Consumption and Saving Behaviour under Uncertainty with Unorthodox Preferences

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly with me is clearly identified in it). The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgement is made. This thesis may not be reproduced without the prior written consent of the author. I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I certify that Chapter 3 of my thesis, "Consumption and Portfolio Rules with Stochastic Quasi-Hyperbolic Discounting", was co-authored with Ignacio Palacios-Huerta. I, Alonso Perez-Kakabadse, contributed in excess of 50 percent to the genesis of the project, the work on the model, and the writing of the text.

H. Alonso Pérez-Kakabadse

Abstract

This thesis consists of three theoretical essays on the consumption and saving behavior of agents with unorthodox preference specifications in an uncertain environment. The first paper puts forward a model in which agents have heterogeneous priors with regard to their assessment of the underlying systemic risk. It considers the particular case of domestic agents being more pessimistic than financial markets. The second paper studies the effects of dynamic inconsistency in the consumption and saving decisions under systemic risk, assuming naïve hyperbolic agents. The third paper investigates the joint consumption-savings and portfolio-selection problem under capital risk, assuming sophisticated hyperbolic discounting agents.

Chapter 1 introduces an economy exposed to external stochastic shocks capable of triggering a crisis. We show that under the assumption of heterogeneity of beliefs, and in particular of pessimistic domestic consumers, it is possible to explain demand booms that arise on the back of policy responses even when the latter were not wealth improving. Quite the opposite, such an expenditure boom could be sparked in conjunction with a cycle of persistent current account deficits and debt accumulating dynamics that would result in higher future risk of collapse.

Chapter 2 considers a setting in which time inconsistent agents discount utility flows with a hyperbolic function instead of a classic, exponential one. This feature effectively characterizes a consumer that is "present biased" or short-term impatient. The agent is assumed to be naïve, in the sense that she does not internalize her time inconsistency problem. As opposed to the orthodox, exponential discounting model, our model is able to generate a negative relationship between the saving rate (or the current account) and the underlying risk premium.

Chapter 3 solves the classic Merton (1969, 1971) problem of optimal consumption-saving and portfolio-selection in continuous time, assuming sophisticated but time-inconsistent agents with hyperbolic preferences as specified in Harris and Laibson (2008). We find closed-form solutions for the optimal consumption and portfolio allocation rules. The portfolio rule remains identical to the time-consistent solution with power utility with no borrowing constraints. However, the marginal propensity to consume out of wealth is unambiguously greater than the time-consistent, exponential case.

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Chapter 1

A Model of Self-fulfilling Scepticism about Stabilization Policy

1.1 Introduction

When stabilization plans have been implemented in highly indebted economies in the hope that situations of severe economic and financial distress could be redressed, a brand new regime has typically been introduced, often involving the exchange-rate system, to provide the country with sounder economic fundamentals.

By and large, the incumbents tend to claim victory when, following the implementation of a new policy set, the economy enjoys a boom in demand. They seem to believe that a consumption boom is almost irrefutable evidence of policy success.

As a matter of fact, conventional wisdom would have anticipated that sounder economic fundamentals should bring about more sustainable macro dynamics, a lower sovereign risk premium, renewed access to capital markets and, in general, a positive wealth effect.

However, in a number of instances, the initial recovery has proved to be only temporary and has been followed by inflationary cycles, over-appreciation, debt accumulation and, ultimately, by economic and financial collapse.

In this paper, we claim that such an optimistic view may be precipitate and flawed by shortsightedness. Although, admittedly, the emergence of expansionary dynamics could result from a wealth effect ensuing an improvement in fundamental systemic risk, we argue that the observation of a boom could also be the result of an intertemporal tilt of consumption stemming from heterogeneity of perceptions of risk between domestic consumers and international capital markets. Specifically, a *negative tilt* effect implying a short-term consumption surge would be consistent with local agents being more *pessimistic* than capital markets with regard to the sustainability of the new regime, i.e. if the former attached a greater probability to a collapse event than the latter.

In particular, if after a new regime is introduced local consumers anticipate a greater probability of a crisis than capital markets do, the model would predict a consumption surge and the appearance of current account deficits. The intuition is that consumers' scepticism about the sustainability of a new regime makes borrowing rates look relatively inexpensive, which induces local agents to build up debt in order to finance higher current expenditure at the expense of future consumption.

In sum, this paper argues that an initial surge in demand constitutes no proof of the soundness of a recently inaugurated regime. An economic boost could also ensue from the domestic consumers' scepticism–relative to the capital markets' own assessment–about the sustainability of the new regime notwithstanding the direction of the wealth effect. Under such an heterogeneity of beliefs, an initial apparent short-run recovery could instead be the prelude of perverse macro dynamics, persistent debt accumulation and higher future risk of collapse.

As an illustration, recall Argentina's experience with a currency-board. The regime was installed in 1991 under the auspices of the IMF and with the consent of the international financial community. It was nonetheless abandoned in 2002 in the midst of painful economic and financial turmoil. This instance illustrates the pattern: a boost in consumption right after the adoption of the new regime was followed by eight years of persistent current account deficits together with relentless inflation and loss of competitiveness.

Arguably, the sustainability of the regime was put in jeopardy by high levels of external debt and the undermined competitiveness of the real economy, which made the financial system vulnerable to external shocks. In effect, the deteriorating fundamentals cast doubt on the sustainability of the regime and ultimately triggered an attack on the currency that brought about the abandonment of the system in January 2002 along with an announcement of sovereign debt default.

The idea that demand dynamics are affected by agents' perceptions beyond the fundamental vulnerabilities of an economic system is certainly not new. From the seminal work of Calvo (1986) and Drazen and Helpman (1987), the literature has given particular attention to the implications of imperfect credibility on the part of domestic agents with regard to the durability of newly implemented policies or regimes. Further extensions and applications include Calvo and Vegh (1994), Velasco (1996), Calvo and Drazen (1998), and Velasco and Neut (2003), among others.

The literature has endeavored to explain temporary consumption booms following the introduction of a new regime based on a limited-policy-durability problem. The models have generally assumed incomplete markets and perfect-foresight, in the sense that the time of the policy reversal is deterministic and common knowledge. The logic is that the anticipation of a collapse of a newly implemented regime implies that agents expect that inflation (or another form of tax) will be higher in the future. Therefore, the opportunity cost of present consumption is relatively lower, which induces higher current levels.

Drazen and Helpman (1988, 1990) and Calvo and Drazen (1998) introduced a more realistic setup. Rather than assuming that agents have a defined belief over the precise time of collapse, as is commonplace in the literature, they allow local agents to attach certain positive hazard rate to the regime-switching event. However, these models were also deterministic in the sense that agents are assumed to know with certainty that a regime switch is due to occur on or before a predetermined maximum time $T_{\max}$ at which the collapse would have happened with probability one.

In these latter models, a temporary consumption surge after a new regime is implemented results from a set of policies that are known to yield an increase in income. The initial consumption boom is followed by current account *surpluses*. In turn, the wealth effect coming from wealth accumulation results in increasing consumption. When a collapse arrives, consumption declines to the lower level that corresponds to an inferior regime.

Importantly, however, without the $T_{\max}$-assumption, these models could generate no consumption boom or current account imbalances.

In contrast, our model does not assume a $T_{\max}$ and instead allows for the new regime to be able to survive an undetermined length of time. It tackles the problem from a purely stochastic approach: a collapse event, associated with the occurrence of a generalized financial crisis and sovereign debt default, could happen at any point in time with positive probability given by a hazard rate (conditional on it not having already happened). We allow the hazard rate to be a function of the current state of the economy and, in particular, of time-varying indices that reflect the vulnerabilities of the economy, such as the net debt position. The latter captures the notion that a highly leveraged entity is more vulnerable to exogenous shocks.

Importantly, contrary to Drazen and Helpman (1988, 1990) and particularly to Calvo and Drazen (1998) (in which the hazard rate increases exogenously with no apparent reason despite the fact that the country runs current account *surpluses)*, in our model a boost in consumption is followed by current account *deficits* and a gradual deterioration of the net foreign asset position. Accordingly, the sovereign risk premium increases and converges to the subjective risk level perceived by domestic consumers as a matter of an asymptotic self-fulfilling prophecy.

To abstract from dynamic considerations that are not pertinent to the discussion in this paper, we assume that output remains constant at all times, so that the smoothing channel to explain current account imbalances has been deliberately closed.

The remainder of the Chapter is structured in four additional sections. Section 1.2 introduces the model. Section 1.3 solves a saddle path stable system for the general case of a CRRA - CES utility function. Section 1.4 experiments with regime shocks and identifies the bearings of the wealth effect and intertemporal substitution effect on the consumption and debt dynamics. Section 1.5 concludes, underscores a number of policy issues and proposes possible directions for further research.

1.2 The Model

1.2.1 Probability of Collapse

In the model, a collapse can be thought of as an event potentially triggered by a generalized banking crisis that would subsequently result in liquidity constraints, external debt default, exclusion from access to international capital markets and a situation of virtual autarky.

The timing of this occurrence is uncertain. For a given regime i, the probability that an event of collapse arrives in the next time interval dt –conditional on it not having already happened– is determined by the hazard rate

$$\phi_t^i = \phi(\theta^i, b_t), \qquad \text{with } \partial\phi/\partial b_t \leq 0 \tag{1.1}$$

where θ^i is a set of fixed economic fundamentals inherent in the regime i and b_t is the state of a set of time varying macroeconomic indicators. Both variables determine a country's economic performance and capacity to pay. In particular, we let b_t denote the country's aggregate *net foreign asset position* at any given time t. Hence, risk ϕ_t^i increases as the foreign asset position deteriorates, i.e. as external debt accumulates.

We assume that capital markets are neutral and that the residual value of a defaulted bond after collapse is zero. In consequence, the premium above the risk-free rate requested on any new loans to the country at hand is equal to sovereign risk of collapse, so that the forward interest rates schedule would be given by $\{r + \phi_t^i\}$, where r is the risk free rate.[1] We assume that contracts are enforceable and that there are no other sources of risk, such as income shocks or counterparty risk.

1.2.2 Heterogeneity of beliefs

We assume that agents are characterized by heterogeneity of beliefs. In particular, domestic consumers and capital markets differ in their perception of risk; the former are assumed to be relatively more pessimistic and attach a higher probability to a collapse than the latter.[2]

This assumption is central in our model and can be justified as the reduced form to a problem of asymmetry of information that could arise due to the costs involved in accessing and interpreting market data by domestic agents. In particular, domestic consumers may be unable to accurately assess the country risk since monitoring the day-to-day macro indicators would involve informational barriers or access costs; therefore, they would have to guesstimate the current probability of collapse. Calvo and Mendoza (2000) have neatly described this situation:

> *"Trading emerging-markets securities requires the collection of detailed information about the countries involved. This information is costly and*

[1] See the Appendix 2.C.10 for a proof that the hazard rate ϕ_t^i that determines the default probability is in effect the risk premium over the risk free interest rate r when the zero recovery value is assumed to be zero.

[2] Our assumption is similar to Fostel and Geanakoplos (2009), who assume heterogeneity of priors about the probability of default between a small group of optimists and the general public, the pessimists. Their model provides an explanation for the volatile access of emerging economies to international financial markets.

"depreciates" quickly. Moreover, fixed information costs are large, because assessing country risk requires gathering and processing information about all key macroeconomic and political variables on a recurrent basis, independently of investment size."

On the other hand, capital markets' institutions are able to closely monitor the evolution of all relevant economic statistics because they are in a much better position to afford the fixed information costs. Accordingly, we assume that capital markets can monitor and interpret the economy's developments permanently and accurately.

Additionally, we assume market incompleteness, so that consumers are allowed to disagree with the market's risk estimate, i.e. consumers do not deduce the probability of collapse from the forward rate curve priced by the financial markets.

In the context of our model, it could be argued that fundamentals θ^i, distinctive of regime i, are public information for all agents, whereas the state of the time varying indicators (in this case the aggregate net foreign asset position b_t), is private information to capital markets and is therefore not internalized by the domestic consumers' risk assessment.[3]

Formally, under given fundamentals θ^i the representative domestic consumer assigns a Poisson process distribution to a fatal financial crisis event and attaches probability $\kappa^i = \kappa(\theta^i)$, with $\partial\kappa/\partial b\,(t) = 0$, to the occurrence that a crisis occurs during the immediate one time period, conditional on it not having occurred before. Accordingly, the random variable T (or time until collapse) is exponentially distributed with density function $f_T(t) = \kappa^i \exp(-\kappa^i t)$, and the expected time until collapse at any point in time during a given exchange rate regime will be $E(T) = (\kappa^i)^{-1}$.

The probability of experiencing a crisis at some point during the interval from

[3]Although this is not a necessary assumption, it simplifies the analysis as it will allow the sovereign risk premium to converge to the constant subjective risk level perceived by the local agents in a given regime i. Alternatively, an assumption that κ varies with $b\,(t)$ could also lead to a steady state equilibrium so long $|\partial\phi/\partial b\,(t)| > |\partial\kappa/\partial b\,(t)|$.

present to t periods ahead is given by the cumulative distribution $F_T(t) = 1 - \exp(-\kappa^i t)$, for all $t \geq 0$. Conversely, the probability of survival or of *not* experiencing a crisis during the time interval from present to t periods ahead is given by $1 - F_T(t) = \exp(-\kappa^i t)$.

Technically, for the purpose of obtaining the qualitative results in our model, we do not need to specify whose risk assessment is *correct*. But for motivational purposes, we shall assume that markets have an accurate, correct assessment of the country's risk while domestic consumers would need to guesstimate it and are inherently more *pessimistic* i.e. they assign a greater probability to a regime collapse event.

In other words, domestic consumers are assumed to systematically overestimate the risk of collapse –at least with respect to the capital markets' inference. This situation may arise from the fact that domestic agents have *sceptical priors* with regard to the benefits of a newly implemented set of policies sponsored by their incumbent, and since they have to rely on softer information than capital markets do, their guesstimate is biased towards a more pessimistic assessment.

1.2.3 Consumer's Maximization Problem

A country is assumed to have full access to capital markets at international risk-free interest rates r plus the sovereign risk premium ϕ_t^i in a given regime i. However, we assume that upon arrival of a financial crisis an indebted economy is forced to default on its external debt and is excluded from accessing international capital markets, which would bring it to a situation of virtual autarky; i.e. post-collapse consumption of tradables and non-tradables is bound by the corresponding domestic production levels, forever, after the collapse event.

The representative consumer maximizes the expected discounted utility drawn from her consumption flow. The objective function at any time z is

$$E_z \left\{ \int_z^{\infty} U\left(\tilde{c}_t^T, \tilde{c}_t^N\right) e^{-\rho(t-z)} dt \right\}, \tag{1.2}$$

where E_z stands for the unconditional expectation operator and $\tilde{c}_t^T$ and $\tilde{c}_t^N$ are random variables that stand for consumption levels in tradables and non-tradables respectively. They can take values c_t^T and c_t^N if a collapse has not occurred, but are constrained to autarkic production levels y^T and y^N if an event of collapse had arisen. Specifically, at a given time z, the distributions of both variables are

$$\tilde{c}_t^T, \tilde{c}_t^N = \begin{cases} c_t^T, c_t^N & \text{wp} \quad e^{-\kappa^i(t-z)} \\ y^T, y^N & \text{wp} \quad 1 - e^{-\kappa^i(t-z)} \end{cases} \tag{1.3}$$

where y^N and y^T are constant output values.

Using (1.3), the objective function (1.2) becomes:

$$\begin{aligned} & \int_z^{\infty} E_z \left\{ U\left(\tilde{c}_t^T, \tilde{c}_t^N\right) \right\} e^{-\rho(t-z)} dt \\ = & \int_z^{\infty} \left(U\left(c_t^T, c_t^N\right) e^{-\kappa^i(t-z)} + U\left(y^T, y^N\right) \left(1 - e^{-\kappa^i(t-z)}\right) \right) e^{-\rho(t-z)} dt \\ = & \int_z^{\infty} U\left(c_t^T, c_t^N\right) e^{-(\rho+\kappa^i)(t-z)} dt + U\left(y^T, y^N\right) \int_z^{\infty} \left(1 - e^{-\kappa^i(t-z)}\right) e^{-\rho(t-z)} dt \end{aligned}$$

We note that the value shown in the second term of the last expression is a constant. Therefore, the consumer's optimal behavior is tantamount to maximizing the first term

$$\int_z^{\infty} U\left(c_t^T, c_t^N\right) \; e^{-(\rho+\kappa^i)(t-z)} dt \tag{1.4}$$

subject to the intertemporal budget constraint (IBC).

In other words, the consumer's problem has reduced to maximizing the flows of consumption ad-infinitum as if a crisis was never due to occur, similar to a problem in a non-stochastic environment. However, we note that in our stochastic problem the discount rate has incorporated the perceived risk of collapse. In fact, the consumer *behaves as if prevailing regime i were to remain in status-quo forever (immune to crises), but with the subjective discount rate being augmented by risk factor κ^i.* Blanchard (1985) refers to this occurrence in the context of "lifetime utility maximization with constant probability of death."

1.2.4 Intertemporal Budget Constraint

Much of the literature related to demand booms after the introduction of new policies or regimes has focused on the dynamics of *nominal* variables and relies on the assumption of cash-in-advance constraints or other sources of demand for money.[4] Instead, our model focuses on the dynamics of real variables and dispenses with real balances.

A country's net financial asset position could take the form of foreign bond holdings or foreign currency denominated external debt. Thus, the increase in a country's net financial asset position in terms of tradables is determined by the current account:

$$\dot{b}_t = (r + \phi_t^i)b_t + y_t^T + \frac{y_t^N}{e_t} - c_t^T - \frac{c_t^N}{e_t}, \tag{1.5}$$

where b_t is the *aggregate* net foreign asset position at time t (i.e. $b_t < 0$ means *net debt*) and e_t is the real exchange rate defined as the price of tradables in terms of non-tradables,

$$e_t = \frac{P_t^T}{P_t^N} = \frac{S_t P_t^{T*}}{P_t^N}$$

[4]Important exceptions are the works of Calvo and Drazen (1998), Velasco (1996) and Velasco and Neut (2003).

where S_t is the nominal exchange rate and P^{T*} is the price of tradables in foreign currency. Prices of non-tradables P_t^N are assumed to be fully flexible. The dollar-denominated nominal interest rate charged by capital markets is $r + \phi_t^i$, $\forall t$, where the risk-free interest rate r is assumed to be constant, and ϕ_t^i is the premium associated with the sovereign risk of default under regime i.

Intertemporal consumption is bounded by a resource constraint that presupposes access to capital markets as long as a collapse event has not arrived. Integration of the current account (1.5) with respect to time under the transversality condition $\lim_{t \longrightarrow \infty} b_t e^{-\int_t^s (r+\phi(\tau))d\tau} = 0$, delivers the intertemporal budget constraint (IBC):

$$\int_z^\infty \left(c_t^T + \frac{c_t^N}{e_t}\right) e^{-\int_z^t (r+\phi(\tau))d\tau} dt = b_z + \int_z^\infty \left(y_t^T + \frac{y_t^N}{e_t}\right) e^{-\int_z^t (r+\phi(\tau))d\tau} dt \qquad (1.6)$$

This equation indicates that the market value of intertemporal consumption in the LHS is constrained by total wealth W_t in the RHS. Wealth at any point in time t would be defined by the net asset position b_t plus the *market's valuation* of the future income stream discounted at the rates at which capital markets are willing to lend, i.e. $\{r + \phi_t^i\}$ for all $t \geq z$. [5]

1.2.5 Characterization of the Problem

Each consumer maximizes the objective function (1.4) subject to her own individual intertemporal budget constraint (1.5). The current value Hamiltonian for the represen-

[5]Note that neither government transfers nor revenues from seigniorage appear in the IBC as we have assumed absence of monetary policy. Note further that the consumers' resource constraint and the IBC are identical. However, this needs not to be the case in general. For instance, other monetary approaches of the problem, such as the model of temporariness of credibility in Calvo and Vegh (1993), introduce a cash-in-advance framework where authorites use the revenues from the inflation tax for undertaking wealth transfers to consumers. Atomistic consumers choose optimal consumption in consideration of the opportunity cost implied by inflation when holding real balances. However, consumers take wealth transfers as parametric. This setting results in the *individual's* IBC separating the parametric transfers from the endogenous cost of holding money for consumption. On the other hand, at the level of the overall economy the revenues and transfers from the inflation-tax unequivocally cancel out and the *aggregate* IBC (or Resource Constraint) looks exactly like (1.6).

tative consumer would be

$$\underset{c_t^T, c_t^N, \lambda_t, b_t,}{H} = U\left(c_t^T, c_t^N\right) + \lambda_t\left((r + \phi_t^i)b_t + y_t^T + \frac{y_t^N}{e_t} - c_t^T - \frac{c_t^N}{e_t}\right), \tag{1.7}$$

where the shadow price of the budget constraint λ_t is the marginal utility of wealth valued at time t, or the extra current utility generated by a marginal increment of the stock of financial assets at time t.[6]

Without loss of generality, we assume a measure-one continuum of identical agents. Note that the risk premium ϕ_t^i is a function of the *aggregate* net foreign asset position b_t in line with definition (1.1), and not a function of the *individual* b_t optimally set by the individual consumer. This implies that the nominal interest rate $r + \phi(b_t, \theta^i)$ is taken as *parametric* in the optimization problem of an atomistic representative agent, as individual actions have a negligible effect on the aggregate.

To be sure, the representative consumer maximizes intertemporal utility with respect to her *individual* b_t (a control variable) but takes *aggregate* b_t embedded in the risk premium function $\phi(b_t, \theta^i)$ as parametric. However, by construction, aggregate b_t appears with the same value as individual b_t.

Fist Order Conditions

We can solve the consumer's problem by maximizing (1.7) with respect to the control variables, the state variable and the shadow price, in account of the transversality condition $\lim_{t\to\infty} \lambda_t b_t e^{-(\rho+\kappa^i)t} = 0$. The first order conditions are:

1) $H_{c_t^T} = 0$ which delivers

$$U_{c^T}\left(c_t^T, c_t^N\right) = \lambda_t \tag{1.8}$$

[6]More generally, λ_t is the extra utility, valued at time t, of relaxing the IBC by one marginal unit.

2) $H_{c_t^N} = 0$ which delivers

$$U_{c^N}\left(c_t^T, c_t^N\right) = \frac{1}{e_t}\lambda_t \tag{1.9}$$

Using (1.8) and (1.9) we can write

$$\frac{U_{c^T}\left(c_t^T, c_t^N\right)}{U_{c^N}\left(c_t^T, c_t^N\right)} = e_t \tag{1.10}$$

which shows that at the optimum, at every point in time, the marginal rate of substitution $-\frac{dc_t^N}{dc_t^T} = \frac{U_{c^T}\left(c_t^T, c_t^N\right)}{U_{c^N}\left(c_t^T, c_t^N\right)}$ equals the marginal rate of transformation $e_t = \frac{P^T}{P^N}$.

3) $H_{b_t} = (\rho + \kappa)\lambda_t - \dot{\lambda}_t$ which delivers

$$\dot{\lambda}_t = \lambda_t(\rho + \kappa^i - (r + \phi_t^i)) \tag{1.11}$$

Notice that in the above equation, the derivative of ϕ_t^i with respect to time is zero because the individual consumer takes the value ϕ_t^i as parametric.

4) $H_{\lambda_t} = \dot{b}_t$ which simply restates the IBC (1.5).

Equilibrium conditions

Supply Side We assume constant output of traded and non-traded goods $\forall\ t$, before and after a possible collapse event:

$$y_t^T = y^T \tag{1.12}$$

$$y_t^N = y^N \tag{1.13}$$

Demand Side Domestic prices of non-tradables are assumed to be fully flexible, i.e. P_t^N is such that the non-traded market clears at all times:

$$c_t^N = y^N \tag{1.14}$$

Given the conditions (1.12), (1.13) and (1.14), the current account (1.5) reduces to

$$\dot{b}_t = (r + \phi_t^i)b_t + y^T - c_t^T \tag{1.15}$$

Accordingly, the IBC (1.6) can be simplified to

$$\int_t^{\infty} c_s^T e^{-\int_t^s (r+\phi(\tau))d\tau} ds = b_t + y^T \int_t^{\infty} e^{-\int_t^s (r+\phi(\tau))d\tau} ds \tag{1.16}$$

1.3 The CES Utility Function Case

Suppose that the instantaneous utility function in equation (1.4) takes the constant-elasticity-of-substitution form

$$U\left(c_t^T, c_t^N\right) = \left(\frac{\sigma}{\sigma - 1}\right) c_t^{\frac{\sigma-1}{\sigma}}$$

where σ is the *intertemporal* elasticity of substitution and $c_t = c(c_t^T, c_t^N)$ is a consumption index defined as

$$c_t \equiv \left[\gamma^{\frac{1}{\eta}} \left(c_t^T\right)^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} \left(c_t^N\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, \tag{1.17}$$

Parameter η is the *intratemporal* elasticity of substitution between tradables and non-tradables. Note that we can find a price index q_t such that the product $q_t c_t$ transforms total expenditure of the consumption composite c_t in terms of units of tradables, i.e.

$$q_t c_t = c_t^T + \frac{1}{e_t} c_t^N. \tag{1.18}$$

The relevant price index is a function of the real exchange rate $q_t = q(e_t)$ with $q' < 0$ and takes the form[7]

$$q_t = \left[\gamma + (1-\gamma) \left(\frac{1}{e_t} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{1.19}$$

The representative consumer optimization problem boils down to maximizing

$$\int_z^\infty \left(\frac{\sigma}{\sigma-1} \right) c_t^{\frac{\sigma-1}{\sigma}} e^{-(\rho+\kappa^i)(t-z)} dt \tag{1.20}$$

subject to her IBC.

The Hamiltonian corresponding to the maximization of (1.20) subject to the IBC in equation (1.6) is

$$\underset{c_t^T, c_t^N, \lambda_t, b_t,}{H} = \left(\frac{\sigma}{\sigma-1} \right) c_t^{\frac{\sigma-1}{\sigma}} + \lambda_t \left((r + \phi_t^i) b_t + y_t^T + \frac{y_t^N}{e_t} - q_t c(c_t^T, c_t^N) \right).$$

Recalling FOC (1.8), the solution for $H_{c_t^T} = 0$ results in

$$\left[c_t^{\frac{\sigma-1}{\sigma}-1} \frac{\partial c_t}{\partial c_t^T} \right] = \lambda_t \tag{1.21}$$

where $\frac{\partial c_t}{\partial c_t^T} = c_t^{\frac{1}{\eta}} \gamma^{\frac{1}{\eta}} (c_t^T)^{-\frac{1}{\eta}}$. [8] Similarly, from (1.9), solving for $H_{c_t^N} = 0$ renders

$$\left[c_t^{\frac{\sigma-1}{\sigma}-1} \frac{\partial c_t}{\partial c_t^N} \right] = \lambda_t \frac{1}{e_t} \tag{1.22}$$

[7]Price index $q(e_t)$ is in fact the minimum expenditure $Z = c_t^T + \frac{1}{e_t} c_t^N$ in terms of units of tradables that would be needed if an agent desired to acquire one unit of the consumption index c_t, for a given real exchange rate $e_t = \frac{P^T}{P_t^N}$.

[8]See the Appendix for details on the derivation of $\partial c_t / \partial c_t^T$ and $\partial c_t / \partial c_t^N$.

where $\dfrac{\partial c_t}{\partial c_t^N} = c_t^{\frac{1}{\eta}} (1-\gamma)^{\frac{1}{\eta}} \left(c_t^N\right)^{-\frac{1}{\eta}}$. Dividing (1.21) by (1.22), we get the relation

$$\frac{\gamma}{(1-\gamma)} \frac{c_t^N}{c_t^T} = e_t^{\eta}, \tag{1.23}$$

which, in consideration of equilibrium condition (1.14), can be written as

$$\frac{\gamma}{(1-\gamma)} \frac{y^N}{c_t^T} = e_t^{\eta}, \tag{1.24}$$

This last equation establishes the direct relationship between consumption of tradables and the real exchange rate. Namely, for any level of c_t^T, the real exchange rate e_t will be such that the non-tradable market clears at all times. Recall that the real exchange rate can adjust at any time thanks to the flexibility of prices of non-tradables.

Replacing c_t^N in (1.18) by equation (1.23) we obtain

$$c_t^T = \gamma c_t q_t^{\eta} \tag{1.25}$$

Also, from (1.21) and (1.22) the following expression for c_t can be obtained:[9]

$$c_t^{-1/\sigma} = \lambda_t q_t, \tag{1.26}$$

and replacing c_t in (1.25) by equation (1.26) results in the following expression for demand of tradables as a function of the price index q_t :

$$c_t^T = \gamma \left(\lambda_t\right)^{-\sigma} q_t^{\eta-\sigma} \tag{1.27}$$

[9]See the Appendix for details of the derivations of (1.25) and (1.26).

Finally, FOC (1.11) implies

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho + \kappa^i - (r + \phi_t^i) \tag{1.28}$$

1.3.1 Equations of Motion

By log-differentiating (1.26) with respect to time, we obtain

$$\frac{\dot{c}_t}{c_t} = -\sigma \left. \frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{q}_t}{q_t} \right), \tag{1.29}$$

which, using equation (1.28), results in the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = \sigma \left(r + \phi_t^i - \frac{\dot{q}_t}{q_t} - (\rho + \kappa^i) \right)$$

The terms $r + \phi_t^i - \frac{\dot{q}_t}{q_t}$ can be regarded as the *real consumption-index-based interest rate.* Therefore, the tilt of the real consumption-index is determined by the difference between the consumption-index based rate of return and the subjective discount rate $\rho + \kappa^i$, and its sensitivity is positively related with the elasticity of intertemporal substitution σ. Note that a higher price-index inflation $\frac{\dot{q}_t}{q_t}$ implies a lower consumption-index interest rates and a lower intertemporal tilt of the consumption bundle c_t.

Log-differentiation of (1.25) with respect to time results in

$$\frac{\dot{c}_t^T}{c_t^T} = \frac{\dot{c}_t}{c_t} + \eta \frac{\dot{q}_t}{q_t}, \tag{1.30}$$

which, together with (1.29) and with log-differentiation of (1.27), can be written as

$$\frac{\dot{c}_t^T}{c_t^T} = -\sigma \left. \frac{\dot{\lambda}_t}{\lambda_t} \right) + (\eta - \sigma) \frac{\dot{q}_t}{q_t}, \tag{1.31}$$

Using equation (1.28), equation (1.31) can be expressed as

$$\frac{\dot{c}_t^T}{c_t^T} = \eta\frac{\dot{q}_t}{q_t} + \sigma\left(r + \phi_t^i - \frac{\dot{q}_t}{q_t} - \left(\rho + \kappa^i\right)\right) \tag{1.32}$$

This last equation reveals the tension between two effects of the rate of change of the (traded-denominated) price index q_t onto the determination of the rate of growth of traded goods consumption. Each effect is associated with one of the two types of elasticity of substitution. On the one hand, the greater the intertemporal elasticity of substitution σ, the more important the *negative* effect that lower real interest rates will have on the saving rate, and on the consumption growth rate. This occurs, as stated above, as the consumption-index interest rate decreases with price-index inflation $\dot{q}_t/q_t$.

On the other hand, higher elasticity of substitution η between tradables and non-tradables implies a greater switching effect between the two types of goods as their relative prices changes. Note from (1.19) that increasing q_t translates in increasing $1/e_t = P_t^N/P_t^T$. Thus, as q_t increases, consumption is diverted away from non-traded goods towards traded goods, thereby encouraging the growth rate of c_t^T.

In turn, relative prices are a function of demand. In particular, their dynamics are such that equilibrium in the non-traded market is preserved at all times. In order to see this, we log-differentiate (1.24) to obtain

$$\frac{\dot{e}_t}{e_t} = -\frac{1}{\eta}\frac{\dot{c}_t^T}{c_t^T} \tag{1.33}$$

The relation between the growth rates between the price index and the real exchange-rate can be found by log-differentiating (1.19) to get:

$$\frac{\dot{q}_t}{q_t} = -\Psi_t\frac{\dot{e}_t}{e_t}, \tag{1.34}$$

which together with (1.33) and (1.30) returns

$$\frac{\dot{e}_t}{e_t} = -\frac{1}{\eta}\frac{1}{1-\Psi_t}\frac{\dot{c}_t}{c_t} \tag{1.35}$$

where $\Psi_t(e_t) \in [0,1]$ with $\Psi' < 0$.[10] This last equation evidences the *negative* relationship between the real exchange rate appreciation and the growth rate of consumption demand. Intuitively, as total demand increases, relative prices of non-tradables ought to increase in order to divert the demand away from these goods towards the tradable goods.

In order to solve for the equation of motion of c_t^T, we use (1.33), (1.34) and (1.31) to obtain

$$\frac{\dot{c}_t^T}{c_t^T} = -\Sigma_t \left. \frac{\dot{\lambda}_t}{\lambda_t}\right) \tag{1.36}$$

where $\Sigma_t = \Sigma(\eta, \sigma, e_t) > 0$ and $\frac{\partial \Sigma}{\partial \eta} > 0$, $\frac{\partial \Sigma}{\partial \sigma} > 0$, $\frac{\partial \Sigma_t}{\partial e_t} > 0$ if $\sigma < \eta$, and $\frac{\partial \Sigma_t}{\partial e_t} < 0$ if $\sigma > \eta$.[11] . Since Σ_t is positive, $\frac{\dot{c}_t^T}{c_t^T}$ has the opposite sign than $\frac{\dot{\lambda}_t}{\lambda_t}$. Inserting equation (1.28) in the last equation, and assuming that $\rho = r$, we can rewrite it as

$$\frac{\dot{c}_t^T}{c_t^T} = \Sigma_t \left(\phi_t^i - \kappa^i\right) \tag{1.37}$$

where $\phi_t^i = \phi(b_t, \theta^i)$ is defined in (1.1).

Note that, ultimately, the sign of $\dot{c}_t^T / c_t^T$ will be solely determined by the differential between market interest rates and the subjective discount rate, and more specifically by $\phi_t^i - \kappa^i$.

[10] See details of this expression and its derivation in the Appendix.
[11] See the Appendix for details on the derivation of (1.36) and the characterization of Σ_t.

1.3.2 Steady State

Let's rewrite the system of equations of motion (1.15) and (1.37) for the variables c_t^T and b_t:

$$\dot{c}_t^T = \Sigma_t \left(\phi_t^i - \kappa^i\right) c_t^T \tag{1.38}$$

$$\dot{b}_t = \left(r + \phi(b_t, \theta^i)\right) b_t + y^T - c_t^T. \tag{1.39}$$

Consider the steady state (SS) such that $\dot{c}_{SS}^T = 0$ and $\dot{b}_{SS} = 0$. The SS condition corresponding to (1.38) becomes

$$\dot{c}_{SS}^T = 0 = \Sigma_t \left(\phi_{SS}^i - \kappa^i\right) c_{SS}^T, \tag{1.40}$$

where $\phi_{SS}^i = \phi(b_{SS}^i, \theta^i)$ is set in accordance with definition (1.1). Equation (1.40) implies that at the SS the following must hold:

$$\phi(b_{SS}^i, \theta^i) = \kappa^i, \tag{1.41}$$

where b_{SS}^i is the steady state level of net foreign asset position in a given regime i such that the sovereign risk premium has converged to the consumers' perceived risk of collapse.[12] Note from (1.28) that under the assumption that $\rho = r$ equation (1.41) is also consistent with a constant shadow price of wealth and a constant level of wealth throughout time. Correspondingly, as the model assumes constant output levels for tradables and non-tradables, we can verify from (1.6) that constant wealth requires a constant net foreign asset position, which in steady state is b_{SS}^i.

Given a constant shadow price of wealth, steady state consumption of tradables is implicitly determined by (1.21), so that at every point in time the marginal utility of

[12]Note that if the net asset position deteriorated below b_{SS}^i, further loans would have brought interest rates higher than the consumers' subjective discount factor $\rho + \kappa^i$, making them no longer attractive for consumers. Therefore, the continuing deterioration of net foreign asset position below the level b_{SS}^i can be ruled out.

consumption equals its opportunity cost.

From (1.39) and (1.41), the SS in foreign asset accumulation requires a balanced current account:

$$\dot{b}_{SS} = 0 = (r + \kappa^i) b^i_{SS} + y^T - c^T_{SS} \tag{1.42}$$

or,

$$c^T_{SS} = \left(r + \kappa^i\right) b^i_{SS} + y^T \tag{1.43}$$

That is, the SS consumption of tradables is equal to the permanent (constant) income y^T plus the interests payments (positive or negative) on the SS net foreign asset position, so that the current account (1.5) is in balance.[13]

Recall from condition (1.14) that domestic prices, and through them the real exchange rate, are such that the non-traded market remains in equilibrium. At the SS, the real exchange rate will be determined by (1.24) and (1.43). Specifically:

$$e^{\eta}_{SS} = \frac{\gamma}{(1-\gamma)} \frac{y^N}{y^T + (r + \kappa^i) b^i_{SS}} \tag{1.44}$$

or the weighted ratio of non-tradable wealth to tradable wealth, where the weights come from the shares in the consumption composite (1.17).

1.3.3 Phase Diagram

The dynamic system (1.38)–(1.39) is characterized by the loci $\dot{c}^T_t = 0$, scheduled at a level $b^i_t = b^i_{SS}$ as determined in (1.41) and the loci $\dot{b}_t = 0$, determined by (1.43), which has with slope $r + \phi^i_t + b_t \dfrac{\partial \phi}{\partial b_t} > 0$ as depicted in Figure 1.1.

[13]Note that (1.43) is also consistent with the resource constraint (1.16) at the steady state, at which point the risk premium $\{\phi^i_s\}$ remains constant at κ^i. Therefore, the resource constraint reduces to:

$$\frac{c^T_{SS}}{r + \kappa^i} = b_{SS} + \frac{y^T}{r + \kappa^i}.$$

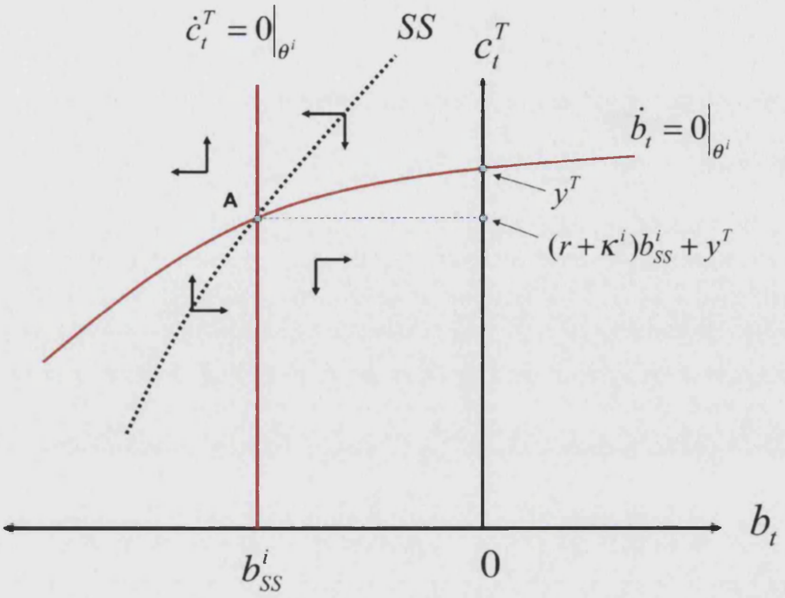

Figure 1.1: Saddle-path stable system

A linearization of the system allows us to write

$$\begin{bmatrix} \dot{c}_t^T \\ \dot{b}_t \end{bmatrix} = \begin{bmatrix} (\phi(\cdot) - \kappa^i)\dfrac{\partial \Sigma_t}{\partial e_t}\dfrac{\partial e_t}{\partial c_t^T} & \dfrac{\partial \phi}{\partial b_t}\Sigma_t c_t^T \\ -1 & r + \phi(\cdot) + b_t\dfrac{\partial \phi}{\partial b_t} \end{bmatrix} \begin{bmatrix} c_t^T \\ b_t \end{bmatrix}$$

and since around the steady state $\phi_{SS}^i = \kappa^i$, the linearized system becomes

$$\begin{bmatrix} \dot{c}_t^T \\ \dot{b}_t \end{bmatrix}_{SS} = \begin{bmatrix} 0 & \dfrac{\partial \phi}{\partial b_t}\Sigma_t c_t^T \\ -1 & r + \kappa^i + b_t\dfrac{\partial \phi}{\partial b_t} \end{bmatrix} \begin{bmatrix} c_t^T \\ b_t \end{bmatrix} \tag{1.45}$$

The Jacobian $|J| = \dfrac{\partial \phi}{\partial b_t}\Sigma_t c_t^T$ is negative, so the system (1.45) is saddle-path stable around the steady state equilibrium.

Note however that since at every point in time there is a positive conditional probability that a collapse could occur over the next period ahead, the system's dynamics could stop before they reach a SS.[14]

1.4 Regime Shock

In this paper we will not discuss the issues relative to the circumstances that could have led a given country to adopt a new regime. Instead, the interest of the present analysis is to establish the behavior of macroeconomic aggregates once an economy faces a new set of fundamentals. In particular, we endeavor to identify the effects that heterogeneous priors about the risk perceptions between domestic agents and capital markets would have on the dynamics of macro variables.

