspring a couple can have. When there is no mutual consent, both i and j proceed unmated into the next period where their type increases by 1, becoming $i+1$ and $j+1$ for i and j respectively. In each period, there are a_i males of age i and b_j females of age j. We call $R = \frac{a_1}{b_1}$ the sex ratio of the incoming population and r the population sex ratio, where $R, r > 0$. In our analysis, we consider that the normalised male population is r and the normalised female population is 1.

A male strategy s_M denotes the minimum utility that each male of age i is willing to accept. The female strategy s_F is defined similarly. This strategy formulation facilitates the analysis and it is equivalent to a specification of the highest type (oldest partner) that one should accept. If the strategy of player of age i is $s(i)$, i accepts any player at least as young as $\lambda - s(i) + 1$.

Let U_i be the utility that a male of age i expects to receive in the game and V_j the utility that a female of age j expects to receive in the game. The utility U_i that a male of age i expects to receive is a weighted average of a the utility he expects to receive if he is mated in age i (probability $\sum_{j=1}^{j=\tau} k_{i,j}\frac{1}{r}b_j$) and the utility he expects to receive if he enters in age $i+1$ unmated (probability $\sum_{j=1}^{j=\tau}(1-k_{i,j})\frac{1}{r}b_j + \frac{r-1}{r}$). The utility V_j that a female of age j expects to receive in the game is defined similarly.

Thus when $r > 1$

$$\begin{aligned} U_i &= \sum_{j=1}^{j=\tau} k_{i,j}\frac{1}{r}b_j u_{i,j} + \left(\sum_{j=1}^{j=\tau}(1-k_{i,j})\frac{1}{r}b_j + \frac{r-1}{r}\right) U_{i+1} \\ V_j &= \sum_{i=1}^{i=\lambda} k_{i,j}\frac{a_i}{r}u_{i,j} + \sum_{i=1}^{i=\lambda}(1-k_{i,j})\frac{a_i}{r}V_{j+1} \end{aligned} \tag{70}$$

and when $r \leq 1$

$$\begin{aligned} U_i &= \sum_{j=1}^{j=\tau} k_{i,j}b_j u_{i,j} + \left(\sum_{j=1}^{j=\tau}(1-k_{i,j})b_j\right) U_{i+1} \\ V_j &= \sum_{i=1}^{i=\lambda} k_{i,j}a_i u_{i,j} + \left(1 - r + \sum_{i=1}^{i=\lambda}(1-k_{i,j})a_i\right) V_{j+1} \end{aligned} \tag{71}$$

where

$$k_{i,j} = \begin{cases} 1 & \text{if both } i,j \text{ accept each other} \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

It is interesting to note that when $r \geq 1$, $\frac{1}{r}$ males and all females participate in the matching in each period. ($r-1$ males remain unpaired.) In the same way, when $r < 1$ all males and only r females participate in the matching is each period. ($1-r$ females remain unpaired.) As a result, when $r \geq 1$ the probability that a male of age i meets a female of age j is $\frac{1}{r}b_j$ while when $r < 1$ it is b_j. Similarly, when $r \geq 1$, the probability that a female of age j meets a male of age i is $\frac{a_i}{r}$ while when $r < 1$ it is a_i.

We assume that a male of age λ and a female of age τ leaving the game without finding a partner receive a utility less than 1.

The equilibrium notion used is the same as the one used in the Alpern and Reyniers (1999, 2005) and in all the previous non steady state models we presented. At equilibrium every player accepts those types providing him with utility as least as a high as his expected utility in the next period.

Hence at equilibrium the male strategy $\bar{s}_M$ is

$$\bar{s}_M = [\lceil U_2 \rceil, \lceil U_3 \rceil, ..., \lceil U_\lambda \rceil, 1]$$

since each male of age i is willing to accept any female of age j providing him with a utility of at least U_{i+1}.

In the same way, the female equilibrium strategy $\bar{s}_F$ is

$$\bar{s}_F = [\lceil V_2 \rceil, \lceil V_3 \rceil, ..., \lceil V_\tau \rceil, 1]$$

For instance if a male of age i, expects to receive a utility of 2.3 if he enters unmated in the next period, then his strategy is $\lceil 2.3 \rceil = 3$ and he should accept any female of age $\tau - 2$ or younger.

Since it is a steady state model, the total population must remain unchanged during time. Hence the number of males of age i should be equal with the number of males of age $i-1$ which remain unmated. (Similarly for females of age j.) Hence at equilibrium, when $r > 1$, it must be true that

$$\begin{aligned} a_i(1 - \sum k_{i,j} b_j \frac{1}{r}) &= a_{i+1} && (73) \\ b_l(1 - \sum k_{s,l} \frac{a_i}{r}) &= b_{l+1} && (74) \end{aligned}$$

and when $r \leq 1$, it must be true that

$$\begin{aligned} a_i(1 - \sum k_{i,j} b_j) &= a_{i+1} && (75) \\ b_l(1 - \sum k_{s,l} a_i) &= b_{l+1} && (76) \end{aligned}$$

4.3 General Observations at Equilibrium

In order to facilitate the reader, we illustrate the male strategy in a table $\lambda \times \tau$, where each row i represents a male of age i and each column j a female of age j. A cell (i, j) is filled with an AT when a male of age i accepts a female of age j and with an RT otherwise. In a similar table $\lambda \times \tau$ we can illustrate the female strategy. In a female strategy table, a cell (i, j) is filled with an at when a female of age j accepts a male of age i and with an rt otherwise.

The following general observations are true at equilibrium, independently of

τ, λ, R and r.

(i) Males of age λ and females of age τ are universal acceptors. Independently of the age of their partner, the utility they receive from a mating is always 1, hence equal with the minimum utility any mating in the game can produce; immediate consequence is that they are indifferent between all types with which they can be mated.

(ii) Males of age $\lambda - 1$ and females of age $\tau - 1$ are universal acceptors as well. They expect than if they enter unmated in the last period, the maximum utility they can get is 1, and therefore they are willing to accept even the oldest players (of age λ or τ).

(iii) If $\lambda \geq \tau$, all males of age $\lambda - \tau + 1$ or younger are accepted by all females, since they provide to them the maximum possible utility the latter can get. If $\lambda < \tau$, all males accept all females of an age $\tau - \lambda + 1$ or younger.

	1	2	3	4	5
1	AT				
2	AT				
3	AT				
4	AT				
5	AT	AT	AT	AT	AT
6	AT	AT	AT	AT	AT

male strategy

	1	2	3	4	5
1	at	at	at	at	at
2	at	at	at	at	at
3				at	at
4				at	at
5				at	at
6				at	at

female strategy

Figure 4.1: Application of observations (*i*), (*ii*) and (*iii*) in a game where $\lambda = 6$ and $\tau = 5$.

(iv) Equilibrium strategies are non increasing sequences of numbers. At equilibrium, no player is choosier than any younger player of the same sex. If a male of age x is not willing to accept any partner older than y, then any male younger than x never accepts any female older than y as well. The same is true for females.

Let a male x not accept any female older than y at equilibrium. Then, a male younger than x, expects that if he reaches age $x+1$ unmated, he will receive a utility higher than $u_{x+1,y+1}$ but at most $u_{x+1,y}$. Therefore, at equilibrium, a male younger than x is not willing to accept anyone older than y, in other words anyone offering him a utility lower than $u_{x+1,y}$. (Or he can be even choosier than that). Accordingly, any female younger than

y, knowing that at equilibrium a female type y rejects any male older than x, she will accept only those males offering her a utility at least as high as $u_{x,y+1}$(or even higher).

For instance, let's assume that males are fertile for 4 periods and females are fertile for 3 periods and at equilibrium a male of age 2 only accepts females of age 1 and 2. Then a male of age 1 knows that if he reaches age 3 unmated, the minimum payoff he can receive is higher than what a female of age 3 would offer him (hence it is higher than 1) but it is at most 2. (If it was higher than 2, a male of age 2 would also reject a female of age 2). As a result, at equilibrium, a male of age 1 has to reject any female older than 2 (and can be even choosier).

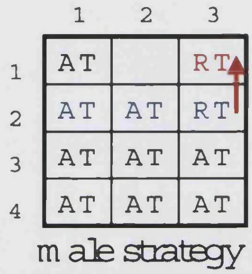


Figure 4.2: Application of observation (iv) in a game where $\lambda = 4$, $\tau = 3$.

In general, in a male strategy table, if a cell (i, j) is filled with an RT, all cells (z, j) for $z < i$, are also filled with an RT; if a cell (i, j) is filled with an AT, all cells (i, z') for $z' < j$, are also filled with an AT.

In a female strategy table, if a cell (i, j) is filled with an rt, all cells (i, w) for $w < j$ are also filled with an rt; if a cell (i, j) is filled with an at, all cells (w', j) for $w' < i$, are also filled with an at.

(v) A male of age x is always willing to accept any female at least as old as $\tau + 1 - \lambda + x$ when either $\tau \geq \lambda$ or $\lambda > \tau$ and $\lambda - x \leq \tau$; and at least female of age 1 when $\lambda > \tau$ and $\lambda - x > \tau$. Accordingly, any female of age y is always willing to accept any male at least as old as $\lambda + 1 - \tau + y$ when either $\lambda \geq \tau$ or $\tau > \lambda$ and $\tau - y \leq \lambda$; and at least male of age 1 when $\tau > \lambda$ and $\tau - y > \lambda$.

This is an immediate consequence of the fact that all players should accept types which offer them a utility at least as high as the utility they expect to receive when they enter the next period unmated. The maximum utility that a male of age x and a female of age y expect to receive if they enter unmated the next period depends on the maximum fertile life of the opposite sex. When the $\tau \geq \lambda$, the utility that a male of age $x+1$ expects is at most $\lambda - x$ and the maximum utility that a female of age $y+1$ expects is λ if $\tau - y > \lambda$ or $\tau - y$ otherwise. When $\lambda \geq \tau$ the maximum utility that a female of age $y+1$ expects is $\tau - y$ and the maximum utility that a male of age $x+1$ expects is τ if $\lambda - x > \tau$ or $\lambda - x$ otherwise.

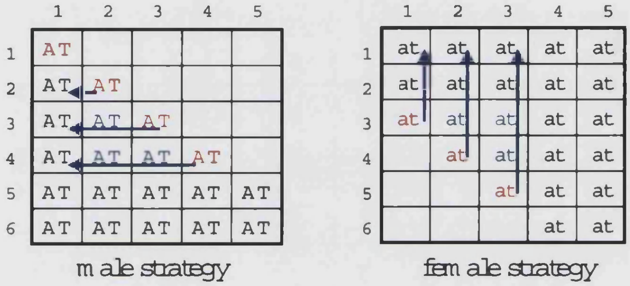


Figure 4.3: Application of observation (v) in a game where $\lambda = 6$ and $\tau = 5$.

(vi) If the players of age z of the limiting sex (the sex that has the smaller population) are non choosy and are accepted by all players of the opposite sex, then at equilibrium there are no players of the limiting sex that are older than z.

Since a type z is always accepted and non choosy, all pairings in which he participates lead to matings. Therefore, no player of age z will remain unmated.

(vii) When $r = 1$, then $R = 1$ when either none leaves the game unmated or when the number of females and males $a_{\lambda+1}$and $b_{\tau+1}$ who leave the game without a partner is the same ($a_{\lambda+1} = b_{\tau+1} \neq 0$). In the first case, at equilibrium, players of age 1 accept all existing types of the opposite sex and therefore no player enters in the second period of his fertile life ($a_1 = b_1 = 1$). In the second case, at equilibrium, $a_1 = b_1 < 1$, young players are not universal acceptors and males of age $\lambda - 1$ and females of age $\tau - 1$ are always rejected by some players of the opposite sex.

The second case can never be true when at least one sex is fertile for at most 3 periods (say males), since in this case a female of age $\tau - 1$ is always accepted at equilibrium.

(viii) When $R = 1$ and $a_1 = b_1 = 1$, then $r = 1$ and as a result nobody enters in the second period of his life unmated. Assuming that females are the limiting sex and the female population is 1, then $b_1 = 1$ only when females are non choosy. Hence in this case all males who are paired with females are mated. Hence from (73)

$$\begin{aligned}
\sum a_i &= r \text{ where } a_i = \frac{r-1}{r} a_{i-1} \\
\Leftrightarrow & \sum a_i \left(1 + \frac{r-1}{r} + \left(\frac{r-1}{r}\right)^2 + \ldots + \left(\frac{r-1}{r}\right)^{\lambda-1}\right) = r \\
\Leftrightarrow & \left(1 - \left(\frac{r-1}{r}\right)^{\lambda}\right) r = r \\
\Leftrightarrow & \; r = 1
\end{aligned}$$

(ix) In our model, both males and females are taken to have the same preferences and we regard the utility that males and females receive from a mating to be the same. Hence, it is obvious that a model where $r \geq 1$ and males are fertile for λ periods and females are fertile for τ periods is identical to a model where $r \leq 1$, males are fertile for τ periods and females are fertile for λ periods.

In the next sections, we provide analytical results for $\tau \in \{2, 3\}$ and various λ. We provide the equilibria, without showing how we calculate them. Therefore, it is not expected that the formulas are obvious to the reader. The complete analysis of the equilibria can be found in Appendix C, where the algebra involved is presented.

4.4 $\tau = 2$: Females are Fertile for 2 Periods

When females are fertile for 2 periods, it is known from observation (*i*) and observation (*ii*) that at equilibrium all females are unchoosy, independently from the number λ of male types and of the male strategy s_M. Knowing the female equilibrium strategy $\bar{s}_F = (1, 1)$, we need to find the male equilibrium strategy $\bar{s}_M$ and a distribution supporting $(\bar{s}_M, \bar{s}_F)$ in order to complete the description of the equilibrium. The male strategy and therefore the equilibrium distribution depend on our assumptions on λ and on r.

4.4.1 $\lambda = \tau = 2$

If there are only two male types, it is a trivial case, since both sexes are unchoosy at equilibrium and the equilibrium profile is $((1,1),(1,1))$.

When $r = 1$, populations are identical since both sexes are fertile for the same number of periods and have the same preferences. We add an asymmetry to the model, assuming that at equilibrium the population sex ratio r must be male biased (it is $r = 1$ in the symmetric case) and we check how this affects the equilibrium distribution.

Independently of r and R, all matchings in each period end up in matings. No female enters age 2 and only the males remaining unpaired in a period enter the next period unmated. Hence the distribution of males and females at equilibrium $((1,1),(1,1))$ is such that

$$\begin{aligned} a_1 &= \frac{r^2}{2r-1} \text{ and } a_2 = \frac{r^2 - r}{2r-1} \\ b_1 &= 1 \text{ and } b_2 = 0 \end{aligned}$$

The sex ratio R of the incoming population is

$$R = a_1 = \frac{r^2}{2r-1} \Leftrightarrow r(R) = R + \sqrt{R^2 - R} \tag{77}$$

The population sex ratio r is an increasing function of the incoming population sex ratio R. It is clear that in order to have a sex ratio that is male biased ($r \geq 1$), we must have $R \geq 1$. If we had a female biased incoming population sex ratio R, given that males and females have the same features, we would end up having a female biased population sex ratio r at equilibrium.

4.4.2 $\lambda \geq 3, \tau = 2, r < 1$ (Females Are Fertile for 2 Periods and Males Are Fertile for $\lambda \geq 3$ Periods; The Population Sex Ratio r Is Female Biased.)

All male types and female types in the last two periods of their fertile life are non choosy (from observations (i) and (ii)). Therefore our only concern is whether males younger than $\lambda - 1$ accept females of age 2. It is proved (further details can be found in Appendix $C.2$) that when $r < 1$ the unique equilibrium profile is $((2,2,2,...,2,1,1),(1,1))$. At equilibrium, we have a female biased population sex ratio ($r < 1$) at equilibrium only when $R < 1$.

At equilibrium, all males younger than $\lambda - 1$ are selective. They reject any potential partner of age 2, in their effort to maximise their chances to meet a female of age 1 and receive the maximum utility of 2. However, when males reach age $\lambda - 1$, this strategy ceases to have any meaning and they become universal acceptors. If males reached age λ unmated, the chances of receiving a utility greater than 1 would have been 0 given that even a female of age 1 would offer them the minimum possible utility 1.

Since females are non choosy and males of an age $\lambda - 1$ or higher are also unchoosy, all pairings of females with a male of age $\lambda - 1$ lead to matings. Consequently no male enters age λ unmated and so no male leaves the game unmated.

The distribution of males and females at equilibrium $((2, 2, 2, ..., 2, 1, 1), (1, 1))$ is such that

$$\begin{aligned}
a_1 &= \frac{(-1+r)r}{(-2+r)\left(1-2\left(\frac{1-r}{2-r}\right)^{\lambda}+\left(-1+\left(\frac{1-r}{2-r}\right)^{\lambda}\right)r\right)} \\
a_i &= a_1\left(\frac{1-r}{2-r}\right)^{j-1} \text{ for } i < \lambda \text{ and } a_\lambda = 0 \\
b_1 &= \frac{1}{2-r} \text{ and } b_2 = \frac{1-r}{2-r}
\end{aligned}$$

All females of age 1 which are paired with a male get mated. As a result only $(1 - r)b_1$ of females reach age 2. Males younger than $\lambda - 2$ are only mated if they are paired with a female of age 1; hence only b_2 of males of age $i \leq \lambda - 2$ enter age $i + 1$.

When $\lambda = 3$, then the incoming population sex ratio is

$$R = \frac{\frac{(2-r)r}{3-2r}}{\frac{1}{2-r}} = \frac{(2-r)^2 r}{3-2r}$$

and r is the a root of the polynomial A_p and an increasing function of R, where

$$A_p = -3R + (4 + 2R)r - 4r^2 + r^3$$

For $r < 1$, it is necessary that $R < 1$.

4.4.3 $\lambda \geq 3, \tau = 2, r \geq 1$ (Females Are Fertile for 2 Periods and Males Are Fertile for $\lambda \geq 3$ Periods, The Population Sex Ratio r Is Male Biased.)

In the previous section, we examined the equilibrium behaviour when the population sex ratio r is female biased ($r < 1$). We can also have equilibria where $r \geq 1$; at these equilibria the incoming population sex ratio R is male biased as well ($R \geq 1$).

As noted in the previous section, females of all ages and males of age 2 and 3 are non choosy, while females of age 1 are accepted by all males. So, it is expected that at equilibrium there will be no females of age 2 and therefore no female will leave the game without finding a partner (observation v)). Furthermore, only those males who are not paired with a female can enter the next period unmated.

Thus, independently of the strategy pair at equilibrium, the distribution has to be of the form

$$\begin{aligned} a_1 &= \frac{1}{1-\left(\frac{r-1}{r}\right)^{\lambda}} \\ a_i &= a_1\left(1-\frac{1}{r}\right) \text{ for } i>1 \\ b_1 &= 1 \end{aligned}$$

Hence

$$R = \frac{1}{1-\left(\frac{r-1}{r}\right)^{\lambda}}$$

and

$$r = \frac{1}{1-\left(\frac{R-1}{R}\right)^{\frac{1}{\lambda}}}$$

When $\lambda = 3$, then the incoming population sex ratio is

$$R = \frac{1}{1-\left(\frac{r-1}{r}\right)^{3}} = \frac{r^3}{1-3r+3r^2}$$

and

$$r = \frac{1}{1-\sqrt[3]{\frac{1}{R}\left(R-1\right)}}$$

For $r \geq 1$, it is necessary that $R \geq 1$.

Since the strategy of females and the population distribution at equilibrium are known, we need to find the best response of males in order to complete the equilibrium description. It is known from observations (iii) and (iv) that if a male of age x is selective (accepts only females of age 1), then all males younger than x are equally selective.

In order to check whether a strategy is an equilibrium, we just have to find r such that the youngest males which are not choosy (accept all females) have an expected utility that is higher than 1. A male of age $x \leq \lambda - 3$ expects that if he enters in the next period unmated, he will receive a utility of

$$U_{x+1} = \frac{2}{r}\left(\sum_{i=0}^{x-2}\left(\frac{r-1}{r}\right)^{i}\right) + \frac{1}{r}\left(\frac{r-1}{r}\right)^{x-1}$$

A strategy profile

$$((\underbrace{2,2,2,2,.........,2,2,2}_{\text{a number } x \text{ of 2's}},\underbrace{1,1,1,...,1,1,1}_{\text{a number } \lambda - x \text{ of 1's}}),(1,1))$$

is an equilibrium if

$$U_{x+1} > 1 \text{ and } U_{x+2} \leq 1$$

Hence for λ large,

- $((2,2,..........,2,1,1),(1,1))$ is an equilibrium for $1 \leq r < 2.618$
- $((2,2,.........,2,1,1,1),(1,1))$ is an equilibrium for $2.618 \leq r < 4.08$
- $((2,2,.......,2,1,1,1,1),(1,1))$ is an equilibrium for $4.08 \leq r < 5.53$
- $((2,2,.....,2,1,1,1,1,1),(1,1))$ is an equilibrium for $5.53 \leq r < 6.977$
- $((2,2,...,2,1,1,1,1,1,1),(1,1))$ is an equilibrium for $6.977 \leq r < 8.423$

  etc

It becomes clear that for given λ, as r increases the age of the youngest male which is unchoosy decreases.

When $\lambda = 3$, strategy pair $((2,1,1),(1,1))$ is an equilibrium for $r \leq 2.618$ and strategy pair $((1,1),(1,1,1))$ is an equilibrium for $r > 2.618$.

4.5 $\lambda = 3$: Females Are Fertile for 3 Periods

When females are fertile for 3 periods, females of age 2 and 3 are not choosy. However the strategy of females of age 1 depends on the male behaviour and the equilibrium distribution.

4.5.1 $\lambda = \tau = 3$

When $\lambda = \tau = 3$, not only females but also males of age 2 and 3 are universal acceptors (from observations (i) and (ii)). Therefore, we only need to check whether females and males of age 1 are choosy. We add an asymmetry to the game, by assuming that at equilibrium the population sex ratio is male biased ($r = 1$ when the game is symmetric).

For $r \geq 1$, females of age 1 always reject males of age 3. For males of age 1 to have a similar behaviour, population sex ratio at equilibrium must not exceed $r = 2$. Hence if $1 \leq r \leq 2$, the equilibrium strategy profile is $((2,1,1),(2,1,1))$. When $r > 2$ though, males have no incentive to be choosy any more (as it is shown in Appendix C.4) and the strategy profile at equilibrium becomes the $((1,1,1),(2,1,1))$.

At both equilibria, no female ever reaches age 3 (from observation (v)). So, the distribution that supports both equilibrium strategy profiles is the following

$$\begin{aligned}
a_1 &= \frac{r^3}{1-3r+3r^2} \\
a_2 &= \frac{(-1+r)r^2}{1-3r+3r^2} \\
a_3 &= \frac{(-1+r)^2 r}{1-3r+3r^2} \\
b_1 &= \frac{1-3r+3r^2}{2-5r+4r^2} \\
b_2 &= \frac{(-1+r)^2}{2-5r+4r^2} \\
b_3 &= 0
\end{aligned}$$

Only those females of age 1 which are paired with a male of age 3 reach age 2 and only those males of age $x < 3$ which are not paired with a female enter age $x+1$. Males of age 3 are mated only when they are paired with a female of age 2.

At both equilibria,

$$R = \frac{\frac{r^3}{1-3r+3r^2}}{\frac{1-3r+3r^2}{2-5r+4r^2}} = \frac{2r^2 - 5r^3 + 4r^4}{15r^2 - 6r - 18r^3 + 9r^4 + 1}$$

and r is a root of the polynomial B_p; $r(R)$ is an increasing function of R.

$$B_b = -R + 6Rr - 15Rr^2 + 2r^3 + 18Rr^3 - 5r^4 - 9Rr^4 + 4r^5$$

In order for the male population to be higher than the female population at equilibrium ($r \geq 1$), the incoming population sex ratio has to be male biased ($R \geq 1$).

To summarise, the equilibria are

$((2,1,1),(2,1,1))$ for $1 \leq R \leq 1.30612$ hence for $1 \leq r \leq 2$
$((1,1,1),(2,1,1))$ for $R > 1.30612$ hence for $r > 2$

Any combination between males and females appears in the couple distribution at both equilibria, except from matings between females of age 1 and males of age 3. Males of age 3 are the most underrepresented in the couple distribution since they have the highest probability to remain unmated in a period.

4.5.2 $\lambda = 4, \tau = 3$ (Males Are Fertile for 4 Periods, Females Are Fertile for 3 Periods.)

Males and females in the last two periods of their fertile life are universal acceptors (from observations (*i*) and (*ii*)). Hence we need to check whether males and females of age 1 and males of age 2 are choosy. From observation (*v*) if females of age 1 are choosy, they can only reject males of age 4, since the maximum expected utility they can get in age 2 is 2. In the same way, if males of age 2 are choosy, they can only reject females of age 3; if males of age 1 are choosy they may reject females of age 2 and 3 or only females of age 3. From observation (*iv*), it is known that if females of age 2 are choosy, then females of age 1 have to be equally choosy or even choosier. Hence there are 10 potential equilibrium strategy pairs. We show at which equilibria the population sex ratio is male biased or female biased.

4.5.2.1 $r \leq 1$: *Searching for Equilibria Where The Population Sex Ratio r Is Female Biased*

When male population is lower than female population at equilibrium, there is a unique equilibrium for all $r \leq 1$, where females are non choosy while males are the choosiest they can be. Hence $((3,2,1,1),(1,1,1))$ is the unique equilibrium strategy pair and the equilibrium distribution is of the form

a_1 is a root of the polynomial D

$$a_2 = \frac{-2a_1 - a_1^2 + a_1 r}{2(1 + a1 - r)} + \frac{1}{2}\sqrt{\frac{\begin{array}{c}-4a_1^2 - 8a_1^2 - 3a_1^3 + 8a1r + 20a_1^2 r \\ +10a_1^3 r - 12a_1 r^2 - 11a_1^2 r^2 + 4a_1 r^3\end{array}}{(1 + a1 - r)^2}}$$

$$a_3 = r - a_1 - a_2$$

$$a_4 = 0$$

$$b_1 = \frac{a_1 - a_2}{a_1}$$

$$b_2 = 1 - b_1 - \frac{a_3}{a_2}$$

$$b_3 = \frac{a_3}{a_2}$$

where

$$D = -9r + 18r^2 - 15r^3 + 6r^4 - r^5 + \left(\left(17 - 46r + 50r^2 - 24r^3 + 5r^4\right) R \right.$$
$$\left. + (14 - 32r + 25r - 7r^3)R^2 + (3 - 6r + 3r^2)R^3\right.$$

No male enters in the last period of his fertile life, as follows from observation (*iv*). Males of type 1 end up mated only with females of age 1 and females of type 3 end up mated only with males of age 3. Only those females of age $x < 3$ which are not paired with a male enter age $x + 1$ and only those males of age 2 which are paired with females of age 3 enter age 3. Males of age 3 and females of age 1 accept all possible pairings, so they can have partners of different ages.

Let

$$\begin{aligned} E &= 4R + 32R^2 + 9R^3 + (-54 - 151R - 92R^2 - 18R^3)r \\ &\quad +(189 + 278R + 107R^2 + 9R^3)r^2 + (-297 - 273R - 62R^2)r^3 \\ &\quad +(270 + 158R + 17R^2)r^4 + (-153 - 55R - 2R^2)r^5 + (54 + 11R)r^6 \\ &\quad +(-11 - R)r^7 + r^8 \end{aligned}$$

Then r is a root of the polynomial E; $r(R)$ is an increasing function and it is proved that in order to have a female biased population sex ratio ($r \leq 1$) at equilibrium, it is necessary that the incoming population is also female biased ($R \leq 1$).

In the more general case, where $\lambda > 4$ but $\tau = 3$, the equilibrium strategy profile is $((3, 3, ...3, 2, 1, 1), (1, 1, 1))$ according to observations (*iv*) and (*v*). Since females are non choosy, males of age $\lambda - 3$ or younger are not willing to compromise and accept any female of an age older than 1. Nevertheless, when males reach age $\lambda - 3$ they adapt their behaviour and become less choosy, under the threat of entering in the last period of their life without a partner.

4.5.2.2 $r \geq 1$: *Searching for Equilibria Where The Population Sex Ratio r Is Male Biased*

When female population is lower than the male at equilibrium, different equilibria may appear depending on the sex ratio r or on the incoming population sex ratio R. As r increases males tend to become less choosy, while females become choosier.

When females are unchoosy at equilibrium, all females get mated in the first period of their fertile life, hence no female ever reaches ages 2 and 3. Therefore, when the female strategy is $(1, 1, 1)$, the population distribution is the following

$$\begin{aligned} a_1 &= \frac{r^4}{-1 + 4r - 6r^2 + 4r^3} \\ a_i &= \frac{-a_{i-1} + a_{i-1}r}{r} \text{ for } 1 < i \leq 4 \\ b_1 &= 1 \text{ and } b_2 = b_3 = 0 \end{aligned} \tag{78}$$

Only those males of age $x < \lambda$ which are not paired with a female enter age $x+1$.

Given the distribution (78), we have

$$R = \frac{r^4}{-1+4r-6r^2+4r^3}$$

and

$$r(R) = R - \sqrt{R+R^2} + \frac{1}{2}\sqrt{-8R+8R^2 - \frac{32R-96R^2+64R^3}{8\sqrt{-R+R^2}}}$$

Hence $r \leq 1$ when $R \leq 1$.

On the other hand, when females of age 1 are choosy at equilibrium, for any $r \geq 1$ females of both age 1 and age 2 are accepted by all males. Therefore from observation (*vi*) it follows that at equilibrium there are females of age 1 and age 2 but no females ever reach age 3. When the equilibrium female strategy is $(2,1,1)$, the distribution is the following

$$\begin{aligned} a_1 &= \frac{r^4}{-1+4r-6r^2+4r^3} \\ a_i &= \frac{-a_{i-1}+a_{i-1}r}{r} \text{ for } 1 < i \leq 4 \\ b_1 &= \frac{-1+4r-6r^2+4r^3}{-2+7r-9r^2+5r^3} \\ b_2 &= 1-b_1 \text{ and } b_3 = 0 \end{aligned} \tag{79}$$

As before, only those males of age $x < \lambda$ which are not paired with a female enter age $x+1$. Furthermore, only those females of age 1 which are paired with males of age λ enter age 2 unmated.

