Problems in Rendezvous Search

A Dissertation for The Degree

of

Doctor of Philosophy

By

Wei Shi Lim

Department of Mathematics

London School of Economics and Political Science

The University of London

February 1996

Abstract

Suppose n players are placed randomly on the real line at consecutive integers, and faced in random directions. Each player has maximum speed one and cannot see the others. The least expected time required for $m(\leq n)$ of them to meet together at a single point, if all players have to use the same strategy, is the symmetric rendezvous value $R^s_{m,n}$. If the players can use different strategies, the least expected meeting time is the asymmetric rendezvous value $R^a_{m,n}$. We show that $R^a_{3,2}$ is 47/48 and $R^s_{n,n}$ is asymptotic to $n/2$. If the minimax rendezvous time M_n is the minimum time required to ensure that all players can meet together at a single point regardless of their initial placement, we prove that M_2 is 3, M_3 is 4 and M_n is asymptotic to $n/2$. If players have to stick together upon meeting, we prove that three players require 5 time units to ensure a meeting.

We also consider a problem proposed by S. Alpern (in his joint paper with A. Beck, *Rendezvous Search on the Line with Bounded Resources, LSE Math Preprint Series, 92 (1995)*) of how two players can optimally rendezvous while at the same time evading an enemy searcher. We model this rendezvous-evasion problem as a two-person, zero-sum game between the rendezvous team R (with agents R_1, R_2) and the searcher S and consider a version which is discrete in time and space. R_1, R_2 and S start at different locations among n identical locations and no two of them share a common labelling of the locations. Each player can move between any two locations in one time step (this includes the possibility of staying still) until at least two of them are at the same location together, at which time the game ends. If S is at this location, S (maximizer) wins and the

payoff is 1; otherwise the team R (minimizer) wins and the payoff is 0. The value of the game v_n is the probability that S wins under optimal play. We assume that R_1 and R_2 can jointly randomize their strategies and prove that v_3 is $47/76 \approx 0.61842$ and v_4 is at least $31/54 \approx 0.57407$. If all the players share a common notion of a directed cycle containing all the n locations (while still able to move between any two locations), the value of the game d_n is $((1-2/n)^{n-1}+1)/2$. In particular, d_3 is less than v_3 and d_4 is less than v_4. We also compare some of these results with those obtained when the rendezvous-evasion game is modelled as a multi-stage game with observed actions (W. S. Lim, *Rendezvous-Evasion As a Multi-Stage Game With Observed Actions, LSE Math-CDAM Research Report Series, 96-05 (1996)*). In all instances considered, we find that obligatory announcement of actions at the end of each step either does not affect the value of the game or helps the rendezvous team secure a lower value.

Contents

List of Figures

List of Tables

Acknowledgements

I am grateful to Professor Steve Alpern for his supervision which rendered this thesis possible.

I am also greatly indebted to those who have been a generous source of encouragement.

Finally, I am glad to acknowledge financial support from a National University of Singapore Overseas Postgraduate Scholarship.

Statement of Originality

Chapter 2 (*Rendezvous Search On The Line With More Than Two Players*) is based on my paper (*LSE Mathematics Preprint Series, 81 (1995)*) and also includes joint work with Professors S. Alpern and A. Beck (*LSE Mathematics Preprint Series, 82 (1995)*).

Chapter 3 (*Minimax Rendezvous On The Line*) is joint work with Professor S. Alpern (*LSE Mathematics Preprint Series, 83 (1995)*). We obtained independent proofs of the main results and agreed which proofs were best for presentation. This work has been accepted for publication by *SIAM Journal of Control and Optimization.*

Chapter 4 (*A Rendezvous-Evasion Game On Discrete Locations With Joint Randomization*) is my original work. It is currently under revision for *Journal of Applied Probability.*

Chapter 1

Introduction

1.1 Introduction To The Rendezvous Search Problem

The problem of rendezvous search asks how two people placed in a known search region can find each other in least expected time. This problem arises typically when two people wish to meet each other but fail to specify an exact location and communication is not possible. The scenario where two people lose each other in a departmental store and wish to meet up again is one such example. Essentially, the rendezvous search problem is concerned with the study of coordination without communication.

This problem is first mentioned by Schelling in his early classic work on game theory [22] where the emphasis is on coordination with *focal points*; a focal point is a location or signal identified by the parties involved as unique. For example, Schelling asked a group of respondents to imagine that each of them was one of two individuals trying to meet one another in New York under the circumstances where communication is not possible. Each individual is to choose some place in New York and any location is as good as the other so long as both of them pick the same location. It seems an impossible task as there are an infinite number of choices for the location. However, the majority of the respondents chose the same place - Grand Central Station. In this problem, Grand Central Station has certain prominence which renders it to 'stand out' among all other locations to be chosen by the respondents; it provided a 'focal point for each individual's expectation of what the other expects him to expect to be expected to do' (Schelling [22], p. 57). In another example, Schelling considered the problem faced by two parachutists landing in a field with some known feature (house, river, bridge etc) and asked which of these should they use as focal points to meet at. He argued that a player should not simply predict where the other will go, since the other player will go to where he predicts the first player would go, which is where the first player predicts the second to predict the first to go, and so on ad infinitum. Schelling's approach was considered as a paradigm for coordination problem and parallel processing. However, the treatment of the subject has been non-mathematical.

The rendezvous search problem was first formulated in a mathematical framework by Alpern [1]. The approach differs from that of Schelling's by assuming that the search region is homogeneous (absence of focal points) and players move so as to meet (i.e., when the distance between them comes within a given detection radius) in the least expected time, i.e., this is not a one-shot game. In his paper, Alpern discusses the notion of a given group G of isometries (of the search region) and its connection with the spatial symmetries of the search region (from the point of view of the players). Consider the example where the search region is the circumference of a circle. In the extreme case where the given group G is the trivial group consisting of only the identity element, players regard all points on the circle as unique. We may thus interpret the search region as a clock face which both players can read so that the strategy *Go to 12 o' clock* is a legitimate strategy. If G is the group consisting of all rotations, one possible interpretation is that the players have a common notion of 'up' (and thus a common notion of clockwise and anticlockwise directions along the cycle). Then they could agree to move in opposite angular directions after their random placement. However, for the same circular region where the group G consists of all isometries of the cycle (rotations and reflections), a strategy pair such as that described above (which dictates that players move in opposite angular directions) would not be feasible since players here no longer share a common notion of a clockwise (or anticlockwise) direction. In general, by a random placement of the players onto the search region, Nature chooses a random element g from the isometry group G such that if players are using strategies s_1 and s_2, the meeting time $T(gs_1, s_2)$ is the first time when the paths $g(s_1(t))$ and $s_2(t)$ come within the detection radius. There is no need for two random elements g_1, g_2 since $T(g_1s_1, g_2s_2)$ is equals to $T(g_2^{-1}g_1s_1, s_2)$. The expected meeting time for the players is thus $T(gs_1, s_2)$ averaged over all elements g in the given isometry group G. Alpern [1] also introduces two versions of the rendezvous search problem, namely, the *asymmetric* and the *symmetric*. In the asymmetric version, players are distinguishable and thus can use different strategies. The least expected time for the players to meet is referred to as the *asymmetric rendezvous value* R^a. In the symmetric version however, players are indistinguishable and must therefore adopt the same mixed strategy. This scenario is equivalent to one where a controller broadcasts the players' strategies and since he has no means of identifying the players, he announces a single mixed strategy which is adopted by both players. The choice of a *mixed* strategy is necessary here because there is a possibility that players will not meet at all if they are to use the same pure strategy. For example, two players placed on the circumference of a circle will never meet if both of them use the same pure strategy and were pointed in the same direction initially since the distance between them is preserved. When players are constrained to use the same mixed strategy, the expected meeting time is the expectation of $T(gs_1, s_2)$, where the expectation is taken over all elements g in the isometry group G, and over all pure strategies s_1, s_2 chosen independently according to the common mixed strategy. The least expected time required for the players to meet is referred to as the *symmetric rendezvous value* R^s. Anderson and Weber [8] considered symmetric rendezvous on n locations in discrete time and proved that the players can do better than visit each location at random when n is at least 3. Subsequent work concentrated on the case where the search region is the line. In [5], Alpern and Gal showed that when two players are placed at a known distance of one apart and have no common notion of a positive direction along the line, the asymmetric rendezvous value R^a is given by 13/8 (= 1.625). In the analogous symmetric problem, Anderson and Essegaier [7] proved that the symmetric rendezvous value R^s is bounded above by 2.28. This estimate has recently been improved upon by V. Baston [9]. A similar problem has been considered in [19] and [23] where two players must coordinate to find a third party who is stationary and whose distribution is known. Alpern and Beck [4] solved the rendezvous search problem on the line where the players are constrained by fuel resources, quantified in terms of given bounds on the total distance that each player may travel. When players can ensure that they meet, the problem of minimizing the expected meeting time is considered. In the instance where both players run out of fuel before they meet, the problem of minimizing this probability was studied. J. Howard [12] solved the rendezvous problem on the interval, where the players can see how far they are from each end (of the interval), but cannot see each other. He assumed that the players are placed independently according to a common distribution which has a monotone density (for example, the uniform distribution). A simulation approach to Schelling's problem is considered in [24].

1.2 Overview of Thesis

In Chapter 2, we extend the analysis of the rendezvous search problem on the line to more than two players. This work is motivated by one of the questions posed in [1]. We consider the scenario where n blind, speed one, players are placed by a random permutation onto the integers 0 to $(n-1)$ on the line, and each is pointed randomly to the right or left. We seek to find the least expected time required for $m(\leq n)$ of them to meet together at a single point and denote this time by the asymmetric rendezvous value $R^a_{n,m}$ (if players can use different strategies) and the symmetric rendezvous value $R^s_{n,m}$ (otherwise). We prove that $R^a_{3,2}$ is 47/48 (= 0.97917). A general algorithm for solving $R^a_{n,m}$ is presented in [17]. We also show that $R^s_{n,n}$ is asymptotic to $n/2$. These results respectively extend those for two players given by Alpern and Gal [5], and Anderson and Essegaier [7]. The results of this chapter is based on my paper [14] and joint work with Alpern and Beck [17].

In Chapter 3, we consider a rendezvous problem on the line with the same setting as that in Chapter 2 except that instead of asking for the least *expected* meeting time, we find the minimum time M_n required to ensure that all the n players can meet together at a single point, *regardless of their initial placement.* We prove that M_2 is 3, M_3 is 4 and that M_n is asymptotic to $n/2$. We also consider a variant of this problem which requires players who meet to stick together and show that with this limitation on the players' motions, three players require five time units to ensure a meeting. This chapter presents the minimax version of the rendezvous search problem, which has hitherto been studied only in terms of minimizing the expected meeting time. The results of this chapter will appear in [18].

In Chapter 4, we study a problem proposed by Alpern, in his joint paper with Beck [4] of how two players can optimally rendezvous while at the same time evading an enemy searcher. We model this problem as a two-person, zero-sum game between the rendezvous team R (with agents R_1, R_2) and the searcher S. This chapter gives the first solution to such a *rendezvous-evasion* game by considering a version which is discrete in time and space, as in a pure rendezvous problem of Anderson and Weber [8]. The three players R_1, R_2 and S start at different locations among n identical locations and no two of them share a common labelling of the locations. At each integer time they can either stay where they are or move to any one of the other locations until some location is occupied by more than one player, at which time the game ends. If S is at this location, S (maximizer) wins and the payoff is 1; otherwise the rendezvous team R (minimizer) wins and the payoff is 0. The value of the game is the probability that S wins under optimal play. We assume that the agents R_1 and R_2 can jointly randomize their strategies. The case where the agents are restricted to use the same mixed strategy is considered in a separate paper with Alpern [6], and has not been included here. When n equals 3, the value of the game is $47/76$ (≈ 0.61842).

When n equals 4, the value is at least $31/54$ (≈ 0.57407). If in addition, the players share a common notion of a directed cycle containing all the n locations (while still able to move between any two locations), the value of the game is $((1 - 2/n)^{n-1} + 1)/2$. Comparing the values of this game for n equal 3 and 4 with those of their counterparts described above (without common knowledge of the directed cycle), we find that this extra information helps the rendezvous team secure a lower value in both instances. In the final section, we compare some of the results obtained in this chapter with that in [16], which models the rendezvous-evasion game as a multi-stage game with observed actions (the results in the paper has not been included in this thesis). In all cases that we consider, the announcement of actions at the end of each step by all players either do not change the value of the game, or is favourable towards the rendezvous team, helping them achieve a lower value.

Chapter 2

Rendezvous Search On The Line With More Than Two Players

2.1 Introduction

In this chapter, we extend the study of rendezvous search on the line to n (≥ 3) players. This is motivated by one of the questions raised in [1]. In general, when two (or more) players meet (before the game ends) they may exchange information about where they have been and who they have met (which is their private information) so that the rendezvous search problem with more than two players exhibits an added dimension of complexity.

At the start of the game, n players are placed by a random permutation onto the integers 0 to $(n-1)$ on the line and each is pointed randomly to the left or right. The players thus have no common notion of a positive direction along the line. They move at speed at most one, exchanging information freely with those they meet until the game ends, which is the first time when m $(2 \leq m \leq n)$ of them are at a single point. We denote this class of games by $\Gamma_{n,m}$. The objective of the players is to meet in the least expected time. We denote the asymmetric and symmetric rendezvous values for the problem $\Gamma_{n,m}$ as $R^a_{n,m}$ and $R^s_{n,m}$ respectively. When n equals m equals 2, it has been proved that $R^a_{2,2} = 13/8$ [5] and $R^s_{2,2} \leq 2.28388$ [7]. The upper bound has since been improved upon by V. Baston [9].

This chapter is organized as follows. In Section 2.2, we formalize the framework for the asymmetric version of the rendezvous search problem $\Gamma_{3,2}$. We further illustrate this framework in Section 2.3 by considering a particular strategy triple. In Section 2.4, we establish an optimality condition which we use in Section 2.5 to devise an algorithm, which allows us to compute the value of $R^a_{3,2}$ and find all optimal strategy triples. We prove that $R^a_{3,2}$ is 47/48. Although most of the work concentrates on 3-player rendezvous, similar results can be extended to the general asymmetric rendezvous search problem $\Gamma_{n,2}$ [17]. In Section 2.6, we apply a modified version of the algorithm established in Section 2.5 to the rendezvous problem $\Gamma_{2,2}$. The intent of this section is to provide an alternative proof of the result $R^a_{2,2} = 13/8$ first proven in [5]. Lastly, in Section 2.7, we analyze the rendezvous problem $\Gamma_{n,n}$ for large n and show that both rendezvous values are asymptotic to $n/2$.

2.2 Formulation of $\Gamma_{3,2}$

We consider the asymmetric version of the rendezvous search problem $\Gamma_{3,2}$ where two of the three players are required to meet for the game to end. At the start of the game, players 1, 2 and 3 are placed randomly on the real line at unit distance apart. Each player is equally likely to be pointed in either directions so that the players do not have a common notion of a positive direction along the line. We may represent the progress of the game on a Cartesian plane, where the horizontal axis denotes time, and the vertical axis denotes the position of the player on the real line. For $i = 1, 2, 3$, the *initial position* π_i of player i is such that (π_1, π_2, π_3) is a random permutation of the integers $0, 1, 2$ and the *initial orientation* ω_i of player i is defined as

$$\omega_i = \begin{cases} +1 & \text{if player } i \text{ is pointing in the upward} \\ & \text{direction at the start of the game,} \\ -1 & \text{if player } i \text{ is pointing in the downward} \\ & \text{direction at the start of the game.} \end{cases}$$

The set $\mathcal{C}$ containing all elements $c = (\pi_1, \pi_2, \pi_3, \omega_1, \omega_2, \omega_3)$ denotes the set of all possible ways that the game may begin. We refer to the elements of $\mathcal{C}$ as *cases.* We observe that a game which begins as case $(\pi_1, \pi_2, \pi_3, -1, \omega_2, \omega_3)$ proceeds in exactly the same way as one which begins as $(2 - \pi_1, 2 - \pi_2, 2 - \pi_3, +1, -\omega_2, -\omega_3)$. For example, Figures 2.1a and 2.1b show the games starting as cases $(1, 0, 2, -1, +1, -1)$ and $(1, 2, 0, +1, -1, +1)$ on the Cartesian plane. And we see that one can be obtained from the other by reflecting the strategy curves of the players about the time axis, followed by a translation of two units along the position axis. Hence, we shall assume that $\omega_1 = +1$ and thus the set $\mathcal{C}$ has $24 (= 3! \times 2^2)$ elements. Clearly, the game ends when the player initially placed at position 1 meets either the player initially placed at position 0 or the player initially placed at position 2, or when all three players meet at the same time. We shall later see that the last scenario is not possible when all the players move optimally.

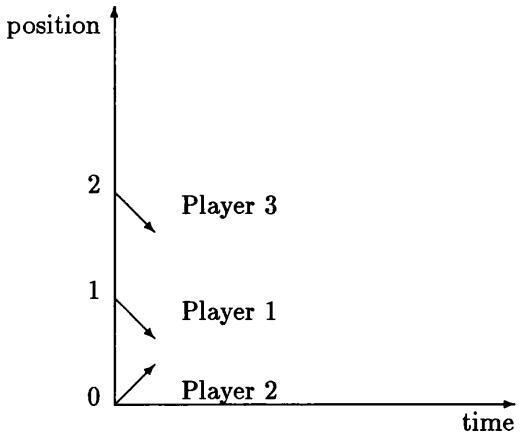


Figure 2.1a: Case $(1, 0, 2, -1, +1, -1)$

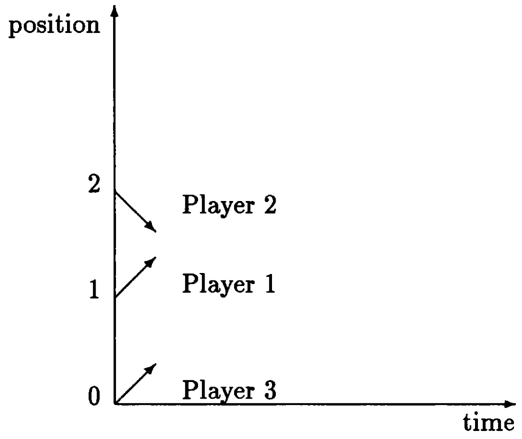


Figure 2.1b: Case $(1, 2, 0, +1, -1, +1)$

A strategy triple for the rendezvous problem $\Gamma_{3,2}$ is given by $\{f, g, h\}$ where Player 1 uses strategy path f, Player 2 uses strategy path g and Player 3 uses strategy path h. Each of f, g, h is chosen from the set P which consists of paths with speed bounded by one, i.e.,

$$P = \{p : \Re^+ \longrightarrow \Re, p(0) = 0, |p(t_1) - p(t_2)| \leq |t_2 - t_1|\}.$$

Let $T_c(f, g, h)$ denote the meeting time that corresponds to case c; it is the first time that two distinct players occupy the same point on the line. The expected meeting time $T(f, g, h)$ is thus

$$T(f, g, h) = \frac{1}{24} \sum_{c \in C} T_c(f, g, h),$$

and the asymmetric rendezvous value $R^a_{3,2}$ is given by

$$R^a_{3,2} = \min_{f,g,h \in P} T(f, g, h). \tag{2.1}$$

The existence of the minimum in equation (2.1) and of an optimal strategy triple is assured as T_c (and thus T) is lower semi-continuous in its variables and P is a compact set under the topology of uniform convergence on compact intervals. Given any strategy triple $\{f, g, h\}$, it is always possible to construct the following eight curves so that every $T_c(f, g, h)$ can be represented as the time when some two particular curves intersect:

$$\begin{aligned} L_{0,\alpha}(t) &= \alpha h(t), \\ L_{1,\alpha}(t) &= \alpha f(t) + 1, \\ L_{2,\alpha}(t) &= \alpha g(t) + 2, \\ L_{3,\alpha}(t) &= \alpha h(t) + 3, \qquad \alpha = \pm 1. \end{aligned} \tag{2.2}$$

More specifically, recall that the game ends when either (i) the player initially placed at position 0 meets the player initially placed at position 1, or (ii) the player initially placed at position 1 meets the player initially placed at position 2. Thus $T_c(f, g, h)$ is the minimum of the two intersection times of two pairs of associated curves. For example, corresponding to case $(1, 2, 0, +1, -1, +1)$ depicted in Figure 2.1b, the two pairs of associated curves are $(L_{0,1}, L_{1,1})$ and $(L_{1,1}, L_{2,-1})$, whose meeting times correspond to the meeting times of Player 1 with Player 3, and Player 1 with Player 2 respectively. We abbreviate the pair of curves associated with case $(1, 2, 0, +1, -1, +1)$ as $(0, +1), (1, +1)$ and $(1, +1), (2, -1)$. The relationship between all the cases and the associated pairs of curves are given in Table 2.1 below (case $(1, 2, 0, +1, -1, +1)$ explained above is listed as case c_{11}).

$c \in \mathcal{C}$	$(\pi_1, \pi_2, \pi_3, \omega_1, \omega_2, \omega_3)$	Two Pairs of curves associated with c	
c_1	$(0,1,2,+1,+1,+1)$	$(1,+1),(2,+1)$	$(2,+1),(3,+1)$
c_2	$(0,1,2,+1,+1,-1)$	$(1,+1),(2,+1)$	$(2,+1),(3,-1)$
c_3	$(0,1,2,+1,-1,+1)$	$(1,+1),(2,-1)$	$(2,-1),(3,+1)$
c_4	$(0,1,2,+1,-1,-1)$	$(1,+1),(2,-1)$	$(2,-1),(3,-1)$
c_5	$(0,2,1,+1,+1,+1)$	$(0,-1),(1,-1)$	$(2,-1),(3,-1)$
c_6	$(0,2,1,+1,+1,-1)$	$(0,+1),(1,-1)$	$(2,-1),(3,+1)$
c_7	$(0,2,1,+1,-1,+1)$	$(0,-1),(1,-1)$	$(2,+1),(3,-1)$
c_8	$(0,2,1,+1,-1,-1)$	$(0,+1),(1,-1)$	$(2,+1),(3,+1)$
c_9	$(1,2,0,+1,+1,+1)$	$(0,+1),(1,+1)$	$(1,+1),(2,+1)$
c_{10}	$(1,2,0,+1,+1,-1)$	$(0,-1),(1,+1)$	$(1,+1),(2,+1)$
c_{11}	$(1,2,0,+1,-1,+1)$	$(0,+1),(1,+1)$	$(1,+1),(2,-1)$
c_{12}	$(1,2,0,+1,-1,-1)$	$(0,-1),(1,+1)$	$(1,+1),(2,-1)$
c_{13}	$(1,0,2,+1,+1,+1)$	$(0,-1),(1,-1)$	$(1,-1),(2,-1)$
c_{14}	$(1,0,2,+1,+1,-1)$	$(0,+1),(1,-1)$	$(1,-1),(2,-1)$
c_{15}	$(1,0,2,+1,-1,+1)$	$(0,-1),(1,-1)$	$(1,-1),(2,+1)$
c_{16}	$(1,0,2,+1,-1,-1)$	$(0,+1),(1,-1)$	$(1,-1),(2,+1)$
c_{17}	$(2,0,1,+1,+1,+1)$	$(0,+1),(1,+1)$	$(2,+1),(3,+1)$
c_{18}	$(2,0,1,+1,+1,-1)$	$(0,-1),(1,+1)$	$(2,+1),(3,-1)$
c_{19}	$(2,0,1,+1,-1,+1)$	$(0,+1),(1,+1)$	$(2,-1),(3,+1)$
c_{20}	$(2,0,1,+1,-1,-1)$	$(0,-1),(1,+1)$	$(2,-1),(3,-1)$
c_{21}	$(2,1,0,+1,+1,+1)$	$(1,-1),(2,-1)$	$(2,-1),(3,-1)$
c_{22}	$(2,1,0,+1,+1,-1)$	$(1,-1),(2,-1)$	$(2,-1),(3,+1)$
c_{23}	$(2,1,0,+1,-1,+1)$	$(1,-1),(2,+1)$	$(2,+1),(3,-1)$
c_{24}	$(2,1,0,+1,-1,-1)$	$(1,-1),(2,+1)$	$(2,+1),(3,+1)$

Table 2.1: Cases and Associated Pairs of Curves

The rationale for presenting Table 2.1 here is two-fold. Firstly, it verifies our earlier claim that each $T_c(f,g,h)$ can be represented as an intersection point between some two curves. Secondly, this observation provides the intuition for the optimality condition, which will be stated and proved in Section 2.4.

2.3 Strategy Triple $\{\bar{f},\bar{g},\bar{h}\}$

In this section, we compute the expected meeting time achieved by the strategy triple $\{\bar{f},\bar{g},\bar{h}\}$, where

$$\text{Let} \quad \bar{f}(t) = \begin{cases} t & t \in [0,1], \\ 2-t & t \in [1,5/2], \end{cases}$$

$$\bar{g}(t) = \begin{cases} t & t \in [0,1/2], \\ 1-t & t \in [1/2,2], \\ t-3 & t \in [2,5/2], \end{cases}$$

$$\bar{h}(t) = \begin{cases} t & t \in [0,1/2], \\ 1-t & t \in [1/2,3/2], \\ t-2 & t \in [3/2,5/2]. \end{cases}$$

The strategy triple $\{\bar{f},\bar{g},\bar{h}\}$ is shown in Figure 2.2. For each case c, we obtain the meeting times $T_c(\bar{f},\bar{g},\bar{h})$ by taking the minimum of the two meeting times of the pairs of associated curves. The set of curves $L_{k,\alpha}(t)$ $(k=0,1,2,3, \alpha=\pm 1)$ as defined in (2.2) for the strategy triple $(\bar{f},\bar{g},\bar{h})$ is shown in Figure 2.3 and the meeting times are summarized in Table 2.2, where cases c_i are as defined in Table 2.1 of Section 2.2.

$c \in \mathcal{C}$	First Pair of Associated Curves	Meeting Time	Second Pair of Associated Curves	Meeting Time	$T_c(\bar{f}, \bar{g}, \bar{h})$
c_1	$(1,+1),(2,+1)$	1	$(2,+1),(3,+1)$	$> 5/2$	1
c_2	$(1,+1),(2,+1)$	1	$(2,+1),(3,-1)$	$1/2$	$1/2$
c_3	$(1,+1),(2,-1)$	$1/2$	$(2,-1),(3,+1)$	$3/2$	$1/2$
c_4	$(1,+1),(2,-1)$	$1/2$	$(2,-1),(3,-1)$	2	$1/2$
c_5	$(0,-1),(1,-1)$	1	$(2,-1),(3,-1)$	2	1
c_6	$(0,+1),(1,-1)$	$1/2$	$(2,-1),(3,+1)$	$3/2$	$1/2$
c_7	$(0,-1),(1,-1)$	1	$(2,+1),(3,-1)$	$1/2$	$1/2$
c_8	$(0,+1),(1,-1)$	$1/2$	$(2,+1),(3,+1)$	$> 5/2$	$1/2$
c_9	$(0,+1),(1,+1)$	$5/2$	$(1,+1),(2,+1)$	1	1
c_{10}	$(0,-1),(1,+1)$	$> 5/2$	$(1,+1),(2,+1)$	1	1
c_{11}	$(0,+1),(1,+1)$	$5/2$	$(1,+1),(2,-1)$	$1/2$	$1/2$
c_{12}	$(0,-1),(1,+1)$	$> 5/2$	$(1,+1),(2,-1)$	$1/2$	$1/2$
c_{13}	$(0,-1),(1,-1)$	1	$(1,-1),(2,-1)$	$> 5/2$	1
c_{14}	$(0,+1),(1,-1)$	$1/2$	$(1,-1),(2,-1)$	$> 5/2$	$1/2$
c_{15}	$(0,-1),(1,-1)$	1	$(1,-1),(2,+1)$	2	1
c_{16}	$(0,+1),(1,-1)$	$1/2$	$(1,-1),(2,+1)$	2	$1/2$
c_{17}	$(0,+1),(1,+1)$	$5/2$	$(2,+1),(3,+1)$	$> 5/2$	$5/2$
c_{18}	$(0,-1),(1,+1)$	$> 5/2$	$(2,+1),(3,-1)$	$1/2$	$1/2$
c_{19}	$(0,+1),(1,+1)$	$5/2$	$(2,-1),(3,+1)$	$3/2$	$3/2$
c_{20}	$(0,-1),(1,+1)$	$> 5/2$	$(2,-1),(3,-1)$	2	2
c_{21}	$(1,-1),(2,-1)$	$> 5/2$	$(2,-1),(3,-1)$	2	2
c_{22}	$(1,-1),(2,-1)$	$> 5/2$	$(2,-1),(3,+1)$	$3/2$	$3/2$
c_{23}	$(1,-1),(2,+1)$	2	$(2,+1),(3,-1)$	$1/2$	$1/2$
c_{24}	$(1,-1),(2,+1)$	2	$(2,+1),(3,+1)$	$> 5/2$	2

Table 2.2: Summary of $T_c(\bar{f}, \bar{g}, \bar{h})$ for all case $c \in \mathcal{C}$

For example, for case $c_{11} = (1, 2, 0, +, -, +)$, the two pairs of curves associated with it are $(0, +1)$, $(1, +1)$ and $(1, +1)$, $(2, -1)$, which intersect at times $5/2$ and $1/2$ respectively so that $T_{c_{11}}(\bar{f}, \bar{g}, \bar{h}) = 1/2$. From Table 2.2, it is easy to see that $T(\bar{f}, \bar{g}, \bar{h})$ being the average of $T_c(\bar{f}, \bar{g}, \bar{h})$ (for all cases c) is $47/48$. This result gives rise to the following lemma:

Lemma 2.1 *The asymmetric rendezvous value $R^a_{3,2}$ satisfies $R^a_{3,2} \leq 47/48$.*

We shall show in Section 2.4 that the asymmetric rendezvous value $R^a_{3,2}$ is indeed $47/48$, so that the strategy triple $\{\bar{f}, \bar{g}, \bar{h}\}$ is optimal.

