Essays on Matching Models of the Labour Market

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Abstract

This thesis is divided into three parts, all related to matching models of the labour market. In the first part, I analyze wage determination in search equilibrium. In the second part, I study human capital acquisition and depreciation when the labour market contains frictions. In the last part, I discuss various issues related to search and matching. Below follows a brief description of each paper.

Part 1: Wage Determination In *A Matching Model with Wage Announcement*, I study a matching model where heterogeneous firms publicly announce wage offers. I derive a Walrasian type of equilibrium, which is constrained efficient. In *Bargaining Over the Business Cycle*, I assume that wages are determined by strategic bargaining. This makes wages more and unemployment less volatile than when the conventional Nash solution is applied. In *Bargaining and Matching*, I design an alternative extensive form bargaining game, where a third agent may arrive and Bertrand competition take place. The resulting wage schedule is of the same form as the one that prevails from Nash bargaining.

Part 2. Human Capital and Matching In *Human Capital Investments and Market Imperfections*, I analyze how frictions in the labour market can distort the incentives to invest in human capital, and lead to sub-optimal investments and multiple equilibria. In *Education and Competition for Jobs*, each vacancy can get more than one applicant, and several workers may compete for the same job. Depending on parameter values, workers may or may not diversify and choose different levels of education. In *Loss of Skills During Unemployment*, workers gradually lose skills during unemployment. As a result, multiple equilibria may exist, and unemployment benefits to the long-term unemployed can reduce unemployment.

Part 3. Other Topics In *Optimal Unemployment*, I study the efficiency of matching models using techniques from optimal control theory. In *A Search Model with Hiring Costs*, I introduce hiring costs in the model, and show that this makes the vacancy rate less volatile and the adjustment process after a shock smoother.

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Introduction

Matching models are models where the focus is on market frictions. In matching models of the labour market, both unemployed workers and firms with vacancies have to undertake costly and time-consuming search to find a trading partner. These frictions can be due to informational imperfections about the existence and location of potential trading partners, and to the fact that announcing jobs, producing and processing applications, selecting the right applicant etc are time consuming activities. All factors are summarized in the *matching function*, which maps stocks of searching workers and jobs into a stream of new matches.

In models with frictions, the concept of market clearing is not well-defined, and the usual "supply equal demand" rule for determining wages does not apply. The reason is that when finding new trading partners are costly, it is mutually beneficial for a worker-firm pair to stay together rather than to continue searching. Thus there is a surplus associated with each match, and how to split this surplus between the worker and the firm is a bilateral decision.

In the literature on matching models of the labour market, it is common to assume that the match surplus is shared according to the (asymmetric) Nash sharing rule. The disagreement points in the Nash bargaining is then the agents' outside options, i.e. their expected income with no trading partner at hand. The agents' bargain power, or their shares of the surplus, are exogenously given. However, there also exists a separate and distinct branch of the literature, dealing with more theoretical issues concerning decentralized trade, where wages are determined by strategic bargaining between workers and firms. The key element in these models is the Stål-Rubinstein bargaining game, where the opponents alternate to make proposals on prices / wages.

When wages are not determined by market clearing, the welfare properties of the market solution become an open issue. Since this receives much attention, both in the literature and in my thesis, I will discuss it here in some detail. I focus on the question of whether the private and social gains by opening new jobs coincide. Entry of a new job creates positive externalises for workers and negative externalities for other searching firms, since it increases the unemployed workers job-fining rate and reduces the rate at which each job is recruiting. If wages are high, the positive externality for unemployed workers is strong, since the value of finding a job is high, while the negative externality is weak, since the value of an occupied job is low. Thus the positive externality dominates, the private benefits of opening a job is smaller than the social benefits, and we get suboptimal entry and too high unemployment rate. On the other hand, if the wage rate is low, the negative externality dominates, and we get over-optimal entry and too many resources used on the search process. Optimality is achieved when the wages are such that the positive and the negative externality exactly balance.

When wages are determined by Nash bargaining, there is a one to one correspondence between the workers' share of the surplus created by the job and the wage that prevails in the market. There exists a value of this share that leads to an efficient wage. However, since the share is exogenous, there are no reasons to believe that the actual and the optimal values of the share will coincide.

In the first part of my theses I address this and other questions, by studying alternative ways of determining wages. In the second part, I endogenize the workers' productivity by introducing human capital. Finally, in a third part, I study various topics on matching models, including the efficiency properties of the standard matching model, and also the effects of introducing convex hiring costs. In most of the papers, the starting point is the search equilibrium model presented in Pissarides (1990). I first briefly summarise the main features of this model and then discuss the points of departure taken in each chapter.

Pissarides' model is set in continuous time, and the labour market consists of a continuum of homogenous workers and firms. The agents are modeled in a rather unsophisticated way, workers are either unemployed and searching for a job, or employed and working, while jobs are idle and searching for a worker or occupied (by one worker) and producing. To ensure that there is unemployment in steady state, job matches are destroyed at a constant and exogenous rate.

The most important innovation in the model is the matching function, which gives the number of matches as a function of the number of unemployed workers and vacant jobs, and thus captures the frictions in the market. The matching technology is assumed to be concave, and, like the aggregate production technology, to exhibit constant returns to scale. Workers exit unemployment at random, and their exit rates are thus constant (in steady state), and depends (positively) on the relative number of vacant jobs to unemployed workers, or the labour market tightness, but not on the size of the market. For vacant jobs, the arrival rate of workers depends negatively on the labour market tightness.

Workers thus move between unemployment and employment according to a Poisson process, with endogenous transition rate form unemployment to employment (depending on the labour market tightness), and an exogenous transition rate from employment to unemployment. Analogously, jobs moves between the states of being occupied and vacant. Since the transition rates are constant in time, this makes it relatively straightforward to calculate the expected discounted incomes for the agents in the different states using the Bellman equation (or the asset value equation). Wages are determined by decentralized Nash bargaining. The match surplus is defined as the difference in the joint expected income when matched and when unmatched for a job-worker pair. and is divided according to the Nash sharing rule.

The only decision that is made in the standard model is the firms' decision whether or not to open vacancies. Free entry of firms ensures that the value (or expected income) of a vacant job is zero. The value of a vacancy depends on the labour market tightness; if it is high, the arrival rate of workers is low. Furthermore, since high labour market tightness means, *ceteris paribus*, a high expected income for unemployed workers, the match surplus and thus the value of an occupied job is low as well. The value of a vacancy thus falls with the labour market tightness, and the free entry condition determines it so that a vacancy has zero value. Given the labour market tightness, the transition rate from unemployment is known, and the steady state unemployment rate is easy to calculate.

The first part of the thesis concerns wage determination, and consists of three papers. The titles are "A Matching Model with Wage Announcement", "Bargaining over the Business Cycle" , and finally "Bargaining and Matching".

In "A Matching Model with Wage Announcements", which is the main paper in this part of thesis, we assume that wages are not set by Nash bargaining after the firm and the worker meat but that they are announced publicly by the firm before workers arrive. Consequently, workers know the wages in the jobs they are applying for. The main source for frictions in the model is not imperfect information about the location of jobs, but rather the costs and time lags associated with producing job advertisements, producing and processing job applications, selection of workers etc. When firms announce different wages, we can think of the labour market as divided into submarkets, with different wages in each submarket. Since all workers are homogeneous, they must be indifferent between which of the (non-empty) submarkets to enter. Submarkets with high wages therefore face lower labour market tightness than those with low wages. A vacant firm faces a trade-off between wage costs and search costs, and chooses the submarket that maximizes value of the vacancy. Since the opportunity costs of searching is higher the higher the productivity is, high-productivity firms join submarkets with higher wage than low-productivity firms do.

Hence when a vacant firm decides on wages, it chooses the one that maximizes profit, contingent on providing a certain level of (expected) income for the workers. As a result, the announced wages are optimal in the sense that they yield an efficient allocation of searching workers on submarkets and the vacancies the right incentives to enter the market.

It is not trivial to define the equilibrium of the model since almost all submarkets typically are empty. However, by using a refinement of the rational expectations equilibrium concept along the lines of Gale (1994), I am able to pin down the expectations about arrival rates of workers in empty submarkets uniquely. As a result, I obtain a Walrasian type of equilibrium, where firms maximize profit given their beliefs about the relationship between wages and search costs.

In the two other papers on wage determination, I retain the assumption that wages are determined after workers and firms are matched. However, instead of applying the Nash solution, I assume that wages are determined by strategic bargaining.

In the first paper, "Bargaining Over the Business Cycle", my main goal is to study whether the results derived in the literature on decentralized trade mentioned above can give new insights when applied to matching models of the labour market. My analysis focus on the effects of productivity shocks. I assume that wages are determined by a bargaining game with alternating offers of the Stål Rubinstein type. The outcome of this game depends on the arrival of new trading partners to the players. If the arrival rate of job offers is high, the risk is high that the worker abandon the incumbent firm. This increases the workers bargaining power, and vice versa for the firm. As a result, the share of the surplus allocated to the worker is an increasing function of the labour market tightness, and since the tightness is higher during booms than during recessions, the worker's share of the match surplus fluctuates procyclically. Thus the model predicts wages to be more volatile and the unemployment rate to be less volatile than when the wages are determined by Nash-bargaining.

I also analyze the effects of the shocks being anticipated. My main finding is, that if we allow for renegotiations of the wage after a shock, this typically increases the volatility of wages even further. After say a negative shock, the firm finds itself with higher bargaining power, and starts to renegotiate the wage. This is anticipated by the worker, who is compensated by getting a higher wage before the shock.

In the third paper, "Bargaining and Matching", I introduce a new extensive wage bargaining game. I assume that firms and workers can write binding contracts. In the bargaining game, the firm has all the bargaining power, in the sense that it makes all the proposals. However, the worker can delay the response to an offer, hoping that a second firm shows up. If this happens, the two firms compete in a Bertrand fashion for the worker. On the other hand, if a worker shows up, the two workers engage in Bertrand competition for the job. I show that all equilibrium wage offers are accepted immediately, and that the solution is similar to the solution of the alternating offer game described above.

The second part of the thesis, where I focus on the role of human capital, consists of three papers: "Human Capital Investments and Market Imperfections", "Education and Competition for Jobs", and finally "Loss of Skills During Unemployment".

In "Human Capital Investments and Market Imperfection", I show how frictions in the market can create hold-up problems and lead to under-investment in general human capital. The wages are determined by Nash bargaining. Since the match surplus depends on the productivity of the worker in question, the workers bears all the costs of education, while they only receive parts of the return. As a consequence, we get under-investments in education. Furthermore, this positive externality from education for firms can lead to multiple equilibria; the more firms that enter the market, the more profitable it is for workers to invest in human capital. This again may increase the incentives for vacancies to enter the market. I also study investments in general human capital undertaken after workers and firms are matched (training). When there is turnover in the market, for instance due to on-the-job search, similar results are obtained.

However, these inefficiency problems can be resolved by allowing for non-standard debt contracts, where repayment is contingent on the worker in question being employed. In this case, the repayments become a part of the match surplus, and the firm in effect pays the same share of the investment costs as it receives from the return. The externality is thereby eliminated, and the level of education and training become optimal.

In the second paper, "Education and Competition for Jobs", I also study the incentives to invest in human capital. In addition, I alter the matching technology, and allow for more than one applicant per vacancy. Applicants for a job thus compete "face to face", and since the absence of binding contracts rule out Bertrand competition, a worker can be turned down for a job he strictly wants. Now more education speeds up the transition to employment. Furthermore, an agent's investment in human capital creates a negative externality for other workers, as it reduces their transition rates to employment. I show that the equilibrium of the model can take different forms, depending on parameter values. If the competition effect is not too strong, *ex ante* homogeneous workers diversify and choose different levels of education. If the competition for jobs is very strict, all workers again behave equally, and choose a level of education above the socially optimal level. At this point, the hold-up problem described above is eliminated, while the negative externality from education on other workers prevails.

In the third article, I assume that workers loose skills during unemployment. The idea is not new, Pissarides (1992) makes the same assumption. What is new is to implement loss of skills into a standard matching model. I show that the model may exhibit multiple equilibria, and that the unemployment rate tends to be over-optimal. Furthermore, as unemployed workers stay too long in the market, unemployment benefits directed exclusively towards long-term unemployed can reduce overall unemployment.

The last part of the thesis consists of the two articles "Optimal Unemployment" and "A Matching Model with Hiring Costs". The first one gives an analysis the welfare properties of matching models, more thorough and general than the existing literature provides, using optimal control theory. I show that the optimality rule derived in Pissarides (1990) and other places (that optimality is achieved when the workers' bargaining power β is equal to the elasticity of the arrival rate of workers with respect to the labour market tightness) holds out of steady state and for time dependent exogenous variables, but not if the agents are heterogeneous or if workers do on-the-job search.

The free entry of vacancies described above implies that the vacancy rate is extremely volatile, and substantially overshoots its new equilibrium value after a shock. In "A Matching Model with Hiring Costs", I deal with this by assuming that expanding a firm by employing more workers is costly, and that the costs are convex in the rate of change. I show that this alter the dynamic properties of the model substantially, the vacancy rate becomes less volatile and the transition after a shock to the new steady state equilibrium goes more smoothly and slowly.

Although most of the chapters in the thesis are interrelated, they are presented as autonomous papers. Thus they can be read independently of eachother, an advantage I think that more than outweighs the costs of some repetition.

Part I

Wage Determination

Chapter 1

A Matching Model With Wage Announcement

1.1 Introduction

Search models are widely used as an example of how decentralized markets may fail to produce an efficient allocation of resources. When wages are determined by *ex post* bargaining, a wage which yields the correct incentives *ex ante* exists (Hosios 1990), but in general there are no mechanisms, or "forces", which lead to the optimal wage. Hence the equilibrium wage is typically inefficient.

In this article I assume that firms are able to communicate wage offers to potential workers before they are matched. More specifically, I assume that a firm, when advertising the vacancy, also announces the offered wage. Thus workers know the wages in the jobs for which they apply. This turns out to resolve the inefficiency problems described above: the equilibrium wage offers lead to (constrained) optimal allocation of resources.

Compared to the early search literature, publicly available information about wages may seem a strange assumption. In early search theory (Mortensen 1971, Phelps 1971), the main reason for search activity was the collection of information about wage offers in different firms. Diamond (1971) was the first to show that wage posting with sequential search leads to a unique equilibrium wage, removing the information gathering role of search. In Pissarides (1985), (1987) the role of search in revealing information about wages is also removed; firms and workers are homogeneous, and in equilibrium all firms pay the same wage. The costs and time delay associated with the search process are due to unmodelled frictions and costs when trading, represented by the matching function. Costs and time delay due to information gathering about wages or to costly wage negotiations are not part of the trading frictions underlying the matching function. This point is made clear in Pissarides (1990).

In this paper I explicitly remove the information-gathering function of search by assuming that all wages are publicly announced and so available to all workers prior to search. This is motivated by the fact that public wage announcement is common in parts of the labour market. Often job advertisements for low-skilled workers (and high skilled as well) give information about the wage offered.

The idea that information gathering about wages plays a minor role in labour market search is consistent with the description of the search process given in Layard *et al.* (1991). They argue that a worker's job search can be divided into two parts: First he collects information about vacancies, which come with different pre-assigned wages and conditions. Then he applies for some of the vacancies he has heard off. In general the worker accepts the offer for any job for which he has applied[1].

I focus on the second part of the process (writing and processing of appli-

[1]Akerlof *et al.* (1988) finds that only 8 percent of job-seekers have rejected a previous job offer.

cations, interviews, time lags due to selection by the firm etc.) as the main contributor to costs and time-delays. I therefore assume that information collection about wages happens instantaneously and without any costs.

Note also that similar assumptions can be found in the search models of the retail market, see f.i. Butters(1977) or Robert and Stahl(1993). Here the shops can, through (costly) advertising, give potential customers information about their price. The buyers thus receive price information prior to search.

A brief description of the model

I use a standard search model, where the basics are taken from Pissarides (1990). Each firm has either one worker and is producing, or one vacancy and is searching for a worker. Firms may differ in productivity, while all workers are equally productive. Workers are either employed and not searching, or unemployed and searching. There is a fixed cost $k \geq 0$ associated with the opening of a vacancy, and the productivity of the vacancy is drawn from a distribution F after the cost k is incurred.

The main innovation in this paper is in the wage setting process. In Pissarides'(1990) model, as in the models of Diamond(1982) and Mortensen(1982), wages are set after the firm and the worker meet, so as to split the surplus from the job match. There is then a unique wage outcome for each match productivity, which is in general inefficient. In my model the firms choose the wage and announce it before workers arrive.

If different vacant firms announce different wages, we can think of them as separating the labour market into submarkets. In each "submarket" the announced wage is the same, while it differs between submarkets[2]. The number of matches in each submarket is determined by the number of jobseekers and vacancies in each market. In equilibrium unemployed workers must be

[2]The term submarket is more fully defined below.

indifferent between submarkets, so, because they know the wage associated with each submarket, the labour market tightness must be higher in markets where the wage is higher. Thus, when deciding its wage, the firm faces a trade-off between search costs and wage costs.

I show that the equilibrium is separating, high productivity firms always offer strictly higher wage than low productivity firms. If the firms are homogeneous, they generally announce the same wage. Furthermore, the wages announced are optimal in the sense that they give an efficient allocation of unemployed workers across submarkets, and also give vacancies the right incentives to enter the market.

Discussion

Most retail market search models with price advertising are different from my model, since they do not have congestion effects on both sides of the markets. The suppliers (who advertise) are typically ready to satisfy any demand. Hence, customers always visit the supplier who announces the lowest price. In my model this does not hold because frictions are present on both sides of the market, and the well-paid jobs are more difficult to get.

An exception is the model by Peters(1991). Here a non-stationary market with a continuum of homogeneous agents is studied. There is no entry, and matched agents leave the market. His focus is on the construction of a matching technology when the agents are matched randomly, but when the match probability for a seller is influenced by the price that he advertises.

To achieve this, Peters partitions the sets of agents into a finite numbers of subsets. In each time period, a two-stage matching game between the sets of buyers and sellers is constructed, where the "sellers" first announce prices, and the "buyers" decide which "seller" to visit. A matching technology is constructed, which for a given "seller" gives the probability of getting matched as a function of announced price, given the other (symmetric) price announcements.

In contrast, my paper follows a Walrasian approach. The atomistic firms choose the wage to announce given their *beliefs* about the relationship between the wage and the arrival rate of workers. The beliefs are exogenous to the firm.

However, in order to obtain a reasonable outcome, we must restrict the firms' out-of-equilibrium beliefs (i.e. the beliefs about the arrival rates of workers in empty submarkets). We therefore proceed along the lines of Gale (1992,1994), and apply a refinement of the rational expectations model similar to his concept of a stable walrasian equilibrium. This refinement resemblances the stable equilibrium concept derived in Kohlberg and Mertens(1986). In our context, an equilibrium is stable (loosely speaking) if any deviation from the equilibrium wage, announced by a tiny set of firms, only has a small impact on the equilibrium.

The walrasian approach simplifies the analysis, and enables the treatment OF wage announcements in A stationary and continuous time search model with entry, commonly used in labour economics. The simplicity of the model also makes efficiency considerations tractable, both with respect to the distribution of searching workers over firms with different productivity, and to the entry decision of vacancies. This is absent in Peters' model.

Wage announcement in models of the labour market is rather uncommon. An exception can be found in Montgomery (1991). In a different setting, Montgomery uses wage announcement to explain inter-industry differences in wages.

Efficiency wages

The idea that firms may increase the wage to attract more workers is commonly used as a rationale for efficiency wages, see for instance Layard *et al.* (1991). The issue was important in early search theory, and is also central in for instance Burdet and Mortensen(1989). In this literature the workers typically have different reservation wages, and therefore the number of workers accepting a job increases with the posted wage (although the arrival rate of workers is independent of the wage).

As already mentioned, empirical findings suggest that there is very little rejection of job-offers. Another empirical finding is that higher wages do attract more applicants. Holzer *et al.* (1988) finds that higher wage leads to more applicants per vacancy (although the effect is weak). Kaufman(1984) finds that employers certainly believe there is a relationship between the wage and the arrival rate of workers.

In my model, firms are able to manipulate the arrival rate of workers through the announced wage. The efficiency wage argument thus holds even with homogeneous workers. The model is also consistent with the fact that workers rarely reject jobs.

1.2 The Model

I assume that all workers are identical and risk-neutral. The number (measure) of workers is constant and normalized to 1. The number of jobs are endogenously determined through entry, with a sunk cost $k \geq 0$ associated with the opening of a vacancy. When the cost is incurred, the productivity of the vacancy is drawn from a discrete probability distribution F with mass points at $y_1, , , y_n$. Firms are risk neutral, and the free entry condition therefore implies that the expected value of a vacancy equals the creation cost.

Let $x(u,v)$ denote the stream of new worker-firm matches, where u is the measure of unemployed workers searching for a measure v of (advertised) vacancies. The matching function $x(u,v)$ captures the frictions in the market. As described in the introduction the sources of the frictions are time delays when producing and proceeding applications etc., and to unmodelled heterogeneities and information imperfections (though not concerning wages).

Following standard assumptions, let x be concave, and homogeneous of degree 1 in (u,v). We also assume that x has continuous derivatives. Let $p = x(u,v)/u = x(1,\theta) = p(\theta)$ denote the transition rate from unemployment to employment for an unemployed worker, and $q = x(u,v)/v = q(\theta)$ the arrival rate of workers for a vacancy, where θ is the labour market tightness v/u. Let

$$\lim_{\theta\to 0} p(\theta) = \lim_{\theta\to\infty} q(\theta) = 0$$

and

$$\lim_{\theta\to\infty} p(\theta) = \lim_{\theta\to 0} q(\theta) = \infty$$

When the labour market tightness goes to zero, the arrival rate of trading partners for firms and workers go to infinity and zero respectively. When θ goes to infinity, the opposite holds.

When matched, the worker-firm pair start to produce immediately, and the worker receives the announced wage. Following Pissarides (1990) there is a constant and exogenous probability rate s of match destruction. When separated, the worker joins the unemployment pool, while the remaining vacancy is worthless and therefore destroyed.

Before we continue to study the behaviour of the firms and the workers, let us define a submarket in the following way: A submarket is defined in the following way:

Definition 1 *A submarket with wage w_i (or submarket i) consists of all firms announcing wage w_i and all workers that apply for jobs with this wage.*

1.2.1 Workers

In this subsection we study the workers' behaviour in steady state when the announced wages are given by a finite-dimensional vector $(w_1, , , , w_m)$.

Let U_i denote the expected discounted income (or asset value) for an unemployed worker in submarket i. Then we have that

$$rU_i = z + p(\theta_i)(E_i - U_i) \tag{1.1}$$

Here z denotes the unemployment income, θ_i the labour market tightness in submarket i, r the discount factor, and E_i the expected income when employed at wage i. The latter can be written as

$$rE_i = w_i - s(E_i - U) \tag{1.2}$$

where s denote separation rate. Substituting out E gives

$$rU_i = \frac{(r+s)z + w_i p(\theta_i)}{r + s + p(\theta_i)} \tag{1.3}$$

for $w \geq z$. If $w < z$ the workers do not search, and $rU = z$.

The workers enter the submarkets that yield the highest expected income. All submarkets that attract workers must therefore give the same expected income. Denote this income by U. Substituting in for U in (1.3) and rearranging gives

$$p(\theta_i) = \frac{rU - z}{w_i - rU}(r+s) \tag{1.4}$$

For a given U the equation defines a unique relationship between the wage and the labour market tightness in each submarket. We write this relationship as $\theta(w;U)$. For later reference we also define the correspondence $G(U)$ as the set of pairs $(w,1/\theta)$[3] which give workers an expected utility U. Formally, $G(U)$ is defined by

$$G: R_+ \to R_+^2,\; G(U) = \{(w,1/\theta) : p(\theta) = \frac{rU-z}{w-rU}(r+s); w > U\} \cup (rU,0) \tag{1.5}$$

In the appendix we show that $G(U)$ has the following properties:

Lemma 1 *$G(U)$ is continuous, bounded, and non-empty*

1.2.2 Firms

As already mentioned, the productivity of a vacancy is determined after the fixed cost is incurred. The vacancy is maintained (and announced) if and only if the is profitable, if not it is destroyed immediately. Below I first calculate the value of an announced vacancy.

Denote by $V(y_i,w,)$ and $J(y_i,w)$ the expected discounted value, or asset value, of an announced vacancy and a filled job with productivity y_i respectively, when the posted wage is w. Let $q^e(w)$ denote the firms' beliefs about the arrival rate of workers when announcing w. The (perceived) asset value equation for a vacant job is then

$$rV(y_i,w) = -c + q^e[J(y_i,w) - V(y_i,w)]$$

The (perceived) asset value equation of a filled job is

$$rJ(y_i,w) = y_i - w - sJ(y_i,w) \tag{1.6}$$

[3]It turns out that some of the proofs are simpler when working with $1/\theta = u/v$ rather than θ.

If we substitute the expression for J into the value equation for V we get

$$(r+q)V(y_i,w) = q^e(w)\frac{y_i - w}{r+s} - c \tag{1.7}$$

The question is now how the expectations $q^e(w)$ are formed. For all wages actually announced in equilibrium, a standard rational expectations argument implies that $q^e(w_i) = q(\theta(w_i))$, where $\theta(w_i)$ is defined by (1.4). However, this is not enough to give the model predictive power. With no restrictions on the out-of-equilibrium beliefs, any wage can be optimal to announce if the beliefs are "right". We therefore make the following assumption, rationalized in the next section:

Firms beliefs': The firms' beliefs are such that

$$q(w) = q(\theta(w))$$

for all w, where $\theta(w)$ is defined by (1.4)

An equivalent way of expressing this is to say that $(w, q^{-1}(q^e(w)) \in G(U)$. The arrival rate of workers to firms and the arrival rate of jobs to workers are linked through the matching function. The assumption therefore implies that the firms expect to give potential applicants an expected income U also if they announce an out of equilibrium wage.

The firms' maximization problem can be written as

$$\max_w V(y_i,w) = \max_{(w,1/\theta)\in G(U)} \frac{q(y_i - w)/(r+s) - c}{r+q} \tag{1.8}$$

For all w, $V(y_i,w) \leq y_i/r$, so V_i is bounded for all i. Let $\bar{V}_i$ denote the supremum of $V(y_i,w)$. We then get the following result:

Proposition 1 *When $\bar{V}_i \geq 0$ and $c > 0$, there exists a solution w_i^* to the problem given by (1.8), with $rU < w_i^* < y_i$. When $c = 0$, the result holds iff $\bar{V}_i > 0$.*

Proof: For all $w \geq y_i$ we have that $V(w) \leq -c/(r+q)$. For all $w < rU$ we have that $V = -c/r$, since the firm then search forever without getting a worker. Since $\bar{V} \geq 0$, this implies that $\bar{V} = \sup_{w \in [rU, y_i]} V(w)$. Since the interval is closed, $V(y_i, w)$ is continuous and $\bar{V} > 0$, we know that an optimum exists and lies in the interior of $[rU, y_i]$. The argument runs in the same way when $c = 0$ and $V > 0$

□

Since all firms with the same productivity face the same optimization problem, they choose the same wage unless (1.8) has more than one solution. In the appendix I give sufficient conditions for uniqueness of the optimal solution, but I do not assume uniqueness in the exposition below. However, to simplify the analysis I assume that if there is more than one optimal solution for vacancies with a given productivity, all the firms with this productivity still choose the same wage[4]. Hence, for a given value of U, the number of announced wages can at most be equal to the number $\tilde{n}$ of different productivity levels among operating vacancies. The following proposition ensures that we get exactly $\tilde{n}$ different wages:

Proposition 2 *Let w_i be a solution to (1.8) when the productivity is y_i. Then $w_i > w_j$ if and only if $y_i > y_j$*

1.2.3 Entry

From (1.8) it follows that we can write the maximum value of a maintained and announced vacancy as a function $\tilde{V}(y_i, U)$. The vacancy will be an-

[4]Note, however, that since the functional forms in the maximization problem are exogenous, more than one solution is in some sense "unlikely".

nounced if and only if $\tilde{V}(y_i, U) \geq 0$, otherwise it is destroyed. The value of any vacancy can therefore be written as

$$V(y, U) = \max[0, \tilde{V}(y, U)]$$

Let $\hat{\imath}$ denote the cut-off productivity for destruction of the vacancy, so that $V(y_i, U) \geq 0$ for $i \geq \hat{\imath}$ and $V(y_i, U) < 0$ for $i < \hat{\imath}$. The expected value of opening a vacancy is then given by

$$V(U) = \sum_{i=\hat{\imath}}^{n} V(y_i, U) f_i$$

where $f_i = Pr[y = y_i]$. Since k denotes the cost of creating a vacancy, entry implies that

$$V(U) = k$$

Intuitively, the value of an (announced) vacancy decreases with U, since by (1.4) higher U means higher labour market tightness and thereby higher search costs for all w. This and some other important properties of V are stated in the lemma below. The proof is given in the appendix.

