The Minimal Entropy Martingale Measure and Hedging in Incomplete Markets

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A thesis submitted for the degree of
Doctor of Philosophy

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Thesis examination: 09 October 2009

Abstract

The intent of these essays is to study the minimal entropy martingale measure, to examine some new martingale representation theorems and to discuss its related Kunita-Watanabe decompositions.

Such problems arise in mathematical finance for an investor who is confronted with the issues of pricing and hedging in incomplete markets. We adopt the standpoint of a rational investor who principally endeavours to maximize her expected exponential utility. Resolving this issue within a semimartingale framework leads to a non-trivial martingale problem equipped with an equation between random variables but not processes.

It is well known that utility maximization admits a dual formulation: maximizing expected utility is equivalent to minimizing some sort of distance to the physical probability measure. In our setting, this is compatible to finding the entropy minimizing martingale measure whose density process can be written in a particular form. This minimal entropy martingale model has an information theoretic interpretation: if the physical probability measure encapsulates some information about how the market behaves, pricing financial instruments with respect to this entropy minimizer corresponds to selecting a martingale measure by adding the least amount of information to the physical model.

We present a method of solving the non-trivial martingale problem within models which exhibit stochastic compensator. Several martingale representation theorems are established to derive an apparent entropy equation. We then verify that the conjectured martingale measure is indeed the entropy minimizer.

Contents

Chapter 1

Introduction

The intent of this thesis is to study the minimal entropy martingale measure, to examine some new martingale representation theorems and to discuss its related Kunita-Watanabe decompositions.

In a complete market, there is no more than one arbitrage free way to value an option: the value is defined as the cost of replicating it. Hence, options are in principle redundant in that their exercise values can be replicated by trading in the underlying. If a contingent claim is not redundant, perfect hedges do not exist and the notion of pricing by replication falls apart because there are some risks that one cannot hedge. Due to this, we consider hedging in the sense of approximating the terminal payoff with a trading strategy. We also specify ways to measure this risk and to minimize it. Numerous approaches to measuring risk lead to different ways to hedging in the literature: super-hedging, utility maximization, variance-optimal hedging and mean-variance hedging, to mention a few. Financial models considered in these essays are incomplete: there exists infinitely many martingale measures equivalent to the physical measure, each compatible with the no arbitrage constraint. Each of them corresponds to a set of derivatives prices respecting the no arbitrage argument. One methodology to find 'the' equivalent martingale measure consists in identifying a utility function describing the investors preferences. It has been shown by Delbaen et al (2002) that utility maximization admits a dual formulation: maximizing expected utility is equivalent to minimizing some sort of distance to the physical probability measure $\mathbb{P}$. The minimal distance martingale for the distance f means the following minimization problem

$$\widehat{J}_f(\mathbb{Q}) := \inf_{\mathbb{Q}} \mathbb{E}_{\mathbb{P}}\left[f\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

where $f : [0, \infty) \to \mathbb{R}$ is some strict convex function over all equivalent martingale measures $\mathbb{Q}$. Examples are the minimal Hellinger martingale measure: $f(x) = -\sqrt{x}$, the variance-optimal martingale measure: $f(x) = |x-1|^2$ and the minimal entropy martingale measure: $f(x) = x\log x$, which we shall *focus* in this dissertation. The method of fixing one specific probability measure amongst all diverse probability measures to price and hedge financial products is a commonly researched issue in the literature. It is worth pointing out that the relationships between these different martingale measures are themselves a field of study, see for instance Miyahara (1999), Arai (2001), Esche & Schweizer (2005) and Monoyios (2007).

Before we get into the details as to what this thesis 'does', we first present an overview of existing results concerning the entropy minimizing martingale measure across asset classes. The paper of Grandits & Rheinländer (2002) proved that the density of the minimal entropy martingale measure Z for general stochastic volatility models driven by an independent noise process Y

$$\frac{dS_t}{S_t} = a(Y_t)dt + \sigma(Y_t)dB_t \tag{1.1}$$

has the explicit form

$$\begin{aligned}\left.\frac{d\mathbb{Q}^E}{d\mathbb{P}}\right|_{\mathcal{G}_T} = Z_T &= c\exp\left(-\int \lambda dS\right)_T \\ &= c\exp\left(-\frac{1}{2}K_T\right)\mathcal{E}\left(-\int_0^{\cdot}\frac{a(Y_t)}{\sigma(Y_t)}dB_t\right)_T \end{aligned}\tag{1.2}$$

where $K := \int \frac{a^2}{\sigma^2}dt$, $\lambda = \frac{a}{\sigma^2 S}$, c being the normalizing constant given by

$$c^{-1} = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_0^T \frac{1}{2}\frac{a^2(Y_t)}{\sigma^2(Y_t)}dt\right)\right]$$

and B a standard Brownian motion under $\mathbb{P}$. The density process for this model (1.1) or that of the variants have been studied by Becherer (2004), Benth & Meyer-Brandis (2005) as well as Hobson (2004) and Rheinländer (2005) where they generalize Y to the stochastic volatility model

$$dY_t = \alpha(\beta - Y_t)dt + \gamma dW_t, \quad d[W,B]_t = \rho dt, \quad \rho \in [-1,1]$$

The following scheme lists the authors that study the entropy minimizing density process under the stochastic volatility model (1.1) with different stochastic processes Y:

Y	**Author(s)**
Multivariate Point Process	Becherer (2004)
Pure Jump Levy	Benth & Meyer-Brandis (2005)
Brownian $\rho = 0$ $\rho \neq 0$	Hobson (2004), Rheinländer (2005)

The techniques used to evaluate the density process include the optimal martingale measure methodology (Rheinländer, 1999), the construction of dependent intensities via change of measure (minimal martingale measure) as well as exploiting some results from portfolio optimization.

Another popular approach to identifying the entropy minimizing measure for incomplete models has been related to the construction of Esscher martingale transform. As pointed out in Kallsen & Shiryaev (2002), two sorts of Esscher martingale transforms exists according to the choice of the parameter which defines this measure: the first turns the ordinary exponential process into a martingale and another turning into a martingale the stochastic exponential $\mathcal{E}(\cdot)$. They have been coined the Esscher martingale transform for the exponential process and the Esscher martingale transform for the linear process respectively. Recent results related to the calculation of Esscher transforms to a non-Lévy setting are in Bellini & Mercuri (2007) and Dassios & Jang (2003).

The minimal entropy martingale measure for geometric Lévy processes

$$dS_t = \pi S_{t-}dt + \sigma S_{t-}dB_t + \int_{\mathbb{R}} (e^x - 1)\mu(dt, dx) \tag{1.3}$$

where π, σ are constants, μ is the jump measure associated to the Lévy process and the compensator $\nu(dt, dx) = \nu(dx)dt$ have been solved by Chan (1999), Goll & Ruschendorf (2001), Fujiwara & Miyahara (2003) (†) and Esche & Schweizer (2005) (‡). Note that the results of (†) and (‡) are extremely interesting since they established that under very weak conditions, the entropy minimizing measure can be defined and furthermore represented explicitly. Furthermore, it has been shown by Esche & Schweizer (2005) that for exponential Lévy models, the Esscher martingale transform for linear process coincides with the minimal entropy martingale measure.

The entropy calculations beyond Lévy processes have also been studied relatively recently. Rheinländer & Steiger (2006) looked at the entropy minimizer where they considered the general jump-diffusion model

$$\begin{aligned}
\frac{dS_t}{S_{t-}} &= \eta^M(t, V_t)dt + \sigma^M(t, V_{t-})dB_t + \int_{\mathbb{R}} W^M(\cdot, V_{t-}, x)(\mu(dt, dx) - \nu(dt, dx)) \\
dV_t &= \eta^V(t, V_t)dt + \int_{\mathbb{R}} W^V(t, V_{t-}, x)\mu(dt, dx)
\end{aligned}$$

with the functions W^M, W^V, σ^M satisfying some integrability assumptions as well as a deterministic compensator. On the other hand, Ceci & Gerardi (2009) computed the minimal entropy measure for a geometric market point process of the form

$$\begin{aligned}
S_t &= S_0 \exp(X_t^1) \\
dX_t &= b(X_t)dt + \sigma(X_t)dB_t + \int_{\mathbb{R}} U_0(t, X_{t-})\mu(dt, dx) \\
dX_t^1 &= \int_{\mathbb{R}} U_1(t, X_{t-})\mu(dt, dx)
\end{aligned}$$

also with deterministic compensator as well as U_1, U_2 satisfying some conditions. Fujiwara (2009) investigated the entropy minimizer when the stock price follows a time inhomogeneous Lévy process

$$\frac{dS_t}{S_{t-}} = \pi_t dt + \sigma_t dB_t + \int_{\mathbb{R}} (e^x - 1)\mu(dt, dx)$$

with the compensator taking the form $\nu(dt, dx) = \int_0^t \int_{\mathbb{R}} \nu_s(dx)ds$, which is time inhomogenous but deterministic.

The following table gives a status of existing results:

Asset model	**Deterministic compensator** $\nu(dt, dx)$	**Author(s)**
Lévy	Yes ✓	Fujiwara & Miyahara (2003) Esche & Schweizer (2005)
General Barndorff-Nielsen & Shepherd	Yes ✓	Rheinländer & Steiger (2006)
Inhomogeneous Lévy	Yes ✓	Rheinländer & Steiger (2006) Fujiwara (2009)
Marked Point Process	Yes ✓	Ceci & Gerardi (2009)

One observes that the above entropy minimizers involve models that exhibit *deterministic* compensating measures. However, the minimal entropy martingale measure for the above models when their associated compensators are stochastic has not been discussed at all. Given the facts above, one is tempted to ask: How would the entropy minimizing martingale measure look like when the compensator is *stochastic*?

These essays study this issue.

We begin with a preliminary discussion in Chapter 2 where we recollect some notions on relative entropy and martingale measures as well as some facts from stochastic calculus. The concept of relative entropy of a probability measure with respect to another probability measure and its minimization over some convex set of measures stems from information theory, see, e.g., Csiszár (1975). It is now widely used in the area of financial mathematics owing to its intimate connection with maximizing expected utility in the event when the investor exhibits exponential utility function.

In Chapter 3, we revisit the main example given in Grandits & Rheinländer (2002). There it was shown that the density Z of the minimal entropy martingale measure takes the form as in equation (1.2). Given this density, we present a new method of calculating the density process of the minimal entropy martingale measure by means of martingale techniques. We feel that this method is essentially faster and more intuitive compared to the procedure outlined by Benth & Meyer-Brandis (2005). We end the section by drawing connections to other entropic models that has been highlighted in the literature.

Martingale representation theorems are vital to the calculation of the entropy minimizing measure. In Chapter 4, we establish two versions of the martingale representation theorem for a certain enlarged filtration $\mathbb{G}$. We first deal with the representation concerning the One-Jump process and a Brownian motion before moving on to proving the martingale representation involving Markov Additive Processes. The martingale representation theorem with respect to a Brownian filtration $\overline{\mathcal{F}}$ is well known. It states that any $\overline{\mathcal{F}}-$martingale M_t with $M_t \in L^2(\mathbb{P})$ can be represented as

$$M_t(\omega) = \mathbb{E}_{\mathbb{P}}[M_0] + \int_0^t g(s,\omega)dB_s$$

for some predictable process g such that $\mathbb{E}_{\mathbb{P}}[\int_0^T g^2(t,\omega)dt] < \infty$. Lokka, Oksendal & Proske (2004) derived the formula for the martingale representation when the driving noise is a pure jump process with independent and stationary increments. Another interesting result due to Jacod & Shiryaev (2003, Theorem III.4.34) tells us that every $(\mathbb{P},\mathbb{F})-$local martingale M can be written as

$$M = M_0 + \int HdB + \int\int W(t,x)(\mu_Y - \nu_Y)$$

where H, W satisfy some integrability conditions. One could then ask: What can we say about the martingale representations for a certain enlarged filtration $\mathbb{G}$? The filtration $\mathbb{G}$ can be seen as the product of two or even more filtrations, e.g.,

$$\mathbb{G} = \mathbb{H} \vee \mathbb{F}$$

where $\mathbb{H}$ and $\mathbb{F}$ can be filtrations generated by the one-jump process and Brownian motion respectively, say. It could well be that $\mathbb{H}$ denotes the filtration generated by the Markov chain C rather than the one-jump process. We derive martingale representations for these such cases under the immersion property.

Chapter 5 sees the calculation of the entropy minimizing measure when the stock price process is driven by a Brownian motion and a one-jump process,

$$\mathbb{P}: \qquad \frac{dS_t}{S_{t-}} = a_t dt + b_t dB_t + c_t d\widehat{M}_t$$

This model is proposed by Biagini & Rheinländer (2009). It turns out that the intensity of the one-jump process with respect to the minimal entropy martingale measure is random but bounded. This is in contrast to general Lévy processes where the compensator remains deterministic under the entropy measure. The Esscher martingale transforms for one-jump process is also studied in this chapter. As explained earlier, two different Esscher martingale transforms exists for semimartingales depending on the choice of parameter which defines the measure: we focus on the Esscher transform for the exponential process.

Chapter 6 looks at the entropy minimizer for a general jump diffusion model. In particular, the model proposed can be seen as a Markov switching Lévy processes where the drift, volatility and the Lévy measure is a function of the continuous time Markov chain. If the drift and volatility were not present, it reduces to the model proposed by Elliot & Osakwe (2006). Hence the contribution of this chapter is two-fold; we have solved the minimal entropy martingale measure for the model proposed by Elliot & Oskawe (2006) as well as the entropy minimizer for a generalization for their model. We also compute the basic L^2−decomposition of martingales associated to this measure $\mathbb{Q}^E$.

Miyahara (2000) worked out the minimal entropy martingale measure for Birth and Death processes by proving that the existence for this entropy minimizer is reduced to solving the corresponding Hamilton-Jacobi-Bellman equation. His concluding remarks suggested the generalization to a more general Markov process and semimartingale price processes extending the Birth and Death process. Chapter 7 answers these questions. Using the price process introduced in Norberg (2003) as well as the martingale representation theorem of Markov chains in Rogers & Williams (2000) together with the framework outline in Delbaen et al (2002), we evaluate the entropy minimizer when the stock price process is driven by exponential Markov chains. It turns out that this puzzle presents its very own problem due to the stochasticity of the compensator associated to the martingales of transitions.

Of late, the density process of the minimal entropy for Markov modulated geometric Brownian motion or more complicated stochastic volatility models has been evaluated by several authors, for instance Elliot, Chan & Siu (2005) and Song & Bo (2009), to mention a few. In our view, the method used by these authors to evaluate the entropy minimizer relies on an interesting fact for Lévy processes: if the Esscher martingale transform for the linear exists, it coincides with the minimal entropy martingale measure. Due to the work of Esche & Schweizer (2005), one can infer that this analogy with Lévy models breaks down for complex models beyond exponential Lévy models. We claim and show that the results given in the paper of Elliot, Chan & Siu (2005) and Song & Bo (2009) are incorrect. On another issue, we apply our findings on the entropy minimizing measure for quasi-Markov Additive processes to the valuation options written on the prices of carbon emissions to the field of carbon finance. These matters will be dealt with in the final chapter.

Chapter 2

Preliminaries

To begin with, we state some general assumptions and list several fundamental known results needed throughout this text. We present a short extraction of commonly used notations and notions in stochastic calculus. Here we will find excerpts from the literature on relative entropy, jump processes and optimal martingale measure which are tailored to fit into a consistent format within our essays.

2.1 Semimartingale framework and martingale measures

The mathematical framework is given by a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, a finite time horizon $T > 0$ and a filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness. We further assume that $\mathcal{G}_0$ is trivial and $\mathcal{G}_T = \mathcal{G}$. All semimartingales are taken to have right continuous paths with left limits. Expectation taken with respect to $\mathbb{P}$ is denoted by $\mathbb{E}_{\mathbb{P}}$.

Let S be an $\mathbb{R}-$valued $(\mathbb{P}, \mathbb{G})-$semimartingale. We consider S as the discounted price of a risky asset in a financial market and we assume that interest rates are zero, i.e. $r = 0$. In addition, we make the standing assumption that

$$S \text{ is } \mathbb{G} - \text{locally bounded.}$$

Since S is $\mathbb{G}-$adapted, it has the following canonical decompostion

$$S = S_0 + M + A$$

where M is a locally bounded local martingale with $M_0 = 0$ and A is a process of locally finite variation. Moreover we assume that the asset price process S satisfies the following:

Assumption 2.1 (Structure condition). There exists a predictable process λ satisfying

$$A = \int \lambda d\langle M, M \rangle$$

with

$$K_T := \int_0^T \lambda_t^2 d\langle M, M\rangle_t < \infty \quad \mathbb{P} - a.s.$$

Let us present the notion of a martingale measure taken from Rheinländer (1999):

Definition 2.2. *Let $\mathcal{V}$ be the linear subspace of $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ spanned by the elementary stochastic integrals of the form $f = h(S_{T_2} - S_{T_1})$, where $0 \leq T_1 \leq T_2 \leq T$ are stopping times such that the stopped process S^{T_2} is bounded and h is a bounded $\mathcal{G}_{T_1}-$measurable random variable. A martingale measure is a probability measure $\mathbb{Q} \ll \mathbb{P}$ with $\mathbb{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} f\right] = 0$ for all $f \in \mathcal{V}$.*

An alternative but equivalent definition for martingale measure:

Definition 2.3. *A probability measure $\mathbb{Q}$ absolutely continuous to $\mathbb{P}$ is a martingale measure for S if S is a local $\mathbb{Q}-$martingale. It is called an equivalent martingale measure if it is equivalent to $\mathbb{P}$.*

Definition 2.4. *The relative entropy $I(\mathbb{Q}, \mathbb{P})$ of a probability measure $\mathbb{Q}$ w.r.t. a probability measure $\mathbb{P}$ is given as*

$$I(\mathbb{Q}, \mathbb{P}) = \begin{cases} \mathbb{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right], & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty, & \text{otherwise} \end{cases}$$

It is well known that $I(\mathbb{Q}, \mathbb{P}) \geq 0$, $I(\mathbb{Q}, \mathbb{P}) = 0$ if and only if $\mathbb{Q} = \mathbb{P}$ and $\mathbb{Q} \to I(\mathbb{Q}, \mathbb{P})$ is strictly convex.

The sets of absolutely continuous and equivalent local martingale measure for S with respect to $\mathbb{G}$, and those with finite relative entropy are defined as

$$\begin{aligned} \mathcal{M} &:= \{\mathbb{Q} \ll \mathbb{P} | S \text{ is a local } (\mathbb{Q}, \mathbb{G})\text{-martingale}\} \\ \mathcal{M}^e &:= \{\mathbb{Q} \sim \mathbb{P} | S \text{ is a local } (\mathbb{Q}, \mathbb{G})\text{-martingale}\} \\ \mathcal{M}^f &:= \{\mathbb{Q} \in \mathcal{M} | I(\mathbb{Q}, \mathbb{P}) < \infty\} \end{aligned}$$

We assume throughout this text that our financial model is arbitrage free in the sense that

$$\mathcal{M}^e \cap \mathcal{M}^f \neq \emptyset.$$

Definition 2.5. *The minimal entropy martingale measure $\mathbb{Q}^E$ is the solution of*

$$I\left(\mathbb{Q}^E, \mathbb{P}\right) = \min_{\mathbb{Q} \in \mathcal{M}} I(\mathbb{Q}, \mathbb{P})$$

Theorems 1,2 and Remark 1 of Fritelli (2000) as well as the fact that $\mathcal{V} \subset L^\infty(\mathbb{P})$, yield the following result of Fritelli (2000) :

Theorem 2.6. *If there exists $\mathbb{Q} \in \mathcal{M}^e$ such that $I(\mathbb{Q}, \mathbb{P}) < \infty$, then the minimal entropy martingale measure exists, is unique and moreover equivalent to $\mathbb{P}$.*

We now state a Theorem from Grandits & Rheinländer (2002) which provides a criterion for a martingale measure to coincide with the minimal entropy martingale measure.

Theorem 2.7. *Assume there exists a* $\mathbb{Q} \in \mathcal{M}^e$ *with* $I(\mathbb{Q}, \mathbb{P}) < \infty$. *Then* $\overline{\mathbb{Q}}$ *is the minimal entropy martingale measure if and only if there exists a constant* c^E *and an* $S-$*integrable predictable process* ϕ^E

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} = \exp\left(c^E + \int_0^T \phi_t^E dS_t\right) \tag{2.1}$$

such that $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \phi_t^E dS_t\right] = 0$ *for all* $\mathbb{Q} \in \mathcal{M}^e$ *with finite relative entropy.*

We will henceforth pursue by finding some potential candidate measure $\overline{\mathbb{Q}}$ which can be represented as in equation (2.1). One would then verify that this potential measure $\overline{\mathbb{Q}}$ is indeed the entropy minimizer. To carry out this last step, we appeal to the Verification Procedure of Theorem 2.1.5 in Grandits & Rheinländer (2002).

2.2 Verification procedure

Let us now describe the procedure in full, consisting of four steps, for verifying that a given probability measure is indeed the minimal entropy martingale measure $\mathbb{Q}^E$.

Step 1: $\overline{\mathbb{Q}}$ is an equivalent probability measure. To show this, we have to show that $\exp\left\{c^E + \int_0^T \phi_t^E dS_t\right\}$ is integrable with $\mathbb{E}_{\mathbb{P}}\left[\exp\left\{c^E + \int_0^T \phi_t^E dS_t\right\}\right] = 1$.

Step 2: $\overline{\mathbb{Q}}$ is a martingale measure, i.e. S is a local $\overline{\mathbb{Q}}-$martingale.

Step 3: The probability measure $\overline{\mathbb{Q}}$ has finite relative entropy with respect to $\mathbb{P}$, i.e.

$$I\left(\overline{\mathbb{Q}}, \mathbb{P}\right) = \mathbb{E}_{\mathbb{P}}\left[\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} \log \frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right] = \mathbb{E}_{\overline{\mathbb{Q}}}\left[\log \frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right] < \infty$$

Step 4: $\int \phi^E dS$ is a true $\mathbb{Q}-$martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with $I\left(\mathbb{Q}, \mathbb{P}\right) < \infty$

If none of these conditions are violated, then $\overline{\mathbb{Q}}$ is the minimal entropy martingale measure $\mathbb{Q}^E$.

To carry out **Step 1**, we need the following generalization of the Novikov condition for discontinuous processes:

Theorem 2.8. *Let* N *be a locally bounded local* $\mathbb{P}-$*martingale. Let* $\mathbb{Q}$ *be a measure defined by*

$$Z_t := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(N)_t$$

where $\Delta N > -1$. *If the process*

$$U_t = \frac{1}{2}\left\langle N^c, N^c\right\rangle_t + \sum\nolimits_{s\leq t}\left\{(1+\Delta N_s)\log(1+\Delta N_s) - \Delta N_s\right\}$$

admits predictable compensator A as well as

$$\mathbb{E}_{\mathbb{P}}\left[\exp A_T\right] < \infty$$

then $\mathcal{E}(N)$ is a uniformly integrable martingale and $\mathcal{E}(N)_T > 0$ $\mathbb{P}-a.s.$

Proof. Theorem III.1. of Lepingle & Mémin (1978). □

Finally, to cope with **Step 4**, we shall use the following result presented in Rheinländer (2005):

Proposition 2.9. *Let $\mathbb{Q}$ be an equivalent martingale measure with finite relative entropy. Let $\int \psi dS$ be a local $\mathbb{Q}-$martingale. Then $\int \psi dS$ is a square-integrable $\mathbb{Q}-$martingale if, for some $\beta > 0$, $\exp\left\{\beta \int_0^T \psi_t^2 d\left[S,S\right]_t\right\}$ is $\mathbb{P}-$integrable.*

Proof. Proposition 3.2 of Rheinländer (2005). □

2.3 Random measures

Random measures and their compensators are important tools to encapsulate the behaviour of the jumps of the semimartingale S. We begin with a general definition of random measures before stating the definition for random measures associated with the jumps of a general semimartingale. All of these definitions are taken from Jacod & Shiryaev (2003).

Definition 2.10. *A random measure $\mathbb{R} \times \mathbb{R}$ is a family*

$$\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$$

of measures on $(\mathbb{R}_+ \times \mathbb{R}, \widetilde{\mathcal{B}}_+ \times \widetilde{\mathcal{B}})$ satisfying $\mu(\omega; \{0\} \times \mathbb{R}) = 0$ for all $\omega \in \Omega$, where $\widetilde{\mathcal{B}}$ denotes the Borel field on $\mathbb{R}$.

Definition 2.11. *To any measurable function W on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ we introduce the integral process*

$$W * \mu(\omega) := \begin{cases} \int \int_{\mathbb{R}} W(\omega,t,x)\mu(\omega;dt,dx), & \text{if } \int \int_{\mathbb{R}} |W(\omega,t,x)|\mu(\omega;dt,dx) < \infty \\ +\infty, & \text{otherwise} \end{cases}$$

We now identify the random measures which are essential in our context, that is to say, random measures associated with the jumps of a semimartingale and their associated compensators.

Definition 2.12. *Let Y be a semimartingale. The random measure μ_Y associated with the jumps of Y is defined by*

$$\mu_Y(dt,dx) = \sum_{s>0} \mathbb{1}_{\{\Delta Y_s \neq 0\}} \delta_{\{(s,\Delta Y_s)\}}(dt,dx)$$

where δ_q is the Dirac measure at point q. It is also called the jump measure of Y.

Remark 2.13.
1. Note that μ_Y is integer-valued.
2. Let μ_Y be the jump measure of Y. For any measurable function W we have

$$W * \mu_Y = \sum_{0<s\leq t} W(s, \Delta Y_s) \mathbb{1}_{\{\Delta Y_s \neq 0\}}.$$

3. These definitions concerning the jump measure of Y are taken from Jacod & Shiryaev (2003), Proposition II.1.16 & II.1.15.

Theorem 2.14. *Let μ_Y be the jump measure of Y. The predictable $\mathbb{P}-$compensator of μ_Y, denoted by ν_Y is the predictable random measure which satisfies one of the two following equivalent conditions:*

(i) $\mathbb{E}_{\mathbb{P}}[W * \nu_Y] = \mathbb{E}_{\mathbb{P}}[W * \mu_Y]$ *for every non-negative predictable function W.*

(ii) *For every predictable function W such that $|W| * \mu_Y < \infty$ and locally $\mathbb{P}-$integrable (which is equivalent to $|W| * \nu_Y < \infty$ and locally $\mathbb{P}-$integrable), $W * \mu_Y - W * \nu_Y$ is a local $\mathbb{P}-$martingale.*

Proof. Jacod & Shiryaev (2003), Theorem II.1.8. □

Let us turn our attention to stochastic integrals with respect to a compensated random measure. The next theorem is an important one.

Theorem 2.15. *If the increasing process $|W| * \mu_Y$ or equivalently $|W| * \nu_Y$ is locally $\mathbb{P}-$integrable, then W is integrable with respect to $\mu_Y - \nu_Y$ and*

$$W * (\mu_Y - \nu_Y) = W * \mu_Y - W * \nu_Y.$$

Proof. Jacod & Shiryaev (2003), Theorem II.1.28. □

2.4 Class of equivalent martingale measures

Let us state a result from Steiger (2005, Lemma 3.2.1). The elegant proof given below is constructed by Steiger and we illustrate it here for completeness and because of its importance.

Proposition 2.16. *Let $\mathbb{Q} \in \mathcal{M}^e$. Then the density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}$ is given by the Doléans Dade exponential process*

$$Z = \mathcal{E}\left(-\int \lambda dM + L\right)$$

where L and $[M, L]$ are local $\mathbb{P}-$martingales.

Proof. Our aim is to show that $[M, L]$ is a local $\mathbb{P}$-martingale. Define $N := -\int \lambda dM + L$. From He et al. (1992), we get due to Girsanov's Theorem that S is a local $\mathbb{Q}$−martingale if and only if

$$S + \int \frac{1}{Z_-} d[S, Z]$$

is a local $\mathbb{P}$-martingale. Let us write

$$\begin{aligned} S + \int \frac{1}{Z_-} d[S, Z] &= M + \int \lambda d\langle M\rangle + \int \frac{1}{Z_-} d[S, 1 + \int Z_- dN] \\ &= M + \int \lambda d\langle M\rangle + \int \frac{1}{Z_-} d\Big(\int Z_- d[S, N]\Big) \end{aligned}$$

Examining

$$\begin{aligned} [S, N] &= \Big[M + \int \lambda d\langle M\rangle, L - \int \lambda dM\Big] \\ &= [M, L] + \Big[M, -\int \lambda dM\Big] + \Big[\int \lambda d\langle M\rangle, L - \int \lambda dM\Big] \\ &= [M, L] - \int \lambda d[M] + \Big[\int \lambda d\langle M\rangle, L - \int \lambda dM\Big] \end{aligned}$$

since

$$\Big[\int dM, -\int \lambda dM\Big] = \Big[M, -\int \lambda dM\Big]$$

so that

$$\begin{aligned} S + \int \frac{1}{Z_-} d[S, Z] &= M + \int \lambda d\langle M\rangle - \int \lambda d[M] + [M, L] \\ &= \Big[\int \lambda d\langle M\rangle, L - \int \lambda dM\Big] \end{aligned}$$

Thus $[M, L]$ must be a local $\mathbb{P}$-martingale to ensure that $S + \int \frac{1}{Z_-} d[S, Z]$ is a local $\mathbb{P}$-martingale. □

2.5 The minimal entropy martingale measure

We shall give a systematic plan for finding the minimal entropy martingale measure in a general semimartingale context. The idea is to first look for a candidate measure whose density can be written as in (2.1). One has then to show that this candidate measure is indeed the entropy minimizer by performing the verifications as outlined earlier.

We know by (2.1) that the density of the minimal entropy martingale measure $\mathbb{Q}^E$ can necessarily be written as

$$\frac{d\mathbb{Q}^E}{d\mathbb{P}} = \exp\left(c^E + \int_0^T \phi_t^E dS_t\right)$$

for some constant c^E and some $(\mathcal{G}_t)$−predictable process ϕ^E. We look for some candidate measure which can be represented in this way and execute the verifications. As a consequence of Proposition 2.16, every martingale measure $\overline{\mathbb{Q}}$ can be written as

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^{\cdot} \lambda dM + L\right)_T$$

where L is some locally square local $(\mathbb{P}, \mathcal{G})$−martingale strongly orthogonal to M, i.e. $\langle M, L\rangle = 0$ with $L_0 = 0$. The constant c^E and some $\mathbb{G}$−predictable process ϕ^E that satisfy the following equation

$$\exp\left(c^E + \int_0^T \phi_t^E dS_t\right) = \mathcal{E}\left(-\int_0^{\cdot} \lambda dM + L\right)_T$$

such that $\mathbb{E}_{\mathbb{P}}[\int \phi^E dS] = 0$ is called the minimal entropy martingale measure equation.

Chapter 3

Grandits & Rheinländer revisited

3.1 A review of risk minimization

In this section, we give a brief review of the concept of risk-minimizing hedging of Föllmer & Sondermann (1986) where they extended the theory of hedging for complete markets to the case of incomplete markets. Let the contingent claim H satisfy $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q}^E)$. To replicate this contingent claim, we consider a strategy which involves the stock S_t and the money market account r_t which yields the terminal payoff H at time T. Let ξ_t and η_t denote the amounts invested in S_t and the money market account respectively at time t; where ξ_t satisfies

$$\mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \xi_t^2 d[S,S]_t\right] < \infty$$

and η is a $\mathcal{G}-$adapted process. We let $L^2(\mathbb{Q}_S^E)$ be the space of square-integrable predictable processes $(\xi_t)_{0\le t\le T}$ satisfying the above integrability condition. A trading strategy at t is of the form $\varphi = (\xi, \eta)$ where $\varphi = \{\varphi_t; 0 \le t \le T\} = \{(\xi_t, \eta_t); 0 \le t \le T\}$. The value of the portfolio at time t is given by

$$V_t^\varphi = \xi_t S_t + \eta_t, \quad 0 \le t \le T$$

A strategy $\varphi = (\xi, \eta)$ is said to be admissible if $V_T^\varphi = H$.

Definition 3.1. *A $\mathcal{G}-$strategy is any process $\varphi = (\xi, \eta)$ with $\xi \in L^2(\mathbb{Q}_S^E)$ and η is $\mathcal{G}-$adapted such that the value process V^φ is right-continuous and $V_t^\varphi \in L^2(\mathbb{Q}^E)$ for all $t \in [0,T]$.*

Definition 3.2. *The cost process C^φ associated with the strategy φ is defined by*

$$C_t^\varphi = V_t^\varphi - \int_0^t \xi_u dS_u, \quad 0 \le t \le T$$

We say that a contingent claim H can be perfectly replicated if there exists a self-financing strategy portfolio φ such that $V_t^\varphi = H$. In this case the claim is said to be attainable. Since the market under consideration is incomplete, not every contingent claim H may be attainable. Hence instead of looking for an admissible self financing strategy, we look for admissible strategies which minimize, at each time t, the remaining residual risk, defined by

Definition 3.3. *The residual risk associated with the strategy φ is given by*

$$R_t^\varphi = \mathbb{E}_{\mathbb{Q}^E}\left[(C_T^\varphi - C_t^\varphi)^2 \,|\, \mathcal{G}_t\right]$$

over all admissible strategies. The admissible strategy φ^* is said to be risk minimizing if

$$R_t \leq R_t^{\varphi^*} \quad \text{for any other admissible strategy } \varphi \quad \forall t \in [0,T]$$

The construction of strategies is based on an application of the Kunita-Watanabe decomposition, see Föllmer & Sondermann (1986). Define the intrinsic value process associated to the claim as V^H by

$$V_t^H := \mathbb{E}_{\mathbb{Q}^E}[H \,|\, \mathcal{G}_t]$$

Note that V^H is a $(\mathcal{G}, \mathbb{Q}^E)-$martingale. Let us define the following space

$$\Theta(\mathbb{G}) := \left\{ \vartheta \in \mathbb{G} \,\middle|\, \mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \vartheta_t^2 d[S,S]_t\right] < \infty \right\} \tag{3.1}$$

Theorem 3.4 (Entropic Kunita-Watanabe decomposition). *Let $(S_t)_{t\in[0,T]}$ be a square integrable martingale under $\mathbb{Q}^E$, i.e. $\sup_{t\in[0,T]} \mathbb{E}_{\mathbb{Q}^E}[S_t^2] < \infty$. For any random variable $H \in L^2(\mathbb{Q}^E)$, there exists a predictable stochastic process $\vartheta^H \in \Theta(\mathbb{G})$ and a square-integrable $\mathcal{G}_T-$measurable random variable N_T^H such that*

$$H = \mathbb{E}_{\mathbb{Q}^E}[H] + \int_0^T \vartheta_t^H dS_t + N_T^H$$

Define $N_t^H := \mathbb{E}_{\mathbb{Q}^E}[N_T^H|\mathcal{G}_t]$. Moreover N^H is strongly orthogonal to S in the sense that $\langle N^H, S\rangle = 0$. Furthermore the martingale $N_t^H := \mathbb{E}_{\mathbb{Q}^E}[N_T^H|\mathcal{G}_t]$ is strongly orthogonal to all stochastic integrals with respect to S: for any $\vartheta \in \Theta(\mathbb{G})$, $N_t^H \int_0^t \vartheta dS$ is a martingale or equivalently, $\langle N^H, S\rangle_t = 0$ for all $t \in [0,T]$.