2.4 An Optimality Condition

The main objective of this section is to prove a necessary condition for a strategy triple to be optimal. Informally, this optimality condition can be expressed as follows.

Each player when moving optimally uses some strategy path that is linear with slope ± 1 between his consecutive meeting times.

We shall see that as a consequence of this necessary condition, we can reduce the rendezvous problem $\Gamma_{3,2}$ to a finite problem. This enables us to devise an algorithm in Section 2.5 to compute the value of $R^a_{3,2}$.

For any strategy triple $\{f, g, h\}$, let $\Upsilon(f, g, h)$ denote the set of meeting times $T_c(f, g, h)$ for all c in $\mathcal{C}$, i.e.,

$$\Upsilon(f, g, h) = \{T_c(f, g, h), c \in \mathcal{C}\}.$$

Since only some of $T_c(f, g, h)$ involve the meeting of player i with another player, we use $\Upsilon_i(f, g, h)$ to denote this subset of $\Upsilon(f, g, h)$. The following lemma asserts that if two strategy triples $\{f, g, h\}$ and $\{f', g', h'\}$ are such that they differ only along the path of player i at times not in $\Upsilon_i(f, g, h)$, then the strategy triple $\{f', g', h'\}$ cannot be worse than the strategy triple $\{f, g, h\}$, i.e., $T(f', g', h')$ is at most $T(f, g, h)$. One implication of this lemma is that if we modify any strategy triple $\{f, g, h\}$ along the path of one player (say i) off times in $\Upsilon_i(f, g, h)$, the expected meeting time would not increase with such modifications. We shall later apply this lemma in the proof of the optimality condition.

Lemma 2.2 *Suppose $\{f, g, h\}$ and $\{f', g', h'\}$ are strategy triples which agree except for the ith player, and that the strategy paths of player i agree for all times t in $\Upsilon_i(f, g, h)$. Then $T_c(f', g', h')$ is at most $T_c(f, g, h)$ for all c in $\mathcal{C}$.*

Proof:

Given any c in $\mathcal{C}$, if $T_c(f', g', h')$ does not involve player i meeting some other player, $T_c(f', g', h')$ is equals to $T_c(f, g, h)$ since the strategy triples agree off the path of player i. If $T_c(f', g', h')$ involves player i, then since the strategy triple $\{f', g', h'\}$ agrees with the strategy profile $\{f, g, h\}$ for all times t in $\Upsilon_i(f, g, h)$, $T_c(f', g', h')$ is at most $T_c(f, g, h)$.
□

Now we are ready to state the optimality condition formally.

Theorem 2.1 *Any optimal strategy triple $\{f, g, h\}$ satisfies the following condition:*

Each player i moves with a strategy path that is linear with slope ± 1 between consecutive meeting times in $\Upsilon_i(f, g, h)$. (2.3)

Proof:

Without loss of generality, assume that f (the strategy of player 1) fails condition (2.3) in the time interval $[t_1, t_2]$, t_1, t_2 in $\Upsilon_1(f, g, h)$. If $L_{1,\alpha}$, $L_{2,\beta}$ are the two paths responsible for the meeting time t_2 $(= T_{c'}(f, g, h)$ for some case c'), then $L_{1,\alpha}(t_1) < L_{2,\beta}(t_1)$. Modify the path $L_{1,\alpha}$ to $\hat{L}_{1,\alpha}$ (and thus strategy f to $\hat{f}$) in the time interval $[t_1, t_2]$ in the following manner: $\hat{L}_{1,\alpha}$ moves 'upwards' at speed one until time $t_2'(< t_2)$ when the paths $\hat{L}_{1,\alpha}(\cdot)$ and $L_{2,\beta}(\cdot)$ meet (i.e., $T_{c'}(\hat{f}, g, h) = t_2' < t_2 = T_{c'}(f, g, h)$), after which $\hat{L}_{1,\alpha}(\cdot)$ follows the trajectory of $L_{2,\beta}(\cdot)$ to ensure reaching $L_{1,\alpha}(t_2)(= L_{2,\beta}(t_2))$ by time t_2. Since f and $\hat{f}$ agree on all times t in $\Upsilon_i(f, g, h)$, by the above lemma, $T_c(\hat{f}, g, h) \leq T_c(f, g, h)$ for all c in $\mathcal{C}$. In particular, our modification guarantees that $T_{c'}(\hat{f}, g, h) < T_{c'}(f, g, h)$. Hence the strategy triple $\{f, g, h\}$ cannot be optimal. This is illustrated in Figure 2.4a. Likewise, if $L_{1,\alpha}, L_{0,\gamma}$ are the two paths responsible for the meeting time $t_2 = T_{c'}(f, g, h)$, we have $L_{1,\alpha}(t_1) > L_{0,\gamma}(t_1)$. Here, we modify the path $L_{1,\alpha}$ to $\hat{L}_{1,\alpha}$ (and hence strategy f to $\hat{f}$) in the time interval $[t_1, t_2]$ in the following manner: $\hat{L}_{1,\alpha}$ moves 'downwards' at speed one until time $t_2'(< t_2)$ when the paths $\hat{L}_{1,\alpha}(\cdot)$ and $L_{0,\gamma}(\cdot)$ meet with $T_{c'}(\hat{f}, g, h) = t_2' < t_2 = T_c(f, g, h)$. After which, $\hat{L}_{1,\alpha}(\cdot)$ follows the trajectory of $L_{0,\gamma}(\cdot)$ to ensure reaching $L_{1,\alpha}(t_2)(= L_{0,\gamma}(t_2))$ by time t_2. Again, $T_c(\hat{f}, g, h) \leq T_c(f, g, h)$ for all c in $\mathcal{C}$ and $T_{c'}(\hat{f}, g, h) < T_{c'}(f, g, h)$. Hence the result holds. □

One implication of the above theorem is that it is not optimal for any player to stay still, and so when moving optimally, the scenario where all three players meet at the same time at a single point on the line is not possible. As a corollary, the following result further limits the type of movement that any player can adopt when moving optimally. It states that each player changes its direction of movement only at times which are integer multiples of 1/2.

Corollary 2.1 *Suppose* $\{f, g, h\}$ *is an optimal strategy triple. Let* $t_0 = 0$ *and* $t_1 < t_2 < \ldots < t_\kappa$ *be the natural ordering of the elements of* $\Upsilon(f, g, h)$. *Then for each* $i = 1, 2, \ldots, \kappa$, *we have the following statements:*

$S_1(i)$: $2t_i$ *is an integer.*

$S_2(i)$: $|L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)|$ *is an integer* $\forall\, k \in \{0, 1, 2\}$, α, $\beta \in \{\pm 1\}$.

Proof:

We proceed by proving the joint statement $S(i)$ using induction on i. When $i = 0$, by definition, $t_0 = 0$ so $2t_0 = 0$ is an integer and $S_1(0)$ holds. At time $t_0 = 0$, the players are placed at positions 0, 1 and 2 respectively, so $S_2(0)$ holds. Assume that $S(i)$ is true for some i. Then $S_2(i)$ being true implies that $|L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)|$ is an integer for all k, α, β. By Theorem 2.1, every player moves at speed one till meeting occurs. As a result, $2(t_{i+1} - t_i)$ is an integer. By our assumption that $S_1(i)$ holds, we know that $2t_i$ is an integer. Hence, so is $2t_{i+1}$. This proves $S_1(i+1)$. Since therorem 2.1 says that when players search optimally, they move at speed one between the time interval $[t_i, t_{i+1}]$, the distance between the players is either (i)preserved (if they move in the same direction), or (ii) increased by $2(t_{i+1} - t_i)$ (if they move away from each other), or (iii) decreased by $2(t_{i+1} - t_i)$ (if they move towards each other). Hence, for all k, α, β,

$$|L_{k,\alpha}(t_{i+1}) - L_{k+1,\beta}(t_{i+1})| - |L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)| = \pm 2(t_{i+1} - t_i) \text{ or } 0.$$

Rearranging the terms, we have

$$|L_{k,\alpha}(t_{i+1}) - L_{k+1,\beta}(t_{i+1})| = \begin{cases} \pm 2(t_{i+1} - t_i) + |L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)| & \text{or} \\ |L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)|. \end{cases} \tag{2.4}$$

Since we are assuming that $S_2(i)$ holds, $|L_{k,\alpha}(t_i) - L_{k+1,\beta}(t_i)|$ is an integer for all k, α, β. Also, we have shown above that $2(t_{i+1} - t_i)$ is an integer. It is thus clear from (2.4) that $|L_{k,\alpha}(t_{i+1}) - L_{k+1,\beta}(t_{i+1})|$ is an integer, and so we establish $S_2(i+1)$. $\square$

Combining the results of Theorem 2.1 and Corollary 2.1, we have the following conclusion on the properties of any optimal strategy triple.

Corollary 2.2 *Any optimal strategy triple $\{f, g, h\}$ is such that each player moves with a strategy path that has slope $+1$ or -1 and changes direction only at times that are integer multiples of $1/2$.*

Proof:

The result is immediate from Theorem 2.1 and Corollary 2.1. $\square$

2.5 An Algorithm To Find $R^a_{3,2}$

In the previous section, we established a necessary condition for optimality: Each player follows speed one paths and changes direction only between his consecutive meeting times. As a corollary, we reduce the set of potential optimal paths to those speed one paths which change direction only at times which are integer multiples of $1/2$; this is summarised in Corollary 2.2. In this section, we apply this corollary to devise an algorithm to find all optimal strategy triples and the asymmetric rendezvous value $R^a_{3,2}$.

Let $s(k)$ denote a potential optimal strategy triple defined up to time $(k+1)/2$. We know from Corollary 2.2 that each player has to move with speed one paths which change direction only at times which are integer multiples of $1/2$. In other words, we can describe $s(k)$ by merely specifying if it changes direction at times which are integer multiples of $1/2$. If we assume without loss of generality that all players move in the initial direction that they are pointing (given by $\omega_1, \omega_2, \omega_3$ as defined in Section 2.2) for the time interval $[0, 1/2]$, we can write $s(k)$ as a $3 \times k$ matrix such that for $i = 1, 2, 3$ and $j = 1, 2, \ldots, k$,

$$s_{ij}(k) = \begin{cases} +1 & \text{if Player } i \text{ continues in his previous} \\ & \text{direction during the time interval } [\frac{j}{2}, \frac{j+1}{2}], \\ -1 & \text{if Player } i \text{ changes his direction of} \\ & \text{motion during the time interval } [\frac{j}{2}, \frac{j+1}{2}]. \end{cases}$$

At time $j/2$ ($j \geq 1$), $s_{ij}(k)$ indicates if player i changes direction in the next $1/2$ unit of time. As an example, we illustrate in Figure 2.5 the strategy triple represented by the 3×2 matrix

$$s(2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Given a fixed k, the total number of possible $s(k)$'s is 2^{3k}. Clearly, if the players use a strategy triple $s(k)$ which defines the strategy paths up to time $(k+1)/2$, we would only be able to determine if the game has ended by time $(k+1)/2$, and nothing can be said about what happens after that. Thus, for all $c \in \mathcal{C}$, let

$$T(c, s(k)) = \begin{cases} t & \text{if the game ends at time } t \ (\leq (k+1)/2), \\ \infty & \text{otherwise.} \end{cases}$$

Let $T^*(c, s(k)) = \min(T(c, s(k)), (k+1)/2 + 1/2)$. One may interpret $T(c, s(k))$ as the actual meeting time when $s(k)$ is used, assuming that all players stay still from time $(k+1)/2$ onwards. $T^*(c, s(k))$ may then be taken as the best possible meeting time achievable by the strategy triple $s(k)$, since if the game has not ended by time $(k+1)/2$, the most optimistic estimate of the actual meeting time is $(k+1)/2 + 1/2$. This is because we have seen from Corollary 2.1 that all meeting times are integer multiples of $1/2$ when players search optimally. Let $M(s(k))$ be an indicator of whether the game has ended by time $(k+1)/2$ for all twenty four cases when strategy $s(k)$ is used, i.e.,

$$M(s(k)) = \begin{cases} 1 & \text{if } \max_{c \in \mathcal{C}} T(c, s(k)) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

For example, $M(s(1)) = 0$ for all possible $s(1)$'s since not all cases can end at time 1. Note that $M(s(k)) = 1$ if $T(c, s(k)) = T^*(c, s(k))$ for all $c \in \mathcal{C}$.

The main idea of the algorithm is that instead of computing the expected meeting time for all potential optimal strategy triples (there are only finitely *many* of them as proved in the previous section), we compute, at each stage k, the value of $ET^*(s(k))$ (which is the average value of $T^*(c, s(k))$ taken over all c in $\mathcal{C}$) for each $s(k)$ and partition these $s(k)$'s into potential optimal triples (i.e. those $s(k)$'s with $ET^*(s(k))$ not exceeding a known upper bound of $R^a_{3,2}$) and otherwise. Such partition at each stage k significantly reduces the number of matrices $s(k+1)$ that one has to consider in subsequent stages. If an $s(k)$ is such that $ET^*(s(k))$ is less than the value of the known upper bound of $R^a_{3,2}$ and $M(s(k)) = 1$, we replace the current upper bound of $R^a_{3,2}$ by this smaller value. If an $s(k)$ is such that $ET^*(s(k))$ is not larger than the upper bound of $R^a_{3,2}$ and $M(s(k)) = 0$, we cannot draw any exact conclusion about the strategy triple $s(k)$ at stage k (except that it may be optimal) and for such $s(k)$'s, we append an additional $(k+1)$th column to the matrix so that it becomes a $3 \times (k+1)$ matrix and advances into stage $(k+1)$ of the algorithm. Each of these $s(k)$'s can be extended to eight possible $s(k+1)$'s and the algorithm ends when no such $s(k)$ needs to be extended. The formal description of the algorithm is given below.

The algorithm is defined in stages such that at each stage k, only strategy triples $s(k)$'s are considered. Let B_k denote the least upper bound for $R^a_{3,2}$ obtained by stage k. In Lemma 2.1 we showed that $R^a_{3,2}$ is bounded above by 47/48, so we shall take B_0 to be 47/48 and every strategy triple $s(k)$ is a potential optimal strategy triple if $ET^*(s(k))$ is at most B_{k-1}. Let P_1 denote the set of all $s(1)$ matrices (distinct up to row permutation). Let R_k denote the subset of P_k where each $s(k)$ in R_k has an expected meeting time (or an estimate of it) which is more than B_{k-1} and thus these $s(k)$'s cannot be optimal; R_k in essence, contains the $s(k)$'s in P_k that are rejected in stage k. We use E_k to denote the subset of P_k where all $s(k)$'s in E_k have achieved an expected meeting time of at most B_{k-1} by stage k (i.e., $M(s(k)) = 1$ and $ET^*(s(k))$ is at most B_{k-1}). For such $s(k)$'s, there is no need to consider any extension of it as a $3 \times (k+1)$ matrix since its exact expected meeting time is known and we end any further operations on these $s(k)$'s in E_k. Let A_k denote the subset of strategy triples $s(k)$ in P_k with $M(s(k)) = 0$ and $ET^*(s(k))$ is at most B_{k-1}. These $s(k)$'s are those strategy triples where precise conclusions about their optimality (or the lack of it) cannot be drawn. To each of these $s(k)$'s (distinct up to row permutation) in A_k, we append a $(k+1)$th column to it so that it becomes a $3 \times (k+1)$ matrix. The set of such $s(k+1)$'s that is obtained by extending all $s(k)$'s in A_k is the set P_{k+1}. P_{k+1} thus constitutes the set of potential optimal strategy triples. To summarize, at each stage k, the set P_k is partitioned into the sets R_k, E_k and A_k and each $s(k)$ in A_k (distinct up to row permutation) is then extended to eight $s(k+1)$'s which form the set P_{k+1} that enters stage $(k+1)$. That is,

$A_k = \{s(k) \in P_k : ET^*(s(k)) \leq B_{k-1} \text{ and } M(s(k)) = 0\}$,

$R_k = \{s(k) \in P_k : ET^*(s(k)) > B_{k-1}\}$,

$E_k = \{s(k) \in P_k : ET^*(s(k)) \leq B_{k-1} \text{ and } M(s(k)) = 1\}$, and

$B_k = \min\{B_{k-1}, \min_{s(k)\in E_k} ET^*(s(k))\}$.

The steps involved in the algorithm are summarized in Figure 2.6. We apply the algorithm and the result is given in the following theorem.

Theorem 2.2 *The asymmetric rendezvous value for three person search on the line, where two are required to meet, is given by* $R^a_{3,2} = 47/48$. *Up to permutations of the labels of the players, there are six optimal strategy triples. These are given in matrix notations as:*

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

Proof: We implement the above algorithm and the process stops at $k = 4$, with $B_4 = 47/48$. Hence $R^a_{3,2} = 47/48$. A summary of the results is given in Table 2.3. In stage 1, P_1 has four elements (distinct up to row permutation) and two of them are rejected, namely,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The two $s(1)$'s in the set A_1 as shown in Table 2.3 are

$$\tilde{s}(1) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \hat{s}(1) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

To further illustrate this algorithm, we shall consider, in particular, what happens to $\tilde{s}(1)$ in the subsequent stages of the algorithm. Eight 3×2

matrices (representing strategy triples) can be extended from the matrix $\tilde{s}(1)$ and for each of these strategy triple $s(2)$, the value of $ET^*(s(2))$ is computed. All but two (distinct up to row permutation) have an estimated expected meeting time exceeding 47/48. The remaining two matrices are

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Again, in stage 3, each of these two $s(2)$'s is extended to eight $s(3)$'s. However, upon computation of the value of $ET^*(s(3))$, it is found that all these $s(3)$'s have estimated expected meeting times larger than 47/48, so all of them are rejected from the algorithm and are partitioned into the set R_3. As we can see from Table 2.3, all the $s(3)$'s in A_3 have their first column as

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \text{ and not } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

The same reasoning follows for $\hat{s}(1)$. Eventually, six $s(4)$'s achieve an expected meeting time of 47/48 in stage 4 and no matrix remains in A_4 so that the algorithm ends at stage 4, giving $R^a_{3,2} = 47/48$.

The six optimal triples obtained are exactly those in the statement of the theorem, the first of which corresponds to the strategy triple $\{\bar{f}, \bar{g}, \bar{h}\}$ given in Section 2.3. □

k	B_k	$s(k) \in A_k$	$\mid E_k \mid$	$\mid P_{k+1} \mid$
1	47/48	$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	0	$2 \times 8 = 16$
2	47/48	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}$	0	$6 \times 8 = 48$
3	47/48	$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$	0	$3 \times 8 = 24$
4	47/48	$\emptyset$	6	0

Table 2.3: Summary of the results of the Algorithm

2.6 Asymmetric 2-Person Rendezvous Revisited

In this section, we provide an alternative approach to finding the value of $R^a_{2,2}$ (which was first obtained in [5]) by applying a revised version of the algorithm described in Section 2.5.

At the start of the game, two players are placed randomly at positions 0 and 1 randomly and they are equally likely to be pointed in either directions. As in three-person rendezvous, we use π_i and ω_i to denote the initial position and orientation of each player. For the same reason as given in Section 2.2, we assume without loss of generality that $\omega_1 = +1$ so that there are four $(2! \times 2)$ ways that the game may begin, given by the set $\{(\pi_1, \pi_2, \omega_1, \omega_2) : \{\pi_1, \pi_2\} = \{0, 1\}, \omega_1 = +1, \omega_2 \in \{+1, -1\}\}$. As before, P_1 is the set of all $s(1)$ matrices, and in this case where $n = 2$, P_1 consists of the 2×1 matrices (distinct up to row permutations)

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{and} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The sets R_k, E_k, A_k and P_k are defined as before, except that every $s(k)$ in A_k when extended, gives rise to four $s(k+1)$'s. B_k is again the least upper bound obtained by stage k. And in order to be able to apply the algorithm more efficiently, the optimistic estimate $T^*(c, s(k))$ is varied depending on the number of cases c with $T(c, s(k)) = \infty$. This is because in 2-person rendezvous, each case must have an unique meeting time. This observation is based on Theorem 3 of [5]. Note that this, however, need not be the case for n-person rendezvous, if n is strictly greater than 2. For example, in Table 2.2, we see that $T_{c_2}(\bar{f}, \bar{g}, \bar{h}) = T_{c_3}(\bar{f}, \bar{g}, \bar{h}) = 1/2$. In addition, an analogue of Corollary 2.1 for n-person rendezvous is proved in [17]: That the meeting times are integer multiples of 1/2. As a consequence, we know that each meeting time differs from the others by $z/2$, z being a positive integer. As an example, if $T(c, s(k))$ is equals to infinity for all the four cases, the meetings occur earliest at times $(k+1)/2 + 1/2$, $(k+1)/2 + 2 \times 1/2$, $(k+1)/2 + 3 \times 1/2$ and $(k+1)/2 + 4 \times 1/2$ respectively and $ET^*(s(k))$ is thus taken to be $(k+1)/2 + 5/4$.

$$\text{Let } \sigma_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Use $(u_1, u_2, \ldots, u_k)$ to denote a $2 \times k$ matrix obtained by appending σ_{u_k} to the matrix given by $(u_1, u_2, \ldots, u_{k-1})$. For example, $(1, 2, 3)$ denotes the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Suppose we take B_0 to be 13/8, which is the expected meeting time when strategy pair $(2, 3, 1, 2, 1)$ is used (this is obtained from [5]). At stage 1, we need only to consider matrices (1), (2) and (4) since (2) and (3) are identical under row permutation. Due to constraint of space, we show only part of the analysis on a tree diagram in Figure 2.7 where all the branches and nodes are labelled. For example, a node that arises from branches with labels $u_1, u_2, \ldots, u_k$ at stage k and labelled t means that the node represents $ET^*((u_1, u_2, \ldots, u_k)) = t$. For example, node Q at the end of stage two means that $ET^*((1,1)) = 2$, and since $B_1 = 13/8$, $(1,1)$ belongs to the set R_2. Upon implementation of the algorithm it is found that $E_k = \emptyset$ for $k = 1, 2, 3, 4$ and $A_5 = \emptyset$, which means that an optimal strategy must be a 2×5 matrix. Indeed, there are two of them, given by $(2, 3, 1, 2, 1)$ and $(2, 4, 1, 2, 1)$ (Figures 2.8a and 2.8b).

2.7 Symmetric n-Person Rendezvous

We now consider the problem $\Gamma_{n,n}$ faced by n players who must *all* meet at a single point in the least expected time and have to use the same mixed strategy (if they used the same *pure* strategy, then if all of them were initially pointed in the same direction the distance between adjacent players would forever remain as one, and they would never meet in this case). We observe that in this situation, two players upon meeting may exchange any information known to either of them at the time.

The aim of this section is to establish that the symmetric rendezvous value $R^s_{n,n}$ is asymptotic to $n/2$. It is clear that the rendezvous time T can never be less than $(n-1)/2$, since the distance between the players who are initially placed at positions 0 and $(n-1)$ is already $(n-1)$ at the start of the game. So the problem is to find a mixed strategy which if universally adopted produces an expected rendezvous time (for *all* players to meet) of $n/2 + K$, for some constant K independent of n. The following mixed strategy (given behaviorally) achieves this aim.

A strategy for symmetric n-player rendezvous:

The strategy consists of three stages, which we outline below. In these, the word *start* refers to a player's initial position, which he will always know. However, overriding these actions spelt out in the various stages is the following rule:

Rule 1: If someone you meet says 'follow me', then follow him, i.e., adopt the same strategy path.

The three stages of the strategy are as follows.

- Stage 1: Pick a random direction, independently and equiprobably each time. Go a distance $1/2$ at speed one in that direction and then return to *start*. Then go a distance $1/2$ (again with speed one) in the opposite direction and return to *start*. This takes a total time of 2, and you are back at *start* at all integer times. Repeat until you are back at *start* and have met another player. Then proceed to Stage 2.

- Stage 2: Go at speed one a maximum distance one in the direction away from that in which you have met another player at Stage 1. If you meet another player during this time, return immediately to your starting point and wait. If after one unit of time and you have not met another player, then proceed to Stage 3.

- Stage 3: Proceed at unit speed towards and then past *start*, continuing until the game ends, instructing everyone you meet to follow you.

Figure 2.9 shows a typical set of paths resulting from the use of this strategy where all but the player at position 4 is pointed downwards initially (n is 5 in this instance). In this figure, the paths of the players never cross over, though they may merge. So the paths can be read, labelling the players according to their initial positions. Consider player 1. He meets player 0 at time 2.5, so for him Stage 1 ends (and Stage 2 begins) at time 3. At time 3, he is at his initial position and he goes away from where he met player 0 towards player 2, who he meets at time 3.5. Since he has now met someone on either side, he returns to his start and waits. At time 6, player 0 tells him to follow, and he does. At time 6.5, everyone meets at position 1.5 on the line. Next look at player 4. At time 2, Stage 2 begins for him. He goes one unit in the upwards direction (away from where he met player 3). Finding nobody there, he knows that he is an end-player so he reverses and picks up players 2 and 3 until he gets to position 1.5 on the line with everyone else at time 6.5.

The basic idea of this strategy is that the two players at the ends (i.e., with initial positions at 0 and $(n-1)$ respectively) realize that they are the end-players by local interaction. They then proceed towards each other, gathering up everyone during the process.

The analysis of the expected meeting time is straightforward. Let all the chance moves, by Nature in picking the initial positions and initial orientations of the players be denoted by θ in Θ. Let $t_i = t_i(\theta)$ denote the time when player i (the player who starts at position i) leaves Stage 1. That is, t_i is the first time that player i is at his start after having met an adjacent player at time $t_i - 1/2$ (if this adjacent player is in Stage 1) or at time t_i (if this adjacent player is in Stage 2). If no player ever meet, then all the t_i's are infinity, but this event has probability zero. In Figure 2.9, t_0 is 3 and t_{n-1} $(= t_4)$ is 2. Note that for all i, $\mid t_i - t_{i+1} \mid \leq 1$ and hence by induction $\mid t_0 - t_{n-1} \mid \leq n-1$.

To compute the time T $(= T(\theta))$ required for all the players to meet, we focus our attention on the two end players 0 and $n-1$. Observe that player 0 is at position -1 at time t_0+1 and player $(n-1)$ is at position n at time $t_{n-1}+1$. This follows from the definition of the Stage 2 strategy, since their first meetings are with players 1 and $(n-2)$ respectively. By the definition of the Stage 3 strategy, player 0's position $p_0(t)$ at time t $(\geq t_0+1)$ is given by $p_0(t) = -1+t-(t_o+1)$ and player $(n-1)$'s position $p_{n-1}(t)$ at time t $(\geq t_{n-1}+1)$ is given by $p_{n-1}(t) = n-(t-(t_{n-1}+1))$. It follows that these two players meet (together with all other players) at the time T when $p_0(t)$ equals $p_{n-1}(t)$, or

$$T = \frac{n+3+t_0+t_{n-1}}{2}.$$

Denoting the expected value of T by $E(T)$ and the expected value of t_0

(and also t_{n-1}) by $E(t_0)$ we have the estimate

$$R^s_{n,n} \leq E(T) = \frac{n+3}{2} + E(t_0). \tag{2.5}$$

To estimate the value $E(t_0)$, let t_0^* denote the random variable t_0 in the case for n equals 2, or equivalently the meeting time of player 0 with player 1 when the motion of player 1 is unaffected by players initially placed at positions other than 0 and 1. Thus t_0^* gives an overestimate of t_0 since any meeting of player 1 with player 2 will in the actual situation send player 1 for any earlier meeting with player 0. Let E^* denote the expected value of t_0^*. Clearly t_0^* equals 1 with probability 1/4 and equals 2 with probability 1/4. Given that t_0^* is strictly greater than 2, its expected value is $2 + E^*$. Hence E^* satisfies the equation

$$E^* \;=\; \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{2}(2 + E^*), \text{or } E^* = \frac{7}{2} > E(t_0).$$

It now follows from (2.5) that

$$R^s_{n,n} \leq E(T) \leq \frac{n+3}{2} + \frac{7}{2} = \frac{n}{2} + 5.$$

To obtain a lower bound on $R^s_{n,n}$, first observe that even if all the players see each other, it takes at least time $(n-1)/2$ for them to meet at their center of mass. Thus $(n-1)/2$ is a lower bound. However, since players 0 and $(n-1)$ cannot meet anyone on the first 1/2 unit of time (and hence cannot receive any information during this time), it follows (by pairing cases with the player at 0 pointing upward and downward) that for any strategy, the expected distance between these two players at any time t ($\leq 1/2$) is the same as their initial distance, $n-1$. Hence, even if all positions are revealed at time 1/2, it will still take an expected additional time $(n-1)/2$ for them to meet. As a result, $1/2 + (n-1)/2$ $(= n/2)$ is a lower bound for the expected meeting time and also for the least expected meeting time $R^s_{n,n}$. Combining our upper and lower bounds on $R^s_{n,n}$, we obtain the following result.

Theorem 2.3 *The least expected time $R^s_{n,n}$ required for n (≥ 2) players to meet togther using identical mixed strategies, starting from a a random placement on the integers 0 to $(n-1)$, is asymptotic to $n/2$. In particular, it satifies the inequality*

$$\frac{n}{2} \leq R^s_{n,n} \leq \frac{n}{2} + 5.$$

(The same bounds obviously hold also for $R^a_{n,n}$ since it is less than $R^s_{n,n}$ and also satisfies the lower bound).

We conclude with a few remarks regarding the bounds obtained for $R^s_{n,n}$. First, these bounds are not the best possible. In fact the techniques used to obtain these can get better bounds. But since this requires much more time (and paper), and still does not get the two bounds equal, we adopt the current treatment. For n equals 2, the given bounds place $R^s_{2,2}$ between 1 and 6, whereas the strategy of Anderson and Essegaier [7] gives the bound $R^s_{2,2} \leq 2.29$. Of course the bound of 6 is not the true expected meeting time for our strategy when n equals 2, since in this instance $T = t_0 - 0.5$ (the first time the two players meet) which has expected value $E^* - 0.5$ $(= 3)$.