Given the distribution (79), we have

$$R = \frac{\left(5r^7-9r^6+7r^5-2r^4\right)}{\left(4r^3-6r^2+4r-1\right)^2}$$

Solving the previous equation in terms or r, we have

$$F = -R+8Rr-28Rr^2+56Rr^3+(-2-68R)r^4+(7+48R)r^5+(-9-16R)r^6+5r^7$$

and r is a root of the polynomial F; r is an increasing function of R. $r(R)$ takes its minimum value $r=0$ when $R=0$ and its maximum value $r=1$ when $R=1$.

The following table summarises the equilibria.

$((3,2,1,1),(1,1,1))$	for $1 \leq R \leq 1.078$	hence for $1 \leq r \leq 2.078$
$((2,2,1,1),(2,1,1))$	for $1.131 \leq R < 1.145$	hence for $1.969 \leq r < 2.033$
$((2,1,1,1),(2,1,1))$	for $1.145 \leq R < 1.837$	hence for $2.033 \leq r < 4.5$
$((1,1,1,1),(2,1,1))$	for $R \geq 1.837$	hence for $r \geq 4.5$

Looking at the above table, we note that no equilibrium was found for $1.078 \leq R < 1.131$.

If we try to find an equilibrium for $1.0781 \leq R < 1.131$ where females are not choosy, it is known that the distribution should be of the form of (78), and males best response should be either strategy $(2,2,1,1)$ or strategy $(3,2,1,1)$ depending on the value of R. Nevertheless when males use either of these strategies, females are better off by being choosy, hence by using strategy $(2,1,1)$. Against $(2,1,1)$, neither $(2,2,1,1)$ or $(3,2,1,1)$ are best responses and the equilibrium collapses. On the other hand, if we assume that there is an equilibrium where females are choosy (they use strategy $(2,1,1)$) for $1.078 \leq R < 1.131$ then the population distribution is of the form of (79) and the best response of males is to use strategy $(3,2,1,1)$. However, against this strategy, females are better off by not being choosy anymore and the equilibrium collapses again. As it becomes obvious, when $1.078 \leq R < 1.131$ it is expected that male and female pure strategies will alternate constantly and no pure strategy equilibrium will ever been reached.

Although an equilibrium is not always attained for all $R > 0$, there is at least an equilibrium for any $r > 0$. It is particularly interesting that to any r in $[1.969, 2.078]$ correspond two possible equilibria. Hence, there are two different incoming population sex ratios R that may lead to a particular $\bar{r}$ in $[1.969, 2.078]$ at equilibrium; for $R \in [1.062, 1.078]$, females are unchoosy at equilibrium while for $R \in [1.131, 1.156]$ females are choosy at equilibrium.

In Figure 4.4 are illustrated the male and female population distributions at equilibrium when $r \in [1.969, 2.078]$. It is shown that for given r, male distribution is the same independently of the equilibrium. However, this is not true for females. For given r, the fraction of females of age 1 is higher at equilibrium when females are non choosy. Consequently, for given r, R has to be lower at the equilibrium when females are non choosy, as it is shown in Figure 4.5. At equilibrium $((3,2,1,1),(1,1,1))$ there are no type 2 females, hence

$$b_{1((3,2,1,1),(1,1,1))} = \begin{cases} b_{1((2,2,1,1),(2,1,1))} + b_{2((2,2,1,1),(2,1,1))} & \text{for } r \in [1.969, 2.033) \\ b_{1((2,1,1,1),(2,1,1))} + b_{2((2,1,1,1),(2,1,1))} & \text{for } r \in [2.033, 2.078) \end{cases}$$

while

$$a_{i((3,2,1,1),(1,1,1))} = \begin{cases} a_{i((2,2,1,1),(2,1,1))} & \text{for } r \in [1.969, 2.033) \text{ and } 1 \le i \le 4 \\ a_{i((2,1,1,1),(2,1,1))} & \text{for } r \in [2.033, 2.078) \text{ and } 1 \le i \le 4 \end{cases}$$

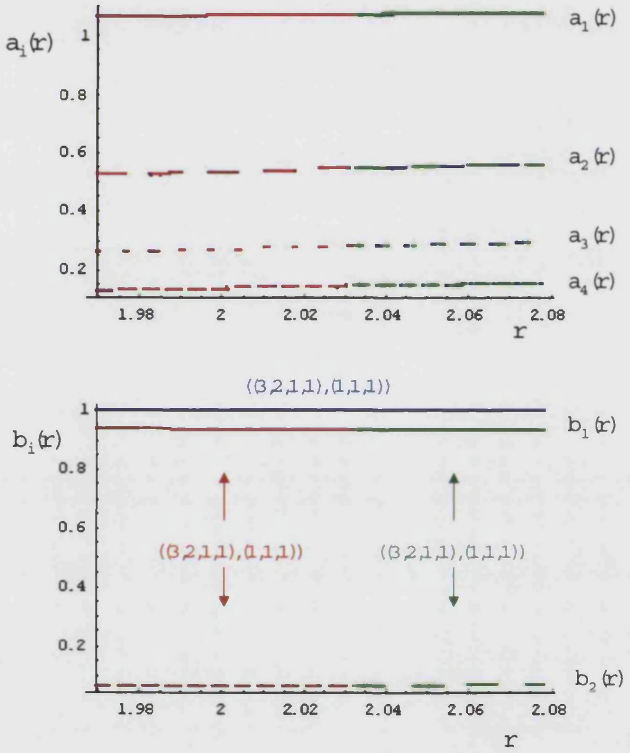


Figure 4.4: Comparison of the male distribution (top picture) and the female distribution (bottom picture) at different equilibria when $\lambda = 4$, $\tau = 3$ and $r \in [1.969, 2.078]$.

In the following figure we draw the incoming population sex ratio R as a function of the population sex ratio r at $((3,2,1,1),(1,1,1))$, $((2,2,1,1)(2,1,1))$ and $((2,1,1,1),(2,1,1))$. It is apparent that R is an increasing function of r at all equilibria.

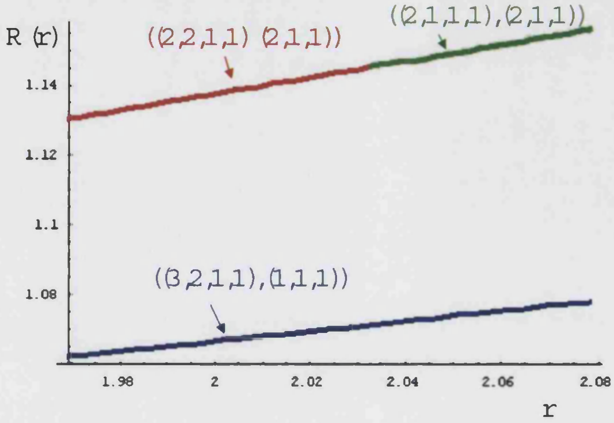


Figure 4.5: Comparison of the incoming popualtion sex ratio R at different equilibria when $\lambda = 4$, $\tau = 3$ and $r \in [1.969, 2.078]$.

Having analysed the equilibrium behaviour for $\lambda = 4$ and $\tau = 3$, we can focus on the more general case, where $\lambda > 4$ and $\lambda = 3$. From the analysis of $\lambda = 4$, it is known that for $\lambda > 4$, when $1 \leq r \leq 2.078$, the strategy pair $((3, 3, ..., 3, 2, 1, 1), (1, 1, 1))$ is always an equilibrium and the equilibrium distribution is the following

$$
\begin{aligned}
a_1 &= \frac{1}{1 - (\frac{r-1}{r})^{\lambda}} \\
a_i &= b_{i-1}(\frac{r-1}{r}) \text{ for } 1 < i \leq \lambda \\
b_1 &= 1
\end{aligned}
$$

Nevertheless, when $r > 2.078$, it is not possible to find the equilibrium strategies based on our results of the $\lambda = 4$ and $\tau = 3$ model and a complete analysis of each specific case is needed.

4.6 Analysis of The Sex Ratio r and of The Male and Female Distributions at Equilibrium

In general, in all models examined, r is proved to be increasing with R. In each case analysed, r is male (female) biased only when R is male (female) biased and $r = 1$ only when $R = 1$. However, the rate of increase of r varies, depending on the particular game. The following figures show the population sex ratio

$r(R)$ at different equilibria when $\tau \in \{2,3\}$ and $\lambda \in \{2,3,4\}$. The population sex ratio r for $\lambda = \tau = 2$ is described is red, for $\lambda = 3, \tau = 2$ is described in green (the dashed lines illustrate change of equilibrium strategies), for $\lambda = \tau = 3$ is described in blue (the dashed lines illustrate change of equilibrium strategies) and for $\lambda = 4, \tau = 3$ is described in purple, black and grey (where each colour describes different equilibrium strategies). In Appendix C, there are more detailed figures for each particular model.

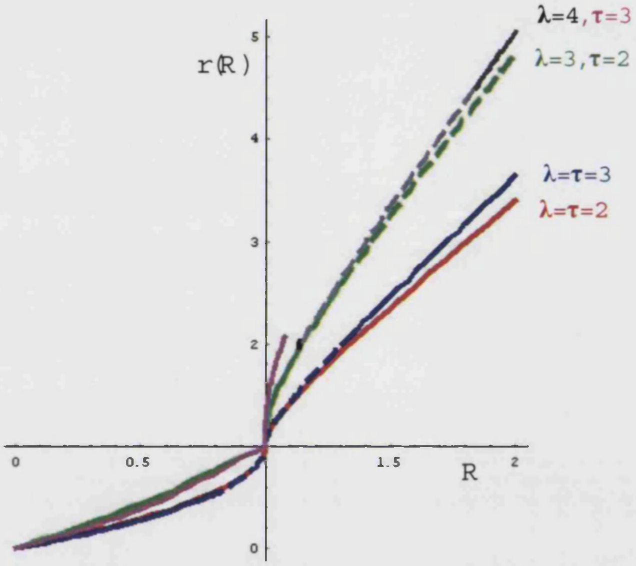


Figure 4.6: Population sex ratio r for different λ and τ.

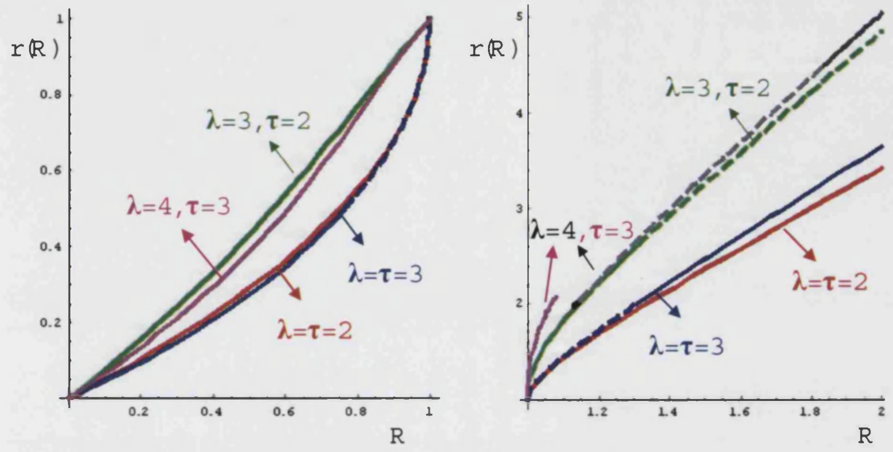


Figure 4.7: Population sex ratio r for different λ and τ. The left figure focuses on the behaviour of r when $R < 1$ and the right picture focuses on the behaviour of r when $R \geq 1$.

Having a close look at the male and female distribution in the different games we studied, we see that there are common characteristics between the models, although the rate of change of a_i and b_j may be different depending on the case. Analysing $\frac{a_i}{r}$ in each model, it becomes obvious that the fraction $\frac{a_1}{r}$ of males of age 1 is increasing for $R < 1$ and decreasing for $R \geq 1$. The opposite is true for $\frac{a_i}{r}$, where $i > 1$. Focusing on the models where $R < 1$, we observe that the number a_i of male i ,where $i > 1$, can reach its maximum for $R \neq 1$ (as in the model $\lambda = 3, \tau = 2$, Figure $C.4$, illustrated in Appendix $C.2$). The number of males of age 1 is increasing for all R.

Focusing on the female distribution, b_1 is increasing for $R < 1$ and decreasing otherwise, while b_j, where $j > 1$, is decreasing for $R < 1$ and increasing otherwise. (Exception is the case where $b_1 = 1$, since b_1 remains constant for $R \geq 1$).

At $R = 1$, all males and females are of type 1. For $R \geq 1$, as R increases all $\frac{a_i}{r}$ and correspondingly all b_j tend to come closer, but they never become equal; it is always true that $\frac{a_i}{r} > \frac{a_{i+1}}{r}$ and $b_j > b_{j+1}$. When $R \leq 1$, $\frac{a_i}{r}$ and b_j behave similarly as R decreases and moves closer to 0.

The behaviour of a_i and b_i is explained with the help of equations (73) and (74). When $R = 1$, the game is symmetric and all matings are between males and females of age 1. Nevertheless, as R moves away from 1, it is expected that more players of the sex that has a higher population will remain unmated and will move to the next period, especially when the players are choosy. The choosier the players are, the more players will end up in the final periods of their life unmated.

Further analytical results and figures describing the behaviour of a_i (and $\frac{a_i}{r}$) and b_j in each model are given in Appendix C.

4.7 Population Productivity

We assume that players seek partners so that they maximise the number of periods they are mated with a partner who is reproductive. We are interested in analysing the population productivity pp. Since we used the terms males and females in the analysis of the previous sections, we are going to use the term offspring for the output of a mating. In this section, the sex of an offspring is not relevant.

It is expected that the productivity of a couple (i, j) is connected to the number of periods $u_{i,j}$ the partners (i, j) expect to be both fertile. If we assume that a couple (i, j) has on average J offspring every period over which both partners are able to reproduce, i and j are expected to produce Ju_{ij} offspring during their lifetime.

Since we have a steady state model, we find the population productivity pp by calculating the total number of offspring produced by the couples created in a particular period, over the lifetime of each couple. The population productivity also coincides with the total number of offspring expected to be produced in a particular period, by all the couples for which both partners are still fertile, independently of the period during which they were created.

Hence the total number pp of offspring that is produced as a result of the mating of the population in any period (hence the population productivity) is

$$pp_{(s_M,s_F)}(r) = \begin{cases} \sum_{i=1}^{\lambda} a_i(\sum_{j=1}^{\tau} k_{i,j} b_j u_{ij}) \text{ if } r \leq 1 \\ \sum_{j=1}^{\tau} b_j(\sum_{i=1}^{\lambda} k_{i,j} \frac{a_i}{r} u_{ij}) \text{ if } r \geq 1 \end{cases}$$

We remind to the reader, that in each period, when $r < 1$, the number of males of age i matched with females of age j are $a_i b_j$ (respectively when $r \geq 1$ it is $a_i b_j \frac{1}{r}$). k_{ij} was defined earlier in (72) as a binary variable taking the value 1 only when both i and j accept each other.

We expect that the population "satisfaction" or the population welfare is maximised when the productivity takes its maximum value. In the models we analyse, the pp at equilibrium is maximised when $r = 1$, so when $R = 1$, while it tends to decrease as the incoming population sex ratio R (so the population sex ratio r) becomes male or female biased. When $r = R = 1$, all matings are between males and females of age 1, hence the number of offspring each couple produces is maximum.

When $J = 1$, the population productivity expresses the average number of periods over which are fertile both the male and the female forming a random couple in a period.

Figure 4.8 shows how the population productivity varies with R. We assume $J = 1$. We describe *pp* in red for $\lambda = \tau = 2$, in green for $\lambda = 3, \tau = 2$ (the dashed lines illustrate change of equilibrium strategies), in blue (the dashed lines illustrate change of equilibrium strategies) for $\lambda = \tau = 3$ and in purple, black and grey (where each colour describes different equilibrium strategies) for $\lambda = 4, \tau = 3$.

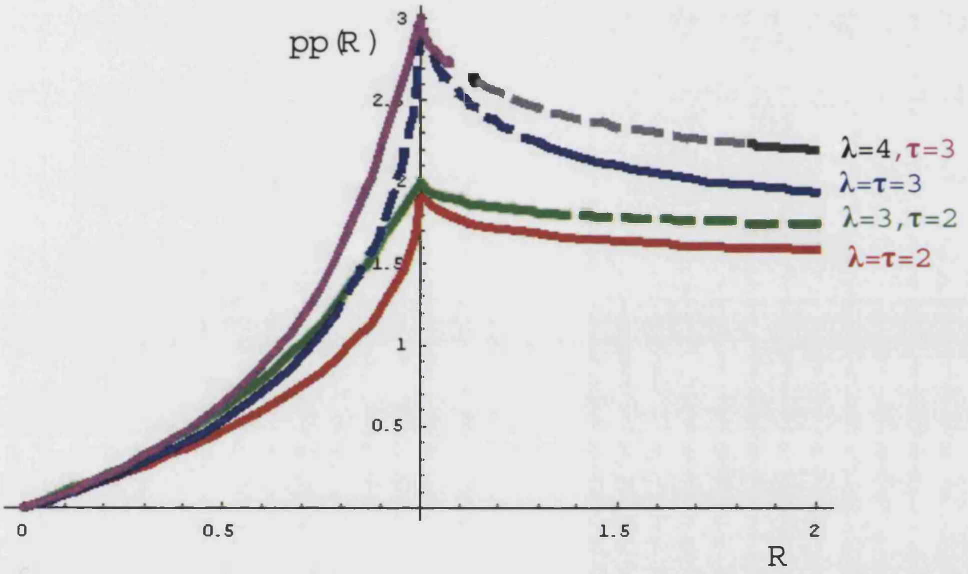


Figure 4.8: Population productivity *pp* for different λ and τ.

4.7.1 4.7.1 Stable population

We assume that at equilibrium the population remains constant over time. We are interested in finding under which conditions at equilibrium, the population input is equal with the population output. Hence we want to find for which r, or for which R, the number of offspring produced in each period is the same as the total number of males and females of age 1, so that the population distribution does not change and the population remains "stable".

We assume that the ratio of male offspring over female offspring produced in each year is the same as the ratio R of the incoming population. Of course, this assumption may not be true in a real life situation. However, let's assume that offspring are sexless and we can choose their sex.

In biology, the productivity of a population depends on various factors, such as presence of predators and other species, abundance of food and other environmental factors. In this section, we analyse for which R the productivity pp is such that the population remains stable, under the absence of any other factor that can affect it. Hence we search for R such that

$$pp_{(s_M,s_F)}(r) = a_1(r) + b_1(r) \tag{92}$$

In our analysis we assume that $J = 1$, hence a couple produces 1 offspring per period. This analysis could be done for any value of J.

When $\tau = 2$, for $\lambda \epsilon \{2, 3\}$, the number of offspring born in each period is equal with the number of new individuals entering in game in each period only in the trivial case when $r = 1$ and $R = 1$ and therefore when the maximum number of offspring (1 male and 1 female) is produced in every period.

When $\lambda = \tau = 3$ and $r \geq 1$, the equation (92) is satisfied when $r = 2.43756$, hence when $R = 1.48389$ and the equilibrium strategy is $((1, 1, 1), (2, 1, 1))$. Under these conditions, the number of offspring produced is 2.106 (61.261% of the total population), consisting of 1.258 males (51.611% of the total male population) and 0.848 females (84.781% of the total female population). Correspondingly, assuming that $r < 1$, then the equation(92) is satisfied when $r = 0.41$, hence when $R = 0.674$ and the equilibrium strategy is $((2, 1, 1), (1, 1, 1))$. Under these conditions, the number of offspring produced is 0.864, consisting of 0.348 males and 0.516 females.

When $\lambda = 3$, but the females are productive for longer, so $\tau = 4$ and $r \geq 1$, the number of offspring produced is the same as the number of males and females of type 1 entering in the game in each period when $r = 2.131$, so when $R = 1.734$. The equilibrium strategy is $((1, 1, 1), (3, 2, 1, 1))$. In this case, the number of offspring per period is 1.707 (54.532% of the total population), consisting of 1.083 males (50.814 % of the total male population) and 0.625 females (62.455% of the total female population). Comparing this result with the case where $\lambda = \tau = 3$, it becomes obvious that the number of offspring produced is a higher percentage of the population when $\lambda = 3$, $\tau = 4$, and the difference between the number of males and females produced is higher when $\lambda = \tau = 3$.

When $\lambda = 3$ and $m = 4$ and $r < 1$, the equation (92) is satisfied in two cases. Either when $r = 0.597$, so $R = 0.974$ and the equilibrium strategy pair is $((1, 1, 1), (3, 2, 1, 1))$ or when $r = 0.264$, so $R = 0.615$ and the equilibrium strategy pair is $((1, 1, 1), (2, 1, 1, 1))$. When males are unchoosy, the number of offspring produced is 1.21 (75.777% of the total population), consisting of 0.597

(100% of the male population) males and 0.613 females (61.308% of the female population) while when males are choosy, 0.603 offspring (47.733% of the total population) are produced per period, consisting of 0.23 males (87.033% of the male population) and 0.374 females (37.36% of the female population). Comparing the two cases, it is obvious that when males are unchoosy, the number of offspring produced is much higher, while the difference between the number of young males and females is significantly smaller.

4.8 Average Male and Female Age at Marriage

In the models we analyse, all males and females are mated at age 1 when $R = 1$. It is interesting to examine what is the average age at marriage of a male and of a female when $R \neq 1$. We name $\overline{mma}$ and $\overline{mfa}$ the mean male and mean female ages at marriage. We have

$$\begin{aligned}
\overline{mma}(r) &= \begin{cases} \frac{1}{k_{r\geq 1}}\left(\sum_{i=1}^{i=\lambda} i a_i \sum_{j=1}^{j=\tau} k_{i,j}\frac{1}{r}b_j\right) & \text{for } r \geq 1 \\ \frac{1}{k_{r<1}}\left(\sum_{i=1}^{i=\lambda} i a_i \sum_{j=1}^{j=\tau} k_{i,j}b_j\right) & \text{for } r < 1 \end{cases} \\
\overline{mfa}(r) &= \begin{cases} \frac{1}{k_{r\geq 1}}\left(\sum_{j=1}^{j=\tau} j b_j \sum_{i=1}^{i=\lambda} k_{i,j}a_i\right) & \text{for } r \geq 1 \\ \frac{1}{k_{r\geq 1}}\left(\sum_{j=1}^{j=\tau} j b_j \sum_{i=1}^{i=\lambda} k_{i,j}a_i\right) & \text{for } r < 1 \end{cases} \\
&\text{where} \\
k_{r\geq 1} &= \sum_{i=1}^{i=\lambda} a_i \sum_{j=1}^{j=\tau} k_{i,j}\frac{1}{r}b_j \text{ and } k_{r<1} = \sum_{i=1}^{i=\lambda} a_i \sum_{j=1}^{j=\tau} k_{i,j}b_j
\end{aligned}$$

The average age at marriage and the population productivity can be used as measures of the satisfaction of players at equilibrium.

In Figures 4.9 and 4.10, we present the average age at marriage of a male and a female respectively as functions of R. Clearly when $\lambda = \tau = 2$ and $R < 1$, all males are mated at age 1, since $r = a_1$. Similarly when $\lambda = 3$, $\tau = 2$ and $R > 1$, all females are mated at age 1, since $b_1 = 1$. In all models examined, the number of periods that both males and females have to wait before they get mated is minimum when $R = 1$, since as we already noted there are no players which are not mated at age 1 when $R = 1$. However, as the incoming population becomes male or female biased, the average age at marriage of both males and females increases, but it never exceeds a maximum limit. Figure 4.11 describes the maximum age at marriage of a male or of a female at any equilibrium, for R large ($R \rightarrow \infty$) or small ($R \rightarrow 0$). It is interesting to note that for given R and λ, male average age at marriage is lower when τ is higher. We also observe that for given R and τ, female average age at marriage is lower when λ is higher.

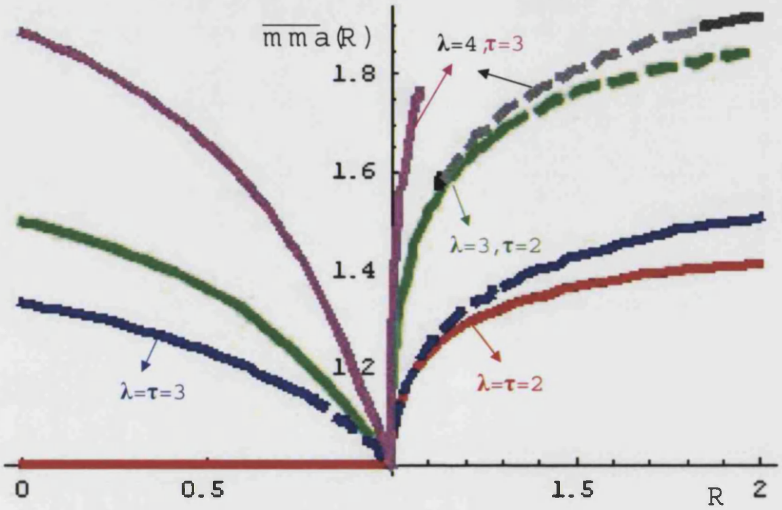


Figure 4.9: Average age of males at marriage.

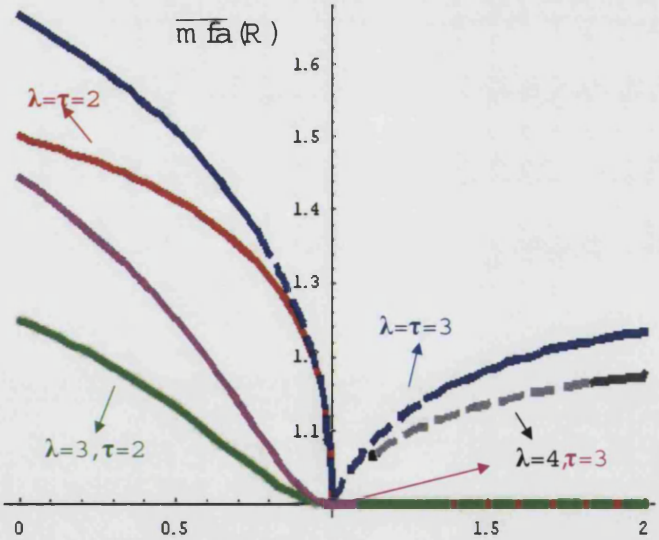


Figure 4.10: Average age of females at marriage.

Game			Max $\overline{mma}$	Max $\overline{mfa}$
$\lambda = \tau = 2$		R large/small	1.5	1
$\lambda = 3, \tau = 2, r < 1$		R small	1.5	1.25
$\lambda = 3, \tau = 2, r > 1$	((2,1,1),(1,1))	R=1.309	1.691	1
	((1,1,1),(1,1))	R large	2	1
$\lambda = \tau = 3$		R large/small	1.667	1.333
$\lambda = 4, \tau = 3, r < 1$	((3,2,1,1),(1,1,1))	R small	1.889	1.444
$\lambda = 4, \tau = 3, r > 1$	((3,2,1,1),(1,1,1))	R= 1.078	1.078	1
	((2,2,1,1),(2,1,1))	R= 1.145	1.139	1.002
	((2,1,1,1),(2,1,1))	R=1.837	1.531	1.504
	((2,1,1,1),(2,1,1))	R large	2.125	1.25

Figure 4.11: Maximum number of periods that a male and a female need to wait before they get mated.

4.9 Conclusions

We analysed a model where the utility of a mating depends on the age of both partners: the utility that a male and a female forming a couple receive is the number of periods they are both able to reproduce. Players are interested in finding a young partner, but their choosiness depends on their age. Players demonstrate a less selective behaviour as they get older; close to the end of their fertile life they are always willing to accept any partner they are paired with.

The decrease of choosiness of females with age is observed in nature. When the breeding season is finite and a mating has to take place in a specific time frame, females often present an adaptive mating behaviour (Bateson and Healy (2005)). As Uetz and Norton (2007) note, in an annual species where reproduction has time constraints, older females become less choosy, since if they risk waiting for a higher quality mate, they may end up with no mate at all (Moore and Moore (2001)). Furthermore, choosier behaviour of young compared to old females has been described in studies of crickets and fish (Gray (1999), Kodric-Brown and Nicoletto (2001)), although it is not always clarified if it is connected with physiological reasons or with the motivation of the animal.

In our model, choosiness does not depend on age only, but also on the availability of potential partners. Assuming that females are the limiting sex, we noted that as the incoming population becomes more male biased, females demonstrate a choosier behaviour while males become less choosy. The opposite is true when males are the limiting sex. In the models described in the previous chapter, we did not observe an analogous behaviour. On the contrary, the choosiness of both sexes moved to the same direction as r varied. However, given that the model described in this chapter is a steady state model whilst the models in chapters 3 were not, any previous results do not contrast with our current observations.

We also examined the population productivity and the average age at marriage as functions of the incoming population sex ratio R. In the models examined, we found that when $R = 1$ the population productivity is maximum and the average age at marriage of both males and females is minimum. As the bias of the incoming population sex ratio R increases, the population productivity decreases and the average age at marriage of both males and females increases. Furthermore, making some specific assumptions, we examined for which R, the total input of the population is the same with the total output.

Population productivity and mean age at marriage can be considered as indicators of the "satisfaction" of male and female populations at the equilibrium. In this case, players are better off when the number of males equals the number of females entering in the game in each period.

In our game we did not allow "remarrying" or divorcing, even though these are phenomena that may appear in nature. If divorcing and remarrying are allowed, without assuming any searching or other costs, all equilibria will be trivial; in each period, anyone who has no partner will accept the player with whom he is mated and will abandon her when he finds a better partner who is willing to accept him. However, if searching costs, cost of divorcing or remarrying are added to our model, new conclusions may be attained.

Another possible extension of the model would be to examine the equilibrium when different types have different preferences or when all types prefer older partners.

In the analysis of this chapter, we focused mostly on the biological implications of our model. Nevertheless, our model can also have applications in non biological problems and it is not built to be used only in a strict biological or psychological context.

Imagine a situation where two teams of children in a camp are asked to form couples to participate in a game taking place during the whole year. We assume that there are two possible programs in the camp and children can either stay λ or τ weeks. Children are allocated in a team according to the duration of their stay in the camp. It is not possible to have all children of both programs to be involved in the game, but the game is very popular and every child wants to participate. The organisers, trying to deal with this problem, have established a rule that does not permit to any child without a partner or to any child that had a partner but the latter left him, to take part in the game. Couples must be created between children from different teams. Each child wants to participate in the game the longest possible. Hence he wants to find a partner that stays as long as he does in the camp (or longer) or, if this is not possible, a partner that stays the longest possible in the camp. Our model can predict what strategy should a child use in each week of his stay in each program so that to maximise the number of weeks participating in the game.

