A Pair of Explicitly Solvable Impulse Control Problems

Fares M M S AlAzemi

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics
London School of Economics
and Political Science

September 2010

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, with the following exceptions.

The contents of Chapter 2 are well-known results due to various authors not including myself.

Abstract

This thesis is concerned with the formulation and the explicit solution of two impulse stochastic control problems that are motivated by applications in the area of sequential investment decisions. Each of the two problems considers a stochastic system whose uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. In the first of the two problems, the control that can be applied to the system takes the form of one-sided impulsive action, and the associated objective is to maximise a performance criterion that rewards high values of the utility derived from the system's controlled state and penalises the expenditure of any control effort. Potential applications of this model arise in the area of real options where one has to balance the sunk costs incurred by investment against their resulting uncertain cashflows. The second model is concerned with the so-called buy-low and sell-high investment strategies. In this context, an investor aims at maximising the expected discounted cash-flow that can be generated by sequentially buying and selling one share of a given asset at fixed transaction costs. Both of the control problems are solved in a closed analytic form and the associated optimal control strategies are completely characterised. The main results are illustrated by means of special cases that arise when the uncontrolled system dynamics are a geometric Brownian motion or a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model.

Acknowledgements

I am indebted to my PhD supervisor Professor Mihail Zervos. His excellent supervision, patienent and friendly support has been invaluable. He has always been available with advice. Most of all, I have had great fun working with him. I must also thank Professor Zervos for introducing me to many of the people whose research is cited in this thesis.

I thank the members of the Mathematics Department for all their support. In particular, Simon Jolly, Jackie Everid, David Scott, Hessah Al-Motairi, Polly Lon, Ovidiu Precup, Ahmad Abu-Khazneh, and Neofytos Rodosthenous. I would also like to thank Kuwait University (KU), for financial support and all my colleagues at the Department of Mathematics at KU for their guidance; in particular Professor Chawla for initiating me into this exciting fields. Special thanks also to Mr.Saif Basheer and Mrs.Faiza Ahmed and all the staff of the Kuwait Cultural Office in London.

My gratitude extends further to thank all my professors and my numerous friends for their endless inspiration and motivation. Special thanks goes to Tariq Salah for his support.

Finally, I thank my family, especially my mother, my brother Naser and my wife for all their love and encouragement. This thesis couldn't have been completed if it wasn't for the sweet company of my two daughters Ghala and Sara. This thesis is dedicated in the memory of my father, Marzouq Mohammed Al-Azami.

Contents

1

Introduction

1.1 Introduction

This thesis is concerned with the formulation and the explicit solution of two impulse stochastic control problems that are motivated by applications in the area of sequential investment decisions. Each of the two problems considers a stochastic system whose uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. In the first of the two problems, the control that can be applied to the system takes the form of one-sided impulsive action, and the associated objective is to maximise a performance criterion that rewards high values of the utility derived from the system's controlled state and penalises the expenditure of any control effort. The second model is concerned with the so-called buy-low and sell-high investment strategies.

In our first problem we consider a stochastic system whose state is modelled by the controlled, one-dimensional, positive Itô diffusion

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x > 0,$$

where W is a standard one-dimensional Brownian motion and the controlled process Z is a càglàd increasing process. The objective of the optimisation problem is to maximise the performance criterion

$$J_x(Z) = E\left[\int_0^\infty e^{-\Lambda_t^{r(X)}} h(X_t)dt - \sum_{t\geq 0} e^{-\Lambda_t^{r(X)}} \left(c_f + \int_0^{\Delta Z_t} k(X_t+s)ds\right) 1_{\{\Delta Z_t > 0\}}\right], \quad (1.1)$$

over all admissible choices of Z, where

$$\Lambda_t^{r(X)} = \int_0^t r(X_v)\,dv,$$

and $c_f > 0$ is a fixed cost. This stochastic impulse control problem arise in various fields. In the context of mathematical finance, economics and operations research, notable contributions include Harrison, Sellke and Tayor [HST83], Mundaca and Øksendal [MO98], Korn [KORN99], Cadenillas [?], Bar-Ilan, Sulem and Zanello [BSZ02], Bar-Ilan, Perry and Stadje [BPS02], Ohnishi and Tsujimura [OT06], Cadenillas, Sarkar and Zapatero [CSZ07], and other references therein. Also, models motivated by the optimal management of renewable resources have been studied by Alvarez [ALV04], and Alvarez and Koskela [AK07]. In view of the wide range of applications, the mathematical theory of stochastic impulse control is well-developed: for instance, see Lepeltier and Marchal [LM84], Perthame [PER84], Djehiche, Hamadène and Hdhiri [DHH10], as well as the books by Bensoussan and Lions [B68], Øksendal and Sulem [OS07], Pham [PHAM07], and several references therein. The stochastic control problem that we solve is motivated by the following application that arises in the context of the so-called goodwill problem. A company considers the timing of launching a new product that they have developed. Prior to launching it in a given market, the company attribute an image to the product based on the market's attitudes to similar products, the new product's quality differences from existing products, and the company's own image in the market. We use the process X to model the evolution in time of the product's image. In this context, the process Z represents the effect of costly interventions, such as advertising, that the company can make to raise the product's image. The company's objective is to maximise their utility from launching the product minus their "dis-utility" associated with the cost of intervention and the cost of waiting. In particular, the company aims at maximising the performance index defined by (1.1) over all intervention strategies Z and launching times τ .

Optimal control problems addressing this type of application have attracted significant interest in the literature for about half a century. Most of the models that have been studied in this area involve deterministic control and can be traced back to Nerlove and Arrow [NA62] (see Buratto and Viscolani [BV02] and the references therein). More realistic models in which the product's image evolves randomly over time have also been proposed and studied (see Feichtinger, Hartl and Sethi [FHS94] for a review and Marinelli [?] for some more recent references). In particular, Marinelli [?] considers extensions of the classical Nerlove and Arrow model, and studies a class of problems that involve linear dynamics of the state process, absolutely continuous control and linear or quadratic payoff functions. Also, Jack, Johnson and Zervos [JJZ08] study the singular stochastic control version of the problem that we solve here.

In the second problem we consider an asset with price process X that is modelled by the one-dimensional Itô diffusion

$$dX_t = b(Xt)dt + \sigma(X_t)\, dW_t, \quad X_0 = x > 0, \tag{1.2}$$

where W is a standard one-dimensional Brownian motion. An investor follows a strategy that consists of sequentially buying and selling one share of the asset. We use a controlled finite variation càglàd process Y that takes values in $\{0, 1\}$ to model the investors position in the market. In particular, $Y_t = 1$ (resp., $Y_t = 0$) represents the state where the investor holds (resp., does not hold) the asset, while, the jumps of Y occur at the sequence of times $(\tau_n, n \geq 1)$ at which the investor buys or sells. Given an initial condition $(Y_0, X_0) = (y, x) \in \{0, 1\}\times]0, \infty[$, the investor objective is to select a strategy that maximises the performance criterion

$$J_{y,x}(Y) = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_x \left[e^{-\Lambda_{\tau_j}} [(X_{\tau_j} - c_s)\mathbf{1}_{\{\Delta Y_{\tau_j} = -1\}} - (X_{\tau_j} + c_b)\mathbf{1}_{\{\Delta Y_{\tau_j} = 1\}}]\mathbf{1}_{\{\tau_j < \infty\}} \right]. \tag{1.3}$$

Where the state-dependent discounting factor Λ is defined by

$$\Lambda_t = \int_0^t r(X_s) ds, \tag{1.4}$$

for some function $r > 0$, and the constants $c_b > 0$ (resp., $c_s > 0$) represents the transaction cost of buying (resp., selling) one share of the asset. Accordingly, we define the problem's value function v by

$$v(y, x) = \sup_{Y \in \mathcal{A}_{y,x}} J_{y,x}(Y), \quad \text{for} \quad y \in \{0, 1\} \quad \text{and} \quad x > 0, \tag{1.5}$$

where $\mathcal{A}_{y,x}$ is the set of admissible investment strategies, which is introduced by Definition 2 in Section 4.2.

In the presence of the general assumptions that we make on b, σ and r, the optimisation problem defined by (4.1)–(4.4) is well-posed, in particular, the limit in (4.2) exists (see Theorem 12, our main result). We solve this problem in a closed analytic form and we characterise fully the optimal strategy. It turns out that, if 0 is a natural boundary point of the diffusion X, e.g., if X is a geometric Brownian motion, then it is optimal to never enter the market, and to sell it as soon as its price exceeds a given level $\alpha > 0$ if the investor holds the asset at time 0 (see Remark 4 and Theorem 12). The situation is different if 0 is an entrance boundary point of the diffusion X. In this case, the strategy of just exiting the market appropriately may still be optimal. However, depending on the problem data, it may be optimal for the investor to sequentially buy as soon as the asset price falls below a given level $\beta > 0$ and then sell the asset as soon as its price rises above another level $\gamma > \beta$.

Despite the fascinating nature of buy-low and sell-high investment strategies, there have been few papers studying models with sequential buying and selling decision strategies. The reason for this observation can be attributed to the fact that, as we have briefly discussed above, the prime example of an asset price process, namely, the geometric Brownian motion, does not allow for optimal buying and selling strategies that have a sequential nature. The results presented in Shiryaev, Xu and Zhou [SXZ08] and Dai, Jin, Zhong and Zhou [DJZZ10] support such a conclusion: assuming that a stock price follows a geometric Brownian motion, it is optimal for an investor to either sell the stock immediately or hold it until their planning horizon. Other related models involving optimal selling decisions with a geometric Brownian motion type of price model have been studied by Zhang [Z01] and Guo and Zhang [GZ05].

In the context of one-dimensional Itô diffusions other than a geometric Brownian motion, sequentially buying and selling investment strategies can indeed be optimal. Such a result has already been established by Zhang and Zhang [ZZ08] and Song, Yin and Zhang [SY09] who model the underlying asset price dynamics by means of a mean-reverting Ornstein-Uhlenbeck process such as the one appearing in Vasicek's interest rate model. Apart from highlighting the significance of the classification of the diffusion's X boundary point 0 in determining the character of the optimal strategy, our results have a substantially more general nature. Furthermore, our results can account for a rather large family of stochastic processes that includes the so-called mean-reverting constant elasticity of variance processes, which have been proposed in the empirical finance literature as better models for a range of asset prices, particularly, in the commodity markets (e.g., see Geman and Shih [GS09] and the references therein).

The problem that we solve has the characteristics of an entry and exit decision problem. Stochastic optimal control problems involving sequential switching decisions have attracted considerable interest in the literature, particularly, in relation to the management of commodity production facilities. Following Brennan and Schwartz [BS85], Dixit and Pindyck [DP94], and Trigeorgis [T96], who were the first to address this type of a decision problem in the economics literature, Brekke and Øksendal [BØ94], Bronstein and Zervos [BZ06], Costeniuc, Schnetzer and Taschini [CST08], Djehiche and Hamadène [DH09], Djehiche, Hamadène and Popier [DHP09/10], Duckworth and Zervos [DZ01], Guo and Pham [GP05], Guo and Tomecek [GT08], Hamadène and Jeanblanc [HJ07], Hamadène and Zhang [HZ10], Lumley and Zervos [LZ01], Ly Vath and Pham [LP07], Pham [P07], Pham, Ly Vath and Zhou [PLZ09], Tang and Yong [TY93], and Zervos [Z03], provide an incomplete, alphabetically ordered, list of authors who have studied a number of related models by means of rigorous mathematics. The contributions of these authors range from explicit solutions to characterisations of the associated value functions in terms of classical as well as viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations, as well as in terms of backward stochastic differential equation characterisations of the optimal strategies. Chapter 5 of Pham [P09] provides a nice introduction in the area.

This thesis is organised as follows. Chapter 2 is concerned with the study of the homogeneous and non-homogeneous ODE that plays a fundamental role in our analysis. In Chapter 3, we describes the problem as follows. In Section 3.2, we formulate the problem that we solve and we list all of the assumptions that we make, which are the same as the ones in Jack, Johnson and Zervos [JZ07]. Section 3.3 is concerned with a review of certain results that we need as well as with a new result of a technical nature. We solve the control problem that we consider in Section 3.4 , and we illustrate our results by means of the two special cases that arise when X is a geometric Brownian motion or a square-root mean-reverting process, such as the one in the Cox-Ingersoll-Ross model, in Section 3.5. Finally, in Chapter 4, we describe the problem as follows. Section 4.2 is concerned with the setting of the problem that we study. We derive the solution to this problem in Section 4.3. In Section 4.4, we consider a couple of examples that illustrate our result.

2

Study of an ordinary differential equation

2.1 Introduction

In this chapter we study an ODE that plays a fundamental role in our analysis in the following chapters.

In Section 2.2 we consider a one-dimensional Itô diffusion that is closely related with the ODE that we use to model the state process driving the economy in the following chapters. Most of the results presented here have been established by Feller [F52] and can be found in various forms in several references that include Breiman [B68], Mandl [Man68], Itô and McKean [IM74], Karlin and Taylor [KT94], Rogers and Williams [RW94] and Borodin and Salminen [BS02]. Our presentation, which is based on modern probabilistic techniques, has largely been inspired by Rogers and Williams [RW94, Sections V.3, V.5, V.7] and includes ramifications not found in the literature.

2.2 The properties of the underlying diffusion

We consider a stochastic system, the state process X of which satisfies the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{2.1}$$

where W is a one-dimensional standard Brownian motion and $b, \sigma :]0, \infty[\to \mathbb{R}$ are given deterministic functions satisfying the conditions (ND)′ and (LI)′ in Karatzas and Shreve [KS91, Section 5.5C], and given in the following assumption.

Assumption 1. The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0, \text{ for all } x \in]0, \infty[,$$
$$\text{for all } x \in]0, \infty[, \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(s)|}{\sigma^2(s)} \, ds < \infty.$$

□

This assumption guarantees the existence of a unique, in the sense of probability law, solution to (2.1) up to an explosion time. In particular, given $c > 0$, the *scale function* p_c and the *speed measure* $m_c(dx)$, given by

$$p_c(x) = \int_c^x \exp\left(-2 \int_c^s \frac{b(u)}{\sigma^2(u)} \, du\right) ds, \quad \text{for } x > 0, \tag{2.2}$$

$$m_c(dx) = \frac{2}{\sigma^2(x) p_c'(x)} \, dx, \tag{2.3}$$

are well-defined. In what follows, we assume that the constant $c > 0$ is fixed.

We also assume that the diffusion X is *non-explosive*. In particular, we impose the following assumption (see Karatzas and Shreve [KS91, Theorem 5.5.29]).

Assumption 2. If we define

$$u_c(x) = \int_c^x \left[p_c(x) - p_c(y)\right] m_c(dy), \tag{2.4}$$

then $\lim_{x \downarrow 0} u_c(x) = \lim_{x \to \infty} u_c(x) = \infty$. □

2.3 The solution to the homogeneous ODE

The objective is to show that the general solution to the ODE

$$\tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x > 0. \tag{2.5}$$

is given by

$$w(x) = A\psi(x) + B\phi(x). \tag{2.6}$$

Here, $A, B \in \mathbb{R}$ are constants and the functions ϕ, ψ are defined by

$$\phi(x) = \begin{cases} 1/\mathbb{E}_c[e^{-\Lambda_{\tau_x}}], & \text{for } x < c, \\ \mathbb{E}_x[e^{-\Lambda_{\tau_c}}], & \text{for } x \geq c, \end{cases} \tag{2.7}$$

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\Lambda_{\tau_c}}], & \text{for } x < c, \\ 1/\mathbb{E}_c[e^{-\Lambda_{\tau_x}}], & \text{for } x \geq c. \end{cases} \tag{2.8}$$

respectively, and

$$\Lambda_t = \int_0^t r(X_s)ds.$$

In these definitions, as well as in what follows, given a weak solution to (2.1) and a point $a \in]0, \infty[$, we denote by τ_a the first hitting time of $\{a\}$, i.e.,

$$\tau_a = \inf\{t \geq 0 \,|\, X_t = a\},$$

with the usual convention that $\inf \emptyset = \infty$.

Since X is continuous, a simple inspection of (2.7) (resp., (2.8)) reveals that ϕ (resp., ψ) is strictly decreasing (resp., increasing). Also, since X is non-explosive, these definitions imply

$$\lim_{x \downarrow 0} \phi(x) = \lim_{x \to \infty} \psi(x) = \infty.$$

We also need the following assumption.

Assumption 3. The function $r :]0, \infty[\to]0, \infty[$ is locally bounded. □

One purpose of the following result is to show that the definitions of ϕ, ψ in (2.7), (2.8), respectively, do not depend, in a non-trivial way, on the choice of $c \in]0, \infty[$.

Lemma 1. *Suppose that Assumptions 1-3 hold. Given any $x, y \in]0, \infty[$ the functions ϕ, ψ defined by (2.7), (2.8), respectively, satisfy*

$$\phi(y) = \phi(x)\mathbb{E}_y[e^{-\Lambda_{\tau_x}}] \quad \textit{and} \quad \psi(x) = \psi(y)\mathbb{E}_x[e^{-\Lambda_{\tau_y}}], \quad \textit{for all } x < y. \tag{2.9}$$

Moreover, the processes $(e^{-\Lambda_t}\phi(X_t),\ t \geq 0)$ and $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ are both local martingales.

Proof. Given any points $a, b, c \in]0, \infty[$ such that $a < b < c$, we calculate

$$\mathbb{E}_a[e^{-\Lambda_{\tau_c}}] = \mathbb{E}_a\left[e^{-\Lambda_{\tau_b}}\, \mathbb{E}_a[e^{-(\Lambda_{\tau_c} - \Lambda_{\tau_b})} | \mathcal{F}_{\tau_b}]\right] \tag{2.10}$$

$$= \mathbb{E}_a[e^{-\Lambda_{\tau_b}}]\mathbb{E}_b[e^{-\Lambda_{\tau_c}}], \tag{2.11}$$

Note that

$$\mathbb{E}_a[e^{-(\Lambda_{\tau_c} - \Lambda_{\tau_b})} | \mathcal{F}_{\tau_b}] = \mathbb{E}_a\left[e^{-\int_{\tau_b}^{\tau_c} r(X_u)du} | \mathcal{F}_{\tau_b}\right] \tag{2.12}$$

$$= \mathbb{E}_a\left[e^{-\int_0^{\tau_c - \tau_b} r(X_{u+\tau_b})du} | \mathcal{F}_{\tau_b}\right] \tag{2.13}$$

$$= \mathbb{E}_a\left[e^{-\int_0^{u} r(X_{u+\tau_b})du} | X_{\tau_b}\right] \tag{2.14}$$

$$= \mathbb{E}_b\left[e^{-\int_0^{u} r(X_u)du}\right] \tag{2.15}$$

where the second equality follows thanks to the strong Markov property of X. In view of this result, given any $x < z < y$, the choice $a = x$, $b = z$ and $c = y$ yields

$$\mathbb{E}_x[e^{-\Lambda_{\tau_y}}] = \mathbb{E}_x[e^{-\Lambda_{\tau_z}}]\mathbb{E}_z[e^{-\Lambda_{\tau_y}}], \tag{2.16}$$

which, combined with the definition of ψ, implies the second identity in (2.9). We can verify the first identity in (2.9) by appealing to similar arguments.

Now, given any initial condition x and any sequence (x_n) such that $x < x_1$ and $\lim_{n \to \infty} x_n = \infty$, we observe that the second identity in (2.9) implies

$$\psi(X_t)1_{\{t \leq \tau_{x_n}\}} = \psi(x_n)\mathbb{E}_{X_t}[e^{-\Lambda_{\tau_{x_n}}}]1_{\{t \leq \tau_{x_n}\}}, \quad \text{for all } t \geq 0.$$

In view of this identity, we appeal to the strong Markov property of X, once again to calculate

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau_{xn}}}\psi(X_{\tau_{xn}})\middle|\mathcal{F}_t\right] = e^{-\Lambda_t}\psi(x_n)\mathbb{E}_x\left[e^{-(\Lambda_{\tau_{xn}}-\Lambda_t)}\middle|\mathcal{F}_t\right]1_{\{t<\tau_{xn}\}} + e^{-\Lambda_{\tau_{xn}}}\psi(x_n)1_{\{\tau_{xn}<t\}} \tag{2.17}$$

$$= e^{-\Lambda_t}\psi(X_t)1_{\{t<\tau_{xn}\}} + e^{-\Lambda_{\tau_{xn}}}\psi(x_n)1_{\{\tau_{xn}<t\}} \tag{2.18}$$

$$= e^{-\Lambda_{(t\wedge\tau_{xn})}}\psi(X_{t\wedge\tau_{xn}}). \tag{2.19}$$

However, this calculation and the tower property of conditional expectation implies that, given any times $s < t$,

$$\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{xn}}}\psi(X_{t\wedge\tau_{xn}})|\mathcal{F}_s\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{t\wedge\tau_{xn}}}\psi(X_{t\wedge\tau_{xn}})\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right] \tag{2.20}$$

$$= e^{-\Lambda_{s\wedge\tau_{xn}}}\psi(X_{s\wedge\tau_{xn}}), \tag{2.21}$$

which proves that $\left(e^{-\Lambda_t}\psi(X_t),\, t \geq 0\right)$ is a local-martingale. Proving that $\left(e^{-\Lambda_t}\phi(X_t),\, t \geq 0\right)$ is a local-martingale follows similar arguments. □

We can now prove the following result.