For n equals 3, our bounds place $R^s_{3,3}$ between 1.5 and 6.5. However this upper bound can be lowered by the addition of the following Rule 2 to our strategy: 'If on your first meeting with another player you learn (from that player) that there are $n-1$ players to one side of you, then proceed in that direction, telling everyone you meet to follow you. That is, skip Stage 2 and go directly to Stage 3'. In the case when n equals 3, this rule definitely applies to either player 0 or player 2. To calculate the expected meeting time $\tilde{T}$, we consider that following equally likely scenarios at time zero (all players are using the same mixed strategy).

Scenario 1: All three players move in the same direction.

Scenario 2: Player 0 and Player 2 move towards each other.

Scenario 3: Two adjacent players move towards each other while the third player (at one of the ends) moves in the direction away from them.

Scenario 4: Player 0 and Player 2 move in opposite directions away from each other.

Using τ_i ($i = 1, 2, 3, 4$) to denote the expected meeting time for each of the scenarios, we observe that no meeting between any players is possible in the first 2 units of time in Scenario 1 so that $\tau_1 = \tilde{T} + 2$. We analyze the game in the other three scenarios in Figure 2.10 and we have $\tau_2 = 3.3$, $\tau_3 = 3.0$ and $\tau_4 = 4.25$. In Scenario 4, player 0 is equally likely to move upwards or downwards at time 2. This randomization means that the expected meeting time τ_4 is $1/2(4.0 + 4.5)$ $(= 4.25)$. Thus,

$$\begin{aligned} \tilde{T} &= \frac{1}{4}\sum_{i=1}^{4}\tau_i \\ &= \frac{1}{4}(\tilde{T} + 2 + 3.5 + 3 + 4.25) \\ \Longrightarrow \quad \tilde{T} &= 4.25. \end{aligned}$$

Hence, $R^s_{3,3} \leq 4.25$.

As n becomes large, the probability of Rule 2 coming into force (for player 0 or $(n-1)$) becomes negligible. Note that our original strategy (without Rule 2) does not require that the players know the value of n, so that it is an effective strategy when the number of players is unknown.

In the next chapter, we consider a variant of the rendezvous search problem where instead of minimizing the expected meeting time, players wish to guarantee meeting within the shortest time regardless of their initial placement. As in this chapter, we focus on the case where the search region is the real line.

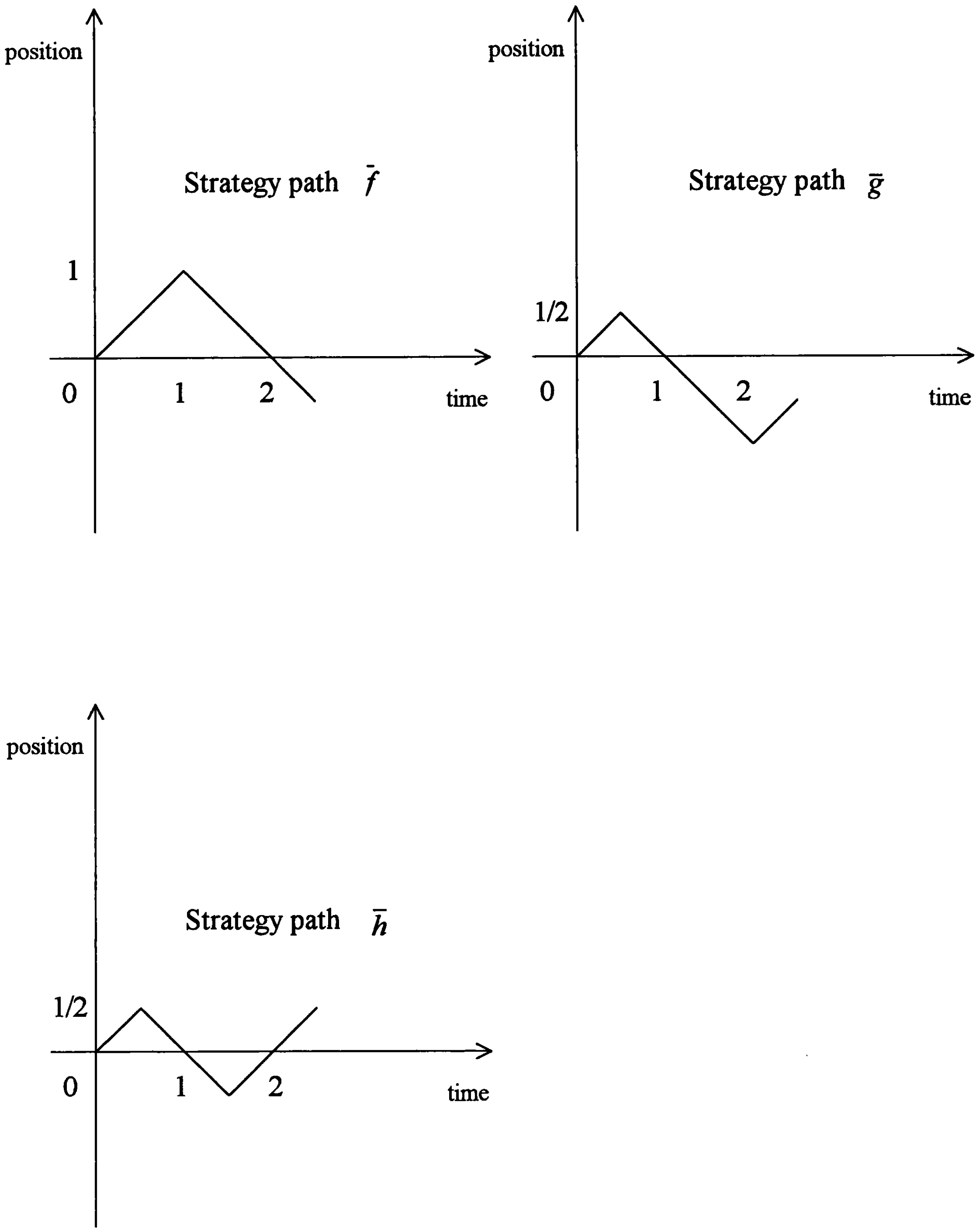


Figure 2.2: Strategy Triple $\{\overline{f}, \overline{g}, \overline{h}\}$

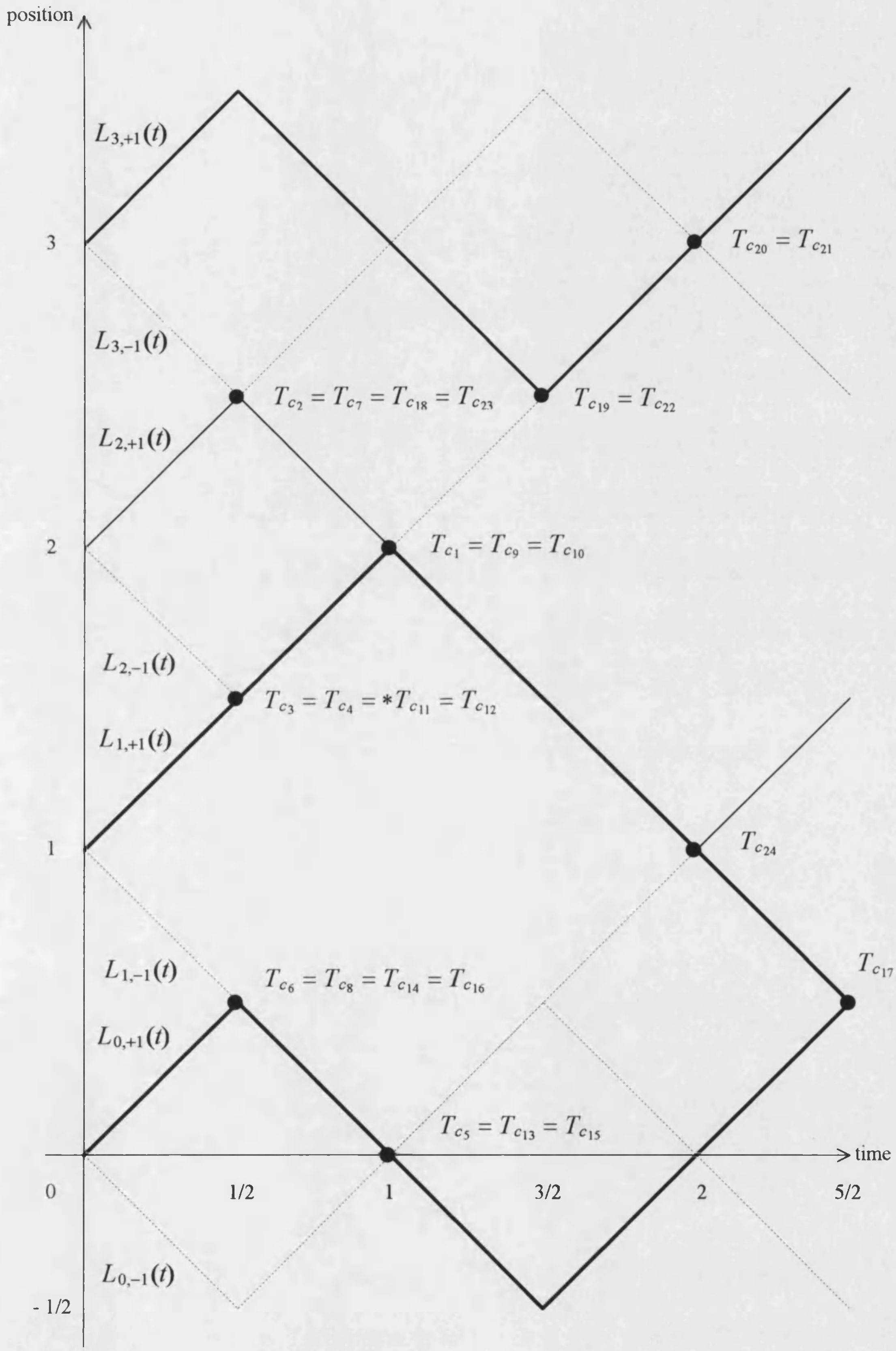


Figure 2.3: Curves $L_{k,i}(t)$ (k=0,1,2,3, i=+1,-1) for strategy triple $\{\overline{f},\overline{g},\overline{h}\}$

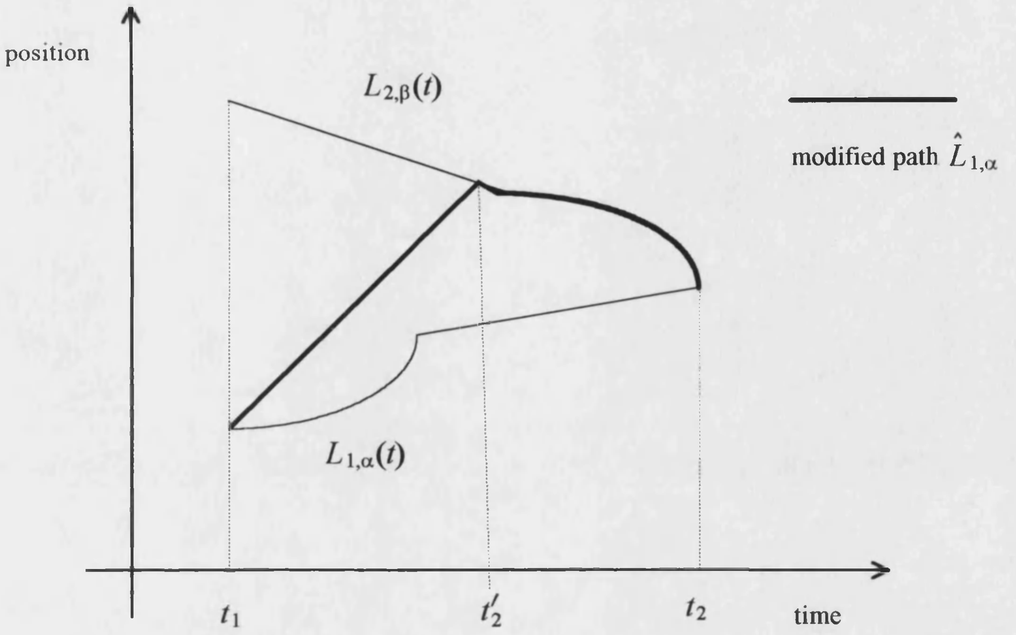


Figure 2.4 a: Illustration for the proof of theorem 2.1

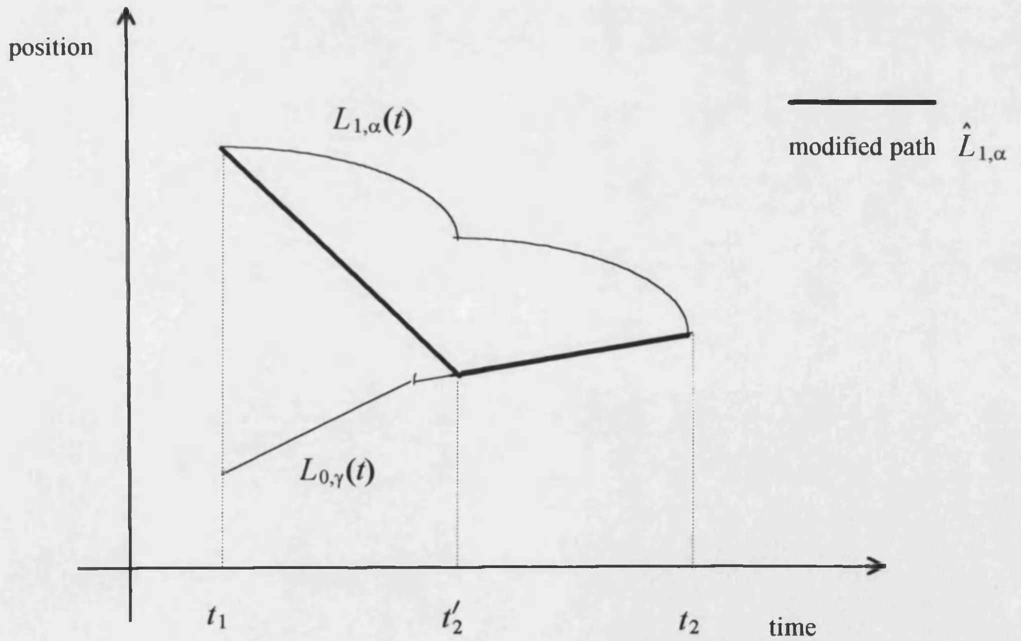


Figure 2.4b: Illustration for the proof of theorem 2.1

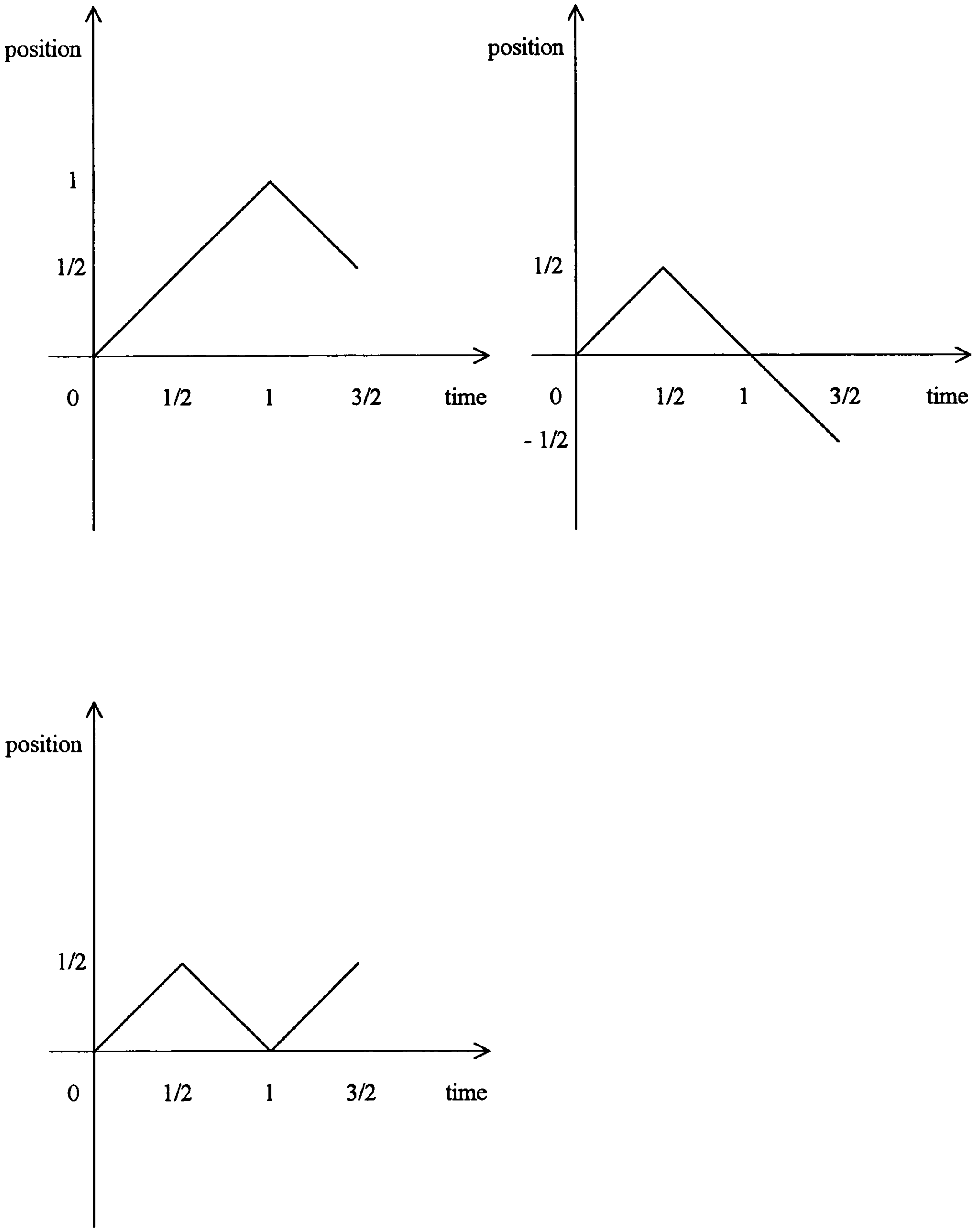


Figure 2.5: Strategy triple represented by s(2)

$$P_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}; B_0 = 47/48;$$

Begin with k=1;

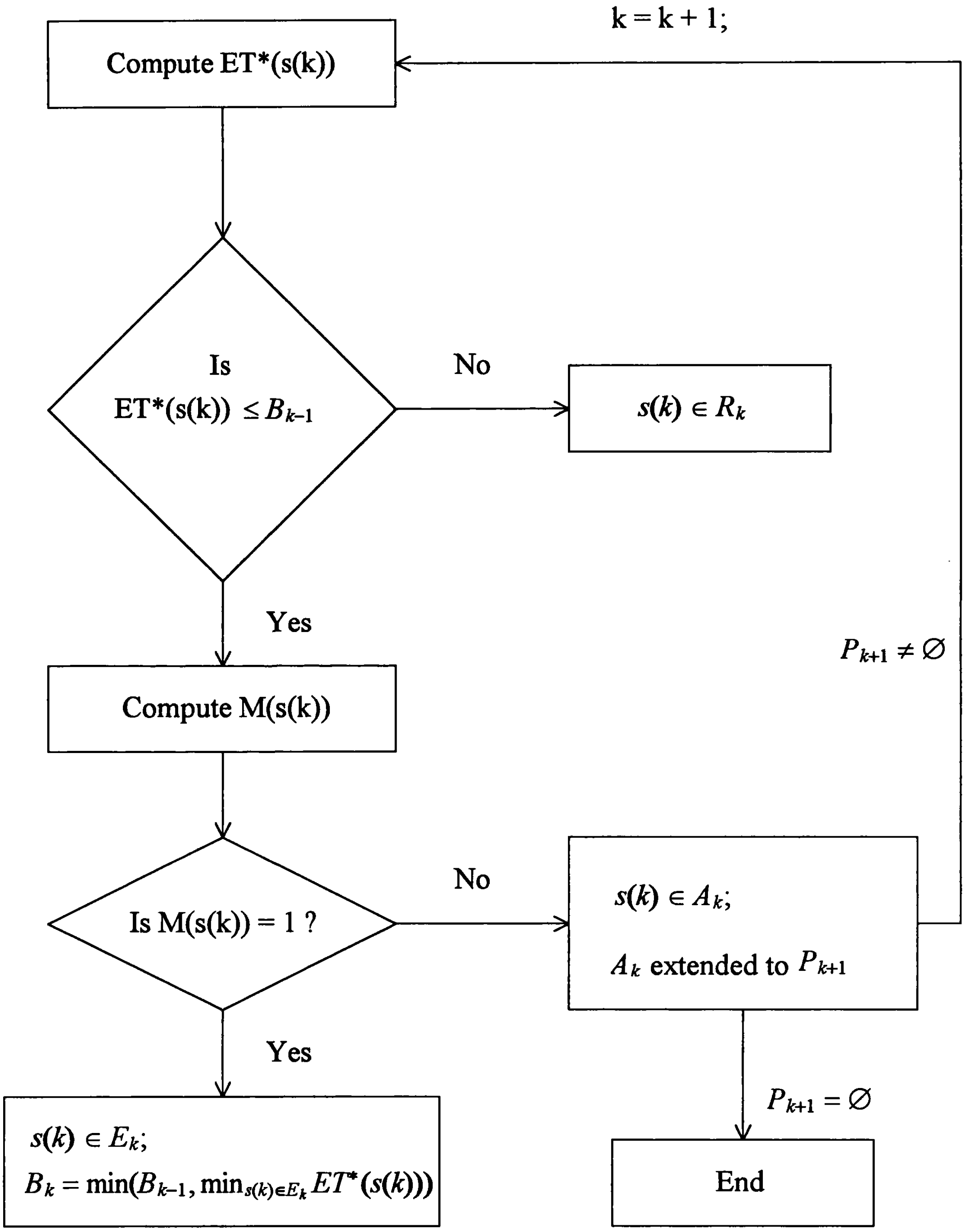


Figure 2.6: Flow Chart For The Algorithm

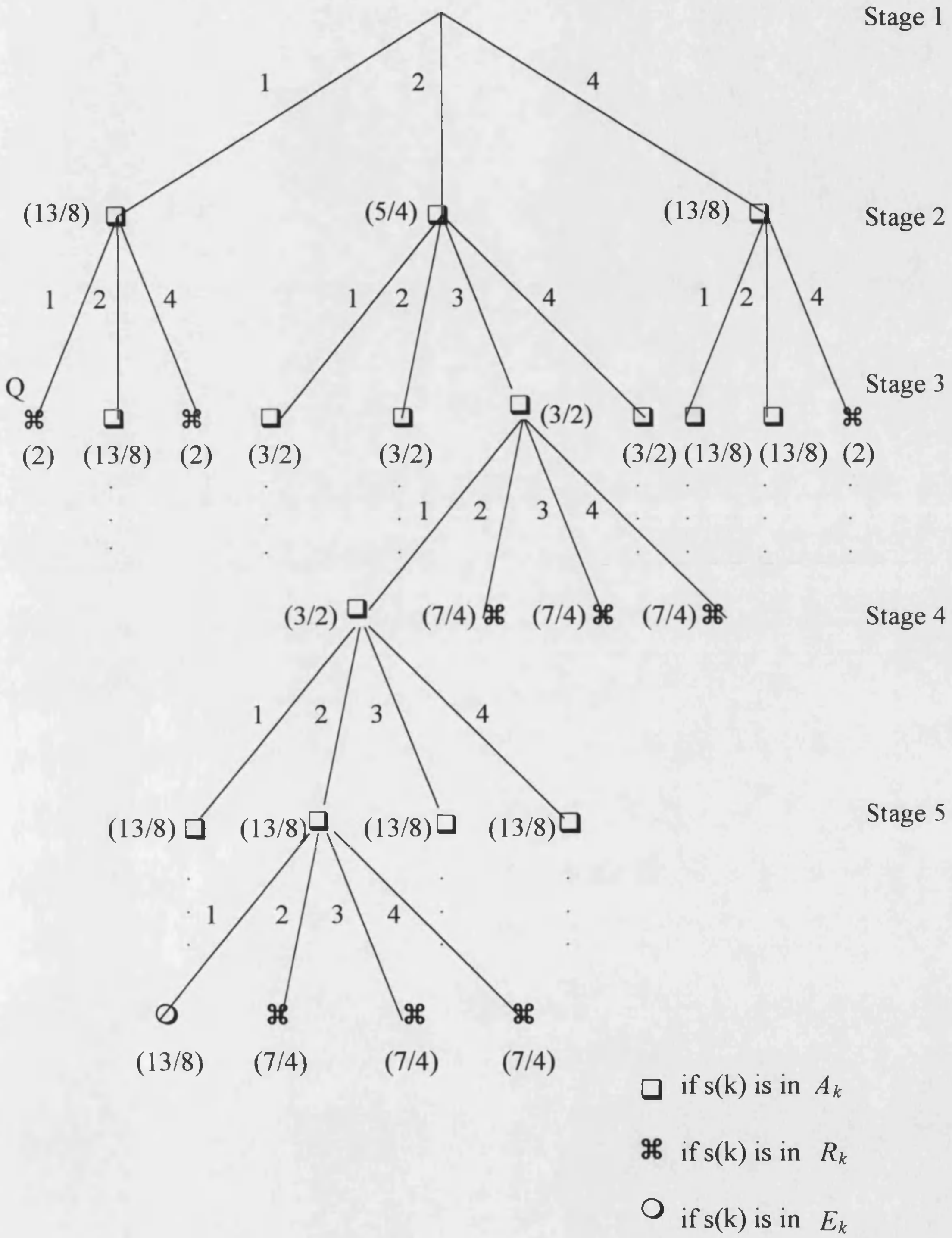


Figure 2.7: Tree Diagram Showing The Computation of $R^a_{2,2}$

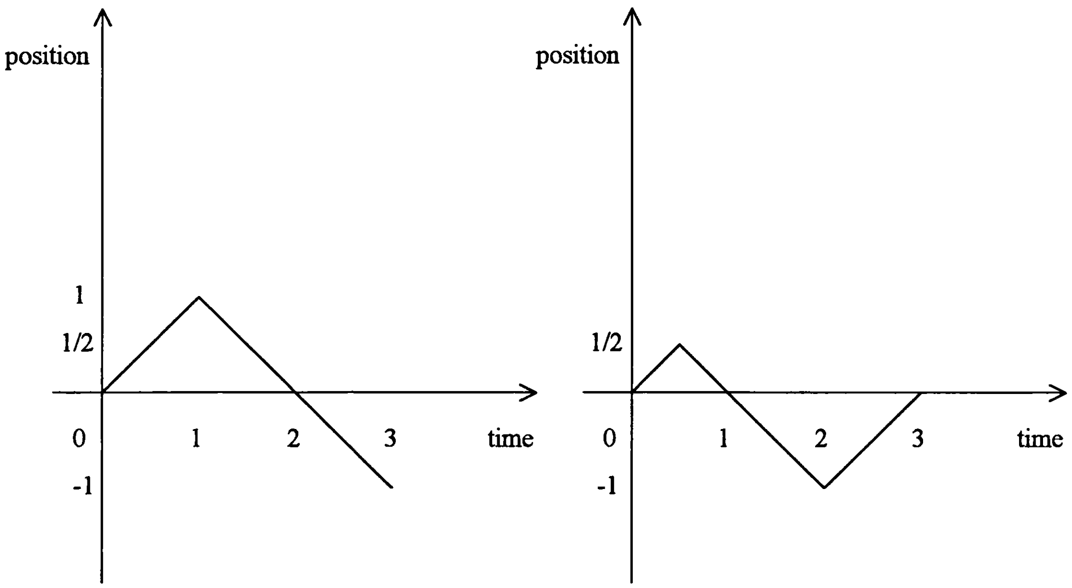


Figure 2.8 a: Optimal strategy pair (2,3,1,2,1)

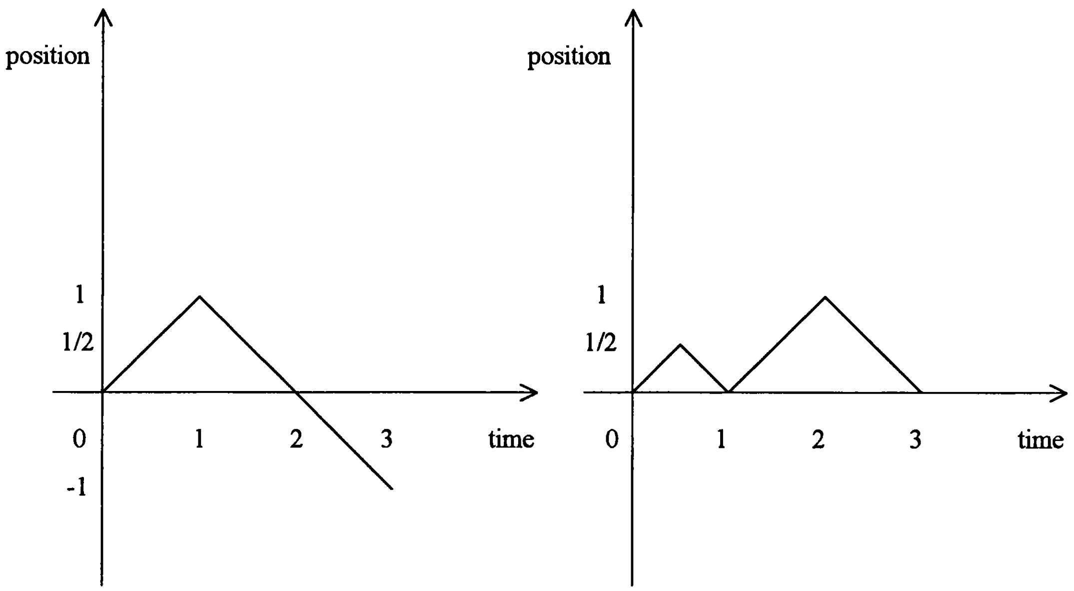


Figure 2.8 b: Optimal strategy pair (2,4,1,2,1)