Another application of our model could be a situation where the students in two different departments (universities) are asked to find roommates to share their rooms in the halls in the beginning of each year. Each room has to be occupied by two students from different departments (universities). Students may be in different years of their degree. If a student finishes his degree and leaves the halls, his roommate has to leave the room as well and does not have the right to find a new roommate, due to the popularity of the rooms. Our model can describe how students should choose their potential roommates at equilibrium.

Further examples can be found in different fields, where our model can be used to predict the equilibrium behaviour. Age can prove to be an important determinant of the utility of a matching and its role needs to be analysed further.

5 Literature review

5.1 Introduction

We have taken the unusual approach of placing this review after presenting our work. However, in the introduction we have already described to the reader the basic models that our work is based on. In chapter 1 we offered a detailed description of the models of Alpern and Reyniers (1999, 2005), giving the reader an insight into the background of our own research. The goal of this chapter is to offer a general description of the area of dynamic mutual choice mating games.

Firstly, we comment on the importance of the matching problem in different sciences, outlining the general area of mate selection. Thereby it becomes easier to understand the multiple applications of models we have described in the previous chapters and the various applications of our results. Next, we focus on the area of game theory and briefly present different approaches to the analysis of mating selection and compare them to our models.

We think it is helpful to present the simulation model of Kalick and Hamilton (1999, 2005) which motivated Alpern and Reyniers (1999, 2005) to develop their game theoretical mating models. Examining the background of Alpern and Reyniers' (1999, 2005) models helps to a better understanding of the ideas on which our models were built.

Finally we refer to some game theoretical models in biology, operational research and economics which are relevant to our results, allowing the reader to make comparisons with our own work. However, it is important to note that these models present important differences from our own. Often they are based on very different assumptions and serve different purposes. Therefore, even though they are relevant to our research field, it is not easy to make connections with our own results.

5.2 Importance of Mate Selection

The importance and complexity of mate selection process was initially noted by biologists. Darwin was the first to talk about sexual selection, but even after his revolutionary theories were known, it took biologists several decades before they generally accepted that animals have preferences and actively select their partners. Nowadays the importance of mate selection is widely recognised and in the last fifty years mate selection has been a pivotal research field in evolutionary biology. Experimental research either in the field or in the laboratory has proved that different kinds of animals can indeed be selective when it comes to choosing a mate, and they exhibit their selectiveness in many possible ways

(Kvarnemo and Simmons (1999), Beeching and Hopp (1999), Cunningham an Birkhead (1998), Paul (2002)). In addition to experiments and observation, analytical, computational and simulation models have been used, especially lately, to explain and predict mating behaviour (Parker (1983), Johnstone et al (1996), Johnstone (1997), Kokko and Monaghan (2001), Kokko and Johnstone (2002)). The analytical models developed in biology and evolution have become more and more sophisticated and often they are based on complicated mathematical theories and practices.

Psychology and social sciences accept that many findings concerning animals have application in humans (Doosje et al (1999), Buunk et al (2002), Kenrick et al (1990)). Hence, following the example of biological sciences, in the last two decades social sciences have shown an interest in evolution and human courting habits. Even though there are some mathematical models describing human courtship (Bergstrom and Bagnoli (1993), Sozou and Seymour (2005)), most social scientists use demographic research, experiments and simulations as tools (Kalick and Hamilton (1986), Buunk et at (2002), Kenrick and Keefe (1992)), given that human behaviour is extremely complicated and difficult to model.

In parallel to biological and sociological research, especially after 1980, the importance of real life problems involving the couple formation (such as the matching of companies and job seekers or the matching of colleges and students) has motivated mathematicians and economists to develop game theoretical models of mate choice (Gale and Shapley (1962), Roth and Sotomayor (1990). Several game theoretical approaches exist to mate selection. We will briefly present the game theoretical approaches used in the analysis of mutual choice models.

5.2.1 Game Theoretical Approaches of Mate Selection

The matching procedure is in most cases analysed with the help of the Gale-Shapley (1962) algorithm. Gale and Shapley first formulated the **two-sided matching** problem. In their model, two populations need to be matched (men and women or universities and applicants). Preferences of all players are known and therefore every member of each population can list the members of the opposite population in terms of desirability. Gale and Shapley developed a **deferred acceptance matching procedure**[3] which produces at least one

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[3] During a deferred acceptance procedure, the members of one of the two populations (suppose they are males) make an offer to the members of the other population (let them be females) they best prefer. Females accept these offers or reject them according to their own preferences. Then, the males who are rejected, make a new offer to the females they prefer next and again females decide whether to accept or reject them. Each female has the right to accept temporarily a mate who makes an offer to her that is better than the ones she has encountered so far, and leave him when a better partner makes her a new offer. At the end of the procedure, a stable matching is produced, so there is no player who wants to leave his

stable matching based on the preference lists of the players. Roth and Sotomayor (1990) review and comment on the applications of stable matching in the classical two-sided matching model (as described by Gale and Shapley) and the assignment model[4] (Shapley and Shubik (1972)). Apart from its economic applications, the two-sided matching approach has also been used in the analysis of human and animal courtship and marriage (Knuth (1976); Becker (1981); Bergstrom and Lam (1989); Bergstrom and Bagnoli (1993), Bergstrom and Real (1990)).

The problem of Gale and Shapley and its extensions are not relevant to our research. In our models, the mating is considered as a dynamical process and the players' strategies cannot be described as preference lists. The mating process is modelled as a sequential game. Furthermore, we assume a non atomic distribution of players; there exists a continuum of players and the behaviour and the decisions of each player are made independently of the decisions made by others at the same stage of the game. Finally, in contrast with the Gale and Shapley problem, in our models players are assumed to leave the game when they mate and it is not possible to put "on hold" or defer an offer made in any period. Players in each period have to decide whether they accept or reject an offer, and they have to commit to their decision for the rest of the game.

Another approach that is used in economics and mathematics for the analysis of mating problems, is based on the **secretary** or **marriage problem**. In the standard form of the problem, as described by Freeman (1983), a list of n objects (secretaries) is presented in a random order to an observer. The quality of an object is not revealed to the latter until he sees it; hence at any time the observer can rank only the objects he has seen up to that time in terms of desirability. Until all n objects have been presented to the observer, he does not know which one is the best for him. Each object can be presented to him once only. Each time he is presented with an object, he can either accept it or reject it. In the first case, the game finishes. In the second case, a new object is offered to him and he faces once again the same option. When the last object is presented to the observer, he is obliged to accept it. The observer wants to maximize the probability that the object he accepts is the most desirable out of the n objects available.

The secretary model can easily be adapted and used to analyse a variety of mate choice problems where players do not know the types of their potential partners from the beginning and consequently do know their preferences from the beginning. In our models, we assume that players are aware of the types

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partner for someone who prefers him more than her current partner.

[4] The assignment problem allows both matching and wage determination to be considered together since the players' objective is to maximise the profit they receive from a matching.

of their potential partners and therefore know their preferences from the initial stage of the game. Furthermore, as Ferguson (1989) notes, in secretary problems "the payoff depends on the observations only through their relative ranks and not otherwise on their actual values". In our analysis the maximisation of the rank of a partner is no longer the objective; players seek to maximise their expected utility that is a function of the actual values of their partners.

Our models belong in the general class of dynamic game theoretical models.

5.3 Kalick and Hamilton's (1986) Simulation Model

In biology and sociology it is commonly observed that individuals find partners who are similar to themselves. This phenomenon of assortative mating has been by the focus of multiple biological and sociological studies.

Kalick and Hamilton (1986) tried to find which type of preferences leads to a high correlation of physical attractiveness in humans. They examined two types of preferences: homotypic preferences, assuming that humans prefer partners similar to themselves, and common preferences, assuming that humans prefer partners of a high attractiveness. Their work was the inspiration for research in operational research and evolutionary theory and has been extended by game theorists and biologists.

They built a computer simulation model in which two very large equal populations of males and females of various levels of attractiveness were matched randomly over sequential periods. They ran two separate simulations. In the first simulation they tested the effect of homotypic preferences by programming the individuals to accept partners of similar levels of attractiveness to themselves with a high probability. In the second simulation they focused on the impact of common preferences by programming the individuals to accept individuals of a high attractiveness with a high probability. In both cases they assumed that players become less choosy over time and therefore they programmed the probabilities of accepting a potential partner to increase over time.

They assumed that a couple is created only when there is mutual consent from both paired partners, and they did not allow remarrying. Hence, in every period in both simulations, when two individuals were paired and they both accepted each other, they formed a couple and they no longer participated in the rest of the matchings taking place in the next periods.

After all individuals were mated in both simulations, Kalick and Hamilton examined the correlation of the levels of attractiveness in the couples created. The results were rather unexpected, since both simulations resulted in high correlations of attractiveness in the couples.

Alpern and Reyniers (1999, 2005), motivated by these results, built two game theoretical models in order to test and explain Kalick and Hamilton's results and to examine how preferences affect strategies and equilibrium behaviour. Kalick and Hamilton did not offer any information on the relation between preferences and strategies and how the first form the latter, since they did not distinguish between preference, choice and motivation (Kalick an Hamilton (1986), Alpern and Reyniers (1999)). Hence, they did not offer any insight on how an equilibrium can be reached when players are free to decide on the strategies they should use according to their preferences. Alpern and Reyniers (1999, 2005) tried to fill this gap.

5.4 Further Research in Game Theory

A variety of game theoretical models, trying to present the mating selection as a dynamical process has been developed, especially in recent years. In general, the models focusing in labour economics and the marriage problem have great differences from our models and are based on very different assumptions. As a result, in many cases it is difficult to make comparisons between them and our work. We would categorise our models as being focused mostly on applications in operational research and possibly in biology rather than in economics. This will become clearer if we briefly present some models that are related to our work and comment on their connection with the models we developed.

5.4.1 Related Models Focusing on Biological Applications

A game theoretical model that presents many similarities with Alpern and Reyniers (2005) was developed earlier by **Johnstone (1997)**. This presents computational results on a discrete type model where the initial distribution has the form of a truncated normal. Johnstone (1997) assumes a $1:1$ sex ratio (as Alpern and Reyniers(1999,2005)) and he focuses on the influence of time constraints, choice costs and competition for mates. He concludes that high quality players become less choosy over time, while the low quality players may be the most selective in the middle periods of the mating season when choice is costly. He notes that a model where the population of males and females are unequal should be developed.

Before publishing this work, **Johnstone et al (1996)** developed a game theoretical model examining the effect of costs of choice on the strategies of males and females. They commented on the role of parental care in the choosiness of animals and noted that when parental care offered by both sexes is similar, the differences in quality have a strong impact on mating strategies.

Johnstone's note on the possible importance of the sex ratio and the earlier observation of Johnstone et al on the role of quality variation, in combination with further comments in the biology literature, motivated us to adapt our non symmetrical model with common preferences in order to incorporate these two factors. Examining the influence of quality variation and a biased sex ratio in the development of mating behaviour, we found that when variation is small and the population is male biased, multiple equilibria exist. On the contrary, when the variation is large, a unique equilibrium exists in the two period model.

5.4.2 Related Models, Focusing on Operational Research Applications

The models of **Eriksson et al (2007, 2008a)** are related to our work, even though they base their research on the two-sided choice secretary problem. Eriksson et al (2007) examine a game where preferences are independent (non uniform) and each player can rank his potential partners according to his preferences . As in Alpern and Reyniers (2005), cohorts of males and females are matched over n periods. However, in contrast to the Alpern and Reyniers model, the objective of players is to maximise the rank of their potential partner. Eriksson et al (2008a) also focus on a game where preferences are independent, building a steady state model where populations of males and females are equal. They show that asymmetric equilibria may exist when the game is perfectly symmetric and investigate the effect of the option to stay single. They also study how the equilibria change if both men and women are given the option of staying single, but value this option in a different way. They show that if the utility of staying single is higher for males than for females, males are better off at equilibrium, while when staying single is not very bad for any sex, there is a unique equilibrium.

Ramsey (2006) focuses on a non-atomic game where males and females have only two types. At random times, players of the opposite sex are paired and form couples when they both accept each other. Players have maximising preferences (preferring high type partners) and there are costs of searching, expressed by discounting of the expected utilities over time. Ramsey derives the equilibrium of this game and shows that the choosiness of the players does not decrease over time.

5.4.3 Related Models, Focusing on Economic Applications

In parallel with the game theoretical models focusing on biology and operational research, some models applied in labour markets are related to ours. The game theoretical model developed by **McNamara and Collins (1990)** has similarities with the research produced by Alpern and Reyniers (2005) and Johnstone (1997) and with our model focusing on common preferences. McNamara and Collins examine the two-sided choice job search problem assuming discounting of utilities over time. However, they examine a steady state model and therefore their results have great differences from ours. They show there is a unique equilibrium in which couples are created between employers and candidates who belong to the same level, where levels in this case are defined as intervals.

Steady state models related to our common preferences work were also developed by **Burdett an Coles (1997, 1999)**. The latter focus on the marriage problem and (as McNamara and Collins) they prove that men and women form couples only if they are in the same "class". Furthermore, they find multiple equilibria under some conditions, but their assumptions and analysis are very different from ours.

5.5 Final Remarks

The goal of this chapter was to give an overview of the area of mutual choice games. We have briefly described approaches and models that have been developed to analyse mate selection. The reader, being familiar now with our models, should be able to understand the relationship of our models to other work in the area. In that way, he will understand better what our models offer to the analysis of the dynamic mating games, and what are their possible applications.

6 Conclusions - Further Research

6.1 Introduction

The area of mate choice is very wide and is a subject of interest to different scientific fields, such as biology, social sciences and economics. A variety of methods have been used for its analysis. We use a game theoretical approach to examine equilibrium mating behaviour and its implications. As shown in the previous chapter, various other game theoretical mating models have been developed. Our models are different from secretary and stable matching problems. We have analysed the mate selection process as a sequential game, where males and females randomly meet over successive periods and form couples only when there is mutual consent.

Our analysis extended and generalised Alpern and Reyniers' (1999, 2005) and Alpern' (2008) models. However, our work differs from the models on which it is based in several ways. Two most important differences between our models and the relevant literature are as follows. Firstly, in Alpern and Reyniers analysis, and in previous work done on dynamic mating games, a sex ratio $r = 1$ is assumed. We in contrast examined the effect of a biased sex ratio on the equilibrium behaviour (chapters 3 and 4). A sex ratio deviating from equality is more likely to be realistic (Johnstone (1997)). Secondly, we found multiple equilibria in different games under different preferences (chapters 2 and 3). In most of the relevant literature, a unique equilibrium is found or is assumed. The existence of multiple equilibria is not often observed. A natural question is which equilibrium would appear in reality. Possible answers to this question can be based on the results of a comparison of the different equilibria in terms of stability and of the benefits they offer to different players.

Even when there are not multiple equilibria, the outcome of the analysis of the equilibrium stability and the welfare of players at equilibrium is useful. The analysis of the basins of attraction of a stable equilibrium, show us how we expect the players to react against different strategies and how we expect the strategies to change till we reach an equilibrium. This can be useful in a real life application, where predictions on the population behaviour need to be made. On the other hand, welfare can be thought as an indicator of the satisfaction of players. It is possible that the change of some parameters can lead to another equilibrium where welfare is improved. This can help to explain changes appearing in reality. Therefore, we find it necessary to focus especially on the analysis of the stability and of the welfare of the equilibria found in all cases.

We studied the equilibrium strategies under homotypic, mixed (combination of homotypic and common), common and age dependent preferences. The goal of our research was to analyse how the equilibrium behaviour is affected by different preferences and to examine the influence of other factors such as the sex ratio, the variation of types and their number (in case they are discrete) on the equilibrium strategies. It would be interesting if our results were compared to observed behaviours and real life examples. The aim of this chapter is to provide a summary of our main findings, comment on them and examine any possible extensions of our work.

6.2 Homotypic Preferences

The basis of our analysis is the model of Alpern and Reyniers (1999). We provided some limited results on the continuous type game, extending the equilibrium analysis of the uniform distribution two period model of Alpern and Reyniers (1999). It would be interesting to further analyse the n period continuous type game and compare the mutual choice and the one sided choice models. However, it can be difficult to find explicit results, due to the complexity of the analysis.

In chapter 2, we mostly focused on the discrete type game. We developed a method for reducing the number of potential equilibrium strategies needed to be examined in the two period discrete type game. Using this method, we found and studied the equilibria in several discrete type games with different numbers of types. We observed that choosiness increases as the number of types increases and we showed that it is possible for multiple equilibria to appear. We compared the equilibria we found, analysing the intra-couple correlation, the marital stability and the expected payoff that each equilibrium offers to different types of players. At all equilibria, the type correlation in the couples formed is low. However, in a game played for a larger number of periods ($n > 2$), the intra-couple correlation is expected to be higher. The marital stability of equilibria in all models examined is much higher than the stability in the corresponding one sided choice models (marital stability of 0.5), a fact noted in the common preferences model as well. In the cases of multiple equilibria, we found that different types are better off in different equilibria. It would be interesting to observe under which conditions each equilibrium appears in reality and whether types are able to recognise which equilibrium is more beneficial for them.

In the discrete type games we studied, two types of symmetry were observed at equilibrium: symmetry between males and females and between positive and negative types, even when it was not assumed that at equilibrium strategies are symmetric. Further research could focus on proving the existence or non existence of asymmetric equilibria in the continuous and discrete type games.

During our analysis of both the continuous and the discrete type games sex ratio is $r = 1$ and players' types are uniformly distributed in the first period.

Relaxing these two assumptions might lead to interesting results.

6.3 Mixed Preferences (Combination of Homotypic and Maximising Preferences)

One of the basic factors affecting the selection procedure is the preferences of players. Usually, players are assumed to seek partners similar to themselves (homotypic preferences) or partners who possess a characteristic in a high degree (common preferences). Animals and humans exhibit both kinds of preferences or a mixture of them, depending on the context. However, it is not always possible to find a specific pattern of mating preferences in all species (Bakker et al (1999), Widemo and Sæther (1999)).

We examined the equilibrium in a model with mixed preferences. We analysed the two and three period discrete type model under a combination of homotypic and common preferences. We studied the possible changes in the equilibrium behaviour and in the mating pattern as homotypic preferences become less important than common preferences in the choice of mate. In both the two and three period models more couples are expected to be formed before the last period when homotypic preferences are stronger. This is because all types have the opportunity to find a partner before the last period, whereas under common preferences only high types are mated in the first periods of the game. In the three period game we compared the mating patterns in the first and second period. Under all combinations of preferences, the range of mutual acceptance is wider in the second period, with the exception of purely homotypic preferences. Under homotypic preferences, mating takes place between the same pairs of types in both periods.

It would be interesting to check how equilibrium behaviour would be affected if players were characterised by more than one index (multidimensional types) and had different types of preferences for each index. Furthermore, a biased sex ratio could have important impact on the equilibrium behaviour.

6.4 Common Preferences

We extended the common preferences model of Alpern and Reyniers (2005) by assuming a biased sex ratio r and a cost c for remaining unmated after the end of the game. It is easy to show that our model is equivalent to a model where there is a biased sex ratio and quality variation. We proved the existence of multiple equilibria under certain conditions and we managed to give one possible answer to the question of "when do we have mutual and when do we have one sided choice". Our results are not based only on simulation and numerical analysis, in fact we have analytical results for the two and partially for the three period model, a fact that differentiates our work greatly from other research in the field.

The role of sex ratio as the principal factor determining differences in choosiness has been criticised by researchers in biology (Kokko and Monaghan (2001)). Nevertheless, examining the simultaneous effect of a biased sex ratio and a non mating penalty offers a new insight into the role of sex ratio in the formation of the equilibrium strategies and complements the research done so far (Alpern and Reyniers (2005), Johnstone (1997)).

Of course, our model is based on simplicity in order to keep it general, so some assumptions may not be realistic. Criticisms of the model may be based on the assumptions of simultaneous arrival of all players in the beginning of each period and of the absence of learning. The last assumption though, could be realistic in a short game of 2 to 4 periods.

We assumed that types are uniformly distributed in the first period. Further research could focus on some other type distributions. We have some limited results, using a truncated normal distribution, in the discrete type three period model; we found multiple equilibria, where both mutual and female choice appear. Our analysis is presented in Appendix $B.6$.

Our main results are is accordance with general theory. For a very large sex ratio r, males are not willing to be selective, unless the non mating cost c is extremely low. The results for a very high non mating cost cost c are analogous; unless the sex ratio is close to 1, males are better off by not being choosy. Our analytical results in the two period model and our numerical results in the three and four period models showed that multiple equilibria are possible. This result may be of significant importance and it deserves further examination.

In the n period problem, where $n > 2$, we noted that both female and male tend to become less choosy over time, a fact that accords with results of previous one sided choice models (Real 1990)). Johnstone (1997) in his mutual choice model notes that high type individuals become less choosy as time passes by but low type individuals tend to become choosier in intermediate periods. A result like this could not possibly be observed in our model. In our analysis only the strategies of the highest types matter. The strategy of the lowest types is not important since these types would never be accepted before the last period. Hence, even if the lowest types of a sex choose to be more selective than high types in an intermediate period, they will be rejected and thus not manage to increase their expected utility. Consequently, Johnstone's results do not contradict our own.

In the case where multiple equilibria exist in the two period game, we examined the dynamical and marital stability of each equilibrium and the expected payoff each equilibrium offers to players. It is obvious that marital stability decreases as players become less choosy at equilibrium. The choosiness of the players is associated to the extent that the mating pattern is assortative. When a female choice equilibrium appears, males have no say in the mating, so a male type has equal chances to be mated with a high or a low female type. On the other hand, when both sexes are selective, high males tend to mate with high females with a higher probability.

Females welfare is related to the degree that a mating is assortative. When males are universal acceptors, low type and high type females have equal chances to find a high type partner; consequently the average male type in the couples created is higher than at mutual choice equilibria and low type females are happier. However, high type females have individual preferences for mutual choice equilibria. Mutual choice favours assortative mating. As a result it allows high type females to have access to higher type males, without having to compete with low type females, at least before the last period. At female choice equilibrium, a conflict of social and individual interest of high type females appears. This can be beneficial for males, since the majority of male types (both high and very low types) prefer mutual choice equilibria to female choice.

We are particularly interested on the biological applications of our model. In section 3.4 we briefly presented a modified version of our model, showing the effect of sex ratio and female type variation on equilibrium behaviour. Work in progress shows that further analysis of this modified model should offer results that can have significant implications for biology.

6.5 Age Dependent Preferences

Based on an idea of Alpern and Reyniers (1999) and Alpern (2008) we analysed the equilibrium strategies in a steady state model where individuals have age dependent preferences and they seek partners to maximise the number of periods that both they and their partners can produce offspring. We examined the equilibrium strategies as the sex ratio of the incoming population changes. Furthermore, we studied the changes in the total population and in the average number of years a couple is mated when the maximum number of periods that males and females are able to reproduce varies.

The availability of potential partners for each player depends on the player's type (age) and the sex ratio. Choosiness of players decreases as their type (age) increases. This is intuitive, since as players become older, a wider range of types offers them the same utility and the risk of remaining unmated increases. Examples of the tendency of individuals to become less choosy as they get older has been observed in several animals (Uetz and Norton (2007), Gray (1999), Kodric-Brown and Nicoletto (2001)). The effect of sex ratio on the choosiness of the two sexes is different. It was shown that the limiting sex, let it be females, becomes choosier as r increases, while the opposite sex (males in this case) becomes less choosy. This observation contrasts with our conclusions in the previous models, where the choosiness of both sexes moved to the same direction as r varied. However, it needs to be noted that in the previous models all females were mated after the end of the game, whereas in the current model it is possible for members of both sexes to leave the game without finding a partner.

We used the population productivity as a measure of the population welfare. In the models examined, the social welfare is maximised when the number of males and females is the same ($r = 1$). It is interesting to note that Fisher (1930) in a more general situation with a non monogamous setting stated that Natural Selection leads to equality of the number of males and females at maturity. Fisher argued that "the sex ratio will so adjust itself, under of influence of Natural Selection, that the total parental expenditure incurred in respect of children of each sex, shall be equal". It would be of particular interest to check whether social welfare is maximised when $r = 1$ when both males and female are fertile for longer than 3 years.

We assumed that another measure of the "satisfaction" of players at equilibrium is the average age of players at marriage. We found that the average age at marriage is minimum for both males and females when $r = 1$, since in that case all players get mated in the first period that they enter the game. However, as the population becomes male or female biased, the average age at marriage increases for both males and females.

In all the games analysed an equilibrium always exists if we assume that the total population is constant. When males are fertile for $\lambda = 4$ periods and females are fertile for $\tau = 3$ periods, multiple equilibria exist for some sex ratio r. Thus when multiple equilibria exist for a sex ratio r, one can achieve this r by two different incoming population sex ratios R. However, in the $\lambda = 4$, $\tau = 3$ model, an equilibrium does not always exist for all incoming population sex ratios R. Further research could focus in examining whether an equilibrium exist for all r for larger λ and τ and whether multiple equilibria may appear in other cases.

A possible extension of our model could incorporate remarrying and divorcing. This could make it more realistic. Remarrying is frequent in nature. Albatross whose mates have died usually mate with other widowed birds (Jouventin et al (1999)). Furthermore in many monogamous birds examples of divorcing appear (Green et al (2004)). Examples of "divorcing" and "remarrying" can also appear in sociology and economics between humans or organisations.

Further research could also be devoted into incorporating search costs in our model. In economics, there are examples of frictionless markets (Eeckhout

(1999)); nevertheless, in most cases search, lost opportunity or signalling costs are taken into account (Burdett and Coles (1999)). In biology often costs are not taken into account during the analysis of mating behaviour (Alpern and Reyniers (2005), since it is very difficult to measure them empirically (Kokko et al (2003)) and draw conclusions on their exact effect.

Finally our model could be modified to assume a different type of preferences. As was argued in chapter 4, in many cases animals prefer older partners. Hence a new model could try to analyse the equilibrium behaviour given a preference for older types.

6.6 Final Remarks

We have analysed the effect of different preferences on equilibrium strategies. We tried to examine the effects of different factors on equilibrium behaviour. In all our models, we had to make simplifications. The addition of new factors possibly important for couple formation, can lead to an increase in complexity that can make it impossible to reach general results. Even in our analysis, many results are computational since it is difficult to find analytical results.

Our work has applications in different fields and we are particularly interested to check whether examples verifying our results appear in real life. However, we take into account that models are always approximations of reality, and their results simply provide an intuition into what might happen in real circumstances.

Further research can extend and modify our analysis. We are already planning to modify the common preferences model. We hope our work can provide the motivation for new research that will reveal new interesting results. Furthermore, our results should be combined with observations and experimental results to help with the analysis of real life problems in different fields.

Appendices

Appendix A. Homotypic Preferences

Appendix A provides material that can facilitate the understanding of chapter 2.

A.1 Continuous Type Model

In the continuous type 2 period model in sections 2.2 and 2.3, types are initially distributed in $[-1,1]$. According to (1) $s(x)$ denotes the strategy of a type x.

Let $x_L = \max\{x : -x + s(x) = x\}$.

Lemma A.1 *If* $0 < x \leq x_L$ *then* $x - s(x)$ *always accepts* x.

Proof. $x - s(x)$ is an increasing function in $[0,1]$, since its derivative is always nonnegative. Therefore since

$$\begin{gathered} x \leq x_L \Longrightarrow x - s(x) \leq x_L - s(x_b) \\ \text{and } x_L - s(x_L) = x_L \\ \text{Then } x - s(x) \leq -x_L. \end{gathered}$$

We know that s in decreasing in the interval $[-1,0]$, thus

$$\begin{aligned} s(x - s(x)) &\geq s(-x_L) \\ s(x - s(x)) &\geq 2x_L. \end{aligned}$$

In order to prove that $x - s(x)$ accepts x, we just have to prove that

$$(x - s(x)) + s(x - s(x)) \geq x$$

Hence it is sufficient to prove that

$$(x - s(x)) + 2x_L \geq x$$

Hence

$$\begin{aligned} (x - s(x)) + 2x_L \geq x & \\ &\Longleftrightarrow 2x_L \geq s(x) \\ &\Longleftrightarrow s(x_L) \geq s(x), \end{aligned}$$

which is true, since $x_L \geq s(x)$. ■

A.2 Discrete Type 2 Period Games - More Significant Figures of the Analysis of the Mating Probability and of the Expected Cost at Equilibrium

We analysed the discrete type two period games where non negative types m range from 1 to 9. In section 2.6, we presented in Figure 2.7 the probability that a type is mated in period 1 as a percentage and we illustrated in Figure 2.8 the normalised cost and a type expects to receive when he enters the game. Figures $A.1$ and $A.2$ present more significant figures of the probability that a type is mated in the first period and of the cost that a type expects to receive when he enters the game respectively.

		type0	type1	type2	type3	type4	type5	type6	type7	type8	type9
m=1		1	0.667								
m=2		0.6	0.6	0.4							
m=3	s1	0.429	0.429	0.429	0.286						
	s2	0.429	0.714	0.429	0.429						
	s3	0.714	0.714	0.571	0.429						
m=4		0.556	0.556	0.556	0.444	0.333					
m=5	s1	0.455	0.455	0.545	0.455	0.364	0.364				
	s2	0.455	0.636	0.636	0.455	0.455	0.364				
	s3	0.636	0.636	0.636	0.545	0.455	0.364				
m=6		0.538	0.538	0.538	0.538	0.462	0.385	0.308			
m=7		0.6	0.6	0.6	0.6	0.533	0.467	0.4	0.467		
m=8		0.529	0.529	0.529	0.588	0.529	0.471	0.412	0.353	0.353	
m=9		0.579	0.579	0.579	0.579	0.579	0.526	0.474	0.421	0.368	0.316

Figure $A.1$: Probability of getting mated in the first period at equilibrium.

			type0	type1	type2	type3	type4	type5	type6	type7	type8	type9	average
		1rst period	0.667	0.333									0.444
m=1		2nd period	0	0.333									0.222
		total	0.667	0.667									0.667
		1rst period	0.4	0.4	0.2								0.32
m=2		2nd period	0.533	0.6	1.2								0.827
		total	0.933	1	1.4								1.147
		1rst period	0.286	0.286	0.286	0.143							0.245
	s1	2nd period	1.029	1.105	1.333	2.143							1.456
		total	1.314	1.39	1.619	2.286							1.701
		1rst period	0.286	0.857	0.286	0.429							0.49
m=3	s2	2nd period	1.048	0.571	1.333	1.714							1.184
		total	1.333	1.429	1.619	2.143							1.673
		1rst period	0.857	0.857	0.571	0.429							0.653
	s3	2nd period	0.571	0.6	1.029	1.714							1.037
		total	1.429	1.457	1.6	2.143							1.69
		1rst period	0.667	0.667	0.667	0.444	0.333	0.545					0.543
m=4		2nd period	1.079	1.122	1.249	1.825	2.667	3.182					1.645
		total	1.746	1.788	1.915	2.27	3	3.727					2.188
		1rst period	0.545	0.545	0.818	0.545	0.364	0.545					0.562
	s1	2nd period	1.556	1.604	1.457	1.973	2.676	3.182					2.122
		total	2.102	2.15	2.275	2.519	3.04	3.727					2.684
		1rst period	0.545	1.09	1.09	0.545	0.636	0.545					0.76
m=5	s2	2nd period	1.618	1.115	1.2	2	2.309	3.182					1.93
		total	2.164	2.206	2.291	2.545	2.945	3.727					2.69
		1rst period	1.09	1.09	1.09	0.818	0.636	0.545					0.86
	s3	2nd period	1.117	1.143	1.221	1.688	2.318	3.182					1.838
		total	2.208	2.234	2.312	2.506	2.955	3.727					2.698
		1rst period	0.923	0.923	0.923	0.923	0.692	0.538	0.462				0.757
m=6		2nd period	1.62	1.651	1.744	1.897	2.465	3.2	4.154				2.449
		total	2.544	2.574	2.667	2.82	3.157	3.738	4.615				3.207
		1rst period	1.333	1.333	1.333	1.333	1.067	0.867	0.733	0.667			1.067
m=7		2nd period	1.658	1.68	1.745	1.855	2.342	2.948	3.709	4.667			2.637
		total	2.992	3.013	3.079	3.188	3.408	3.815	4.442	5.333			3.703
		1rst period	1.176	1.176	1.176	1.47	1.176	0.941	0.765	0.647	0.882		1.038
m=8		2nd period	2.149	2.173	2.248	2.075	2.539	3.1	3.785	4.623	5.176		3.152
		total	3.325	3.35	3.424	3.546	3.715	4.041	4.55	5.27	6.059		4.19
		1rst period	1.579	1.579	1.579	1.579	1.579	1.316	1.105	0.947	0.842	0.789	1.274
m=9		2nd period	2.198	2.216	2.272	2.364	2.494	2.993	3.586	4.294	5.143	6.158	3.433
		total	3.777	3.795	3.851	3.943	4.073	4.309	4.691	5.242	5.985	6.947	4.708

Figure A.2: Expected cost at equilibrium.