As a matter of example, consider an economy that lies at the steady state under certain given regime I (say, a floating exchange rate regime). Then, consider the announcement of a new regime II (say, a fixed exchange rate regime) that is introduced as a surprise to all agents at a given time $t = 0$ and brings with it a new set of fundamentals θ^{II}.[15]

1.4.1 Wealth Effect

When an *indebted* emerging economy enjoys a decrease in its risk premium as a result of positive news following the installation of a new, sounder regime, a wealth effect would, not surprisingly, bring on higher level of consumption (in tradables, as non-tradables are bound by domestic fixed production). The direct channel of the wealth is the alleviation of the debt service burden. Conversely, a negative regime shock involving higher systemic risk would have entailed a decrease in consumption.

[14]However, for a decreasing hazard rate, it is not necessarily true that an event of collapse *must* happen in the future. In other words, if the hazard rate ϕ_t decreases fast enough, the probability that a collapse event will never take place is positive.

[15]If the economy has entered regime I from a situation of financial distress, it is assumed that the new regime II has been introduced as part of a package of measures that secures access to borrowing in the international capital markets. Typically, new regimes are introduced in conjunction with a set of policies that often include the renegotiation of the external debt overhang. The latter is expected to achieve substantial debt alleviation and the reinsertion of the country in the international capital markets.

As a benchmark, consider the case of complete markets, perfect insurance, symmetry of information and homogeneity of beliefs across all agents. After a regime shock, the subjective perception of risk by local consumers shifts at par with the change in the market pricing of the sovereign risk of default: $\Delta\kappa = \Delta\phi$. It is clear from (1.41) that, under the assumption that $r = \rho$, a situation of perfect insurance would imply constant levels of consumption and intertemporal wealth and that there would be no change in the steady state level b_{SS}.[16] From (1.43) we observe that consumption of tradables (the *jumpy* variable) will immediately reach its new SS, permanent level and its change will be given by $\Delta c^T_{SS} = \Delta\kappa b_{SS}$. Figure 1.2 depicts the case of a negative regime shock $\Delta\kappa = \Delta\phi > 0$ (negative news means higher premium) in a given indebted country with negative initial net foreign asset position $b_{SS} < 0$. The economy would immediately shift from point **A** to the new steady state **C**, with lower consumption of tradables and unchanged level of net foreign assets. Intuitively, as interest rates increase, a lower consumption of tradable goods is required in order to service more onerous interest payments.

1.4.2 Intertemporal Substitution Effect

The gap between actual risk and subjective risk assessment, motivated in subsection (1.2.2), drastically changes the pattern of intertemporal consumption and wealth dynamics. In particular, subject to resource constraint (1.6), wealth and consumption intertemporal paths would depend on the difference between market's forward interest rates $r + \phi^i_t$ and the consumer's subjective discount rate $\rho + \kappa^i$.

In order to illustrate the tension between the wealth effect (WE) and the intertemporal substitution effect (ISE), we consider the case where domestic consumers are *more sceptical* than capital markets in regard to the fate and sustainability of a newly intro-

[16]This would be the case in a perfect insurance setting similar to Blanchard (1985) , in which both consumers and markets take account of the same risk of "death". The former will internalize such risk in the their subjective discount rates, and the latter will internalize that risk in the market interest rates.

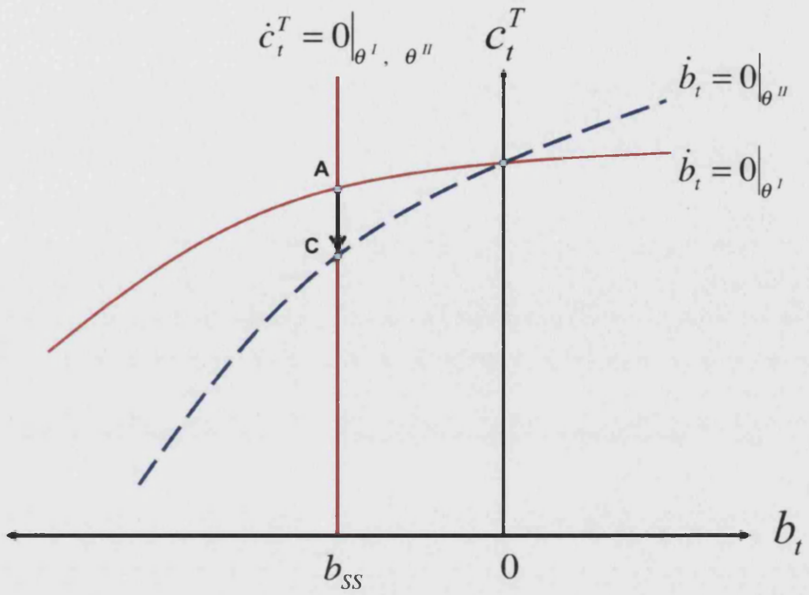

Figure 1.2: Negative regime shock under the assumption of homogeneity of believes

duced set of policies. Namely, consumers subjectively assign a higher risk of collapse than capital markets at a given time $t = 0$ when a new regime II is introduced:

$$\kappa^{II} \geq \phi(\theta^{II}, b_0) \tag{1.46}$$

where θ^{II} is the new set of fundamentals, $\phi(\theta^{II}, b_0)$ is the initial sovereign risk premium of the new regime at $t = 0$ and $b_0 = b_{SS}^I$ is net foreign asset position inherited from previous regime I. Recall that by assumption the domestic consumers' risk assessment is time-independent, i.e. $\kappa^{II} = \kappa(\theta^{II})$.

A Positive Regime Shock

Suppose that the inauguration of a new regime II brings about apparent good news and lower risk of collapse relative to the previous regime I, i.e. $\phi(\theta^{II}, b_0) < \phi(\theta^I, b_0)$.

The introduction of a less risky regime II results in a positive wealth effect stem-

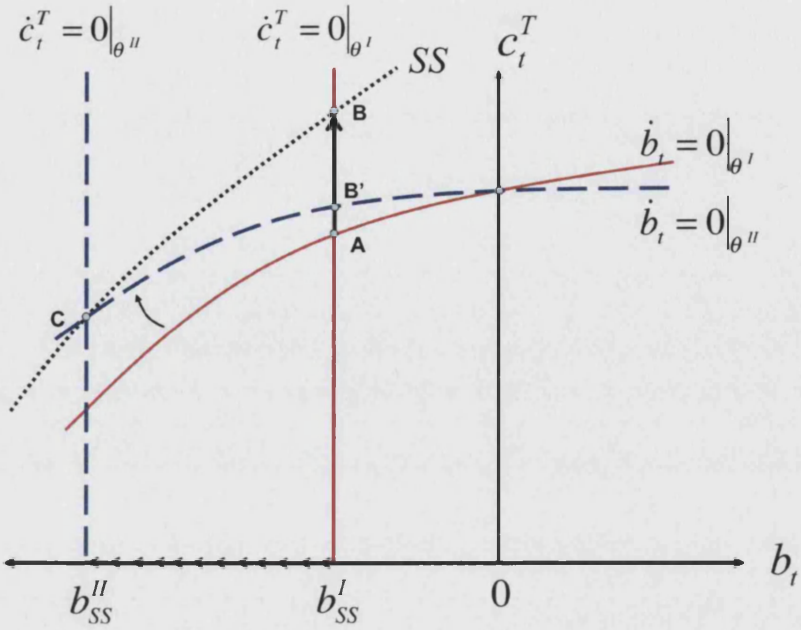

Figure 1.3: Positive regime shock with *fully* sceptical domestic consumers

ming from the alleviation of the debt service. A positive WE can be visualized as a clockwise rotation of the $\dot{b} = 0$ schedule. Additionally, consumers' relative scepticism results in an ISE that boosts current expenditure even further (at the expense of future consumption). Graphically, the latter effect is reflected in a leftward shift of the $\dot{c}^T = 0$ and the SS schedules.

As an example, consider the case of *complete lack of credibility*, in which after the positive regime shock domestic consumers did not believe in *any* fundamental improvement of the county's inherent vulnerabilities and, instead, keep their risk assessment just identically as before the regime shock, i.e. $\Delta\kappa = 0$. As depicted in Figure 1.3, the dynamics show an instantaneous boost in consumption thanks to both effects: the wealth effect pushes consumption from **A** to **B'**, and the intertemporal substitution effect advances it further from **B'** to **B**. From **B** onwards, consumption follows a downward trend towards the new steady state level at point **C**. Sovereign risk ϕ_t^{II} will gradually increase together with net debt accumulation and at the new SS the country's sovereign risk would have returned to her previous level κ^I.

Note that despite the *positive* wealth effect and the initial consumption boost, the lack of credibility and asymmetric perceptions of risks lead to an *unambiguously inferior* long-term consumption level. From (1.43), the new steady state consumption in regime II will be $c_{SS}^T = y^T + (r + \kappa)\, b_{SS}^{II}$, clearly lower than before as κ had remained unchanged while the level of net debt would have deteriorated, i.e. $b_{SS}^{II} < b_{SS}^{I} < 0$.[17]

A Negative Regime Shock

Interestingly, under the assumption of heterogeneity of beliefs and of sceptical domestic consumers, the arrival of bad news and a switch towards a *higher-risk* regime, i.e. $\phi(\theta^{II}, b_0) > \phi(\theta^{I}, b_0)$, could also entail a short-term boost in current expenditure and prices, despite the negative WE.

After the implementation of a new, riskier regime, higher risk premium and interest rates would have a negative effect on wealth and on the net present value of intertemporal consumption. However, as the subjective discount rate of sceptical consumers would be higher than the post-shock interest rates (see Figure 1.4), the ISE would induce consumers to tilt their consumption towards the present.

Therefore, the direction of the initial jump of consumption would depend on which one of the two effects dominates as can be seen in Figure 1.5 andFigure 1.6. A negative WE would result in an anti-clockwise rotation of the $\dot{b} = 0$ schedule, and, as before, the ISE that corresponds to the sceptical consumers case is reflected in a shift to the left of the $\dot{c}^T = 0$ and the SS schedules.

Since by assumption $\rho = r$, the new SS risk premium ϕ_{SS}^{II} would be equal to κ^{II} only when the net asset position reaches steady state b_{SS}^{II}. At this point, the consumer's subjective discount rate $\rho + \kappa^{II}$ is at par with the market lending rates $r + \phi_{SS}^{II}$. The new SS net asset position would be unambiguously worsened. This implies that, since

[17]Note, however, that if consumers are only partially sceptical, with κ^{II} such that $\phi_0^{II} < \kappa^{II} < \kappa^{I}$, then c_{SS}^T in regime II is not unambigously lower than in regime I.

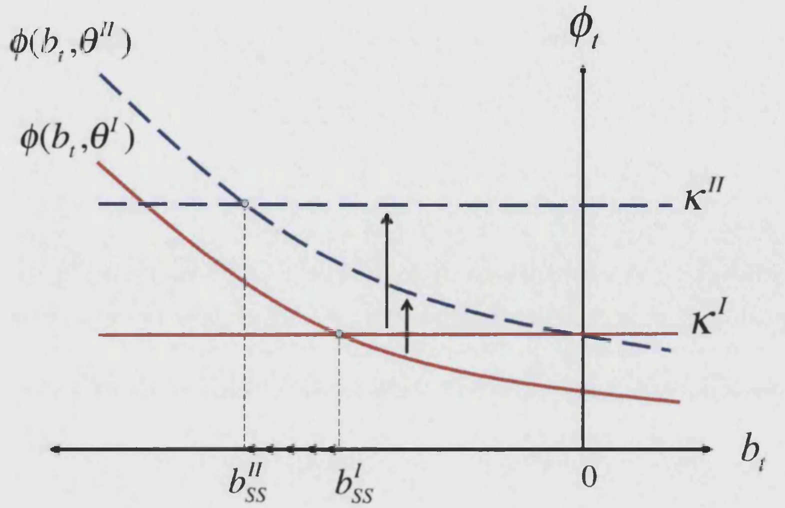

Figure 1.4: Introduction of a riskier new regime with partially sceptical, pessimistic agents

$\kappa^{II} > \kappa^{I}$, the new steady-state consumption will also be unambiguously lower and equal to $c_{SS}^{T} = (r + \kappa^{II})b_{SS}^{II} + y^{T}$.

In this saddle-path stable system, c_t^T is jumpy while b_t is sticky and (dis)accumulates following the current account equation (1.39). Therefore, at time $t = 0$ when a change in regime is announced, consumption would shift on impact onto the stable arm of the new saddle-path system. The actual direction of the jump of c_0^T would depend on the relative strength of the WE versus the ISE. The case where WE dominates is depicted in Figure 1.5: at the time of the introduction of a new, riskier regime, consumption lowers from **A** to **B.** The negative wealth effect from higher risk premium and interest rates has a greater negative impact on current consumption, equal to the distance from **A** to **B'**, than the positive ISE, which slightly shifts up consumption from **B'** to **B**. From point **B** on, consumption will continue to decrease gradually along the stable arm towards the new steady state in point **C**.

But it may also be that the intertemporal substitution effect is strong enough to offset

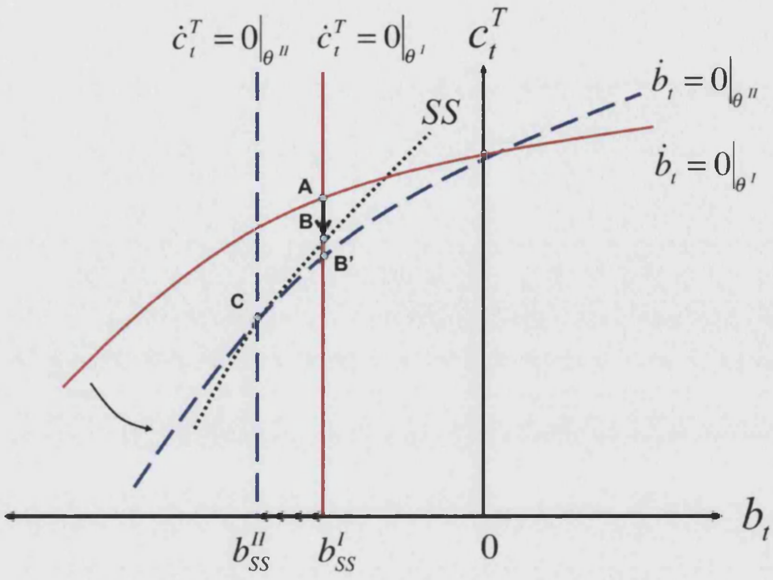

Figure 1.5: Negative regime shock: WE dominates

the negative WE in the short-run, and that immediate consumption actually increases on impact. This is an interesting case of study where *bad news* about a switch to a new regime with weaker fundamentals could result in expansionary dynamics, and shows that such an expenditure boost is not proof of wealth improving policies. Graphically, in Figure 1.6, consumption would jump from **A** to **B** (despite the negative wealth effect, given by the distance **A** to **B'**) followed by a gradual cooling-off period towards the steady state point **C,** along with a process of further debt accumulation. The greater the discrepancy between κ^{II} and $\phi(\theta^{II}, b_0)$, the stronger the ISE. In other words, the greater the "degree of scepticism" of local agents about the sustainability of a newly announced regime, the more likely the *reverse-overshooting* of domestic consumption.

Note that in all cases prices and the real exchange rate are determined by the non-traded goods market clearing condition (1.14). From the first order condition (1.24) it follows that

$$(P_t^N)^\eta = \left(\frac{(1-\gamma)}{\gamma}\frac{1}{y^N}\right)(S_t\, P^{T*})^\eta\, c_t^T$$

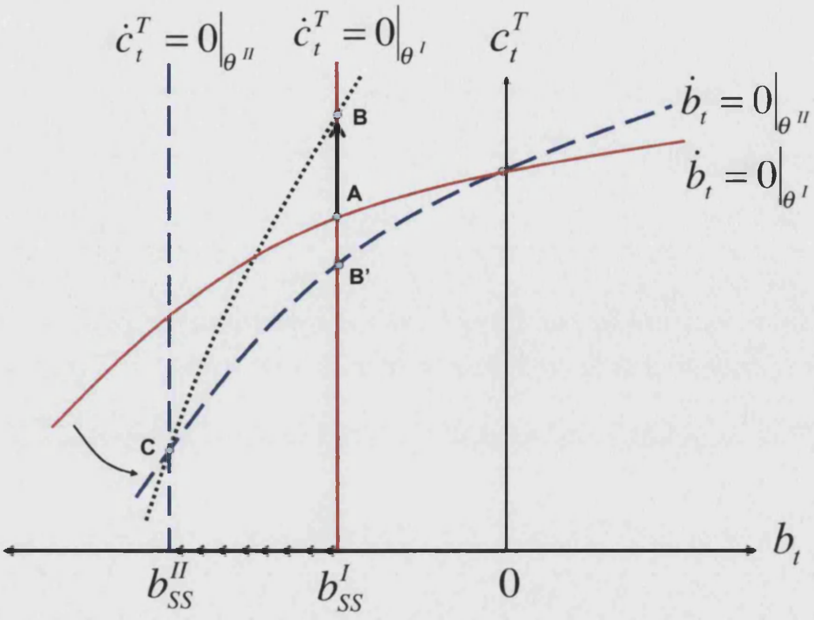

Figure 1.6: Negative regime shock: ISE dominates

where S_t is the nominal exchange rate and P^{T*} is the (constant) exogenous price of tradables in foreign currency. Thus, assuming a fixed nominal exchange rate, domestic prices P_t^N follow a path that is positively related with c_t^T. Intuitively, when the opportunity cost of consumption λ_t is low the consumer is eager to increase current expenditure. Accordingly, higher non-tradables prices (or an appreciation of the real exchange rate e_t) would divert the excess demand towards tradable goods and out of non-tradables.

A Note on the Shadow Price

The expansionary dynamics can also be understood by observing the effects of a shock on the opportunity cost of consumption, reflected in the shadow price λ_t. Note from (1.36) that tradable consumption growth is inversely related to the growth of λ_t. Following a regime shock, the path of the shadow price will also be determined by the WE and the ISE. On the one hand, the WE of the implementation of a new, riskier regime would result in a positive impact on λ_t because the value of wealth decreases on

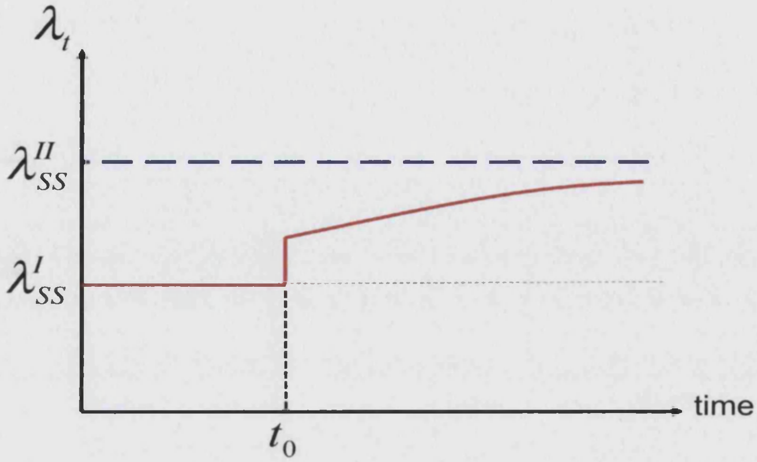

Figure 1.7: Negative regime shock: WE dominates

impact (due to the heavier debt service profile) so the marginal value of consumption increases. On the other hand, the ISE would tilt up the opportunity cost schedule.

Figure 1.7 illustrates the case of a negative regime shock where WE dominates the initial reaction of the shadow price, leading to an immediate drop in tradable consumption, while Figure 1.8 displays the case where ISE dominates, leading to an initial drop of the shadow price of wealth and a corresponding boost in consumption, despite the negative wealth effect of the shock.

1.4.3 The Role of Beliefs

In the long-run, domestic agents' beliefs become (asymptotically) self-fulfilling. If the risk perceived by locals is higher than the sovereign risk premium, i.e. if $\kappa^{II} > \phi_t^{II}$, then perverse current account dynamics would lead to a process of debt accumulation and the economy's sovereign risk would converge towards the higher subjective risk level, irrespective of the sense of the shock and of the direction of the initial jump of consumption.

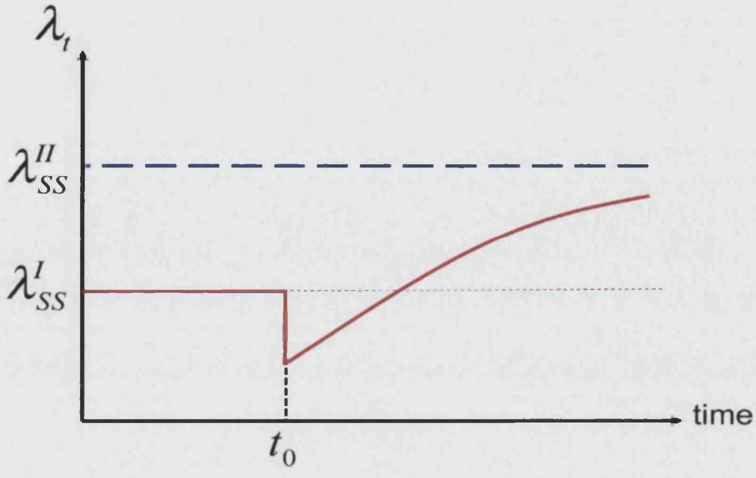

Figure 1.8: Negative regime shock: ISE dominates

A mirror analysis could be undertaken for consumers taking a *less* sceptical assessment regarding a regime shock than capital markets. In such a case, a change in regime will tilt up consumption and the ISE would favor current account surpluses that would result in a gradual improvement of the net foreign asset position. Again, the sovereign risk premium would converge to the subjective risk assessment as the economy gradually becomes a better subject of credit.

1.5 Conclusion

Time and again, the announcement of a regime change has been a common course of action when policy makers aim to redress an economy that struggles with weak fundamentals. Too often though, the incumbents do not hesitate to congratulate themselves soon after the domestic demand appears to pick up. As a matter of fact, an initial consumption surge has by and large been considered as an indisputable sign of success. Even disregarding the effect on production, conventional wisdom would anticipate the that lower sovereign risk premium and less expensive access to capital markets would bring about a positive wealth effect that should be reflected in greater consumption.

In contrast, we have argued that an immediate expansion following a regime shock does not necessarily mean that an economy had entered in a sustainable path. In particular, if such asymmetries of perception of risk exist and local consumers attach a greater probability of collapse than markets do, a consumption surge could instead be the prelude of perverse external account dynamics that would lead to a debt build-up and an escalation of the risk of collapse.

A crucial assumption in our model is that beliefs are heterogeneous. In particular, we look into the case of study where capital markets price uncertainty based on the entire available information set while domestic consumers assess sovereign risk based on a subset of incomplete information due, for instance, to costly barriers to access information. Thus, domestic agents ought to guesstimate the systemic risk.

In addition of being less well informed than capital markets, domestic consumers are assummed to attach a greater probability to a systemic collapse than markets to.[18]

We have analyzed the cases of positive and negative regime shocks assuming sceptical local consumers. If a new regime is fundamentally favorable, implying a reduction in the systemic risk, the WE and the ISE would reinforce each other to set off a surge in consumption.

More interesting perhaps is the case of the inauguration of an inherently *riskier* regime that could also be followed by an initial expansion of domestic demand despite the negative WE. This situation could obtain insofar as local agents were "sufficiently sceptical" to guarantee that the ISE is the dominant effect. In these circumstances, the initial consumption surge would be followed by current account deficits and a gradual contraction of demand towards an unambiguously inferior steady state. In addition, debt accumulation and the deterioration of the net asset position would result in higher future

[18]Alternatively, they anticipate a lower expected duration of the current regime.

systemic vulnerabilities. Accordingly, the sovereign risk premium is set to converge towards the higher subjective risk level perceived by local consumers, as a matter of an asymptotic self-fulfilling prophecy.

In all, irrespective of whether the policy change had initially generated positive or negative wealth effects, the condition of heterogeneous beliefs and relatively more sceptical domestic consumers is, in the present model, at the basis of the persistent current account deficits.

Recent experiences seem suitable to illustrate the predictive dynamics of the model. In the past few decades, a number of countries have adopted radical ultra-fixed exchange rate regimes as credibility shocks aimed at restoring stability amidst situations of economic and financial distress. Such instances include Argentina's currency board (1991 - 2002) and Ecuador' official dollarization regime (2000 - date). Arguably, in both cases capital markets initially welcomed the inauguration of the regimes and interpreted the ensuing demand booms as credible signals of recovery.

The regimes were supposed to be in place for good, especially given that, ex-ante, a reversal was thought to potentially entail extremely damaging consequences. Allegedly, the idea of a very high cost of a collapse event should have dissuaded policymakers from pursuing loose economic policies. Specifically, a required condition to insure the stability was the commitment to tight fiscal policy.

Unfortunately, in both instances events gave due credit to the local consumers' scepticism: governments eventually became unwilling to comply with their committed conservative fiscal stance. Arguably, in retrospect, anyone who had attached a high probability to this scenario would have proved to be correct.

In the terms of our model, if locals distrusted local authorities' ability to commit to sustainable policies, the regime would be perceived as involving a serious systemic risk. In turn, the fate of a regime would also be affected by credibility itself. As would have been predicted by the model with sceptical local agents, after the inauguration of each of the above mentioned regimes, both economies experienced a sharp acceleration of demand and persistent current account deficits.

In Argentina, a surge in domestic demand, an aggressive debt accumulation and real appreciation coming from inflationary dynamics undermined the economy's fundamentals and made this country increasingly vulnerable to shocks and speculative attacks. Eventually, a reversal of the regime occurred in January 2002 together with a unilateral declaration of default.

In the case of Ecuador, although the dollarization regime, introduced in 2000 is still underway,[19] the system is going through a very dangerous current account pattern. The success of dollarization depends on the economy's ability to maintain its competitiveness over time. A necessary condition is that authorities pursue a disciplined conservative fiscal stance. Otherwise, real over-appreciation and debt overhang would follow, and the possibility of a regime reversal could be terribly harmful.[20] Regrettably, though, it is apparent that actual policymaking in Ecuador is seriously flawed by time-inconsistency problems. A fiscal spending spree began shortly after the regime's inauguration and has significantly accelerated since 2007. To be sure, if it were not for the unexpectedly high oil prices and the weakening of the US dollar since 2006 – admittedly both completely unanticipated external shocks back in the early 2000s– the stability of the regime would have been under serious threat. From the point of view of our model, local agents' scepticism would have been perfectly consistent with the rapid expansion in demand and low saving rates that have characterized Ecuador since

[19] Full official dollarization was implemented in early 2000, whereby the country gave up its monetary policy and adopted a foreign currency (the US dollar) as the legal tender. Adopting the US dollar got rid of exchange-rate noise which was meant to make the economy more attractive to foreign investment flows. The price to pay was the abandonment of a discretionary monetary policy and the "apparent" irreversibility of the regime. The expected benefits of the dollarization regime were believed to outweight the costs stemming from the possibility of a regime breakdown in the future, a catastrophic event with very small probability. See Mendoza (2001), Chang and Velasco (2003) for an analysis of advantages and disadvantages of dollarization.

[20] A crisis in dollarization becomes particularly treacherous as the financial system has no longer support from a Lender of Last Resort (LLR), which is de-facto absent in this regime.

the outset of the dollarization regime.[21]

In sum, the fate of a regime depends not only on its inherent characteristics but also on how much credibility it enjoys. In the empirical cases recalled above, it can inferred that the scepticism of local agents may have contributed to the perverse dynamics that ultimately increased the chances of catastrophe.

This paper suggests a number of policy lessons. First, it stresses the fact that scepticism or lack of credibility has first order effects and can be *self-fulfilling.* Low credibility implies low saving rates and perverse external account dynamics until the actual systemic risk converges to the local consumer's subjective estimation.

Second, it makes the point that an initial demand boom may not necessarily be good news. For instance, a fundamentally *positive* new regime flawed by serious lack of credibility could move towards an unambiguously inferior steady state in the future. The difficulty, however, is that these perverse dynamics could be masked for the injudicious analyst who observes the positive effect on short-run demand. Even worse, a fundamentally *negative* new regime could also display a short-run consumption boost on the back of the IES that stems from the lack of credibility of local agents. Therefore, following the implementation of a set of policies, close attention must be given to the current account pattern.

And third, policymakers may need to consider the implementation of policies that counterbalance eventual excessive consumption, such as a counter-cyclical policy fiscal rule. This would in turn help build up credibility of local agents and alleviate perverse current account dynamics.

To be fair, in this paper, we have paid special attention to the case of domestic con-

[21] On the other hand, it was apparent that markets received the news of the switch in regime with relative optimism. Only one year after having defaulted on all external debt categories, Ecuador regained access to capital markets and enjoyed a rapid decrease of EMBI premium from 60% by end 1999 to 12% over LIBOR by end 2000. Right after the inauguration of dollarization, economic indicators appeared encouraging. Debt-to-GDP ratios decreased, partly due to the debt reduction achieved by the renegotiation of the Brady debt in 2000 but also due to major real appreciation that took place during the first 24 months of dollarization, when USD denominated domestic prices increased by a cumulative 150%.

sumers being more sceptical about the sustainability of a new regime relative to capital markets. A mirror image analysis would study the case where local consumers take a less sceptical stance, i.e. them being more optimistic than capital markets. Under this assumption, the general results would be reversed: a change of regime would increase domestic savings, contract aggregate demand and generate current account surpluses. Accordingly, these dynamics would trigger a gradual improvement of the net foreign asset position and the sovereign risk premium.[22]

Further empirical research to quantify the magnitude of the effects leading to consumption booms would be most wanted. We have hereby provided with a qualitative approach, but it would be key to quantify the degree of heterogeneity of risk perceptions and how much of a consumption surge is prompted by an intertemporal substitution effect in order to estimate the importance of this channel. We leave for further research a more thorough analysis of the empirical evidence when reform programs or new regimes have been introduced and credibility has been a determinant factor of the demand and current account behavior.

[22]This could have been the case of Brazil and the "Lula effect" in 2002. Immediately after the regime shock, there was a recession followed by increase in saving rates and current account surpluses.

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1.A Appendix: Derivations and characterizations

1.A.1 Derivation of $\frac{\partial c_t}{\partial c_t^T}$ in equation (1.21)

From equation (1.17), we can take the partial derivative of c_t with respect to c_t^T and obtain

$$\begin{aligned}
\frac{\partial c_t}{\partial c_t^T} &= \frac{\eta}{\eta-1}\left[\gamma^{\frac{1}{\eta}}(c_t^T)^{\frac{\eta-1}{\eta}}+(1-\gamma)^{\frac{1}{\eta}}(c_t^N)^{\frac{\eta-1}{\eta}}\right]^{\frac{1}{\eta-1}}\frac{\eta-1}{\eta}\gamma^{\frac{1}{\eta}}(c_t^T)^{\frac{\eta-1}{\eta}-1} \\
&= \left[\gamma^{\frac{1}{\eta}}(c_t^T)^{\frac{\eta-1}{\eta}}+(1-\gamma)^{\frac{1}{\eta}}(c_t^N)^{\frac{\eta-1}{\eta}}\right]^{\frac{1}{\eta-1}}\gamma^{\frac{1}{\eta}}(c_t^T)^{\frac{\eta-1}{\eta}-1} \\
&= c_t^{\frac{1}{\eta}}\gamma^{\frac{1}{\eta}}(c_t^T)^{\frac{\eta-1}{\eta}-1} \\
&= c_t^{\frac{1}{\eta}}\gamma^{\frac{1}{\eta}}(c_t^T)^{-\frac{1}{\eta}}
\end{aligned}$$

A similar derivation is used later for $\frac{\partial c_t}{\partial c_t^N}$. **QED.**

1.A.2 Derivation of equation (1.25):

Equation (1.23) can be rewritten as

$$c_t^N = e_t^{\eta} c_t^T \frac{(1-\gamma)}{\gamma}$$

Introducing this last expression into (1.18), we get

$$\begin{aligned}
q_t c_t &= c_t^T + \frac{1}{e_t}\left(e_t^{\eta} c_t^T \frac{(1-\gamma)}{\gamma}\right) \\
&= c_t^T\left(1+e_t^{\eta-1}\frac{(1-\gamma)}{\gamma}\right) \\
&= c_t^T \frac{1}{\gamma}\left(\gamma+(1/e_t)^{1-\eta}(1-\gamma)\right) \\
&= c_t^T \frac{1}{\gamma} q_t^{1-\eta} \\
&= c_t^T q_t^{1-\eta},
\end{aligned}$$

$$\iff \quad c_t^T = \gamma c_t q_t^\eta$$

QED.

1.A.3 Derivation of equation (1.26):

We can write (1.21) as

$$\left[c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} \gamma^{\frac{1}{\eta}} (c_t^T)^{\frac{\eta-1}{\eta}-1}\right] = \lambda_t$$

$$\iff \quad \left[c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} \gamma^{\frac{1}{\eta}} (c_t^T)^{\frac{\eta-1}{\eta}}\right] = \lambda_t c_t^T \tag{1.47}$$

Similarly, we can write FOC (1.22) as

$$\left[c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} (1-\gamma)^{\frac{1}{\eta}} (c_t^N)^{\frac{\eta-1}{\eta}-1}\right] = \lambda_t \frac{1}{e_t}$$

$$\iff \left[c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} (1-\gamma)^{\frac{1}{\eta}} (c_t^N)^{\frac{\eta-1}{\eta}}\right] = \lambda_t \frac{1}{e_t} c_t^N \tag{1.48}$$

By summing up (1.47) and (1.48), we get

$$\begin{aligned}
c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} \left[\gamma^{\frac{1}{\eta}} (c_t^T)^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} (c_t^N)^{\frac{\eta-1}{\eta}}\right] &= \lambda_t \left[c_t^T + \frac{1}{e_t} c_t^N\right] \\
c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} \left[\gamma^{\frac{1}{\eta}} (c_t^T)^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} (c_t^N)^{\frac{\eta-1}{\eta}}\right] &= \lambda_t q_t c_t \\
c_t^{\frac{\sigma-1}{\sigma}-1} c_t^{\frac{1}{\eta}} c_t^{\frac{\eta-1}{\eta}} &= \lambda_t q_t c_t \\
c_t^{\frac{\sigma-1}{\sigma}-1} c_t &= \lambda_t q_t c_t \\
c_t^{\frac{\sigma-1}{\sigma}} &= \lambda_t q_t c_t,
\end{aligned}$$

$$\iff \quad c_t^{-1/\sigma} = \lambda_t q_t$$

QED.

Alternatively, we could have expressed the Hamiltonian in terms of consumption index $c_t = c(c_t^T, c_t^N)$ and taken FOC with respect to c_t^T, c_t^N. Specifically

$$\underset{c_t^T, c_t^N, \lambda_t, b_t,}{H} = \left(\frac{\sigma}{\sigma-1}\right) c_t^{\frac{\sigma-1}{\sigma}} + \lambda_t \left((r + \phi_t^i) b_t + + y_t^T + \frac{y_t^N}{e_t} - q_t c_t \right)$$

where

$$q_t = \left[\gamma + (1-\gamma) \left(\frac{1}{e_t} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

$$c_t \equiv \left[\gamma^{\frac{1}{\eta}} (c_t^N)^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} (c_t^T)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}},$$

FOCs would be

$$\frac{\partial H}{\partial c_t^T} = 0 \Longrightarrow \quad \left[c_t^{\frac{\sigma-1}{\sigma}-1} \frac{\partial c_t}{\partial c_t^T} \right] = \lambda_t q_t \frac{\partial c_t}{\partial c_t^T} \tag{1.49}$$

$$\frac{\partial H}{\partial c_t^N} = 0 \Longrightarrow \quad \left[c_t^{\frac{\sigma-1}{\sigma}-1} \frac{\partial c_t}{\partial c_t^N} \right] = \lambda_t q_t \frac{\partial c_t}{\partial c_t^N} \tag{1.50}$$

Note that either (1.49) or (1.50) allow us to write

$$c_t^{-1/\sigma} = \lambda_t q_t$$

which is equation (1.26) . **QED.**

1.A.4 Derivation of (1.35):

Recall (1.19)

$$q_t = \left[\gamma + (1-\gamma) (1/e_t)^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

$$\Longleftrightarrow \qquad q_t^{1-\eta} = \gamma + (1-\gamma) (1/e_t)^{1-\eta}$$

Then log-differentiate and get

$$(1-\eta) \frac{\dot{q}_t}{q_t} = -\frac{(1/e_t)^{1-\eta}}{q_t^{1-\eta}} (1-\gamma)(1-\eta) \frac{\dot{e}_t}{e_t}$$

or

$$\frac{\dot{q}_t}{q_t} = - \left. \frac{(1-\gamma)\,(1/e_t)^{1-\eta}}{\gamma + (1-\gamma)\,(1/e_t)^{1-\eta}} \right) \frac{\dot{e}_t}{e_t}.$$

or

$$\frac{\dot{q}_t}{q_t} = -\Psi_t \frac{\dot{e}_t}{e_t}$$

where

$$\Psi_t\,(e_t) \equiv \frac{(1-\gamma)\,(1/e_t)^{1-\eta}}{\gamma + (1-\gamma)\,(1/e_t)^{1-\eta}} \in [0,1]$$

and $\Psi' < 0$.

Recall (1.33)

$$\frac{\dot{e}_t}{e_t} = -\frac{1}{\eta} \frac{\dot{c}_t^T}{c_t^T}$$

The previous two equations lead to

$$\frac{\dot{q}_t}{q_t} = \frac{\Psi_t}{\eta} \frac{\dot{c}_t^T}{c_t^T},$$

and using the last three equations together with equation (1.30), results in

$$\frac{\dot{e}_t}{e_t} = -\frac{1}{\eta} \frac{1}{1-\Psi_t} \frac{\dot{c}_t}{c_t}$$

QED.

1.A.5 Derivation of (1.36):

$$\begin{aligned} \frac{\dot{c}_t^T}{c_t^T} &= -\sigma \left. \frac{\dot{\lambda}_t}{\lambda_t} \right) + (\eta - \sigma) \frac{\dot{q}_t}{q_t} \\ &= -\sigma \left. \frac{\dot{\lambda}_t}{\lambda_t} \right) + (\eta - \sigma) \frac{\Psi_t}{\eta} \frac{\dot{c}_t^T}{c_t^T} \end{aligned}$$

$$\Longrightarrow \quad \frac{\dot{c}_t^T}{c_t^T} = - \ \Sigma_t \frac{\dot{\lambda}_t}{\lambda_t}$$

where

$$\Sigma_t \equiv \frac{\sigma}{\left(1 - (\eta - \sigma)\frac{\Psi_t(e_t)}{\eta}\right)}$$

QED.

1.A.6 Characterization of $\Sigma(\eta, \sigma, e_t)$

Σ_t can also be written as

$$\Sigma_t \equiv \frac{\sigma}{\left(1 + (\frac{\sigma}{\eta} - 1)\Psi_t(e_t)\right)}$$

It follows that $\frac{\partial \Sigma_t}{\partial \eta} > 0$ and $\frac{\partial \Sigma_t}{\partial \sigma} > 0$, so long σ, $\eta > 0$ and $0 < \Psi_t < 1$. Note that $\frac{\partial \Sigma_t}{\partial e_t} > 0$ if $\sigma < \eta$ and $\frac{\partial \Sigma_t}{\partial e_t} < 0$ otherwise.

Proof that Σ_t is positive

Σ_t is positive if

$$\left(1 - (\eta - \sigma)\frac{\Psi_t}{\eta}\right) > 0$$

$$\begin{aligned}
&\Longleftrightarrow \quad (\eta - \sigma)\frac{\Psi_t}{\eta} < 1 \\
&\Longleftrightarrow \quad \left(1 - \frac{\sigma}{\eta}\right)\Psi_t < 1 \\
&\Longleftrightarrow \quad \frac{\sigma}{\eta} > 1 - \frac{1}{\Psi_t},
\end{aligned}$$

which is true since $\sigma, \eta > 0$ and $\Psi_t \in [0, 1]$. **QED.**

Chapter 2

Consumption and saving behavior of a hyperbolic discounting agent under systemic risk

2.1 Introduction

The economic literature has extensively studied the effects of uncertainty on the decisions of economic agents. Part of this literature focused on the study of the particular type of risk consisting of a hazard rate that determines the probability of arrival of a crisis.

As an example of such event, we consider a country exposed to infrequent exogenous, "lethal" shocks that would result in a systemic crisis involving a default on its sovereign debt and its exclusion from capital markets. So long the event has not occurred, economic agents are allowed to set their optimal intertemporal decisions, knowing that if and when a crisis arrives their plans would become obsolete and they will be constrained to a new economic environment worth a certain "residual value".

In this paper we endeavour to study whether dynamic inconsistency could affect the way in which consumption and saving decisions are made under systemic risk. We do this by introducing time inconsistent agents who discount utility flows with a hyperbolic function instead of an exponential one. This feature effectively characterizes a consumer that is impatient when she undertakes short-term trade-offs and more patient when she evaluates long-term ones. O'Donoghue and Rabin (1999, 2001) refer to this peculiarity as "present bias." It was Robert Strotz's (1955) who first formally conjectured that people are more impatient when they make short-run trade-offs than when they make long-run ones:

> "When two rewards are both far away in time, decision-makers act relatively patiently. But when both rewards are brought forward in time, preferences exhibit a reversal, reflecting more impatience."

The example put forward by Thaler (1981) also nicely illustrates this behaviour:

> "I prefer two apples in 101 days, rather than one apple in 100 days. But I prefer one apple right now, rather than two apples tomorrow."

In our setup, the agent is naïve in the sense that she is unable to realize her time-inconsistency problem. To be sure, she believes, in a wishful-thinking manner, that she will behave according to her present optimal saving-consumption plan.[1] Alas, as time goes by, she changes her mind and reassesses the situation by formulating a new, different plan. Despite this element of surprise, she never adjusts her expectations about her future behaviour.