We see from the above that the Kunita-Watanabe decomposition for the intrinsic value process is of the form

$$V_t^H = \mathbb{E}_{\mathbb{Q}^E}[H] + \int_0^t \xi_t^H dS_u + L_t^H$$

where $L^H = (L_t^H)_{0\leq t\leq T}$ is a zero mean $(\mathcal{G}, \mathbb{Q}^E)-$martingale, L^H and S are strongly orthogonal to each other, i.e. $\langle L^H, S\rangle = 0$ and ξ^H is a predictable process satisfying $\mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \xi_t^2 d[S,S]_t\right] < \infty$. Föllmer & Sondermann (1986, Theorem 2) states:

Theorem 3.5. *There exists a unique admissible risk minimizing strategy $\varphi = (\xi, \eta)$ given by*

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^H - \xi_t^H S_t), \quad 0 \le t \le T$$

The associated risk process R_t^H is given by

$$R_t^H = \mathbb{E}_{\mathbb{Q}^E}\left[(L_T^H - L_t^H)^2 \middle| \mathcal{G}_t \right]$$

Remark 3.6. Some comments are in place.
(i) The portfolio ξ^H defined by this Kunita-Watanabe decomposition minimizes the residual risk process among all admissible strategies and the minimum risk is given by R_t^H.
(ii) **Mean-variance hedging.** As mentioned in Föllmer & Sondermann (1986), the risk-minimizing strategies are typically not self-financing, i.e. an admissible portfolio satisfying $V_T^\varphi = H$ cannot be self-financing. A related criterion that still leads to self-financing strategies is therefore to minimize over all self-financing strategies the quantity

$$\|H - V_T^\varphi\|^2_{L^2(\mathbb{Q}^E)} \to \min_{\varphi, V_0} \|H - V_T^\varphi\|^2_{L^2(\mathbb{Q}^E)}$$

This is different to the risk-minimizing definition where Föllmer & Sondermann (1986) proposed not to have the restriction to self-financing strategies but persisted on retaining the condition of admissibility, $V_T^\varphi = H$

3.2 Grandits & Rheinländer revisited

Grandits & Rheinländer (2002) considered a class of diffusion models where an additional random factor Y is present and thus influencing the coefficients of the diffusion, which is independent of the driving Brownian motion B. Denote by $\mathbb{H}$ the associated filtration $\mathcal{H}_t = \sigma(Y_s : s \le t)$. Define further by $\mathbb{F}$ the associated filtration $\mathcal{F}_t = \sigma(B_s : s \le t)$ and assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ i.e., $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$, $\forall t \in \mathbb{R}_+$. All filtrations are augmented to satisfy the usual conditions of right continuity and completeness. We fix a finite time horizon $T > 0$. We consider the stochastic differential equation

$$\frac{dS_t}{S_t} = a(Y_t)dt + \sigma(Y_t)dB_t$$

We assume that $\rho := \frac{a}{\sigma}$ is uniformly bounded. This assumption is consistent with that remarked in Grandits & Rheinländer (2002). Other integrability conditions can be placed but we assume boundedness for simplicity.

3.2.1 Brownian-Lévy case

Let Y be a pure jump Lévy process. Y can be written as

$$Y_t = \int_0^t \int_{\mathbb{R}} y \, \mu_Y$$

where μ_Y, ν_Y are the associated jump measure of Y and the predictable compensator respectively. Grandits & Rheinländer (2002) showed that the density of the minimial entropy martingale measure $\mathbb{Q}^E$ for general stochastic volatility models driven by an independent noise process is of the form

$$Z_T = c\exp\left(-\int \lambda dS\right)_T = c\exp\left(-\frac{1}{2}K_T\right)\mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_t)}{\sigma(Y_t)}dB_t\right)_T$$

where $K := \int \frac{a^2}{\sigma^2}dt$ and c being the normalizing constant is given by

$$c^{-1} = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_0^T \frac{a^2(Y_t)}{\sigma^2(Y_t)}dt\right)\right]$$

Recall that Novikov's criterion tells us if M is a continuous local martingale and suppose that $\mathbb{E}\left\{e^{\frac{1}{2}[M,M]_T}\right\} < \infty$, then $\mathcal{E}(M)_t$ is a uniformly integrable martingale for $t \in [0,T]$. Hence we see that for $M = -\int \frac{a}{\sigma}dB$, we have

$$\mathbb{E}_{\mathbb{Q}^E}\left[\exp\left\{\int_0^T \frac{a^2(Y_t)}{\sigma^2(Y_t)}dt\right\}\right] < \exp(c_1 T) < \infty$$

for some constant c_1 since $\rho := \frac{a}{\sigma}$ is bounded by hypothesis. Hence the stochastic exponential process $\left(\mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y)}{\sigma(Y)}dB\right)_t\right)$ is a $\overline{\mathcal{H}}$−martingale, $t \in [0,T]$ where $\overline{\mathcal{H}}_t := \mathcal{F}_t \vee \mathcal{H}_T$. We now proceed to calculate the density process $Z = (Z_t)_{t\geq 0}$

$$\begin{aligned}
Z_t &:= \mathbb{E}_{\mathbb{P}}\left[Z_T|\mathcal{G}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[c\exp\left(-\frac{1}{2}K_T\right)\mathcal{E}\left(-\int_0^T \frac{a(Y_t)}{\sigma(Y_t)}dB_t\right)\middle|\mathcal{G}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[c\exp\left(-\frac{1}{2}K_T\right)\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_t)}{\sigma(Y_t)}dB_t\right)_T\middle|\mathcal{F}_t \vee \mathcal{H}_T\right\}\middle|\mathcal{G}_t\right] \\
&= c\mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_s)}{\sigma(Y_s)}dB_s\right)_t \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}K_T\right)\middle|\mathcal{G}_t\right]
\end{aligned}$$

We now examine the quantity

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}K_T\right)\middle|\mathcal{G}_t\right] &\overset{(i)}{=} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_0^T \frac{a^2(Y_t)}{\sigma^2(Y_t)}dt\right)\middle|\mathcal{G}_t\right] \\
&\overset{(ii)}{=} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_0^T \frac{a^2(Y_t)}{\sigma^2(Y_t)}dt\right)\middle|\mathcal{H}_t\right] \\
&\overset{(iii)}{=} \exp\left(-\frac{1}{2}K_t\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{a^2(Y_s)}{\sigma^2(Y_s)}ds\right)\middle|\mathcal{H}_t\right] \\
&\overset{(iv)}{=} \exp\left(-\frac{1}{2}K_t\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{a^2(Y_s)}{\sigma^2(Y_s)}ds\right)\middle|Y_t\right]
\end{aligned}$$

where (i) can be compared to the function $H(t,y)$ of equation (4.1) in the paper by Benth & Meyer Brandis (2005); (ii) follows from the independence of Y and B, since $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$; (iii) seperates the 'past' from the 'future' and (iv) is due to the Markov property of Y that states: conditional on the present, the future is independent of the past. We now define the deterministic function $A_t^y := A(t,y)$ as

$$A_t^y := A(t,y) := \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{a^2(Y_s)}{\sigma^2(Y_s)}ds\right)\middle| Y_t = y\right] \tag{3.2}$$

For convenience, we also use the notation $A_t := A(t,Y_t)$. Recall that $\rho := \frac{a}{\sigma}$ and define $\Gamma_t := \mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_s)}{\sigma(Y_s)}dB_s\right)_t$ and $M_t^\Gamma := -\int_0^t \frac{a(Y_s)}{\sigma(Y_s)}dB_s$ so that $d\Gamma_t = -\Gamma_t dM_t^\Gamma = -\Gamma_t\rho_t dB_t$. Z_t can be neatly recast as

$$Z_t = ce^{-\frac{1}{2}K_t}\Gamma_t A_t$$

We further assume that A is one time differentiable with respect to t, i.e. $A \in \mathcal{C}^1$ and so by the Itô formula we obtain

$$\begin{aligned} A(t,Y_t) &= \int_0^t \frac{\partial}{\partial s}A(s,Y_{s-})ds + \sum_{s\le t} A(s,Y_s) - A(s,Y_{s-}) \\ &= \int_0^t \frac{\partial}{\partial s}A(s,Y_{s-})ds + \int_0^t\int_{\mathbb{R}} (A(s,Y_{s-}+x) - A(s,Y_{s-})\,\mu(ds,dx) \end{aligned}$$

We work with the following assumption:

Assumption 3.7. *Let* $Y_{t-} = y$,

$$\int\int_{\mathbb{R}} |A(t,y+x) - A(t,y)|\nu(dx)dt < \infty, \quad t \in [0,T]$$

From Theorem II.1.2.8 in Jacod & Shiryaev (2003), this condition allows us to decompose

$$\begin{aligned} &\int\int_{\mathbb{R}} (A(t,y+x) - A(t,y))(\mu_Y - \nu_Y) \\ &= \int\int_{\mathbb{R}} (A(t,y+x) - A(t,y))\mu_Y - \int\int_{\mathbb{R}} (A(t,y+x) - A(t,y))\nu_Y \end{aligned}$$

Formal calculations then yield

$$dZ_t = -Z_{t-}\rho_{t-}dB_t + Z_{t-}\int_{\mathbb{R}} \frac{A(t,y+x) - A(t,y)}{A(t,y)}(\mu(dt,dx) - \nu(dx)dt)$$

Define $A_t^{\Delta *} := \frac{A(t,Y_{t-}+x) - A(t,Y_{t-})}{A(t,Y_{t-})}$ and $A_t^{\Delta y} := \frac{A(t,y+x)-A(t,y)}{A(t,y)}$. This gives

$$dZ_t = Z_{t-}\left(\rho_{t-}dB_t + \int_{\mathbb{R}} A_t^{\Delta *}(\mu(dt,dx) - \nu(dx)dt)\right), \quad Z_0 = 1$$

which is equivalent to

$$Z_t = 1 - \int_0^t \rho_{s-} Z_{s-} dB_s + \int_0^t \int_{\mathbb{R}} Z_{s-} A_s^{\Delta *}(\mu(dt,dx) - \nu(dx)dt))$$

Taking the uniqueness of solutions of the above stochastic differential equation for granted, we conclude that the unique solution to this SDE is given by the expression

$$Z_t = \mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_s)}{\sigma(Y_s)} dB_s\right)_t \mathcal{E}\left(\int_0^{\cdot} \int_{\mathbb{R}} A_s^{\Delta *}(\mu_Y - \nu_Y)\right)_t$$

We can sum up our arguments by the following:

Proposition 3.8 (Density process of $\mathbb{Q}^E$ for Brownian-Lévy). *Fix $T > 0$. Consider the stock price process of the form*

$$\mathbb{P}: \qquad \frac{dS_t}{S_t} = a(Y_t)dt + \sigma(Y_t)dB_t$$

where Y is a Lévy process which satisfies the conditions of Assumption 3.7. Let the probability measure $\mathbb{Q}^E$ be given by

$$Z_T \;=\; c\exp\left(-\int \lambda dS\right)_T$$

Then the Radon-Nikodým density process Z_t, $t \in [0,T]$ satisfies

$$Z_t = \mathcal{E}\left(-\int_0^{\cdot} \frac{a(Y_s)}{\sigma(Y_s)} dB_s\right)_t \mathcal{E}\left(\int_0^{\cdot} \int_{\mathbb{R}} A_s^{\Delta *}(\mu_Y - \nu_Y)\right)_t$$

where

$$A_t^{\Delta *} := \frac{A(t, Y_{t-} + x) - A(t, Y_{t-})}{A(t, Y_{t-})}$$

with $Z_0 = 1$.

Corollary 3.9. *Let $\mathbb{Q}^E$ be given as above. Then*
(i) The process $B_t^{\mathbb{Q}^E} := B_t + \int_0^t \rho_s ds$, $t \in [0,T]$ follows a Brownian motion under $\mathbb{Q}^E$.
(ii) The process Y_t, $t \in [0,T]$ follows a jump process with the predictable compensating measure $\int_0^t \int_{\mathbb{R}} (1 + A_s^{\Delta *})\nu(dx)ds = \int_0^t \int_{\mathbb{R}} \frac{A(s, Y_{s-}+x)}{A(s, Y_{s-})}\nu(dx)ds$ under $\mathbb{Q}^E$.
(iii) The function $A(t,y)$ satisfies the following integro-partial differential equation since for $t \in [0,T]$, Z is a martingale.

$$\frac{\partial}{\partial t}A(t,y) - \frac{1}{2}A(t,y)\rho^2(y) + \int_{\mathbb{R}}((A(t,y+x) - A(t,y))\nu(dx)dt = 0$$

with the terminal condition $A(T,\cdot) = 1 \quad \forall y \in \mathbb{R}$.
(iii) The dynamics of (S,Y) under $\mathbb{Q}^E$ are given by

$$\mathbb{P}: \begin{cases} dS_t = a(Y_t)S_t dt + \sigma(Y_t)S_t dB_t \\ dY_t = \int_{\mathbb{R}} x\mu(dt,dx) \end{cases} \xrightarrow{\left.\frac{d\mathbb{Q}^E}{d\mathbb{P}}\right|_{\mathcal{G}_t}} \mathbb{Q}^E: \begin{cases} dS_t = \sigma(Y_t)S_t dB_t^{\mathbb{Q}^E} \\ dY_t = \int_{\mathbb{R}} x\mu(dt,dx) \end{cases}$$

where the compensator under $\mathbb{P}: \nu_Y = \nu(dx)dt$ and $\mathbb{Q}^E: \nu_Y^{\mathbb{Q}^E} := A_t^{\Delta y}\nu_Y$.

3.2.2 Brownian-Brownian framework

Following the same approach outlined before, we can also derive the density process of the minimal entropy martingale measure $\mathbb{Q}^E$ for the following model

$$\mathbb{P}: \qquad \begin{cases} dS_t = a(\widehat{Y}_t)S_t dt + \sigma(\widehat{Y}_t)S_t dB_t \\ \\ d\widehat{Y}_t = \alpha(\beta - \widehat{Y}_t)dt + \gamma dW_t \end{cases}$$

where W and B are strongly orthogonal in the sense that $[W, B] = 0$. Note that we could not have incorporated this W term in the Lévy process Y because it violates one of the conditions stated in Grandits & Rheinländer (2002), i.e. Y is allowed to have only *one* additional random factor which is independent of B. However, as we shall see later, the density process of $\mathbb{Q}^E$ for complicated models could be analyze using the techniques of the optimal martingale measure. The minimal entropy martingale measure for the general case when $[W, B] = \widetilde{\rho}dt$, $\widetilde{\rho} \in [-1, 1]$ has been solved by Rheinländer (2005). Furthermore, the entropy minimizer for the case where $\widetilde{\rho} = 0$ has been evaluated and generalized by Hobson (2004). It is worth pointing out that both the authors Rheinländer (2005) and Hobson (2004) attacked the problem by means of the optimal martingale measure technique. Here we do not use the optimal martingale measure methodology. We however used a hint given in Grandits & Rheinländer (2002) that K is $\mathcal{G}-$measurable and basic martingale techniques. To our knowledge, the *method* of calculating the minimal entropy martingale measure in this setting has not been highlighted previously. We feel that this approach is extremely flexible and simple as we saw earlier in the Brownian-Lévy case and shall henceforth observe for the Brownian-Brownian framework.

Analogously to the previous section, define the deterministic function $\widehat{A}_t^y := \widehat{A}(t, y)$ as

$$\widehat{A}_t^y := \widehat{A}(t,y) := \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{a^2(\widehat{Y}_s)}{\sigma^2(\widehat{Y}_s)}ds\right)\middle| \widehat{Y}_t = y\right]$$

For convenience, we also use the notation $\widehat{A}_t := \widehat{A}(t, \widehat{Y}_t)$. Recall that the mean-variance tradeoff process $K := \int \lambda^2 d\langle M^*\rangle = \int \frac{a^2(Y)}{\sigma^2(Y)}dt$. Assume further that $\rho := \frac{a}{\sigma}$ is uniformly bounded. Define $\widehat{\Gamma}_t := \mathcal{E}\left(-\int_0^{\cdot} \frac{a(\widehat{Y}_s)}{\sigma(\widehat{Y}_s)}dB_s\right)_t$ so that $d\widehat{\Gamma}_t = -\widehat{\Gamma}_t dM_t^{\widehat{\Gamma}} = -\widehat{\Gamma}_t\rho_t dB_t$. Z_t can be neatly recast as

$$Z_t = ce^{-\frac{1}{2}K_t}\widehat{\Gamma}_t\widehat{A}_t$$

Apply the Itô formula to Z gives

$$\begin{aligned} dZ_t &= c\left\{-\widehat{A}_t e^{-\frac{1}{2}K_t}d\widehat{\Gamma}_t + \Gamma_t d\left(e^{-\frac{1}{2}K_t}A_t\right) + d[\Gamma_t, e^{-\frac{1}{2}K_t}A_t]\right\} \\ &= c\left\{-A_t e^{-\frac{1}{2}K_t}\Gamma_t\rho_t dB_t\right\} + c\Gamma_t\left\{-\frac{1}{2}\rho(\widehat{Y}_t)A_t dt\right\} \\ &\quad + c\Gamma_t e^{-\frac{1}{2}K_t}\left(\frac{\partial}{\partial t}A_t dt + \frac{\partial}{\partial y}(\alpha(\beta - \widehat{Y}_t))dt + \frac{1}{2}\frac{\partial^2}{\partial y^2}A_t\gamma^2 dt + \frac{\partial}{\partial y}A_t\gamma dW_t\right) \end{aligned}$$

This is because $d\left(e^{-\frac{1}{2}K_t}A_t\right) = -\frac{1}{2}\rho(\widehat{Y}_t)A_t + e^{-\frac{1}{2}K_t}dA_t + d[e^{-\frac{1}{2}K}, A]_t$ and that

$$\begin{aligned} A(t,\widehat{Y}_t) &= A(0,\widehat{Y}_0) + \int_0^t \frac{\partial}{\partial s}A(s,\widehat{Y}_s)ds + \int_0^t \frac{\partial}{\partial y}A(s,\widehat{Y}_s)(\alpha(\beta - \widehat{Y}_s))ds \\ &+ \frac{1}{2}\frac{\partial^2}{\partial y^2}A(s,\widehat{Y}_s)\gamma^2 ds + \int_0^t \frac{\partial}{\partial y}\gamma A(s,\widehat{Y}_s)dW_s \end{aligned}$$

since the $dt-$terms are continuous processes with finite variation, we must have

$$\begin{aligned} dZ_t &= -Z_t\rho_t dB_t + ce^{-\frac{1}{2}K_t}\Gamma_t\frac{\partial}{\partial y}A(t,\widehat{Y}_t)\gamma dW_t \\ &= -Z_t\rho_t dB_t + Z_t\frac{\gamma\frac{\partial}{\partial y}A(t,\widehat{Y}_t)}{A(t,\widehat{Y}_t)}dW_t \end{aligned}$$

Define the function $\widetilde{\kappa}_t^y := \frac{\gamma\frac{\partial}{\partial y}A(t,y)}{A(t,y)}$ and so we get

$$dZ_t = Z_t\left(-\frac{a(\widehat{Y}_t)}{\sigma(\widehat{Y}_t)}dB_t + \widetilde{\kappa}_t^{\widehat{Y}_t}dW_t\right)$$

Again taking the uniqueness of solutions of the above stochastic differential equation for granted, we conclude that the unique solution to this SDE is given by the expression

$$Z_t = \mathcal{E}\left(-\int_0^\cdot \frac{a(\widehat{Y}_s)}{\sigma(\widehat{Y}_s)}dB_s\right)_t \mathcal{E}\left(\int_0^\cdot \widetilde{\kappa}_s^{\widehat{Y}_s}dW\right)_t$$

We can sum up our arguments by the following:

Proposition 3.10 (Density process of $\mathbb{Q}^E$ for Brownian-Brownian). *Fix $T > 0$. Consider the stock price process of the form*

$$\mathbb{P}: \quad \begin{cases} dS_t = a(\widehat{Y}_t)S_t dt + \sigma(\widehat{Y}_t)S_t dB_t \\ \\ d\widehat{Y}_t = \alpha(\beta - \widehat{Y}_t)dt + \gamma dW_t \end{cases}$$

where B and W are strongly orthogonal in the sense that $[B,W] = 0$. Furthermore the probability measure $\mathbb{Q}^E$ is given by

$$Z_T = c\exp\left(-\int \lambda dS\right)_T$$

Then the Radon-Nikodým density process Z_t, $t \in [0,T]$ satisfies

$$Z_t = \mathcal{E}\left(-\int_0^\cdot \frac{a(\widehat{Y}_s)}{\sigma(\widehat{Y}_s)}dB_s\right)_t \mathcal{E}\left(\int_0^\cdot \widetilde{\kappa}_s^{\widehat{Y}_s}dW_s\right)_t$$

where

$$\widetilde{\kappa}_t^{\widehat{Y}_t} := \frac{\gamma\frac{\partial}{\partial y}A(t,\widehat{Y}_t)}{A(t,\widehat{Y}_t)}$$

with $Z_0 = 1$.

Corollary 3.11. *Let $\mathbb{Q}^E$ be given as above. Then*
(i) The process $B_t^{\mathbb{Q}^E} := B_t + \int_0^t \rho(\widehat{Y}_s)ds$, $t \in [0,T]$ follows a Brownian motion under $\mathbb{Q}^E$.
(ii) The process $W_t^{\mathbb{Q}^E} := W_t - \int_0^t \widetilde{\kappa}_s^{\widehat{Y}_s} ds$, $t \in [0,T]$ follows a Brownian motion under $\mathbb{Q}^E$.
(iii) The processes $B_t^{\mathbb{Q}^E}$ and $W_t^{\mathbb{Q}^E}$, $t \in [0,T]$ are independent.
(iv) The dynamics of $(S,\widehat{Y})$ under $\mathbb{Q}^E$ are given by

$$\mathbb{P}: \begin{cases} dS_t = a(\widehat{Y}_t)S_t dt + \sigma(\widehat{Y}_t)S_t dB_t \\ d\widehat{Y}_t = \alpha(\beta - \widehat{Y}_t)dt + \gamma dW_t \end{cases} \quad \xrightarrow{\frac{d\mathbb{Q}^E}{d\mathbb{P}}\big|_{\mathcal{G}_t}} \quad \mathbb{Q}^E: \begin{cases} dS_t = \sigma(\widehat{Y}_t)S_t dB_t^{\mathbb{Q}^E} \\ dY_t = \Xi_t dt + \gamma dW_t^{\mathbb{Q}^E} \end{cases}$$

where

$$\Xi_t := \alpha(\beta - Y_t) + \gamma \kappa_t^{\widehat{Y}_t}$$

(v) The function $A(t,y)$ satisfies the following partial differential equation since for $t \in [0,T]$, Z is a martingale:

$$\frac{\partial}{\partial t}A(t,y) - \frac{1}{2}A(t,y)\rho^2(y) + \frac{\partial}{\partial y}A(t,y)\alpha(\beta - y) + \frac{1}{2}\gamma^2 \frac{\partial^2}{\partial y^2}A(t,y) = 0$$

with the terminal condition $A(T,\cdot) = 1 \quad \forall y \in \mathbb{R}$.

3.2.3 Connection to some entropic models in the literature

Becherer (2004). Becherer considered a model of the following form

$$\mathbb{P}: \quad \begin{cases} dS_t = a(t,Y_t)S_t dt + \sigma(t,Y_t)S_t dB_t \\ dY_t = \sum_i \sum_j (j-i)\mathbb{1}_i(Y_{t-})dH_t^{ij} \end{cases}$$

where $i,j \in \mathcal{Y}$, a and σ are functions of class C^1 with respect to $t \in [0,T]$, $\mathbb{1}_i$ denotes the indicator function on $\{i\}$ and $H = (H^{ij})$ is a multivariate point process. Using our framework, we can see that the function $A(t,y)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}A(t,i) - \frac{1}{2}\frac{a^2(i)}{\sigma^2(i)}A(t,i) + \sum_{j\in\mathcal{Y}} \left(A(t,j) - A(t,i)\right)\mu^{ij}\mathbb{1}_{j\neq i} &= 0 \\ A(T,j) &= 1 \end{aligned}$$

for $j \in \mathcal{Y}$. Performing an Euler transformation $u(t,i) := \log A(t,i)$ yields

$$A(t,j) - A(t,i) = e^{u(t,j)} - e^{u(t,i)}, \quad \frac{\partial}{\partial t}u(t,i) = \frac{1}{A(t,i)}\frac{\partial}{\partial t}A(t,i)$$

so that $A(T,\cdot) = 1 \Rightarrow u(T,\cdot) = 0$ and we get the following differential equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,i) - \frac{1}{2}\frac{a^2(i)}{\sigma^2(i)} + \sum_{j\in\mathcal{Y}} \left(e^{u(t,j)-u(t,i)} - 1\right)\mu^{ij}\mathbb{1}_{j\neq i} &= 0 \\ u(T,j) &= 0 \end{aligned}$$

for $j \in \mathcal{Y}$ which is in complete agreement with equation (4.29) in Becherer (2004) when there there is no claim, c.f. Remark (6.4.2) from Steiger (2005).

Benth & Meyer-Brandis (2005). The authors, occasionally abbreviated with B/MB from this time forth, investigated the density process of the minimal entropy martingale measure for the stochastic volatility model introduced by Barndorff-Nielsen & Shephard (2001). The dynamics of Y takes the form

$$dY_t = -\eta Y_t dt + dL_{\eta t}, \qquad Y_0 = y > 0$$

where L_t is a pure jump subordinator, i.e. an increasing pure jump Lévy process with no drift. As mentioned by B/MB, the key ingredient in the description of the density process of the minimal entropy martingale measure is the function $H(t, y)$ defined as

$$H(t,y) = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{a^2(Y_u)}{\sigma^2(Y_u)}du\right)\middle| Y_t = y\right], \qquad (t,y) \in [0,T] \times \mathbb{R}_+$$

where $\mathbb{R}_+ = (0, \infty)$. Benth & Meyer-Brandis remarked that their motivation for considering the function $H(t, y)$ comes from portfolio optimization with an exponential utility function. They state that the difference between the value function of the utility maximization problem and the utility function itself can be represented as $H(t, y)$ and can be seen by considering the Hamilton-Jacobi-Bellman equation of the stochastic control problem. As stated in their paper, the function $H(t, y)$ arises from the logarithmic transform of the value function in a similar fashion as demonstrated in Musiela & Zariphopoulou (2003). The function $H(t, y)$ in their paper can be compared with the function $A(t, y)$ defined in (3.2) of ours. B/MB do not assume that ρ is uniformly bounded so in this case, their results are more general compared to ours. However, the basic and central point of this chapter is to illustrate that B/MB introduced the density process of $\mathbb{Q}^E$ via the function $H(t, y)$ while in ours, the function $H(t, y)$ or its equivalence $A(t, y)$, appears *effortlessly.*

Note that one has still to carry out the verification procedure to show that Z is indeed the minimal entropy martingale measure.

Chapter 4

Martingale Representation Theorems

Our goal in this chapter is to establish two versions of the martingale representation theorem with respect to some enlarged filtration $\mathbb{G}$ which we shall state explicitly when the prerequisites are in place. We first deal with the representation involving a one-jump process and a Brownian motion. This problem was first solved by Kusuoka (1999) and reappeared in Bielecki & Rutkowski (2002). We present an alternative proof of Kusuoka's result while filling in the gaps of the proof given in Bielecki & Rutkowski (2002). The second martingale representation concerns that of the Quasi-Markov Additive processes. To our knowledge, it has not been highlighted previously.

Definition 4.1. *Let $C^k(\mathbb{R}^n)$ be the space of all functions on $\mathbb{R}^n$ whose partial derivatives of order $\leq k$ all exist and are continuous and we set $C^\infty(\mathbb{R}^n) = \bigcap_1^\infty C^k(\mathbb{R}^n)$. Furthermore we denote by $C_c^\infty(\mathbb{R}^n)$ the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and contained in $\mathbb{R}^n$.*

4.1 Martingale representation involving $\mathbb{1}_{\{\tau\leq t\}}$

Let us denote by τ a non-negative random variable defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau \geq t) > 0$ for any $t \in [0,T]$. Define H by $H_t := \mathbb{1}_{\{\tau\leq t\}}$ and denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t = \sigma(H_s : s \leq t)$. Let B be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by B i.e., $\mathcal{F}_t = \sigma(B_r : r \leq t)$. Introduce the filtration $\mathbb{G}$ on $(\Omega, \mathcal{G}, \mathbb{P})$ as

$$\mathbb{G} = \mathbb{H} \vee \mathbb{F} \quad \text{i.e.,} \quad \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t \quad \text{for any } t \in [0,T]. \tag{4.1}$$

All filtrations are augmented to satisfy the usual conditions of right-continuity and completeness. Let $M_t^H := H_t - \int_0^t \mu_s \mathbb{1}_{\{\tau>s\}} ds$ be the martingale associated with the one-jump process H. We make the following assumptions:

Assumption 4.2. *The intensity μ of the random time τ is deterministic and bounded.*

Assumption 4.3. *The filtrations* $\mathbb{F}$ *and* $\mathbb{H}$ *are immersed in* $\mathbb{G}$.

Proposition 4.4. *The collection of random variables of the form*

$$\{f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n}), \quad n=1,2,... \mid t_1,t_2,...,t_n \in [0,T], \ f \in C_c^\infty(\mathbb{R}^{2n})\} \tag{4.2}$$

is dense in $L^2(\mathcal{G}_T,\mathbb{P})$.

Proof. As a starting point, let $\mathcal{G}_n := \sigma(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})$ so that $\bigvee_{n=1}^\infty \mathcal{G}_n = \mathcal{G}_T$. Fix $m \in L^2(\mathcal{G}_T,\mathbb{P})$. Define $R_n := \mathbb{E}_\mathbb{P}[m|\mathcal{G}_n]$. By Doob's martingale convergence theorem, $R_n \to m$ a.s. and in L^2. Note that $R_n \in \mathcal{G}_n$ so by Doob-Dynkin's Lemma we have $R_n = g_n(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})$ for some measurable $g_n : \mathbb{R}^{2n} \to \mathbb{R}$. By Folland (1999, Chapter 8), we see that each such $g_n(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})$ can be approximated in $L^2(\mathcal{G}_T,\mathbb{P})$ by functions $f_n(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n}) \in C_c^\infty(\mathbb{R}^{2n})$. This in turn means that for each n, R_n can be approximated in $L^2(\mathcal{G}_T,\mathbb{P})$ by some $f_n(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n}) \in C_c^\infty(\mathbb{R}^{2n})$, the statement follows. □

For a square integrable $\mathcal{G}_T-$measurable random variable X, i.e. $X \in L^2(\mathbb{P})$, define the $\mathbb{G}-$martingale M by $M_t := \mathbb{E}_\mathbb{P}[X|\mathcal{G}_t]$. Since $\mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$, we are inspired to consider a random variable X of the following form

$$X = \exp\left(-\int_0^T q(t)dB_t\right) D \tag{4.3}$$

for deterministic step functions q and a bounded $\mathcal{H}_T-$measurable random variable D. Define the process L as

$$L_t := \mathcal{E}\left(-\int_0^\cdot q(s)dB_s\right)_t \tag{4.4}$$

Lemma 4.5. *L is a martingale with respect to the larger filtration* $\mathbb{G}$.