Lemma 2 *$V(y, U)$ is continuous, strictly increasing in y for $y > rU$ and strictly decreasing in U on the interval $(z, y]$. $V(U)$ is strictly decreasing in U on the same interval.*

1.3 Equilibrium

To close the model we must include the steady state relationship between the stock of unemployed and the streams into and out of the various submarkets. This gives a multi-dimensional version of the Beveridge curve. Note that in steady state, the stream of workers entering the unemployment pool is given by $(1-u)s$, and is equal to the stream of vacancies entering the market. Furthermore, the inflow of vacancies must be equal to the outflow in each submarket, the latter given by $u_i\theta_i$ where u_i denotes the measure of unemployed in the submarket in question. We therefore have that

$$u_i p(\theta_i) = (1-u)s\tilde{f}_i$$

where $\tilde{f}_i = f_i/(1-F_{\hat{i}-1})$, with $F_{\hat{i}} = Pr[y \leq y_{\hat{i}}]$. Together with the fact that $\sum_i u_i = u$, this equation determines $u_{\hat{i}}, ..., u_n$ and u given $\theta_{\hat{i}}, ..., \theta_n$.

An equilibrium E^* of the model is defined in following way:

Definition 2 *An equilibrium of the model is two scalars U and u, and vectors $(w_{\hat{i}}, ..., w_n)$, $(\theta_{\hat{i}}, ..., \theta_n)$ and $(u_{\hat{i}}, ..., u_n)$ such that*

$$\begin{aligned}
V(U) &= k && (1.9)\\
w_i &= argmax_{w\in G(U)} V(y_i, w) \;\; i \geq \hat{i} && (1.10)\\
rU &= \frac{(r+s)z + p(\theta_i)w_i}{r+s+p(\theta_i)} \;\; i \geq \hat{i} && (1.11)\\
u_i\theta_i q(\theta_i) &= \tilde{f}_i(1-u)s \;\; i \geq \hat{i} && (1.12)\\
\sum_{i=\hat{i}}^{n} u_i &= u && (1.13)
\end{aligned}$$

where $G(U)$ is defined by (1.5) and $\hat{i}$ is the smallest i such that $\tilde{V}(y_i, U) \geq 0$.

Proposition 3 *If $\sum_{i=1}^{n} \max[y_i - z, 0] f_i/(r+s) > k$, the equilibrium defined above exists.*

Note that the structure of the equilibrium is quite simple, since it is almost recursive. The key variable U is determined in the first equation by the entry condition. Given U, the second equation determines the announced wages, and the third equation the corresponding values of θ. The two last equations determine the unemployment rate and the distribution of unemployed over submarkets given the labour market tightness in each submarket. Note also that although the equilibrium is not necessarily unique (since (1.10) for each i can have more than one solution), the value of U is, i.e. in all equilibria the unemployed workers get the same expected income.

1.3.1 Characterization of Equilibrium

If we substitute $p(\theta) = \theta q(\theta)$ into (1.4) we get that

$$\theta(w)q(\theta(w)) = \frac{rU - z}{w - rU}(r + s) \tag{1.14}$$

Since the maximization problem always has an interior solution, the optimal w must be such that $V'(w)|_{\theta(w)} = 0$. Taking the derivative of (1.7) with $\theta^e = \theta(w)$ gives:

$$q'(\theta_i)\frac{d\theta}{dw}(\frac{y_i - w}{r + s} - V_i) = \frac{q(\theta_i)}{r + s} \tag{1.15}$$

The left hand side gives the value of increased q when w increases, the right hand side the costs of increasing the wage as a result of lower profit when a worker is found. Taking the derivative of (1.14) with respect to w gives

$$\frac{d\theta}{dw}q(1 - \eta) = -\frac{rU - z}{(w - rU)^2}(r + s)$$

where $\eta = \eta(\theta_i) = -\theta q'(\theta)/q$ and so $\frac{d}{d\theta}\theta q(\theta) = q(1 - \eta)$. Substituting out $\frac{rU-z}{w-rU}$ by virtue of (1.14) gives

$$(1 - \eta)\frac{d\theta}{dw} = -\frac{\theta}{w - rU}$$

Inserting this into (1.15) yields

$$\frac{\eta_i}{1 - \eta_i} = \frac{w - rU}{y_i - w - (r + s)V(y_i)} \tag{1.16}$$

$$= \frac{E_i - U}{J_i - V_i} \tag{1.17}$$

Where, as before, J and E denote the asset value of an occupied job and of an employed worker, given by the equations (1.2) and (1.6) respectively. Defining the match surplus S_i as $S_i = J_i - V_i + E_i - U$, we find that $J - V = (1 - \eta_i)S_i$ and $E - U = \eta_i S_i$. Thus the division of surplus between the worker and the firm is the same as when the wage is determined by ex post bargaining and the worker' bargaining power is η_i.

The equilibrium can thus be characterized by the following equations (with $\eta_i = \eta(\theta_i)$)

$$\begin{aligned}
k &= \sum_{i=\hat{i}}^{n} f_i V_i && (1.18)\\
rU &= z + \theta_i q(\theta_i)\eta_i S_i \quad i \geq \hat{i} && (1.19)\\
rV_i &= -c + q(\theta_i)(1 - \eta_i)S_i \quad i \geq \hat{i} && (1.20)\\
S_i &= \frac{y + sU}{r + s} - U - V \quad i \geq \hat{i} && (1.21)
\end{aligned}$$

where the last equation is derived from the asset value equation for S. In addition the equilibrium must satisfy the equations (1.12) and (1.13). Note that the equilibrium is equivalent to the corresponding equilibrium with ex post bargaining when the workers' bargaining power in a firm of type i is η_i.

1.3.2 Rationalization of the out-of equilibrium beliefs

The equilibrium defined above rests heavily on the assumptions about the out-of equilibrium beliefs. Clearly other beliefs lead to a different equilibrium. In this section I show how a refinement of the rational expectations equilibrium concept, similar in spirit to Gale's(1992) concept of stable equilibrium, rationalizes our out-of equilibrium beliefs and delivers the equilibrium determined by (1.9)-(1.13) as the only equilibrium.

The idea is to study the effects of exogenous deviations by small sets of firms. This has strong bite in the model. If some firms deviate and announce a wage that was not announced in the original equilibrium, the labour market tightness in this submarket "becomes observable", and this effectively rules out "crazy" beliefs.

A (unrefined) rational expectations equilibrium E^r is defined by the equations (1.9)-(1.13), but with $G(U)$ substituted out with an arbitrary relationship $q^e(w)$ between q and w. The only restriction we pose on q^e is that $q^e(w) = q(w)$ for all wages announced in equilibrium.

A deviation is any finite dimensional[5] vector $(w_1^d,,,, w_m^d,\ \alpha_1,,,,\alpha^m, \varepsilon)$ such that $0 \leq \alpha_i \leq \varepsilon$ and $\sum_{i=1}^m \alpha_i = \varepsilon$. A rational expectations equilibrium with deviations $E^r(w^d, \alpha)$ is a rational expectations equilibrium with the restriction that w^d is announced by at least α firms. We now define stability in the following way[6]:

Definition 3 *A rational expectations equilibrium E^r is stable iff, for all sequences $\{(w_i^d, \alpha_i, \varepsilon_i\}_{i=0}^\infty$ such that $\lim_{i\to\infty} \varepsilon_i = 0$ we have that*

$$\lim_{\varepsilon \to 0} E^r(w^d, \alpha, h) = E^r \tag{1.22}$$

We get the following result:

Proposition 4 *1. The equilibrium defined by (1.9)-(1.13) is stable.*

2. All other rational expectations equilibria are not stable.

Proof: The proof of the first part is simple, since the existence of deviating firms only influence the equilibrium in the sense that the submarkets with

[5]For technical reasons we do not require that the tremble includes all possible wages. This contrasts Gale(1992) and Kohlberg and Mertens(1986)

[6]In the definition below I implicitly assume that when considering entry, the potential vacancy does not take into account that it may "tremble"

wage w^d are non-empty, with labour market tightness given by the corresponding element in $G(U)$[7]. The firms' maximization problem is therefore unaltered by the deviating firms, and thereby also $V(U)$.

To prove the second part, let E^r be a rational expectations equilibrium that does not satisfy equations (1.9)-(1.13). Let w^r denote the vector of announced wages, and U^r the equilibrium value of U. Furthermore, let w^G denote the vector of wages that maximizes $V(w^G, y, U^r)$. Clearly, for some i we must have that $w_i^r \neq w_i^G$, and therefore also that $EV(w_i^G, y_i, U^r) > EV(w_i^r, y_i, U^r)$. Define U^G to be the solution to the equation $EV(w_i^G, y_i, U^G) = 0$. Then $U^G > U^r$.

Now we study deviations from the rational expectations equilibrium of the form $(w^G, \alpha/n, \alpha)$ with $\alpha > 0$. In any such deviation equilibrium the value of U must be at least U^G, and hence the deviation equilibrium does not converge to E^r as $\alpha \to 0$. Therefore the rational expectations equilibrium is not stable.

$\square$

1.4 Optimality

In this section we look at the welfare properties of the model, and analyze whether the equilibrium gives a socially optimal number of vacancies entering the market, and an optimal allocation of unemployed workers across submarkets. The optimality criteria we use is the same as in Pissarides(1990), maximizing the discounted aggregate production net of search costs.

Let a denote the stream of new vacancies created. The social optimum then

[7]This is of cause only the case if the value of U, given that only the deviating firms are in the market, is lower than the equilibrium value U^*. However, this is always the case for small enough values of α.

maximizes

$$W = \int_0^\infty e^{-rt} \sum_{i=\hat{i}}^{n} [N_i y_i + z u_i - c v_i - ak] dt \tag{1.23}$$

with respect to $a, \hat{i}$, and $u_{\hat{i}}, ..., u_n$, given that the paths of the state variables $N_{\hat{i}}, ..., N_n, v_{\hat{i}}, ..., v_n$ are are governed by the differential equations

$$\dot{N}_i = v_i q(v_i/u_i) - s(1 - N_i) \quad i \geq \hat{i} \tag{1.24}$$

$$\dot{v}_i = aF_i - v_i q(v_i/u_i) \quad i \geq \hat{i} \tag{1.25}$$

and given the constraint

$$\sum_{i=\hat{i}}^{n} (u_i + N_i) = 1 \tag{1.26}$$

We now get the following result

Proposition 5 *All equilibria satisfying (1.9)-(1.13) are optimal.*

The proof is given in the appendix. Note that if there is more than one equilibrium solution, the proposition implies that they are equivalent from a welfare point of view. To get some intuition why the model leads to an optimal allocation, first notice that in sequential search models with ex post bargaining, the incurred search costs are sunk when bargain takes place. Thus the costs will not influence the bargaining game directly, and there are no reasons why the bargaining outcome should reflect the expected search costs *ex ante* and thereby lead to the optimal number of vacancies.

In my model the wage determination is centralized in the sense that all wage offers are evaluated simultaneously by the workers before they are matched. When the firms choose the wage, they maximize profit given the worker's indifferent constraint, and thereby internalize the effects of their own decision for the unemployed workers.

1.5 Examples and extensions

In this section we first study the relationship between productivity and wage (insider-outsider effects). Then we extend the model to allow for heterogeneous workers.

1.5.1 Wage distributions

We can write the optimal wage announcement for a firm with productivity y as $w = \delta(y; U)$. The function δ relates the exogenous distributions of productivities over firms and the distribution of wages. We know from proposition 2 that δ is strictly increasing in y. We can also show the following result:

Lemma 3 $\lim_{y \to \infty} \delta(y; U) = \infty$. *When* $c = 0$ *we also have that* $\lim_{y \to rU} \delta(y; U) = rU$

For a general functional form of the matching function we can not say much specific about the properties of δ. It might even be discontinuous in y, since (1.8) can have more than one solution.

In the literature, the matching functions are often assumed to be Cobb-Douglas, a functional form that fits data reasonably well. Therefore let $x(u,v) = u^{\beta}v^{1-\beta}$. Then $q = \theta^{-\beta}$, $\eta = \beta$ (since η is the absolute value of the elasticity of q with respect to θ), and finally $q = p^{-\frac{\beta}{1-\beta}}$. Since $E - U = (w - rU)/(r+s) = \beta S$ (where E still denote the expected discounted income when unemployed), (1.19) gives

$$w(y) = rU + (r+s)\beta S(y) \tag{1.27}$$

Taking the derivative of V with respect to y in (1.8) using the envelope theorem, gives $V'(y) = q/[(r+s)(r+q)]$. Taking the derivative of S in (1.21) thus gives

$$S'(y) = \frac{1}{r+s} - \frac{q}{(r+s)(r+q)} = \frac{r}{(r+s)(r+q)} \tag{1.28}$$

Since the announced wage increases with y, so does q. We therefore immediately get that the effect on the announced wage of increased productivity y is decreasing in y, i.e. $\delta'(y)$ is decreasing in y. Further, by the virtue of Lemma 3 we find that $\lim_{y\to\infty}\delta'(y) = 0$. In the case with $c = 0$ we find that $\lim_{y\to rU+}\delta(y) = \beta$ (since θ in the two cases goes to infinity and zero respectively). Thus, loosely speaking, we can say that the "insider effect" (the effect of a firm's productivity on the wage) is stronger for low values of y than for high values of y.

For the special (but interesting) case where $\beta = 1/2$, we can easily calculate δ by solving for S in (1.28). We then find that the announced wage is a function of the square root of the productivity y.

1.5.2 Extensions

In this section we extend the model to allow for heterogeneous workers. More specifically, we assume that the workers differ in z, their income (or utility) when unemployed. We also assume that η is non-decreasing, which implies that the equilibrium is unique in the previous model with homogeneous workers.

Denote each worker's unemployment income by z_i, $i = 1,,,k$, with z_i increasing in the index i, and let U_i denote the expected discounted income for a worker of type i. Then U_i is increasing with i. Define $G_i(U)$ and $\theta_i(U)$ for each i in the same way as we did in (1.5) and (1.4). Finally define $G(U_1,...,U_k)$ as the set of pairs $(1/\theta, w)$ where for each w, $\theta = \min_i \theta_i(w)$.

We extend the equilibrium given by (1.9)-(1.13) in the obvious way, such that the firms choose w to maximize profit given $G(U_1,...,U_m)$, and each type of unemployed enters the submarkets that maximize their expected income. In addition, we have to equal flows of workers into and out of unemployment for each type of workers.

Proposition 6 *1. With homogeneous firms, the equilibrium described above is unique. The equilibrium vectors $(w_1^*, ..., w_k^*)$, $(U_1^*, ..., U_k^*)$, are such that for all i, (w_i^*, U_i^*) corresponds to the (unique) equilibrium values in the original model with workers of type i only.*

2. In any equilibrium with heterogeneous firms, all w announced by firms with productivity y_i are strictly greater than all w announced by firms with productivity $y_j < y_i$.

Furthermore, workers with unemployment income i apply for jobs with strictly higher wage than workers with unemployment income $z_j < z_i$.

The first part of the proposition says that with homogeneous firms, the equilibrium with heterogeneous workers can be obtained by pooling the corresponding set of homogenous-worker equilibria. Since the labour market tightness in each submarket is socially efficient (and this is the only allocative variable in this case), the allocation for the economy as a whole is also efficient. Note also that since the wage in the homogeneous workers equilibrium is increasing in z, the workers with high unemployment income get higher wages than the workers with low unemployment income. The intuition is clear, it "hurts" more for the worker with low unemployment income to stay unemployed than for the worker with high unemployment income. Therefore the former is relatively more concerned about getting a job quickly than the latter is.

The last part of the proposition says that with both heterogeneous firms and workers, the market is still completely separating. This implies that an unemployed with high z always apply for jobs with at least as high productivity as workers with lower z. In other words, workers with low waiting costs enter the submarkets with productive firms and therefore low transition rate to employment. This seems reasonable from an efficiency point of view.

1.6 Conclusion

In this paper I studied the effects of wage announcements in the context of a simple search model. The solution I obtained is different from Diamonds'(1971), who also studied a search equilibrium with wage announcements. In Diamond's model, a firm announces the wage after it is matched with a worker. The firm then has an "advantage" compared to other firms, since the worker must incur search costs to get another job offer. Therefore, the firm can offer the worker a lower wage than offered by other firms, driving the wage down to the monopsony wage in symmetric equilibrium.

In my model this mechanism does not work, because the wage is announced prior to the match. Each firm, when deciding a wage to announce, faces competition from other firms on equal terms to obtain workers. The solution that I obtain in symmetric equilibrium is constrained-efficient.

Burdett and Mortensen(1989) study a search model with wage announcements and find a nondegenerate wage distribution in equilibrium. The reason is that in their model the announcements are made after the contact, but the workers may have more than one job offer at one time.

I have treated the wage offers as binding, and excluded that workers may start bargaining with the firm when they are matched. Note however that the announced wage is equal to the Nash bargaining solution when the workers' bargaining strength β equals η. If $\beta \leq \eta$, the announced wage is higher than what the worker can obtain by bargaining, if $\beta > \eta$, the opposite holds. This suggests that wage announcement is more likely to occur in labor markets where the workers' bargaining power is relatively small.

Appendix

Proof of Lemma 1

Define $\tilde{\theta} = 1/\theta$, and let $\tilde{\theta}(w,U)$ be defined by (1.4) for $y_n > w > rU$. For $w = rU$, let $\tilde{\theta} = 0$. Note that since

$$\lim_{w \to rU+} \tilde{\theta}(w,U) = \lim_{U \to w/r^-} \tilde{\theta}(w,U) = 0$$

this implies that $\tilde{\theta}(w,U)$ is continuous at $[rU, y_n]$. In addition, $\tilde{\theta}$ is decreasing in U, increasing in w, bounded, and non-empty for $y_n > w$.

We know that $G(U)$ is continuous if it is lower hemicontinuous and has closed graph. To show lower hemicontinuity, let $\{U^n\}$ be an arbitrary sequence in $[z, y_n]$ which converges to U, and let $(w, \tilde{\theta})$ be an element in $G(U)$. We have to show that there exists a sequence $\{(w^n, \theta^n)\}$, where $(w^n, \theta^n) \in G(U^n)$ which converges to (w, θ).

The proof is done by construction. First assume that $w > rU$. Then there exists an $\bar{n}$ such that for all $n \geq \bar{n}$, $rU^n < w$. Fix $w^n = w$ for $n \geq \bar{n}$, and define $\tilde{\theta}^n = \tilde{\theta}(w, U^n)$. Since $\tilde{\theta}()$ is continuous at (w,U), the sequence $(w^n, \tilde{\theta}^n)$ converges to $(w, \tilde{\theta})$.

If $w = rU$ we know that $\tilde{\theta} = 0$. Define the sequence $\{rU^n, 0\}$, which obviously converges to $(rU, 0)$ when U^n converges to U. This shows lower hemicontinuity.

To show that $G(U)$ has closed graph, let $\{U^n\}$ be a sequence in $(z, y]$ which converges to a point U^* in the same set (i.e. not to z). Let $\{(w^n, \tilde{\theta}^n)\}$ be an arbitrary sequence with $(w^n, \tilde{\theta}^n) \in G(U^n)$. We have to show that if $\{(w^n, \tilde{\theta}^n)\}$ converges to $(w^*, \tilde{\theta}^*)$, then $(w^*, \tilde{\theta}^*) \in G(U^*)$. Rewrite $(w^n, \tilde{\theta}^n)$ to $(w^n, \tilde{\theta}(w^n, U^n))$. Since $\tilde{\theta}(,)$ is continuous in w^n and U^n, we have that $\tilde{\theta} = \tilde{\theta}(w^*, U^*)$, and by definition $(w^*, \tilde{\theta}^*) \in G(U^*)$. This shows that $G(U)$

has closed graph.

Proof of Proposition 2

Let $y_1 > y_2$ and $w_1 > w_2$, and define $\Delta_i = V(w_1, y_i) - V(w_2, y_i),\ i = 1,2$. Then we have

$$\Delta_1 - \Delta_2 = \frac{q_1}{r+q_1}(y_1 - y_2)/(r+s) - \frac{q_2}{r+q_2}(y_1 - y_2)/(r+s) > 0$$

where $q_i = q(\theta(w_i))$ and thus $q_1 > q_2$. This means that a firm with high productivity allays earns strictly more than a firm with lower productivity when increasing the wage, thus $\delta(y)$ is nondecreasing in y. Further, since by assumption the derivative of V with respect to w is continuous, the optimal wages cannot be equal, and hence $\delta(y)$ is strictly increasing in y.

Proof of Lemma 2

V(U) is continuous in U Define

$$V(w, \tilde{\theta}) = \frac{q(\tilde{\theta}(w))(y-w)/(r+s) - c}{r + \tilde{q}(\tilde{\theta})} \tag{1.29}$$

which is continuous for all $w, \tilde{\theta} \geq 0$. $V(U)$ can then be written as

$$V(U) = \max_{(w,\tilde{\theta}) \in G(U)} V(w, \tilde{\theta})$$

Since G is continuous, bounded and nonempty we know that $V(U)$ is continuous in U (See Stokey and Lucas (1989) lemma 3.6)

V(U) is strictly decreasing in U Let $U^1 > U^2$, and denote by w^1 the optimal wage to announce given U^1. Since $\tilde{\theta}(w, U)$ is strictly decreasing in U and $V(w, \tilde{\theta})$ is strictly increasing in $\tilde{\theta}$ we thus have that

$$V(U^1) < V(w^1, \tilde{\theta}(w^1, U^2)) \leq V(U^2)$$

and hence V is strictly increasing in U.

Proof of proposition 3

It is sufficient to show that equation (1.9) is well-defined, i.e. that there exists a value of U such that $V(U) = k$. The existence of optimal wages then follows from proposition (1), and the rest of the equations are well-defined by definition.

First, note that for any y and any $w > z$ we must have that $\lim_{rU \to z} V(w, \tilde{\theta}(w, U)) = (y - w)/(r + s)$, and therefore that $\lim_{rU \to z} V(y) = (y - z)/(r + s)$. Thus we have that

$$\lim_{rU \to z^+} V(U) = E \max[y - z, 0]/(r + s)$$

Then if $E \max[y - z, 0]/(r + s) > k$, there exists an U such that $V(U) > k$. Since obviously $V(y_n) < k$, the continuity of $V(U)$ and the fact that $V(U)$ is strictly decreasing in U implies that there exists a unique U solving $V(U) = k$. Thus equation (1.9) is well-defined. For a given value of U, proposition 1 tells us that equation (1.10) is well defined.

Uniqueness of the optimal solution:

The right-hand side of (1.16) is increasing in w. Since θ is decreasing in w, (1.16) is unique if $\frac{\eta}{1-\eta}$ is nondecreasing in θ. Hence a sufficient (but not necessary) condition for uniqueness is that $\eta = |\frac{\theta q'(\theta)}{q}|$ is nondecreasing in θ. This holds for the Cobb-Douglas matching function, where η is constant.

Proof of proposition 5

I first show that the market solution satisfies the necessary conditions for optimality in the special case where the elasticity of η is non-decreasing. Then I show necessary conditions in the general case. Finally I give sufficient conditions.

Let $\tilde{i}$ denote a given cut-off level, so that a vacancy is announced iff $i \geq \tilde{i}$. The current value Hamiltonian associated with the maximization of (1.23) subject to (1.24)-(1.25) is then given by (where λ_i and γ_i are the adjoint functions corresponding to N_i and v_i respectively, and α denotes the multiplier for the constraint (1.26)):

$$\begin{aligned} H &= \sum_{i=\tilde{i}}^{n} N_i y_i + z\sum_{i=\tilde{i}}^{n} u_i - c\sum_{i=\tilde{i}}^{n} v_i - ak \\ &+ \sum_{i=\tilde{i}}^{n} \lambda_i(v_i q(v_i/u_i) - sN_i) \\ &+ \sum_{i=\tilde{i}}^{n} \gamma_i(aF_i - v_i q(v_i/u_i)) \\ &+ \alpha(1 - \sum_{i=\tilde{i}}^{n}(u_i + N_i)) \end{aligned} \tag{1.30}$$

Necessary conditions for the steady state optimal solution are given by

$$u_i = arg\max_{u_i} H(u, a, v, N) \quad \forall i \tag{1.31}$$

$$a = arg\max_{a} H(u, a, v, N) \tag{1.32}$$

$$\partial H/\partial N_i = r\lambda_i \Rightarrow \frac{y-\alpha}{r+s}\forall i \tag{1.33}$$

$$\partial H/\partial v_i = r\gamma_i \Rightarrow r\gamma_i = -c + q(1-\eta)(\lambda_i - \gamma_i) \tag{1.34}$$

When η is non-decreasing in θ, (1.31) is determined by the unique set of first order conditions. Since H is linear in a we can thus write (remember that $\eta = -q'(\theta)\theta/q$, and thus $\frac{\partial v q(v/u)}{\partial u} = \theta q(\theta)\eta$ etc.)

$$\partial H/\partial a = 0 \Rightarrow k = \sum_{i=\tilde{i}}^{n} \gamma_i F_i \tag{1.35}$$

$$\partial H/\partial u_i = 0 \Rightarrow z + \eta_i \theta_i q(\theta_i)(\lambda_i - \gamma_i) = \alpha \forall i \tag{1.36}$$

To determine $\tilde{i}$, note that the derivative of W with respect to v_i is given by γ_i, therefore we must have that a vacancy is announced if and only if

$$\gamma_i \geq 0 \Rightarrow -c + q_i(1-\eta_i)(\lambda_i - \gamma_i) \geq 0 \tag{1.37}$$

The set of first order conditions is thus equivalent to the equilibrium conditions (1.19)-(1.21), with α, γ_i and λ_i substituted in for rU, V_i and $S_i + \gamma_i$. Since we know that the equilibrium is unique when η is non-decreasing, this means that the market solution satisfy the first-order conditions.

□

The proof in the general case follows the same lines as in the case when η is non-decreasing. However, a solution to (1.36) is not necessarily solving (1.31) and we therefore have to work with (1.36) directly. Thus

$$u_i = arg\max_{u_i} H \Rightarrow u_i = arg\max_{u_i}[zu_i + (\lambda_i - \gamma_i)v_i q(v_i/u_i) - \alpha u_i] \quad (1.38)$$

We want to show that this problem is equivalent to the problem facing the firm when choosing wage to announce. The firms problem can be rewritten to

$$\max_{\eta} -c + q(\theta)(1-\eta)S$$

subject to

$$rU = z + \theta q(\theta)\eta S$$

where the last equation is the workers' indifferent constraint. Since there is a one to one relationship between η and θ, we can substitute out η from the maximand, and maximize with respect to θ instead. The maximization problem can thus be written as

$$\max_{\theta}(-c + q(\theta)S - (rU - z)/\theta \quad (1.39)$$

which can be rewritten as

$$-cv_i + v_i(\max_{u_i}[zu_i + v_i qS - u_i rU]$$

Note that the problem has the same form as (1.38) The set of first order conditions is therefore the again equivalent to the market equilibrium conditions. Therefore all market equilibria satisfies the necessary conditions for optimality.

To show sufficiency, we use Arrow's sufficiency theorem (see Seierstad and Sydsæter(1987), theorem 6, page 289 for details). Write $\hat{H}(N,v) = H(N,v,\lambda^*,\gamma^*)$, where the star indicates that we are using the values derived by the necessary conditions. It is then sufficient to show that $\hat{H}$ is concave in N,v. First note that $\hat{H}$ is linear in N and that $\hat{H}_{Nv} = 0$. Further we know from (1.34) that $\hat{H}_v$ is positive, and (since $vq(\theta) = x(u,v)$) $\hat{H}_v = -c + x_v(u,v)(\lambda^* - \gamma^*)$. Since x is concave in v this gives sufficiency.

$\square$

Proof of lemma 3

We want to prove that $\lim_{y\to\infty}\delta(y) = \infty$ and that $\lim_{y\to rU}\delta(y) = rU$. First assume that the set of announced wages is bounded from above. We will utilize the first-order conditions given by equation (1.16) to show that this leads to a contradiction. Let $\bar{w}$ denote the supremum of the announced wages.

Further, let

$$\tilde{\eta} = \min_{w\in[\bar{w}-a,\bar{w}]} \frac{\eta(\theta(w))}{1-\eta(\theta(w))} > 0$$

where a is an arbitrary value such that $\bar{w} - a > rU$. From the first order condition (1.16) we then have that for all $y \geq a$

$$\tilde{\eta} \leq \frac{\delta(y) - rU}{y - \delta(y) - k} \leq \frac{\bar{w} - rU}{y - \bar{w} - rV} \tag{1.40}$$

We want to show that $\lim_{y\to\infty} y - rV = \infty$ i.e. that the denominator in the last expression goes to infinity when y does. Let $\bar{q} = q(\theta(\bar{w})) < \infty$. Then

$$V(y) = q\frac{(y-\delta(y))/(r+s)}{r+q} - \frac{c}{r+q} < \bar{q}\frac{y/(r+s)}{r+\bar{q}}$$

Thus we get

$$y - rV(y) > y - r\bar{q}\frac{y/(r+s)}{r+\bar{q}q} > y\frac{r}{r+\bar{q}}$$

Hence $y - rV$ goes to infinity when y goes to infinity, and the right hand side of (1.40) converges to zero when y goes to infinity. Hence the inequality is violated and we have a contradiction.