We put forward a full-fledged continuous time hyperbolic function as opposed to the quasi-hyperbolic discounting (QHD) setup typically studied in the literature[2]. The

[1] The literature has provided evidence of at least partial naivety (Strotz 1955, Phelps and Pollak 1968, O'Donoghue and Rabin 1999), which more recently has been emphasized in the theories of default options (Choi et al 2005, and other references therein) and excess borrowing behaviour (Skiba and Tobacman 2007).

[2] See, for example, Laibson (1994, 1997) and Harris and Laibson (2003),

discrete-time, QHD version was introduced as an approximation to the hyperbolic problem mainly for tractability purposes. To be sure, the QHD model is appealing for its recursivity, which makes it suitable for being dealt with Dynamic Programming methods.[3] Instead, our HD model recalls the original characterization of the discount factor as initially proposed by Ainslie (1975) and generalized by Loewenstein and Prelec (1992) and makes use of the Pontryagin's Maximum Principle.[4]

To abstract from considerations that are extraneous to our discussion, we consider an economy with constant output, absence of frictions, price rigidities, transaction costs and credit constraints, and impose that all agents are identical and well informed.

The literature has shown that in the case of a frictionless economy with an infinitely lived representative agent and perfect and homogeneous information, the orthodox exponential model delivers no interaction between the saving rate and changes in the systemic risk. This result was put forward by Yaari (1965) and further developed by Blanchard (1985) in the context of "lifetime utility maximization with constant probability of death". They show how an increase in systemic risk has two opposite effects that cancel each other out: higher risk premium implies higher returns on savings, but it also implies that the discount rate is being augmented by the corresponding hazard rate. The net effect on the saving rate turns out to be nil.

Another trend in the literature dealing with systemic risk pays attention to the shape of the utility function. For instance, Barro (2009) departed from the non-recursive utility setup and introduced Epstein-Zin-Weil preferences. He shows how a small hazard rate that determines the probability of a catastrophic "rare event" can help resolve the equity premium puzzle. However, his setting do not generate a change in the agent's risk attitude as it was pointed out in his paper: "In an endowment economy, agents do not

[3]Harris and Laibson (2008) introduced a continuous time, more refined version of the QHD model where the transition from present to future is stochastic. Their specification recovers recursivity and continuity of the policy functions.

[4]Other references that used a pure hyperbolic discounting characterization include Barro (1999) and Luttmer and Mariotti (2002).

react to changes in uncertainty by altering saving and investment."[5]

The main result of our paper is that the saving behaviour of naïve, hyperbolic discounting agents is indeed affected by the probability of arrival of a crisis. In particular, we obtain the result that higher systemic risk effectively lowers the HD agent's saving rate.

The relationship between the probability of a crisis and the saving rate of a time inconsistent agent can be understood by recalling the HD agent's "present-bias", i.e. she is short-term impatient but anticipates becoming more patient over time. Formally, a hyperbolic discount factor implies that the subjective discount rate is high in the short-term but declines with the time horizon, in clear contrast with the ED case in which the discount rate is set to be constant.

As we noted before, the higher risk premium enters additively in the interest rate schedule and in the subjective discount rate. Contrary to the ED case, however, in the HD setup a change in risk premium has an asymmetric effect along the term structure of the subjective discount rate because the latter is not flat (as in the ED model) but decreases with the time horizon. Consequently, the longer-term, lower subjective discount rates change proportionally more than the short-term, higher discount rates. For that reason, an increase in risk premium would result in the longer-term utility flows being discounted relatively more heavily than the short-term utility flows, which would be reflected in a lower desire to save now for future consumption.

Given the insensitivity of the saving rate to permanent changes in systemic risk in the frictionless ED model, the economic theory has enriched the basic setup by taking account of the temporary nature of shocks,[6] or by incorporating frictions, asymmetries and heterogeneities. Our present model intends to complement this literature by putting forward a new mechanism that generates such a negative relationship and pro-

[5]In Barro (2009), Section V, he introduces an AK growth model and allows for endogenous saving and investment. The saving ratio in that case increases with uncertainty so long the intertemporal elasticity of substitution is greater than one.

[6]See, for example Kraay and Ventura (2000, 2002).

vides testable implications for the behaviour of the saving rate and the current account.

For instance, observational evidence suggests that the current accounts of emerging economies often improve after a positive external shock has reduced their risk of default and helped them gain access to capital markets at cheaper rates. Some of these economies have even been in a position to change their saving patterns completely and switch their current account from persistent deficits to surpluses. Our model is able to provide an explanation for these sign reversals if the positive external shock is large enough.

In addition, our model would make it consistent that two countries with similar growth outlooks and dissimilar fundamental vulnerabilities could present opposite current account patterns.

The remainder of this Chapter is structured in five additional sections. Section 2.2 introduces the model and characterizes the problem. Section 2.3 solves the model assuming a constant hazard rate. We present the main result obtained in the hyperbolic discounting setup, namely the negative relationship between the saving rate and the risk premium, and provide some intuition. We undertake comparative statics and analyze the effects of unexpected permanent shocks that affect the risk premium and contrast the characteristics of the HD setup with those pertaining to the exponential discounting model. In Section 2.4 we extend the analysis to the case of a variable risk premium, which we allow to be a function of the inherent vulnerabilities of the economy and, in particular, of time varying indicators such as the debt-to-income ratio. We show that the system is unstable, which results in what we call "world polarization"—a system in which there will always be economies that have entered into exploding debt accumulating dynamics that would necessarily lead them towards high indebtedness ratios, wealth exhaustion and high probability of arrival of a systemic crisis. In Section 2.5 we discuss the general equilibrium implications. Section 2.6 concludes, underscores a number of policy issues and proposes possible directions for further research.

2.2 The Model

2.2.1 The event of default

We assume that a "credit event" or a country's default on its sovereign debt could occur only as a result of exogenous shocks that would force the country to discontinue the service of its external debt payments. As in Chapter 1, we assume away the possibility of strategic defaults.[7]

In addition, for simplicity we assume that if and when a country defaults it would be excluded from the international capital markets forever and would be led to a situation of virtual autarky. This assumption is technically convenient as it imposes a *constant* residual value for the future income stream after a default event.

Default probability

The timing of a crisis is uncertain. At any future point in time s, the actual probability of a crisis occurring in the immediate time period dt –conditional on it not having happened before– is determined by the *hazard rate* $\phi(s) \in [0, +\infty)$.

Without loss of generality, we assume zero recovery value of the defaulted debt. Therefore, the premium ϕ above the risk-free rate r would compensate risk-neutral investors for the default risk and would make them indifferent between investing at the risk-free rate r and holding a risky bond yielding $r + \phi$.[8]

Perceived default probability

A representative consumer standing at time t assigns probability $\kappa(s)ds$ to the occurrence that a crisis could occur in a future interval $(s; s + ds)$, $\forall s > t$, conditional on

[7] In order to rule out voluntary defaults, we assume that a credit event involves a fixed cost that would always make the country worse off compared to a pre default situation.

[8] See the Appendix 2.C.10 for a proof that the hazard rate $\phi(s)$ that determines the default probability is in effect the risk premium over the risk free interest rate r, assuming zero recovery value.

it not having occurred before. The hazard function is defined by $\kappa(s) = \frac{f_T(s)}{1-F_T(s)}$ where $f_T(s)$ is the density function of the random variable T, the "time until collapse". Therefore, the survival probability, or the likelihood of not experiencing a crisis during the time interval from the present time t to some future time s is $1 - F_T(s)$ or, equivalently, $\exp\left[-\int_t^s \kappa(\tau)d\tau\right]$.

Homogeneity of beliefs

In contrast with the model developed in Chapter 1, in which we assumed heterogeneity of beliefs between different type of agents, here we assume forward-looking agents with homogeneous beliefs and symmetry of information. This implies that the consumers' perceived probability of collapse $\kappa(s)$ is identical to the markets' estimation of sovereign risk of default $\phi(s)$.

Notation 1 *Denote $\{x(s)\}_t$ the time path values that any given variable $x(s)$ takes from time t to infinity.*

In practice, a representative agent is assumed to trust the market's assessment of the underlying default risk and takes these values parametrically from the forward-rate curve or, alternatively, from the credit default swap (CDS) market. The forward-rate curve at time t is determined by the points $\{r + \phi(s)\}_t$, for all $s \geq t$. Thus,

$$\kappa(s) = \phi(s), \qquad \forall s \geq t. \tag{2.1}$$

2.2.2 Consumer's maximization problem

A representative agent optimizes her intertemporal consumption path knowing that post-default consumption would be bound to the country's autarkic production. She maximizes the expected discounted intertemporal utility drawn from her consumption flow

$$E_t\left\{\int_t^{\infty} U\left(\tilde{c}(s)\right) F(t,s)ds\right\}, \tag{2.2}$$

where $F(t,s)$ is the consumer's discount factor, $E_t\{\cdot\}$ is the unconditional expectation operator, and $\tilde{c}(s)$ is consumption, which is a random variable: it could take values $c(s)$ if a crises has not arrived, but would be constrained to the autarkic constant production y if a credit event has arisen. Specifically, for any given time $t < s$, the density function would be

$$\tilde{c}(s) = \begin{cases} c(s) & \text{wp} \quad \exp\left[-\int_t^s \kappa(\tau)d\tau\right] \\ y & \text{wp} \quad 1-\exp\left[-\int_t^s \kappa(\tau)d\tau\right] \end{cases} \tag{2.3}$$

where $\kappa(s)$ is the hazard rate that reflects the subjective probability of default $\forall s \geq t$.

Using the density function (2.3) into (2.2) we can rewrite the objective function as

$$\begin{aligned} &\int_t^{\infty}\left(U\left(c(s)\right)e^{-\int_t^s \kappa(\tau)d\tau} + U\left(y\right)\left(1-e^{-\int_t^s \kappa(\tau)d\tau}\right)\right)F(t,s)ds \\ =& \int_t^{\infty} U\left(c(s)\right)F(t,s)e^{-\int_t^s \kappa(\tau)d\tau}ds \\ &\qquad + U\left(y\right)\int_t^{\infty} F(t,s)\left(1-e^{-\int_t^s \kappa(\tau)d\tau}\right)ds \end{aligned}$$

The second term in the RHS is merely a terminal value that does not involve the control $c\left(s\right)$. Therefore, the consumer's optimal consumption plan reduces to maximizing the first term, or

$$\int_t^{\infty} U\left(c(s)\right)\chi(t,s)ds \tag{2.4}$$

where

$$\chi(t,s) = F(t,s)\exp\left[-\int_t^s \kappa(\tau)d\tau\right] \tag{2.5}$$

subject to the intertemporal budget constraint. Notice that the reduced objective function (2.4) conveniently deals with a non-stochastic type of problem. In sum, the representative agent's problem reduces to maximizing the sum of discounted utility flows from present time t to infinity as if it were a deterministic problem. Note, however, that utility flows are not discounted with the discount factor $F(t, s)$ but instead with the *augmented discount factor* $\chi(t, s)$, which effectively takes account of the risk of collapse.

2.2.3 Intertemporal Budget Constraint

Prices are fully flexible so the non-traded goods market clears at all times. Tradable output y is assumed to be constant. The traded goods market clears via the current account. The current account determines the country's savings or the change in its net foreign asset position:

$$\dot{b}(s) = (r + \phi(s))b(s) + y - c(s) \tag{2.6}$$

where $b(s)$ is the net foreign asset position at time s and $r+\phi(s)$ is the *forward* instantaneous interest rate. Implicitly, the maturity of each debt instrument is ds and rolls-over continuously at the current rates $r + \phi(s)$, $\forall s$.

The IBC presumes endless access to capital markets insofar a credit event has not arrived, in which case the current account would become zero. The risk-neutral financial markets already take account of the inherent risk in a loan extended to a particular sovereign borrower by charging her a premium over the risk free rate. The intertemporal budget constraint (IBC) is obtained by integrating (2.6) with respect to time in consideration of the transversality condition $\lim_{s \longrightarrow \infty} b(s)\ e^{-\left(r+\int_t^s \phi(\tau)d\tau\right)(s-t)} = 0$. The IBC reduces to

$$\int_t^{\infty} c(s) \exp\left[-r(s-t) - \int_t^s \phi(\tau)d\tau\right] ds \; = \; W(t) \tag{2.7}$$

where $W(t)$ stands for intertemporal wealth, defined as the net foreign assets position $b\,(t)$ plus the present value of the future income stream discounted at market lending rates:

$$W\,(t) \equiv b_t + \int_t^{\infty} y \exp\left[-r\,(s-t) - \int_t^s \phi(\tau)d\tau\right] ds. \tag{2.8}$$

The IBC (2.7) shows that the market value of the planned consumption flow can be no greater than total wealth. Note that the time path of the risk premium $\{\phi(s)\}_t$ is known *ex-ante* as it is provided by the forward-rate curve $\{r + \phi(s)\}_t, \forall s \geq t$.

It follows from (2.8) and the current account (2.6) that the equation of motion for wealth can be stated as the difference between the return on wealth, or *annuity value of wealth*, and instantaneous consumption:[9]

$$\dot{W}\,(s) = (r + \phi(s))W\,(s) - c\,(s)\,, \quad \forall \phi\,(s) \tag{2.9}$$

Definition 1 *The consumption rate $C\,(t)$ is defined as the marginal propensity to consume out of wealth:*

$$C\,(t) \equiv c\,(t)\,/W\,(t)$$

Definition 2 *Similarly, the saving rate $S\,(t)$ is defined as the rate of total wealth accumulation*

$$S\,(t) \equiv \dot{W}\,(t)\,/W\,(t)\,. \tag{2.10}$$

From the above definition and from (2.9) we can write:

$$S\,(t) = r + \phi\,(t) - C\,(t)\,, \tag{2.11}$$

which reveals that the saving rate is equal to the *wealth annuity rate* $(r + \phi)$ minus the consumption rate.

[9]See the Appendix 2.C.7 for the derivation of the wealth equation of motion.

2.2.4 Characterization of the problem

From (2.4) and (2.6), the present value Hamiltonian can be written as

$$\underset{c_s,\mu_s,b_s,}{H} = U\left(c_s\right)\chi_s + \mu_s\left((r+\phi_s)b_s + y - c_s\right) \tag{2.12}$$

where $\chi_s \equiv \chi(t,s)$ is the representative agent's *augmented* discount factor given by (2.5), in which the discount rate may be time-varying.[10] The shadow price of the budget constraint μ_s is the marginal utility of wealth valued at present time t or, equivalently, the present value (valued at time t) of the utility generated by releasing the constraint by one unit .[11]

For convenience, we can write the problem in the form of a current value Hamiltonian

$$\underset{c_s,\lambda_s,b_s,}{\hat{H}} = U\left(c_s\right) + \lambda_s\left((r+\phi_s)b_s + y - c_s\right) \tag{2.13}$$

where $\lambda_s = \mu_s\chi_s^{-1}$ is the shadow price valued at time s. Note from (2.5) that $\chi_s^{-1} = \digamma(t,s)^{-1}\exp\left[\int_t^s \kappa(\tau)d\tau\right], \quad \forall s \in [t,\infty)$.

2.3 Constant Hazard Rate

In this Section we solve the model assuming that the risk of default is *fixed* at certain level $\phi > 0$. Most of the insights of the paper come from this Section, and in particular the negative relationship between the saving rate of a hyperbolic agent and the hazard rate that determines the probability of a arrival of a crisis.

[10]See the Appendix for the derivation of the Pontryagin' Maximum Principle conditions for the general dynamic optimization problem with *time-varying* discount rates. The time-independent, exponential discounting is a special case.

[11]More generally, μ_s is the extra utility valued at time t of releasing the IBC by one marginal unit.

2.3.1 Intertemporal Budget Constraint

If the risk premium were constant, equation (2.7) would reduce to

$$\int_t^{\infty} c(s) \exp\left[-\left(r+\phi\right)(s-t)\right] ds = b(t) + \frac{y}{r+\phi} \tag{2.14}$$

where the RHS is the intertemporal wealth defined as

$$W(t) = b(t) + \frac{y}{r+\phi} \tag{2.15}$$

Recalling (2.9) we can write

$$\dot{W}(t) = (r+\phi)\, W(t) - c(t) \tag{2.16}$$

where $(r+\phi)\, W(t)$ is the annuity value of wealth.

The current account (2.6) simplifies to

$$\dot{b}(t) = (r+\phi)\, b\,(t) + y - c(t)$$

or, in terms of wealth, given that y and $r+\phi$ are constant,

$$\dot{b}(t) = (r+\phi)\, W(t) - c(t) \tag{2.17}$$

and therefore

$$\dot{W}(t) = \dot{b}(t) \tag{2.18}$$

Note that only if the representative agent consumed the same amount as the wealth annuity, the current account would be in balance and wealth would remain constant over time.

2.3.2 Exponential Discounting

At the risk of being over-explanatory, we solve the exponential, time-consistent case in a step-by-step manner, which will make it readily comparable to the hyperbolic discounting case analysed in Section 2.3.3.

The discount factor

In the classic exponential discounting (ED) case, the consumer's discount factor is

$$F(t,s) = \exp\left[-\rho(s-t)\right]$$

and the corresponding augmented discount factor defined in (2.5) is

$$\chi_s = \exp\left[-\rho(s-t) - \int_t^s \kappa(\tau)d\tau\right]$$

where $\kappa(\tau) = \kappa = \phi$, $\forall\tau$ in line with our assumption of homogeneity of beliefs formulated in (2.1).

The objective function (2.4) takes the form

$$\int_t^\infty U(c(s))\exp\left[-(\rho+\kappa)(s-t)\right]ds$$

where we note that the ED consumer solves her intertemporal consumption problem as if the status-quo ante situation were non-stochastic and were due to last forever, immune to financial crises, but with the subjective discount rate being augmented by the subjective risk of default κ.[12]

[12] As noted in Chapter 1, Section 1.2.3, this result is reminiscent of Blanchard (1985), who refers to this occurrence in the context of the "lifetime utility maximization with constant probability of death."

First order conditions

We want to determine the consumption and wealth dynamics for given parameters r, ρ and ϕ. For tractability purposes, we assume log instantaneous utility[13]

$$U(c(s)) = \ln c(s). \tag{2.19}$$

The consumer's optimal intertemporal choice will be determined by the Maximum Principle conditions. The first-order-condition (FOC) of the current value Hamiltonian (2.13) with respect to $c(s)$ is

$$\frac{1}{c(s)} = \lambda(s) \tag{2.20}$$

and the FOC with respect to the net foreign asset position $b(s)$ is

$$\frac{\dot{\lambda}(s)}{\lambda(s)} = \rho + \kappa - (r + \phi). \tag{2.21}$$

Together, both FOCs result in the Euler equation:

$$\frac{\dot{c}(s)}{c(s)} = r + \phi - (\rho + \kappa), \qquad \forall s \geq t \tag{2.22}$$

where ϕ is assumed to be constant throughout the current Section, implying that the forward rate curve also remains flat at the level $r + \phi$, which is common knowledge to all forward-looking agents who set $\kappa = \phi$. So the Euler equation simplifies to

$$\frac{\dot{c}(s)}{c(s)} = r - \rho \tag{2.23}$$

In order to find the instantaneous consumption at time t, we use the equation of

[13]The qualitative results in this paper are also valid for the more general CRRA utility, as will be verified later in this Section.

motion (2.23) into the resource constraint (2.14) and get

$$c(t)\int_{t}^{\infty} e^{(r-\rho)(s-t)} \cdot e^{-(r+\phi)(s-t)} ds = b(t) \; + \; y\int_{t}^{\infty} e^{-(r+\phi)(s-t)} ds \tag{2.24}$$

or,

$$\frac{c(t)}{\rho+\phi} = b(t) + \frac{y}{r+\phi}, \tag{2.25}$$

where the RHS is simply the intertemporal wealth $W(t)$ defined in (2.15). Thus, the last equation can be written as

$$c(t) = C\,(t)\,W(t) \tag{2.26}$$

where the consumption rate is

$$C(t) = \rho + \phi \tag{2.27}$$

and the saving rate (2.11) becomes

$$\begin{aligned} S(t) &= \frac{\dot{W}\,(t)}{W\,(t)} = (r+\phi) - (\rho+\phi) \\ &= r - \rho \end{aligned} \tag{2.28}$$

which is given by the difference between the annuity rate of wealth $r + \phi$ and the augmented subjective discount rate $\rho + \kappa$, where $\kappa = \rho$ due to the heterogeneity of beliefs assumption.

Interpretation The last equation indicates that wealth growth rate is determined by the difference between the rate of return on savings and the rate at which wealth is depleted by consumption. In other words, wealth would increase (decrease) over time if the wealth annuity $(r+\phi)\,W_t$ were greater (lower) than per period consumption $(\rho+\phi)\,W(t)$. A *patient* agent is a natural saver and would be characterized by a *pos-*

itive saving rate $S(t)$. Thus, her consumption would be lower than the annuity value of wealth, which would allow her to run current account surpluses to ensure wealth accumulating dynamics.

The opposite would be true for an *impatient* agent characterized by a *negative* saving rate. Her per period consumption would be higher than the wealth annuity and would run current account deficits resulting in wealth being depleted over time. Incidentally, note that if the annuity rate and the consumption rate were equal, i.e. $\rho + \phi = r + \phi$, the saving rate would be zero and the current account would be in balance implying that both wealth and the consumption level would remain constant over time.

Notice from (2.28) that, in the ED case, the saving rate $S(t)$ is *constant* and insensitive to changes in risk ϕ.[14] That is, the current account is set to remain an invariable share of wealth over time, regardless of the probability of a crisis arrival. In contrast, as we shall see in Section 2.3.3, this insensitivity property would not hold when agents are time-inconsistent. On the contrary, the level ϕ will have important implications for the consumption and saving behavior of hyperbolic discounting agents.

Shocks in productive capacity y and in risk premium ϕ

The effect of unanticipated shocks in y or in ϕ on consumption depend on the agent's degree of impatience, reflected in her saving rate. Consider a *patience-neutral* representative agent for whom the saving rate $S = r - \rho$ is equal to zero. In this case, the economy's current account would be in balance before and after the shock.

An unanticipated shock in production y or in risk premium ϕ would make the system jump on impact from one steady state to another, while the current account would remain in balance. In particular, an increase in permanent income y would result in a one-on-one shift in consumption, i.e. $\Delta c = \Delta y$.

[14]The properties regarding the linear sensitivity of the consumption rate and insensitivity of the saving rate to changes in ϕ is not exclusive of the log utility case. In particular, they also hold in the more general CRRA utility. See the Appendix.

In turn, a permanent shock in risk premium $\Delta\phi$ would shift the patience-neutral agent's consumption to a new steady state that would be either higher or lower depending on whether the agent is a net debtor or a net creditor. In particular, the steady state consumption would shift by $\Delta c = \Delta\phi\ b$. For instance, if the risk premium rose by $\Delta\phi > 0$ and the agent were a net debtor, i.e. $b < 0$, the steady state consumption would fall in order to give way to a higher share of disposable income that needs to be reallocated to servicing higher interest payments.

Next, consider the effects of unanticipated shocks when the representative agent is *not* patient-neutral, i.e. when the saving rate $S \neq 0$. Let's recall (2.26) and rewrite it as

$$c(t) = (\rho + \phi)\left(b(t) + \frac{y}{r + \phi}\right).$$

Take a *patient* consumer, characterized by $S > 0$ and positively tilted consumption. A positive shock in permanent income Δy would result in higher consumption on impact but of *lesser* magnitude than Δy; specifically, $\Delta c = \frac{\rho+\phi}{r+\phi}\Delta y < \Delta y$. This *undershooting* of consumption reflects the patient consumer's saving-prone behavior, as she reallocates some of the wealth windfall to increase longer-term consumption at the expense of immediate consumption.[15]

A mirror-image effect would happen for an *impatient* representative agent with a negative saving rate, i.e. $S < 0$, and a negatively tilted intertemporal consumption path. In such a case, a positive shock Δy would result in an *overshooting* of consumption, i.e. $\Delta c = \frac{\rho+\phi}{r+\phi}\Delta y > \Delta y$.

As for a risk premium shock, it would also bring about a non-neutral wealth effect

[15] In particular, a permanent increase in the productive capacity Δy would allow consumers to boost their consumption schedule by an equal amount Δy in every future period. However, a "parallel" upward shift of the intertemporal consumption schedule would violate the Euler equation $\dot{c}(t)/c(t) = r - \rho$ as it would imply a decline of the rate or growth. Therefore, the optimal consumption path is such that a higher share of the windfall is being allocated to future consumption at the expense of short-term consumption.

if an agent is not patient-neutral. Let's differentiate (2.26) to get

$$\begin{aligned}\frac{dc(t)}{d\phi} &= \underbrace{C'W}_{\text{I.E.}} + \underbrace{CW'}_{\text{W.E.}} \\ &= W(t) - C\frac{y}{(r+\phi)^2},\end{aligned}$$

where $C = r + \phi - S$ by equation (2.11). An increase in ϕ implies a *positive income effect* (higher consumption rate C) captured by the first term of the RHS and a *negative wealth effect* (lower intertemporal wealth W) captured by the second term of the RHS.

We can rewrite this last equation as

$$\frac{dc(t)}{d\phi} = b(t) + S\frac{y}{(r+\phi)^2}. \tag{2.29}$$

Leaving aside the net asset position $b(t)$, the second term reveals that the net effect of higher ϕ on consumption would depend on the relative magnitudes of ρ versus r, and specifically on the saving rate $S = r - \rho$. This term reflects the tension between the income and the wealth effects. As for the first term, we note that if $b(t) > 0$, the income effect would be strengthened as higher ϕ would imply higher revenues from a positive net foreign asset position, while if $b(t) < 0$, the income effect could be hampered as higher ϕ would imply a more burdensome debt service.

In general $\frac{dc(t)}{d\phi}$ would be positive for a *patient*, saving-prone agent (unless the outstanding debt $-b(t)$ were sufficiently large) as an unanticipated rise in ϕ would imply higher greater returns on savings, which would allow for greater future *and* present consumption, in conformity with the Euler equation (2.23).

The case of an *impatient* agent would be the mirror image of the one described above: a rise in risk premium would imply an increase in the cost of borrowing, which would result in lower future and present consumption (unless the initial net asset position is positive and sufficiently large).

Importantly, note that for both patient and impatient consumers, the consumption rate $C = \rho + \phi$ increases linearly in ϕ while the saving rate $S = r - \rho$ is insensitive to changes in ϕ, as can be checked in (2.27) and (2.28).

2.3.3 Hyperbolic Discounting

"Lord, make me chaste, but not yet." St. Augustine (354-386)

In this Section we depart from the orthodox approach and let the representative agent discount intertemporal utility flows with a hyperbolic factor rather than with an exponential one. We show how, when the representative agent is *naïve*, the hyperbolic model could account for a negative relationship between the saving behavior and an exogenous probability of a crisis arrival. Specifically, the saving rate would deteriorate when the systemic risk ϕ increases and the sign of the current account would depend on the level of ϕ.

Given the insensitivity of the saving rate to permanent changes in systemic risk in the frictionless ED model, the economic theory has enriched the basic setup by taking account of the temporary nature of shocks, or by incorporating frictions, asymmetries and heterogeneities.

We propose a new mechanism that generates such a negative relationship and may shed some light on the behaviour of the saving rate and the current account observed in different occurrences in the empirical world. Our model would be consistent with a case of study in which two economies with similar growth outlooks present opposite current account dynamics on the back of their dissimilar fundamental vulnerabilities.

The hyperbolic model can also account for current account *sign reversals* when an exogenous shock has affected a country's fundamental probability of a crisis. This result could help explain how an economy can be in a position to change its saving patterns completely and switch its current account from deficits to surpluses even if the nature of the shock were permanent.

In what follows, we continue to have in mind a representative agent in a country that faces an exogenous probability of a systemic crisis. We leave for future research additional applications of this mechanism to other settings and agents. I

Naivety and time inconsistency

We introduce agents who discount utility flows with a hyperbolic function instead of an exponential one. An agent with hyperbolic discounting preferences faces a time inconsistency problem characterized by a higher degree of impatience in short term trade-offs than in long term ones.

In the context of this paper, the representative HD consumer is *naïve*, as opposed to sophisticated.[16] A *naive* agent draws an optimal intertemporal plan for saving and consumption, which she believes she will stick to. However, as time goes by, she would decide to depart from her plan and reset her consumption and saving strategy, and she will do it continually without ever realizing her time-inconsistency problem.

In other words, once the naive hyperbolic agent is one instant away, she reassesses her intertemporal maximization problem and draws a new consumption-saving plan that is effectively *inconsistent* with the previous one. To be sure, every consumption plan will prove to be nothing but "wishful-thinking", as it will be inevitably violated by the agent's future *selves* who will choose to divert from it (unless we assumed some type of intertemporal commitment device). As a result, the *actual* consumption path differs from all those paths that an agent had originally anticipated.

Notation 2 ***Anticipated versus actual variables.*** *The expression* ${}_tx(s)$ *denotes the*

[16]Note that although empirical evidence suggests at least a degree of naivity, most of the theoretical work with hyperbolic discounting agents have assumed fully sophisticated agents, including the classic references of Laibson (1994, 1997) and Harris and Laibson (2001a, 2001b, 2008). These papers assume that consumers are sophisticated in the sense that they realize that in the next period her own *self* will not align to the planned consumption path but will instead tend to overconsume. Therefore, in that sophisticated setting, the optimal decision is to consume even more at present time, so that future selves have less wealth at their disposal.

value that at time t an agent anticipates that a variable x will take at time s, $\forall s > t$. For the sake of handiness, if a variable is parametric from a representative agent's viewpoint –implying that its anticipated and actual future values are identical– we will dispense with the anticipation operator altogether and simply express the variable as $x(s)$.

The discount factor

We put forward a generalized hyperbolic function as initially defined by Harvey (1986) and derived axiomatically by Prelec (1989) and Loewenstein and Prelec (1992). The hyperbolic discounting agent (HD) discounts events occurring $(s-t)$ periods away with weight

$$F(t,s) = (1+\alpha(s-t))^{-\gamma/\alpha} \tag{2.30}$$

where parameters γ and α are positive and are assumed to be such that

$$\gamma > r > \gamma - \alpha > 0.$$

Contrary to the ED factor that declines at constant rate ρ, the rate of decay of the HD factor (2.30) *falls* with the time horizon $s-t$:[17]

$$-\frac{\dot{F}(t,s)}{F(t,s)} = \frac{\gamma}{1+\alpha(s-t)} \tag{2.31}$$

[17]Note that the exponential discount factor is the limit of the hyperbolic discount factor (2.30) when α tends to 0:

$$\begin{aligned} \lim_{\alpha \longrightarrow 0}\left[(1+\alpha(s-t))^{-\gamma/\alpha}\right] &= \lim_{\alpha \longrightarrow 0}\left[\left((1+\alpha(s-t))^{1/\alpha}\right)^{-\gamma}\right] \\ &= (\exp[s-t])^{-\gamma} \\ &= \exp[-\gamma(s-t)] \end{aligned}$$

where the discount rate (2.31) is constant at γ.

Following (2.5), the corresponding augmented discount factor takes the form

$$\chi_s = \exp\left[-\int_t^s {}_t\kappa(\tau)d\tau\right]\,(1+\alpha(s-t))^{-\gamma/\alpha}$$

where ${}_t\kappa(\tau) = \kappa$ is constant and equal to ϕ in accordance with (2.1).

First order conditions

Let's determine the consumption and wealth dynamics for given parameters r, γ, α, and ϕ. As in the previous Section, we assume instantaneous log utility function, so the consumer's objective function (2.4) can be written as

$$\int_t^{\infty} \ln\left[{}_tc(s)\right]\exp\left[-\kappa\,(s-t)\right](1+\alpha(s-t))^{-\gamma/\alpha}\,ds$$

where ${}_tc(s)$ stands for *anticipated* consumption.

The consumer's optimal consumption-saving paths are found by applying the Maximum Principle conditions.[18] The FOC with respect to *anticipated* consumption is

$$\frac{1}{{}_tc(s)} = {}_t\lambda(s) \tag{2.32}$$

The FOC with respect to the *anticipated* net asset position ${}_tb(s)$ renders[19]

$$\frac{{}_t\dot{\lambda}(s)}{{}_t\lambda(s)} = \frac{\gamma}{1+\alpha(s-t)} + \kappa - (r+\phi) \tag{2.33}$$

$\forall s \geq t$, where ϕ is assumed to be constant. Equations (2.32) and (2.33) combine to bring about the equation of motion of *anticipated* consumption, or the anticipated Euler

[18]See Appendix 2.B for the derivation of the Maximum Principle conditions for the general case of a time-varying discount factor $F(t,s)$.

[19]See the Appendix for details of the derivation of the FOC with respect to ${}_tb(s)$ in the HD case.

equation

$$\frac{{}_t\dot{c}(s)}{{}_tc(s)} = r + \phi - \left(\frac{\gamma}{1+\alpha(s-t)} + \kappa\right) \tag{2.34}$$

$\forall s \geq t$, which, under the assumption of homogeneity of beliefs and perfect information, reduces to

$$\frac{{}_t\dot{c}(s)}{{}_tc(s)} = r - \frac{\gamma}{1+\alpha(s-t)}. \tag{2.35}$$

From the Euler equation we note that the assumption that $\gamma > r$ essentially reflects the consumer's relative short-run impatience. In particular, the term $\frac{\gamma}{1+\alpha(s-t)}$ is simply the rate of decline of the discount function $F(t,s)$ specified in (2.31), which falls with horizon $s-t$, for all $s \geq t$.[20] Accordingly, the HD agent anticipates that the negative present consumption growth ${}_t\dot{c}(t)/{}_tc(t) = r - \gamma$ becomes *less negative* and eventually turns *positive* in the future. Effectively, the optimal anticipated consumption plan displays a U-shape where the inflection point occurs at a time $\breve{s}$ such that (See Figure 2.1):

$$r = \frac{\gamma}{1+\alpha(\breve{s}-t)}. \tag{2.36}$$

The FOC with respect to the shadow price results in the *anticipated* current account

$${}_t\dot{b}(s) = (r+\phi)\,{}_tb(s) + y - {}_tc(s) \tag{2.37}$$

which we integrate with respect to time, from $s \geq t$ to infinity, to obtain IBC anticipated at time t :

$$\begin{aligned} \int_s^\infty {}_tc(\tau)e^{-(r+\phi)(\tau-s)}d\tau &= {}_tb(s) + y\int_s^\infty e^{-(r+\phi)(\tau-s)}d\tau \\ &\quad - \lim_{T\longrightarrow\infty} {}_tb_T\, e^{-(r+\phi)(T-s)} \end{aligned} \tag{2.38}$$

[20]The ED model would have this term replaced by a constant subjective rate of discount ρ as in equation (2.23).

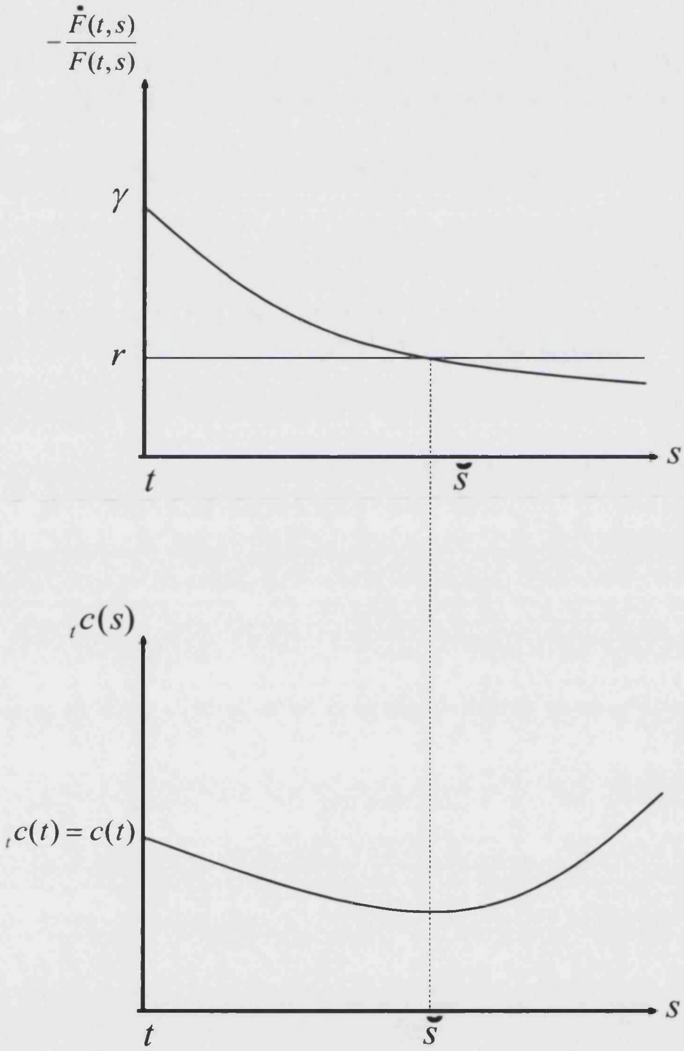

Figure 2.1: Anticipated consumption path of a hyperbolic discounting agent.

$\forall s \geq t$, where the last term $\lim_{T \longrightarrow \infty} {}_tb_T \, e^{-(r+\phi)(T-s)} = 0$ by the Transversality condition.

The saving and consumption plans

Intuition In order to determine the dynamics of the *actual* variables in the model, it is essential to understand how the naive, time-inconsistent agent *anticipates* the evolution of her control variables. The HD consumer is short run impatient but more patient in the long run. She would be eager to spend a higher share of wealth in immediate consumption and, at the same time, she plans to become a better saver in the future, which would allow her to accumulate wealth for the longer term. Thank to this wealth accumulation the consumer anticipates that she would be able to afford higher future consumption despite the higher saving *rate* and lower consumption *rate.* In terms of levels, high consumption in the short term and in the long term would come at the expense of medium term consumption.

More specifically, the HD agent chooses high levels of consumption at the present time t and anticipates (naively, in a wishful thinking manner) a higher future saving rates ${}_tS(s)$ and a decreasing consumption rate ${}_tC(s)$. When the consumption rate ${}_tC(s)$ falls below the annuity rate $r+\phi$ the saving rate would turn positive, as would also do the growth rate of wealth[21]. At a later point in time $\breve{s}$, defined by equation (2.36), the growth rate of wealth would have overtaken the rate of decline of the consumption rate and consumption ${}_tc(s) = {}_tC(s) \; {}_tW(s)$ itself would start growing at positive rates. Therefore, the saving plan is consistent with the U-shape consumption schedule set in the Euler equation (2.35). (See Figure 2.2).

The HD agent's saving behavior is intuitive from the point of view of the *opportunity cost of consumption.* The instantaneous subjective discount rate is $\gamma / (1 + \alpha(s-t))$; it takes the value $\gamma > r$ at present time t and declines with time horizon $s-t$. This implies that the discount factor declines at a relatively fast pace in the short term but slowly in

[21]Recall that the wealth growth rate is $\frac{\dot{W}(t)}{W(t)} = S(t)$ in line with (2.10).

the long term, so that the consumer would be increasingly indifferent about consuming in either two consecutive periods if the time horizon is far away. As the opportunity cost of consumption is anticipated to increase in the future, it would also make sense to anticipate a more aggressive future saving plan.

Characterization Let's characterize more formally the saving and consumption path described above. Using the IBC (2.35) and the Euler equation (2.38) we obtain

$$\frac{{}_tc(s)}{\tilde{\Lambda}(t,s;\phi)} = {}_tb(s) + \frac{y}{r+\phi},$$

where the RHS is *anticipated wealth* ${}_tW(s)$ defined as the net present value of the future income plus the net foreign asset position that at time t the consumer anticipates will prevail in a future point in time s:

$${}_tW(s) = {}_tb(s) + \frac{y}{r+\phi}$$

Thus, the *anticipated consumption* $s-t$ periods ahead would be given by[22]

$${}_tc(s) = \tilde{\Lambda}(t,s;\phi)\,{}_tW(s)$$

and the *anticipated consumption rate* ${}_tC(s)$ becomes

$${}_tC(s) = \tilde{\Lambda}(t,s;\phi) \qquad (2.39)$$

where $\tilde{\Lambda}(t,s;\phi)$ is given by

$$\tilde{\Lambda}(t,s;\phi) \equiv \left(\int_s^{\infty} e^{-\phi(\tau-s)}\left(1+\frac{\alpha}{1+\alpha(s-t)}(\tau-s)\right)^{-\frac{\gamma}{\alpha}} d\tau\right)^{-1}. \qquad (2.40)$$

[22]See the Appendix for details on the derivation of (2.39) and the characterization of $\tilde{\Lambda}(t,s;\hat{\phi})$.