A.3 Mixed Preferences- Plots of the equilibrium strategies

We analysed the discrete 10-type two and three period games with mixed preferences, where the weight ω of homotypic preferences takes values between 0 and 1. In section 2.8, we presented in Figures 2.11 and 2.14 the strategies at equilibrium in the two and three period models respectively. In the following figures ($A.3$ and $A.4$) we plot the equilibrium strategies in order to help the reader to understand better the behaviour of different types in different games.

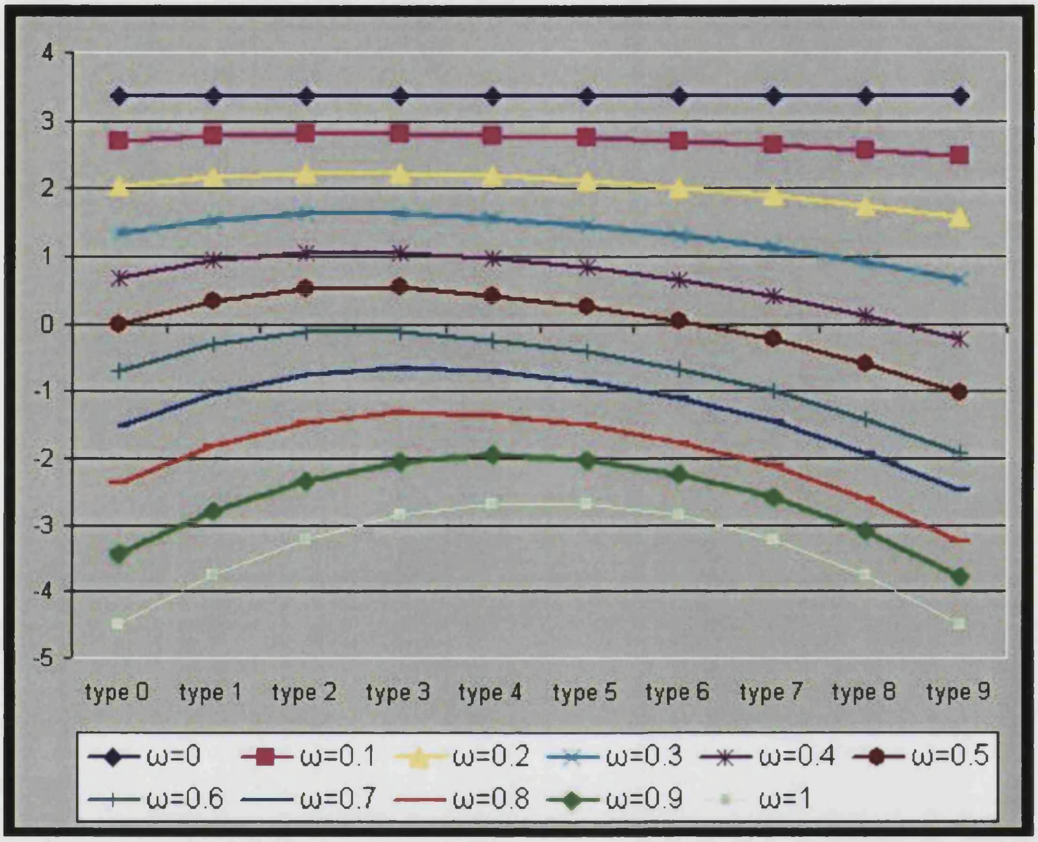


Figure $A.3$: Plots of equilibrium strategies in the 2 period game for $m = 9$.

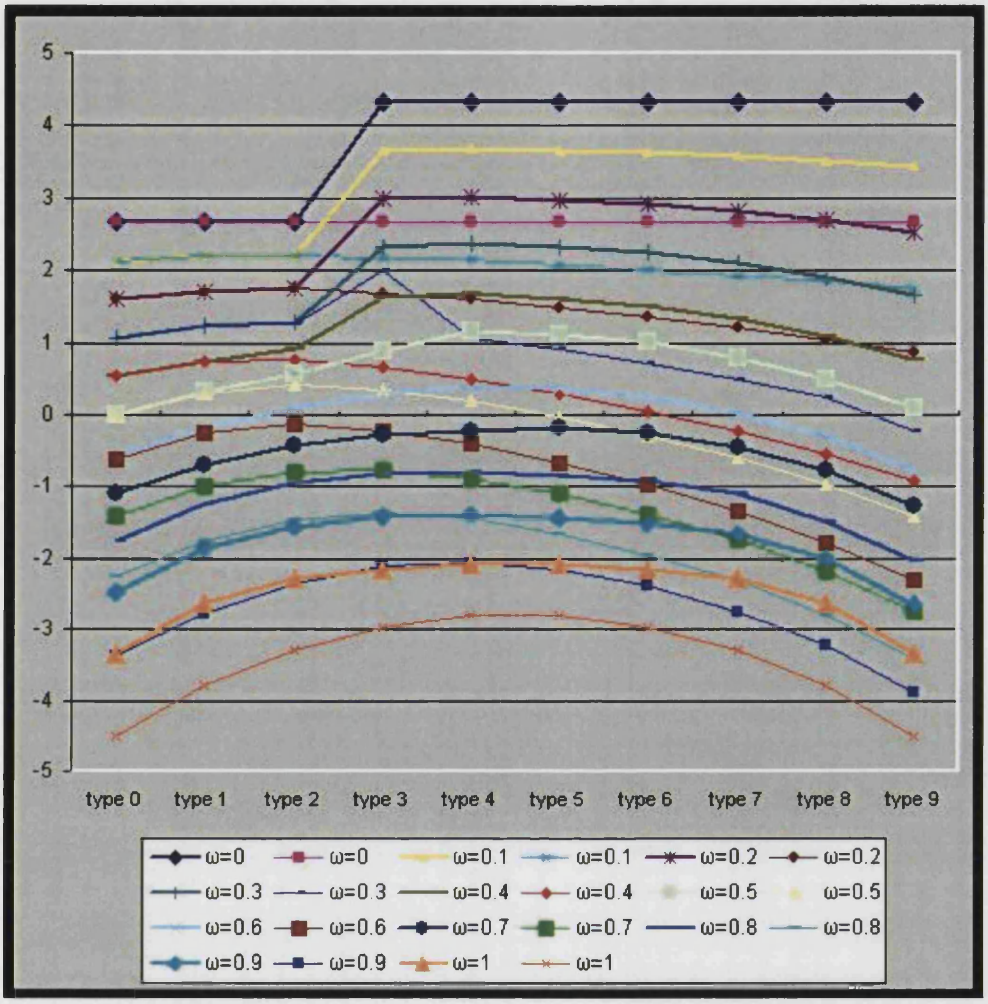


Figure $A.4$: Plots of equilibrium strategies in the 3 period game for $m = 9$.

Appendix B. Common Preferences

This Appendix provides additional material that can help the better understanding of the results presented in chapter 3.

B.1 Equilibria on the F, M Lines and on Their Point of Intersection ((9/8),1)

The following figures (*B*.1 and *B*.2) illustrate some special cases of equilibria appearing on the *F* and *M* lines or on their intersection. We analyse the equilibria

- at the point $(\frac{9}{8}, 1)$ in the (r, c) space; at this point F and M lines coincide; hence this point is the intersection of regions I, II and III
- on the part of the F line where $c < 1$, being the intersection of regions I and II
- on the part of the F line where $c > 1$, being the intersection of regions II and III, and
- on the M line where $c > 1$, which is the intersection of regions II and I.

In figure *B*.1, the equilibria are given as the intersection of the female equilibrium condition (29) drawn in red (thin) and the male equilibrium condition (23) drawn in green (thick). In figure *B*.2 though, the equilibria are described as intersections of L_1 and L_2 and the polynomial $q(u)$ (42) as described in the Proof of Theorem 3.1.

$$\begin{aligned} L_1 &= \{(0, y), -\infty < y \leq 0 \\ L_2 &= \{(0, u), 0 < u < 1 \end{aligned}$$

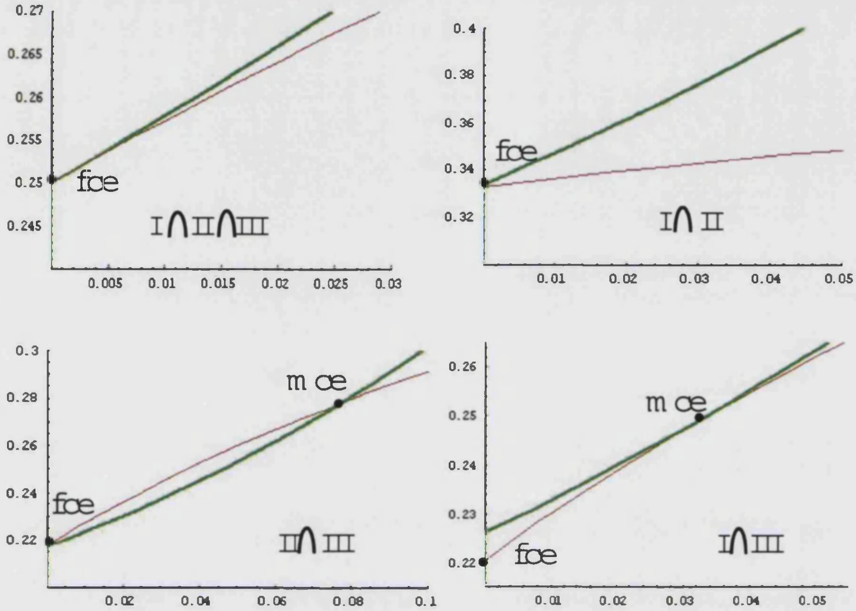


Figure B.1: Equilibria as intersection points of the female equilibrium condition (23) drawn in red (thin) and the male equilibrium condition (29) drawn in green (thick)

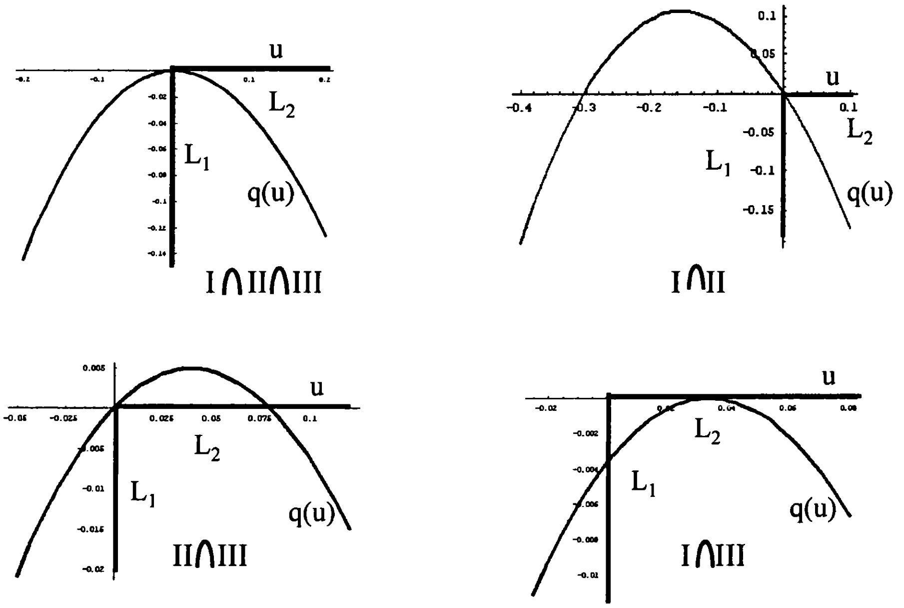


Figure $B.2$: Equilibria as intersections of L_1 and L_2 and the polynomial $q(u)$(42)

At $(\frac{9}{8}, 1) = I{\cap}II{\cap}III$ the easy, the choosy and the female equilibria coincide; hence there exist a unique female choice equilibrium of the form $(0, 0.25)$.

For $c < 1$, on the F line, $(I \cap II \cap \overline{III} = \{(r_F, c) : c < 1\})$, the choosy equilibrium coincides with the female choice equilibrium; hence there exist a unique female choice equilibrium.

For $c > 1$, on the F line, $(II \cap III \cap \overline{II} = \{(r_F, c) : c > 1\})$, the easy equilibrium coincides with the female choice equilibrium; as a result there exist two equilibria: one female choice equilibrium and one mutual choice equilibrium (the choosy equilibrium).

On the M line, $(I \cap III \cap \overline{II} = \{(r_M, c) : c > 1\})$, the easy and the choosy equilibrium coincide; as a result there exist two equilibria: one female choice equilibrium and one mutual choice equilibrium.

B.2 Numerical Results on the Dynamical Stability of the Equilibria

In chapter 3, equilibria are defined as fixed points of the mapping T (52). A fixed point is described as dynamically stable if $N^j \leq 1$ (53).

If an equilibrium is dynamically stable, iterations of T applied to nearby points, converge back to it in j steps where

$$j = min\ \{i : N^j(u,v) \leq 1\}$$

We consider a point dynamically stable if and only if $N^j \leq 1$ for $j \leq 2$.

As it is already noted in chapter 3, only the female and the choosy equilibrium seem to be dynamically stable while the easy equilibrium shows unstable behaviour. For the female and the choosy equilibrium, we found that $N^2(u,v) \leq 1$ for all values of (r,c) in $(1,0) \times (2.5,2.5)$, fact that is not true for $N^1(u,v)$. By noting (r,c) such that $N^1(e^1(r,c)) > 1$ and (r,c) such that $N^1(e^1(r,c)) \leq 1$ we were able to approximate the areas of III and I and II where female choice and choosy equilibria are attracting in one step or in two steps.

B.2.1 Female Choice Equilibrium

We analysed the behaviour of the female equilibrium in I and III. We divided the area $I \cup III$ of $(1,0) \times (2.5,2.5)$ in (r,c) space into 40,000 points and we calculated $N^1(e^1(r,c))$. We found that in the bigger part of III and around the M line in I, we have $N^1(e^1(r,c)) > 1$. Our results are illustrated in figure $B.3$, where female equilibria having $N^1(e^1(r,c)) > 1$ are represented as pink points. In order to check our results, we focused further on the area where $N^1(e^1(r,c)) > 1$, calculating the $N^1(e^i(r,c))$ for 10,000 points where $1 \leq r \leq r_M + 0.2$ and $0 \leq c \leq 2.5$; our further examination confirmed our initial results. Our additional analysis is shown in figure $B.4$, where all points where $N^1(e^1(r,c)) > 1$ are illustrated in pink as well. It is worth mentioning that it appears that in I there exist an almost parallel line (where $r \simeq 1.092$) below which we have $N^1(e^1(r,c)) > 1$.

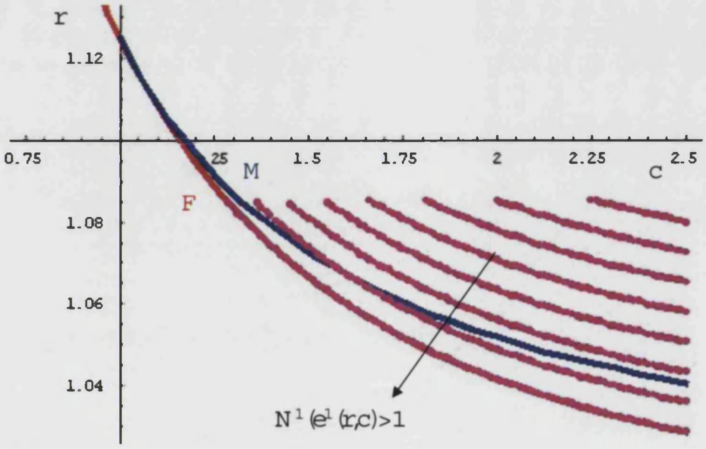


Figure $B.3$: Points where $N^1(e^1(r,c)) > 1$ for
$1 \leq r \leq 2.5$ and $0 \leq c \leq 2.5$ (illustrated in pink)

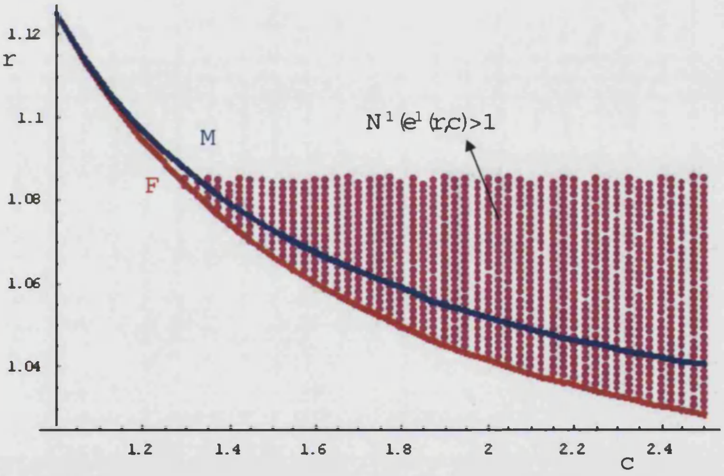


Figure $B.4$: Points where $N^1(e^1(r,c)) > 1$ for $1 \leq r \leq r_M$
and $0 \leq c \leq 2.5$ (illustrated in pink)

Having pointed out the areas where $N^1(e^1(r,c)) < 1$, it is interesting to examine how the value of the norm $N^1(e^1(r,c))$ changes as r and c vary. Figures $B.5$, $B.6$ and $B.7$ show the values of $N^1(e^1(r,c))$ in different areas where female equilibrium appears. For $c \leq 1$, $maxN^1(e^1(r,c)) < 0.4$ as shown in figure $B.4$. However, this is not always true for $c > 1$; in I for $c > 1$ we have $N^1(e^1(r,c) > 1.7$ in some points and in III we observe that $N^1(e^1(r,c))$ can even be higher, reaching the value of 2 in some cases. For illustration purposes, we normalised the value of r in all figures so that it ranges between 0 and 1. In figure $B.5$, $r\prime = 0$ corresponds to $r = r_F$ and $r' = 1$ corresponds to $r = 2.5$. In figure $B.6$, the normalised r takes its minimum value for $r = r_M$ and its maximum for $r = r_M + 0.2$. In figure $B.7$, $r\prime = 0$ corresponds to $r = r_F$ and $r' = 1$ corresponds to $r = r_M$.

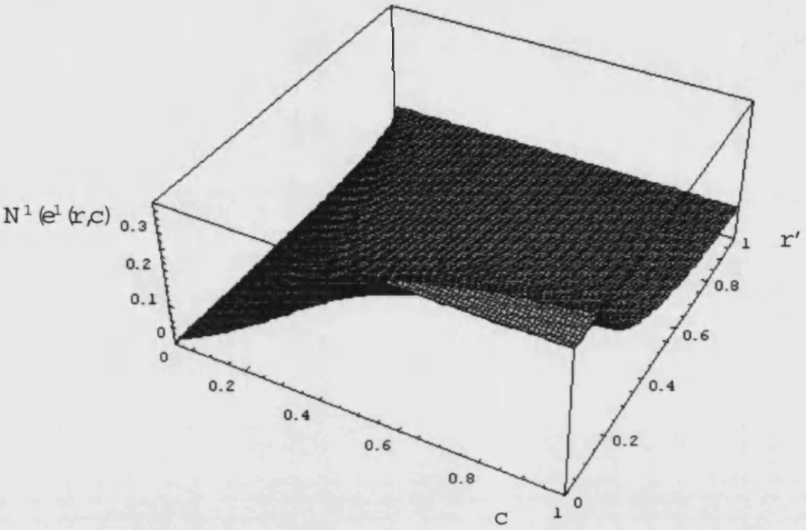


Figure $B.5$: $N^1(e^1(r, c))$ for $r_F \leq r \leq 2.5$ and $0 \leq c \leq 1$

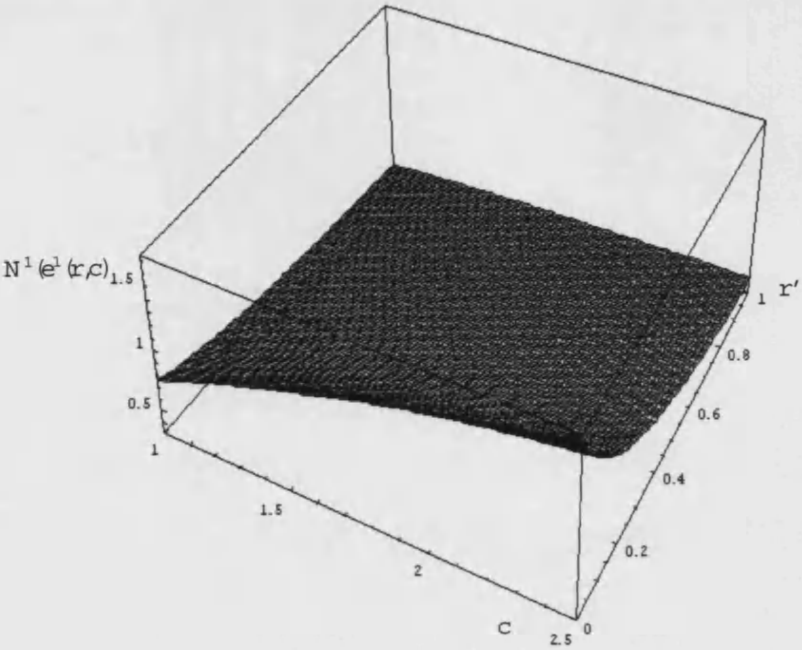


Figure $B.6$: $N^1(e^1(r,c))$ for $r_M \leq r \leq r_M + 0.2$ and $1 \leq c \leq 2.5$

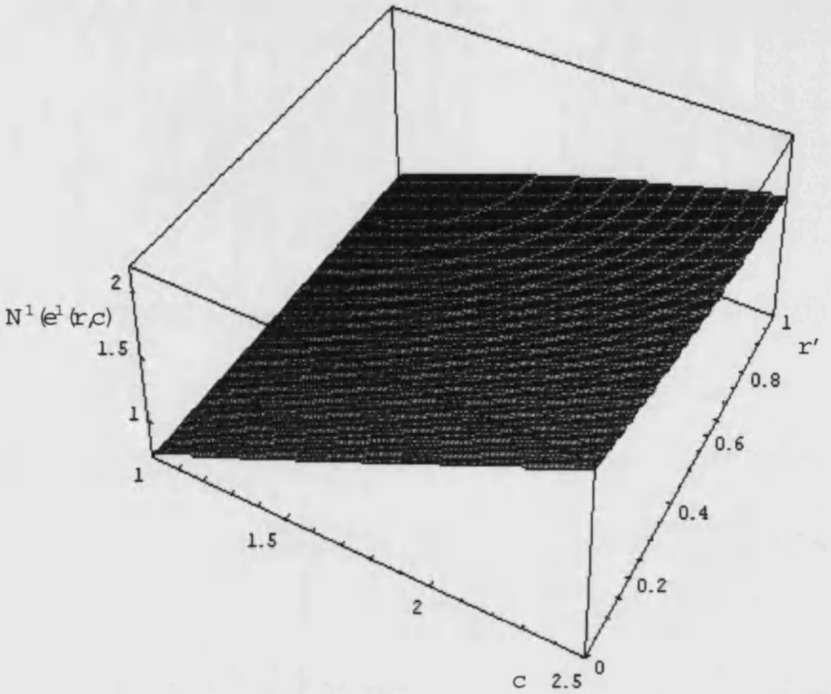


Figure $B.7$: $N^1(e^1(r,c))$ for $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$

Although we have $N^1(e^1(r,c)) > 1$ at many points around and between the F line and between the F and the M lines, female equilibrium is always stable. As it is shown in the figure $B.8$, we have $N^2(e^1(r,c)) < 1$ in all cases. As before, in figure $B.8$ r is normalised so that $r' = 0$ corresponds to $r = 1$ and $r' = 1$ corresponds to $r = 2.5$. Taking into considerations the values of $N^1(e^1(r,c))$ in III, it is expected that the higher values of $N^2(e^1(r,c))$ are reached close to the F line.

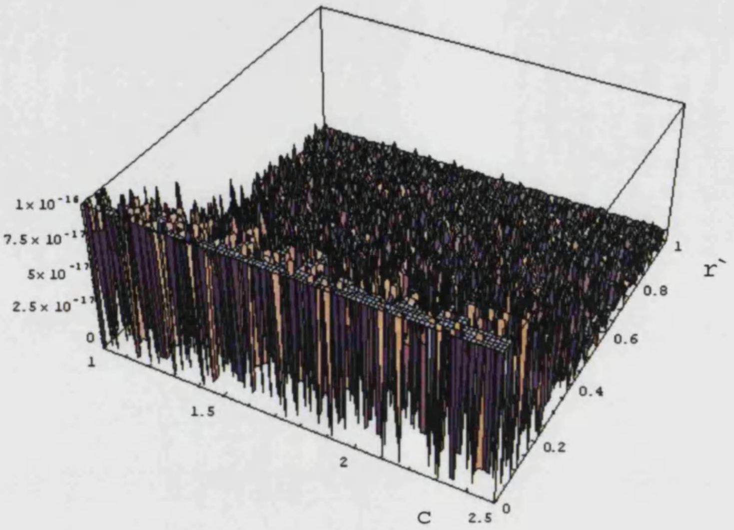


Figure $B.8$: $N^1(e^1(r,c))$ for $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$

B.2.2 Easy Equilibrium

The easy equilibrium is not a stable equilibrium. We calculated $N^1(e^2(r,c))$ and $N^2(e^2(r,c)$ for $1 \leq c \leq 2.5$ and $r_F \leq r \leq r_M$. For illustration purposes, for given c, we rescaled r so that $r' = 0$ corresponds to $r = r_F$ and $r' = 1$ corresponds to $r = r_M$. Hence as in the previous figures for any two points A and B in (c, r, N) space, where $r\prime_A = r'_B$ but $c_A \neq c_B$, then we have r$_A \neq r_B$, since the value of r depends on the particular c it is related to.

As we can see from the two figures below (figures $B.9$ and $B.10$), both norms $N^1(e^2(r,c))$ and $N^2(e^2(r,c)$ are higher than 1 in all cases, apart from the limiting cases, where either $c = 1$ and $r = r_F = r_M$ (represented by the point $(1,0,1)$) or where $r = r_F$ (represented by the points of the form $(c,0,N)$) or where $r = r_M$ (represented by the points of the form $(c,1,1)$). As we are going to show in the next section, our proposition on the instability of the easy equilibrium is supported by analysis of the basins of attraction of the each equilibrium is region *III*.