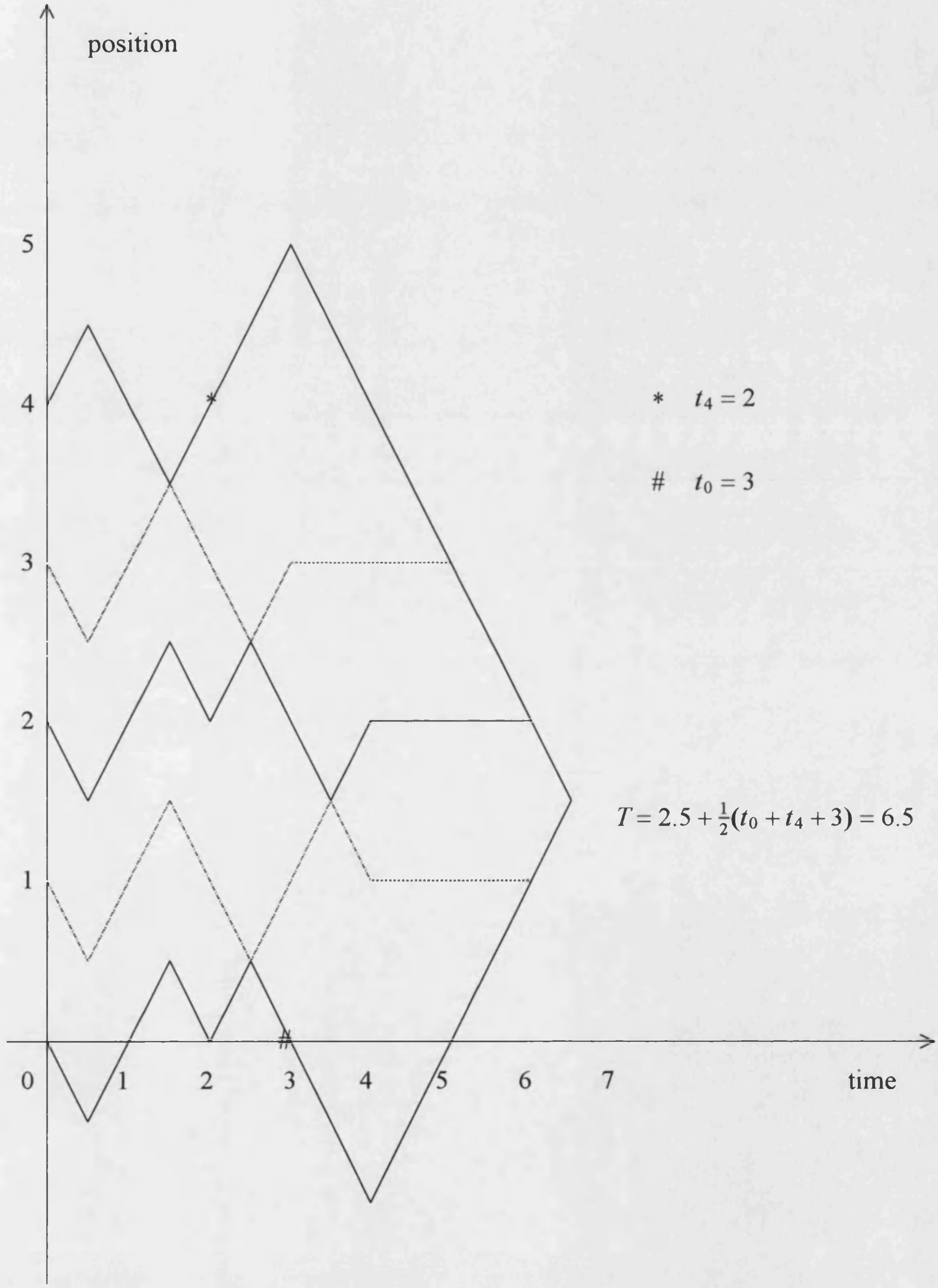
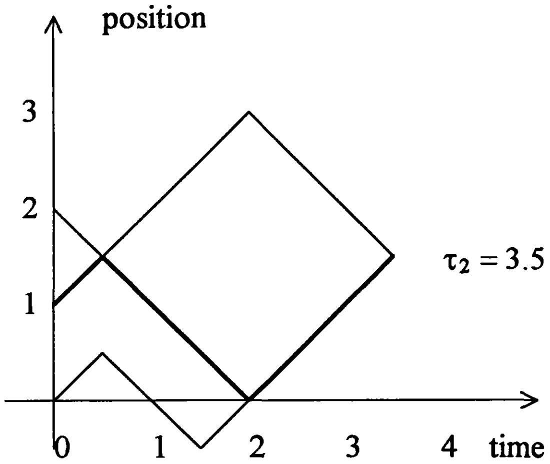
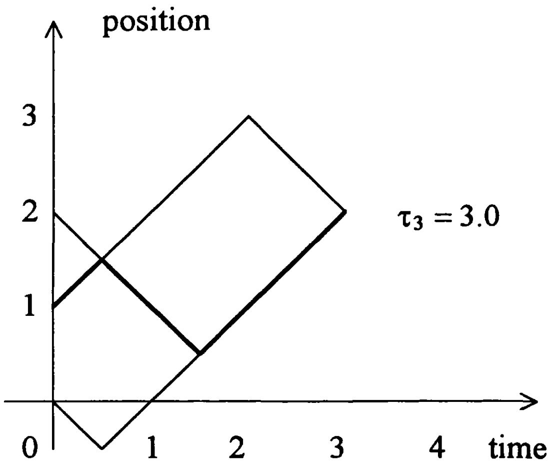


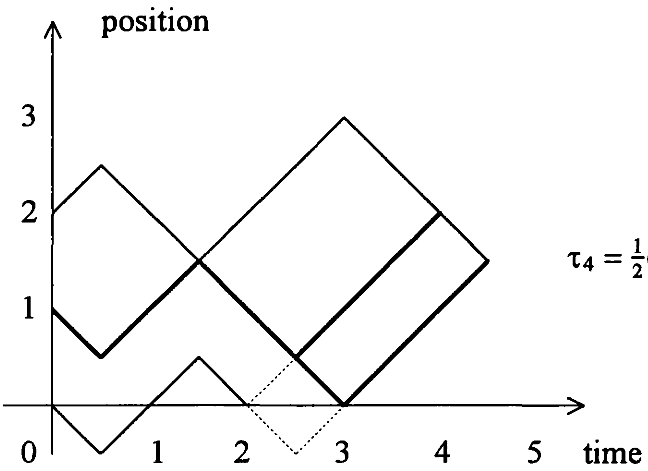
Figure 2.9: A typical instance of the symmetric n-person rendezvous strategy (n=5)



Scenario 2: Players 0 and 2 move towards each other at time 0



Scenario 3: Players 1 and 2 move towards each othher; while Player 0 moves away from 1 at time 0



$\tau_4 = \frac{1}{2}(4.0+4.5) = 4.25$

Scenario 4: Players 0 and 2 move away from each other at time 0

Figure 2.10: Computation of meeting times $\tau_i (i = 2, 3, 4)$

Chapter 3

Minimax Rendezvous On The Line

3.1 Introduction

In this chapter we ask how much time is needed to ensure that n players, placed randomly onto consecutive integers on the line, can all meet together at a single point. We assume that they cannot see each other and can move at unit speed. We also assume that they have no common notion of a positive direction on the line, or equivalently are each placed pointing in a random direction. This setup is similar in essence to that of the rendezvous problem considered in Chapter 2 *except* that players here have a different objective. We seek the minimum time M_n by which some n-tuple of strategies guarantees a group meeting regardless of the initial placement of the players. For example, it is clear that $M_2 \leq 3$ because if one player remains still while the other goes say one unit to the right followed by two to the left, then the meeting time is 1 or 3.

The problem considered here is a search game, and fits into the framework of [10] and [20], except that this problem is far from being zero sum.

More specifically, the problem that we consider here can be seen as a minimax version of the *rendezvous search problem* [1] which has previously been analyzed only in terms of expected time minimization ([1], [5], [8], [7], [4], [17] and as in Chapter 2). However there are many applications in search theory where expected time minimization is not the most appropriate solution concept, and minimax is one of several others that has been studied in various contexts. It seems that minimax is appropriate even in the original rendezvous problem of Schelling [22] where two parachutists have to meet after a simultaneous landing in a large field. If the overall plan involves their moving out together from this field at time t_1, then they must be dropped into the field no later than t_1 minus the minimax rendezvous time. Clearly, the symmetric version of the rendezvous problem (where all players use the same mixed strategy) is not appropriate to the minimax context because it allows the possibility that all players might use the same pure strategy. (They would never meet in the case where they are initially pointed in the same direction.) Hence, only the asymmetric version (where players are allowed to use different pure strategies) is of interest here.

This chapter finds the first exact value for a full three player rendezvous problem, *i.e.* the time needed for all three to meet at a single point. We do this in a minimax context and find that 4 is the least time required to ensure that three players placed at unit distances apart can meet, that is $M_3 = 4$. In order for all three to meet this quickly, the two players who first meet must in some cases split up to find the remaining player, before regrouping in a threesome. The problem facing the two who meet first is thus similar to that studied in a different context ([23], [19]).We also consider an important variant of the problem, namely "sticky" rendezvous, where players who meet are required to remain together. We find that when players' strategies are thus restricted, they need 5 time units to ensure full three player rendezvous. Our final result concerns the behavior of M_n for large values of n, the time required for large numbers of players to meet. We find that M_n is asymptotic to $n/2$. This asymptotic value is the same as for the least expected time in the symmetric problem, as proved in Section 2.7, although of course the problem and solution concept are entirely different.

This chapter is organized as follows. In Section 3.2 we give a precise formulation of the minimax rendezvous problem, in its unrestricted and 'sticky' forms. In Section 3.3 we analyze the two player problem in an extended form that will arise for three players after two of them meet. In particular we establish that $M_2 = 3$. In Section 3.4 we solve the sticky version of the three person problem, proving that sticky players require 5 time units to guarantee a three way meeting. In Section 3.5 we use similar arguments to show that without this restriction three players can meet in 4 time units, i.e., $M_3 = 4$. The final Section 3.6 establishes the asymptotic result that $\lim_{n\to\infty} M_n/n = 1/2$.

3.2 The Minimax Rendezvous Time

In this section we give the definitions of the n-player rendezvous problems Γ_n and the associated minimax rendezvous time M_n. We also define their 'sticky' counterparts, in which players who meet must thenceforth remain together.

The problem (or game) Γ_n begins with a random placement of the n players onto the integers $0, 1, \cdots, n-1$ (or equivalently any n consecutive integers). The initial position of player i is denoted by π_i where $\{\pi_1, \pi_2, \ldots, \pi_n\}$ is equals to $\{0, 1, \ldots, n-1\}$. Since the players do not have a common notion of a positive direction on the real line, we assume they are initially faced in independent random directions. We use ω_i to denote the direction that player i is initially pointing, from an observer's fixed global view, where $\omega_i \in \{+1, -1\}$. Let $C = C_n$ denote the set of all $n!2^n$ *initial configurations* or *cases* of the form $c = \{\pi_1, \pi_2, \ldots, \pi_n, \omega_1, \ldots, \omega_n\}$, where the first n coordinates denote positions, while the next n denote directions.

A strategy for player i in the game Γ_n is a rule that gives his motion (relative to his starting point and starting direction) as a function of the information he receives from players he may meet. A strategy profile is simply an n-tuple of strategies, one for each player. A strategy profile together with an initial configuration determines completely the motions of all the players. We define $T_c\left(\bar{S}\right)$ to be the first time (if any) that all n players following the profile $\bar{S}$ meet together at a single point, when the initial configuration is case c.

In particular, a strategy for player i must say how that player should move before he meets anyone. This part of the strategy is called the Stage 1 strategy. A Stage 1 strategy s_i is a path with speed bounded by one belonging to the set

$$P = \left\{p : \Re^+ \to \Re,\, p(0) = 0,\, |p(t_1) - p(t_2)| \le |t_1 - t_2|\right\}.$$

The Stage 1 path of a player following strategy s_i when the initial configuration is c is given by $\pi_i + \omega_i s_i(t)$. In a recent paper [17] it is shown that when inital distances between adjacent players are integers, any strategy profile can be modified to one in a subset of P called P^* where players have piecewise linear paths, with slopes ± 1, and which turn only at times $k/2$ where k is an integer. This modification does not postpone any meeting between players, and hence does not postpone the final meeting of all the players. For this reason we will further assume that throughout the game all players are restricted to paths in P^*. Observe that this assumption implies that $T_c\left(\bar{S}\right)$ is always half of an integer. A useful notational device for describing Stage 1 strategies in P^* is simply to list the slopes in successive half units of time. Thus $s = [+1, -1, +1, -1, \ldots]$ describes a path that oscillates between its starting point and a point a half unit above the start. More generally, if $s = [x_1, x_{2,\ldots}]$ then we have (with [ ] denoting integer part),

$$s(t) = \sum_{m=1}^{[2t]} x_m + x_{[2t]+1}\left(\frac{2t - [2t]}{2}\right).$$

Note. One should notice that this notation for describing Stage 1 strategies in P^* differs from the convention introduced in Section 2.5. In the latter, a strategy triple where all three players adopt the path that oscillates between its starting point and a point a half unit above the start for the first $(k+1)/2$ units of time is denoted by a $3 \times k$ matrix where all entries are -1.

One final remark regarding player strategies in Γ_n is that we may assume without loss of generality that they all begin with $+1$, that is they go in the forward direction for the first half unit of time. This assumption is valid because it does not restrict the actual player motion, since the player may be initially pointed either way.

The maximum rendezvous time for a strategy profile $\bar{S}$, denoted simply $\bar{T}\left(\bar{S}\right)$ is defined as

$$\bar{T}\left(\bar{S}\right) = \max_{c \in C} T_c\left(\bar{S}\right).$$

Then we may define the minimax rendezvous time M_n as

$$M_n = \min_{\bar{S}} \bar{T}\left(\bar{S}\right),$$

where the minimum is taken over all strategy profiles $\bar{S}$. We note that the index n does not appear on the right side of the above equation, but of course the player number parameter n is implicit in all the definitions in this section. The existence of the minimum follows from our assumption that all player paths must belong to P^*.

In Section 3.4 we will consider a further restriction on player paths, namely that when players meet they stick together. All the above definitions remain valid, with this assumption. The sticky version of the n-player game is denoted $\tilde{\Gamma}_n$ and the sticky minimax time is denoted $\tilde{M}_n$.

3.3 Two Player Minimax Rendezvous

In this section we consider the minimax rendezvous problem on the line for the case of two players. In passing, we will prove that the minimax rendezvous time for the standard two player problem Γ_2 is 3, but we will mainly be concerned with a more general two person problem denoted by $\Gamma(\alpha, \beta)$. We are forced to consider this more general problem because this is what the three player 'sticky' problem (discussed in the next section) collapses to after two players meet.

The problem $\Gamma(\alpha, \beta)$ is an asymmetric information rendezvous game defined as follows. Player I is placed at some point on the line (which we take as the origin 0) and pointed facing up (the line is taken to be vertical). Player II is then faced in a random direction either a distance α above player I or a distance β below player I (*i.e.* at α or at $-\beta$). Thus player II knows only that his partner is a distance α or β away, while player I knows that his partner is either α above (forward) or β below. If $\alpha = \beta = 1$ then the information is symmetric and indeed $\Gamma(1, 1)$ is the same as the problem we called Γ_2. If player I chooses a strategy $s_1 = f \in P$ and player II chooses a strategy $s_2 = g \in P$ (we cannot restrict strategies to P^* unless α and β are integers) then the maximum rendezvous time $T^{\alpha,\beta}(f,g)$ is the first time t when the path $f(t)$ has intersected the four paths given below.

$$\begin{aligned} Z_1(t) &= \alpha - g(t), \\ Z_2(t) &= \alpha + g(t), \\ Z_3(t) &= -\beta - g(t), \\ Z_4(t) &= -\beta + g(t). \end{aligned}$$

That is, $T^{\alpha,\beta}(f,g)$ is the maximum of $t_j = t_j(f,g) = \min\{t : f(t) = Z_j(t)\}$, $\forall j = 1,\ldots,4$. The minimax rendezvous time for $\Gamma(\alpha,\beta)$ is the minimum of $T^{\alpha,\beta}(f,g)$, $\forall (f,g) \in P \times P$.

Note that any strategy pair (f,g) determines an ordering of the meeting times t_j, *i.e.* there is a permutation $\sigma = \sigma(f,g)$ of $\{1,2,3,4\}$ such that

$$t_{\sigma(1)} \leq t_{\sigma(2)} \leq t_{\sigma(3)} \leq t_{\sigma(4)} = T^{\alpha,\beta}(f,g).$$

In such a case we will say the (f,g) has permutation type σ. If α and β are integers and $(f,g) \in P^* \times P^*$ then the permutation type is unique and $t_{\sigma(j+1)} \geq t_{\sigma(j)} + 1/2$.

There is a complementary notion (introduced in [5]) by which each permutation σ determines a canonical strategy pair (F_σ, G_σ), such that in the interval $t_{\sigma(j-1)} < t < t_{\sigma(j)}$, $j = 1,\ldots,4$ (with $\sigma(0)$ defined as 0 and t_0 as 0) the path F_σ and the path $Z_{\sigma(j)}$ (which depends on G_σ) are moving towards each other at maximum speed. It is shown in [5, Theorem 3 and its proof] that *the canonical strategy pair (F_σ, G_σ) minimizes all the meeting times t_j, within the class of strategy pairs of permutation type σ*. It follows that the minimax rendezvous time, as well as the leasted expected rendezvous time, will be attained for some canonical strategy. However there may also be non-canonical strategy pairs which acheive the minimax rendezvous time.

To see how this definition defines a unique strategy pair (F_σ, G_σ), we illustrate the construction for the identity permutation $\bar{\sigma}$, using the simpler notation $\bar{f} = F_{\bar{\sigma}}$ and $\bar{g} = G_{\bar{\sigma}}$. The path $\bar{f}$ is pictured in Figure 3.1 for the parameters α, β where $\alpha \leq \beta$. The strategies begin with the player I path $\bar{f}$ and the player II possible path $Z_1(t) = \alpha - \bar{g}(t)$ moving towards each other at maximum speed. Since Z_1 is above $\bar{f}$ at time zero, this means that $\bar{f}' = 1$ and $Z_1' = -1$, or $\bar{g}' = +1$, from time zero until the first meeting time $t_1 = \alpha/2$. At this time Z_2 is above $\bar{f}$ and so $\bar{f}' = +1$ and $Z_2' = -1$, or $\bar{g}' = -1$, from time $t_1 = \alpha/2$ until $\bar{f}$ meets Z_2 at time $t_2 = \alpha$. Note that $\bar{f}(\alpha) = \alpha$ and $\bar{g}(\alpha) = 0$. At time $t_2 = \alpha$, $Z_3(\alpha) = -\beta - \bar{g}(\alpha) = -\beta$ is below $\bar{f}(\alpha) = \alpha$ and so for the next time interval,

$$t_2 < t < t_3 = t_2 + (\alpha - (-\beta))/2 = (3\alpha + \beta)/2,$$

$\bar{f}' = -1$ and $Z_3' = +1$, or $\bar{g}' = -1$. At time t_3, $\bar{f}$ is at $(\alpha - \beta)/2$, while Z_4 is lower, at $-(\alpha + 3\beta)/2$. Hence $\bar{f}$ goes down and Z_4 goes up (or $\bar{g}$ goes up) throughout the interval

$$t_3 < t < t_4 = t_3 + \left(\frac{\alpha - \beta}{2} - \left(-\frac{\alpha + 3\beta}{2}\right)\right)/2 = 2\alpha + \beta.$$

The data just derived are presented in the first data row of the following table which gives the four meeting times $t_{\sigma(j)}$ for the canonical strategies (F_σ, G_σ) corresponding to various permutations. While this table contains just six of the 4!=24 possible permutations, it will in fact be sufficient to calculate the minimax rendezvous time (this will be evident in the proof of the following theorem).

σ	$t_{\sigma(1)}$	$t_{\sigma(2)}$	$t_{\sigma(3)}$	$t_{\sigma(4)} = T^{\alpha,\beta}(F_\sigma, G_\sigma)$
$(1,2,3,4)$	$\alpha/2$	α	$(3\alpha+\beta)/2$	$2\alpha+\beta$
$(1,2,4,3)$	$\alpha/2$	α	$(3\alpha+\beta)/2$	$2\alpha+\beta$
$(1,3,2,4)$	$\alpha/2$	$(2\alpha+\beta)/2$	$(3\alpha+\beta)/2$	$2\alpha+\beta$
$(1,3,4,2)$	$\alpha/2$	$(2\alpha+\beta)/2$	$\alpha+\beta$	$(3\alpha+3\beta)/2$
$(1,4,2,3)$	$\alpha/2$	$(\alpha+\beta)/2$	$\alpha+\beta$	$(3\alpha+3\beta)/2$
$(1,4,3,2)$	$\alpha/2$	$(\alpha+\beta)/2$	$\alpha+\beta$	$(3\alpha+3\beta)/2$

Table 3.1: Meeting times for canonical strategies (F_σ, G_σ).

The data in this table will be useful in the following.

Theorem 3.1 *The minimax rendezvous time for the problem* $\Gamma(\alpha,\beta)$, $\alpha \leq \beta$, *is*

$$\min_{f,g\in P\times P} T^{\alpha,\beta}(f,g) = 2\alpha+\beta.$$

Furthermore, if $T^{\alpha,\beta}(f,g) = 2\alpha+\beta$ *and* $\alpha < \beta$, *then the permutation type of* (f,g) *is (1,2,3,4), (1,2,4,3), or (1,3,2,4).*

Proof. According to the result of [5] quoted above, the minimax rendezvous time is the minimum of $T^{\alpha,\beta}(F_\sigma, G_\sigma)$ as σ varies over the permutations of $\{1,2,3,4\}$. We claim that it is sufficient to consider only the six permutations listed in the table above (those where the player II path $\alpha - g(t)$ is intersected first), where the minimum (given that $\alpha \leq \beta$) is $2\alpha+\beta$. The cases which intersect the player II path $\alpha + g(t)$ first will give the same results, that is the same meeting times $t_{\sigma(j)}$, the same F, and a sign reversal for G. The cases which intersect either of the paths $-\beta \pm g(t)$ first will give a similar table with α and β interchanged. So the only new maximum rendezvous time appearing in such a table would be $2\beta + \alpha$, which is not less than $2\alpha + \beta$. When $\alpha < \beta$, only the first three data rows of the table give the minimum $2\alpha + \beta$. □

As a special case of the above result when $\alpha = \beta = 1$, we have the following solution to the standard two person minimax rendezvous problem.

Corollary 3.1 $M_2 = 3$

The three person 'sticky' rendezvous problem $\tilde{\Gamma}_3$ to be analyzed in the next section may reduce to the problem $\Gamma(1,2)$ in certain cases. For this reason we explicitly give the four optimal player I strategies in P^* for $\Gamma(1,2)$ in the following result.

Corollary 3.2 *The minimax rendezvous value for the problem* $\Gamma(1,2)$ *is* 4. *Furthermore if* $T^{1,2}(f,g) = 4$, *for* $(f,g) \in P^* \times P^*$ *then* f *is one of the four strategies in* P^* *which satisfy*

$$f(1/2) = 1/2, \;\; f(5/2) = -1/2, \;\; \textit{and} \;\; f(4) = -2, \;\; \textit{i.e.}$$

$$\begin{aligned} f_1 &= [+1,+1,-1,-1,-1,-1,-1,-1], \\ f_2 &= [+1,-1,+1,-1,-1,-1,-1,-1], \\ f_3 &= [+1,-1,-1,+1,-1,-1,-1,-1], \\ f_4 &= [+1,-1,-1,-1,+1,-1,-1,-1]. \end{aligned}$$

Each of these has maximum rendezvous time of 4 when paired with the player II strategy $g = [+1,-1,-1,-1,-1,+1,+1,+1]$. *(See Figure 3.2.)*

Proof. The reader should first verify the obvious fact that the f_k' s are indeed the only four strategies in P^* which satisfy the three conditions (including of course $f(0) = 0$, which is part of the definition of P^*). According to Theorem 3.1 we can only find optimal strategies for the permutation types (1,2,3,4), (1,2,4,3), and (1,3,2,4).

Case 1, σ is (1,2,3,4) or (1,2,4,3): Suppose $T^{1,2}(f,g) = 4$ for $(f,g) \in P^* \times P^*$, and

$$t_1 < t_2 < t_3, t_4.$$

We may assume that $t_1 = 1/2$ and $f(1/2) = g(1/2) = 1/2$ because in the alternative case that $f(1/2) = -1/2$ the resulting problem at time 1/2 is either $\Gamma(1,2)$(if I is told that II moved down) or $\Gamma(2,1)$ (if I is told that II moved up), and therefore (by Theorem 3.1) requires an *additional* 4 units of time to ensure meeting. Since $g(t) - g(1/2) \geq 1/2 - t$ and $g(1/2) = 1/2$ it follows that $g(t) \geq 1/2 - (t - 1/2) = 1 - t$, for $t \geq 1/2$. Hence after f has intercepted $1 + g(t)$ at time t_2, we must have that

$$f(t) \geq 2 - t, \text{ for } t \geq t_2. \tag{3.1}$$

Since f intersects $1+g$ at time t_2 and $-2+g$ at time $t_4 \leq 4$, and speeds are bounded by 1, we must have

$$\begin{gathered} t_4 - t_2 \geq |(1 + g(t_2)) - (-2 + g(t_2))|/2 = 3/2, \\ \text{or } t_2 \leq t_4 - 3/2 \leq 5/2. \end{gathered} \tag{3.2}$$

Suppose that strict inequality holds in (3.1), that $f(s) = 2-s+p$ for some $p > 0$ and some time $s < 4$. Then a player I starting at position $2+p$ at time zero, following path $2-t+p$ until time $t = s$, and then following path f, could ensure meeting paths $-2 \pm g$ by time 4. But this would mean that the minimax rendezvous time for the problem $\Gamma(0, 4+p) = \Gamma(4+p, 0)$ is not more than 4, whereas Theorem 3.1 says it is equal to $4+p$. Hence our assumption that $f(t)$ could be larger than $2-t$ was false, and (3.1) must hold with equality, that is,

$$f(t) = 2-t, \text{ for } t \geq 5/2. \tag{3.3}$$

We showed earlier that $f \in P^*$ must go through (1/2,1/2) and (3.3) shows further that it must go through (5/2, -1/2) and (4, -2). It follows that it must be one the four strategies f_k. (Actually it cannot be f_4 in this case, but we do not need to prove this fact.)

Case 2, σ is (1,3,2,4): Suppose $T^{1,2}(f,g) = 4$ for $(f,g) \in P^* \times P^*$, and

$$t_1 < t_3 < t_2 < t_4 = 4.$$

Since speeds are bounded by 1, it follows that

$$\begin{gathered} 4-t_2 = t_4 - t_2 \geq |Z_4(t_2) - Z_2(t_2)|/2 = |-2+g(t_2) - (1+g(t_2))| = 3/2, \\ \text{hence we have } \; 5/2 \geq t_2 \geq t_3 + 1/2, \\ \text{i.e., } t_3 \leq 2. \end{gathered} \tag{3.4}$$

By the same reasoning as (3.4) , we have

$$\begin{array}{rcl} t_3 - t_1 & \geq & |Z_3(t_1) - Z_1(t_1)|/2 \\ & = & |-2-g(t_1) - (1-g(t_1))|/2 \\ & = & 3/2, \\ \text{and so} \qquad t_3 & \geq & t_1 + 3/2 \geq 2. \end{array} \tag{3.5}$$

The only solution to (3.4) and (3.5) is $t_1 = 1/2$, $t_3 = 2$, $t_2 = 5/2$, $t_4 = 4$, which are the times for the canonical strategy pair with this permutation. Hence f must be $F_\sigma = f_4$. □

The optimal strategies for the problem $\Gamma(1,2)$ are drawn in Figure 3.2. They are also drawn in a stacked form in Figure 3.4, which will be discussed later.

We conclude this section with an analysis of some two person rendezvous problems where one of the players (taken to be I) knows the direction of the other. Since this person will clearly move at speed one in this direction, these are really one person problems. The only strategic variable is the path of player II. These results are called lemmas because they will be used in Section 3.5 in the following way: When two players meet and do not know the direction of the other, they will each be assigned a direction and will assume the remaining player lies in that direction. A lower bound on the maximum time taken to find the remaining player, assuming he is in this direction, is the minimax value of the game in which the direction is known. It is these minimax values that we now calculate. These are all very simple results.

Lemma 3.1 *Suppose player I is placed facing up at* 0 *and player II is either placed facing down at* 1 *or facing up at* 2*. Then the minimax rendezvous time is* 3/2*. Furthermore this maximum meeting time occurs if and only if I moves up at speed one and II uses a strategy* $h \in P$ *satisfying* $h(3/2) = -1/2$*. There are three strategies* $h \in P^*$*, defined up to time* 3/2*, satisfying this condition. In the notation giving the slopes of the paths in successive time intervals of length* 1/2 *these paths (as shown in Figure 3.3) are as follows:*

$$\begin{aligned} h_1 &= [+1,-1,-1], \\ h_2 &= [-1,-1,+1], \\ h_3 &= [-1,+1,-1]. \end{aligned}$$

Proof. If player II uses strategy h, the maximum meeting time $T(h)$ is the time required for the path t (of player I) to meet both possible paths of II, that is $1-h(t)$ and $2+h(t)$. This is the same as the time required for $h \in P$ to meet both $1-t$ and $t-2$ (See Figure 3.3). Clearly if $h(3/2) = -1/2$ then it meets both these paths at time $3/2$. If it meets either of these paths before this time, the earliest it can meet the other is $3/2$, since these two paths are approaching each other at combined speed 2. Furthermore if $h(3/2) > -1/2$ then it cannot yet have intersected $t-2$; if $h(3/2) < -1/2$ then it cannot yet have intersected $1-t$. Thus only paths with $h(3/2) = -1/2$ can acheive a maximum meeting time of $3/2$. The three paths stated in the lemma are the only ones in P^* satisfying this condition. These paths are illustrated in thick lines in Figure 3.3.□

The following two lemmas are even easier, as they give minimax times when player I knows not only the direction but also the initial distance to player II. They may appear too obvious to bother stating, but we do so because they will be used repeatedly in Section 3.5, without specific mention. (The first is actually a corollary of Theorem 3.1, with $\alpha = 0$).