The proof of the claim that $\inf_{y\in\{z,\infty\}} \delta(y) = rU$ is simpler. Let $c = 0$. We know that $V(y) > 0$ for all $y > rU$. Assume that the infimum over posted wages $\underline{w}$ is greater than rU. Then there exists an y such that $rU < y < \underline{w}$. But then $V(\underline{w}, y) < 0$, and we have a contradiction.

Proof of Proposition 6:

PART 1. First we show that the proposed equilibrium actually is an equilibrium. Notice that in the proposed equilibrium, all the firms maximizes profit given $G(U_1,,,U_k)$. To see this, note first that since $V(w_i^*) = k$ for all i, the firms are indifferent between which of the equilibrium wages w_i^* to offer. Suppose now that a wage $\tilde{w} \neq w_i^*$ gives higher profit. Then $(\theta(\tilde{w}, w) \in G_i$ for some i. But since w_i^* is the unique wage that maximizes profit given G_i, this leads to a contradiction.

Assume now that a worker of type i gets higher expected income by going for wage w_j, $j \neq i$. Since the firms are indifferent between which of the wages to announce we get by arguing as above that w_i^* can not be the unique wage that maximizes profit given $G(U_j)$. We know from proposition 3 that the equilibrium exists, so the only thing left to prove is uniqueness.

In another equilibrium, at least some workers of one type i must apply for jobs that offer a wage w' different from w_i^*. These workers must get at least U_i^*, if not a profit oportunity would arise by announcing slightly less than w_i^*. But since the firms, when announcing w_i^* maximizes profit given U_i^* this means that the firms announcing w' gets negative profit. Therefore this can not be an equilibrium.

PART 2: Let $y_i > y_j$. By the revealed preference argument given in the proof of proposition 2 we know that vacancy of type i never announce a lower wage than the vacancy of type j. Suppose they announce the same wage. Then this wage must maximize the value of both sorts of vacancies given $G_l(U_l)$ for some l. But then we know by apply proposition 2 that the vacancy with the higher productivity announces a strictly higher wage.

To prove the last part of the proposition, note that if a worker of type i prefers w^2 to w^1, $w_2 > w^1$, any worker of type $j > i$ prefers w^2 to w^1. We therefore only have to prove that in equilibrium, different types of workers never enter the same sub-market. Assume that workers of types l and m do, with $l \neq m$. The wage w in this submarket must then maximize $V(w; y_i)$ for some i subject to both $G_1(U_1)$ and $G_2(U_2)$, which is impossible since the two indifferent curves have different slopes.

□

Chapter 2

Bargaining Over the Business Cycle

2.1 Introduction

Equilibrium models with frictions, or matching models, are widely used to explain various features of the labour market, see Pissarides (1985), (1987), 1990, Diamond (1981), Mortensen(1982 a and b) and others. Lately, such models have also been used to explain and predict the dynamics of aggregate variables over the business cycle (Mortensen and Pissarides (1994)). In matching models, the wages are determined by bargaining between workers and firms. Since it is costly to find trading partners, there is a surplus associated with each match, and the surplus is split according to the Nash sharing rule.

There also exists a distinct literature on decentralized trade, where wages (or prices) are determined by strategic bargaining. A key element here is the Stål Rubinstein bargaining game, where the agents in question give offers and counteroffers as to how to share the surplus. It is shown that market conditions, such as the availability of alternative trading partners, strongly influence the outcome, see Rubinstein (1982), Rubinstein and Wolinsky (1985), Binmore and Hererro (1988 b), Osborne and Rubinstein (1990).

In this article, we combine these two branches of the literature. We introduce strategic form bargaining in a matching model, developed by Pissarides (1990) and Mortensen and Pissarides (1994), and show how this alters the effects of aggregate shocks to the economy. We find that the workers bargaining power (his share of the surplus) fluctuates procyclically. This implies that the wage rate is more and the unemployment rate less volatile over the cycle than is predicted when the wage is determined by the Nash sharing rule. Our results suggest that matching models may not explain sticky wages to the extent suggested in Pissarides(1987).

The intuition behind the result is clear. Both discounting, and the possibility that his partner switch partner, makes an agent want to settle the negotiations quickly. A worker is more likely to be rematched in a boom than in a recession, and vice versa for the firm. As a result the workers' bargaining power vary procyclically, and this increases the wage flexibility.

We also study the effects of the shocks being anticipated. We show that this tends to make the worker's bargaining power less volatile. However, wages tend to be even more volatile. This is to compensate for renegotiations after a shock, at which stage the bargaining power of the agents have changed.

2.2 The Model

Our matching framework is standard, and can be described like this: There is a continuum of homogeneous workers with constant measure normalized to one. Workers exit the market at a constant and exogenous rate s. New workers enter the economy at the same rate, and join the market as unemployed.

Firms are either vacant and searching for a worker, or occupied (by one worker) and producing. All jobs are homogeneous. Firms are free to enter the market at any time to search for workers, and in equilibrium the value of a vacancy is zero[1]. When a worker and a firm meet, they start to bargain over the wage. We model the wage bargaining game as an alternating offer game of the Stål rubinstein type. The bargaining game is studied in the next section. In equilibrium, the negotiations settle immediately, and production starts. The match then lasts until the worker exits the market.

The economy is in one of two states: boom or recession. The economy moves between these to states according to a Markow process. The transition rates between the states are the same in both directions, and given by a parameter λ. All firms' productivities are y^h during booms and y^l during recessions. After a transition, all wages are renegotiated.

The number of matches in the economy is given by a concave, constant returns to scale matching function $x(u, v)$. The transition rate to employment for a searching worker is given by $p = x(u, v)/v = p(\theta)$, where θ denotes the labour market tightness θ. The arrival rate of workers to firms is given by $q = x(u, v)/v = q(\theta)$.

Let E^i, U^i J^i, and V^i denote the expected discounted income (asset values) for employed and unemployed workers and searching and occupied jobs, respectively, where $i \in \{h, l\}$ indicates boom and recession, respectively. The associated asset value equations are given by

$$\begin{aligned}
(r+s)U^i &= z + p(E^i - U^i) + \lambda(U^j - U^i) \\
(r+s)E^i &= w + \lambda(E^j - E^i) \\
rV &= -c + (J^i - V) = 0 \\
(r+s)J^i &= y^i - w^i + \lambda(J^j - J^i) \qquad (2.1)
\end{aligned}$$

[1]This is not important for most of the analyses, but simplify the expressions (which actually become quite long

where $i \in \{h, l\}$ and $j \in \{l, h\}$ and where z denotes the unemployment benefits.

To close the model, we have to determine the unemployment rate. Equalizing flows to and from unemployment we get the Beveridge curve

$$u = \frac{s}{s+p} \tag{2.2}$$

Finally, denote by M the joint expected income for a worker-firm pair. It follows that

$$(r+s)M^i = y^i + \lambda(M^j - M^i) \tag{2.3}$$

It follows that M^h decreases and M^l increases in λ.

Wages are determined by a modified version of the bargaining game in Osborne / Rubinstein (1990). Workers and firms bargain over how to share the expected discounted income M. We first model the game in discrete time, and then take the limit as the length of the periods converges to zero.

Matching takes place in the beginning of each period. A random device determines who is going to give the first offer, assigning a probability π to the event that it is the worker. If the offer is accepted the game ends, and production starts immediately. If the opponent rejects, a new offer can not be made before the next period. In the mean time, the agents can be rematched. If this happens, they abandon their former opponents and start bargaining with a new one. The match can also be resolved for exogenous reasons (the production opportunity or the demand for the good may disappear). If nothing like this happens, the worker and the firm in question continue to bargain. A random device chooses a proposer, with the same assigned probabilities, and the game continues in the same way as described above, potentially forever.

2.3 No anticipated shocks

In this section we assume that there are no anticipated shocks, so that $\lambda = 0$. The superscripts on the variables are thus superfluous, so we write E instead if E^h etc.

2.3.1 Wage determination

Let E^f and E^e denote the share the worker receives when the firm and the worker proposes respectively, and let $E = \pi E^e + (1-\pi)E^f$ represent the expected value of E. Furthermore, let A denote the probability that the worker is matched in the next period, given that he has not exited the market, and let B the probability that the firm is matched next period.

The expected income for a worker if he rejects an offer by the firm is $\delta(1-S)(AE+(1-A)U)$, where δ is the discount factor and S the probability that the worker exits the market between the periods. Thus the worker rejects all proposals that give him less than this. By a similar argument, we find that the expected income for a firm if the firm rejects an offer is $\delta B(M-E)$ (since $V = 0$). Hence the firm rejects all offers less than this. The equilibrium of the game is, therefore, given by the equations[2]

$$\begin{aligned} E^f &= (1-S)\delta(AE+(1-A)U)+b \\ M-E^e &= \delta B(M-E) \\ E &= \pi E^e+(1-\pi)E^f \\ J+E &= M \end{aligned} \tag{2.4}$$

[2]In equilibrium, it is always better for the firm to make a proposal the worker accepts, rather than waiting one period, and vice versa for the worker. Hence the equations below follows.

where b denotes the worker's income during the bargaining process. Substituting in for E^e and E^f in the expression for E gives

$$E = \frac{\pi M(1-\delta B) + (1-\pi)b + \delta(1-A)(1-S)(1-\pi)U}{1-\pi\delta B - (1-\pi)(1-S)\delta A} \tag{2.5}$$

Let P and Q denote the probability of finding a new trading partner before the next period for the worker and the firm, respectively. Further, let X represent an exogenous probability of job destruction. Then

$$\begin{aligned} A &= P + (1-P)(1-X)(1-Q) && (2.6) \\ B &= Q + (1-P)(1-X)(1-Q)(1-S) && (2.7) \end{aligned}$$

We assume that the arrival rates of trading partners for agents that bargain is the same as for idle agents. We thus write $P = 1 - e^{-p\Delta t}$, $Q = 1 - e^{-q\Delta t}$, $\delta = e^{-(r+s)\Delta t}$, $X = 1 - e^{-x\Delta t}$, $S = 1 - e^{-s\Delta t}$, and $b = b\Delta t$, where Δt is the time lag between two wage offers in the bargaining game. Inserting this into (2.5) and taking the limit as $\Delta t \to 0$ gives, using l'Höopital's rule:

$$E = \frac{\pi(r+s+x+p)M + (1-\pi)(x+q)U + (1-\pi)b - \pi(x+q)V}{\pi(r+s+x+p) + (1-\pi)(r+x+q)} \tag{2.8}$$

Thus the following holds:

1. If the main driving force in the bargaining game is the exogenous risk x of job destruction, i.e. if $q \approx p \approx r \approx 0$, we find that

   $$E = \pi M + (1-\pi)U$$

   The solution corresponds to the Nash solution when the workers bargaining power equals π.

2. If the main driving force in the bargaining game is the impatience, that is if $r > 0, p \approx q \approx x \approx 0$, then the expected income for the worker is

   $$E = \pi M$$

In this case, the worker's outside option (his expected income when unemployed) does not influence E.

3. If $r > 0$ and $x + p \approx x + q = \tilde{x}$, the worker's expected income is

$$E = \pi M + (1-\pi)\frac{\tilde{x}}{\tilde{x}+r}\tilde{x} + r$$

This is equal to the Nash bargaining solution when the worker's "disagreement point" is a fraction $\tilde{x}/(\tilde{x}+r)$ of U and V respectively.

When using that $w = (r+s)E$ and that $M = y/(r+s)$, the wage equation (2.8) can be written as

$$w = \beta_s y + \beta_u U(r+s) \tag{2.9}$$

where

$$\begin{aligned} \beta_s &= \pi\frac{(r+p+s+x)}{\pi(r+p+s+x)+(1-\pi)(r+q+s+x)} \\ \beta_u &= (1-\pi)\frac{q+x}{\pi(r+s+p+x)+(1-\pi)(r+s+q+x)} \end{aligned} \tag{2.10}$$

We call β_s the worker's bargaining power, and β_u the influence of the outside option. We note that $B_u + B_s \leq 1$, with strict inequality if $s + x > 0$. By inspecting (2.10), we obtain the following result:

Proposition 7 *The worker's bargaining power β_s increases with θ, while the influence of the outside option β_u decreases in θ.*

The proposition follows directly from equation (2.10) and the fact that p increases and q decreases in θ. Hence β_s increases and β_u decreases in θ. It is therefor natural to ask whether the decrease in β_u can outweigh the increase in β_s so much that the worker's expected income decreases with θ. By inspecting (2.10), we find that $\beta_s'(\theta) < \beta_u'(\theta)$ when $\pi p + (1-\pi)q$ is increasing in θ, which may well happen. Therefore we can not rule out *a priori* that E increases in θ unless $r+s$ is small compared to p and q, in which case $\beta_u + \beta_s \approx 1$ (see below).

2.3.2 Equilibrium

In this subsection we incorporate the wage formulas in the matching model described in the previous section. To simplify the analysis, we assume that both p and q are large compared to $r+s$, and that $\pi = 1/2$. We then get that

$$\begin{aligned} \beta_s &= p/(p+q) \\ \beta_u &= 1 - \beta M = q/(p+q) \end{aligned} \tag{2.11}$$

From Pissarides (1990), we know that the equilibrium value of w is given by

$$w = \beta y - (1-\beta)z + \beta\theta c$$

Inserting this in the asset value equations for J thus gives

$$c(r+s)/q = (1-\beta)(y-z) - \beta\theta c \tag{2.12}$$

and inserting $\beta = p/(p+q)$ thus yields

$$c(r+s)/q = \frac{y - z - \theta^2 c}{1+\theta} \tag{2.13}$$

Assume now that $\beta = \beta_s$ initially. Then the right-hand side of (2.13) falls more quickly with θ than does the right-hand side of (2.12). Consequently, a positive shock in y gives a bigger response in θ when the equilibrium is given by the first equation in (2.12), rather than by (2.13). We get the following result:

Proposition 8 *Suppose the situation is as described above, with $\beta = \beta_s$ initially. Then a shift in y implies a bigger response in w and a smaller response in θ and in the unemployment rate when the worker's bargaining power is given as in (2.11), rather than as an exogenous and constant β.*

Proof: That the response in θ is smaller when the worker's bargaining power is β_s rather than a constant β is shown above. Now

$$(r+s)c/q = y - w$$

In the model with an endogenous β, the left hand side increases less when y shifts (since θ shifts less), and it follows that w must change by more in order to restore equality. Finally, since θ shifts by less, it follows from the beveridge curve that u shifts less as well.

□

2.4 Anticipated productivity shocks

In this section, we assume that the productivity shocks are anticipated by the agents in the model, i.e. we assume that $\lambda > 0$. The analyses is only partial, in the sense that all market parameters are treated as exogenous.

The bargaining game proceeds as in the last section. However, the agents now have to take into account that the economy can be hit by an aggregate shock between two successive offers. We assume that the aggregate conditions are determined just after the matching process has taken place and just before the new offer is to be made[3]. Let Λ denote the probability that a shock occurs before the next offer is made. By arguing as in the last section, we find that the equilibrium of the game is given by

[3]As we will see, the timing here is actually irrelevant in the limit when the time lag between the offers converges to zero.

$$
\begin{aligned}
E^f &= (1-\Lambda)\delta(1-S)(AE^h+(1-A)U^h) \\
&+ \Lambda\delta(1-S)(AE^l+(1-A)U^l) \\
M-E^e &= (1-\Lambda)\delta(B(M^h-E^h)+(1-\beta)V^h) \\
&+ \Lambda\delta(\beta(M^l-E^l)+(1-\beta)V^l \\
E^h &= \pi E^e+(1-\pi)E^f
\end{aligned}
\tag{2.14}
$$

Solving for E, this gives

$$
\begin{aligned}
E &= \frac{\pi M(1-\delta(1-\Lambda)B)+\delta(1-\Lambda)(1-S)(1-A)(1-\pi)U}{\chi} \\
&+ \frac{\Lambda[(1-\pi)(AE^l+(1-A)U^l)+(1-\pi)(B(M^l-E^l)}{\chi}
\end{aligned}
\tag{2.15}
$$

where $\chi = 1-\pi\delta(1-S)(1-\Lambda)A-(1-\pi)\tilde{\delta}B$. The first term corresponds to the right-hand side of (2.5) (with Λ taken into account), while the last term incorporates the effects from the shocks. Now we proceed as above, and define A and B as in (2.7). However, to simplify the expression slightly we assume that $x=0$. In addition we write $\Lambda = 1-e^{\lambda\Delta t}$. Inserting this and taking the limits as Δt goes to zero gives

$$
E^h = \frac{\pi(r+s+p+\lambda)M^h+(1-\pi)qU^h+\lambda[E^l+\pi M^l]}{\pi(r+s+q)+(1-\pi)(r+s+p)+\lambda} \tag{2.16}
$$

which we can rewrite as

$$
E^h = \frac{\pi(r+s+p)M^h+(1-\pi)qU^h-\lambda(E^h-E^l-\pi(M^h-M^l)}{\pi(r+s+p)+(1-\pi)(r+s+q)} \tag{2.17}
$$

By using the equations in (2.10) we can express this as

$$
E = \beta_s M+\beta_u U-\frac{\lambda}{\rho}(E^h-E^l-\pi(M^h-M^l)) \tag{2.18}
$$

with $\rho = \pi(r+s+q)+(1-\pi)(r+s+p)$. The last term captures the effects of the potential shocks to the worker's bargaining power (the effects of the shock on the joint expected income is already captured in E). Note that we can write the last term as

$$-\lambda[(1-\pi)(E^h - E^l) - \pi(J^h - J^l)$$

Since the worker loses relatively more by the negative shock than the firm does (since his bargaining power falls when θ falls), we would expect this expression to be negative. However, if the firm loses more in absolute terms, E may still increase. It turns out to be difficult to give sharp results as to when this will happen. Note, however, the following:

Lemma 4 *Suppose a negative shock reduces the labour market tightness, and that this ceteris paribus increases E. Furthermore, assume that $\beta_s^h > \pi$, or equivalently that $p^h > q^h$. Then, for a given M and given market parameters θ and U, E^h falls with λ.*

Proof: Suppose the presence of the shock increases E. Then we know that $E^l > \beta_s^h M^h + \beta_u^h U^h$, and, by symmetry, that $E^l < \beta_s^l M^l + \beta_u^l U^l$. This gives

$$\begin{aligned} E^h - E^l &- \pi(M^h - M^l) \\ &> M^h\beta_s^h - M^l\beta_s^l + \beta_u^h U^h - \beta_u^l U^l + \pi(M^h - M^l) \\ &> (\beta_s^h - \beta_s^l)M^l + (\beta_u^h - \beta_s^h)U^l + (\beta_s^h - \pi)(M^h - M^l) + \beta^h(U^h - U^l) \\ &> 0 \end{aligned}$$

But then an increase in λ reduces E, and we have derived a contradiction

□

The lemma is not very sharp, the result obviously holds in many other situations as well. More precise results may be possible to obtain if we endogenize the market parameters U and θ, but we leave this issue aside for future work.

Hence, for given market parameters, the worker's share of the expected income falls with λ.

Finally, we want to analyze how the anticipation of shocks influence the wage rate, and compare this with results obtained when wages are determined by Nash-bargaining. To simplify the exposition, we assume that $r+s$ are small compared to p and q, so that we can write $\beta_s = (1-\beta_u)$.

To get an expression for the wages, we use the equations for E, M and U given by (2.1). When the wages are determined by Nash-bargaining, we find that

$$\begin{aligned} w^n &= (r+s)\beta M + (r+s)(1-\beta)U + \lambda(E^h - E^l) \\ &= \beta y - (r+s)\beta\lambda(M^h - M^l) + (1-\beta)[z + p(E^h - E^l) \\ &+ \lambda(E^h - E^l)] + \lambda[E^h - E^l] \\ &= \beta y + (1-\beta)[z + p(E^h - U^h)] \end{aligned} \tag{2.19}$$

where E^h, U^h, p and q are perceived as exogenous variables for the worker-firm pair in question. Note that λ does not alter the expression for w^n.

Now we turn to strategic bargaining. The last term in (2.18) can be ignored when p and q are large compared to $r+s$. The expression for the wage is then still given by the second line in (2.19). However, since $\beta_h \neq \beta_l$, the expression does not simplify in the same way. Instead we get

$$w = \beta^h y + (1-\beta^h)[z + p(E-U)] + \lambda(\beta^h - \beta^l)[M^h - U^h - (M^l - U^l)] \tag{2.20}$$

The last term is strictly positive and increasing in λ. Hence we have shown the following proposition:

Proposition 9 *Suppose $r+s$ is small compared to p and q. Then anticipation of a negative shock shifts the wage schedule (2.20) up.*

The worker knows that if the economy is hit by a negative shock, the firm will start renegotiating the wages downward. In this renegotiation game the worker's bargaining power is lower than it was before the shock, and he ends up getting a lower fraction of the surplus. The wages before the shock have to compensate for this.

2.5 Conclusion

We have studied wage determination by strategic bargaining in the context of a matching model of the labour market. We find that the worker gets a bigger share of the surplus during booms than during recessions. Thus, wages become more and unemployment less volatile compared to the model where wages are determined by the Nash bargaining solution. We also study the effects of anticipated shocks. We show that the anticipation of shocks tends to make the worker's bargaining power less volatile. Still the wages tends to be even more volatile, to compensate for the effects of renegotiation after a shock has occurred.

Chapter 3

Bargaining and Matching

3.1 Introduction

In models of decentralized trade, where wages are determined by strategic bargaining, the agents are assumed to switch trading partner if they are re-matched during the negotiations. After the switch, they start the bargaining game over again with the new partner (Binmore and Herrero (1988), Osborne and Rubinstein (1990), Rubinstein and Wolinsky (1985)). Similar assumptions can be found in matching models of the labour market, where workers do on-the-job search. A worker then accepts job offers from all firms where he is more productive than he is in his current job. After a job-switch, the wages are determined by Nash bargaining, independently of the worker's wage and productivity in his previous job (Mortensen and Pissarides (1994), Pissarides (1994))[1].

Since an incumbent agent is willing to offer the entire match surplus to prevent his trading partner from quitting, a more plausible assumption can be that Bertrand competition takes place when a third agent appear. With

[1]This contrasts Bertola and Felli (1993), where wages are determined by Bertrand competition at any point in time.

on-the-job search, this would mean that the worker in his new job receives a wage equal to his net productivity in the previous job.

Mortensen and Pissarides rule this out by assuming that contracts are not binding, so that all wage agreements costlessly can be renegotiated. Since a worker seldom has two job offers for a long period of time, any wage offer above the wage that prevails in a bargaining game between the two will be renegotiated later.

However, it seems to be a fact that a worker increases his wage if he is in contact with more than one employer during the wage negotiations. Furthermore, job contracts do contain agreements on wages, and are often legally enforceable. Therefore, firms seem to be able to commit to wage offers, at least to some extent and in the short to medium run.

In this paper, we assume that workers and firms are able to write binding contracts, so that Bertrand competition can take place. However, the model is set in continuous time, and the arrivals of trading partners are modeled as Poisson processes. Hence an agent never find two new trading partners at any one time.

When a worker and a firm are matched, the bargaining game proceeds like this: The firm proposes a wage, and the worker accepts or rejects the offer. First we assume that the worker responds immediately to an offer, but that a fixed amount of time elapses between two successive offers. Then we assume that the worker can delay his response as long as he wants (which is essentially the same as letting the delay between two successive offers go to zero). When agreement is reached, binding contracts are written, and production starts immediately. If the worker rejects the offer, or delay the response, a new trading partner may arrive and Bertrand competition takes place.

We find, that although the firm has all the bargaining power, in the sense that it makes all the proposals, the equilibrium wage is generally above the monopsony wage. Hence the famous Diamond paradox is absent in our model (see Diamond (1971)). Furthermore, under reasonable assumptions, the wage schedule has the same form as when the wages are determined by conventional Nash bargaining.

The analysis is partial in the sense that arrival rates of trading partners are exogenous. Our exposition is informal, and focus on the ideas rather than on technicalities. We do not specify the agents strategy sets, and in the proofs some details are omitted.

3.2 Homogeneous firms

Assume first that all workers are equally productive and have the same productivity in all firms. If a worker is in contact with two firms, and Bertrand competition takes place, he receives a wage y (equal to his productivity). If a firm is in contact with two workers, the wage is bid down to $z < y$, the expected discounted per period income when unemployed (hence $z = rU$, where U is the expected discounted income and r the relevant discount factor). For simplicity we assume that when a contract is written, workers and firms stay together forever.

3.2.1 Fixed time delay between offers

First we assume that there is a fixed time delay between two successive wage offers. Let P and Q denote the probabilities that the that the worker and the firm is matched with a second trading partner during the time delay respectively. Let E denote the worker's expected discounted per income if he rejects all offers until a new trading partner arrives. We ignore all terms containing PQ, and find that

$$
\begin{aligned}
E &= \delta[E(1-P-Q)+Py/r+QU] \\
&\Downarrow \\
rE &= \frac{Py+Qz}{1-\delta(1-P-Q)}
\end{aligned}
$$

This suggests the following result.

Proposition 10 *The wage-setting game described above has a unique equilibrium outcome, where the firm proposes w^* given by*

$$w^* = \max[z, \delta\frac{Py+Qz}{1-\delta(1-P-Q)}] \tag{3.1}$$

and the worker accepts the proposal.

Proof: Suppose the firm's strategy is to offer w^* in every period. Then it is optimal for the worker to accept the offer, and we have a subgame perfect equilibrium. Furthermore, the worker will never accept a wage below w^*. Thus the best the firm can achieve is agreement in the first period at this wage. Since the firm is the proposer, uniqueness of the outcome follows.

□

Hence if $(Py + Qz/(1 - \delta(1 - P - Q) > z$, the Diamond paradox is violated. With homogenous firms and no unemployment benefits, this is always satisfied in equilibrium.

Assume now that we can write $P = 1 - e^{-p\Delta t}$, $Q = 1 - e^{-q\Delta t}$, and $\delta = e^{-r\Delta t}$. Then we find, using l'Hôpitals rule, that the equilibrium wage converges to

$$w^* = \frac{py+qz}{p+q+r} \tag{3.2}$$

as Δt goes to zero.

3.2.2 Endogenous time delay between offers

Assume now that the worker can choose to delay his response to the firm's wage offer. The expected income for the worker if he always rejects all offers is given by:

$$
\begin{aligned}
rE &= p(y/r - E) - q(E - U) \\
&\Downarrow \\
E &= \frac{p}{r+p+q} y/r + \frac{q}{r+p+q} U
\end{aligned}
$$

where p is the arrival rate of jobs, and q the arrival rate of workers. We get the following result:

Proposition 11 *The wage setting game has a unique outcome, where the firm offers the worker a wage*

$$w^* = \max[z, y\frac{p}{r+p+q} + z\frac{q}{r+p+q}] \tag{3.3}$$

which is accepted immediately.

Proof: The proof follows the same line as the proof of the first equilibrium. The worker accepts w^*, but never anything less. Thus the best the firm can do is to propose w^*, which is accepted immediately.

□

The equilibrium outcome is thus the same as in the previous model, when the delay between the wage offers goes to zero. Note that if q and p are large compared to r, the wage can be written as

$$w = \frac{p}{p+q} y + \frac{q}{p+q} z$$

This solution is equal to the Nash bargaining solution when the worker's bargaining power is equal to $p/(p+q)$. It is also equal to the solution to the alternating offer game in Chapter 2, (when workers and firms have propose with the same probability). Note also that with this solution the Diamond paradox is always violated.

3.3 Extensions

In this section we assume that different worker-firm pair have different productivities. First we assume that the differences are due to match-specific factors, i.e. that all agents are identical *ex ante*, before they are matched. Then we briefly discuss a model with heterogeneous firms. The wage setting game is like in the previous section, where a worker can choose to delay his response to a wage offer. To simplify the analysis, we assume that only the worker can switch partner, and that he has to accept the Bertrand wage if he decides to change trading partner. Finally we assume that no on-the-job search takes place after an agreement is reached.

3.3.1 Match-specific differences in productivity

In this subsection we retain the assumption that the value of a vacancy is zero. All matches with productivity less than z thus dissolve immediately. Let $\bar{y}$ denote the productivity for the match in question. We assume that the value of a vacancy is zero. If Bertrand competition takes place, the worker is allocated to the firm where his productivity is highest, and receives a wage equal to his productivity in the other firm. Let F denote the cumulative distribution of match-specific productivities, and f the corresponding density. The expected discounted income when the worker delay the response until a second trading partner arrives is thus given by

$$rE(\bar{y}) = p\int_{E}^{y^1} (\min[y, \bar{y}]/r - E(\bar{y})f(y)dy \tag{3.4}$$

where y^1 is the top of the support of F. Taking derivatives with respect to $\bar{y}$ yields

$$rE'(\bar{y}) = p(1 - F(\bar{y}))(1 - E'(\bar{y})) \tag{3.5}$$

or

$$rE' = \frac{p(1 - F(\bar{y}))}{r + p(1 - F(\bar{y}))} \tag{3.6}$$

It follows that E' is positive but decreasing, and 0 at the supremum of the support. Thus if $rE(y^1) > z$ while $rE(y^0) < 0$ (where y^0 is the infimum of the support of F) there exists a unique $y^* \in (z, y^1)$ which solves $E(y^*)r = z$. This leads us to the following result:

Proposition 12 *Assume that $E(y^1) > z/r$, and that $y^0 \leq z$. Then the wage setting game described above has a unique equilibrium outcome. For matches with productivity on the interval $[z, y^*]$, where y^* is defined above, the firm offer a wage z. The worker immediately accepts the offer. If the productivity is above y^*, the firm offer the worker a wage $rE(\bar{y})$, which the worker immediately accepts.*

Proof: From the calculations above, it follows that the worker always can obtain $\max[z/r, E]$ by not accepting the firm's offer. By arguing as in the proof of the previous proposition the result follows.