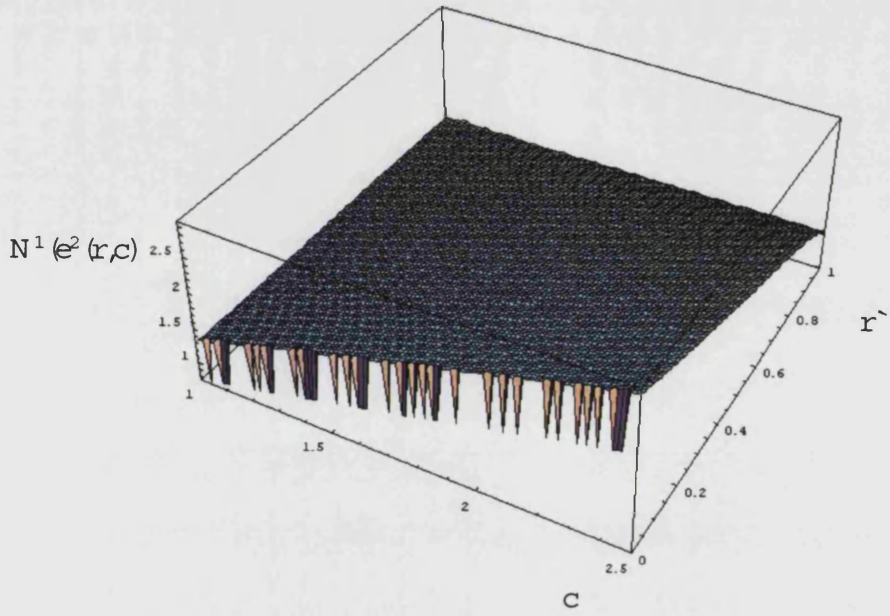


Figure *B*.9: $N^1(e^2(r,c))$ for $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$

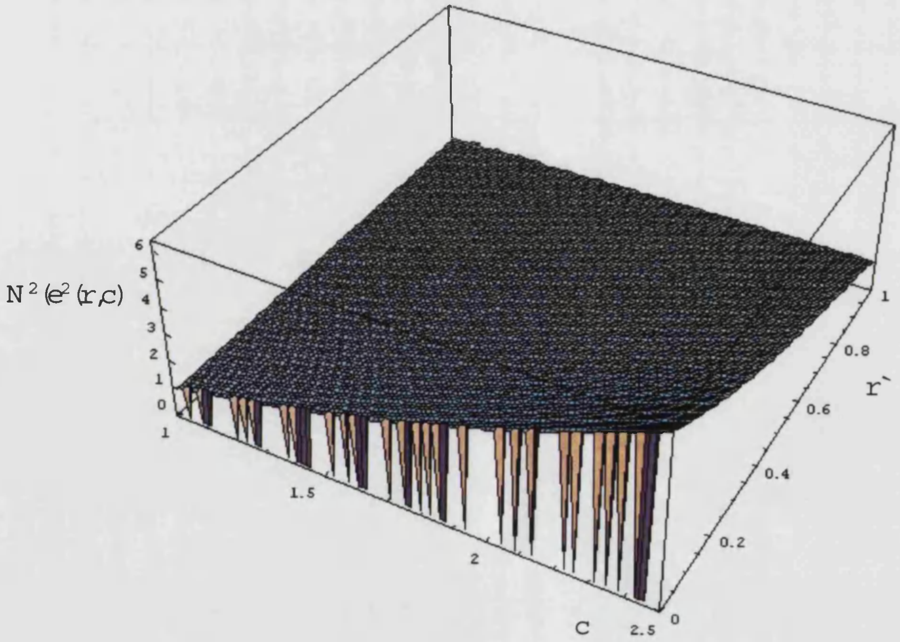


Figure *B*.10: $N^2(r,c)$ for $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$.

B.2.3 Choosy Equilibrium

Choosy equilibrium is attracting in one step in most of the cases; however, there seems to exist a small area in regions II and III where it is attracting in two steps. In order to examine its behaviour in detail, we analysed separately regions I and III.

The following figures ($B.11$ and $B.12$) describe the behaviour of $N^1(e^3(r,c))$ in region III. From figure $B.11$, it becomes apparent that for points close to the F line (when c is close to 1) and for points close to the M line (when c is close to 2.5), we observe that $N^1(e^3(r,c)) > 1$ and $N^1(e^3(r,c))$ even reaches values higher than 1.2 in some cases. Given these results, we analysed the values of $N^1(e^3(r,c))$ for $10,000$ points in region III so that we get a better approximation of the line above which $N^1(e^3(r,c)) > 1$ in region III (illustrated in green in figure $B.12$).

In figure $B.13$ it is graphed $N^2(e^3(r,c))$ and it is shown that $N^2(e^3(r,c) < 1$ for all points in region III. Therefore we can conclude that the choosy equilibrium is stable in region III.

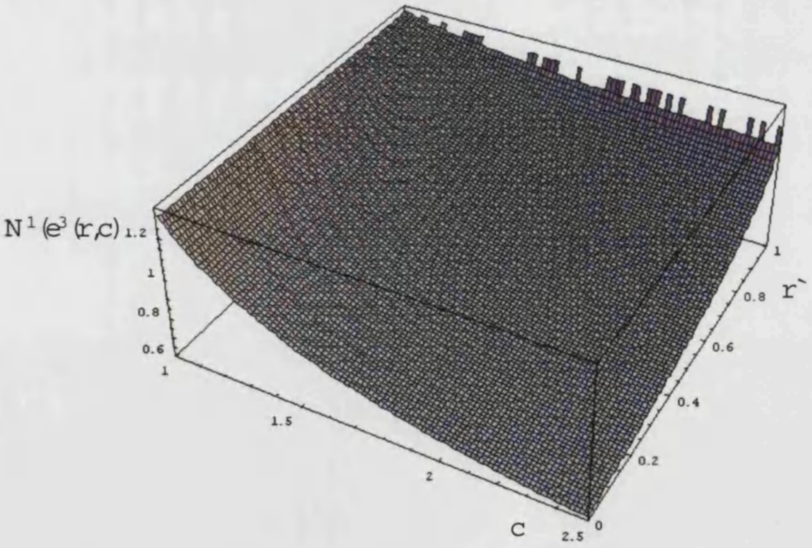


Figure $B.11$: $N^1(e^3(r,c))$ in region III, where $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$

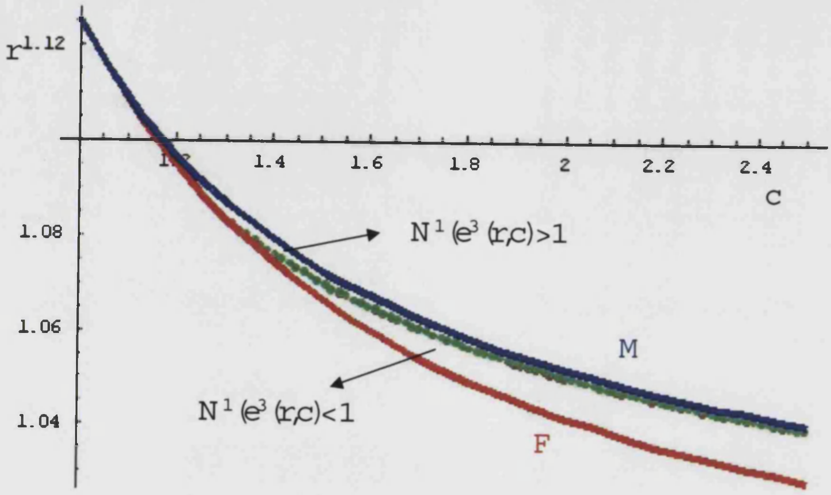


Figure $B.12$: Line above which we have $N^1(e^1(r,c)) > 1$ in region III

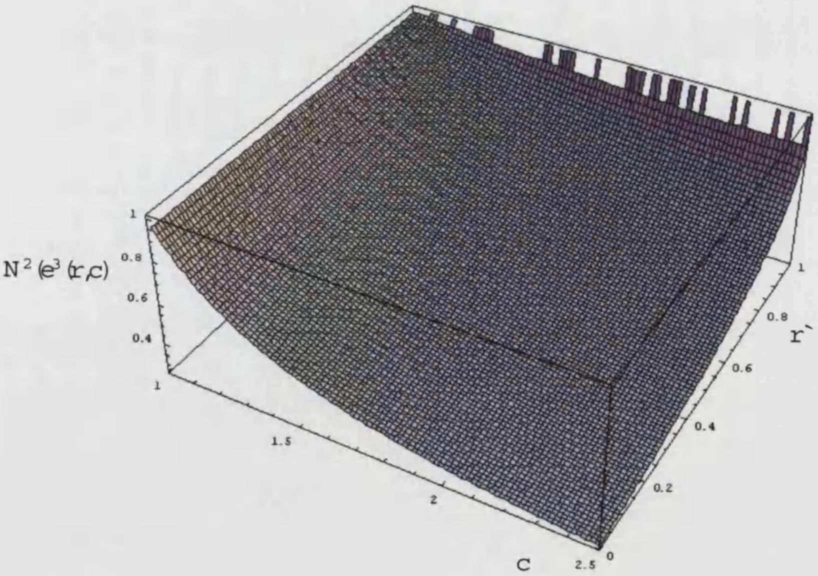


Figure $B.13$: $N^2(e^3(r,c))$ in region III, where $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$

Figures $B.14$, $B.15$ and $B.16$ describe region II; there seem to exist an area close to the F line where $N^1(e^3(r,c)) > 1$; the size of this area is quite small though as illustrated in the second graph of figure $B.14$. In the second graph of Figure $B.14$, we duplicated the first graph of figure $B.14$ and we added a plane that is parallel to (r,c) and it includes all points where $N^1(e^3(r,c)) = 1$ (it can be seen in blue colour in figure $B.14$). In that way, it is easier to focus on the region where $N^1(e^3(r,c)) > 1$ and appreciate how high are the values of the $N^1(e^3(r,c))$. A clearer image of the points where $N^1(e^3(r,c)) > 1$ is given in figure $B.15$. Analysing the values of $N^1(e^3(r,c))$ for $10,000$ points in region II gave as a better estimation of the area where $N^1(e^3(r,c)) > 1$ in region II. The points where $N^1(e^3(r,c)) > 1$ and $c < 1$ are illustrated in dark green and the points where $N^1(e^3(r,c)) > 1$ and $c \geq 1$ are illustrated in purple in figure $B.15$.

Next we illustrate $N^2(e^3(r,c))$ in figure $B.16$ where it is shown that is is less than 0.5 for all points in region II.

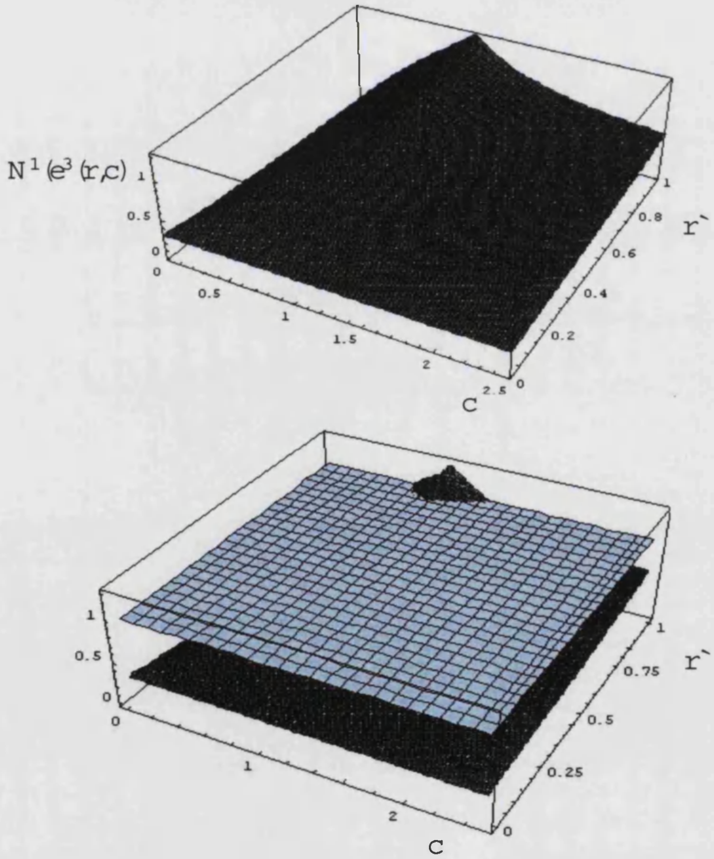


Figure $B.14$: $N^1(e^3(r,c))$ in region II, where $r_F \leq r \leq 2.5$ and $0 \leq c \leq 2.5$

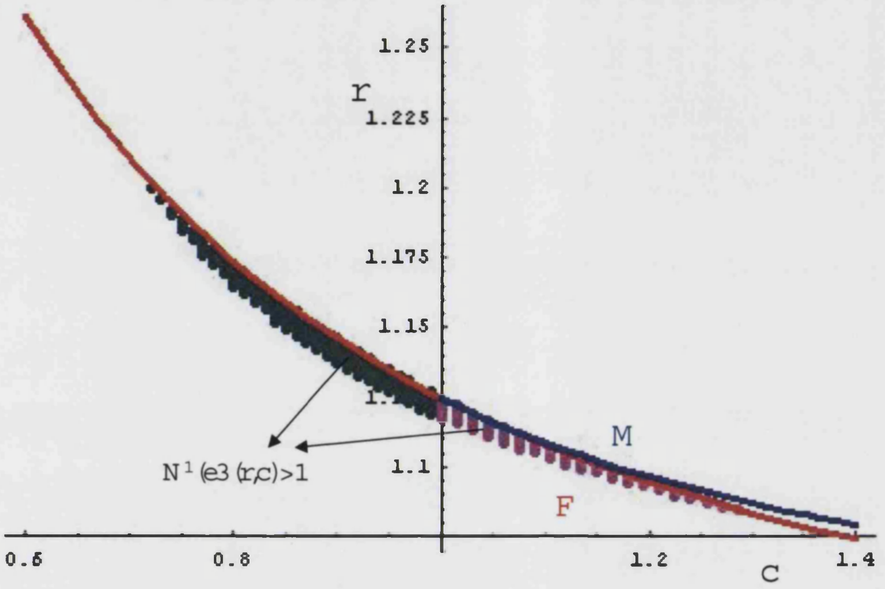


Figure B.15: Points where $N^1(e^1(r,c)) > 1$ in region III (in green and purple colour).

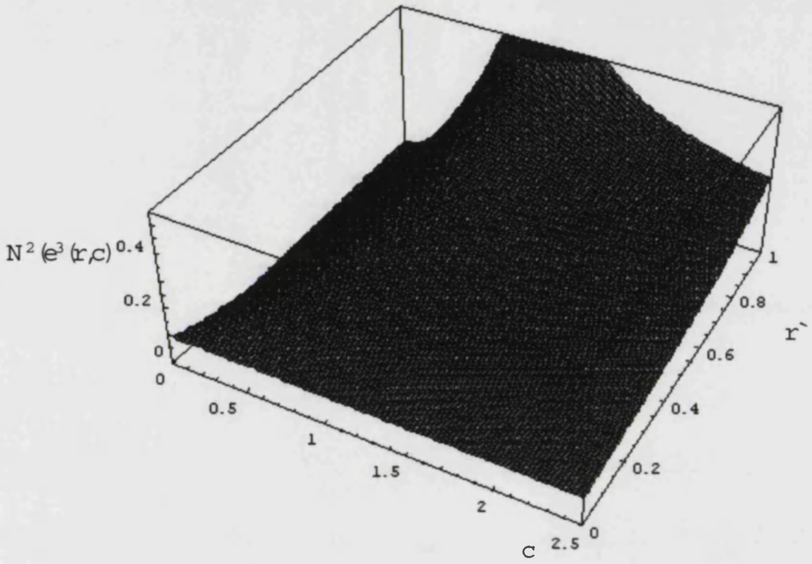


Figure $B.16$: $N^2(r,c)$ in region II, where $r_F \leq r \leq 2.5$ and $0 \leq c \leq 2.5$

Figure $B.17$ combines figures $B.12$ and $B.15$, showing all points where $N^1(e^3(r,c)) > 1$ in regions II and III.

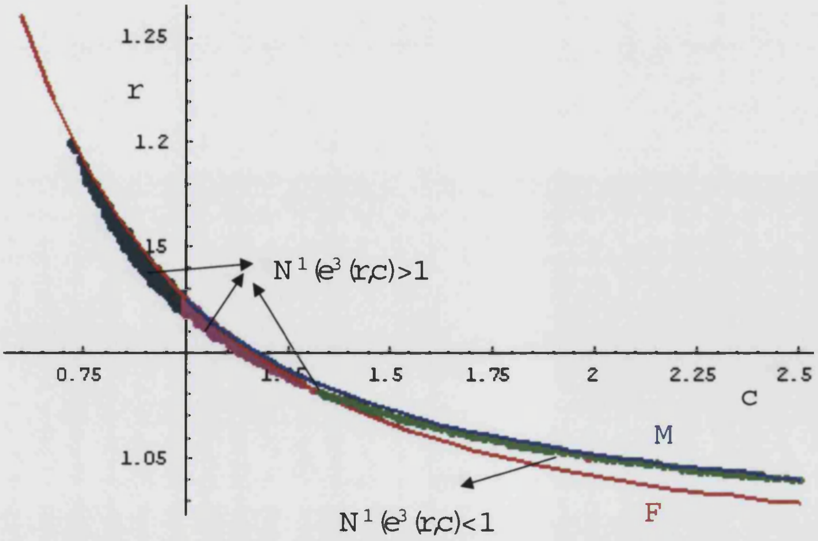


Figure $B.17$: Points where $N^1(e^3(r,c)) > 1$ in regions III and II

Figure $B.18$ combines figures $B.4$, $B.12$ and $B.15$, showing all points where $N^1(e^1(r,c)) > 1$ and $N^1(e^3(r,c)) > 1$ in I, II and III.

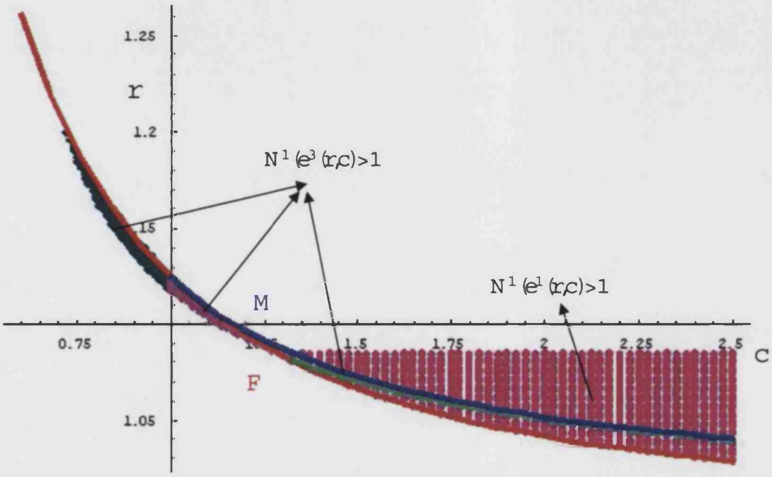


Figure $B.18$: Points where $N^1(e^1(r,c)) > 1$ and $N^1(e^3(r,c)) > 1$ in regions I, II and III

B.2.4 Basins of Attraction

We proved that multiple equilibria exist in region III and we showed that both female and choosy equilibria are stable. Therefore, it would be interesting to examine which equilibrium would be more probable to appear in reality. We analysed the basins of attraction of each equilibrium, assuming that there exist a positive relation between the size of the basin of attraction of an equilibrium and the probability it appears in real life.

A basin of attraction of a point $(\bar{u}, v)$ is defined as the set of points (u, v) (the region) such that any iterations of the mapping $T(u, v)$ (52) will finally converge to $(\bar{u}, v)$. For given (r, c) in region III, we chose 10,000 uniformly distributed points (u, v) in the $[0, 0.25] \times [0, 0.25]$ space and we examined whether a mapping T starting from any of these points converges to an equilibrium point[5]; for the points that converge to an equilibrium, we noted what type of equilibrium they converge to.

---

[5] In reality we checked whether they converge to a point $(\hat{u}, \hat{v})$ such that

$$|\hat{u} - \breve{u}| < 0.00001 \text{ and } |\hat{v} - \breve{v}| < 0.00001$$

where $(\breve{u}, \breve{v})$ is an equilibrium point.

For any pair (r, c) we examined in III, all mappings starting any of the (u, v) points analysed, converged to an equilibrium. In no case though this equilibrium was the easy, apart from the mapping of the easy equilibrium point to itself. This observation comes into accordance with our claim that the easy equilibrium is unstable. Our claims of stability of the female choice and the choosy equilibrium were also confirmed since in all examined cases a number of mappings converged to the female choice equilibrium and all the rest of the mappings to the choosy equilibrium. It is important to note that the basin of attraction of the female choice equilibrium was significantly smaller than the one of the choosy equilibrium in all cases examined. This fact may indicate that choosy equilibrium is more probable to be found in real circumstances. The basin of attraction of the female choice equilibrium included mostly points were the male strategy was very close to 0.

We present some examples in order to permit to the reader to gain an intuition on our partial results on the basins of attraction.

- *Example B.1:* $r = 1.08388$ *and* $c = 1.32$

In this case $N^1(e^1(1.08388, 1.32)) > 1$ and $N^1(e^3(1.08388, 1.32)) > 1$ while $N^2(e^1(1.08388, 1.32)) < 1$ and $N^2(e^3(1.08388, 1.32)) < 1$

The following graph (figure $B.19$) shows in green all the points belonging to the basin of attraction of the female equilibrium. The female equilibrium is marked with red and the easy equilibrium is marked in blue. All the rest of the points, marked in yellow belong to the basin of attraction of the choosy equilibrium; the choosy equilibrium is not appearing in the graph (the female strategy in this case is higher than 0.25). The second part of the graph permits us to have a closer look to the basin of attraction of the female equilibrium.

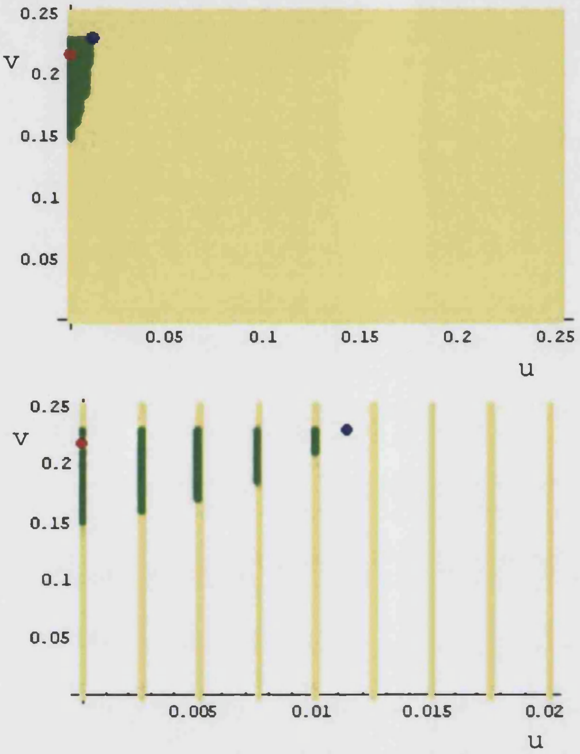


Figure $B.19$: Basins of attraction of the female and the choosy equilibrium in $[0, 0.25] \times [0, 0.25]$

We chose two random initial points and starting from those, we repeatedly calculated T up to the point that it converged to an equilibrium. In the following figure ($B.20$), we illustrate the steps of the iteration of T from these points to the equilibria.

In the upper graph, we chose a point very close to the easy equilibrium (left blue point) as a starting point and we proved that the iteration of T converges to the choosy equilibrium (right blue point).

In the lower graph we chose a point (marked in pink) that is not part of points proved to belong to the basin of attraction of the female equilibrium (marked in green) but it lies between them. As it becomes obvious from the graph, iterations of T starting from this point converge to the female equilibrium (marked in red). This and analogous results permit to us to claim that all points in the area between the green points (which we know that belong in the basin of attraction of the female equilibrium) belong to the basin of attraction of the female equilibrium as well.

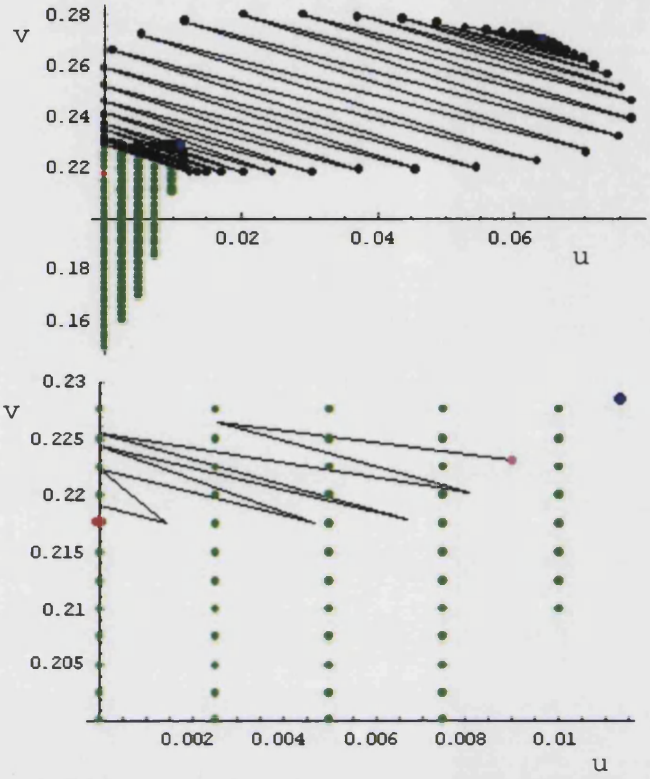


Figure B.20: Examples of iterations of T converging to the choosy and the female equilibrium

The following figure (B.21) allows us to gain an insight of how the iteration of T converges to the equilibria points starting from different points in the (u, v) space. The equilibria are marked in red (female equilibrium) and in blue (easy (lower dot) and choosy (upper dot)). As it becomes apparent, the majority of the points tend to converge to the choosy equilibrium and a few points, mostly below the female choice equilibrium, converge to the latter.

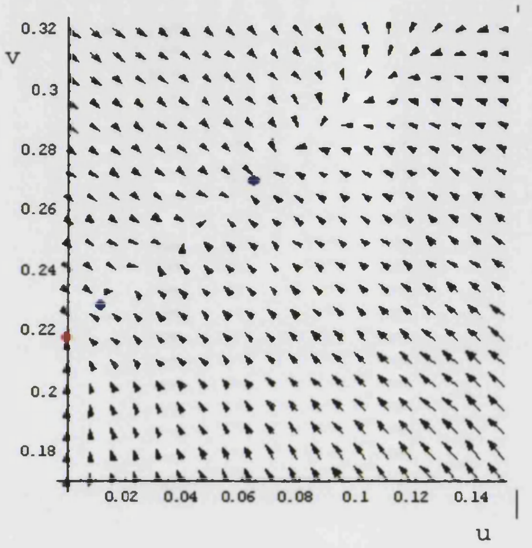


Figure $B.21$: Vectorfield of T for
$r = 1.08388$ and $c = 1.32$

- *Example B.2:* $r = 1.0817$ *and* $c = 1.32$

In this case $N^1(e^1(1.08388, 1.32)) > 1$ and $N^1(e^3(1.08388, 1.32)) < 1$ while $N^2(e^1(1.08388, 1.32)) < 1$ and $N^2(e^3(1.08388, 1.32)) < 1$

The following graph (figure $B.22$) illustrates the steps of the iteration of T from a random point (marked in pink) to the choosy equilibrium (represented by the right blue point). The female equilibrium is marked in red and the green points represent its area of attraction. All the rest of the points belong to the basin of attraction of the choosy equilibrium, apart from the point corresponding to the easy equilibrium (the left blue point). The area of attraction of the female equilibrium is smaller in this case that in the previous example we analysed.

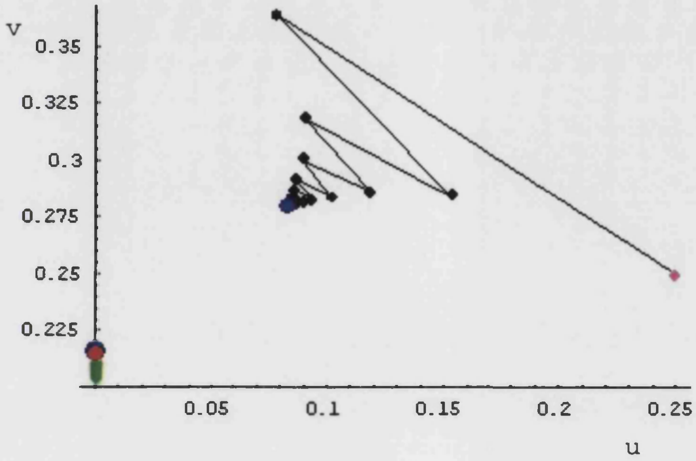


Figure $B.22$: Example of iterations of T converging to the choosy equilibrium

The size of the basin of attraction of each equilibrium can also be analysed by looking at the vector field of T, shown in the next figure ($B.23$).

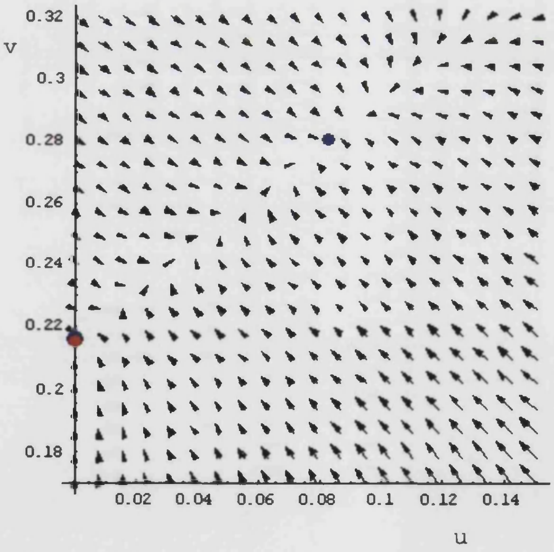


Figure $B.23$: Vectorfield of T for $r = 1.0817$ and $c = 1.32$

- *Example B.3:* $r = 1.086$ *and* $c = 1.3$

In this case $N^1(e^1(1.08388, 1.32)) < 1$ and $N^1(e^3(1.08388, 1.32)) > 1$ while $N^2(e^1(1.08388, 1.32)) < 1$ and $N^2(e^3(1.08388, 1.32)) < 1$

Figure $B.24$ shows in green all the points belonging in the basin of attraction of the female equilibrium. The female equilibrium is marked with red and the easy equilibrium is marked in blue. The rest of the points (yellow area) belong to the basin of attraction of the choosy equilibrium; the latter is not appearing in the graph (since the female strategy in this case is higher than 0.25). The second part of the graph focuses on a smaller area of the (u, v) space in order to allow a more detailed look at the female basin of attraction.

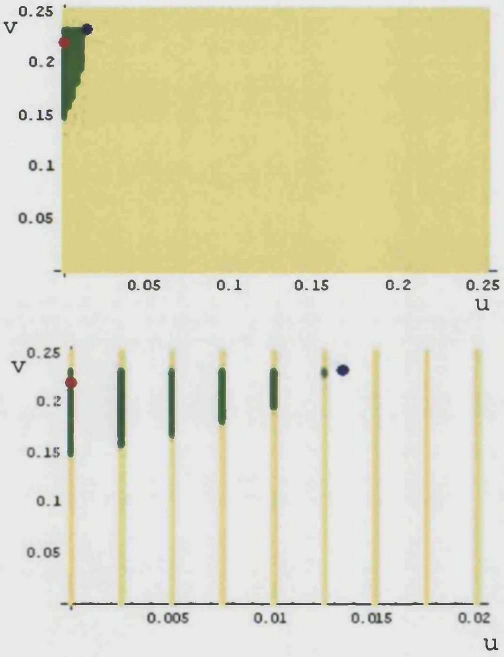


Figure $B.24$: Basins of attraction of the female and the choosy equilibrium in $[0, 0.25] \times [0, 0.25]$

In the following figure ($B.25$), we chose 2 random initial points and we illustrate the steps of the iteration of T from the these points to the equilibria.