Theorem 2. *Suppose that Assumptions 1–3 hold. The general solution to the ordinary differential equation (2.5) exists in the* classical sense, *namely there exists a two dimensional space of functions that are C^1 with absolutely continuous first derivatives, and that satisfy (2.5) Lebesgue-a.e.. This solution is given by (2.6), where $A, B \in \mathbb{R}$ are constants and the functions ϕ, ψ are given by (2.7), (2.8), respectively. Moreover, ϕ is strictly decreasing, ψ is strictly increasing, and, if the drift $b \equiv 0$, then both ϕ and ψ are strictly convex.*

Proof. First, we recall that, given $l < x < m$,

$$\mathbb{P}_x(\tau_l < \tau_m) = \frac{p_c(x) - p_c(m)}{p_c(l) - p_c(m)} \tag{2.22}$$

(e.g., see Karatzas and Shreve [KS91, Proposition 5.5.22] or Rogers and Williams [RW94, Definition V.46.10]). Also in view of the second identity in (2.9), we can see that

$$\begin{aligned}\psi(x) &< \psi(m)\mathbb{E}_x\left[1_{\{\tau_m<\tau_l\}}\right] + \psi(m)\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}1_{\{\tau_l<\tau_m\}}\right] \\ &= \psi(m)\mathbb{P}_x(\tau_m < \tau_l) + \psi(m)\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}\middle|\mathcal{F}_{\tau_l}\right]1_{\{\tau_l<\tau_m\}}\right].\end{aligned}$$

To see this, we recall (2.9)

$$\begin{aligned}\psi(x) =&\psi(m)\mathbb{E}_x[e^{-\Lambda_{\tau_m}}], \quad \text{for all } x < m\\ <&\psi(m)\mathbb{E}_x[e^{-\Lambda_{\tau_m}} 1_{\{\tau_l<\tau_m\}}] + \psi(m)\mathbb{E}_x[1_{\{\tau_m<\tau_l\}}]\\ =&\psi(m)\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}|\mathcal{F}_{\tau_l}\right] 1_{\{\tau_l<\tau_m\}}\right] + \psi(m)\mathbb{P}_x(\tau_m < \tau_l)\end{aligned}$$

since

$$0 < e^{-\Lambda_{\tau_m}} < 1 \tag{2.23}$$

Now, since X has the strong Markov property we can see that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_{\tau_m}}|\mathcal{F}_{\tau_l}\right] 1_{\{\tau_l<\tau_m\}} &= e^{-\Lambda_{\tau_l}}\mathbb{E}_x\left[e^{-\Lambda_{(\tau_m-\tau_l)}}|\mathcal{F}_{\tau_l}\right] 1_{\{\tau_l<\tau_m\}}\\ &= e^{-\Lambda_{\tau_l}}\frac{\psi(l)}{\psi(m)}1_{\{\tau_l<\tau_m\}},\end{aligned}$$

with the last equality following thanks to (2.9). Combining these calculations we can see that

$$\psi(x) <\psi(m)\mathbb{P}_x(\tau_m < \tau_l) \;+\; \psi(l)\mathbb{E}_x\left[e^{-\Lambda_{\tau_l}} 1_{\{\tau_l<\tau_m\}}\right] \tag{2.24}$$

$$<\psi(m)\mathbb{P}_x(\tau_m < \tau_l) \;+\; \psi(l)\mathbb{P}_x(\tau_l < \tau_m). \tag{2.25}$$

Now, let us assume that $b \equiv 0$, so that the diffusion X defined by (2.1) is in *natural scale*, in which case $p_c(x) = x - c$. Combining this fact with (2.22), it is straightforward to verify that

$$x = l\mathbb{P}_x(\tau_l < \tau_m) + m\mathbb{P}_x(\tau_m < \tau_l).$$

However, this calculation and (2.24) imply that ψ is strictly convex. In this case, we have also that

$$\mathbb{P}_x(\tau_l < \tau_m) = \frac{x-m}{l-m}. \tag{2.26}$$

Under the assumption that $b \equiv 0$, which implies that ψ is strictly convex, we can use the Itô-Tanaka and the occupation times formula to calculate

$$\begin{aligned}\psi(X_t) - \int_0^t r(X_s)\psi(X_s)\,ds =&\psi(x) + \int_{]0,\infty[} L_t^a\frac{1}{\sigma^2(a)}\left[\tfrac{1}{2}\sigma^2(a)\,\mu''(da) - r(a)\psi(a)\,da\right]\\ &+ \int_0^t \psi'_-(X_s)\sigma(X_s)\,dW_s,\end{aligned}$$

where ψ'_- is the left-hand-side first derivative of ψ, $\mu''(da)$ is the distributional second derivative of ψ, and L^a is the local time process of X at level a. With regard to the integration by parts formula, this implies

$$\begin{aligned} e^{-\Lambda_t}\psi(X_t) =& \psi(x) + \int_0^t e^{-\Lambda_s}\, d\int_{]0,\infty[} L^a_s \frac{1}{\sigma^2(a)} \left[\tfrac{1}{2}\sigma^2(a)\, \mu''(da) - r(a)\psi(a)\, da\right] \\ &+ \int_0^t e^{-\Lambda_s}\psi'_-(X_s)\, \sigma(X_s)\, dW_s. \end{aligned}$$

Since $(e^{-\Lambda_t}\psi(X_t),\ t \geq 0)$ is a local-martingale (see Lemma 1), this identity implies that the finite variation process Q defined by

$$Q_t = \int_0^t e^{-\Lambda_s} d\int_{]0,\infty[} L^a_s \frac{1}{\sigma^2(a)} \left[\tfrac{1}{2}\sigma^2(a)\, \mu''(da) - r(a)\psi(a)\, da\right], \quad \text{for } t \geq 0,$$

is a local martingale. Since finite-variation local martingales are constant, it follows that $Q \equiv 0$, which implies

$$\int_{]0,\infty[} L^a_t\, \nu(da) = 0, \quad \text{for all } t \geq 0, \tag{2.27}$$

where the measure ν is defined by

$$\nu(da) = \tfrac{1}{2}\mu''(da) - \frac{r(a)\psi(a)}{\sigma^2(a)}. \tag{2.28}$$

To proceed further, fix any points $l < a < m$, define

$$\tau_{l,m} = \inf\left\{t \geq 0 \mid X_t \notin\,]l,m[\right\},$$

and let (T_j) be a localising sequence for the local martingale $\int_0^\cdot \operatorname{sgn}(X_s - a) dX_s$. With regard to the definition of local times and Doob's optional sampling theorem, we can see that

$$\begin{aligned} \mathbb{E}_x\Big[\left|X_{\tau_{l,m}\wedge T_j} - a\right|\Big] &= |x-a| + \mathbb{E}_x\left[\int_0^{\tau_{l,m}\wedge T_j} \operatorname{sgn}(X_s - a)\, dX_s\right] + \mathbb{E}_x\left[L^a_{\tau_{l,m}\wedge T_j}\right] \\ &= |x-a| + \mathbb{E}_x\left[L^a_{\tau_{l,m}\wedge T_j}\right]. \end{aligned}$$

However, passing to the limit using the dominated convergence theorem on the left hand side and the monotone convergence theorem on the right hand side, we can see that this identity implies

$$\mathbb{E}_x\left[L^a_{\tau_{l,m}}\right] = \mathbb{E}_x\Big[\left|X_{\tau_{l,m}} - a\right|\Big] - |x-a| \tag{2.29}$$

$$= \frac{(m-a)(x-l)}{m-l} + \frac{(a-l)(m-x)}{m-l} - |x-a|. \tag{2.30}$$

Now to see (2.30),

$$
\begin{aligned}
\mathbb{E}_x\Big[\,|X_{\tau_{l,m}}-a|\,\Big] =&\mathbb{E}_x\Big[\,|X_{\tau_{l,m}}-a|\,1_{\{\tau_l<\tau_m\}}\Big]+\mathbb{E}_x\Big[\,|X_{\tau_{l,m}}-a|\,1_{\{\tau_m<\tau_l\}}\Big]\\
=&\mathbb{E}_x\Big[\,|X_{\tau_l}-a|\,1_{\{\tau_l<\tau_m\}}\Big]+\mathbb{E}_x\Big[\,|X_{\tau_m}-a|\,1_{\{\tau_m<\tau_l\}}\Big]\\
=&\mathbb{E}_x\Big[\,|l-a|\,1_{\{\tau_l<\tau_m\}}\Big]+\mathbb{E}_x\Big[\,|m-a|\,1_{\{\tau_m<\tau_l\}}\Big]\\
=&|l-a|\mathbb{E}_x\Big[1_{\{\tau_l<\tau_m\}}\Big]+|m-a|\mathbb{E}_x\Big[1_{\{\tau_m<\tau_l\}}\Big]\\
=&(a-l)\mathbb{P}_x(\tau_l<\tau_m)+(m-a)\mathbb{P}_x(\tau_m<\tau_l)\\
=&\frac{(a-l)(m-x)}{m-l}+\frac{(m-a)(x-l)}{m-l}.
\end{aligned}
$$

the second equality following thanks to (2.26). Now, (2.27), the fact that $t\mapsto L^a_t$ increases on the set $\{t\geq 0\,|\,X_t=a\}$ and Fubini's theorem, imply

$$
\begin{aligned}
0=\mathbb{E}_x\Big[\int_{]0,\infty[}L^a_{\tau_{l,m}}\,\nu(da)\Big]=&\mathbb{E}_x\Big[\int_{]l,m[}L^a_{\tau_{l,m}}\,\nu(da)\Big]\\
=&\int_{]l,m[}\mathbb{E}_x\big[L^a_{\tau_{l,m}}\big]\,\nu(da).
\end{aligned}
$$

Combining this calculation with (2.30), it is a matter of algebraic calculation to verify that

$$
\int_l^m h(a;l,x,m)\,\nu(da)=0, \tag{2.31}
$$

where $h(\cdot;l,x,m)$ is the *tent-like* function of height $2(x-l)(m-x)/(m-l)$ defined by

$$
h(a;l,x,m)=\begin{cases}2(a-l)(m-x)/(m-l), & \text{for } a\in[l,x],\\ 2(m-a)(x-l)/(m-l), & \text{for } a\in[x,m].\end{cases}
$$

Now, fix any points $x_l<x_m$ in $]0,\infty[$ and let (l_j) and (m_j) be strictly decreasing and strictly increasing, respectively, sequences such that

$$
l_1<\frac{x_l+x_m}{2}<m_1,\quad \lim_{j\to\infty}l_j=x_l\quad\text{and}\quad \lim_{j\to\infty}m_j=x_m.
$$

We can see that

$$
\mathbf{1}_{]x_l,x_m[}(a)=\lim_{j\to\infty}H_j(a),\quad \text{for all } a\in]0,\infty[,
$$

where the increasing sequence of functions (H_j) is defined by

$$\begin{aligned} H_j(a) = &\frac{2}{(x_m - x_l)} \quad h\left(a; x_l, \frac{x_l + x_m}{2}, x_m\right) \\ &+ \frac{1}{2(l_j - x_l)} \quad h\left(a; x_l, l_j, \frac{x_l + x_m}{2}\right) \\ &+ \frac{1}{2(x_m - m_j)} \quad h\left(a; \frac{x_l + x_m}{2}, m_j, x_m\right), \quad \text{for } a \in]0, \infty[\text{ and } j \geq 1. \end{aligned}$$

Using the monotone convergence theorem and (2.31), it follows that

$$\nu(]x_l, x_m[) = \lim_{j \to \infty} \int_{x_l}^{x_m} H_j(a)\, \nu(da) = 0,$$

which proves that the signed measure ν assigns measure 0 to every open subset of $]0, \infty[$. However, this observation and the definition of ν in (2.28) imply that the total variation of ν is zero, and, therefore, $\mu''(da)$ is an absolutely continuous measure. It follows that there exists a function ψ'' such that

$$\mu''(da) = \psi''(a)\, da \quad \text{and} \quad \tfrac{1}{2}\sigma^2(a)\psi''(a) = r(a)\psi(a), \quad \text{Lebesgue-a.e..}$$

However, the second identity here shows that ψ is a classical solution to (2.5).

Now, let us consider the general case where the drift b does not vanish. In this case, we use Itô's formula to verify that, if $\bar{X} = p_c(X)$, then

$$d\bar{X}_t = \bar{\sigma}(\bar{X}_t) dW_t, \quad \bar{X}_0 = p_c(x),$$

where

$$\bar{\sigma}(\bar{x}) = p_c'\left(p_c^{-1}(\bar{x})\right) \sigma\left(p_c^{-1}(\bar{x})\right), \quad \text{for } \bar{x} \in]0, \infty[.$$

Since $\bar{X}$ is a diffusion in natural scale, the associated function $\bar{\psi}$ defined as in (2.8) is a classical solution of

$$\tfrac{1}{2}\sigma^2(a)\bar{\psi}''(a) - r(a)\bar{\psi}(a) = 0. \tag{2.32}$$

Now, recalling that p_c is twice differentiable in the classical sense, we can see that if we define $\tilde{\psi}(x) = \bar{\psi}(p_c(x))$ then

$$\begin{aligned} \tilde{\psi}'(x) = &\bar{\psi}'(p_c(x)) p_c'(x), \\ \tilde{\psi}''(x) = &\bar{\psi}''(p_c(x)) \left[p_c'(x)\right]^2 + \bar{\psi}'(p_c(x)) p_c''(x). \end{aligned}$$

However, combining these calculations with (2.32), we can see that $\tilde{\psi}$ satisfies the ODE (2.5).

To prove that $\tilde{\psi}$, namely the classical solution to (2.5), as constructed above, identifies with ψ defined by (2.8), we apply Itô's formula to $e^{-\Lambda(\tau_y \wedge T)}\tilde{\psi}(X_{\tau_y \wedge T})$, where $T > 0$ is a constant, to show that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau_y \wedge T}}\tilde{\psi}(X_{\tau_y \wedge T})\right] = \tilde{\psi}(x), \qquad \text{for all } x < y.$$

Since $\psi > 0$ is increasing, the monotone and the dominated convergence theorems imply

$$\lim_{T \to \infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_y \wedge T}}\tilde{\psi}(X_{\tau_y \wedge T})\right] = \tilde{\psi}(y)\mathbb{E}_x\left[e^{-\Lambda_{\tau_y}}\right], \qquad \text{for all } x < y.$$

However, these calculations, show that $\tilde{\psi}$ satisfies the first identity in (2.9) and therefore identifies with ψ defined by (2.8). Proving all of the associated claims for ϕ follows similar reasoning. □

Remark 1. Although we have chosen to undertake this analysis for a positive diffusion, similar results can be obtained for a regular Itô diffusion with values in any interval $\mathcal{I}$ where $\mathcal{I} \subseteq \mathbb{R}$.

Using the fact that ϕ and ψ satisfy the ODE (2.5), it is a straightforward exercise to verify that the scale function, p_c, defined by (2.2) satisfies

$$p_c'(x) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{\mathcal{W}(c)}, \quad \text{for all } x > 0, \tag{2.33}$$

where $\mathcal{W}$ is the Wronskian of ϕ and ψ, defined by

$$\mathcal{W}(x) = \phi(x)\psi'(x) - \phi'(x)\psi(x)$$

and $\mathcal{W}(x) > 0$ for all $x > 0$.

2.4 Study of a non-homogeneous ordinary differential equation

We now study the non-homogeneous ODE

$$\tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) = 0, \quad x \in \left]0, \infty\right[. \tag{2.34}$$

We need to impose the following assumptions, which are stronger than Assumption 1 and Assumption 3, respectively.

Assumption 1$'$ *The conditions of Assumption 1 hold true, and*

$$\text{the function } \sigma \text{ is locally bounded.}$$

Assumption 3$'$ *The conditions of Assumption 3 hold true, and there exists a constant r_0 such that*

$$0 < r_0 \le r(x) < \infty, \quad \text{for all } x > 0. \tag{2.35}$$

We can now prove the following propositions.

Proposition 3. *Suppose that Assumption 1$'$, Assumption 2 and Assumption 3$'$ hold. The following statements are equivalent:*

(I) Given any initial condition $x > 0$ and any weak solution $\mathbb{S}_x$ to (2.1),

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} |h(X_t)| \, dt \right] < \infty.$$

(II) There exists an initial condition $y > 0$ and a weak solution $\mathbb{S}_y$ to (2.1) such that

$$\mathbb{E}_y \left[\int_0^\infty e^{-\Lambda_t} |h(X_t)| \, dt \right] < \infty.$$

(III) Given any $x > 0$,

$$\int_0^x |h(s)| \, \psi(s) \, m(ds) < \infty \quad \text{and} \quad \int_x^\infty |h(s)| \, \phi(s) \, m(ds) < \infty.$$

(IV) There exists $y > 0$ such that

$$\int_0^y |h(s)| \, \psi(s) \, m(ds) < \infty \quad \text{and} \quad \int_y^\infty |h(s)| \, \phi(s) \, m(ds) < \infty.$$

If these conditions hold, then the function

$$R_h(x) = \frac{\phi(x)}{\mathcal{W}(c)} \int_0^x h(s)\psi(s) \, m(ds) + \frac{\psi(x)}{\mathcal{W}(c)} \int_x^\infty h(s)\phi(s) \, m(ds), \quad x \in]0, \infty[, \tag{2.36}$$

is well-defined, is twice differentiable in the classical sense and satisfies the ODE (2.34), Lebesgue-a.e.. In addition, R_h admits the expression

$$R_h(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_s} h(X_s) \, ds \right], \quad \text{for all } x > 0. \tag{2.37}$$

Proof. Suppose that (IV) is true and let y be such that

$$C_1 := \int_0^y |h(s)|\psi(s)\, m(ds) < \infty \quad \text{and} \quad C_2 := \int_y^\infty |h(s)|\phi(s)\, m(ds) < \infty.$$

and so

$$\int_0^x |h(s)|\psi(s)\, m(ds), \quad \int_x^y |h(s)|\psi(s)\, m(ds) \leq C_1, \quad \text{for all } x \in]0, y[, \tag{2.38}$$

$$\int_y^x |h(s)|\phi(s)\, m(ds), \quad \int_x^\infty |h(s)|\phi(s)\, m(ds) \leq C_2, \quad \text{for all } x \in]y, \infty[. \tag{2.39}$$

Combining these inequalities with the fact that ψ is increasing and ϕ is decreasing, we can see that

$$\int_x^\infty |h(s)|\phi(s)\, m(ds) = \int_x^y |h(s)|\phi(s)\, m(ds) + \int_y^\infty |h(s)|\phi(s)\, m(ds) \tag{2.40}$$

$$\leq \frac{\phi(x)}{\psi(x)} \int_x^y |h(s)|\psi(s)\, m(ds) + C_2 \tag{2.41}$$

$$\leq \frac{\phi(x)}{\psi(x)} C_1 + C_2 \tag{2.42}$$

$$< \infty \tag{2.43}$$

Now to see (2.43), we use the fact that

$$\phi(s) \leq \phi(x) \quad \text{for all } s \geq x. \tag{2.44}$$

and,

$$\psi(x) \leq \psi(s) \quad \text{for all } s \geq x. \tag{2.45}$$

Therefore,

$$\int_x^\infty |h(s)|\phi(s)\, m(ds) = \int_x^y |h(s)|\phi(s)\, m(ds) + \int_y^\infty |h(s)|\phi(s)\, m(ds) \tag{2.46}$$

$$\leq \int_x^y |h(s)|\phi(x)\, m(ds) + C_2 \tag{2.47}$$

$$\leq \frac{\phi(x)}{\psi(x)} \int_x^y |h(s)|\psi(x)\, m(ds) + C_2 \tag{2.48}$$

$$\leq \frac{\phi(x)}{\psi(x)} \int_x^y |h(s)|\psi(s)\, m(ds) + C_2 \tag{2.49}$$

$$\leq \frac{\phi(x)}{\psi(x)} C_1 + C_2 \tag{2.50}$$

Thanks to (2.44) and (2.45).

Similarly we have

$$\begin{aligned}
\int_0^x |h(s)|\psi(s)\, m(ds) &= \int_0^y |h(s)|\psi(s)\, m(ds) + \int_y^x |h(s)|\psi(s)\, m(ds) && (2.51)\\
&\leq C_1 + \frac{\psi(x)}{\phi(x)} \int_y^x |h(s)|\phi(s)\, m(ds) && (2.52)\\
&\leq C_1 + \frac{\psi(x)}{\phi(x)} C_2 && (2.53)\\
&< \infty. && (2.54)
\end{aligned}$$

However, (2.38)–(2.54) imply that (III) holds. The reverse implication is obvious.

To proceed further, assume that $h = h_{\mathrm{B}}^{+}$ where h_{B}^{+} is a positive and bounded measurable function. In this case, it is straightforward to verify that (I) and (II) are both satisfied. Since ϕ and ψ are continuous functions satisfying $\lim_{x\to\infty}\phi(x), \lim_{x\downarrow 0}\psi(x) < \infty$ and m is a locally finite measure, the function $R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}} :]0,\infty[\to \mathbb{R}_+$ given by (2.36), or, equivalently, by

$$\begin{aligned}
R_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(x) =& \frac{\phi(x)}{\mathcal{W}(c)} \mathbf{1}_{[1/k,k]}(x) \int_{1/k}^x h_{\mathrm{B}}^{+}(s)\psi(s)\, m(ds) && (2.55)\\
&+ \frac{\psi(x)}{\mathcal{W}(c)} \mathbf{1}_{[1/k,k]}(x) \int_x^k h_{\mathrm{B}}^{+}(s)\phi(s)\, m(ds), && (2.56)
\end{aligned}$$

is well-defined and bounded for all $k > 1$. In light of the calculations

$$\begin{aligned}
R'_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(x) =& \frac{\phi'(x)}{\mathcal{W}(c)} \int_0^x h_{\mathrm{B}}^{+}(s)\mathbf{1}_{[1/k,k]}(s)\psi(s)\, m(ds)\\
&+ \frac{\psi'(x)}{\mathcal{W}(c)} \int_x^\infty h_{\mathrm{B}}^{+}(s)\mathbf{1}_{[1/k,k]}(s)\phi(s)\, m(ds),\\
R''_{h_{\mathrm{B}}^{+}\mathbf{1}_{[1/k,k]}}(x) =& \frac{\phi''(x)}{\mathcal{W}(c)} \int_0^x h_{\mathrm{B}}^{+}(s)\mathbf{1}_{[1/k,k]}(s)\psi(s)\, m(ds)\\
&+ \frac{\psi''(x)}{\mathcal{W}(c)} \int_x^\infty h_{\mathrm{B}}^{+}(s)\mathbf{1}_{[1/k,k]}(s)\phi(s)\, m(ds)\\
&- \frac{2h_{\mathrm{B}}^{+}(x)}{\sigma^2(x)} \mathbf{1}_{[1/k,k]}(x)
\end{aligned}$$

we can see that $R_{h_B^+ \mathbf{1}_{[1/k,k]}}$ is twice differentiable in the classical sense, because this is true for the functions ψ and ϕ, and satisfies the ODE

$$\tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h_B^+(x)\mathbf{1}_{[1/k,k]}(x) = 0, \tag{2.57}$$

Lebesgue-a.e. in $]0, \infty[$.

Now, fix any $k > 1$, any initial condition $x > 0$, any weak solution $\mathbb{S}_x$ to (2.1) and define the sequence of $(\mathcal{F}_t)$-stopping times (τ_l) by

$$\tau_l = \inf \{t \geq 0 \mid \ X_t \notin [1/l, l]\},$$

Using integration by parts formula and the fact that $R_{h_B^+ \mathbf{1}_{[1/k,k]}}$ satisfies (2.57), we calculate

$$e^{-\Lambda_{\tau_l \wedge T}} R_{h_B^+ \mathbf{1}_{[1/k,k]}}(X_{\tau_l \wedge T}) + \int_0^{\tau_l \wedge T} e^{-\Lambda_s} h_B^+(X_s)\mathbf{1}_{[1/k,k]}(X_s)\, ds \tag{2.58}$$

$$= R_{h_B^+ \mathbf{1}_{[1/k,k]}}(x) + M_t^{(l)}, \tag{2.59}$$

where $M^{(l)}$ is defined by

$$M_t^{(l)} = \int_0^{\tau_l \wedge t} e^{-\Lambda_s} \sigma(X_s) R'_{h_B^+ \mathbf{1}_{[1/k,k]}}(X_s)\, dW_s.$$

Combining the fact that $R'_{h_B^+ \mathbf{1}_{[1/k,k]}}$ is locally bounded, because $R_{h_B^+ \mathbf{1}_{[1/k,k]}}$ is continuous, with the the fact that σ is locally bounded from Assumption 1′, we can see that the quadratic variation of the local martingale $M_t^{(l)}$ satisfies

$$\begin{aligned}
\mathbb{E}_x\left[\langle M^{(l)}\rangle_\infty\right] &= \int_0^\infty \mathbb{E}_x\left[1_{\{s \leq \tau_l\}}\left(e^{-\Lambda_s}\sigma(X_s)R'_{h_B^+ \mathbf{1}_{[1/k,k]}}(X_s)\right)^2\right] ds \\
&\leq \sup_{y\in[1/l,l]}\left[\sigma(y)R'_{h_B^+ \mathbf{1}_{[1/k,k]}}(y)\right]^2 \int_0^\infty \mathbb{E}_x\left[e^{-2\Lambda_s}\right] ds \\
&\leq \frac{1}{2r_0}\sup_{y\in[1/l,l]}\left[\sigma(y)R'_{h_B^+ \mathbf{1}_{[1/k,k]}}(y)\right]^2 \\
&< \infty,
\end{aligned}$$

the second inequality following as a consequence of (2.35) in Assumption 3′.