□

Hence the model predicts a distribution of wages among workers, with an atom at $w = z$. For matches with productivity above y^*, wages are strictly increasing in the match productivity, although the derivative decreases in y and is zero at the top of the support.

3.3.2 Heterogeneous firms

Now we assume that the firms are heterogeneous. This makes the analysis more complicated, since the value of a vacancy then depends on its productivity.

Let $V(y)$ denote the value of a vacancy with productivity y. We assume that $V(y) = a(y-w)/r$, where a is strictly less than one[2]. The highest wage a firm is willing to offer in Bertrand competition is then $g(y) = y - rV(y)$. The expected income for a worker who delays his response until a second firm turns up is thus

$$rE(\bar{y}) = p\int_{g^{-1}(E)}^{y^1} (\min[g(y), g(\bar{y})]/r - E(\bar{y}))dF \tag{3.7}$$

For simplicity we assume that $E(\bar{y}) > U$ for all y. By arguing as above we then find that

$$\begin{aligned} g(y) &= y - a(y - w(y)) \\ &= (1-a)y - w(y) \end{aligned}$$

Now $g^{-1}(w(y)) = (1-a)y$, which gives

$$w(y) = p\int_{(1-a)y}^{y^1} (\min[(1-a)y - w(y), (1-a)\bar{y} - w(\bar{y})]/r - w(y)/r)dF \tag{3.8}$$

Thus the wage schedule $w(y)$ is a fixed-point of the mapping given by (3.8). Although the solution is generally not analytic, it can be solved numerically on a computer.

Before we conclude, we also discuss briefly the situation where a potential

[2] This is typically the case if the firm faces no direct search costs. If the arrival rate of workers to a vacant job is q, and the discount rate is r, we find that $a = q/(r+q)$.

second firm has imperfect information about the incumbent firms' productivity. We assume that the new firm only can make one wage offer, and the incumbent firm choose whether or not to match it.

Like in the previous models, all equilibrium wage offers are accepted immediately. The new firm's beliefs about the incumbent firm's productivity is therefore an open issue. Here we assume that the beliefs are given by the original distribution $F(y)$. For simplicity, we also assume that the value of any vacancy is zero[3]. If the new firm offers a wage w, the probability that he gets the worker is thus $F(w)$.

The wage offer $\tilde{w}(y)$ by a third firm with productivity y is thus given by

$$\begin{aligned} \tilde{w}(\bar{y}) &= argmax(y-w)F(w) \\ &\Downarrow \\ y-\tilde{w}(y) &= F(w(\bar{y}))/f(w(\bar{y})) \end{aligned} \tag{3.9}$$

If F/f is non-decreasing, this equation has a unique solution. The lowest initial offer the worker accepts is thus given by[4].

$$w^* = \delta(p\int \tilde{w}(y) - w^*(y))dF(y) \tag{3.10}$$

In equilibrium, all firm's offer w^*, and the workers accept this offer immediately. Thus the workers' wages are independent of the employers productivity.

[3]Now this is an innocent assumption since, as we will see, the equilibrium wage is independent of productivity. Thus if $V(y) = a(y-w)$, the maximum wage the firm will match is $(1-a)y - aw$, which is just a linear transformation of y.

[4]All firms in equilibrium are willing to offer w^*, if not they will never find a worker.

3.4 Conclusion

In this paper we have studied wage bargaining games between workers and firms. The firm makes all wage proposals, while the worker can choose to delay a settlement. If he does so, new trading partners may arrive, and since we allow for binding contracts, this leads to Bertrand competition among the agents. If the new agent is a firm (worker), the wage is bid up (down). In all the models, equilibrium wage proposals are accepted immediately.

We first study a model with homogeneous firms, and find that the outcome of the game is similar to the Nash bargaining solution. We then proceed to allow for heterogeneous firms, under the simplifying assumption that only workers can meet new trading partners if agreement is delayed. If the firms' productivities are observable, high-productivity firms offer higher wage than low-productivity firms. If the productivities are not observable, all firms offer the same wage.

Part II

Human Capital and Matching

Chapter 4

Human Capital Investments and Market Imperfections

4.1 Introduction

When the labour market is competitive, workers receive a wage equal to their productivity. If a worker's stock of general human capital increases, this is fully reflected in his wages. Since the worker in question receives all benefits from education, a socially optimal level prevails if he also carries all costs. For the same reason, firms are never willing to finance the investments. These results where first shown in Becker (1964).

With relationship-specific investments the situation is different since hold-up problems may occur. Hold-up problems are studied by Grout (1984) among others. Grout constructs a model with a single firm and a trade union, where the wages are determined by bargaining. Irreversible investments are undertaken prior to the wage determination by one (or both) of the agents. The investor therefore pays all the costs of the investments, while only receiving a share of the return. As a result we get sub-optimal investments, and in the case of human capital acquisition too little education.

In the present article we study human capital investments in an equilibrium model with frictions. Following Pissarides (1985, 87) and Mortensen (1986) among others, we assume that the agents have to do costly and time-consuming search to find a trading partner. When a worker and a firm are matched, the wage is determined by Nash bargaining.

A worker's productivity can be increased by investments in human capital. As in Becker, we focus on general human capital, i.e. skills that increase productivity by an equal amount in all firms. However, we show that the frictions in the market give room for hold-up problems. When a worker and a firm are matched, they have a mutual interest in staying together, since finding new trading partners is costly. Furthermore, the size of the rent depends on the worker's productivity. Since this happens in all firms, even small frictions can imply that the firm gets a substantial fraction of the gains from education if the workers' share of the surplus is low.

We study two different situations, where workers undertake the investments prior to entering the labour market (education), and where they are taken after the workers are matched and have formed relationships with firms. In both cases, the equilibrium depends on the way the investments are financed. In the first two sections we study investments in education and training when the costs are covered by equity or by standard loans. In the last section we allow for state-contingent debt contracts, where repayment depends on the worker's status in the labour market. Before we give a formal presentation of the model, we briefly discuss some of the results obtained.

Education

As mentioned above, the frictions in the market imply that there is a rent associated with a match, and the size of the rent depends on the worker's productivity. This gives rise to hold-up problems, workers pay for all the investments, while they only receive a share of the surplus. The situation is similar to the one with relationship-specific investments, and we get underinvestment. The extent of the underinvestments is smaller than in Grout's model though, since higher productivity increases a worker's outside option and thereby his bargaining position. This feedback effect through the market becomes stronger when the frictions in the market are reduced. When the frictions vanish, any increase in productivity is fully reflected in the outside option, and the hold-up problem disappears.

Since firms receive some of the gains from education, we have positive externalities in the model, and this gives rise to multiplier effects and possibly multiple equilibria. If the labour market tightness is low, the transition rate out of unemployment is low. Hence the returns from education are low, and the workers that enter the market have low productivity. As a result, firms have low incentives to open vacancies, and the labour market tightness and transition rate from unemployment stay low. Similarly, high labour market tightness leads to high education, productive workers, and strong incentives for firms to enter the market. Hence the labour market tightness stays high, and we have multiple equilibria. The different equilibria are Pareto-rankable, with the high-education equilibrium as the superior one [1].

Training

When the investments are undertaken after the match, the level of human capital is determined so as to maximize the firm's and worker's joint expected income. If the worker's productivity only influences the agents involved in

[1] A similar result, though in a different setting, is conjectured in Acemuglu (1993). The mechanism that leads to multiple equilibria are similar to mechanisms found in Pissarides (1992), who studies loss of skills during unemployment.

the bargaining game, the efficiency properties of the Nash bargaining solution implies that the level of training is both privately and socially optimal. However, this does not hold if there is turnover in the market. Potential new employers are influenced by the investments, and since they are not present when the decisions are made, their profit is not taken into account. The situation is similar to the one above where the investments are undertaken prior to the match, and we get underinvestments in training.

We focus on the effects of endogenous turnover due to on-the-job search. We model on-the-job search in a similar way to Pissarides (1994) and Mortensen and Pissarides (1994), but deviate from them when we assume that on-the-job search is unobservable. This implies that wages can not be contingent on workers' search behaviour, and on-the-job search may take place even when it reduces joint expected income.

The effect of on-the-job search on the investments in training is ambiguous. If a worker quits for a better job, this increases his own income and reduces the income in the initial firm. If the difference in marginal productivity of training between the firms is high enough, the first effect dominates. Then on-the-job search increases the joint expected return from training, and therefore increases the investments. If the difference between the new and the old firm is small, on-the-job search reduces the gains from training and thereby the human capital investments. In other words, low unemployment means more training.

The role of financing

In the two first sections we assume that the costs of education are financed by equity or by standard loans, where the repayments are independent of the lender's status in the labour market. In the last section we change this assumption, and introduce debt contracts where repayment only takes place when the worker is employed. This means that the repayments are included in the match surplus, and reflected in the wage.

We show that if education is financed with this sort of loan, the hold-up problem is resolved and social efficiency restored. Since the repayments are included in the match surplus, the firms in effect pay a share of the costs equal to the share it receives from the returns. A similar result holds for training. Joint surplus is now maximized when the worker finance the investments, and the level is socially optimal even in the presence of turnover. Hence Becker's (1975) results are restored.

With this in mind it is interesting to note that repayments of student loans in many countries are contingent on employment (f.i. in Great Britain and in Scandinavia).

4.2 The Model

In this section we present the basic model. Workers are unemployed and searching for a job, or employed and working (and eventually searching for a new job). All workers are identical and face an exogenous probability rate s of exiting the market. Firms are homogeneous, and either vacant and searching for a worker at a cost $c \geq 0$ or occupied (by one worker) and producing a stream of H units of output. Both firms and workers are risk neutral, and have identical discount factors r. In addition , jobs are destroyed at an exogenous and time independent rate t (In most of the paper, $t = 0$).

To determine the transition rates to employment, we introduce the standard matching function $x(u, v)$, which maps stocks of unemployed (u) and vacancies (v) into a flow of matches. We follow standard assumptions, and let x be increasing, concave and homogeneous of degree one in the two dependent variables. Let θ denote the labour market tightness v/u. We can then write the transition rates for unemployed and vacancies as $p = x(u, v)/u = x(1, \theta)$

and $q = x(u,v)/v = x(1/\theta, 1)$ respectively, the former increasing and the latter decreasing in θ. We also assume that $p(\theta)$ approaches infinity when θ goes to infinity and zero when θ goes to zero, while the opposite holds for $q(\theta)$.

We want to derive the wage for a worker with productivity H. Let p denote the transition rate from unemployment to employment. Furthermore, let E and U denote the expected discounted income for an employed and an unemployed worker respectively. The corresponding asset value equations are then given by

$$(r+s)U = z + p(E-U) \tag{4.1}$$
$$(r+s)E = w - t(E-U) \tag{4.2}$$

The equations, which can be derived from the Bellman equation, determine the expected discounted income (asset value) when unemployed and employed respectively, by comparing the actual return to the return of assets with a value equal to the discounted income. The return on the assets (compensated for risks) are given by the left hand sides of the equations. The right-hand sides give the actual returns. For an unemployed worker, the return consists of the income when unemployed z and the expected capital gain associated with finding a job. For an employed worker, the return consists of the wage, and the capital loss associated with job destruction[2].

The expected income V for a vacant and J for a filled job are calculated in the same way, leading to the asset value equations

[2]Below we are only interested in the difference between E and U. Therefore the expected income when exiting the market is irrelevant, and we set it equal to 0. Furthermore, the relevant unemployment income is the income (or value of spare time) that is lost when becoming employed.

$$
\begin{aligned}
rV &= -c + q(J - V) && (4.3) \\
rJ &= H - w - (s + t)J && (4.4)
\end{aligned}
$$

where H denotes production per unit of time. The equations can be interpreted in a similar way as we interpreted E and U above. Adding the asset equations for J and E now gives

$$J + E = \frac{H + tU}{r + s + t}$$

We follow standard assumptions in the literature, and let the expected income when a trading partner is not present be the agents' threat points in the bargaining game. Thus the workers' and the firms' threat points are given by U and V respectively. The match surplus S is then $E + J - U - V$. Denote by β the worker's share of the surplus. Then $E - U = \beta(J + E - U - V)$, or when substituting in for $E + J$:

$$E - U = \beta \frac{H - (r + s)U}{r + s + t} - \beta V \tag{4.5}$$

which inserted into the asset value equation for U gives

$$U = \frac{z + p\beta(\bar{H} - V)}{r + s + p\beta(1 - \hat{t})} \tag{4.6}$$

where $\bar{H} = H/(r + s + t)$ and $\hat{t} = t/(r + s + t)$. When $z = V = 0$ this gives

$$U = \kappa \bar{H} \tag{4.7}$$

with $\kappa = \beta p/(r + s + \beta p(1 - \hat{t})$.

The worker's productivity H depends on his investments in education or training. If we denote by k the amount (or the value of the effort) invested, we write his productivity as $H(k)$. We assume that H is increasing and concave, with $H(0) = 0$, $H'(0) = \infty$, $H'(\infty) = 0$. Investments in human capital are assumed to happen instantaneously, or the time spells are independent of k[3].

4.3 Education

In this section we assume that the investments are undertaken and financed by the workers before they enter the labour market. We are employing a traditional matching framework, where the number of contacts between workers and firms only depends on the labour market tightness. Furthermore, since applicants arrive to a vacancy according to a Poisson process, two workers never show up at the same time. This implies that the arrival rate to employment for an unemployed worker is independent of his education, as long as his productivity is above the reservation productivity of firms.

We assume that workers finance the investments by equity, or alternatively by a contract neutral with respect to the outcome of bargaining with a future employer. This holds for all debt contracts where the annual payments are independent of income and employment status. The workers' objective is then to maximize $U(H) - k$, where $H = H(k)$, where $U(H)$ is given by (4.6). To simplify some of the expressions we assume that $t = 0$. The first order condition for the workers is then given by

$$H'(k) = (r + s)\kappa \tag{4.8}$$

where, as before, $\kappa = p\beta/(r + s + p\beta)$. This gives us our first result:

Lemma 5 *For a given value of p, the unique equilibrium value of k is given by (4.8)*

[3]This is the case if k represents effort at school.

Proof: First note that since H is concave and $H'(0) = \infty, H'(\infty) = 0$, we know that (4.8) has one unique solution k^*. We only have left to show that $U(k^*) - k^* > 0$. Suppose not, i.e. that V was sufficiently high to make $U(k^*) < k^*$. Then no workers would take any education, in which case the equilibrium value of V is zero, a contradiction.

□

Since κ increases with p, so does the optimal investment level k^*. The reason for this is twofold. Firstly, higher p means that a worker gets a job and thus receives the return from the investments more quickly. Secondly, increased p means that the share κ of the return allocated to the worker increases as well. When p goes to infinity, equation (4.8) converges to $(r + s)H' = 1$, the competitive solution.

Note that, for a given p, neither the asset value V of the firm, nor the unemployment benefit z, influence the choice of k. Higher investments in human capital only influence the wage, not the unemployment spell, so z and V have no impact on the return from education.

Optimality of the market solution

We want to compare the market solution k^* derived above with the socially optimal level of investment. First we hold the labour market tightness θ and hence also the transition rate p constant. Optimal investments are then obtained when the workers receive all the benefits from the investments. This happens when $\beta = 1$. The optimal k, k^s, is thus given by

$$H'(k^s)/(r + s) = \frac{p}{r + s + p} \tag{4.9}$$

By comparing (4.8) and (4.9) we get the following result:

Proposition 13 *Suppose the human capital investments are undertaken prior to search, and financed by the workers as described above. Then, for a given value of θ, the investments are too low compared to the optimal level*

The intuition is clear: The worker bears all the costs of the investments, while the firm receives a part of the benefits. The worker does not take the positive externality for the firm into account when the decision is made, and as a result we get suboptimal investments.

We have already seen that κ approaches 1 when p goes to infinity. By comparing (4.8) and (4.9) we then find that k^* approaches k^s. In the limit, when the frictions are zero, the market solution and the optimal solution coincide.

The effects of entry

In the rest of this section we assume free entry of vacancies. New vacancies enter the market until all profit opportunities are exploited, and the value V of a vacancy is zero.

With free entry the welfare analysis above is more complicated. The aggregate level of human capital influences the firms' entry decisions and hence the labour market tightness. This can give second best effects.

From Pissarides (1990) and Hosios (1991) we know that the social and private benefits from entry coincide in the special case where the workers' bargaining power β is equal to η. Here η denotes the elasticity of q (the arrival rate of workers to firms) with respect to the labour market tightness θ. In this case the results above carry over to the model with entry.

However, if β is different from η, so that the social value of entry deviates from the private value, k^s given by (4.9) is no longer optimal. Suppose $\beta < \eta$. Then the firms' private gains from entry outweigh the social benefit, and we get overoptimal entry. Suppose initially that $k = k^s$. A small reduction in

k^s only leads to a second-order effect on production less investment costs for a given value of θ. However, this has first order effects for firms' profit, since they only receive the benefits. We thus get first order (negative) effect on entry as well, and since entry is superoptimal initially this gives a first order positive effect on social welfare.

Note, however, that even when the effect on entry is taken into account, the market solution k^* is suboptimal. A marginal increase in k has only a second order effect on workers' income (for given p), while it has a first order effect on firms' profit and on entry, and hence on p. This again has a positive first order effect on workers expected income. Hence an increase in k increases the expected income for unemployed and employed workers and for filled jobs, while the expected income for vacancies remain constant (equal to zero). It follows that k^* is suboptimal.

Multiple equilibria

When entry is included, the positive externalities from workers' education on firms profit give rise to multiplier effects and possibly multiple equilibria. The relationship between positive externalities and multiplier effects and multiple equilibria in macromodels is analyzed in Cooper and John (1988).

To see why we have multiplier effects, consider a small shift in the marginal cost of education, for instance due to a subsidy. This leads to an increase in H. It becomes more profitable for firms to enter the market, and θ and hence also p increase. This again leads to a new increase in H and then in θ etc. If the model exhibits multiple equilibria, the process may not converge: an infinitesimal shift in k may change the equilibrium substantially.

To be more specific, note from (4.8) that we can write $k^* = H'^{-1}((r+s)\kappa)$, and hence $H^* = H(k^*) = H(H'^{-1}(\kappa(r+s))$. Since $\kappa = p\beta/(r+s+p\beta)$ we can write $\kappa = \kappa(\theta)$, and thus $H = \tilde{H}(\theta)$. By taking derivatives of k in (4.8) with respect to θ gives us that

$$\tilde{H}'(\theta) = -(r+s)H'/H'' * p'(\theta)(1-\beta)/(p^2\beta) > 0$$

Recall that $J = (1-\beta)(H/(r+s) - U(H))$. In equilibrium all workers are equally productive, and the Nash sharing rule thus gives that $E - U = (J - V)\beta/(1-\beta)$. Since $V = 0$ implies $J = c/q$, we get $(r+s)U = z + \theta c\beta/(1-\beta)$ when utilizing that $p(\theta) = \theta q(\theta)$. Inserted into the equation for J gives

$$(1-\beta)\tilde{H}(\theta) - \frac{r+s+\beta\theta q(\theta)}{q(\theta)}c = 0 \qquad (4.10)$$

The equation determines the values of θ which make firms indifferent between entering or not, i.e. which imply $V = 0$. The first term is increasing in θ, reflecting that higher labour market tightness leads to more productive workers, which *ceteris paribus* is good for the firms. The second term is decreasing in θ (increasing in absolute value), and reflects both higher search costs and a better bargaining position for the worker.

Since $\tilde{H}(0) = 0$, we trivially get multiple equilibria. If $H = 0$ no firms enter the market, and $\theta = 0$. When $\theta = 0$ we know that $p(\theta) = 0$, and from (4.8) we know that this implies $k = H(k) = 0$. Since $H(k) - k > 0$ for some values of k there also exists at least one equilibrium where $k^* > 0$, and we have multiple equilibria. Since we have not imposed any restrictions on neither the second derivatives of the transition rates nor the third derivative of H, non-trivial multiple equilibria may occur as well.

The intuition why we may have multiple non-trivial equilibria is similar to the intuition behind the multiplier effects. If θ is high, the expected return from education is high, and the workers invest much in human capital and are productive. This makes it profitable for the firms to enter the market, since high profit when matched makes costly search worthwhile. When θ is low , the workers' productivities are low as well, and more vacancies do not enter even though search costs are low.

Since the multiple equilibria result is driven by a positive externality, the equilibria can be pareto-ranked, with high-activity equilibria dominating low-activity equilibria. In the former, both employed and unemployed workers, and also occupied jobs are better off in expected terms than in the latter, while the expected incomes for vacancies are unaltered.

4.4 Training

Now we assume that the investments in human capital take place when workers and firms are matched. We call this sort of investment training. We still assume that the human capital is completely general, the increase in productivity due to training is the same in all firms of the same type.

Since workers and firms bargain efficiently over the joint expected income, and have access to perfect capital markets, they choose the level of training that maximizes joint expected income. The costs are included in the match surplus, which is shared according to the Nash bargaining rule.

Since the Nash solution implies that the joint surplus is maximized, the amount of training chosen is optimal if and only if only the training decision has no external effects. However, if there is turnover in the market, externalities are typically present. A third party, typically a future new employer, also gains from training. Since his profit is not included in the match surplus the level of training in equilibrium is too low compared to the socially optimal level.

4.4.1 The model with on-the-job search

In this section we focus on the impact of turnover due to on-the-job search on the level of training. We assume that firms' productivity differ, and since wages are determined by bargaining, workers in high productivity firms are better paid than workers in low productivity firms. Hence the latter have incentives to do on-the-job search. When the search is successful, the worker changes job, and the old employer loses his part of the match surplus.

The search framework we employ is similar to the framework in Mortensen and Pissarides (1994) and in Pissarides (1994). We assume that all new vacancies have the same productivity indicator y^h. There is a constant probability μ that a filled job is hit by a negative productivity shock. When this happens, the productivity indicator falls to y^l. The output for an occupied job is equal to yH. Hence we assume that the marginal impact from training on output is proportional to the productivity indicator (hereafter productivity).

In contrast with Mortensen and Pissarides, we assume that the workers' search activity is unobservable to the firm. Contracts can therefore not be contingent upon the workers search behaviour. When on-the-job search costs are zero, a worker can credibly promise not to search only when his expected income is as high as in the high-productivity firms.

After a negative shock, a firm is willing to pay the worker sufficiently to prevent him from searching if the cost of doing so is less than the expected loss from on-the-job search. The worker always accepts such an offer, since it gives him more.

Alternatively, the firm and the worker can start bargaining over the expected income in the usual way. The expected income must then incorporate the effects of the worker's on-the-job search.

Let $J(y)$ and $E(y)$ denote the asset value of an occupied job and an employed worker respectively, as a function of the productivity indicator. Define $M(y) = J(y) + E(y)$ to be the joint expected income. To simplify the analysis, we assume that a vacant job only can advertise once (for instance due to reputation effects), so that the firms' threat points in the bargaining game with the worker is always zero. hence the asset value of a vacancy is zero. We also assume that t is zero. Let s denote the workers' exit rate as before, and λ the arrival rate of jobs when doing on the job search. If the worker do search, the joint expected incomes are determined by the asset value equations

$$\begin{aligned} (r+s+\gamma)M^h &= y^h H + \gamma M^l \\ (r+s)M^l &= y^l H + \lambda(E^h - M^l) \end{aligned} \tag{4.11}$$

where $E^h = E(y^h)$ etc. When using that $V = 0$, and for the moment assuming that $z = 0$ (no unemployment benefits), we show in the appendix that $E = \hat{\beta} M$, where $\hat{\beta} = \beta + (1-\beta)p\beta/(r+s+p\beta)$. Note that $\hat{\beta}$ is the fraction of the joint income received by the worker. When we insert $E^h = \hat{\beta} M^h$ into (4.11), we get two equations and two unknown variables M^h and M^l. Solving them gives:

$$\begin{aligned} M^h &= \tilde{y}^h H \\ M^l &= \tilde{y}^l H \end{aligned}$$

where[4]

$$\begin{aligned} \tilde{y}^h &= \frac{y^h + \gamma y^l/(r+s+\lambda)}{r+s+\gamma(1-\frac{\lambda\hat{\beta}}{r+s+\lambda})} \\ \tilde{y}^l &= \frac{y^l + \lambda\hat{\beta}\tilde{y}^h}{r+s+\lambda} \end{aligned} \tag{4.12}$$

[4]To see this, note that $M^l = (y^l + \lambda\hat{\beta}M^h)/(r+s+\lambda)$. Hence $(r+s+\gamma(1-\hat{\beta}\lambda/(r+s+\lambda))M^h = M^l$, and the expression follows.

If the worker is not doing on-the-job search the asset value equations for the joint expected income are

$$
\begin{aligned}
(r+s)M^h &= Hy^h - \gamma(M^h - M^l) \\
(r+s)M^l &= Hy^l
\end{aligned}
\tag{4.13}
$$

Hence we can write $M^h = H\hat{y}$, where

$$\hat{y} = \frac{y^h + \gamma y^l/(r+s)}{r+s+\gamma}$$

A comparison of (4.11) and (4.13) immediately gives that on-the-job search increases the joint expected income if and only if $E^h > M^l$. Hence the surplus increases if the expected income for a worker in a high-productive firm is higher than the joint expected income in the low-productivity firm. The next lemma expresses this in terms of the exogenous variables in the model, the proof is given in the appendix.

Lemma 6 *On-the-job search increases the joint surplus if and only if y^l is lower than y^*, the latter given by*

$$y^* = \hat{\beta}\frac{y^h}{1+\gamma'(1-\hat{\beta})} \tag{4.14}$$

where $\gamma' = \gamma/(r+s)$

As we have discussed already, a firm may want to increase the wage after a shock to prevent the worker from doing on-the-job search. If y^l is close to y^h, the costs of increasing the wage is low, and losses caused by on-the-job search are great, hence we would expect the firms to go for this option. On the other hand, if y^l is close to y^*, the opposite holds, and we would expect the firm to ignore the worker's search behaviour. The intuition is confirmed in the next lemma:

Lemma 7 *There exists a value $y', y^* < y' < y^h$ such that firms choose the bargaining solution if and only if $y^l < y'$, where y' is given by*

$$\frac{y^h - y^l}{y^l - y^*} = \frac{1-\beta}{\beta} \frac{\lambda(r+s+\gamma)}{(r+s)(r+s+\lambda)} \tag{4.15}$$

where y^* is given by (4.14).

The lemma states that if the productivity after a shock is relatively close to the productivity before the shock, the firm offers the same wage after the shock as before, to prevent the worker from doing on-the-job search. Hence the worker gets more than his share after a shock has occurred. This will be taken into account in the wage bargaining before the shock. Hence the worker's wage is reduced before the shock to compensate the firm for the losses after the shock[5].

If the productivity of the firm after the shock is below y', the wages are reduced after the shock, and the worker starts on-the-job search. If $y^* < y < y'$, on the job search reduces the joint expected income, and we are in a prisoners dilemma situation. As already discussed above, the worker cannot commit not to search. Since the wage is continuously renegotiated, the firm cannot commit to a contract giving the worker a high wage after a shock either. These inabilities to commit creates inefficiencies in terms of reduced joint surplus.

4.4.2 Training and turnover

The level of training is set to maximize $M^h - k$. Since the critical values y' and y^* are independent of H, the optimal value for k is characterized by

[5]The presence of shocks thus reduces the wage in high-productivity firms, and increases the wage in low-productivity firms. The model thus predicts that the wages are independent of the firms' productivity on some intervals. Note the similarity with implicit contract models of the labour market, see f.i. Hart (1983).

$H'(k) = 1/\tilde{y^H}$ if $y^l < y'$ and $H'(k) = 1/\hat{y}$ for $y^l > y'$. Hence we can show the following proposition:

Proposition 14 *Given that on-the-job search is unobservable, the following holds (for a given value of θ):*

1. *If $y^l \in [y', y^h]$, H is independent of λ*
2. *If $y^l \in [y^*, y')$, H is decreasing in λ*
3. *If $y^l < y^*$, H is increasing in λ*

The proof follows directly from the discussion in the last subsection together with the fact that the marginal return on investments in human capital is proportional to M^h.

Note that if on-the-job search increases joint expected income, both the worker and the firm are better off in terms of expected income. The Nash bargaining solution implies that the expected gain from on-the-job search is included as part of the pay-off for the worker. In other words, more on-the-job search means a lower wage. If the expected income increases with on-the-job search, this means that the gain for the firm due to the lower wage is higher in expected terms than the loss if the worker quits[6].