In the upper graph, we chose a point below the area of attraction of the female equilibrium (pink point) as a starting point and we proved that the iteration of T converges to the choosy equilibrium (right blue point).

In the lower graph we chose a point (marked in pink) lying between the points that belong to the basin of attraction of the female equilibrium (green points). Iteration of starting from this point converge to the female equilibrium (marked in red).

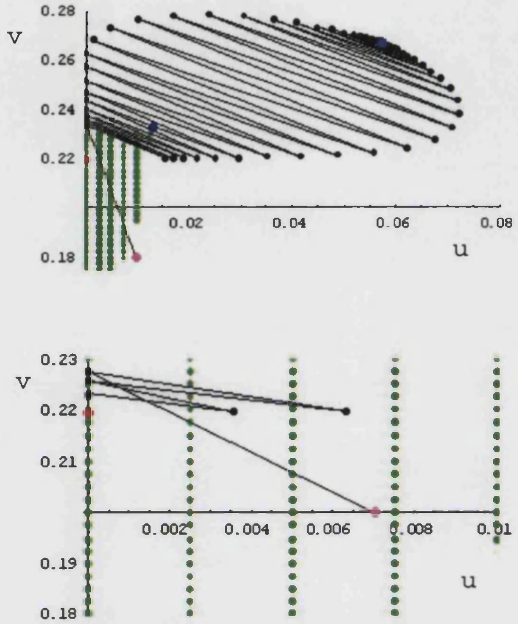


Figure B.25: Examples of iterations of T converging to the choosy and the female equilibrium

The vector field of T, shown in the figure (B.26) gives us the opportunity to understand better how the mapping T converges to the equilibria.

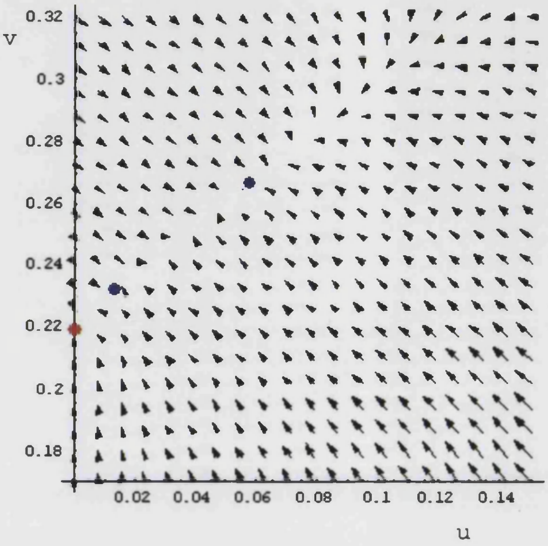


Figure $B.26$: Vectorfield of T for
$r = 1.0817$ and $c = 1.32$

B.3 Distribution of Couples

In chapter 3, we examine the marital stability of the couples created at equilibrium. Before analysing the marital stability of a mating game, we need to calculate first the distribution of mated couples when any pair of strategies (u, v) is used by the male and the female population respectively.

For reasons of simplicity, we call males (females) of a type lower than $v(u)$ low type males (females) and males (females) of a type higher than v (u) high types males (females).

The fraction π_1 of couples (x, y) where $x < v$ and $y < u$ is

$$\begin{aligned}\pi_1 &= \Pr(x < v, y < u) \\ &= u\frac{vr}{r-k} \\ &= \frac{uvr}{v + u(1-v) + r - 1}\end{aligned}$$

In the first period no lower type male or female is accepted, hence they all enter the second period unmated. Hence in the second period there are u low type females and vr low type males. Hence, the probability that a low type female is mated with a low type male is $\frac{vr}{r-k}$, where $r - k$ is the total male population in the second period, since k are the number of pairs created in the first period (21).

The fraction π_2 of couples (x, y) where $x < v$ and $y \geq u$ is

$$\begin{aligned}\pi_2 &= \Pr(x < v, y \geq u)\\ &= v(1-u)\frac{vr}{r-k}\\ &= \frac{v^2(1-u)r}{v+u(1-v)+r-1}\end{aligned}$$

All high females who meet low type males in the first period, enter the second period unmated. Hence in the second period there are $v(1-u)$ high type females who have a chance of $\frac{vr}{r-k}$to be mated with a low type male.

The fraction π_3 of couples (x, y) where $x \geq v$ and $y \geq u$ is

$$\begin{aligned}\pi_3 &= \Pr(x \geq v, y \geq u)\\ &= k + v(1-u)\frac{u(1-v)+(r-1)(1-v)}{r-k}\\ &= \frac{(u-1)(v-1)}{r+u+v-uv-1}(r+u+rv-1)\end{aligned}$$

The fraction of couples where $x \geq v$ and $y \geq u$ created in the first period is k while the faction of couples of high type created in the second period is $\frac{v(1-u)(v+u(1-v)+r-1-rv)}{r-k}$, since there are $v(1-u)$ high type females and $(u(1-v)+(r-1)(1-v))$ males having a type greater than v in the second period.

The fraction π_4 of couples (x, y) where $x \geq v$ and $y < u$

$$\begin{aligned}\pi_4 &= \Pr(x \geq v, y < u)\\ &= u\frac{u(1-v)+(r-1)(1-v)}{r-k}\\ &= \frac{u(1-v)(r+u-1)}{v+u(1-v)+r-1}\end{aligned}$$

All high type males who either meet low type females or remain unpaired in the first period enter the second period unmated. Hence in the second period there are $(u(1-v)+(r-1)(1-v))$ of high type males and u low type females.

The following table summarises our results.

$$\begin{array}{rl}
\pi_1 = \frac{uvr}{v+u(1-v)+r-1} & for\ x < v\ and\ y < u \\
\pi_2 = \frac{v^2(1-u)r}{v+u(1-v)+r-1} & for\ x < v\ and\ y \geq u \\
\pi_3 = \frac{(u-1)(v-1)}{r+u+v-uv-1}\,(r+u+rv-1) & for\ x \geq v\ and\ y \geq u \\
\pi_4 = \frac{u(1-v)(r+u-1)}{v+u(1-v)+r-1} & for\ x \geq v\ and\ y < u \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 &
\end{array}$$

Taking into account the fractions $(\pi_1, \pi_2, \pi_3, \pi_4)$, we can estimate the distribution $f(x, y)$ of mated couples. So,

$$\begin{aligned}
f(x,y) &= \begin{cases} \frac{1}{(u)(v)}\pi_1 & for\ x < v\ and\ y < u \\ \frac{1}{(1-u)(v)}\pi_2 & for\ x < v\ and\ y \geq u \\ \frac{1}{(1-v)(1-u)}\pi_3 & for\ x \geq v\ and\ y \geq u \\ \frac{1}{(1-v)(u)}\pi_4 & for\ x \geq v\ and\ y < u \end{cases} \\
&= \begin{cases} \frac{r}{v+u(1-v)+r-1} & for\ x < v\ and\ y < u \\ \frac{vr}{v+u(1-v)+r-1} & for\ x < v\ and\ y \geq u \\ \frac{r+u+rv-1}{r+u+v-uv-1} & for\ x \geq v\ and\ y \geq u \\ \frac{r+u-1}{v+u(1-v)+r-1} & for\ x \geq v\ and\ y < u \end{cases}
\end{aligned}$$

B.3.1 Distribution of Mated Males and Females

All females are mated after the end of the second period. Hence the distribution of females y in the couples created under the pair of strategies(u, v) is the same as the initial distribution of females. Hence after the game is finished, mated females are uniformly distributed in $[0, 1]$ and

$$f(y) = 1 \text{ for every } 0 \leq y \leq 1$$

The distribution of the mated male population though differs from the initial male distribution. Only $\frac{1}{r}$ of males is mated after the end of the second period. From those mated males, k males are of high type mated in the first period, and $(r - 1 + u)(1 - v)\frac{1-k}{r-k}$ are males of high type mated in the second period. Therefore, there are

$$(1-v)(1-u) + (r-1+u)(1-v)\frac{1-k}{r-k} = \frac{(1-v)\,(r+u+rv-ruv-1)}{r+u+v-uv-1}$$

males having a type higher than v, in the couples created.

No male having a type less than v is mated in the first period. Hence the number of males having a type less than v in the couples created is equal with the number of males who are mated in the second period and have a type less than v and it is

$$vr\frac{1-k}{r-k} = \frac{(u+v(1-u))vr}{v+u(1-v)+(r-1)}$$

Hence the distribution of males in the couples created is

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{v}\frac{(u+v(1-u))vr}{v+u(1-v)+(r-1)} & for\ x < v \\ \frac{1}{1-v}\frac{(1-v)(r+u+rv-ruv-1)}{r+u+v-uv-1} & for\ x \geq v \end{cases} \\ f(x) &= \begin{cases} \frac{(u+v(1-u))r}{v+u(1-v)+(r-1)} & for\ x < v \\ \frac{r+u+rv-ruv-1}{r+u+v-uv-1} & for\ x \geq v \end{cases} \end{aligned}$$

B.4 Marital Stability of Couples

The marital stability σ of a given distribution of couples is defined in chapter 3 as the probability that given two random couples x_1y_1 and x_2y_2, either the pairing x_1y_2 provides a greater utility to both x_1 and y_2 than x_1y_1 and x_2y_2 respectively, or the pairing x_2y_1 provides a greater utility to both x_2 and y_1 than x_2y_2 andx_1y_1 respectively.

We saw that $s_{i,j}$ is the probability that a pair of couples is stable when one belongs in R_i and the other one belongs in R_j. Hence we constructed in chapter 3 the matrix S (54), where

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So, given that the distribution over the table is $\pi = \pi(u,v) = (\pi_1, \pi_2, \pi_3, \pi_4)$ then the stability index σ is

$$\begin{aligned} \sigma &= \sigma(u,v) = \pi S \pi = \frac{1}{2}(1 - 2\pi_2\pi_4 + 2\pi_1\pi_3) \\ &= \frac{1}{2}\left(\begin{array}{c} 1 - 2\frac{v^2(1-u)r}{v+u(1-v)+r-1}\frac{u(1-v)(r+u-1)}{v+u(1-v)+r-1} \\ +2\frac{uvr}{v+u(1-v)+r-1}\frac{(u-1)(v-1)}{r+u+v-uv-1}\left(r+u+rv-1\right) \end{array} \right) \\ &= \frac{r+u+v-uv-2ruv^2-2ru^2v+2ru^2v^2+2ruv-1}{2\left(r+u+v-uv-1\right)} \end{aligned}$$

It is obvious that the female choice equilibrium has marital stability $\sigma(e^1(r,c) = \frac{1}{2}$ for all (r,c).

$$\sigma(e^1(r,c) = \sigma(0,v_1) = \frac{r+v-1}{2\left(r+v-1\right)} = \frac{1}{2}$$

Focusing in region III, it is interesting to find which equilibrium has the highest marital stability for given (r, c). We divided the region III where $r_F \leq r \leq r_M$ and $1 \leq c \leq 2.5$ into $10,000$ points and we compared the marital stability of the choosy equilibrium with that of the easy equilibrium; in all cases (apart from the extreme case where $r = r_M$ and choosy and easy equilibrium coincide) choosy equilibrium had a higher marital stability index than the choosy equilibrium. Next we compared the marital stability of the easy equilibrium with the one of the female choice equilibrium. In no case (apart from the extreme case where $r = r_F$ and easy and female choice equilibrium coincide) the stability of the easy equilibrium was larger or equal with $\frac{1}{2}$.

Therefore, we may end up into the conclusion that for any examined (r, c) in III the choosy equilibrium is the most stable one and the female equilibrium the most unstable one.

$$0.5 = \sigma(e^1(r, c) \leq \sigma(e^2(r, c) \leq \sigma(e^3(r, c)$$

We plotted the stability index of the mutual choice equilibria in order to get some upper limits for it. As it can be seen in the following figures

$$\sigma(e^2(r, c) \leq 0.54 \text{ and } \sigma(e^3(r, c) \leq 0.59$$

Figure $B.27$ focuses in region II; the first graph refers to the equilibrium marital stability for $c \leq 1$ and the second graph for $c \geq 1$. In both graphs r is normalised in such a way that $r\prime = 0$ corresponds to $r = 1$ and $r' = 1$ to $r = r_F$. It becomes apparent from the two graphs that $\sigma(e^3(r, c) \leq 0.59$ in region II. Stability σ takes its highest values for r close to 1. However, for large c , σ also takes high values for higher r. As we see from the graphs, σ takes high values for all (c, r') below the line $\frac{1}{2.5}c = r'$ (and above this line for c higher than 1) while it takes its lowest values near the F line and above the line $\frac{1}{2.5}c = r'$ (for c low).

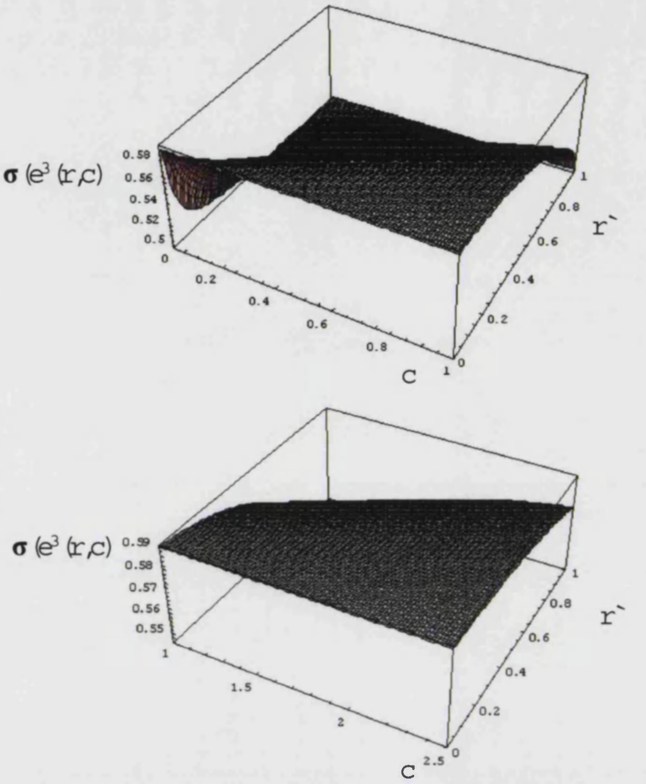


Figure $B.27$: Marital stability index $\sigma(e^3(r,c)$ of choosy equilibrium in region II

Figures $B.28$ and $B.29$ graph the marital stability index of the choosy and the easy equilibria respectively in region III. Figure $B.29$ illustrates the difference $\sigma(e^3(r,c)) - \sigma(e^2(r,c)$ between the marital stability of the choosy and of the easy equilibria for given (r,c) in region III. In all figures, $r\prime = 0$ corresponds to $r = r_F$ and $r' = 1$ to $r = r_M$.

As we can observe from figure $B.28$ and $B.29$, $\sigma(e^3(r,c))$ is the lowest for $c = 1$ and the highest for r close to F (especially for c close to 2.5), while $\sigma(e^2(r,c))$ is the lowest close to the F line and the highest close to the M line, (especially for $c = 2.5$). From the analysis of chapter 3 we know that for given c, choosy equilibrium strategies take their maximum values on the F line and their minimum values on the M line, while the opposite is true for the easy equilibrium. This observation in combination with Figures $B.28$ and $B.29$ indicates that the choosier the populations are at equilibrium, the highest is the marital stability of the couples created.

This claim is also supported by Figure $B.30$, where we can see that the marital index of the choosy and the easy equilibrium is the same on the M line and on the F line for $c = 1$, since in these cases the two equilibria coincide, while in all other cases, the marital stability of the choosy equilibrium is higher, taking its maximum value (>0.06) for large c close to the F line.

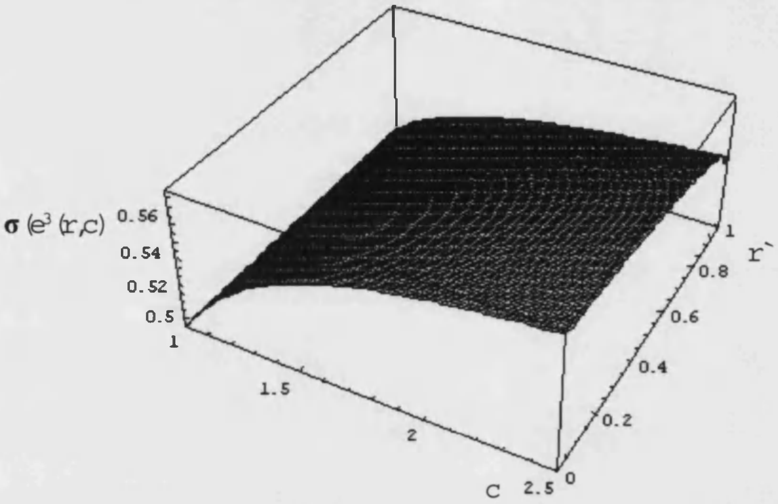


Figure B.28: Marital stability index $\sigma(e^3(r,c))$ of choosy equilibrium in region III

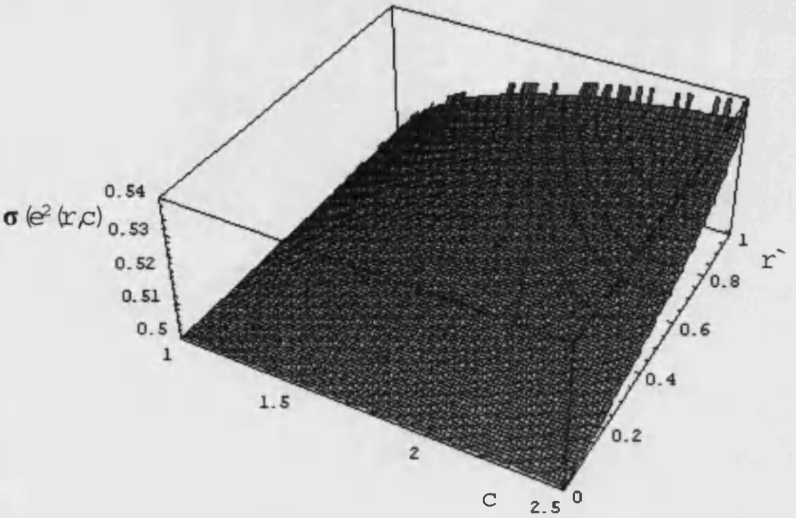


Figure $B.29$: Marital stability index $\sigma(e^2(r,c))$ of easy equilibrium in region III

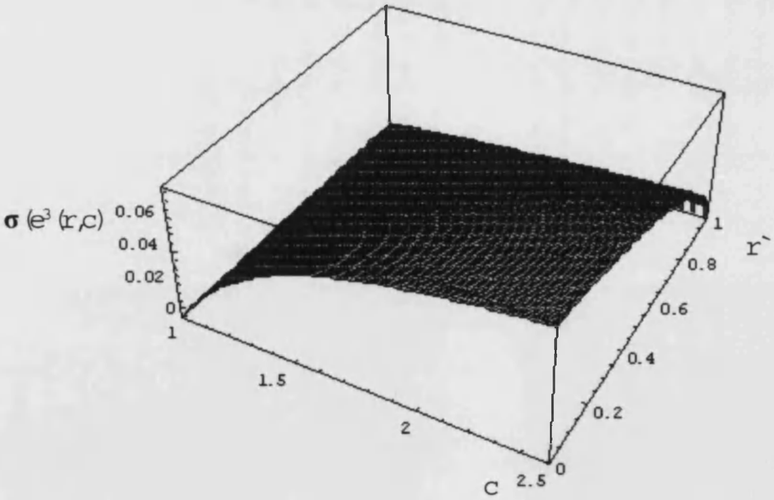


Figure $B.30$: Difference of marital stability indices $\sigma(e^3(r,c)) - \sigma(e^2(r,c))$ of the choosy and the easy equilibrium in region III

B.5 Discrete Type n Period Model, Where n>2

Assuming that the game is played a small number of periods $n > 2$ and there are m male and m female types, it is easy to calculate all the equilibria for given (r, c).

Let males use strategy $s_m = (s_{m1}, s_{m2}, ..., s_{mn-1})$, where s_{mi} denotes the strategy males use in period i. In the same way, let female strategy be $s_f = (s_{f1}, s_{f2}, ..., s_{fn-1})$. Given strategy $s = (s_m, s_f)$ it is easy to calculate the expected utilities V_{mi} and V_{fi} that a male and a female expect to receive in period i.

Let $V_m = V_{m,s}$ and $V_f = V_{f,s}$ denote the utilities that males and females respectively expect to receive in each period when strategy s is used. $V_m = (V_{m2}, V_{m3,} ..., V_{mn})$ and $V_f = (V_{f2}, V_{f3,} ..., V_{fn})$

At equilibrium, it must be true that

$$s_{mi} = V_{mi+1} \text{ and } s_{fi} = V_{fi+1}$$

Therefore, we can define equilibria as fixed points of the mapping

$$\begin{aligned} &T_n((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1})) = \\ &= (V_m^+((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1})), \\ &\quad V_f((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1}))) \end{aligned}$$

Since types are discrete, there exist a finite number of strategies we need to check in order to specify whether they are equilibria or not. Hence for given pair (r, c), we just have to analyse $m^{2(n-1)}$ strategies and point out the equilibria where

$$\begin{aligned} &T_n((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1})) = \\ &= (\lceil V_m^+((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1}))\rceil, \\ &\quad \lceil V_f((s_{m1}, s_{m2}, ..., s_{mn-1}), (s_{f1}, s_{f2}, ..., s_{fn-1}))\rceil) \end{aligned}$$

Taking under consideration that in the last period, $V_{mn} < \frac{m}{2}$ and that strategy cutoff points are decreasing in each period, the number of potential equilibrium strategies we need to check becomes even smaller.

In order to conclude to the results concerning the discrete type **3** period model presented in chapter **3**, we found all the fixed points of given (r, c) of the mapping T_3 (52).

B.6 Truncated Normal Distribution

We analyse the discrete 8 type model played over 3 periods. We assume a truncated normal distribution of the population in the first period, so that middle types are more common than extreme types. The following figure ($B.31$) describes the distribution of types in the first period.

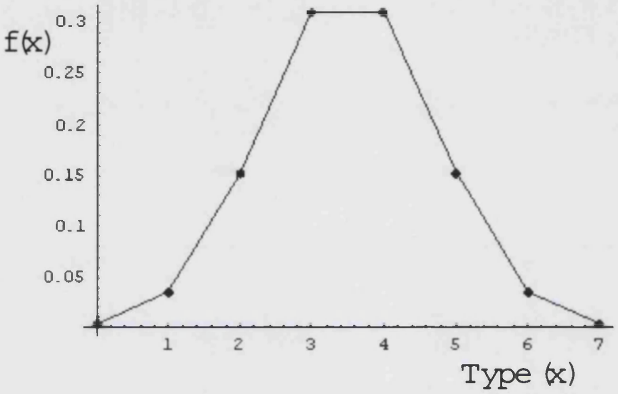
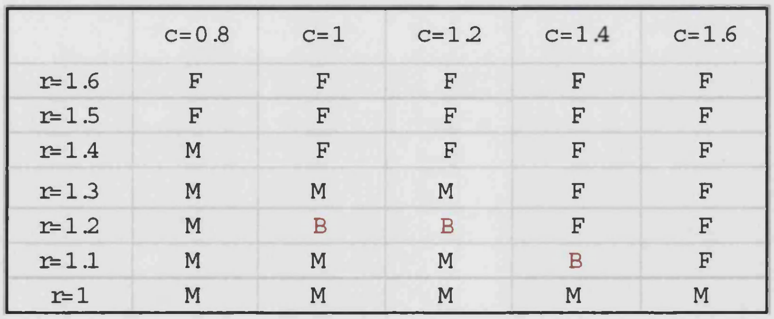


Figure $B.31$: Truncated normal distribution $f(x)$
in the first period.

Using exhaustive search, we found the equilibria for different (r, c). The equilibria are described in the following figure and table.

Figure $B.31$ charts, for $m = 8$, the qualitative aspects of this search, for r and c in the grid. Here, F represents just a female choice equilibrium, M just a male choice equilibrium, and B the presence of both types.



	c=0.8	c=1	c=1.2	c=1.4	c=1.6
r=1.6	F	F	F	F	F
r=1.5	F	F	F	F	F
r=1.4	M	F	F	F	F
r=1.3	M	M	M	F	F
r=1.2	M	B	B	F	F
r=1.1	M	M	M	B	F
r=1	M	M	M	M	M

Figure $B.32$:. Equilibria for $n = 3$, $m = 8$

To obtain a more quantitative analysis of the equilibria, as functions of r and c, we describe in Figure $B.33$ the equilibria corresponding to a grid of r

and c values. We construct Figure $B.33$ in the same way as Figure 3.16. The reader can first study Figure 3.15 in order to be able to understand better how to interpret Figure $B.33$.

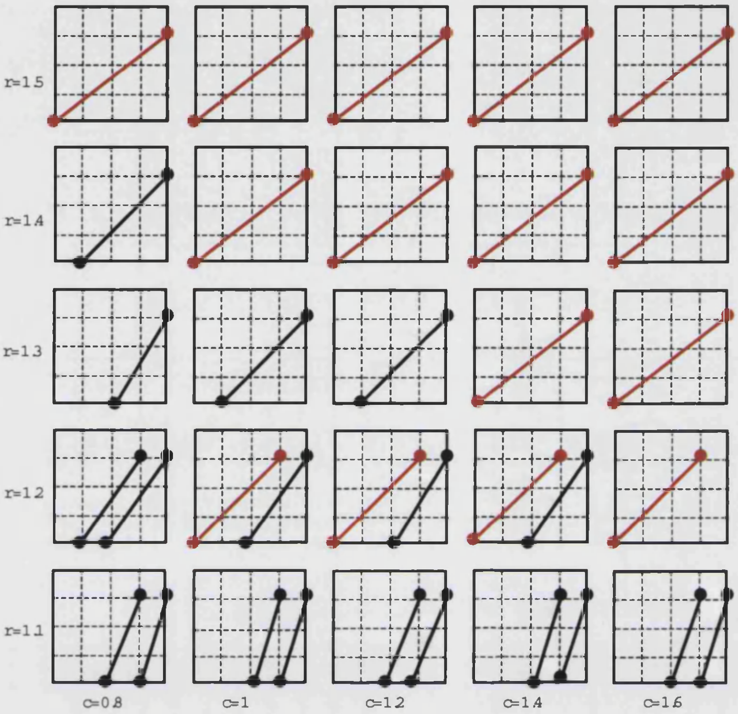


Figure $B.33$: Equilibria for discrete truncated normal distribution, $m = 8$.

It is obvious that multiple equilibria appear as well when the distribution is not uniform and it is possible to have both female and mutual choice appearing.

Appendix C: Age Dependent Preferences

This Appendix offers the complete analysis of the equilibria presented in chapter 4.

C.1 $\lambda = \tau = 2$ and $r \geq 1$

The equilibrium strategy profile is $((1,1),(1,1))$ and the distribution of males and females should be such that

$$\begin{aligned} a_1(1 - \frac{1}{r}(b_1 + b_2) &= a_2 \Rightarrow a_2 = a_1\frac{r-1}{r} \Rightarrow a_1 = \frac{r^2}{2r-1} \text{ and } a_2 = \frac{r^2 - r}{2r-1} \\ b_1(1 - (\frac{a_1}{r} + \frac{a_2}{r}) &= b_2 \Rightarrow b_2 = 0 \text{ and } b_1 = 1 \end{aligned}$$

The ratio R of the incoming population is

$$R = a_1 = \frac{r^2}{2r-1} \Leftrightarrow r(R) = R + \sqrt{R^2 - R}$$

The population sex ratio r is an increasing function of the incoming population sex ratio R. Given that we assume the sex ratio is male biased $(r \geq 1)$, $R \geq 1$.

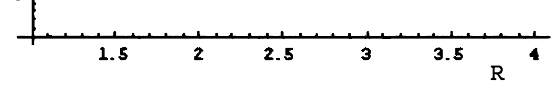


Figure $C.1$: Population sex ratio $r(R)$, when $\lambda = \tau = 2$, $r \geq 1$.

As expected, both a_1 and a_2 are increasing functions of R. Nevertheless, the fraction of males of age 1, decreases as R increase, while the fraction of males of age 2 increases as R increase. As we can see from figure $C.2$, as we move further from $R = 1$, the faction of males of age 1 and age 2 come very close; however, the number of males in the first period of their fertile life will always exceed the number of males of age 2.

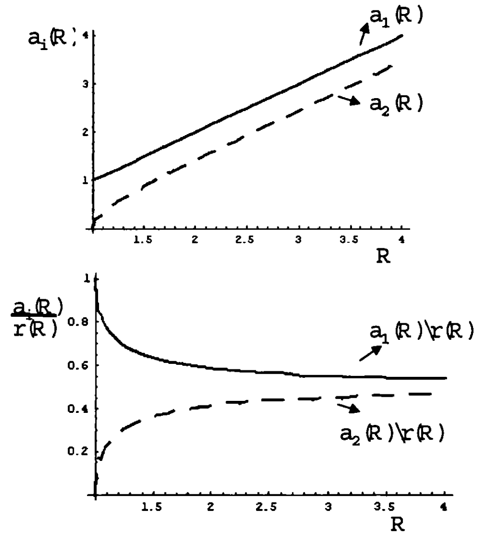


Figure $C.2$: Number a_i of males of type i in the male population (top picture) and male distribution (bottom picture) at equilibrium, when $\lambda = 2$, $\tau = 2$, $r \geq 1$.

C.2 $\lambda = 3$, $\tau = 2$ and $r < 1$

All male types and female of age 2 and age 3 are non choosy (from observation (i) and observation(ii)). Therefore our only concern, is whether male of age 1 accepts females of age 2 or not.

The expected utility of males in the second period of their fertile life is

$$U_2 = b_1 2 + b_2 1 > 1$$

Since $U_2 > 1$, males of age 1 are not willing to accept any type offering them a utility less than 2, hence they only accept females of age 1.

The equilibrium strategy profile is $((2, 1, 1), (1, 1))$.