Proof. The fact that $L_t \in L^1(\mathbb{P})$ for $t \in [0,T]$ follows from

$$\mathbb{E}_\mathbb{P}\left|\mathcal{E}\left(-\int_0^\cdot q(s)dB_s\right)_t\right| = 1 < \infty$$

By Novikov's condition, L is an $\mathbb{F}-$martingale, hence a $\mathbb{G}-$martingale by the immersion property in Assumption 4.3. □

Proposition 4.6. *The set*

$$\left\{\mathcal{E}\left(-\int_0^\cdot q(t)dB_t\right)_T \widetilde{D}\right\} \tag{4.5}$$

with

$$\widetilde{D} := \exp\left(\frac{1}{2}\int_0^T q^2(t)dt\right) D \in L^\infty(\mathcal{H}_T,\mathbb{P})$$

spans a dense subspace in $L^2(\mathcal{G}_T,\mathbb{P})$ *where* q *are deterministic step functions.*

Proof. That the random variables of the form (4.5) are in $L^2(\mathbb{P})$ follows from

$$\mathbb{E}_{\mathbb{P}}\left[\left(\exp\left(-\int_0^T q(t)dB_t - \int_0^T \frac{1}{2}q^2(t)dt\right)\right)^2 \widetilde{D}\right]$$

$$\le C\exp\left(-\int_0^T q^2(t)dt\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-2\int_0^T q(t)dB_t\right)\right] < \infty,\ C := \|\widetilde{D}\|_{L^\infty(\mathcal{H}_T,\mathbb{P})}$$

since $\widetilde{D}\in L^\infty(\mathcal{H}_T,\mathbb{P})$. Let $L^\perp \in L^2(\mathcal{G}_T,\mathbb{P})$ be orthogonal to all the random variables in equation (4.5). Then for any choice of q and $\widetilde{D}\in L^\infty(\mathbb{P})$,

$$\mathbb{E}_{\mathbb{P}}\left[\widetilde{D}\exp\left(\frac{1}{2}\int_0^T q^2(t)dt\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_0^T q(t)dB_t\right)L^\perp\middle|\mathcal{H}_T\right]\right]$$

$$\mathbb{E}_{\mathbb{P}}\left[\mathcal{E}\left(-\int_0^\cdot q(t)dB_t\right)_T \widetilde{D}L^\perp\right] = 0$$

This implies that for $q(t)=\sum_{i=1}^n k_i \mathbb{1}_{\{0,t_i\}}$,

$$\mathcal{Q}(k) := \mathbb{E}_{\mathbb{P}}\left[\exp\left(k_1 B_{t_1}+\ldots+k_n B_{t_n}\right)L^\perp\middle|\mathcal{H}_T\right] = 0 \tag{4.6}$$

for all $k=(k_1,\ldots,k_n)$ and $n\in\mathbb{N}$. However the above equation (4.6) can be seen as

$$\frac{\mathbb{E}_{\mathbb{P}}[\exp\left(k_1B_{t_1}+\ldots+k_nB_{t_n}\right)L^\perp\mathbb{1}_{\{\tau\le T\}}]}{\mathbb{P}[\tau\le T]}\mathbb{1}_{\{\tau\le T\}}$$
$$+\frac{\mathbb{E}_{\mathbb{P}}[\exp\left(k_1B_{t_1}+\ldots+k_nB_{t_n}\right)L^\perp\mathbb{1}_{\{\tau> T\}}]}{\mathbb{P}[\tau> T]}\mathbb{1}_{\{\tau> T\}} = 0$$

Here we used the fact that $\mathcal{H}_T$ is generated by $\{\tau\le T\},\{\tau>T\}$. Since the exponential function is analytic, it enjoys analytic continuation to the complex space $\mathbb{C}^n$, in particular $\mathcal{Q}(iy_1,iy_2,\ldots,iy_n)=0$ for $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. Our aim now is to investigate

$$\int_\Omega f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})L^\perp d\mathbb{P} = \mathbb{E}_{\mathbb{P}}[f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})L^\perp]$$

for $f\in C_c^\infty(\mathbb{R}^{2n})$. First of, note that

$$f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})$$
$$= \underbrace{\int\int\cdot\cdot\int}_{2n}\widehat{f}(y_1,.,y_n,y_{n+1},.,y_{2n})e^{i(y_1B_{t_1}+..+y_nB_{t_n}+y_{n+1}H_{t_n}+..y_{2n}H_{t_n})}dy_1..dy_{2n} \tag{4.7}$$

where $\widehat{f}$ is the Fourier transform of f and $(y_1,..y_n,y_{n+1},\ldots,y_{2n})\in\mathbb{R}^{2n}$. Hence

$$\mathbb{E}_{\mathbb{P}}\{f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})L^\perp\}$$
$$= \mathbb{E}_{\mathbb{P}}\Big(\underbrace{\int\int\cdots\int}_{2n}\widehat{f}(y_1,..,y_{2n})e^{i(y_{n+1}H_{t_1}+..+y_{2n}H_{t_n})}\mathcal{Q}(iy)dy_1..dy_{2n}\Big)$$
$$= 0$$

Hence $L^\perp$ is orthogonal to $f(B_{t_1},..,B_{t_n},H_{t_1},..,H_{t_n})$. From Proposition 4.4, $f(\cdot)$ is dense in $L^2(\mathcal{G}_T,\mathbb{P})$ so $L^\perp$ is orthogonal to a dense subset of $L^2(\mathcal{G}_T,\mathbb{P})$. It follows that $L^\perp = 0$. From Steele (1991, p.210), we conclude that the set of random variables in (4.5) is dense in $L^2(\mathcal{G}_T,\mathbb{P})$. □

Due to (4.3), X can also be written as $X = L_T\widetilde{D}$ with $\widetilde{D}$ defined in Proposition 4.6. Note that $\widetilde{D}$ is $\mathcal{H}_T$−measurable. Introduce the square-integrable $\mathbb{H}$−martingale V

$$V_t := \mathbb{E}_{\mathbb{P}}[\widetilde{D}|\mathcal{H}_t] = \mathbb{E}_{\mathbb{P}}[\widetilde{D}] + \int_0^t h_s dM_s^H \tag{4.8}$$

where h is $\mathbb{H}$−predictable stochastic processes such that $\int hdM^H$ is square-integrable. The second equality in equation (4.8) is a consequence of the martingale representation theorem for the one-jump process associated to the filtration $\mathbb{H}$, see Bielecki & Rutkowski (2002, p.131). By Assumption 4.3, M^H is also a martingale under the enlarged filtration $\mathbb{G}$. Clearly $V_T = \widetilde{D}$ so that $X = L_T V_T$. The Itô integration by parts gives

$$\begin{aligned} X &= L_0V_0 + \int_0^T L_t dV_t + \int_0^T V_t dL_t + [L,V]_T \\ &= \mathbb{E}_{\mathbb{P}}[\widetilde{D}] + \int_0^T L_{t-}h_t dM_t^H + \int_0^T V_t L_t(-q(t))dB_t \end{aligned}$$

Observe that $[L,V] = 0$ since the Brownian motion B, the process H are strongly orthogonal to each other, i.e. $\langle B,H\rangle = 0$ since

$$[L,V]_t = \langle L^c,V^c\rangle_t + \sum_{s\le t}\Delta L_s\Delta V_s = \langle L^c,V^c\rangle_t = 0$$

As L is $\mathbb{G}$−adapted, $b_t := -V_tL_tq(t)$ and $c_t := L_th(t)$ are $\mathbb{G}$−adapted stochastic processes. X can then be neatly recast as

$$X = \mathbb{E}_{\mathbb{P}}[\widetilde{D}] + \int_0^T b_t dB_t + \int_0^T c_t dM_t^H$$

Recall that $M_t := \mathbb{E}_{\mathbb{P}}[X|\mathcal{G}_t]$. Since $X \in L^2(\mathbb{P})$ we have

$$\mathbb{E}_{\mathbb{P}}\{[M,M]_T\} \overset{(i)}{\le} C_1\mathbb{E}_{\mathbb{P}}\Big(\sup_{0\le t\le T}(M_t)^2\Big) \overset{(ii)}{\le} C_2\mathbb{E}_{\mathbb{P}}[(X_T)^2] \overset{(iii)}{<} \infty$$

with the enumerations (i), (ii) and (iii) owing to the Burkholder-Davis-Gundy inequality, Doob and $X \in L^2(\mathbb{P})$ respectively. Now

$$\mathbb{E}_{\mathbb{P}}[(X)^2] = \text{const.} + \mathbb{E}_{\mathbb{P}}\left[\int_0^T b_t^2 dt\right] + \mathbb{E}_{\mathbb{P}}\left[\int_0^T c_t^2\mu_t \mathbb{1}_{\{\tau>t\}}dt\right]$$

since the expectations of the cross terms vanish due to Lemma 2 and Theorem 36 of Protter (2004). Define the following spaces

$$\begin{aligned}\mathcal{L}^2(B) &:= \left\{ b \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T b_t^2 dt\right) < \infty \right\}\\ \mathcal{L}^2(M) &:= \left\{ c \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T c_t^2 \mu_t \mathbb{1}_{\{\tau>t\}} dt\right) < \infty \right\}\end{aligned}$$

Because X is square-integrable, we have that $b \in \mathcal{L}^2(B)$ and $c \in \mathcal{L}^2(M)$ where b and c are $\mathbb{G}-$progressively measurable and $\mathbb{G}-$predictable processes respectively. Since $X \in L^2(\mathcal{G}_T, \mathbb{P})$ is arbitrary, we can by Proposition 4.6 approximate $X \in L^2(\mathcal{G}_T, \mathbb{P})$ by linear combinations $X^{(n)}$ of random variables of the form (4.5). Then for each n, we have

$$X^{(n)} = \mathbb{E}_{\mathbb{P}}[X^{(n)}] + \int_0^T b_t^{(n)} dB_t + \int_0^T c_t^{(n)} dM_t^H$$

where

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T (b_t^{(n)})^2 dt\right] < \infty, \quad \mathbb{E}_{\mathbb{P}}\left[\int_0^T (c_t^{(n)})^2 \mu_t \mathbb{1}_{\{\tau>t\}} dt\right] < \infty$$

The Itô isometry yields, for any $m, n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[X^{(n)} - X^{(m)}]^2 &= (\mathbb{E}_{\mathbb{P}}[X^{(n)} - X^{(m)}])^2 + \int_0^T \mathbb{E}_{\mathbb{P}}(b_t^{(n)} - b_t^{(m)})^2 dt\\ &\quad + \int_0^T \mathbb{E}_{\mathbb{P}}(c_t^{(n)} - c_t^{(m)})^2 \mu_t \mathbb{1}_{\{\tau>t\}} dt\\ &\quad \longrightarrow 0 \quad \text{as } n, m \longrightarrow 0\end{aligned}$$

due to the fact that the expectations of the cross terms vanish by Lemma 2 and Theorem 36 of Protter (2004). So $\{b^{(n)}\}$ and $\{c^{(n)}\}$ are Cauchy sequences in $\mathcal{L}^2(B)$ and $\mathcal{L}^2(M)$ and thus converge to some b, c in $\mathcal{L}^2(B)$ and in $\mathcal{L}^2(M)$ respectively. Hence the spaces of stochastic integrals with respect to B and M^H given by

$$\begin{aligned}\mathcal{O}^B &:= \left\{ \int b\, dB \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T b_t^2 dt\right) < \infty \right\}\\ \mathcal{O}^{M^H} &:= \left\{ \int c\, dM^H \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T c_t^2 \mu_t \mathbb{1}_{\{\tau>t\}} dt\right) < \infty \right\}\end{aligned}$$

are closed. Thus we get

$$\begin{aligned}X = \lim_{n\to\infty} X^{(n)} &= \lim_{n\to\infty}\left(\mathbb{E}_{\mathbb{P}}[X^{(n)}] + \int_0^T b_t^{(n)} dB_t + \int_0^T c_t^{(n)} dM_t^H\right)\\ &= \mathbb{E}_{\mathbb{P}}[X] + \int_0^T b_t dB_t + \int_0^T c_t dM_t^H\end{aligned}$$

where the limit is taken in $L^2(\mathcal{G}_T, \mathbb{P})$. Also note that the spaces of stochastic integrals $\mathcal{O}^B$ and $\mathcal{O}^{M^H}$ are orthogonal to each other since $[B, M^H] = 0$. Summing up, we obtain

Theorem 4.7 (Kusuoka (1999)). *Every square integrable* $(\mathbb{P},\mathbb{G})-$*martingale* M *admits the following representation*

$$M_t = M_0 + \int_0^t b_s dB_s + \int_0^t c_s dM_s^H$$

for some $\mathbb{G}-$*predictable stochastic processes* b *and* c *satisfying*

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T b_t^2 dt\right] < \infty, \quad \mathbb{E}_{\mathbb{P}}\left[\int_0^T c_t^2 \mu_t \mathbb{1}_{\{\tau>t\}} dt\right] < \infty \tag{4.9}$$

Proof. Let $X = M_T$ and since M is a square integrable martingale, we take conditional expectations and we are done. □

Corollary 4.8. *Every local* $(\mathbb{P},\mathbb{G})-$*martingale* M *admits the above martingale representation but* (4.9) *need not hold in this case.*

4.2 Martingale representation for quasi-Markov additive processes

Let $\{C_t\}_{0\leq t\leq T}$ be a continuous time Markov chain on $(\Omega,\mathcal{G},\mathbb{P})$ with finite state space $\mathcal{Y} = \{1,....,m\}$. Let $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T} = \sigma(C_s; 0\leq s\leq t)$, $0 \leq t \leq T$ be the filtration generated by this Markov chain. Let B and Y be stochastic processes living on $(\Omega,\mathcal{G},\mathbb{P})$ where B is a Brownian motion and Y a real valued pure jump process given by

$$Y_t := \int_0^t \int_{\mathbb{R}} g(x)\mu_Y$$

where g is measurable, deterministic and bounded while μ_Y denotes the associated jump measure of Y defined by

$$\mu_Y(dt,dx) = \sum_{s>0} \mathbb{1}_{\{\Delta Y_s\neq 0\}} \delta_{\{(s,\Delta Y_s)\}}(dt,dx)$$

We assume that the predictable $\mathbb{P}-$compensator of μ_Y, denoted by ν_Y is given by

$$\nu_Y(dx,dt) := \nu_{C_{t-}}(dx)dt = \sum_i H_{t-}^i \nu^i(dx)dt \tag{4.10}$$

where $\nu^i(dx)$ denotes the transition kernel in state $i \in \mathcal{Y}$. In addition, we work under the assumption that $\nu^i(\mathbb{R}) < \infty$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by B and Y i.e., $\mathcal{F}_t = \sigma(B_s, Y_s : s \leq t)$. We assume the filtration $\mathbb{G}$ is given by

$$\mathbb{G} = \mathbb{H}\vee\mathbb{F} \quad \text{i.e.,} \quad \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t \quad \text{for any } t \in [0,T]. \tag{4.11}$$

We make the following assumption

Assumption 4.9. *The filtrations* $\mathbb{F}$ *and* $\mathbb{H}$ *are immersed in* $\mathbb{G}$.

Define $\mathscr{B}_n := B_{t_1}, ..., B_{t_n}$, $\mathscr{Y}_n := Y_{t_1}, ..., Y_{t_n}$ and $\mathscr{C}_n := C_{t_1}, ..., C_{t_n}$ so that

$$(\mathscr{B}_n, \mathscr{Y}_n, \mathscr{C}_n) = (B_{t_1}, ..., B_{t_n}, Y_{t_1}, ..., Y_{t_n}, C_{t_1}, ..., C_{t_n})$$

Proposition 4.10. *The collection of random variables of the form*

$$\{f(\mathscr{B}_n, \mathscr{Y}_n, \mathscr{C}_n), \quad n = 1, 2, ...|t_1, .., t_n \in [0, T], \ f \in C_c^\infty(\mathbb{R}^{3n})\}$$

is dense in $L^2(\mathcal{G}_T, \mathbb{P})$.

Proof. As a starting point, let $\mathcal{G}_n := \sigma(\mathscr{B}_n, \mathscr{Y}_n, \mathscr{C}_n)$ so that $\bigvee_{n=1}^\infty \mathcal{G}_n = \mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$. Fix $m \in L^2(\mathcal{G}_T, \mathbb{P})$. Define $R_n := \mathbb{E}_\mathbb{P}[m|\mathcal{G}_n]$. By Doob's martingale convergence theorem, $R_n \to m$ a.s. and in L^2. Note that $R_n \in \mathcal{G}_n$ so by Doob-Dynkin we have $R_n = g_n(\mathscr{B}_n, \mathscr{Y}_n, \mathscr{C}_n)$ for some measurable $g_n : \mathbb{R}^{3n} \to \mathbb{R}$. By Folland (1999, Chapter 8), each $g_n \in L^2$ by some $f_n \in C_c^\infty(\mathbb{R}^{3n})$. This means that for each n, R_n can be approximated in L^2 by some $f_n \in C_c^\infty(\mathbb{R}^{3n})$, the statement follows. □

For a square integrable $\mathcal{G}_T$−measurable random variable X, i.e. $X \in L^2(\mathbb{P})$, define the $\mathbb{G}$−martingale M by $M_t := \mathbb{E}_\mathbb{P}[X|\mathcal{G}_t]$. Since $\mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$, we are inspired to consider a random variable X of the following form

$$X = \exp\left(-\int_0^T q(t)dB_t + \int_0^T \int_\mathbb{R} p(t)g(x)\mu_Y(dt, dx)\right) D \tag{4.12}$$

where q, p are deterministic step functions, g is deterministic, measurable, bounded and some $\mathcal{H}_T$−measurable random variable D which is bounded. Define the process $\widetilde{L}$ as

$$\widetilde{L}_t := \mathcal{E}\left(-\int_0^\cdot q(s)dB_s + \int_0^\cdot \int_\mathbb{R} (e^{p(s)g(x)} - 1)(\mu_Y - \nu_Y)\right)_t \tag{4.13}$$

We get from Assumption 4.9 the following:

Lemma 4.11. *$\widetilde{L}$ is a martingale with respect to the the larger filtration $\mathbb{G}$.*

Proof. That $\widetilde{L}$ is a martingale follows from Theorem 2.8. □

Proposition 4.12. *The set*

$$\left\{\mathcal{E}\left(-\int_0^\cdot q(t)dB_t + \int_0^\cdot \int_\mathbb{R} (e^{p(t)g(x)} - 1)(\mu_Y - \nu_Y)\right)_T \widetilde{D}\right\} \tag{4.14}$$

with

$$\widetilde{D} = \exp\left(\frac{1}{2}\int_0^T q^2(t)dt + \int_0^T \int_\mathbb{R} (e^{q(t)g(x)} - 1)\nu_{C_{t-}}(dx)dt\right) D \in L^\infty(\mathbb{P})$$

spans a dense subspace in $L^2(\mathcal{G}_T, \mathbb{P})$ where q are deterministic step functions, g is bounded, deterministic and measurable.

Proof. The random variables of the form (4.14) are in L^2 since p, q are deterministic step functions as well as $D \in L^\infty(\mathbb{P})$. Let $\widehat{L}^\perp \in L^2(\mathcal{G}_T, \mathbb{P})$ be orthogonal to all the random variables in equation (4.14). Define $\alpha_T := \exp(-\frac{1}{2}\int_0^T q^2(t)dt - \int_0^T \int_{\mathbb{R}}(e^{p(t)g(x)} - 1)\nu_Y)$. Then $\forall \widetilde{D} \in L^\infty(\mathbb{P})$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[\alpha_T \widetilde{D}\mathbb{E}_{\mathbb{P}}\left\{\exp\left(\left(\int_0^T q(t)dB_t + \int_0^T p(t)dY_t\right)\widehat{L}^\perp \middle| \mathcal{H}_T\right\}\right] &= \\ \mathbb{E}_{\mathbb{P}}\left[\mathcal{E}\left(-\int_0^\cdot q(t)dB_t + \int_0^\cdot \int_{\mathbb{R}}(e^{p(t)g(x)} - 1)(\mu_Y - \nu_Y)\right)_T \widetilde{D}\widehat{L}^\perp\right] &= 0 \end{aligned}$$

This implies that

$$\mathcal{Q}(k) := \mathbb{E}_{\mathbb{P}}\left[\exp\left(k_1 B_{t_1} + ... + k_n B_{t_n} + k_{n+1}Y_{t_1} + ... + k_{2n}Y_{t_n}\right)\widehat{L}^\perp \middle| \mathcal{H}_T\right] = 0$$

where $q(t) = \sum_{i=1}^n k_i \mathbb{1}_{\{0,t_i\}}$, $p(t) = \sum_{i=1}^n k_{n+i}\mathbb{1}_{\{0,t_i\}}$ for $k = (k_1, ., k_n, k_{n+1}, ., k_{2n})$. By conditioning on every path, the above expression is a function and by analytic continuation, we have $\mathcal{Q} = 0$ on the complex space $\mathbb{C}^{2n}$, in particular $\mathcal{Q}(iy_1, iy_2, ..., iy_{2n}) = 0$ for $y = (y_1, ..., y_{2n}) \in \mathbb{R}^{2n}$. Recall that the Fourier inverse transform states that $w(x) = \int \widehat{w}(r)e^{ixr}dr$ where $\widehat{w}$ is the Fourier transform of w. We proceed to investigate

$$\int_\Omega f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n)\widehat{L}^\perp d\mathbb{P} = \mathbb{E}_{\mathbb{P}}[f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n)\widehat{L}^\perp]$$

First of, note that

$$f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n) = \underbrace{\int\int\cdots\int}_{3n} \widehat{f}(y_1, ..., y_{3n})\exp(i(\pi(y)))dy_1...dy_{3n}$$

where $\pi(y) := y_1 B_{t_1}, .., y_n B_{t_n}, y_{n+1}Y_{t_1}, ..., y_{2n}Y_{t_n}, y_{2n+1}C_{t_1}, ..., y_{3n}C_{t_n}$ and $\widehat{f}$ is the Fourier transform of f. In addition, define $\overline{\pi}(y) := y_1 B_{t_1}, ., y_n B_{t_n}, y_{n+1}Y_{t_1}, ., y_{2n}Y_{t_n}$. Hence

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}[f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n)\widehat{L}^\perp] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n)\widehat{L}^\perp | \mathcal{H}_T]] \\ = \ &\mathbb{E}_{\mathbb{P}}\Big(\underbrace{\int\int\cdots\int}_{3n} \mathcal{Q}(iy)dy_1...dy_{3n}\Big) = 0 \end{aligned}$$

Hence $\widehat{L}^\perp$ is orthogonal to $f(\mathcal{B}_n, \mathcal{Y}_n, \mathcal{C}_n)$. From Proposition 4.10, $f(\cdot)$ is dense in $L^2(\mathcal{G}_T, \mathbb{P})$ so $\widehat{L}^\perp = 0$. From the fact of Hilbert space outlined in Steele (1991, p. 210), we conclude that the set of random variables in equation (4.14) is dense in $L^2(\mathcal{G}_T, \mathbb{P})$. □

From (4.12), X can be written as $X = \widetilde{L}_T\widetilde{D}$. Note that $\widetilde{D}$ is $\mathcal{H}_T$ measurable since $\nu_{C_-}(dx)$ is $\mathbb{H}-$measurable. We call ν_{C_-} the transition kernel *driven* or *modulated* by the continuous time Markov chain. Denote U^{ij} as the martingale associated with transitions from i to j. Introduce the square-integrable $\mathbb{H}-$martingale

$$V_t := \mathbb{E}_{\mathbb{P}}[\widetilde{D}|\mathcal{H}_t] = \mathbb{E}_{\mathbb{P}}[\widetilde{D}] + \int_0^t \sum_i \sum_j \eta_s^{ij} dU_s^{ij} \tag{4.15}$$

such that $\int \sum_i \sum_j \eta^{ij} dU^{ij}$, is square integrable for $i \neq j$. The second equality in equation (4.15) is a consequence of the martingale representation theorem of the Markov chain's filtration $\mathbb{H}$, see Norberg (2003). We make the following assumption:

Assumption 4.13. *The processes Y, B and C are strongly orthogonal to one another.*

Clearly $V_T = \widetilde{D}$ so that $X = \widetilde{L}_T V_T$. The Itô integration by parts gives

$$\begin{aligned} X &= \widetilde{L}_0 V_0 + \int_0^T \widetilde{L}_t dV_t + \int_0^T V_t d\widetilde{L}_t + [\widetilde{L}, V]_T \\ &= \mathbb{E}_{\mathbb{P}}[\widetilde{D}] \int_0^T \sum_i \sum_j \widetilde{L}_{t-} \eta_t^{ij} dU_t^{ij} + \int_0^T V_t(-q(t)\widetilde{L}_t) dB_t \\ &\quad + \int_0^T \int_{\mathbb{R}} V_{t-} q(t) g(x) \widetilde{L}_{t-} (\mu_Y - \nu_Y)_t \end{aligned}$$

First of, note that $[\widetilde{L}, V] = 0$ due to Assumption 4.13. $\widetilde{L}$ is $\mathbb{G}-$measurable so that $\widetilde{\varphi}_t^{ij} := \widetilde{L}_{t-}\eta_t^{ij}$, $\sigma_t := V_t(-q(t)\widetilde{L}_t)$ and $W_t(x) := V_{t-}q(t)g(x)\widetilde{L}_{t-}$ are $\mathbb{G}-$adapted stochastic processes. X can then be neatly recast as

$$X = \mathbb{E}_{\mathbb{P}}[\widetilde{D}] + \int_0^T \sum_i \sum_j \widetilde{\varphi}_t^{ij} dU_t^{ij} + \int_0^T \sigma_t dB_t + \int_0^T \int_{\mathbb{R}} W_t(x)(\mu_Y - \nu_Y)_t$$

Recall that $M_t := \mathbb{E}_{\mathbb{P}}[X|\mathcal{G}_t]$. Since $X \in L^2(\mathbb{P})$ we have

$$\mathbb{E}_{\mathbb{P}}\{[M, M]_T\} \overset{(i)}{\leq} C_1 \mathbb{E}_{\mathbb{P}}\Big(\sup_{0 \leq t \leq T} M_t^2\Big) \overset{(ii)}{\leq} C_2 \mathbb{E}_{\mathbb{P}}[X_T^2] \overset{(iii)}{<} \infty$$

where the enumerations $(i), (ii)$ and (iii) are due to the Burkholder-Davis-Gundy inequality, Doob and $X \in L^2(\mathbb{P})$ respectively. Define the following spaces

$$\begin{aligned} \mathcal{L}^2(B) &:= \left\{ \sigma \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T b_t^2 dt \right) < \infty \right\} \\ \mathcal{L}^2(U) &:= \left\{ \widetilde{\varphi}^{ij} \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T \sum_i \sum_j (\widetilde{\varphi}^{ij})^2 \mu^{ij} H_t^i dt \right) < \infty \right\} \\ \mathcal{L}^2(Y) &:= \left\{ W \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T \int_{\mathbb{R}} (W_t(x))^2 \nu_{C_{t-}}(dx) dt \right) < \infty \right\} \end{aligned}$$

Since $X \in L^2(\mathbb{P})$ we have $\widetilde{\varphi}^{ij} \in \mathcal{L}^2(U)$, $\sigma \in \mathcal{L}^2(B)$ and $W \in \mathcal{L}^2(Y)$. The expectations of the cross terms vanish due to Lemma 2 and Theorem 36 of Protter (2004). Since $X \in L^2(\mathcal{G}_T, \mathbb{P})$ is arbitrary, we can by Proposition 4.12 approximate $X \in L^2(\mathcal{G}_T, \mathbb{P})$ by linear combinations $X^{(n)}$ of random variables of the form (4.14). Then for each n, we have

$$X^{(n)} = \mathbb{E}_{\mathbb{P}}[X^{(n)}] + \int_0^T \sum_i \sum_j \widetilde{\varphi}_t^{ij,(n)} dU_t^{ij} + \int_0^T \sigma_t^{(n)} dB_t + \int_0^T \int_{\mathbb{R}} W_t^{(n)}(x)(\mu_Y - \nu_Y)_t$$

The Itô isometry yields, for any $m, n \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{P}}[X^{(n)} - X^{(m)}]^2 \quad \longrightarrow 0 \quad \text{as } n, m \longrightarrow 0$$

by similar reasoning since the expectations of the cross terms vanish due to Lemma 2 and Theorem 36 of Protter (2004). We fetch our original problem to see that $\sigma^{(n)}$, $W_t^{(n)}(x)$ and $\widetilde{\varphi}^{ij,(n)}$ are Cauchy sequences in $\mathcal{L}^2(B)$, in $\mathcal{L}^2(Y)$ and in $\mathcal{L}^2(U)$ respectively and thus converge to some predictable processes σ, $W_t(x)$ and $\widetilde{\varphi}^{ij}$ in $\mathcal{L}^2(B)$, in $\mathcal{L}^2(Y)$ and in $\mathcal{L}^2(U)$ respectively. Hence the spaces of stochastic integrals given by

$$\begin{aligned}
\mathcal{Q}^B &:= \left\{ \int \sigma \, dB \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T \sigma_t^2 dt\right) < \infty \right\} \\
\mathcal{Q}^C &:= \left\{ \int \sum_i \sum_j \widetilde{\varphi}^{ij} dU^{ij} \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T \sum_i \sum_j (\widetilde{\varphi}^{ij})^2 \mu^{ij} H_t^i dt\right) < \infty \right\} \\
\mathcal{Q}^Y &:= \left\{ \int \int_{\mathbb{R}} W_t(x)(\mu_Y - \nu_Y)_t \,\middle|\, \mathbb{E}_{\mathbb{P}}\left(\int_0^T \int_{\mathbb{R}} (W_t(x))^2 \nu_{C_{t-}}(dx) dt\right) < \infty \right\}
\end{aligned}$$

are closed. Thus we get

$$\begin{aligned}
X &= \lim_{n\to\infty} X^{(n)} \\
&= \lim_{n\to\infty} \left(\mathbb{E}_{\mathbb{P}}[X^{(n)}] + \int_0^T \sum_i \sum_j \widetilde{\varphi}_t^{ij,(n)} dU_t^{ij} + \int_0^T \sigma_t^{(n)} dB_t \right. \\
&\qquad \left. + \int_0^T \int_{\mathbb{R}} W_t^{(n)}(x)(\mu_Y - \nu_Y)_t \right) \\
&= \mathbb{E}_{\mathbb{P}}[X] + \int_0^T \sum_i \sum_j \widetilde{\varphi}_t^{ij} dU_t^{ij} + \int_0^T \sigma_t dB_t + \int_0^T \int_{\mathbb{R}} W_t(x)(\mu_Y - \nu_Y)_t
\end{aligned}$$

with the limit being taken in $L^2(\mathcal{G}_T, \mathbb{P})$. Also note that the spaces of stochastic integrals $\mathcal{Q}^B, \mathcal{Q}^C$ and $\mathcal{Q}^Y$ defined above are mutually orthogonal to each other since $[B, C] = [B, Y] = [C, Y] = 0$. Summing up, we obtain

Theorem 4.14. *Every square integrable $(\mathbb{P}, \mathbb{G})$−martingale M admits the following representation*

$$M_t = M_0 + \int_0^t \sum_i \sum_j \widetilde{\varphi}_s^{ij} dU_s^{ij} + \int_0^t \sigma_s dB_s + \int_0^t \int_{\mathbb{R}} W_t(x)(\mu_Y - \nu_Y)_s$$

for some $\mathbb{G}-$*predictable stochastic processes* $\widetilde{\varphi}^{ij}$, σ *and* W *satisfying*

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \sum_i \sum_j (\widetilde{\varphi}^{ij})^2 \mu^{ij} H_t^i dt\right] < \infty, \quad \mathbb{E}_{\mathbb{P}}\left[\int_0^T \sigma_t^2 dt\right] < \infty,$$
$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}} W_t^2(x) \nu_{C_{t-}}(dx) dt\right] < \infty. \tag{4.16}$$

Proof. This is now trivial. Let $X = M_T$ and since M is a square integrable martingale, we take conditional expectations and we are done. □

Corollary 4.15. *Every locally square-integrable* $(\mathbb{P}, \mathbb{G})-$*martingale* M *admits the the above martingale representation but* (4.16) *need not hold in this case.*

Chapter 5

Esscher Transforms and Entropy for Hazard Processes

5.1 One-jump process H and asset model

Let us denote by τ a non-negative random variable defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau \geq t) > 0$ for any $t \in [0, T]$. Define H by $H_t := \mathbb{1}_{\{\tau \leq t\}}$ and denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t = \sigma(H_s : s \leq t)$. Let B be a Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by B i.e., $\mathcal{F}_t = \sigma(B_r : r \leq t)$. Introduce the filtration $\mathbb{G}$ on $(\Omega, \mathcal{G}, \mathbb{P})$ as

$$\mathbb{G} = \mathbb{H} \vee \mathbb{F} \quad \text{i.e.,} \quad \mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t \quad \text{for any } t \in [0, T].$$

All filtrations are augmented to satisfy the usual conditions of right-continuity and completeness. Let

$$\widehat{M}_t := H_t - \int_0^t \mu_s \mathbb{1}_{\{\tau > s\}} ds$$

be the martingale associated with the one-jump process H where μ is the deterministic and bounded intensity of the random time τ. Let the conditions of Assumption 4.3 hold, i.e. the filtrations $\mathbb{F}$ and $\mathbb{H}$ are immersed in $\mathbb{G}$. Recall that we consider S as the discounted price of a risky asset in a financial market which contains a riskless asset with discounted price constant at 1. To exclude arbitrage opportunities, our stock price process S must be a semimartingale of the form $S = S_0 + M + \int \lambda d\langle M, M \rangle$ where M is a locally bounded local martingale null at zero and some predictable process λ. We further assume $K_T := \int_0^T \lambda_s^2 d\langle M, M \rangle_s < \infty$ $\mathbb{P}-a.s.$. By Corollary 4.8 we may write M as

$$M_t = M_0 + \int_0^t b_s^M dB_s + \int_0^t c_s^M d\widehat{M}_s$$

for some $\mathbb{G}-$predictable stochastic processes b^M and c^M.

5.2 Change of probability measure

By Proposition 2.16, the density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}$ is given by the Doléans Dade exponential process

$$Z = \mathcal{E}\left(-\int \lambda dM + L\right)$$

where L and $[M, L]$ are local $\mathbb{P}$–martingales. Since L is a local martingale we may write as a consequence of Corollary 4.8

$$L = L_0 + \int b^L dB + \int c^L d\widehat{M}$$

for some $\mathbb{G}$–predictable stochastic processes b^L and c^L.

Proposition 5.1 (Equivalent martingale measure). *Let $\mathbb{Q}$ be a probability measure whose density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}$ is given by the Doléans Dade exponential process $Z = \mathcal{E}\left(-\int \lambda dM + L\right)$ where L and $[M, L]$ are local $\mathbb{P}$–martingales. Define $\mu^{\mathbb{Q}} := (1 + c^L - \lambda c^M)\mu$. Then the following processes 1. $\widetilde{B}_t^{\mathbb{Q}} := B_t - \int_0^t (b_u^L - \lambda_u b_u^M) du$, 2. $\widehat{M}_t^{\mathbb{Q}} = H_t - \int_0^t \mu_u^{\mathbb{Q}} \mathbb{1}_{\{\tau > u\}} du$ are local $\mathbb{Q}$-martingales.*

Proof. To see that $\widetilde{B}^Q$ and $\widehat{M}^{\mathbb{Q}}$ are local $\mathbb{Q}$–martingales, it suffices to check that the products $\widetilde{B}^Q Z$ and $\widehat{M}^{\mathbb{Q}} Z$ respectively are local $\mathbb{P}$–martingales.
1.

$$\begin{aligned}
\widetilde{B}_t^Q Z_t &= \int_0^t Z_{u-} d\widetilde{B}_u^{\mathbb{Q}} + \int_0^t \widetilde{B}_{u-}^{\mathbb{Q}} dZ_u + [Z, \widetilde{B}^{\mathbb{Q}}]_t \\
&= \int_0^t Z_{u-} dB_u + \int_0^t \widetilde{B}_{u-} Z_{u-} (b_u^L - \lambda_u b_u^M) dB_u \\
&\quad + \int_0^t \widetilde{B}_{u-} Z_{u-} (c_u^L - \lambda_u c^M) d\widehat{M}_u
\end{aligned}$$

2.

$$\begin{aligned}
\widehat{M}_t^{\mathbb{Q}} Z_t &= \int_0^t \widehat{M}_{u-}^{\mathbb{Q}} dZ_u + \int_0^t Z_{u-} d\widehat{M}_u^{\mathbb{Q}} + [\widehat{M}^{\mathbb{Q}}, Z]_t \\
&= \int_0^t \widehat{M}_{u-}^{\mathbb{Q}} Z_{u-} (b_u^L - \lambda_u b_u^M) dB_u + \int_0^t \widehat{M}_{u-}^{\mathbb{Q}} Z_{u-} (b_u^L - \lambda_u b_u^M) d\widehat{M}_u \\
&\quad + \int_0^t Z_{u-} (c_u^L - \lambda_u c_u^M) d\widehat{M}_u
\end{aligned}$$

Note that $\widehat{M}_-^{\mathbb{Q}}$ and Z_- are adapted processes which are left continuous with right limits hence they are locally bounded processes (page 166, Protter (2004)). Since they are locally bounded, we have that $Z\widetilde{B}^{\mathbb{Q}}$ and $\widehat{M}^{\mathbb{Q}} Z$ and are local martingales since they are stochastic integrals with respect to local martingales, see Theorem 29, Chapter IV of Protter (2004). To finish off the proof, we note that the integrators are local $\mathbb{P}$-martingales. $\square$

Proposition 5.2. *Let $\mathbb{Q} \in \mathcal{M}^e$. Then the random variable Z evaluated up to $T \wedge \tau$ can be written as*

$$\begin{aligned} Z_{T\wedge\tau} &= \exp\left(\int_0^{T\wedge\tau}(b_t^L - \lambda_t b_t^M)dB_t - \int_0^{T\wedge\tau}(c_t^L - \lambda_t c_t^M)\mu_t dt \right. \\ &\left. -\frac{1}{2}\int_0^{T\wedge\tau}(b_t^L - \lambda_t b_t^M)^2 dt + \int_0^{T\wedge\tau}\log(1 + c_t^L - \lambda_t c_t^M)dH_t\right) \end{aligned} \tag{5.1}$$

Proof. We first evaluate $\mathcal{E}(N)$ where $N = -\int \lambda dM + L$ hence

$$N_t := \int_0^t (b_u^L - \lambda_u b_u^M)dB_u + \int_0^t (c_u^L - \lambda_u c_u^M)d\widehat{M}_u$$

so that from the Doléans-Dade exponential we know that

$$\mathcal{E}(N)_t = \exp\left(N_t - \frac{1}{2}\langle N, N\rangle_t^c\right)\prod_{u\leq t}(1 + \Delta N_u)\exp(-\Delta N_t)$$

Observe that $\Delta N_t = (c_t^L - \lambda_t c_t^M)\Delta H_t$ so inserting the appropriate quantities yields equation (5.1). □

We seek to find L via b^L and c^L such that the corresponding martingale measure $\overline{\mathbb{Q}}$ has the form of equation (2.1) and further proceed to carry out the Verification Procedure outlined in Section 2.2.

5.3 Entropic equation involving a one-jump process

Theorem 2.7 provides an equation which is our starting point to identify the minimal entropy martingale measure:

Theorem 5.3. *The strategy ϕ^E and the constant c^E in equation (2.1) satisfy the following equation*

$$\begin{aligned} c^E &+ \int_0^{T\wedge\tau}\frac{1}{2}(b_t^L - \lambda_t b_t^M)^2 dt + \int_0^{T\wedge\tau}\phi_t^E\lambda_t(b_t^M)^2 dt + \int_0^{T\wedge\tau}\phi_t^E\lambda_t(c_t^M)^2\mu_t dt \\ &+ \int_0^{T\wedge\tau}(c_t^L - \lambda_t c_t^M - \phi_t^E c_t^M)\mu_t dt \\ = &\int_0^{T\wedge\tau}(b_t^L - \lambda_t b_t^M - \phi_t^E b_t^M)dB_t + \int_0^{T\wedge\tau}(\log(1 + c_t^L - \lambda_t c_t^M) - \phi_t^E c_t^M)dH_t \end{aligned} \tag{5.2}$$

where b^L and c^L have to be chosen such that

$$b^M b^L = -c^M c^L \mu \tag{5.3}$$

Proof. From Proposition 2.16, we see that L and $[M, L]$ are local $\mathbb{P}$−martingales. Observe that

$$[M, L]_{\cdot\wedge\tau} = \int_0^{\cdot\wedge\tau} b_t^M b_t^L dt + \int_0^{\cdot\wedge\tau} c_t^M c_t^L dH_t \tag{5.4}$$

We further know from Dellacherie & Meyer (1980) VII.39 that the predictable bracket process

$$\langle M, L\rangle_{\cdot\wedge\tau} = \int_0^{\cdot\wedge\tau} b_t^M b_t^L dt + \int_0^{\cdot\wedge\tau} c_t^M c_t^L \mu_t dt$$

exists, since M and L are locally bounded. However $\langle M, L\rangle = 0$ because $[M, L]$ is a local martingale. We therefore get equation (5.3). Equating equations (2.1) and (5.1) yields equation (5.2). □

5.4 Entropy minimizing candidate for one-jump process

Let us consider the following stock price process

$$\mathbb{P}: \qquad \frac{dS_t}{S_{t-}} = a_t dt + b_t dB_t + c_t d\widehat{M}_t \tag{5.5}$$

where the real-valued functions $a, b > 0$ and $c > -1$ are deterministic but are in general time-inhomogenous. We also assume that they are bounded. Given the price process (5.5) and our fundamental entropy equation (5.2), we see that this corresponds to $b_t^M = b_t S_{t-}$, $c_t^M = c_t S_{t-}$ and

$$\lambda_t = \frac{a_t}{S_{t-}(b_t^2 + \mu_t c_t^2)}$$

Define $\widehat{\lambda}_t := \lambda_t S_{t-}$ and $\widehat{\phi}_t = \phi_t^E S_{t-}$. Let us introduce some further notations.