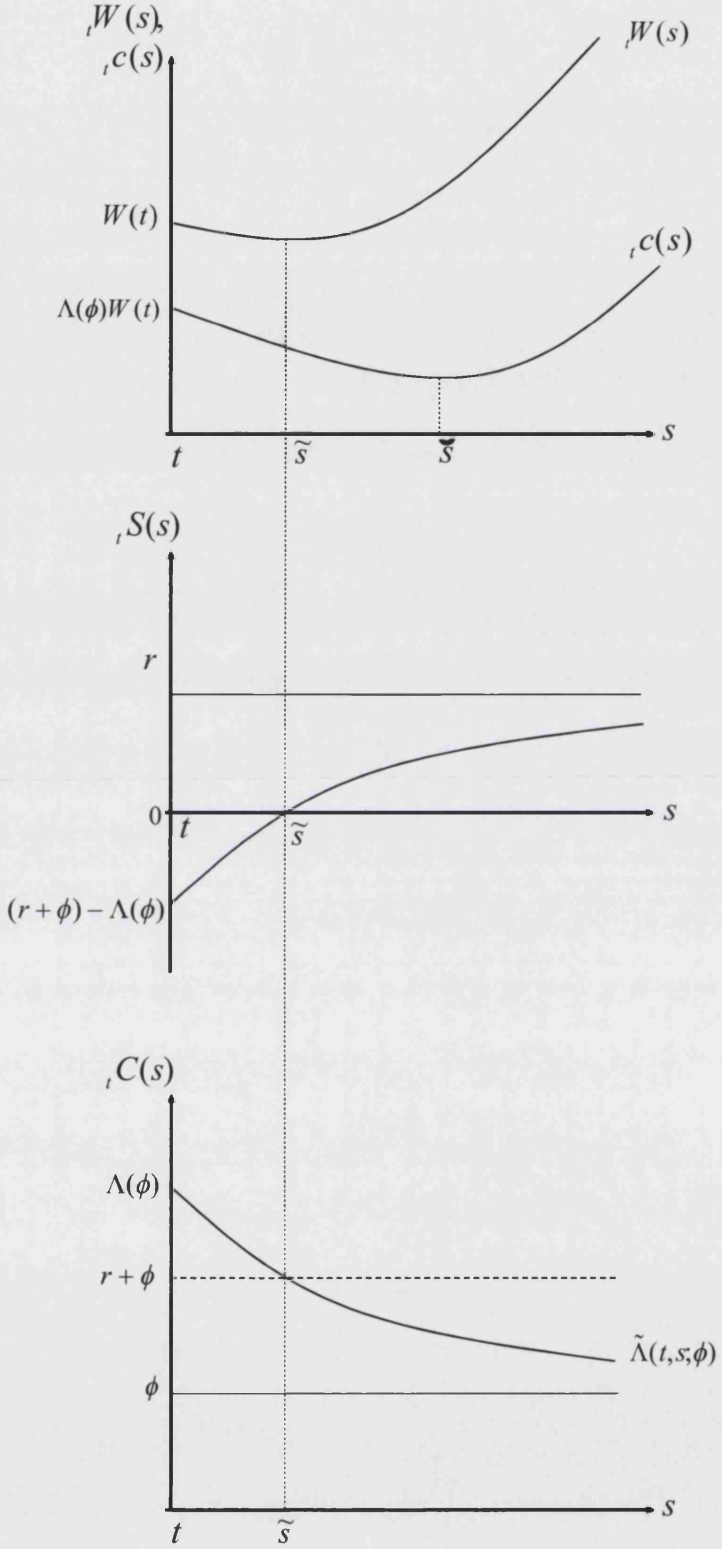

Figure 2.2: Anticipated consumption and saving rates of a naive hyperbolic discounting agent. In this case the initial saving rate $S(t)$ is negative but is expected to turn positive after $\tilde{s}$.

Proposition 2.1 *For any constant risk premium level $\phi \in [0, \infty)$, the HD representative agent's optimal anticipated path is such that the consumption rate ${}_tC(s)$ declines over time and asymptotically tends to ϕ.*

Proof. *See the Appendix.*

The Appendix gives the characterization of $\tilde{\Lambda}(t, s; \phi)$. See the bottom diagram in Figure 2.2. Under the assumption that $\gamma > \alpha > 0$, the function $\tilde{\Lambda}(t, s; \phi)$ is greater than ϕ, declines with time horizon $s - t$ and has the following limits:

$$\lim_{s \longrightarrow \infty} \tilde{\Lambda}(t, s; \phi) = \phi \quad \text{and} \quad \lim_{s \longrightarrow t} \tilde{\Lambda}(t, s; \phi) = \Lambda(\phi),$$

where

$$\Lambda(\phi) \equiv \left(\int_t^{\infty} e^{-\phi(\tau - t)} (1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} d\tau \right)^{-1} > \phi \tag{2.41}$$

Therefore, recalling (2.39), the *long-term anticipated consumption rate* is

$$\lim_{s \longrightarrow \infty} {}_tC(s) = \phi, \tag{2.42}$$

and the *present consumption rate* is

$$\lim_{s \longrightarrow t} {}_tC(s) = C(t) = \Lambda(\phi) > \phi. \tag{2.43}$$

Remark 1 *The declining anticipated path of the consumption rate put forward in Proposition 2.1 is in clear contrast with the consumption rate path of an ED representative agent formulated in equation* (2.27). *In particular, note that an ED agent would consume a* constant *share of wealth $\rho + \phi$ at all times.*

Corresponding to a declining consumption rate is an increasing saving rate. It follows from (2.11) and (2.37) that the anticipated current account is given by

$$\begin{aligned} {}_t\dot{b}\,(s) &= (r+\phi)\ {}_tW\,(s) - {}_tc\,(s) \\ &= \left[(r+\phi) - \tilde{\Lambda}(t,s;\phi)\right]\ {}_tW\,(s) \end{aligned}$$

and given that $\dot{W}(s) = \dot{b}(s)$ from (2.18), the anticipated saving rate is

$$ {}_tS\,(s) = (r+\phi) - \tilde{\Lambda}(t,s;\phi) \tag{2.44}$$

where expression $\tilde{\Lambda}(t,s;\phi)$ is given by (2.40). Recalling the limits (2.42) and (2.43), the long-term anticipated saving rate becomes

$$\lim_{s \longrightarrow \infty} {}_tS\,(s) = \lim_{s \longrightarrow \infty} \frac{{}_t\dot{b}\,(s)}{{}_tW\,(s)} = r$$

That is, in the long run, as the anticipated consumption rate tends towards ϕ, wealth growth itself would be anticipated to tend to r, i.e. $\lim\limits_{s \longrightarrow \infty} \frac{{}_t\dot{W}(s)}{{}_tW(s)} = r$. Similarly, the *present saving rate* $S\,(t)$, such that $s = t$, would be

$$S\,(t) = \lim_{s \longrightarrow t} \frac{{}_t\dot{b}\,(s)}{{}_tW\,(s)} = r + \phi - \Lambda(\phi). \tag{2.45}$$

where $\Lambda\,(\phi)$ is given by (2.41). Note that the initial saving rate and the current account could be either positive or negative depending on whether $\Lambda(\phi)$ is greater or lower than $r + \phi$.

Proposition 2.2 *For any initial current account (either in surplus or in deficit), an increase in the anticipated saving rate specified in (2.44) would generate enough future wealth growth to turn future consumption growth positive.*

Proof. *See Appendix.*

Proposition 2.2 implies that the U-shape determined by the Euler equation is consistent with the declining path of the consumption rate specified in Proposition 2.1 and equation (2.39). In particular, even if the saving rate were initially negative (see the middle diagram in Figure 2.2), the consumer would anticipate to turn it positive at some point in time $\tilde{s}$ and to generate enough pace of wealth accumulation that would allow consumption to start increasing at a certain point in time $\breve{s} > \tilde{s}$, despite the declining consumption rate. Similarly, if the saving rate were initially positive, the agent expects it to improve it in the future so that the pace of wealth accumulation outpaces the rate of decline of the consumption rate (see Figure 2.3).

The actual consumption path

The representative consumer is dynamically time-inconsistent. That is, the values of consumption for future dates $t+s$, $\forall s$ chosen at date t do not solve the maximization problem of the same consumer standing at a point in time other than t. The implication is that the anticipated consumption path $\{{}_tc(s)\}$ would differ from the *actual* consumption path $\{c(s)\}$, except for $s = t$.

Characterization The actual consumption path would be drawn by a continuum of *present* consumption values $\{{}_tc(t)\}$ that solve the consumer's intertemporal optimization problem at every point in time t, subject to the anticipated IBC, i.e. the continuum of *initial* points ${}_tc(t) = c(t)$ that solve the anticipated Euler equation (2.35) subject to (2.38).

In order to find the expression for ${}_tc(t)$ in the consumer's optimization problem, let's rewrite (2.35) as

$$\frac{d}{ds}(\ln_t c(s)) = r - \frac{\gamma}{1 + \alpha(s - t)}$$

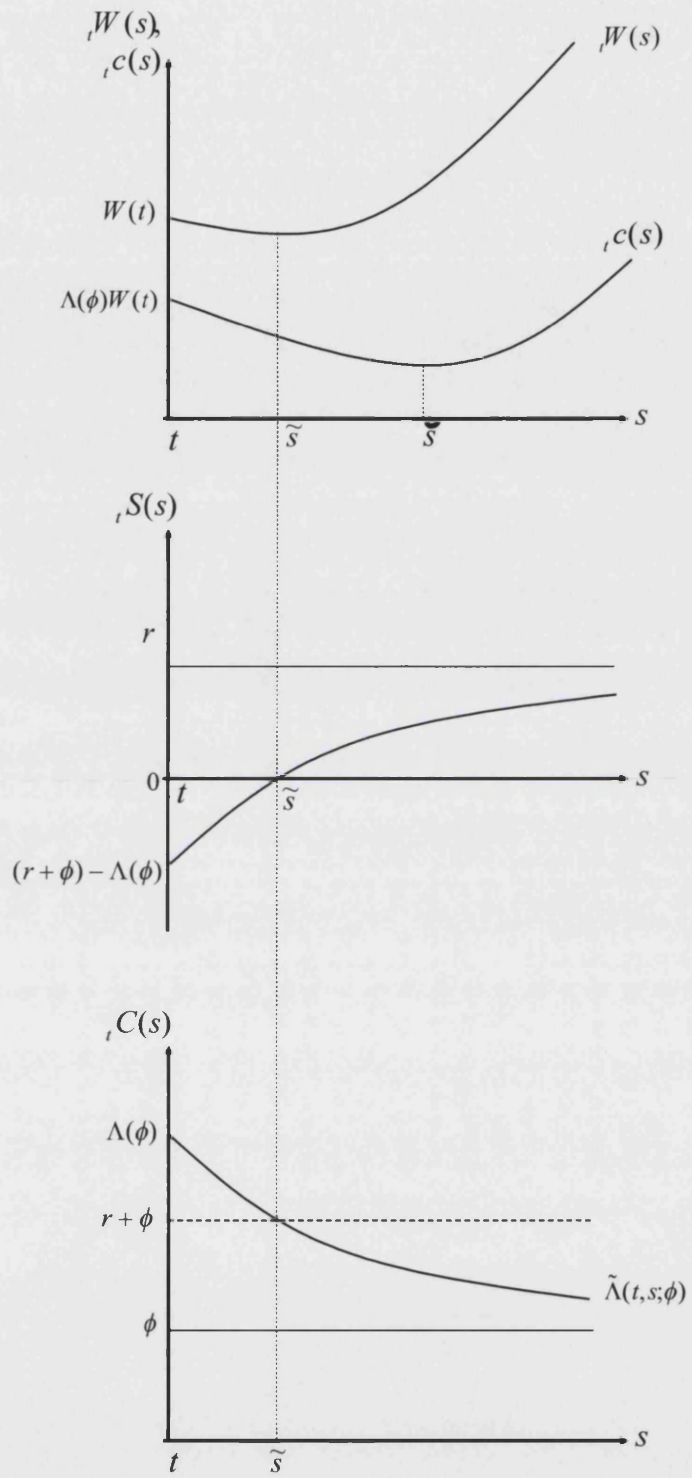

Figure 2.3: Anticipated consumption and saving rates of a naive hyperbolic discounting agent. In this case the initial saving rate $S(t)$ is already positive, and is anticipated to improve in the future so that wealth accumulation outpaces the rate of decline of the consumption rate.

and integrate both sides over time to get[23]

$$
{}_t c(\tau) = {}_t c(t)\, e^{r(\tau-t)} \left(1+\alpha(\tau-t)\right)^{-\frac{\gamma}{\alpha}}
$$

$\forall \tau \geq t$, where ${}_t c(t) = c(t)$. Plug this expression in the LHS of the *anticipated* budget constraint (2.38) for starting time t to obtain

$$
c(t)\int_t^{\infty} \left(1+\alpha(\tau-t)\right)^{-\frac{\gamma}{\alpha}} \exp\left[-\phi\left(\tau-t\right)\right] d\tau = b(t) + y\int_t^{\infty} \exp\left[-\left(r+\phi\right)\left(\tau-t\right)\right] d\tau,
$$

or

$$
\frac{c(t)}{\Lambda\left(\phi\right)} = b(t) + \frac{y}{r+\phi} \tag{2.46}
$$

So, actual consumption rate becomes

$$
C(t) = \Lambda\left(\phi\right). \tag{2.47}
$$

which defines the share of consumption for a given risk premium ϕ and is perfectly consistent with (2.43).

The function $\Lambda\left(\phi\right)$ specified in (2.41) is characterized by the following properties (see Figure 2.4):[24]

(i) under the assumptions that $\gamma, \alpha, r > 0$ and $0 < \gamma - \alpha < r$:

$$
\Lambda\left(\phi\right) > \phi > 0, \qquad \Lambda'\left(\phi\right) > 1, \qquad \Lambda''\left(\phi\right) < 0,
$$

(ii) the partial derivatives with respect to the parameters are $\partial\Lambda/\partial\gamma > 0$ and $\partial\Lambda/\partial\alpha < 0$,

[23]See Appendix for details of the derivation.

[24]See the Appendix for details of the derivation of these properties.

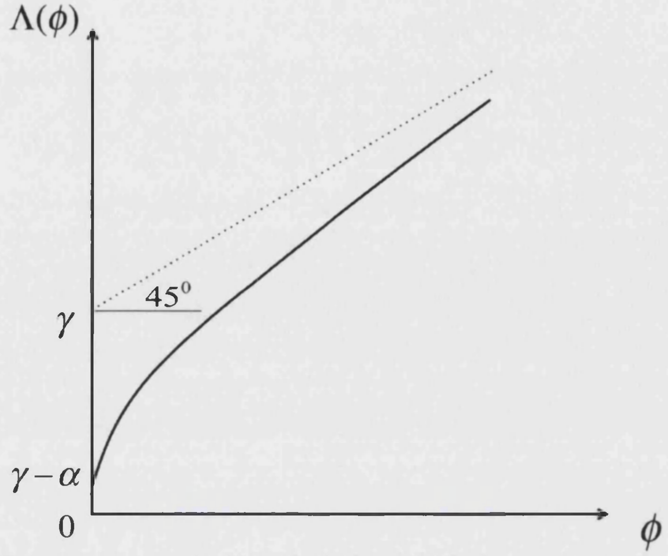

Figure 2.4: Function $\Lambda(\phi)$ has slope greater than 1.

(iii) the limits are

$$\Lambda(0) = \gamma - \alpha \qquad \lim_{\phi(t)\longrightarrow\infty} [\Lambda(\phi(t)) - \phi(t)] = \gamma$$

So, actual consumption will be a constant share of wealth given by (2.47) for a given risk premium ϕ. Correspondingly, wealth dynamics will be given by the saving rate $S(t)$. Recalling equation (2.17), we note that the actual current account and the saving rate would be positive (negative) if the wealth annuity rate is greater (lower) than the consumption rate given by (2.47), that is

$$\begin{aligned} S(t) &\equiv \dot{W}(t)/W(t) = \dot{b}(t)/W(t) \\ &= (r+\phi) - C(t) \end{aligned}$$

or

$$S(t) = r + \phi - \Lambda(\phi). \tag{2.48}$$

As opposed to the ED case, when utility flows are discounted hyperbolically, the saving rate and the sign and magnitude of the current account depend on the risk premium. Specifically, the saving rate is decreasing in ϕ. The derivative of the saving rate $S(t)$ with respect to ϕ is

$$\frac{\partial S}{\partial \phi} = 1 - \Lambda'(\phi) < 0$$

where $\Lambda'(\phi) > 1$.

For *low* levels of systemic risk ϕ, consumption would be higher than the wealth annuity and the saving rate $S(t)$ would be *positive*, reminiscent of the behavior of an ED *patient* consumer. In turn, for high levels of risk premium the saving rate $S(t)$ would be *negative* and the agent's behavior would evoke an ED *impatient* agent. However, in the HD framework the attributes of "patient" and "impatient" do not qualify the nature of the agent's utility function. To be sure, all agents are assumed to be identical, enjoying the same utility function and discount factor, and following the same Euler equation. Instead, a patient (impatient) agent is defined in terms of her saving behavior being *observationally equivalent* to a patient (impatient) ED agent.

Intuition The negative relationship between the probability of a crisis and the saving rate of a time inconsistent agent can be understood by recalling the HD agent's "present-bias", i.e. she is short-term impatient but anticipates becoming more patient over time. Formally, a hyperbolic discount factor implies that the subjective discount rate is high in the short-term but declines with the time horizon, contrary to the ED case in which the discount rate is set to be constant.

As we noted elsewhere, a higher risk premium enters *additively* in the interest rate schedule and in the subjective discount rate. So higher ϕ implies greater interests on future savings, and it also implies greater augmented discount rates. In the ED model, these two effects cancel out because interest rates are $r+\phi$ and the subjective augmented discount rate is $\dot{\chi}(s)/\chi(s) = \rho + \phi$, so the saving rate remains at $S = r - \rho$.

However, in the HD setup a change in risk premium has an asymmetric effect along the term structure of the subjective discount rate because the latter is not flat (as in the ED model) but is instead decreasing in the time horizon. Specifically, the anticipated subjective *augmented* discount rate is

$$\frac{{}_t\dot{\chi}(s)}{{}_t\chi(s)} = \frac{\gamma}{1+\alpha(s-t)} + \phi$$

Consequently, the longer-term, lower subjective discount rates change proportionally more than the short-term, higher discount rates. For that reason, an increase in risk premium would result in the longer-term utility flows being discounted relatively more heavily than the short-term utility flows, which would be reflected in a lower desire to save now for future consumption.

All in all, the higher interest rates $r + \phi$ do not compensate for the increase in the longer term subjective impatience rates. Thus, when ϕ increases, the agent behaves as if she had become more impatient (in terms of an ED agent's observationally equivalent behavior) as she reduces her current saving rates.

Definition 3 *Define ϕ^* as the level of risk premium such that the saving rate is zero:*

$$S(\phi^*) = r + \phi^* - \Lambda(\phi^*) = 0 \tag{2.49}$$

An implication of Definition 3 is that when risk premium takes value ϕ^*, consumption $c(t)$ would be equal to the value of wealth annuity $(r+\phi)W(t)$, the current account would be in balance and wealth would remain constant over time. However wealth growth would be positive (negative) if $\phi < \phi^*$ ($> \phi^*$), as the consumption rate $\Lambda(\phi)$ would be higher (lower) than the annuity rate (see Figure 2.5). In other words, an "impatient" representative agent would consume more than the annuity value of wealth. This economy would run current account deficits, resulting in wealth deteriorating dynamics. The opposite would be true for an economy with a "patient" representative agent consuming less than the wealth annuity.

Shocks in permanent income y and in risk premium ϕ

Similarly to the case of an ED consumer analysed in Section 2.3.2 above, the effect on consumption of an unanticipated shock in y or ϕ would depend on the prevailing saving rate $S(\phi)$. However, as opposed to the ED case, in the HD model unanticipated shocks affect consumption differently depending on the level of systemic risk in a given economy.

Let's recall (2.46) and rewrite it as

$$\begin{aligned} c(t) &= C(\phi) \cdot W(t) \\ &= \Lambda(\phi) \left(b(t) + \frac{y}{r+\phi} \right) \end{aligned}$$

We note that $\Delta c = \frac{\Lambda(\phi)}{r+\phi} \Delta y$. If $\phi < \phi^*$, the agent behaves as an impatient consumer with $S(\phi) < 0$, implying that $\Lambda(\phi) > (r + \phi)$, and an unexpected rise in permanent income y would result in an *overshooting* of consumption, i.e. $\Delta c > \Delta y$. The opposite would happen if $\phi > \phi^*$, as the consumer would behave as a patient consumer with saving rate $S(\phi) > 0$. In this latter case, consumption would have undershot the change in income.

Consider now the effect of a shock in ϕ. The derivative of (2.46) with respect to ϕ would be

$$c' = C'W + C\,W'$$

where $C = \Lambda(\phi)$ and $C' = \Lambda'(\phi)$. The first term of the RHS is the (positive) *income effect* and the second term is the (negative) *wealth effect*. More precisely, we can write[25]

$$\frac{dc(t)}{d\phi} = C'(\phi) W + (S(\phi) - r + \phi) \frac{y}{(r+\phi)^2}$$

[25]See Appendix 2.C.8 for a detailed derivation of the income and wealth effects.

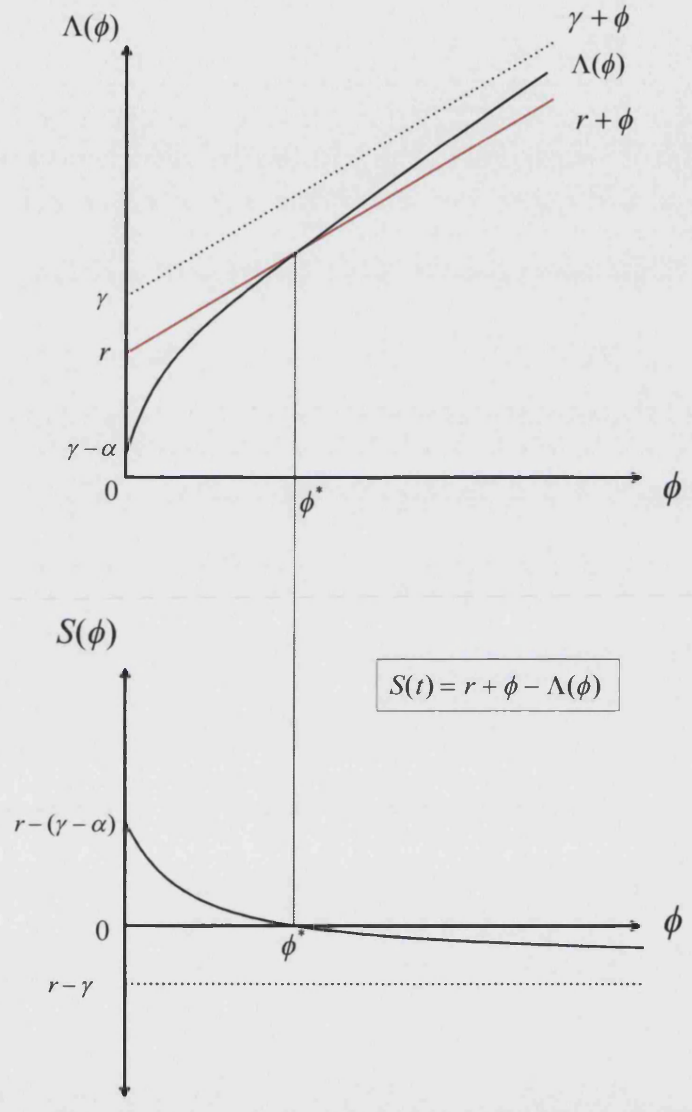

Figure 2.5: Actual saving rate as a function of ϕ. The saving rate is negative if $\phi > \phi^*$ as the consumption rate $\Lambda(\phi)$ is greater than the annuity rate $r + \phi$. Otherwise, it is positive.

where $S(\phi) = r + \phi - C(\phi)$. The above equation can be written as

$$\frac{dc(t)}{d\phi} = b + S(\phi)\,\frac{y}{(r+\phi)^2} - S'(\phi)\,W(t) \qquad (2.50)$$

where $S'(\phi) = 1 - \Lambda'(\phi) < 0$. The interpretation of the first and the second terms is identical to the ED case described in Section 2.3.2. In particular, the saving rate $S(\phi)$ reflects the tension between the income and the wealth effects, i.e. a higher $S(\phi)$ would strengthen the income effect of a saving prone consumer and vice versa. We note, however, that compared to equation (2.29) in the ED model the saving rate is dependent on ϕ, so the sign of $\frac{dc(t)}{d\phi}$ would ultimately depend on the magnitude of ϕ. Note, also, that the third term of the RHS was absent in the ED model because in the latter case S was constant and $S'(\phi) = 0$. In the HD case, this third term is positive and reflects the effect of the change of ϕ onto the saving rate: if ϕ increases, the saving rate deteriorates giving way to a further boost of consumption, thereby reinforcing the income effect.

The HD income effect will be particularly strong for low values of ϕ, where $S(\phi)$ is the steepest. It would be particularly relevant for understanding changes in the sign of the current account for large changes of ϕ. For instance, as can be seen in Figure 2.6, the current account would switch from surplus to deficit if ϕ increased from point **A** to **B** (accordingly, the saving rate $S(\phi) = \dot{W}(t)/W(t)$ would become negative). This change in sign would be consistent with an overshooting of consumption (relative to the ED case) required to generate a current account deficit.

Note also that this switch of saving patterns is observationally equivalent to an ED consumer changing her behaviour from "patient", or wealth accumulating, to "impatient", or wealth depleting, if confronted to an increase in the systemic risk of collapse.

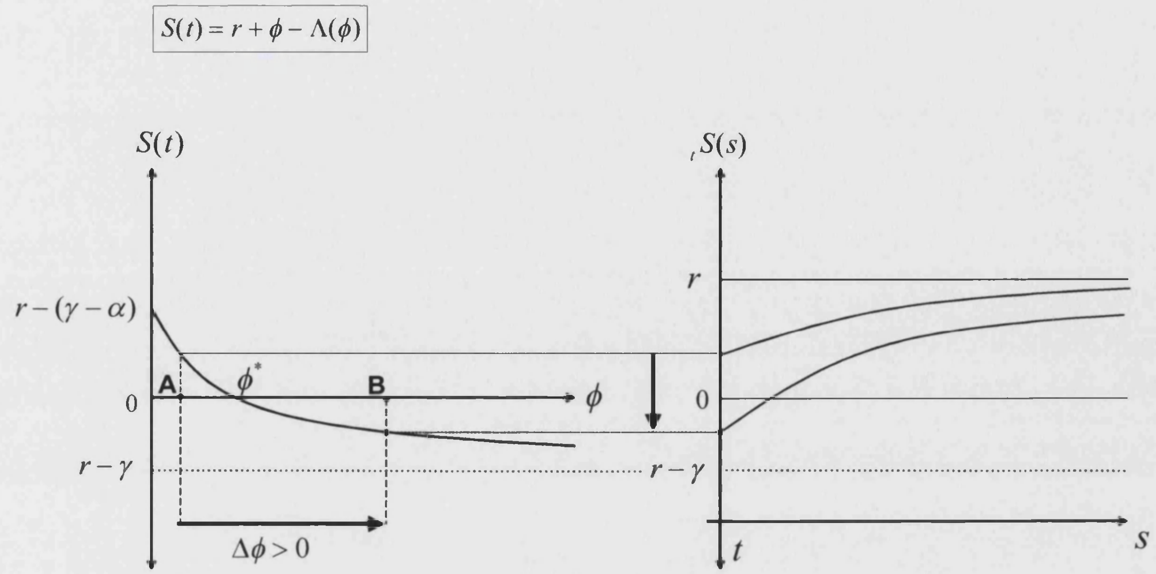

Figure 2.6: A large unanticipated increase in risk premium ϕ could change the sign of the saving rate.

Actual Solvency Condition

It is in order now to verify that the *actual* solvency condition is satisfied when the consumption path is determined by a continuum of selves that solve the HD intertemporal consumption problem at every point in time, subject to the anticipated intertemporal budget constraint.

Consumption is anticipated to follow a U-shape following the Euler equation (2.35). However, from (2.47) we note that actual consumption remains a constant share of wealth. Therefore, actual consumption growth rate is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{C}(t)}{C(t)} + \frac{\dot{W}(t)}{W(t)} \tag{2.51}$$

where $\frac{\dot{C}(t)}{C(t)} = 0$ for constant ϕ and where the second term is given by (2.10), in particular

$\frac{\dot{W}(t)}{W(t)} = S(t)$. Thus, consumption growth is

$$\frac{\dot{c}(t)}{c(t)} = S(t) = r + \phi - \Lambda(\phi) \tag{2.52}$$

and the consumption level is

$$c(\tau) = c(t) \exp\left[r + \phi - \Lambda(\phi)\right] (\tau - t), \ \forall \tau \geq t. \tag{2.53}$$

The actual solvency condition (2.7) for constant risk premium is

$$\int_t^\infty c(\tau) \exp\left[-(r+\phi)(\tau - t)\right] d\tau \leq b(t) + \frac{y}{r+\phi} \tag{2.54}$$

and using (2.53) into the last equation, we obtain

$$\int_t^\infty c(t) \exp\left[-\Lambda(\phi)\right] d\tau \leq b(t) + \frac{y}{r+\phi} \tag{2.55}$$

or

$$\frac{c(t)}{\Lambda(\phi)} \leq b(t) + \frac{y}{r+\phi} \tag{2.56}$$

which defines the consumption values that would satisfy the solvency condition for given $b(t)$ and risk premium ϕ.

Therefore, from (2.46) it is apparent that the actual consumption path resulting from the hyperbolic optimization process satisfies the actual solvency condition, as the net present value of consumption is simply the binding solution of the solvency condition (2.56).

2.4 Variable Risk Premium

In this Section we extend the previous case and get similar, although richer, results when we allow the risk premium be a function of two sets of variables: a time-invariant variable that captures the fundamental vulnerabilities inherent in the structure of the economy θ, and the time-varying variable that captures the dynamics of vulnerability indices.

At any point in time s, the actual probability of a crisis occurring in the immediate time period dt –conditional on it not having happened before– is determined by the *hazard rate* $\phi(s) \in [0, +\infty)$, defined as

$$\phi(s) = \phi\left(\delta(s), \theta\right),$$

where θ is a set of *time-independent* variables that account for the *fundamental vulnerabilities* inherent in a particular economy, and $\delta(s)$ is a vector of *time-varying* macroeconomic indicators. In our particular setting, $\delta(s)$ denotes the country's net foreign debt-to-output ratio at time s :

$$\delta(s) \equiv \frac{-b(s)}{y},$$

where $b(s)$ stands for the country's net foreign asset position at time s (i.e., negative $b(s)$ is equivalent to a net debtor position) and y stands for the national income flow, which we assume to be constant and parametric. Arguably, a crisis is more likely to occur in a country that is highly leveraged or indebted. Together, $\delta(s)$ and θ determine the country's likelihood of a crisis event as well as its ability to continue servicing the debt.

Without loss of generality, we assume that the likelihood of default for a net *creditor*

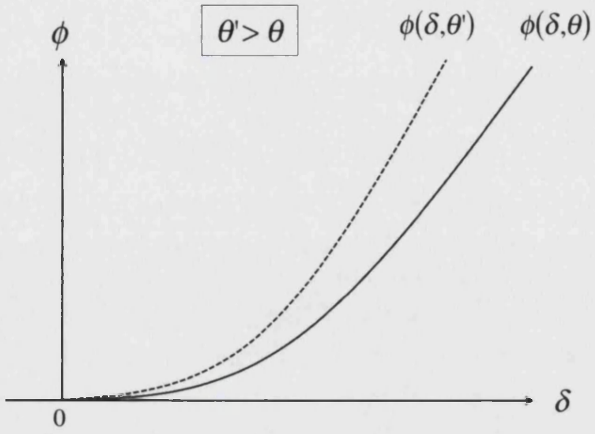

Figure 2.7: Default probability function. If fundamentals deteriorate, i.e. $\theta' > \theta$, the curve shifts up.

country is zero. Thus, the hazard rate function takes values

$$\phi\left(\delta(s),\theta\right)\begin{cases} >0 & \text{if } \delta(s)>0 \\ =0 & \text{if } \delta(s)\leq 0 \end{cases} \tag{2.57}$$

and is continuously differentiable for all positive values of debt $\delta \in (0,\infty)$. The derivatives of $\phi\left(\delta(s),\theta\right)$ are

$$\phi_1 \geq 0 \quad \text{and} \quad \phi_2 \geq 0, \;\; \forall \delta > 0.$$

The implication of the first derivative $\phi_1 > 0$ is that the risk of default would increase if the level of indebtedness increases over time.[26] Similarly, $\phi_2 > 0$ implies that the arrival of a shock that worsens the economy's fundamentals would result in a discrete shift of the default probability function on impact. (See Figure 2.7).

[26]The point is that in an economy that is subject to random exogenous systemic shocks the probability that a negative shock leads to a systemic bankruptcy is higher if the country's balance sheet is highly leveraged.

2.4.1 Intertemporal Budget Constraint

When the risk premium is time-varying, the IBC (2.7) can be written as

$$\int_t^\infty c(s)\exp\left[-r(s-t)-\int_t^s \phi(\tau)d\tau\right]ds = W(t) \tag{2.58}$$

where wealth is

$$W(t) \equiv b(t) + \frac{y}{\Pi(t,r,\{\phi(s)\})} \tag{2.59}$$

and the function Π is defined as

$$\Pi(t,r,\{\phi(s)\}_t) \equiv \left(\int_t^\infty \exp\left[-r(\tau-t)-\int_t^\tau \phi(s)ds\right]d\tau\right)^{-1} \tag{2.60}$$

where $\phi(\tau)$ is the *forward* instantaneous risk premium. Accordingly, the instantaneous *forward interest rate* is given by $r+\phi(\tau)$.

Note that the function Π has the following properties: (i) it has positive partial derivative[27] $\partial\Pi/\partial r > 0$ and (ii) it would reduce to $r+\phi$ if the risk premium were constant.

Note that the schedule $\{\phi(s)\}_t$ depends on the forecasted path of the aggregated foreign asset position $\{b(s)\}_t$. However, when the infinitesimal representative agent solves her intertemporal maximization problem, she takes the forward rates curve as parametric and does not consider the impact of her own decision on the aggregate asset position.

2.4.2 Solvency condition

The transversality condition implies that wealth and consumption are restricted to be non-negative. An agent has access to capital markets inasmuch as her net wealth

[27] See the Appendix for the proof of the sign of the derivatives of $\Pi(\cdot)$.

remains positive, that is so long the following condition holds:

$$b(t) + \frac{y}{\Pi(t)} \geq 0$$

or,

$$\delta(t) \leq \frac{1}{\Pi(t)} \tag{2.61}$$

where $\delta(t) \equiv -\frac{b(t)}{y}$ is the debt-to-output ratio. This condition defines the *upper bound* of indebtedness consistent with non-negative consumption.

Since the risk premium $\phi(\cdot)$ is a function of the debt-to-output ratio δ, we can write the inverse function of (2.57) as

$$\delta(s) = \begin{cases} \delta(\phi(s), \theta)) > 0 & \text{if } \phi(s) > 0 \\ \leq 0 & \text{if } \phi(s) = 0 \end{cases} \tag{2.62}$$

where the function $\delta(\phi(s), \theta)$ is continuously differentiable in ϕ for all $\phi \in [0, +\infty]$. The signs of the partial derivatives are $\delta_1 > 0$ and $\delta_2 < 0$, $\forall\ \phi(s) \geq 0$.

Notation 3 *Consider any given function $G(\cdot)$ that depends on the entire time path of a variable $\{v(s)\}_t$, where s is the time index and t is the present time. The expression $G|_v$ denotes the value that $G(\cdot)$ would take if $v(s) = v$, for all $s \geq t$.*

Recall the solvency condition (2.61) and consider a situation of a current account deficit. This implies net debt accumulation, which would increase the LHS of (2.61). In turn, higher debt ratios would result in higher risk premia, which raises the term $\Pi(t)$ and lower the value of the RHS of (2.61). It can be verified that there is a maximum level of indebtedness $\delta^{**} = \delta(\phi^{**}, \theta)$ for which the solvency condition is binding. Therefore, the domain of possible risk premium values is $\phi(s) \in [0, \phi^{**}(r, \theta)]$, where the maximum possible level of risk premium ϕ^{**} is characterized by:

$$\delta(\phi^{**}, \theta) = \frac{1}{\Pi|_{\phi^{**}}} \tag{2.63}$$

This equality implicitly defines ϕ^{**} as a function of parameters r and θ. If the risk premium reached level ϕ^{**}, wealth and consumption would be zero, meaning that the wealth annuity would be utilized in full for servicing the debt, i.e. for paying interests equal to $-(r + \phi^{**})\, b^{**}$.

In order to represent the system's dynamics graphically, we note that the solvency constraint (2.61) can be written as

$$\delta(t)^{-1} \geq \Pi(t), \tag{2.64}$$

where both sides are functions of the risk premium: $\delta(t)$ is defined by (2.62) and $\Pi(t)$ is defined by (2.60). Figure 2.8 displays the two equations as functions of $\phi\,(t)$: schedule **ZZ** is just the inverse of function (2.62), corresponding to the LHS of (2.64):

$$\delta\,(t)^{-1} = \frac{1}{\delta(\phi(t), \theta)} \text{ for } \phi(t) \in [0, 1]. \tag{ZZ}$$

and schedule **RR** represents the values of Π under the particular assumption that the latter remains *constant* at the present level $\phi\,(t)$. Recall from the properties of the function (2.60) that if ϕ is constant, the function reduces to $r + \phi$. Hence, we can write

$$\Pi(r)|_{\phi(t)} = r + \phi(t) \tag{RR}$$

The horizontal coordinate of the intersection of **ZZ** and **RR** corresponds to ϕ^{**}, which is the maximum possible risk premium value for given parameters r and θ. Correspondingly, the maximum net debt level consistent with non-negative wealth and con-

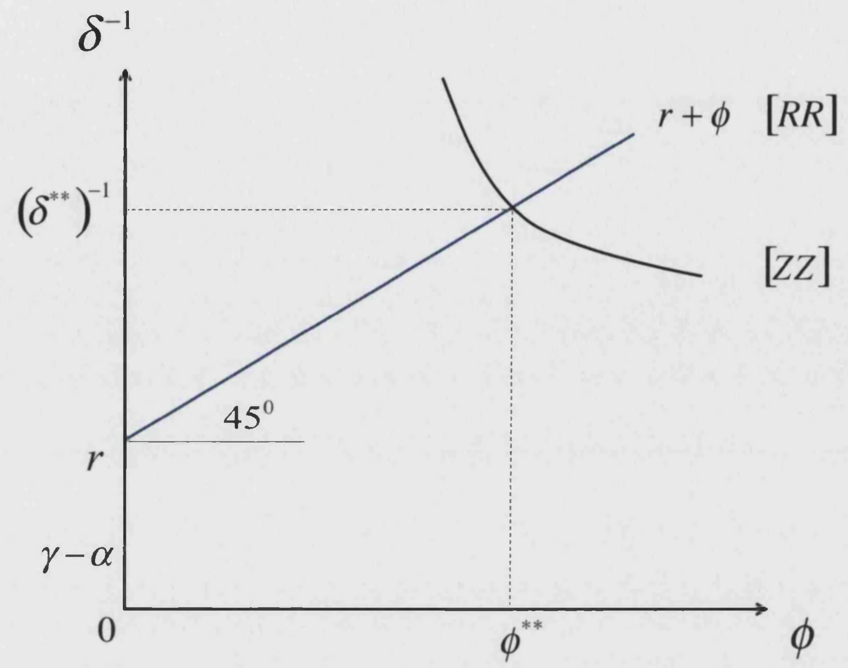

Figure 2.8: The maximum possible risk premium is ϕ^{**}, for which the solvency condition is binding.

sumption is such that

$$\delta(\phi^{**}, \theta)^{-1} = r + \phi^{**}, \tag{2.65}$$

which corresponds to the vertical coordinate of the intersection of both schedules.

Note that if r increases, **RR** shifts up and ϕ^{**} decreases. Similarly, ϕ^{**} reduces as fundamentals deteriorate, i.e. as the variable θ (the fundamental vulnerabilities) increases. This would in turn shift the schedule **ZZ** to the right and increase the maximum value ϕ^{**}. Hence, the signs of the partial derivatives are $\partial\phi^{**}/\partial r < 0$ and $\partial\phi^{**}/\partial\theta > 0$.

2.4.3 Exponential Discounting

As in Section 2.3 with constant ϕ, we first treat the orthodox time-consistent, exponential case in order to compare it later with the hyperbolic discounting case developed later in Section 2.4.4. The system of differential equations cannot be solved explicitly because the IBC depends on the entire anticipated path of risk premium, which is in turn determined by the aggregate of individually optimized paths of the net foreign asset position. However, we are able to characterize the model's dynamics by examining the stability at the steady state.

The discount factor

As before, the consumer's intertemporal discount factor in the ED model is

$$F(t,s) = \exp\left[-\rho(s-t)\right]$$

and the augmented discount factor is

$$\chi_s = \exp\left[-\rho(s-t) - \int_t^s \kappa(\tau)d\tau\right]$$

where $\kappa(s) = \phi(s)$ under the assumption of homogeneity of beliefs and perfect information.

However, in this Section the risk premium is not a constant. Accordingly, the objective function (2.4) takes the form

$$\int_t^\infty U(c_s) \exp\left[-\rho(s-t) - \int_t^s \kappa(\tau)d\tau\right] ds. \tag{2.66}$$

Remark 2 *As in Section 2.3 with constant ϕ, the ED consumer solves her intertemporal consumption problem as if the status-quo ante situation were non-stochastic and were due to last forever, immune to financial crises. Here, however, the discount rate is augmented by the "spot" subjective risk of default $\bar{\kappa}(t,s) \equiv \int_t^s \kappa(\tau)d\tau = \int_t^s \phi(\tau)d\tau$.*

First order conditions

For convenience, we assume log instantaneous utility $U(c(s)) = \ln c(s)$. Proceeding in similar way as in the previous Section, the first-order-conditions result in the following Euler equation

$$\frac{\dot{c}(s)}{c(s)} = r + \phi(s) - (\rho + \kappa(s)), \qquad \forall s \geq t$$

which, under the assumption of homogeneity and perfect information, simplifies to

$$\frac{\dot{c}(s)}{c(s)} = r - \rho \tag{2.67}$$

The forward-rate curve at time t is given by $\{r + \phi(s)\}_t$ where r is constant and $\{\phi(s)\}_t^{\infty}$ is common knowledge to all forward-looking agents.