Given $((2, 1, 1), (1, 1))$, the distribution of males and females should be such that

$$\begin{aligned}
a_1(1-b_1) &= a_2 \Rightarrow \frac{r}{2-b_1} = a_1 \\
a_2(1-b_1-b_2) &= a_3 \Rightarrow a_3 = 0 \\
b_1(1-(a_1+a_2+a_3)) &= b_2 \Rightarrow (1-r)b_1 = b_2 \text{ hence } b_1 = \frac{1}{2-r} \text{ and } b_2 = \frac{1-r}{2-r} \\
\text{Hence } a_1 &= \frac{(2-r)r}{3-2r} \text{ and } a_2 = \frac{(1-r)r}{3-2r}
\end{aligned}$$

The incoming population sex ratio is

$$R = \frac{\frac{(2-r)r}{3-2r}}{\frac{1}{2-r}} = \frac{r}{3-2r}(r-2)^2$$

Let

$$A_p = -3R + (4+2R)r - 4r^2 + r^3$$

Then r is the first root given by Mathematica for the polynomial A and it is an increasing function of R. Hence

$$\begin{aligned}
r(R) &= \frac{4}{3} - \frac{\sqrt[3]{2}(-4+6R)}{3\sqrt[3]{-16+9R+3\sqrt{3}\sqrt{32R-61R^2+32R^3}}} \\
&\quad + \frac{\sqrt[3]{-16+9R+3\sqrt{3}\sqrt{32R-61R^2+32R^3}}}{3\sqrt[3]{2}}
\end{aligned}$$

Since, we assume that the population sex ratio r is male biased, at equilibrium $R \geq 1$.

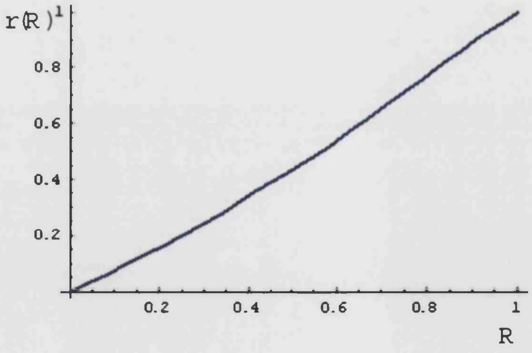


Figure C.3: Population sex ratio $r(R)$,
when $\lambda = 3$, $\tau = 2$ and $r < 1$.

The proportion of males of age 1 increases as the incoming population sex ratio R increases, while the opposite is true for the proportion of males of age 2, which is decreasing in R. At $R = 1$, the proportion os males of age 1 becomes maximum ($a_1 = 1$) while the proportion of males of age 2 becomes minimum ($a_2 = 0$).

Female distribution behaves similarly. It is interesting to note that for R close to 0, female population is almost equally divided between females of age 1 and 2, while in the male population males of age 1 are significantly more.

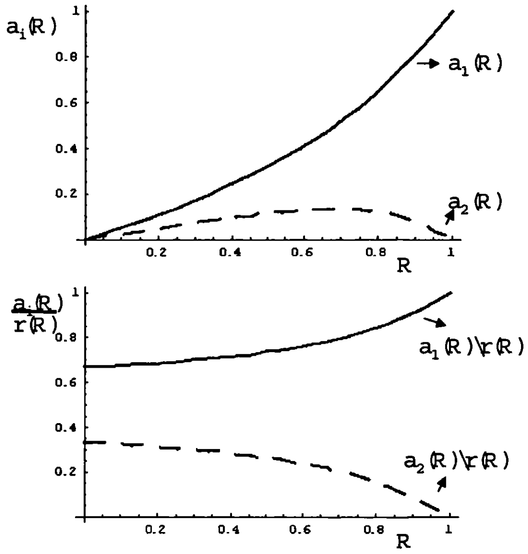


Figure $C.4$: Number a_i of males of type i in the male population (top picture) and male distribution (bottom picture) at equilibrium, when $\lambda = 3$, $\tau = 2$ and $r < 1$.

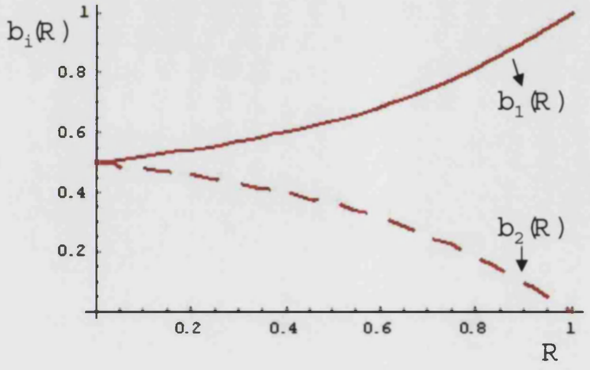


Figure $C.5$: Female distribution at equilibrium, when $\lambda = 3$, $\tau = 2$ and $r < 1$.

C.2.1 Extension of the Model $\lambda \geq 3$, $\tau = 2$ and $r < 1$

We have found that the equilibrium strategy pair when $\lambda = 3$ and $\tau = 2$ is $((2,1,1),(1,1))$. If we assume that males are fertile for $\lambda > 3$, then from observations (i) and observation (ii) we know that males of age λ and $\lambda - 1$ and female of age 1 and 2 are non choosy. From observation (iv) we know that strategies cannot be decreasing sequences. Hence any type lower than $\lambda - 2$ has to be at least as choosy as $\lambda - 2$. From $\lambda = 3$, $\tau = 2$ model, we know that male of age $\lambda - 2$ accepts only females of age 1. Hence any male younger than $\lambda - 2$ has to accept only females of age 1. So the unique equilibrium strategy pair has to be the $((2,2,2,...,2,1,1),(1,1))$.

The distribution of the population is such that

$$
\begin{aligned}
a_i(1 - b_1) &= a_{i+1} \text{ for } i \leq \lambda - 2 \\
a_{\lambda-1}(1 - b_1 - b_2) &= a_\lambda \Rightarrow a_\lambda = 0 \\
b_1(1 - r) &= b_2 \\
& \text{Hence} \\
a_1 &= \frac{(-1+r)r}{(-2+r)\left(1 - 2\left(\frac{1-r}{2-r}\right)^{\lambda} + \left(-1 + \left(\frac{1-r}{2-r}\right)^{\lambda}\right)r\right)} \\
a_i &= a_1\left(\frac{1-r}{2-r}\right)^{j-1} \quad \text{for } i < \lambda \text{ and } a_\lambda = 0 \\
b_1 &= \frac{1}{2-r} \text{ and } b_2 = \frac{1-r}{2-r}
\end{aligned}
$$

C.3 $\lambda = 3$, $\tau = 2$ and $r \geq 1$

All females and males of ages 2 and 3 are non choosy (from observations (i) and (ii)). We want to find conditions so that males of age 1 are choosy.

The expected utilities of a male of age 3 and age 2 are

$$U_3 = \frac{1}{r}(b_1 + b_2)\,1 = \frac{1}{r}$$
$$U_2 = \frac{1}{r}b_1 2 + \frac{1}{r}b_2 1 + \frac{r-1}{r}U_3 = \frac{1}{r}b_1 2 + \frac{1}{r}b_2 1 + \frac{r-1}{r}\frac{1}{r}$$

Male of age 1 rejects females of age 2 iff $U_2 > 1$. Hence

$$U_2 > 1 \Leftrightarrow 1 < r \leq 2.61803 \text{ and } 0 < b_1 \leq \frac{1-2r+r^2}{r}$$

- We will check if there is distribution where $0 < b_1 \leq \frac{1-2r+r^2}{r}$ that supports the strategy profile $((2,1,1),(1,1))$ when $1 < r \leq 2.61803$.

For strategy pair $((2,1,1),(1,1))$ to be an equilibrium, the distribution has to be such that

$$\begin{aligned}
a_1(1 - \frac{1}{r}b_1) &= a_2 \\
a_2(1 - \frac{1}{r}b_1 - \frac{1}{r}b_2) &= a_3 \Rightarrow a_2\frac{r-1}{r} = a_3 \\
b_1(1 - \frac{a_1}{r} - \frac{a_2}{r} - \frac{a_3}{r}) &= b_2 \Rightarrow b_2 = 0 \text{ thus } b_1 = 1 \\
& \text{Hence} \\
a_1 &= \frac{r^3}{1-3r+3r^2} \\
a_2 &= a_1(1 - \frac{1}{r}) \\
a_3 &= a_1\left(\frac{r-1}{r}\right)^2
\end{aligned}$$

Since $b_1 = 1 > \frac{1-2r+r^2}{r}$, $((2,1,1),(1,1))$ is an equilibrium when $r \leq 2.61803$.

- We will also check whether there is a distribution that supports the strategy profile $((1,1,1),(1,1))$ for $r > 2.61803$.

For the trivial strategy pair $((1,1,1),(1,1))$ to be an equilibrium, the distribution has be such that

$$
\begin{aligned}
a_1(1-\frac{1}{r}b_1-\frac{1}{r}b_2) &= a_2 \Rightarrow a_1\frac{r-1}{r}=a_2 \\
a_2(1-\frac{1}{r}b_1-\frac{1}{r}b_2) &= a_3 \Rightarrow a_2\frac{r-1}{r}=a_3 \\
b_1(1-\frac{a_1}{r}-\frac{a_2}{r}-\frac{a_3}{r}) &= b_2 \Rightarrow b_2=0 \text{ thus } b_1=1 \\
& \text{Hence} \\
a_1 &= \frac{r^3}{1-3r+3r^2} \\
a_2 &= a_1(1-\frac{1}{r}) \\
a_3 &= a_1\left(\frac{r-1}{r}\right)^2
\end{aligned}
$$

Since $b_1=1>\frac{1-r-r^2+r^3}{r^3}$, the pair of strategies $((1,1,1),(1,1))$ is an equilibrium for all $r>2.61803$.

Since the distribution is of the same form at both equilibria, the incoming population sex ratio at both equilibria is

$$R=\frac{r^3}{1-3r+3r^2}$$

Let

$$B=-R+3Rr-3Rr^2+r^3$$

Thus r is the first root given by Mathematica for the polynomial B and it is an increasing function of R. Hence

$$
\begin{aligned}
r(R) &= \frac{\sqrt[3]{2}(9R-R^2)}{9\sqrt[3]{R-3R^2+2R^3+\sqrt{R^2-2R^3+R^4}}} \\
&\quad +\frac{\sqrt[3]{R-3R^2+2R^3+\sqrt{R^2-2R^3+R^4}}}{\sqrt[3]{2}}
\end{aligned}
$$

Since we assume that the population is male biased, $R\geq 1$.

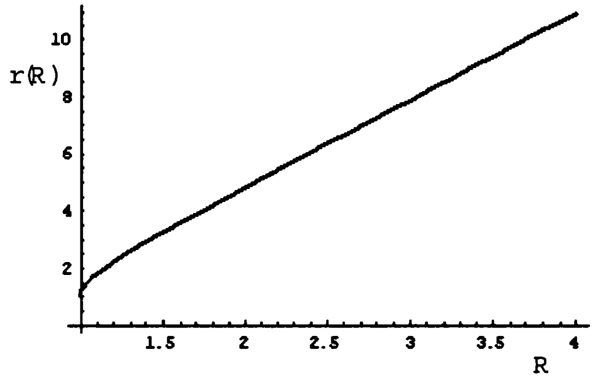


Figure $C.6$: Population sex ratio $r(R)$, when $\lambda = 3$, $\tau = 2$ and $r \geq 1$.

The following picture reveals the behaviour of the number a_i of males of age i in terms or R . In each period, independently of R, all females are of age 1 at both equilibria, as expected from observation (vi). Therefore, both strategy pairs $((2,1,1),(1,1))$ and $((1,1,1),(1,1))$ result in the same couple distribution.

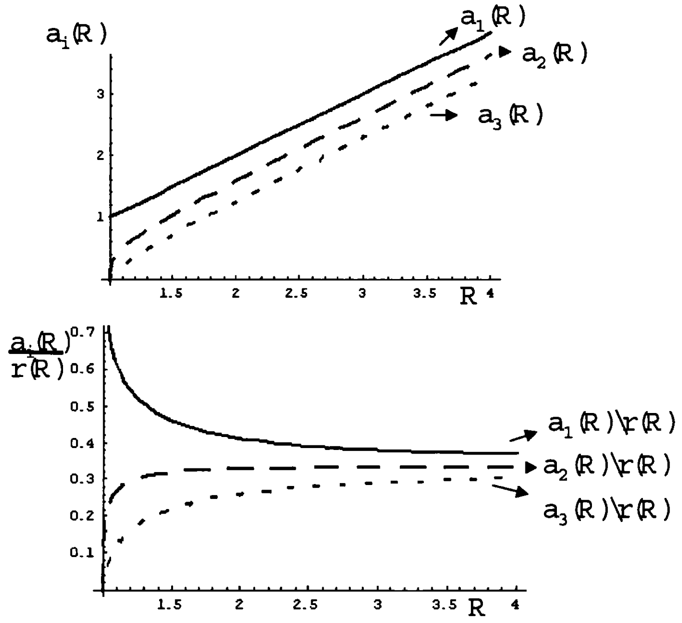


Figure $C.7$: Number a_i of males of type i in the male population (top picture) and male distribution (bottom picture) at equilibrium, when $\lambda = 3$, $\tau = 2$ and $r \geq 1$.

Hence the equilibria are the following

$$\begin{array}{lll} ((2,1,1),(1,1)) & \text{for } 1 \leq R \leq 1.30902 & \text{hence for } r \leq 2.61803 \\ ((1,1,1),(1,1)) & \text{for } R > 1.30902 & \text{hence for } r > 2.61803 \end{array}$$

and the distribution has the following form at both equilibria

$$\begin{aligned} a_1 &= \frac{r^3(R)}{1-3r(R)+3r^2(R)} \\ a_2 &= a_1(1-\frac{1}{r(R)}) \\ a_3 &= a_1\left(\frac{r(R)-1}{r(R)}\right)^2 \\ b_1 &= 1 \end{aligned}$$

C.3.1 Extension of the Model $\lambda \geq 3$, $\tau = 2$ and $r \geq 1$

We have found the equilibrium strategies when $\lambda = 3$ and $\tau = 2$. If we assume that males are fertile for $\lambda > 3$, then obviously males of ages λ and $\lambda - 1$ and all females are universal acceptors at equilibrium (from observation (*i*) and observation (*ii*)).

For $r \leq 2.61803$, for any $\lambda \geq 3$, there is an equilibrium strategy pair of the form $((2,2...,2,1,1),(1,1))$ (from observations (*iv*) and (*v*)), while the distribution of the population is such that

$$\begin{aligned} b_1 &= 1 \\ a_1 &= \frac{1}{1-(\frac{r-1}{r})^\lambda} \\ a_i &= a_1\left(1-\frac{1}{r}\right) \text{ for } i > 1 \end{aligned} \tag{81}$$

(from observation (*vi*)).

For $r > 2.61803$, knowing that females are universal acceptors, the distribution at equilibrium has to be of the same form (81)(from observation (*vi*)).

If $r > 2.61803$, when a type $x < \lambda - 3$ accepts female type 2, then all types higher than x must be universal acceptors as well (from observation (*iv*)). In order to check whether a strategy is an equilibrium, we just have to find r such that males one period older than the maximum male type that is choosy, have an expected utility that is higher than 1. A male type $x < \lambda - 3$ expects that if he enters in the next period unmated, he will receive

$$U_{x+1} = \frac{2}{r}\left(\sum_{i=0}^{x-2}\left(\frac{r-1}{r}\right)^i\right) + \frac{1}{r}\left(\frac{r-1}{r}\right)^{x-1}$$

A strategy

$$((\underbrace{2,2,2,2,.........,2,2,2}_{\text{a number } x \text{ of 2's}},\underbrace{1,1,1,...,1,1,1}_{\text{a number } \lambda - x \text{ of 1's}}),(1,1))$$

is an equilibrium if

$$U_{x+1} > 1 \text{ and } U_{x+2} \leq 1$$

With the help of Mathematica, we found the equilibrium strategy for λ large for different r. Hence for λ large,

- $((2,2,.........,2,1,1,1),(1,1))$ is an equilibrium for $2.618 \leq r < 4.08$
- $((2,2,.......,2,1,1,1,1),(1,1))$ is an equilibrium for $4.08 \leq r < 5.53$
- $((2,2,.....,2,1,1,1,1,1),(1,1))$ is an equilibrium for $5.53 \leq r < 6.977$
- $((2,2,...,2,1,1,1,1,1,1),(1,1))$ is an equilibrium for $6.977 \leq r < 8.423$

  etc

C.4 $\lambda = \tau = 3$ and $r \geq 1$

Both males and females of ages 2 and 3 are universal acceptors (from observations (*i*) and (*ii*)). We want to check whether both females and males of age 1 are choosy.

The expected utility of a female of age 2 is

$$V_2 = \frac{a_1}{r}2 + \frac{a_2}{r}2 + \frac{a_3}{r}1 > 1$$

Therefore females of age 1 never accepts males of age 3.

The utility that a male of age i expects to receive if he does not find a partner and enters in the next period $i+1$ unmated is the following

$$U_3 = \frac{1}{r}(b_2 + b_3)1 < \frac{1}{r}$$
$$U_2 = \frac{b_1}{r}2 + \frac{b_2}{r}2 + \frac{b_3}{r}1 + \frac{r-1}{r}\frac{1}{r}U_3 \Leftrightarrow$$
$$U_2 = \frac{b_1}{r}2 + \frac{b_2}{r}2 + \frac{b_3}{r}1 + \frac{r-1}{r}\frac{1}{r}(b_2 + b_3) \leq 2$$

A male of age 2 does not accept female of age 3 if $U_2 > 1$.

$$U_2 > 1 \Leftrightarrow \begin{cases} 1 \leq r \leq 2 \\ \text{or} \\ 2 < r \leq 2.61803 \quad \text{and } \frac{b_1+b_2}{r} < \frac{-1+3r-r^2}{r-1} \end{cases}$$

We will check if there is a distribution that supports strategy profile $((2,1,1),(2,1,1))$ when $1 \leq r \leq 2.61803$.

For the strategy $((2,1,1),(2,1,1))$ to be an equilibrium, the distribution has to be such that

$$\begin{aligned}
a_1(1-\frac{1}{r}(b_1+b_2)) &= a_2 \Rightarrow \frac{r-1+b_3}{r}a_1 = a_2 \\
a_2(1-\frac{1}{r}(b_1+b_2+b_3)) &= a_3 \Rightarrow \frac{r-1}{r}a_2 = a_3 \\
b_1(1-\frac{a_1}{r}-\frac{a_2}{r}) &= b_2 \\
b_2(1-\frac{a_1}{r}-\frac{a_2}{r}-\frac{a_3}{r}) &= b_3 \Rightarrow b_3 = 0 \\
\text{Hence } a_1 &= \frac{r^3}{1-3r+3r^2} \\
a_2 &= \frac{(-1+r)r^2}{1-3r+3r^2} \\
a_3 &= \frac{(-1+r)^2 r}{1-3r+3r^2} \\
b_1 &= \frac{1-3r+3r^2}{2-5r+4r^2} \text{ and } b_2 = \frac{(-1+r)^2}{2-5r+4r^2}
\end{aligned}$$

Since $\frac{b_1+b_2}{r} = 1 \Rightarrow \frac{b_1+b_2}{r} > \frac{-1+3r-r^2}{r-1}$, the strategy pair $((2,1,1),(2,1,1))$ is an equilibrium for $1 \leq r \leq 2$.

We will check if strategy pair $((1,1,1),(2,1,1))$ is an equilibrium for $r > 2$.

For the strategy profile $((1,1,1),(2,1,1))$ to be an equilibrium, the distribution has to be such that

$$
\begin{aligned}
a_1(1-\frac{1}{r}(b_1+b_2+b_3)) &= a_2 \Rightarrow \frac{r-1}{r}a_1 = a_2 \\
a_1(1-\frac{1}{r}(b_1+b_2+b_3)) &= a_3 \Rightarrow \frac{r-1}{r}a_2 = a_3 \\
b_1(1-\frac{a_1}{r}-\frac{a_2}{r}) &= b_2 \Rightarrow \frac{a_3}{r}b_1 = b_2 \\
b_2(1-\frac{a_1}{r}-\frac{a_2}{r}-\frac{a_3}{r}) &= b_3 \Rightarrow b_3 = 0 \\
\text{Hence } a_1 &= \frac{r^3}{1-3r+3r^2} \\
a_2 &= \frac{(-1+r)r}{1-3r+3r^2} \\
a_3 &= \frac{(-1+r)^2}{1-3r+3r^2} \\
b_1 &= \frac{1-3r+3r^2}{2-5r+4r^2} \text{ and } b_2 = \frac{(-1+r)^2}{2-5r+4r^2}
\end{aligned}
$$

Since $\frac{b_1+b_2}{r} = 1 \Rightarrow \frac{b_1+b_2}{r} > \frac{-1+3r-r^2}{r-1}$, the strategy profile $((1,1,1),(2,1,1))$ is indeed an equilibrium for $r > 2$.

At both equilibria,

$$
R = \frac{\frac{r^3}{1-3r+3r^2}}{\frac{1-3r+3r^2}{2-5r+4r^2}} = \frac{2r^2-5r^3+4r^4}{15r^2-6r-18r^3+9r^4+1}
$$

Let

$$
C = -R + 6Rr - 15Rr^2 + 2r^3 + 18Rr^3 - 5r^4 - 9Rr^4 + 4r^5
$$

Then r is the third root given by Mathematica for the polynomial C when $1 \leq R \leq 1.21403$ and the first root given by Mathematica for the polynomial C otherwise; $r(R)$ is an increasing function.

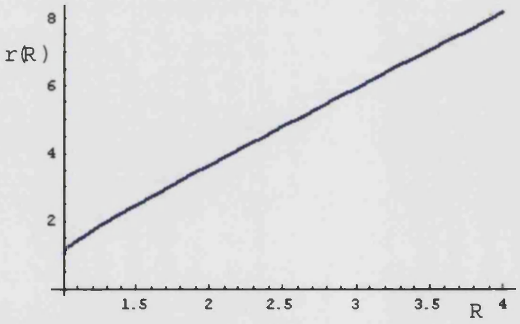


Figure C.8: Population sex ratio $r(R)$, when $\lambda = \tau = 3$ and $r \geq 1$.

Figures $C.9$ and $C.10$ describe the behaviour of the number a_i of males of type i and the number b_i of females of type i in terms of R. Figure $C.9$ is similar to Figures $C.2$ and $C.7$.

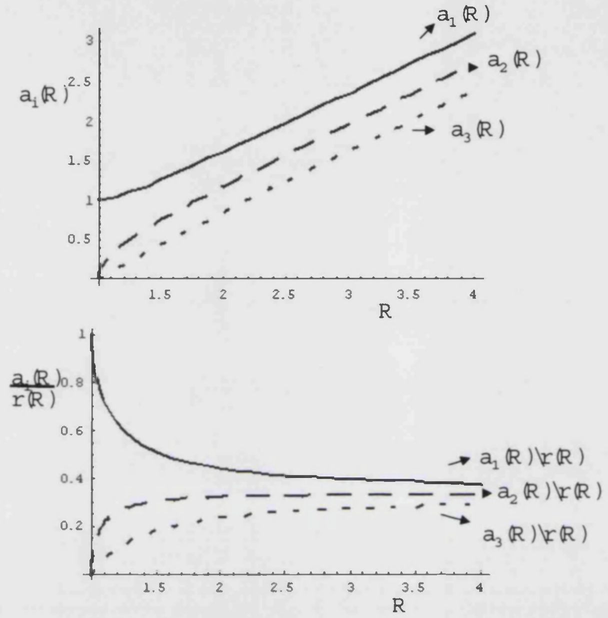


Figure $C.9$: Number a_i of males of type i in the male population (top picture) and male distribution (bottom picture) at equilibrium, when $\lambda = \tau = 3$ and $r \geq 1$.

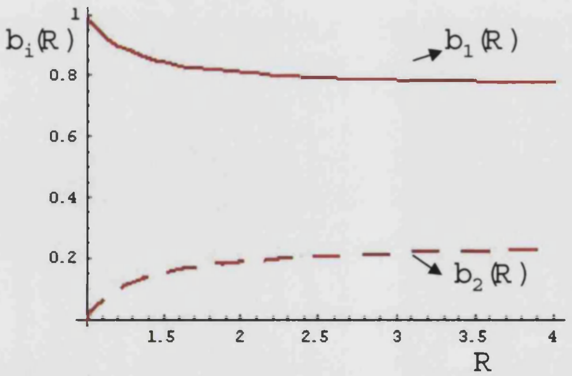


Figure $C.10$: Female distribution at equilibrium, when $\lambda = \tau = 3$ and $r \geq 1$.

Hence the equilibria are

$$\begin{array}{llll} ((2,1,1),(2,1,1)) & \text{for } 1 \leq R \leq 1.30612 & \text{hence for } 1 \leq r \leq 2 \\ ((1,1,1),(2,1,1)) & \text{for } R > 1.30612 & \text{hence for } r \geq 2 \end{array}$$

and the distribution at both equilibria is

$$\begin{aligned} a_1 &= \frac{r^3(R)}{1-3r(R)+3r^2(R)} \\ a_2 &= \frac{(-1+r(R))r(R)}{1-3r(R)+3r^2(R)} \\ a_3 &= \frac{(-1+r(R))^2}{1-3r(R)+3r^2(R)} \\ b_1 &= \frac{1-3r(R)+3r^2(R)}{2-5r(R)+4r^2(R)} \\ b_2 &= \frac{(-1+r(R))^2}{2-5r(R)+4r^2(R)} \end{aligned}$$

C.5 $\lambda = 4$, $\tau = 3$ and $r < 1$

The expected utility of a male of age i is

$$\begin{aligned} U_3 &= (b_1+b_2)\,2+b_3 > 1 \\ U_2 &= b_1 3+b_2 2+b_3 U_3 = b_1 3+b_2 2+b_3\left((b_1+b_2)\,2+b_3\right) > 1 \qquad (82) \end{aligned}$$

Males of age 2 do not accept females of age 3, expecting that if they enters in age 3 they will receive a utility higher than 1. Consequently, males of age 1 reject females of age 3 as well (from observation (*iv*)). If $U_2 > 2$, males of age 1 reject females of both age 3 and age 2, otherwise they reject only females of age 3. Since we know that at equilibrium males of age 2 are always choosy and as a result that males of age 1 have only two possible equilibrium strategies, there are only 4 potential strategy pairs left for us to check.

C.5.1 Assuming that Males of Age 1 Accept Females of Age 1 Only (2<U$_2$ $\leq$ 3)

If males of age 1 accepts only females of age 1, the expected utility of a female entering in period i of her fertile life is

$$
\begin{aligned}
V_3 &= a_3 + a_4 < 1 \\
V_2 &= (a_2 + a_3)\,2 + a_4 + (1 - r + a_1)\,V_3 \Leftrightarrow \qquad (83) \\
V_2 &= (a_2 + a_3)\,2 + a_4 + (1 - r + a_1)\,(a_3 + a_4) < 2
\end{aligned}
$$

If $V_2 > 1$, females of age 1 reject males of age 4; otherwise they are universal acceptors.

- If $V_2 > 1$ and females of age 1 reject males of age 4, the equilibrium strategy pair is of the form $((3, 2, 1, 1), (2, 1, 1))$ and the distribution that supports it is the following

$$
\begin{aligned}
a_1(1 - b_1) &= a_2 \\
a_2(1 - b_1 - b_2) &= a_3 \\
a_3(1 - (b_1 + b_2 + b_3)) &= a_4 \Leftrightarrow a_4 = 0 \\
b_1(1 - (a_1 + a_2 + a_3)) &= b_2 \Leftrightarrow b_1(1 - r) = b_2 \\
b_2(1 - (a_2 + a_3 + a_4)) &= b_3
\end{aligned}
$$

For $((2, 1, 1), (3, 2, 1, 1))$ to be an equilibrium under the support of the above distribution, it is necessary that $V_2 > 1$ and $2 < U_2 \leq 3$.

Given (83) and (82) then

$$
\begin{aligned}
V_2 &< 1 \text{ when } r \leq 1 \\
U_2 &> 2 \text{ when } r < 1
\end{aligned}
$$

Since $V_2 < 1$ for $r \leq 1$, the strategy pair $((2, 1, 1), (3, 2, 1, 1))$ is not an equilibrium.

- If $V_2 \leq 1$ and females of age 1 are universal acceptor, the equilibrium strategy pair is of the form $((3, 2, 1, 1), (1, 1, 1))$ and the distribution that supports it is the following

$$
\begin{aligned}
a_1(1 - b_1) &= a_2 \\
a_2(1 - b_1 - b_2) &= a_3 \\
a_3(1 - (b_1 + b_2 + b_3)) &= a_4 \Leftrightarrow a_4 = 0 \\
b_1(1 - (a_1 + a_2 + a_3 + a_4)) &= b_2 \Leftrightarrow b_1(1 - r) = b_2 \\
b_2(1 - (a_2 + a_3 + a_4)) &= b_3
\end{aligned}
$$

For $((1,1,1),(3,2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $U_2 \leq 1$ and $2 < V_2 \leq 3$.

Given (83) and (82) then

$$\begin{aligned} U_2 \leq 1 \text{ when } r \geq 1 \\ 2 < V_2 \leq 3 \text{ when } r \geq 1 \end{aligned}$$

Hence for $r \geq 1$ the strategy profile $((3,2,1,1),(1,1,1))$ is an equilibrium and the distribution at equilibrium is such that

a_1 is the first root given by Mathematica for the polynomial D, where

$$\tag{84}$$

$$\begin{aligned} D &= -9r + 18r^2 - 15r^3 + 6r^4 - r^5 + \left(\left(17 - 46r + 50r^2 - 24r^3 + 5r^4\right) R \right. \\ &\quad + (14 - 32r + 25r - 7r^3)R^2 + (3 - 6r + 3r^2)R^3 \qquad (85) \\ a_2 &= \frac{3a_1 + a_1^2 - 3r - 4a_1 r - a_1^2 r + 3r^2 + 2a_1 r^2 - r^3}{-4 - 2a_1 + 5r + 2a_1 r - 2r^2} \\ a_3 &= r - a_1 - a_2 \\ a_4 &= 0 \\ b_1 &= \frac{a_1 - a_2}{a_1} \\ b_2 &= 1 - b_1 - \frac{a_3}{a_2} \\ b_3 &= \frac{a_3}{a_2} \end{aligned}$$

C.5.2 Assuming that Males of Age 1 Reject Females of Age 3 Only $(1 < U_2 \leq 2)$

If males of age 1 reject females of type 3 (but accept the rest of the females), the expects utility of a female of age i is following

$$\begin{aligned} V_3 &= (a_3 + a_4) < 1 \\ V_2 &= (a_1 + a_2 + a_3)\, 2 + a_4 + (1 - r) V_3 \Leftrightarrow \\ V_2 &= (a_1 + a_2 + a_3)\, 2 + a_4 + (1 - r)\,(a_3 + a_4) \leq 2 \end{aligned} \tag{86}$$

If $V_2 > 1$, females of age 1 reject males of age type 4; otherwise they are a universal acceptors.