To see this, we recall the definition of

$$M_t^{(l)} = \int_0^{\tau_l \wedge t} e^{-\Lambda_s} \sigma(X_s) R'_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}(X_s)\, dW_s.$$
$$= \int_0^t 1_{\{s \le \tau_l\}} \quad e^{-\Lambda_s} \sigma(X_s) R'_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}(X_s)\, dW_s.$$

Also we know that

$$\left[\langle M^{(l)} \rangle_\infty\right] = \int_0^\infty \left[1_{\{s \le \tau_l\}} \left(e^{-\Lambda_s} \sigma(X_s) R'_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}(X_s)\right)^2\right] ds$$

and we use the fact that

$$\begin{aligned} \mathbb{E}_x\left[\int_0^\infty e^{-2\Lambda_t}\, dt\right] &< \mathbb{E}_x\left[\int_0^\infty e^{-2r_0 t}\, dt\right] \\ &= \mathbb{E}_x\left[\frac{1}{2r_0}\right] \\ &= \left[\frac{1}{2r_0}\right] \\ &< \infty. \end{aligned}$$

This proves that $M^{(l)}$ is a martingale bounded in L^2, so, $\mathbb{E}_x\left[M_t^{(l)}\right] = 0$, for all $t \ge 0$. This observation and (2.58) imply

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau_l \wedge T}} R_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}(X_{\tau_l \wedge T}) + \int_0^{\tau_l \wedge T} e^{-\Lambda_s} h_{\mathrm{B}}^{+}(X_s) \mathbf{1}_{[1/k,k]}\, ds\right] = R_{h_{\mathrm{B}}^{+}} \mathbf{1}_{[1/k,k]}(x). \tag{2.60}$$

Since $R_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}$ is bounded, the dominated convergence theorem implies

$$\lim_{T \to \infty} \lim_{l \to \infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_l \wedge T}} R_{h_{\mathrm{B}}^{+} \mathbf{1}_{[1/k,k]}}(X_{\tau_l \wedge T})\right] = 0,$$

while the monotone convergence theorem yields

$$\lim_{T \to \infty} \lim_{l \to \infty} \mathbb{E}_x\left[\int_0^{\tau_l \wedge T} e^{-\Lambda_s} h_{\mathrm{B}}^{+}(X_s) \mathbf{1}_{[1/k,k]}(X_s)\, ds\right] = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s} h_{\mathrm{B}}^{+}(X_s) \mathbf{1}_{[1/k,k]}(X_s)\, ds\right].$$

These limits and (2.60) imply

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\mathbf{1}_{[1/k,k]}(X_s)\,ds\right] = R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}(x).$$

Recalling the definition of $R_{h_{\mathrm{B}}^+\mathbf{1}_{[1/k,k]}}$ as in (2.56), we can pass to the limit $k \to \infty$ in this identity to obtain

$$R_{h_{\mathrm{B}}^+}(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s} h_{\mathrm{B}}^+(X_s)\,ds\right]. \tag{2.61}$$

Note that, since, h_{B}^+ plainly satisfies conditions (I) and (II), this identity also implies that h_{B}^+ satisfies conditions (III) and (IV).

Now assume that $h = h^+$, where h^+ is a positive measurable function. Using (2.61) with $h_{\mathrm{B}}^+ = h^+ \wedge n$, for $n \geq 1$, and applying the monotone convergence theorem, we can see that, given any initial condition $x > 0$ and any weak solution $\mathbb{S}_x$ to (2.1),

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_s} h^+(X_s)\,ds\right] = R_{h^+}(x), \tag{2.62}$$

where both sides may be equal to infinity. However, with reference to the definition of R_{h^+}, this proves that (I) and (III) are equivalent and that (II) and (IV) imply each other. Recalling the equivalence of (III) and (IV) that we proved above, it follows that statements (I)–(IV) are all equivalent. Furthermore, given any h satisfying (I)–(IV), we can immediately see that (2.62) implies (2.37) once we consider the decomposition $h = h^+ - h^-$ of h to its positive and its negative parts h^+ and h^-, respectively. □

The following result is concerned with a number of properties of the function R_h studied in the previous proposition.

Proposition 4. *Suppose that Assumption 1′, Assumption 2 and Assumption 3′ hold.. Let $h :]0,\infty[\to \mathbb{R}$ be a measurable function satisfying Conditions (I)–(IV) in Proposition 3. The function R_h given by (2.36) or (2.37) satisfies*

$$\lim_{x\downarrow 0} \frac{R_h(x)}{\phi(x)} = \lim_{x\to\infty} \frac{R_h(x)}{\psi(x)} = 0, \tag{2.63}$$

$$\inf_{x>0} \frac{h(x)}{r(x)} \leq R_h(x) \leq \sup_{x>0} \frac{h(x)}{r(x)}, \quad \textit{for all } x > 0, \tag{2.64}$$

$$R_h'(x)\phi(x) - R_h(x)\phi'(x) = p_c'(x)\int_x^\infty h(s)\phi(s)\,m(ds), \quad \textit{for all } x > 0, \tag{2.65}$$

$$R_h'(x)\psi(x) - R_h(x)\psi'(x) = -p_c'(x)\int_0^x h(s)\psi(s)\,m(ds), \quad \textit{for all } x > 0, \tag{2.66}$$

if h/r is increasing (resp., decreasing), then R_h is increasing (resp., decreasing). Also,

$$R_r(x) = 1, \quad \text{for all } x > 0. \tag{2.67}$$

Furthermore, given a solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to (2.1) and an $(\mathcal{F}_t)$-stopping time τ,

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} R_h(X_\tau) 1_{\{\tau<\infty\}} \right] = R_h(x) - \mathbb{E}_x \left[\int_0^\tau e^{-\Lambda_t} h(X_t)\, dt \right], \tag{2.68}$$

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} R_h(X_\tau) 1_{\{\tau<\infty\}} \right] = \mathbb{E}_x \left[\int_\tau^\infty e^{-\Lambda_t} h(X_t)\, dt \right], \tag{2.69}$$

while, if (τ_n) is a sequence of stopping times such that $\lim_{n\to\infty} \tau_n = \infty$, $\mathbb{P}_x$-a.s., then

$$\lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_n}} \left| R_h(X_{\tau_n}) \right| 1_{\{\tau_n<\infty\}} \right] = 0. \tag{2.70}$$

Proof. Fix a solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ to (2.1) and let τ be an $(\mathcal{F}_t)$-stopping time. Using the definition of Λ and the strong Markov property of X, we can see that (2.37) implies

$$\begin{aligned} R_h(x) &= \mathbb{E}_x \left[\int_0^\tau e^{-\Lambda_t} h(X_t)\, dt + e^{-\Lambda_\tau} \mathbb{E}_x \left[\int_\tau^\infty e^{-(\Lambda_s - \Lambda_\tau)} h(X_s)\, ds \mid \mathcal{F}_\tau \right] 1_{\{\tau<\infty\}} \right] \\ &= \mathbb{E}_x \left[\int_0^\tau e^{-\Lambda_t} h(X_t)\, dt + e^{-\Lambda_\tau} \mathbb{E}_{X_\tau} \left[\int_0^\infty e^{-\Lambda_s} h(X_s)\, ds \right] 1_{\{\tau<\infty\}} \right] \\ &= \mathbb{E}_x \left[\int_0^\tau e^{-\Lambda_t} h(X_t)\, dt + e^{-\Lambda_\tau} R_h(X_\tau) 1_{\{\tau<\infty\}} \right], \end{aligned}$$

which establishes (2.68). Also, this expression and (2.37) imply immediately (2.69), while (2.70) follows from the observation that $|R_h| \le R_{|h|}$ (note that h satisfies conditions (I)–(IV) of Proposition 3 if and only in $|h|$ does), (2.69) and the dominated convergence theorem.

Now, let $c > 0$ be the point that we used in (2.2) and (2.3) to define the scale function and the speed measure of the diffusion X. Given a solution $\mathbb{S}_c = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_c, W, X)$ to (2.1) and any $x > 0$, we denote by τ_x the first hitting time of $\{x\}$, and we note that $\lim_{x\downarrow 0} \tau_x = \lim_{x\to\infty} \tau_x = \infty$, $\mathbb{P}_c$-a.s., because the diffusion X is non-explosive by Assumption 2. In view of this observation and the fact that $r(x) \ge r_0 > 0$, for all $x > 0$, by Assumption 3′, we have that (2.70) implies

$$\lim_{x\downarrow 0} \mathbb{E}_c \left[e^{-\Lambda_{\tau_x}} \right] R_h(x) = \lim_{x\to\infty} \mathbb{E}_c \left[e^{-\Lambda_{\tau_x}} \right] R_h(x) = 0.$$

However, these limits and the definitions (2.7) and (2.8) of the functions ϕ and ψ imply (2.63).

Also, a simple inspection of the ODE (2.34) reveals that if we set $h = r$ then R_r satisfies (2.67), noting that R_r is well-defined thanks to (2.35) in Assumption 3′. We can verify (2.65) and (2.66) by a straightforward calculation using the definition (2.36) of R_h and (2.33).

To proceed further, let us assume that h/r is increasing, let us fix any $x > 0$, and let us define $C_x = h(x)/r(x)$. In view of (2.68), the monotonicity of h/r, and the definition (2.8) of ψ we calculate

$$\begin{aligned} R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon) &= \mathbb{E}_{x-\varepsilon}\left[\int_0^{\tau_x} e^{-\Lambda_t}\left[h(X_t) - C_x r(X_t)\right] dt\right] + R_{h(\cdot)-C_x r(\cdot)}(x)\mathbb{E}_{x-\varepsilon}\left[e^{-\Lambda_{\tau_x}}\right] \\ &\le R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\psi(x-\varepsilon)}{\psi(x)}, \quad \text{for all } \varepsilon > 0, \end{aligned} \tag{2.71}$$

To see (2.71), we recall (2.68)

$$\begin{aligned} R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon) =& \mathbb{E}_{x-\varepsilon}\left[e^{-\Lambda_{\tau_x}} R_{h(\cdot)-C_x r(\cdot)}(x_\tau) 1_{\{\tau<\infty\}}\right] + \mathbb{E}_{x-\varepsilon}\left[\int_0^{\tau_x} e^{-\Lambda_t}\left[h(X_t) - C_x r(X_t)\right] dt\right] \\ =& R_{h(\cdot)-C_x r(\cdot)}(x)\mathbb{E}_{x-\varepsilon}\left[e^{-\Lambda_{\tau_x}}\right] + \mathbb{E}_{x-\varepsilon}\left[\int_0^{\tau_x} e^{-\Lambda_t} r(X_t)\left[\frac{h(X_t)}{r(X_t)} - \frac{h(x)}{r(x)}\right] dt\right] \\ \le& \ 0, \end{aligned}$$

because $\quad X_t \le x \quad$ for all $t \le \tau_x$.

Therefore,

$$\begin{aligned} R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon) \le& \ R_{h(\cdot)-C_x r(\cdot)}(x)\mathbb{E}_{x-\varepsilon}\left[e^{-\Lambda_{\tau_x}}\right] \\ =& \ R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\psi(x-\varepsilon)}{\psi(x)}. \end{aligned}$$

With the last equality thanks to (2.9)

Therefore, (2.71)shows that

$$\frac{R_{h(\cdot)-C_x r(\cdot)}(x) - R_{h(\cdot)-C_x r(\cdot)}(x-\varepsilon)}{\varepsilon} \ge \frac{R_{h(\cdot)-C_x r(\cdot)}(x)}{\psi(x)} \frac{\psi(x) - \psi(x-\varepsilon)}{\varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Recalling that $R_{h(\cdot)-C_x r(\cdot)}$ is C^1, we can pass to the limit $\varepsilon \downarrow 0$ in this inequality to obtain

$$R'_{h(\cdot)-C_x r(\cdot)}(x) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\psi'(x)}{\psi(x)}. \tag{2.72}$$

Making a calculation similar to the one in (2.71) using the definition (2.7) of ϕ this time, we can see that

$$R_{h(\cdot)-C_x r(\cdot)}(x+\varepsilon) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\phi(x+\varepsilon)}{\phi(x)}, \quad \text{for all } \varepsilon > 0.$$

Rearranging terms and passing to the limit $\varepsilon \downarrow 0$, we can see that this inequality implies

$$R'_{h(\cdot)-C_x r(\cdot)}(x) \geq R_{h(\cdot)-C_x r(\cdot)}(x)\frac{\phi'(x)}{\phi(x)}. \tag{2.73}$$

Recalling that the strictly positive function ψ is strictly increasing and that the strictly positive function ϕ is strictly decreasing, we can see that (2.72) implies $R'_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ if $R_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$, while, (2.73) implies $R'_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ if $R_{h(\cdot)-C_x r(\cdot)}(x) \leq 0$. Now, combining the inequality $R_{h(\cdot)-C_x r(\cdot)}(x) \geq 0$ with the identities

$$\begin{aligned}
R'_{h(\cdot)-C_x r(\cdot)}(x) =& \frac{\phi'(x)}{\mathcal{W}(c)}\int_0^x [h(s) - C_x r(s)]\,\psi(s)\,m(ds) \\
&+ \frac{\psi'(x)}{\mathcal{W}(c)}\int_x^\infty [h(s) - C_x r(s)]\,\phi(s)\,m(ds) \\
=& R'_h(x) - C_x R'_r(x) \\
=& R'_h(x)
\end{aligned}$$

that follow from the definition (2.36) of R_h and (2.67), we can see that $R'_h(x) \geq 0$. However, since the point $x > 0$ has been arbitrary, this analysis establishes the claim that, if h/r is increasing, then R_h is increasing. Proving the claim corresponding to the case when h/r is decreasing follows similar symmetric arguments.

Finally, to show (2.64), let us assume that $\underline{h} := \inf_{x>0} h(x)/r(x) > -\infty$. In this case, we can use (2.67) and the representation (2.37) to calculate

$$\begin{aligned}
R_h(x) - \inf_{x>0}\frac{h(x)}{r(x)} &= R_h(x) - R_{\underline{h}r(\cdot)}(x) \\
&= \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda t}\left[h(X_t) - \inf_{x>0}\frac{h(x)}{r(x)}r(X_t)\right]dt\right] \\
&\geq 0,
\end{aligned}$$

which establishes the lower bound in (2.64). The upper bound in (2.64) can be established in exactly the same way, and the proof is complete. □

The following lemma gives result useful for practical applications.

Lemma 5. *Suppose that Assumption 1′, Assumption 2 and Assumption 3′ hold. Let* h : $]0,\infty[\to\mathbb{R}$ *be a measurable function satisfying Conditions (I)–(IV) in Proposition 3. If, in addition,* h/r *is increasing, then if* 0 *is a natural boundary (not an entrance boundary) then*

$$\lim_{x\downarrow 0} R_h(x) = \lim_{x\downarrow 0} \frac{h(x)}{r(x)}, \tag{2.74}$$

and if ∞ *is a natural boundary then*

$$\lim_{x\to\infty} R_h(x) = \lim_{x\to\infty} \frac{h(x)}{r(x)}. \tag{2.75}$$

Proof We have

$$\begin{aligned} R_h(x) &\le -\left(\sup_{x>0}\frac{h(x)}{r(x)}\right)\mathbb{E}_x\left[\int_0^\infty d\big(e^{-\Lambda_t}\big)\right] && (2.76)\\ &= \sup_{x>0}\frac{h(x)}{r(x)} && (2.77)\end{aligned}$$

and

$$\begin{aligned} R_h(x) &\ge -\left(\inf_{x>0}\frac{h(x)}{r(x)}\right)\mathbb{E}_x\left[\int_0^\infty d\big(e^{-\Lambda_t}\big)\right] && (2.78)\\ &= \inf_{x>0}\frac{h(x)}{r(x)}. && (2.79)\end{aligned}$$

Note that, in the above inequalities we use the fact that

$$\frac{h(X_t)}{r(X_t)} \le \sup_{x>0}\frac{h(x)}{r(x)} \quad \text{and} \quad \frac{h(X_t)}{r(X_t)} \ge \inf_{x>0}\frac{h(x)}{r(x)} \quad \text{for all} \quad t.$$

For $b < x$

$$
\begin{aligned}
R_h(x) =& \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + \mathbb{E}_x\left[\int_{\tau_b}^{\infty} h(X_t)e^{-\Lambda_t}dt\right] \\
=& \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
\overset{(2.79)}{\geq}& \inf_{x>b}\frac{h(x)}{r(x)}\mathbb{E}_x\left[\int_0^{\tau_b} d(e^{-\Lambda_t}))\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
=& \inf_{x>b}\frac{h(x)}{r(x)}\left(1-\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right]\right) + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
\overset{(2.9)}{=}& \inf_{x>b}\frac{h(x)}{r(x)}\left(1-\frac{\phi(x)}{\phi(b)}\right) + R(b)\frac{\phi(x)}{\phi(b)}
\end{aligned}
$$

while for $x < b$

$$
\begin{aligned}
R_h(x) =& \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + \mathbb{E}_x\left[\int_{\tau_b}^{\infty} h(X_t)e^{-\Lambda_t}dt\right] \\
=& \mathbb{E}_x\left[\int_0^{\tau_b} h(X_t)e^{-\Lambda_t}dt\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
\overset{(2.77)}{\leq}& \sup_{x<b}\frac{h(x)}{r(x)}\mathbb{E}_x\left[\int_0^{\tau_b} d(e^{-\Lambda_t}))\right] + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
\leq& \sup_{x<b}\frac{h(x)}{r(x)}\left(1-\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right]\right) + R(b)\mathbb{E}_x\left[e^{-\Lambda_{\tau_b}}\right] \\
\overset{(2.9)}{=}& \sup_{x<b}\frac{h(x)}{r(x)}\left(1-\frac{\psi(x)}{\psi(b)}\right) + R(b)\frac{\psi(x)}{\psi(b)}.
\end{aligned}
$$

If ∞ is a natural boundary (not an entrance boundary) then $\lim_{x\to\infty}\phi(x) = 0$ and we can say

$$
\begin{aligned}
\lim_{x\to\infty} R_h(x) \leq& \limsup_{x\to\infty}\frac{h(x)}{r(x)} \\
\geq& \liminf_{x\to\infty}\frac{h(x)}{r(x)}
\end{aligned}
$$

and we have the result for the limit as x tends to infinity. Similarly if 0 is a natural boundary then $\lim_{x\downarrow 0}\psi(x) = 0$ and we can say

$$
\begin{aligned}
\lim_{x\downarrow 0} R_h(x) \leq& \limsup_{x\downarrow 0}\frac{h(x)}{r(x)} \\
\geq& \liminf_{x\downarrow 0}\frac{h(x)}{r(x)}
\end{aligned}
$$

and we have the result for the limit as x tends to zero. □

Remark 2. In the cases where we do not have a natural boundary point, R_h does not converge to a value determined in a straightforward way by h and r in the limit. To see this, consider the case of the so-called square root mean reverting process, defined by

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t, \quad X_0 = x > 0,$$

where κ, θ and σ are positive constants satisfying $\kappa\theta - \frac{1}{2}\sigma^2 > 0$. Note that this diffusion has an entrance boundary point at zero. It is a standard exercise to calculate that

$$\begin{aligned}
\mathbb{E}_x[X_t] &= \theta + (x-\theta)e^{-\kappa t}, \\
\mathbb{E}_x\left[X_t^2\right] &= \left(\frac{\sigma^2\theta}{2\kappa} + \theta^2\right) + e^{-\kappa t}\left(2\theta(x-\theta) - 2\frac{\sigma^2\theta}{2\kappa} + \frac{\sigma^2 x}{\kappa}\right) \\
&\quad + e^{-2\kappa t}\left(\frac{\sigma^2\theta}{2\kappa} - \frac{\sigma^2 x}{\kappa} + (x-\theta)^2\right).
\end{aligned}$$

Now, let us consider the following three cases for the payoff function, h:

$$h_1(x) = 0, \quad h_2(x) = x \quad \text{and} \quad h_3(x) = x^2.$$

and assume that r is a constant. In all these cases $\lim_{x\downarrow 0} h(x)/r(x) = 0$. However, we can see that

$$\begin{aligned}
\lim_{x\downarrow 0} R_{h_1}(x) &= 0, \\
\lim_{x\downarrow 0} R_{h_2}(x) &= \frac{\theta\kappa}{r(r+\kappa)} > 0, \\
\lim_{x\downarrow 0} R_{h_3}(x) &= \left(\frac{\sigma^2\theta}{2\kappa} + \theta^2\right)\left[\frac{2\kappa^2}{r(r+\kappa)(r+2\kappa)}\right] > 0.
\end{aligned}$$

3

Impulse Control of a General One-dimensional Ito Diffusion: an Explicitly Solvable Problem

3.1 Introduction

We consider a stochastic system whose uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. The control that can be applied to this system takes the form of one-sided impulsive action. The objective of the control problem is to maximise a performance criterion that rewards high values of the utility derived from the systems controlled state and penalises any expenditure of control effort. We derive the solution to this problem in a closed analytic form under rather general assumptions. We illustrate our results by means of the special cases that arise when the uncontrolled system dynamics are a geometric Brownian motion or a mean reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model. A potential application of the model that we study arises in the context of the so-called goodwill problem in which the systems state is used to represent the image that a product has in a market, while control expenditure is associated with raising the products image, e.g., through advertising.

3.2 Problem Formulation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a stochastic system whose uncontrolled dynamics are modelled by the Itô diffusion associated with the stochastic differential equation (SDE)

$$dX_t^0 = b(X_t^0)\,dt + \sigma(X_t^0)\,dW_t, \quad X_0^0 = x > 0, \tag{3.1}$$

and we make the following assumption.

Assumption 4. The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are C^1, σ' is locally Lipschitz, and

$$\sigma^2(x) > 0, \quad \text{for all } x > 0,$$

□

This assumption implies that (3.1) has a unique strong solution. It also implies that the scale function p_{X^0} and the speed measure m_{X^0} given by

$$p_{X^0}(c) = 0, \quad p'_{X^0}(x) = \exp\left(-2\int_c^x \frac{b(s)}{\sigma^2(s)}\,ds\right), \tag{3.2}$$

and

$$m_{X^0}(dx) = \frac{2}{\sigma^2(x)p'_{X^0}(x)}\,dx, \tag{3.3}$$

respectively, for some $c > 0$ fixed, are well-defined. Additionally, we assume that the solution to (3.1) is non-explosive, so that, given any initial condition x, $X_t^0 \in]0, \infty[$, for all $t \geq 0$, with probability 1.