Lemma 3.2 *If player I is placed (at time* 0*) facing up at position* 0 *and player II is placed in a random direction at position* $\beta > 0$*, at any time prior to time* β*, then the minimax rendezvous time is* β*. Call this problem* $\Gamma'(\beta)$.

Lemma 3.3 *If player I is placed (at time* 0*) facing up at position* 0 *and player II is placed in a known direction (say up) at position* $\beta > 0$*, at time* δ*,* $-\beta \le \delta \le \beta$*, then the minimax rendezvous time is* $(\beta + \delta)/2$*. Call this problem* $\Gamma''(\beta, \delta)$.

3.4 Sticky Three Person Rendezvous

We are now in a position to attack the problem $\tilde{\Gamma}_3$. Recall that in this problem three players are randomly placed onto the integers $0, 1$, and

2, and faced in random directions. Once two players meet, they must stick together while trying to locate the third. The players' strategy paths are assumed to belong to P^*. The main result of this section is the determination of the minimax rendezvous time $\tilde{M}_3$ for this problem.

Theorem 3.2 *The minimax rendezvous time $\tilde{M}_3$ for the sticky three person problem $\tilde{\Gamma}_3$ is* 5.

Proof. We first show that $\tilde{M}_3 \leq 5$ by exhibiting a simple strategy triple which guarantees three player rendezvous by time five. The simplest version is that two of the players remain still (until they are met by the moving player) while the third moves forward, taking along any player he meets, until he reaches an integer location with no player on it. He then reverses direction, similarly taking along any player he meets, until he has accumulated both of the other players. The case with maximum rendezvous time is when the moving player starts in the middle, and in this case the rendezvous time is 5. Since the strategy of staying still in Stage 1 does not belong to P^*, it has to be modified. The modification is simply to oscillate between the starting point and a point 1/2 unit forward. The analysis for the modified strategy is essentially the same and it also has a maximum rendezvous time of 5.

To demonstrate that $\tilde{M}_3 \geq 5$, we assume that there is a strategy triple S^* with $T(S^*) = 9/2$, and then show that this assumption leads to a contradiction. Since for strategies involving paths in P^* intersections can occur only at integer multiples of 1/2, this will establish that $\tilde{M}_3 \geq 5$.

Let $S = (s_1, s_2, s_3)$ be the Stage 1 strategies for S^*. We may assume that each is simply the identity function t for $t \leq 1/2$. Observe that for any of the three players $j = 1, 2, 3$, there is an initial configuration $c = c(j)$ for which the two players other than j meet at time 1/2. (For example $c(2) = (0, 2, 1, +1, +1, -1)$.) Let χ_j denote the strategy (path) followed by the two who meet in case $c(j)$ from time $1/2$ onwards. We normalize this so that the position of these two players at time $t + 1/2$ is $\chi_j(t)$ plus their position when they meet at time $1/2$. Thus χ_j belongs to P^* (takes value 0 at 0). Similarly let $\tilde{s}_j$ denote the remainder of player j's path from time $1/2$ onwards, $\tilde{s}_j(t) = s_j(t + 1/2) - s_j(1/2)$. Thus $\tilde{s}_j$ also belongs to P^* (takes value 0 at 0). Note that the situations of player j and of the remaining two players are the same as that of players I and II in the game $\Gamma(1, 2)$; the paired players are either 1 unit above player j (that is in the direction he was initially pointed) or 2 units below him. Hence it follows that

$$T_{c(j)}(S^*) = 1/2 + T^{1,2}(\tilde{s}_j, \chi_j),$$

where $T^{1,2}$ is the minimax rendezvous time defined in the previous section for the game $\Gamma(1, 2)$. Our assumption that $T(S^*) = 9/2$ implies that $T_{c(j)}(S^*) \leq 9/2$ and by the above that $T^{1,2}(\tilde{s}_j, \chi_j) \leq 4$. It follows from Corollary 3.2 that $T^{1,2}(\tilde{s}_j, \chi_j) = 4$ and that $\tilde{s}_j$ belongs to the set of optimal strategies for player I in $\Gamma(1, 2)$, that is, to the set $\{f_1, f_2, f_3, f_4\}$

.

Since the above argument holds for each player $j = 1, 2, 3$, we have shown that the Stage 1 paths of S^* must be optimal for the problem $\Gamma(1, 2)$ from time $1/2$ onwards, that is

$$\tilde{s}_j \in \{f_1, f_2, f_3, f_4\}, \quad \text{for } j = 1, 2, 3.$$

It now follows that there is a case $\bar{c}$ for which none of the three players meet (not even two of them) by time $9/2$, that is $T_{\bar{c}}(S^*) > 9/2$, which contradicts our assumption. To see that such a case (initial configuration) $\bar{c}$ exists, first look at Figure 3.4. This shows a drawing of the four paths f_1, f_2, f_3, f_4 with each preceeded by a slope 1 diagonal for time

1/2. The lower indexed functions are started at higher positions on the line, and there are no intersections by time 9/2. The general algorithm for choosing $\bar{c}$ as a function of (s_1, s_2, s_3) is very simple: Point all the players up, and place the players using lower indexed f_k's higher. If two players are using the same f_k then of course it doesn't matter which of these is placed higher. For example, if $(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) = (f_1, f_4, f_3)$, then $\bar{c} = (2, 0, 1, +1, +1, +1)$. □

3.5 Three Player Rendezvous (Unrestricted)

In this section we show that three players placed on adjacent integers can ensure a three way meeting by time 4, that is $M_3 = 4$. This is a savings of one time unit over the 5 needed in the sticky case considered in the previous section. The novel feature considered here is that players who meet can separate to find the third (although the game does not end until all three are together).

Lemma 3.4 $M_3 \leq 4$.

Proof. We exhibit a strategy profile with a maximum rendezvous time of 4. The Stage 1 strategies are the same as for the optimal sticky rendezvous: Player 1 follows the path (given in slope form for half unit time interval) $[+1, +1, -1, -1, -1, -1]$ up to time 3. The other two players follow a path which oscillates between their start and a point half a unit away, such as $[+1, -1, +1, -1, +1, -1, +1, -1]$. If player 1 has not met another player by time 1, then he can conclude the other players were both behind him, so when he reverse direction at time 1 he continues forever in this direction, bringing with him the first player he meets (at time 3), and meeting the second at time 4. If he first meets another player at time 1, the two who meet know that the remaining player is either 1 above or 2 below, and will be there at every integer time. So one of them (say player 1) goes 1 above and then reverses, while the other goes 2 below and then reverses. If either finds the remaining player he asks that player to stick with him. Thus the two who originally met will meet again in not more than three time units, at player 1's starting point. Furthermore one of the two is sure to have brought the remaining player along with him. Finally, if player 1 first meets a player at time 1/2, he can ignore this and bring that player back to that player's start. This puts the two who met in the situation analyzed above. Thus in any case the rendezvous time is not more than 4. □

Lemma 3.5 *$M_3 \geq 7/2$. Furthermore any strategy profile for the game Γ_3 which has a maximum rendezvous time of 7/2 must have all its stage 1 paths, up to time 2, belonging to the set $\left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right\}$ defined as follows. The path $\tilde{h}_k$ is the path h_k of Lemma 3.1, preceeded by a forward speed one motion for $t \leq 1/2$. That is, $\tilde{h}_k(t) = t$, $t \leq 1/2$, and $\tilde{h}_k(t) = h_k(t - 1/2) + 1/2$, $t \geq 1/2$. In the notation giving the slope in each half unit of time, these paths are*

$$\begin{aligned}
\tilde{h}_1 &= [+1 + 1, -1, -1], \\
\tilde{h}_2 &= [+1, -1, -1, +1], \\
\tilde{h}_3 &= [+1, -1, +1, -1].
\end{aligned}$$

Proof. As in the proof of Theorem 3.2, we begin by assuming an initial configuration such that the two players other than player j meet each other at time 1/2. One of these players (call each of these player I), must go up to find the remaining player j (call him player II). Renormalize the line so that the origin (0) is where the two players have met. Assuming II is above this, he is either at 1 facing down (if he started facing down) or at 2 facing up (if he started facing up). Hence by Lemma 3.1 the earliest that the player I who goes up can guarantee finding II, assuming he is up, is (additional) time 3/2. It follows that the earliest the two agents of Player I (the one going up and the one going down) can meet together, bringing along player II, is $\bar{T} = 1/2 + 2\,(3/2) = 7/2$. Furthermore, it follows from the second part of Lemma 3.1 that in order for this time to be acheived, player II must be following one of the paths h_k from time 1/2. Since we are assuming that strategies for Γ_3 begin by going up for time 1/2, it follows that the player we are calling II and j must use a strategy $\tilde{h}_k$ up to time 2. But since this argument applies to any player $j = 1, 2, 3$, we are done. □

Lemma 3.6 *Any strategy for the game* Γ_3, *whose Stage 1 paths (up to time* 2) *belong to the set* $\left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right\}$, *has maximum rendezvous time at least* $M = 4$.

Proof. Since order does not matter, there are ten strategy triples in $\left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right\}^3$. We divide these into four types. For each type we stop the action at some time T_0 and assume a certain set of initial configurations. We then give a lower bound on the maximum remaining time, which when added to T_0 is at least 4.

Type 1: All three use same strategy. This type covers the three strategy profiles $\left(\tilde{h}_k, \tilde{h}_k, \tilde{h}_k\right)$, $k = 1, 2, 3$. For strategy profiles of this type the first integer q such that some players have a different Stage 1 strategy for the time interval $[q/2, (q+1)/2]$ satisfies $q \geq 4$.

Assume that all three players start facing up. Then at time $T_0 = q/2$ they are back at their original positions, and the top and bottom players are at distance 2. By the definition of q, there are two players who move in opposite directions in the interval $[q/2, (q+1)/2]$. So for some initial configuration the player starting at 2 will move up throughout this interval and the player starting at 0 will move down. Hence at time $(q+1)/2$ the players at the ends will be at distance 3. Therefore the earliest these two player can meet is at time $(q+1)/2 + 3/2 \geq (4+1)/2 + 3/2 = 4$.

Type 2: The strategy $\left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_2\right)$ is used. Consider two initial configurations case $c' = (2, 1, 0, +1, +1, +1)$ and case $c'' = (2, 1, 0, +1, -1, -1)$. In each of these configurations player 1, who we now call player I, starts at 2 and is back at 2 at time $T_0 = 2$. Player 3, who we now call player II, starts at 0 and is back at 0 at time 2. In both cases no players have met, so they cannot determine by time 2 which of these cases (c' or c'') is the actual initial configuration. In case c', player II is pointing up at time T_0, while in case c'' he is pointing down. Hence by Lemma 3.2 with $\beta = 2$, players cannot meet before time $T_0 + \beta = 2 + 2 = 4$. This situation is illustrated in Figures 3.5(i) and 3.5(ii).

For the remaining types 3 and 4, the analysis will be as follows. For each of the two types we give a set of initial configurations. Then for each strategy profile of that type, we give a time T_0 at which two players meet (who we then call player I) but cannot distinguish between the configurations in the given set. The problem of finding player II above them is called Γ_{up} and the minimax time to find him (or turn back) is denoted by T_{up} ,which can be calculated using Lemmas 3.2 or 3.3 depending on the nature of Γ_{up}. If the remaining player (II) is below then the associated problem and minimax time are denoted by Γ_{down} and T_{down}. Thus the maximum meeting time $\bar{T}$ satisfies

$$\bar{T} \geq T_0 + T_{up} + T_{down}. \tag{3.6}$$

Examples of types 3 and 4 are illustrated in Figures 3.5(iv) and 3.5(iii) respectively. The paths of the two players who meet at time T_0 are drawn in bold up to time T_0; the three possible paths of the remaining player (each corresponding to some initial configuation in the given set) is drawn in dashed up to time $2 = T_0 + \delta$; the parameters β and δ of the games Γ' and Γ'' of Lemmas 3.2 and 3.3 are drawn from player I's position at time T_0 in thin lines.

Type 3: The strategies $\left(\tilde{h}_2, \tilde{h}_2, \tilde{h}_3\right)$ or $\left(\tilde{h}_2, \tilde{h}_3, \tilde{h}_3\right)$ are used. Consider the set of configurations

$$C_3 = \{(0,2,1,+1,+1,+1)\,,\,(2,0,1,+1,-1,-1)\,,(2,1,0,+1,+1,+1)\}\,.$$

For these two strategy profiles and any of these configurations, a player using $\tilde{h}_2$ placed above a player using $\tilde{h}_3$ will meet at time $T_0 = 3/2$, and they will then be unable to distinguish between these three initial configurations. Call this pair player I. For either profile we can take Γ_{up} to be $\Gamma''(3/2, 1/2)$ and so $T_{up} = 1$ by Lemma 3.3. Similarly for either profile $\Gamma_{down} = \Gamma'(3/2)$ so $T_{down} = 3/2$. Thus by (3.6) we have $\bar{T} \geq 3/2 + 1 + 3/2 = 4$. To aid the reader we give the full analysis for the profile $\left(\tilde{h}_2, \tilde{h}_2, \tilde{h}_3\right)$, which is illustrated in Figure 3.5(iv). At time T_0 we normalize the time back to zero and let the meeting point be the new origin. With respect to this framework, the position of player II if above is 3/2 units above player I (i.e., the two players who met) at time 1/2. (at actual time 2) and facing up. Hence as claimed above, $\Gamma_{up} = \Gamma''(3/2, 1/2)$. If player II is below player I, then he is 3/2 units below in additional time 1/2, and can be facing either way, depending on the configuration. Hence as claimed, $\Gamma_{down} = \Gamma'(3/2)$.

Type 4: One of the strategies $\left(\tilde{h}_1, \tilde{h}_1, \tilde{h}_2\right)$, $\left(\tilde{h}_1, \tilde{h}_1, \tilde{h}_3\right)$, $\left(\tilde{h}_1, \tilde{h}_3, \tilde{h}_3\right)$, $\left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right)$is used. For these strategy profiles, consider the following set of configurations,

$$C_4 = \{(0,2,1,+1,+1,+1), (1,0,2,+1,+1,+1), (1,0,2,+1,-1,+1)\}$$

In each of these profiles, two players meet at time $T_0 = 1$, $\Gamma_{up} = \Gamma''(1,1)$ so $T_{up} = 1$ by Lemma 3.3, and $\Gamma_{down} = \Gamma'(2)$, so $T_{down} = 2$ by Lemma 3.2. Hence in all these cases, $\bar{T} \geq 1 + 1 + 2 = 4$. □

Theorem 3.3 $M_3 = 4$. *That is, the minimax rendezvous time for three players placed in random directions on consecutive integers is* 4.

Proof. Lemma 3.4 says that $M_3 \leq 4$, so we need only show that $M_3 \geq 4$. Since we are assuming that all paths belong to P^*, it is sufficient to show that $M_3 > 7/2$. Lemma 3.5 says that any strategy with maximum rendezvous time $\leq 7/2$ must have all of its Stage 1 paths belonging to the set $\left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right\}$. However Lemma 3.6 says that any strategy triple with this property must have maximum rendezvous time of at least 4. □

3.6 Asymptotic value of M_n

In this section we estimate the value of the minimax rendezvous time M_n for the n-player rendezvous game Γ_n when n is large. Clearly a lower bound for M_n is $(n-1)/2$, since the distance between the players initially placed at 0 and $(n-1)$ is $(n-1)$. The main work of this section is the presentation and analysis of a class of strategy profiles $S(n,m)$ for the games Γ_n which have maximum rendezvous times asymptotic to $n/2$. This analysis thus gives the main result of this section (Theorem 3.4), that M_n is asymptotic to $n/2$.

We now define the strategy profile $S(n, m)$ for the games Γ_n. Up to time $3m + 1$ the players adopt *adjacency search* paths called g_k . (The paths g_1, g_2, g_3 are drawn in Figure 3.6, for players initially pointed to the right.) These paths remain at a player's starting point except during the time interval $[3(k-1), 3k+1]$ of length four, when they search first forward one unit, then backwards two units, and then forward again to return to the starting point. This path will meet any adjacent player who is stationary at their starting point during this period (in particular at times $3(k-1)+1$ and $3(k-1)+3$.) More formally these *adjacency search* paths are defined as

$$g_k(t) = \begin{cases} t - 3(k-1) & \text{if } t \in [3(k-1), 3(k-1)+1], \\ -t + (3k-1) & \text{if } t \in [3(k-1)+1, 3k], \\ t - (3k+1) & \text{if } t \in [3k, 3k+1], \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if $k < k'$ and two adjacent players are using *adjacency search* paths g_k and $g_{k'}$, then they will meet at time $3(k-1)+1$ or $3(k-1)+3$, and in any case by time $3(k-1)+3$.

In the strategy profile $S(n, m)$ the players use the first m *adjacency search* paths $g_1, \ldots, g_m$, in as equal numbers as possible. We will take as an example $S(8, 3)$, which is illustrated in Figure 3.6. Due to constraint of space, we use the vertical axis to represent time and the horizontal axis to represent the players' positions on the line. Let $n = am+b$, $0 \leq b < m$, and let exactly $a+1$ players use each strategy g_k, $k = 1, \ldots, b$ and let exactly a players use each *adjacency search* path g_k, $k = b+1, \ldots, m$, for times $0 \leq t \leq 3m+1$, disregarding (for the time being) any players they may meet. Note that $a = \text{Int}(n/m)$. (In the example, $a = 2$ and three players use g_1, three use g_2, and two use g_3.) Observe that at time $3m+1$ (10, in the example) all the players are back at their starting points and that any pair of adjacent players who are using distinct strategies g_k will have already met each other, regardless of the directions in which they are initially pointed. Once the players have been placed on the integers $0, \ldots, n-1$, name them according to the integer where they start. We use a horizontal description of the line on which the players are placed. Let L denote the leftmost player for which the adjacent player on the right is using a different initial strategy g_k. Let R denote the rightmost player such that the player on his left is using a different strategy. (In the example $L = 2$ and $R = 5$.) Since there are at most a players to the left of L (who are using the same strategy g_k as player L) and similarly at most a players to the right of player R, we have

$$L \leq a, \text{ and } R \geq n - a - 1. \tag{3.7}$$

Note that equality holds in the above if and only if the first $a+1$ players are all using the same strategy and the last $a+1$ players are all using the same strategy. If these end groups are initially pointed in a common direction, the respective players at L and R would have only met one player by time $3m+1$. This configuration (shown in Figure 3.6) produces the maximum meeting time $\bar{T}$.

We now describe the strategies the players adopt from time $3m+1$. At this time the players have either met no adjacent players, two adjacent players, or exactly one adjacent player. (In the example of Figure 3.6, players 0,1,6, and 7 are of the first type, nobody is of the second type, and players 2,3,4 and 5 are of the third type.) Players of the first two types should remain still at their starting points until they meet a player who says 'follow me'. Players of the third type, who have met an adjacent player in only one direction, should go in the opposite direction (at speed one) until they either, (A) meet another moving player or, (B) reach an unoccupied integer location (relative to their starting point). In case

(A) they stop and remain still until someone says 'follow me'. In case (B) they can conclude that they are then at position -1 or n and hence they reverse direction and go at speed one, telling anyone they meet to follow them, until the game ends. (In the example of Figure 3.6, players L=2 and R=5 reach situation B at time 13, while players 3 and 4 reach situation A at time 21/2.) In general, it follows from the inequalities (3.7) that players L and R reach positions -1 and n respectively (situation B) by time $(3m+1)+(a+1)$. They then meet each other, together with everyone else, by maximum $\bar{T}$ where

$$\bar{T} = \bar{T}_n \leq (3m+1)+(a+1)+(n+1)/2. \tag{3.8}$$

(In the example of Figure 3.6, this gives a worst case of 35/2)

Suppose we define m =Int(log n) so that a =Int(n/Int (log n)). It follows that

$$\frac{\bar{T}_n}{n} \leq \frac{3\,\mathrm{Int}\,(\log\ n)+2+\mathrm{Int}\,(n/\mathrm{Int}\,(\log\ n))+1/2}{n}+\frac{1}{2} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\,.$$

Consequently the minimax rendezvous time M_n satisfies

$$\frac{1}{2} = \lim_{n\to\infty} \frac{(n-1)/2}{n} \leq \lim_{n\to\infty} \frac{M_n}{n} \leq \lim_{n\to\infty} \frac{\bar{T}_n}{n} = \frac{1}{2}, \text{ or } \lim_{n\to\infty} \frac{M_n}{n} = \frac{1}{2}.$$

Thus we have proved our final result.

Theorem 3.4 *The minimax rendezvous time M_n, required for n players placed on adjacent integers to meet together at a single point, is asymptotic to $n/2$.*