Instantaneous consumption at present time t can be found by using the equation of motion (2.67) into the resource constraint (2.7) to get

$$c(t) \int_t^{\infty} e^{(r-\rho)(s-t)} e^{-r(s-t)-\int_t^s \phi(\tau)d\tau} ds = b(t) \; + \; y \int_t^{\infty} e^{-r(s-t)-\int_t^s \phi(\tau)d\tau} ds$$

or

$$c(t) \int_t^{\infty} e^{-\rho(s-t)-\int_t^s \phi(\tau)d\tau} ds = b(t) \; + \; y \int_t^{\infty} e^{-r(s-t)-\int_t^s \phi(\tau)d\tau} ds \tag{2.68}$$

which can be written as

$$\frac{c(t)}{\Pi(t,\rho)} = b(t) + \frac{y}{\Pi(t,r)} \tag{2.69}$$

or

$$c(t) = \Pi\,(t,\rho)\, W(t) \tag{2.70}$$

where the function $\Pi\,(t,\cdot)$ has been defined in (2.60) and $W\,(t)$ is intertemporal wealth.

Correspondingly, the consumption rate is

$$C(t) = \Pi(t, \rho)$$

Note that both $\Pi(t, r)$ and $\Pi(t, \rho)$ depend on the entire time path of the risk premium $\{\phi(s)\}_t$. In turn, every level of default risk $\phi(s)$ also depends on the debt level as per equation (2.57). So, consumption $c(t)$ is governed by the expected time schedule of the *aggregate* net foreign asset position $\{b(s)\}_t$ for the entire time set $s \in [t, \infty)$.

Steady State

In order to characterize the steady state (SS), let's recall three key equations: (ZZ), or the (inverse of the) indebtedness level as a function of risk premia; (RR), or the value of $\Pi(t, r)$ under the assumption that the risk premium remained constant at $\phi(t)$; and equation (EE), which is the value of function $\Pi(t, \rho)$ if the risk premium remained constant at $\phi(t)$. Specifically,

$$\delta(t)^{-1} = \frac{1}{\delta(\phi(t), \theta)} \tag{ZZ}$$

$$\Pi(r)|_{\phi(t)} = r + \phi(t) \tag{RR}$$

$$\Pi(\rho)|_{\phi(t)} = \rho + \phi(t) \tag{EE}$$

At the steady state the equations of motion are such that $\dot{b} = \dot{c} = \dot{\phi} = 0$ so that all state variables are constant at SS levels $\{b^*, c^*, \phi^*\}$. The SS resource constraint can be written using (2.69), (RR) and (EE) as

$$\frac{c^*}{\rho + \phi^*} = b^* + \frac{y}{r + \phi^*}; \tag{2.71}$$

or, in terms of SS consumption,

$$c^* = (\rho + \phi^*)\, W^* \tag{2.72}$$

where W^* is the SS wealth:

$$W^* = b^* + \frac{y}{r + \phi^*}.$$

Accordingly, the consumption rate at the steady state is

$$C^* = \rho + \phi. \tag{2.73}$$

In turn, the current account (2.6) at the steady state becomes

$$\dot{b}^* = (r + \phi^*)\, W^* - c^* = 0$$

or, using (2.72),

$$\dot{b}^* = (r - \rho)\, W^* = 0.$$

This last equation implies that the steady state $\{\phi^*, b^*, c^*\}$ is consistent with either $W^* = 0$ or with $r = \rho$. As for the former possibility, we note that wealth exhaustion would be ruled out since it would correspond to zero consumption, infinite marginal utility and $u(0) = \ln[0] = -\infty$. In other words, if $r < \rho$, the system would approach to that equilibrium only asymptotically. The latter possibility, $r = \rho$, corresponds to a non-tilt consumption path, as can be verified in the Euler equation (2.67). In other words, the steady state would require the saving rate to be zero:

$$\dot{S}^* = 0 = r - \rho$$

The SS therefore requires $r = \rho$ at all times. Note that in order to abstract from dynamic considerations that are not pertinent to our discussion, we have assumed that output remains constant at all times so that the smoothing channel to explain current account imbalances has been deliberately closed. Therefore, if all agents across the globe had the same impatience rate, there would be no tilt in consumption and the current account should always remain in balance.

Out of Steady State dynamics

The system of differential equations cannot be solved explicitly because the IBC depends on the entire anticipated path of risk premium, which is in turn determined by the aggregate of individually optimized paths of the net foreign asset position. However, we are able to characterize the model's dynamics by examining the system's stability at the steady state.

Proposition 2.3 *In the ED model with constant output, the current account is in surplus iff $r > \rho$, and in deficit iff $r < \rho$ (or if the domestic interest rate $r + \phi(t)$ is greater than the subjective discount rate $\rho + \phi(t)$ for any given $\phi(t)$). Graphically, the current account is in surplus when the schedule* ***RR*** *lies above* ***EE****, and it's in deficit when* ***RR*** *lies below* ***EE*** *(see Figures 2.9 and 2.10)*

Proof. *(By contradiction.). Recall the current account (2.6) and use (2.59) and (2.70) to find the following expression:*

$$\dot{b}(t) = ((r + \phi(t)) - \Pi(t, r))\, b(t) - (\Pi\,(t, \rho) - \Pi(t, r))\, W(t) \qquad (2.74)$$

Consider the possibility of $r > \rho$ being consistent with $\dot{b}(t) < 0$ or, equivalently, with $\dot{\phi}(t) > 0$. First, from inspection of (2.60) we note that $\dot{\phi}(t) > 0$ implies $\Pi(t, r) > r + \phi(t)$. Since by assumption $b(t) < 0$, the first term of the RHS of (2.74) is positive. Second, it follows from Property (i) of function (2.60) *that $r > \rho$ implies $\Pi(t, r) > \Pi(t, \rho)$. Since wealth $W(t)$ is non-negative, the second term of the RHS of (2.74) is*

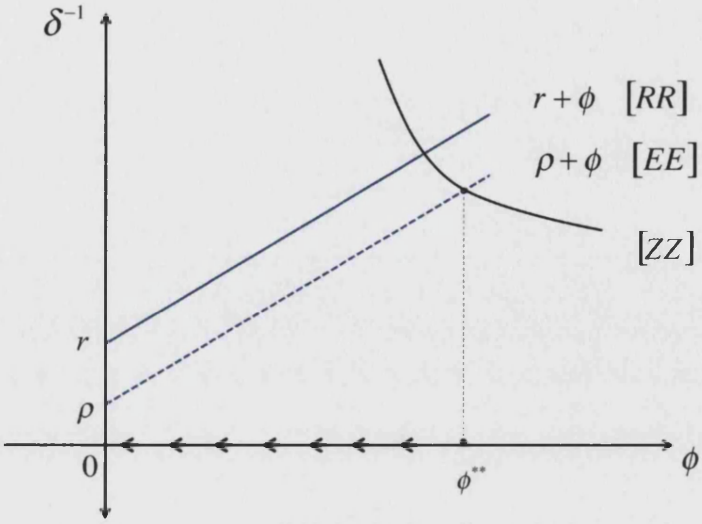

Figure 2.9: Case of positive current account: $\dot{b} > 0$. As the net asset position improves, the systemic risk declines and eventually vanishes.

positive. Therefore, the sign of (2.74) would be positive, which contradicts the prior that $\dot{b}(t) < 0$. *Hence,* $r > \rho$ *can only be consistent with* $\dot{b}(t) > 0$ *(or, equivalently* $\dot{\phi}(t) < 0$, *for all* $\phi(t) \in (0, \phi^{**})$.[28]

A similar proof shows that $r < \rho$ *iff* $\dot{b}(t) < 0$ *or* $\dot{\phi}(t) > 0$, *for all* $\phi(t) \in (0, \phi^{**})$. ***QED.***

The implication of Proposition 2.3 is that if $r > \rho$ an economy would experience an improvement in its net foreign asset position over time, which would eventually eliminate the risk of default. In the opposite case, if $r < \rho$, the economy will steadily accumulate debt, thereby exhausting her net wealth over time as the sovereign risk asymptotically approaches ϕ^{**} (see Figure 2.9).

[28]More precisely, the solution $\dot{b}(t) > 0$ for the case $r > \rho$ implies that $r + \phi(t) \geq \Pi(t, r)$. Thus, although the first term of the RHS of (2.74) would be negative, its absolute value must be lower than the second term, which is positive. The opposite argument is true if $r < \rho$.

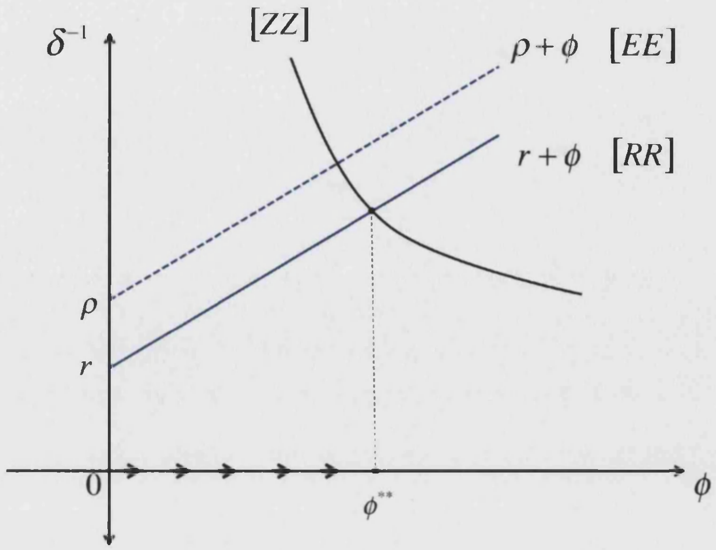

Figure 2.10: Case of a positive current account: $\dot{b} < 0$. As the net asset position deteriorates, the systemic premium approaches to ϕ^{**}.

2.4.4 Hyperbolic Discounting

The HD model with endogenous risk premium is consistent with most of the results of the previous Section 2.3.3, which develops the HD case assuming constant risk premium. In particular, we still obtain the result that the magnitude and *sign* of the saving rate could change when an economy is confronted to exogenous shocks.

Discount factor

As in the previous Section, the consumer's discount factor is

$$F(t, s) = (1 + \alpha(s - t))^{-\gamma/\alpha}$$

where $\gamma > 0$ and $\alpha > 0$ are such that $0 < \gamma - \alpha < r$, and the augmented discount factor is

$$\chi_s = \exp\left[-\int_t^s \kappa(\tau)d\tau\right](1 + \alpha(s - t))^{-\gamma/\alpha}.$$

where, as before $\kappa(s) = \phi(s)$, $\forall s \geq t$, under the assumptions of homogeneity of beliefs and perfect information.

First order conditions

Assuming log-utility and time-varying $\kappa(s)$, the consumer's objective function (2.4) can be written as

$$\int_t^\infty \ln\left[{}_tc(s)\right] \exp\left[-\int_t^s {}_t\kappa(s)d\tau\right] (1+\alpha(s-t))^{-\gamma/\alpha} ds$$

The first order conditions determine the consumer's anticipated Euler equation

$$\frac{{}_t\dot{c}(s)}{{}_tc(s)} = r + \phi(s) - \left(\frac{\gamma}{1+\alpha(s-t)} + {}_t\kappa(s)\right), \qquad \forall s \geq t, \tag{2.75}$$

where, under the assumption of homogeneity of beliefs ,the (infinitesimally small) representative agent adopts the markets' pricing of risk $\phi(s)$, which is taken as parametric:

$${}_t\kappa(s) = \phi(s) \tag{2.76}$$

Accordingly, equation (2.75) simplifies to

$$\frac{{}_t\dot{c}(s)}{{}_tc(s)} = r - \frac{\gamma}{1+\alpha(s-t)} \tag{2.77}$$

where $\gamma > r$, just as in Section 2.3.3 with constant risk premium ϕ. In other words, the anticipated U-shape consumption is identical to the case of constant ϕ.

However, in the present case with time varying $\phi(s)$, the intertemporal budget constraint is determined by the entire path $\{\phi(s)\}_t$. Let's recall the anticipated current account

$${}_t\dot{b}(s) = (r + \phi(t))\, {}_tb(s) + y - {}_tc(s)$$

and integrate it with respect to time, from any time $s \geq t$ to infinity, to get the anticipated IBC:

$$\begin{aligned}\int_s^{\infty} {}_tc(\tau)e^{-r(\tau-s)-\int_s^{\tau}\phi(z)dz}d\tau &= {}_tb(s)+y\int_s^{\infty} e^{-r(\tau-s)-\int_s^{\tau}\phi(z)dz}d\tau \\ &\quad -\lim_{T\longrightarrow\infty} {}_tb_T\, e^{-r(T-s)-\int_s^{T}\phi(z)dz}\end{aligned} \qquad (2.78)$$

$\forall s \geq t$, where the last term is zero under the Transversality condition.

The risk premium forward schedule $\{\phi(s)\}_t$ is implicit in the forward-rate curve $\{r+\phi(s)\}_t$ provided by the financial markets. Note that a risk premium future level $\phi(s)$ depends on the aggregate actual future $b(s)$, and not on the future asset position that an individual anticipates for herself. To be sure, the individual's anticipated path $\{{}_tb(s)\}$ is inconsistent with the *actual aggregate path* $\{b(s)\}$.[29]

In similar way as in Section 2.3.3 with constant ϕ, the hyperbolic naive agent anticipates a U-shaped consumption plan following the Euler equation (2.77) and subject to the IBC (2.78). However, due to the time-inconsistency problem, actual consumption would differ from the planned consumption path..

Actual consumption path

The actual consumption path is drawn by the continuum of initial consumption values ${}_sc(s)$, $\forall s \geq t$ that solve the consumer's intertemporal optimization problem at every point in time, subject to the anticipated IBC. Specifically, the continuum of initial points ${}_tc(t) = c(t)$ that solve the anticipated Euler equation (2.77) subject to (2.78).

[29]Consumers are assumed to be able to extract the probability of a crisis from the market and are assumed to agree with it. A naive agent however would still optimistically believe that she will be able to commit to her consumption and debt accumulation paths, although by accepting the path $\{\phi\}_t$ determined at the (actual) aggregate demand level, she is implicitly acknowledging that all other agents are time-inconsistent.

Integrating (2.77) with respect to time we obtain[30]

$$ {}_t c(\tau) = {}_t c(t)\, e^{r(\tau-t)} \left(1+\alpha(\tau-t)\right)^{-\frac{\gamma}{\alpha}} \tag{2.79}$$

$\forall\tau \geq t$, where ${}_t c(t) = c(t)$. Inserting this in the LHS of the anticipated IBC (2.78) results in

$$ c(t)\int_t^\infty \left(1+\alpha(\tau-t)\right)^{-\frac{\gamma}{\alpha}} e^{-\int_t^\tau \phi(s)ds} d\tau = b(t) + y\int_t^\infty e^{-r(\tau-t)-\int_t^\tau \phi(s)ds} d\tau, \tag{2.80}$$

or

$$ \frac{c(t)}{\Psi(t,\alpha,\gamma)} = b(t) + \frac{y}{\Pi(t,r)} \tag{2.81}$$

or

$$ c(t) = \Psi(t,\alpha,\gamma)\, W(t) \tag{2.82}$$

where, as before, wealth $W(t)$ is given by

$$ W(t) = b(t) + \frac{y}{\Pi(t,r)}, \tag{2.83}$$

and function Π has been defined in equation (2.60). Function Ψ is defined as follows

$$ \Psi(t,\alpha,\gamma) \equiv \left(\int_t^\infty \left(1+\alpha(\tau-t)\right)^{-\frac{\gamma}{\alpha}} \exp\left[-\int_t^\tau \phi(s)ds\right] d\tau\right)^{-1} $$

where $\phi(s)$ is the forward instantaneous risk premium. The function Ψ has the following properties : (i) it has partial derivatives[31]

$$ \partial\Psi/\partial\gamma > 0 \quad \text{and} \quad \partial\Psi/\partial\alpha < 0 $$

[30]See Appendix for details on the derivation.
[31]See the Appendix for the proof of the sign of the derivatives of $\Psi(\cdot)$.

and, (ii) Ψ would reduce to $\Lambda(\phi)$ if the risk premium were a constant, where $\Lambda(\phi)$ is defined by (2.41).

Note that expressions Ψ above and Π in (2.81) depend on the entire risk premium path $\{\phi(s)\}$, $\forall s \in [t, \infty)$, which in turn is governed by the path $\{b(s)\}$. Therefore, $c(t)$ is also governed by the entire path of aggregate $\{b(s)\}$, for $s \in [t, \infty)$.

Steady State

In order to characterize the steady state (SS) let us recall three key equations: (ZZ), or the (inverse of the) indebtedness level as a function of risk premia; (RR), or the value of $\Pi(t, r)$ under the assumption that the risk premium remained constant at $\phi(t)$; and equation (HH), which states the values of function $\Psi(t, \alpha, \gamma)$ if the risk premium remained constant at $\phi(t)$. Specifically,

$$\delta(t)^{-1} = \frac{1}{\delta(\phi(t), \theta)} \tag{ZZ}$$

$$\Pi(r)|_{\phi(t)} = r + \phi(t) \tag{RR}$$

$$\Psi(t, \alpha, \gamma)|_{\phi(t)} = \Lambda(\phi(t)) \tag{HH}$$

Figure 2.11 displays the SS equilibrium $\{\phi^*, b^*, c^*\}$ such that $\dot{b} = \dot{c} = \dot{\phi} = 0$. Recalling equations (RR) and (HH), we can write the budget constraint (2.81) at the SS as

$$\frac{c^*}{\Lambda(\phi^*)} = b^* + \frac{y}{r + \phi^*};$$

so SS consumption (2.82) becomes

$$c^* = \Lambda(\phi^*) W^*. \tag{2.84}$$

where SS wealth would simply be $W^* = b^* + \frac{y}{r+\phi^*}$. Or, in terms of the SS consumption

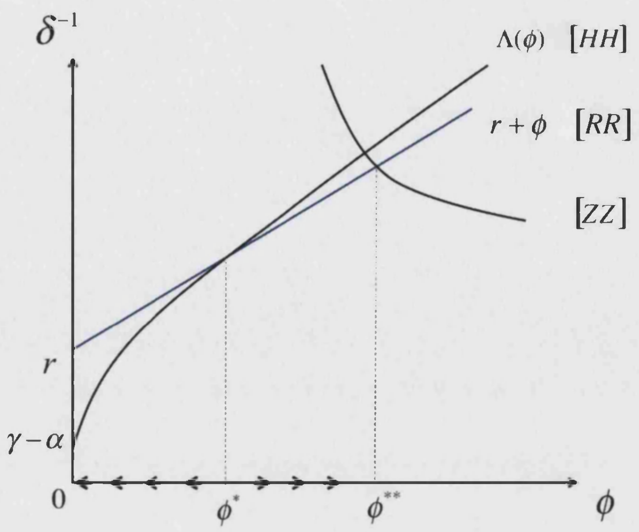

Figure 2.11: Steady state equilibrium ϕ^* is unstable

rate:

$$C^* = \Lambda(\phi^*).$$

Recalling the current account (2.6), in the steady state it becomes

$$\dot{b}^* = 0 = (r + \phi^*) W^* - c^*$$

or,

$$\dot{b}^* = 0 = (r + \phi^* - \Lambda(\phi^*)) \; W^* \qquad (2.85)$$

or, in terms of the SS saving rate,

$$S^* = r + \phi^* - \Lambda(\phi^*) = 0$$

Therefore, since $S^* = \dot{b}^* = 0$ at the SS equilibrium it must be that

$$\Lambda(\phi^*) = r + \phi^*, \qquad (2.86)$$

which defines the steady state variables $\{\phi^*, b^*, c^*\}$ as a function of parameters r, α and γ (that characterize function Λ). A condition for the existence of ϕ^* is that $r \geq \gamma - \alpha$.[32] In accordance to the steady state condition (2.86), the level $\phi^* = \phi^*(r, \alpha, \gamma)$ corresponds to the horizontal coordinate of the intersection between schedules **HH** and **RR**. Notice that if r increases, schedule **RR** shifts up and ϕ^* increases. On the contrary, an increase in $\gamma - \alpha$ would shift schedule **HH** up and ϕ^* would decrease. It is then clear that the signs of the partial derivatives of $\phi^*(r, \alpha, \gamma)$ are $\partial\phi^*/\partial r > 0$, $\partial\phi^*/\partial\gamma < 0$ and $\partial\phi^*/\partial\alpha > 0$.

Stability

Proposition 2.4 *The steady state equilibrium* $\{\phi^*, b^*, c^*\}$ *that fulfills condition* (2.86) *is unstable.*

Proof. *See Appendix.*

Proposition 2.4 implies that the current account would be in deficit if the initial $\phi(t) > \phi^*$, which would imply further deterioration of the net asset position and greater future risk premium. Similarly, the current account would be in surplus if $\phi(t) < \phi^*$, which would imply a decreasing risk premium. Therefore, the risk premium would tend to diverge either towards 0 or ϕ^{**} depending on its initial position, and would remain unchanged if and only if $\phi(t) = \phi^*$.

Note from (2.85) that there is another *stable* steady state solution $\{\phi^{**}, b^{**}, c^{**}\}$ that corresponds to a situation of zero wealth or wealth exhaustion. Recall from (2.63) that at this point the risk premium would reach its maximum possible value ϕ^{**}, as income would be entirely devoted to interest payments and no resources would be left

[32] This condition will always hold in general equilibrium where r is endogenous: if $r < \gamma - \alpha$, then *all* countries would run current account deficits. Therefore r would rise so that global financial market is in balance. See Section 2.5.

for consumption. Since ϕ^{**} is a constant, the budget constraint (2.81) would become

$$\frac{c^{**}}{\Lambda(\phi^{**})} = b^{**} + \frac{y}{r + \phi^{**}},$$

or

$$c^{**} = \Lambda\left(\phi^{**}\right) W^{**} \tag{2.87}$$

where $W^{**} = b^{**} + \frac{y}{r+\phi^{**}}$. This last equation indicates that despite of the fact that the consumption rate is positive at $C^{**} = \Lambda\left(\phi^{**}\right)$, actual consumption would be zero. At equilibrium $\{\phi^{**}, b^{**}, c^{**}\}$ income is entirely devoted to interest payments and no resources are left for consumption, i.e. ϕ^{**} is such that $y = -b^{**}\left(r + \phi^{**}\right)$ or the solution of equation (2.65)

$$\frac{1}{\delta(\phi^{**}, \theta)} = r + \phi^{**},$$

which corresponds to the horizontal coordinate of the intersection of schedules **ZZ** and **RR** in Figure 2.11.

World Polarization

The instability of the steady state implies that countries would tend to polarize in net debtor and net creditor economies. The equilibrium ϕ^* corresponds to a balanced current account $\dot{b}^* = 0$. However, Proposition 2.4 states that such an equilibrium is unstable. Recalling Figure (2.11), if an economy departed towards a risk premium lower than ϕ^* the current account would turn into surplus, resulting in *ever improving net foreign asset position*, which would eventually turn positive as the economy turns into a *net creditor country*. As such, a country running current account surpluses would eventually enjoy zero risk premium, as per equation (2.57).

Conversely, if an economy posted a risk premium greater than ϕ^* the current account would be in deficit as the representative agent would post negative saving rates

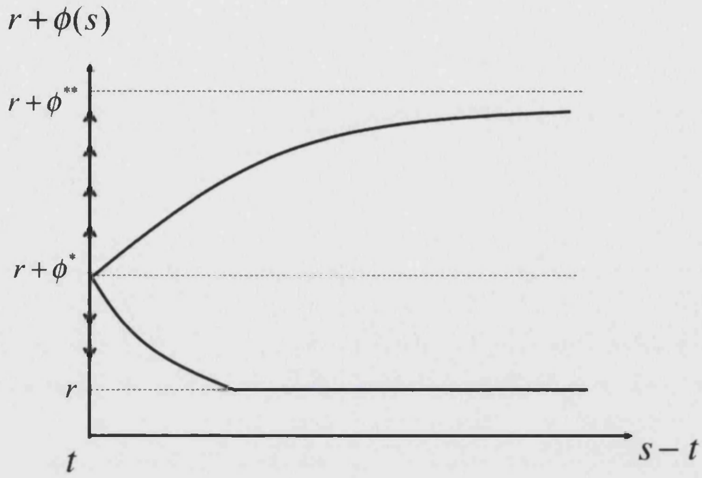

Figure 2.12: Two possible forward-rate curves depending on the initial rate is higher or lower than $r + \phi^*$.

and would consume more than her disposable income. This would result in deteriorating wealth and declining consumption. Thus, *debt accumulating* economies are led towards the maximum level of over-indebtedness that corresponds to a situation of *wealth exhaustion*, i.e. equilibrium $\{\phi^{**}, b^{**}, c^{**}\}$, in which income is entirely used for servicing interests payments.

Forward interest rates

Proposition 2.5 *If the risk-free rates* r *are assumed to be constant, the forward curve that corresponds to an economy that presents* $\phi > \phi^*$ *is upward-sloping and has its long end anchored at* $r + \phi^{**}$. *Similarly, the forward curve that corresponds to an economy with* $\phi < \phi^*$ *is inverted, and its long end is simply interest rate* r *(see Figure 2.12).*

The above Proposition provides a fixed point for the long end of the forward curve depending on the starting level of interest rates. Thus, there are two mutually exclusive forward-rate curves that could prevail depending on whether the initial interest rates is above or below the steady state interest rate level $r+\phi^*$. Accordingly, the interest rate of a debt accumulating country that runs current account deficits would converge towards the maximum possible rate $r + \phi^{**}$, whereas interest rates of a wealth accumulating economy would reach the risk-free rate r corresponding to a zero risk of default.

Shocks in permanent income y and in risk premium ϕ

The HD setup seems to be adequate for explaining reversals of current account patterns when economies experience shocks at the level of permanent income y or on other fundamental variables θ that affect the risk premium.

Note that changes in the income level y affect the debt-to-income ratio and through it the risk premium. Let's recall the schedules **ZZ, RR, HH** and consider an unexpected rise in permanent income y for an overindebted country that has reached the bad equilibrium ϕ^{**}. As y increases, the debt-to-income ratio δ decreases on impact, thereby lowering the country's risk premium. This would allow the debtor country to regain access to capital markets. If the shock were small, the agent will continue to behave impatiently and will overshoot consumption (i.e. $\Delta c > \Delta y$) as the current account would become negative (from having been in balance, as $\dot{b}^{**} = 0$). As debt accumulates, the risk premium would resume its debt risk deteriorating pattern and will be set to converge back to ϕ^{**} with an identical debt-to-income ratio as before the positive income shock.

Interestingly, and in contrast with the ED case, if the magnitude of the shock were large enough to lower the debt-to-income ratio to a level that corresponds to a risk premium below ϕ^{*}, the saving behavior of the representative agent would change. The consumer would start posting positive saving rates and current account surpluses. Thus, a big enough positive shock could indeed help a country break the perverse recurrent debt cycles. (See Figure 2.13)

Consider an unexpected improvement of the fundamental vulnerabilities of an over-

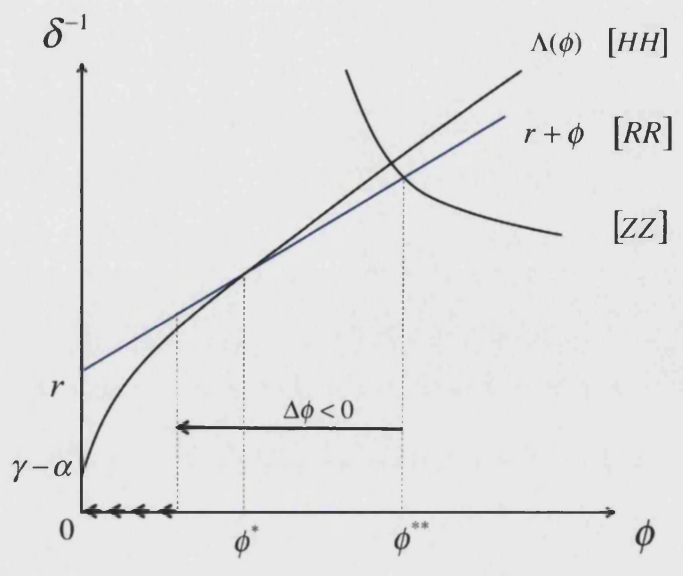

Figure 2.13: A positive shock on ϕ of sufficient magnitude could lead to an immediate decline in risk premium resulting in the country changing its current account pattern toward wealth accumulating dynamics.

indebted country, i.e. a declin in θ (for example an increase in permanent income y), and through it a decline in ϕ. As can be seen in Figure 2.14, schedule **ZZ** would shift inwards, leading to an increase of the maximum level of indebtedness δ^{**} and a decline of the maximum permissible risk premium ϕ^{**}. As ϕ declined and the economy became a better credit, it regains access to capital markets. If the post-shock risk premium is still higher than ϕ^*, debt would start to accumulate and the risk would increase again. Note however that, thanks to the improvement in fundamentals, the maximum level of indebtedness δ^{**} has been extended.

Again, in contrast with the ED case, if the fundamental shock were sufficiently important to lower the risk premium below ϕ^* threshold, the saving behavior of the representative agent would change to a net wealth improving one, as she would start posting positive saving rates.

Similar examples can address the cases of economies with positive initial saving rates, i.e. less indebted economies with positive current accounts or even net creditor shock

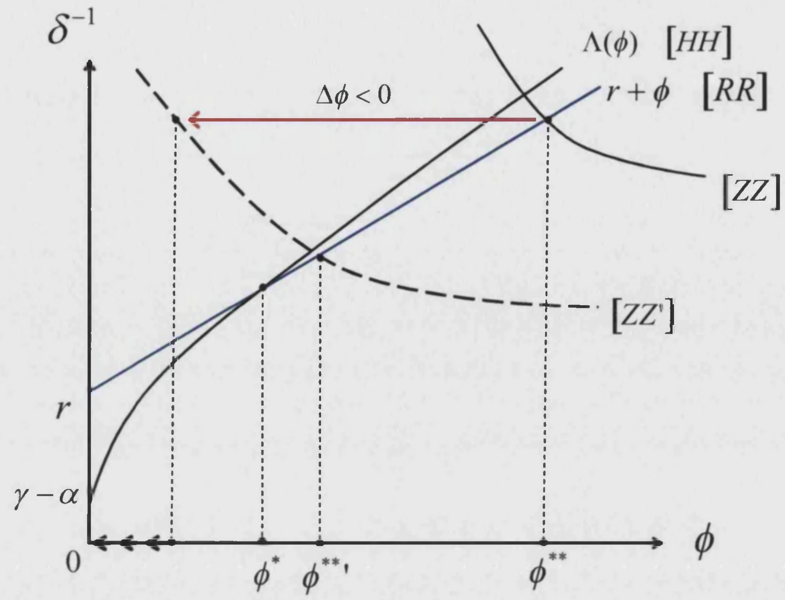

Figure 2.14: A permanent income y shock could lead to a decline in risk premium large enough that could to switch the country's current account from deficit to surplus.

countries. Incidentally, note that in the case of a net creditor country with $\phi = 0$, a rise in y or an improvement in fundamentals θ would simply translate in greater present and future consumption, and an improvement of its current account surpluses, but the saving rate would remain constant at $r - \Lambda(0) = r - (\gamma - \alpha)$.

2.5 General Equilibrium

2.5.1 World polarization and long term risk-free interest rate

In general equilibrium r would be endogenous and such that financial markets at all times clear. Assuming identical agents across countries and frictionless economies with the same economic fundamentals and growth rates normalized to zero, the ED model would set r be equal to ρ; otherwise all economies would run either current account surpluses or deficits, which is naturally impossible. In terms of the **EE**-**RR**-**ZZ** system, schedule **RR** would always overlay schedule **EE** regardless of the level of debt or risk

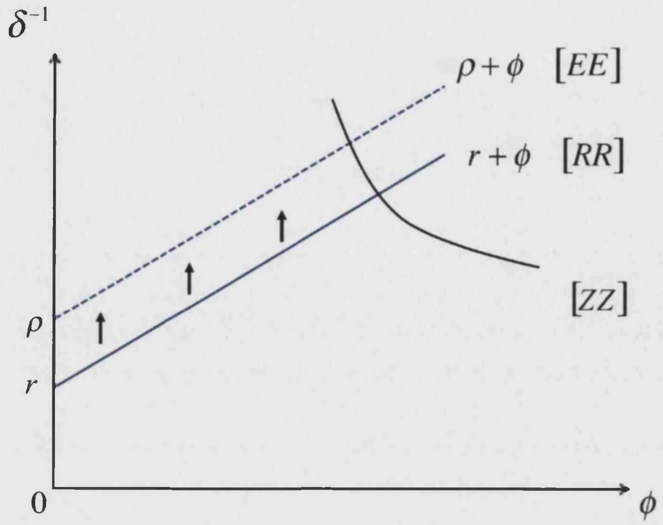

Figure 2.15: Exponential Discounting in General Equilibium

premia prevailing in the different countries.[33] For example, in Figure 2.15 , if $r < \rho$ and the current account were negative in all countries, the worldwide excess demand of funds would make r to jump immediately so that financial markets are cleared.

The ED model requires that in general equilibrium $r = \rho$ at all times. In a frictionless environment, this implies that in a ED setup in order to explain differences in current account patterns of economies with similar growth prospects we would need to assume that their representative agents are endowed with different utility functions or subjective discount rates.

However, the HD model provides a new channel through which two countries with identical growth prospects (although different leverage ratios) could post opposite current account dynamics, while still preserving the assumptions of homogeneity of agents and frictionless economies.

In general equilibrium, the HD model implies a tendency of polarization of seem-

[33]Note that if we assumed income to grow at rate g, the interest rate would be such that the goods market clear and that consumption also grows at rate g. In particular, the Euler equation $\frac{\dot{c}(s)}{c(s)} = r - \rho$ implies that $r = \rho + g$.

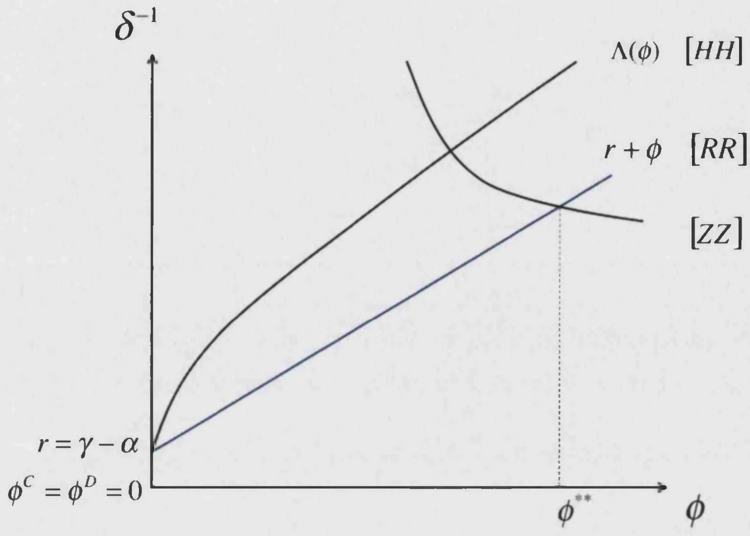

Figure 2.16: Countries C and D, with zero net debt, before the balance-sheet shock

ingly similar economies between net creditor or wealth accumulating economies and net debtor or wealth depleting ones.

As a matter of example, let's assume a world integrated by two countries, C and D, which are originally at the steady state SS with *zero debt* and no risk premium, i.e. $\phi^* = 0$. As can be seen in Figure 2.16 that recalls the hyperbolic **HH**-**RR**-**ZZ** system, this steady state requires that no country runs a current account imbalance, and the corresponding world interest rate would be $r_{ss} = \gamma - \alpha$.

Now, consider a balance-sheet shock: imagine that a court sentenced country D to be liable for past damages incurred against country C. From that instant, country C has got a claim on country D equal to the amount of reparations, i.e. $b^C = -b^D > 0$. While risk premium in country C remains at zero, country D experiences a deterioration of its risk premium, which jumps to $\phi^{D\prime} > 0$ (see Figure 2.17).

In this situation, interest rates r_{SS} would no longer clear the markets. As the risk premium for country D increased, the representative agent would tend to run current account deficits. But at interest rates r_{SS} there are no incentives for country C to lend, so the excess demand of funds will be cleared by a rise in world interest rates from r_{SS}

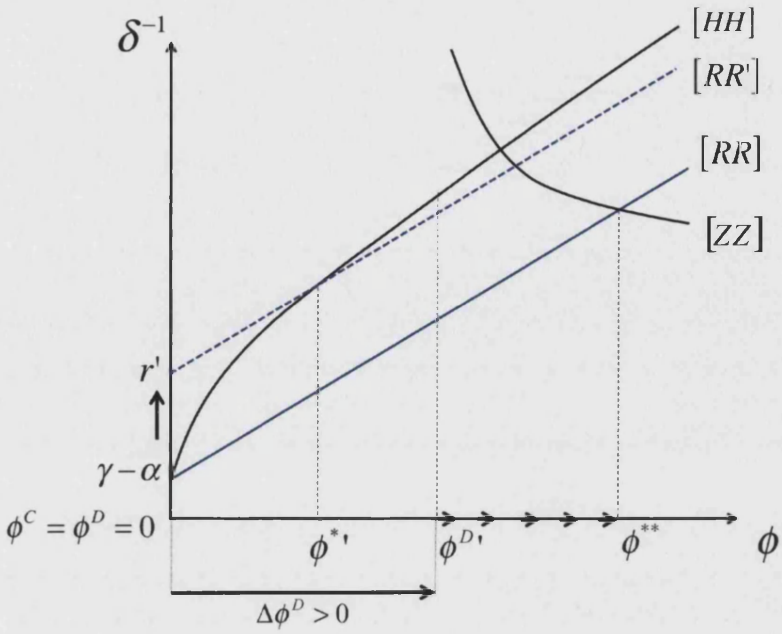

Figure 2.17: Balance-sheet shock: country C has got a claim on country D. The latter's risk premium increases, which makes it run current account deficits.

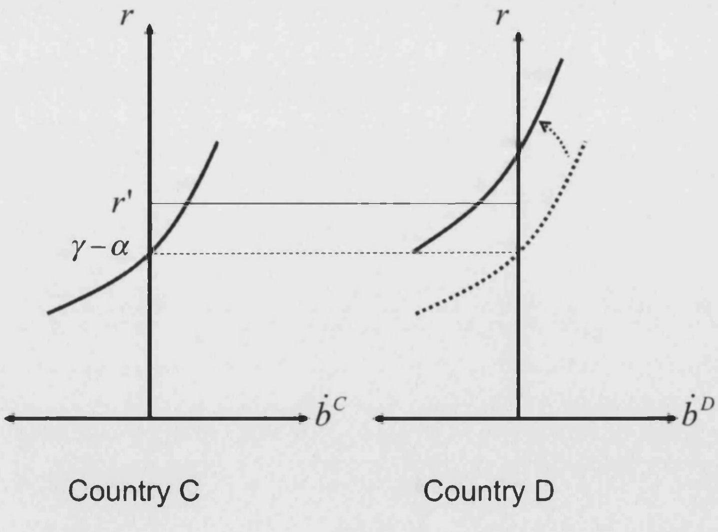

Figure 2.18: Balance-sheet shock: country C has got a claim on country D. The excess demand for credit results in an increase in world interst rates.

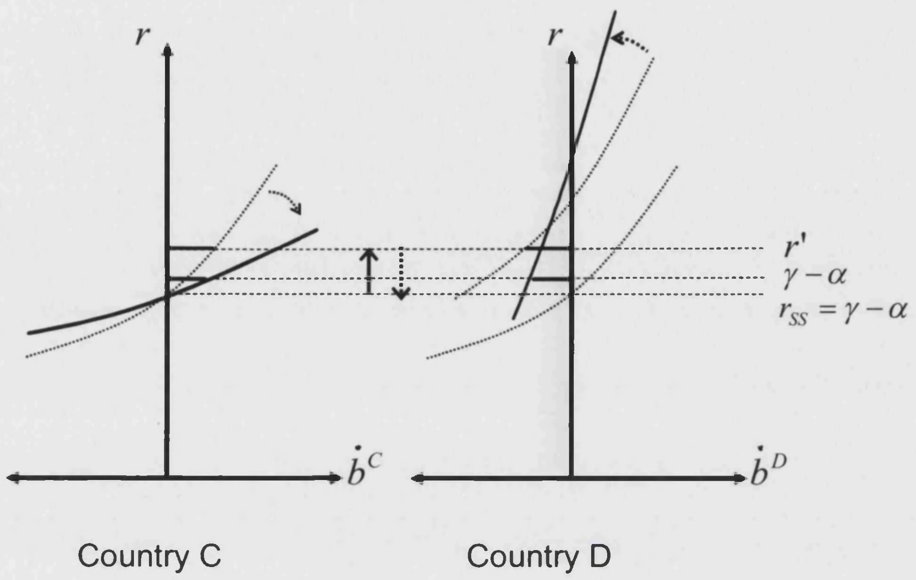

Figure 2.19: Wealth improvement versus wealth deterioration.

to r' as can be seen in the Metzler diagrams in Figure 2.18. Correspondingly, the **RR** schedule in Figure 2.17 shifts up. Since $\phi^{D\prime}$ is to the right of $\phi^{*\prime}$ the current account for country D is negative, while C would turn its current account into surplus as its risk premium remains at $\phi^C = 0$.[34]

As time goes by, country D will continue to exhaust its wealth. Eventually, negative current accounts would become narrower as consumption asymptotically tends to zero and income is increasingly used for servicing interest payments. So country D gradually becomes of "smaller" importance for international capital markets as its demand for credit fades away. On the contrary, economy C had entered into wealth-improving dynamics, and would tend to post increasingly large current account surpluses. Therefore, the excess supply of funds would need to be cleared by declining interest rates as can be seen in the Metzler diagrams in Figure 2.19.