Assumption 5. The Itô diffusion X^0 defined by (3.1) is non-explosive. □

Feller's test for explosions (see Theorem 5.5.29 in Karatzas and Shreve [KS91]) provides a necessary and sufficient condition for this assumption to hold true. Indeed, if we define

$$l_{X^0}(x) = \int_c^x [p_{X^0}(x) - p_{X^0}(s)]\, m_{X^0}(ds),$$

then Assumption 5 is satisfied if and only if $\lim_{x\downarrow 0} l_{X^0}(x) = \lim_{x\to\infty} l_{X^0}(x) = \infty$.

Now, we model the system's controlled dynamics by the SDE

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{3.4}$$

where the controlled process Z is an increasing process. With each admissible intervention strategy Z (see Definition 3.8 below), we associate the performance criterion

$$\begin{aligned}J_x(Z) =& E\left[\int_0^\infty e^{-\Lambda_t^{r(X)}} h(X_t)dt - \sum_{t\geq 0} e^{-\Lambda_t^{r(X)}} \left(\int_0^{\Delta Z_t} k(X_t+s)ds + c_f\right) 1_{\{\Delta Z_t > 0\}}\right] \\ =& E\left[\int_0^\infty e^{-\Lambda_t^{r(X)}} h(X_t)dt - \sum_{t\geq 0} e^{-\Lambda_t^{r(X)}} \left(K(X_t+\Delta Z_t) - K(X_t) + c_f\right) 1_{\{\Delta Z_t > 0\}}\right],\end{aligned} \tag{3.5}$$

where

$$\Lambda_t^{r(X)} = \int_0^t r(X_v)\, dv, \tag{3.6}$$

and

$$K(x) = \int_0^x k(s)\, ds, \tag{3.7}$$

for some functions $h :]0,\infty[\to \mathbb{R}$ and $k, r :]0,\infty[\to \mathbb{R}_+$ and $c_f > 0$ is a fixed cost.

Definition 1. The family $\mathcal{A}$ of all *admissible intervention strategies* is the set of all $(\mathcal{F}_t)$-adapted càglàd processes Z with increasing piece-wise constant sample paths such that $Z_0 = 0$, (3.4) has a unique non-explosive strong solution, and

$$\mathbb{E}\left[\sum_{t\geq 0} e^{-\Lambda_t^{r(X)}} \left(K(X_t+\Delta Z_t) - K(X_t) + c_f\right) 1_{\{\Delta Z_t > 0\}}\right] < \infty, \quad \text{for all } T > 0. \tag{3.8}$$

The objective of our control problem is to maximise J_x over all admissible strategies. Accordingly, we define the problem's value function v by

$$v(x) = \sup_{Z\in\mathcal{A}} J_x(Z), \quad \text{for } x > 0.$$

Remark 3. Every admissible intervention strategy $Z \in A$ can be characterised by the collection

$(T_0, ..., T_n, ...; \Delta Z_0, ..., \Delta Z_n, ...)$, where (T_n) is the sequence of $(\mathcal{F}_t)$-stopping times defined by

$$T_0 = 0 \quad \text{and} \quad T_n = \inf\{t \geq T_{n-1} | Z_{t+} - Z_t > 0\}, \quad \text{for} \quad n \geq 1, \tag{3.9}$$

with usual convention that $\inf \emptyset = \infty$, and

$$\Delta Z_{T_n} = \begin{cases} Z_{T_n+} - Z_{T_n}, & \text{if} \quad T_n < \infty, \\ 0, & \text{if} \quad T_n = \infty, \end{cases} \quad \text{for} \quad n \geq 0. \tag{3.10}$$

□

For our optimisation problem to be well-posed and to admit a solution that conforms with economic intuition, we need to make additional assumptions.

Assumption 6. The discounting factor r is C^1, r' is locally Lipschitz, and there exists a constant $r_0 > 0$ such that $r(x) \geq r_0$, for all $x > 0$. □

Our analysis will also involve the SDE

$$dY_t^0 = \mu(Y_t^0)\, dt + \sigma(Y_t^0)\, dW_t, \quad Y_0^0 = x > 0, \tag{3.11}$$

where

$$\mu(x) = b(x) + \sigma(x)\sigma'(x) - \tfrac{1}{2}\sigma^2(x)\frac{r'(x)}{r(x)}. \tag{3.12}$$

In the presence of Assumptions 4 and 6, this SDE has a unique weak solution. Also, a short calculation shows that the scale function p_{Y^0} of this diffusion satisfies

$$p'_{Y^0}(x) := \exp\left(-\int_c^x \frac{2\mu(s)}{\sigma^2(s)}\, ds\right) = \frac{\sigma^2(c)}{r(c)}\frac{r(x)}{\sigma^2(x)} p'_{X^0}(x), \tag{3.13}$$

where p_{X^0} is the scale function of the diffusion X^0 defined by (3.2). We make the following additional assumption.

Assumption 7. The Itô diffusion Y^0 defined by (3.11) is non-explosive. □

Given a C^2 function w, we denote by $\mathcal{L}_X w$ the function given by

$$\mathcal{L}_X w(x) = \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x). \tag{3.14}$$

Also, we define

$$Q(x) = h(x) + \mathcal{L}_X K(x). \tag{3.15}$$

We can now complete the list of assumptions made in the paper.

Assumption 8. The following conditions hold:
(a) The running payoff function h is C^1 and the function h/r is bounded from below.
(b) The running cost function k is C^2. Also, $k(x) \geq 0$, for all $x > 0$, and the function K defined by (3.7) is real-valued.
(c) The problem data is such that

$$\rho(x) := \frac{r^2(x) - r(x)b'(x) + r'(x)b(x)}{r(x)} \geq r_0, \quad \text{for all } x > 0, \tag{3.16}$$

where the constant r_0 is the same as in Assumption 6, without loss of generality.
(d) There exists a real $x^* \geq 0$ such that

$$\mathcal{D}_r Q(x) := \frac{r(x)Q'(x) - r'(x)Q(x)}{r(x)} \begin{cases} \geq 0, & \text{for } x \leq x^*, \text{ if } x^* > 0, \\ < 0, & \text{for } x > x^*, \end{cases} \tag{3.17}$$

where the function Q is defined by (3.15). Also, if $x^* = 0$, then $\lim_{x \downarrow 0} Q(x)/r(x) < \infty$.
(e) The integrability condition

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \left[|h(X_t^0)| + |\mathcal{L}_X K(X_t^0)|\right] dt\right] < \infty \tag{3.18}$$

is satisfied, and

$$K(x) = -\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \mathcal{L}_X K(X_t^0)\, dt\right], \tag{3.19}$$

for every initial condition $x > 0$.
(f) There exists a constant $x^\dagger > x^*$ such that

$$\frac{\mathcal{D}_r Q(x)}{\rho(x)} \leq -\varepsilon, \quad \text{for all} \quad x > x^\dagger > x^*. \tag{3.20}$$

□

3.3 Preliminary Considerations

In the presence of Assumptions 4, 5 and 6, the general solution to the homogeneous ODE

$$\mathcal{L}_X w(x) \equiv \tfrac{1}{2}\sigma^2(x) w''(x) + b(x) w'(x) - r(x) w(x) = 0, \tag{3.21}$$

is given by

$$w(x) = A\phi(x) + B\psi(x),$$

for some constants $A, B \in \mathbb{R}$, where ϕ and ψ are C^2 functions such that

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x > 0, \tag{3.22}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x > 0, \tag{3.23}$$

and

$$\lim_{x \downarrow 0} \phi(x) = \lim_{x \to \infty} \psi(x) = \infty. \tag{3.24}$$

These functions are unique, modulo multiplicative constants; to simplify the notation we assume, without loss of generality, that

$$\phi(c) = \psi(c) = 1, \tag{3.25}$$

where $c > 0$ is the same constant as the one that we used in the definition (3.2) of the scale function p_{X^0}. Also, these functions satisfy

$$\phi(x)\psi'(x) - \phi'(x)\psi(x) = Cp'_{X^0}(x), \tag{3.26}$$

where

$$C := \left[\psi'(c) - \phi'(c)\right] > 0. \tag{3.27}$$

Furthermore, the fact that ϕ and ψ satisfy the ODE $\mathcal{L}_X w(x) = 0$ can be used to verify that

$$\begin{aligned}\phi''(x)\psi'(x) - \phi'(x)\psi''(x) &= \frac{2r(x)}{\sigma^2(x)}\left[\phi(x)\psi'(x) - \phi'(x)\psi(x)\right] \\ &= \frac{2Cr(x)}{\sigma^2(x)} p'_{X^0}(x). \end{aligned} \tag{3.28}$$

With each pair of points x, $y > 0$, we associate the $(\mathcal{F}_t)$-stopping time $T_{x,y}$ that is the first hitting time of $\{y\}$ by the solution of the SDE (3.1) with initial condition $X_0^0 = x$, which is defined by

$$T_{x,y} = \inf\{t \geq 0 | X_t^0 = y\}. \tag{3.29}$$

The function ϕ and ψ then admit the representations

$$\phi(x) = \phi(y)\mathbb{E}\left[e^{-\Lambda^{r(X^0)}_{T_{x,y}}}\right] \quad \text{for all} \quad y < x, \tag{3.30}$$

and

$$\psi(x) = \psi(y)\mathbb{E}\left[e^{-\Lambda^{r(X^0)}_{T_{x,y}}}\right] \quad \text{for all} \quad x < y. \tag{3.31}$$

Given a Borel-measurable function G satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \left|G(X_t^0)\right| dt\right] < \infty \tag{3.32}$$

we denote by $R_{X^0,G}$ the function defined by

$$R_{X^0,G}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} G(X_t^0)\, dt\right]. \tag{3.33}$$

This function admits the analytic expression

$$R_{X^0,G}(x) = \frac{2}{C}\phi(x)\int_0^x \frac{G(s)\psi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds + \frac{2}{C}\psi(x)\int_x^\infty \frac{G(s)\phi(s)}{\sigma^2(s)p'_{X^0}(s)}\, ds, \tag{3.34}$$

where $C > 0$ is the constant defined by (3.27), and is a special solution of the non-homogeneous ODE

$$\mathcal{L}_X w(x) + G(x) \equiv \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + G(x) = 0, \tag{3.35}$$

Also,

$$R_{X^0,G}(x) = -\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^0)}} \mathcal{L}_X R_{X^0,G}(X_t^0) dt\right] = R_{X^0,-\mathcal{L}_X R_{X^0,G}}(x) \tag{3.36}$$

$$\lim_{x\downarrow 0} \frac{R_{X^0,G}(x)}{\phi(x)} = \lim_{x\to\infty} \frac{R_{X^0,G}(x)}{\psi(x)} = 0, \tag{3.37}$$

and, given any $(\mathcal{F}_t)$-stopping time τ,

$$-\mathbb{E}\left[\int_0^\tau e^{-\Lambda_t^{r(X^0)}} \mathcal{L}_X R_{X^0,G}(X_t^0) dt\right] = \mathbb{E}\left[\int_0^\tau e^{-\Lambda_t^{r(X^0)}} G(X_t^0)\, dt\right] \tag{3.38}$$

$$= R_{X^0,G}(x) - \mathbb{E}\left[e^{-\Lambda_\tau^{r(X^0)}} R_{X^0,G}(X_\tau^0) 1_{\{\tau<\infty\}}\right]. \tag{3.39}$$

For future reference, we also note that these results and the integrability condition (3.18) imply that

$$R_{X^0,h}(x) - K(x) = R_{X^0,Q}(x) \quad \text{for all } x > 0, \tag{3.40}$$

where h is the running payoff function appearing in the performance criterion defined by (3.5), K is given by (3.7) and Q is defined by (3.15).

We also need the following results that have been established in Jack, Johnson and Zervos [JJZ08, Section3]. The functions $\tilde{\phi}$, $\tilde{\psi}$ defined by

$$\tilde{\phi}(x) = \frac{1}{\phi'(c)}\phi'(x) \quad \text{and} \quad \tilde{\psi}(x) = \frac{1}{\psi'(c)}\psi'(x), \tag{3.41}$$

where $c > 0$ is the same constant as in the definition (3.2)of the scale function p_{X^0}, are C^2

$$0 < \tilde{\phi}(x) \quad \text{and} \quad \tilde{\phi}'(x) < 0, \quad \text{for all } x > 0, \tag{3.42}$$

$$0 < \tilde{\psi}(x) \quad \text{and} \quad \tilde{\psi}'(x) > 0, \quad \text{for all } x > 0, \tag{3.43}$$

$$\lim_{x \downarrow 0} \tilde{\phi}(x) = \lim_{x \to \infty} \tilde{\psi}(x) = \infty, \tag{3.44}$$

and

$$\tilde{\phi}(x)\tilde{\psi}'(x) - \tilde{\phi}'(x)\tilde{\psi}(x) = \tilde{C}p'_{Y^0}(x), \tag{3.45}$$

where

$$\tilde{C} = -\frac{2Cr(c)}{\sigma^2(c)\phi'(c)\psi'(c)}. \tag{3.46}$$

and C is given by (3.27). Also, these functions span the solution space of the ODE

$$\mathcal{L}_Y u(x) := \tfrac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) - \rho(x)u(x) = 0, \tag{3.47}$$

where μ and ρ are given by (3.12) and (3.16). In the presence of Assumption 8.(d), the function $R_{Y^0,\mathcal{D}_rQ}$ given by

$$R_{Y^0,\mathcal{D}_rQ}(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{\rho(Y^0)}} \mathcal{D}_rQ(Y_t^0)\, dt\right]$$

where ρ and $\mathcal{D}_rQ$ are defined by (3.16) and (3.17), is well-defined and real-valued. Furthermore,

$$R'_{X^0,Q}(x) \quad = R_{Y^0,\mathcal{D}_rQ}(x) \tag{3.48}$$

$$R_{Y^0,\mathcal{D}_rQ}(x) = \frac{1}{C}[-\phi'](x)\int_0^x \frac{(\mathcal{D}_rQ)(s)\psi'(s)}{r(s)p'_{X^0}(s)}\,ds + \frac{1}{C}\psi'(x)\int_x^\infty \frac{(\mathcal{D}_rQ)(s)[-\phi'](s)}{r(s)p'_{X^0}(s)}ds, \quad (3.49)$$

and

$$\lim_{x\downarrow 0}\frac{R_{Y^0,\mathcal{D}_rQ}(x)}{\phi'(x)} = \lim_{x\to\infty}\frac{R_{Y^0,\mathcal{D}_rQ}(x)}{\psi'(x)} = 0. \quad (3.50)$$

In the presence of assumptions that we have made, Lemma 6 in Jack, Johnson and Zervos [JJZ08] proves that there exists a unique point $a \geq 0$ such that

$$g(x) := \int_x^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{r(s)p'_{X^0}(s)}\,ds \begin{cases} < 0, & \text{if } a > 0, \text{ and } \ x < a, \\ > 0, & \text{if } x > a. \end{cases} \quad (3.51)$$

Furthermore, $a > 0$ if and only if

$$x^* > 0 \quad \text{and} \quad \lim_{x\downarrow 0} g(x) < 0, \quad (3.52)$$

where $x^* \geq 0$ is as in Assumption 8.(d).

We can now establish the following technical result on which part of our analysis in the next section will be based.

Lemma 6. *In the presence of Assumptions 4–8, the function $R_{X^0,Q}$ given by (3.40) satisfies*

$$\lim_{x\to\infty}\frac{R'_{X^0,Q}(x)}{\phi'(x)} = \infty \quad (3.53)$$

and

$$\frac{d}{dx}\frac{R'_{X^0,Q}(x)}{\phi'(x)} = \frac{2r(x)p'_{X^0}(x)}{\sigma^2(x)[\phi'(x)]^2}g(x)\begin{cases} < 0, & \text{if} \quad a > 0 \quad \text{and} \quad x < a, \\ > 0, & \text{if} \quad x > a, \end{cases} \quad (3.54)$$

where $a \geq 0$ is the unique point appearing in (3.51).

Proof. Using (3.48)–(3.49), we can calculate

$$R''_{X^0,Q}(x) = -\frac{1}{C}\phi''(x)\int_0^x \frac{(\mathcal{D}_rQ)(s)\psi'(s)}{r(s)p'_{X^0}(s)}\,ds - \frac{1}{C}\psi''(x)\int_x^\infty \frac{(\mathcal{D}_rQ)(s)\phi'(s)}{r(s)p'_{X^0}(s)}\,ds.$$

In view of this identity and (3.48)–(3.49), we can see that

$$\begin{aligned}\frac{d}{dx}\frac{R'_{X^0,Q}(x)}{\phi'(x)} &= \frac{\phi''(x)\psi'(x) - \phi'(x)\psi''(x)}{C[\phi'(x)]^2}g(x), \\ &\overset{(3.28)}{=} \frac{2r(x)p'_{X^0}(x)}{\sigma^2(x)[\phi'(x)]^2}g(x). \end{aligned} \quad (3.55)$$

Combining this calculation with (3.51), we obtain (3.54).

To prove (3.53), we first note that (3.42)–(3.43) and (3.45) imply that

$$0 < -\frac{\tilde{\phi}'(x)\tilde{\psi}'(x)}{\tilde{C}p'_{Y^0}(x)} < 1.$$

Combining these inequalities with (3.13) and the definition of $\tilde{\phi}$ in (3.41) we can see that

$$\lim_{x\to\infty} \frac{\sigma^2(x)\phi''(x)}{r(x)p'_{X^0}(x)} = \frac{\sigma^2(c)\phi'(c)}{r(c)} \lim_{x\to\infty} \frac{\tilde{\phi}'(x)}{p'_{Y^0}(x)} = 0 \tag{3.56}$$

Because $\lim_{x\to\infty} \tilde{\psi}(x) = \infty$ (see(3.44). Also, the identity

$$\frac{d}{dx}\left(\frac{1}{p'_{Y^0}(x)}\right) = \frac{2\mu(x)}{\sigma^2(x)p'_{Y^0}(x)}$$

which follows from the definition (3.13) of the scale function p_{Y^0}, and the fact that $\tilde{\phi}$ satisfies the ODE $\mathcal{L}_Y u(x) = 0$ imply that

$$\begin{aligned}\frac{d}{dx}\left(\frac{\tilde{\phi}'(x)}{p'_{Y^0}(x)}\right) &= \frac{2}{\sigma^2(x)p'_{Y^0}(x)}\left[\tfrac{1}{2}\sigma^2(x)\tilde{\phi}''(x) + \mu(x)\tilde{\phi}'(x)\right] \\ &= \frac{2\rho(x)\tilde{\phi}(x)}{\sigma^2(x)p'_{Y^0}(x)}.\end{aligned} \tag{3.57}$$

This calculation, (3.13) and the definition of $\tilde{\phi}$ in (3.41) imply that

$$\frac{d}{dx}\left(\frac{\sigma^2(x)\phi''(x)}{r(x)p'_{X^0}(x)}\right) = \frac{2\rho(x)\phi'(x)}{r(x)p'_{X^0}(x)}. \tag{3.58}$$

Combining this calculation with (3.56), we can see that

$$\int_x^\infty \frac{\rho(s)\phi'(s)}{r(s)p'_{X^0}(s)}ds = -\frac{\sigma^2(x)\phi''(x)}{2r(x)p'_{X^0}(x)}. \tag{3.59}$$

In view of (3.20) in Assumption (8).(f) and the fact that $\phi' < 0$, we can see that the function g defined by (3.51) satisfies

$$g(x) \geq -\varepsilon \int_x^\infty \frac{\rho(s)\phi'(s)}{r(s)p'_{X^0}(s)}ds \overset{(3.59)}{=} \varepsilon\frac{\sigma^2(x)\phi''(x)}{2r(x)p'_{X^0}(x)} \quad \text{for all } x > x^\dagger. \tag{3.60}$$

Combining this calculation with (3.53), we can see that, given any $x > x^\dagger$,

$$\begin{aligned}\frac{R'_{X^0,Q}(x)}{\phi'(x)} &= \frac{R'_{X^0,Q}(x^\dagger)}{\phi'(x^\dagger)} + \int_{x^\dagger}^x \frac{2r(s)p'_{X^0}(s)}{\sigma^2(s)[\phi'(s)]^2}g(s)ds \\ &\geq \frac{R'_{X^0,Q}(x^\dagger)}{\phi'(x^\dagger)} + \frac{\epsilon}{\phi'(x^\dagger)} - \frac{\epsilon}{\phi'(x)},\end{aligned}$$

which implies that (3.53) because $\lim_{x\to\infty}\phi'(x) = 0$ and $\phi'(x) < 0$, for all $x > 0$. □

3.4 The solution to the control problem

In view of standard theory of stochastic control (e.g., see Chapter VIII in Fleming and Soner [FS93]), we expect that the value function v identifies with an appropriate solution w to the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\left\{\mathcal{L}_X w(x) + h(x), \quad \sup_{z>0}\left[w(x+z) - K(x+z)\right] - \left[w(x) - K(x)\right] - c_f\right\} = 0, \tag{3.61}$$

where $\mathcal{L}_X$ is the operator defined by (3.14).

In order to get some intuition around equation (3.61) consider the following argument. At time zero the management can choose between two possible alternatives:

OPTION 1: intervene immediately with an impulse of size z and then continue optimally. Since this is not necessarily the optimal strategy the following inequality should hold

$$w(x) \geq \sup_{z>0}\left[w(x+z) - K(x+z)\right] - K(x) - c_f. \tag{3.62}$$

OPTION 2: do nothing for an interval of time $\Delta t > 0$ and then continue optimally. According to Bellman's principle of optimality, we have

$$w(x) \geq \mathbb{E}\left[\int_0^{\Delta t} e^{-\Lambda_t} h(X_t)dt + e^{-\Lambda_{\Delta t}} w(X_{\Delta t})\right]. \tag{3.63}$$

Applying the Itô's formula for semi-martingales to the term $e^{-\Lambda_{\Delta t}} w(X_{\Delta t})$ dividing (3.63) by Δt and finally letting $\Delta t \to 0$, we obtain

$$\mathcal{L}_X w(x) + h(x) \leq 0. \tag{3.64}$$

Since the alternatives described are the only two available, at least one of (3.62) or (3.64) should hold with equality, which is equivalent to state that $w(x)$ has to satisfy equation (3.61).

We conjecture that the optimal strategy of our problem can take one of two qualitatively different forms, depending on the problem data. The first of these arises when it is optimal to never exert any control effort. In this case, we expect that the value function v identify with

$$J_x(0) = \mathbb{E}\left[\int_0^{\infty} e^{-\Lambda_t^{r(X^0)}} h(X_t^0)\, dt\right],$$

and the function $R_{X^0,h}$ defined by (3.33) in with $G = h$ should satisfy (3.61). The following result is concerned with a necessary and sufficient condition that determines the optimality of this case.