Definition 5.4.
1. Let $\mathcal{K} := \{0, 1\}$
2. The space $\mathcal{C}_b^{\mathcal{K}} := \mathcal{C}_b([0, T] \times \{0, 1\})$ *consists of continuous bounded functions* $u : [0, T] \times \{0, 1\} \to \mathbb{R}$.
3. For any $u \in \mathcal{C}_b^{\mathcal{K}}$, $u_t := u(t, \cdot) : \mathcal{K} \to \mathbb{R}$.
4. For any $u \in \mathcal{C}_b^{\mathcal{K}}$, $\Delta u_t := u(t, H_t) - u(t, H_{t-})$.

We work with the *ansatz* that there exists a smooth function $u \in \mathcal{C}_b^{\mathcal{K}}$ such that

$$\log(1 + c_t^L - \widehat{\lambda}_t c_t) - \widehat{\phi}_t c_t = u(t, H_t) - u(t, H_{t-}) \tag{5.6}$$

and set

$$u(T \wedge \tau, h) = 0, \qquad h \in \mathcal{K} \tag{5.7}$$

With these, we can then write

$$\int_0^{T\wedge\tau} (\log(1 + c_t^L - \widehat{\lambda}_t c_t) - \widehat{\phi}_t c_t) dH_t = -\int_0^{T\wedge\tau} \frac{\partial}{\partial t} u(t, H_{t-}) dt - u(0, H_0) \tag{5.8}$$

We then rewrite the minimal entropy martingale measure equation (5.2) as

$$\begin{aligned}
&c^E + u(0, H_0) \\
= &\int_0^{T\wedge\tau} \left\{ \frac{\partial}{\partial t} u(t, H_{t-}) + \frac{1}{2}(b_t^L - \widehat{\lambda}_t b_t)^2 \right. \\
&\qquad \left. + \widehat{\phi}_t \widehat{\lambda}_t b_t^2 + (c_t^L - \widehat{\lambda}_t - \widehat{\phi}_t c_t + \widehat{\lambda}_t \widehat{\phi}_t c_t^2)\mu_t \right\} dt \\
&\qquad + \int_0^{T\wedge\tau} (b_t^L - (\widehat{\lambda}_t + \widehat{\phi}_t) b_t) dB_t
\end{aligned} \tag{5.9}$$

To ensure that the RHS of equation (5.9) is non-stochastic, one possible avenue is to choose

$$\frac{\partial}{\partial t} u(t, H_{t-}) + \frac{1}{2}(b_t^L - \widehat{\lambda}_t b_t)^2 + \widehat{\lambda}_t \widehat{\phi}_t b_t^2 + (c_t^L - \widehat{\lambda}_t c_t - \widehat{\phi}_t c_t + \widehat{\lambda}_t \widehat{\phi}_t c_t^2)\mu_t = 0 \tag{5.10}$$

as well as

$$b_t^L = b_t(\widehat{\lambda}_t + \widehat{\phi}_t) \tag{5.11}$$

Inserting the orthogonality equation (5.3) into (5.11) yields

$$\widehat{\phi}_t = -\widehat{\lambda}_t - \frac{c_t^L c_t \mu_t}{b_t^2} \tag{5.12}$$

We also get from our *ansatz* (5.6) that

$$c_t^L = \exp(\Delta u_t + \widehat{\phi}_t c_t) + \widehat{\lambda}_t c_t - 1 \tag{5.13}$$

We see that $\widehat{\phi}_t, b_t^L$ and c_t^L are functions of u_t. Introduce

$$g^h(t, u_t) := \frac{1}{2}(b_t^L - \widehat{\lambda}_t b_t)^2 + \widehat{\lambda}_t \widehat{\phi}_t b_t^2 + (c_t^L - \widehat{\lambda}_t c_t - \widehat{\phi}_t c_t + \widehat{\lambda}_t \widehat{\phi}_t c_t^2)\mu_t \tag{5.14}$$

and arrive by (5.10) at a system of two coupled ordinary differential equations for u of the form

$$\begin{aligned}
\frac{\partial}{\partial t} u(t, h) + g^h(t, u_t) &= 0 \\
u(T, h) &= 0 \quad \text{for every } h \in \{0, 1\}
\end{aligned} \tag{5.15}$$

The next proposition is due to Rheinländer & Steiger (2006).

Proposition 5.5 (Rheinländer & Steiger, 2006). *Let $u \in C_b^K$ such that Δu is bounded. Then u uniquely defines a function c_L solving equation* (5.13) *after inserting* (5.12). *Furthermore c_L, $\widehat{\phi}$ and b_L are bounded.*

Proof. Following the lines of Corollary 3.4 in Rheinländer & Steiger (2006) □

The next question which naturally arises is that of the existence of u. The following theorem provides an existence result:

Theorem 5.6. *Let us consider the differential equation with boundary condition:*

$$\frac{\partial}{\partial t}u(t,h) + g^h(t,u_t) = 0 \tag{5.16}$$

$$u(T,h) = 0 \tag{5.17}$$

for every $h \in \mathcal{K}$. We shall assume that g is a Lipschitz-continuous function in the second argument, uniformly in t, i.e. there exists a constant $c < \infty$ such that $\forall t \in [0,T]$

$$|g(t,w_t) - g(t,z_t)| \leq c|w_t - z_t| \tag{5.18}$$

Then there exists a unique solution $\widehat{u} \in \mathcal{C}_b([0,T] \times \{0,1\})$ which solves the boundary problem (5.16)-(5.17). *It can be written as*

$$\widehat{u}(t,h) = \int_t^T g^h(s,\widehat{u}_s)ds \qquad \textit{for every} \quad h \in \mathcal{K}$$

Proof. Note that

$$u(t,h) = \int_t^T g^h(s,u_s)ds$$

solves the boundary problem (5.16)-(5.17). Introduce the operator $\Phi : \mathcal{C}_b([0,T]\times\{0,1\}) \to \mathcal{C}_b([0,T] \times \{0,1\})$ which is defined as follows:

$$\Phi[u](t,h) := \int_t^T g^h(s,u_s)ds \quad \text{for every } h \in \mathcal{K}$$

Then the above equation is simply $\Phi[u] = u$ and any solution to this equation must be a fixed point of Φ. Let us consider the norm

$$\|u\|_L := \sup_{(t,h)\in[0,T]\times\{0,1\}} e^{-L(T-t)}|u(t,h)|$$

which is equivalent to the supremum-norm $\|u\|_\infty$. Due to condition (5.18), we obtain for $u, v \in \mathcal{C}_b^{\mathcal{K}}$,

$$\begin{aligned}
& e^{-L(T-t)}|\Phi[u](t,h) - \Phi[v](t,h)| \\
= \;& e^{-L(T-t)}\left|\int_t^T (g^h(s,u_s) - g^h(s,v_s))ds\right| \\
\leq \;& e^{-L(T-t)}\int_t^T |g^h(s,u_s) - g^h(s,v_s)|e^{-L(T-s)}e^{L(T-s)}ds \\
\leq \;& e^{-L(T-t)}c\|u-v\|_L \int_t^T e^{L(T-s)}ds \\
\leq \;& c\frac{\|u-v\|_L}{L}(1-e^{-L(T-t)}) \\
\leq \;& \frac{c}{L}\|u-v\|_L
\end{aligned}$$

for all $t \in [0,T]$ and $h \in \{0,1\}$. Thus Φ is a contraction on the normed space $(\mathcal{C}_b^{\mathcal{K}}, \|\cdot\|)$ with contraction constant $c/_L$ with $L > c$. Therefore there exists a unique fixed point $\widehat{u} \in \mathcal{C}_b([0,T] \times \{0,1\})$ which satisfies (5.16)-(5.17). □

However, the function g defined in equation (5.14) is in general not Lipschitz continuous. In the following we shall work on several aspects so that Theorem 5.6 can be invoked.

Step 1: Local Lipschitz continuity of the function g. This part will be devoted to show that $g^h(t,u_t)$ is locally Lipschitz continuous in the second argument. In the following equations, we drop the index h. It can be shown that equation (5.14) can be written as

$$\begin{aligned} g(t,u_t) = \frac{1}{2}\left(\left(\frac{c_t\mu_t c^L(t,u_t)}{b_t}\right)^2 - (\widehat{\lambda}_t b_t)^2\right) + \left(c^L(t,u_t) - (\widehat{\lambda}_t c_t)^2\right)\mu_t \\ + \left(\frac{c_t\mu_t - \mu_t\widehat{\lambda}_t c^2}{b_t^2}\right) c_t\mu_t c^L(t,u_t) \end{aligned} \tag{5.19}$$

Lemma 5.7. *For $u, v \in C_b^{\mathcal{K}}$, there is a constant K^{**} such that*

$$|g(t,u_t) - g(t,v_t)| \leq K^{**}|c^L(t,u_t) - c^L(t,v_t)|$$

Proof. From equation (5.19), we see that (recall that we drop the index h, otherwise it would have been $g^h(t,u_t)$)

$$\begin{aligned} g(t,u_t) - g(t,v_t) &= \frac{1}{2}\left[\left(\frac{c_t\mu_t}{b_t}\right)^2 (c^L(t,u_t))^2 - (c^L(t,v_t))^2\right] \\ &\quad + (c^L(t,u_t) - c^L(t,v_t))\mu_t \\ &\quad + c_t\mu_t\left(\frac{c_t\mu_t - \mu_t c_t^2\widehat{\lambda}_t}{b_t^2}\right)(c^L(t,u_t) - c^L(t,v_t)) \end{aligned}$$

so that

$$\begin{aligned} |g(t,u_t) - g(t,v_t)| &\leq \frac{1}{2}\left(\frac{c_t\mu_t}{b_t}\right)^2 |(c^L(t,u_t))^2 - (c^L(t,v_t))^2| \\ &\quad + \mu_t|c^L(t,u_t) - c^L(t,v_t)| \\ &\quad + \left|\frac{c_t\mu_t(c_t\mu_t - \mu_t\widehat{\lambda}_t c_t^2)}{b_t^2}\right| |c^L(t,u_t) - c^L(t,v_t)| \end{aligned}$$

but by the elementary identity $x^2 - y^2 \leq 2\max(|x|,|y|)|x-y|$, we obtain

$$|g(t,u_t) - g(t,v_t)| \leq K^{**}|c^L(t,u_t) - c^L(t,v_t)|$$

for some K^{**} due to the boundedness of b, μ and c. □

Before we state our next exposition, note that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = {}^{e^x-1}/_x$ is increasing in x. To see this, note that $f'(x) = (e^x(x-1)+1)x^{-2} > 0$ for all x since $e^x > 0$ for all x so that $e^x(x-1) > 0$ when $x > 1$ and $e^x(x-1) < 0$ when $x < 1$.

Lemma 5.8 (Local Lipschitz continuity of g). *For $(t,h) \in [0,T] \times \{0,1\}$ fixed, $c^L(t,u_t)$ is locally Lipschitz-continuous with respect to the second argument, uniformly with respect to $t \in [0,T]$.*

Proof. Our aim is to show that for any $K > 0$, there exists a constant L_K such that for fixed $(t,h) \in [0,T] \times \{0,1\}$ we have

$$|c^L(t,u_t) - c^L(t,v_t)| \leq L_K |u_t - v_t| \tag{5.20}$$

such that $\|u - v\|_\infty \leq \frac{K}{2}$ for all $u, v \in \mathcal{C}_b^K$. For this purpose we consider $u_t = v_t + z\theta_t$ so that $u_t - v_t = z\theta_t$ where $\theta \in \mathcal{C}_b^K$ with $\|\theta\|_\infty = \frac{K}{2}$ and $z \in [0,1]$. Our goal now is to show that there is a constant K_2 such that for all $z \in [0,1]$

$$|c^L(t, v_t + z\theta_t) - c^L(t,v_t)| \leq K_2 |z\theta_t| \leq K_2 \|z\theta\|_\infty \tag{5.21}$$

Recall that $c^L(t,u_t) = \exp(\Delta u_t + \widehat{\phi}_t c_t) + \widehat{\lambda}_t c_t - 1$ so that

$$\begin{aligned}
|c^L(t, v_t + z\theta_t) - c^L(t,v_t)| &= \left| e^{\widehat{\phi}_t c_t} \left(e^{\Delta v_t + z\Delta\theta_t} - e^{\Delta v_t} \right) \right| \\
&\leq \left(e^{\widehat{\phi}_t c_t + \Delta v_t} \right) \left| \frac{e^{z\Delta\theta_t} - 1}{z\Delta\theta_t} z\Delta\theta_t \right| \\
&\leq K^* \|z\Delta\theta\|_\infty \leq 2K^* \|z\theta\|_\infty
\end{aligned}$$

for some constant K^* due to the boundedness of $\widehat{\phi}, c, v, \theta$. Hence one concludes that $g^h(t,u_t)$ is locally Lipschitz-continuous in u_t. □

Step 2: Passing from local to global Lipschitz via a truncation function. Define a truncation function $\kappa : [0,T] \times \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\kappa(t,x) = \max(\min(C(T-t), x), -C(T-t))$$

for some constant $C > 0$ so that the function κ truncates u_t in the sense

$$\kappa(t,u_t) \quad := \quad \max(\min(C(T-t), u_t), -C(T-t))$$

We now define the function

$$\widetilde{g}(t,u_t) \quad := \quad g(t, \kappa(t,u_t))$$

Note that κ takes values in a compact interval $[-C(T-t), C(T-t)]$ and $\widetilde{g}$ is defined on a compact set. Hence we conclude that local Lipschitz continuity of g in u_t passes to global Lipschitz continuity of $\widetilde{g}$ in u_t.

Step 3: Solution with $\widetilde{g}$ and also that of g. We conclude that there exists a unique bounded solution $\widehat{u}$ for (5.16)-(5.17) with $\widetilde{g}$ instead of g. Moreover $\widehat{u}$ is the fixed point of Φ defined with $\widetilde{g}$ instead of g. We shall now show that there exists a constant C such that $\forall (t,h) \in [0,T] \times \{0,1\}$,

$$|\widehat{u}(t,h)| \leq (T-t)C$$

To prove that $\widehat{u}(t,h) \leq (T-t)C$, we fix an arbitrary $(t,h) \in [0,T] \times \{0,1\}$. Fix a C. Define the random variable π as

$$\pi := \inf\{s \in [t,T] | \widehat{u}(s,H_s) < (T-s)C\} \wedge T$$

Then $\widehat{u}(s,H_s) \geq (T-s)C$ for all $s \in [t,\pi)$ and $\widehat{u}(\pi,H_\pi) \leq (T-\pi)C$. Since $\widehat{u}(s,H_s) \geq (T-s)C$ for all $s \in [t,\pi)$ we get that $\Delta\kappa = 0$ ($\Delta\kappa := \Delta u$ if $\kappa = u$ or 0 otherwise) so that $c^L(\kappa(s,\widehat{u}_s))$ is bounded by some constant and that consequently $\forall s \in [t,\pi)$, $\widetilde{g}(s,\widehat{u}_s) < C_1$ for some C_1. It results that

$$\begin{aligned}
\widehat{u}(t,h) &= \int_t^T \widetilde{g}^h(s,\widehat{u}_s)ds \\
&= \int_t^\pi \widetilde{g}^h(s,\widehat{u}_s)ds + \int_\pi^T \widetilde{g}^h(s,\widehat{u}_s)ds \\
&= \int_t^\pi \widetilde{g}^h(s,\widehat{u}_s)ds + \widehat{u}(\pi,H_\pi) \\
&\leq (\pi - t)C_1 + (T-\pi)C
\end{aligned}$$

One can then choose $C \geq C_1$ and we are done. Analogously, one can define

$$\pi := \inf\{s \in [t,T] | \widehat{u}(s,H_s) > -(T-s)C\} \wedge T$$

and proceed by the same lines to show that $\widetilde{g}(s,\widehat{u}_s) > -C_1$. This gives us $|\widehat{u}(t,h)| \leq (T-t)C$. We then conclude that $\kappa(t,\widehat{u}(t,h)) = \widehat{u}(t,h)$ and therefore $g(t,\widehat{u}_t) = \widetilde{g}(t,\widehat{u}_t)$. Hence $\widehat{u}$ also solves the system of differential equations (5.16)-(5.17) with g this time round.

5.5 Entropy minimizing martingale measure for a one-jump process

Having proved the existence of a solution to the differential equations (5.16) and (5.17) we can thus determine the triplet $(\widehat{\phi}, c^L, b^L)$ which solves equation (5.2).

Theorem 5.9. *The density process Z^E associated with $\mathbb{Q}^E$ defined by*

$$Z_t^E = \frac{d\mathbb{Q}^E}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = \mathcal{E}\left(-\left[\int_0 \left(\widehat{\lambda}_u b_u - b_u^L\right) dB_u + \int_{0+} \left(\widehat{\lambda}_u c_u - c_u^L\right) d\widehat{M}_u\right]\right)_t$$

is the density process of the minimal entropy martingale measure.

Proof.
Step 1: $\mathbb{Q}^E$ is an equivalent probability measure. We shall check the conditions of Theorem 2.8. Let the local martingale N be

$$N = -\int_0 \left(\widehat{\lambda}_u b_u - b_u^L\right) dB - \int_{0+} \left(\widehat{\lambda}_u c_u - c_u^L\right) d\widehat{M}_u$$

We have to show that the process U defined by

$$\begin{aligned} U_t &= \frac{1}{2}\int_0^t \left(\widehat{\lambda}_u b_u - b_u^L\right)^2 du \\ &+ \int_0^t \left\{\left(1 - \widehat{\lambda}_u c_u + c_u^L\right)\log\left(1 - \widehat{\lambda}_u c_u + c_u^L\right) + \widehat{\lambda}_u c_u - c_u^L\right\} dH_u \end{aligned}$$

admits a predictable compensator A. To see this note that

$$\int_0^t \left|(1 - \widehat{\lambda}_u c_u + c_u^L)\log(1 - \widehat{\lambda}_u c_u + c_u^L) + \widehat{\lambda}_u c_u - c_u^L)\right| \mu_u \mathbb{1}_{\{\tau>u\}} du < \infty$$

since $\widehat{\lambda}_u, c, c^L, \mu$ and the indicator function $\mathbb{1}_{\{\cdot\}}$ are all bounded. From Theorem II.1.28 of Jacod & Shiryaev (2003), this implies the integrability of $(1-\widehat{\lambda}_u c_u + c_u^L)\log(1-\widehat{\lambda}_u c_u + c_u^L) + \widehat{\lambda}_u c_u - c_u^L) =: \jmath(t)$ with respect to the martingale associated with the one-jump process $\widehat{M}$ and

$$\int_0^t \jmath(u) d\widehat{M}_u = \int_0^t \jmath(u) dH_u - \int_0^t \jmath(u)\mu_u \mathbb{1}_{\{\tau>u\}} du$$

This yields that the compensator A of U is given by

$$\begin{aligned} A_t &= \frac{1}{2}\int_0^t \left(\widehat{\lambda}_u b_u - b_u^L\right)^2 du \\ &+ \int_0^{t\wedge\tau} \left\{\left(1 - \widehat{\lambda}_u c_u + c_u^L\right)\log\left(1 - \widehat{\lambda}_u c_u + c_u^L\right) + \widehat{\lambda}_u c_u - c_u^L\right\} \mu_u du \end{aligned}$$

A is obviously predictable. That $\mathbb{E}_{\mathbb{P}}\left[\exp\left(A_{T\wedge\tau}\right)\right] < \infty$ is obvious since c^L, $\widehat{\phi}$ and b^L are bounded.

Step 2: We want to show $I(\mathbb{Q}^E, \mathbb{P}) < \infty$. We have that

$$\begin{aligned} I\left(\mathbb{Q}^E, \mathbb{P}\right) &= \mathbb{E}_{\mathbb{Q}^E}\left[c + \int_0^{T\wedge\tau} \frac{\widehat{\phi}_t}{S_{t-}} dS_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^E}\left[c + \int_0^{T\wedge\tau} \widehat{\phi}_t \left(a_t S_t dt + b_t S_t dB_t + c_t S_t d\widehat{M}_t\right)\right] \end{aligned}$$

Our aim is to show that

$$\mathbb{E}_{\mathbb{Q}^E}\left[\int_0^{T\wedge\tau} \frac{\widehat{\phi}_t}{S_{t-}} dS_t\right] = 0$$

since this implies $I\left(\mathbb{Q}^E, \mathbb{P}\right) = c$, which is finite from previous section. We define

$$\mu_t^{\mathbb{Q}^E} := \left(\widehat{\lambda} c_t - c_t^L + 1\right)\mu_t,$$

it follows from Girsanov's theorem that $\int bdB + \int b\left(\widehat{\lambda}b - b^L\right)dt$ and $\int cdM^{\mathbb{Q}^E}$ are local $\mathbb{Q}^E$−martingales, where $M^{Q^E} := H - \int_0^{\cdot\wedge\tau}\mu^{Q^E}\,dt$. In fact they are square integrable $\mathbb{Q}^E$−martingales since their quadratic variations are $\mathbb{Q}^E$−integrable. This is because

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\int cdM^{\mathbb{Q}^E}, \int cdM^{\mathbb{Q}^E}\right]_T &= \mathbb{E}_{\mathbb{P}}\left(\int_0^T c_t^2 dH_t\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^T c_t^2\mu_t^{\mathbb{Q}^E}\mathbb{1}_{[\tau\geq t]}dt\right) \leq \mathbb{E}_{\mathbb{P}}\left(\int_0^T c_t^2\mu_t^{\mathbb{Q}^E}dt\right) \\
&< \infty
\end{aligned}
$$

since c, μ and $\widehat{\lambda}c - c^L + 1$ are bounded. The dynamics of S under $\mathbb{Q}^E$ can then be written as

$$
\mathbb{Q}^E: \qquad \frac{dS_t}{S_{t-}} = b_t dB_t + b_t\left(\widehat{\lambda}_t b_t - b_t^L\right)dt + c_t dM_t^{\mathbb{Q}^E}
$$

since we have

$$
\begin{aligned}
& a_t - b_t\left(\widehat{\lambda}_t b_t - b_t^L\right) + c_t\left(\lambda_t c_t - c_t^L\right)\mu_t\mathbb{1}_{[\tau>t]} \\
=\; & \widehat{\lambda}_t\left(b_t^2 + c_t^2\mu_t\mathbb{1}_{[\tau>t]}\right) - b_t\left(\widehat{\lambda}_t b_t - b_t^L\right) + c_t\left(\widehat{\lambda}_t c_t - c_t^L\right)\mu_t\mathbb{1}_{[\tau>t]} \\
=\; & b_t b_t^L + c_t c_t^L\mu_t\mathbb{1}_{[\tau>t]} \\
=\; & 0
\end{aligned}
$$

where the last line follows from the orthogonality condition (5.3).

Step 3: Before we proceed, we consider the following stochastic differential equation:

$$
dZ^\dagger = kZ_-^\dagger d\widehat{M}
$$

where $\widehat{M}_t := H_t - \int_0^{t\wedge\tau}\mu_s ds$ and some deterministic function k. The solution of this SDE is given by

$$
Z_t^\dagger = \exp\left(\int_0^t \log\left(1 + k_u\right)dH_u - \int_0^{t\wedge\tau} k_u\mu_u du\right)
$$

From the above SDE, $Z_-^\dagger$ is an adapted process which is cadlag, so that $Z_-^\dagger$ is locally bounded (page 166, Protter (2004)). Since $Z_-^\dagger$ is locally bounded, we have that $Z^\dagger$ is a local martingale (Theorem 29, Chapter IV of Protter (2004)). Further observe that $Z^\dagger \geq 0$ so that $Z^\dagger$ is a supermartingale. Hence one gets

$$
\begin{aligned}
1 = \mathbb{E}_{\mathbb{P}}[Z_0^\dagger] \geq \mathbb{E}_{\mathbb{P}}[Z_t^\dagger] &= \mathbb{E}_{\mathbb{P}}\Big[\exp\left(\int_0^t \log(1+k_u)dH_u - \int_0^{t\wedge\tau} k_u\mu_u du\right)\Big] \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \log\left(1+k_u\right)dH_u - \int_0^t k_u\mu_u\mathbb{1}_{[\tau\geq t]}du\right)\right] \\
&\geq \exp\left(-\int_0^t k_u\mu_u du\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \log\left(1+k_u\right)dH_u\right)\right]
\end{aligned}
$$

Rearranging one gets

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \log\left(1+k_u\right)dH_u\right)\right] \leq \exp\left(\int_0^t k_u\mu_u du\right). \tag{5.22}$$

Now, we proceed to the actual verification for Step 3. We would like to show that $\int \phi^E dS$ is a true $\mathbb{Q}^E$−martingale. Recall that ϕ^E is defined in the fundamental entropic equation (5.2). According to Proposition 2.9, this can be accomplished if we can find some $\beta > 0$, such that

$$\mathbb{E}_{\mathbb{P}}\Big[\exp\left\{\beta\int_0^{T\wedge\tau}(\phi_t^E)^2 d[S]_t\right\}\Big] < \infty.$$

Define $\delta := \sup \|\widehat{\phi}\|_{L^\infty}$. From the boundedness of b, c, μ we have

$$\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\beta\int_0^{T\wedge\tau}\widehat{\phi}_t^2 d[S]_t\right\}\right] \\
= \; & \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\int_0^T \frac{1}{2}b_t^2 \mathbb{1}_{[\tau>t]}dt + \int_0^{T\wedge\tau}\frac{1}{2}c_t^2 dH_t\right\}\right] \\
< \; & \infty
\end{aligned}$$

by the bound obtained in equation (5.22) together with $\beta := \left(2\delta^2\right)^{-1}$. □

Hence we conclude that $\mathbb{Q}^E$ fulfills all the conditions to being the entropy minimizing martingale measure for a one-jump process with a Brownian component.

5.6 Esscher martingale transforms for a one-jump process

An additional approach to option pricing for models that are incomplete has been related to the mathematical construction of the Esscher martingale transform. As explained in Kallsen & Shiryaev (2002) there are two different Esscher martingale measures namely the Esscher martingale transform for the exponential process and the Esscher martingale transform for the linear process. It has been shown in Esche & Schweizer (2005) that for exponential Lévy models, the minimal entropy martingale measure and the Esscher martingale transform for the linear process coincides. The study of Esscher martingale measure transforms for general semimartingales has been carried out by Kallsen & Shiryaev (2002). The intent of this section is to analyse the Esscher martingale measure for the one-jump process. We start with giving basic definitions from Kallsen & Shiryaev (2002). If X is a semimartingale, then $L(X)$ represents the set of predictable X−integrable processes.

Definition 5.10. *We denote by $\mathcal{M}_{\text{loc}}$ the class of all local martingales.*

Definition 5.11. *For any real-valued semimartingale X with $X_0 = 0$, we call $\widetilde{X} := \mathcal{L}(\exp(X))$ the exponential transform of X. Conversely we call $X := \log(\mathcal{E}(\widetilde{X}))$ the logarithmic transform of any real-valued semimartingale $\widetilde{X}$.*

Recall that a real-valued semimartingale X is called *special* if it can be written as $X = X_0 + M + A$ for some local martingale M and some predictable process A of finite variation, null at 0. In other words, X is a special semimartingale if there exists a predictable process A such that $X - X_0 - A \in \mathcal{M}_{\text{loc}}$.

Definition 5.12. *Let X be a real-valued semimartingale. X is called exponentially special if* $\exp(X - X_0)$ *is a special semimartingale.*

Definition 5.13. *Let X be a real-valued semimartingale. A predictable process A is called the exponential compensator of X if* $\exp(X - X_0 - A) \in \mathcal{M}_{\text{loc}}$.

Definition 5.14. *The Laplace cumulant process $\widetilde{K}^X(\vartheta)$ of X in ϑ is defined as the compensator of the semimartingale $\mathcal{L}(\exp(\int \vartheta dX))$.*

Definition 5.15. *The modified Laplace cumulant process $K^X(\vartheta)$ of X in ϑ is the logarithmic transform of $\widetilde{K}^X(\vartheta)$, i.e. $K^X(\vartheta) := \log(\mathcal{E}(\widetilde{K}^X(\vartheta)))$.*

5.6.1 Specification of the model

Denote by τ a non-negative random variable satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. Introduce the one-jump process $H_t := \mathbb{1}_{\{\tau \le t\}}$ and denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t : \sigma(H_s : s \le t)$. We further assume that the one-jump process has deterministic intensity μ_t. Let B be a Brownian motion. B generates $\mathbb{F}$. Let $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ i.e., $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in [0, T]$. All filtrations are augmented to satisfy the usual conditions of right continuity and completeness. We consider the price process S of the form

$$S_t = \exp(X_t) \tag{5.23}$$

where

$$X_t := \int_0^t \widetilde{a}_s ds + \int_0^t \widetilde{b}_s dB_s + \int_0^t \widetilde{c}_s dH_s \tag{5.24}$$

The functions $\widetilde{a}, \widetilde{b}$ and $\widetilde{c} \neq 0$ are assumed to be deterministic up to $t \wedge \tau$, continuous and bounded but can possibly be time-inhomogeneous. Recall that the jumps of the processes X and H are related by $\Delta X_t = \widetilde{c}_t \Delta H_t$ and therefore from Theorem II.2a in Jacod & Shiryaev (2003), the semimartingale characteristics of X are given by $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ satisfying

$$\widetilde{B}_t = \int_0^t \widetilde{a}_s ds, \quad \widetilde{C}_t = \int_0^t \widetilde{b}_s^2 dB_s, \quad \widetilde{\nu}_t = \int_0^t \widetilde{c}_s \mathbb{1}_{\{\tau > s\}} \mu_s ds$$

We can state the following:

Lemma 5.16. *Let $\vartheta \in L(X)$ be such that $\int \vartheta dX$ is exponentially special. The modified Laplace cumulant process of X in ϑ evaluated up to $t \wedge \tau$ is then given by*

$$K^X(\vartheta)_{t\wedge\tau} = \int_0^{t\wedge\tau} \widetilde{\kappa}^X(\vartheta)_s ds$$

where

$$\widetilde{\kappa}^X(\vartheta)_t = \widetilde{a}_t \vartheta_t + \frac{1}{2} \widetilde{b}_t^2 \vartheta_t^2 + (e^{\widetilde{c}_t \vartheta_t} - 1) \mathbb{1}_{\{\tau > t\}} \mu_t \tag{5.25}$$

Proof. This is immediate from the semimartingale characteristics outlined above and an application of Theorem 2.18-(1) in Kallsen & Shiryaev (2002). □

Our approach begins by finding the solution $\vartheta^\sharp$ evaluated up to $t \wedge \tau$ as described in Theorem 4.1 of Kallsen & Shiryaev (2002) and Jacod & Shiryaev (2003, Theorem VII.28) to the equation

$$K^X(\vartheta + 1) - K^X(\vartheta) = 0 \tag{5.26}$$

Suppose now that there is a solution $\vartheta^\sharp$ to equation (5.26) and suppose that

$$G^\sharp_{t\wedge\tau} := \exp\left(\int_0^{t\wedge\tau} \vartheta^\sharp_s dX_s - K^X(\vartheta^\sharp)_{t\wedge\tau}\right) \tag{5.27}$$

is a martingale $(G^\sharp_{t\wedge\tau})_{0\le t\le T}$, then we can define a probability measure $\mathbb{Q}^\sharp$ by

$$\frac{d\mathbb{Q}^\sharp}{d\mathbb{P}} = \exp\left(\int_0^{T\wedge\tau} \vartheta^\sharp_t dX_t - K^X(\vartheta^\sharp)_{T\wedge\tau}\right) \tag{5.28}$$

The measure $\mathbb{Q}^\sharp$ is then called the Esscher martingale transform for the exponential process e^X. If no such $\vartheta^\sharp$ exists, we say the Esscher martingale transform for e^X does not exist. It follows from Theorem 4.1 of Kallsen & Shiryaev (2002) that the density in (5.28) defines an equivalent local martingale measure for e^X.

5.6.2 Existence and martingale conditions for the exponential Esscher martingale transforms

The aim of this section is two-fold. First we single out conditions such that there exists a bounded solution $\vartheta^\sharp$ to equation (5.26). We then further show that the process $(G^\sharp_{t\wedge\tau})_{0\le t\le T}$ defined in equation (5.27) is a martingale.

From equation (5.26), one sees that we are required to find a solution $\vartheta^\sharp$ to the equation

$$\widetilde{a}_t(\vartheta_t+1) + \frac{1}{2}\widetilde{b}_t^2(\vartheta_t+1)^2 + (e^{(\vartheta_t+1)\widetilde{c}_t} - 1)\mu_t = \widetilde{a}_t\vartheta_t + \frac{1}{2}\widetilde{b}_t^2(\vartheta_t)^2 + (e^{\vartheta_t\widetilde{c}_t} - 1)\mu_t \tag{5.29}$$

Simplifying the above we present

Lemma 5.17. *Define the following*

$$\widetilde{f}(\vartheta) := \widetilde{a}_t + \frac{1}{2}\widetilde{b}_t^2, \quad \widetilde{g}(\vartheta) := -\widetilde{b}_t^2\vartheta - e^{\widetilde{c}_t\vartheta}(e^{\widetilde{c}_t} - 1)\mu_t$$

Then there exists a unique bounded function $\vartheta^\sharp : [0,T] \to \mathbb{R}$ with $\vartheta^\sharp_t := \vartheta^\sharp(t)$ solving equation (5.29).

Proof. By hypothesis $\widetilde{a}$ and $\widetilde{b}$ are bounded hence $\widetilde{f}$ is bounded. Also, $\widetilde{g}'(\vartheta) = -\widetilde{b}_t^2 - (e^{\widetilde{c}_t} - 1)\mu_t\widetilde{c}_t e^{\widetilde{c}_t\vartheta} < 0$, $\widetilde{g}(-\infty) = \infty$ and $\widetilde{g}(\infty) = -\infty$. This means that $\widetilde{g}$ is decreasing and continuous so we conclude that there exists a value $\vartheta^\sharp_t$ such that $\widetilde{g}(\vartheta^\sharp_t) = \widetilde{f}(\vartheta^\sharp_t)$. To see that $\vartheta^\sharp_t$ is bounded, note that since $\widetilde{g}$ is decreasing, the inverse function $\widetilde{g}^{-1}$ exists so that $\widetilde{g}^{-1}(\sup_{0\le t\le T}\widetilde{f}) \le \vartheta^\sharp_t \le \widetilde{g}^{-1}(\inf_{0\le t\le T}\widetilde{f})$. □

Having proved the existence of $\vartheta^\sharp$, we now turn our attention to proving that the process $(G^\sharp_{t\wedge\tau})_{0\le t\le T}$ defined in equation (5.27) is a martingale.