We know that $\hat{\beta}$ is increasing in p, and converges to 1 when p goes to infinity. By inspecting (4.14) and the definition of $\tilde{y}$ we get the following result:

Lemma 8 *The level of human capital investments are increasing in p. Furthermore, for all y^h, y^l such that $y^h > y^l$, there exists a $\bar{p}$ such that on-the-job search increases M for all $p \geq \bar{p}$*

[6]In an earlier version of the paper, the wages where explicitly calculated. Since the wages are not directly relevant for the solution of the model the calculations are now omitted

The intuition for the proposition is that the higher is p, the higher is the fraction of M that goes to the worker, and therefore the higher is E^h compared to M^l. Note also that high p in steady state is equivalent to low unemployment. The lemma thus states that lower unemployment means more training.

We want to find the share of the investment costs carried by the worker. We know that his expected income when matched, investment costs included, is given by $\hat{\beta}(M^{h*} - k^*)$, where k^* is the optimal k and M^{h*} the corresponding value of M^h. His expected income after the investments is $\hat{\beta}M^{h*}$. By comparing the two we get the following lemma:

Lemma 9 *The worker's share of the investment costs is $\hat{\beta} < 1$. The share is increasing in p, and converges to 1 when p goes to infinity*

The last proposition states that the level of on-the-job search is too low compared to the socially optimal level.

Proposition 15 *For a given value of θ, and given that the workers do on-the-job search, the market solution described above gives a level of training that is too low compared to the socially optimal level.*

Proof: The optimal level maximizes M^h for $\beta = 1$, which also implies $\hat{\beta} = 1$. Since M^h and thereby k increases in $\hat{\beta}$, the result follows.

□

As with education, free entry may lead to multiple equilibria in the model. If $y^l < y^*$ we know that on-the-job search increases the incentives to invest in human capital. This increases firms' incentives to open vacancies, which increases the incentives to do training etc. Note also that on-the-job search may increase unemployment if the search for employed and unemployed workers goes through different channels. If $y^* < y' < y'$, increased turnover reduces the value of finding a worker, fewer firms enter the market, and hence unemployment rises.

4.5 Optimal Debt Contracts

In the analysis above we have assumed that the costs of education were financed in a neutral way, by equity or a standard loan contract. In this section we alter this assumption, and assume that all investments are financed by loans where repayment is contingent on the status in the labour market[7]. More specifically, we assume that the worker is responsible for all loans, and that loans are infinitely long-lived, with constant repayments when the worker is employed and no repayments when he is unemployed[8].

4.5.1 Education

First we study the model from chapter 3, where the worker makes investments prior to search. In order to highlight the generality of the results, we allow for job-separations due to productivity shocks ($t > 0$).

Let a denote repayments per unit of time when employed for a one unit loan. Since the worker only repays the loan when employed, it reduces the match surplus. The effects of an increase in a on match surplus is equal to the same reduction in productivity H. The workers' objective function can thus be written as (from equation (4.6)):

$$U(k) = \frac{z + p\beta(\bar{H} - ak - V)}{r + s + p\beta(1 - \hat{t}} \tag{4.16}$$

[7]Strictly speaking it is sufficient that the investments are financed this way on the margin.

[8]We have assumed that the unemployed workers' search intensities are exogenous and homogenous firms, and thereby ruled out moral hazard problems that otherwise can arise

where still $\bar{H} = H/(r+s+t)$ and $\hat{t} = t/(r+s+t)$. The first order condition for optimal human capital investments is thus

$$H'(k)/(r+s) = 1/a \tag{4.17}$$

Now we turn to the determination of a. We assume that the capital market is perfect, and the lenders risk neutral. The expected discounted value of the repayment must then be equal to the amount lent. Let A be the asset value of a loan when the worker is employed. A and a are then determined by the asset value equations

$$\begin{aligned} r+s &= p(A-1) \\ (r+s)A &= a - t(A-1) \end{aligned}$$

Solving for a gives

$$a = \frac{r+s+p(1-\hat{t})}{p} \tag{4.18}$$

We are now able to show the following result

Proposition 16 *With the loan contract described above, and for a given θ, the investments in education are socially optimal.*

Proof: The optimal value of k maximizes $U - k$ when $\beta = 1$. From (4.6) we therefore find that the first order condition for optimality is given by

$$H'(k)/(r+s) = \frac{r+s+p(1-\hat{t})}{p}$$

which is the same as (4.17). Since the solution is unique, the result follows

□

4.5.2 Training

Now we return to investments made after the match has occurred. We still assume that the loan follows the worker. We then has that[9] $a = r + s$. The joint expected income can now be written as

$$\begin{aligned} M^h &= \frac{H(k)y^h - (r+s)k + \gamma M^l}{r+s+\gamma} \\ M^l &= \frac{H(k)y^l - a + \lambda E^h}{r+s+\lambda} \end{aligned}$$

We can now show the following proposition:

Proposition 17 *Suppose the worker searches on the job after a negative shock. Then the loan contract specified above leads to a socially optimal level of training.*

Proof: M^h is maximized when $M^{h\prime}(k) = 0$. Since $E^h = \hat{\beta} M^h$, we know that this is equivalent to $E^{h\prime}$ being zero. Hence

$$M^{l\prime}(k) = \frac{y^l H'(k) - (r+s)}{r+s+\lambda}$$

Taking the derivative of $M^{h\prime}$ and setting it equal to zero thus gives

$$0 = y^h H'(k) + \gamma \frac{y^l H'(k) - (r+s)}{r+s+\lambda}$$

or

$$1/H'(k) = \frac{y^h + \gamma y^l/(r+s+\lambda)}{r+s+\gamma(r+s)/(r+s+\lambda)}$$

[9] When a worker is matched, he does not become unemployed later. However, it is crucial that the contract specifies zero repayments during unemployment, since this influences the worker's outside option in the bargaining game.

The right-hand side of the expression is equal to $\tilde{y}^h$ when $\hat{\beta} = 1$. Since the socially optimal value of k is given by $1/H'(k) = \tilde{y}^h$ for $\hat{\beta} = 1$, this completes the proof

□

Since maximizing E^h and M^h leads to the same value of k, the training decision can be decentralized to the worker. If the worker decides k, and finances the investments by a loan of the type described above, optimality is obtained, and the division of the surplus corresponds to the Nash-solution. This is similar to Becker's results stating that optimality is achieved when the worker pays for all the training.

4.5.3 Discussion

To get intuition why we get optimality, note that when the investments are financed by equity, and the costs are sunk, the usual hold-up problems known from the literature arise. Since the costs do not influence the investor's *ex ante* gains from trade, they do not influence the bargaining outcome either.

If the costs are financed by a loan, this may be looked upon as a delay in the payment of the investments. The investor may then end up paying for the costs after he is matched (or has found a new partner after successful on-the-job search). This does not help, however, if the repayments are independent of whether the agents are matched. What matters for the outcome of the bargaining game is the difference in income when matched and unmatched, a neutral loan reduces expected income both when employed and when unemployed, and leaves the difference unchanged.

This is not the case when the repayments are contingent on employment. The repayment is then a part of the match surplus. Hence the bargaining solution implies that the repayments are shared. The costs of the investments are split between the worker and the firm. The firm thus pays the same share of the costs as he receives of the benefits. The externalities from the investments are internalized, and optimality achieved.

Formally, this is equivalent to the situation where the human capital is rented, and where the rental price (in the case of education) also covers the costs that arise because the capital is idle before the worker is matched.

4.6 Conclusion

We have studied the implications for education and training of frictions in the labour market. We have seen that with normal loan contracts, the presence of frictions creates hold-up problems and suboptimal investments. Education and training create positive externalities which may lead to multiple equilibria. In the case of training, a lower unemployment rate increases the level of investments. If we allow for loan contracts where repayments are contingent on the worker's status in the labour market, the hold-up problems are resolved, and the level of investments becomes socially optimal.

Appendix

Proof of the claim that $E = \hat{\beta} M$ When $V = z = 0$ E^h is given by

$$E^h = \beta(M^h - U) + U \tag{4.19}$$

Thus

$$(r+s)U = p(E^h - U) = p\beta(M^h - U)$$

since only high-productivity firms open vacancies. Hence

$$U = \frac{p\beta}{r+s+p\beta} M^h$$

Which gives $E^h = (\beta + (1-\beta)p\beta/(r+s+p\beta))M^h = \hat{\beta} M^h$

$$E^h = \hat{\beta} M^h$$

where $\hat{\beta} = \beta + (1-\beta)\frac{p\beta}{r+s+p\beta} < 1$ is the share of the expected income the worker receives .

Proof of Lemma 6 First insert $E^h = M^l$ into (4.11) to obtain $M^l = y^l/(r+s)$ and

$$M^h = \frac{y^h + \gamma y^*/(r+s)}{r+s+\gamma}$$

We know that $E(y^*) = \hat{\beta} M^h = M^l$, which then gives

$$\hat{\beta} \frac{y^h + \gamma y^*/(r+s)}{r+s+\gamma} = y^*/(r+s)$$

Rearranging gives

$$\frac{\hat{\beta} y^h}{r+s+\gamma} = y^* \frac{r+s+(1-\hat{\beta})\gamma}{(r+s)(r+s+\gamma)} \tag{4.20}$$

Hence $E^h > M^l$ if and only if

$$y^l < \hat{\beta} \frac{y^h}{1+(1-\hat{\beta})\gamma'} \tag{4.21}$$

□

Proof of lemma 7 For a given set of parameters, let y^* denote the value of y^l which makes the expected income independent of whether the worker searches. Then we know that on-the job search reduces the joint expected income if and only if $y^l > y^*$. Denote by M^{ln} the joint income when the worker does not search and M^{ls} the joint income when he does. Then

$$M^{ln} - M^{ls} = \lambda H \frac{y^l - y^*}{(r+s)(r+s+\lambda)}$$

To prevent the worker from searching, the firm must offer him an expected income E^h. Let M^{ln} and E^{ln} denote joint expected income and the worker's expected income after a shock if the worker does not search. Then

$$M^h - M^{ln} = H \frac{y^h - y^l}{r+s+\gamma}$$

The firm chooses to offer the worker E^h if this increases his profit, or if $\beta(M^h - M^l) = E^h - E^{ls} < (1-\beta)M^{ln} - M^{ls}$, or if

$$\frac{y^h - y^l}{y^l - y^*} < \frac{1-\beta}{\beta} \frac{\lambda(r+s+\gamma)}{(r+s)(r+s+\lambda} \tag{4.22}$$

and the lemma follows.

Chapter 5

Education and Competition for Jobs

5.1 Introduction

A well-known result in the literature is that hold-up problems can distort the incentives to invest, and lead to underinvestment. Hold-up problems arise when the agents that form a relationship and bargain over the output undertake relationship-specific investments before the bargaining takes place. Since sunk investment costs do not influence the game, the investors must bear all the costs, while they only receive a fraction of the return from the investments. As a result, we get underinvestment(see Grout(1984) for an example from the labour market).

In the last section we studied investment in general human capital when the labour market contains frictions. The frictions are captured by the matching function, as in Mortensen(1986) and Pissarides (1990), and wages are determined by Nash bargaining. Since the costs of education are sunk before wage bargaining takes place hold-up problems arise, and we get underinvestment in education.

What analyses of hold-up problems typically fail to consider is that *ex ante* investments may influence the likelihood of finding a trading partner. Because the investments benefit a trading partner, they can increase the probability of finding one. In Grout this is not an issue, a trading partner is always at hand. In our analysis in Chapter 2, the costs of finding a trading partner (a job) play an important role. However, there we assume that the transition rate to employment is independent of a worker's productivity. Hence, investment in human capital does not speed up the process of finding a job.

In almost all of the literature on matching, the arrival of trading partners are modelled as Poisson processes. Thus a firm never gets more than one applicant at any one time, and a worker never experiences face to face competition with another worker for the same job. Any match that is mutually beneficial is therefore sustained.

As pointed out in Blanchard and Diamond(1994), this may not be an accurate description of the labour market, where competition between workers for the same job seems to be the norm. Firms usually get more than one applicant per vacancy, and the difficult task for unemployed workers is not to find jobs to apply for, but to be accepted for the jobs they are applying for.

Blanchard and Diamond derive a matching technology that allows for more than one applicant per vacancy. However, their technology implies, among other things, that the workers transition rates to employment are not time invariant. In this paper, we first derive a new matching technology. Like Blanchard and Diamond, we start out with the urn-ball process. By modelling the frictions differently we obtain constant transition rates. Furthermore, the transition rate to employment in our model depends explicitly on the productivity of the worker in question. Finally, our framework makes it possible to parameterize the degree of competition for jobs.

Thus, in our model, a firm that announces a vacancy can get more than one applicant. If it does, it chooses one of them and starts bargaining with him over the wage. As in most of the literature, we assume that all wage contracts must be renegotiation proof. That is, workers cannot commit to accept a lower wage in order to get the job. The absence of binding contracts thus prevents Bertrand competition among the applicants.

Workers invest in general human capital (education) before they enter the labour market. We show that if the competition for jobs is not too severe (i.e. not too many applicants per job on average), firms strictly prefer workers with high education to workers with low education. Having a marginally higher education than the other applicants then gives a discrete increase in income. We show that as a result, workers diversify and choose various levels of education. Therefore, we obtain a non-degenerate, continuous and connected distribution of investment in education.

Ex ante identical workers thus become heterogeneous *ex post*, after the human capital investments are made. The logic (though not the model) resemblance that in Burdet and Mortensen (1989) (who obtain a non-degenerate distribution of wages) and in Butter (1977) (who obtains a non-degenerate distribution of prices).

If the competition for a given job is sufficiently fierce, firms do not always choose the most productive applicant. If this where the case, the expected the wage would increase faster than the productivity, due to a strong effect of productivity on the expected income for an unemployed (which influence the Nash solution). Hence, firms select randomly among applicants with education on certain intervals. In this case, we find that the distribution of educations contains an atom. If the competition for jobs is sufficiently fierce, the atom has mass one. The wages then fully reflect any increase in productivity, and the hold-up problem described above is eliminated.

Competition among applicants for the same job has interesting welfare implications. With competition for jobs, we have introduced a new, negative externality from education, since it has a negative impact on other workers' transition rates to employment. We also have a positive externality from education on firms' profit (the source of hold-up problems). However, which of the two externalities is stronger is, in general, ambiguous.

Note, however, that firms around the atom are indifferent about whom to hire. The positive externality from education on firms profit is internalized at this point, and the hold-up problem is eliminated. Only the negative externality prevails, and we have over-optimal investment levels. Workers at the atom have too much education compared to the socially optimal level[1].

5.2 The Matching Function

Butters (1977) and Hall (1979) are the first to use the urn-ball process to derive a specific matching function. Blanchard and Diamond (1994) show how the urn-ball matching process can be interpreted in a continuous time setting. Their basic assumption is that vacancies are pending a on a fixed amount of time, but are evenly staggered. The pending time is the only source of frictions in the model.

We start out with a simple random matching model set in discrete time. We partition each unit of time into n periods. In each period, a fraction γ/n of the vacancies advertise for one period only, and a fraction a/n of the workers responds and apply for one of them at random. The continuous time version of the model is obtained at the limit, when n goes to infinity.

[1]Similar results are obtained in signalling games, where the firms have incomplete information about worker productivity (see Spence (1974), Hart and Holmstrom (1987)). These models are very different from the present one.

The parameters a and γ capture the frictions in the market, a high value of a relative to γ implies that the competition for each job is severe, with many applicant per job. If a is low relative to γ, the model is similar to the standard matching framework with at most one applicant per job.

Let the unemployed workers' productivity be continuously distributed and without atoms. We want to calculate the probability that a worker who is more productive than a fraction π of the unemployed workers gets a job when applying, given that firms always choose the applicant with the highest productivity. Let u denote the number of unemployed workers, and v the number of vacant jobs. In each period, au/n workers apply at random for one of the $\gamma v/n$ advertised jobs. Let $\lambda = au/\gamma v$. Thus λ gives the relative number of applicants to vacancies. The number of applicants for each job is Poisson distributed with parameter λ.

First we want to calculate the probability $Pr(x)$ that an applicant applies for a job which receives a total of x applicants. The number of such workers is x times the number of firms that have x applicants. $Pr(x)$ is thus given by:

$$Pr(x) = x\frac{\lambda^x e^{-\lambda}}{x!}\frac{\gamma v/n}{au/n} = \frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!}$$

With $x-1$ other applicants, the probability that the worker in question gets the job is π^{x-1}. The probability of getting a job is thus:

$$\begin{aligned} Pr(\pi) &= \sum_{x=1}^{\infty}\frac{\lambda^{x-1}e^{-\lambda}\pi^{x-1}}{(x-1)!} \\ &= e^{-\lambda(1-\pi)}\sum_{x=0}^{\infty}\frac{(\lambda\pi)^x e^{-\lambda\pi}}{x!} \\ &= e^{-\lambda(1-\pi)} \qquad (5.1) \end{aligned}$$

Note that $Pr(\pi)$ is independent of n. We know that a fraction a/n of the workers applies for a job each period. The probability that a worker apply in a small interval Δt is thus approximately $a/n\ \Delta t/n = a\Delta t$. In the continuous time version of the model obtained when $n \to \infty$, the transition rate to employment is therefore given by

$$p(\pi; \lambda, a) = ae^{-(1-\pi)\lambda} \tag{5.2}$$

The derivation of (5.2) is based on one particular matching process, the urn-ball process. Still, we think that the resulting matching function possess some important properties that we would expect to hold more generally. We list them below:

1. $p(1; \lambda)$ is independent of λ. This captures the fact that the best worker is preferred to all other workers. Hence he does not face any congestion effects from other workers.

2. $p(\pi)$ is convex in π. This makes sense, since the probability of having the highest productivity among m workers is π^{m-1}. The derivative with respect to π is $(m-1)\pi^{m-2}$, which is independent of π for $m = 1$, constant for $m = 2$, and increasing in π for $m > 2$.

3. The sign of the cross derivative $p_{\pi\lambda}$ is ambiguous, but positive for values of π close to 1. The intuition is again rooted in the fact that the probability of getting a job when there are m competitors is π^{m-1}. The derivative with respect to π is $(m-1)\pi^{m-2}$, which is increasing in m for high values of π. This rationalizes the claim, since high values of m are associated with high values of λ.

4. $p(\pi)$ is finite for $\pi = 1$. There are "real" frictions in the model, since producing and processing applications are time consuming.

Before we start presenting the model, we want to derive the exit rates for workers with identical productivities. If a firm gets many applicants, we assume that it chooses one at random. Arguing as above, we find that the number of vacancies per applicant is Poisson distributed with parameter λ. The probability of getting at least one applicant is then one minus the probability of getting zero applicants. Multiplying by the number of vacancies and dividing by the number of workers, then gives the probability that a worker find employment. The transition rate to employment is therefore

$$p_a(a,\lambda) = a\frac{1-e^{-\lambda}}{\lambda} \tag{5.3}$$

In (5.2) and (5.3), both parameters λ and a influence the transition rate to employment. A high value of λ means there are many applicants per job. The parameter a captures other frictions in the market, low a means large frictions.

We want to index the degree of competition for a given level of friction. To this end, we define $a = a(\lambda)$ by the equation

$$a(\lambda)\frac{1-e^{-\lambda}}{\lambda} = c \tag{5.4}$$

where c is a constant Thus $(\lambda, a(\lambda))$, $\lambda > 0$ shows the combinations of parameters which keep the number of matches constant[2]. It follows that $a'(\lambda) = (c - ae^{-\lambda})/(1-e^{-\lambda}) > 0$ and that $\lim_{\lambda\to\infty} a(\lambda) = \infty$. A high value of λ (and a correspondingly high value of a) means that the competition for jobs is severe while other frictions are small and vice versa. In the the rest of the article, we will refer to an increase in λ with $a = a(\lambda)$ for a (balanced) increase in the competition for jobs. For later reference we also state and prove the following lemma:

Lemma 10 *Let $p(\pi; \lambda, a)$ be given by (5.2). Then $p(0, \lambda, a(\lambda))$ is strictly decreasing in λ, and converges to zero when λ goes to infinity. Furthermore,*

[2]Remember that the total number of matches is independent of the selection strategies used by the firms.

$p(1, a) = a$ is strictly increasing in λ and goes to infinity when λ does. Finally, $p(\pi, \lambda, \lambda(\pi))$ converges to $p_a = c$ (given by equation (5.4) for all π when λ goes to zero.

Proof: We know that

$$\frac{p(0, \lambda, a(\lambda))}{p_a(\lambda, a(\lambda))} = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} \tag{5.5}$$

Obviously, the right-hand side goes to zero when λ goes to infinity. Since p_a is constant, this implies that $p(0)$ converges to zero when λ goes to infinity. To prove that $p(0)$ is decreasing in λ, note that the derivative of the right-hand side of (5.5) is

$$\frac{e^{-\lambda}(1 - \lambda - e^{-\lambda})}{(1 - e^{-\lambda})^2}$$

which is strictly negative for $\lambda > 0$. Again, since p_a is constant, the claim follows.

Since $p(1) = a$, $p(1)$ goes to infinity when λ does. Finally, note that

$$\frac{p(\pi, \lambda, a(\lambda))}{p_a(\lambda, a(\lambda))} = \frac{\lambda e^{-\lambda(1-\pi)}}{1 - e^{-\lambda}} \tag{5.6}$$

When $\lambda \to 0$ we find, using l'Hôpital's rule, that the right-hand side converges to 1.

□

Thus, a balanced increase in the competition for jobs reduces the transition rate for workers with low education. On the other hand, since $p(1) = a$, a balanced increase in the competition for jobs increases the transition rate for the best educated workers .

5.3 The model

Except for the matching function, our model of the labour market is standard, and follows along the lines of Mortensen (1986), Pissarides (1985) and (1987), among others.

There is a constant number of *ex ante* homogeneous workers in the economy. Workers are replaced at a constant exogenous rate s. We normalize the number (measure) of workers to 1. Before new workers join the labour market, they invest an amount k in education, and obtain a productivity $H(k)$, where $H' > 0$, $H'' < 0$, $H(0) = 0$, $H'(0) = \infty$ and $H'(\infty) = 0$. Then they join the unemployment pool and start searching for a job.

The workers are equally productive in all firms. Since the wages are determined by bargaining, a worker gets the same wage in all firms, and thus accepts all job offers. The match lasts until the worker in question exits the market.

For simplicity, we assume that the incomes when unemployed is zero. The expected discounted income U for an unemployed and E for an employed worker are then

$$\begin{aligned}(r+s)U &= p(\pi(k))[E-U] \\ (r+s)E &= w\end{aligned}$$

where, as before, $\pi(k)$ denotes the distribution of k over searching workers. Unless otherwise stated, we assume that $p()$ is a general matching function satisfying properties (1)-(4) in the last section.

A constant number of firms are either vacant and searching for a worker, or occupied (by one worker) and producing. The wage w is determined by Nash bargaining between workers and firms. Let β denote the worker's share of the surplus. A firm-worker pair's joint expected income is given by $M(H) = H/(r+s)$. We assume that a vacancy can be advertised only once, and therefore its disagreement point in the bargaining game is zero[3]. The

[3]Alternatively we could have assumed free entry. However, since the effects of education on entry is not an issue here paper, we went for the assumption in the text.

Nash-solution thus implies that $E - U = \beta(M - U)$, which inserted into the expression for U, gives (after some rearranging):

$$\begin{aligned} U &= U(H(k), \pi(k)) = \frac{p(\pi)\beta}{r + s + p(\pi)\beta} H/(r + s) \\ &= \kappa H(k)/(r + s) \end{aligned} \tag{5.7}$$

where $\kappa = p(\pi)\beta/(r + s + p(\pi)\beta)$. An increase in the transition rate to employment due to higher education thus increases a worker's expected income both through shorter unemployment spells and through higher productivity, and both are reinforced due to higher outside option (expected income when unemployed) when bargaining with the firm.

5.4 Wage Distributions

In this section, we characterize the equilibrium under the assumption that firms, if they have more than one applicant, choose the one with the highest education. Conditions under which this assumption holds are derived later, together with a more formal treatment of the model. All workers choose a level of education that solves the problem

$$\max_k U(H(k), p(\pi(k))) - k \tag{5.8}$$

We immediately get the following result:

Proposition 18 *If all firms prefer to employ the most productive worker among its applicants, the distribution $\pi(k)$ of human capital investment is without atoms and has connected support.*

Proof: Suppose first that the distribution has an atom. Then, by increasing human capital investment marginally, some workers could get a discrete increase in p, thus increasing the expected income.

Then suppose the support is disconnected, so that no worker invests an amount on the interval (k_1, k_2). This implies $p(k_1) = p(k_2)$. Since all workers are *ex ante* identical, we must have that $U(k_1) - k_1 = U(k_2) - k_2$. Since $H(k)$ is strictly concave in k and U linear in H for a given p, it follows that $U(k) - k > U(k_1) - k_1$ for all $k \in (k_1, k_2)$, which is a contradiction

□

To characterize π, again remember that all values of k in the support give the same value of $U - k$. Thus we must have that:

$$\begin{aligned} \frac{dU}{dk} &= 1 \\ &\Downarrow \\ U(H(k), p(\pi(k)) &= U_0 + k \end{aligned} \tag{5.9}$$

where U_0 is a constant. Given U_0, (5.9) together with (5.2) and (5.7) define $\pi(k)$ uniquely. To determine U_0, note that the infimum k_0 of the distribution characterized by (5.9) solves the problem $\max_k U(H(k), 0) - k$. Hence $\frac{\partial U}{\partial H} H'(k) = 1$, and we can write (with $\bar{H} = H/(r+s)$):

$$k_0 = arg \max_k \bar{H}(k) \frac{p(0)\beta}{r + s + p(0)\beta} - k \tag{5.10}$$

It turns out to be difficult to give interesting characteristics of the distribution $\pi(k)$. We know, however, that the density $\pi'(k)$ is zero at $k = k_0$. To see this, we take the derivative of (5.9) with respect to k to get:

$$U_H H'(k) + U_p p'(\pi) \pi'(k) = 1 \tag{5.11}$$

Since the first term is equal to 1 at k_0, we find that $\pi'(k_0)$ is equal to zero. However, as k grows, $H'(k)$ falls, and $p()$ increases. A higher p means increased returns to investment, so $\pi'(k)$ may be non-monotonic in k.

When it comes to the support of π, we are able to show the following:

Lemma 11 *A balanced increase*[4] *in λ reduces k_0, raises k_1 (the supremum of the support) and thus increases the support of π.*

Proof: A higher λ means a lower $p(0)$ and thereby k_0, so that the worker at the infimum of the support is worse off. Thus the worker at the supremum of the support must be worse off as well. Since $p(1) = a(\lambda)$ increases with λ, this means that k_1 must increase (since $U(k_1) - k_1$ is decreasing in k_1).

□

The more competition there is for jobs, the more the workers diversify in terms of education. This is intuitive, since more competition means that there is more to gain by being on the top of the distribution.

When $p(\pi)$ is given by (5.2), we are able to derive $(\pi(k))$ explicitly. In the appendix, we show that $\pi(k)$ is given by

$$\pi(k) = 1 - \ln[a\beta \frac{\bar{H}(k) - U_0 - k}{(r+s)(U_0 + k)}]/\lambda \tag{5.12}$$

Now we will show that there is an upper bound on how far the upper tail of the distribution of education levels may reach. If the level of education becomes too high, firms no longer strictly prefer workers with higher education.

We know that firms strictly prefer the best educated worker if $H'(k) > w'(k)$, i.e. if productivity increases more quickly than wages. Since wages depend on p as well as on H, this is not necessarily the case. The wages can be written as

$$w(k) = \beta H(k) + (1 - \beta)U(k)(r + s)$$

[4]Recall that a balanced increase in λ is one where $a = a(\lambda)$ increases such that the aggregate number of matches is held constant, see equation (5.4).

Thus, $H'(k) > w'(k)$ if and only if

$$H'(k) > U'(k)(r+s)) \tag{5.13}$$

Let Π denote the support π defined by (5.11). Then the following holds:

Lemma 12 *Firms strictly prefer a worker with education k' to a worker with education k'', $k' > k''$, $k', k'' \in \Pi$ if and only if the supremum of Π is below k^*, where k^* is the solution to $H'(k)/(r+s) = 1$.*

Proof: For all k in the support of π, we know that $U'(k) = 1$. Hence we know from (5.13) that $H'(k) > w'(k)$ iff $H'(k) > r + s$.

□

We have already seen that the support of the distribution increases with a balanced increase in λ. We would therefore expect that k_1 is below k^* if the competition effect is not too strong. This is confirmed by the next proposition:

Lemma 13 *Let $a(\lambda)$ be defined by (5.4) for an arbitrary constant. Then there exists a $\lambda = \bar{\lambda}$ such that k_1, the support of the distribution defined by (5.11), is less than k^* for all $(\lambda, a(\lambda))$, $\bar{\lambda}$*

Proof: From Lemma 10 we know that

$$\lim_{\lambda \to 0} p(\pi) = c$$

for all π, where c is a constant. From (5.11), it follows that $k_1 - k_0 = U(p(1)) - U(p(0))$, and hence $k_1 \to k_0$ when $\lambda \to 0$. Furthermore, k_0 converges to $arg \max_k U(c,k) - k$, which is strictly smaller than k^*, and the proposition follows

□

The proposition just states that if the competition for jobs is not too fierce, the entire distribution given by (5.11) is below k^*. We know that $k_1 < k^*$ if and only if a worker is worse off investing k^* and facing an arrival rate of jobs equal to a than he is investing an amount k_0 and facing an arrival rate of jobs equal to $ae^{-\lambda}$. Using (5.2) and (5.7), we find that this holds if and only if

$$\beta \bar{H}(k^*)\frac{a}{r+s+\beta a} - k^* \leq \beta \bar{H}(k_0)\frac{ae^{-\lambda}}{\beta ae^{-\lambda}+r+s} - k_0 \tag{5.14}$$

5.5 Randomizing equilibrium

In this section, we generalize the analysis above, and allow for the possibility that firms do not necessarily prefer the best educated workers. We will also give the equilibrium concept, and specifically the firms' hiring strategies, a more careful treatment.