[34]Recall the fact that expected return of a risky position in a foreign country is r. Alternatively, we could imagine that market provides full insurance - via credit default swaps - to cover for the default risk.

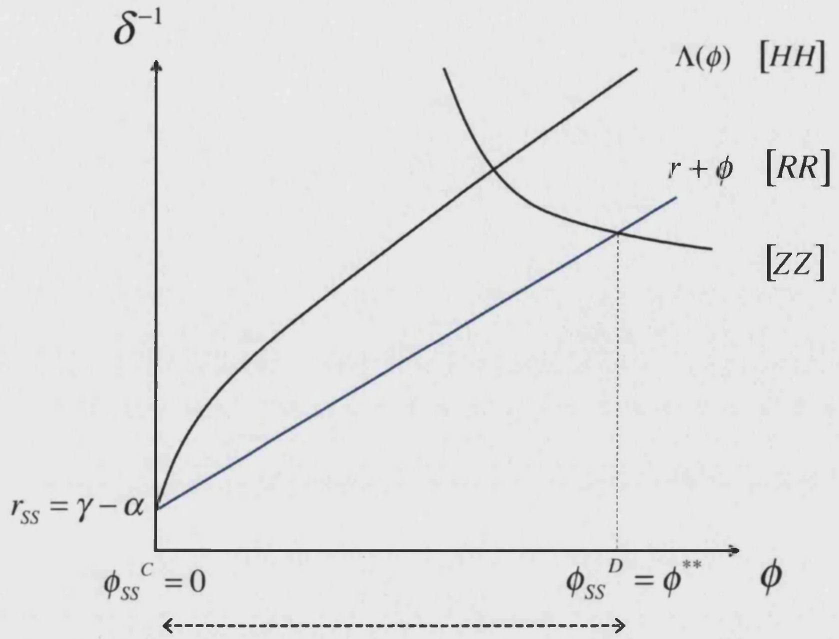

Figure 2.20: "World polarization".

The steady state in general equilibrium is one in which country D has accumulated debt up to a point where all income is allocated to interest payments and nothing is left for consumption. Correspondingly, demand of funds in international markets is nil. On the other hand, any further supply of funds from country C is closed as interest rates should *return* to $r_{SS} = \gamma - \alpha$: this is what we call "world polarization". (See Figure 2.20).

2.5.2 Shocks in permanent income y and in risk premium ϕ

In general equilibrium it is still also possible for a debtor country to switch its current account patterns. An unexpected improvement in the growth prospects (in our case, an increase in permanent income y) or in the fundamental vulnerabilities of net debtor countries (a decline in θ) could signify not only a resumption of access to capital markets but also the opportunity for some of them to make the leap towards wealth accumulating dynamics.

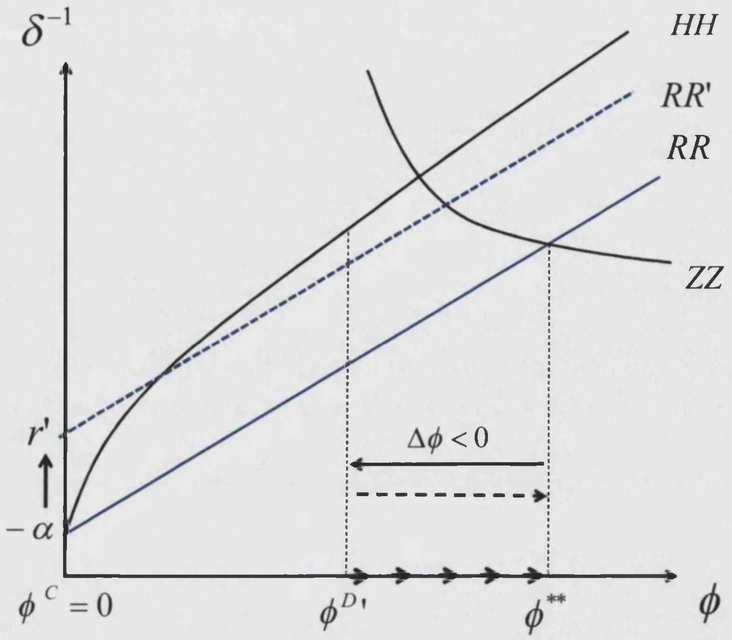

Figure 2.21: A positive shock on permanent income y of over-indebted economies in general equilibrium. Some economies would make the transition towards a wealth accumulating patter. Others, like this one, would not.

Start from a steady state of "world polarization". Consider for example a positive shock on permanent income benefiting a number of *indebted* economies. Such a positive shock would imply excess demand of credit that would increase the world interest rate (See Figure 2.21). Interestingly, while some indebted countries that see their access to capital markets renewed would just resume their credit overhang pattern, other countries may actually become on impact part of the current account surplus countries club.

2.5.3 Conclusion

In this paper we studied how dynamic inconsistency could affect an agent's consumption and saving decisions under systemic risk. By introducing naive, hyperbolic discounting agents characterized for being short-term impatient or "present-biased", our model has been able to generate a negative relationship between the saving rate (or the current account) and the risk premium.

Given the insensitivity of the saving rate to permanent changes in systemic risk in the frictionless ED model, the economic theory has enriched the basic setup by taking account of the temporary nature of shocks, or by incorporating frictions, asymmetries and heterogeneities. Our theoretical contribution intends to complement that literature.

A recurrent critique that has been posed against models of time inconsistency has been that, from a macroeconomic point of view, their results are observationally equivalent to those pertaining to the exponential discounting setup.[35] However, our results are surely not equivalent. On the contrary, our model provides with testable implications that can be contrasted with empirical evidence and tested against alternative models.

Although the scope of our paper is purely theoretical, we would like to note that there is international evidence that suggests that a number of economies that have experienced recurrent cycles of debt accumulations have frequently suffered systemic crises and defaults. The question is whether the framework presented in our paper could provide any insights beyond the classic intertemporal approach to the current account explained under the ED paradigm, which would predict no changes in the saving rate unless an economy has been affected by a temporary shocks. In contrast, our model is able to explain such changes in the saving rate even when shocks are of permanent nature.

One example of positive external shock is the commodities boom that surged as a result of China's exceptional demand for natural resources in the past decade. It could be argued that the nature of this shock is permanent rather than temporary, and nevertheless several emerging markets have been able to switch their current account pattern from persistent deficits to surpluses.[36]

We have focused, mainly for illustrative purposes, on the case of a representative

[35] See, for example, Barro (1999).

[36] See for example the case of Brazil, which turned its current account to a surplus in 2003, in tandem with a drastic decline of its risk premium, after a decade a current account deficits.

agent in a country that faces an exogenous probability of a crisis. We leave for future research further applications of our results to other settings and agents, such as individuals or corporations.

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2.A Proofs of Propositions

2.A.1 Proof of Proposition 2.1

1. The equation of motion for the anticipated consumption path is dictated by the Euler Equation (2.35) which implies that a consumption level $\tau - t$ periods ahead is expected to take value

$$ {}_t c_\tau = c_t e^{r(\tau-t)} (1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} \tag{2.88}$$

2. Let's define anticipated wealth $\tau - t$ periods ahead of time t as the net present value of future income plus the net foreign asset position

$$ {}_t W_\tau \equiv {}_t b_\tau + \frac{y}{r + \phi} \tag{2.89}$$

where $r + \phi$ is the relevant steady state interest rate.

3. Therefore, the anticipated current account

$$ {}_t \dot{b}_\tau = (r + \phi) \, {}_t b_\tau + y - {}_t c_\tau \tag{2.90}$$

can be rewritten as

$$ {}_t \dot{b}_\tau = (r + \phi) \, {}_t W_\tau - {}_t c_\tau \tag{2.91}$$

4. Since y is constant, wealth changes with net foreign asset position

$$ {}_t \dot{W}_\tau = {}_t \dot{b}_\tau \tag{2.92}$$

5. We can rewrite (2.91) as

$$
{}_t\dot{W}_\tau = (r+\phi)\,{}_tW_\tau - {}_tc_\tau
$$

or

$$
{}_tc_\tau = -\left({}_t\dot{W}_\tau - (r+\phi)\,{}_tW_\tau\right) \tag{2.93}
$$

6. Multiplying both sides of (2.93) by $e^{-(r+\phi)(\tau-t)}$ and integrating it with respect to time from time s to ∞, $\forall s \geq t$, we obtain

$$
\int_s^\infty e^{-(r+\phi)(\tau-t)}\,{}_tc_\tau d\tau = -\int_s^\infty e^{-(r+\phi)(\tau-t)}\left({}_t\dot{W}_\tau - (r+\phi)\,{}_tW_\tau\right)d\tau
$$

or

$$
\begin{aligned}
\int_s^\infty e^{-(r+\phi)(\tau-t)}\,{}_tc_\tau d\tau &= -\, e^{-(r+\phi)(\tau-t)}\,{}_tW_\tau\Big]_s^\infty \\
&= -\left[\lim_{\tau\longrightarrow\infty}\left(e^{-(r+\phi)(\tau-t)}\,{}_tW_\tau\right) - e^{-(r+\phi)(s-t)}\,{}_tW_s\right] \\
&= -\left[\lim_{\tau\longrightarrow\infty} e^{-(r+\phi)(\tau-t)}\left({}_tb_\tau + \frac{y}{r+\phi}\right) - e^{-(r+\phi)(s-t)}\,{}_tW_s\right] \\
&= -\left[\lim_{\tau\longrightarrow\infty} e^{-(r+\phi)(\tau-t)}\,{}_tb_\tau - e^{-(r+\phi)(s-t)}\,{}_tW_s\right]
\end{aligned}
$$

where $\lim_{\tau\longrightarrow\infty} e_t^{-(r+\phi)(\tau-t)}\,{}_tb_\tau = 0$ by the transversality condition. Therefore,

$$
\int_s^\infty e^{-(r+\phi)(\tau-t)}\,{}_tc_\tau d\tau = e^{-(r+\phi)(s-t)}\,{}_tW_s \tag{2.94}
$$

7. Recall equation (2.88), which states anticipated consumption taking place $\tau - t$ periods away of period t:

$$
{}_tc_\tau = c_t e^{r(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}} \tag{2.95}
$$

and, similarly, for consumption taking place $s-t$ periods away of period t :

$$ {}_tc_s = c_t e^{r(s-t)}(1+\alpha(s-t))^{-\frac{\gamma}{\alpha}} \tag{2.96} $$

with $\tau > s$.

8. Using (2.95) and (2.96) we can write ${}_tc_\tau$ in terms of ${}_tc_s$, $\forall\tau \geq s$. Let's rewrite (2.96) as

$$ c_t = {}_tc_s e^{-r(s-t)}\left(\frac{1}{1+\alpha(s-t)}\right)^{-\frac{\gamma}{\alpha}} $$

and introduce it into (2.95) to get

$$ {}_tc_\tau = {}_tc_s e^{r(\tau-s)}\left(\frac{1+\alpha(\tau-t)}{1+\alpha(s-t)}\right)^{-\frac{\gamma}{\alpha}} \tag{2.97} $$

or

$$ {}_tc_\tau = {}_tc_s e^{r(\tau-s)}\left(1+\frac{\alpha(\tau-s)}{1+\alpha(s-t)}\right)^{-\frac{\gamma}{\alpha}} \tag{2.98} $$

9. We can insert this last equation into (2.94) and get

$$ e^{(r+\phi)(s-t)}\;{}_tc_s\int_s^\infty e^{-(r+\phi)(\tau-t)}\quad e^{r(\tau-s)}\left(1+\frac{\alpha(\tau-s)}{1+\alpha(s-t)}\right)^{-\frac{\gamma}{\alpha}}\Bigg)\,d\tau = {}_tW_s \tag{2.99} $$

or

$$ {}_tc_s\int_s^\infty e^{-\phi(\tau-s)}\left(1+\frac{\alpha}{1+\alpha(s-t)}(\tau-s)\right)^{-\frac{\gamma}{\alpha}} d\tau = {}_tW_s \tag{2.100} $$

10. Therefore, we can write the anticipated consumption path as

$$ {}_tc_s \;=\; \tilde{\Lambda}(t,s;\phi)\,{}_tW_s \tag{2.101} $$

or

$$\frac{{}_tc_s}{{}_tW_s} = \tilde{\Lambda}(t,s;\phi) \tag{2.102}$$

where

$$\tilde{\Lambda}(t,s;\phi) \equiv \left(\int_s^{\infty} e^{-\phi(\tau-s)} \left(1+\tilde{\alpha}(t,s)(\tau-s)\right)^{-\frac{\gamma}{\alpha}} d\tau \right)^{-1} \tag{2.103}$$

and where $\tilde{\alpha}(t,s) \equiv \frac{\alpha}{1+\alpha(s-t)}$.

Characterization of function $\tilde{\Lambda}(t,s;\phi)$:

1. Since $\tilde{\alpha}(t,s)$ is decreasing in s, and the integrand is decreasing in function $\tilde{\alpha}(t,s)$, then $\frac{{}_tc_s}{{}_tW_s} = \tilde{\Lambda}(t,s;\phi)$ is **decreasing** in time s.

2. **Limits**: At present time $s=t$, we have $\lim_{s\longrightarrow t} \tilde{\alpha}(t,s) = \alpha$ therefore

$$\lim_{s\longrightarrow t} \tilde{\Lambda}(t,s;\phi) = \Lambda(\phi) \tag{2.104}$$

where $\Lambda(\phi) \equiv \int_s^{\infty} e^{-\phi(\tau-s)} \left(1+\alpha(\tau-s)\right)^{-\frac{\gamma}{\alpha}} d\tau$.

Also note that when time horizon $s-t$ goes to infinity, $\lim_{s\longrightarrow\infty} \tilde{\alpha}(t,s) = 0$, thus

$$\lim_{s\longrightarrow\infty} \tilde{\Lambda}(t,s;\phi) = \phi \tag{2.105}$$

QED.

2.A.2 Proof of Proposition 2.2

Claim 1 *The current account is anticipated to run increasingly large surpluses in the future regardless of the initial sign of the current account.*

1. Similar to the actual current account, the anticipated current account can be written as

$$_tCA_s = \, _t\dot{b}_s = \, (r+\phi)\, _tb_s - \, _tc_s$$

2. Let's use the anticipated wealth definition (2.89) and equation (2.101) into the anticipated current account equation to obtain

$$_tCA_s = \, _t\dot{b}_s = \, (r+\phi)\left(_tb_s + \frac{y}{r+\phi}\right) - \, \tilde{\Lambda}(t,s;\phi)\left(_tb_s + \frac{y}{r+\phi}\right)$$

or

$$\frac{_tCA_s}{_tW_s} = \frac{_t\dot{b}_s}{_tW_s} = \, r+\phi - \tilde{\Lambda}(t,s;\phi), \qquad (2.106)$$

which increases with time horizon $s - t$. (Recall from our previous discussion around equation (2.103) that expression $\tilde{\Lambda}(t,s;\phi)$ declines with time horizon $s - t$).

3. By differentiating (2.89) with respect to time we find that $_t\dot{b}_s = \, _t\dot{W}_s$, which allow us to write (2.106) as follows:

$$\frac{_t\dot{W}_s}{_tW_s} = \, r+\phi - \tilde{\Lambda}(t,s;\phi) \qquad (2.107)$$

This clearly differentiates from the ED model, where ratio $\frac{CA_s}{W_s} = \frac{\dot{W}_s}{W_s} = r - \rho$ at all times.

4. Recall from (2.104) that $\tilde{\Lambda}\,(t,s;\phi)$ for present time $s = t$ simplifies to $\Lambda\,(\phi)$. If at time $s = t$ we had $\Lambda\,(\phi) < r + \phi$ and the current account were initially in surplus, it is clear from (2.106) that as $\tilde{\Lambda}(t,s;\phi)$ declines over time current account surpluses become increasingly large.

5. Similarly, if initially $\Lambda\,(\phi) > r + \phi$, the current account would initially be in deficit and equal to $CA_t = \, ((r+\phi) - \Lambda\,(\phi))\, W_t$, but it is anticipated to turn

into surplus at some future point in time $\tilde{s}$ such that

$$\tilde{\Lambda}(t, \tilde{s}; \phi) = r + \phi.$$

Note that this point in time $\tilde{s}$ exists because, as has been shown in (2.105), as time horizon $s - t$ goes to infinity, then $\lim_{s \longrightarrow \infty} \tilde{\Lambda}(t, s; \phi) = \phi$, which is lower than $r + \phi$.

6. It also follows that the current account to wealth ratio asymptotically converges to risk-free rate r, i.e. $\lim\limits_{s \longrightarrow \infty} \frac{{}_tCA_s}{{}_tW_s} = \frac{{}_t\dot{W}_s}{{}_tW_s} = \; r > 0.$

7. Since wealth is positive and increasing if $\Lambda\left(\phi\right) < r + \phi$ (or turns positive after time $\tilde{s}$ if initial current account were in deficit or if $\Lambda\left(\phi\right) > r + \phi$) it must be that the anticipated current account runs increasingly large surpluses after time $\tilde{s}$.

Claim 2 *A declining anticipated consumption-to-wealth ratio as specified in (2.101) must allow anticipated wealth to grow so that at certain future point in time $\breve{s}$ the anticipated rate of wealth accumulation outpaces the rate of decline of the consumption-to-wealth ratio, thereby allowing anticipated consumption to increase.*

1. Recalling the anticipated Euler equation (2.35), we know that, under the assumption that $\gamma > r$, declining anticipated consumption would bottom at future time $\breve{s}$, such that

$$\frac{\gamma}{1 + \alpha(\breve{s} - t)} = r,$$

after which time consumption is anticipated to increase.

2. We aim to prove that at such a future point in time $\breve{s}$ the wealth positive growth rate outpaces the rate of declining consumption-to-wealth ratio. We can state

equation (2.101) in terms of growth rates as

$$\begin{aligned}\frac{{}_t\dot{c}_s}{{}_tc_s} &= \frac{{}_t\dot{W}_s}{{}_tW_s} + \frac{(\dot{{}_tc_s/{}_tW_s})}{({}_tc_s/{}_tW_s)} \\ &= \frac{{}_t\dot{W}_s}{{}_tW_s} + \frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}}\end{aligned}$$

where $\frac{{}_t\dot{W}_s}{{}_tW_s} > 0$ and the rate of chance of the consumption-to-wealth ratio is $\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}} < 0$. So anticipated consumption would turn positive when $\frac{{}_t\dot{W}_s}{{}_tW_s} > -\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}}$.

3. Let's first find $\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}}$ and rewrite $\tilde{\Lambda}(t,s;\phi)$ as

$$\tilde{\Lambda}(t,s;\phi) = \left(e^{\phi(s-t)}(1+\alpha(s-t))^{\frac{\gamma}{\alpha}} \int_s^{\infty} e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}} d\tau \right)^{-1}$$

which we can rewrite for convenience as

$$\tilde{\Lambda}(t,s;\phi) = F(t,s)\breve{\Lambda}(t,s;\phi)$$

where $F(t,s) \equiv \left(e^{\phi(s-t)}(1+\alpha(s-t))^{\frac{\gamma}{\alpha}}\right)^{-1}$ and
$\breve{\Lambda}(t,s;\phi) \equiv \left(\int_s^{\infty} e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}} d\tau \right)^{-1}$

4. Thus, the share of anticipated consumption in wealth $\frac{{}_tc_s}{{}_tW_s} = \tilde{\Lambda}$ grows at rate

$$\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}} = \frac{\dot{F}(t,s)}{F(t,s)} + \frac{\dot{\breve{\Lambda}}(t,s;\phi)}{\breve{\Lambda}(t,s;\phi)}$$

where

$$\frac{\dot{F}(t,s)}{F(t,s)} = -\left(\phi + \frac{\gamma}{1+\alpha(s-t)}\right)$$

and

$$
\begin{aligned}
\frac{\dot{\breve{\Lambda}}(t,s;\phi)}{\breve{\Lambda}(t,s;\phi)} &= -\frac{\frac{\partial}{\partial s}\int_s^\infty e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}}d\tau}{\int_s^\infty e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}}d\tau} \\
&= -\frac{e^{-\phi(s-t)}(1+\alpha(s-t))^{-\frac{\gamma}{\alpha}}}{\int_s^\infty e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}}d\tau} \\
&= -e^{-\phi(s-t)}(1+\alpha(s-t))^{-\frac{\gamma}{\alpha}}\cdot\left(\int_s^\infty e^{-\phi(\tau-t)}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}}d\tau\right)^{-1} \\
&= -e^{-\phi(s-t)}(1+\alpha(s-t))^{-\frac{\gamma}{\alpha}}\cdot\breve{\Lambda}(t,s;\phi) \\
&= \tilde{\Lambda}
\end{aligned}
$$

5. Therefore,

$$
\frac{\dot{({}_tc_s/{}_tW_s)}}{({}_tc_s/{}_tW_s)} = \frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}} = -\left(\phi+\frac{\gamma}{1+\alpha(s-t)}\right)+\tilde{\Lambda}
$$

which, incidentally , is negative, as per the above discussion about (2.103).

6. Since consumption share on wealth is declining and wealth itself is improving, consumption growth will be positive if and only if $\frac{{}_t\dot{W}_s}{{}_tW_s}+\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}}>0$

7. Recalling (2.107), we can write

$$
\begin{aligned}
\frac{{}_t\dot{c}_s}{{}_tc_s} &= \frac{{}_t\dot{W}_s}{{}_tW_s}-\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}} = (r+\phi)-\tilde{\Lambda}-\left(\phi+\frac{\gamma}{1+\alpha(s-t)}\right)+\tilde{\Lambda} \\
&= r-\frac{\gamma}{1+\alpha(s-t)}>0
\end{aligned}
$$

which is independent of ϕ and defines the time $\breve{s}$ beyond which anticipated consumption growth is positive. This is perfectly consistent with Euler Equation (2.35).

Claim 3 *If initial current account is negative, it is anticipated to become positive at some future time $\tilde{s}$ before time $\breve{s}$ at which anticipated consumption growth rate turns from negative to positive.*

For consumption to increase requires that wealth increases at a faster rate than the rate of decline of the share of consumption in wealth. A necessary condition is that the current account is positive (so that rate of growth rate of wealth is also positive).We must show that the minimum time $\tilde{s}$ at which the current account becomes positive must take place before time $\breve{s}$ at which consumption growth rate turns from negative to positive.

1. By definition, time $\tilde{s}$ is such that $\tilde{\Lambda}(t, \tilde{s}; \phi) = r + \phi$. From (2.107) it follows that $\tilde{s}$ corresponds to $\frac{{}_t\dot{W}_s}{{}_tW_s} = 0$.

2. From (2.101) we can verify that for anticipated consumption ${}_tc_s = \tilde{\Lambda}(t, s; \phi)\, {}_tW_s$ to we positive it must be that $\frac{{}_t\dot{W}_s}{{}_tW_s} > -\frac{\dot{\tilde{\Lambda}}}{\tilde{\Lambda}} > 0$. From (2.107) we notice that this requires $\tilde{\Lambda}(t, s; \phi) < (r + \phi)$. Since $\tilde{\Lambda}(t, s; \phi)$ is decreasing with time horizon $s - t$, it must be the case that the time $\breve{s}$ at which anticipated consumption growth turns from negative to positive, (i.e. when $\frac{{}_t\dot{c}(\breve{s})}{{}_tc(s)} = 0$) takes place some time after $\tilde{s}$. **QED.**

2.A.3 Proof of Proposition 2.4

1. Steady state ϕ^* is stable iff

$$\left.\frac{d\dot{\phi}}{d\phi}\right|_{\phi^*} < 0$$

or unstable if otherwise, where $\phi = \phi\left(\delta(t),\ \theta\right)$ is given by equation (2.57), were $\delta(t) = \frac{-b(t)}{y}$.

2. Differentiating $\phi(\cdot)$ with respect to time, we get

$$\dot{\phi} = -\phi_\delta \frac{1}{y}\dot{b}(t)$$

where $\phi_\delta > 0$ and $y > 0$ is constant. Therefore,

$$\frac{d\dot{\phi}}{d\phi} = -\phi_\delta \frac{1}{y}\frac{d\dot{b}(t)}{d\phi}$$

we note that the sign of $\frac{d\dot{\phi}}{d\phi}$ is the *opposite* of the sign of $\frac{d\dot{b}(t)}{d\phi}$.

3. Recall the equation of motion for $b(t)$ given by (2.6):

$$\dot{b}(t) = (r + \phi)\, b(t) + y - c(t)$$

4. Taking derivatives with respect to ϕ, we obtain

$$\frac{d\dot{b}(t)}{d\phi} = b(t) - \frac{dc(t)}{d\phi} \tag{2.108}$$

5. Therefore, we need to find

$$\left.\frac{d\dot{\phi}}{d\phi}\right|_{\phi^*} = -\phi_\delta \frac{1}{y}\left.\frac{d\dot{b}(t)}{d\phi}\right|_{\phi^*}$$

where

$$\left.\frac{d\dot{b}(t)}{d\phi}\right|_{\phi^*} = b^* - \left.\frac{dc(t)}{d\phi}\right|_{\phi^*}$$

where, at the steady state, $b^* < 0$ and consumption is given by (2.84) which we recall:

$$c^* = \Lambda(\phi^*)\, W^*.$$

6. Deriving $c(t)$ with respect to any constant ϕ obtains

$$\begin{aligned}\frac{dc(t)}{d\phi} &= \frac{d}{d\phi}\left[\Lambda(\phi) W(t,\phi)\right] \\ &= \Lambda'(\phi) W(t,\phi) + \Lambda(\phi)\frac{d}{d\phi}\left[W(t,\phi)\right] \\ &= \Lambda'(\phi)\left(b(t) + \frac{y}{r+\phi}\right) - \frac{\Lambda(\phi)}{r+\phi}\frac{y}{r+\phi}\end{aligned}$$

where $\Lambda'(\phi) > 1$ for any constant ϕ as was already proved by the facts that (i) $\Lambda'(0) > 1$, (ii) $\Lambda'(\phi) > 0$ and (iii) $\Lambda''(\phi) < 0$ and (iv) $\lim_{\phi \longrightarrow \infty} \Lambda'(\phi) = 1$ or $\lim_{\phi \longrightarrow \infty} [\Lambda(\phi) - \phi] = \gamma$.

7. Thus, we can write

$$\left.\frac{dc(t)}{d\phi}\right|_{\phi^*} = \Lambda'(\phi^*) b^* + \Lambda'(\phi^*) \frac{y}{r+\phi^*} - \frac{\Lambda(\phi^*)}{r+\phi^*}\frac{y}{r+\phi^*}$$

and since at the steady state $\Lambda(\phi^*) = r + \phi^*$,

$$\left.\frac{dc(t)}{d\phi}\right|_{\phi^*} = \Lambda'(\phi^*) b^* + \left(\underbrace{\Lambda'(\phi^*) - 1}_{>0}\right)\frac{y}{r+\phi^*} > 0$$

which is unambiguously positive.

8. Therefore, $\left.\frac{d\dot{b}(t)}{d\phi}\right|_{\phi^*} < 0$ and $\left.\frac{d\dot{\phi}}{d\phi}\right|_{\phi^*} > 0$, which proves that the steady state is *unstable*.

9. Additionally, the sign of the steady state cannot reverse in the space $\phi(s) \in [0, \phi^{**})$ because in that space there in only one possible steady state, i.e. $\{\phi^{**}, b^{**}, c^{**}\}$.

QED.

2.B Derivation of First Order Conditions and Transversality Condition for General Dynamic Optimization with Hyperbolic Discounting

This development generalizes the first-order-conditions and transversality condition for a dynamic optimization problem where the discount factor is not necessarily exponential, as is typically the case in the current-value Hamiltonian literature. In particular, we set up the maximization conditions for the case of hyperbolic discounting.

2.B.1 The general finite-horizon case

The general problem is to

$$\text{maximize} \qquad V = \int_0^T F(t,b,c)dt \tag{2.109}$$

$$\text{s.t.} \quad c(t) \in C \qquad \forall t \in [0,T] \tag{2.110}$$

$$\dot{b} = f(t,b,c) \quad \forall t \in [0,T] \tag{2.111}$$

$$b_T \geq 0 \qquad \text{Non-Ponzi-Game condition} \tag{2.112}$$

$$T \text{ given} \tag{2.113}$$

$$b_0 \text{ given} \tag{2.114}$$

Pontryagin' Maximum Principle conditions:

Define the Hamiltonian as

$$H(t,b,c,\mu) \equiv F(t,b,c) + \mu(t)f(t,b,c) \tag{2.115}$$

Assuming that the Hamiltonian is differentiable with respect to c, $\forall t \in [0, T]$, and that there is an interior solution, Pontryagin' Maximum Principle dictates that the maximum value of the functional (2.109) subject to (2.110) - (2.114) requires that the following first order conditions and transversality condition hold:

$$\frac{\partial H}{\partial c} = 0 \tag{2.116}$$

$$\frac{\partial H}{\partial b} = -\dot{\mu} \tag{2.117}$$

$$\frac{\partial H}{\partial \mu} = \dot{b}_t \tag{2.118}$$

$$\mu(T) b_T = 0 \tag{2.119}$$

$$\text{with complementary slackness} \quad : \quad \mu(T) \geq 0; \qquad b_T \geq 0$$

Proof of the Maximum Principle conditions

If the equation of motion $\dot{b} = f(t, b, c)$ is strictly adhered to for the entire time period $[0, T]$, we can incorporate it into the objective functional V and create a Lagrangian. Maximization of objective functional (2.109) subject to (2.110) - (2.114) is therefore equivalent to maximizing a properly defined Lagrangian in the following manner:

1. Recalling the Kuhn-Tucker theorem, we can construct a Lagrangian function of the form

$$L = \int_0^T F(t, b, c) dt + \int_0^T \mu(t) \left[f(t, b, c) - \dot{b} \right] dt + \nu b_T \tag{2.120}$$

The second term is just the integration of constraint (2.111) that must hold for $\forall t \in [0, T]$, i.e.

$$\int_0^T \mu(t) \left[f(t, b, c) - \dot{b} \right] dt = 0$$

where $\mu(t)$ is the shadow price of the constraint $\forall t \in [0, T]$.

The third term comes from the inequality (Non-Ponzi-Game) condition (2.112), which implies

$$\nu b_T = 0 \tag{2.121}$$

with complementary slackness i.e. $\nu \geq 0$ and $b_T \geq 0$, where ν is the Langrange multiplier associated with constraint (2.112).

2. We can rewrite (2.120) as

$$L = \int_0^T \left\{ F(t, b, c) + \mu(t) \left[f(t, b, c) - \dot{b} \right] \right\} dt + \nu b_T$$

3. Integration by parts of expression $\int_0^T \mu(t)\dot{b}dt$ allows us to write the above expression as

$$L = \int_0^T [F(t, b, c) + \mu(t)f(t, b, c) + \dot{\mu}b(t)]\, dt + \mu(0)b_0 - \mu(T)b_T + \nu b_T$$

or,

$$L = \int_0^T [H(t, b, c, \mu) + \dot{\mu}b(t)]\, dt - \mu(T)b_T + \mu(0)b_0 + \nu b_T \tag{2.122}$$

where $H(\cdot)$ is defined as the Hamiltonian function of the form

$$H(t, b, c, \mu) \equiv F(t, b, c) + \mu(t)f(t, b, c) \tag{2.123}$$

where $\mu(t)$ is the costate variable[37].

Notice that the Lagrangian (2.122) depends on the paths of b, c and μ.

[37]Economic interpretation: μ_t is the shadow price of the budget constraint or the marginal utility of wealth W_t valued at time 0. It is the extra utility in present value units of a marginal increment of the stock of financial assets b at time t. Alternatively, μ_t is the extra utility (valued at time 0) of releasing the IBC by one marginal unit.

4. We can generate paths for c and b to be compared with neighboring optimal paths. To do this we take known paths $c^*(t)$ and $b^*(t)$ and adopt perturbing "arbitrary" curves $p(t)$ and $q(t)$ so that

$$c(t,\epsilon) = c^*(t) + \epsilon p(t)$$

$$b(t,\epsilon) = b^*(t) + \epsilon q(t)$$

Similarly, if T and b_T were free variables, we can write

$$T(\epsilon) = T^* + \epsilon \Delta T$$

$$b_T(\epsilon) = b_T^* + \epsilon \Delta b_T$$

5. We cannot choose μ to be a function of ϵ or an arbitrary curve. Notice from definition (2.123) that $\frac{\partial H}{\partial \mu} = f(t,b,c)$. Therefore, so long the equation of motion of the state variable $\dot{b} = f(t,b,c)$ is strictly adhered to, as per condition (2.110), the optimal path $\mu(t)$ must be such that

$$\frac{\partial H}{\partial \mu} = \dot{b}_t \tag{2.124}$$

6. The Lagrangian (2.122) can be written in terms of the artificially generated paths $c(t,\epsilon)$ and $b(t,\epsilon)$ and definitions for $T(\epsilon)$ and $b_T(\epsilon)$:

$$\begin{aligned} L &= \int_0^{T(\epsilon)} \left[H(t, b(t,\epsilon), c(t,\epsilon), \mu) + \dot{\mu}\left[b(t,\epsilon)\right]\right] dt \\ &\quad + \mu(0)b_0 - \mu(T(\epsilon))b_T(\epsilon) + \nu b_T(\epsilon) \end{aligned} \tag{2.125}$$

Notice that L has been transformed into a function of ϵ only.

7. We can take first order conditions and maximize (2.125) with respect to ϵ, i.e.

setting

$$\frac{dL}{d\epsilon} = 0$$

8. Using Leibniz rule, we find

$$0 = \tfrac{dL}{d\epsilon} = \int_0^{T(\epsilon)} \left[\tfrac{\partial H}{\partial b} q(t) + \tfrac{\partial H}{\partial c} p(t) + \dot{\mu} q(t)\right] dt + \left[H(t,\cdot) + \dot{\mu} b(t)\right]_{t=T} \tfrac{dT}{d\epsilon}$$
$$- \mu(T) \tfrac{db_T}{d\epsilon} - \dot{\mu}(T) b_T \tfrac{dT}{d\epsilon} + \nu \tfrac{db_T}{d\epsilon} \tag{2.126}$$

9. Noting that $\frac{dT}{d\epsilon} = \Delta T$ and $\frac{db_T}{d\epsilon} = \Delta b_T$ and that $\left[\dot{\mu} b(t)\right]_{t=T} \frac{dT}{d\epsilon} = \dot{\mu}(T) b_T \Delta T$, the above equation simplifies to

$$\begin{aligned} 0 \ = \ & \frac{dL}{d\epsilon} = \underbrace{\int_0^T \left[\left[\frac{\partial H}{\partial b} + \dot{\mu}\right] q(t) + \frac{\partial H}{\partial c} p(t)\right] dt}_{\Omega_1} \\ & \underbrace{+ \left[H\right]_{t=T} \Delta T}_{\Omega_2} \ + \underbrace{\left[\nu - \mu(T)\right] \Delta b_T}_{\Omega_3} \end{aligned} \tag{2.127}$$

10. Since we chose arbitrary perturbing curves and $\frac{dV}{d\epsilon} = 0$ must hold, it must be that the term $\Omega_1$vanishes for any curves $p(t)$ and $q(t)$. This requires that

$$\frac{\partial H}{\partial c} = 0 \tag{2.128}$$

and

$$\frac{\partial H}{\partial b} = -\dot{\mu} \tag{2.129}$$

Equation (2.128), invoked in place of "$\max_c H$", assumes that the Hamiltonian is differentiable with respect to c and that there is an interior solution. Equation (2.129) gives the equation of motion for the costate variable.

11. As for the second component, $\Omega_2 = 0$ is true if T is by assumption fixed, i.e. $\Delta T = 0$

12. In order to make $\Omega_3 = 0$, it must be the case that

$$\nu = \mu(T) \tag{2.130}$$

that is, the costate variable $\mu(T)$ at the terminal date must equal the static Lagrange multiplier ν associated with the non-negativity constraint on b_T at the terminal date.

13. From (2.130) we can rewrite the boundary complementary slackness condition (2.121) as

$$\mu(T)b_T = 0 \tag{2.131}$$

This is the the Transversality Condition.

Pontryagin' Maximum Principle therefore dictates that the maximum value of the functional (2.109) subject to (2.110) - (2.114) require that first order conditions (2.124), (2.128) and (2.129), and transversality condition (2.131) hold.

2.B.2 The general infinite-horizon case

In an infinite-horizon framework, the general problem is to

$$\text{maximize} \qquad V = \int_0^\infty F(t, b, c)dt \tag{2.132}$$

$$\text{s.t.} \quad c(t) \in C \qquad \forall t \in [0, \infty) \tag{2.133}$$

$$\dot{b} = f(t, b, c) \quad \forall t \in [0, \infty) \tag{2.134}$$

$$\lim_{T \to \infty} b_T \exp\left(-\int_0^T i(t)dt\right) \geq 0 \qquad \text{Non-Ponzi-Game condition} \tag{2.135}$$

$$b_0 \text{ given} \tag{2.136}$$

Pontryagin' Maximum Principle conditions:

Define the Hamiltonian as

$$H(t,b,c,\mu) \equiv F(t,b,c) + \mu(t)f(t,b,c) \tag{2.137}$$

Assuming that the Hamiltonian is differentiable with respect to c, $\forall t \in [0,\infty)$, and that there is an interior solution, Pontryagin' Maximum Principle dictates that the maximum value of the functional (2.132) subject to (2.133) - (2.136) requires that the following first order conditions and transversality condition hold:

$$\frac{\partial H}{\partial c} = 0 \tag{2.138}$$

$$\frac{\partial H}{\partial b} = -\dot{\mu} \tag{2.139}$$

$$\frac{\partial H}{\partial \mu} = \dot{b}_t \tag{2.140}$$

$$\begin{aligned} \lim_{T\to\infty} \mu(T)b_T &= 0 \\ \text{with complementary slackness} &: \lim_{T\to\infty} \mu(T) \geq 0; \quad \lim_{T\to\infty} b_T \geq 0 \end{aligned} \tag{2.141}$$

Proof of the Maximum Principle conditions for the infinite-horizon case

1. If the equation of motion $\dot{b} = f(t,b,c)$ is strictly adhered to for the entire period of time $[0,\infty)$, we can incorporate it into the objective functional V and create a Lagrangian. Maximization of objective functional (2.132) subject to (2.133) - (2.136) is therefore equivalent to maximizing a properly defined Lagrangian.

2. Recalling the Kuhn-Tucker theorem, we can construct a Lagrangian function of

the form

$$L = \int_0^{\infty} F(t,b,c)dt + \int_0^{\infty} \mu(t)\left[f(t,b,c) - \dot{b}\right] dt + \nu \lim_{T\to\infty} b_T \exp\left(-\int_0^T i(t)dt\right) \tag{2.142}$$

The second term is just the integration of constraint (2.134) that must hold for $\forall t \in [0,\infty)$, i.e.

$$\int_0^{\infty} \mu(t)\left[f(t,b,c) - \dot{b}\right] dt = 0$$

where $\mu(t)$ is the shadow price of the constraint $\forall t \in [0,\infty)$.

The third term comes from the inequality (Non-Ponzi-Game) condition (2.135), which implies

$$\nu \lim_{T\to\infty} b_T \exp\left(-\int_0^T i(t)dt\right) = 0 \tag{2.143}$$

with complementary slackness i.e. $\nu \geq 0$ and $\lim_{T\to\infty} b_T \exp\left(-\int_0^T i(t)dt\right) \geq 0$, where ν is the Langrange multiplier associated with constraint (2.135).

3. When the heuristic proof for the finite-horizon case is adapted to the infinite-horizon framework with $T \to \infty$, the first order condition (2.127) of the Lagrangian becomes

$$\begin{aligned} 0 \;=\; \frac{dL}{d\epsilon} = & \underbrace{\int_0^{\infty}\left[\left[\frac{\partial H}{\partial b} + \dot{\mu}\right] q(t) + \frac{\partial H}{\partial c} p(t)\right] dt}_{\Omega_1} \\ & \underbrace{+\lim_{T\to\infty} [H]_{t=T}\,\Delta T}_{\Omega_2} \; \underbrace{-\lim_{T\to\infty}\left[\nu \exp\left(-\int_0^T i(t)dt\right) - \mu(T)\right]\Delta b_T}_{\Omega_3} \end{aligned} \tag{2.144}$$

4. As for the finite-horizon case, the Lagrangian maximizing condition requires that each of components $(\Omega_1, \Omega_2, \Omega_3)$ vanishes individually.

5. Vanishing of term Ω_1 give rise to two first order conditions. Specifically, the maximization of the Hamiltonian with respect to control variable c and the equation

of motion for the costate variable:

$$\frac{\partial H}{\partial c} = 0 \tag{2.145}$$

and

$$\frac{\partial H}{\partial b} = -\dot{\mu} \tag{2.146}$$

6. It is the vanishing of Ω_2 and Ω_3 that give rise to the Transversality Conditions.

7. The second term Ω_2, notice that ΔT is non-zero in an infinite-horizon problem. To make Ω_2 vanish we must impose the first infinite-horizon transversality condition

$$\lim_{T\to\infty} [H]_{t=T} = 0$$

   Economic interpretation: If H function sums up the overall (current plus future) utility prospect associated with each admissible value of control variable c, so long as H remains positive, there is yet some utility to be made by the appropriate choice of control c. This conditions is not in dispute.