- If $V_2 > 1$ and females of age 1 reject males of age 4, the equilibrium strategy pair are of the form $((2,2,1,1),(2,1,1))$ and the distribution that supports it is the following

$$\begin{aligned}
a_1(1-(b_1+b_2)) &= a_2 \\
a_2(1-(b_1+b_2)) &= a_3 \\
a_3(1-(b_1+b_2+b_3)) &= a_4 \Leftrightarrow a_4 = 0 \\
b_1(1-(a_1+a_2+a_3)) &= b_2 \Leftrightarrow b_1(1-r) = b_2 \\
b_2(1-(a_1+a_2+a_3+a_4)) &= b_3 \Leftrightarrow b_2(1-r) = b_3
\end{aligned}$$

For $((2,2,1,1),(2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $1 < V_2 \leq 2$ and $1 < U_2 \leq 2$.

Given (82) and (86) then

$$\begin{aligned}
&V_2 > 1 \text{ when } 0.497781 < r \leq 1 \\
&U_2 > 2 \text{ when } r \leq 1
\end{aligned}$$

Since for all $r \geq 1$, $U_2 > 2$, the strategy profile $((2,2,1,1),(2,1,1))$ cannot be an equilibrium.

- If $V_2 \leq 1$ and females of age 1 are non choosy, the equilibrium strategy pair is of the form $((2,2,1,1),(1,1,1))$ and the distribution that supports it is the following

$$\begin{aligned}
a_1(1-(b_1+b_2)) &= a_2 \\
a_2(1-(b_1+b_2)) &= a_3 \\
a_3(1-(b_1+b_2+b_3)) &= a_4 \Leftrightarrow a_4 = 0 \\
b_1(1-(a_1+a_2+a_3+a_4)) &= b_2 \Leftrightarrow b_1(1-r) = b_2 \\
b_2(1-(a_1+a_2+a_3+a_4)) &= b_3 \Leftrightarrow b_2(1-r) = b_3
\end{aligned}$$

For $((2,2,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$ and $1 < U_2 \leq 2$.

Given (82) and (86) then

$$V_2 = (a_1 + a_2 + a_3)\, 2 + a_4 + (1 - r)\,(a_3 + a_4) \leq 1 \text{ when } r \geq 0.497781$$
$$U_2 = b_1 3 + b_2 2 + b_3\,((b_1 + b_2)\, 2 + b_3) > 2 \text{ when } r \leq 1$$

Since $U_2 > 2$ for $r \leq 1$, the strategy profile $((2,2,1,1),(1,1,1))$ is not an equilibrium.

Hence strategy profile $((3,2,1,1),(1,1,1))$ is the unique equilibrium for $r \leq 1$ and the equilibrium distribution is the one described in (84).

Let

$$\begin{aligned} E &= 4R + 32R^2 + 9R^3 + (-54 - 151R - 92R^2 - 18R^3)r \\ &\quad +(189 + 278R + 107R^2 + 9R^3)r^2 + (-297 - 273R - 62R^2)r^3 \\ &\quad +(270 + 158R + 17R^2)r^4 + (-153 - 55R - 2R^2)r^5 + (54 + 11R)r^6 \\ &\quad +(-11 - R)r^7 + r^8 \end{aligned}$$

Then r is the first root given by Mathematica for the polynomial E; $r(R)$ is an increasing function.

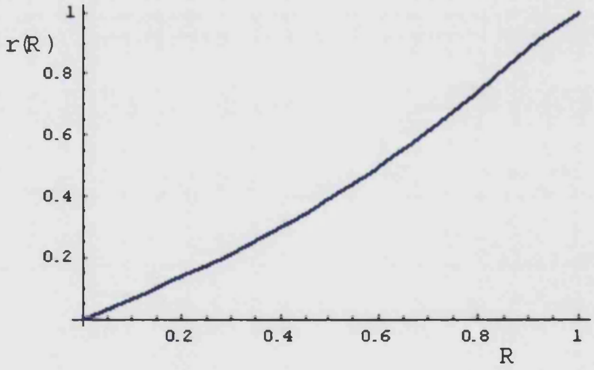


Figure C.11: Population sex ratio $r(R)$, when $\lambda = 4$, $\tau = 3$ and $r < 1$.

The behaviour of a_i and b_i in terms of R is similar to the ones described in Figures C.4 and C.5. It is worth noting that the number a_2 of males of type 2 increases for $0 \leq R \leq 0.7658$, but decreases for $R > 0.7658$, a fact that is connected with the very high rate of increase of a_1 for R very close to 1.

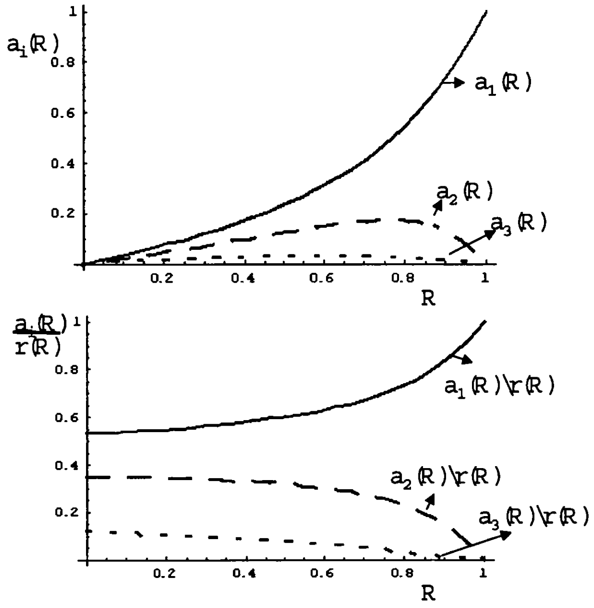


Figure C.12: Number a_i of males of type i in the male population (top picture) and male distribution (bottom picture) when $\lambda = 4$, $\tau = 3$ and $r < 1$.

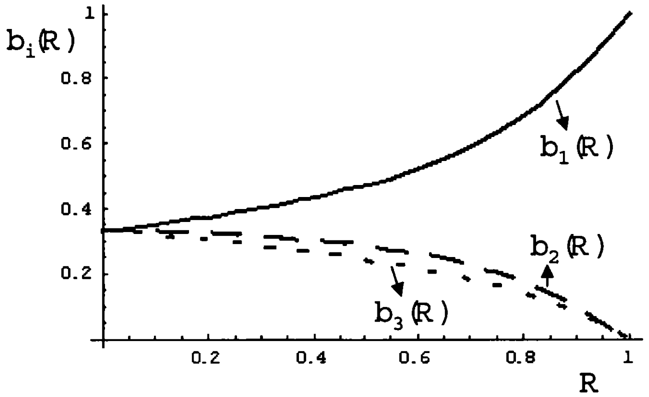


Figure C.13: Female distribution when $\lambda = 4$, $\tau = 3$ and $r < 1$.

C.6 $\lambda = 4$, $\tau = 3$ and $r \geq 1$

The expected utility of a female in age i is

$$V_2 \leq \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} < 2$$

If $V_2 > 1$, then females of age 1 do not accept males of age 4; otherwise they are universal acceptors.

C.6.1 Assuming that Females of Age 1 Reject Males of Age 4 ($V_2 > 1$)

If females of age 1 do not accept male of age 4 then the expected utility of females of age 2 is the following

$$V_2 \leq \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} V_3 \tag{87}$$

where

$$V_3 \leq \frac{a_1 + a_2 + a_3 + a_4}{r} = 1$$

The utility that a male of age i expects is the following

$$\begin{aligned} U_4 &= \frac{1}{r}\left(b_2 + b_3\right) 1 < \frac{1}{r} \\ U_3 &= \frac{1}{r}\left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} U_4 \Leftrightarrow \\ U_3 &= \frac{1}{r}\left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} \frac{b_2 + b_3}{r} \Leftrightarrow \end{aligned} \tag{88}$$

If $U_3 > 1$, then males of age 2 reject females of age 3, otherwise they are a universal acceptors.

C.6.1.1 *Assuming that Males of Age 2 Reject Females of Age 3* ($1 < U_3$)

The expected utility of a male of age i is the following

$$\begin{aligned} U_2 &= \frac{1}{r} b_1 3 + \frac{1}{r} b_2 2 + \frac{r - 1 + b3}{r} U_3 > 1 \\ U_2 &= \frac{1}{r} b_1 3 + \frac{1}{r} b_2 2 + \frac{r - 1 + b3}{r} \left(\frac{1}{r}\left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} \frac{b_2 + b_3}{r} \right) \end{aligned} \tag{89}$$

If $U_2 > 2$ then males of age 1 accept only female of age 1, otherwise they reject females of age 3 and accept all younger females. We are going to check whether males of age 1 can use either of these strategies at equilibrium.

- When males of age 1 accept only females of age 1, the equilibrium strategy pair is of the form $((3,2,1,1),(2,1,1))$ and the distribution that supports it is such that

$$b_1\left(1-\frac{a_1+a_2+a_3}{r}\right)=b_2$$
$$b_2\left(1-\frac{a_2+a_3+a_4}{r}\right)=b_3$$
$$a_1\left(1-\frac{b_1}{r}\right)=a_2$$
$$a_2\left(1-\frac{1}{r}(b_1+b_2)\right)=a_3$$
$$a_3\left(1-\frac{1}{r}(b_1+b_2+b_3)\right)=a_4 \Leftrightarrow a_3\frac{r-1}{r}=a_4$$

For $((3,2,1,1),(2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2>1$, $U_3>1$ and $U_2>2$.

Given (87), (88), (89), then

$$V_2=\frac{a_2+a_3}{r}2+\frac{a_4}{r}+\frac{a_1}{r}\frac{a_3+a_4}{r}>1 \text{ when } r>1.93489$$
$$U_3>1 \text{ when } 1<r<2.01676$$
$$U_2>2 \text{ when } 1<r<1.92264$$

There is no r where all necessary conditions are true simultaneously, hence the strategy pair $((3,2,1,1),(2,1,1))$ is not an equilibrium.

- When males of age 1 reject females of age 3 and accept all younger females, the equilibrium strategy pair is of the form $((2,2,1,1),(2,1,1))$ and the distribution that supports it is such that

$$b_1\left(1-\frac{a_1+a_2+a_3}{r}\right)=b_2$$
$$b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right)=b_3 \Leftrightarrow b_3=0$$
$$a_1\left(1-\frac{b_1+b_2}{r}\right)=a_2$$
$$a_2\left(1-\frac{1}{r}(b_1+b_2)\right)=a_3$$
$$a_3\left(1-\frac{1}{r}(b_1+b_2+b_3)\right)=a_4 \Leftrightarrow a_3=\frac{r-1}{r}a_4$$

For $((2,2,1,1),(2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 > 1$, $U_3 > 1$ and $1 < U_2 \leq 2$.

Given (87), (88), (89), then

$$\begin{aligned}
V_2 &= \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} > 1 \text{ when } r > 1 \\
U_3 &> 1 \text{ when } 1 < r < 2.03285 \\
1 &< U_2 \leq 2 \text{ when } 1.96931 \leq r < 4.49959
\end{aligned}$$

Hence for $1.96931 \leq r < 2.03285$ the strategy profile $((2,2,1,1),(2,1,1))$ is an equilibrium with a support of a distribution such that

$$\begin{aligned}
b_1 &= \frac{-1 + 4r - 6r^2 + 4r^3}{-2 + 7r - 9r^2 + 5r^3} \\
b_2 &= 1 - b_1 = 1 - \frac{-1 + 4r - 6r^2 + 4r^3}{-2 + 7r - 9r^2 + 5r^3} \\
b_3 &= 0 \\
a_1 &= \frac{b_2 r^4}{-b_1 + 3b_1 r - 3b_1 r^2 + b_1 r^3} \\
a_2 &= \frac{-a_1 + a_1 r}{r} \\
a_3 &= \frac{-a_2 + a_2 r}{r} \\
a_4 &= \frac{-a_3 + a_3 r}{r}
\end{aligned} \tag{90}$$

Hence

$$\begin{aligned}
R &= \frac{\frac{b_2 r^4}{-b_1 + 3b_1 r - 3b_1 r^2 + b_1 r^3}}{\frac{-1+4r-6r^2+4r^3}{-2+7r-9r^2+5r^3}} \\
&= \frac{5r^7 - 9r^6 + 7r^5 - 2r^4}{\left(4r^3 - 6r^2 + 4r - 1\right)^2}
\end{aligned}$$

Let

$$F = -R + 8Rr - 28Rr^2 + 56Rr^3 + (-2 - 68R)r^4 + (7 + 48R)r^5 + (-9 - 16R)r^6 + 5r^7 \tag{91}$$

Then r is the first root given by Mathematica for the polynomial F and it is an increasing function of R.

C.6.1.2 *Assuming that Males of Age 2 Accept Females of Age 3 ($U_3 < 1$)*

The expected utility of a male of age i is

$$U_2 = \frac{1}{r}b_1 3 + \frac{1}{r}b_2 2 + \frac{b_3}{r}1 + \frac{r-1}{r}U_3 \Leftrightarrow$$
$$U_2 = \frac{1}{r}b_1 3 + \frac{1}{r}b_2 2 + \frac{1}{r}b_3 1 + \frac{r-1}{r}\left(\frac{1}{r}\left(b_1+b_2\right)2 + \frac{1}{r}b_3 + \frac{r-1}{r}\frac{b_2+b_3}{r}\right) \quad (92)$$

If $U_2 < 1$ males of age 1 are universal acceptor; if $1 < U_2 \leq 2$ they reject females of type 3 and accept all younger females; otherwise they accept only female of age 1. We will check whether it is possible for males of age 1 to use any of these strategies at equilibrium.

- When males of age 1 accept only females of age 1, the equilibrium strategy pair is of the form $((3,1,1,1),(2,1,1))$ and the distribution that supports it is such that

$$\begin{aligned}
b_1\left(1 - \frac{a_1+a_2+a_3}{r}\right) &= b_2 \\
b_2\left(1 - \frac{a_2+a_3+a_4}{r}\right) &= b_3 \\
a_1\left(1 - \frac{1}{r}b_1\right) &= a_2 \\
a_2\left(1 - \frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3 \\
a_3\left(1 - \frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}$$

For $((3,1,1,1),(2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 > 1$, $U_3 \leq 1$ and $U_2 > 2$.

Given (87), (88), (92), then

$$\begin{aligned}
&V_2 > 1 \text{ when } r > 1.64991 \\
&U_3 \leq 1 \text{ when } r \geq 2.01622 \\
&U_2 > 2 \text{ when } 1 < r < 1.92324
\end{aligned}$$

There is no r where all necessary conditions are true simultaneously, hence the strategy pair $((3,1,1,1),(2,1,1))$ is not an equilibrium.

- When males of age 1 accept only females of ages 1 and 2, the equilibrium strategy pair is of the form $((2,1,1,1),(2,1,1))$ and the distribution that supports it is such that

$$
\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3}{r}\right) &= b_2 \\
b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0 \\
a_1\left(1-\frac{1}{r}(b_1+b_2)\right) &= a_2 \\
a_2\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3 \\
a_3\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}
$$

For $((2,1,1,1),(2,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 > 1$, $U_3 \leq 1$ and $1 < U_2 \leq 2$.

Given (87), (88), (92), then

$$
\begin{aligned}
&V_2 = \frac{a_1+a_2+a_3}{r}2 + \frac{a_4}{r} > 1 \text{ when } r > 1 \\
&U_3 \leq 1 \text{ when } r \geq 2.03285 \\
&\quad 2 \geq U_2 > 1 \text{ when } 1.96931 \leq r < 4.49959
\end{aligned}
$$

Hence for $2.03285 \leq r < 4.49959$ the strategy profile $((2,1,1,1),(2,1,1))$ is an equilibrium with the support of a distribution as the one described in (90).

- When males of age 1 are not choosy, the equilibrium strategy pair is of the form $((1,1,1,1),(2,1,1))$ and the distribution that supports it is such that

$$
\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3}{r}\right) &= b_2 \\
b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0 \\
a_1\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_2 \Leftrightarrow a_1\frac{r-1}{r} = a_2 \\
a_2\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3 \\
a_3\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}
$$

For $((2,1,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 > 1$, $U_3 \leq 1$ and $U_2 \leq 1$.

Given (87), (88), (92), then

$$
\begin{aligned}
&V_2 = \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} > 1 \text{ when } r > 1 \\
&U_3 \leq 1 \text{ when } r \geq 2.03285 \\
&U_2 \leq 1 \text{ when } r \geq 4.49959
\end{aligned}
$$

Hence for $r \geq 4.49959$ the strategy profile $((2,1,1,1),(2,1,1))$ is an equilibrium with the support of a distribution as the one described in (90).

C.6.2 Assuming that Females of Age 1 Accept Males of Age 4 ($\mathbf{V_2 \leq 1}$)

The expected utility of a female of age 2 is at most 1. We know that

$$
V_2 \leq \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} 1 \tag{93}
$$

The expected utility of a male of age i is the following

$$
\begin{aligned}
U_4 &= \frac{1}{r} \left(b_1 + b_2 + b_3\right) 1 = \frac{1}{r} \\
U_3 &= \frac{1}{r} \left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} U_4 \Leftrightarrow \\
U_3 &= \frac{1}{r} \left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} \frac{1}{r}
\end{aligned} \tag{94}
$$

If $1 < U_3$ then males of age 2 reject females of age 3; otherwise they are universal acceptors.

C.6.2.1 *Assuming that Males of Age 2 Reject Females of Age 3* $(1 < U_3)$

If males of age 2 reject females of age 3, their expected utility is the following

$$
\begin{aligned}
U_2 &= \frac{1}{r} b_1 3 + \frac{1}{r} b_2 2 + \frac{r-1+b_3}{r} U_3 > 1 \\
&= \frac{1}{r} b_1 3 + \frac{1}{r} b_2 2 + \frac{r-1+b_3}{r} \left(\frac{1}{r} \left(b_1 + b_2\right) 2 + \frac{1}{r} b_3 + \frac{r-1}{r} \frac{1}{r} \right)
\end{aligned} \tag{95}
$$

If $2 < U_2 \leq 3$ then males of age 1 do not accept females of age 2; otherwise they only reject females of age 3. We are going to check whether males of age 1 can use either of these strategies at equilibrium.

- When males of age 1 accept only females of age 1, the equilibrium strategy pair is of the form $((3,2,1,1),(1,1,1))$ and the distribution that supports is such that

$$\begin{aligned} b_1\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_2 \Leftrightarrow b_2 = 0 \\ b_2\left(1-\frac{a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0 \\ a_1\left(1-\frac{b_1}{r}\right) &= a_2 \\ a_2\left(1-\frac{1}{r}(b_1+b_2)\right) &= a_3 \\ a_3\left(1-\frac{1}{r}(b_1+b_2+b_3)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4 \end{aligned}$$

For $((3,2,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$, $1 < U_3 \leq 2$ and $2 < U_2 \leq 3$.

Given (93), (94), (95), then

$$V_2 = \frac{a_2+a_3}{r}2 + \frac{a_4}{r} + \frac{a_1}{r}\frac{a_3+a_4}{r} \leq 1 \text{ when } 1 \leq r \leq 2.0781$$

$$1 < U_3 \leq 2 \text{ when } 1 \leq r < 2.61803$$

$$2 < U_2 \leq 3 \text{ when } \text{ when } 1 \leq r < 2.19149$$

Hence for $1 \leq r \leq 2.0781$ the strategy profile $((3,2,1,1),(1,1,1))$ is an equilibrium with a support of a distribution such that

$$\begin{aligned} b_1 &= 1 \\ b_2 &= 0 \\ b_3 &= 0 \\ a_1 &= \frac{r^4}{-1+4r-6r^2+4r^3} \\ a_2 &= \frac{-a_1+a_1r}{r} \\ a_3 &= \frac{-a_2+a_2r}{r} \\ a_4 &= \frac{-a_3+a_3r}{r} \end{aligned} \tag{96}$$

Hence

$$R = \frac{r^4}{-1+4r-6r^2+4r^3} \Leftrightarrow r = R-\sqrt{R+R^2}+\frac{1}{2}\sqrt{-8R+8R^2-\frac{32R-96R^2+64R^3}{8\sqrt{-R+R^2}}}$$

Thus $r(R)$ is increasing in R and $R \geq 1$ so that $r \geq 1$.

- When males of age 1 reject females of age 3 and accept all younger females, the equilibrium strategy pair is of the form $((2,2,1,1),(1,1,1))$ and the distribution that supports it is such that

$$
\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_2 \Leftrightarrow b_2 = 0 \\
b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0 \\
a_1\left(1-\frac{b_1+b_2}{r}\right) &= a_2 \\
a_2\left(1-\frac{1}{r}\left(b_1+b_2\right)\right) &= a_3 \\
a_3\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}
$$

For $((2,2,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$, $1 < U_3 \leq 2$ and $1 < U_2 \leq 2$.

Given (93), (94), (95), then

$$
\begin{aligned}
V_2 &= \frac{a_1+a_2+a_3}{r}2+\frac{a_4}{r} > 1 \text{ when } r \geq 1 \\
U_3 &> 1 \text{ when } r > 2.618 \\
1 &< U_2 \leq 2 \text{ when } 2.19149 \leq r < 5.27804
\end{aligned}
$$

Since $V_2 > 1$ for all $r \geq 1$, the strategy pair $((2,2,1,1),(1,1,1))$ is not an equilibrium.

C.6.2.2 *Assuming that Males of Age 2 Accept Females of Age 3 ($U_3 < 1$)*

If males of age 2 are universal acceptors, their expected utility (taking into account (94)) is the following

$$
\begin{aligned}
U_2 &= \frac{1}{r}b_1 3+\frac{1}{r}b_2 2+\frac{1}{r}b_3+\frac{r-1}{r}U_3 \\
&= \frac{1}{r}b_1 3+\frac{1}{r}b_2 2+\frac{1}{r}b_3+\frac{r-1}{r}\left(\frac{1}{r}\left(b_1+b_2\right)2+\frac{1}{r}b_3+\frac{r-1}{r}\frac{1}{r}\right) \qquad (97)
\end{aligned}
$$

If $U_2 \leq 1$ then males of age 1 are universal acceptors; if $1 < U_2 \leq 2$ then they reject females of age 3 and accept all younger females; otherwise they accept only female of age 1. We are going to check whether males of age 1 can use either of these strategies at equilibrium.

- When males of age 1 accept only females of age 1, the equilibrium strategy pair is of the form $((3,1,1,1),(1,1,1))$ and the distribution that supports it is such that

$$\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_2 \Leftrightarrow b_2 = 0\\
b_2\left(1-\frac{a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0\\
a_1\left(1-\frac{1}{r}b_1\right) &= a_2\\
a_2\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3\\
a_3\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}$$

For $((3,1,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$, $U_3 \leq 1$ and $2 < U_2 \leq 3$

Given (93), (94), (97), then

$$V_2 = \frac{a_2+a_3}{r}2 + \frac{a_4}{r} + \frac{a_1}{r}\frac{a_2+a_3+a_4}{r} \leq 1 \text{ false when } r \geq 1$$

$U_3 \leq 1$ when $r \geq 2.618$

$2 < U_2 \leq 3$ when $1 < r < 2.19149$

Since $V_2 > 1$ for $r \geq 1$, the strategy pair $((3,1,1,1),(1,1,1))$ is not an equilibrium.

- If males of age 1 reject females of age 3 and accept younger females, the equilibrium strategy pair is of the form $((2,1,1,1),(1,1,1))$ and the distribution that supports it is such that

$$
\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_2 \Leftrightarrow b_2 = 0\\
b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0\\
a_1\left(1-\frac{1}{r}\left(b_1+b_2\right)\right) &= a_2\\
a_2\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3\\
a_3\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}
$$

For $((2,1,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$, $U_3 \leq 1$ and $1 < U_2 \leq 2$

Given (93), (94), (97), then

$$
\begin{aligned}
V_2 = \frac{a_1+a_2+a_3}{r}2+\frac{a_4}{r} &> 1 \text{ when } r \geq 1\\
U_3 \leq 1 &\text{ when } r \geq 2.618\\
1 < U_2 \leq 2 &\text{ when } 2.19149 \leq r < 5.27804
\end{aligned}
$$

Since $V_2 > 1$ for $r \geq 1$, the strategy pair $((2,1,1,1),(1,1,1))$ is not an equilibrium.

- If males of age 1 are non choosy, the equilibrium strategy pair is of the form $((1,1,1,1),(1,1,1))$ and the distribution that supports it is such that

$$
\begin{aligned}
b_1\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_2 \Leftrightarrow b_2 = 0\\
b_2\left(1-\frac{a_1+a_2+a_3+a_4}{r}\right) &= b_3 \Leftrightarrow b_3 = 0\\
a_1\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_2 \Leftrightarrow a_1\frac{r-1}{r} = a_2\\
a_2\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_3 \Leftrightarrow a_2\frac{r-1}{r} = a_3\\
a_3\left(1-\frac{1}{r}\left(b_1+b_2+b_3\right)\right) &= a_4 \Leftrightarrow a_3\frac{r-1}{r} = a_4
\end{aligned}
$$

For $((1,1,1,1),(1,1,1))$ to be an equilibrium under the support of the above distribution it is necessary that $V_2 \leq 1$, $U_3 \leq 1$ and $U_2 \leq 1$

Given (93), (94), (97), then

$$V_2 = \frac{a_1 + a_2 + a_3}{r} 2 + \frac{a_4}{r} > 1 \text{ when } r \geq 1$$
$$U_3 \leq 1 \text{ when } r \geq 2.618$$
$$U_2 \leq 1 \text{ when } r \geq 5.27804$$

Since $V_2 > 1$ for $r \geq 1$, strategy pair in not an equilibrium.

When the female equilibrium strategy is the $(2,1,1)$, the population distribution is of the form of (90), while when the female equilibrium strategy is the $(1,1,1)$, the population distribution is of the form of (96).

The following table summarises the equilibria. Hence

$((1,1,1),(3,2,1,1))$	for $1 \leq R \leq 1.078$	hence for $1 \leq r \leq 2.078$
$((2,1,1),(2,2,1,1))$	for $1.131 \leq R < 1.145$	hence for $1.969 \leq r < 2.033$
$((2,1,1),(2,1,1,1))$	for $1.145 \leq R < 1.837$	hence for $2.033 \leq r < 4.5$
$((2,1,1),(1,1,1,1))$	for $R \geq 1.837$	hence for $r \geq 4.5$

The following figures present the population sex ratio r, the number of males of age i and the male and female distributions at equilibrium in terms of R. The different colours represent different strategies.

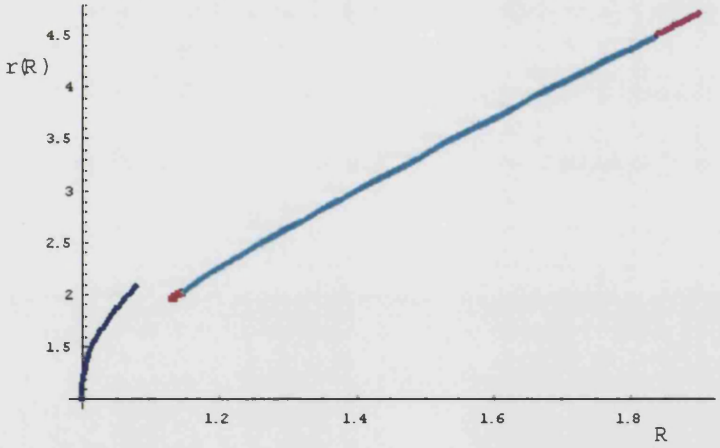


Figure C.14: Population sex ratio $r(R)$ when $\lambda = 4$, $\tau = 3$ and $r \geq 1$.

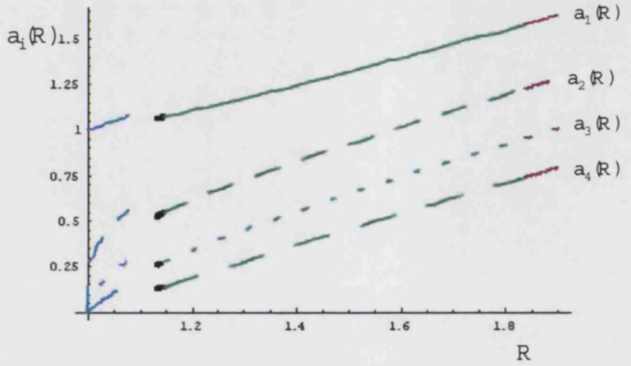


Figure C.15: Number a_i of males of type i when $\lambda = 4$, $\tau = 3$ and $r \geq 1$.

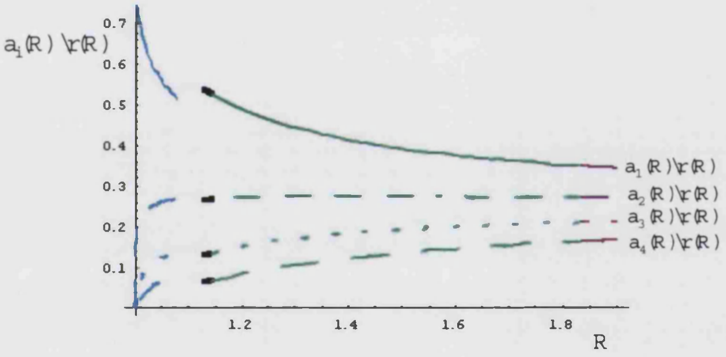


Figure C.16: Male distribution when $\lambda = 4$, $\tau = 3$ and $r \geq 1$.

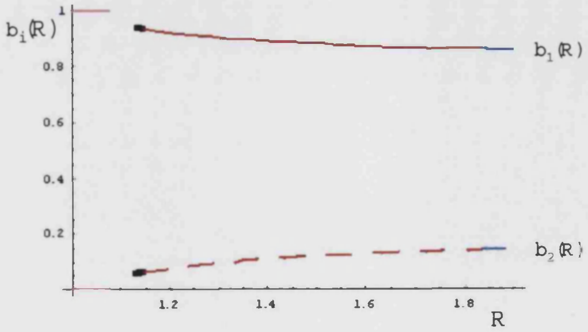


Figure $C.17$: Female distribution when $\lambda = 4$, $\tau = 3$ and $r \geq 1$.

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