A *selection rule* applied by a firm with a vacant job is a vector specifying the choice probability for each worker as a function of all the applicants' education levels and the number of applicants[5]. In a *selection equilibrium*, we require that the selection rules for all firms are optimal, given the other firms' selection rules, for all k undertaken in equilibrium. This is the usual rational expectations requirement. In addition, we require that the selection rules are optimal for any deviation by subset of workers with measure zero[6].

[5]Since I do not need it for the analysis, I will not go into technicalities. Note, however, that a selection rule can be defined mathematically in the following way: If a firm gets n applicants, define the infinite dimensional vector K as $(k_0,,,,,k_n,0,0...)$, where the first n elements denote the education level of the applicants. A selection rule is a functional $P(K) : R_+^\infty \rightarrow R_+^\infty$, where element m in $P(K)$ is zero if the same element in R is zero, and where the sum of elements in P is less than or equal to 1.

[6]The refinement resemblance those in Kohlberg and Mertens (1986), Gale (1992), (1994), and the one we introduced in Chapter 1.

As we will see, there exists a continuum of selection equilibria. However, the distribution of k that can be sustained by a selection equilibrium is unique.

An equilibrium of the model is a distribution $\pi(k)$ of investments in education with support Π such that:

1. All $k \in \Pi$ maximize the workers' expected income given the firms' selection strategies
2. The firms' selection strategies form a selection equilibrium

Before we go on to characterize the equilibrium of the model, we show the following result:

Lemma 14 *Assume that an atom of workers choose an education k'. Then in all selection equilibria there exist an open interval I, containing k', such that the firms are indifferent between workers with education k' and k'' for all $k \in I$.*

Proof: Let $k'' > k'$ be arbitrarily close to k', and suppose the firms strictly prefer a worker with education k'' to a worker at the atom. By the definition of selection equilibria this means that all firms choose the worker with education k'' if selecting between workers with educations k' and k''. Thus $p(k'') - p(k') > e > 0$ for some fixed e, and $\lim_{k'' \to k'} H(k'') = H(k')$, while $\lim_{k'' \to k'} w(k'') > w(k')$ (since U is strictly increasing in p). But then the firms prefer workers with education k', and we have derived a contradiction. An analogous argument shows that the workers at the atom can not be preferred either.

□

Therefore, the selection strategies around an atom must be such that all firms are indifferent between workers on some interval containing the atom.

On an interval where firms select randomly, we must have that $J'(k)$ is constant. Hence

$$\begin{aligned} J'(k) &= (1-\beta)(H'(k)-(r+s)U'(k)) = 0 \\ &\Downarrow \end{aligned}$$

$$H'(k) = U'(k)(r+s) \tag{5.15}$$

In equilibrium, workers choose k to maximize expected income, and at any atom, k maximizes $U - k$ subject to (5.15). This leads us to the next lemma:

Lemma 15 *In equilibrium, the only possible location of an atom is at k^* defined by $H'(k^*) = r + s$.*

Proof: The first order condition for the worker's problem is that $U'(k) = 1$, and hence $H'(k) = r + s$.

□

Before we go on to the main result in this section, we also show the following lemma, which states that no workers choose an education above k^*.

Lemma 16 *There does not exist any selection equilibria which supports educations above k^*.*

Proof: We know from lemma 15 that there are no atoms above k^*. Thus if a positive measure of workers invest more than k^*, the support of k is connected at some intervals. But then we know from Lemma 12 that firms select randomly over this interval in all selection equilibria. Hence optimal behaviour of the worker implies that $U'(k) = H'(k)/(r + s)$ for all k on the connected intervals. Since this is satisfied only for $k = k^*$, we have derived a contradiction.

Suppose now that a single worker invests $k' > k^*$. If there is not an atom at k^* this can not be optimal, since $U'(k, a) < H'(k^*)/(r + s)$ for all $k > k^*$. Suppose there is an atom at k^*. Then workers strictly prefer k^* to any $k \in I, k \neq k^*$, so k'' can obviously not be on the interval Furthermore, $U'(k)$ is less than 1 for all k on the interval between the supremum of I and k' (since p is constant at this interval). Hence k' cannot maximize $U(k) - k$.

□

There exists an infinite set of selection equilibria which support this equilibrium. Here we define a simple one: Let a fraction $\tau(k)$ of the firms draw at random when choosing between a worker with education $k \in I$ and workers at the atom. Let the rest choose the worker with highest productivity. τ is chosen so that $w'(k) = H'(k)$. Obviously, $\tau(k)$ is then 1 at the atom and 0 at the boundaries of the interval I where the firms randomize[7]. Let ξ denote the mass of the atom at x^*. For later reference, we define the set of selection rules $\Psi(\xi)$ to be as follows: Firms choose the best educated worker unless they choose among workers in the interval I given in lemma 14. If the firm chooses between workers at the atom, they pick one at random. If they choose between a worker with education $k' \in I$ and workers at the atom, a fraction $\tau(k)$ select randomly as described above.

From Lemmas 15 and 16 and from the analysis in the previous section, it follows that an equilibrium must be of one of the following types:

1. *A pure distribution equilibrium.* The equilibrium is characterized by a distribution Π with connected support $[k_0, k_1]$, where $k_1 \leq k^*$.

2. *A hybrid equilibrium.* The support of the equilibrium consists of an atom at k^* and a continuous tail on $[k_0, k_1]$, $k_1 < k^*$.

[7]Since we are only interested in deviations by one worker, we do not have to specify selection rules for two applicants who are in I but not at the atom.

3. *A one-point equilibrium.* The support consists of k^* only.

The transition rate to employment for workers at an atom with probability mass ξ is (by (5.3))

$$p = \frac{a(1 - e^{-b\xi\lambda})}{\xi\lambda} \tag{5.16}$$

The transition rate is decreasing in ξ. Hence the expected income of an unemployed worker at the atom is also falling in ξ. The expected income at k_0 is independent of the mass at the atom . All workers therefore choose the atom if it yields higher expected income than at k_0 for $\xi = 1$, or if

$$\beta\bar{H}(k^*)\frac{\lambda/(\xi b); (1 - e^{-b\xi/\lambda})}{r + s + \beta\lambda\gamma/a\,(1 - e^{-b/\lambda})} - k^* \leq \beta\bar{H}(k_0)\frac{a}{(e^{b/\lambda}(r + s) + a} - k_0 \tag{5.17}$$

If the expected income at k_0 is lower than at the atom for $\xi = 1$, but higher than for ξ close to zero, the equilibrium is a hybrid with both a tail and an atom. We thus get the following result

Proposition 19 *Assume that $(\lambda, a(\lambda))$ satisfies (5.4) for a value of c such that $U(H(k^*), c) - k^* > 0$. Then the equilibrium defined above is unique. Furthermore, there exists two values $\lambda_1 > \lambda_2$ of λ, and corresponding values for a, such that the equilibrium is*

- *A pure distribution equilibrium for $\lambda \geq \lambda_1$*
- *A mixed equilibrium for $\lambda_2 < \lambda < \lambda_1$*
- *A one-point equilibrium for $\lambda \leq \lambda_2$*

Proof: Define λ_1 as the value of λ which makes $k^1 = k^*$, and λ_2 as the value which makes (5.17) an equality for $\xi = 1$. Since $U(k_0) - k_0$ decreases and k^1

increases in in λ, and since $U(k^*, c) - k^* > 0$ by assumption, it follows that both λ_1 and λ_2 are well defined and unique.

First let $\lambda < \lambda_1$. We want to show that the pure distribution equilibrium given by (5.11) is indeed an equilibrium. Let the set of selection rules be given by $\Psi(0)$, so that all firms choose the best educated worker. Given $\Psi(0)$ it follows from the analysis in the last section that (5.11) implies that all workers behave optimally, and from Lemma 12 that $\Psi(0)$ forms a selection equilibrium. Hence existence follows.

Now we turn to uniqueness. By construction, we know that the distribution (5.11) is unique given $\Psi(0)$. By the definition of λ_1, it follows that we cannot have an atom at k^*, and therefore that the equilibrium distribution can have no atoms. Suppose $\Upsilon \neq \Psi$ supports a different equilibrium distribution than (5.11). Since $\Psi \neq \Upsilon$, the latter must imply that the firms are indifferent between workers with different education levels on some intervals below k^*. Since this is ruled out by Lemma 12, uniqueness follows.

Now let $\lambda_1 < \lambda < \lambda_2$. First note that since the left-hand side is strictly decreasing and the right-hand side is independent of ξ, there exists a unique $\xi^* \in (0,1)$ which makes (5.17) into an equality. We want to show that a hybrid equilibrium exists, where a fraction ξ^* of the searching workers invests k^*, while the rest of the distribution is given by (5.11), with $\pi(k^1) = 1 - \xi^*$. Let the set of selection strategies be given by $\Psi(\xi^*)$. By construction, workers then maximize their expected income given the selection strategies. We also know by construction that the selection strategies are optimal at the interval I around the atom at which the firms randomize. Because of Lemma 12, we only have left to show that k^1 is less than k', the infimum of I. But this follows from the fact that, by construction, $U(k^1) - k^1 = U(k^*) - k^* > U(k') - k'$.

Uniqueness now follows easily. First note that from the definition of λ, it follows that the equilibrium must be a hybrid. Hence it follows that the mass at the atom must be ξ^*. By using a similar argument as above, we then find that the tail is unique as well, and the result follows.

Finally, let $\lambda > \lambda_2$. We want to show that $\xi = 1$ is an equilibrium. Let the selection strategies be given by $\Psi(1)$. By the construction of λ_2, it follows that $U(k^*) - k^* > U(k_0) - k_0$. This proves both existence and uniqueness.

□

The level of education in the model can be overoptimal or suboptimal, depending on the parameter values. As mentioned in the introduction, education has a positive externality for firms, which get higher profit when workers are more productive. This is the hold-up effect. At the same time, education creates a negative externality for other workers, since it reduces their transition rate to employment. Which of these effects is stronger depends on parameter values.

If λ converges to zero, the model converges to the one in the Chapter 2, which yields suboptimal investment in education. This makes sense: when the competition effect vanishes, we are left with only the positive externality from education. On the other hand, we know that the education level at the atom is overoptimal. k^* maximizes $H-(r+s)k$, which means overinvestment since there is a time lag between education and production.

5.6 Conclusion

We analyze the incentives to invest in human capital when the labour market contains frictions. We develop a matching model where firms can have more than one applicant for a vacant job. The equilibrium of the model can take different forms depending on the degree of competition for jobs. If the competition is not to severe, workers choose to diversify, and the distribution of education levels among the workers is continuous and without atoms. If the competition is severe enough, all workers choose the same education level. A hybrid equilibrium exists for intermediate levels of competition for jobs.

We show that when the competition for jobs is stiff, the well-known hold-up problem that often arises in models with ex post bargaining is eliminated. The investments in human capital is then overoptimal, due to negative externalites from education on other workers.

Appendix

To get a closed form solution for π we first insert (5.7) into (5.9) and get

$$\frac{\beta p \bar{H}(k)/(r+s)}{r+s+\beta p} = U_0 + k$$

Rearranging gives

$$\begin{aligned} \beta p &= \beta a e^{-\lambda(1-\pi(k)} \\ &= \frac{(r+s)(U_0+k)}{\bar{H}(k) - U_0 - k} \end{aligned} \tag{5.18}$$

Taking logarithms and rearrange give

$$\pi(k) = 1 - \ln[a\beta \frac{\bar{H}(k) - U_0 - k}{(r+s)(U_0+k}]/\lambda \tag{5.19}$$

Chapter 6

Loss of Skills During Unemployment

6.1 Introduction

A well documented result in applied labour economics is that long unemployment spells reduce a worker's prospects in the labour market (see f.i. Layard *et al* (1991) . One possible explanation for this is that long unemployment spells reduce a worker's productivity.

The idea that workers may loose skills during unemployment is not a new one. In a resent paper, Pissarides (1992) makes the same assumption. What is new in this paper, is that we implement loss of skills in a standard matching model of the labour market, with wages determined by Nash-bargaining. New workers enter the market as unemployed, and their skills depreciate geometrically during the search process.

We show that the model may have multiple equilibria. The reason is, that if the exit rate from unemployment is high, the unemployed workers average productivity is high. This encourages firms to open vacancies, and the exit rate remains high. On the other hand, if the exit rate from unemployment is low, the average productivity is low as well, and this discourages firms from open vacancies. Hence the exit rate remains low, and we have multiple equilibria[1].

Thus, if workers loose skills during unemployment, a high labour market tightness both increases workers' productivity and reduces the unemployment rate. We would therefore expect the socially optimal level of entry to increase (and the unemployment rate to decrease) when we introduce depreciation of skills. On the other hand, loss of skills reduce the workers outside option in the wage bargaining, and thus reduces his wage (for a given productivity) and thus encourage entry of jobs. However, we show that this market response is too weak to accommodate fully for the increased social gains from entry. Thus loss of skills tends to make the labour market tightness too low and the unemployment rate too high compared to the socially optimal level.

We also argue that workers continue to search for jobs too long. The social gain from a match is low when the worker's productivity is low, while the social cost in terms of congestion for other agents remains constant. Unemployment benefits encourage workers to stop searching, and can thus be welfare improving. We show that unemployment benefits directed exclusively towards long-term unemployed can reduce overall unemployment.

6.2 The model

Our matching framework is a simplified version of the one in Pissarides (1990). The model consists of a continuum of workers and firms. Both workers and firms have to undertake costly and time-consuming search to

[1]Pissarides(1992) also obtains multiple equilibria. That his result carries over to our model is far from obvious. For one reason, the wages in our model are sensitive to market conditions through the workers' outside options in the bargaining game between workers and firms. This is not the case in Pissarides' model.

find a trading partner.

When entering the market, all workers have the same productivity, normalized to one. They immediately start searching for a job, and during the search process their productivity depreciate with a rate δ. When matched, the workers stay with the same firm until they exit the market, which they do at a constant and exogenous rate s (both when employed and when unemployed). New workers enter the market at the same rate, so the size of the labour force is constant. We normalize the number (measure) of workers to 1. In order to simplify the analysis, we assume that the workers receive no income when unemployed[2].

Firms are either vacant and searching for a worker, or occupied (by one worker) and producing. All firms are homogeneous. There is free entry in the model, and the zero profit condition implies that the expected discounted income (asset value) of a vacancy is zero.

The matching technology is characterized by a concave and constant returns to scale matching function $x(u, v)$, where u is the unemployment rate and v the vacancy rate. The transition rate from unemployment to employment for a searching worker is given by $p = x(u,v)/u = p(\theta)$, where θ denotes the labour market tightness v/u. The arrival rate of workers to searching firms is $q = x(u,v)/v = q(\theta)$. It follows that p is increasing and q decreasing in θ.

6.2.1 Workers

Let $E(y)$ and $U(y)$ represent the expected discounted income when employed and unemployed respectively, as a function of current productivity y. The asset value equation for U is then given by

[2] In the appendix, the model is solved for a more general version of the model.

$$\begin{aligned}(r+s)U(y) &= p(E(y)-U(y))-U'(y)\dot{y} \\ &= p(E(y)-U(y))-U'(y)\delta y \qquad (6.1)\end{aligned}$$

where r is the discount rate and δ the rate of depreciation of y. The equation is standard except for the last term, which captures the capital loss due to reduced productivity. Let $S(y)$ denote the joint match surplus when the worker in question has productivity y. Since the value of a vacancy is zero, the surplus is given by $S(y) = y/(r+s) - U$. The worker and the firm split the surplus according to the nash bargaining rule, hence $E - U = \beta S$, where β is the worker's bargaining power. We substitute this into (6.1) to obtain

$$(r+s+p\beta)U = p\beta y/(r+s) - U'(y)\delta y \qquad (6.2)$$

This is a first order linear differential equation. In addition we know that $U(0) = 0$. The system therefore has a unique solution, we guess that it is of the form $U = Ay$. Inserted into (6.2) this gives

$$(r+s+p\beta)Ay = p\beta y/(r+s) - A\delta y \qquad (6.3)$$

which is satisfied if A is given by

$$A = \frac{p\beta/(r+s)}{r+s+p\beta+\delta} \qquad (6.4)$$

Thus $U(y)$ can be written as

$$U(y) = Ay = \frac{p\beta/(r+s)}{r+s+p\beta+\delta}y \qquad (6.5)$$

The effect of depreciation of human capital on expected income when unemployed is the same as the effect of harder discounting (higher r). Note that

A is decreasing in δ and that $\delta = \infty$ implies $A = 0$ as we would expect. Note also that A increases in p, and $p = \infty$ gives $U = y/(r+s)$, the expected income in a competitive market.

6.2.2 Distribution of skills

In this section we calculate the distribution $F(y)$ of productivities among the unemployed. In steady state F is constant. Equalizing streams to and from the pool of workers with productivity less than y gives (with $f(y) = F'(y)$)

$$\delta y f(y) = (s+p)F(y) \tag{6.6}$$

or

$$\frac{f(y)}{F(y)} = \frac{s+p}{\delta y} \tag{6.7}$$

Integrating and transforming each side, using the exponential function, give

$$F(y) = Cy^{\frac{s+p}{\delta}} \tag{6.8}$$

where C is a constant of integration. Substituting in $F(1) = 1$ gives $C = 1$. It follows that $f(y) = y^{\frac{s+p}{\delta}-1}(s+p)/\delta$. Let $\bar{y}$ denote the expected value of y. Then

$$\begin{aligned} \bar{y} &= \int_0^1 y f(y) dy \\ &= \int_0^1 \frac{s+p}{\delta} y^{\frac{s+p}{\delta}} dy \\ &= \frac{s+p}{s+p+\delta} \end{aligned} \tag{6.9}$$

As expected, $\bar{y}$ increases in p and decreases in δ.

6.2.3 Firms

Let $J(y)$ denote the asset value of a filled job. Since the asset value of a vacancy is zero, we have that $J(y) = (1-\beta)S(y)$, where $S(y)$ still denotes the match surplus. As we showed above, $S(y) = y/(r+s) - U$. Substituting in for U from (6.5) gives

$$J(y) = (1-\beta)y[\frac{1}{r+s}(1 - \frac{p\beta}{r+s+p\beta+\delta})]$$

When we take expectations, using (6.9), this yields

$$(r+s)\bar{J} = (1-\beta)\frac{p+s}{p+s+\delta}\,\frac{r+s+\delta}{r+s+p\beta+\delta} \tag{6.10}$$

where $\bar{J} = E^y J(y)$. The first factor increases in p. This reflects that the average productivity increases in p. The second term decreases in p, and this reflects that higher p increases U and thus improves the worker's bargaining position. If β is low and δ is high, the first effect will dominate, and J increases with p and with the labour market tightness θ.

We now turn to entry. The asset value equation for a vacant job is given by

$$rV = -c + q(J - V)$$

where c is search costs while q still denotes the arrival rate of workers. Since $V = 0$ this gives

$$J = c/q$$

Inserting this into (6.10) gives

$$(r+s)c/q = (1-\beta)\frac{p+s}{p+s+\delta}\,\frac{r+s+\delta}{r+s+p\beta+\delta} \tag{6.11}$$

where $p = p(\theta)$ and $q = q(\theta)$. This equation determines the labour market tightness θ.

6.2.4 Equilibrium

The last variable we have to solve for is the unemployment rate. Equalizing streams into and out of unemployment gives us the Beveridge curve:

$$u = \frac{s}{p+s} \tag{6.12}$$

We want to investigate whether or not the model can have multiple equilibria. From (6.11) we know that both sides of the equation are increasing in θ, and this suggests that multiple equilibria may exist in some situations.

To get a sharper result, we assume that the matching function is Cobb-Douglas, so that $q(\theta) = \theta^{-\nu}$, $0 < \nu < 1$. Then we can show the following proposition:

Proposition 20 *If $\nu < 1/2$, and the workers' bargaining power β not too high, some sets of parameters give three equilibria.*

To realize this, note first that the recursive structure of the model makes it sufficient to cheque that (6.11) can have three real, positive roots. For this purpose, assume first that $\beta = 0$. Inserting this and the Cobb-Douglas matching function into (6.11) give

$$(r+s)c\theta^{\nu} = \frac{\theta^{1-\nu} + s}{\theta^{1-\nu} + s + \delta} \tag{6.13}$$

Rearranging gives

$$(r+s)c(s+\delta) = s\theta^{-\nu} - c(r+s)\theta^{1-\nu} + \theta^{1-2\nu} \tag{6.14}$$

For small values of θ, the first term on the right-hand side dominates (since $\nu < 1$), and the right-hand side falls with θ. For medium sized values of θ, the last term dominates (if $c(r+s)$ is relatively small compared to 1, the maximum productivity), and the right- hand side of the equation increases in θ. For large values of θ, the medium term dominates, and the right-hand side falls in θ. Hence there exist values of r and δ such that we get multiple equilibria. Since (6.11) is continuous in β, we also get multiple equilibria for values of β close to 0.

Note that when $\beta = 0$, the last coefficient in (6.10) is equal to 1. Thus the expected value of a match is increasing in θ in this case. In generally, an increase in p increases wages and hence reduces profit, but the effect is weak when β is small.

We now turn to the welfare properties of the model. High labour market tightness implies high average productivity. Hence we would expect the optimal labour market tightness to be higher than in the original model without depreciation. On the other hand, the depreciation of human capital reduces wages, and hence increases entry.

In order to simplify the analysis, we follow Hosios(1990), and assume that $r = 0$. The optimal value of θ then solves the problem

$$\max_{\theta} \bar{y}(1-u) - cu\theta \tag{6.15}$$

subject to (6.12) and (6.9). We also assume that the matching function is Cobb-Douglas, i.e. that $x(u,v) = u^{\nu}v^{1-\nu}$ as above. From Pissarides (1990) and Hosios (1990) we know that optimality in the original model is obtained whenever $\nu = \beta$. We want to find out whether this rule still leads to optimality, and the next proposition tells us that it does not. The proof is given in the appendix

Proposition 21 *Suppose the matching function is Cobb-Douglas, and $\beta = \nu$ as described above. Then (6.11) implies too low θ when $\bar{y}$ is given by (6.9).*

From the equations (6.24) and (6.25) in the appendix, it follows that the distortions converge to zero when β converges to 1, while they increases as β converges to zero (for $\beta = \nu$). Hence, loosely speaking, multiple equilibria are most likely when the distortions are large (given that $\beta = \nu$).

6.3 Extensions

Our assumptions about no unemployment benefits and no cost of creating vacancies simplify the analysis, since they imply that the expected income when unemployed is proportional to a worker's current productivity. However, the simplifications come at some costs, since the presence of especially unemployment benefits have interesting implications for the behaviour of the model.

If workers are eligible to unemployment benefits, they stop searching when their productivity have fallen below a certain cut-off level. Long-term unemployed workers thus "fall out" of the market. From a social point of view this can be a good thing. The social (and private) value of getting a worker matched goes to zero as his productivity vanish. However, the negative congestion effect for other workers stay the same, so after a certain period of time (a "cut-off" time, or alternatively a "cut-off" productivity), it is socially optimal that the worker leaves the market.

In this section we introduce unemployment benefits for the long-term unemployed. More specifically, we assume that workers receive a payment ω when $y \leq \omega$, contingent on not being employed. Hence workers leave the market when their productivity falls below ω. The unemployment benefit has two important implications. Firstly, it increases the average productivity in the pool of unemployed workers searching for a job. Secondly, it increases the workers' outside option (his expected income when unemployed) when they bargain with their employers, and thereby also the wages.

6.3.1 Workers

The introduction of unemployment benefits for long-term unemployed does not influence the differential equation (6.2) determining a worker's expected income when unemployed, only the terminal conditions. In the last section we found that $U(0) = 0$, now the relevant terminal condition is that $U(\omega) = \omega/(r+s)$. The general solution to (6.2) is on the form

$$U(y) = Ay + Cy^{-\rho/\delta}$$

where $\rho = r + s + \beta p$. (see Sydsæter (1978)). Inserting this into (6.2) and rearrange gives

$$Ay + zCy^{-\rho/\delta} = \frac{p\beta}{r+s+p\beta+\delta} y/(r+s) + zCy^{-\rho/\delta}$$

where z is an uninteresting constant[3]. Hence A is still given by (6.4). Inserting $U(\omega) = \omega/(r+s)$ gives

$$\omega/(r+s) = A\omega + C(\omega)^{-\rho/\delta}$$

Hence

$$C = \frac{r+s+\delta}{(\rho+\delta)(r+s)} \omega^{\frac{\rho+\delta}{\delta}} \tag{6.16}$$

6.3.2 Equilibrium

First we want to calculate the distribution $F(y;\omega)$ of productivities among the unemployed, given ω. We again use the fact that workers leave the market when $y = \omega$. Now[4].

[3]We have that $z = (r+s+p\beta)/(r+s+p\beta+\delta)$.

[4]To see this, recall the expressions for $f(y;0)$ and $F(y;0)$ from the previous section.

$$f(y;\omega) = \frac{f(y;0)}{1-F(\omega,0)} = \frac{y^{\frac{s+p-\delta}{\delta}}(s+p)/\delta}{1-\omega^{\frac{s+p}{\delta}}} \tag{6.17}$$

For $F(y;\omega)$ we thus get (by integration)

$$F(y) = \frac{y^\alpha - \omega^\alpha}{1-\omega^\alpha} \tag{6.18}$$

where $\alpha = (s+p)/\delta$.

We want to calculate the expected value of a match for a firm. Since $J = \beta(S-U)$, and $S = y/(r+s)$, we get that

$$J = \int_\omega^1 (1-\beta)[\frac{r+s+\delta}{\rho+\delta}y/(r+s) - Cy^{-\rho/\delta}]\frac{\alpha y^{\alpha-1}}{1-\omega^\alpha}dy \tag{6.19}$$

where C is given by (6.16).

Proposition 22 *Suppose $r+\delta > (1-\beta)p$. Then there exists an interval $[0,\bar{\omega}]$ where $J(\omega)$ is increasing. Furthermore, if $\alpha = (s+p)/\delta < 1$, $J'(\omega)$ goes to infinity when ω converges to zero.*

Proof: Taking the derivative of (6.19) with respect to ω gives (recall that C depends on ω):

$$\begin{aligned} J'(\omega) &= -\omega^{\rho/\delta}\frac{r+s+\delta}{\delta(r+s)}(1-\beta)\int_\omega^1 y^{-\rho/\delta}\frac{\alpha y^{\alpha-1}}{1-\omega^\alpha}dy \\ &+ \frac{\alpha\omega^{\alpha-1}}{1-\omega^\alpha}J(\omega) \end{aligned} \tag{6.20}$$

If $\alpha - 1 < \rho/\delta$ we know that $\lim_{\omega\to 0}\omega^{\alpha-1}/\omega^{\rho/\delta} = \infty$, and hence we know from (6.20) that there exists an interval $(0,\bar{\omega})$ where $J'(\omega) > 0$. From this the first part of the proposition follows. If $\alpha - 1 < 0$ it follows that $\lim_{\omega\to 0}\omega^{-\alpha} = \infty$, and from (6.20) we then get that $\lim_{\omega\to 0} J'(\omega) = \infty$ as well.

□

As we have discussed earlier, the introduction of benefits for long-term unemployed increases the average productivity among unemployed and shifts the wage schedule upwards. The first effect is positive and the second negative for firms profit. The proposition tells us that the positive effect outweighs the negative when $\delta + r > p(1-\beta)$.

The effect of ω on the average productivity is strong if the lower tail of the distribution is thick, so that cutting off the tail has a large impact on the average productivity. The tail is thick when p and s are small while δ is large.

The effect on workers outside option is strong when the discount rate r and the exit rate s are small. The effects of p is more complex. High p makes it less likely that the worker will cash in the unemployment benefit. However, if the worker becomes employed, his increased outside option increases his wage, and this reduces the negative impact of p on U. Finally high δ increases the value of the benefits to the worker, but the effect is less in relative terms than the effects of δ on the size of the tail. In sum, we find that the positive impact dominates whenever $(1-\beta)p < r + \delta$.

If $(s+p)/\delta$ is less than one, the density $f(\omega)$ goes to infinity as ω goes to zero. As a result, $J'(\omega)$ goes to infinity as well. Since θ is given by $c/q(\theta) = J$, the marginal impact on θ is very large for small ω, and the introduction of a small benefit for the long-term reduces overall unemployment.

6.4 Conclusion

We have studied a matching model where workers loose skills during unemployment. We calculate the impact of this on the asset value of unemployed workers, and find that it is similar to harder discounting. We introduce entry, and show that depreciation of productivity during unemployment leads to multiplier effects and possibly multiple equilibria. Furthermore, the number of firms entering the market tends to be too low compared to the socially optimal level. In the last part of the paper, we introduce unemployment benefits for long-term unemployed, and show how this can increase welfare and reduce unemployment.