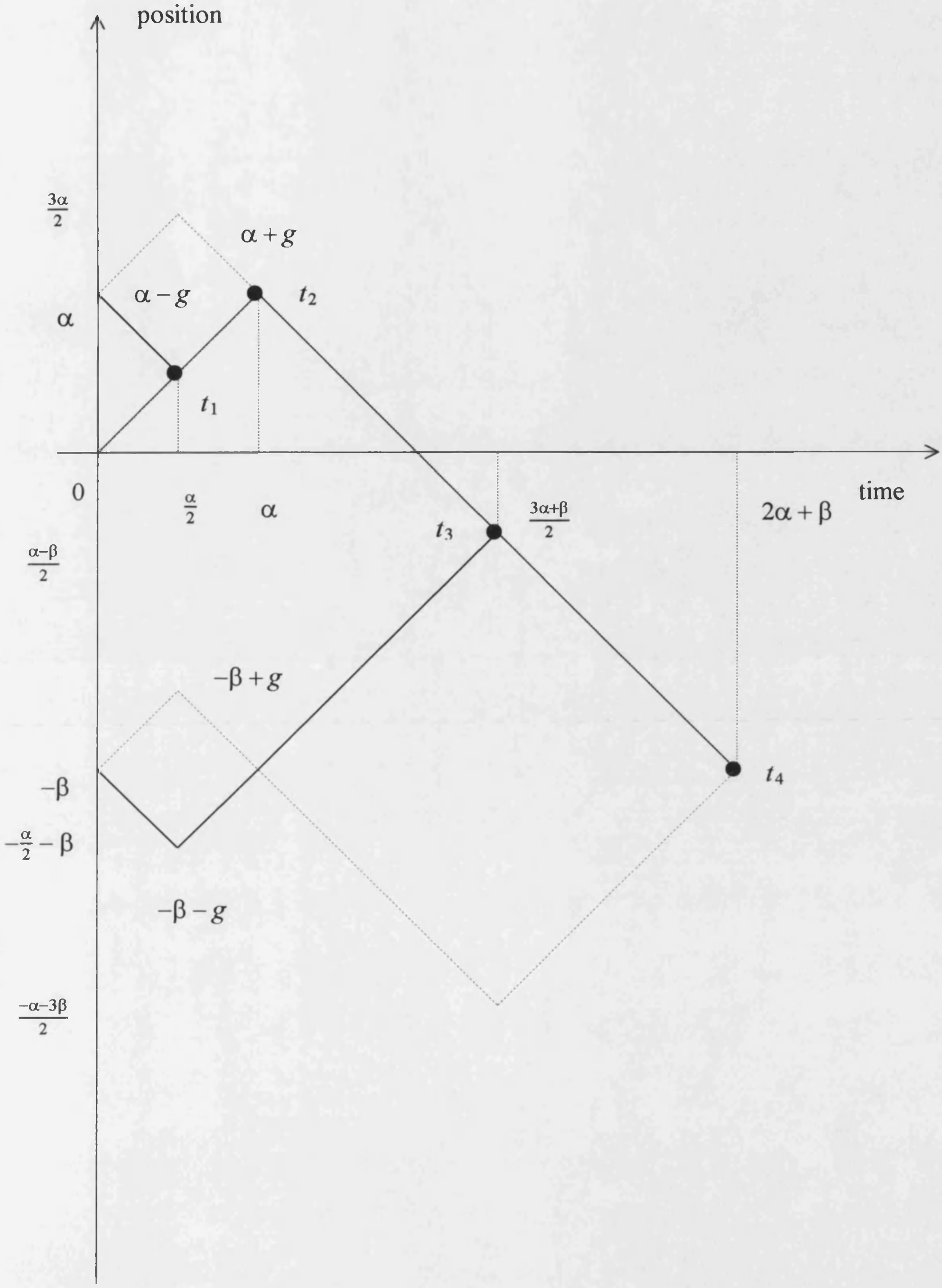


Figure 3.1: Meeting times for the canonical strategy pair

$(F_{\bar{\sigma}}, G_{\bar{\sigma}})$, $\alpha \leq \beta$

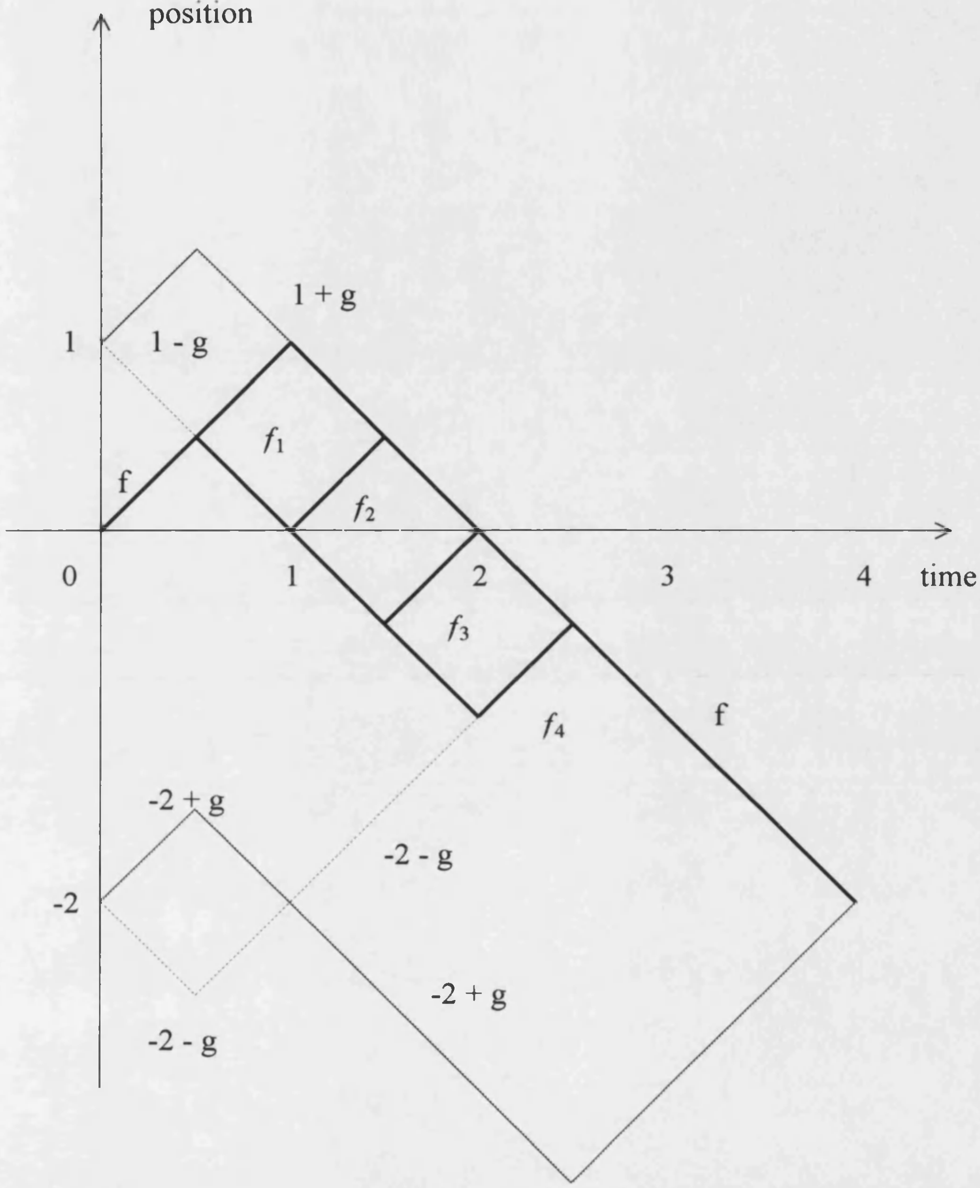


Figure 3.2: Optimal strategies for the problem $\Gamma(1,2)$

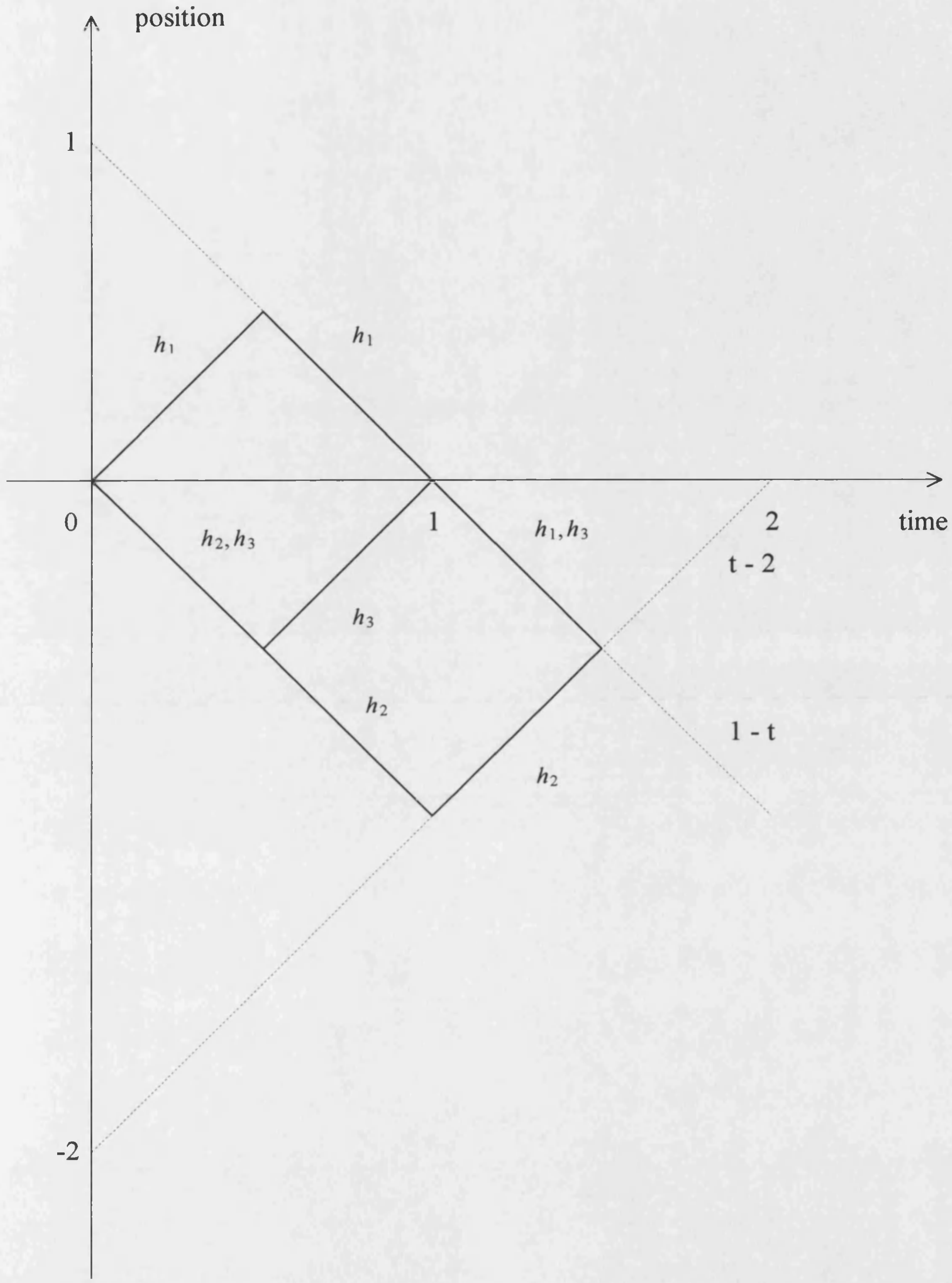


Figure 3.3: Optimal paths derived in Lemma 3.1

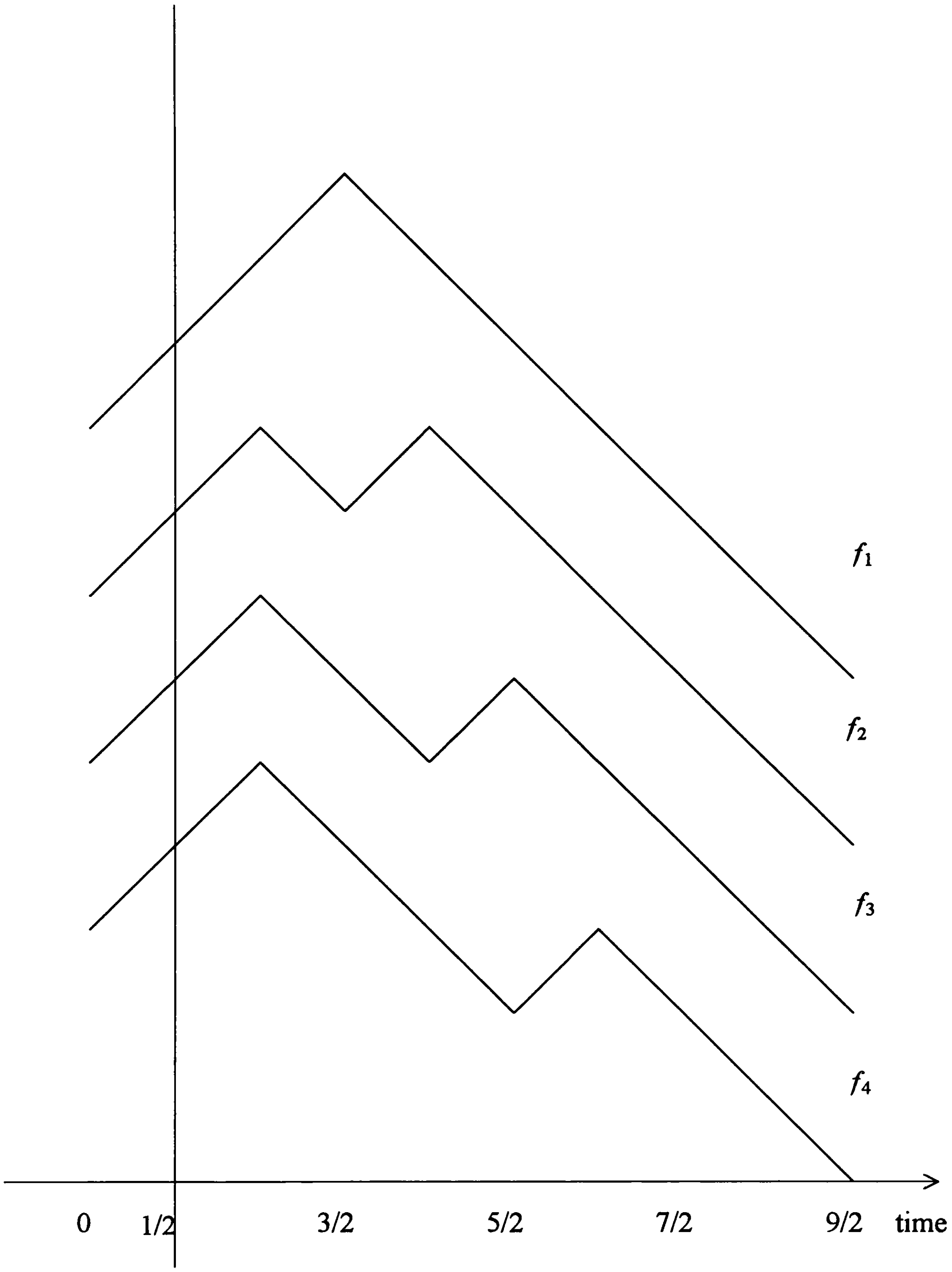


Figure 3.4: A non-intersecting stacking of the strategies f_k, starting at time 1/2

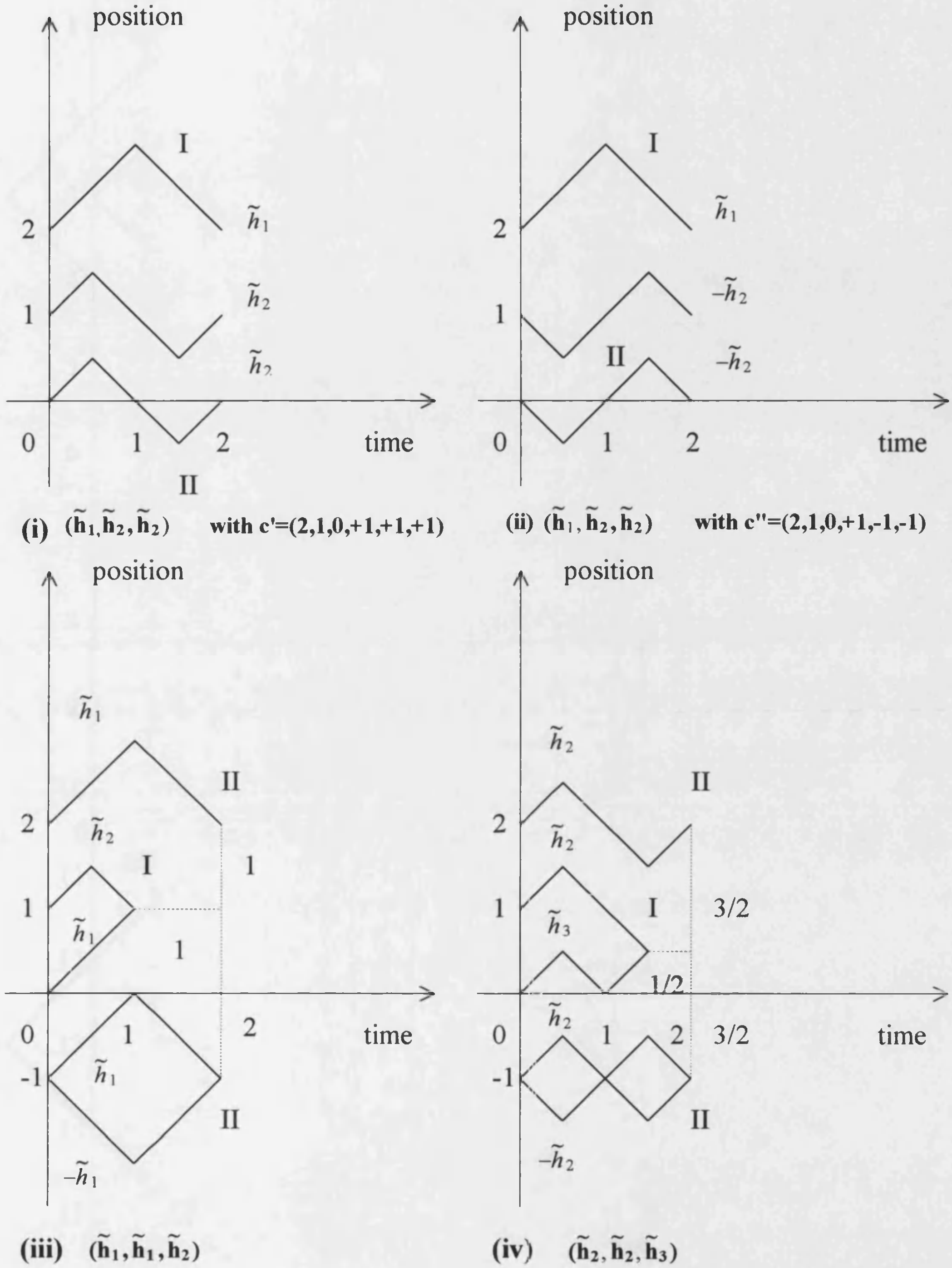


Figure 3.5: Strategies with $s_j \in \{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}$ have $T \geq 4$

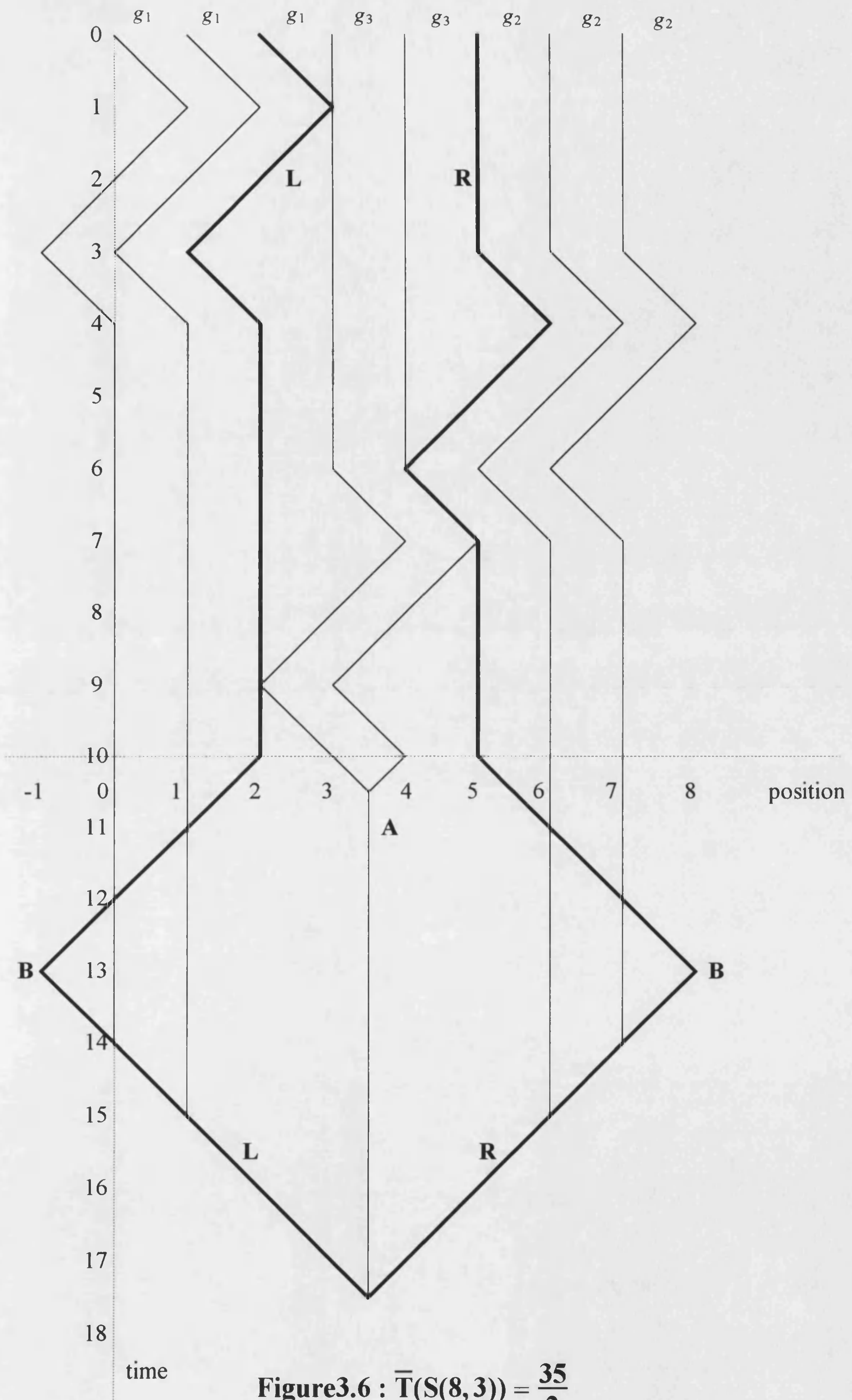


Figure3.6 : $\overline{\mathrm{T}}(\mathrm{S}(8,3)) = \frac{35}{2}$

Chapter 4

A Rendezvous-Evasion Game On Discrete Locations With Joint Randomization

4.1 Introduction

In this chapter we solve a discrete version of the rendezvous-evasion game proposed by S. Alpern (in his paper with Beck [4]) for general search spaces. This type of game is a two-person zero-sum game between a team R which comprised of two agents R_1 and R_2, and another player called S. All three of them are randomly placed in a known search region where they cannot see the others until they come within a specified 'meeting' or 'capture' distance, at which time the game ends. The team R is a pair of 'rendezvousers' whose objective is to meet each other before either of them is captured by the opposing searcher S. That is, R wins the game if the agents successfully *rendezvous* while *evading* the enemy searcher. Otherwise, S wins. The payoff of this game is taken to be 1 if S (maximizer) wins and 0 if R (minimizer) wins, so that the value of the game is simply the probability that S wins under optimal play.

The rendezvous-evasion game was proposed as a link between two related but separate branches of search theory: (i) search games with mobile hiders, and (ii) rendezvous search. The first of these, as proposed by R. Isaacs [13], studies the zero-sum game played between a mobile searcher (minimizer) and a mobile hider (maximizer) in a given region with capture time as the payoff. Extensive work has been done on these games in the last few decades ([2], [3], [10], [11], [13], [21]). The more recent problem of rendezvous search asks how two players randomly placed in a known region can meet in the least expected time. There are two versions, namely the asymmetric version where the two players can agree on distinct strategies and the symmetric version where the players cannot distinguish themselves and must therefore use the same mixed strategy [1]. Both versions have received attention recently ([1], [4], [5], [7], [8], [17]), particularly for the case where the search region is the line. In the rendezvous-evasion game with joint randomization the agents are involved in an asymmetric rendezvous problem with each other, while at the same time playing the role of the hider in a search game with mobile hider against the searcher S. However, unlike the continuous payoff (i.e., capture or meeting time) games it is based on, the rendezvous-evasion game is one of kind, in that the outcome depends on which of the two times (rendezvous or capture) occurs earlier. In [6], the symmetric analogue of the rendezvous-evasion game where the rendezvous agents have to use the same mixed strategy is considered. At present, we are aware of no rendezvous-evasion games which have been solved, although the case of a circle as the search region has been considered by S. Alpern and S. Gal.

As a preliminary study of the rendezvous-evasion game, we consider the case where the search region is comprised of n identical locations and the players can move between any two locations in one time step (this includes the possibility of staying still). This is the setting of the first solution of a pure rendezvous problem by Anderson and Weber [8] who analyzed the symmetric version of the problem. We consider a scenario which describes both problems: In some room in New York, there are n telephones randomly strewn about. These are pairwise connected to n telephones in a room in San Francisco. The rendezvous problem studied by Anderson and Weber [8] may be illustrated as follows. In each time period, the New Yorker and the San Franciscan each picks up a phone and says hello. They wish to minimize the expected number of time periods required for them to pick up paired telephones. Anderson and Weber [8] prove that the strategy of randomly picking up the telephones is not optimal when n is greater than 2. To demonstrate the present problem, consider the situation where the New Yorker and the San Franciscan are the 'rendezvousers' and all these n cables pass through some room in Chicago, where S can listen in on any single cable in each time period. If he hears someone says hello on that cable, he wins the game, even if all three are on the same line. Since there is no common labelling of the phones, the question of how the telephoners can choose their phones so that they can convey information to each other without being eavesdropped by the listener is of a rendezvous-evasion nature.

An important general question concerning rendezvous-evasion games is whether the optimal strategy for pure asymmetric rendezvous is still optimal. In other words, can the rendezvous team when playing optimally, ignore the presence of S by simply following strategies that minimize the expected time for them to meet? We shall show that the answer is 'no' when n is 3, as the optimal rendezvous strategy takes a maximum of 3 steps to ensure a meeting whereas in asymmetric rendezvous, the strategy of one remaining stationary while the other searches requires only at most 2 steps. The reason that the (latest possible) meeting must be delayed a step is that if the rendezvous team ensured that they would meet at the second step, a searcher S who moved at step 1 could also conclude where this meeting would have to occur, and would be there himself.

Another general question regarding rendezvous-evasion games is whether the rendezvous team can coordinate their movements so as to deter S from winning more than 2/3 of the time - 2/3 is the probability that the first pairwise meeting is between S and an agent of R, if all such meetings were equally likely. (In the discrete model given here, the probability 2/3 should be further increased because we ascribe a three way meeting as a win for S). We find that with joint randomization for the rendezvous team, S wins with probability less than 2/3 in all the versions considered in this paper. Our model of rendezvous-evasion with joint randomization assumes that the two agents R_1 and R_2 get together before the game begins and perform a randomization experiment which determines the pure strategy (describing the actions of both agents) that they will employ during the game.

This chapter is organized as follows. In Section 4.2, we give a precise formulation of the rendezvous-evasion game Γ^n on n locations and a simplified description of the strategies. We denote the value of this game by v_n. We solve the game for n equals 3, specifying optimal strategies and proving that v_3 is 47/76 (≈ 0.61842) (decimals are given for comparisons). As noted above, this game ends after at most three steps under optimal play though two would suffice to ensure that the rendezvous players meet. We also give a partial analysis for the case n equals 4, and prove that v_4

is at least 31/54 (≈ 0.57407). In Section 4.3, we consider a general class of rendezvous-evasion games Γ_D^n in which some additional information about the search region (containing n identical locations as before) is available to all the players; they share a common notion of a directed cycle D which contains all the n locations. For example, we can regard the locations as docks on a lake, with the players travelling on boats. The players can still move between any two locations (not only those adjacent on the directed cycle). This information structure is formalized using the isometry group consisting of the n rotations of this cycle, as explained in [1]. It turns out that this game is easier to solve and we show that the value d_n is given by the general formula $d_n = ((1-2/n)^{n-1}+1)/2$, which increases monotonically to $(e^{-2}+1)/2$ ($\approx 0.56767 < 2/3$). In particular, we have $d_3 = 5/9 \approx 0.55556 < v_3$ and $d_4 = 9/16 \approx 0.5625 < v_4$. Thus in these cases the additional information on the search region (given by the directed cycle D) is of relatively more benefit to the rendezvous team. In the final section, we compare the results obtained here with that obtained in [16], where the rendezvous-evasion game is studied as a multi-stage game with observed actions.

4.2 The Rendezvous-Evasion Game Γ^n

In this section, we formalize the rendezvous-evasion zero-sum game Γ^n between R (with agents R_1, R_2) and S, where the search region is comprised of n identical locations. The main objective of this section is to prove that the value of the game v_n is 47/76 when n is 3. We subsequently establish that v_4 is at least 31/54.

At the start of the game, R_1, R_2 and S are placed randomly on n locations so that no two of them occupy the same location (our analysis applies even if the initial placement allows two players to be at the same location). They do not share a common labelling of the n locations and they cannot see the others. We assume that all the players adopt the following convention when labelling the locations: The location where he or she is initially placed is referred to as 1, the next new location where he or she moves to is labelled as 2, and the third location which is neither 1 nor 2 is labelled as location 3 and so on. We use $S(i)$, $R_1(i)$ and $R_2(i)$ to denote the respective locations which S, R_1 and R_2 each labels as i. We consider the discrete version of the game where at every time step, each player can stay still or move between any two locations and we say that two players meet if they are at the same location at the same time. A pure strategy for S is a rule which spells out his motion at every step of the game. For example, 'move to location 2 at step 1 and stay there throughout the rest of the game' is a pure strategy for S and is denoted by $(2, 2, \ldots)$. A pure strategy for R is a rule which describes what each of its agents does at every step of the game. For example, 'R_1 stays still for $(n-1)$ steps while R_2 visits all the $(n-1)$ locations (other than the one which he is initially placed) for the first $(n-1)$ steps' is a legitimate strategy for R since this strategy ensures that the game ends by $(n-1)$ steps. In general, we use $(x_1, y_1, \ldots, x_k, y_k)$ to denote a k-step action for R, where R_1 visits location $R_1(x_i)$ while R_2 visits location $R_2(y_i)$ at step $i (i \leq k)$. The above strategy is thus represented by $(1, 2, 1, 3, \ldots, 1, n)$. In the usual sense, a mixed strategy of S (R) is a randomization of some pure strategies of S (R). We use $\mathcal{S}_n$ and $\mathcal{R}_n$ to denote the set of mixed strategies of S and R respectively. If R_1 meets R_2 before either one of them is captured by S, the *payoff* is 0 and we say that R (minimizer) wins. Otherwise, S (maximizer) wins and the *payoff* is 1. The payoff of the game $\pi_n(\sigma, \rho)$ when S uses strategy σ and R uses strategy ρ is the expected probability that S wins. Strategy σ^* of S and strategy ρ^* of R is an optimal pair if

$$\pi_n(\sigma^*, \rho) \geq \pi_n(\sigma^*, \rho^*) \ \geq \pi_n(\sigma, \rho^*) \ \forall \sigma \in \mathcal{S}_n, \rho \in \mathcal{R}_n.$$

The value v_n of the game Γ_n is defined to be

$$v_n = \max_{\sigma \in \mathcal{S}_n} \min_{\rho \in \mathcal{R}_n} \pi_n(\sigma, \rho) = \min_{\rho \in \mathcal{R}_n} \max_{\sigma \in \mathcal{S}_n} \pi_n(\sigma, \rho).$$

4.2.1 The Rendezvous-Evasion Game Γ^3

In this subsection, we consider the game Γ^3 and prove that its value v_3 is 47/76. We shall first describe a strategy ρ^* for R and prove that it ensures a payoff of no more than 47/76.

An optimal strategy for R

We begin by giving an outline of the semi-strategy $\bar{\rho}$ for R, which describes the actions of agents R_1 and R_2 at steps 1 and 2.

Semi-Strategy $\bar{\rho}$

- **Step** 1 R_1 moves to location $R_1(2)$, while R_2 stays at location $R_2(1)$;
- **Step** 2 There are three types of actions, namely

  Type 1: R_1 visits location $R_1(3)$ while R_2 stays at location $R_2(1)$;

  Type 2: R_1 visits location $R_1(1)$ while R_2 visits location $R_2(2)$;

  Type 3: R_1 stays at location $R_1(2)$ while R_2 visits location $R_2(2)$;

  Team R plays Type 1 action with probability 7/19; Type 2 action with probability 6/19 and Type 3 action with probability 6/19.

The following lemma describes the conclusions that R can draw at the end of step 2 when using semi-strategy $\bar{\rho}$.

Lemma 4.1 *If R uses semi-strategy $\bar{\rho}$ for the first two steps and the game has not ended by step 2, the agents R_1 and R_2 can deduce a common labelling of the three locations.*

Proof:
The proof of this lemma follows immediately from a step by step analysis of the semi-strategy $\bar{\rho}$. If the game does not end by step 1, agents R_1 and R_2 would be able to deduce that $R_1(3) = R_2(1)$. If Type 1 action is taken at step 2, $R_1(3)$ and $R_2(1)$ are the locations visited by agents R_1 and R_2 and the game is sure to end. If Type 2 action is followed at step 2 and the game does not end, the agents would be able to conclude that $R_1(1) = R_2(3)$. Since there are only three locations, together with the analysis of step 1, the agents' labellings of the locations are related in the following manner: $R_1(1) = R_2(3)$, $R_1(2) = R_2(2)$ and $R_1(3) = R_2(1)$. If Type 3 action is adopted at step 2 and the game does not end, it must be that $R_1(2) = R_2(3)$. Then agents R_1 and R_2 share the following common labelling of the three locations: $R_1(1) = R_2(2)$, $R_1(2) = R_2(3)$ and $R_1(3) = R_2(1)$. □

Now we are ready to describe a full strategy ρ^* for team R, which we shall later prove to be optimal.

Strategy ρ^* for team R

- Adopt the semi-strategy $\bar{\rho}$ for the first two steps. At step 3, co-coordinate to meet at each of the locations with probability 1/3. The actions of the agents at step 3 is justified by Lemma 4.1.

Next we shall go on to prove that strategy ρ^* of R guarantees a payoff of not more than 47/76 (which we later prove to be v_3) against all strategies of S. It is worthwhile to note that in order to prove the optimality of a strategy in a two-person zero-sum game, it is sufficient to compare the strategy against all pure strategies of the opponent. This is the traditional approach. Here, we adopt an alternative procedure.

Theorem 4.1 *For all* $\sigma \in \mathcal{S}_3$, $\pi_3(\sigma, \rho^*) \leq 47/76$.

Proof:
We proceed with the proof by analyzing each step of strategy ρ^*. Use c_i to denote the event that the game ends at step i with S meeting at least one of R_1 or R_2 (S wins); m_i to denote the event that the game ends at step i with R_1 meeting R_2 without either of them meeting S (R wins); and t_i to denote the event that the game does not end by step i. These three events are mutually exclusive and for each i, $\text{Prob}(c_i|t_{i-1}) + \text{Prob}(m_i|t_{i-1}) + \text{Prob}(t_i|t_{i-1}) = 1$.
Corresponding to every strategy σ of S, there is some β ($0 \leq \beta \leq 1/2$) such that at step 1, S moves to location $S(2)$ with probability 2β and stays at location $S(1)$ with probability $(1-2\beta)$. If S chooses to stay at location $S(1)$ at step 1, the game is sure to end by step 1 since by the definition of strategy ρ^*, only agent R_1 moves at step 2, while both R_2 and S stay still. S and R each wins with a probability of 1/2 (see Figure 4.1a). If S chooses to move to location $S(2)$ at step 1, S wins when location $S(2)$ coincides with $R_2(1)$, and this occurs with probability 1/2; while R wins with probability 1/4 (See Figure 4.1b). Thus, we have

$$\begin{aligned}\text{Prob}(c_1) &= (1/2)\,(1-2\,\beta) + (1/2)\,(2\,\beta) = 1/2,\\ \text{Prob}(m_1) &= (1/2)\,(1-2\,\beta) + (1/4)\,(2\,\beta)\\ &= 1/2\,(1-\beta) \geq 1/4.\end{aligned}$$

As a consequence, Prob(t_1) is at most 1/4. We observe from the analysis of step 1 that the game proceeds into step 2 if and only if R_1 and S traded positions at step 1, i.e., $S(1) = R_1(2)$ and $S(2) = R_1(1)$. Since there are only three locations, it must be that $R_1(3) = S(3) = R_2(1)$. Suppose at step 2, S moves to location $S(i)$ with probability β_i ($i = 1, 2, 3$) so that $\sum_{i=1}^{3} \beta_i = 1$. To analyze step 2, we consider what happens when R uses each type of action. If R uses Type 1 action (see Figure 4.2a), the game is sure to end and R wins if S is not at location $S(3)$. Hence, R wins with probability $(1 - \beta_3)$ while S wins with probability β_3. If R uses Type 2 action at step 2 (see Figure 4.2b), there are two equally likely scenarios and S can guarantee winning if S visits location $S(2)$ $(= R_1(1))$; or S wins with a probability of 1/2 if S visits location $S(1)$. That is, S can win with a probability of $(\beta_2 + (1/2)\ \beta_1)$. R wins if $R_1(1) = R_2(2)$ and S is not at location $S(2)$ $(= R_1(1))$, i.e., R wins with a probability of $(1/2)(1 - \beta_2)$. In this case, the only way that the game does not end is when S visits location $S(3)$ $(= R_1(3))$ and the agents R_1 and R_2 do not meet; this occurs with probability $(1/2)\beta_3$. Similarly, if R uses Type 3 action at step 2 (see Figure 4.2c), there are two equally likely scenarios and S can guarantee winning if S visits location $S(1)$ $(= R_1(2))$; or S wins with probability 1/2 if S visits location $S(2)$, which sums to a total probability of $(\beta_1 + (1/2)\beta_2)$. R wins with probability 1/2 if S is not at location $S(1)$ $(= R_1(2))$, i.e., R wins with a probability of $(1/2)(1 - \beta_1)$. The only way that the game does not end at step 2 and proceeds to step 3 is when S visits location $S(3)(= R_1(3))$ and the agents R_1, R_2 do not meet. This occurs with probability $(1/2)\beta_3$. Summing, we have

$$\begin{aligned}
\text{Prob}(c_2|t_1) &= 7/19\ \beta_3\ +\ 6/19\ (1/2\ \beta_1\ +\ \beta_2)\ + \\
&\quad 6/19\ (1/2\ \beta_2\ +\ \beta_1) \\
&= 7/19\ +\ 2/19\ (\beta_1\ +\ \beta_2),
\end{aligned}$$

$$\begin{aligned}\text{Prob}(t_2|t_1) &= 1/2 \times 6/19\, \beta_3 + 1/2 \times 6/19\, \beta_3 \\ &= 6/19\,(1 - \beta_1 - \beta_2).\end{aligned}$$

At step 3, strategy ρ^* ensures that the agents R_1 and R_2 can coordinate to meet at each location with probability 1/3. As a result, regardless of the action taken by S at step 3, $\text{Prob}(c_3|t_2) = 1/3$. Hence, for all $\sigma \in \mathcal{S}_3$,

$$\begin{aligned}\pi_3(\sigma, \rho^*) &= \text{Prob}(c_1) + \text{Prob}(t_1)(\text{Prob}(c_2|t_1) + \text{Prob}(t_2|t_1)(1/3)) \\ &\leq 1/2 + 1/4\,(7/19 + 2/19\,(\beta_1 + \beta_2) + \\ &\quad 6/19\,(1 - \beta_1 - \beta_2)\,1/3) = 47/76. \quad \square\end{aligned}$$

Next, we describe strategy σ^* for the searcher S and prove that S can secure a payoff of at least 47/76 with this strategy against all strategies of R.