8. In order to make $\Omega_3 = 0$, the following must hold

$$\lim_{T\to\infty} \nu \exp\left(-\int_0^T i(t)dt\right) = \lim_{T\to\infty} \mu(T) \tag{2.147}$$

   that is, when $T \to \infty$ the costate variable $\mu(T)$ must equal the discounted value of the Lagrange multiplier ν associated with the non-negativity constraint of the discounted value of b_T (Non-Ponzi-Game condition).

9. From equation (2.147) we can rewrite the boundary complementary slackness condition (2.121) simply as

$$\lim_{T\to\infty} \mu(T) b_T = 0 \tag{2.148}$$

This is the Transversality Condition.

2.B.3 Generalized current value Hamiltonian

Assume that functional (2.132) above can be written in such form that the integrand $F(t, b, c)$ comprises the present utility function $U(t, b, c)$ multiplied by discount factor $\chi(t)$, i.e.

$$F(t, b, c) = U(t, b, c)\chi(t)$$

The dynamic problem becomes:

$$\text{maximize} \quad V = \int_0^\infty U(t, b, c)\chi(t)dt \tag{2.149}$$

$$\text{s.t.} \quad c(t) \in C \qquad \forall t \in [0, \infty) \tag{2.150}$$

$$\dot{b} = f(t, b, c) \quad \forall t \in [0, \infty) \tag{2.151}$$

$$\lim_{T\to\infty} b_T \exp\left(-\int_0^T i(t)dt\right) \geq 0 \qquad \text{Non-Ponzi-Game condition} \tag{2.152}$$

$$b_0 \ \text{ given} \tag{2.153}$$

The corresponding *present value* Hamiltonian can be written as

$$H(t, b, c, \mu) \equiv U(t, b, c)\chi(t) + \mu(t)f(t, b, c) \tag{2.154}$$

Definition 4 *Define now the Current Value Hamiltonian as*

$$\hat{H} = H\chi_t^{-1}$$

Therefore, we can write

$$\hat{H}(t,b,c,\mu) = U(t,b,c) + \lambda(t)f(t,b,c) \tag{2.155}$$

where

$$\lambda_t = \mu_t \chi_t^{-1} \tag{2.156}$$

is the shadow price valued at time t.

Maximum Principle conditions for a *current value* Hamiltonian

The Maximum Principle conditions applied to the current value Hamiltonian, become

$$\hat{H}_c = 0 \tag{2.157}$$

$$\hat{H}_{b_t} = -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1} \dot{\chi}_t\right) \tag{2.158}$$

$$\hat{H}_{\lambda_t} = \dot{b}_t \tag{2.159}$$

$$\begin{aligned} \lim_{T\to\infty} \lambda_T \chi_T b_T &= 0 \\ \text{with complementary slackness} &: \lim_{T\to\infty} \lambda_T \geq 0; \lim_{T\to\infty} b_T \chi_T \geq 0 \end{aligned} \tag{2.160}$$

Let's find each of the three first order conditions and the transversality condition above, making use of definition (2.155):

Maximization of H with respect to c : Recalling (2.155) and FOC (2.138) we can restate $H_c \equiv \frac{\partial H}{\partial c} = 0$ in terms of $\hat{H}$ as follows:

$$\begin{aligned} H_c &= 0 \\ \frac{\partial}{\partial c}\left(\hat{H}\chi_t\right) &= 0 \end{aligned}$$

Since χ_t is constant for any given t, it follows that

$$\chi_t \hat{H}_c = 0$$

$\Rightarrow$

$$\hat{H}_c = 0 \tag{2.161}$$

Equation of motion for the costate variable λ **:** Recalling (2.155), FOC (2.139) and (2.156), we can restate $H_b \equiv \frac{\partial H}{\partial b} = -\dot{\mu}$ in terms of $\hat{H}$, χ_t and λ_t as follows:

$$\begin{aligned} H_b &= -\dot{\mu} \\ \chi_t \frac{\partial}{\partial b}\left(\hat{H}\right) &= -\dot{\mu} \\ \hat{H}_{b_t}\chi_t &= -\frac{d}{dt}\left(\lambda_t \chi_t\right) \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} \hat{H}_{b_t} &= \chi_t^{-1}\left(-\frac{d}{dt}\left(\lambda_t \chi_t\right)\right) \\ &= -\chi_t^{-1}\frac{d}{dt}\left(\lambda_t \chi_t\right) \\ &= -\chi_t^{-1}\left(\dot{\lambda}_t \chi_t + \lambda_t \frac{\partial}{\partial t}\chi_t\right) \\ &= -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1}\left(\frac{\partial}{\partial t}\chi_t\right)\right) \end{aligned}$$

$\Longleftrightarrow$

$$\hat{H}_{b_t} = -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1}\dot{\chi}_t\right) \tag{2.162}$$

Equation of motion for the state-variable b **:** Recalling (2.155), FOC (2.140) and (2.156), we can restate $H_\mu \equiv \frac{\partial H}{\partial \mu} = \dot{b}_t$ in terms of $\hat{H}$ and λ_t as follows:

$$
\begin{aligned}
H_{\mu_t} &= \dot{b}_t \\
\chi_t \frac{\partial}{d\mu_t}\left(\hat{H}\right) &= \dot{b}_t \\
\chi_t \frac{\partial \hat{H}}{\partial \lambda_t}\frac{\partial \lambda_t}{\partial \mu_t} &= \dot{b}_t \\
\chi_t \hat{H}_{\lambda_t}\chi_t^{-1} &= \dot{b}_t
\end{aligned}
$$

$\Longrightarrow$

$$\hat{H}_{\lambda_t} = \dot{b}_t \tag{2.163}$$

Transversality condition: Recall FOC (2.141)

$$\lim_{T\to\infty} \mu_T b_T = 0$$

We can restate the above equation in terms of χ_t and λ_t simply by inserting definition (2.156) in place of μ_T. The Transversality Condition therefore becomes

$$
\begin{aligned}
\lim_{T\to\infty} \lambda_T \chi_T b_T &= 0 \qquad (2.164)\\
\text{with complementary slackness} &: \lim_{T\to\infty} \lambda_T \geq 0; \lim_{T\to\infty} b_T \chi_T \geq 0
\end{aligned}
$$

2.B.4 Case of Hyperbolic Discounting with no risk premium

Current value Hamiltonian

Recall the current value Hamiltonian (2.155)

$$\hat{H}(t, b, c, \mu) = U(t, b, c) + \lambda(t) f(t, b, c) \tag{2.165}$$

Define the particular case of a discount factor taking the hyperbolic form

$$\chi_t \equiv (1+\alpha t)^{-\gamma/\alpha} \tag{2.166}$$

with $\alpha, \gamma > 0$

Maximum Principle conditions

For the particular case of hyperbolic discounting, we realize that the FOCs and Transversality Condition are the same as those stated for the general case in equations (2.157) - (2.160), except that now equation (2.159) must consider the hyperbolic discount factor as defined by equation (2.166). In particular, the Maximum Principle conditions will be

$$\hat{H}_c = 0 \tag{2.167}$$

$$\hat{H}_{b_t} = \frac{\gamma}{1+\alpha t}\lambda_t - \dot{\lambda}_t \tag{2.168}$$

$$\hat{H}_{\lambda_t} = \dot{b}_t \tag{2.169}$$

$$\begin{aligned} \lim_{T\to\infty} \lambda_T \chi_T b_T &= 0 \\ \text{with complementary slackness} &: \lim_{T\to\infty} \lambda_T \geq 0; \lim_{T\to\infty} b_T \chi_T \geq 0 \end{aligned} \tag{2.170}$$

Derivation of (2.168) We can use the specification of the hyperbolic discount factor (2.166) to evaluate FOC (2.162):

$$\begin{aligned} \hat{H}_{b_t} &= -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1} \dot{\chi}_t\right) \\ &= -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1} \left(\frac{\partial}{\partial t}\chi_t\right)\right) \\ &= -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1} \left(\frac{\partial}{\partial t}\left[(1+\alpha t)^{-\gamma/\alpha}\right]\right)\right) \end{aligned}$$

$$= -\left(\dot{\lambda}_t + \lambda_t \chi_t^{-1}\left(-\frac{\gamma\chi_t}{(1+\alpha t)}\right)\right)$$

$\iff$

$$\hat{H}_{b_t} = \frac{\gamma}{1+\alpha t}\lambda_t - \dot{\lambda}_t \tag{2.171}$$

where the term $\frac{\gamma}{1+\alpha t}$ falls with t, as opposed to what would have been the case if the discount function were exponential.

2.B.5 References

The following references were used in this section of the Appendix:

1. Barro, Robert J. and Sala-i-Martin, X. (1995). "Economic Growth". New York: McGraw-Hill.

2. Chiang, A. (1992). "Elements of Dynamic Optimization". McGraw-Hill. Chapter 7, 8 and 9.

3. Dixit, A. K. (1990). "Optimization in Economic Theory". Oxford University Press, 2nd edition.

4. Kamien, Moront I. and Schwartz, Nancy L. (1991). "Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management". North-Holland (Advanced Textbooks in Economics), Amsterdam and New York

5. Pontriagyn, Lev S., et al. (1962). "The Mathematical Theory of Optimal Processes". New York. Interscience Publishers.

6. Simon and Blume (1994). "Mathematics for Economists". New York, London. W.W. Norton & Company

2.C Derivations

2.C.1 Derivatives of function Π

Recall function $\Pi(t, x, \{\phi(s)\}_t)$:

$$\Pi(t, x, \{\phi(s)\}_t) \equiv \left(\int_t^{\infty} \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau \right)^{-1}$$

where $\bar{\phi}(t,\tau) = \int_t^{\tau} \phi(s) ds$ is the *spot* risk premium for period $(t, \tau]$. The partial derivatives of Π with respect to x is

$$\frac{\partial \Pi(\cdot)}{\partial x} = -\frac{1}{\Pi(\cdot)^2} \frac{\partial}{\partial x} \left[\int_t^{\infty} \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau \right].$$

Therefore,

$$\text{Sign} \, \frac{\partial \Pi(\cdot)}{\partial x} = - \text{Sign} \, \frac{\partial}{\partial x} \left[\int_t^{\infty} \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau \right]$$

Using the Leibniz's rule to derive the expression in the RHS, we get

$$\begin{aligned} \tfrac{\partial}{\partial x} \left[\int_t^{\infty} \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau \right] &= \int_t^{\infty} - \tfrac{\partial [x(\tau - t)]}{\partial x} \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau \\ &= - \int_t^{\infty} (\tau - t) \exp\left[- \left(x + \bar{\phi}(t,\tau) \right) (\tau - t) \right] d\tau < 0 \end{aligned}$$

Therefore, $\frac{\partial \Pi(t,x)}{\partial x} > 0$. **QED.**

2.C.2 Derivation of (2.20) – (2.21)

Recall that the Maximum Principle conditions for a current value Hamiltonian for any modified discount factor χ_s correspond to equations (2.157) - (2.160) in the Appendix. Applied to an ED problem enjoying utility function $U_{c^N}(c_s, c_s^N)$ and intertemporal discount $F(t,s) = \exp\left[-\rho(s-t)\right]$, the Maximum Principle conditions are the following

1. $\hat{H}_{c_s^N} = 0$,

$$U_{c^N}(c_s, c_s^N) = \frac{1}{e_s}\lambda_s \tag{a}$$

2. $\hat{H}_{c_s} = 0$,

$$U_c(c_s, c_s^N) = \lambda_s \tag{b}$$

The ratio of (b) to (a) is

$$\frac{U_c(c_s, c_s^N)}{U_{c^N}(c_s, c_s^N)} = e_s \tag{c}$$

3. $\hat{H}_{b_s} = -\left(\dot{\lambda}_s + \lambda_s \chi_s^{-1}\dot{\chi}_s\right)$,

From (2.13) we note that the LHS is given by $\hat{H}_{b_s} = \lambda_s(r+\phi(s))$. As for the RHS, recall that the ED modified discount factor is $\chi_s = \exp\left[-\rho(s-t) - \int_t^s \kappa(\tau)d\tau\right]$. So, $\dot{\chi}_s$ becomes

$$\begin{aligned}
\dot{\chi}_s &= \frac{d}{ds}\left(e^{-\rho(s-t)-\int_t^s \kappa(\tau)d\tau}\right) \\
&= \frac{d}{ds}\left(-\rho(s-t) - \int_t^s \kappa(\tau)d\tau\right)\chi_s \\
&= \left(-\rho - \kappa(s)\right)\chi_s
\end{aligned}$$

Therefore, this FOC is

$$\lambda_s(r+\phi(s)) = -\dot{\lambda}_s + \lambda_s\left(\rho + \kappa(s)\right)$$

or, written differently,

$$\frac{\dot{\lambda}_s}{\lambda_s} = \rho + \kappa(s) - (r + \phi(s)) \tag{d}$$

4. $\hat{H}_{\lambda_s} = \dot{b}_s$,

   which restates the current account (2.6).

5. Transversality Condition

$$\lim_{s\to\infty} \lambda_s \chi_s b_s = 0 \tag{TVC}$$

2.C.3 Derivation of (2.24)

1. Rewrite (2.23) as

$$\frac{d}{ds}(\ln c_s) = r - \rho$$

   which can be integrated over time,

$$\left[\frac{d}{ds}(\ln c(s))\right]_t^{\tau} = \int_t^{\tau} (r-\rho)\, ds$$

   or,

$$\ln c(\tau) - \ln c(t) = (r-\rho)(\tau - t)$$

   or,

$$\ln c(\tau) = \ln c(t) + (r-\rho)(\tau - t)$$

   Taking exponential of both sides of the last equation, we obtain

$$c(\tau) = c(t)\exp\left[(r-\rho)(\tau - t)\right]$$

   $\forall \tau \geq t$.

2. Insert this last expression in the LHS of the resource constraint (2.7), and obtain

$$
\begin{aligned}
& c(t)\int_t^\infty \exp\left[(r-\rho)(\tau-t)\right]\exp\left[-r(\tau-t)-\int_t^\tau \phi(s)ds\right]d\tau \\
= \; & b(t)+y\int_t^\infty \exp\left[-r(\tau-t)-\int_t^\tau \phi(s)ds\right]d\tau
\end{aligned}
$$

or,

$$
c(t)\int_t^\infty \exp\left[-\rho(\tau-t)-\int_t^\tau \phi(s)ds\right]d\tau = b(t)+y\int_t^\infty \exp\left[-r(\tau-t)-\int_t^\tau \phi(s)ds\right]d\tau
$$

where $\phi(s)=\phi(b(s),\theta)$. **QED.**

2.C.4 Derivation of (2.33)

Notation: For convenience, throughout this derivation we dispense with the anticipation operator.

1. Following equation (2.158) in the Appendix corresponding to the general case of current value Hamiltonian, the FOC for with respect to ${}_tb(s)$ is $\hat{H}_{b_s} = -\left(\dot{\lambda}_s + \lambda_s \chi_s^{-1}\dot{\chi}_s\right)$.

2. Recall that the augmented discount factor for the HD case is $\chi_s = \exp\left[-\int_t^s \kappa(\tau)d\tau\right]\digamma(t,s)$, where $\digamma(t,s)=(1+\alpha(s-t))^{-\gamma/\alpha}$.

3. First, find $\dot{\chi}_s$. Given definition (2.5), it can be derived as

$$
\begin{aligned}
\dot{\chi}_s &= \frac{d}{ds}\left(e^{-\int_t^s \kappa(\tau)d\tau}\digamma(s,t)\right) \\
&= \frac{d}{ds}\left(e^{-\int_t^s \kappa(\tau)d\tau}\right)\digamma(s,t) + e^{-\int_t^s \kappa(\tau)d\tau}\frac{d}{ds}\left(\digamma(s,t)\right) \\
&= \frac{d}{ds}\left(-\int_t^s \kappa(\tau)d\tau\right)e^{-\int_t^s \kappa(\tau)d\tau}\digamma(s,t) + e^{-\int_t^s \kappa(\tau)d\tau}\frac{d}{ds}\left((1+\alpha(s-t))^{-\gamma/\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\kappa(s)e^{-\int_t^s \kappa(\tau)d\tau}F(s,t) + e^{-\int_t^s \kappa(\tau)d\tau} \quad -\frac{\gamma}{\alpha}\alpha\frac{(1+\alpha(s-t))^{-\gamma/\alpha}}{1+\alpha(s-t)}\Bigg) \\
&= -\kappa(s)e^{-\int_t^s \kappa(\tau)d\tau}F(s,t) + e^{-\int_t^s \kappa(\tau)d\tau}\left(\frac{\gamma}{1+\alpha(s-t)}F(s,t)\right) \\
&= -\left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\left(e^{-\int_t^s \kappa(\tau)d\tau}F(s,t)\right) \\
&= -\left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\chi_s
\end{aligned}
$$

4. Therefore, the FOC can be rewritten as:

$$
\begin{aligned}
\hat{H}_{b_s} &= -\left(\dot{\lambda}_s + \lambda_s\chi_s^{-1}\dot{\chi}_s\right) \\
&= -\left(\dot{\lambda}_s - \lambda_s\chi_s^{-1}\chi_s\left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\right) \\
&= -\left(\dot{\lambda}_s - \lambda_s\left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\right) \\
&= \left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\lambda_s - \dot{\lambda}_s
\end{aligned}
$$

5. Note from (2.13) that the LHS of the FOC is given by $\hat{H}_{b_s} = \lambda_s(r + \phi(s))$.

6. Therefore, the FOC can be solved as follows:

$$
\lambda_s(r+\phi(s)) = \left(\kappa(s) + \frac{\gamma}{1+\alpha(s-t)}\right)\lambda_s - \dot{\lambda}_s
$$

or, reintroducing the use of the anticipation operator,

$$
\frac{{}_t\dot{\lambda}(s)}{{}_t\lambda(s)} = \frac{\gamma}{1+\alpha(s-t)} + {}_t\kappa(s) - (r + \phi(s))
$$

QED.

2.C.5 Derivation of (2.79):

1. Integrate (2.77) over time

$$\begin{aligned}\left[\frac{d}{ds}(\ln {}_t c(s))\right]_t^\tau &= \int_t^\tau \left(r - \frac{\gamma}{1+\alpha(s-t)}\right) ds \\ &= \left(r(\tau - t) - \int_t^\tau \left(\frac{\gamma}{1+\alpha(s-t)}\right) ds\right) \quad (2.172)\end{aligned}$$

2. Change variables of the last term in the RHS of the above equation such that $v = s - t$, $ds = dv$, when $s = t$, $v = 0$ and when $s = \tau$, $v = \tau - t$. Then, solve this term as

$$\begin{aligned}\int_t^\tau \left(\frac{\gamma}{1+\alpha(s-t)}\right) ds &= \int_0^{\tau-t} \left(\frac{\gamma}{1+\alpha v}\right) dv \\ &= \left[\frac{\gamma}{\alpha}\ln[1+\alpha v]\right]_0^{\tau-t} \\ &= \frac{\gamma}{\alpha}\ln[1+\alpha(\tau-t)]\end{aligned}$$

3. Therefore, equation (2.172) can be written as

$$\ln_t c(\tau) - \ln_t c(t) = \left(r(\tau-t) - \frac{\gamma}{\alpha}\ln[1+\alpha(\tau-t)]\right)$$

or,

$$\ln_t c(\tau) = \ln_t c(t) + r(\tau-t) - \frac{\gamma}{\alpha}\ln[1+\alpha(\tau-t)]$$

$$\ln_t c(\tau) = \ln_t c(t) + r(\tau-t) - \frac{\gamma}{\alpha}\ln[1+\alpha(\tau-t)]$$

4. Taking exponential of both sides of the equation, we get (2.79):

$$ {}_t c(\tau) = {}_t c(t)\exp[r(\tau-t)](1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}} \quad (2.173)$$

QED.

2.C.6 Characterization of function Λ

Partial derivatives of (2.41)

Recall Λ from (2.41)

$$\Lambda(\phi) \equiv \left(\int_t^{\infty} e^{-\phi(\tau - t)} (1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} d\tau \right)^{-1}$$

which can be rewritten by changing variables: $v = \tau - t$, $dv = dz$; when $\tau = t$, $v = t$ and when $\tau = \infty$, $v = \infty$. Therefore,

$$\Lambda^{-1} = \int_0^{\infty} \frac{e^{-v\phi}}{(v\alpha + 1)^{\frac{\gamma}{\alpha}}} dv$$

The partial derivatives of Λ^{-1} are:

1.
$$\frac{\partial \Lambda^{-1}}{\partial \phi} = -\int_0^{\infty} v e^{-\phi v} (1 + \alpha v)^{-\frac{\gamma}{\alpha}} dv < 0$$

2.
$$\begin{aligned}
\frac{\partial \Lambda^{-1}}{\partial \alpha} &= \frac{d}{d\alpha} \left[\int_0^{\infty} \frac{e^{-v\phi}}{(v\alpha + 1)^{\frac{\gamma}{\alpha}}} dv \right] \\
&= \int_0^{\infty} e^{-v\phi} \left(\frac{1}{\alpha^2} \gamma \frac{\ln(v\alpha + 1)}{(v\alpha + 1)^{\frac{1}{\alpha}\gamma}} - \frac{v}{\alpha} \frac{\gamma}{(v\alpha + 1)^{\frac{1}{\alpha}\gamma + 1}} \right) dv \\
&= \int_0^{\infty} e^{-v\phi} \left(\frac{\gamma}{\alpha^2 (v\alpha + 1)^{\frac{1}{\alpha}\gamma}} \ln(v\alpha + 1) - \frac{\gamma}{\alpha^2 (v\alpha + 1)^{\frac{1}{\alpha}\gamma}} \frac{v\alpha}{(v\alpha + 1)} \right) dv \\
&= \int_0^{\infty} e^{-v\phi} \left(\frac{1}{\alpha^2} \frac{\gamma}{(v\alpha + 1)^{\frac{1}{\alpha}\gamma}} \left(\ln(v\alpha + 1) - \frac{v\alpha}{(v\alpha + 1)} \right) \right) dv > 0
\end{aligned}$$

which is positive since $\ln(v\alpha + 1) > \frac{v\alpha}{(v\alpha+1)}$.

3.

$$
\begin{aligned}
\frac{\partial \Lambda^{-1}}{\partial \gamma} &= \frac{d}{d\gamma}\left[\int_0^{\infty} \frac{e^{-v\phi}}{(v\alpha+1)^{\frac{\gamma}{\alpha}}}\, dv\right] \\
&= -\frac{1}{\alpha}\int_0^{\infty} \left. e^{-v\phi}\frac{\ln(v\alpha+1)}{(v\alpha+1)^{\frac{\gamma}{\alpha}}}\right)\, dv < 0
\end{aligned}
$$

Therefore, the derivatives of Λ with respect to the three parameters will be just the *opposite* sign of the derivatives with respect to Λ^{-1}. **QED.**

Limiting properties of

Proof of $\lim_{\phi \longrightarrow 0}\left[\Lambda(\phi) - \phi\right] = \gamma - \alpha$

1. The limit can be written as

$$
\begin{aligned}
\lim_{\phi \longrightarrow 0}\left[\Lambda(\phi) - \phi\right] &= \lim_{\phi \longrightarrow 0}\left[\frac{1}{\Lambda(\phi)}\right] - \lim_{\phi \longrightarrow 0}\left[\phi\right] \\
&= \frac{1}{\lim_{\phi \longrightarrow 0}\left[\Lambda(\phi)\right]} \qquad (2.174)
\end{aligned}
$$

2. In order to solve the denominator, recall the definition $\Lambda(\phi)$ from (2.41) and solve for the limit

$$
\begin{aligned}
\lim_{\phi \longrightarrow 0}\left[\Lambda(\phi)\right] &= \lim_{\phi \longrightarrow 0}\left[\int_t^{\infty} e^{-\phi(\tau - t)}(1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} d\tau\right] \\
&= \int_t^{\infty} (1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} d\tau
\end{aligned}
$$

3. The last integral can be easily solved by changing variables: $z = 1 + \alpha(\tau - t)$,

$d\tau = \frac{1}{\alpha}dz$; when $\tau = t$, $z = 1$ and when $\tau = \infty$, $z = \infty$. Therefore,

$$\begin{aligned}
\lim_{\phi \longrightarrow 0} [\Lambda(\phi)] &= \int_1^\infty \frac{1}{\alpha} z^{-\frac{\gamma}{\alpha}} dz \\
&= \frac{1}{\alpha}\left[\frac{z^{1-\frac{\gamma}{\alpha}}}{1-\frac{\gamma}{\alpha}}\right]_1^\infty \\
&= \frac{1}{\gamma - \alpha} > 0
\end{aligned}$$

4. Using this result in equation (2.174) result in

$$\begin{aligned}
\lim_{\phi \longrightarrow 0} [\Lambda(\phi) - \phi] &= \frac{1}{\lim_{\phi \longrightarrow 0} [\Lambda(\phi)]} \\
&= \frac{1}{\frac{1}{\gamma - \alpha}} \\
&= \gamma - \alpha.
\end{aligned}$$

QED.

Proof of $\lim_{\phi \longrightarrow \infty} [\Lambda(\phi) - \phi] = \gamma$.

1. Recall $\Lambda(\phi)$ from (2.41) and write

$$\lim_{\phi \longrightarrow \infty} [\Lambda(\phi) - \phi] = \lim_{\phi \longrightarrow \infty} \left[\left(\int_t^\infty e^{-\phi(\tau - t)}(1 + \alpha(\tau - t))^{-\frac{\gamma}{\alpha}} d\tau\right)^{-1} - \phi\right]$$

2. Change variables such that $v = \frac{\phi}{\alpha}(1 + \alpha(\tau - t))$, so $d\tau = \frac{1}{\phi}dv$; if $\tau = t$, then $v = \frac{\phi}{\alpha}$. Therefore,

$$\lim_{\phi \longrightarrow \infty} [\Lambda(\phi) - \phi] = \lim_{\phi \longrightarrow \infty} \left[\left(\int_{\frac{\phi}{\alpha}}^\infty e^{\frac{\phi}{\alpha} - v}\left(\frac{\alpha}{\phi}v\right)^{-\frac{\gamma}{\alpha}} \frac{1}{\phi} dv\right)^{-1} - \phi\right]$$

$$= \lim_{\phi\longrightarrow\infty}\left[\alpha^{\frac{\gamma}{\alpha}}e^{-\frac{\phi}{\alpha}}\phi^{1-\frac{\gamma}{\alpha}}\left(\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv\right)^{-1}-\phi\right]$$

$$= \lim_{\phi\longrightarrow\infty}\left[\phi\left(\alpha^{\frac{\gamma}{\alpha}}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}}\left(\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv\right)^{-1}-1\right)\right]$$

$$= \lim_{\phi\longrightarrow\infty}\left[\frac{\left(\alpha^{\frac{\gamma}{\alpha}}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}}-\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv\right)}{\phi^{-1}\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv}\right] = \lim_{\phi\longrightarrow\infty}\left[\frac{0}{0}\right]$$

where $\lim_{\phi\longrightarrow\infty}\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv = 0.$

3. Use L'Hopital: if $\left[\frac{\lim_{\phi\longrightarrow\infty}f(\phi)}{\lim_{\phi\longrightarrow\infty}g(\phi)}\right] = \left[\frac{0}{0}\right]$, then $\lim_{\phi\longrightarrow\infty}\left[\frac{f(\phi)}{g(\phi)}\right] = \lim_{\phi\longrightarrow\infty}\left[\frac{f'(\phi)}{g'(\phi)}\right]$, where

$$f(\phi) = \alpha^{\frac{\gamma}{\alpha}}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}} - \int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv$$

and

$$g(\phi) = \phi^{-1}\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv$$

4. The first derivatives are

$$\begin{aligned} f'(\phi) &= -\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}} - \gamma\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}-1} + \alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}} \\ &= -\gamma\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}-1} \end{aligned}$$

$$g'(\phi) = -\phi^{-2}\int_{\frac{\phi}{\alpha}}^{\infty}e^{-v}v^{-\frac{\gamma}{\alpha}}dv - \alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}-1}$$

5. So,

$$
\begin{aligned}
\lim_{\phi \longrightarrow \infty} \left[\frac{f(\phi)}{g(\phi)} \right] &= \lim_{\phi \longrightarrow \infty} \left[\frac{f'(\phi)}{g'(\phi)} \right] \\
&= \lim_{\phi \longrightarrow \infty} \left[\frac{-\gamma \alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{-\frac{\gamma}{\alpha}-1}}{-\phi^{-2} \int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v} v^{-\frac{\gamma}{\alpha}} dv - \alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{-\frac{\gamma}{\alpha}-1}} \right] \\
&= \lim_{\phi \longrightarrow \infty} \left[\frac{\gamma \alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{1-\frac{\gamma}{\alpha}}}{\int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v} v^{-\frac{\gamma}{\alpha}} dv + \alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{1-\frac{\gamma}{\alpha}}} \right] \\
&= \lim_{\phi \longrightarrow \infty} \left[\frac{\gamma}{\frac{\int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v} v^{-\frac{\gamma}{\alpha}} dv}{\alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{1-\frac{\gamma}{\alpha}}} + 1} \right]
\end{aligned}
$$

where we note that $1 - \frac{\gamma}{\alpha} < 0$ by assumption.

6. The limit of the term in the denominator is $\lim_{\phi \longrightarrow \infty} \left[\frac{\int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v} v^{-\frac{\gamma}{\alpha}} dv}{\alpha^{\frac{\gamma}{\alpha}-1} e^{-\frac{\phi}{\alpha}} \phi^{1-\frac{\gamma}{\alpha}}} \right] = \left[\frac{0}{0} \right]$.

7. Again, use L'Hopital for this last term alone, so that

$$\lim_{\phi\longrightarrow\infty}\left[\frac{\int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v}v^{-\frac{\gamma}{\alpha}}dv}{\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{1-\frac{\gamma}{\alpha}}}\right] = \lim_{\phi\longrightarrow\infty}\left[\frac{-\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}}}{-\alpha^{\frac{\gamma}{\alpha}-2}e^{-\frac{\phi}{\alpha}}\phi^{1-\frac{\gamma}{\alpha}}-\left(1-\frac{\gamma}{\alpha}\right)\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{-\frac{\gamma}{\alpha}}}\right]$$

$$= \lim_{\phi\longrightarrow\infty}\left[\frac{1}{\phi+\left(\frac{\gamma}{\alpha}-1\right)}\right]=0$$

8. Therefore,

$$\begin{aligned}
\lim_{\phi\longrightarrow\infty}\left[\frac{f(\phi)}{g(\phi)}\right] &= \lim_{\phi\longrightarrow\infty}\left[\frac{f'(\phi)}{g'(\phi)}\right] \\
&= \lim_{\phi\longrightarrow\infty}\left[\frac{\gamma}{\frac{\int\limits_{\frac{\phi}{\alpha}}^{\infty} e^{-v}v^{-\frac{\gamma}{\alpha}}dv}{\alpha^{\frac{\gamma}{\alpha}-1}e^{-\frac{\phi}{\alpha}}\phi^{1-\frac{\gamma}{\alpha}}}+1}\right] \\
&= \lim_{\phi\longrightarrow\infty}\left[\frac{\gamma}{0+1}\right] \\
&= \gamma.
\end{aligned}$$

QED.

Proof of $\lim_{\phi\longrightarrow 0}\Lambda(\phi)=\frac{1}{\gamma-\alpha}$

$$\Lambda(0)=\int\limits_{t}^{\infty}(1+\alpha(\tau-t))^{-\frac{\gamma}{\alpha}}d\tau$$

Proof. The integral can be easily solved by changing variables: $z=1+\alpha(\tau-t)$,

$d\tau = \frac{1}{\alpha}dz$; when $\tau = t$, $z = 1$ and when $\tau = \infty$, $z = \infty$. Therefore,

$$\begin{aligned}
&= \int_1^\infty \frac{1}{\alpha} z^{-\frac{\gamma}{\alpha}} dz \\
&= \frac{1}{\alpha}\left[\frac{z^{1-\frac{\gamma}{\alpha}}}{1-\frac{\gamma}{\alpha}}\right]_1^\infty \\
&= \frac{1}{\gamma-\alpha} > 0
\end{aligned}$$

2.C.7 Derivation of the wealth equation of motion from the current account equation

Recall the current account equation:

$$\dot{b}(t) = r(t)\, b(t) + y(t) - c(t) \tag{2.175}$$

So the question is whether we are able to write

$$\dot{W}(t) = r(t)\, W(t) - c(t) \tag{2.176}$$

where $r(t)\, W(t)$ is the wealth annuity at time t.

We define wealth as the net asset position plus the present value of the net income stream, or

$$W(t) = b(t) + \int_t^T e^{-\int_t^s r(v)dv} y(s)\, ds \tag{2.177}$$

so, using Leibniz rule

$$\dot{W}(t) = \dot{b}(t) + \int_t^T \frac{d}{dt}\left[e^{-\int_t^s r(v)dv} y(s)\right] ds - e^{-\int_t^t r(v)dv} y(t)$$

$$= \dot{b}(t) + r(t)\int_t^T e^{-\int_t^s r(v)dv} y(s)\, ds - y(t)$$

inserting 2.175 we get

$$\begin{aligned}\dot{W}(t) &= \{r(t)\, b(t) + y(t) - c(t)\} + r(t)\int_t^T e^{-\int_t^s r(v)dv} y(s)\, ds - y(t) \\ &= r(t)\, b(t) + r(t)\int_t^T e^{-\int_t^s r(v)dv} y(s)\, ds - c(t)\end{aligned}$$

and recalling 2.177:

$$\dot{W}(t) = r(t)\, W(t) - c(t)$$

QED.

Remark 3 *Note that as long as (i) the agent has access to capital markets at all times and (ii) we define wealth as the sum of current net asset positions plus the NPV of the net income stream, discounted at the variable rate* $r(s)$ *that will be faced by that particular agent in the future, the wealth equation can be stated as the difference of the return on wealth, called the annuity value of wealth, and the instantaneous consumption rate.*

2.C.8 The income effect and wealth effect of a change in ϕ

Recall the solution for current consumption (2.46) and derive it with respect to ϕ to obtain

$$c' = C'W + C\, W'$$

where $W = b + \frac{y}{r+\phi}$. Specifically,

$$\frac{dc(t)}{d\phi} = C'(\phi)\, W - C(\phi)\, \frac{y}{(r+\phi)^2}$$

or

$$\frac{dc(t)}{d\phi} = C'(\phi) W + (S(\phi) - r + \phi) \frac{y}{(r+\phi)^2}$$

where $S(\phi) = r + \phi - C(\phi)$ and $S'(\phi) = 1 - C'(\phi)$. So we can write

$$\frac{dc(t)}{d\phi} = C'(\phi) W + (S(\phi) - r + \phi) \frac{y}{(r+\phi)^2}$$

$$\frac{dc(t)}{d\phi} = C'(\phi) W + S(\phi) \frac{y}{(r+\phi)^2} - \frac{y}{r+\phi}$$

$$\frac{dc(t)}{d\phi} = C'(\phi) W + S(\phi) \frac{y}{(r+\phi)^2} - W + b$$

$$\frac{dc(t)}{d\phi} = \underbrace{C'(\phi) W}_{\text{IE}} \underbrace{- W + b + S(\phi) \frac{y}{(r+\phi)^2}}_{\text{WE}} \tag{2.178}$$

The above equation can be written as

$$\frac{dc(t)}{d\phi} = -S'(\phi) W + b + S(\phi) \frac{y}{(r+\phi)^2}$$

which is equivalent to (2.50).

Note that:

$$\frac{dc(t)}{d\phi} = \Lambda'(\phi) W - \Lambda(\phi) \frac{y}{(r+\phi)^2}$$

The first term in (2.178) is the IE, and the 2nd, 3rd and 4th terms determine the WE:

- ED
  - WE is less negative if S is higher (and positive)
  - IE: $C' = 1$, so the IE is always positive and equal to W
- HD
  - WE is less negative for higher values of S (i.e. for low values of ϕ)

– IE is more positive if $C'(\phi) = \Lambda'(\phi)$ is higher or equivalently if $S'(\phi)$ is more negative (i.e. for low values of ϕ).

2.C.9 Saving rate in exponential discounting with CRRA utility

The properties regarding the linear sensitivity of the consumption rate and the insensitivity of the saving rate to changes in ϕ is not exclusive to the log utility case. In particular, they also hold in the more general CRRA utility where

$$u(c(s)) = \left(\frac{\sigma}{\sigma - 1}\right) c(s)^{\frac{\sigma-1}{\sigma}} \tag{2.179}$$

The consumption rate is

$$C = \sigma\rho + (1-\sigma) r + \phi,$$

which increases linearly in ϕ. Also, the saving rate is

$$S = \sigma(r - \rho),$$

which is insensitive to changes in ϕ.

Proof. Consider the utility function (2.179). The corresponding Euler equation is

$$\frac{\dot{c}(s)}{c(s)} = \sigma(r - \rho)$$

which, together with the intertemporal budget constraint, results in

$$\frac{c(t)}{\sigma\rho + (1-\sigma) r + \phi} = b(t) + \frac{y}{r + \phi}$$

or,

$$c(t) = (\sigma\rho + (1-\sigma) r + \phi) W(t)$$

where $W(t) = b(t) + \frac{y}{r+\phi}$; or, in terms of the consumption rate:

$$C(t) = \sigma\rho + (1-\sigma)r + \phi \tag{2.180}$$

which is linear in ϕ.

The current account becomes

$$\dot{b}(t) = (r+\phi)W(t) - c(t)$$

$\Longleftrightarrow$

$$\dot{b}(t) = ((r+\phi) - (\sigma\rho + (1-\sigma)r + \phi))W(t)$$

or, in terms of the saving rate:

$$\begin{aligned} S &= (r+\phi) - (\sigma\rho + (1-\sigma)r + \phi) \\ &= \sigma(r-\rho). \end{aligned} \tag{2.181}$$

which is insensitive to ϕ.

Q.E.D.

2.C.10 Risk premium and probability of default

Preamble

Consider a perpetuity, i.e. a bond with maturity equal to infinity. There is a spot TSIR $\{i_t\}$ that the market uses for discounting a schedule of future flows $\{X_t\}$ corresponding to a given security. If there is some risk of default inherent in such a security, a spot interest rate i_t [38] with which the flow will be discounted is likely to be higher than

[38] Assume there is no devaluation risk; e.g.the security in question is issued in the same FX currency as the units of r_t.

the risk-free rate r_t, for every maturity t. The difference between prevailing spot rates i_t and risk free rates r_t is generally interpreted as the default probability ϕ_t, so that

$$i_t = r_t + \phi_t$$

An inattentive market player, could utilize spot rates $\{i_t\}$ to discount the discounting schedule and calculate the security's market value as

$$\int_{t=0}^{\infty} X_t \exp\left[-it\right] dt$$

However, the market player should at all times take the recovery value R in consideration in order to properly assess the security's actual value.

Constant Probability of Default

No recovery value Consider the case where a credit event could occur in the immediate one time period ahead with constant likelihood κ, conditional on it not having occurred before. Correspondingly, random variable T -time until collapse- is exponentially distributed with density function $f(t) = \kappa \exp\left[-\kappa t\right]$. The expected time until collapse at any point in time, conditional on it not having occurred yet, is $E(T) = \kappa^{-1}$.

The conditional probability of a credit event occurring at some point from present time to t periods ahead is given by the cumulative distribution $F(t) = 1 - \exp\left[-\kappa t\right]; \forall$ $t \geq 0$. Conversely, the conditional probability of **not** having experienced a crisis during the time interval from present time to t periods ahead (i.e. the probability of survival of the current regime) is given by $1 - F(t) = \exp\left[-\kappa t\right]$.