Lemma 5.18. *Let*

$$N^\sharp_{t\wedge\tau} := \int_0^{t\wedge\tau} \vartheta^\sharp_s dX_s - K^X(\vartheta^\sharp)_{t\wedge\tau}$$

and

$$G^\sharp_{t\wedge\tau} = \exp\left(\int_0^{t\wedge\tau} \vartheta^\sharp_s dX_s - \int_0^{t\wedge\tau} \widetilde{\kappa}^X(\vartheta)_s ds\right)$$

Then $(G^\sharp_{t\wedge\tau})_{0\le t\le T}$ is a martingale.

Proof. Observe that $G^\sharp_{t\wedge\tau}$ can be written as

$$G^\sharp_{t\wedge\tau} = \mathcal{E}\left(\int_0^{\cdot} \vartheta^\sharp_s \widetilde{b}_s dB_s + \int_0^{\cdot} \left(e^{\vartheta^\sharp_s \widetilde{c}_s} - 1\right) d\widehat{M}_s\right)_{t\wedge\tau} \tag{5.30}$$

since $\int_0^{t\wedge\tau} \mu_s \mathbb{1}_{\{\tau>s\}} ds = \int_0^t \mu_s \mathbb{1}_{\{\tau>s\}} ds = \int_0^{t\wedge\tau} \mu_s ds$ where $\widehat{M}_t := H_t - \int_0^{t\wedge\tau} \mu_s ds$. To prove this Lemma, we invoke the integrability condition of Theorem 2.8. We have to show that

$$U_{t\wedge\tau} := \frac{1}{2}\int_0^{t\wedge\tau} (\vartheta^\sharp_s \widetilde{b}_s)^2 ds + \int_0^{t\wedge\tau} (\exp(\vartheta^\sharp_s \widetilde{c}_s)\vartheta^\sharp_s \widetilde{c}_s - (\exp(\vartheta^\sharp_s \widetilde{c}_s) - 1)) dH_s$$

admits a predictable compensator A such that $\mathbb{E}_{\mathbb{P}}[\exp(A_{T\wedge\tau})] < \infty$. Observe that

$$\int_0^{t\wedge\tau} \left|(\exp(\vartheta^\sharp_s \widetilde{c}_s)\vartheta^\sharp_s \widetilde{c}_s - (\exp(\vartheta^\sharp_s \widetilde{c}_s) - 1)\right| \mu_s ds < \infty$$

since $\widetilde{c}$ and μ is bounded. From Jacod & Shiryaev (2003, Theorem II.1.2.8) we conclude that $(\exp(\vartheta^\sharp_s c_s)\vartheta^\sharp_s c_s - (\exp(\vartheta^\sharp_s c_s) - 1))$ is integrable with respect to $\widehat{M}$ so that the predictable compensator is given by

$$A_{t\wedge\tau} = \frac{1}{2}\int_0^{t\wedge\tau} (\vartheta^\sharp_s b_s)^2 ds + \int_0^{t\wedge\tau} (\exp(\vartheta^\sharp_s \widetilde{c}_s)\vartheta^\sharp_s \widetilde{c}_s - (\exp(\vartheta^\sharp_s \widetilde{c}_s) - 1)\mu_s ds$$

The fact that $\mathbb{E}_{\mathbb{P}}[\exp(A_{T\wedge\tau})] < \infty$ is obvious due to the boundedness of $\widetilde{c}, \widetilde{b}$ and μ. □

5.6.3 Main result on Esscher martingale transform

Having the existence of solutions verified and integrability conditions of martingale ensured, we are now ready for our main theorem.

Theorem 5.19. *Let the conditions of Lemma* 5.17 *and Lemma* 5.18 *hold. Then*

$$\frac{d\mathbb{Q}^\sharp}{d\mathbb{P}} = \mathcal{E}\left(\int_0^{\cdot} \vartheta^\sharp_s \widetilde{b}_s dB_s + \int_0^{\cdot} \left(e^{\vartheta^\sharp_s \widetilde{c}_s} - 1\right) d\widehat{M}_s\right)_{T\wedge\tau} \tag{5.31}$$

defines a probability measure $\mathbb{Q}^\sharp \sim \mathbb{P}$ *on* $\mathcal{F}_{T\wedge\tau}$. *Moreover, under the Esscher martingale measure* $\mathbb{Q}^\sharp$, *we have*

$$B_t^{\mathbb{Q}^\sharp} := B_t - \int_0^t \vartheta_s^\sharp \widetilde{b}_s ds$$

is a $\mathbb{Q}^\sharp$ *standard Brownian motion and the compensator of* H *under* $\mathbb{Q}^\sharp$ *is then given by*

$$\int_0^t e^{\vartheta_s^\sharp \widetilde{c}_s} \mu_s \mathbb{1}_{\{\tau>s\}} ds$$

Proof. The existence of solutions and the integrability conditions have been discussed in Section 5.6.2; in particular we have shown that there is a $\vartheta^\sharp$ that solves

$$K^X(\vartheta^\sharp + 1) - K^X(\vartheta^\sharp) = 0$$

and that

$$G_{t\wedge\tau}^\sharp = \mathcal{E}\left(\int_0^\cdot \vartheta_s^\sharp \widetilde{b}_s dB_s + \int_0^\cdot \left(e^{\vartheta_s^\sharp \widetilde{c}_s} - 1\right) d\widehat{M}_s\right)_{t\wedge\tau}$$

is a martingale for $0 \le t \le T$. We apply Theorem 4.1 in Kallsen & Shiryaev (2002) and conclude that the density in (5.31) defines an equivalent local martingale measure for $\exp(X)$. However, by Lemma (5.18), $G^\sharp$ is a proper martingale and thus a density process. The dynamics under $\mathbb{Q}^\sharp$ can be seen by showing that the product $B^\sharp G$ is a $\mathbb{G}$−local martingale under the original probability measure $\mathbb{P}$. The dynamics of the compensator for H under $\mathbb{Q}^\sharp$ follows from similar arguments. □

5.6.4 Biagini-Rheinländer model

We now turn our attention to finding the Esscher martingale transform for the following model proposed by Biagini & Rheinländer (2009):

$$\frac{dS_t}{S_{t-}} = a_t dt + b_t dB_t + c_t dM_t$$

where B is a Brownian motion, M is the counting process martingale associated to the one-jump process $H_t = \mathbb{1}_{\{\tau\le t\}}$ which we assume has deterministic intensity μ_t. The functions a, b and c are as well assumed for simplicity to be deterministic but can depend on time. We further assume that they are bounded. From our preceding discussions, we get

$$\widetilde{a}_t = a_t - \frac{1}{2}b_t^2 - c_t\mu_t\mathbb{1}_{\{\tau>t\}}, \quad \widetilde{b}_t = b_t, \quad \widetilde{c}_t = \log(1+c_t)$$

To see this note that $dS_t = S_{t-}d\widehat{L}_t$ or equivalently $S_t = \mathcal{E}(\widehat{L})_t$ where $\widehat{L}_t := \int_0^t a_s ds + \int_0^t b_s dB_s + \int_0^t c_s dM_s$. One can then expand this stochastic exponential to compare with the parameters in Section 5.6.1. Strictly speaking $\widetilde{a}_t$ is not deterministic due to the presence of τ within the indicator function. However, this does not matter since we are evaluating the modified Laplace cumulant process K up to and including $t\wedge\tau$. To check for existence of solutions, define

$$\begin{aligned} f(\vartheta) &:= a_t - \frac{1}{2}b_t^2 - c_t\mu_t + \frac{1}{2}b_t^2 = a_t - c_t\mu_t \\ g(\vartheta) &:= -b_t^2\vartheta - (\exp(\log(1+c_t)) - 1)\exp(\vartheta\log(1+c_t)) \end{aligned}$$

By hypothesis, $\widetilde{a}, \widetilde{b}$ and $\widetilde{c} \neq 0$ are deterministic and bounded. From the above definition, this translates to the conditions

$$c > -1$$

Provided the above holds, $g(\infty) = -\infty$ and $g(-\infty) = +\infty$. This is meaningful because we can either choose c to be positive, i.e. $c > 0$ (upward jump for S) or c to be negative, i.e. $-1 < c < 0$ (downward jump for S). By similar reasoning, one sees that there exists a solution to (5.26). By similar reasoning, the integrability conditions for the process $(G^{\sharp}_{t\wedge\tau})_{0\leq t\leq T}$ to be a martingale are fulfilled since we only perform a change of variable to the quantities $\widetilde{a}, \widetilde{b}$ and $\widetilde{c}$. We can now state

Corollary 5.20. *Let the integrability and existence conditions in Section* 5.6.4 *hold. Then*

$$\frac{d\mathbb{Q}^{\sharp}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^{\cdot} \vartheta^{\sharp}_s b_s dB_s + \int_0^{\cdot} \left(e^{\vartheta^{\sharp}_s \log(1+c_s)} - 1\right) d\widehat{M}_s\right)_{T\wedge\tau} \tag{5.32}$$

defines a probability measure $\mathbb{Q}^{\sharp} \sim \mathbb{P}$ *on* $\mathcal{F}_{T\wedge\tau}$. *Moreover, under the Esscher martingale measure* $\mathbb{Q}^{\sharp}$, *we have*

$$B^{\mathbb{Q}^{\sharp}}_t := B_t - \int_0^t \vartheta^{\sharp}_s b_s ds$$

is a $\mathbb{Q}^{\sharp}$ *standard Brownian motion and the compensator of* H *under* $\mathbb{Q}^{\sharp}$ *is then given by*

$$\int_0^t e^{\vartheta^{\sharp}_s \log(1+c_s)} \mu_s \mathbb{1}_{\{\tau>s\}} ds$$

Proof. The proof of this goes along the lines of Theorem 5.19. $\square$

Chapter 6

The Minimal Entropy Martingale Measure for Quasi-Markov Additive Processes

6.1 Introduction

In the finance literature at large, the Markov modulated geometric Brownian motion or commonly known as switching Black-Scholes is one popular common choice and it has been used for many applications. Some papers consider hidden Markov models for interest rates and insurance applications, see for instance Elliot, Chan & Siu (2005) and Jobert & Rogers (2006). The goal of this chapter is to compute the minimal entropy martingale measure for Markov additive processes, where the price process takes the form as in Section 6.6. Loosely speaking, this process can be thought of as a geometric Lévy process where the triplet follows a continuous time Markov chain.

Sections 6.2 to 6.4 gives the basic tools needed to evaluate the minimal entropy martingale measure.

6.2 Continuous time Markov chains

Let $\{C_t\}_{0\leq t\leq T}$ be a continuous time Markov chain on $(\Omega, \mathcal{G}, \mathbb{P})$ with finite state space $\mathcal{Y} = \{1, ..., m\}$. Let $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T} = \sigma(C_s; 0 \leq s \leq t)$, $0 \leq t \leq T$ be the filtration generated by this Markov chain. The paths of C are taken to be right continuous and we make the following standard assumption

Assumption 6.1. *$\mathcal{H}_0$ is trivial*

Assume further that C is time homogeneous so that we have

$$\mathbb{P}(C_{t+s} = j | C_s = i) = p_{ij}(t) \quad \forall i, j \in \mathcal{Y}, \; s, t \in \mathbb{R}^+, \; 0 \leq s \leq t \leq T$$

The following limit exists for $i,j \in \mathcal{Y}$ (Rolski, et. al. (1998) Theorem 8.1.2)

$$\mu^{ij} := \lim_{t \searrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t}$$

We note that for every $i \neq j$, we have $\mu^{ij} \geq 0$ and $\mu^{ii} = -\sum_{j=1, j\neq i}^{m} \mu^{ij}$. Hence μ^{ij} represents the intensity of transition from state i to state j and is a constant. The matrix $\Lambda := \left[\mu^{ij}\right]_{i,j\in\{1,\ldots,m\}}$ is called the infinitesimal generator matrix for a Markov chain. It is also commonly known as the intensity matrix. This intensity matrix together with the matrix of transition probabilities $\mathcal{P}(t) := [p_{ij}(t)]_{i,j\in\{1,\ldots,m\}}$ are related by the Kolmogorov Backward Equation

$$\frac{d\mathcal{P}(t)}{dt} = \Lambda \mathcal{P}(t), \quad \mathcal{P}(0) = \mathbb{I}$$

and the Kolmogorov Forward Equation

$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t)\Lambda, \quad \mathcal{P}(0) = \mathbb{I}$$

where $\mathbb{I}$ is the identity matrix. It is known that both these equations have the unique solution:

$$\mathcal{P}(t) = e^{t\Lambda} := \sum_{j=0}^{\infty} \frac{\Lambda^j t^j}{j!}, \quad \forall t \in \mathbb{R}^+,\ t \leq T$$

We assume throughout that all states intercommunicate, i.e. there are no absorbing states. Furthermore we introduce for $i \neq j$ and $\forall s,t \in \mathbb{R}^+$, $t \leq T$

$$\begin{aligned} H_t^i &: = \mathbb{1}\{C_t = i\} \\ H_t^{ij} &: = \sharp\{s : 0 < s \leq t | C_{s-} = i, C_s = j\} = \sum_{0<s\leq t} \mathbb{1}\{C_{s-} = i\}\mathbb{1}\{C_s = j\} \end{aligned}$$

Thus H_t^i is the indicator of the event that C is in state i at time t. On the other hand H_t^{ij} is the number of jumps from i to j during $]0,t]$. We state some results from Last & Brandt (1995) and Rogers & Williams (2000).

Lemma 6.2. *For every $i,j \in \mathcal{Y}$, $i \neq j$, the processes*

$$U_t^{ij} = H_t^{ij} - \int_0^t \mu^{ij} H_u^i du$$

Proof. Theorem 7.5.5 in Last & Brandt (1995) □

Theorem 6.3. *Any arbitrary $(\mathbb{P}, \mathbb{H})-$local martingale $\widetilde{Y}$ can be written as*

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t \sum_i \sum_j g_s(i,j)\, dU_s^{ij}$$

where g is locally bounded and predictable $\forall i,j \in \mathcal{Y}$ and $g(i,i) = 0$ $\forall i \in \mathcal{Y}$, $0 \leq t \leq T$

Proof. Lemma 21.13 and Theorem 21.15 in Rogers & Williams (2000). □

We now introduce a real valued process $\widetilde{\varphi}^{ij}$, $i, j \in \mathcal{Y}$, $i \neq j$ of bounded $\mathbb{H}$−predictable stochastic processes such that $\widetilde{\varphi}^{ij} > -1$ with $\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$. Let us define $g(i, j)$ of Theorem 6.3 as

$$g_t(i, j) := \widetilde{\varphi}^{ij}_t$$

Hence we get from Theorem 6.3 that for $0 \leq t \leq T$,

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t \sum_i \sum_j \widetilde{\varphi}^{ij}_s dU^{ij}_s \tag{6.1}$$

6.3 Asset model and general results

We consider S as the discounted price of a risky asset in a financial market which contains a riskless asset with discounted price constant at 1. To exclude arbitrage opportunities, S must be a semimartingale of the form $S = S_0 + M + \int \lambda d\langle M, M\rangle$ where M is a locally bounded local martingale null at zero and some predictable process λ. We further assume $K_T := \int_0^T \lambda_s^2 d\langle M, M\rangle_s < \infty$ $\mathbb{P}-a.s.$ By Corollary 4.15 we may write M as

$$M = M_0 + \int \sigma^M dB + \int\int W^M(x)(\mu_Y - \nu_Y) + \int \sum_i \sum_j {}^M\widetilde{\varphi}^{ij} dU^{ij}$$

for some $\mathbb{G}$−predictable stochastic processes σ^M, W^M and ${}^M\widetilde{\varphi}^{ij}$, $i, j \in \mathcal{Y}$, $i \neq j$, $\nu_Y = \sum_i H^i_{t-}\nu^i(dx)dt$.

6.4 Equivalent martingale measures for quasi-Markov additive process

By Proposition 2.16, the density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}$ is given by the Doléans Dade exponential process

$$Z = \mathcal{E}\left(-\int \lambda dM + L\right)$$

where L and $[M, L]$ are local $\mathbb{P}$−martingales. Hence we may as a consequence of Corollary 4.15 write

$$L = L_0 + \int \sigma^L dB + \int\int W^L(x)(\mu_Y - \nu_Y) + \int \sum_i \sum_j {}^L\widetilde{\varphi}^{ij} dU^{ij}$$

for some $\mathbb{G}$−predictable stochastic processes σ^L, W^L and ${}^L\widetilde{\varphi}^{ij}$, $i, j \in \mathcal{Y}$, $i \neq j$, $\nu_Y = \sum_i H^i_{t-}\nu^i(dx)dt$ with ${}^L\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$.

Proposition 6.4 (Equivalent martingale measure). *Let $\mathbb{Q}$ be a probability measure whose density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}$ is given by the Doléans Dade exponential process $Z = \mathcal{E}\left(-\int \lambda dM + L\right)$ where L and $[M, L]$ are local $\mathbb{P}-$martingales. Define*

$$\nu_Y^{\mathbb{Q}} := \int\int (1 + W^L(x) - \lambda W^M(x))\nu_Y, \ {}^{\mathbb{Q}}\mu^{ij} := (1 + {}^L\widetilde{\varphi}^{ij} - \lambda^M\widetilde{\varphi}^{ij})\mu^{ij}$$

Then the following 1. $\widetilde{B}^{\mathbb{Q}} := B - \int(\sigma^L - \lambda\sigma^M)dt$, 2. $Y^{\mathbb{Q}} := \int\int W^L(x)(\mu_Y - \nu_Y^{\mathbb{Q}})$ and 3. ${}^{\mathbb{Q}}U^{ij} := H^{ij} - \int {}^{\mathbb{Q}}\mu^{ij} H^i_- dt$ are local $\mathbb{Q}$-martingales.

Proof. $\widetilde{B}^Q$, $Y^{\mathbb{Q}}$ and ${}^{\mathbb{Q}}U^{ij}$ are local $\mathbb{Q}-$martingales if and only if $\widetilde{B}^Q Z$, $Y^{\mathbb{Q}}Z$ and ${}^{\mathbb{Q}}U^{ij}Z$ respectively are local $\mathbb{P}-$martingales.

1.

$$\begin{aligned}
\widetilde{B}^{\mathbb{Q}}Z &= \int Z_- d\widetilde{B} + \int \widetilde{B}_- dZ + [Z, \widetilde{B}] \\
&\quad + \int\int Z_-\widetilde{B}(W^L(x) - \lambda W^M(x))(\mu_Y - \nu_Y) \\
&= \int Z_- dB \\
&\quad + \int \sum_i \sum_j \widetilde{B}Z_-({}^L\widetilde{\varphi}^{ij} - \lambda^M\widetilde{\varphi}^{ij})dU^{ij} \\
&\quad + \int\int Z_-\widetilde{B}(W^L(x) - \lambda W^M(x))(\mu_Y - \nu_Y)
\end{aligned}$$

2.

$$\begin{aligned}
Y^{\mathbb{Q}}Z &= \int Y_-^{\mathbb{Q}} dZ + \int Z_- dY^{\mathbb{Q}} + [Y^{\mathbb{Q}}, Z] \\
&= \int Y_-^{\mathbb{Q}} dZ + \int\int Z_- W^L(x)(\mu_Y - \nu_Y^{\mathbb{Q}}) + [L, Y^{\mathbb{Q}}] \\
&= \int Y_-^{\mathbb{Q}} dZ + \int\int Z_- W^L(x)(\mu_Y - \nu_Y) \\
&\quad + \int\int Z_- W^L(x)(W^L(x) - \lambda W^M(x))(\mu_Y - \nu_Y)
\end{aligned}$$

3.

$$\begin{aligned}
{}^{\mathbb{Q}}U^{ij}Z &= \int {}^{\mathbb{Q}}U_-^{ij} dZ + \int Z_-(dH^{ij} - {}^{\mathbb{Q}}\mu^{ij}H^i dt) \\
&\quad + \int \sum_i \sum_j Z_-({}^L\widetilde{\varphi}^{ij} - \lambda^M\widetilde{\varphi}^{ij})dH^{ij} \\
&= \int {}^{\mathbb{Q}}U_-^{ij} dZ + \int Z_- dU^{ij} + \int \sum_i \sum_j Z_-({}^L\widetilde{\varphi}^{ij} - \lambda^M\widetilde{\varphi}^{ij})dU^{ij}
\end{aligned}$$

Note that ${}^{\mathbb{Q}}U^{ij}_{-}, Y^{\mathbb{Q}}_{-}$ and Z_{-} are adapted processes which are left continuous with right limits, so ${}^{\mathbb{Q}}U^{ij}_{-}, Y^{\mathbb{Q}}_{-}$ and Z_{-} are locally bounded processes (page 166, Protter (2004)). Since ${}^{\mathbb{Q}}U^{ij}_{-}, Y^{\mathbb{Q}}_{-}$ and Z_{-} are locally bounded, we have that $Z\widetilde{B}^{\mathbb{Q}}, Y^{\mathbb{Q}}Z$ and ${}^{\mathbb{Q}}U^{ij}Z$ are local martingales since they are stochastic integrals with respect to local martingales, see Theorem 29, Chapter IV of Protter (2004). To finish off the proof, we note that the integrators are local $\mathbb{P}$-martingales. □

Proposition 6.5. *Let $\mathbb{Q} \in \mathcal{M}^e$. Then $Z_T = \mathcal{E}\left(-\int \lambda dM + L\right)_T$ is given explicitly as*

$$\begin{aligned}
Z_T \;=\; & \exp\Bigg(\int_0^T (\sigma_t^L - \lambda_t\sigma_t^M)dB - \int_0^T \int_{\mathbb{R}} (W_t^L(x) - \lambda_t W_t^M(x))\nu_Y \\
& -\frac{1}{2}\int_0^T (\sigma^L - \lambda\sigma^M)^2 dt - \int_0^T \int_{\mathbb{R}} \log(1 + W^L(x) - \lambda W^M(x))\mu_Y\Bigg) \\
& \exp\left\{\int_0^T \sum_i \sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} -{}^L \widetilde{\varphi}_t^{ij}\right)\mu^{ij} H_t^i dt\right\} \\
& \prod_i \prod_{\substack{j \\ i\neq j}} \prod_{0<t\leq T} \left(1 + \left(-\lambda_t^M \widetilde{\varphi}_t^{ij} +{}^L \widetilde{\varphi}_t^{ij}\right)\Delta H_t^{ij}\right)
\end{aligned} \tag{6.2}$$

Proof. We first evaluate $\mathcal{E}(N)$ where $N = -\int \lambda dM + L$

$$\begin{aligned}
N \;&=\; -\int \lambda dM + L \\
&=\; -\int \lambda\Big(\sum_i \sum_j {}^M\widetilde{\varphi}^{ij} dU^{ij} + \sigma^M dB + \int W^M(x)(\mu_Y - \nu_Y)\Big) \\
&\quad + \int {}^L\widetilde{\varphi}^{ij} dU^{ij} + \int \sigma^L + \int\int W^L(x)(\mu_Y - \nu_Y) \\
&=\; \int (\sigma^L dB - \lambda\sigma^M)dB + \int\int (W^L(x) - \lambda W^M(x))(\mu_Y - \nu_Y) \\
&\qquad + \int \sum_i \sum_j ({}^L\widetilde{\varphi}^{ij} - \lambda^M \widetilde{\varphi}^{ij})dU^{ij} \\
&=:\; M^\dagger + M^\ddagger
\end{aligned}$$

where

$$\begin{aligned}
M^\dagger \;&=:\; \int (\sigma^L - \lambda\sigma^M)dB + \int\int (W^L(x) - \lambda W^M(x))(\mu_Y - \nu_Y) \\
M^\ddagger \;&=:\; \int \sum_i \sum_j ({}^L\widetilde{\varphi}^{ij} - \lambda^M \widetilde{\varphi}^{ij})dU^{ij}
\end{aligned}$$

so that the Doléans-Dade exponential of N is given by

$$\mathcal{E}(N) = \mathcal{E}(M^\dagger + M^\ddagger) = \mathcal{E}(M^\dagger + M^\ddagger + [M^\dagger, M^\ddagger]) \overset{(b)}{=} \mathcal{E}(M^\dagger)\mathcal{E}(M^\ddagger)$$

where the equality ($\flat$) follows since $[M^\dagger, M^\ddagger] = 0$ by Yor's formula. Let us evaluate each of them separately, Note that $M^{\dagger c} = \int(\sigma^L - \lambda\sigma^M)dB$, $\langle M^{\dagger c}\rangle = \int(\sigma^L - \lambda\sigma^M)^2 dt$, $\Delta M^\dagger = (W^L(\Delta Y) - \lambda W^M(\Delta Y))\mathbb{1}_{\{\Delta Y \neq 0\}}$, we get

$$\begin{aligned}\mathcal{E}(M^\dagger)_T &= \exp\left(\int_0^T (\sigma_t^L - \lambda_t\sigma_t^M)dB - \int_0^T\int_{\mathbb{R}}(W_t^L(x) - \lambda_t W_t^M(x))\nu_Y\right.\\ &\left. -\frac{1}{2}\int_0^T(\sigma^L - \lambda\sigma^M)^2 dt - \int_0^T\int_{\mathbb{R}}\log(1 + W^L(x) - \lambda W^M(x))\mu_Y\right)\end{aligned}$$

Similarly, $\mathcal{E}(M^\ddagger)$ so that routine calculations yield

$$\begin{aligned}\mathcal{E}(M^\ddagger)_T &= \exp\left\{\int_0^T\sum_i\sum_j\left(\lambda_t^M\widetilde{\varphi}_t^{ij} -^L\widetilde{\varphi}_t^{ij}\right)\mu^{ij}H_t^i dt\right\}\times\\ &\prod_{\substack{i\\ i\neq j}}\prod_j\prod_{0<t\leq T}\left(1 + \left(-\lambda_t^M\widetilde{\varphi}_t^{ij} +^L\widetilde{\varphi}_t^{ij}\right)\Delta H_t^{ij}\right)\end{aligned}$$

where the last line follows from the fact that for a fixed t and for every $i \neq j$ and $k \neq l$, $(i,j) \neq (k,l)$, the processes H_t^{ij} and H_t^{kl} have no common jumps (see Proof of 22.9 in Rogers & Williams (2000)). Combining completes the proof. □

We now have the basic tools needed to proceed to our main goal: to find the entropy minimizing entropy measure. To do this, we find L via σ^L, W^L and ${}^L\widetilde{\varphi}^{ij}$ for $i \neq j$ such that the corresponding martingale measure $\overline{\mathbb{Q}}$ has the form of equation (2.1) and further proceed to carry out the Verification Procedures outlined in Section 2.2.

6.5 Fundamental equation for quasi-Markov additive processes

Theorem 6.6. *The strategy ϕ^E and the constant c^E in equation* (2.1) *satisfy the following equation*

$$\begin{aligned}c^E &= \int_0^T(\sigma_t^L - \lambda_t\sigma_t^M - \phi_t^E\sigma_t^M)dB - \int_0^T\left(\frac{1}{2}(\sigma_t^L - \lambda_t\sigma_t^M)^2 + \phi_t^E\lambda_t(\sigma_t^M)^2\right)dt\\ &- \int_0^T\int_{\mathbb{R}}(W_t^L(x) - \lambda_t W_t^M(x) + \phi_t^E\lambda_t(W_t^M(x))^2 - \phi_t^E W_t^M(x))\nu_Y\\ &- \int_0^T\sum_i\sum_j({}^L\widetilde{\varphi}_t^{ij} - \lambda_t{}^M\widetilde{\varphi}_t^{ij} + \phi_t^E\lambda_t({}^M\widetilde{\varphi}_t^{ij})^2 - \phi_t^{E\,M}\widetilde{\varphi}_t^{ij})dt \qquad (6.3)\\ &+ \int_0^T\int_{\mathbb{R}}(\log(1 + W_t^L(x) - \lambda_t W_t^M(x)) - \phi_t^E W_t^M(x))\mu_Y\\ &+ \int_0^T\sum_i\sum_j(\log(1 + {}^L\widetilde{\varphi}_t^{ij} - \lambda_t{}^M\widetilde{\varphi}_t^{ij}) - \phi_t^{E\,M}\widetilde{\varphi}_t^{ij})dH_t^{ij}\end{aligned}$$

where for every $i \neq j$, ${}^{L}\widetilde{\varphi}_t^{ij}$, $W_t^L(x)$ and σ_t^L has to be chosen such that for all $t \in [0,T]$

$$\sigma_t^M \sigma_t^L dt + \int_{\mathbb{R}} W_t^M(x) W_t^L(x) \nu_Y + \sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij} {}^{L}\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt = 0 \tag{6.4}$$

Proof. From Proposition 2.16, we see that L and $[M,L]$ are local $\mathbb{P}$−martingales. Observe that

$$[M,L]_\cdot = \int_0^\cdot \sigma_t^M \sigma_t^L dt + \int_0^\cdot \int_{\mathbb{R}} W_t^M(x) W_t^L(x) \mu_Y + \int_0^\cdot \sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij} {}^{L}\widetilde{\varphi}_t^{ij} dH_t^{ij} \tag{6.5}$$

We further know from Dellacherie & Meyer (1980) VII.39 that the predictable bracket process

$$\langle M,L\rangle_\cdot = \int_0^\cdot \sigma_t^M \sigma_t^L dt + \int_0^\cdot \int_{\mathbb{R}} W_t^M(x) W_t^L(x) \nu_Y + \int_0^\cdot \sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij} {}^{L}\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt \tag{6.6}$$

exists, since M and L are locally bounded. However $\langle M,L\rangle = 0$ because $[M,L]$ is a local martingale. By equation (2.1) we have

$$\begin{aligned}
\log Z_T &= c^E + \int_0^T \phi_t^E dS_t \\
&= c^E + \int_0^T \sum_i \sum_j \phi_t^E {}^{M}\widetilde{\varphi}_t^{ij} dU^{ij} + \int_0^T \phi_t^E \sigma_t^M dB \\
&\quad + \int_0^T \int_{\mathbb{R}} \phi_t^E W_t^M(x)(\mu_Y - \nu_Y) + \int_0^T \phi_t^E \lambda_t ({}^{M}\widetilde{\varphi}_t^{ij})^2 \mu^{ij} H_t^i dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \phi_t^E (W_t^M(x))^2 \nu_Y + \int_0^T \phi_t^E \lambda_t (\sigma_t^M)^2 dt
\end{aligned}$$

Equating equations (2.1) and (6.2) while using the martingale $U_t^{ij} = H_t^{ij} - \int_0^t \mu^{ij} H_s^i ds$ yields equation (6.3). □

6.6 Quasi-Markov additive processes: a candidate as an entropy minimizer

Let us consider a stock price process which is driven by a quasi-Markov additive process:

$$\mathbb{P}: \quad \frac{dS_t}{S_{t-}} = \eta(C_{t-})dt + \sigma(C_{t-})dB_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y)_t \tag{6.7}$$

where $\nu_Y(dt,dx) := \nu_{C_{t-}}(dx)dt$. The above Markov additive process (6.7) can be recast as

$$\frac{dS_t}{S_{t-}} = \sum_i H_{t-}^i \eta^i dt + \sum_i H_{t-}^i \sigma^i dB_t + \int_{\mathbb{R}} \sum_i H_{t-}^i f(x)(\mu_Y - \nu^i(dx)dt))$$

where $\eta^i := \eta(i)$ and $\sigma^i := \sigma(i)$ are constants for a fixed $i \in \mathcal{Y}$. We work under the following assumptions

Assumption 6.7.
1. For each $i \in \mathcal{Y}$, $\nu^i(\mathbb{R}) < \infty$.
2. $f(\cdot) : supp(\nu) \to (-1,\infty)$ is uniformly bounded. This is needed to ensure the local boundedness of S.
3. For a fixed $i \in \mathcal{Y}$, $\sigma^i > 0$.

Given the price process (6.7) and our fundamental entropy equation (6.3), we see that for every $i \in \mathcal{Y}$ and $i \neq j$, $\sigma_t^M = \sigma(C_{t-})S_{t-}$, $W_t^M(x) = f(x)S_{t-}$ and

$$\begin{aligned} \lambda_t &= \frac{\eta(C_{t-})}{S_{t-}\left(\sigma^2(C_{t-}) + \int_{\mathbb{R}} f^2(x)\nu_{C_{t-}}(dx)\right)} \\ \Longleftrightarrow \quad \lambda_t S_{t-} &= \frac{\eta(C_{t-})}{\sigma^2(C_{t-}) + \int_{\mathbb{R}} f^2(x)\nu_{C_{t-}}(dx)} \end{aligned} \tag{6.8}$$

Observe that the RHS of equation (6.8) is a function of the chain C. This observation hints us to define

$$\widehat{\lambda}(C_{t-}) := \lambda_t S_{t-} = \frac{\eta(C_{t-})}{\sigma^2(C_{t-}) + \int_{\mathbb{R}} f^2(x)\nu_{C_{t-}}(dx)} \tag{6.9}$$

or equivalently for every $i \in \mathcal{Y}$,

$$\lambda_t S_{t-} = \sum_i H_{t-}^i \left(\frac{\eta^i}{(\sigma^i)^2 + \int_{\mathbb{R}} f^2(x)\nu^i(dx)}\right) = \sum_i H_{t-}^i \widehat{\lambda}^i \tag{6.10}$$

as a consequence of (6.9) where

$$\widehat{\lambda}^i := \frac{\eta^i}{(\sigma^i)^2 + \int_{\mathbb{R}} f^2(x)\nu^i(dx)} \tag{6.11}$$

with the usual shorthand $\widehat{\lambda}^i := \widehat{\lambda}(i)$. Similarly, we have $\sigma_t^M = \sum_i H_{t-}^i \sigma^i S_{t-}$. Let us introduce the following notations.

Definition 6.8.
1.The space $\overline{C_b} := C_b([0,T] \times \{1,...,m\})$ consists of continuous bounded functions $u : [0,T] \times \{1,...,m\} \to \mathbb{R}$.
2. For any $u \in \overline{C_b}$, $u_t := u(t,\cdot) : \mathcal{Y} \to \mathbb{R}$.
3. For any $u \in \overline{C_b}$, $\Delta u_t = \Delta u_t(C_{t-}) := u(t,C_t) - u(t,C_{t-})$.