Strategy σ^* for S

- **Step** 1 S moves to a new location $S(2)$;
- **Step** 2 Randomize over three actions:

  Action 1: Return to location $S(1)$;

  Action 2: Stay at location $S(2)$;

  Action 3: Go to location $S(3)$;

S chooses these actions according to the probabilities 5/19, 5/19 and 9/19.

- **Step** k **(k** ≥ 3**)** S visits each of the locations with probability 1/3.

Theorem 4.2 *For any strategy $\rho \in \mathcal{R}_3$, $\pi_3(\sigma^*, \rho) \geq 47/76$.*

Proof:

We first note that for all pure strategies of R, there are exactly three possible actions at step 1, namely, both R_1 and R_2 stay still, only one of them stays still, or both of them move to a new location. We proceed with the proof by first considering in each case, a possible action of R at step 1 and show that S can always guarantee an expected payoff of at least 47/76. Any strategy of R that involves randomization in the first step has an expected payoff which is a linear combination of the payoffs of these three cases. Hence it has to be at least 47/76 as well. The events c_i, m_i and t_i are as defined in the proof of Theorem 4.1.

Case 1: At step 1, R_1 stays at location $R_1(1)$ and R_2 stays at location $R_2(1)$.

By using strategy σ^*, S must meet either R_1 or R_2 at step 1.

Case 2: At step 1, R_1 and R_2 move to locations $R_1(2)$ and $R_2(2)$ respectively.

Since R_1 and R_2 cannot coordinate at their first step to ensure that they meet, there are altogether eight possible scenarios (See Figure 4.3) and $\text{Prob}(c_1) = 1/2$, $\text{Prob}(m_1) = 1/4$ and $\text{Prob}(t_1) = 1/4$. If the game does not end by step 1, it has to be the case that the labellings of the players are cyclic permutations of each other, i.e., for each $u = 1, 2, 3$, the three locations $S(u), R_1(u)$ and $R_2(u)$ are distinct. (See Figure 4.3(iii), (vi)). To be more specific, it is equally likely that the labellings are related in the following ways:

$$\begin{cases} S(1) = R_1(2) = R_2(3); \\ S(2) = R_1(3) = R_2(1); \\ S(3) = R_1(1) = R_2(2), \end{cases} \qquad \begin{cases} S(1) = R_1(3) = R_2(2); \\ S(2) = R_1(1) = R_2(3); \\ S(3) = R_1(2) = R_2(1). \end{cases}$$

R_1 has three possible actions at step 2, i.e., to move to locations $R_1(1)$, $R_1(2)$, or $R_1(3)$. This is also true for agent R_2. In other words, there are altogether nine possible actions for R at step 2. The probabilities that S wins are given in the matrix C_2 written below, where the actions of S moving to locations $S(1), S(2)$, and $S(3)$ are represented by the rows in that order, and the actions of R are arranged as columns in the order $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$. For example, since the labellings of the players are cyclic permutations of one another, if all players visit the location which all of them labels as x, no two players meet. These are the three 0 entries. If S visits location $S(1)$ and R uses action $(1,2)$, R_2 meets S or R_1 with an equal probability of $1/2$; and this is the $(1,2)$ entry in matrix C_2.

$$C_2 = \begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 & 1 & 1/2 & 1/2 & 1/2 & 1 \\ 1 & 1/2 & 1/2 & 1/2 & 0 & 1/2 & 1/2 & 1/2 & 1 \\ 1 & 1/2 & 1/2 & 1/2 & 1 & 1/2 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

By deleting repeated columns in the matrix C_2, we obtain a reduced version of the matrix C_2, which we denote by C_2', where

$$C_2' = \begin{pmatrix} 0 & 1/2 & 1 & 1 \\ 1 & 1/2 & 0 & 1 \\ 1 & 1/2 & 1 & 0 \end{pmatrix}.$$

Let α_i $(i = 1,2,3,4)$ denote the probability that R chooses the action represented by column i of the reduced matrix ($\sum_{i=1}^{4} \alpha_i = 1$). We have

$$\begin{aligned} \text{Prob}(c_2|t_1) &= \begin{pmatrix} 5/19 & 5/19 & 9/19 \end{pmatrix} C_2' \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \\ &= 14/19\, \alpha_1 + 1/2\, \alpha_2 + 14/19\, \alpha_3 + 10/19\, \alpha_4 \\ &\geq 1/2. \end{aligned}$$

The expected payoff in this case is given by

$$\begin{aligned}\pi_3(\sigma^*,\rho) &\geq \text{Prob}(c_1) + \text{Prob}(t_1)\times\text{Prob}(c_2|t_1)\\ &= 1/2 + 1/4\times 1/2\\ &= 5/8 > 47/76.\end{aligned}$$

Case 3: At step 1, R_1 moves to location $R_1(2)$ while R_2 stays at location $R_2(1)$.

The case where R_2 visits location $R_2(2)$ while R_1 stays at location $R_1(1)$ at step 1 can be treated in the same way to obtain the same result. Here, there are four equally likely scenarios and $\text{Prob}(c_1) = 1/2$, $\text{Prob}(m_1) = 1/4$ and $\text{Prob}(t_1) = 1/4$ (See Figure 4.4). Observe that the game proceeds to step 2 if and only if S and R_1 swapped locations at step 1, i.e., $S(2) = R_1(1)$, $S(1) = R_1(2)$ and $S(3) = R_1(3) = R_2(1)$. This scenario is illustrated as (iii) in Figure 4.4. We note that it is then equally likely that the labellings of R_1, R_2 and S are related in the following ways:

$$\begin{cases} S(1) = R_1(2) = R_2(2);\\ S(2) = R_1(1) = R_2(3);\\ S(3) = R_1(3) = R_2(1), \end{cases} \quad \begin{cases} S(1) = R_1(2) = R_2(3);\\ S(2) = R_1(1) = R_2(2);\\ S(3) = R_1(3) = R_2(1). \end{cases}$$

R_1 has three possible actions at step 2, i.e., go to $R_1(1)$, $R_1(2)$ or $R_1(3)$. Since R_2 has stayed at location $R_2(1)$ at step 1, he has only two possible actions at step 2, namely, to stay at location $R_2(1)$ or to move to location $R_2(2)$. Altogether, there are six possible actions for R at step 2, that is $(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)$. Suppose the kth action of R in the list above is chosen with probability γ_k ($\sum_{k=1}^{6}\gamma_k = 1$). The probability that S wins when S visits location $S(i)$ at step 2 and R uses the kth action is given as the (i,k) component of the matrix $\tilde{C}_2$. In a similar manner, the matrix $\tilde{M}_2$ gives the respective probabilities that R wins at step 2. For example, if S visits location $S(1)$ while R uses action $(1,1)$, all three players occupy different locations and so the $(1,1)$ entry of both $\tilde{C}_2$ and $\tilde{M}_2$ is 0. If S visits location $S(1)$ while R chooses action $(1,2)$, R_2 meets either R_1 or S with an equal probability of $1/2$. Thus the $(1,2)$ component of $\tilde{C}_2$ and $\tilde{M}_2$ are both $1/2$.

$$\tilde{C}_2 = \begin{pmatrix} 0 & 1/2 & 1 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad \tilde{M}_2 = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$\begin{aligned}
\mathrm{Prob}(c_2|t_1) &= \begin{pmatrix} 5/19 & 5/19 & 9/19 \end{pmatrix} \tilde{C}_2 \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \\
&= 14/19\,\gamma_1 + 15/38\,\gamma_2 + 14/19\,\gamma_3 + 15/38\gamma_4 + \\
&\quad 9/19\,\gamma_5 + 14/19\,\gamma_6, \\
\mathrm{Prob}(m_2|t_1) &= \begin{pmatrix} 5/19 & 5/19 & 9/19 \end{pmatrix} \tilde{M}_2 \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \\
&= 7/19\,\gamma_2 + 7/19\,\gamma_4 + 10/19\,\gamma_5.
\end{aligned}$$

Since S visits each of the location at step 3 with an equal probability of $1/3$, it has to be true that $\mathrm{Prob}(c_3|t_2) \geq 1/3$. The expected payoff in this case is at least

$$\begin{aligned}
&\text{Prob}(c_1) \;+\; \text{Prob}(t_1)(\text{Prob}(c_2|t_1) \;+\; \text{Prob}(t_2|t_1)(1/3)) \\
=\;& \text{Prob}(c_1) \;+\; \text{Prob}(t_1)(\text{Prob}(c_2|t_1) \;+\; (1 \;-\; \text{Prob}(c_2|t_1) \;-\; \text{Prob}(m_2|t_1))(1/3)) \\
=\;& 7/12 \;+\; 1/(57)(7\gamma_1 \;+\; 2\gamma_2 \;+\; 7\gamma_3 \;+\; 2\gamma_4 \;+\; 2\gamma_5 \;+\; 7\gamma_6) \\
\geq\;& 7/12 \;+\; 2/57 = 47/76.
\end{aligned}$$

If R randomizes over these three cases at step 1, the expected payoff is a linear combination of the payoffs of the above cases, so it is still true that the expected payoff is at least 47/76. Hence, for all strategies $\rho \in \mathcal{R}_3$, $\pi_3(\sigma^*, \rho) \geq 47/76$. □

By combining Theorems 4.1 and 4.2, we obtain the following.

Theorem 4.3 *The value* v_3 *is* 47/76.

At this point, we make a comparsion between the optimal strategy ρ^* of R_1, R_2 in this rendezvous-evasion context with that in the absence of S. It has been mentioned in [8] that when the rendezvous players are allowed to use different pure strategies to minimize the expected meeting time, it is optimal for one of them to stay still while the other searches all other locations in some random order. If R restricts to this strategy (which is described by Type 1 action at step 2 of strategy ρ^*), it can ensure a successful rendezvous by two steps. Nevertheless, a searcher who has moved to a new location at step 1 would also be able to deduce exactly where the agents will meet at step 2 (in the same way that the moving rendezvous agent can) and S would be there himself to win the game. This explains the need for R to possibly delay their meeting by one more step so that there is no way that S can have a sure win at step 2, and thus the use of mixed strategies by both R and S. It is crucial to observe that the presence of S in the rendezvous-evasion context cannot be ignored by R under optimal play here.

The reader might now be wondering how the optimal strategy pair (σ^*, ρ^*) was derived. (σ^*, ρ^*) was initially obtained by the author while considering a different game, in which players have to announce their actions at the end of each step [16]. However, it turns out that such truthful announcement does not change the value of the game. We provide an explanation for this. If we examine the strategy pair (σ^*, ρ^*), we find that step 1 is a pure action for both S and R, so that the announcement of actions at the end of step 1 is not critical. At step 2, if the game does not end, we see from Lemma 4.1 that R would be able to coordinate in the following step to ensure a successful rendezvous; this is regardless of what S has done at step 2 so that knowing the action of S at step 2 does not help R at all. Such coordination on the part of R also means that whatever extra information that S can deduce from knowing the action of R at step 2 really does not give S any extra edge in the game at step 3; the best that S can do is to pick each of the three locations equiprobably. To summarize, the obligatory announcement of actions at the end of each step by R does not prevent the team from securing the payoff of 47/76, neither does announcement on the part of S deter the searcher himself from sustaining the same payoff.

4.2.2 The Rendezvous-Evasion Game Γ^4

In this subsection, we consider the game Γ^4 where the search region comprises of four identical locations. We propose a strategy for S which secures a payoff of 31/54 against all strategies of R. We include this result here solely for comparison purposes later on and the lower bound of 31/54 for the value v_4 suffices.

Strategy $\hat{\sigma}$ of S

Consider the following strategy $\hat{\sigma}$ of S:

- **Step** 1 Move to a new location $S(2)$;
- **Step** 2 Move to a third location $S(3)$;
- **Step** k **($k \geq 3$)** Visit each of the four locations with probability 1/4.

Lemma 4.2 *For all $\rho \in \mathcal{R}_4$, $\pi_4(\hat{\sigma}, \rho) \geq 31/54$.*

Proof:
Since S adopts a random choice of locations from step 3 onwards, S wins with a probability of at least 1/4 given that the game continues into step 3. We shall see that in order to obtain the lower bound of 31/54, it is sufficient to compare strategy $\hat{\sigma}$ against all two-step actions of R. We note that when S uses strategy $\hat{\sigma}$ and both R_1 and R_2 stay still at step 1, the probability that S captures one of the agents at step 1 is $2/3 (> 31/54)$. Hence, we may restrict our analysis to two-step actions of R where at least one of the agents moves to a new location at step 1. We use c_i to denote the event that S wins at step i and t_i to denote the event that the game does not end by step i. For all such two-step actions of R, we compute the expression Prob(c_1)+ Prob(c_2) + 1/4Prob(t_2) as a lower bound for the payoff. By symmetry of R_1 and R_2, there are twelve such actions for R. We summarize the lower bounds for the payoffs corresponding to all such actions, (x_1, y_1, x_2, y_2) in Table 4.1. For example, when R uses action $(1, 2, 1, 1)$, we can see from Figure 4.5 that Prob$(c_1) = 4/9$. In the three cases ((i), (vii), (viii)) where the game proceeds to step 2, we note that other than in (i) where S wins with probability 1 at step 2, S can only manage a winning probability of 1/2, and the game continues

with another probability of 1/2 into step 3. Thus Prob(c_2) $= 2/9(= 1/9 + 1/2 \times 2/9)$, Prob($t_2$) $= 1/9(1/2 \times 2/9)$ and the lower bound of $25/36(= 4/9 + 2/9 + 1/4 \times 1/9)$ is obtained.

(x_1, y_1, x_2, y_2)	Prob(c_1)	Prob(c_2)	Prob(t_2)	Lower Bound of Payoff (=Prob(c_1)+ Prob(c_2)+ 1/4 Prob(t_2))
(1,2,1,1)	4/9	2/9	1/9	25/36 (≈ 0.69444)
(1,2,1,2)	4/9	2/9	1/9	25/36 (≈ 0.69444)
(1,2,1,3)	4/9	7/36	1/18	47/72 (≈ 0.65278)
(1,2,2,1)	4/9	5/54	4/27	31/54 (≈ 0.57407)
(1,2,2,2)	4/9	5/54	4/27	31/54 (≈ 0.57407)
(1,2,2,3)	4/9	17/108	7/54	137/216 (≈ 0.63426)
(2,2,1,1)	11/27	7/27	4/27	19/27 (≈ 0.70370)
(2,2,1,2)	11/27	5/27	4/27	17/27 (≈ 0.62963)
(2,2,1,3)	11/27	19/108	7/54	133/216 (≈ 0.61574)
(2,2,2,2)	11/27	7/27	4/27	19/27 (≈ 0.70370)
(2,2,2,3)	11/27	19/108	7/54	133/216 (≈ 0.61574)
(2,2,3,3)	11/27	5/36	11/54	129/216 (≈ 0.59722)

Table 4.1: Lower Bounds of the Payoff $\pi_4(\hat{\sigma}, (x_1, y_1, x_2, y_2))$

From Table 4.1, we see that all payoffs are bounded below by 31/54. Hence, we must have $\pi_4(\hat{\sigma}, \rho) \geq 31/54$ for all strategies ρ of R. □

4.3 The Rendezvous-Evasion Game Γ_D^n

In this section we analyze a different version of the rendezvous-evasion game Γ_D^n, where all players share a common notion of a directed cycle D which contains all the n locations. This information structure can be formalized using the isometry group of the complete graph which consists of all the n rotations of this cycle [1]. Here, we adopt all the terminologies defined for the game Γ^n in Section 4.2 except for the labellings of the locations. We assume that the location which the player is initially placed is labelled as 1 and the remaining locations are labelled in an increasing order along the given direction of the cycle D. We use $\pi_n^D(\cdot,\cdot)$ and d_n to denote the payoff and the value of the game. We shall first describe a strategy pair (σ_n^*, ρ_n^*) for S and R before we prove that it is an optimal strategy pair for the game Γ_D^n and establish the result $d_n = ((1-2/n)^{n-1} + 1)/2$.

Strategy σ_n^* for S

- At each step, S visits each of the n locations with probability $1/n$, independent of previous choices.

Essentially, σ_n^* is the random strategy. However, we shall see that R_1 and R_2 can do better than using their version of the random strategy. We provide the intuition for strategy ρ_n^* which is described below. The main idea is that in the absence of S, we would expect that in order to minimize the expected meeting time, R cannot do better than adopt the 'one-stays-still-the-other-searches' strategy. Strategy ρ_n^* is fundamentally equivalent to this, although there is an additional randomization element, the purpose of which is to introduce noise into the search process to confuse the searcher.

Strategy ρ_n^* for R

- Before the start of the game, R_1 and R_2 together choose at random an element $(\tau_1, \tau_2, \cdots, \tau_{n-1})$ from the set $\{1, 2, \cdots, n\}^{n-1}$.
- At each step k ($k = 1, \ldots, n-1$), R_1 visits location $R_1(\tau_k)$ while R_2 visits location $R_2(\lambda_k))$, where λ_k is congruent (mod n) to $k + \tau_k$.

In essence, strategy ρ_n^* is a randomization of n^{n-1} pure strategies. For example, when n is 3 and the random element chosen is $(1, 2)$, the strategy adopted by R would be $(1, 2, 2, 1)$.

Before we proceed to prove the optimality of the strategy pair (σ_n^*, ρ_n^*), we evaluate the payoff $\pi_n^D(\sigma_n^*, \rho_n^*)$. Due to the construction of strategy ρ_n^*, the game has to end by step $(n-1)$ and the rendezvous team is equally likely to meet at any one step so that the payoff is given by

$$\begin{aligned}
\pi_n^D(\sigma_n^*, \rho_n^*) &= 1 - \sum_{k=1}^{n-1} \text{Prob (R wins at step } k) \\
&= 1 - \sum_{k=1}^{n-1} \frac{1}{n-1}(1 - \frac{1}{n})(1 - \frac{2}{n})^{k-1} \\
&= ((1 - \frac{2}{n})^{n-1} + 1)/2. \qquad (4.1)
\end{aligned}$$

Now, we shall prove that strategy ρ_n^* of R ensures that the payoff is no more than $\pi_n^D(\sigma_n^*, \rho_n^*)(= ((1 - 2/n)^{n-1} + 1)/2)$ for all strategies of S.

Theorem 4.4 *For all* $\sigma \in \mathcal{S}_n, \pi_n^D(\sigma, \rho_n^*) \leq ((1 - 2/n)^{n-1} + 1)/2$.

Proof:

By virtue of strategy ρ_n^*, where each τ_i is chosen independently and equiprobably from the set $\{1, 2, \ldots, n\}$, R_1 is equally likely to be at any one of the n locations at every step independent of all previous actions. The same can be said about R_2. Hence, S cannot do any better than choosing each of the n locations with probability $1/n$ at each step. And this is precisely strategy σ_n^* as described above. Thus the theorem follows immediately from (4.1). □

Theorem 4.5 *For all $\rho \in \mathcal{R}_n$, $\pi_n^D(\sigma_n^*, \rho) \geq ((1-2/n)^{n-1}+1)/2$.*

Proof:

We shall first prove that if S is using strategy σ_n^*, it is not optimal for R_1 and R_2 to engage in 'repeated' search, i.e., they do not search locations which are at the same distance apart for a second time. This is not surprising indeed since if the game continues after the first attempt, the agents will not meet in their subsequent attempts and would have wasted the steps. Let $r = (x_1, y_1, x_2, y_2, \cdots)$ denote a pure strategy of R which engages the agents in 'repeated' search at some step. Let Δ denote the set of integers k ($\leq n-1$) such that $(x_k - y_k)$ is congruent (mod n) to $(x_j - y_j)$ or 0 for some $j < k$. The set Δ is non-empty by the choice of r. Modify strategy r to strategy $\tilde{r}$ by changing the actions of R_1 and R_2 at step k for all k in the set Δ. For each k, choose a pair of locations (α_k, δ_k) so that instead of visiting locations $R_1(x_k)$ and $R_2(y_k)$ at step k, R_1 and R_2 visit locations $R_1(\alpha_k)$ and $R_2(\delta_k)$ respectively, and $\alpha_k - \delta_k$ is neither congruent (mod n) to any of the $(x_j - y_j)$ for all $j < k$ nor 0. Such a pair of (α_k, δ_k) exists for each k since k is at most $n-1$. Let c_j^r and $c_j^{\tilde{r}}$ denote the events that S wins at step j when R uses strategy r and $\tilde{r}$ respectively (and S uses strategy σ_n^*). We first observe that the modification of strategy r to strategy $\tilde{r}$ ensures that

$$\text{Prob}(c_j^r) = \text{Prob}(c_j^{\tilde{r}}), \forall\, j \notin \Delta,\ j \leq n-1. \tag{4.2}$$

Secondly, when R uses strategy r, the agents do not meet at step k (for all k in Δ). However, with strategy $\tilde{r}$, the agents meet with a positive probability, so that we have

$$\text{Prob}(c_k^r) > \text{Prob}(c_k^{\tilde{r}}), \forall\, k \in \Delta. \tag{4.3}$$

Thirdly, strategy $\tilde{r}$ guarantees that the game ends by step $n-1$. This implies that

$$\text{Prob}(c_j^r) \geq \text{Prob}(c_j^{\tilde{r}})(=0), \forall\, j > n-1. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we have

$$\pi_n^D(\sigma_n^*, r) > \pi_n^D(\sigma_n^*, \tilde{r}).$$

Hence, strategy r is not a best response to strategy σ_n^* of S. As a consequence, any mixed strategy of R which plays r with a positive probability cannot be a best response as well. Since the starting distance between the agents is equally likely to be $1, 2, \ldots, n-1$, the agents meet with a probability of $1/(n-1)$ if they adopt any one strategy which does not involve 'repeated' search, and in particular, if they adopt strategy ρ_n^*. Hence, for all strategy ρ of R, the payoff $\pi_n^D(\sigma_n^*, \rho)$ is bounded below by $\pi_n^D(\sigma_n^*, \rho_n^*)$, which is $((1-2/n)^{n-1}+1)/2$, as given in (4.1). Hence the theorem holds. □

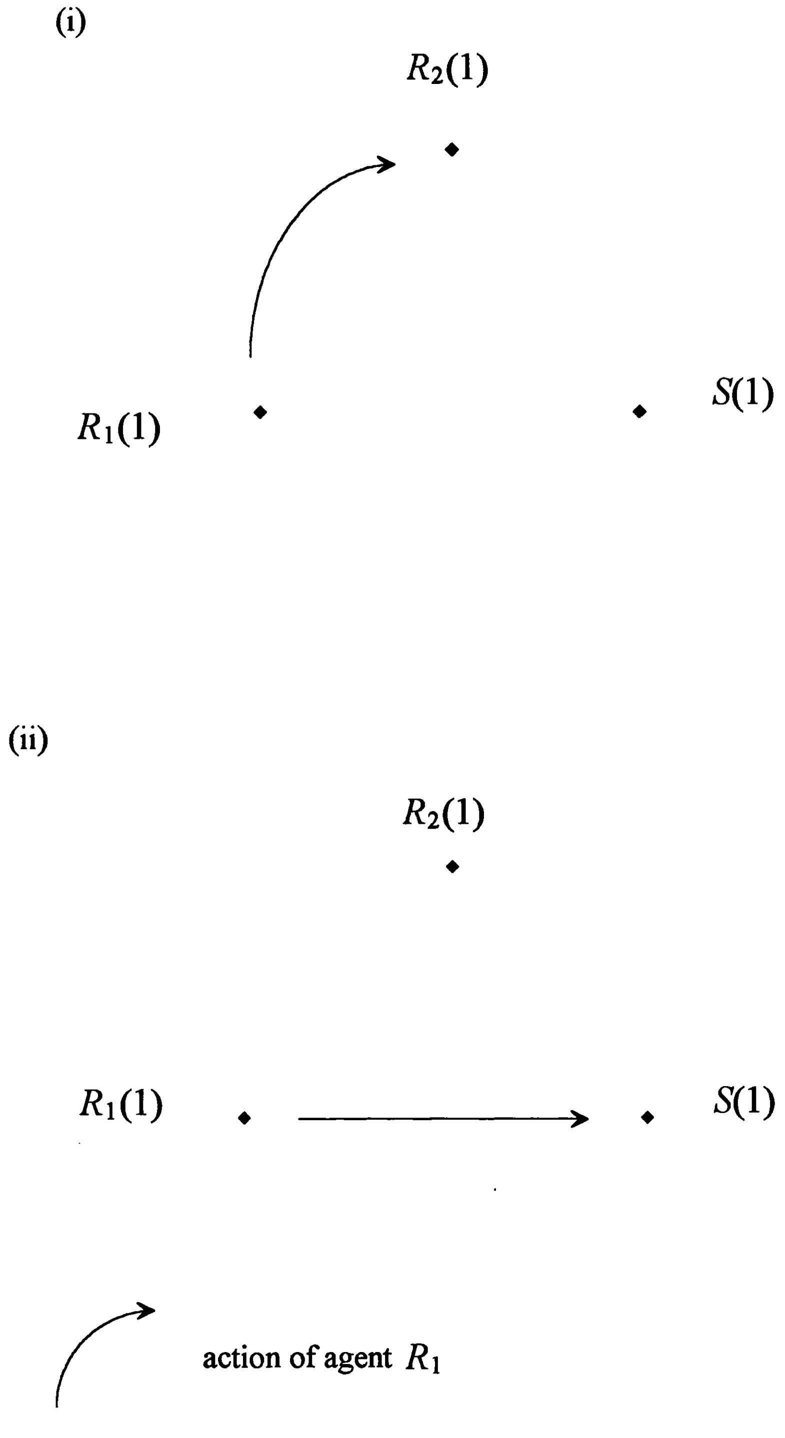
From Theorems 4.4 and 4.5, it is immediate that we have the following result.

Theorem 4.6 *The value d_n is $((1-2/n)^{n-1}+1)/2$.*

First, we note that $d_3(=5/9) < v_3$ and $d_4(=9/16) < v_4$; the common knowledge of the cycle D helps the rendezvous team secure lower values in these two instances. We further note that d_n increases monotonically to $(e^{-2}+1)/2 \approx 0.56767$.

4.4 Comparison With Rendezvous-Evasion As a Multi-Stage Game With Observed Actions

We use this final section to compare the results obtained here with those in [16], where the rendezvous-evasion game is modelled and solved as a multi-stage game with observed actions. By this we mean that (1) all players knew the actions chosen at all previous steps $0, 1, 2, \ldots, k-1$ when choosing their actions at step k, and that (2) all players move simultaneously at each step k. In this latter setting, an action for any player may depend on the history of play. We interpret this version of rendezvous-evasion game as one where all players are obliged to announce their actions truthfully at the end of each step. We use $\tilde{v}_n$ and $\tilde{d}_n$ to denote the respective values of the corresponding games Γ^n and Γ^n_D, where actions have to be announced. We prove in [16] that $\tilde{v}_3 = 47/76, \tilde{d}_3 = 5/9$ and $\tilde{d}_4 = 17/32$. Thus we have $\tilde{v}_3 = v_3$, $\tilde{d}_3 = d_3$, $d_3 < v_3$ and $\tilde{d}_4 < d_4 < v_4$. To summarize, in all the instances that we consider, extra information either does not affect the value of the game (as when n is 3) or helps the rendezvous team secure a lower value.



In (i), R wins;
In (ii), S wins.