Lemma 7. *Suppose that Assumptions 4–8 hold, and let $a \geq 0$ be the point associated with (3.51) in the previous section. Then there exists a unique point $\hat{x} \geq 0$ such that*

$$\hat{x} = 0, \quad if \quad a = 0, \quad \hat{x} > 0, \quad if \quad a > 0, \tag{3.65}$$

and

$$R'_{X^0,Q}(x) \begin{cases} > 0, & if \quad \hat{x} > 0 \quad and \quad x < \hat{x}, \\ = 0, & if \quad \hat{x} > 0 \quad and \quad x = \hat{x}, \\ < 0, & if \quad x > \hat{x}. \end{cases} \tag{3.66}$$

Furthermore, the function $R_{X^0,h}$ satisfies the HJB equation (3.61) if and only if

$$\hat{x} = 0 \quad or \quad \hat{x} > 0 \quad and \quad R_{X^0,Q}(\hat{x}) - \lim_{x \downarrow 0} R_{X^0,Q}(x) - c_f \leq 0. \tag{3.67}$$

The second and more interesting possibility is to intervene. In this case, we postulate that the optimal strategy is characterised by two strictly positive points $\beta < \gamma$, and can be described as follows. If the initial condition x of the state process is less than or equal to β, then control effort should be exerted to cause a jump to γ. As long as the state process takes values in $]\beta, \gamma[$, the system's controller should take no action, i.e., wait. Each time the state process hits the boundry point β, the system's state should be immediately repositioned at γ (see Figure 1 below). In view of Remark 1, this strategy can be described by the collection $(T_0^*, ..., T_n^*, ...; \Delta Z_0^*, ..., \Delta Z_n^*, ...)$ defined by

$$T_0^* = 0, \quad \Delta Z_0^* = \begin{cases} \gamma - x, & \text{if} \quad x \leq \beta, \\ 0, & \text{if} \quad x > \beta, \end{cases} \tag{3.68}$$

$$T_n^* = \inf\{t \geq T_{n-1}^* | X_t^* \leq \beta\} \quad \text{and} \quad \Delta Z_{T_n}^* = \gamma - \beta, \quad \text{for} \quad n \geq 1. \tag{3.69}$$

We will prove that this strategy is indeed optimal by showing that its payoff, as a function of the initial condition x, identifies with an appropriate solution of the HJB equation (3.61). In light of the standard heuristic arguments that explain the structure of (3.61), this solution should satisfy

$$\mathcal{L}_X w(x) + h(x) = 0, \quad \text{for} \quad x > \beta,$$

and

$$w(x) = w(\gamma) - [K(\gamma) - K(x)] - c_f, \quad \text{for} \quad x \leq \beta.$$

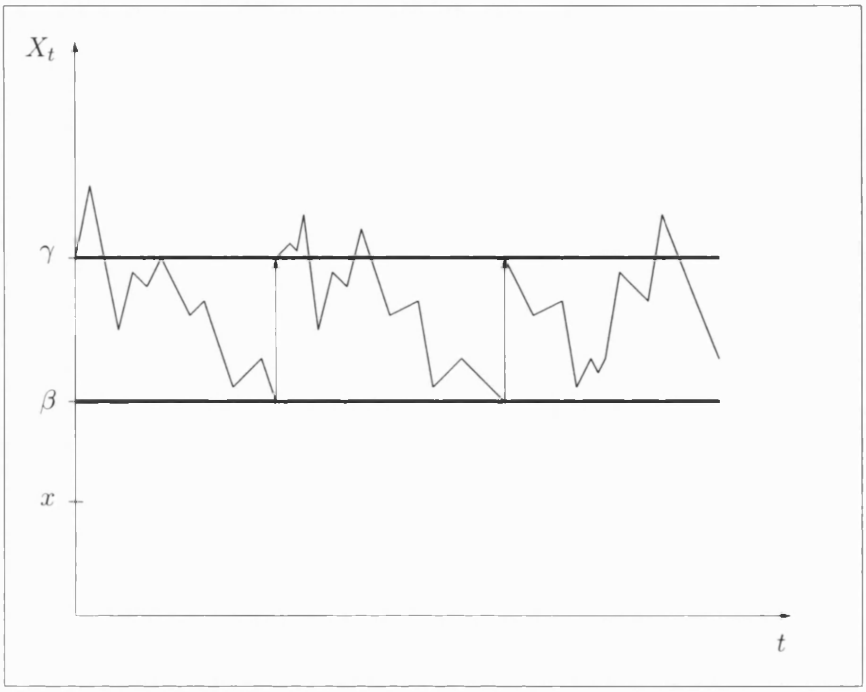

Figure 1.

Recalling the discussion at the beginning of Section 3, w admits the expression

$$w(x) = A\phi(x) + B\psi(x) + R_{X^0,h}(x), \quad \text{for} \quad x > \beta,$$

for some constants $A, B \in \mathbb{R}$, where the functions ϕ and ψ have been introduced at the beginning of Section 3 and the function $R_{X^0,h}$ is defined by (3.33)–(3.34) for $G = h$. It turns out that the arguments that we use to establish Theorem 9 below, which is our main result, remain valid only for the choice $B = 0$. For this reason, we look for a solution to the HJB equation (3.61) of the form

$$\begin{aligned} w(x) &= \begin{cases} A\phi(x) + R_{X^0,h}(x), & \text{for } x > \beta, \\ w(\gamma) - [K(\gamma) - K(x)] - c_f, & \text{for } x \le \beta. \end{cases} \\ &= \begin{cases} A\phi(x) + R_{X^0,h}(x), & \text{for } x > \beta, \\ K(x) + A\phi(\gamma) + R_{X^0,h}(\gamma) - K(\gamma) - c_f, & \text{for } x \le \beta. \end{cases} \end{aligned} \tag{3.70}$$

To specify the parameter A , β and γ , we appeal to the so-called "smooth pasting" condition of stochastic control that requires that the value function should be C^1, in particular, at the point β. This requirement gives rise to the system of equations

$$A\phi(\beta) + R_{X^0,h}(\beta) = K(\beta) + A\phi(\gamma) + R_{X^0,h}(\gamma) - K(\gamma) - c_f, \tag{3.71}$$

$$A\phi'(\beta) + R'_{X^0,h}(\beta) = K'(\beta). \tag{3.72}$$

Also, combining the identity

$$w(\gamma) - K(\gamma) - [w(\beta) - K\beta] - c_f = 0,$$

which follows from the definition (3.70), with the inequality

$$w(\beta + z) - K(\beta + z) - [w(\beta) - K(\beta)] - c_f \leq 0, \quad \text{for all } z > 0,$$

that is associated with (3.61), we can see that the function

$$z \mapsto w(\beta + z) - K(\beta + z) - w(\beta) + K(\beta) - c_f$$

has a local maximum at $z = \gamma - \beta$. Combining this observation with the structure of the function w given by (3.70), we can see that

$$w'(\gamma) - K'(\gamma) \equiv A\phi'(\gamma) + R'_{X^0,h}(x) - K'(\gamma) = 0 \tag{3.73}$$

should hold true.

Recalling (3.40) from the previous section, we can see that (3.72) and (3.73) are equivalent to

$$A = -\frac{R'_{X^0,Q}(\beta)}{\phi'(\beta)} \quad \text{and} \quad A = -\frac{R'_{X^0,Q}(\gamma)}{\phi'(\gamma)}. \tag{3.74}$$

Therefore, the free-boundary point $\beta < \gamma$ should satisfy the equation

$$\eta(\beta, \gamma) := -\frac{R'_{X^0,Q}(\beta)}{\phi'(\beta)} + \frac{R'_{X^0,Q}(\gamma)}{\phi'(\gamma)} = 0. \tag{3.75}$$

Furthermore, (3.40), (3.71) and (3.74) imply that the points $\beta < \gamma$ should also satisfy the equation

$$\begin{aligned}
\theta(\beta, \gamma) :&= -\left[R_{X^0,Q}(\gamma) - \frac{\phi(\gamma)}{\phi'(\gamma)} R'_{X^0,Q}(\gamma)\right] + \left[R_{X^0,Q}(\beta) - \frac{\phi(\beta)}{\phi'(\beta)} R'_{X^0,Q}(\beta)\right] + c_f \\
&= \int_\beta^\gamma \phi(s) \left(\frac{R'_{X^0,Q}}{\phi'}\right)'(s) ds + c_f \\
&= 0,
\end{aligned} \tag{3.76}$$

where the second identity follows from the calculation

$$\frac{d}{dx}[R_{X^0,Q}(x) - \frac{\phi(x)}{\phi'(x)}R'_{X^0,Q}(x)] = -\phi(x)\frac{d}{dx}\left(\frac{R'_{X^0,Q}(x)}{\phi'(x)}\right), \tag{3.77}$$

while, (3.74) and (3.76) imply that A admits the expression

$$A = \frac{R_{X^0,Q}(\gamma) - R_{X^0,Q}(\beta) - c_f}{\phi(\beta) - \phi(\gamma)}. \tag{3.78}$$

The following result, which we prove in the Appendix, is concerned with the considerations above regarding the solvability of the HJB equation (3.61).

Lemma 8. *Suppose that Assumptions 4–8 are satisfied. The system of equations (3.75)–(3.76) has a unique solution $0 < \beta < \gamma$ if and only if $\hat{x} > 0$ and (3.67) fails, i.e., if and only if*

$$\hat{x} > 0 \quad \text{and} \quad R_{X^0,Q}(\hat{x}) - \lim_{x\downarrow 0} R_{X^0,Q}(x) - c_f > 0, \tag{3.79}$$

where $\hat{x} \in [0,\infty[$ is as in Lemma 7. If (3.79) hold true, then the function w defined by (3.70) with A given by (3.74) or (3.78) is a C^1 with absolutely continuous first derivative bounded from below solution of the HJB equation (3.61).

We can now prove the main result of the paper.

Theorem 9. *Suppose that Assumptions 4–8 hold. The following two cases hold true:*

(I) If (3.67) is true, then the value function v identifes with the function $R_{X^0,Q}$, and the optimal intervention strategy is $Z^0 = 0$, namely, it is optimal to never exert any control effort.

(II) If (3.79) is true, namely, if (3.67) is false, then the value function v identifies with the function w defined by (3.70) with A given by (3.74) or (3.78), and the optimal intervention strategy is described by (3.68)–(3.69).

Proof. We fix any initial condition $x > 0$ and any admissible intervention strategy $Z \in \mathcal{A}$. Also, depending on whether (3.67) is true or false, we let w be the solution of the HJB equation (3.61) that is as in Lemma 7 or Lemma 8, respectively. Using Itô's formula and the fact that $X_{t+} = X_t + \Delta Z_t$, we calculate

$$\begin{aligned} e^{-\Lambda_T^{r(X)}} w(X_{T+}) &= w(x) + \int_0^T e^{-\Lambda_t^{r(X)}} \mathcal{L}_X w(X_t)\, dt + M_T \\ &\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} [w(X_t + \Delta Z_t) - w(X_t)] \end{aligned}$$

where the operator $\mathcal{L}_X$ is defined by (3.14), and

$$M_T = \int_0^T e^{-\Lambda_t^{r(X)}} \sigma(X_t) w'(X_t)\, dW_t.$$

In view of this calculation, and the fact that w satisfies the HJB equation (3.61), we obtain

$$\begin{aligned}
&\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left[K(X_t + \Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t > 0\}}, \\
&\quad = w(x) - e^{-\Lambda_T^{r(X)}} w(X_{T+}) + \int_0^T e^{-\Lambda_t^{r(X)}} \left[\mathcal{L}_X w(X_t) + h(X_t)\right] dt + M_T \\
&\quad\quad + \sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left(w(X_t + \Delta Z_t) - K(X_t + \Delta Z_t) - [w(X_t) - K(X_t)] - c_f\right) 1_{\{\Delta Z_t > 0\}} \\
&\quad \le w(x) - e^{-\Lambda_T^{r(X)}} w(X_{T+}) + M_T. \qquad (3.80)
\end{aligned}$$

Given a localising sequence (τ_n) for the stochastic integral M, we can take expectations in (3.80), to obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_n \wedge T} e^{-\Lambda_t^{r(X)}} h(X_t)\, dt - \sum_{t\in[0,\tau_n\wedge T]} e^{-\Lambda_t^{r(X)}} \left[K(X_t + \Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t > 0\}}\right] \\
&\qquad \le w(x) + \mathbb{E}\left[e^{-\Lambda_{\tau_n \wedge T}^{r(X)}} w^-\left(X_{(\tau_n \wedge T+)}\right)\right], \qquad (3.81)
\end{aligned}$$

where $w^- = -(w \wedge 0)$. The assumption that h/r is bounded from below, the fact that the process Θ defined by $\Theta_t = -\exp\left(-\Lambda_t^{r(X)}\right)$, for $t \ge 0$, is increasing, and Fatou's lemma imply that

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{-\Lambda_t^{r(X)}} h(X_t)\, dt\right] &= \mathbb{E}\left[\int_0^T \frac{h(X_t)}{r(X_t)}\, d\Theta_t\right] \\
&\le \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \mathbf{1}_{\{t \le \tau_n\}} \frac{h(X_t)}{r(X_t)}\, d\Theta_t\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^{\tau_n \wedge T} e^{-\Lambda_t^{r(X)}} h(X_t)\, dt\right], \qquad (3.82)
\end{aligned}$$

while the monotone convergence theorem implies that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t\in[0,T]} e^{-\Lambda_t^{r(X)}} \left[K(X_t + \Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t > 0\}}\right] \\
&\quad = \lim_{n\to\infty} \mathbb{E}\left[\sum_{t\in[0,\tau_n\wedge T]} e^{-\Lambda_t^{r(X)}} \left[K(X_t + \Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t > 0\}}\right]. \qquad (3.83)
\end{aligned}$$

Also, the fact that w^- is bounded and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda^{r(X)}_{\tau_n\wedge T}} w^-(X_{(\tau_n\wedge T+)})\right] = \mathbb{E}\left[e^{-\Lambda^{r(X)}_T} w^-(X_{T+})\right].$$

In view of these observations, we can pass to the limit $n\to\infty$ in (3.81) to obtain

$$\mathbb{E}\left[\int_0^T e^{-\Lambda^{r(X)}_t} h(X_t)\,dt - \sum_{t\in[0,\tau_n\wedge T]} e^{-\Lambda^{r(X)}_t}\left[K(X_t+\Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t>0\}}\right]$$
$$\leq w(x) + \mathbb{E}\left[e^{-\Lambda^{r(X)}_T} w^-(X_{T+})\right]. \tag{3.84}$$

Combining this inequality with the limit

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\Lambda^{r(X)}_T} w^-(X_{T+})\right] = 0,$$

which follows from Assumption 8.(c) and the fact that w^- is bounded, we can see that

$$J_x(Z) \equiv \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T e^{-\Lambda^{r(X)}_t} h(X_t)\,dt - \sum_{t\in[0,T]} e^{-\Lambda^{r(X)}_t}\left[K(X_t+\Delta Z_t) - K(X_t) + c_f\right] 1_{\{\Delta Z_t>0\}}\right]$$
$$\leq w(x)$$

It follows that

$$v(x) \leq w(x) \quad \text{for all } x > 0. \tag{3.85}$$

If (3.67) is true, then $w = R_{X^0,h}$. Combining this fact with (3.85) and the observation that the control strategy $Z^0 \equiv 0$ has payoff $J_x(0) = R_{X^0,h}(x) = w(x)$ for all $x > 0$, which follows from the definition (3.33) of $R_{X^0,h}$, we obtain part I of the theorem.

In view of (3.85), we will establish case II of the theorem if we prove that the payoff function $x \to J_x(Z^*)$associated with the strategy Z^* given by (3.68)–(3.69) is equal to the function w given by (3.70) when the problem data satisfies (3.79) in Lemma 8. To this end, we first note that, given any $n \geq 0$, the process X^* satisfies

$$\begin{aligned} X^*_t &= \beta + \int_{T_n}^t b(X^*_u)du + \int_{T_n}^t \sigma(X^*_u)dW_u \\ &= \beta + \int_0^{t-T_n} b(X^*_{u-T_n})du + \int_0^{t-T_n} \sigma(X^*_{u-T_n})dW^{(n)}_u \end{aligned}$$

on the event $\{T_n < \infty\}$, where $W_t^{(n)} = (W_{T_n^*+t} - W_{T_n^*})\ 1_{\{T_n^*<\infty\}}$ and the second equality follows from Revuz and Yor [RY94, Propositions V.1.5] The process $W^{(n)}$ is a standard $(\mathcal{F}_{T_n^*+t})$-Brownian motion that is independent of $\mathcal{F}_{T_n^*}$ under the conditional probability measure $P(.|T_n^* < \infty)$ (see Revuz and Yor [RY94, Exercise IV.3.21]). Therefore, the process $X^{(n)}$ defined by

$$X_t^{(n)} = X^*_{T_n+t} 1_{\{T_n^*<\infty\}} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad n \geq 0, \tag{3.86}$$

is independent of $\mathcal{F}_{T_n^*}$ under $P(.|T_n^* < \infty)$ and its distribution under $P(.|T_n^* < \infty)$ is the same as the distribution of X^0 under P. In view of this observation and the definition of conditional expectation, we can see that

$$\mathbb{E}\left[\mathbb{F}(X^{(n)})|\mathcal{F}_{T_n^*}\right] 1_{\{T_n^*<\infty\}} = \mathbb{E}\left[\mathbb{F}(X^{(n)})\right] 1_{\{T_n^*<\infty\}} \tag{3.87}$$

for every operator $\mathbb{F}$ mapping continuous functions on $\mathbb{R}_+$ to continuous functions on $\mathbb{R}_+$. Indeed, given any event $\Gamma \in \mathcal{F}_{T_n^*}$, we can calculate

$$\begin{aligned}
\frac{1}{\mathbb{P}(T_n^* < \infty)}\mathbb{E}\left[\mathbb{E}\left[\mathbb{F}(X^0)\right] 1_{\{T_n^*<\infty\}} 1_\Gamma\right] &= \mathbb{E}\left[\mathbb{F}(X^{(0)})\right] \mathbb{P}(\{T_n^* < \infty\} \cap \Gamma)\frac{1}{\mathbb{P}(T_n^* < \infty)} \\
&= \mathbb{E}^{\mathbb{P}(\cdot|T_n^*<\infty)}\left[\mathbb{F}(X^n)\right] \mathbb{E}^{\mathbb{P}(\cdot|T_n^*<\infty)}\left[1_{\{T_n^*<\infty\}\cap\Gamma}\right] \\
&= \mathbb{E}^{\mathbb{P}(\cdot|T_n^*<\infty)}\left[\mathbb{F}(X^n) 1_{\{T_n^*<\infty\}\cap\Gamma}\right] \\
&= \frac{1}{\mathbb{P}(T_n^* < \infty)}\mathbb{E}\left[\mathbb{F}(X^n) 1_{\{T_n^*<\infty\}} 1_\Gamma\right],
\end{aligned}$$

where we denote by $\mathbb{E}^{\mathbb{P}(\cdot|T_n^*<\infty)}[.]$ expectation computed under the conditional probability measure $P(.|T_n^* < \infty)$, and the last equality follows because the Radon-Nikodym derivative of $P(.|T_n^* < \infty)$ with respect to P is given by

$$\frac{d\mathbb{P}(.|T_n^* < \infty)}{d\mathbb{P}} = \frac{1}{\mathbb{P}(T_n^* < \infty)} 1_{\{T_n^*<\infty\}}. \tag{3.88}$$

In view of the definition (3.86) of $X^{(n)}$, (3.87), the fact that

$$X_0^{(n)} = \gamma \quad \text{and} \quad T_{n+1}^* - T_n^* = \inf\{t \geq 0|\quad X_t^{(n)} = \beta\} \quad \text{for all} \quad n \geq 1, \tag{3.89}$$

and the definition (3.29) of the stopping time $T_{\gamma,\beta,}$ we can see that

$$\begin{aligned}
\mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_{n+1}}}\right] &= \mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_n}}\mathbb{E}\left[\exp\left(-\int_{T^*_n}^{T^*_{n+1}} r(X^*_t)dt\right)|\mathcal{F}_{T^*_n}\right]1_{\{T^*_n<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_n}}\mathbb{E}\left[\exp\left(-\int_0^{T^*_{n+1}-T^*_n} r(X^{(n)}_t)dt\right)|\mathcal{F}_{T^*_n}\right]1_{\{T^*_n<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_n}}\mathbb{E}\left[e^{-\Lambda^{r(X^0)}_{T_{\gamma,\beta}}}\right]1_{\{T^*_n<\infty\}}\right] \\
&\overset{(3.30)}{=} \mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_n}}\right]\frac{\phi(\gamma)}{\phi(\beta)} \quad \text{for all} \quad n \geq 1.
\end{aligned}$$

Combining this calculation with the observation that

$$\mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_1}}\right] = \begin{cases} \mathbb{E}\left[e^{-\Lambda^{r(X^0)}_{T_{\gamma,\beta}}}\right], & \text{if} \quad x \leq \beta, \\ \mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T_{x,\beta}}}\right], & \text{if} \quad x > \beta, \end{cases} \qquad \overset{(3.30)}{=} \begin{cases} \frac{\phi(\gamma)}{\phi(\beta)} & \text{if} \quad x \leq \beta, \\ \frac{\phi(x)}{\phi(\beta)} & \text{if} \quad x > \beta, \end{cases}$$

which follows from the description of the intervention strategy Z^* in (3.68)–(3.69), we can see that, given an initial condition $x > 0$ of the state process X^*,

$$\mathbb{E}\left[e^{-\Lambda^{r(X^*)}_{T^*_n}}\right] = \begin{cases} \left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^n & \text{if} \quad x \leq \beta, \\ \frac{\phi(x)}{\phi(\beta)}\left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^{n-1} & \text{if} \quad x > \beta, \end{cases} \quad \text{for all} \quad n \geq 1. \tag{3.90}$$

Similarly, we can use (3.86)–(3.89), the definition (3.29) of the $(\mathcal{F}_t)$-stopping time $T_{\gamma,\beta}$ and the structure of the intervention strategy Z^* given by (3.68)–(3.69) to calculate

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{T_n^*<\infty\}}\int_{T_n^*}^{T_{n+1}^*} e^{-\Lambda_t^{r(X^*)}}|h|(X_t^*)dt\right] \\
&= \mathbb{E}\left[e^{-\Lambda_{T_n^*}^{r(X^*)}}\mathbb{E}\left[\int_0^{T_{n+1}^*-T_n^*}\exp\left(-\int_0^{t-T_n^*} r(X_s^{(n)})ds\right)|h|(X_t^{(n)})dt|\mathcal{F}_{T_n^*}\right]1_{\{T_n^*<\infty\}}\right] \\
&= \mathbb{E}\left[e^{-\Lambda_{T_n^*}^{r(X^*)}}\right]\mathbb{E}\left[\int_0^{T_{\gamma,\beta}} e^{-\Lambda_t^{r(X^0)}}|h|(X_t^0)dt\right] \\
&= \begin{cases} R_{X^0,|h|}(x) - R_{X^0,|h|}(\beta)\mathbb{E}\left[e^{-\Lambda_{T_{x,\beta}}^{r(X^0)}}\right], & \text{if} \quad n=0 \quad \text{and} \quad x>\beta, \\ \mathbb{E}\left[e^{-\Lambda_{T_n^*}^{r(X^*)}}\right]\left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\mathbb{E}\left[e^{-\Lambda_{T_{x,\beta}}^{r(X^0)}}\right]\right], & \text{if} \quad n\ge 1 \quad \text{or} \quad n=0 \quad \text{and} \quad x\le\beta, \end{cases} \\
&= \begin{cases} R_{X^0,|h|}(x) - R_{X^0,|h|}(\beta)\frac{\phi(x)}{\phi(\beta)}, & \text{if} \quad n=0 \quad \text{and} \quad x>\beta, \\ \left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^n\left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\frac{\phi(\gamma)}{\phi(\beta)}\right], & \text{if} \quad n\ge 0 \quad \text{and} \quad x\le\beta, \\ \frac{\phi(x)}{\phi(\beta)}\left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^{n-1}\left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\frac{\phi(\gamma)}{\phi(\beta)}\right], & \text{if} \quad n\ge 1 \quad \text{and} \quad x>\beta, \end{cases}
\end{aligned}
\tag{3.91}
$$

the last equality following from (3.30) and (3.90).