We proceed with the *ansatz* that there exists a sufficiently smooth and bounded function $u \in \overline{C_b}$ such that

$$\log(1 + {}^L\widetilde{\varphi}^{C_{t-},C_t}) = \Delta u_t(C_{t-}) \tag{6.12}$$

Furthermore we set

$$u(T,\cdot) = 0 \quad \text{on } \mathcal{Y}. \tag{6.13}$$

We see that each $u \in \overline{C_b}$ defines via Δu a unique bounded function

$$
{}^{L}\widetilde{\varphi}^{C_-,C} := {}^{L}\widetilde{\varphi}^{C_-,C}(u) : [0,T] \times \mathbb{R} \to (-1,\infty) \tag{6.14}
$$

which fulfills equation (6.12). Note that ${}^{L}\widetilde{\varphi}$ is bounded because $|\Delta u| \leq K^*$ for some constant K^* since each $u \in \overline{C_b}$ is bounded. Recall that ${}^{M}\widetilde{\varphi}^{ij} = 0$. With the *ansatz* in (6.12) observe that we can write the last term on the RHS of equation (6.3) as

$$
\begin{aligned}
\int_0^T \sum_i \sum_j \log(1 + {}^{L}\widetilde{\varphi}^{ij}) dH_t^{ij} &= \sum_{0<t\leq T} \sum_i \sum_j (u(t,j) - u(t,i)) \Delta H_t^{ij} \\
&= -u(0,C_0) - \int_0^T \sum_i H_{t-}^i \frac{\partial}{\partial t} u(t,i) dt
\end{aligned}
$$

In light of equation (6.9), we postulate that $\phi^E S_-$ is of the form $\widehat{\phi}(C_-)$ and this inspires us to define $\widehat{\phi}(C_-) := \phi^E S_-$. This definition allows us to recast $\widehat{\phi}(C_-)$ as

$$
\widehat{\phi}(C_{t-}) = \sum_i H_{t-}^i \widehat{\phi}^i, \quad \text{with} \quad \widehat{\phi}^i := \widehat{\phi}(i)
$$

We can then rewrite the minimal entropy martingale measure equation (6.3) as

$$
\begin{aligned}
& c^E + u(0,C_0) \qquad\qquad (6.15) \\
&= -\int_0^T (\sigma_t^L - \widehat{\lambda}(C_{t-})\sigma(C_{t-})\widehat{\phi}(C_{t-})\sigma(C_{t-}))dB_t \\
&\quad - \int_0^T \int_{\mathbb{R}} (W_t^L(x) - \widehat{\lambda}(C_{t-})f(x) + \widehat{\lambda}(C_{t-})f^2(x) - \widehat{\phi}(C_{t-})f(x))\nu_Y \\
&\quad - \int_0^T \left(\frac{1}{2}(\sigma_t^L - \widehat{\lambda}(C_{t-})\sigma(C_{t-}))^2 + \widehat{\lambda}(C_{t-})\widehat{\phi}(C_{t-})\sigma^2(C_{t-}) \right) dt \\
&\quad - \int_0^T \sum_i \sum_j {}^{L}\widetilde{\varphi}^{ij} \mu^{ij} H_{t-}^i dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \log(1 + W_t^L(x) - \widehat{\lambda}(C_{t-})f(x)) - \widehat{\phi}(C_{t-})f(x))\mu_Y \\
&\quad - \int_0^T \frac{\partial}{\partial t} u(t, C_{t-}) dt \qquad\qquad (\heartsuit\heartsuit)
\end{aligned}
$$

To ensure that the RHS of equation ($\heartsuit\heartsuit$) is non-stochastic, one possible avenue is to choose

$$
\sigma_t^L = \sigma^L(C_{t-}) = (\widehat{\lambda}(C_{t-}) + \widehat{\phi}(C_{t-}))\sigma(C_{t-}) \tag{6.16}
$$

$$
W_t^L(x) = W^{L,C_{t-}}(x) = \widehat{\lambda}(C_{t-}) - 1 + \exp(\widehat{\phi}(C_{t-})f(x)) \tag{6.17}
$$

and

$$\frac{\partial}{\partial t}u(t,C_{t-})+\frac{1}{2}\left(\sigma_t^L-\widehat{\lambda}\sigma\right)^2+\widehat{\phi}\widehat{\lambda}\sigma^2+\sum_j(e^{u(t,j)-u(t,C_{t-})}-1)\mu^{ij}$$
$$+\int_{\mathbb{R}}\left(W^{L,C_{t-}}(x)-(\widehat{\phi}+\widehat{\lambda})f(x)+\widehat{\phi}\widehat{\lambda}f^2(x)\right)\nu^i(dx) \;=\; 0 \tag{6.18}$$

where we have used the *ansatz* from equation (6.12). To make expressions more compact, note that for every $i\in\mathcal{Y}$ equations (6.16), (6.17) and (6.18) can be recast as

$$\sigma^{L,i}=(\widehat{\lambda}^i+\widehat{\phi}^i)\sigma^i \tag{6.19}$$
$$W^{L,i}=\widehat{\lambda}^i-1+\exp(\widehat{\phi}^i f(x)) \tag{6.20}$$

and

$$\frac{\partial}{\partial t}u(t,i)+\frac{1}{2}(\sigma^{L,i}-\widehat{\lambda}^{L,i}\sigma^i)^2+\widehat{\phi}^{L,i}\widehat{\lambda}^{L,i}(\sigma^i)^2+\sum_j(e^{u(t,j)-u(t,i)}-1)\mu^{ij}$$
$$+\int_{\mathbb{R}}(W^{L,i}-(\widehat{\phi}^{L,i}+\widehat{\lambda}^{L,i})f(x)+\widehat{\phi}^{L,i}\widehat{\lambda}^{L,i}f^2(x)\nu^i(dx)=0 \tag{6.21}$$

respectively together with (6.13) and

$$c^E=-u\,(0,C_0)$$

We introduce

$$g^i(t,u_t):=\frac{1}{2}(\sigma^{L,i}-\widehat{\lambda}^{L,i}\sigma^i)^2+\widehat{\phi}^{L,i}\widehat{\lambda}^{L,i}(\sigma^i)^2+\sum_j(e^{u(t,j)-u(t,i)}-1)\mu^{ij}$$
$$+\int_{\mathbb{R}}(W^{L,i}-(\widehat{\phi}^{L,i}+\widehat{\lambda}^{L,i})f(x)+\widehat{\phi}^{L,i}\widehat{\lambda}^{L,i}(f(x))^2) \tag{6.22}$$

and arrive at an ordinary differential equation for u of the form

$$\begin{aligned}\frac{\partial}{\partial t}u(t,i)+g^i(t,u_t) &= 0\\ u\,(T,i) &= 0 \quad \text{for every } i\in\mathcal{Y}\end{aligned}$$

We further get using our *ansatz* that upon transition from i to j, $i\neq j$,

$${}^L\widetilde{\varphi}^{ij}=e^{u(t,j)-u(t,i)}-1$$

Replacing the above ${}^L\widetilde{\varphi}^{ij}_t$ in equation (6.4) culminates in the following condition for $\widehat{\phi}$:

$$\int_0^{\cdot} S_{t-}H^i_{t-}\left\{\widehat{\lambda}^i\sigma^i+\widehat{\phi}^i\sigma^i+\int_{\mathbb{R}}f(x)(e^{\widehat{\phi}^i f(x)}+\widehat{\lambda}^i f(x)-1)\nu^i(dx)\right\}=0$$

We see that the above is true if the following holds

$$\eta^i+(\sigma^i)^2\widehat{\phi}^i+\int_{\mathbb{R}}f(x)(e^{\widehat{\phi}^i f(x)}-1)\nu^i(dx)=0 \quad \text{for every } i\in\mathcal{Y} \tag{6.23}$$

We shall call equation (6.23) the $\overline{\mathbb{Q}}$*−martingale equation.* Hence any strategy ϕ^E that fulfills the $\overline{\mathbb{Q}}$*−martingale equation* is a potential candidate for the minimal entropy martingale measure.

6.6.1 Existence of solutions

Let us discuss the existence of $\widehat{\phi}$ in equation (6.23). We present the following:

Lemma 6.9. *For each $i \in \mathcal{Y}$ there exists a $\widehat{\phi}^i_* \in \mathbb{R}$ which solves the $\overline{\mathbb{Q}}-$martingale equation* (6.23).

Proof. For a fixed $i \in \mathcal{Y}$, we introduce two functions m^i_1, m^i_2 with

$$\begin{aligned} m^i_1 &: \mathbb{R} \to \mathbb{R} \\ &\widehat{\phi}^i \to \eta^i + (\sigma^i)^2 \widehat{\phi}^i \end{aligned}$$

$$\begin{aligned} m^i_2 &: \mathbb{R} \to \mathbb{R} \\ &\widehat{\phi}^i \to -\int_{\mathbb{R}} f(x)(e^{\widehat{\phi}^i f(x)} - 1)\nu^i(dx) \end{aligned}$$

One sees that for each $i \in \mathcal{Y}$, m^i_1 is continuous, $\uparrow$ with

$$\lim_{\widehat{\phi}^i \searrow -\infty} m^i_1 = -\infty \qquad \lim_{\widehat{\phi}^i \nearrow \infty} m^i_1 = \infty$$

On the other hand, m^i_2 is continuous, $\downarrow$ with

$$(m^i_2)' = -\int_{\mathbb{R}} (f(x))^2 e^{\widehat{\phi}^i f(x)} \nu^i(dx) < 0$$

so that $m^i_2 > 0$ when $\widehat{\phi}^i < 0$; $m^i_2 < 0$ when $\widehat{\phi}^i > 0$ and $m^i_2 = 0$ when $\widehat{\phi}^i = 0$. Hence we see that for each $i \in \mathcal{Y}$, $\exists \widehat{\phi}^i_* \in \mathbb{R}$ such that $m^i_1(\widehat{\phi}^i_*) = m^i_2(\widehat{\phi}^i_*)$. We see that $\widehat{\phi}^i_*$ is a constant for a fixed $i \in \mathcal{Y}$. Note that $\widehat{\phi}_*$ is a function of the chain C as well, i.e. $\widehat{\phi}_*(C)$. □

The next question which naturally arises is that of the existence of u. The following theorem provides an existence result:

Theorem 6.10. *Let us consider the differential equation with boundary condition:*

$$\frac{\partial}{\partial t} u(t,i) + g^i(t, u_t) = 0 \tag{6.24}$$

$$u\,(T,i) = 0 \tag{6.25}$$

for every $i \in \mathcal{Y}$. We shall assume that g is a Lipschitz-continuous function in the second argument, uniformly in t, i.e. $\exists$ a constant $c < \infty$ such that

$$|g(t, w_t) - g(t, z_t)| \leq c|w_t - z_t| \tag{6.26}$$

then there exists a unique solution $\widehat{u} \in C_b([0,T] \times \{1, ..., m\})$ which solves the boundary problem (6.24)-(6.25). *It can be written as*

$$\widehat{u}\,(t,i) = \int_t^T g^i\,(s, \widehat{u}_s)\, ds \qquad \textit{for every} \quad i \in \mathcal{Y}$$

Proof. Note that

$$u(t,i) = \int_t^T g^i(s,u_s)ds$$

solves the boundary problem (6.24)-(6.25). Introduce the operator $\Phi : \mathcal{C}_b([0,T]\times\{1,...,m\}) \to \mathcal{C}_b([0,T] \times \{1,...,m\})$ which is defined as follows:

$$\Phi[u](t,i) := \int_t^T g^i(s,u_s)ds \quad \text{for every } i \in \mathcal{Y}$$

Then the above equation is simply $\Phi[u] = u$ and any solution to this equation must be a fixed point of Φ. Let us consider the norm

$$\|u\|_L := \sup_{(t,i)\in[0,T]\times\{1,...,m\}} e^{-L(T-t)}|u(t,i)|$$

which is equivalent to the supremum-norm $\|u\|_\infty$. Due to condition (6.26), we obtain for $u, v \in \overline{\mathcal{C}_b}$

$$\begin{aligned}
& e^{-L(T-t)}|\Phi[u](t,i) - \Phi[v](t,i)| \\
= \; & e^{-L(T-t)}\left|\int_t^T (g^i(s,u_s) - g^i(s,v_s))ds\right| \\
\leq \; & e^{-L(T-t)}\int_t^T |g^i(s,u_s) - g^i(s,v_s)|e^{-2L(T-s)}e^{L(T-s)}ds \\
\leq \; & e^{-L(T-t)}c\|u-v\|_L\int_t^T e^{L(T-s)}ds \\
\leq \; & c\frac{\|u-v\|_L}{L}(1-e^{-L(T-t)}) \\
\leq \; & \frac{c}{L}\|u-v\|_L
\end{aligned}$$

for all $t \in [0,T]$ and $i \in \{1,...,m\}$. Thus Φ is a contraction on the normed space $(\overline{\mathcal{C}_b}, \|\cdot\|)$ with contraction constant $c/_L$ with $c < L$. Therefore there exists a unique fixed point $\widehat{u} \in \mathcal{C}_b([0,T] \times \{1,...,m\})$ which satisfies (6.24) – (6.25). □

However, the function g defined in equation (6.22) is in general not Lipschitz continuous. We define

$$\widetilde{g}^i(t,u_t) := g^i(t,\kappa(t,u_t))$$

where $\kappa : [0,T] \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\kappa(t,x) := \min(\max(x, -(T-t)L), (T-t)L).$$

It turns out that our function g defined in equation (6.22) matches that of the equation (2.23) in Becherer & Schweizer (2005). In fact their equation generalizes ours since they allow for a claim. They showed that $\widetilde{g}$ is Lipschitz so we can conclude using their results that there is a unique bounded solution $\widehat{u}$ with $\widetilde{g}$ instead of g. Moreover $\widehat{u}$ is the fixed point of Φ defined with $\widetilde{g}$ instead of g. We shall show that there exists a constant L such that for all $(t,i) \in [0,T] \times \{1,...,m\}$, $|\widehat{u}(t,i)| \leq (T-t)L$. To prove this, we fix an arbitrary $(t,i) \in [0,T] \times \{1,...,m\}$ and examine the quantity

$$\tau := \inf\{s \in [t,T] : \widehat{u}(s,C_s) < (T-s)L\} \wedge T$$

By the definition of τ, we get $\widehat{u}(s,C_s) \geq (T-s)L$ for $s \in [t,\tau)$ and $\widehat{u}(\tau,C_\tau) \leq (T-\tau)L$. From the truncation function, we see that for $s \in [t,\tau)$, $g(s,\kappa(s,\widehat{u}_s)) = \widetilde{g}(s,\widehat{u}_s) \leq L^{**}$ for some constant L^{**}, so that

$$\begin{aligned}
\widehat{u}(t,i) &= \int_t^T \widetilde{g}(s,\widehat{u}_s)ds \\
&= \int_t^\tau \widetilde{g}(s,\widehat{u}_s)ds + \int_\tau^T \widetilde{g}(s,\widehat{u}_s)ds \\
&\leq L^{**}(\tau-t) + \widehat{u}(\tau,C_\tau) \\
&\leq L^{**}(\tau-t) + (T-\tau)L
\end{aligned}$$

The choice of $L \geq L^{**}$ yields

$$\widehat{u}(t,i) \leq L(T-t).$$

Hence $\widehat{u}$ also solves the differential equation with g.

6.7 Entropy minimizer for quasi-Markov additive processes

Having proved the existence of a solution to the differential equations (6.24) and (6.25) we can thus determine the quadruplet $(\widehat{\phi}^i_*, {}^L\widetilde{\varphi}^{ij}, W^{L,i}, \sigma^{L,i})$ which solves equation (6.3). We will now show that the candidate measure outlined in Section 6.6 is indeed the minimal entropy martingale measure.

Proposition 6.11. *Let $M' := \int\int(\exp(f(x))-1)(\mu_Y - \nu_Y)$ where $\nu_Y(dx,dt) = \nu_{C_{t-}}(dx)dt$. Then we have the following bound*

$$\mathbb{E}_\mathbb{P}\left[\exp\left(\int_0^t \int_\mathbb{R} f(x)\mu_Y\right)\right] \leq \exp\left(t\int_\mathbb{R} \exp(f(x))\sum_i \nu^i(dx)\right) < \infty$$

Proof. Observe that M' is local martingale. Let us now consider the stochastic differential equation

$$dZ'_t = Z'_{t-}dM'_t, \qquad Z'_0 = 1 \tag{6.27}$$

We know that its solution is given by

$$Z'_t = \exp\left\{-\int_0^t\int_\mathbb{R}(\exp(f(x))-1)\nu_Y + \int_0^t\int_\mathbb{R}\log f(x)\mu_Y\right\}$$

Note that Z'_- is an adapted process which is left continuous with right limits, so Z'_- is locally bounded (page 166, Protter (2004)). Since Z'_- is locally bounded, we have that Z' is a local martingale since it is by equation (6.27) the stochastic integral with respect to M' which is a local martingale (Theorem 29, Chapter IV of Protter (2004)). Observe that Z' is non-negative and by Fatou's Lemma we can conclude that Z' is a supermartingale. Hence

$$\begin{aligned} 1 = \mathbb{E}_{\mathbb{P}}[Z'_0] &\geq \mathbb{E}_{\mathbb{P}}[Z'_t] \\ &\geq \exp\left(-\left\{t\int_{\mathbb{R}}(\exp(f(x))-1)\sum_i \nu^i(dx)\right\}\right)\times \\ &\qquad \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t\int_{\mathbb{R}} f(x)\mu_Y\right)\right] \end{aligned}$$

Rearranging yields the result. □

Proposition 6.12. *Define the following processes* $M^+ := \int\int f(x)(\mu_Y - \nu_Y)$, $M^- := \int \widehat{\phi}_*(C_-)\sigma(C_-)dB$ *and* $\nu_Y^{\overline{\mathbb{Q}}} := \exp\left(\widehat{\phi}_*(C_-)f(x)\right)\nu_Y$. *Then*

$$\widetilde{M}^+ := \int\int f(x)(\mu_Y - \nu_Y^{\overline{\mathbb{Q}}}), \qquad \widetilde{M}^- := \int \widehat{\phi}_*(C_-)\sigma(C)(dB - \widehat{\phi}_*(C_-)\sigma(C_-))dt$$

are local $\overline{\mathbb{Q}}$*−martingales.*

Proof. Note that the density $Z = \frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}$ may be written as $c^E + \int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}}dS_t$. Further note that M^+ and M^- are local $\mathbb{P}$−martingales. By Girsanov's Theorem, we have that $M^+ - \int \frac{1}{Z_-}d\langle Z, M^+\rangle$ is a local $\overline{\mathbb{Q}}$−martingale. Since $\int\int_{\mathbb{R}} \widehat{\phi}_*(C_-)(\nu_Y^{\overline{\mathbb{Q}}} - \nu_Y) < \infty$ we may write

$$\begin{aligned} M^+ - \int \frac{1}{Z_-}d\langle Z, M^+\rangle &= \int\int f(x)(\mu_Y - \nu_Y) - \int\int f(x)(e^{\widehat{\phi}_*(C_-)f(x)} - 1)\nu_Y \\ &= \int f(x)\mu_Y - \int\int f(x)(e^{\widehat{\phi}_*(C_-)f(x)} - 1)\nu_Y \\ &\quad - \int\int f(x)\nu_Y \\ &= \int\int f(x)(\mu_Y - \nu_Y^{\overline{\mathbb{Q}}}) = \widetilde{M}^+ \end{aligned}$$

By the same token, we know that $M^- - \int \frac{1}{Z_-}d\langle Z, M^-\rangle$ is a local $\overline{\mathbb{Q}}$−martingale. Now

$$\begin{aligned} M^- - \int \frac{1}{Z_-}d\langle Z, M^-\rangle &= \int \widehat{\phi}_*(C_-)\sigma(C_-)dB - \int (\widehat{\phi}_*(C_-))^2\sigma^2(C_-)dt \\ &= \int \widehat{\phi}_*(C_-)\sigma(C_-)(dB - \widehat{\phi}_*(C_-)\sigma(C_-)dt) \\ &=: \widetilde{M}^- \end{aligned}$$

□

We are now ready for our main theorem.

Theorem 6.13 (Main Theorem). *Let the conditions of Assumption* 6.7 *holds. Let the quadruplet* $(\widehat{\phi}^i_*, {}^L\widetilde{\varphi}^{ij}, W^{L,i}, \sigma^{L,i})$ *solves* (6.3)-(6.4). *Then the process* $Z^{\mathbb{Q}^E} = (Z^{\mathbb{Q}^E}_t)_{t\in[0,T]}$ *defined by*

$$Z^{\mathbb{Q}^E}_t = \left.\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \mathcal{E}(N)_t$$

where N *takes the following form*

$$\begin{aligned} N_t &:= \int_0^t \sum_i H^i_{s-}\widehat{\phi}^i_*\sigma^i dB_s + \int_0^t\int_{\mathbb{R}}\sum_i H^i_{s-}(e^{\widehat{\phi}^i_* f(x)} - 1)(\mu(dx,ds) - \nu^i(dx)ds) \\ &\quad + \int_0^t \sum_i\sum_j (e^{\widehat{u}(s,j)-\widehat{u}(s,i)} - 1)dU^{ij}_s \end{aligned}$$

is the density process of the minimal entropy martingale measure for quasi-Markov additive process.

Proof. We now carry out the verfications as outlined in Section 2.2.

1: $\overline{\mathbb{Q}}$ is an equivalent probability measure:

Let us check the conditions of Theorem 2.8. Let the local martingale N be defined as follows:

$$\begin{aligned} N_t &:= -\int_0^t \lambda_s dM_s + L_t \\ &= \int_0^t \sum_i H^i_{s-}\widehat{\phi}^i_*\sigma^i dB_s \\ &\qquad + \int_0^t\int_{\mathbb{R}}\sum_i H^i_{s-}(e^{\widehat{\phi}^i_* f(x)} - 1)(\mu(dx,ds) - \nu^i(dx)ds) \\ &\qquad\qquad + \int_0^t \sum_i\sum_j (e^{\widehat{u}(s,j)-\widehat{u}(s,i)} - 1)dU^{ij}_s \\ &=: N^\dagger_t + N^\ddagger_t \end{aligned}$$

where

$$\begin{aligned} N^\dagger_t &:= \int_0^t \sum_i H^i_{s-}\widehat{\phi}^i_*\sigma^i dB_s \\ &\qquad + \int_0^t\int_{\mathbb{R}}\sum_i H^i_{s-}(e^{\widehat{\phi}^i_* f(x)} - 1)(\mu(dx,ds) - \nu^i(dx)ds) \\ N^\ddagger_t &:= \int_0^t \sum_i\sum_j (e^{\widehat{u}(s,j)-\widehat{u}(s,i)} - 1)dU^{ij}_s \end{aligned}$$

Note that N is locally bounded since f and $\widehat{\phi}_*^i$ for $i \in \mathcal{Y}$ are bounded. We will show separately that $\mathcal{E}(N^\dagger)$ and $\mathcal{E}(N^\ddagger)$ are uniformly integrable $\mathbb{P}$−martingales and conclude that their product is also uniformly integrable $\mathbb{P}$−martingale. We have to show that

$$\begin{aligned} U_t^\dagger &= \frac{1}{2}\int_0^t \sum_i H_{s-}^i (\widehat{\phi}_*^i \sigma^i)^2 ds \\ &\qquad + \int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i (\widehat{\phi}_*^i f(x) e^{\widehat{\phi}_*^i f(x)} - e^{\widehat{\phi}_*^i f(x)} + 1)\mu(ds, dx) \end{aligned}$$

admits a predictable compensator. To verify this claim, note that

$$\int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i \left| \widehat{\phi}_*^i f(x) e^{\widehat{\phi}_*^i f(x)} - e^{\widehat{\phi}_*^i f(x)} + 1 \right| \nu^i(dx) ds < \infty$$

due to the boundedness of f and $\widehat{\phi}_*^i$ together with our standing assumption that $\nu^i(\mathbb{R}) < \infty$. From Theorem II.1.28 of Jacod & Shiryaev (2003), this implies the integrability of $\widehat{\phi}_*^i f(x) \exp\{\widehat{\phi}_*^i f(x)\} - \exp\{\widehat{\phi}_*^i f(x)\} + 1 =: \widetilde{f}_i(x,t)$ with respect to $\mu_Y - \nu_Y$ and

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i \widetilde{f}_i(x,s)(\mu(ds,dx) - \nu^i(dx)ds) \\ &= \int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i \widetilde{f}_i(x,s)\mu(ds,dx) - \int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i \widetilde{f}_i(x,s)\nu^i(dx)ds \end{aligned}$$

This yields that the $\mathbb{P}$−compensator $B^\dagger$ of $U^\dagger$ is given by

$$B_t^\dagger = \frac{1}{2}\int_0^t \sum_i H_{s-}^i (\widehat{\phi}_*^i \sigma^i)^2 ds + \int_0^t \int_{\mathbb{R}} \sum_i H_{s-}^i \widetilde{f}_i(x,s)\nu^i(dx)ds$$

It is easily seen that $\mathbb{E}_{\mathbb{P}}[\exp(B_T^\dagger)] < \infty$. This concludes that $\mathcal{E}(M^\dagger)$ is a uniformly integrable $\mathbb{P}$-martingale. We now turn to show that $\mathcal{E}(M^\ddagger)$ is also a uniformly integrable $\mathbb{P}$-martingale. By the same reasoning, observe that

$$\int_0^t \sum_i \sum_j \left| e^{\Delta \widehat{u}_s} \Delta \widehat{u}_s - (e^{\Delta \widehat{u}_s} - 1) \right| \mu^{ij} \sum_i H_{s-}^i ds < \infty$$

since $\widehat{u} \in \mathcal{C}_b$ so $\Delta\widehat{u}$ is bounded and that μ^{ij} and H^i are also bounded as well as the fact that there are a finite number of states for the chain. Hence $U^\ddagger$ admits a predictable compensator $B^\ddagger$ and we have $\mathbb{E}_{\mathbb{P}}[B_T^\ddagger] < \infty$ so that $\mathcal{E}(M^\ddagger)$ is a uniformly integrable $\mathbb{P}$−martingale. Hence $\mathcal{E}(N)$ is also a uniformly integrable $\mathbb{P}$−martingale.

2. $\overline{\mathbb{Q}}$ is a martingale measure.

   Since W^L, σ^L and $\widehat{\phi}_*$ are bounded, L and $[M, L]$ are locally bounded. Hence, due to Proposition 6.4, $\overline{\mathbb{Q}}$ is a martingale measure.

3: $I\left(\overline{\mathbb{Q}}, \mathbb{P}\right) < \infty$

Recall that the density $Z = \frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}$ may be expressed as $\exp\left(c^E + \int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}} dS_t\right)$. We have

$$\begin{aligned} I(\overline{\mathbb{Q}}, \mathbb{P}) &= \mathbb{E}_{\overline{\mathbb{Q}}}\left[c^E + \int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}} dS_t\right] \\ &= \mathbb{E}_{\overline{\mathbb{Q}}}\left[c^E\right] + \mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T \widehat{\phi}_*(C_{t-})(\eta(C_{t-})dt + \sigma(C_{t-})dB_t)\right] \\ &\quad + \mathbb{E}_{\overline{\mathbb{Q}}}\left[\widehat{\phi}_*(C_{t-})f(x)(\mu_Y - \nu_Y)\right] \end{aligned}$$

Recall that $\nu_Y^{\overline{\mathbb{Q}}} := \exp\left(\widehat{\phi}_*(C_-)f(x)\right)\nu_Y$. Let us evaluate the third term on the right hand side of the above equation:

$$\begin{aligned} &\mathbb{E}_{\overline{\mathbb{Q}}}\left[\widehat{\phi}_*(C_{t-})f(x)(\mu_Y - \nu_Y)\right] \\ &= \mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T -\widehat{\phi}_*(C_{t-})\eta(C_{t-}) - (\sigma(C_{t-})\widehat{\phi}_*(C_{t-}))^2 dt\right] \end{aligned}$$

on the other hand, due to Proposition 6.12 we can write

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T \widehat{\phi}_*(C_{t-})\sigma(C_{t-})dB - \int_0^T (\widehat{\phi}_*(C_{t-})\sigma(C_{t-}))^2 dt\right] = 0$$

or equivalently,

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T \widehat{\phi}_*(C_{t-})\sigma(C_{t-})dB\right] = \mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T (\widehat{\phi}_*(C_{t-})\sigma(C_{t-}))^2 dt\right].$$

Finally we get

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[\int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}} dS_t\right] = 0$$

which, together with Assumption 6.1 gives

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[c^E + \int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}} dS_t\right] = \mathbb{E}_{\overline{\mathbb{Q}}}[c^E] = c^E = u(0, C_0) < \infty$$

4: $\int \phi^E dS$ is a true $\mathbb{Q}$−martingale for all $\mathbb{Q} \in \mathcal{M}^e$ with $I\left(\mathbb{Q}, \mathbb{P}\right) < \infty$

We require $\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\beta \int_0^T \psi^2 d\left[S, S\right]_t\right\}\right] < \infty$ for some $\beta > 0$. From Proposition 2.9, we have to show that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\beta \int_0^T \frac{(\widehat{\phi}_*(C_{t-}))^2}{S_{t-}^2} d[S]_t\right\}\right] < \infty$$

Denote $k = \sup_{t\in[0,T]} \|\widehat{\phi}_*(C_{t-})\|_{\mathcal{L}^\infty}$. Let us choose $\beta = \frac{1}{k^2} > 0$ so that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\beta\int_0^T \frac{(\widehat{\phi}_*(C_{t-}))^2}{S_{t-}^2}d[S]_t\right\}\right]\\
\leq\ & \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^T (\sigma(C_{t-}))^2dt + \int_0^T\int_{\mathbb{R}} f(x)\mu_Y\right)\right]\\
=\ & \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^T \sum_i H_{t-}^i(\sigma^i)^2dt + \int_0^T\int_{\mathbb{R}} f(x)\mu_Y\right)\right]\\
\leq\ & \exp\left(\sup_{i,j\in\mathcal{Y}} \|\sigma^i\|_\infty^2 mT\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^T\int_{\mathbb{R}} f(x)\mu_Y\right)\right]\\
<\ & \infty
\end{aligned}$$

by Proposition 6.11.

Hence we conclude that

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} = \exp\left(c^E + \int_0^T \frac{\widehat{\phi}_*(C_{t-})}{S_{t-}}dS_t\right)$$

fulfills all the conditions necessary for being the minimal entropy martingale measure, i.e. $\overline{\mathbb{Q}} = \mathbb{Q}^E$.

□

6.8 Integro-differential equations for derivative prices

In this section we investigate the pricing of derivatives when the stock price process follows a Markov additive process under the minimal entropy martingale measure. We illustrate how we can utilize the minimal entropy martingale measure for pricing issues. We begin by considering an arbitrary non-negative, $\mathcal{G}_T$−measurable claim $H \in L^2(\mathbb{Q}^E)$ of the form $H = g(S_T)$ with a maturity $T > 0$. Define the $\mathbb{Q}^E$ martingale $\widetilde{V}^{\mathbb{Q}^E}$ by

$$\begin{aligned}
\widetilde{V}_t^{\mathbb{Q}^E} := \mathbb{E}_{\mathbb{Q}^E}\left[g(S_T)\,|\mathcal{G}_t\right] &\overset{(i)}{=} \mathbb{E}_{\mathbb{Q}^E}\left[g(S_T)|S_t, C_t\right] = V(t, S_t, C_t) =: V^{C_t}(t, S_t)\\
&\overset{(ii)}{=} \sum_i H_t^i V^i(t, S_t)
\end{aligned}$$

The process $\widetilde{V}^{\mathbb{Q}^E}$ is referred to as the *intrinsic value process* in the literature, see Föllmer & Sondermann (1986). We note that (i) is due to the fact that the relevant state variables involved in the conditional value are (t, S_t, C_t). This is due to the Markov property of (C, S) which states that given the present value, the future and the past are independent. The pair (S, C) is Markov because the pair of stochastic differential equations for S and C involves only these processes. It is then easier to rewrite the expectation in terms of (ii), where $V^i(t, s)$ represents the state-wise intrinsic value process

$$V^i(t, s) = \mathbb{E}_{\mathbb{Q}^E}\left[g(S_T)\,|C_t = i, S_t = s\right]$$

We assume that for a fixed $i \in \mathcal{Y}$, the functions $V^i(t,s) \in C^{1,2}$ are continuously differentiable so that an application of Itô's formula to $\widetilde{V}^{\mathbb{Q}^E}$ yields

$$d\widetilde{V}_t^{\mathbb{Q}^E} = \sum_i H_t^i dV^i(t,S_t) + \sum_i V^i(t,S_t)dH_t^i$$

where

$$\begin{aligned} V^i(t,S_t) &= \int_0^t \frac{\partial}{\partial u}V^i(u,S_{u-})du + \frac{1}{2}\int_0^t \frac{\partial}{\partial s^2}V^i(u,S_{u-})(\sigma^i)^2 S_u^2 du \\ &\quad + \sum_{0<u\le t}(V^i(u,S_u) - V^i(u,S_{u-})) \\ &\quad + \int_0^t \frac{\partial}{\partial s}V^i(u,S_{u-})S_{u-}\left(\sigma^i dB_u - \int_{\mathbb{R}} f(x)\nu^i(dx)du\right) \end{aligned}$$

Recall that $H_t^{ij} = \sharp\{\tau \in (0,T] : C(\tau-) = i, C(\tau) = j\}$ and $H_t^i = \mathbb{1}\{C(t) = i\}$. Thus, the indicator processes $(H_t^i)_{t\ge 0}$ and the counting processes $(H_t^{ij})_{t\ge 0}$ are related by the fact that H^i increases or decreases by 1 upon a transition into or out of state $i \in \mathcal{Y}$. Thus

$$dH_t^i = \sum_{j:j\ne i}\left(dH_t^{ji} - dH_t^{ij}\right)$$

Define $\widetilde{V}_t^{\mathbb{Q}^E,*} := \sup_{t\in[0,T]}|\widetilde{V}_t^{\mathbb{Q}^E}| = \sup_{t\in[0,T]}\widetilde{V}_t^{\mathbb{Q}^E}$ since $H \ge 0$. Applying the Doob inequality to the martingale $\widetilde{V}^{\mathbb{Q}^E}$ yields

$$\mathbb{E}_{\mathbb{Q}^E}(\widetilde{V}_t^{\mathbb{Q}^E,*})^2 \le 4\mathbb{E}_{\mathbb{Q}^E}H^2 < \infty \tag{6.28}$$

since $H \in L^2(\mathbb{Q}^E)$. By the Burkholder-Davis-Gundy inequality, there exists a constant $c > 0$ such that

$$c\mathbb{E}_{\mathbb{Q}^E}\{[\widetilde{V}^{\mathbb{Q}^E}, \widetilde{V}^{\mathbb{Q}^E}]_T\} \le \mathbb{E}_{\mathbb{Q}^E}(\widetilde{V}_T^{\mathbb{Q}^E,*})^2 < \infty$$

by equation (6.28). Hence we obtain

$$\mathbb{E}_{\mathbb{Q}^E}\{[\widetilde{V}^{\mathbb{Q}^E}, \widetilde{V}^{\mathbb{Q}^E}]_T\} < \infty$$

Hence $\widetilde{V}^{\mathbb{Q}^E}$ is a square integrable martingale. Note that the difference of martingales remains a martingale but this difference is a continuous process with finite variation. It is therefore a continuous martingale and so by Theorem IV.13-IV.50 of Jacod & Shiryaev (2003) the finite variation terms must vanish $\mathbb{Q}^E$−a.s. and we arrive at the partial differential equations

$$\begin{aligned} &\frac{\partial}{\partial t}V^i(t,s) - s\int_{\mathbb{R}} f(x)\nu^i(dx)\frac{\partial}{\partial s}V^i(t,s) + \frac{1}{2}(\sigma^i)^2 s^2 \frac{\partial^2}{\partial s^2}V^i(t,s) \\ &+ \int_{\mathbb{R}}(V^i(t,sf(x)) - V^i(t,s))\nu^i(dx) + \sum_j (V^j(t,s) - V^i(t,s))\mu^{ij} = 0 \end{aligned}$$

with the terminal conditions

$$V^i(s,T) = g(s)$$

for each $i \in \mathcal{Y}$.