In a future work I plan to extend the welfare analysis, and see whether centralized bargaining can do better than decentralized. I will also simulate a parametrized version of the model on a computer, to find stability properties of the various equilibria, and to asses the qualitative importance of loss of skills during unemployment. Finally, I think it would be interesting to estimate an extended version of the model, with endogenous search intensities included, using unemployment data from U.S., Great Britain, and / or Norway.

Appendix

Proof of proposition 21 First we rewrite equation (6.11). We have that $E^y(E-U) = \beta/(1-\beta)\ c/q$, and since $p = \theta q$ we get $(r+s+\delta)U = \beta/(1-\beta)\ c\theta$. Since

$$\bar{J} = (1-\beta)y/(r+s)\ - U)$$

this gives

$$c/q = J = (1-\beta)y/(r+s) - \beta\theta c/(r+s+\delta) \tag{6.21}$$

Now we want to characterize the optimal solution. Substituting (6.12) and (6.9) into (6.15) gives us

$$\max_{\theta} \frac{p}{s+p+\delta} - c\theta\frac{s}{s+p} \tag{6.22}$$

Taking derivative with respect to θ gives (using that $p'(\theta) = q(1-\nu)$

$$\frac{q(1-\nu)(s+\delta)}{(p+s+\delta)^2} - \frac{s^2c + \nu sc\theta q(\theta)}{(s+p)^2} = 0 \tag{6.23}$$

or

$$\bar{y}^2(1-\nu) - \frac{\nu\theta cs}{s+\delta} = \frac{sc}{q}(1 - \frac{\delta}{s+\delta}) \tag{6.24}$$

Since $\nu = \beta$, we have that $c/q = \bar{J} = \frac{1-\nu}{\nu}\,(E-U)$, and from (6.1) and (6.5) we get that

$$E - U = \frac{s+\delta}{p}U = \beta\frac{s+\delta}{s+p\beta+\delta}\bar{y}/s$$

since $r = 0$. Hence

$$\frac{sc}{q}\,\frac{\delta}{s+\delta} = \frac{(1-\nu)\delta}{s+p\beta+\delta}\bar{y}$$

Inserted into (6.24) this gives

$$\bar{y}(1-\nu)(\frac{\delta}{s+p\beta+\delta} + \frac{s+p}{s+p+\delta}) - \frac{s}{s+\delta}\nu\theta c = sc/q \tag{6.25}$$

Since

$$\frac{\delta}{s+p\beta+\delta}+\frac{s+p}{s+p+\delta}>1$$

the proposition follows when comparing (6.24) and (6.25)

$\square$

We want to calculate the expected income for an unemployed who receives an unemployment benefit b, when the value of a vacancy is V. The differential equation analogous to (6.2) is

$$(r+s+\beta p)U(y)=b+\beta p(y/(r+s)\ -V)-U'(y)\delta y \tag{6.26}$$

where b is the unemployment benefit when the worker is searching. The solution to this differential equation is given by

$$U(y)=\frac{\beta p}{r+s+\beta p+\delta}y/(r+s)+Cy^{-\rho/\delta}-p\beta/\rho V+b/\rho \tag{6.27}$$

where $\rho = 1/r+s+\beta p$ and C is a constant. C can be calculated when we know when the worker stop searching, and what his expected income is at that stage. For this end we assume that the unemployment benefit is time dependent, and changes to ω when a worker has been unemployed longer than t^1, and we assume that $\omega \geq w(t^1)$ so that the worker stops searching at t^1. Let $y^1 = y(t^1)$. Then we must have that $U(y_1) = \omega/(r+s)$, and hence

$$C=[\omega/(r+s)-y^1p\beta/(r+s+p\beta+\delta/\rho+Vp\beta/\rho\frac{Ap\beta}{r+s+p\beta+\delta}+p\beta/\rho]A^{\rho/\delta} \tag{6.28}$$

Part III

Other Topics

Chapter 7

Optimal Unemployment

7.1 Introduction

The discussion of whether the level of unemployment is socially efficient is not new in economics. Since Friedman-Phelps introduced the notion of the natural rate of unemployment, a large body of research has developed on the subject. As demonstrated by Diamond (1982), Mortensen (1982), and Pissarides (1984) among others, matching models with two sided search and with wages determined by decentralized bargaining provide a natural framework when addressing the issue.

The basic idea underlying the matching literature is that finding a trading partner is costly and time consuming. This is captured by the introduction of the matching function $x(u, v)$, which relates the number of matches per unit of time to the stock of searching workers and firms. Since finding new trading partners are costly, the worker and the firm have a mutual interest in staying together. Hence there is a surplus associated with the match. We study models where this surplus is split between workers and firms according to the Nash sharing rule, where the worker gets a share β and the firm a share $1 - \beta$ of the match surplus.

The behaviour of the model depends on the size of β. If $\beta = 1$, the workers get all the surplus, and no firm has an incentive to enter the market and incur search costs to find a trading partner. Hence all the workers are unemployed in steady state. On the other hand, if $\beta = 0$ the firms get all the surplus, and the wage is bid down to the monopsony price (the Diamond(1971) case). Both cases are obviously sub-optimal since they imply that the market does not generate surplus[1].

Hosios(1990) characterizes the (constrained) optimal allocation of resources in steady state. He finds that for a specific value of the workers' bargaining power β, the decentralized market solution generates the same steady state allocation. This happens when β is equal to η, the elasticity of the arrival rate of workers to searching firms with respect to the labour market tightness, $\theta = v/u$.

The $\beta = \eta$ rule for optimality is frequently used as a valid criteria for optimality, although it is derived under rather strong assumptions. Hosios uses techniques from static maximization, the interest rate is set equal to zero, and the steady state path that maximizes average income is derived. Since matching models are inherently dynamic, a dynamic maximization approach seems appropriate. In the first part of this paper we show how optimality conditions for standard search models can be derived by employing optimal control theory. This approach has many advantages compared to the static maximization approach. First, it simplifies many of the welfare comparisons. Second, it facilitates a more general and rigours analysis of the problem, also outside steady state and for positive interest rates. Finally, the disjoint functions associated with the Hamiltonian have interesting economic inter-

[1]When $\beta = 0$, free entry still implies that the the expected profit when opening a vacancy is zero

pretations since they replicate the asset value equations in the model of the decentralized market. As we will see, the set of equations constituting the first order conditions for optimality are identical to the equations for the decentralized economy with η replaced by β. We find that the optimal path is a saddlepath, identical to the market solution when $\beta = \eta$.

Hosios only studies the efficiency of relatively simple models. It is therefore of interest to see whether the $\beta = \eta$ rule also holds in more general settings and not only as a special case. We look at three different extensions of the model. First, we allow the exogenous variables to be continuous functions of time. This is a relevant extension since matching models are frequently used to model business cycles. Then we allow for firm heterogenities. The assumption about homogeneous agents is obviously to simplify, an optimality rule that hinges on it is not very applicable. Finally we introduce on-the-job search. Matching models that includes on-the-job search have recently been developed (Mortensen and Pissarides(1994) and Pissarides(1994)). On-the-job search is highly relevant since it accounts for a substantial part of the observed labour turnover.

We find that the optimality rule $\beta = \eta$ are robust to the first extension, that is, it still holds when the exogenous variables are time dependent. However, it is not robust to the other two extensions. With heterogeneous firms, the market typically overvalues the low productivity and undervalues the high productivity firms when $\beta = \eta$, and this distorts all decisions made during the search process. To achieve optimality the wage must be independent of the productivity of the firm in question. On-the-job search typically leads to too much entry.

To grasp the intuition behind the results, recall the two sources of externalities of entry we identified above. A positive externality for the searching workers and a negative externality for other vacancies. In the basic model they offset each other when $\beta = \eta$. If the firms are heterogenous, wage bargaining implies that the compensation paid to the workers (the positive externality) is higher for high productivity firms than for low productivity firms. The negative externality on other vacancies is independent of the productivity of the firm in question. We find that when $\beta = \eta$ the positive and the negative externalities still cancel each other out for the average firm. For the low productivity firms the negative externality then has to be greater than the positive externality, leading to a market value which is too high. For the high productivity firm we get the opposite result.

With on-the-job search, a new externality is present in the model. A firm that hires an employed worker creates a negative externality for the incumbent firm. The old optimality rule therefore leads to too much search.

7.2 Optimality of the standard model

In this section we first present a standard search model of the Pissarides(1990) type, and then use optimal control theory to derive efficiency results. Firms are either vacant and searching for a worker at a cost $c \geq 0$ or occupied (by one worker) and producing a stream of $y > 0$ units of output. There is a fixed cost k associated with the opening of a vacancy. Furthermore, there is a fixed number $\bar{R}$ of workers in the economy which forego an income z when becoming unemployed (the unemployment benefit).

Following standard assumptions, let $x(u, v)$ denote the stream of new worker-firm matches, where u denotes the measure of unemployed workers and v the measure of unoccupied jobs (vacancies). x is concave and homogeneous of degree 1 in (u, v), and has continuous derivatives. Let $q = x(u, v)/v = q(\theta)$ denote the transition rate for a vacancy, where θ is the labour market tightness v/u. The transition rate for an unemployed worker is then given by $x(u,v)/u = \theta q(\theta)$. We assume that $\lim_{\theta\to\infty} q(\theta) = 0$ and that

$$\lim_{\theta\to 0} q(\theta) = \infty$$

When the labour market tightness goes to zero the match probability rate for firms goes to infinity. When θ goes to infinity, it goes to zero.

There is a constant and exogenous probability rate s of match destruction due to adverse idiosyncratic productivity shocks. When a productivity shock occurs the worker joins the unemployment pool. The remaining vacancy is assumed to be worthless. The key variable in the model is the labour market tightness θ. Given θ, the dynamics of u are given by

$$\dot{u} = \theta q(\theta)u - (R-u)s \tag{7.1}$$

where R denotes the measure of the work force. In steady state, with $\dot{u} = 0$, this gives

$$u = R\frac{s}{s+\theta q(\theta)} \tag{7.2}$$

This last equation is the Beveridge curve, showing the long-run relationship between labour market tightness and unemployment.

7.2.1 The decentralized economy

Denote by U, E, J, and V the expected discounted incomes (asset values) for an employed worker, an employed worker, an occupied job, and a vacant job respectively. Then

$$\begin{aligned}
rU &= z + \theta q(\theta)(J-U) + \dot{U} \\
rE &= w + s(E-U) + \dot{E} \\
rJ &= y - w + s(J-V) + \dot{J} \\
rV &= -c + q(\theta)(J-V) + \dot{V}
\end{aligned} \tag{7.3}$$

The first equation states that the return to a worker when unemployed is equal to the current income plus the expected capital gain associated with getting a job. The other equations can be interpreted in a similar way.

The wage is determined by the cooperative (asymmetric) Nash solution in a bargaining game between a matched worker and a firm, with U and V as disagreement points respectively. Define the match surplus S as $S = E + J - U - V$. Then $E = U + \beta S$ where β denotes the workers' bargaining power. We also know that free entry implies that the value V of a vacancy is equal to the cost k of producing one, and we can thus rewrite equation (7.3) to

$$\begin{aligned} rU &= z + \theta q(\theta)\beta S && (7.4) \\ rV &= k && (7.5) \\ rV &= -c + q(1-\beta)S && (7.6) \\ (r+s)S &= y - rU - (r+s)V && (7.7) \end{aligned}$$

Pissarides(1987) shows that the adjustment path of the model is a saddlepath; the only non-exploding path is such that the asset values are constant along the path. Note the recursive structure of the model. The equations (7.4)-(7.7) determine the labour market tightness, θ, while the dynamics of u are given by (7.1). We end this section by stating the following lemma, proved in the appendix:

Lemma 17 *If $y - z > (r+s)k$, the set of equations (7.4)-(7.7) has a unique solution*

7.2.2 The optimal solution

Following Hosios and Pissarides, we define the socially optimal path as the path that maximizes the discounted value of aggregate production less search costs and costs of creating vacancies. The welfare function is then

$$W = \int_0^\infty e^{-rt}[yN + zu - cv - ak]dt \tag{7.8}$$

where v still denotes the measure of vacancies. W is maximized with respect to a and u, subject to the differential equations

$$\begin{aligned} \dot{N} &= vq(v/u) - sN & (7.9) \\ \dot{v} &= a - vq(v/u) & (7.10) \end{aligned}$$

where (7.9) is a redefinition of (7.1). In addition, the solution must satisfy the constraint

$$R \geq u + N \tag{7.11}$$

The state variables in the problem are N and v [2]. The current value Hamiltonian associated with the problem is given by

$$\begin{aligned} H^c &= yN + zu - cv \\ &+ \lambda[vq(v/u) - sN] \\ &+ \gamma[a - vq(v/u)] \\ &+ \alpha(R - u - N] & (7.12) \end{aligned}$$

Here, λ and γ are adjoint functions associated with N and v respectively, while α is the Lagrangian multiplier associated with the constraint (7.11). Necessary conditions for the problem are given by

[2]Note that $v(t)$ might be discontinuous. However, from Seierstad and Sydsæter(1987), theorem 7, pp 196 we know that this does not influence the first order conditions. In particular the adjoint functions are still continous

$$\begin{aligned}
a^* &= arg\max_a H(N^*, v^*, u^*, a) \\
u^* &= arg\max_u H(N^*, v^*, u, v^*) \\
r\lambda - \dot{\lambda} &= \frac{\delta H^c}{\delta N} \\
r\gamma - \dot{\gamma} &= \frac{\delta H^c}{\delta v} \qquad (7.13)
\end{aligned}$$

in addition to (7.9)-(7.11)[3]. Hence we get the following, with $\eta = \theta q'(\theta)/q$:

$$\gamma = k \qquad (7.14)$$

$$\alpha = z + \theta q(\theta)\eta(\lambda - \gamma) \qquad (7.15)$$

$$(r+s)\lambda = y - \alpha + \dot{\lambda} \qquad (7.16)$$

$$r\gamma = -c + q(\theta)(1-\eta)(\lambda - \gamma) + \dot{\gamma} \qquad (7.17)$$

The system (7.14)-(7.17) only depends on v and u through θ. When the path of θ is determined, v is determined by $v = \theta u$, and a by the differential equation (7.10) (except in the discontinuity points of v). Note also that the first order conditions (7.14) and (7.15) for the maximization problems in the first two equations in (7.13) are only necessary, so a solution to (7.14)-(7.17) does not necessarily satisfy (7.13).

The first lemma shows that the necessary conditions defined by (7.13) are also sufficient

Lemma 18 *The first order conditions given by (7.13) together with the transversality condition stated above are also sufficient conditions for optimality.*

[3]In addition we have the transversality condition $\lim_{t\to\infty} e^{-rt}\lambda = 0$, see Seierstad and Sydsæter(1987), Theorem 16, pp 244 for details

Proof: The proof is a straightforward application of Arrows sufficiency theorem (see Seierstad and Sydsæter (1987), theorem 8, pp 198). Thus it is sufficient to show that the Hamiltonian is concave in the state variables given the optimal choice of control variables. Since the Hamiltonian is linear in N it is also concave in N. Note that $vq(v/u) = x(u,v)$, which is concave in v. Thus H is concave in v if $\lambda \geq \gamma$, and this follows from (7.14) and (7.17)

□

The next lemma shows that an optimal solution exists and has constant adjoint functions along the adjustment path:

Lemma 19 *If $y - z > (r+s)k$ the optimization problem has at least one solution. Along all the solution paths the adjoint functions are constant.*

Proof: For the first maximization problem in (7.13) to have a solution, we must have that $\gamma = k$, and thus $\dot{\gamma} = 0$. Thus we only have to prove that $\dot{\lambda} = 0$ along the adjustment path.

In the first chapter of the thesis we show that a solution with λ constant exists (see section 3.1). Let θ^* denote a corresponding constant value of θ, and $v^*(t)$ and $u^*(t)$ the corresponding values of v and u.

Let u', v' be another solution to the problem, say with $\dot{\lambda} > 0$ for some $t = t'$. From (7.16) we then find that $\lambda'(t') > \lambda^*(t')$. Since λ is continous there exists an interval $[t_0, t_1]$ such that $\lambda' > \lambda^*$. Without loss of generality we assume that $t_0 = 0$.

Now define a third path u'', v'', in the following way: Let $(u'', v'') = (u', v')$ for $t \leq t_1$. Then let the path be such that $\theta = \theta^*$ for $t > t_1$. This path also solves the problem, since for all starting times t and all initial values $N(t)$, the path associated with a labour market tightness θ^* solves the problem of maximizing W from time t onwards. Therefore, any path that switches from

u', v' to the path associated with constant θ at any point in time solves the problem. But since λ is discontinuous at $t = t_1$, (u'', a'') does not satisfy the necessary conditions.

□

With $\dot{\lambda} = \dot{\gamma} = 0$, the set of equations (7.14)-(7.17) are identical to the asset value equations (7.4)-(7.7) with $\beta = \gamma$, with $\alpha = rU, \lambda - \gamma = S$, and $\gamma = V$. Together with the lemmas above, this implies the following proposition:

Proposition 23 *Assume that the set of equations (7.13) with $\dot{\lambda} = 0$ has a unique solution. Let η^* denote the corresponding value of η. Then the market solution coincides with the optimal solution if and only if $\beta = \eta^*$*

If (7.13) has $n > 1$ solutions, with n corresponding values of θ and values of η, they all give rise to optimal paths that are equal from a welfare point of view.

Note that since (7.14)-(7.17) may have solutions that are not solutions to (7.13), $\beta = \eta$ is not a sufficient condition for optimality. Note also that if there is more than one optimal path, any path which switches between them is also optimal.

7.2.3 More about the optimal solution

Let $W(N, v, R, t)$ denote the value function associated with the planners problem. From Seierstad and Sydsæter(1987), pp 210-219, we know that

$$
\begin{aligned}
\frac{\partial W}{\partial N} &= e^{-rt}\lambda \\
\frac{\partial W}{\partial v} &= e^{-rt}\gamma \\
\frac{\partial W}{\partial R} &= e^{-rt}\alpha/r
\end{aligned}
\tag{7.18}
$$

Thus λ, γ and α/r denote the shadow price of one more employed (given the total number R of workers), a vacancy, and an unemployed respectively.

As we have seen, the expressions for λ, γ, and α in (7.14) -(7.17) correspond algebraically to the asset values $S+V$, rU and V in (7.4)-(7.7) when $\beta = \gamma$. Thus U and V are equal to the shadow price of an unemployed and a vacancy, respectively. The agents expected income are equal to their social value.

7.3 Extensions

The model described in the last section is stylised and with many simplifying assumptions. In this section we investigate the welfare properties of two extended versions of the model.

7.3.1 Time-dependent exogenous variables

In this subsection we allow for time-dependent exogenous variables. More specifically, we write $y = y(t), z = z(t)$, $c = c(t)$, and $R = R(t)$, where all the functions are Lipschitz continuous in t. We also assume that there exists a $t^1 > 0$ such that $y(t) = \bar{y}, z(t) = \bar{z}$ etc. for all $t > t^1$. We simplify the analysis by assuming that $k = 0$.

Firms and workers are continuously renegotiating the wage, with constant bargaining power. The modified versions of the asset value equations (7.4)-(7.7) are given by

$$\begin{aligned} rU(t) &= z(t) + \theta(t)\, q(\theta(t))\, \beta S(t) + \dot{S} & (7.19) \\ rV &= -c(t) + q(1-\beta)S(t) \equiv 0 & (7.20) \\ (r+s)S(t) &= y(t) - rU(t) - (r+s)V + \dot{S} & (7.21) \end{aligned}$$

The structure of the model is still recursive, with θ determined by (7.19)-(7.21), and N by (7.9).

Lemma 20 *The function $\theta(t)$ satisfying (7.19)-(7.21) is given by the differential equation*

$$(r+s)\frac{c(t)}{q(\theta(t))} = (1-\beta)[y(t)-z(t)] - \beta\theta c(t) + \frac{q\dot{c} - cq'(\theta)\dot{\theta}}{q^2} \qquad (7.22)$$

and the boundary condition $\theta(t^1) = \theta^*$, where θ^* is the unique equilibrium value of the static model at $t > t^1$.

Proof: Taking the derivative of (7.20) with respect to t after re-arranging gives

$$\frac{q\dot{c} - cq'(\theta)\dot{\theta}}{q^2} = (1-\beta)\dot{S} \qquad (7.23)$$

We also have that

$$r(E-U) = w - z - s(E-U) - \theta q(\theta)(E-U) + (\dot{E} - \dot{U})$$

and

$$rJ = y - w - sJ + \dot{J}$$

Using that $E - U = \beta S$, $J = (1-\beta)S$, and $\dot{S} = \dot{E} + \dot{J} - \dot{U}$ give

$$(r+s)S = y - z - \theta q(\theta)\beta S + \dot{S}$$

Substituting S and $\dot{S}$ by the virtue of (7.20) and (7.23) thus gives (7.22)

□

The socially optimal path is determined in the same way as in the last section. The Hamiltonian is unaltered, save for exogenous variables now being time dependent. Since the first order conditions (7.13) do not involve taking derivatives with respect to time, necessary conditions are still given by (7.14)-(7.17).

Since θ varies along the optimal path, so do in general $\eta(\theta)$, and a simple optimality rule like $\beta = \eta$ is not fruitful. The exception is when the matching technology is Cobb-Douglas. The Cobb-Douglas matching function implies that the elasticity of $q(\theta)$ is independent of θ, and hence also that the first order conditions (7.14)-(7.17) also are sufficient. We get the following result:

Proposition 24 *Suppose that the exogenous variables are time dependent as described above, and that the matching function is Cobb-Douglas. Then the market solution and the optimal solution coincide if and only if $\beta = \eta$.*

Proof. The proof is very similar to the proof of lemma 20. Inserting $\gamma = 0$ in (7.17) gives $\lambda = c/q$. Taking derivatives with respect to t gives an expression like (7.23), with S substituted out for λ. Substituting the value of $\dot{\lambda}$ into (7.16), and substituting out α by (7.14) then gives (7.22). Thus the optimal path for θ is governed by the same differential equation as in the market solution when $\beta = \eta$. Since the terminal conditions are the same (proved in the last section), the proposition follows

□

7.3.2 Heterogeneous agents

In this subsection we assume that the vacancies have different productivity. More specifically, we assume that the productivity can take n different values, determined after the fixed cost k is incurred. Let the indexation be such that the productivity y_i is increasing in the index i, and denote by f_i the probability that the realized productivity is y_i.

A vacancy will start searching for a worker if and only if the asset value of an operating vacancy is at least 0. If the asset value is negative, the vacancy is destroyed immediately. Let $\hat{y}$ denote the cut-off productivity, and $\hat{\imath}$ the lowest i such that $y_i \geq y^m$. Let $\hat{f}_i = Pr\{y_i | i \geq \hat{\imath}\}$. The equilibrium is then characterized by the equations

$$
\begin{aligned}
rU &= z + \theta q(\theta) \sum_{i=\hat{\imath}}^{n} \hat{f}_i \beta S_i \\
rV_i &= -c + q(1-\beta) S_i \quad i \geq \hat{\imath} \\
(r+s) S_i &= y_i - rU - (r+s) V_i \quad i \geq \hat{\imath} \\
\sum_{i=\hat{\imath}}^{n} \hat{f}_i V_i &= k
\end{aligned}
\tag{7.24}
$$

In addition, the unemployment rate is governed by (7.1) as before.

Now we turn to the planner's problem. Let $\tilde{\imath}$ be the lowest i for which it is optimal to maintain the vacancy. Define v_i, $i = \tilde{\imath},,,,n$, as state variables in the optimal control problem. By reasoning as in the last section we then get the following first order conditions for the social optimum:

$$
\begin{aligned}
(r+s)\lambda_i &= y_i - \alpha \quad i \geq \tilde{\imath} \\
r\gamma_i &= -c + q(\lambda_i - \gamma_i) - q\eta \sum_{i \geq \tilde{\imath}} \tilde{f}_i \lambda_j \\
\alpha_i &= z + \theta q(\theta) \eta (\sum_{i=\tilde{\imath}}^{n} \tilde{f}_i (\lambda_i - \gamma_i)) \\
\sum_{i=\tilde{\imath}}^{n} f_i \gamma_i &= k
\end{aligned}
\tag{7.25}
$$

To calculate $\tilde{\imath}$, remember that the shadow value of a vacancy with productivity i is given by γ_i. $\tilde{\imath}$ is therefore the lowest i such that γ_i is nonegative. For later reference we also define y^s as the lowest productivity such that a vacancy with that productivity is not destroyed.

The interesting equation in (7.25) is the second equation. The expression for the shadow value of a vacancy of type i, γ_i, is different from the corresponding asset value equation even when $\beta = \eta$ except when λ is equal to the average value in the market. We get the following result:

Proposition 25 *Assume that $\beta = \eta$ and that $\tilde{i} = \hat{i}$. Then the following holds:*

1. *The labour market tightness and the unemployment rate in the decentralized economy is optimal.*

2. *The social value of a vacancy of type i is higher, equal to or lower than the market value as its productivity is higher, equal to or lower than the average productivity among the operating vacancies.*

Proof: Substituting the second equation into the last equation in (7.25) gives

$$k = \sum_{i=\hat{i}}^{n} \hat{f}_i(1-\eta)(\lambda_i - \gamma_i)$$

Thus, when calculating the expected value of a vacancy, the last term in the second equation in (7.15) cancels out. The set of equations determining the market value and the optimal value of θ are then identical, and the first part of the result follows.

When θ is the same in the social and the market economy, we find that the social value of a vacancy is equal to its market value if and only if the value of λ_i is equal to the average value. This holds if and only if the productivity of the vacancy is equal to the average in the market, and the last part of the result follows

□

The first part of the proposition states that if the "ex post" decision of destroying the vacancy is socially optimal, then $\beta = \eta$ still ensures optimal entry. The second part of the proposition implies that the ex post decision of whether or not to destroy the vacancy can be inoptimal, since the market overvalues low productivity vacancies. When $\beta = \eta$, vacancies that would be optimal to destroy can still be operating in the market solution.

The fact that the asset values of a vacancy differ from the socially optimal value will also distort other types of "ex post" decision (decisions taken after the productivity is realized). For instance, if the cost c is a function $c(e)$ of the effort e, and the matching function is of the form $x(u, ev)$, the effort that maximizes the asset value of a vacancy is given by[4]

$$c_i'(e) = \frac{rV_i - c}{e} \tag{7.26}$$

The corresponding expression for the socially optimal effort is obtained by substituting in γ_i for V_i. We have already seen that V_i is higher (lower) than γ_i when y is high (low). Thus when $\beta = \eta$, low-productivity firms are searching too hard and high-productivity firms too little compared to the socially optimal values.

The intuition here is that the negative externality that a vacancy creates for other vacancies is proportional to the average productivity among vacancies, and independent of the productivity of the vacancy in question. Socially correct prices in the market solution are obtained when the compensation to the workers equals this externality. An optimal compensation rule must imply a wage independent of the productivity of the firm in question. This contrasts with the Nash sharing rule, which implies that the compensation to the worker is increasing in y. The social optimum can be obtained by taxing low-productivity jobs and subsidising high-productivity jobs. Note, however, that this has perverse effects when it comes to income distributions.

[4]To realize this, recall that $rV = -c(e) + eqJ$, so that $rV' = -c' + qJ = -c' + (rV - c)/e = 0$, and the equation follows.

Finally, note that optimality is achieved when all workers get a a wage corresponding to the Nash-solution for the average firm and $\beta = \eta$. This may be thought of as the outcome of centralized wage bargaining.

7.4 Optimality with on-the-job search

Recently, matching models with ex.post bargaining and-on -the-job search have been developed in the literature, see Pissarides(1994) and Pissarides and Mortensen(1994). This encourages an analysis of the efficiency of models with on-the-job search.

We introduce on-the-job search in a simplified version of the models above. Following Mortensen and Pissarides(1994), we assume that the firms can be hit by a negative productivity shock, reducing the productivity from y^1 to $y^2 < y^1$. If this happens, the worker employed by the firm starts searching for a better job. We assume that employed and unemployed workers search for the same jobs equally efficiently. To simplify the calculations, we assume that the cost k of creating a vacancy is zero[5].

The decentralized economy Let μ denote the probability rate of transition from the good to the bad state. The asset value equations for the surplus in the good and the bad state, respectively, can then be written as[6]:

[5]Hence we exclude the situation where y^2 is so close to y^1 that on-the-job search not takes place. See Chapter 3 in this thesis for details.