Figure 4.1 a: Analysis of Step 1 (Player S stays still, $\beta = 0$)

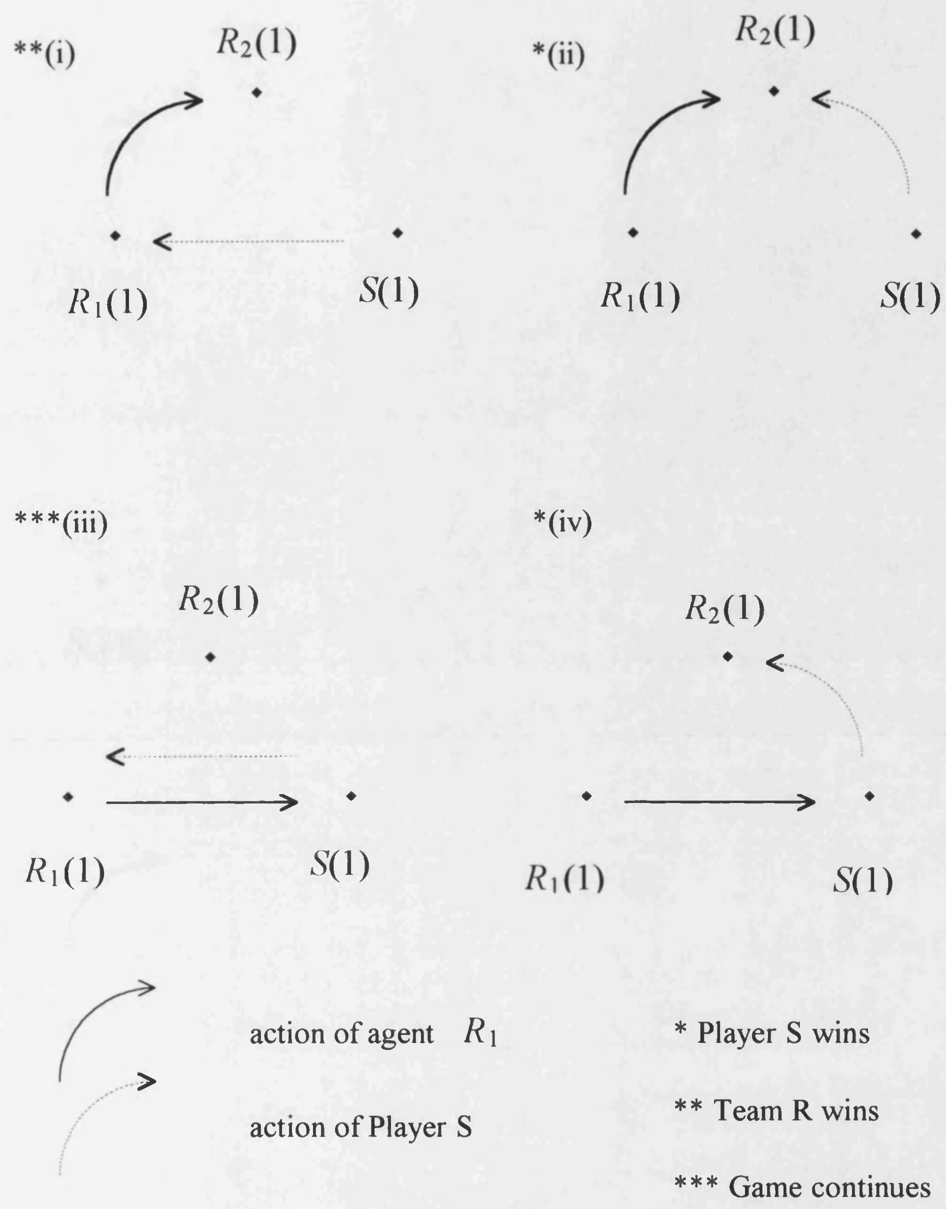


Figure 4.1b: Analysis of Step 1 (Player S moves to location S(2), $\beta = 1$)

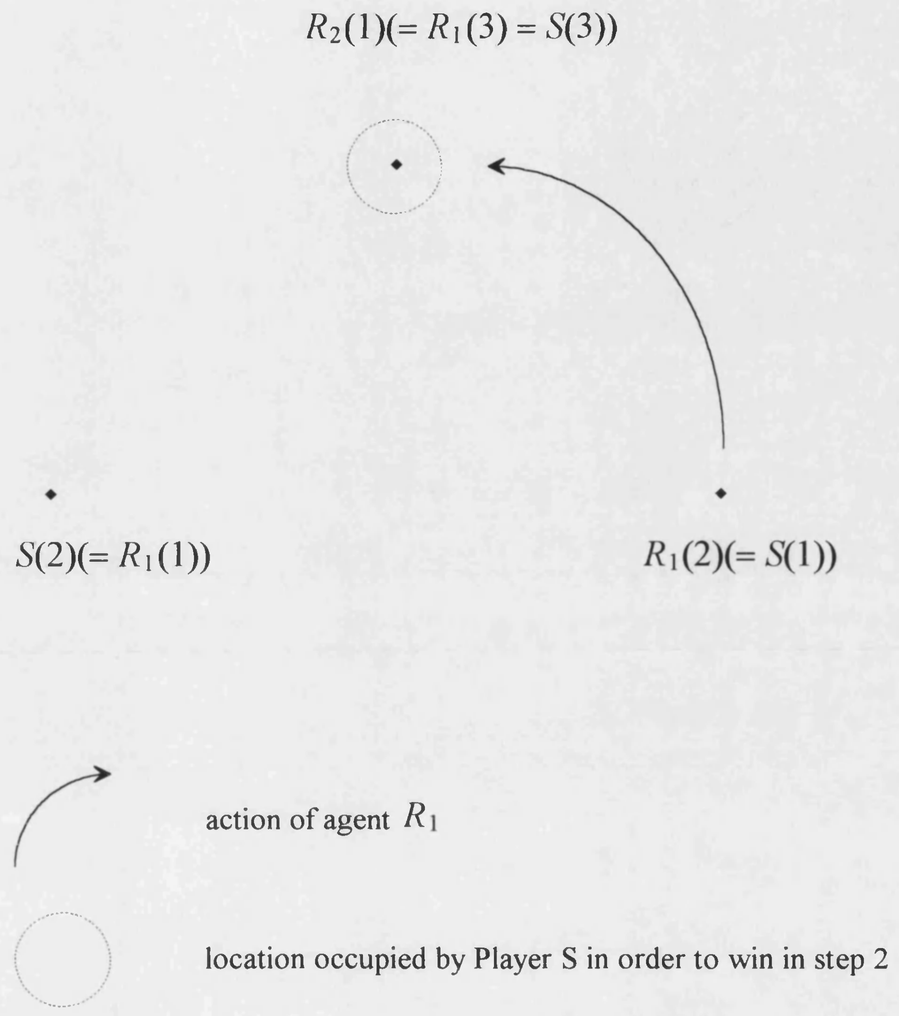


Figure 4.2 a: Analysis of Step 2, Type 1 action of strategy ρ^*

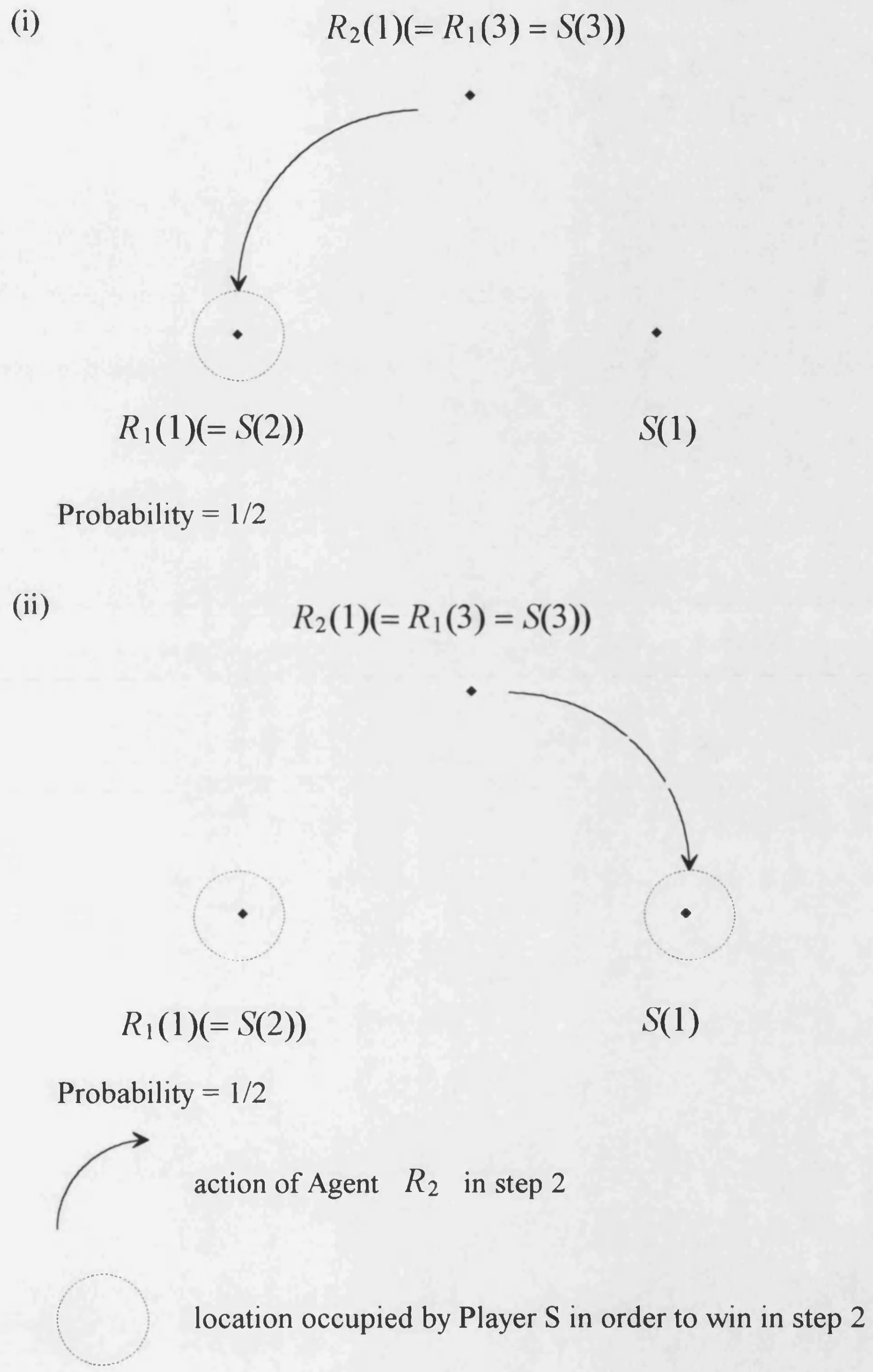


Figure 4.2 b: Analysis of Step 2, Type 2 action of strategy ρ^*

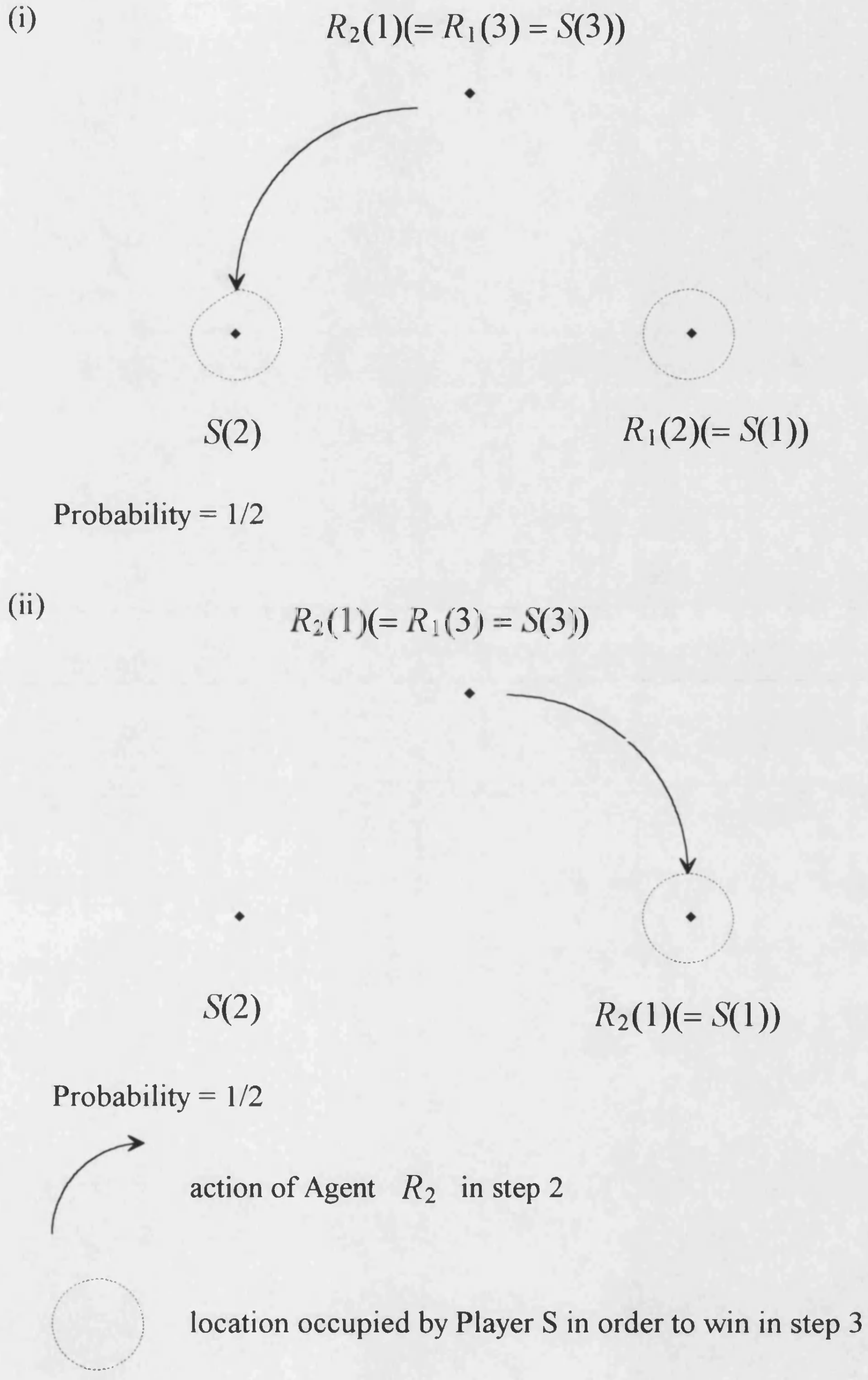


Figure 4.2 c: Analysis of Step 2, Type 3 action of strategy $\rho*$

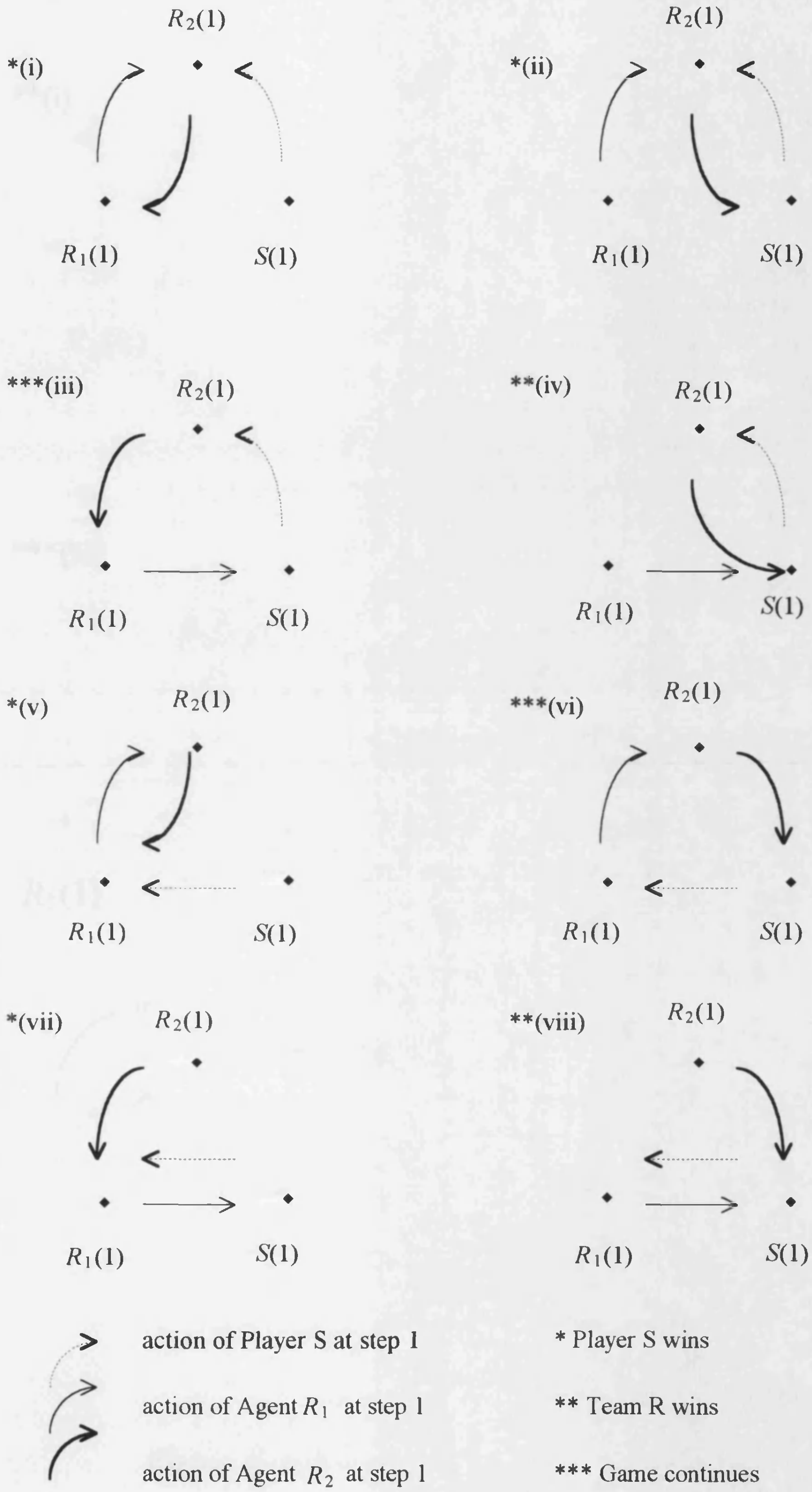


Figure 4.3: Analysis of Theorem 4.1 Case 2 (Step 1)

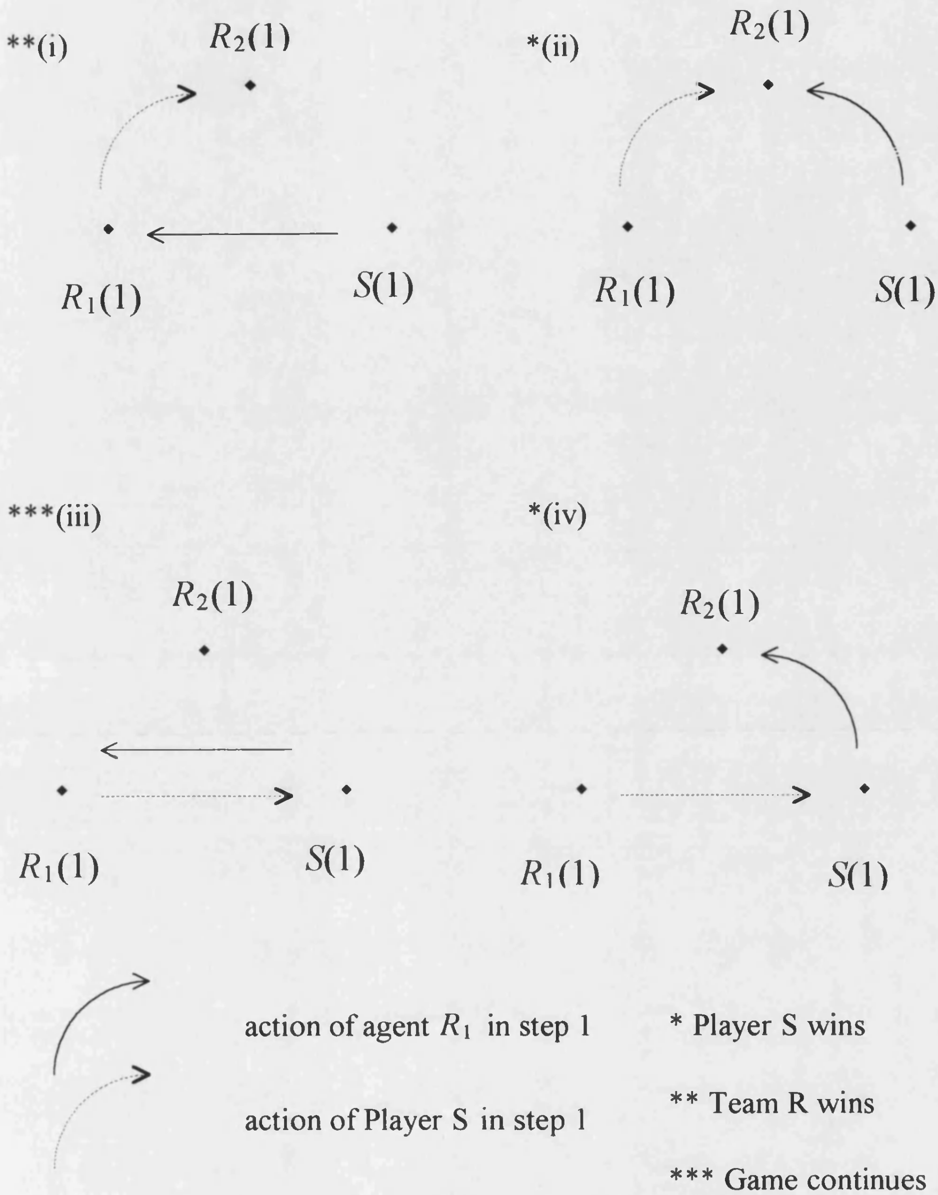


Figure 4. 4: Analysis of Theorem 4.1 Case 3 (Step 1)

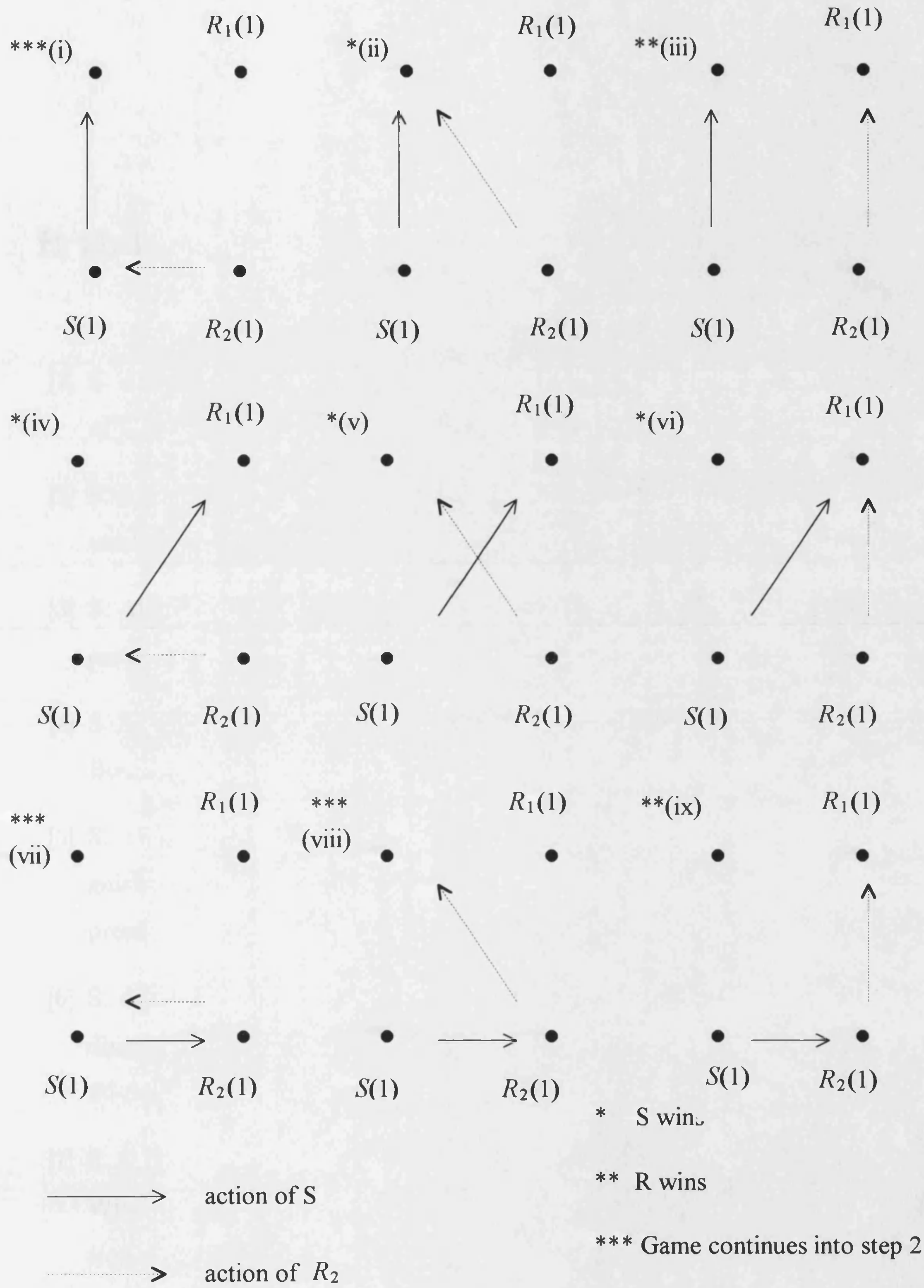


Figure 4.5: Analysis of Step 1 when S uses $\hat{\sigma}$ and R uses action (1,2,1,1)

Bibliography

[1] S. Alpern, (1995) The Rendezvous Search Problem, *SIAM Journal of Control and Optimization, 33*, 673-683.

[2] S. Alpern and M. Asic, (1985) The Search Value of a Network, *Networks, 15*, 229-238.

[3] S. Alpern and M. Asic, (1986) Ambush Strategies in Search Games on Graphs, *SIAM Journal of Control and Optimization, 24*, 66-75.

[4] S. Alpern and A. Beck (1995) Rendezvous Search on The Line With Bounded Resources, *LSE Math Preprint Series, 92.*

[5] S. Alpern and S. Gal, Rendezvous Search on the Line With Distinguishable Players, *SIAM Journal of Control and Optimization,* in press.

[6] S. Alpern and W.S. Lim (1996), Rendezvous-Evasion Games: Coordination Without Focal Points, *LSE-CDAM Research Report Series, 96-06.*

[7] E.J. Anderson, and S. Essegaier, Rendezvous Search on The Line With Indistinguishable Players, *SIAM Journal of Control and Optimization,* in press.

[8] E.J. Anderson and R.R. Weber, (1990) The Rendezvous Problem on Discrete Locations, *Journal of Applied Probability, 28*, 839-851.

[9] V. Baston, (1995) Symmetric Rendezvous On The Line, preprint.

[10] S. Gal, (1960) Search Games. Academic Press, New York.

[11] S. Gal and E.J. Anderson, (1990) Search in a Maze, *Probability in the Engineering and Informational Sciences, 4*, 311-318.

[12] J. Howard, (1995) Rendezvous Search On The Interval, preprint.

[13] R. Isaacs, (1965) Differential Games, Wiley, New York.

[14] W.S. Lim, (1995) Rendezvous Search On The Line With Three Players, *LSE Math Preprint Series, 81.*

[15] W.S. Lim, (1996) A Rendezvous-Evasion Game On Discrete Locations With Joint Randomization, *LSE-CDAM Research Report Series, 96-04.*

[16] W.S. Lim, (1996) Rendezvous-Evasion As a Multi-Stage Game With Observed Actions, *LSE-CDAM Research Report Series, 96-05.*

[17] W.S. Lim, A. Beck and S. Alpern (1995), Rendezvous Search On The Line With More Than Two Players, *LSE Math Preprint Series, 82.*

[18] W.S. Lim and S. Alpern, Minimax Rendezvous on The Line, *SIAM Journal of Control and Optimization,* in press.

[19] D. J. Reyniers, (1994) Coordinating two searchers for a hidden object on the line, *London School of Economics Operational Research Preprint Series*, 1-14.

[20] W. H. Ruckle, (1983) Geometric games and their applications. Pitman, Boston.

[21] W.H. Ruckle, (1983) Pursuit on a cyclic graph, *International Journal of Game Theory, 10*, 91-99.

[22] T. Schelling, (1960) The Strategy of Conflict, Harvard University Press, Cambridge.

[23] L. C. Thomas, (1992) Finding your kids when they are lost, *JORS, 43*, 637-639.

[24] L. C. Thomas and P. B. Hulme, (1993) Searching for targets who want to be found, preprint.

List of Corrections

page 2, line 5, $R^s_{m,n}$ should be $R^s_{n,m}$

page 2, line 7, $R^a_{m,n}$ should be $R^a_{n,m}$

page 17, line -4, ... in the paper **have** not been included ...

page 17, line -2, ... either **does** not change ...

page 20, line 1, ... either **direction** so that ...

page 26, line 7, ... show in **Section 2.5** that ...

page 27, line 13, ... is **equal** to ...

page 29, line 11, ... the players are placed at positions 0,1, and 2 **at random**, so ...

page 29, line 17, Since **Theorem 2.1** says that ...

page 32, line 9, replace the first part of the sentence by:
The algorithm uses the branch and bound technique, i.e., instead of computing the expected meeting time ...

page 33, line 5, ... at most B_{k-1}. **We define P_k by induction on k.** Let P_1 ...

page 37, line -6, ... in 2-person rendezvous, **no two cases have the same meeting time**. This observation is based on ...

page 37, last line, ... proved in [17]: **that** the meeting times ...

page 39, line 12, ... **It suffices to find** a mixed strategy which if ...

page 43, line 2, ... starting from **a** random ... .

page 44, line 10, ... we have $\tau_2 = \mathbf{3.5}$, $\tau_3 = 3.0$ ...

page 53, Scenario 3: ...towards each **other**; while ...

page 56, last line, ... is **equal** to ...

page 60, line 15, ... we will say **that** (f, g) has permutation type ...

page 60, line -3, ... as well as the **least** expected ...

page 60, line -2, ... which **achieve** the minimax ...

page 67, line 6, ... can **achieve** a maximum meeting time ...

page 69, last line, ... with each **preceded** by a slope ...

page 72, line 8, ... be **achieved**, player II ...

page 89, line 4, ... prove that v_3 **is 47/76**. We subsequently ...

page 92, line -3 ... At step 3, **coordinate** to meet ...

page 99, line -3, ... each of the **locations** at step 3 ...

page 115, Figure 4.4, the key of the arrows should be interchanged, i.e., the dotted arrow replaced by the solid arrow and vice versa.

Wei Shi Lim
10 May 1996