6.9 Hedging for models driven by quasi-Markov additive processes

6.9.1 Risk minimization

Let us consider an arbitrary non-negative, $\mathcal{G}_T-$measurable claim $H \in L^2(\mathbb{Q}^E)$ of the following form $H = g(S_T)$ with a maturity $T > 0$. Theorem 3.4 asserts that H exhibits the Kunita-Watanabe decomposition, i.e.

$$H = \mathbb{E}_{\mathbb{Q}^E}[H|\mathcal{G}_0] + \int_0^T \vartheta_t^H \, dS_t + N_T$$

$\mathbb{Q}^E - a.s.$ From this we can see that $\widetilde{V}^{\mathbb{Q}^E}$ defined in Section 6.8 has the representation

$$\widetilde{V}_t^{\mathbb{Q}^E} := \widetilde{V}_0^{\mathbb{Q}^E} + \int_0^t \vartheta_u^H \, dS_u + N_t$$

where $N_t = \mathbb{E}_{\mathbb{Q}^E}[N_T|\mathcal{G}_t]$ is a square-integrable martingale which is strongly orthogonal to S in the following sense $\langle N, S\rangle_. = 0$ where the angle bracket is taken with respect to $\mathbb{Q}^E$. Theorem 2 of Föllmer & Sondermann (1986) states that ϑ^H is the risk minimizing strategy. We seek to find ϑ^H.

$$\begin{aligned}
\langle \widetilde{V}^{\mathbb{Q}^E}, S\rangle &= \int_0^t \vartheta_u^H d\langle S, S\rangle_u + \langle N, S\rangle_t \\
\langle \widetilde{V}^{\mathbb{Q}^E}, S\rangle_t &= \int_0^t \vartheta_u^H d\langle S, S\rangle_u
\end{aligned}$$

where $\langle N, S\rangle_. = 0$ since $[N, S]_. = 0$ giving us

$$\vartheta_t^H = \frac{d\langle \widetilde{V}^{\mathbb{Q}^E}, S\rangle_t}{d\langle S, S\rangle_t}$$

We present the following:

Theorem 6.14. *Consider the evolution of the stock price process driven by Quasi-Markov additive processes*

$$\mathbb{Q}^E : \qquad \frac{dS_t}{S_{t-}} = \sigma(C_{t-})dB_t^{\mathbb{Q}^E} + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y^{\mathbb{Q}^E})_t$$

The risk minimizing hedge, amounts to holding a position in the underlying S equal to $\vartheta_t = \Delta^{C_{t-}}(t, S_{t-})$ where for each $i \in \mathcal{Y}$,

$$\Delta^i(t,s) = \frac{(\sigma^i)^2 \frac{\partial}{\partial s} V^i(t,s) + \frac{1}{s}\int_{\mathbb{R}}(V^i(t, sf(x)) - V^i(t,s))\nu^i(dx)}{(\sigma^i)^2 + \int_{\mathbb{R}} f^2(x)\nu^i(dx)}$$

with $V^i(t,s) = \mathbb{E}_{\mathbb{Q}^E}[H|C_t = i, S_t = s] = \mathbb{E}_{\mathbb{Q}^E}[g(S_T)|C_t = i, S_t = s]$

Proof. We calculate $\langle \widetilde{V}^{\mathbb{Q}^E}, S\rangle$ and $\langle S, S\rangle$

$$\begin{aligned} d\langle \widetilde{V}^{\mathbb{Q}^E}, S\rangle_t &= \sum_i H^i_{t-} S_{t-}(V^i(t, S_{t-}e^x) - V^i(t, S_{t-}))\nu^i(dx)dt \\ &\quad + \sum_i H^i_{t-} S^2_{t-} \left(\sigma^i \frac{\partial}{\partial s} V^i(t, S_t)\right)^2 dt \\ d\langle S, S\rangle_t &= \sum_i H^i_{t-} S^2_{t-}(\sigma^i)^2 dt + \int_{\mathbb{R}} S^2_{t-} \sum_i H^i_{t-} f^2(x)\nu^i(dx)dt \end{aligned}$$

with ϑ^H_t depending on (t, C_{t-}, S_{t-}), i.e.

$$\vartheta^H_t = \frac{\frac{\partial}{\partial s} V^{C_{t-}}(\sigma(C_{t-}))^2 + \frac{1}{S_{t-}} \int_{\mathbb{R}} (V^{C_{t-}}(t, S_{t-} + \Delta S_t) - V^{C_{t-}}(t, S_{t-}))\nu_{C_t}}{(\sigma(C_{t-}))^2 + \int_{\mathbb{R}} f^2(x)\nu_{C_{t-}}}$$

□

As a special case of Theorem 6.14 we obtain the following characterization for the risk minimizing hedge for some well known models.

Corollary 6.15. *By suitable specification of $\mathcal{Y}$, and ν we obtain the following well known optimal hedge under the minimal entropy martingale measure $\mathbb{Q}^E$:*

1. *Cont & Tankov (2004): If $\mathcal{Y} = \{i\}$, then we recapture the optimal hedging strategy for exponential Lévy models.*
2. *Black & Scholes: If $\mathcal{Y} = \{i\}$ and there are no jumps $\nu = 0$, we recover the familiar Black-Scholes delta hedge.*

In fact, we can say more: as a by product, we have worked out the optimal hedging strategy via Theorem 6.14 for several models proposed in the literature.

Theorem 6.16. *The entropic optimal hedging strategy of the following models are in place by suitable specifications of $\mathcal{Y}$, ν and the function g that appears within $H = g(S_T)$:*

1. *Elliot, Chan & Siu (2005): If $\mathcal{Y} = \{1, ..., m\}$ and there are no jumps $\nu = 0$, then we retrieve the optimal hedging strategy for regime switching Black-Scholes model under $\mathbb{Q}^E$.*
2. *Elliot & Osakwe (2006): If $\mathcal{Y} = \{1, ..., m\}$ and $H = g(S_T)$, we get the $\mathbb{Q}^E-$optimal hedging strategy for a jump process with switching compensators.*
3. *Marked point processes: If $\mathcal{Y} = \{i\}$, $\nu(dt, dx) = \overline{\Lambda}(t, dx)dt$ where $\overline{\Lambda}$ is deterministic, then we have the optimal hedging strategy for this specification of marked point processes.*

Remark 6.17. It is important to point out that the above models proposed by Cont & Tankov (2004) and Elliot & Osakwe (2006) do not assume the local boundedness of asset prices.

Chapter 7

Entropy Minimizer for Exponential Markov Chains

7.1 Introduction

The contribution of this chapter is the calculation of the minimal entropy martingale measure for continuous time Markov chains as explicitly as possible in terms of the transition intensities.

The minimal entropy martingale measure for continuous time Markov chains has not been studied. Miyahara (2000) examined the minimal entropy martingale measure for a Birth and Death process by means of the Hamilton-Jacobi-Bellman equation. In this chapter, we extend to the case where the asset price process is modelled by a semimartingale where our dynamics of risky asset follows a continuous time Markov chain.

7.2 Continuous time Markov chains

The mathematical framework is given by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a finite time horizon $T < \infty$. We assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F} = \mathcal{F}_T$. Let $\{C_t\}_{0 \leq t \leq T}$ be a continuous time $(\mathbb{F}, \mathbb{P})$ $-$Markov chain in finite state space $\mathcal{Y} = \{1, ..., m\}$. Further let $\mathbb{F}$ be the completion of the filtration $\mathbb{F}^C = \left(\mathcal{F}_t^C\right)_{0 \leq t \leq T} = \sigma\left(C_s; 0 \leq s \leq t\right), 0 \leq t \leq T$ generated by this Markov chain such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ becomes a complete filtered probability space. The paths of C are taken to be right continuous and C_0 deterministic. Assume further that C is time homogeneous so that we have

$$\mathbb{P}\left(C_{t+s} = j \,|\, C_s = i\right) = p_{ij}(t) \quad \forall i, j \in \mathcal{Y}, \; s, t \in \mathbb{R}^+, \; 0 \leq s \leq t \leq T$$

The following limit exists for $i, j \in \mathcal{Y}$ (Rolski, et. al. (1998) Theorem 8.1.2)

$$\mu^{ij} := \lim_{t \searrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t}$$

We note that for every $i \neq j$, we have $\mu^{ij} \geq 0$ and $\mu^{ii} = -\sum_{j=1,j\neq i}^{m} \mu^{ij}$. Hence μ^{ij} represents the intensity of transition from state i to state j and is a constant. The matrix $\Lambda := \left[\mu^{ij}\right]_{i,j\in\{1,\ldots,m\}}$ is called the infinitesimal generator matrix for a Markov chain. It is also commonly known as the intensity matrix. We say that a state $i \in \mathcal{Y}$ is absorbing for a time-homogeneous $(\mathbb{F},\mathbb{P})$ −Markov chain C_t where $t \in \mathbb{R}^+$ if the following holds:

$$\mathbb{P}(C_t = i|\, C_s = i) = 1, \quad \forall s,t \in \mathbb{R}^+,\ s \leq t \leq T$$

It is clear that if a state $i \in \mathcal{Y}$ is absorbing, then we have $\mu^{ij} = 0$ for every $j = 1, ..., m$. We assume throughout that there are no absorbing states. Furthermore we introduce for $i \neq j$ and $\forall s,t \in \mathbb{R}^+$, $t \leq T$

$$\begin{aligned} H_t^i &:= \mathbb{1}\{C_t = i\} \\ H_t^{ij} &:= \sharp\{s : 0 < s \leq t \ ;\ C_{s-} = i, C_s = j\} = \sum_{0<s\leq t} \mathbb{1}\{C_{s-} = i\}\mathbb{1}\{C_s = j\} \end{aligned}$$

Thus H_t^i is the indicator of the event that C is in state i at time t. On the other hand H_t^{ij} is the number of jumps from i to j during $]0,t]$. We state some results from Last & Brandt (1995) and Rogers & Williams (2000).

Lemma 7.1. *For every $i,j \in \mathcal{Y}$, $i \neq j$, the processes*

$$U_t^{ij} = H_t^{ij} - \int_0^t \mu^{ij} H_u^i du$$

Proof. Theorem 7.5.5 in Last & Brandt (1995) □

Theorem 7.2. *Any arbitrary $(\mathbb{P},\mathbb{F})$ −local martingale M can be written as*

$$M_t = \int_0^t \sum_i \sum_j g_s\,(i,j)\, dU_s^{ij}$$

where g is locally bounded and predictable $\forall i,j \in \mathcal{Y}$ and $g\,(i,i) = 0 \ \ \forall i \in \mathcal{Y}$, $0 \leq t \leq T$

Proof. Lemma 21.13 and Theorem 21.15 in Rogers & Williams (2000). □

We now introduce a real valued process $\widetilde{\varphi}^{ij}$, $i,j \in \mathcal{Y}$, $i \neq j$ of bounded $\mathbb{F}$−predictable such that $\widetilde{\varphi}^{ij} > -1$ with $\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$. Let us define $g\,(i,j)$ of Theorem 2 as

$$\widetilde{\varphi}_t^{ij} =: g_t\,(i,j)$$

Hence we get from Theorem 7.2 that for $0 \leq t \leq T$,

$$M_t = \int_0^t \sum_i \sum_j \widetilde{\varphi}_s^{ij} dU_s^{ij} \tag{7.1}$$

7.3 Specification of asset model for chains

Recall that S is the discounted price of a risky asset in a financial market which contains a riskless asset with discounted price constant at 1. To exclude arbitrage opportunities, our stock price process S must be a semimartingale of the form $S = S_0 + M + \int \lambda d\langle M, M\rangle$ where M is a locally bounded local martingale null at zero and some predictable process λ. We further assume $K_T := \int_0^T \lambda_s^2 d\langle M, M\rangle_s < \infty$ $\mathbb{P}-a.s.$ By Theorem 7.2 and equation (7.1), we may write M as

$$M = \int \sum_i \sum_j {}^M\widetilde{\varphi}^{ij} dU^{ij}$$

where ${}^M\widetilde{\varphi}^{ij}$, $i, j \in \mathcal{Y}$, $i \neq j$ are bounded $\mathbb{F}-$predictable such that ${}^M\widetilde{\varphi}^{ij} > -1$ with ${}^M\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$.

7.4 Change of probability measure for chains

By Proposition 2.16, the density process $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ is given by the Doléans Dade exponential process

$$Z = \mathcal{E}\left(-\int \lambda dM + L\right)$$

where L and $[M, L]$ are local $\mathbb{P}-$martingales. Hence we may write as a consequence of Theorem (7.2) that

$$L = \int \sum_i \sum_j {}^L\widetilde{\varphi}^{ij} dU^{ij}$$

where ${}^L\widetilde{\varphi}^{ij}$, $i, j \in \mathcal{Y}$, $i \neq j$ are bounded $\mathbb{F}-$predictable such that ${}^L\widetilde{\varphi}^{ij} > -1$ with ${}^M\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$. We state some initial results:

Proposition 7.3. *Let $\mathbb{Q} \in \mathcal{M}^e$. Then the following holds:*

(i) $Z_T = \mathcal{E}\left(-\int \lambda dM + L\right)_T$ is given explicitly as

$$\begin{aligned} Z_T \quad &= \quad \exp\left\{\int_0^T \sum_i \sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} -^L \widetilde{\varphi}_t^{ij}\right) \mu^{ij} H_t^i dt\right\} \\ &\quad \prod_{\substack{i \\ i\neq j}} \prod_j \prod_{0<t\leq T} \left(1 + \left(-\lambda_t^M \widetilde{\varphi}_t^{ij} +^L \widetilde{\varphi}_t^{ij}\right) \Delta H_t^{ij}\right) \end{aligned} \tag{7.2}$$

(ii) The local martingales M, L and $[M, L]$ are square integrable martingales.

Proof. (i) Apply Itô's formula to $\log Z$:

$$\begin{aligned}\log Z_T &= \int_0^T \frac{1}{Z_{t-}} dZ_t - \frac{1}{2}\int_0^T \frac{1}{Z_{t-}^2} d\langle Z, Z\rangle_t^c \\ &\quad + \sum_{0<t\leq T}\left\{\log\frac{Z_t}{Z_{t-}} - \frac{1}{Z_{t-}}\Delta Z_t\right\} \\ &= \int_0^T \sum_i\sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} -{}^L \widetilde{\varphi}_t^{ij}\right)\mu^{ij} H_t^i dt \\ &\quad + \sum_{0<t\leq T}\left\{\log\left(1+\sum_i\sum_j(-\lambda_t^M\widetilde{\varphi}_t^{ij} +{}^L \widetilde{\varphi}_t^{ij})\Delta H_t^{ij}\right)\right\}.\end{aligned}$$

Also, $Z_t = \mathcal{E}(N)_t$ and thus $dZ_t = Z_{t-}dN_t$ so that $\Delta Z_t = Z_{t-}\Delta N_t$. Also note that $Z_t/Z_{t-} = 1 + \Delta N_t$. Exponentiate both sides to get

$$\begin{aligned}Z_T &= \exp\left\{\int_0^T \sum_i\sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} -{}^L \widetilde{\varphi}_t^{ij}\right)\mu^{ij} H_t^i dt\right\} \\ &\quad \prod_{0<t\leq T}\left(1+\sum_i\sum_j(-\lambda_t^M\widetilde{\varphi}_t^{ij} +{}^L \widetilde{\varphi}_t^{ij})\Delta H_t^{ij}\right) \\ &= \exp\left\{\int_0^T \sum_i\sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} -{}^L \widetilde{\varphi}_t^{ij}\right)\mu^{ij} H_t^i dt\right\} \\ &\quad \prod_{\substack{i \\ i\neq j}}\prod_j\prod_{0<t\leq T}\left(1+(-\lambda_t^M\widetilde{\varphi}_t^{ij} +{}^L \widetilde{\varphi}_t^{ij})\Delta H_t^{ij}\right)\end{aligned}$$

where the last line follows from the fact that for a fixed t and for every $i \neq j$ and $k \neq l$, $(i,j) \neq (k,l)$, the processes H_t^{ij} and H_t^{kl} have no common jumps, see Proof of 22.9 in Rogers & Williams (2000)).

(*ii*) To show that M is a square-integrable martingale, we have to show that $\mathbb{E}\left[[M,M]_T\right] < \infty$. Note that

$$\begin{aligned}[M,M]_T &= \sum_{0<s\leq T}(\Delta M_s)^2 \\ &= \sum_{0<s\leq T}\left(\sum_i\sum_j {}^M\widetilde{\varphi}_s^{ij}\Delta U_s^{ij}\right)^2 \\ &= \int_0^T \sum_i\sum_j \left({}^M\widetilde{\varphi}_s^{ij}\right)^2 dH_s^{ij}\end{aligned}$$

where the last line follows from the fact that $\widetilde{\varphi}^{ii} = 0$ for $i = 1, ..., m$ and that there are no common jumps, i.e. for a fixed $s \leq T$, $\Delta H_s^{ij}\Delta H_s^{ji} = 0$ for $i \neq j$. We then have

$$\begin{aligned}\mathbb{E}\,[M,M]_T &= \mathbb{E}\left(\int_0^T \sum_i \sum_j \left({}^M\widetilde{\varphi}_s^{ij}\right)^2 dH_s^{ij}\right)\\ &= \mathbb{E}\left(\int_0^T \sum_i \sum_j \left({}^M\widetilde{\varphi}_s^{ij}\right)^2 \mu^{ij} H_s^i ds\right) < \infty\end{aligned}$$

due to the boundedness of ${}^M\widetilde{\varphi}^{ij}$. Further note that

$$[M,L]_T = \sum_{0<s\leq T} (\Delta M_s)(\Delta L_s) = \int_0^T \sum_i \sum_j {}^M\widetilde{\varphi}_s^{ij}\,{}^L\widetilde{\varphi}_s^{ij} dH_s^{ij}$$

so that we have

$$\begin{aligned}\mathbb{E}\,[[M,L],[M,L]]_T &= \mathbb{E}\left(\sum_{0<s\leq T} \left(\Delta\,[M,L]_s\right)^2\right)\\ &= \mathbb{E}\left(\sum_{0<s\leq T}\left(\sum_i \sum_j {}^M\widetilde{\varphi}_s^{ij}\,{}^L\widetilde{\varphi}_s^{ij} \Delta H_s^{ij}\right)^2\right)\\ &= \mathbb{E}\left(\int_0^T \sum_i \sum_j \left({}^M\widetilde{\varphi}_s^{ij}\right)^2 \left({}^L\widetilde{\varphi}_s^{ij}\right)^2 \mu^{ij} H_s^i ds\right) < \infty\end{aligned}$$

due to the boundedness of ${}^M\widetilde{\varphi}^{ij}$ and ${}^L\widetilde{\varphi}^{ij}$ and that there are a finite number of states for the Markov chain. The proof that L is a true martingale follows the same lines as above.

□

7.5 The entropy equation

Theorem 7.4. *The strategy ϕ^E and the constant c^E in equation* (2.7) *satisfy the equation*

$$\begin{aligned}c^E \quad &+ \int_0^T \phi_t^E \lambda_t \sum_i \sum_j \left({}^M\widetilde{\varphi}_t^{ij}\right)^2 \mu^{ij} H_t^i dt - \int_0^T \phi_t^E \sum_i \sum_j {}^M\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt\\ &- \int_0^T \lambda_t \sum_i \sum_j {}^M\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt + \int_0^T \sum_i \sum_j {}^L\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt\\ = \quad &- \int_0^T \phi_t^E \sum_i \sum_j {}^M\widetilde{\varphi}_t^{ij} dH_t^{ij}\\ &\quad + \int_0^T \sum_i \sum_j \log\left(1 -{}^M \widetilde{\varphi}_t^{ij} \lambda_t +{}^L \widetilde{\varphi}_t^{ij}\right) dH_t^{ij} \qquad (7.3)\end{aligned}$$

where for every $i \neq j$, ${}^{L}\widetilde{\varphi}_t^{ij}$ *has to be chosen such that*

$$\sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij\,L}\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt = 0 \qquad \forall t \in [0,T] \tag{7.4}$$

Proof. From Proposition 2.16, we see that $[M, L]$ are local $\mathbb{P}-$martingales. Observe that

$$[M, L]_t = \int_0^t \sum_i \sum_j {}^{M}\widetilde{\varphi}_s^{ij\,L}\widetilde{\varphi}_s^{ij} dH_s^{ij}$$

We further know from Dellacherie & Meyer (1980) VII.39 that the predictable bracket process

$$\langle M, L \rangle_t = \int_0^t \sum_i \sum_j {}^{M}\widetilde{\varphi}_s^{ij\,L}\widetilde{\varphi}_s^{ij} \mu^{ij} H_s^i ds$$

exists, since M and L are locally bounded. However $\langle M, L\rangle = 0$ because $[M, L]$ is a local martingale. We therefore get equation (7.4). Recall that $dS_t = \sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij} dU_t^{ij} + \lambda_t \sum_i \sum_j ({}^{M}\widetilde{\varphi}_t^{ij})^2 \mu^{ij} H_t^i dt$. By equations (2.7) and (7.2) we have

$$\begin{aligned}
\log Z_T &= c^E + \int_0^T \phi_t^E dS_t \\
&= c^E + \int_0^T \phi_t^E \sum_i \sum_j {}^{M}\widetilde{\varphi}_t^{ij} dU_t^{ij} && (7.5)\\
&\quad + \int_0^T \lambda_t \phi_t^E \sum_i \sum_j \left({}^{M}\widetilde{\varphi}_t^{ij}\right)^2 \mu^{ij} H_t^i dt && (7.6)
\end{aligned}$$

and

$$\begin{aligned}
\log Z_T &= \int_0^T \sum_i \sum_j \left(\lambda_t^M \widetilde{\varphi}_t^{ij} - {}^{L}\widetilde{\varphi}_t^{ij}\right) \mu^{ij} H_t^i dt \\
&\quad + \int_0^T \sum_i \sum_j \log\left(1 - \lambda_t^M \widetilde{\varphi}_t^{ij} + {}^{L}\widetilde{\varphi}_t^{ij}\right) dH_t^{ij}
\end{aligned}$$

respectively. Equating the above equations while using the fact that $U_t^{ij} = H_t^{ij} - \int_0^t \mu^{ij} H_s^i ds$ and simplifying the terms would result in equation (7.3). □

7.6 Finding a candidate for entropy minimization

The aim of this section is to find the minimal entropy martingale measure for continuous time Markov chains by employing the techniques discussed above. Our choice of asset price process S under $\mathbb{P}$ is inspired by the asset price process introduced in Norberg (2003) :

$$\frac{dS_t}{S_{t-}} = \sum_i \eta_t^i H_t^i dt + \sum_i \sum_j \theta_t^{ij} dU_t^{ij} \tag{7.7}$$

where η_t^i and θ_t^{ij} are deterministic and bounded functions for a fixed $i \in \mathcal{Y}$. However as we shall show later, restrictions have to be imposed on the function η_t^i. Given the price process (7.7), we see that for every $i \in \mathcal{Y}$ and $i \neq j$,

$$ {}^M\widetilde{\varphi}_t^{ij} = \theta_t^{ij} S_{t-} \tag{7.8}$$

$$\begin{aligned} \lambda_t &= \frac{\sum_i \eta_t^i H_t^i}{S_{t-}\sum_i \sum_j (\theta_t^{ij})^2 \mu^{ij} H_t^i} \\ &\Leftrightarrow \lambda_t S_{t-} = \sum_i H_t^i \left(\frac{\eta_t^i}{\sum_j (\theta_t^{ij})^2 \mu^{ij}} \right) =: \sum_i H_t^i \widehat{\lambda}_t(i,j) \end{aligned} \tag{7.9}$$

where

$$\widehat{\lambda}_t(i,j) := \frac{\eta_t^i}{\sum_j (\theta_t^{ij})^2 \mu^{ij}} \tag{7.10}$$

Note that $\sum_i \sum_j (\theta_t^{ij})^2 \mu^{ij} H_t^i < \infty$ due to the boundedness θ^{ij} and the fact that C has a finite state space. Also note from above that $\theta^{ii} = 0$. With these, equation (7.3) reduces to:

$$\begin{aligned} c^E &= -\int_0^T \lambda_t \phi_t^E S_{t-}^2 \sum_i \sum_j (\theta_t^{ij})^2 \mu^{ij} H_t^i dt + \int_0^T \phi_t^E S_{t-} \sum_i \sum_j \theta_t^{ij} \mu^{ij} H_t^i dt \\ &\quad - \int_0^T \sum_i \sum_j {}^L\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt + \int_0^T \lambda_t S_{t-} \sum_i \sum_j \theta_t^{ij} \mu^{ij} H_t^i dt \\ &\quad + \int_0^T \sum_i \sum_j \left\{ \log\left(1 - \lambda_t \theta_t^{ij} S_{t-} +^L \widetilde{\varphi}_t^{ij}\right) - \theta_t^{ij} S_{t-} \phi_t^E \right\} dH_t^{ij} \end{aligned} \tag{7.11}$$

Define $\mathcal{C}_b := \mathcal{C}_b([0,T] \times \{1,...,m\})$ the space of continuous and bounded functions of $u : [0,T] \times \{1,...,m\} \to \mathbb{R}$, $u_t := u(t,\cdot) : \mathcal{Y} \to \mathbb{R}$ and $\Delta u_t := u(t,C_t) - u(t,C_{t-})$. We proceed with the *ansatz* that there exists a sufficiently smooth function $u \in \mathcal{C}_b$ such that upon transition from i to j at time t :

$$\log\left(1 - \lambda_t \theta_t^{ij} S_{t-} + {}^L\widetilde{\varphi}_t^{ij}\right) - \theta_t^{ij} S_{t-} \phi_t^E = u\,(t,j) - u\,(t,i)$$

With this *ansatz* we observe that we can write

$$\begin{aligned} &\sum_{0<t\leq T} \sum_i \sum_j \left\{ \log\left(1 - \lambda_t \theta_t^{ij} S_{t-} + {}^L\widetilde{\varphi}_t^{ij}\right) - \theta_t^{ij} S_{t-} \phi_t^E \right\} dH_t^{ij} \\ = &\sum_{0<t\leq T} \{u\,(t,C_t) - u\,(t,C_{t-})\} \end{aligned}$$

Furthermore we set

$$u\,(T,\cdot) = 0 \qquad \text{on } \mathcal{Y}. \tag{7.12}$$

We postulate that $\phi_t^E S_{t-}$ takes the form $\sum_i H_t^i \widehat{\phi}_t(i,j) := \phi_t^E S_{t-}$ so that equation (7.3) can be recast as

$$\begin{aligned} c^E + u(0, C_0) &= -\int_0^T \sum_i \widehat{\phi}_t(i,j)\eta_t^i H_t^i dt + \int_0^T \sum_i \sum_j \widehat{\phi}_t(i,j)\theta_t^{ij} \mu^{ij} H_t^i dt \\ &\quad - \int_0^T \sum_i \sum_j {}^L\widetilde{\varphi}_t^{ij} \mu^{ij} H_t^i dt + \int_0^T \sum_i \sum_j \widehat{\lambda}_t(i,j)\theta_t^{ij} \mu^{ij} H_t^i dt \\ &\quad - \int_0^T \sum_i H_t^i \frac{\partial}{\partial t} u(t,i)\, dt \end{aligned} \tag{7.13}$$

Observe that for the RHS of equation (7.13) to be constant, a possible solution might be to require that

$$\frac{\partial}{\partial t} u(t,i) + \widehat{\phi}_t(i,j)\eta_t^i - \sum_j \left\{ \widehat{\lambda}_t(i,j)\theta_t^{ij} + \widehat{\phi}_t \theta_t^{ij} - {}^L\widetilde{\varphi}_t^{ij} \right\} \mu^{ij} = 0$$

together with equation (7.12) and that

$$c^E = -u(0, C_0)$$

We further introduce

$$g^i(t, u_t) := \widehat{\phi}_t(i,j)\eta_t^i - \sum_j \left\{ \widehat{\lambda}_t(i,j)\theta_t^{ij} + \widehat{\phi}_t(i,j)\theta_t^{ij} - {}^L\widetilde{\varphi}_t^{ij} \right\} \mu^{ij}$$

and arrive at a system of coupled ordinary differential equations for u of the form

$$\begin{aligned} \frac{\partial}{\partial t} u(t,i) + g^i(t, u_t) &= 0 \\ u(T,i) &= 0 \quad \text{for every } i \in \mathcal{Y} \end{aligned} \tag{7.14}$$

We further get using our *ansatz* that for $i \neq j$,

$${}^L\widetilde{\varphi}^{ij} = e^{u(\cdot,j) - u(\cdot,i) + \widehat{\phi}(i,j)\theta^{ij}} + \widehat{\lambda}(i,j)\theta^{ij} - 1 \tag{7.15}$$

Replacing ${}^L\widetilde{\varphi}_t^{ij}$ in equation (7.4) culminates in the following condition for $\widehat{\phi}$:

$$S_{t-} \sum_i H_t^i \left\{ \eta_t^i + \sum_j \theta_t^{ij} \left(e^{u(t,j) - u(t,i) + \widehat{\phi}_t(i,j)\theta_t^{ij}} - 1 \right) \mu^{ij} \right\} dt = 0, \qquad \forall t \in [0,T]$$

We see that the above is true if the following holds

$$\eta^i + \sum_j \theta^{ij} \left(e^{u(\cdot,j) - u(\cdot,i) + \widehat{\phi}(i,j)\theta^{ij}} - 1 \right) \mu^{ij} = 0, \quad \text{for every } i \in \mathcal{Y} \tag{7.16}$$

We shall call equation (7.16) the $\overline{\mathbb{Q}}-$*martingale equation.* Hence any strategy ϕ^E that fulfills the $\overline{\mathbb{Q}}-$*martingale equation* is a potential candidate for the minimal entropy martingale measure.

7.6.1 Existence of solutions

Let us discuss the existence of $\widehat{\phi}$ in equation (7.16). We present the following:

Lemma 7.5. *Assume that at least one of the μ^{ij} is strictly positive, i.e. $\mu^{ij} > 0$. Then there is a bounded function $\widehat{\phi} : [0,T] \times \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ with $\widehat{\phi}_t(i,j) := \widehat{\phi}(t,i,j)$ which solves equation the martingale equation (7.3). Note also that $\theta^{ii} = 0$.*

Proof. Let

$$\begin{aligned} f_1(\phi) &: = \eta_t^i, \quad \text{for a fixed } i \in \mathcal{Y} \\ f_2(\phi) &: = -\sum_j \theta_t^{ij} \left(e^{u(t,j)-u(t,i)+\phi\theta_t^{ij}} - 1 \right) \mu^{ij} \end{aligned}$$

Note that $f_2'(\phi) = -\sum_j (\theta_t^{ij})^2 \left(e^{u(t,j)-u(t,i)+\phi\theta_t^{ij}} - 1 \right) \mu^{ij} < 0$, i.e. f_2 is decreasing in ϕ. For any $t \in [0,T]$ we consider three cases;

Case 1. All $\theta_t^{ij} > 0$, for every $i \in \mathcal{Y}$. Note $f_2(0) = -\sum_j \theta_t^{ij} \left(e^{u(t,j)-u(t,i)} - 1\right) \mu^{ij}$ which can be either < 0 or > 0; $f_2(\infty) = -\infty$; $f_2(-\infty) = +\sum_j \theta_t^{ij} \mu^{ij} > 0$.

Case 2. At least one of the $\theta_t^{ij} > 0$; other $\theta_t^{ij} < 0$, for every $i \in \mathcal{Y}$, $i \neq j$. Note $f_2(0) = -\sum_j \theta_t^{ij} \left(e^{u(t,j)-u(t,i)} - 1\right) \mu^{ij}$ which can be either < 0 or > 0; $f_2(\infty) = -\infty$; $f_2(-\infty) = +\infty$.

Case 3. All $\theta_t^{ij} < 0$, for every $i \in \mathcal{Y}$. Note $f_2(0) = -\sum_j \theta_t^{ij} \left(e^{u(t,j)-u(t,i)} - 1\right) \mu^{ij}$ which can be either < 0 or > 0; $f_2(\infty) = +\sum_j \theta_t^{ij} \mu^{ij} < 0$; $f_2(-\infty) = +\infty$.

Hence we see that for any $t \in [0,T]$, $\exists \phi =: \widehat{\phi}_t^{ij} \in \mathbb{R}$ such that $f_1(\widehat{\phi}) = f_2(\widehat{\phi})$ for *all* three cases if

$$\eta_t^i \in [\, \underline{\phi}, \overline{\phi} \,] \tag{7.17}$$

where

$$-\sum_j \theta_t^{ij} \mu^{ij} < \underline{\phi} \leq \overline{\phi} < +\sum_j \theta_t^{ij} \mu^{ij}$$

In other words, existence is guaranteed if we choose η_t^i to be bounded away from $-\sum_j \theta_t^{ij} \mu^{ij}$ and $+\sum_j \theta_t^{ij} \mu^{ij}$ for any $t \in [0,T]$. Furthermore if (7.17) is respected, it is clear that the function $\widehat{\phi}_t(i,j)$ is bounded. Also from equation (7.16), we see that $\widehat{\phi}_t(i,j)$ depends on i and j and for a fix i, $\widehat{\phi}_t(i,j)$ will remain the same for every j. We now define $\widehat{\phi}_t(i,j)$ as the unique function that solves the $\overline{\mathbb{Q}}$–martingale equation (7.16). □

The next question which naturally arises is that of the existence of u. It turns out that the one can follow the lines of Section 5.4 to show that u exists and is unique. Hence we skip the proof here.

7.7 Entropy minimizer for chains

Having proved the existence of a solution to the differential equations (7.14) we can thus determine the duplet $(\widehat{\phi}(i,j), {}^L\widetilde{\varphi}^{ij})$ which solves equation (7.3).

Proposition 7.6. *Let $\widetilde{a}^{ij}$, $i,j \in \mathcal{Y}$, $i \neq j$ be a family of real-valued, bounded $\mathbb{F}-$predictable such that $\widetilde{a}^{ij} > -1$ with $\widetilde{a}^{ii} = \widetilde{a}^{m} = 0$ for $i = 1, ..., m$. Let m be the only absorbing state. Then*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}&\left[\exp\left(\int_0^t \sum_i \sum_j \log\left(1+\widetilde{a}_s^{ij}\right) dH_s^{ij}\right)\right] \\ &\leq \exp\left(\sup_{s\leq t,\ i,j\in\mathcal{Y}} \|\widetilde{a}_s^{ij}\|_{L^\infty} t \sum_i \sum_j \mu^{ij}\right) < \infty\end{aligned}$$

Proof. Let $M' = \int \sum_i \sum_j \widetilde{a}^{ij} dU^{ij}$. Then M' is local martingale. Let us now consider the stochastic differential equation

$$dZ'_t = Z'_{t-} dM'_t, \qquad Z'_0 = 1 \tag{7.18}$$

Thus its solution is given by

$$Z'_t = \exp\left\{-\int_0^t \sum_i \sum_j \widetilde{a}_s^{ij} \mu^{ij} H_s^i ds\right\} \exp\left\{\int_0^t \sum_i \sum_j \log\left(1+\widetilde{a}_s^{ij}\right) dH_s^{ij}\right\} \tag{7.19}$$

Note that Z'_- is an adapted process which is left continuous with right limits, so Z'_- is locally bounded (page 166, Protter (2004)). Since Z'_- is locally bounded, we have that Z' is a local martingale since it is by equation (7.19) the stochastic integral with respect to M' which is a local martingale (Theorem 29, Chapter IV of Protter (2004)). Observe that Z' is non-negative and by Fatou's Lemma we can conclude that Z' is a supermartingale. Hence

$$\begin{aligned} 1 &= \mathbb{E}_{\mathbb{P}}[Z'_0] \geq \mathbb{E}_{\mathbb{P}}[Z'_t] \\ &= \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_0^t \sum_i \sum_j \widetilde{a}_s^{ij} \mu^{ij} H_s^i ds + \int_0^t \sum_i \sum_j \log\left(1+\widetilde{a}_s^{ij}\right) dH_s^{ij}\right)\right] \\ &\geq \exp\left\{-\left(\sup_{s\leq t,\ i,j\in\mathcal{Y}} \|\widetilde{a}_s^{ij}\|_{L^\infty} t \sum_i \sum_j \mu^{ij}\right)\right\} \\ &\qquad \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t \sum_i \sum_j \log\left(1+\widetilde{a}_s^{ij}\right) dH_s^{ij}\right)\right] \end{aligned}$$

Rearranging yields the result. □

Proposition 7.7. *Let* $\widehat{M} := \int \sum_i \sum_j \theta^{ij} dU^{ij}$. *Then* $\widetilde{M} := \widehat{M} - \int \frac{1}{Z_-} d\left\langle Z, \widehat{M} \right\rangle$ *is a local* $\overline{\mathbb{Q}}$*−martingale. Furthermore* $\widetilde{M}$ *is given by*

$$\widetilde{M}_t = \int_0^t \sum_i \sum_j \theta_s^{ij} dH_s^{ij} - \int_0^t \sum_i \sum_j \theta_s^{ij} \exp\left((\Delta \widehat{u}_s + \widehat{\phi}_s(i,j)) \theta_s^{ij} \right) \mu^{ij} H_s^i ds$$

Proof. The density $Z = d\overline{\mathbb{Q}}/d\mathbb{P}$ may be written as $\exp\left(c^E + \int_0^T \sum_i H_t^i \frac{\widehat{\phi}_t(i,j)}{S_{t-}} dS_t\right)$. Further note that $\widehat{M}$ is a local $\mathbb{P}$−martingale. Our aim is to evaluate $\langle Z, \widehat{M} \rangle$. However, observe that

$$\begin{aligned}
[Z, \widehat{M}]_t &= \sum_{0<s\leq t} \Delta Z_s \Delta \widehat{M}_s \\
&= \sum_{0<s\leq t} \left(\sum_i \sum_j Z_{s-} \left({}^L\widehat{\varphi}_s^{ij} - \lambda_s^M \widehat{\varphi}_s^{ij} \right) \Delta H_s^{ij} \right) \left(\sum_i \sum_j \theta_s^{ij} \Delta H_s^{ij} \right)
\end{aligned}$$

One sees that the predictable bracket process is given by

$$\langle Z, \widehat{M} \rangle_t = \int_0^t Z_{s-} \sum_i \sum_j \theta_s^{ij} \left\{ \exp\left(\left(\Delta \widehat{u}_s + \widehat{\phi}_s(i,j) \right) \theta_s^{ij} \right) - 1 \right\} \mu^{ij} H_s^i ds$$

The results follows from an application of Girsanov's Theorem. □

We state the main theorem.