Combining (3.90) with the monotone convergence theorem, we can see that, given any initial condition $x > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t\in[0,T]} e^{-\Lambda_t^{r(X^*)}}\left[K(X_t^*+\Delta Z_t^*) - K(X_t^*) + c_f\right]1_{\{\Delta Z_t^*>0\}}\right] \\
&= \begin{cases} [K(\gamma) - K(x) + c_f] + \sum_{n=1}^\infty \mathbb{E}\left[e^{-\Lambda_{T_n^*}^{r(X^*)}}\right][K(\gamma) - K(\beta) + c_f], & \text{if} \quad x\le\beta, \\ \sum_{n=1}^\infty \mathbb{E}\left[e^{-\Lambda_{T_n^*}^{r(X^*)}}\right][K(\gamma) - K(\beta) + c_f], & \text{if} \quad x>\beta, \end{cases} \\
&= \begin{cases} [K(\gamma) - K(x) + c_f] + \frac{\phi(\gamma)}{\phi(\beta)-\phi(\gamma)}[K(\gamma) - K(\beta) + c_f], & \text{if} \quad x\le\beta, \\ \frac{\phi(x)}{\phi(\beta)-\phi(\gamma)}[K(\gamma) - K(\beta) + c_f], & \text{if} \quad x>\beta, \end{cases}
\end{aligned}
\tag{3.92}
$$

which proves that (3.8) in Definition 1 is satisfied, so Z^* is admissible. In view of (3.18) in Assumption(8), we can see that (3.91) and the monotone convergence theorem imply that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^*)}}|h|(X_t^*)dt\right] \\
&= \lim_{N\to\infty}\sum_{n=0}^{N} \mathbb{E}\left[1_{\{T_n^*<\infty\}}\int_{T_n^*}^{T_{n+1}^*} e^{-\Lambda_t^{r(X^*)}}|h|(X_t^*)dt\right] \\
&= \begin{cases} \left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\frac{\phi(\gamma)}{\phi(\beta)}\right]\sum_{n=0}^{\infty} \left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^n, & \text{if } x \le \beta, \\ R_{X^0,|h|}(x) - R_{X^0,|h|}(\beta)\frac{\phi(x)}{\phi(\beta)} + \frac{\phi(x)}{\phi(\beta)}\left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\frac{\phi(\gamma)}{\phi(\beta)}\right]\sum_{n=1}^{\infty}\left[\frac{\phi(\gamma)}{\phi(\beta)}\right]^{n-1}, & \text{if } x > \beta, \end{cases} \\
&= \begin{cases} R_{X^0,|h|}(\gamma)\frac{\phi(\beta)}{\phi(\beta)-\phi(\gamma)} - R_{X^0,|h|}(\beta)\frac{\phi(\gamma)}{\phi(\beta)-\phi(\gamma)}, & \text{if } x \le \beta, \\ R_{X^0,|h|}(x) + \frac{\phi(x)}{\phi(\beta)-\phi(\gamma)}\left[R_{X^0,|h|}(\gamma) - R_{X^0,|h|}(\beta)\right] & \text{if } x > \beta, \end{cases} \\
&< \infty.
\end{aligned}
\tag{3.93}
$$

This inequality implies that we can use the dominated convergence theorem to make similar calculations with h in place of $|h|$ and obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t^{r(X^*)}}h(X_t^*)dt\right] \\
&= \begin{cases} R_{X^0,h}(\gamma)\frac{\phi(\beta)}{\phi(\beta)-\phi(\gamma)} - R_{X^0,h}(\beta)\frac{\phi(\gamma)}{\phi(\beta)-\phi(\gamma)}, & \text{if } x \le \beta, \\ R_{X^0,h}(x) + \frac{\phi(x)}{\phi(\beta)-\phi(\gamma)}\left[R_{X^0,h}(\gamma) - R_{X^0,h}(\beta)\right] & \text{if } x > \beta. \end{cases}
\end{aligned}
\tag{3.94}
$$

From (3.92)–(3.94) and the fact that $R_{X^0,Q} = R_{X^0,h} - K$ (see (3.40)), we can see that, given any initial condition $x \le \beta$,

$$
\begin{aligned}
J_x(Z^*) &= \frac{R_{X^0,h}(\gamma) - K(\gamma) - [R_{X^0,h}(\beta) - K(\beta) - c_f]}{\phi(\beta) - \phi(\gamma)}\phi(\gamma) + R_{X^0,h}(\gamma) - [K(\gamma) - K(x)] - c_f \\
&\overset{(3.78)}{=} A\phi(\gamma) + R_{X^0,h}(\gamma) - [K(\gamma) - K(x)] - c_f \\
&\overset{(3.70)}{=} w(x),
\end{aligned}
$$

while, given any initial condition $x > \beta$,

$$
\begin{aligned}
J_x(Z^*) &= \frac{R_{X^0,h}(\gamma) - K(\gamma) - [R_{X^0,h}(\beta) - K(\beta) - c_f]}{\phi(\beta) - \phi(\gamma)}\phi(x) + R_{X^0,h}(x) \\
&\overset{(3.78)}{=} A\phi(x) + R_{X^0,h}(x) \\
&\overset{(3.70)}{=} w(x),
\end{aligned}
$$

and the proof is complete. □

3.5 Special cases

We now consider special cases that arise when the running payoff function h is a power utility function and the running cost function k as well as the discounting factor r are constant. In particular, we assume that

$$h(x) = \lambda x^{\nu}, \quad K'(x) \equiv k(x) = \kappa \quad \text{and} \quad r(x) = r_1,$$

for some constants $\kappa, \lambda, r_1 > 0$ and $\nu \in]0, 1[$. Also, we assume that the uncontrolled system's state dynamics are modelled by a geometric Brownian motion (Section 3.5.1) or by a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model (Section 3.5.2).

3.5.1 Geometric Brownian motion

Suppose that X^0 is a geometric Brownian motion, so that

$$dX_t^0 = bX_t^0\, dt + \sigma X_t^0\, dW_t, \quad X_0^0 = x > 0,$$

for some constants b and $\sigma \neq 0$, and assume that $r_1 > b$. In this case, it is a standard exercise to verify that, if we choose $c = 1$, then

$$\phi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p'_{X^0}(x) = x^{n+m-1},$$

where the constants $m < 0 < n$ are the solution to the quadratic equation

$$\tfrac{1}{2}\sigma^2 l^2 + \left(b - \tfrac{1}{2}\sigma^2\right) l - r_1 = 0.$$

Also, it is well-known that

$$r_1 > b \quad \Leftrightarrow \quad n > 1.$$

In this context, Assumptions 4–7 and 8.(a)–8.(c) are all satisfied (see Jack, Johnson and Zervos [JJZ08, section 5.1]). Noting that

$$\rho(x) = r_1 - b > 0 \quad \text{and} \quad Q(x) = \lambda x^{\nu} - (r_1 - b)\kappa x,$$

we can see that

$$\lim_{x \to \infty} \frac{\mathcal{D}_r Q(x)}{\rho(x)} = -\kappa < 0,$$

so Assumption 8.(f) is also satisfied.

In this special case, the point a appearing in (3.51) is given by

$$a = \left[\frac{\lambda\nu(n-1)}{\kappa(r_1-b)(n-\nu)}\right]^{1/(1-\nu)} > 0$$

(see Jack, Johnson and Zervos [JJZ08, section 5.1]). In view of the calculation

$$R_{X^0,Q}(x) = -\frac{\lambda}{\frac{1}{2}\sigma^2\nu^2 + (b-\frac{1}{2}\sigma^2) - r_1}x^\nu - \kappa x, \tag{3.95}$$

we can see the point $\hat{x} > a$ introduced in Lemma 7 is given by

$$\hat{x} = \left[-\frac{\lambda\nu}{\kappa(\frac{1}{2}\sigma^2\nu^2 + (b-\frac{1}{2}\sigma^2) - r_1)}\right]^{1/(1-\nu)},$$

while

$$R_{X^0,Q}(\hat{x}) = \frac{\kappa(1-\nu)}{\nu}\left[-\frac{\lambda\nu}{\kappa(\frac{1}{2}\sigma^2\nu^2 + (b-\frac{1}{2}\sigma^2) - r_1)}\right]^{1/(1-\nu)}. \tag{3.96}$$

Furthermore, we can use (3.75) and the second expression for θ in (3.76) to calculate

$$\eta(\beta,\gamma) = -\frac{2\nu n}{r_1(n-\nu)(\nu-m)}[\gamma^{\nu-m} - \beta^{\nu-m}] + \frac{2\kappa n(r_1-b)}{r_1(n-1)(1-m)}[\gamma^{1-m} - \beta^{1-m}] \tag{3.97}$$

and

$$\theta(\beta,\gamma) = -\frac{2n}{r_1(n-\nu)}[\gamma^\nu - \beta^\nu] + \frac{2\kappa n(r_1-b)}{r_1(n-1)}[\gamma-\beta] + c_f. \tag{3.98}$$

Using (3.95) and (3.96), we can see that the second inequality in (8) is equivalent to

$$\frac{\kappa(1-\nu)}{\nu}\left[-\frac{\lambda\nu}{\kappa(\frac{1}{2}\sigma^2\nu^2 + (b-\frac{1}{2}\sigma^2) - r_1)}\right]^{1/(1-\nu)} > c_f. \tag{3.99}$$

We conclude this special case by observing that, if (3.99) is false, then we are in the context of Case I of Theorem 9 and it is optimal to never exert any control effort, while, if the problem data satisfy (3.99), then we are in the context of Case II of Theorem 9, the system of equations $\eta(\beta,\gamma) = 0$ and $\theta(\beta,\gamma) = 0$, where η and θ are given by (3.97) and (3.98), has a unique solution such that $0 < \beta < a < \gamma$, and the optimal strategy is characterised by (3.68)–(3.69).

3.5.2 Mean-reverting square-root process

Suppose that X^0 is a mean-reverting square-root process, so that

$$dX_t^0 = \alpha(\zeta - X_t^0)\,dt + \sigma\sqrt{X_t^0}\,dW_t, \quad X_0^0 = x > 0,$$

for some constants $\alpha, \zeta, \sigma > 0$, and assume $\sqrt{X_t^0}$

$$\alpha\zeta - \tfrac{1}{2}\sigma^2 > 0, \tag{3.100}$$

which is a necessary and sufficient condition for X^0 to be non-explosive. In this case, Assumptions 4–7 and 8.(a)–8.(e) are all satisfied,

$$\phi(x) = \frac{U\left(\frac{r_1}{\alpha}, \frac{2\alpha\zeta}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)}{U\left(\frac{r_1}{\alpha}, \frac{2\alpha\zeta}{\sigma^2}; \frac{2\alpha}{\sigma^2}\right)}, \quad \psi(x) = \frac{{}_1F_1\left(\frac{r_1}{\alpha}, \frac{2\alpha\zeta}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)}{{}_1F_1\left(\frac{r_1}{\alpha}, \frac{2\alpha\zeta}{\sigma^2}; \frac{2\alpha}{\sigma^2}\right)}$$

and

$$p'_{X^0}(x) = x^{-2\alpha\zeta/\sigma^2} e^{2\alpha(x-1)/\sigma^2},$$

where U and ${}_1F_1$ are the confluent hypergeometric function in Chapter 13 of Abramowitz and Stegun [AS72] , (for a proof of all these statements, see Jack, Johnson and Zervos[JJZ08, section 5.2]). Also, noting that

$$\rho(x) = r_1 + \alpha > 0 \quad \text{and} \quad Q(x) = \lambda x^\nu - (r_1 + \alpha)\kappa x + \alpha\zeta\kappa,$$

we can see that

$$\lim_{x\to\infty} \frac{\mathcal{D}_r Q(x)}{\rho(x)} = -(r_1 + \alpha)\kappa < 0,$$

which implies that Assumption 8.(f) is satisfied as well. We conclude this section by noting that the complexity of this special case is such that we have to resort to numerical techniques to obtain any further results.

3.6 Appendix

Appendix: proof of selected results

Proof of Lemma 7. The claims in (3.65) and (3.65) follow immediately once we combine (3.53)–(3.54) in Lemma 6 with (3.48), (3.50) and the fact that $\phi' < 0$. Since $R_{X^0,h}$ satisfies the ODE $\mathcal{L}_X R_{X^0,h}(x) + h(x) = 0$ (see (3.33)–(3.35)), we can see that the choice of $w = R_{X^0,h}$ is a solution of the HJB equation (3.61) if and only if

$$R_{X^0,h}(x+z) - K(x+z) - [R_{X^0,h}(x) - K(x)] - c_f \leq 0 \quad \text{for all} \quad x, z \geq 0,$$

or, equivalently, if and only if

$$R_{X^0,Q}(x+z) - R_{X^0,Q}(x) - c_f \leq 0 \quad \text{for all} \quad x, z \geq 0,$$

(see (3.40)). In view of (3.66), we can see that this condition is equivalent to (3.67), and the proof is complete. □

Proof of Lemma 8. Given a point $\beta > 0$, (3.53)–(3.54) in Lemma 6 imply that

$$\lim_{\gamma \to \infty} \eta(\beta, \gamma) = \infty \quad \text{and} \quad \frac{\partial}{\partial \gamma} \eta(\beta, \gamma) \begin{cases} < 0, & \text{if} \quad \beta < a \quad \text{and} \quad \gamma \in]\beta, a[, \\ > 0, & \text{if } \gamma > a, \end{cases}$$

Combining these results with the fact that $\eta(\beta, \beta) = 0$, we can see that there exists a unique function L: $]0, a[\to]a, \infty[$ such that

$$\beta < a < L(\beta) \quad \text{and} \quad \eta(\beta, L(\beta)) = 0, \quad \text{for all} \quad \beta \in]0, a[,$$

and that there exists no $\gamma > \beta$ satisfying $\eta(\beta, \gamma)$ if $\beta \geq a$, and that

$$\lim_{\beta \uparrow a} L(\beta) = a. \tag{3.101}$$

In particular,

$$\eta(\beta, \gamma) \begin{cases} < 0, & \text{if} \quad \gamma \in]\beta, L(\beta)[, \\ > 0, & \text{if } \gamma > L(\beta), \end{cases} \quad \text{for all } \beta \in]0, a[. \tag{3.102}$$

Differentiating $\eta(\beta, L(\beta)) = 0$ with respect to β using (3.54), we can see that in Lemma 6

$$L'(\beta) = \frac{r(\beta) p'_{X^0}(\beta)}{r(L(\beta)) p'_{X^0}(L(\beta))} \frac{\sigma^2(L(\beta))[\phi'(L(\beta))]^2}{\sigma^2(\beta)[\phi'(\beta)]^2} \frac{g(\beta)}{g(L(\beta))} < 0, \tag{3.103}$$

where the inequality follows from (3.51) and the strict positivity of the function r, p'_{X^0}.

Furthermore, we can see that the definition of η, (3.48), (3.50) and (3.66) in Lemma 7 imply that

$$\lim_{x \downarrow 0} L(\beta) = \hat{x}. \tag{3.104}$$

Now, we consider the equation $\theta(\beta, L(\beta)) = 0$ for $\beta \in]0, a[$. In view of (3.101), we can see that

$$\lim_{\beta \uparrow a} \theta(\beta, L(\beta)) = \theta(a, a) = c_f > 0.$$

Also, we can use (3.66)in Lemma 6 and (3.103) to calculate

$$\frac{d}{d\beta} \theta(\beta, L(\beta)) = -\frac{2r(\beta)p'_{X^0}(\beta)}{\sigma^2(\beta)[\phi'(\beta)]^2} g(\beta) [\phi(\beta) - \phi(L(\beta))] > 0 \text{ for all } \beta \in]0, a[.$$

These calculations imply that there exists a unique $\beta \in]0, a[$ such that $\theta(\beta, L(\beta)) = 0$ if and only if $\lim_{\beta \downarrow 0} \theta(\beta, L(\beta)) < 0$, which, in view of the first expression in the definition (3.76) of θ, in Lemma 7 and (3.104), is true if and only if

$$-R_{X^0,Q}(\hat{x}) + \lim_{\beta \downarrow 0} \left[R_{X^0,Q}(\beta) - \frac{\phi(\beta)}{\phi'(\beta)} R'_{X^0,Q}(\beta) \right] + c_f < 0. \tag{3.105}$$

If the implications

$$\lim_{x \downarrow 0} R_{X^0,Q}(x) > -\infty \quad \Rightarrow \quad \lim_{x \downarrow 0} \frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x) = 0 \tag{3.106}$$

and

$$\lim_{x \downarrow 0} R_{X^0,Q}(x) = -\infty \quad \Rightarrow \quad \lim_{x \downarrow 0} \left[R_{X^0,Q}(x) - \frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x) \right] = -\infty \tag{3.107}$$

hold true, then we can see that (3.105) is equivalent to (3.79).

To prove (3.106) we argue by contradiction. To this end, we recall that $R'_{X^0,Q}(x) > 0$ for all $x < \hat{x}$ (see(3.66)), and we assume that the problem data is such that

$$\lim_{x \downarrow 0} R_{X^0,Q}(x) > -\infty \quad \text{and} \quad \lim_{x \downarrow 0} \frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x) < 0.$$

In such a case, there exist $\epsilon > 0$ and $\zeta \in]0, \hat{x}[$ such that

$$\frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x) < -\epsilon \quad \text{for all} \quad x \in]0, \zeta[.$$

Using the facts that $\phi' < 0 < \phi$ and $\lim_{x \downarrow 0} \phi(x) = \infty$, we then calculate

$$\begin{aligned}
\infty > & \lim_{x\downarrow 0} \left[R_{X^0,Q}(\zeta) - R_{X^0,Q}(x)\right] \\
= & \lim_{x\downarrow 0} \int_x^\zeta R'_{X^0,Q}(s) ds \\
> & -\epsilon \lim_{x\downarrow 0} \int_x^\zeta \frac{\phi'(s)}{\phi(s)} ds \\
= & -\epsilon \lim_{x\downarrow 0} \left[\ln\phi(\zeta) - \ln\phi(x)\right] \\
= & \infty,
\end{aligned}$$

which is a contradiction.

To prove (3.107) we also argue by contradiction, and we assume that

$$\lim_{x\downarrow 0} R_{X^0,Q}(x) = -\infty \quad \text{and} \quad \lim_{x\downarrow 0}\left[R_{X^0,Q}(x) - \frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x)\right] > -\infty.$$

In this case, there exist $K > 0$ and $\zeta \in]0, \hat{x}[$ such that

$$R_{X^0,Q}(x) < -2k \quad \text{and} \quad R_{X^0,Q}(x) - \frac{\phi(x)}{\phi'(x)} R'_{X^0,Q}(x) > -K \quad \text{for all} \quad x \in]0, \zeta[.$$

Noting that the second inequality here is equivalent to

$$\frac{d}{dx}\frac{R_{X^0,Q}(x)}{\phi(x)} > \frac{K\phi'(x)}{\phi^2(x)},$$

we can see that, in this context,

$$\begin{aligned}
-\frac{2K}{\phi(\zeta)} - \frac{R_{X^0,Q}(x)}{\phi(x)} \geq & \frac{R_{X^0,Q}(\zeta)}{\phi(\zeta)} - \frac{R_{X^0,Q}(x)}{\phi(x)} \\
> & K\int_x^\zeta \frac{\phi'(s)}{\phi^2(s)} ds \\
= & -\frac{K}{\phi(\zeta)} + \frac{K}{\phi(x)}.
\end{aligned}$$

In view of (3.37) with $G = Q$ and the fact that $\lim_{x\downarrow 0}\phi(x) = \infty$, we can pass to the limit as $x \downarrow 0$ to obtain

$$-\frac{2K}{\phi(\zeta)} \geq -\frac{K}{\phi(\zeta)},$$

which is contradiction.

By construction, we will prove that (3.79) holds true, the function w defined by (3.70), with A given by (3.74), satisfies the HJB equation (3.61) if we show that

$$\mathcal{L}_X w(x) + h(x) \equiv \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) + h(x) \leq 0, \quad \text{for } x < \beta, \tag{3.108}$$

and

$$w(x+z) - K(x+z) - [w(x) - K(x)] - c_f \leq 0 \quad \text{for all} \quad x, z > 0. \tag{3.109}$$

To this end, we first use the definition (3.74) of A to calculate

$$\begin{aligned} A\phi'(x) + R'_{X^0,h}(x) &= \phi'(x)\left[-\frac{R'_{X^0,Q}(\beta)}{\phi'(\beta)} + \frac{R'_{X^0,Q}(x)}{\phi'(x)}\right] \\ &\overset{(3.75)}{=} \phi'(x)\eta(\beta, x) \\ &\begin{cases} > 0, & \text{if} \quad x \in]\beta, \gamma[, \\ < 0, & \text{if} \quad x > \gamma, \end{cases} \end{aligned}$$

the inequalities following from (3.102) and the fact that $\gamma = L(\beta)$. This calculation and the definition(3.70) of w imply that

$$w'(x) - K'(x) \begin{cases} = 0, & \text{if } x \in]0, \beta[, \\ > 0, & \text{if } x \in]\beta, \gamma[, \\ < 0, & \text{for } x > \gamma. \end{cases} \tag{3.110}$$

Combining these inequalities with (3.73), we can see that (3.109) holds true. Also, these inequalities imply that

$$\lim_{x \downarrow \beta} [w''(x) - K''(x)] \geq 0. \tag{3.111}$$

Finally, in view of the definition (3.15) of Q, we can see that (3.108) is equivalent to

$$\frac{Q(x)}{r(x)} \leq A\phi(\gamma) + R_{X^0,h}(\gamma) - K(\gamma) - c_f \overset{(3.70)}{=} w(\gamma) - K(\gamma) - c_f \overset{(3.73)}{=} w(\beta) - K(\beta),$$

for $x < \beta$. Since $\beta < a < x^*$, (3.17) in Assumption 8.(d) implies that

$$\frac{d}{dx}\frac{Q(x)}{r(x)} = \frac{1}{r(x)}\mathcal{D}_r Q(x) \geq 0 \quad \text{for all } x \leq \beta.$$

It follows that (3.108) is true if and only if

$$\frac{Q(x)}{r(x)} \leq w(\beta) - K(\beta).$$

To see that this inequality holds true, we use the facts that $\mathcal{L}_X\phi(x) = 0$ and $\mathcal{L}_X R_{X^0,h} + h(x) = 0$, and the definition (3.15) of Q to observe that $\mathcal{L}_X[w(x) - K(x)] + Q(X) = 0$, for $x > \beta$. Combining this with the fact that $w'(\beta) - K'(\beta) = 0$ (see (3.72)), we can see that

$$\begin{aligned} w(\beta) - K(\beta) &= \lim_{x \downarrow \beta} \frac{1}{r(x)} \left[Q(x) - \tfrac{1}{2}\sigma^2(x)[w''(x) - K''(x)] - b(x)[w'(x) - K'(x)] \right] \\ &= \frac{Q(\beta)}{r(\beta)} + \frac{\sigma^2(\beta)}{2r(\beta)} \lim_{x \downarrow \beta} [w''(x) - K''(x)] \\ &\overset{(3.111)}{\geq} \frac{Q(\beta)}{r(\beta)}. \end{aligned}$$

and the proof is complete. □

4

Buy-low and sell-high investment strategies

4.1 Introduction

Buy-low and sell-high investment strategies are a recurrent theme in the considerations of many investors. In this chapter, we consider an investor who aims at maximising the expected discounted cash-flow that can be generated by sequentially buying and selling one share of a given asset at fixed transaction costs. We model the underlying asset price by means of a general one-dimensional Itô diffusion X, we solve the resulting stochastic control problem in a closed analytic form, and we completely characterise the optimal strategy. In particular, we show that, if 0 is a natural boundary point of X, e.g., if X is a geometric Brownian motion, then it is never optimal to sequentially buy and sell. On the other hand, we prove that, if 0 is an entrance point of X, e.g., if X is a mean-reverting constant ellasticity of variance (CEV) process, then it may be optimal to sequentially buy and sell, depending on the problem data.