[6]To see this, first let S^i, $i \in \{h, l\}$ denote the net match surplus. Then obviously $rS = y - \mu(S^1 - S^2) - s(S^1 - U) - rU$ and thus $(r + s + \mu)S^1 = y^1 - rU + \mu S^2$, and the first equation in (7.27) follows. To get the expression for S^2, note that $(r + s)S^2 = y^2 - rU + \theta q(\theta)(\beta S^1 - S^2)$. Since $\theta q(\theta)\beta S^h = rU - z$, the last equation follows.

$$
\begin{aligned}
(r+s+\mu)S^1 &= y^1 - rU + \mu S^1 \\
(r+s+\theta q(\theta))S^2 &= y^2 - z
\end{aligned}
\tag{7.27}
$$

The free entry condition then gives us that

$$
c/q = (1-\beta)S^1 \tag{7.28}
$$

When we insert this into the asset value equation for the unemployed worker (i.e. $rU = z + \theta q(\theta)\beta S^1$) we get that

$$
rU = z + \theta q(\theta)S_1 - \theta c \tag{7.29}
$$

The equations (7.27)-(7.29) determine the labour market tightness θ. To close the model we include the flow equations:

$$
\begin{aligned}
\dot{N}_1 &= \theta q(\theta)(N_2 + u) - (s+\mu)N_1) \\
\dot{N}_2 &= -(\theta q(\theta) - s)N_2 + \mu N_1
\end{aligned}
\tag{7.30}
$$

and finally the identity $N_1 + N_2 + u = R$.

The optimal allocation Since there are no costs of creating vacancies, the welfare function can be written as

$$
W = \int_0^\infty [y^1 N_1 + y^2 N_2 + zu - (u + N_2)\theta c]dt \tag{7.31}
$$

When maximizing W with respect to θ and u, subject to the constraints above, we get the following first order conditions (with the same notation as in the previous maximization problems):

$$
\begin{aligned}
(r+s+\mu)\lambda^1 &= y^1 - rU + \mu\lambda^1 \\
(r+s+\theta q(\theta))\lambda^2 &= y^2 - z \\
\alpha &= z + \theta q(\theta)\lambda_1 - \theta c \\
c/q &= (1-\eta)(\lambda^1\pi + (\lambda^1 - \lambda^2)(1-\pi))
\end{aligned}
\tag{7.32}
$$

where $\pi = u/(N_2 + u)$ is the proportion of the searching workers that is unemployed. Note that when $\beta = \eta$, the three first equations in (7.32) are equivalent to the respective equations (7.27) and (7.29) for the market solution. However, the last equation in (7.32), the entry condition, differs from the entry condition (7.28). For given (and equal) values of λ and S, the optimal solution implies less entry (or lower θ). More generally, since we know that θ is decreasing in the search cost c, we have shown the following result:

Proposition 26 *When $\beta = \eta$, the decentralized economy with on-the-job search leads to too much entry.*

The intuition for the result is the following: We know that when there is no on-the-job search, $\beta = \eta$ leads to optimality, with the negative externality for other firms when entering is exactly offset by the positive externality for the workers. With on-the-job search, a new externality is introduced, since low productivity firms looses profit when the worker quits. lose their worker. Since the firms do not take this into account when entering, the old optimality rule now leads to too high entry and thereby too high turnover.

7.5 Conclusion

Matching models are frequently used to describe the labour market and to explain unemployment. Therefore, the efficiency properties of such models are of great interest.

In the first part of the paper we studied the efficiency properties of a standard matching model using optimal control theory. We found that the optimal path is a saddlepath, corresponding to the market solution when $\beta = \eta$. Thus the optimality criterium derived in the literature using static programming still holds.

We then moved on to study the robustness of the result to changes in the structure of the model. The optimality criterium derived in the literature still holds when the exogenous variables of the model vary continuously. However, it is not robust to the introduction of heterogeneous firms or on-the-job search. When $\beta = \eta$, the market tends to overvalue low-productivity firms and undervalue high-productivity firms. The introduction of on-the-job search leads to too much search in the economy.

Appendix

Proof of lemma 17

Substituting in for U and V from (7.4) and (7.5) in (7.7) gives

$$(r + s + \theta q(\theta)\beta)S = y - z - (r + s)k$$

Substituting in for S from (7.6) gives

$$\frac{r + s + \theta q(\theta)\beta}{q(1 - \beta)}(rk + c) = y - z - (r + s)k \tag{7.33}$$

The left hand side of the equation is increasing in θ, and goes to zero when θ goes to zero and to infinity when θ goes to infinity. Therefore a unique solution exists if and only if $y - z - (r + s)k > 0$.

Chapter 8

A Matching Model with Hiring Costs

8.1 Introduction

In the literature on equilibrium search models of the labour market, two different strands have evolved regarding how to model labour demand. On one hand, Pissarides ((1985) (1987), (1990)) and Mortensen (1986) assume free entry of jobs, while on the other hand Diamond (1971, 1982) and others keep the number of jobs fixed.

Free entry of jobs makes the vacancy rate extremely volatile. After a positive aggregate productivity shock, firms open vacancies until all profit opportunities are exhausted, i.e until the expected search cost incurred before a vacancy is filled, is equal to the expected profit when a worker is found. Since search costs depend on the labour market tightness (vacancies per unemployed), the latter (and wages, as we will see) jumps directly to its new steady state equilibrium value after a shock. The *number* of vacancies therefore overshoot (undershoot) the steady state level considerably after a positive (negative) shock, since the new steady state level of unemployment is lower (higher) than the initial level. As a result, the unemployment rate starts to fall (increase) quickly immediately after the shock.

On the other hand, the assumption that the number of jobs is independent of the overall market conditions, does not seem very appealing and is not supported by data. Davis and Haltivanger (1990), among others, shows that the number of jobs does increase during booms and fall during recessions.

In this paper, we show how the excess volatility caused by free entry can be dampened by introducing convex hiring costs. There is a large, but fixed, number of firms in the model. All firms are free to open vacancies, but there are costs associated with hiring, training, and equipping new workers (i.e. of expanding a firm). We assume that these costs are convex, so that the marginal costs increase with the number of new workers entering the firm at a given time. If, for instance, the firms have a separate department for the recruiting and training of new employees, these may have capacity constraints that leads to decreasing returns. The convex costs imply that firms are reluctant to hire workers too quickly, and thus open less vacancies after a shock than they would have done without such costs. Our model can therefore be viewed as a compromise between the two ways of modelling labour supply described above.

The effects of the hiring costs are similar to the effects of convex adjustment costs in investment theory (see Begg (1982)). We find, that the labour market tightness, the vacancy rate, and the wage rate are still jump variables. However, they do not adjust immediately to their new steady state values, but increase gradually along the adjustment path. The unemployment rate falls less rapidly, and the adjustment process takes more time.

The paper proceeds like this: First, we analyze the effects of shifts in the aggregate productivity level under the simplifying assumption that wages only depend on productivity. Second, we study the bargaining game between a worker and a firm when hiring costs are present, and analyze the effect of productivity shocks with endogenous wages. Third , we simulate the model with different parameter values. In the last section we conclude.

8.2 Exogenous wages

In this section we present the basic model, and solves it under the simplifying assumption that wages are exogenous.

8.2.1 The model

We apply a matching model similar to the one presented in chapter 2 in Pissarides (1990). There is a continuum of workers in the market, who are either unemployed and searching for a job, or employed and working (no on-the-job search). The number (measure) of workers is constant and normalized to one. The production side of the economy consists of a large number of big, identical firms, each employing a continuum of workers, and each having a production technology exhibiting constant returns to scale in the number of workers. All firms can costlessly open as many vacancies as they want[1].

The number of matches in the economy is given by a concave, constant returns to scale matching function $x(u, v)$, where x denotes the number of matches, u the unemployment rate, and v the vacancy rate. The transition rate to employment is given by $p = x(u, v)/u = p(\theta)$, where $\theta = v/u$ is the labour market tightness. We assume that all firms take the aggregate variable θ for given. The arrival rate of workers to a firm is $v_i q$, where v_i is the number (measure) of vacancies posted, and $q = x(u, v)/v = q(\theta)$ is the arrival rate per vacancy. Thus the hiring technology has constant returns

[1]We assume that only existing firms create vacancies. Empirical findings show that two thirds if all new jobs are created in existing firms (see Mortensen and Pissarides (1994).

to scale in each firm. The assumptions about the matching function implies that p is increasing and q decreasing in θ. In addition we assume that p goes to zero and q to infinity when θ goes to zero, and the opposite when θ goes to infinity.

When a firm employs a worker, an amount is invested to make the worker productive. We assume that the investments are fully financed by the firm[2]. We also assume that the investment costs are increasing in the number of workers hired. More specifically, we assume that a firm's hiring costs at any given time is $K = a(v_i q)^2/2$, and consequently marginal hiring costs are $k = av_i q$. Thus the hiring technology has decreasing returns to scale. If, for instance, the firm has a training department, this may have capacity constraints which give rise to decreasing returns[3]. When the investments are undertaken, production starts immediately, and continue until the worker and the job separates. This happens at a constant, exogenous rate s.

Firms announce a positive measure of vacancies, and the stream of new workers can be regarded as deterministic. We study the behaviour of a representative firm. Let N denote the number of workers employed in the firm, y the productivity per worker, c search costs per vacancy, and r the interest rate. The representative firm's maximization problem can then be written as

$$\max_{v(t)} \int_0^\infty e^{-rt}[(y-w)N - cv - a(vq)^2/2]dt \tag{8.1}$$

Subject to the differential equation

$$\dot{N} = vq(\theta) - sN \tag{8.2}$$

[2]This can be because workers are credit constrained.

[3]Since we have decreasing returns, each firm earns profit. Following Diamond we assume that the entry of firms (not vacancies) play a minor role in the short run.

and an initial condition of the form $N(t_0) = N_0$. Let H be the associated current-value Hamiltonian and J the adjoint variable. H can then be written as

$$H = (y - w)N - cv - a(vq)^2/2 + J(vq - sN) \tag{8.3}$$

Necessary conditions are given by (remembering that the firm regards θ as exogenous):

$$\begin{aligned} \frac{\delta H}{\delta v} &= 0 \Rightarrow J = c/q + aqv \\ \frac{\delta H}{\delta N} &= rJ - \dot{J} \Rightarrow J(r+s) = y - w + \dot{J} \end{aligned} \tag{8.4}$$

Since the firm in question is a representative firm we can write $vq = \theta qu = up$ (since $p = \theta q(\theta)$). We know that the dynamics of the aggregate unemployment are given by $\dot{u} = sN - pu = (1-u)s - pu$. The equilibrium of the economy is thus defined by the following set of equations:

$$\begin{aligned} \dot{u} &= s(1-u) - pu \\ \dot{J} &= J - (y - w) \\ J &= apu + c/q \end{aligned} \tag{8.5}$$

Steady state

The model is in steady state when $\dot{J} = \dot{u} = 0$. The steady state is thus characterized by the equations (from equation (8.5)):

$$\begin{aligned} u^* &= \frac{s}{s+p} \\ J^* &= \frac{y-w}{r+s} \\ j^* &= apu + \frac{c}{q} \end{aligned} \tag{8.6}$$

We now prove the following lemma:

Lemma 21 *Suppose $y > w$. Then the steady state equilibrium given by (8.6) exists and is unique.*

Proof: Substituting the two first equations into the third gives

$$\frac{y-w}{r+s} = \frac{asp(\theta)}{s+p(\theta)} + c/q(\theta)$$

The right-hand side is strictly increasing in θ, and goes to zero when θ goes to zero and to infinity when θ goes to infinity. Hence the equation has a unique solution. Given θ, the first equation determines θ uniquely.

□

From the second and the third equation in (8.6) we get that

$$y/(r+s) = c/q + apu \tag{8.7}$$

The equation defines $\theta = f(u, y)$, where θ is falling in u. We call this the vacancy supply curve. Hence there is a falling relationship between the labour market tightness and the unemployment rate in steady state, which implies that if the Beveridge curve shifts out, the steady state value of θ falls. This contrasts Pissarides (1990), where the labour market tightness is independent of u.

Productivity shocks

In this section, we want to study the dynamic behaviour of the model after a productivity shock. We start out showing the following lemma:

Lemma 22 *The steady state equilibrium defined by (8.6) is a saddlepoint.*

Proof: First note that the only non-exploding path for J is the one where $\dot{J} = 0$ almost everywhere. Hence we must have that $J = y/(r+s)$. Substituting this into the last equation in (8.5) gives (8.7), and we know from above that θ is a decreasing function of u. Rearranging (8.7) gives

$$apu = y/(r+s) - c/q$$

Thus pu decreases in θ, and we can write $pu = g(u)$, $g'(u) > 0$ (since θ is increasing in u). Inserted into the expression for $\dot{u}$ this gives

$$\dot{u} = \dot{u}(u) = (1-u)s - g'(u)$$

Obviously, the right-hand side is decreasing in u.

□

Note that since J is constant, the relationship between θ and u, $f(u;y)$, also holds outside the steady state.

The effects of an aggregate productivity shock are now easy to analyze. Suppose y shifts up. Then the vacancy supply curve shifts up as well. Hence θ shifts up, and the unemployment rate starts to fall. The labour market tightness θ increases smoothly along the adjustment path towards the new steady state. If the aggregate productivity shock is negative, the process is reversed. Thus the adjustment costs implies that θ undershoots (overshoots) the new equilibrium value after a positive (negative) shock, and the counter-clockwise movements around the Beveridge curve become smoother. Whether or not the vacancy rate overshoots depends on the parameter values.

8.3 Endogenous Wages

In this section we endogenize the wages. First we analyze the bargaining game, then we incorporate the solution in the model from the last section.

8.3.1 Wage determination

We assume that a firm bargains with each of its employees separately, and that the match-surplus is shared according to the Nash sharing rule. From optimal control theory we know that the adjoint variable J expresses the shadow value, or the expected discounted income to the firm, of each occupied job (see Seierstad and Sydsæter (1987)). We assume that the hiring costs are mostly training costs, and are sunk when the bargaining takes place. The firm's disagreement point is therefore equal to the value of a vacancy, which is zero.

We assume that the investments in training only gives the worker firm-specific skills, in another job he must be retrained at the same cost. Let U and E denote the expected discounted incomes (asset values) for an unemployed and an employed (E) worker respectively. Then

$$\begin{aligned} rU - \dot{U} &= b + p(E-U) \\ rE - \dot{E} &= w - s(E-U) \end{aligned} \tag{8.8}$$

where b is the unemployment benefit. Subtracting the first of the equations from the second gives

$$(r+s)(E-U) = w - b + p(E-U) + \dot{E} - \dot{U} \tag{8.9}$$

The Nash sharing rule implies that the worker gets a share β and the firm a share $1-\beta$ of the surplus, where β is a constant (the worker's bargaining power). Hence $(1-\beta)J = \beta(E-U)$, or

$$E - U = \frac{\beta}{1-\beta} J$$

Since continuous renegotiations require that this holds for all t we must also have that

$$\dot{E} - \dot{U} = \frac{\beta}{1-\beta} \dot{J}$$

Inserted into (8.9) this gives

$$\frac{\beta}{1-\beta}[(r+s)J-\dot{J}] = w - b - p\frac{\beta}{1-\beta}J \tag{8.10}$$

Note that J is both on the left-hand and right-hand side of the equation. We know from (8.4) that $(r+s)J = y - w + \dot{J}$, and substitute this in on the left-hand side. We also know that in other firms, $J = c/q + k$. We insert this into the right-hand side, and rearranging to get

$$w = \beta y + (1-\beta)b + \beta\theta c + \beta pk \tag{8.11}$$

Since k represents the marginal hiring cost in other firms, all the variables on the right-hand side of the wage equation are exogenous to the firm. We have thus proved the following lemma:

Lemma 23 *The wages a firm pay is exogenous to the firm in the sense that it is independent of the firm's behaviour.*

8.3.2 The model with endogenous wages

In this chapter, we introduce the wage equation (8.11) into the matching model presented above. As we will see, this does not qualitatively change the results obtained in the previous section. Since the wages are independent of the firms' hiring policy, the solution to the representative firm's maximization problem is still given by (8.4). As in the last section, we know that $k = avq$ for the representative firm. In equilibrium we can therefore write $k = aup$. When we substitute this and the expression for the wage into the equilibrium conditions (8.5), we get

$$\begin{aligned}
\dot{u} &= (1-u)s - pu \\
\dot{J} &= (r+s)J + c\beta\theta + \beta a p^2 u - \beta(y-b) \\
J &= \frac{c}{q(\theta)} + aup
\end{aligned} \tag{8.12}$$

The corresponding steady state equilibrium is given by the equations

$$\begin{aligned} u^* &= \frac{s}{s+p} \\ J^* &= \frac{(1-\beta)(y-b)-\beta\theta c-\beta a p^2 u^*}{r+s} \\ J^* &= \frac{c}{q}+apu \end{aligned} \tag{8.13}$$

where of cause p and q depends on θ^*. Analogous to the existence result in Lemma 21 we now get the following result:

Proposition 27 *If $y > b$, a unique steady state equilibrium exists.*

Proof: From the first equation we find that $u = s/(s+p)$. Substituting this into the second and the third equation yields

$$\frac{(1-\beta)(y-b)}{r+s}-\frac{\beta\theta c}{r+s}-\frac{\beta a s p^2/(s+p)}{r+s} = c/q + asp^2/(s+p) \tag{8.14}$$

The left-hand side is decreasing in θ, is strictly positive for $\theta = 0$ (since $y > b$), and is negative for sufficiently large values of θ. The right-hand side is strictly increasing in θ, and goes to zero when θ goes to zero. Hence the equation has a unique solution for θ, and the result follows

□

How does an increase in y influence the steady state equilibrium? First note that since the left-hand side of (8.14) is increasing in y, and it follows that θ increases with y as well. From the beveridge curve (the first equation in (8.13)) we know that u falls when θ increases. Since $pu = ps/(s+p)$, which increases in θ, the last equation in (8.13) tells us that J increases as well. Since pu (the number of hirings) increases, so do k, and hence we know from (8.11) that wages increases.

To summarize, an increase in y decreases the unemployment rate, and increases the labour market tightness, the wages, and the marginal value of a job.

Productivity shocks

In this section, we want to study the dynamic properties of the model after a productivity shock. First we start out by showing the following proposition:

Proposition 28 *The steady state equilibrium defined by (8.12) is a saddle-point.*

Proof: If we substitute the third equation in (8.12) into the second we get that

$$\begin{aligned} \dot{J} &= (r+s)J - \beta(y-b) + \beta\theta c + \beta p(J - \frac{c}{q}) \\ &= (r+s)J - \beta(y-b) + \beta pJ \end{aligned} \tag{8.15}$$

Linearising this equation and the first and third equation in (8.12) around the steady state values yields

$$\begin{aligned} \dot{J} &= (r+s+\beta p)(J-J^*) + \beta Jp'\,(\theta-\theta^*) \\ \dot{u} &= -(s+p)(u-u^*) - u^*p'(\theta-\theta^*) \\ (J-J^*) &= (c\hat{q} + ap'u^*)(\theta-\theta^*) + ap(u-u^*) \end{aligned}$$

where $\hat{q} = \frac{d\frac{1}{q}}{d\theta} > 0$. Re-arranging the last equation gives

$$(\theta-\theta^*) = \frac{(J-J^*)}{c\hat{q}+ap'u^*} - \frac{ap}{c\hat{q}+ap'u^*}(u-u^*) \tag{8.16}$$

Inserts this into the other two equations:

$$\begin{aligned} \dot{J} &= (r+s+\beta p + \frac{\beta p'}{c\hat{q}+ap'u^*})(J-J^*) - \frac{\beta p' J}{c\hat{q}+u^*p'a}(u-u^*) \\ \dot{u} &= -(s+p-p\underbrace{(\frac{u^*p'a}{c\hat{q}+ap'u^*})}_{<1})(u-u^*) - p'u(J-J^*) \end{aligned} \tag{8.17}$$

Thus the sign pattern is

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix}$$

and the proposition follows.

□

In a phase diagram in the $u - J$ space, the lines representing $\dot{J} = 0$ and $\dot{u} = 0$ are given by

$$J = J^* + \frac{\beta p' J \kappa}{r + s + \beta p + \beta p' \kappa}(u - u^*) \tag{8.18}$$

$$J = J^* - \frac{s + p - pu^* p' a \kappa}{p' u}(u - u^*) \tag{8.19}$$

respectively, where $\kappa = 1/(c\hat{q} + u^* p' a)$. The first equation (given by $\dot{J} = 0$) is increasing, and the second (given by $\dot{u} = 0$) is decreasing in u. It follows that J is increasing above the curve given by (8.18), while u is increasing below the curve given by (8.19).

Suppose now that we are in the steady state equilibrium initially, and that y shifts up. From a phase diagram it is clear that the only convergent path implies that both u and J are decreasing during the adjustment towards the new steady state. Since the new steady state value of J is higher than the old, this means that J immediately jumps upwards. Furthermore, J overshoots its new steady state value.

Also the wages w and the labour market tightness θ jumps up immediately. To see this, note first that θ must jump up since the unemployment rate starts to fall. From (8.11) we know that we can write

$$w = \beta y + (1 - \beta)b + pJ \tag{8.20}$$

Since both p and J jumps, so do w.

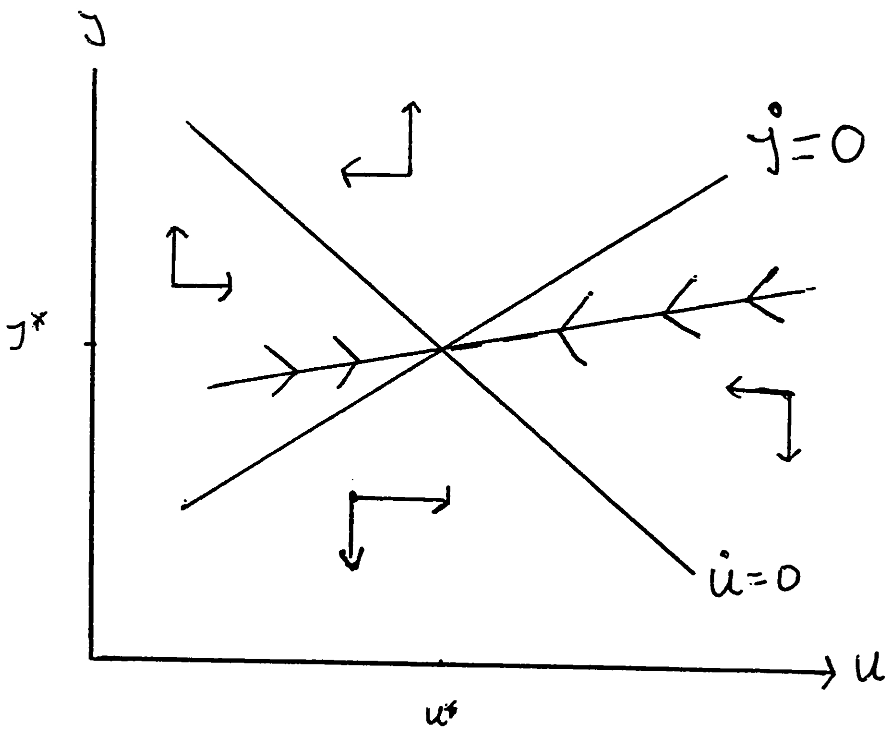

Figure 8.1: The Phase Diagram

Furthermore, the same variables continue to increase on the adjustment path To see this, first note that since J is falling, it follows that w must be increasing[4]. But since u falls it then follows from (8.11) that θ increases (since $k = apu$). Note also that since and since both w and θ first jumps up, and then continue to increase, it follows from the asset value equations that U does the same.

Hence the results from the model with exogenous wages still go through qualitatively. In addition, we obtain a similar result for the wage rate, which first jumps up, and then increases smoothly along the adjustment path to its new steady state value. The intuition for this is straight-forward: Since the marginal hiring costs are increasing in the number of hirings, firms do

[4]Strictly speaking, the argument holds only locally around the new steady state equilibrium, where the signs of the derivatives do not change.

not want to increase their stock of workers too quickly. Hence they do not open as many vacancies initially as in the original model, but instead try to smooth out the adjustment. However, since all firms behave like this, the labour market tightness and thus also the search costs increases along the adjustment path, and the firms face a trade-off between search costs and hiring costs. Furthermore, since the workers' bargaining position are influenced by the conditions in the labour market (the labour market tightness), their wages also increases as the model converges towards its new steady state equilibrium.

8.4 Simulations

In this section, we parameterize the model from section (8.2), to study numerically the effects of introducing hiring costs. We pay special attention to the trajectory of the number of vacancies. For simplicity we use the model with exogenous wages.

The analysis goes like this. First we parameterize the equations in question. Then we calibrate the model, choosing parameters in such a way that we get a suitable equilibrium point as a reference point. Finally we compare the adjustment paths after productivity shocks with varying levels of adjustment costs. We combine the parameters so that the steady state equilibria are (almost) the same in all the simulations.

We assume that the matching function is Cobb-Douglas, and that $x = A(uv)^{1/2}$, $p(\theta) = A(\theta)^{1/2}$ and $q(\theta) = A(\theta)^{-1/2}$. Thus

$$p = \frac{A^2}{q}$$

Without reference to any dataset we choose $s = 0.2$ and $r = 0.05$. Inserted into (8.5) this gives (with $p = A^2/q$ substituted in for q):

$$4y = aup + \frac{p}{A^2}c$$

$$u = \frac{0.2}{0.2+p} \tag{8.21}$$

where y now denotes productivity net of wages. Note that θ only influences the system via p. We therefore solve the system with respect to this variable initially, and then in the end calculate θ.

The parameters are chosen such that the following holds in the initial steady state equilibrium:

1. The unemployment rate is around five percent
2. Labour market tightness θ is 1.
3. The average matching time is a quarter of a period for both workers and firms.

The first two items on the list are satisfied when $A = 4$, which gives $p = 4$ when $\theta = 1$. From (8.21) we then find that $u^* = 4.8$ percent. We normalize c/A^2 to be 1. Furthermore, we define $k = pu^*$. Thus k denotes the ratio of hiring costs to search costs in the initial equilibrium.

In the different scenarios we want to vary k. If $k = 0$ there are no hiring costs, and we are back in the original model. As k increases, so do the relative importance of the hiring costs. To prevent different values of k to yield different initial equilibria, we let $y = 1 + k$. Finally, we let e denote a shift parameter for the productivity, $e = 0$ initially. The system of equations can then be written as

$$\begin{aligned} \dot{u} &= (1-u)s - up \\ 4(1+k+e) &= kup + p \end{aligned} \tag{8.22}$$

The last equation defines v as a function of u. This function, for is plotted below for $e = 0$ and for different values of k: for $k = 1$, the vacancy rate is

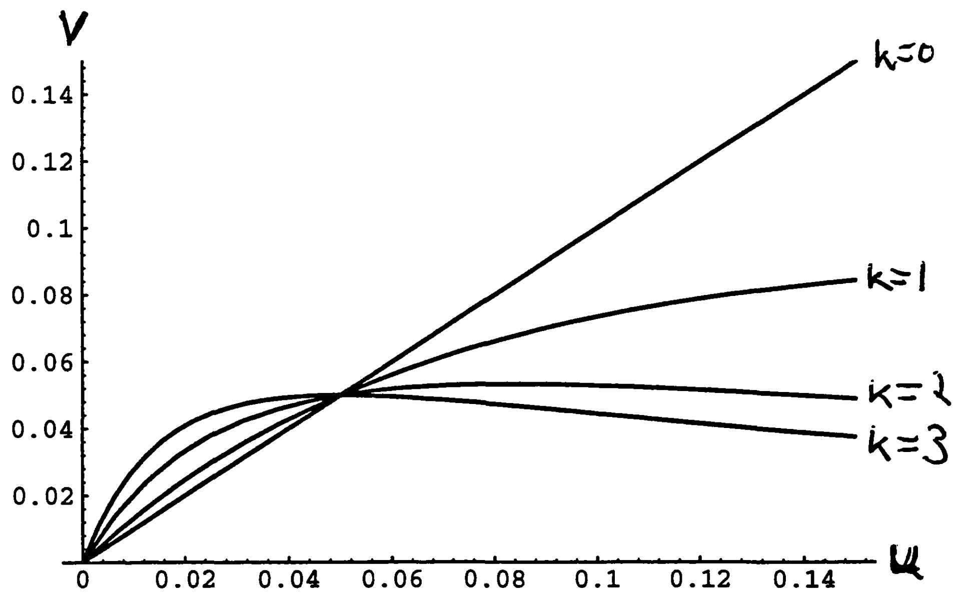

Figure 8.2: The vacancy rate plotted against the unemployment rate.

only slightly increasing in u, around the steady state equilibrium, while it is decreasing in u for $k \geq 2$. Solving for p in the last equation gives $p = \frac{4(1+k+e)}{1+ku}$, which inserted into the first equation yields

$$\dot{u} = (1-u)s + \frac{4u(1+k+e)}{1+ku} \tag{8.23}$$

This is a first order nonlinear difference equation in u, and has a unique solution given the initial value of u. we have solved the equation numerically with *Mathematica* for $e = \pm 0.5$ for various values of k. As the figure above indicates, the vacancy rate undershoots (overshoots) when $k \geq 2$. Already for $k = 1$, the initial jump in *theta* only count for one half of the total change before the new equilibrium is reached, and the speed of the adjustment process is considerably reduced.

8.5 Conclusion

In this paper we have shown that the dynamics of a standard matching model changes considerably when convex adjustment costs are introduced. Both the vacancy rate and wages become less jumpy, and adjust more gradually after a shock. Furthermore, adjustment of the unemployment rate becomes more sluggish. The model can therefore be viewed as a compromise between models with free entry, and models where the number of jobs is exogenous and constant.

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