Theorem 7.8 (Main Theorem - $\mathbb{Q}^E$ for chains). *For every* $i, j \in \mathcal{Y}$ *where* $i \neq j$, *assume that at least one of the* μ^{ij} *is strictly positive, i.e.* $\mu^{ij} > 0$. *Further assume that* $\eta_t^i \in [\, \underline{\phi}, \overline{\phi} \,]$ *where*

$$-\sum_j \theta_t^{ij} \mu^{ij} < \underline{\phi} \leq \overline{\phi} < +\sum_j \theta_t^{ij} \mu^{ij}.$$

Then there exists a bounded function $\widehat{\phi} : [0, T] \to \mathbb{R}$ *which solves the equation*

$$\eta_t^i + \sum_j \theta_t^{ij} \left(e^{\widehat{\phi}_t(i,j) \theta_t^{ij}} - 1 \right) \mu^{ij} = 0$$

In addition, the martingale measure $\overline{\mathbb{Q}}$ *defined by*

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} = \exp\left(c^E + \int_0^T \frac{\widehat{\phi}_t(i,j)}{S_{t-}} dS_t \right)$$

with the normalizing constant

$$c^E = -\widehat{u}(0, C_0)$$

is the minimal entropy martingale measure $\mathbb{Q}^E$.

Proof. Let us now carry out the Verifications as outlined in Section 2.2.

1. $\overline{\mathbb{Q}}$ is an equivalent probability measure: Let us check the conditions of Theorem 2.8. Let the local martingale N be defined as follows:

$$\begin{aligned} N &:= -\int \lambda dM + L \\ &= \int \sum\nolimits_i \sum\nolimits_j \left({}^L\widetilde{\varphi}^{ij} - \lambda^M\widetilde{\varphi}^{ij}\right) dU^{ij} \end{aligned}$$

with

$$N_t = \int_0^t \sum\nolimits_i \sum\nolimits_j \left\{\exp\left(\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}\right) - 1\right\} dU_s^{ij}$$

since $\widehat{\phi}$ and θ are bounded, N is locally bounded. Also observe that

$$\Delta N_t = \sum\nolimits_i \sum\nolimits_j \left\{\exp((\Delta\widehat{u}_t + \widehat{\phi}_t(i,j))\theta_t^{ij}) - 1\right\} \Delta H_t^{ij}$$

so that $\Delta N > -1$ since $e^x > 0$ for all x. We now seek to find the $\mathbb{P}$–compensator of U which we shall denote by B. Note that

$$U_t = \int_0^t \sum\nolimits_i \sum\nolimits_j \left(\widehat{\phi}_s(i,j)\theta_s^{ij} e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} - e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} + 1\right) dH_s^{ij}$$

To show that U admits a predictable compensator, note that

$$\int_0^t \sum_i \sum_j \left|\left(\widehat{\phi}_s(i,j)\theta_s^{ij} e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} - e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} + 1\right)\right| \mu^{ij} H_s^i ds < \infty$$

due to the boundedness of θ^{ij}, $\widehat{\phi}(i,j)$ and $\widehat{u}$. From Theorem II.1.28 of Jacod & Shiryaev (2003), this implies the integrability of

$$\widehat{\phi}_s(i,j)\theta_s^{ij} e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} - e^{\left(\Delta\widehat{u}_s + \widehat{\phi}_s(i,j)\right)\theta_s^{ij}} + 1 =: f_s^{ij}$$

with respect to U^{ij} and

$$\int_0^t \sum_i \sum_j f_s^{ij} dU_s^{ij} = \int_0^t \sum_i \sum_j f_s^{ij} dH_s^{ij} - \int_0^t \sum_i \sum_j f_s^{ij} \mu^{ij} H_s^i ds$$

Hence the $\mathbb{P}$–compensator A of U is given by

$$A_t := \int_0^t \sum_i \sum_j f_s^{ij} \mu^{ij} H_s^i ds$$

It is easily seen that $\mathbb{E}_{\mathbb{P}}[\exp(A_T)] < \infty$ due to the boundedness of the functions $\Delta\widehat{u}$, $\widehat{\phi}$ and θ and the fact that there are a finite number of states for the chain.

2. $\overline{\mathbb{Q}}$ is a martingale measure: Note that ${}^M\widetilde{\varphi}^{ij}$ is bounded. Also, L and $[M, L]$ are locally bounded. Hence, due to Corollary 2.16, $\overline{\mathbb{Q}}$ is a martingale measure.

3. $H\left(\overline{\mathbb{Q}},\mathbb{P}\right)<\infty$: Recall that the density $Z=d\overline{\mathbb{Q}}/d\mathbb{P}$ may be expressed as

$$Z_T=\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}=\exp\left(c^E+\int_0^T\frac{\sum_i H_t^i\widehat{\phi}_t(i,j)}{S_{t-}}dS_t\right).$$

Let us evaluate the following:

$$\begin{aligned}
&\mathbb{E}_{\overline{\mathbb{Q}}}\Big[\int_0^T\sum_i\sum_j\widehat{\phi}_t(i)\theta_t^{ij}dU_t^{ij}\Big]\\
=\quad&\text{housei zero}+\mathbb{E}_{\overline{\mathbb{Q}}}\Big[\int_0^T\sum_i\widehat{\phi}_t(i)\Big(-\eta_t^i+\sum_j\theta_t^{ij}\mu^{ij}\Big)H_t^idt\Big]\\
&-\mathbb{E}_{\overline{\mathbb{Q}}}\Big[\int_0^T\sum_i\sum_j\widehat{\phi}_t(i)\theta_t^{ij}\mu^{ij}H_t^idt\Big] \qquad (7.20)\\
=\quad&-\mathbb{E}_{\overline{\mathbb{Q}}}\Big[\int_0^T\sum_i\widehat{\phi}_t(i)\eta_t^iH_t^idt\Big]
\end{aligned}$$

where the penultimate line (7.20) is obtained by Proposition 7.7 and substituting the $\overline{\mathbb{Q}}$−martingale condition of equation (7.16). Finally we get

$$H\left(\overline{\mathbb{Q}},\mathbb{P}\right)=\mathbb{E}_{\overline{\mathbb{Q}}}[c^E]<\infty$$

4. $\int\phi^E dS$ is a true $\mathbb{Q}$−martingale for all $\mathbb{Q}\in\mathcal{M}^e$ with $H\left(\mathbb{Q},\mathbb{P}\right)<\infty$: We require $\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\beta\int_0^T\psi^2d\left[S,S\right]_t\right\}\right]<\infty$ for some $\beta>0$. We have $\psi_t=\frac{\sum_i H_i^t\widehat{\phi}_t(i,j)}{S_{t-}}$ so that $\psi_t^2=\frac{\sum_i H_t^i\widehat{\phi}_t^2(i,j)}{S_{t-}^2}$. We also have for $i\neq j$, $d\left[S,S\right]_t=S_{t-}^2\sum_i\sum_j(\theta_t^{ij})^2dH_t^{ij}$. We now have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\beta\int_0^T\psi^2d[S,S]_t\Big\}\Big]\\
\leq\;&\mathbb{E}_{\mathbb{P}}\Big[\exp\Big\{\int_0^T\sum_i\sum_j(\theta_t^{ij})^2dH_t^{ij}\Big\}\Big]\mathbb{1}_{\left\{\beta:=\left(\sup_{t\in[0,T],\,i\in\mathcal{Y}}2\|\widehat{\phi}_t^2(i,j)\|_\infty\right)^{-1}>0\right\}}\\
\leq\;&\exp\left(\sup_{t\in[0,T],\,i,j\in\mathcal{Y}}\left\|e^{(\theta_t^{ij})^2}-1\right\|_\infty T\sum_i\sum_j\mu^{ij}\right)\\
\leq\;&\exp\left(\sup_{t\in[0,T],\,i,j\in\mathcal{Y}}\left\|e^{(\theta_t^{ij})^2}\right\|_\infty T\sum_i\sum_j\mu^{ij}\right)<\infty
\end{aligned}$$

due to the boundedness of θ_t^{ij}, μ^{ij} and by setting $\widetilde{a}_t^{ij}:=e^{(\theta_t^{ij})^2}-1$ in Proposition 7.6. Hence we conclude that

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}=\exp\left(c^E+\int_0^T\frac{\sum_i H_t^i\widehat{\phi}_t(i,j)}{S_{t-}}dS_t\right)$$

fulfills all the conditions necessary for being the minimal entropy martingale measure, i.e. $\overline{\mathbb{Q}} = \mathbb{Q}^E$. $\square$

Chapter 8

Examples

The purpose of this chapter is two-fold: First, we make a couple of remarks on several existing results concerning the entropy minimizer for Markov modulated geometric Brownian motion. Furthermore, we claim and explain why these results are incorrect. Second, we apply our prior findings on the entropy minimizing measure for quasi-Markov Additive processes to the valuation of EUA options (which will be made precise later on) to the field of carbon finance.

8.1 Remarks on Elliot, Chan & Siu (2005) and Song & Bo (2009)

The paper of Elliot, Chan & Siu (2005), abbreviated (ECS) and Song & Bo (2009), truncated to be (SB) deals with the entropy minimizers for Markov modulated geometric Brownian motion and Markov switching Lévy processes respectively. Before we proceed, let us note that their model can be transferred to our model in equation (6.7) so we shall use the notations of Chapter 6. Recall that C denotes the continuous time Markov chain.

$$
\begin{aligned}
ECS \quad \mathbb{P}: \quad \frac{dS_t}{S_t} &= \alpha(C_t)dt + \sigma(C_t)dB_t \\
SB \quad \mathbb{P}: \quad \frac{dS_t}{S_{t-}} &= \alpha(C_t)dt + \sigma(C_t)dB_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y)_t
\end{aligned}
$$

where $\nu_Y(dt, dx) := \nu_{C_{t-}}(dx)dt$. For simplicity, let us suppose that there are no jumps in the SB model so that the price processes of ECS and SB are similar,

$$
ECS/SB \quad \mathbb{P}: \quad \frac{dS_t}{S_t} = \alpha(C_t)dt + \sigma(C_t)dB_t \tag{8.1}
$$

Both ECS (Proposition 3.1) and SB (Example 5.2, 'Brown+Poisson' but we remove 'Poisson' due to our assumption that there are no jumps in the SB model) state that the density process of the minimal entropy martingale measure $Z_{\mathbb{Q}^E}^{SB/ECS}(t)$ for the model

(8.1) takes the form

$$Z_{\mathbb{Q}^E}^{SB/ECS}(t) = \mathcal{E}\left(-\int_0^{\cdot} \frac{\alpha(C_s)}{\sigma(C_s)} dB_s\right)_t \tag{8.2}$$

We claim that the entropy minimizer (8.2) is not correct.

To see why this is the case, observe that the density Z_T of the minimal entropy martingale measure for the above model (8.1) has been solved by Grandits & Rheinländer (2002), i.e.

$$\begin{aligned} \left.\frac{d\mathbb{Q}^E}{d\mathbb{P}}\right|_{\mathcal{G}_T} = Z_T &= c\exp\left(-\int \lambda dS\right)_T \\ &= c\exp\left(-\frac{1}{2}K_T\right)\mathcal{E}\left(-\int_0^{\cdot} \frac{\alpha(C_t)}{\sigma(C_t)} dB_t\right)_T \end{aligned} \tag{8.3}$$

where c the normalizing constant is given by

$$c^{-1} = \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\int_0^T \frac{1}{2}\frac{\alpha^2(C_t)}{\sigma^2(C_t)} dt\right)\right]$$

and $K_T := \int_0^T \frac{\alpha^2(C_t)}{\sigma^2(C_t)} dt$. With the usual definitions of $\widehat{\phi} := \phi^E S$ and $\widehat{\lambda} := \lambda S$, Grandits & Rheinländer (2002) further showed that the optimal strategy is given by

$$\widehat{\phi} = -\widehat{\lambda} = -\frac{\alpha(C)}{\sigma^2(C)}$$

Equating equations (8.2) and (8.3) at maturity T yields the following

$$\log c = \frac{1}{2}\int_0^T \frac{\alpha^2(C_t)}{\sigma^2(C_t)} dt = \frac{1}{2}K_T \tag{8.4}$$

However observe that the above quantity (8.4) is a function of the chain and hence it is stochastic. This violates the fact that c has to be a constant. To further elucidate on this issue, let us apply equation (6.3) to our present context:

$$\begin{aligned} \log c &= \int_0^T \sum_i H_t^i(\sigma^L(i) - \widehat{\lambda}_t\sigma(i) - \widehat{\phi}_t\sigma(i))dB_t \\ &\quad - \int_0^T \sum_i H_t^i\left(\frac{1}{2}(\sigma^L(i) - \widehat{\lambda}_t\sigma(i))^2 + (\widehat{\phi}_t\widehat{\lambda}_t\sigma^2(i))\right) dt \\ &\quad - \int_0^T \sum_i\sum_j {}^L\widetilde{\varphi}^{ij}\mu^{ij}H_t^i dt \\ &\quad + \int_0^T \sum_i\sum_j \log(1 + {}^L\widetilde{\varphi}^{ij})dH_t^{ij} \end{aligned} \tag{8.5}$$

Equation (8.4) implies that ECS and SB decided to impose

$${}^L\widetilde{\varphi}^{ij} = 0 \quad \text{for all } i,j \in \mathcal{Y}$$

Clearly the above choice does not make the RHS of (8.5) a constant due to the presence of $H_t^i := \mathbb{1}\{C_t = 1\}$. Hence we conclude that their conjecture of ${}^L\widetilde{\varphi}^{ij} = 0$ does not lead to the entropy minimizing martingale measure for Markov modulated geometric Brownian motion.

8.2 Carbon finance

Background. Carbon trading is a market mechanism determined to deal with the peril of global warming. Although it dates back to the nineties, it only took off as a market after the Kyoto Protocol was ratified. The principal greenhouse gas contributing to global warming is carbon dioxide, which is discharged by burning fossil fuels. Under the Kyoto Protocol, each participating government has its individual local target for reducing carbon dioxide emissions. The *raison d'etre* behind carbon trading is that from the earth's standpoint, the position where the carbon dioxide originates from is considerably less important than aggregate amounts being discharged. This means that its impact on our environment is similar everywhere wherever the carbon dioxide comes from. In other words, carbon dioxide acts *globally* rather than *locally* - yet another example of 'local' passing to 'global'! Consequently, rather than severely pushing every country/company to reduce emissions, the market establishes an option; a country/company can either splurge money to cover the expenses of reducing emissions or continue the harmful emissions but reimburse another to trim down their greenhouse gasses. The solution to global warming set forth by the Kyoto Protocol can be summarized as follows: The Annex 1 nations or the developed world have committed to reducing emissions to 95 percent of their 1990 levels during 2008 and 2012. This is accomplished by a cap and trade system where the Annex 1 nations will be issued with Assigned Amount Units (AAU) equal to their allowed emissions. The Annex 1 nations are expected to do the following: (i) reduce their emissions to equal their AAU allowance or (ii) buy additional AAU from other Annex 1 nations to cover surplus emissions or (iii) purchase emissions reductions certificates CER and ERU to cover surplus emissions. Certified Emissions Reduction (CER) is an example of 'carbon credits' issued in return for a reduction of atmospheric carbon emissions. Emissions reduction units (ERU) are units of greenhouse gas reductions (or, fraction of a country's AAU) that have been generated via Joint Implementation under Article 6 of the Kyoto Protocol - as opposed to CERs - which have been generated and certified under the provisions of Article 12 of the Kyoto Protocol, the Clean Development Mechanism.

EU ETS. The European Union Emission Trading Scheme (EU ETS) is formed to tackle carbon emission in the European Union (EU). The EU ETS has grown to be the largest greenhouse emissions trading scheme in the world. The European Allowance (EUA) are carbon allowances traded in the EU ETS and is analogous to that of the AAU explained earlier. The EU ETS was launched to operate in two phases. Phase I which runs from 2005 to 2007 was regarded as a trial phase, while Phase II takes place from 2008 to 2012. The difference between Phase I and II lies in the fact that the banking of allowances is not permitted during Phase I and thus the allowances expires at their stated maturity. However allowances issued under Phase II can be banked by the installations. To better understand this, let us dwell on the illustration taken from Çetin and Verschuere (2008). Consider the so-called Dec-07 and Dec-08 contracts that had been traded in the EU ETS during 2007. Dec-07 contracts expires at the end of December 2007 and can be used to cover emissions during 2007. Similarly, the Dec-08 contract expires at the end of December 2008 and can only be used to cover emissions during 2008. Since these two contracts were traded during Phase I in which banking is not allowed, Dec-07 contracts cannot be used to cover emission in 2008. Every installations included in the EU ETS has to surrender carbon allowances at the end of every calender year. A company that does not surrender enough allowances will be fined. In the Phase I period, there is a fine of 40 euros per missing allowance. In the Phase II duration, this fine is 100 euros per missing allowance. The company is still obliged to surrender the missing allowances.

Entropic carbon valuation. We adhere to the framework and notations as well as the arbitrage relationship set out by Çetin and Verschuere (2008), henceforth denoted by ÇV. However, we put forward a jump-diffusion model as opposed to a regime switching Black Scholes model proposed by ÇV. We also work under the entropy minimizer as our pricing and hedging measure. We consider a market for the trading of the EUAs. Let us assume that there are two EUAs traded in the market: EUA for the current year, denoted by EUA0 and EUA for the next year, denoted by EUA1. We propose a price process S for the EUA1 contract of the following form

$$\mathbb{P}: \qquad \frac{dS_t}{S_{t-}} = \alpha\eta(C_{t-})dt + \sigma dB_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y)_t \tag{8.6}$$

where $\nu_Y(dt, dx) := \nu(dx)dt$ and C being a continuous time Markov chain. The above price process (8.6) can be recast as

$$\frac{dS_t}{S_{t-}} = \alpha \sum_i H^i_{t-}\eta^i dt + \sigma dB_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu(dx)dt)$$

where $\eta^i := \eta(i)$, σ and α are constants for a fixed $i \in \{-1, 1\}$. We work under the following assumptions that $\nu(\mathbb{R}) < \infty$ and $f(\cdot) : supp(\nu) \to (-1, \infty)$ is uniformly bounded. Observe that we set $\mathcal{Y} = \{-1, 1\}$ here. The idea is that $C_t = 1$ corresponds to the market being long credits and $C_t = -1$ means the market is short credits at time t. Further note that the compensator $\nu_Y(dt, dx) := \nu(dx)dt$ is deterministic. We know that the dynamics of S under $\mathbb{Q}^E$ is given by

$$\mathbb{Q}^E: \qquad \frac{dS_t}{S_{t-}} = \sigma dB^{\mathbb{Q}^E}_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu^{\mathbb{Q}^E}_Y)_t \tag{8.7}$$

The structure of the compensating measure under the entropy minimizer can been revealed upon an application of Girsanov's Theorem. Under the assumption of no banking, contracts will be worthless if the market is long at their respective expiry date, denoted by T. If the market is short at expiry T, then these contracts cannot be purchased since they are not available and companies without sufficient compliance allowances must pay the penalty K of either 40 or 100 Euros/tonne and purchase a corresponding number of allowances for the following year to meet their compliance target. Letting P denote the price process of EUA0 contracts, we have the following relation between S and P at expiry T

$$P_T = \begin{cases} S_T + K, & \text{if } C_T = -1 \\ 0, & \text{if } C_T = +1 \end{cases} \tag{8.8}$$

We now turn to issues of pricing and hedging EUA0 contracts in the EU ETS market. We start by giving a discussion on mean variance hedging before we evaluate the price of EUA0 contracts and the optimal hedging strategy for these contracts.

8.2.1 Mean variance hedging

Consider a contingent claim with maturity $T > 0$, defined by an $\mathcal{G}_T$−measurable random variable H. Define the initial capital V_0 and a trading strategy which will be defined by an adapted process $\phi = (\vartheta_t^0, \vartheta_t)_{t\in[0,T]}$ taking values in $\mathbb{R}^2$. Thus the value at time t of the strategy ϕ is given by $V_t = \vartheta_t^0 e^{rt} + \vartheta_t S_t$ and the strategy is said to be self-financing if $dV_t = \vartheta_t^0 r e^{rt} dt + \vartheta_t dS_t$. Inserting the dynamics of S under the minimal entropy martingale measure $\mathbb{Q}^E$, we get

$$dV_t = \vartheta_t^0 r e^{rt} dt + \vartheta_t(\sigma dB_t^{\mathbb{Q}^E} + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y^{\mathbb{Q}^E})_t)$$

Precisely, the condition of self-financing can be written as

$$V_t = V_0 + \int_0^t \vartheta_u^0 r e^{ru} du + \int_0^t \vartheta_u dS_u$$

Without loss of generality, let us set the interest rate $r \equiv 0$. In mean-variance hedging, we look for a self-financing trading strategy given by an initial capital V_0 and a portfolio $(\vartheta_t)_{t\in[0,T]}$ over the lifetime of the contingent claim which minimizes the shortfall at the terminal date T in a mean square sense:

$$\inf_{\vartheta, V_0} \mathbb{E}_{\mathbb{Q}^E} |H - V_T|^2$$

where

$$H - V_T = H - V_0 - \int_0^T \vartheta_t dS_t$$

We assume now that $H \in L^2(\Omega, \mathbb{G}, \mathbb{Q}^E)$. We further know that $(S_t)_{t\in[0,T]}$ is a square-integrable $\mathbb{Q}^E$−martingale since its quadratic variation is integrable. Let us consider those $(\vartheta_t)_{t\in[0,T]}$ whose terminal values satisfies

$$\mathcal{X} := \left\{ \vartheta : \mathbb{E}_{\mathbb{Q}^E}\Big(\int_0^T \vartheta_t^2 d[S,S]_t\Big) < \infty \right\}$$

Consider now a self-financing strategy $(\vartheta_t^0, \vartheta_t)_{t\in[0,T]}$. For the quadratic hedging criterion to make sense, we must restrict ourselves to portfolios ϑ verifying

$$\mathbb{E}_{\mathbb{Q}^E}\left[\sigma^2 \int_0^T |\vartheta_t S_t|^2 dt + \int_0^T \int_{\mathbb{R}} f^2(x) |\vartheta_t S_t|^2 \nu(dx) dt\right] < \infty \tag{8.9}$$

Define $L^2(S)$ as the set of portfolios ϑ verifying (8.9). Since $\vartheta \in L^2(S)$ and using the fact that S is a martingale under $\mathbb{Q}^E$, the gains process $G(\vartheta) = \int_0^\cdot \vartheta dS$ is also a square integrable martingale given by

$$G_T(\vartheta) = \int_0^T \vartheta_t S_{t-} \sigma dB_t + \int_0^T \int_{\mathbb{R}} f(x) \vartheta_t S_{t-} (\mu_Y - \nu_Y)$$

The mean variance hedging problem can now be recast as:

$$\inf_{\vartheta \in L^2(S), V_0} \mathbb{E}_{\mathbb{Q}^E} \left| H - V_0 - \int_0^T \vartheta_t dS_t \right|^2$$

or equivalently $\inf_{\vartheta \in L^2(S), V_0} \mathbb{E}_{\mathbb{Q}^E} \left| H - V_0 - G_T(\vartheta) \right|^2$. The condition (8.9) implies that the value process V is a square integrable martingale, we have $\mathbb{E}_{\mathbb{Q}^E}[V_T] = V_0$. Applying the elementary identity $\mathbb{E}(Z^2) = (\mathbb{E}(Z))^2 + \mathbb{V}ar(Z)$ to the random variable $Z := H - V_0 - G_T(\vartheta)$, we obtain

$$\begin{aligned} J_0(V_0, \vartheta) &:= \mathbb{E}_{\mathbb{Q}^E} \left| H - V_0 - \int_0^T \vartheta_t dS_t \right|^2 \\ &= \left| \mathbb{E}_{\mathbb{Q}^E}[H] - V_0 \right|^2 + \mathbb{V}ar(H - V_0 - G_T(\vartheta)) \end{aligned}$$

We see that if the writer of the contingent claim tries to minimize the residual risk $J_0(V_0, \vartheta)$, the optimal value that she will ask for is a premium

$$V_0 = \mathbb{E}_{\mathbb{Q}^E}[H]$$

We see that $\mathbb{E}_{\mathbb{Q}^E}[H]$ is the initial value of any strategy $\vartheta \in L^2(S)$ designed to minimize the shortfall at maturity and we take this as the definition of the price associated with our contingent claim H at time 0. By the same token, if the writer sells the option at time $t > 0$ and intends to minimize the remaining risk $J_t(V_t, \vartheta) := \mathbb{E}_{\mathbb{Q}^E}\left(\left| H - V_0 - \int_0^T \vartheta_t dS_t \right|^2 \middle| \mathcal{G}_t \right)$ she will ask a premium $V_t = \mathbb{E}_{\mathbb{Q}^E}(H|\mathcal{G}_t)$. We will take this quantity to define the price of the contingent claim H at time t. This motivates the following definition for the carbon price of EUA0 contracts.

Definition 8.1. *Let $H \in L^2(\Omega, \mathbb{G}, \mathbb{Q}^E)$ be a contingent claim and let $\mathbb{Q}^E$ be the minimal entropy martingale measure for S given by (8.6). The fair price F_t for an EUA0 contract at time t for H is given by*

$$F_t := \mathbb{E}_{\mathbb{Q}^E}[H|\mathcal{G}_t]$$

We state the following.

Proposition 8.2 (Entropy price for EUA0 contracts). *Let $H \in L^2(\Omega, \mathbb{G}, \mathbb{Q}^E)$ be a contingent claim of the form*

$$H = g(C_T, S_T) = (S_T + K)\mathbb{1}_{\{C_T = -1\}} = \frac{S_T + K}{2}(1 - C_T) \tag{8.10}$$

Then the price of H at time t is given by $F(t, S_t, C_t)$ where

$$\begin{aligned} F : [0,T] \times [0,\infty) \times \{-1,1\} &\rightarrow \mathbb{R} \\ (t,s,i) &\mapsto F(t,s,i) = \mathbb{E}_{\mathbb{Q}^E}\left[g(S_T)|S_t = s, C_t = i\right] \end{aligned}$$

Moreover $F(t, S_t, C_t)$ satisfies the partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}F(t,s,i) - s\int_{\mathbb{R}} f(x)\nu(dx)\frac{\partial}{\partial s}F(t,s,i) + \frac{1}{2}s^2\sigma^2\frac{\partial^2}{\partial s^2}F(t,s,i) \\ &+ \int_{\mathbb{R}}(F(t,sf(x),i) - F(t,s,i))\nu(dx) + \sum_j (F(t,s,j) - F(t,s,i))\mu^{ij} = 0 \end{aligned} \tag{8.11}$$

with terminal condition

$$F(T,s,i) = \frac{s+K}{2}(1-i), \quad (s,i) \in \mathbb{R}_+ \times \{-1,1\} \tag{8.12}$$

Proof. Note that due to the Markov property of (S, C) we can write

$$\begin{aligned} F_t &= \mathbb{E}_{\mathbb{Q}^E}\left[g(C_T, S_T)|\mathcal{G}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^E}\left[g(C_T, S_T)|S_t, C_t\right] \\ &=: F(t, S_t, C_t) = \sum_i H_t^i F(t, S_t, i) \end{aligned}$$

where $F(t,s,i) := \mathbb{E}_{\mathbb{Q}^E}\left[g(S_T)|S_t = s, C_t = i\right]$. Assume that for each $i \in \{-1,1\}$, $F \in C^{1,2}$, i.e., the functions $(t,s,i) \mapsto F(t,s,i)$ are continuously differentiable with respect to t and twice continuously differentiable with respect to s. Hence the Itô formula can be applied to $F_t = F(t, S_t, C_t) = \sum_i H_t^i F(t, S_t, i)$ between 0 and T:

$$dF_t = \sum_i H_t^i dF(t, S_t, i) + \sum_i F(t, S_t, i) dH_t^i$$

Define $\widetilde{F}_t^{\mathbb{Q}^E,*} := \sup_{t\in[0,T]}|F_t| = \sup_{t\in[0,T]} F_t$ since $H \geq 0$. Applying the Doob inequality to the martingale F yields

$$\mathbb{E}_{\mathbb{Q}^E}(\widetilde{F}_t^{\mathbb{Q}^E,*})^2 \leq 4\mathbb{E}_{\mathbb{Q}^E}H^2 < \infty$$

since $H \in L^2(\mathbb{Q}^E)$. By the Burkholder-Davis-Gundy inequality, there exists a constant $c > 0$ such that

$$c\mathbb{E}_{\mathbb{Q}^E}\{[F,F]_T\} \leq \mathbb{E}_{\mathbb{Q}^E}(\widetilde{F}_T^{\mathbb{Q}^E,*})^2 < \infty$$

Hence we obtain

$$\mathbb{E}_{\mathbb{Q}^E}\{[F,F]_T\} < \infty$$

This implies that F is a square integrable $\mathbb{Q}^E$−martingale. By Jacod & Shiryaev (2003 Theorem IV.13-IV.50) we conclude that the finite variation terms vanishes giving us the desired PDE

$$\begin{aligned} &\frac{\partial}{\partial t}F(t,s,i) - s\int_{\mathbb{R}} f(x)\nu(dx)\frac{\partial}{\partial s}F(t,s,i) + \frac{1}{2}s^2\sigma^2\frac{\partial^2}{\partial s^2}F(t,s,i) \\ &+ \int_{\mathbb{R}}(F(t,sf(x),i) - F(t,s,i))\nu(dx) + \sum_j (F(t,s,j) - F(t,s,i))\mu^{ij} = 0 \end{aligned}$$

with terminal condition

$$F(T,s,i) = \frac{s+K}{2}(1-i), \quad (s,i) \in \mathbb{R}_+ \times \{-1,1\}$$

□

Having derived the entropy price for EUA0 contracts, we now turn to the problem of mean variance hedging. We can state the following

Theorem 8.3 (Optimal hedging strategy for EUA0 contracts). *Let $H = g(C_T, S_T)$ be of the form in equation (8.10). The minimal risk hedge, amounts to holding a position in the underlying $(S_t)_{t>0}$ equals to $\vartheta_t = \Delta(t, S_{t-}, C_{t-})$ where for each $i \in \{-1,1\}$,*

$$\Delta(t,s,i) = \frac{\sigma^2 \frac{\partial}{\partial s} F(t,s,i) + \frac{1}{s}\int_{\mathbb{R}} (F(t,sf(x),i) - F(t,s,i))\nu(dx)}{\sigma^2 + \int_{\mathbb{R}} f^2(x)\nu(dx)}$$

with $F(t,s,i) = \mathbb{E}_{\mathbb{Q}^E}[H|C_t = i, S_t = s] = \mathbb{E}_{\mathbb{Q}^E}[g(C_T,S_T)|C_t = i, S_t = s]$

Proof. Recall that under $\mathbb{Q}^E$, the stock price process S is a square integrable martingale. Consider now a self-financing trading strategy with $\vartheta \in L^2(S)$, the value process V of the portfolio is also a martingale whose value at maturity $T > 0$ is

$$\begin{aligned}
V_T &= V_0 + \int_0^T \vartheta_t dS_t \\
&= V_0 + \int_0^T \vartheta_t S_{t-}(\sigma dB_t + \int_{\mathbb{R}} f(x)(\mu_Y - \nu_Y)) \\
&= V_0 + \int_0^T \vartheta_t S_{t-}\sigma dB_t + \int_0^T \int_{\mathbb{R}} S_{t-}\vartheta_t f(x)(\mu_Y - \nu_Y) \\
&= V_0 + G_T(\vartheta)
\end{aligned}$$

Also $F(T,S_T,C_T) = g(C_T,S_T) = H$ and $F(0,S_0,C_0) = \mathbb{E}_{\mathbb{Q}^E}[g(C_T,S_T)] = V_0$ so that $H - V_0 - G_T(\vartheta) = F(T,S_T,C_T) - F(0,S_0,C_0) - G_T(\vartheta)$. Taking into consideration equation (8.11), we evaluate the following quantity

$$\begin{aligned}
&F(T,S_T,C_T) - F(0,S_0,C_0) - G_T(\vartheta) \\
&= \int_0^T \sum_i H^i_{t-}\left(\vartheta_t S_{t-} - S_{t-}\frac{\partial}{\partial s}F(t,S_{t-},i)\right)\sigma dB_t \\
&+ \int_0^T \int_{\mathbb{R}} \sum_i H^i_{t-}\left(\vartheta_t S_{t-} f(x) - (F(t,S_{t-}f(x),i) - F(t,S_{t-},i)\right)(\mu_Y - \nu_Y)_t \\
&\quad - \int_0^T \sum_i H^i_{t-} \sum_j (F(t,S_{t-},j) - F(t,S_{t-},i))dU^{ij}_t
\end{aligned}$$

so that

$$
\begin{aligned}
J_0(V_0,\vartheta) &:= \mathbb{E}_{\mathbb{Q}^E}\left|H - V_0 - \int_0^T \vartheta_t dS_t\right|^2 \\
&= \mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \sum_i H^i_{t-}S^2_{t-}\left(\vartheta_t - \frac{\partial}{\partial s}(t,S_{t-},i)dt\right)^2 \sigma^2 dt\right] \\
&\quad +\mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \int_{\mathbb{R}} \sum_i H^i_{t-}(\vartheta_t S_{t-}f(x)\right. \\
&\quad \left. - (F(t,S_{t-}f(x),i) - F(t,S_{t-},i)^2))\nu(dx)dt\right] \\
&\quad -\mathbb{E}_{\mathbb{Q}^E}\left[\int_0^T \sum_i H^i_{t-}\sum_j (F(t,S_{t-},j) - F(t,S_{t-},i))\mu^{ij}dt\right]
\end{aligned}
$$

To obtain the optimal risk-minimizing hedge we minimize the above expression with respect to ϑ:

$$
\begin{aligned}
&S^2_{t-}\sigma^2\left(\vartheta_t - \frac{\partial}{\partial s}(t,S_{t-},i)\right) \\
&\quad + \int_{\mathbb{R}}(S_{t-}\vartheta_t f(x) - (F(t,S_{t-}f(x)) - F(t,S_{t-}))S_{t-}f(x)\nu(dx) \stackrel{!}{=} 0
\end{aligned}
$$

A sanity check for convexity yields $S^2_{t-}\sigma^2 + \int_{\mathbb{R}} S^2_{t-}f^2(x)\nu(dx) > 0$. □

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