4.2 Problem formulation and assumptions

In this problem we consider an asset with price process X that is modelled by the one-dimensional Itô diffusion

$$dX_t = b(Xt)dt + \sigma(X_t)\, dW_t, \quad X_0 = x > 0, \tag{4.1}$$

where W is a standard one-dimensional Brownian motion. An investor follows a strategy that consists of sequentially buying and selling one share of the asset. We use a controlled finite variation càglàd process Y that takes values in $\{0,1\}$ to model the investors position in the market. In particular, $Y_t = 1$ (resp., $Y_t = 0$) represents the state where the investor holds (resp., does not hold) the asset, while, the jumps of Y occur at the sequence of times $(\tau_n, n \geq 1)$ at which the investor buys or sells. Given an initial condition $(Y_0, X_0) = (y, x) \in \{0,1\} \times]0, \infty[$, the investor objective is to select a strategy that maximises the performance criterion

$$J_{y,x}(Y) = \lim_{n \to \infty} \sum_{j=1}^{n} \mathbb{E}_x \left[e^{-\Lambda_{\tau_j}} [(X_{\tau_j} - c_s) \mathbf{1}_{\{\Delta Y_{\tau_j} = -1\}} - (X_{\tau_j} + c_b) \mathbf{1}_{\{\Delta Y_{\tau_j} = 1\}}] \mathbf{1}_{\{\tau_j < \infty\}} \right]. \quad (4.2)$$

Where the state-dependent discounting factor Λ is defined by

$$\Lambda_t = \int_0^t r(X_s) ds, \quad (4.3)$$

for some function $r > 0$, and the constants $c_b > 0$ (resp., $c_s > 0$) represents the transaction cost of buying (resp., selling) one share of the asset. Accordingly, we define the problem's value function v by

$$v(y, x) = \sup_{Y \in \mathcal{A}_{y,x}} J_{y,x}(Y), \quad \text{for} \quad y \in \{0,1\} \quad \text{and} \quad x > 0, \quad (4.4)$$

where $\mathcal{A}_{y,x}$ is the set of admissible investment strategies, which is introduced by Definition 2 in Section 4.2.

We are going to prove later that, in the presence of the general assumptions that we make on b, σ and r, the optimisation problem defined by (4.1)–(4.4) is well-posed, in particular, the limit in (4.2) exists (see Theorem 12, our main result)

We assume that the data of the one-dimensional Itô diffusion given by (4.1) in the introduction satisfy the following assumption.

Assumption 9. The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are Borel-measurable,

$$\sigma^2(x) > 0, \quad \text{for all } x > 0,$$

and

$$\int_{\underline{\alpha}}^{\overline{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)} ds < \infty \quad \text{for all } 0 < \underline{\alpha} < \overline{\beta} < \infty.$$

□

The conditions in this assumption are sufficient for the SDE (4.1) to have a weak solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ that is unique in the sense of probability law up to a possible explosion time (e.g., see Karatzas and Shreve [KS91, Section 5.5.C]). We assume that such a weak solution is fixed for each initial condition $x > 0$ throughout the chapter. In particular, the scale function p and the speed measure m given by

$$p(x) = \int_1^x \exp\left(-2\int_c^s \frac{b(u)}{\sigma^2(u)}\, du\right) ds, \quad \text{for } x > 0, \tag{4.5}$$

$$m(dx) = \frac{2}{\sigma^2(x)p'(x)}\, dx, \tag{4.6}$$

respectively, for some $c > 0$ fixed, are well-defined.

We also assume that the solution of (4.1) is non-explosive, i.e., the hitting time of the boundary $\{0, \infty\}$ of the interval $]0, \infty[$ is infinite with probability 1 (see Karatzas and Shreve [KS91, Theorem 5.5.29] for appropriate necessary and sufficient analytic conditions).

Assumption 10. The solution of (4.1) in non-explosive. □

Relative to the discounting factor Λ defined by (4.3), we make the following assumption.

Assumption 11. The function $r :]0, \infty[\to]0, \infty[$ is Borel-measurable and uniformly bounded away from 0, i.e., there exists $r_0 > 0$ such that $r(x) \geq r_0$, for all $x > 0$. Also,

$$\int_{\overline{\alpha}}^{\overline{\beta}} \frac{r(s)}{\sigma^2(s)}\, ds < \infty \quad \text{for all } 0 < \overline{\alpha} < \overline{\beta} < \infty.$$

□

In the presence of Assumptions 4–11, there exists a pair of C^1 functions $\phi, \psi :]0, \infty[\to]0, \infty[$ with absolutely continuous first derivatives and such that

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x > 0, \tag{4.7}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x > 0, \tag{4.8}$$

$$\lim_{x \downarrow 0} \phi(x) = \lim_{x \to \infty} \psi(x) = \infty. \tag{4.9}$$

$$\phi(x) = \phi(y)\mathbb{E}_y[e^{-\Lambda_{T_y}}] \quad \text{for all } y < x \quad \text{and} \quad \psi(x) = \psi(y)\mathbb{E}_x[e^{-\Lambda_{T_y}}], \quad \text{for all } x < y, \tag{4.10}$$

where T_y, is the first hitting time of $\{y\}$, which is defined by $T_y = \inf\{t \geq 0 \mid X_t = y\}$. Also, the functions ϕ and ψ are classical solutions of the homogeneous ODE

$$\mathcal{L}g(x) := \tfrac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0, \tag{4.11}$$

and satisfy

$$\phi(x)\psi'(x) - \phi'(x)\psi(x) = Cp'(x) \quad \text{for all } x > 0, \tag{4.12}$$

where $C = \phi(1)\psi'(1) - \phi'(1)\psi(1)$ and p is the scale function defined by (4.5).

Now, let g be a C^1 functions with absolutely continuous first derivatives and such that

$$\lim_{x\downarrow 0} \frac{g(x)}{\phi(x)} = \lim_{x\to\infty} \frac{g(x)}{\psi(x)} = 0, \tag{4.13}$$

and

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} |\mathcal{L}g(X_t)| dt \right] < \infty, \tag{4.14}$$

where $\mathcal{L}$ is the operator defined by (4.11). Such a function admits the analytic representation

$$g(x) = -\phi(x) \int_0^x \Psi(s)\mathcal{L}g(s)ds - \psi(x) \int_x^\infty \Phi(s)\mathcal{L}g(s)ds, \tag{4.15}$$

where

$$\Phi(x) = \frac{\phi(x)}{C\sigma^2(x)p'(x)} \quad \text{and} \quad \Psi(x) = \frac{\psi(x)}{C\sigma^2(x)p'(x)}. \tag{4.16}$$

Furthermore, given any $(\mathcal{F}_t)$-stopping time τ, Dynkin's formula

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} g(X_\tau) \mathbf{1}_{\{\tau<\infty\}} \right] = g(x) + \mathbb{E}_x \left[\int_0^\tau e^{-\Lambda_t} \mathcal{L}g(X_t) dt \right] \tag{4.17}$$

holds, and

$$\lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_n}} |g(X_{\tau_n})| \mathbf{1}_{\{\tau_n<\infty\}} \right] = 0, \tag{4.18}$$

for every sequence of $(\mathcal{F}_t)$-stopping times (τ_n) such that $\tau_n \to \infty$, $\mathbb{P}_x$-a.s.. The existence of the functions ϕ, ψ and the various results that we have listed can be found in several references, including Borodin and Salminen [BS02]. For future reference, we also not that a straightforward calculation involving (4.12) and (4.15) implies that

$$\left(\frac{g}{\phi}\right)'(x) = -\frac{Cp'(x)}{\phi^2(x)} \int_x^\infty \Phi(s)\mathcal{L}g(s)ds. \tag{4.19}$$

$$\left(\frac{g}{\psi}\right)'(x) = \frac{Cp'(x)}{\psi^2(x)} \int_0^x \Psi(s)\mathcal{L}g(s)ds. \tag{4.20}$$

We can now complete the set of our assumptions.

Assumption 12. The problem data is such that

$$\lim_{x\to\infty}\frac{x}{\psi(x)}=0 \quad \text{and} \quad \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}|\mathcal{L}H(X_t)|dt\right]<\infty, \tag{4.21}$$

where H is the identity function, i.e., $H(x)=x$. Also, c_b,$c_s>0$, and there exist constants $0\le x_b<x_s$ such that the functions $\mathcal{L}H_b$ and $\mathcal{L}H_s$, defined by

$$H_b(x)=x+c_b \quad \text{and} \quad H_s(x)=x-c_s, \quad \text{for} \quad x>0, \tag{4.22}$$

satisfy

$$\mathcal{L}H_b(x)\begin{cases}>0, & \text{if} \quad x_b>0 \quad \text{and } x<x_b,\\ <0, & \text{if} \quad x>x_b,\end{cases} \quad \text{and} \quad \mathcal{L}H_s(x)\begin{cases}>0, & \text{if} \quad x<x_s,\\ <0, & \text{if} \quad x>x_s,\end{cases} \tag{4.23}$$

□

The following definition introduces the set of all admissible investment strategies over which we maximise the performance criterion $J_{y,x}$ defined by (4.2).

Definition 2. Given an initial condition $(y,x)\in\{0,1\}\times]0,\infty[$, and the associated weak solution $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P}_x,X,W)$ of (4.1), an admissible investment strategy is any $(\mathcal{F}_t)$-adapted finite variation càglàd process Y with values in $\{0,1\}$ such that $Y_0=y$. Given such a process Y, we denote by (τ_n) the strictly increasing sequence of $(\mathcal{F}_t)$-stopping times at which the jumps of Y occur, which can be defined recursively by

$$\tau_1=\inf\{t>0\mid Y_t\neq y\} \quad \text{and} \quad \tau_{j+1}=\inf\{t>\tau_j\mid Y_t\neq Y_{\tau_j}\}, \quad \text{for } j=1,2,\ldots \quad , \tag{4.24}$$

with the usual convention that $\inf\emptyset=\infty$. We denote by $\mathcal{A}_{y,x}$ the set of all admissible strategies. □

We conclude this section with the following remarks.

Remark 4. In the context of Assumption 10, 0 can be a natural boundary point of the diffusion X, in which case, $\lim_{x\downarrow 0}\psi(x)=0$, or an entrance boundary point, in which case, $\lim_{x\downarrow 0}\psi(x)>0$. If 0 is a natural boundary point, then $x_b=0$, where x_b is the constant appearing in Assumption 12. Indeed, if $x_b>0$, then (4.15) and the strict positivity of c_b imply that

$$\infty=\lim_{x\downarrow 0}\frac{H_b(x)}{\psi(x)}\le-\lim_{x\downarrow 0}\int_x^\infty\Phi(s)\mathcal{L}H_b(s)ds<-\int_{x_b}^\infty\Phi(s)\mathcal{L}H_b(s)ds<\infty, \tag{4.25}$$

which is a contradiction. It follows that $x_b > 0$ only if 0 is an entrance boundary point. For future reference, we note that, if 0 is an entrance boundary point and $x_b > 0$, then the last three inequalities in (4.25), the identity $\Phi = \phi\Psi/\psi$, which follows from (4.16), and the fact that the function ϕ/ψ is decreasing imply that

$$\infty > \int_0^x \Phi(s)\mathcal{L}H_b(s)ds = \int_0^x \Psi(s)\frac{\phi(s)}{\psi(s)}\mathcal{L}H_b(s)ds \geq \frac{\phi(x)}{\psi(x)}\int_0^x \Psi(s)H_b(s)ds \geq 0$$

for all $x \in]0, x_b[$. It follows that

$$\lim_{x\downarrow 0}\frac{\phi(x)}{\psi(x)}\int_0^x \Psi(s)\mathcal{L}H_b(s)ds = 0,$$

which combined with (4.15), implies that

$$-\lim_{x\downarrow 0}\int_x^\infty \Phi(s)\mathcal{L}H_b(s)ds = \lim_{x\downarrow 0}\frac{H_b(x)}{\psi(x)}. \tag{4.26}$$

□

Remark 5. In view of the definition (4.11) of the operator $\mathcal{L}$, the definition (4.23) of the functions H_b, H_s and the inequality in (4.21), we can see that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}[|\mathcal{L}H_b(X_t)| + |\mathcal{L}H_s(X_t)|]dt\right] \\
&\qquad \leq 2\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}|\mathcal{L}H(X_t)|dt\right] + (c_b + c_s)\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}r(X_t)dt\right] \\
&\qquad = 2\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}|\mathcal{L}H(X_t)|dt\right] + (c_b + c_s)\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}d\Lambda_t\right] \\
&\qquad = 2\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}|\mathcal{L}H(X_t)|dt\right] + (c_b + c_s) \\
&\qquad < \infty.
\end{aligned}$$

It follows that the functions H_b, H_s satisfy the corresponding requirements of (4.13)–(4.14), and, therefore, they have all of the corresponding properties in (4.15)–(4.18). □

4.3 The solution of the control problem

The existing theory on sequential switching problems suggests that the value function v of our control problem should identify with a classical solution w of the HJB equation

$$\max\Big\{ \tfrac{1}{2}\sigma^2(x)w_{xx}(y,x) + b(x)w_x(y,x) - r(x)w(y,x),$$
$$w(1-y,x) - w(y,x) + yH_s(x) - (1-y)H_b(x) \Big\} = 0, \quad y = 0, 1. \tag{4.27}$$

where H_b and H_s are defined by (4.22). We now solve our control problem by constructing an appropriate solution for this equation. To this end, we have to consider the two qualitatively different cases. The first one arises when it is optimal for the investor to sell as soon as the asset price exceeds a given level $\alpha > 0$, if $y = 1$, and never enter the market otherwise. In this case, we look for a solution of the HJB equation (4.27) of the form given by

$$w(0,x) = 0 \quad \text{and} \quad w(1,x) = \begin{cases} A\psi(x), & \text{if} \quad x < \alpha, \\ H_s(x), & \text{if} \quad x \geq \alpha, \end{cases} \tag{4.28}$$

for some constant A. To determine the parameter A and the free-boundary point α, we appeal to the so-called "principle of smooth fit" of sequential switching and we require that $w(1,\cdot)$ is C^1 along α, which yields the system of equations

$$A\psi(\alpha) = H_s(\alpha) \quad \text{and} \quad A\psi'(\alpha) = H_s'(\alpha).$$

This system is equivalent to

$$A = \frac{H_s(\alpha)}{\psi(\alpha)} = \frac{H_s'(\alpha)}{\psi'(\alpha)}. \tag{4.29}$$

In view of the Remark 5 and (4.20), we can see that these identities imply that $\alpha > 0$ should satisfy the equation

$$q(\alpha) = 0, \tag{4.30}$$

where

$$q(x) = \int_0^x \Psi(s)\mathcal{L}H_s(s)ds. \tag{4.31}$$

The following result which we prove in the Appendix, is concerned with the solvability of (4.30) as well as with a necessary and sufficient condition for the function w given by (4.28) to satisfy the HJB equation (4.27).

Lemma 10. *In the presence of Assumptions 9–12, there exists a unique $\alpha > 0$ satisfying equation (4.30). Furthermore, $\alpha > c_s$ and the function w defined by (4.28), where $A > 0$ is given by (4.29), satisfies the HJB equation (4.27) if and only if the problem data is such that either $x_b = 0$ or*

$$x_b > 0 \quad \textit{and} \quad c_b \geq (\alpha - c_s)\frac{\lim_{x \downarrow 0} \psi(x)}{\psi(\alpha)}. \tag{4.32}$$

The second possibility arises when it is optimal for the investor to sequentially enter and exit the market. In this case, we postulate that the value function of our control problem should identify with a solution w of the HJB equation (4.27) that has the form given by the expressions

$$w(1,x) = \begin{cases} A\psi(x), & \text{if} \quad x < \gamma, \\ B\phi(x) + H_s(x), & \text{if} \quad x \geq \gamma, \end{cases} \tag{4.33}$$

$$w(0,x) = \begin{cases} A\psi(x) - H_b(x), & \text{if} \quad x \leq \beta, \\ B\phi(x), & \text{if} \quad x > \beta, \end{cases} \tag{4.34}$$

for some constants A, B and free-boundary points β,γ such that $0 < \beta < \gamma$. To determine these variables, we conjecture that the functions $w(1, \cdot)$ and $w(0, \cdot)$ should be C^1 at the free-boundary points γ and β, respectively, which yields the system of equations

$$A\psi(\gamma) = B\phi(\gamma) + H_s(\gamma), \quad B\phi(\beta) = A\psi(\beta) - H_b(\beta), \tag{4.35}$$

$$A\psi'(\gamma) = B\phi'(\gamma) + H_s'(\gamma) \quad \text{and} \quad B\phi'(\beta) = A\psi'(\beta) - H_b'(\beta). \tag{4.36}$$

We can check that these equations are equivalent to

$$A = \frac{H_b'(\beta)\phi(\beta) - H_b(\beta)\phi'(\beta)}{\phi(\beta)\psi'(\beta) - \psi(\beta)\phi'(\beta)} = \frac{H_s'(\gamma)\phi(\gamma) - H_s(\gamma)\phi'(\gamma)}{\phi(\gamma)\psi'(\gamma) - \psi(\gamma)\phi'(\gamma)},$$

$$B = \frac{H_b'(\beta)\psi(\beta) - H_b(\beta)\psi'(\beta)}{\phi(\beta)\psi'(\beta) - \psi(\beta)\phi'(\beta)} = \frac{H_s'(\gamma)\psi(\gamma) - H_s(\gamma)\psi'(\gamma)}{\phi(\gamma)\psi'(\gamma) - \psi(\gamma)\phi'(\gamma)}.$$

and then appeal to Remark 5, (4.12) and (4.19)–(4.20)

$$A = -\int_\beta^\infty \Phi(s)\mathcal{L}H_b(s)ds = -\int_\gamma^\infty \Phi(s)\mathcal{L}H_s(s)ds \tag{4.37}$$

$$B = \int_0^\beta \Psi(s)\mathcal{L}H_b(s)ds = \int_0^\gamma \Psi(s)\mathcal{L}H_s(s)ds. \tag{4.38}$$

It follows that the free-boundary points $0 < \beta < \gamma$ should satisfy the system of equations

$$q_\phi(\beta, \gamma) = 0 \quad \text{and} \quad q_\psi(\beta, \gamma) = 0, \tag{4.39}$$

where

$$q_\phi(x,z) = \int_x^\infty \Phi(s)\mathcal{L}H_b(s)ds - \int_z^\infty \Phi(s)\mathcal{L}H_s(s)ds \tag{4.40}$$

and

$$q_\psi(x,z) = \int_0^x \Psi(s)\mathcal{L}H_b(s)ds - \int_0^z \Psi(s)\mathcal{L}H_s(s)ds. \tag{4.41}$$

The following result is concerned with conditions, under which, there exist points $0 < \beta < \gamma$ that satisfy (4.39) and the corresponding function w defined by (4.33)–(4.34) satisfies the HJB equation (4.27).

Lemma 11. *In the presence of Assumptions 9–12, the system of equations (4.39) has a unique solution $0 < \beta < \gamma$ if and only if the problem data is such that*

$$x_b > 0 \quad \textit{and} \quad c_b < (\alpha - c_s)\frac{\lim_{x\downarrow 0}\psi(x)}{\psi(\alpha)}, \tag{4.42}$$

where $\alpha > c_s$ is the unique solution of (4.30). In this case, the function w defined by (4.33)-(4.34) for $A > 0$ and $B > 0$ given by (4.37) and (4.38), satisfies the HJB equation (4.27).

We can now establish our main result.

Theorem 12. *The stochastic optimisation problem formulated in Section 4.2 is well-posed. Furthermore,*

(I) if the problem data is such that either $x_b = 0$ or (4.32) holds true, then $v = w$, where w is as in Lemma 10, and the optimal strategy Y^* is given by

$$Y_t^* = y\mathbf{1}_{[0,\tau_1^*]}(t) + \mathbf{1}_{]\tau_1^*,\infty[}(t), \quad \textit{where} \quad \tau_1^* = \inf\{t \geq 0 \mid X_t \geq \alpha\}, \tag{4.43}$$

with the convention that $\inf\emptyset = \infty$, and $\alpha > 0$ is the unique solution of (4.30);

(II) if the problem data is such that (4.42) holds true, then $v = w$, where w is as in Lemma 11, and the optimal strategy Y^* is given by

$$Y_t^* = \mathbf{1}_{\{y\}}(t) + \sum_{j=0}^{\infty} 1_{]\tau^*_{y+2j-1},\tau^*_{y+2j}]}(t), \tag{4.44}$$

where the $(\mathcal{F}_t)$-stopping times τ_n^*, $n \geq 1$ are defined recursively by

$$\tau^*_{y+2n} = \inf\{t \geq \tau^*_{y+2j-1} \mid X_t \geq \gamma\} \quad \textit{and} \quad \tau^*_{y+2n+1} = \inf\{t \geq \tau^*_{y+2j} \mid X_t \leq \beta\},$$

for the appropriate value of $j = 0, 1, \cdots$, *where* $\tau^*_{-1} = \tau^*_0 = 0$ *and* $0 < \beta < \gamma$ *is the unique solution of the system of equations (4.39).*

Proof. We first note that, in view of (4.7)–(4.8), there exists a constant $K > 0$ such that $|w(y, x)| \leq K(1 + x)$ for all $x > 0$ and $y = 0, 1,$. Therefore, the function $w(y, .)$ satisfy the corresponding in (4.13). Also, the observation that

$$\mathcal{L}w(1, \cdot)(x) = \begin{cases} 0, & \text{if} \quad x < \gamma, \\ \mathcal{L}H_s(x), & \text{if} \quad x > \gamma, \end{cases} \quad \text{and} \quad \mathcal{L}w(0, \cdot)(x) = \begin{cases} -\mathcal{L}H_b(x), & \text{if} \quad x < \beta, \\ 0, & \text{if} \quad x > \beta, \end{cases}$$

and Remark 5 imply that the functions $w(y, \cdot)$, $y = 0, 1$, satisfy the corresponding requirements of (4.14). It follows that

$$w(0, \cdot) \quad \text{and} \quad w(1, \cdot) \quad \text{have all of the corresponding properties in} \quad (4.13) - (4.18). \tag{4.45}$$

In the rest of the analysis, we may assume that the investor is long in the market at time 0, i.e., that $y = 1$; the analysis of the case associated with $y = 0$ follows exactly the same steps. To start with, we consider any admissible investment strategy $Y \in \mathcal{A}_{1,x}$, and we recall that the jumps of Y occur at the times composing the sequence $(\tau_n, n \geq 1)$ defined by (4.24) in Definition 2. For notational simplicity, we define $\tau_0 = 0$, and we note that

$$\tau_j(\omega) < \tau_{j+1}(\omega) \quad \text{for all} \quad \omega \in \{\tau_j < \infty\} \quad \text{and} \quad j \geq 1.$$

Also, we note that $\lim_{j \to \infty} \tau_j = \infty$, $\mathbb{P}_x$-a.s., because Y is a finite-variation process.

Iterating Dynkin's formula (4.17), we obtain

$$\begin{aligned} &\sum_{j=1}^{2n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_j}} [H_s(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j} = -1\}} - H_b(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j} = 1\}}] \mathbf{1}_{\{\tau_j < \infty\}} \right] \\ &= \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2j+1}}} H_s(X_{\tau_{2j+1}}) \mathbf{1}_{\{\tau_{2j+1} < \infty\}} \right] - \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2j}}} H_b(X_{\tau_{2j}}) \mathbf{1}_{\{\tau_{2j} < \infty\}} \right] \\ &= (x - c_s) - (c_s + c_b) - \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2j}}} \mathbf{1}_{\{\tau_{2j} < \infty\}} \right] + \mathbb{E}_x \left[\int_0^{\tau_{2n}} e^{-\Lambda_t} Y_t \mathcal{L}H_s(X_t) dt \right]. \end{aligned} \tag{4.46}$$

Also, (4.14) and the dominated convergence theorem imply that

$$\mathbb{E}_x \left[\int_0^{\infty} e^{-\Lambda_t} Y_t \mathcal{L}H_s(X_t) dt \right] = \lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{\tau_n} e^{-\Lambda_t} Y_t \mathcal{L}H_s(X_t) dt \right] \in \mathbb{R}.$$

Combining this observation with the limit

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2n}}} H_b(X_{\tau_{2n}})\right] = 0,$$

which holds thanks to (4.18) and the fact that $\lim_{n\to\infty} \tau_n = \infty$, we can see that

$$J_{1,x}(Y) = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_x\left[e^{-\Lambda_{\tau_j}} H_s(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_b(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\right] \mathbf{1}_{\{\tau_j<\infty\}}\Big] \tag{4.47}$$

$$\in [-\infty, \infty[. \tag{4.48}$$

It follows that $J_{1,x}(Y)$ is well-defined and our optimisation problem is well-posed.

To proceed further, we recall (4.45) and we iterate Dynkin's formula (4.17) to calculate

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\Lambda_{\tau_{2n}}} w(0, X_{\tau_{2n}}) \mathbf{1}_{\{\tau_{2n}<\infty\}}\right] \\
&= w(1,x) + \sum_{j=0}^{n-1} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2j+1}}} \left[w(0, X_{\tau_{2j+1}}) - w(1, X_{\tau_{2j+1}})\right] \mathbf{1}_{\{\tau_{2j+1}<\infty\}}\right] \\
&\quad + \sum_{j=1}^{n-1} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2j}}} \left[w(1, X_{\tau_{2j}}) - w(0, X_{\tau_{2j}})\right] \mathbf{1}_{\{\tau_{2j}<\infty\}}\right] \\
&\quad + \sum_{j=1}^{n-1} \mathbb{E}_x\left[\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\Lambda_t} \mathcal{L}w(1, X_t) dt\right] + \sum_{j=0}^{n-1} \mathbb{E}_x\left[\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\Lambda_t} \mathcal{L}w(0, X_t) dt\right]
\end{aligned}$$

Adding the term

$$\begin{aligned}
&\sum_{j=1}^{2n-1} \mathbb{E}_x\left[e^{-\Lambda_{\tau_j}} [H_s(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_b(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}] \mathbf{1}_{\{\tau_j<\infty\}}\right] \\
&\quad \equiv \sum_{j=0}^{n-1} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2j+1}}} H_s(X_{\tau_{2j+1}}) \mathbf{1}_{\{\tau_{2j+1}<\infty\}}\right] - \sum_{j=1}^{n-1} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2j}}} H_b(X_{\tau_{2j}}) \mathbf{1}_{\{\tau_{2j}<\infty\}}\right]
\end{aligned}$$

on both sides of this identity, we obtain

$$\sum_{j=1}^{2n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_j}} [H_s(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_b(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}] \mathbf{1}_{\{\tau_j<\infty\}} \right]$$
$$= w(1,x) - \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}}} w(0, X_{\tau_{2n}}) \mathbf{1}_{\{\tau_{2n}<\infty\}} \right]$$
$$+ \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\Lambda_t} \mathcal{L}w(1, X_t) dt \right] + \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\Lambda_t} \mathcal{L}w(0, X_t) dt \right]$$
$$+ \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2j+1}}} \left[w(0, X_{\tau_{2j+1}}) - w(1, X_{\tau_{2j+1}}) + H_s(X_{\tau_{2j+1}}) \right] \mathbf{1}_{\{\tau_{2j+1}<\infty\}} \right]$$
$$+ \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2j}}} \left[w(1, X_{\tau_{2j}}) - w(0, X_{\tau_{2j}}) - H_b(X_{\tau_{2j}}) \right] \mathbf{1}_{\{\tau_{2j}<\infty\}} \right].$$

This calculation and the fact that w satisfies the HJB equation (4.27), imply that

$$\mathbb{E}_x \left[\sum_{j=1}^{2n-1} e^{-\Lambda_{\tau_j}} \left[(X_{\tau_j} - c_s) \mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - (X_{\tau_j} + c_b) \mathbf{1}_{\{\Delta Y_{T_j}=1\}} \right] \right]$$
$$\leq w(1,x) - \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}}} w(0, X_{\tau_{2n}}) \mathbf{1}_{\{\tau_{2n}<\infty\}} \right]. \tag{4.49}$$

In view of (4.48), the fact that

$$\lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}}} w(0, X_{\tau_{2n}}) \mathbf{1}_{\{\tau_{2n}<\infty\}} \right] = 0,$$

which follows from (4.18) and (4.45), we can pass to the limit $n \to \infty$ in (4.49) to obtain the inequality $J_{1,x}(Y) \leq w(1,x)$, which implies that $v(1,x) \leq w(1,x)$.

The nature of the strategy Y^* given by (4.43) or (4.44), depending on the case, is such that (4.49) hold with equality. By passing to the limit $n \to \infty$ as above, we therefore obtain $J_{1,x}(Y) = w(1,x)$, which establishes the inequality $v(1,x) \geq w(1,x)$.

To complete the proof, we still have to show that the process Y^* given by (4.44) is a finite variation process. In particular, we have to show that $\mathbb{P}_x\left(\lim_{n\to\infty} \tau_n^* < \infty\right) = 0$. To this end, we use the definition (4.3) of the discounting factor Λ, the strong Markov property of the process X and (4.10) we obtain

$$\mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n+1}^*}} \right] = \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}^*}} \mathbb{E}_x \left[\exp\left(- \int_0^{\tau_{2n+1}^* - \tau_{2n}^*} r(X_{\tau_{2n}^*} + s)\, ds \right) \mid \mathcal{F}_{\tau_{2n}^*} \right] \right]$$
$$= \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}^*}} \mathbb{E}_\gamma \left[\exp\left(- \int_0^{\tau_\beta} r(X_s)\, ds \right) \right] \right]$$
$$= \frac{\phi(\gamma)}{\phi(\beta)} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{2n}^*}} \right].$$

Similarly, we can see that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n}}}\right] = \frac{\psi(\beta)}{\psi(\gamma)}\mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n-1}}}\right].$$

These calculations and the dominated convergence theorem imply that

$$\mathbb{E}_x\left[\lim_{n\to\infty} e^{-\Lambda_{\tau^*_{2n+1}}}\right] = \lim_{n\to\infty}\mathbb{E}_x\left[e^{-\Lambda_{\tau^*_1}}\right]\left(\frac{\phi(\gamma)\psi(\beta)}{\phi(\beta)\psi(\gamma)}\right)^n = 0,$$

the second equality following from the facts that ϕ (resp., ψ) is strictly decreasing (resp., increasing) and $\beta < \gamma$. This conclusion contradicts the possibility that $\mathbb{P}_x\left(\lim_{n\to\infty}\tau^*_n < \infty\right) > 0$, and the proof is complete. □

4.4 Examples

4.4.1 The underlying is a geometric Brownian motion

Suppose that X is a geometric Brownian motion, so that

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t,$$

for some constants b and $\sigma > 0$, and that $r(x) = r_0$ for all $x > 0$, for some constant $r_0 > b$. In this case, Assumptions 9–11 are all satisfied,

$$\phi(x) = x^m \quad \text{and} \quad \psi(x) = x^n,$$

where the constants $m < 0 < n$ are given by

$$m, n = \frac{-\left(b - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(b - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

In view of the equivalence $r_0 > b \Leftrightarrow n > 1$ (if $r_0 < b \Leftrightarrow n < 1$, then (4.21) will not satisfied), we can see that (4.21) in Assumption 4.22 is satisfied. Also, it is straightforward to check that the inequalities in (4.23) of Assumption 4.22 also hold true for

$$x_b = 0 \quad \text{and} \quad x_s = \frac{rc_s}{r-b}.$$

4.4.2 The underlying is a mean-reverting CEV process

Suppose that X is the mean-reverting CEV process given by

$$dX_t = k(\theta - X_t)dt + \sigma X_t^l dW_t.$$

for some constants $k, \theta, \sigma > 0$ and $l \in [\frac{1}{2}, 1[$ such that $2k\theta > \sigma^2$ if $l = \frac{1}{2}$, and that $r(x) = r_0$ for all $x > 0$, for some constant $r_0 > b$. Assumptions 9–11 are all satisfied, while, the mean-reverting nature of X implies that (4.21) in Assumption 4.22 also holds true. Furthermore, it is straightforward to check that the inequalities in (4.23) of Assumption 4.22 are satisfied for

$$x_b = 0 \vee \frac{k\theta - rc_b}{k + r} \quad \text{and} \quad x_s = \frac{k\theta + rc_s}{k + r}.$$

4.5 Appendix: proof of Lemmas 10 and 11

Proof of Lemma 10. In view of the strict positivity of the function Ψ and (4.23) in Assumption 12, we can see that

$$q'(x) = \Psi(x)\mathcal{L}H_s(x) = \begin{cases} > 0, & \text{if} \quad x < x_s, \\ < 0, & \text{if} \quad x > x_s, \end{cases} \tag{4.50}$$

This observation and the fact that $q(0) = 0$ imply that there exists a unique $\alpha > 0$ satisfying equation (4.30) if and only if $\lim_{x\to\infty} q(x) < 0$. To see that this inequality indeed holds, we note that the definition of H_s and (4.21) in Assumption 12 imply that

$$\frac{H_s(x)}{\psi(x)} \equiv \frac{x - c_s}{\psi(x)} > 0 \quad \text{for all} \quad x > c_s \quad \text{and} \quad \lim_{x\to\infty} \frac{H_s(x)}{\psi(x)} = 0,$$

while (4.20) and the definition (4.31) of q imply that

$$\left(\frac{H_s}{\psi}\right)'(x) = \frac{Cp'(x)}{\psi^2(x)} \int_0^x \Psi(s)\mathcal{L}H_s(s)ds = \frac{Cp'(x)}{\psi^2(x)} q(x). \tag{4.51}$$

Since q is strictly decreasing in $]x_s, \infty[$, we can see that these facts can all be true only if $\lim_{x\to\infty} q(x) < 0$, as required. For future reference, we also note that this conclusion and (4.50) imply that the unique solution $\alpha > 0$ of the equation $q(\alpha) = 0$ is such that

$$x_s < \alpha \quad \text{and} \quad q(x) = \int_0^x \Psi(s)\mathcal{L}H_s(s)ds > 0 \quad \text{for all} \quad x < \alpha. \tag{4.52}$$

By construction, we will show that the function w defined by (4.28) satisfies the HJB equation (4.27) if we show that

$$w(0, x) - w(1, x) + H_s(x) = -A\psi(x) + H_s(x) \leq 0 \quad \text{for all} \quad x < \alpha, \tag{4.53}$$

$$\mathcal{L}w(1, x) = \mathcal{L}H_s(x) \leq 0 \quad \text{for all} \quad x > \alpha, \tag{4.54}$$

$$w(1, x) - w(0, x) - H_b(x) = A\psi(x) - H_b(x) \leq 0 \quad \text{for all} \quad x < \alpha. \tag{4.55}$$

and

$$w(1,x) - w(0,x) - H_b(x) = x - c_s - (x + c_b) \leq 0 \quad \text{for all} \quad x > \alpha. \tag{4.56}$$

The last of these inequalities is plainly true, while (4.54) follows immediately from the first inequality in (4.52) and assumption (4.23). In view of (4.29), we can see that (4.53) is equivalent to

$$\frac{H_s(x)}{\psi(x)} \equiv \frac{x - c_s}{\psi(x)} \leq \frac{\alpha - c_s}{\psi(\alpha)} \equiv \frac{H_s(\alpha)}{\psi(\alpha)} \quad \text{for all} \quad x < \alpha.$$

which is indeed true, thanks to (4.51) and (4.52). Similarly, we can verify that (4.55) is equivalent to

$$\frac{H_b(x)}{\psi(x)} \equiv \frac{x + c_b}{\psi(x)} \geq \frac{\alpha - c_s}{\psi(\alpha)} \equiv \frac{H_s(\alpha)}{\psi(\alpha)} \quad \text{for all} \quad x < \alpha. \tag{4.57}$$

In view of the calculation

$$\left(\frac{H_b}{\psi}\right)'(x) = \frac{Cp'(x)}{\psi^2(x)} \int_0^x \Psi(s)\mathcal{L}H_b(s)ds = \frac{Cp'(x)}{\psi^2(x)} \left[q(x) - (c_b + c_s)\int_0^x \Psi(s)r(s)ds\right],$$

which follows from (4.20) and the definition of q, we can see that, if $x_b = 0$, then the function H_b/ψ is strictly decreasing and (4.57) is true, while, if $x_b > 0$, then there exists $\widetilde{x} \in]x_b, \alpha[$ such that the function H_b/ψ is strictly increasing in $]0, \widetilde{x}[$ and strictly decreasing in $]\widetilde{x}, \alpha]$, in which case (4.57) is true if and only if the inequality (4.32) is true. □

Proof of Lemma 11. We first note that the definition (4.11) of the operator L, the definition (4.22) of the functions H_b and H_s, (4.23) in Assumption 12, and the definition (4.40) of the function q_ϕ imply that

$$q_\phi(x,z) = \int_x^z \Phi(s)\mathcal{L}H_b(s)ds - (c_b + c_s)\int_z^\infty \Phi(s)r(s)ds < 0 \quad \text{for all} \quad x_b \leq x < z. \tag{4.58}$$

This observation implies that the system of equations (4.39) has no solution $0 < \beta < \gamma$ if $x_b = 0$. Furthermore, if the system of equations (4.39) has a solution $0 < \beta < \gamma$, then $\beta \in]0, x_b[$. We therefore assume that the problem data is such that $x_b > 0$ in what follows, and we look for a solution $0 < \beta < \gamma$ of (4.39) such that $\beta \in]0, x_b[$.

In view of the equation (4.30) that $\alpha > 0$ satisfies and the second inequality in (4.52), we can see that

$$\lim_{z \to \infty} q_\psi(x,z) = q(x) - \int_\alpha^\infty \Psi(s)\mathcal{L}H_s(s)ds > 0 \quad \text{for all} \quad x \leq x_b.$$

Also, the strict positivity of the function Ψ and (4.23) in Assumption 12 imply that

$$\frac{\partial q_\psi}{\partial z}(x,z) = -\Psi(z)\mathcal{L}H_s(z) \begin{cases} < 0, & \text{if} \quad z < x_s, \\ > 0, & \text{if} \quad z > x_s, \end{cases}$$

Combining these observations with the fact that

$$q_{\psi}(x,x) = \int_0^x \Psi(s)[\mathcal{L}H_b(s) - \mathcal{L}H_s(s)]ds = -(c_b + c_b)\int_0^x \Psi(s)r(s)ds < 0,$$

we can see that there exists a unique function $L :]0, x_b[\to \mathbb{R}_+$ such that

$$x_s < L(x) \quad \text{and} \quad q_{\psi}(x, L(x)) = 0. \tag{4.59}$$

Furthermore, we can differentiate the identity $q_{\psi}(x, L(x)) = 0$ with respect to x to obtain

$$L'(x) = \frac{\Psi(x)\mathcal{L}H_b(x)}{\Psi(L(x))\mathcal{L}H_s(L(x))} > 0 \quad \text{for all} \quad x < x_b, \tag{4.60}$$

and combine the equation (4.30), which $\alpha > 0$ satisfies, with the definition of q_{ψ} to see that

$$\lim_{x\downarrow 0} L(x) = \alpha. \tag{4.61}$$

In view of (4.7)–(4.8), (4.23) and (4.60), we can see that

$$\begin{aligned}\frac{d}{dx}q_{\phi}(x, L(x)) &= -\frac{\Phi(x)\Psi(L(x)) - \Phi(L(x))\Psi(x)}{\Psi(L(x))}\mathcal{L}H_b(x)\\ &= -\frac{\phi(x)\psi(L(x)) - \phi(L(x))\psi(x)}{C\sigma^2(x)p'(x)\psi(L(x))}\mathcal{L}H_b(x).\\ &< 0 \quad \text{for all} \quad x < x_b.\end{aligned}$$

This inequality and the observation that $\lim_{x\uparrow x_b} q_{\phi}(x, L(x)) < 0$, which follows from (4.58), (4.59) and the assumption that $x_b < x_s$, imply that there exists a unique point $\beta > 0$ such that $q_{\phi}(\beta, L(\beta)) = 0$ if and only if

$$\lim_{x\downarrow 0} q_{\phi}(\beta, L(\beta)) > 0.$$

We conclude this part of the analysis by observing that the unique solution of the system of equations (4.39) satisfies

$$0 < \beta < x_b \quad \text{and} \quad \gamma = L(\beta) > x_s. \tag{4.62}$$

In particular, we note that

By construction, we will show that the function w defined by (4.33)–(4.34) satisfies the HJB equation (4.27) if we show that

$$g_s(x) := w(0,x) - w(1,x) + H_s(x) \le 0 \qquad \text{for all} \quad x < \gamma, \tag{4.63}$$

$$\mathcal{L}w(1,x) = \quad \mathcal{L}H_s(x) \le 0 \quad \text{for all} \quad x > \gamma, \tag{4.64}$$

$$\mathcal{L}w(0,x) = -\mathcal{L}H_b(x) \le 0 \quad \text{for all} \quad x < \beta, \tag{4.65}$$

and

$$g_b(x) := w(1,x) - w(0,x) - H_b(x) \leq 0 \quad \text{for all} \quad x > \beta. \tag{4.66}$$

The inequalities (4.64) and (4.65) follow immediately from (4.23) in Assumption 12 and (4.62). Also, (4.63) for $x \leq \beta$, as well as (4.66) for $x > \gamma$, is equivalent to $-(c_b + c_s) \leq 0$, which is true by assumption. Finally, to establish (4.63) and (4.66) for $x \in [\beta, \gamma]$. We use (4.15) and (4.37)–(4.38) to calculate

$$\begin{aligned} g_s(x) =& - A\psi(x) + B\phi(x) + H_s(x) \\ =& - \psi(x) \int_x^\gamma \Phi(s)\mathcal{L}H_s(s)ds + \phi(x) \int_x^\gamma \Psi(s)\mathcal{L}H_s(s)ds, \\ g_b(x) =& A\psi(x) - B\phi(x) - H_b(x) \\ =& - \psi(x) \int_\beta^x \Phi(s)\mathcal{L}H_b(s)ds + \phi(x) \int_\beta^x \Psi(s)\mathcal{L}H_b(s)ds, \end{aligned}$$

Furthermore, we use the definition (4.16) of the function Φ, Ψ to obtain

$$g_s'(x) = - \psi'(x) \int_x^\gamma \Phi(s)\mathcal{L}H_s(s)ds + \phi'(x) \int_x^\gamma \Psi(s)\mathcal{L}H_s(s)ds, \tag{4.67}$$

$$g_b'(x) = - \psi'(x) \int_\beta^x \Phi(s)\mathcal{L}H_b(s)ds + \phi'(x) \int_\beta^x \Psi(s)\mathcal{L}H_b(s)ds. \tag{4.68}$$

In view of (4.23) in Assumption 12 and the inequalities $\phi' < 0 < \psi'$, we can see that these identities imply that

$$g_s'(x) > 0 \quad \text{for all} \quad x \in [x_s, \gamma[\quad \text{and} \quad g_b'(x) < 0, \quad \text{for all} \quad x \in]\beta, x_b[.$$

These inequalities and the fact that $g_s(\gamma) = g_b(\beta) = 0$, which follows from (4.35), imply that

$$g_s(x) < 0 \quad \text{for all} \quad x \in [x_s, \gamma[\quad \text{and} \quad g_b(x) < 0 \quad \text{for all} \quad x \in]\beta, x_b[$$

Combining these inequalities with the identities

$$g_s(\beta) = H_s(\beta) - H_b(\beta) = -(c_s + c_b) \quad \text{and} \quad g_b(\gamma) = H_s(\gamma) - H_b(\gamma) = -(c_s + c_b),$$

which follow from (4.35), we can see that $g_s(x)$; $g_b(x) \leq 0$ for all x $\in [\beta, \gamma]$, as required, provided that g_s (resp., g_b) does not have a strictly positive local maximum in $]\beta, x_s[(resp.,]x_b, \gamma[)$. We can see that this is indeed the case by noting that

$$\mathcal{L}g_s(x) = \mathcal{L}H_s(x) > 0 \quad \text{for all} \quad x \in]\beta, x_s], \quad \mathcal{L}g_b(x) = -\mathcal{L}H_b(x) > 0 \quad \text{for all} \quad x \in [x_b, \gamma[,$$

and then appealing to the maximum principle. □

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