Consider a security with flows schedule $\{X_t\}_{t=0}^{\infty}$ that is expected to be honored as long as a credit event does not arrive. Each future flow is in fact a random variable

variable $\tilde{X}_t$ that can take values

$$\tilde{X}_t = \begin{cases} X_t & \text{wp} \quad \exp\left[-\kappa t\right] \\ 0 & \text{wp} \quad 1 - \exp\left[-\kappa t\right] \end{cases}$$

Thus, the Expected Present Value (EPV) of the instrument is

$$\begin{aligned} EPV\left(\tilde{X}_t\right) &= \int_{t=0}^{\infty} \exp\left[-rt\right] X_t \exp\left[-\kappa t\right] dt \\ &= \int_{t=0}^{\infty} X_t \exp\left[-\left(r+\kappa\right) t\right] dt \end{aligned}$$

Remarks

1. In the case of constant probability of occurrence of default and zero recovery value $R = 0$, the market would "correctly" interpret $\phi \equiv i - r$ as the instantaneous probability of default κ. Therefore, ϕ can be defined as the no-recovery-value risk implicit in the price P of a security, i.e.

$$P\left(\{X_t\}\right) = \int_{t=0}^{\infty} X_t \exp\left[-\left(r+\phi\right) t\right] dt$$

2. Caveat: Beware of confusing the probability of default "by" time t - which would be given by the cumulative distribution $F(t) = 1 - \exp\left[-\kappa t\right]$ and is always increasing in t - with the conditional probability of default "at" time t, which is simply κ and is assumed to be constant. If you are provided with an estimate $F(t)$ of the probability of collapse by time t, and wanted to find out the implicit (constant) instantaneous probability of default (per one time unit), the simple transformation is

$$\kappa = -\frac{\ln\left(1 - F(t)\right)}{t}$$

Recovery value Consider the case where the recovery value R when the credit event arrives at some future time T is greater than zero. We could then consider the recovery value as a random variable. $\tilde{R}$ expected to take place at time $E(T) = \kappa^{-1}$. Since a default is a one-off event, $\tilde{R}$ would be R at time T, and would be zero before and after T. The EPV of the financial instrument would become

$$EPV\left(\tilde{X}_t\right) = \int_{t=0}^{\infty} X_t \exp\left[-\left(r+\kappa\right)t\right]dt + EPV\left(\tilde{R}\right)$$

where the expected present value of $\tilde{R}$ would be

$$\begin{aligned} EPV\left(\tilde{R}\right) &= E_0\left[R\exp\left[-rT\right]\right] \\ &= RE_0\left[\exp\left[-rT\right]\right] \end{aligned}$$

Since T is a random variable with density function $f(T) = \kappa\exp\left[-\kappa T\right]$, it follows that

$$\begin{aligned} EPV\left(\tilde{R}\right) &= R\int_0^{\infty} \exp\left[-rT\right] f(T)dT \\ &= R\int_0^{\infty} \exp\left[-rT\right] f(T)dT \\ &= R\int_0^{\infty} \exp\left[-rT\right] \kappa\exp\left[-\kappa T\right] dT \\ &= \int_0^{\infty} \kappa R\exp\left[-\left(r+\kappa\right)T\right] dT \\ &= R\frac{\kappa}{r+\kappa} \end{aligned}$$

Therefore $EPV\left(\tilde{X}_t\right)$ becomes

$$\begin{aligned} EPV\left(\tilde{X}_t\right) &= \int_{t=0}^{\infty} \left(X_t\exp\left[-\left(r+\kappa\right)t\right] + R\kappa\exp\left[-\left(r+\kappa\right)t\right]\right)dt \\ &= \int_{t=0}^{\infty} X_t\exp\left[-\left(r+\kappa\right)t\right]dt + R\frac{\kappa}{r+\kappa} \end{aligned}$$

or, written differently

$$EPV\left(\tilde{X}_t\right) = \int_{t=0}^{\infty} (X_t + \kappa R) \exp\left[-(r+\kappa)\, t\right] dt$$

Remarks

1. Note that in this case, with constant instantaneous probability of default κ and **positive** recovery value R, the market should augmented κR to each flow before discounting them with rates $r + \kappa$. Adding κR to each scheduled flow X_t takes account of the residual value effect.

2. Alternatively, the value of the security can be equally calculated as (i) the discounted value of the scheduled flows $\{X_t\}$ using rates $\{r + \kappa\}$ (ii) plus the term $R\frac{\kappa}{r+\kappa}$

3. Notice that both κR and $R\frac{\kappa}{r+\kappa}$ are both increasing in κ, which should be the case: as a credit event becomes more likely, the expected time for it to occur $E(T)$ shortens, and the present value of R increases.

4. An inattentive trader that omitted R in his calculations would be underestimating the value of a security. Similarly, if such a treated attempted to draw the risk of default implied in the market value of a sovereign bond, he would erroneously believe that the market is underestimating the risk of default.

Variable Probability of Default

Consider the case where a credit event could occur in the immediate one period dt ahead with hazard rate $\kappa(t)$, conditional on it not having occurred before, where $\kappa(t)$ is defined as $\kappa(t) = \frac{f(t)}{1-F(t)}$. Correspondingly, random variable T -time until collapse- is exponentially distributed with density function $f(t)$ and cumulative distribution $F(t)$

with $E(T) = \kappa \int_0^\infty T \exp[-\kappa T]\, dT$.[39] In other words, the conditional probability of a credit event occurring at some point from the present time 0 to t periods ahead is given by the cumulative distribution $F(t) = 1 - \exp\left[-\int_0^t \kappa(s)ds\right], \forall\, t \geq 0$.[40] Conversely,

[39]The value $E(T)$ is found as follows:

Since $\tilde{T}$ is a r.v. with density function $f(T) = \kappa \exp[-\kappa T]$, it follows that

$$E(T) = \int_0^\infty T f(T) dT$$

Recalling our definition $\kappa(t) = \frac{f(t)}{1-F(t)}$, we can rewrite

$$E(T) = \int_0^\infty T\kappa(T)\,(1 - F(T))\, dT$$

where $1 - F(t) = \exp\left[-\int_0^t \kappa(s)ds\right]$, so

$$\begin{aligned} E(T) &= \int_0^\infty T\kappa(T) \exp\left[-\int_0^T \kappa(s)ds\right] dT \\ &= \int_0^\infty T\kappa(T) \exp[-\bar{\kappa}(0,T)T]\, dT \end{aligned}$$

If $\kappa(T)$ were constant, then $\bar{\kappa}(0,t) = \kappa$, $\forall t$, and the above equation becomes

$$E(T) = \kappa \int_0^\infty T \exp[-\kappa T]\, dT$$

Integrating by parts, where $v = T$ and $u = -\frac{1}{\kappa}\exp[-\kappa T]$

$$E(T) = \kappa \left\{ -T\frac{1}{\kappa}\exp[-\kappa T]\Big]_{T=0}^{\infty} - \int_0^\infty \left(-\frac{1}{\kappa}\exp[-\kappa T]\right) dT \right\}$$

simplifying

$$E(T) = -\frac{T}{\exp[\kappa T]}\Big]_{T=0}^{\infty} + \int_0^\infty \exp[-\kappa T]\, dT$$

by l'Hopital

$$E(T) = -\frac{1}{\lim T \longrightarrow \infty : \{\kappa \exp[\kappa T]\}} + \int_0^\infty \exp[-\kappa T]\, dT$$

$$E(T) = 0 + \int_0^\infty \exp[-\kappa T]\, dT$$

$$\begin{aligned} E(T) &= -\frac{1}{\kappa}\exp[-\kappa T]\, dT\Big]_{T=0}^{\infty} \\ &= 1/\kappa \end{aligned}$$

[40]Note that from

$$\kappa(s) = \frac{f(s)}{1 - F(s)}$$

the conditional probability of **not** having experienced a crisis during the time interval from present time to t periods ahead (i.e. the probability of survival of the current regime) is given by $1 - F(t) = \exp\left[-\int_0^t \kappa(s)ds\right]$.

Consider a security with flows schedule $\{X_t\}_{t=0}^{\infty}$ that is expected to be honored as long as a credit event does not arrive. Each future flow is in fact a random variable variable $\tilde{X}_t$ that can take values

$$\tilde{X}_t = \begin{cases} X_t & \text{wp} \quad \exp\left[-\int_0^t \kappa(s)ds\right] \\ 0 & \text{wp} \quad 1 - \exp\left[-\int_0^t \kappa(s)ds\right] \end{cases}$$

No Recovery value The Expected Present Value (EPV) of the instrument is

$$\begin{aligned} EPV\left(\tilde{X}_t\right) &= \int_{t=0}^{\infty} \exp\left[-rt\right] X_t \left(1 - F(t)\right) dt \\ &= \int_{t=0}^{\infty} X_t \exp\left[-rt - \int_0^t \kappa(s)ds\right] dt \end{aligned}$$

or

$$EPV\left(\tilde{X}_t\right) = \int_{t=0}^{\infty} X_t \exp\left[-\int_0^t \left(r + \kappa(s)\right) ds\right] dt$$

where $r + \kappa(t)$ is defined as the forward rate for every term t. Correspondingly, $\{r + \kappa(s)\}_0^{\infty}$ would state for the forward curve starting at time 0.

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we could find the relationship between $F(t)$ and $\kappa(t)$ alone. First, note that the RHS can be written as $-\frac{d}{ds}\ln\left[1 - F(s)\right]$, so

$$\kappa(s) = -\frac{d}{ds}\ln\left[1 - F(s)\right]$$

Next, integrate both sides from 0 to t:

$$\begin{aligned} \int_0^t \kappa(s)ds &= -\int_0^t \left(\frac{d}{ds}\ln\left[1 - F(s)\right]\right) ds \\ &= -\ln\left[1 - F(s)\right] |_0^t \\ &= -\ln\left[1 - F(t)\right] \end{aligned}$$

Therefore,

$$1 - F(t) = \exp\left[-\int_0^t \kappa(s)ds\right]$$

Let the spot rate at time t be $r+\bar{\kappa}(0,t)$ and, correspondingly, $\{r+\bar{\kappa}(t)\}_0^\infty$ would be the spot yield curve starting at time 0, where $\bar{\kappa}(0,t)$ is the average risk premium from present time 0 to time t:

$$\bar{\kappa}(0,t)=\frac{\int_0^t \kappa(s)ds}{t}$$

We can then rewrite the expected present value as

$$\begin{aligned} EPV\left(\tilde{X}_t\right) &= \int_{t=0}^{\infty} X_t \exp\left[-rt-\bar{\kappa}(0,t)t\right] dt \\ &= \int_{t=0}^{\infty} X_t \exp\left[-\left(r+\bar{\kappa}(0,t)\right)t\right] dt \end{aligned}$$

Remarks

1. In the case of variable probability of default and zero recovery value $R=0$, an inattentive trader would "incorrectly" interpret $\phi_t=i_t-r$ as the instantaneous probability of default. Some (loosely) define ϕ_t as the zero-recovery value risk implicit in the price P of a given security, i.e.

$$P\left(\{X_t\}\right)=\int_{t=0}^{\infty} X_t \exp\left[-\left(r+\phi_t\right)t\right] dt$$

2. However, $\phi_t=\bar{\kappa}(0,t)$, is NOT the instantaneous probability of default at time t but rather the average probability of default during the time period $(0,t]$.

3. In order to find the risk premium from a known term structure of instantaneous default probability $\{\kappa(s)\}_0^t$, we must apply the definition $\bar{\kappa}(0,t)=\frac{\int_0^t \kappa(s)ds}{t}$. Notice that

$$\frac{\partial\bar{\kappa}(0,t)}{\partial\kappa(t)}=\frac{1}{t}$$

i.e. $\kappa(t)$ adds to average $\bar{\kappa}(0,t)=\phi_t$ with weight $1/t$. Therefore, the further away is t, the less relevant the instantaneous default probability will be for discounting purposes.

4. Conversely, differentiating the average risk premium with respect to time, we obtain

$$\kappa(t) = \bar{\kappa}(0,t) + \frac{\partial \bar{\kappa}(0,t)}{\partial t} t$$

i.e., if we wanted to find the instantaneous implicit default probability $\kappa(t)$ from a given term structure of average risk $\{\bar{\kappa}(0,s)\}^t$, we just need to know how much the term structure $\{r + \bar{\kappa}(0,t)\}$ changes from time t to $t + dt$ and apply the above equation (assuming that r remains constant).

Recovery value Consider the case where some value $R > 0$ can be recovered at date T when the credit event arrives. We could then consider the recovery value as a r.v. $\tilde{R}$ expected to take place at time $E(T)$. Since a default is a one-off event, $\tilde{R}$ would be R at time T, and would be zero before and after T. The EPV of the financial instrument would become

$$\begin{aligned} EPV\left(\tilde{X}_t\right) &= \int_{t=0}^{\infty} X_t \exp\left[-rt - \int_0^t \kappa(s)ds\right] dt + EPV\left(\tilde{R}\right) \\ &= \int_{t=0}^{\infty} X_t \exp\left[-\left(r + \bar{\kappa}(0,t)\right)t\right] dt + EPV\left(\tilde{R}\right) \end{aligned}$$

where the expected present value of $\tilde{R}$ would be

$$\begin{aligned} EPV\left(\tilde{R}\right) &= E_0\left[R\exp\left[-rT\right]\right] \\ &= RE_0\left[\exp\left[-rT\right]\right] \end{aligned}$$

Since T is a r.v. with density function $f(T) = \kappa \exp\left[-\kappa T\right]$, it follows that

$$\begin{aligned} EPV\left(\tilde{R}\right) &= R\int_0^{\infty} \exp\left[-rT\right] f(T)dT \\ &= R\int_0^{\infty} \exp\left[-rT\right] f(T)dT \end{aligned}$$

Recalling our definition $\kappa(t) = \frac{f(t)}{1-F(t)}$, we can rewrite

$$EPV\left(\tilde{R}\right) = R\int_0^\infty \exp\left[-rT\right]\left(\kappa(T)\left(1 - F(T)\right)\right)dT$$

where $1 - F(t) = \exp\left[-\int_0^t \kappa(s)ds\right]$, so

$$\begin{aligned} EPV\left(\tilde{R}\right) &= R\int_0^\infty \exp\left[-rT\right]\left(\kappa(T)\exp\left[-\int_0^T \kappa(s)ds\right]\right)dT \\ &= R\int_0^\infty \kappa(T)\exp\left[-rT\right]\left(\exp\left[-\bar{\kappa}(0,T)\right]\right)dT \\ &= R\int_0^\infty \kappa(T)\exp\left[-\left(r + \bar{\kappa}(0,T)\right)T\right]dT \end{aligned}$$

Therefore $EPV\left(\tilde{X}_t\right)$ becomes

$$EPV\left(\tilde{X}_t\right) = \int_{t=0}^\infty X_t \exp\left[-\left(r + \bar{\kappa}(0,t)\right)t\right]dt + R\int_0^\infty \kappa(t)\exp\left[-\left(r + \bar{\kappa}(0,t)\right)t\right]dt \tag{2.182}$$

or,

$$EPV\left(\tilde{X}_t\right) = \int_{t=0}^\infty \left(X_t + \kappa(t)R\right)\exp\left[-\left(r + \bar{\kappa}(0,t)\right)t\right]dt \tag{2.183}$$

Remarks

1. Note that the market is correct at adding $\bar{\kappa}(t) = \phi_t$ to the risk free interest rate r as long as flows are augmented by $\kappa(t)R$, which takes account of the residual value effect. Alternatively, the actual value of the security will be equal to (i) the discounted value of the scheduled flows $\{X_t\}$ using spot rates $\{r + \bar{\kappa}(0,t)\}$ (ii) plus the value of flows $\{R\kappa(t)\}_0^\infty$ discounted with corresponding spot rates.

2. Note that by definition $\kappa(t)$ is the probability of default conditional on this event not having happened before. So the expected value of variable $\tilde{R}(T)$, conditional on it not having happened before T is $\kappa(T)R + \left(1 - \kappa(T)\right)0 = \kappa(T)R$. We note that $EPV\left(\tilde{R}\right) = \int_0^\infty R\kappa(T)\exp\left[-\left(r + \bar{\kappa}(0,T)\right)T\right]dT$ is simply the

sum of conditional expected values $\{R\kappa(T)\}_{T=0}^{\infty}$ discounted with relevant rates $\{r + \bar{\kappa}(0,t)\}_{T=0}^{\infty}$.

3. Notice that $\kappa(t)R$ and $EPV\left(\tilde{R}\right)$ are both increasing in $\kappa(t)$, which should in fact be the case since as a credit event becomes more likely, the expected time for it to occur $E(T)$ shortens, and the present value of R increases.

4. An inattentive trader that omitted R in his calculations would be underestimating the value of a security. Similarly, if such trader attempted to draw the risk of default implied in the market value of a sovereign bond, he would erroneously believe that the market is underestimating the risk of default.

Chapter 3

Consumption and Portfolio Rules with Stochastic Quasi-Hyperbolic Discounting

3.1 Introduction

In experiments with humans and animals, subjects often exhibit a reversal of preferences when choosing between a smaller, earlier reward and an alternative larger, later reward. The smaller, earlier reward is often preferred when both rewards are near, while the larger, later reward is preferred as they draw more distant.[1] The persistence and robustness of these dynamic inconsistencies has led some economists and psychologists to think "that the problem may not come from some extraordinary condition that impairs the normal operation of intentionality, but rather from the process by which all people, perhaps all organisms, evaluate future goals" (Ainslie and Haslam, 1992, p.58). Recent neurological evidence supports this view (McClure et al, 2004).

Dynamically inconsistent behavior was first analyzed by David Hume, Adam Smith,

[1] See Herrnstein (1997) and other references therein.

and later by William S. Jevons, Alfred Marshall, Wilfredo Pareto, and others in their discussion of passions, sentiments and intertemporal trade-offs. However, it was not until Strotz (1955) that it was first formalized analytically. This first formalization approximates the temporal discount function of individuals by a hyperbola, a function that discounts more heavily than the exponential function for events in the near future, but less heavily for events in the distant future. Beginning with the work of Laibson (1994, 1997), during the last decade an important body of literature has studied the kind of behavior that rational economic agents with hyperbolic discount functions may exhibit.[2] In particular, in order to attain their goals, individuals may prefer to *restrict* their own future choices. The most apparent way for an individual to forestall his change in preferences is to adopt some type of commitment device.

Gul and Pesendorfer (2001) propose an alternative approach to incorporate the evidence on preferences for commitment. They suggest that *temptation* rather than a *preference change* per se (that is, rather than "dynamic inconsistency") may be the cause of these preferences.[3] Gul and Pesendorfer (2004) extend the analysis to an infinite horizon in an attempt to capture the experimental evidence with tractable, dynamically *consistent* preferences.

A particularly important aspect of this research is the extent to which dynamic inconsistency, temptation, and self-control problems may help us understand individuals' consumption-saving decisions, as well as their decisions to allocate savings among available investment opportunities. Understanding these decisions is, after all, at the heart of a large literature spanning the last few decades on consumption, savings, as-

[2]This discount function has been used to model a wide range of behavior, including consumption behavior, contracts, addiction, and others. See Harris and Laibson (2001), O'Donoghue and Rabin (1999), DellaVigna and Malmendier (2004), and other references therein.

[3]They develop a two-period axiomatic model where an ex ante inferior choice may tempt the individual in the second period. Individuals have preferences over sets of alternatives that represent the second period choices. Their representation of preferences identifies the individual's commitment ranking, temptation ranking, and costs of self-control. Moreover, their model yields both different behavioral and normative implications than the change in preferences captured by the hyperbolic discounting approach.

set pricing, macroeconomics and other areas. Households are both consumers and investors, and their decisions reflect these dual roles. As consumer, a household chooses how much of its income and wealth to allocate to current consumption, and thereby how much to save for future consumption including bequests. As investor, the household solves the portfolio-selection problem to determine the allocations of its savings among the available investment opportunities. As the modern finance literature emphasizes, the optimal consumption-saving and portfolio-selection decisions typically *cannot* be made independent of each other (see Merton 1969,1971).

The purpose of this paper is to study the effects of dynamic *inconsistency* on the joint consumption-saving and portfolio-selection problem. Interestingly enough, despite the fact that the consumption-saving problem has received substantial attention in the literature on dynamically inconsistent preferences, the consumption-saving *and* portfolio-selection problem has received virtually no attention.[4] In particular, we will examine the implications of a hyperbolic discount function for the lifetime consumption-saving and portfolio-choice problem of an individual household in a continuous-time setting. We would like to argue that the analysis is relevant for the following reasons.

First, it is important to evaluate whether emotions and self-control play a role in considerations involving time and *risk* preferences, and hence in intertemporal consumption, saving, and portfolio decisions and in asset prices.[5] Hirshleifer (2001), for example, surveys and assesses the theory and evidence regarding investor psychology as a determinant of asset prices, and considers that "this issue is at the heart of a grand debate in finance spanning the last two decades" (p. 1552). Gul and Pesendorfer (2004) have shown that their dynamically *consistent* preferences do have relevant implications for these decisions. In particular, increasing the agents preference for commitment

[4]There is an important amount of work on intertemporal consumption-savings decisions (e.g., Laibson (1994, 1997), Krusell and Smith (2003), Harris and Laibson (2001)). Luttmer and Mariotti (2003) is, to the best of our knowledge, the only paper that also considers households' portfolio decisions. However, they do not study the response of consumption and prices to changes in risk.

[5]Halevy (2005) offers some experimental evidence of the interplay between risk and time preferences.

while keeping self-control constant increases the size of the equity premium.[6] Yet, the extent to which dynamically *inconsistent* preferences have relevant implications for consumption-saving allocations, portfolio choices and asset prices remains unaddressed in the literature.

Second, as emphasized in the finance literature and indicated above, consumption-saving and portfolio-selection decisions typically cannot be made independently of each other. In this sense, the available evidence from the consumption-saving problem need not be sufficient to provide even a partial understanding of these decisions.

Third, these joint decisions have been subject to a great deal of theoretical and empirical scrutiny in the consumption-based asset pricing literature under exponential discounting, in particular in the extensive literature on the equity premium puzzle and the excess volatility puzzle.[7] As a result of these efforts, empirical evidence is readily available to evaluate the implications of a hyperbolic discounting structure for observed market data on consumption and security returns.

Lastly, during the last two decades several attempts have been made in the literature to try to resolve the equity premium and other asset pricing puzzles by departing in increasingly complicated ways from the tractable framework of a representative agent, time-additive isoelastic preferences, and complete frictionless markets.

In this paper, we will examine the intertemporal consumption and portfolio choice problem of an investor with dynamically inconsistent preferences in a stochastic dynamic programming setting. We consider this setting because it offers valuable advantages. First, the use of continuous-time methods has become an integral part of financial economics, and has produced models with a rich variety of testable implications (see Sundaresan (2000) for a review). The adoption of a continuous-time model, in addition, offers a crucial advantage over much of the literature, which has mostly

[6] Similarly, Krusell et al (2002) elaborate on the Gul-Pesendorfer framework which they use to interpret wealth and asset pricing data.

[7] See, for instance, Kocherlakota (1996), Campbell (2000) and Mehra (2008) for reviews.

adopted a discrete-time discount function $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$, $\beta \in (0,1)$, $\delta \in (0,1)$, to model the gap between a high short-run discount rate and a low long-run discount rate. As Harris and Laibson (2001) and other authors have noted, a recurrent problem that plagues most applications of the discrete-time discount function employed in the literature is that strategic interactions among intrapersonal selves often generates counterfactual policy functions where consumption functions are not globally monotonic in wealth, and may even drop discontinuously at a countable number of points.[8] Moreover, hyperbolic Markov equilibria are not unique in deterministic discrete-time settings (Krusell and Smith, 2003).

These problems can be avoided in a continuous-time setting. Our approach is motivated by Harris and Laibson (2008) Instantaneous Gratification (IG) model, which is based on a quasi-hyperbolic stochastic discount function.[9] The IG model is dynamically inconsistent and, while it captures the qualitative properties of the discrete-time $\beta - \delta$ model, it resolves the pathologies of multiplicity of equilibria and non-monotonicity of the consumption function that have flawed previous theoretical advances in the literature of time-inconsistent preferences.[10] Interestingly, our model yields closed-form solutions for the optimal consumption and portfolio rules that makes them readily comparable to those results obtained under a constant rate of time preference as in Merton (1969, 1971).

[8]These pathologies often occur only in a limited region of the parameter space which, as Harris and Laibson (2008) indicate, typically includes defensible calibrations of the parameters. O'Donoghue and Rabin (1999) note that these pathologies arise only to the extent that individuals are sophisticated, i.e. they are aware of their dynamic inconsistencies. It would be needed to assume that individuals are *completely* naive about their dynamic inconsistency problem; otherwise, the pathologies would be reinstated.

[9]They show the existence and uniqueness of a hyperbolic equilibrium, and the equilibrium consumption function is continuous and monotonic in wealth.

[10]See Harris and Laibson (2008, Section 5).

3.2 A Lifetime Consumption-Portfolio Problem with Hyperbolic Discounting

We study the classical Merton (1969,1971) intertemporal consumption-saving and portfolio-selection problem with hyperbolic discounting preferences rather than with exponential discounting ones. An individual is due to make consumption and portfolio decisions that maximize her discounted lifetime utility of consumption. We assume infinite lifetime, complete markets and no borrowing constraints.

Since our setup is an extension of Harris and Laibson (2008)[11], we attempt to adopt their notation whenever possible. The individual's wealth x_t at any time t can be invested in two assets: a riskless bond with value B_t and a risky asset for an amount N_tP_t, where N_t is the quantity held the of risky-asset and P_t is its price at time t; in particular,

$$x_t = N_tP_t + B_t.$$

While the risk-free asset earns a constant rate of return r continuously, the price P_t follows a geometric Brownian motion with drift μ and diffusion parameter σ, where we assume away a dividend process. Specifically,

$$dB_t = rB_tdt$$

$$dP_t = \mu P_tdt + \sigma P_tdz_t \tag{3.1}$$

where z_t is a standard Wiener process. The change in the individual's wealth during a period of infinitesimal duration dt is determined by the investment proceeds minus

[11] Harris and Laibson (2008) consider a setup with one (risky) asset only and impose credit constraints.

consumption $c_t dt$:

$$dx_t = [\mu\theta_t x_t + (1-\theta_t) r x_t - c_t]\, dt + \sigma\theta_t x_t dz_t \tag{3.2}$$

where θ_t is the proportion of wealth invested in the risky asset at time t.

Following Harris and Laibson's (2008) quasi-hyperbolic setup, the consumer-investor seeks to maximize his expected lifetime discounted utility of consumption:

$$E_t\left[\int_t^{t+\tau_t} \delta^{(s-t)} u\left(c\left(x_s\right)\right) ds + \int_{t+\tau_t}^{\infty} \beta\delta^{(s-t)} u\left(c\left(x_s\right)\right) ds\right] \tag{3.3}$$

where $\beta \in (0,1]$ and $\delta \in (0,1]$. The discount function decays exponentially at rate $\gamma = -\ln\delta$ up to time $t+\tau_t$, drops discontinuously at $t+\tau_t$ to a fraction β of its level just prior to $t+\tau_t$, and thereafter decays exponentially at a rate $\gamma = -\ln\delta$ thereafter. The arrival of the "future" is stochastic. In particular, τ_t is distributed exponentially with parameter $\lambda \in [0,\infty)$, which effectively smooths the discount factor and avoids having a kinked or discontinuous discount factor.[12] The Instantaneous Gratification (IG) model in Harris and Laibson (2008) corresponds to the limit $\lambda \to \infty$.

As is well known, a closed-form solution can be found for the case of constant relative risk aversion (CRRA) utility when discounting is exponential (Merton 1969, 1971). For this reason, we consider the utility flow

$$u(c) = \frac{c^{1-b}}{1-b} \tag{3.4}$$

where $b > 0$ is the risk aversion parameter.

Lifetime utility is maximized subject to the budget equation (3.2) and initial wealth

[12] Alternatively, as noted by Harris and Laibson (2003), the value function can be formulated as $w(x_t) = E_t\left[\int_t^{\infty} D_\lambda(t,s)\, u(c(x_s))\, ds\right]$ where the discount factor $D_\lambda(t,s)$ is stochastic and equal to

$$D_\lambda(t,s) = \begin{cases} e^{-\gamma(s-t)} & \text{wp} \quad e^{-\lambda(s-t)} \iff \text{if} \quad s-t \le \tau_t \\ \beta\cdot e^{-\gamma(s-t)} & \text{wp} \quad 1-e^{-\lambda(s-t)} \iff \text{if} \quad s-t > \tau_t. \end{cases}$$

$x_t > 0$. Markets are perfect and there are no taxes, transaction costs, trading restrictions or other impediments to trade. Also that there are no commitment mechanisms. In other words, the introduction of hyperbolic discounting preference is the only difference with respect to the classic formulation of the problem in the literature. As Merton (1969, 1971) shows, this problem can be solved in closed form for optimal consumption and portfolio rules under exponential discounting. We will show next that an explicit solution also exists for the general stochastic hyperbolic-discounting case.[13]

We consider the continuous-time generalization introduced by Harris and Laibson (2008) for two reasons:

i. In order to solve for the Markov perfect Nash equilibrium of the intrapersonal game induced by the hyperbolic discounting structure we use the equivalence result in Barro (1999), Laibson (1997), and Luttmer and Mariotti (2003). They show that in the special case of CRRA utility with no liquidity constraints, and no commitment devices, the equilibrium of the intrapersonal game exists and is *observationally equivalent* to a dynamically consistent optimization problem that shares the same instantaneous utility function and equilibrium policy functions but with a different, higher long-run discount rate.[14] This allows us to solve the model as a pure optimization problem.

ii. The Instantaneous Gratification (IG) model developed in Harris and Laibson (2008) is an important step forward in the treatment of hyperbolic discounting preferences as many of the pathologies of the discrete-time hyperbolic models are eliminated in the continuous time case when $\lambda \to \infty$ as will be discussed later.

[13]Note that the exponential-discounting setup corresponds to the particular hyperbolic-discounting case with $\lambda \to 0$ or $\beta = 0$.

[14]Harris and Laibson (2008) got away with the observationally equivalence critique thank to the liquidity-constraints assumption. The IG model with liquidity-constrained individuals has the same value function of a dynamically consistent optimization problem with the difference that the utility function is wealth contingent. In our problem, however, there are no liquidity constraints.

The problem for the consumer-investor is to maximize the current-value function

$$w(x_t) = E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\,ds + e^{-\gamma\tau_t}\beta v(x_{t+\tau_t})\right] \tag{3.5}$$

where $\gamma \in (0, +\infty)$ is the discount rate and $v(x_{t+\tau_t})$ is the continuation-value defined as

$$v(x_\zeta) \equiv \int_\zeta^\infty e^{-\gamma(s-\zeta)} u(\tilde{c}(x_s))\,ds$$

which discounts utility flows exponentially, and where $\tilde{c}$ stands for the consumption levels *optimally* chosen by future selves. The maximization problem is subject to the budget equation (3.2) and the constraints $c_s \geq 0$ and $x_s \geq 0$, $\forall\, s \geq t$, given an initial wealth endowment $x_t > 0$.

Assumptions

We impose the following set of assumptions:

A1. $b > 1 - \beta$ (feasibility condition),

A2. $\gamma > (1-b)\left(\bar{\mu} - \frac{1}{2} b\bar{\sigma}^2\right)$ (integrability condition),

A3. $\lim_{\zeta \to \infty} E_t \{v(x_\zeta)\} = 0,$ (transversality condition),

where $\bar{\mu}$ and $\bar{\sigma}^2$ in **A2** are the mean and variance of the optimal portfolio return rate, as will be characterized below. **A1** and **A2** ensure that optimal consumption is nonnegative. As discussed in Harris and Laibson (2008, Section 5.1), in practice, these two assumptions are always satisfied at the empirical estimates of the coefficients b, β and γ typically obtained in the literature. **A3** is a convergence condition for integral (3.5).

In what follows, for notational convenience we dispense with the time subscripts unless it became otherwise necessary.

The current-value function (3.5) can be written recursively as

$$w(x) = u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[e^{-\gamma dt} \beta v(x+dx)] \tag{3.6}$$

which satisfies the Bellman equation

$$\gamma w(x) = u(c(x)) + \frac{E[dw(x)]}{dt} + \lambda[\beta v(x) - w(x)], \tag{3.7}$$

where the second term in the right-hand side can be derived applying Ito's Lemma to (3.2) :

$$\frac{E[dw(x)]}{dt} = (rx + (\mu - r)\,\theta x - c)\ w'(x) + \frac{1}{2}\sigma^2\theta^2 x^2 w''(x)$$

As noted by Harris and Laibson (2008), the term $\lambda(\beta v(x) - w(x))$ in (3.6) reflects the hazard rate λ of making the transition from the "present" to the "future", at which point the continuation value $v(x)$ begins. The intertemporal discount function that applies to the utility flows pertaining to the "future" is a fraction β of the function that prevails in the "present". Of course, there is no transition effect if $\beta = 1$. The intuition is that when $\beta = 1$ there is no difference in how present utilities and in future utilities are discounted, in which case we would have obtained the classic expression of the time-consistent Bellman equation. The same would be true if the transition probability from "present" to "future" were nil, i.e. if $\lambda = 0$.

Let $\{c^*, \theta^*\}$ denote the optimal policy set of the problem defined as

$$\{c^*(x), \theta^*(x)\} = \arg\max_{c,\theta} \{w(x)\}$$

Recalling the Bellman equation (3.7), we can write

$$\{c^*(x), \theta^*(x)\} \quad = \quad \arg\max_{c,\ \theta} \{u(c) + (rx + (\mu - r)\,\theta x - c)\, w'(x) \tag{3.8}$$

$$+\frac{1}{2}\sigma^2\theta^2x^2w''(x)+\lambda(\beta v(x)-w(x))\}$$

or, suppressing the terms that do not contain the controls,

$$\{c^*(x), \theta^*(x)\} = \arg\max_{c,\theta}\{u(c)+((\mu-r)\theta x-c)w'(x)+\frac{1}{2}\sigma^2\theta^2x^2w''(x)\}$$

The unique interior optimum from the first order conditions determine the optimal consumption and portfolio policies. In particular, such conditions are

$$0=(\mu-r)xw'(x)+\sigma^2\theta^*x^2w''(x), \tag{3.9}$$

and

$$0=u'(c^*)-w'(x), \tag{3.10}$$

which imply that the optimal policies must satisfy

$$\theta^*(x)=-\frac{w'(x)}{xw''(x)}\frac{(\mu-r)}{\sigma^2},$$

and

$$u'(c^*)=w'(x).$$

The natural candidate solution for $w(x)$ that corresponds to a CRRA, power instantaneous utility (3.4) is

$$w(x_s)=\alpha_H^{-b}\frac{x_s^{1-b}}{1-b}, \ \forall s\geq t \tag{3.11}$$

and the corresponding optimal portfolio and consumption rules are [15]

$$\theta^*=\frac{1}{b}\frac{(\mu-r)}{\sigma^2}, \ \forall x>0 \tag{3.12}$$

[15]Note that equation (3.12) is obtained by inserting the derivatives of the candidate solution (3.11) into (3.9). In particular, the derivatives are $w'(x)=\alpha_H^{-b}x^{-b}$ and $w''(x)=-b\alpha_H^{-b}x^{-b}/x=-bw'(x)/x$.

and

$$c^*(x) = \alpha_H x \tag{3.13}$$

where α_H stands for the *hyperbolic* marginal propensity to consume out of wealth. We note from (3.12) that the portfolio strategy is independent of the current wealth level, and is *identical* to the portfolio rule of the CRRA, exponential, time-consistent agent studied in Merton (1969, 1971).[16] However, as we show below, the wealth dynamics will be affected by a different, *higher* marginal propensity to consume relative to the Merton case.

In order to find an explicit solution for α_H, we expand the terms in the right-hand side of the Bellman equation that characterizes the hyperbolic-discounting problem. First, we note that

$$u(c^*(x)) = \alpha_H w(x) \tag{3.14}$$

Then, inserting (3.12), (3.13), (3.14) and the derivatives of the candidate solution $w'(x) = (1-b)\frac{1}{x}w(x), w''(x) = -b(1-b)\frac{1}{x^2}w(x)$ into the Bellman equation (3.7), we obtain

$$\gamma w(x) = \alpha_H w(x) + (1-b)\left(-\alpha_H + r + \frac{(\mu - r)^2}{2b\sigma^2}\right) w(x) + \lambda(\beta v(x) - w(x)) \tag{3.15}$$

We show in the Appendix that the last term in the right-hand side of (3.15) is

$$\lambda(\beta v(x_t) - w(x_t)) = -\lambda(1-\beta)\alpha_H \int_t^\infty e^{-(\lambda+\gamma)(s-t)} E_t[w(x_s)]\, ds \tag{3.16}$$

[16]Note that technically this is due to the fact that the present self has no control over the variables that determine the continuation value function $v(x)$, as can be verified in (3.8).

where $E_t\left[w\left(x_s\right)\right]$ is determined by

$$E_t\left[\frac{w\left(x_s\right)}{w\left(x_t\right)}\right]=E_t\left[\left(\frac{x_s}{x_t}\right)^{1-b}\right]=\exp\left\{(1-b)\left(-\alpha_H+r+\frac{(\mu-r)^2}{2b\sigma^2}\right)(s-t)\right\}, \tag{3.17}$$

which is derived from the candidate solution (3.11) and the policy rules and by applying Ito's Lemma[17].

Finally, using (3.16) and (3.17) into (3.15), in account of the transversality condition **A3**, we obtain

$$\alpha_H=\frac{1}{b}\left(\underbrace{\left[\gamma+\alpha_H\left(1-\beta\right)\frac{\lambda}{\lambda+\gamma-(1-b)\left(-\alpha_H+\bar{\mu}-b\frac{\bar{\sigma}^2}{2}\right)}\right]}_{\textbf{Effective discount rate}}-(1-b)\left(\bar{\mu}-b\frac{\bar{\sigma}^2}{2}\right)\right) \tag{3.18}$$

where $\bar{\mu}\equiv\theta^*\mu+(1-\theta^*)\,r$ and $\bar{\sigma}^2\equiv\theta^{*2}\sigma^2$ are the mean and variance of the optimal portfolio's rate of return[18]. The terms in brackets account for the *effective discount rate,* which is *greater* than γ under **A1** and **A2**. In the particular case where $\beta=1$ or $\lambda=0$, the effective discount rate would simply be γ and the marginal propensity to consume (MPC) out of wealth would correspond to the exponential discounting model treated in Merton (1969, 1971):

$$\alpha_M=\frac{1}{b}\left[\gamma-(1-b)\left(\bar{\mu}-b\frac{\bar{\sigma}^2}{2}\right)\right]. \tag{3.19}$$

In turn, for the case of interest where $\lambda\rightarrow\infty$ that corresponds to Instantaneous Gratification, the MPC reduces to[19]

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[17]Specifically, the expected growth of wealth when the agent chooses optimal policies (3.12)-(3.13) is given by $E\left[dx/x\right]=\left[-\alpha_H+r+\frac{1}{b}\frac{(\mu-r)^2}{\sigma^2}\right]dt$. Applying Ito's Lemma on the candidate solution (3.11), the expected growth of the value function is $E\left[dw/w\right]=E\left[dx^{1-b}/x^{1-b}\right]=(1-b)\left(-\alpha_H+r+\frac{(\mu-r)^2}{2b\sigma^2}\right)dt$, which immediately implies (3.17). (See the Appendix for details on the derivations).

[18]Note that, the optimal policy $\theta^*=\frac{1}{b}\frac{(\mu-r)}{\sigma^2}$ given in (3.12) implies $\bar{\mu}-b\frac{\bar{\sigma}^2}{2}=r+\frac{(\mu-r)^2}{2b\sigma^2}>0$.

[19]Note that if $\sigma=0$, the Merton and the IG propensities to consume would reduce to:

$$\alpha_{IG} \equiv \alpha_H|_{\lambda\to\infty} = \frac{1}{b-(1-\beta)}\left(\gamma - (1-b)\left(\bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)\right), \qquad (3.20)$$

which is increasing in $\beta \in (0,1)$ and, under assumptions **A1-A2,** is unambiguously *greater* than the Merton's α_M.

Note that the MPC α_{IG} is *linear* in $\bar{\mu}$ and $\bar{\sigma}^2$ and is *not* wealth contingent, contrary to what has been posited in previous research that tackled this problem. See for example Gong et al. (2006, 2007).

3.3 Discussion

The following results are obtained from the analysis:

i. CONSUMPTION, SAVINGS AND PORTFOLIO CHOICES

First, the relative proportion of wealth allocated to stocks and bonds along the equilibrium path is *identical* to that obtained in the exponential discounting case (i.e. where $\beta = 1$ or $\lambda = 0$). This means that the size of the risk premium of stocks over bonds is also identical to that in the exponential case. In other words, the size of the equity premium is no more or less puzzling than what it is under exponential discounting.

Second, since the effective rate of time preference is greater than γ when $\beta \in (0,1)$ and $\lambda > 0$, the optimal marginal propensity to consume out of wealth α_H is unambiguously *greater* than the exponential discounting solution obtained by Merton (1969, 1971).[20] This is in line with the results in the literature that anticipate present bias.[21] Behind this conclusion is the assumption that the hyperbolic discounting model and

$\alpha_R = \alpha_M|_{\sigma=0} = \frac{1}{b}\left[\gamma - (1-b)\,r\right]$ and $\alpha_H|_{\substack{\lambda\to\infty\\ \sigma=0}} = \frac{1}{b-(1-\beta)}\left[\gamma - (1-b)\,r\right]$.

[20] Note that in the particular case of *logarithmic* instantaneous utility, i.e. where $b \to 1$, for which the intertemporal subsitution effect and the wealth effect cancel each other, the marginal propensity to consume out of wealth is simply given by the subjective discount rate γ in the exponential model, and γ/β in the IG, hyperbolic model.

[21] See for instance O'Donoghue and Rabin (1999), Laibson (1994, 1997), Harris and Laibson (2003, 2008) and Luttmer and Mariotti (2003).

the exponential discounting model have the same long-run discount rate. While this is a useful modelization typically followed in the literature that studies the implications of introducing dynamically inconsistent preferences, this needs not be the case.[22] For instance, the structural estimates reported in Laibson et al (2007) indicate that the β-δ quasi-hyperbolic model may have *more* short-run discounting and *less* long-run discounting than the exponential model. An alternative formulation could have followed this route and introduced two different parameters, rather than one, to compare the two models. We would then have reached the same general conclusion: hyperbolic discounting has quantitative implications for consumption-saving allocations and whether the model generates greater or lower consumption than in the exponential case depends on the specific parameters. For instance, the parameter estimates in Laibson et al (2007) indicate that hyperbolic discounting would indeed generate a greater consumption share.

Finally, note that under the feasibility assumption **A1**, α_H increases as $\beta \in (0,1)$ decreases and as $\lambda \in (0,\infty)$ increases. As stated above, the exponential discounting is a limit case where the marginal propensity to consume is $\alpha_M = \alpha_H|_{\beta=1} = \alpha_H|_{\lambda=0}$.

These results imply that outcomes are observationally equivalent to the exponential case with a suitably higher level of discounting. Barro (1999) also finds that the basic properties of the neoclassical growth model under exponential discounting remain intact when allowing for a variable rate of time preference.

ii. LEVEL OF ASSET PRICES AND RETURNS

A lower saving rate than in the exponential case implies a lower demand for financial assets, which in turn implies lower prices and greater rates of return for both stocks and bonds. As a result, hyperbolic discounting preferences may predict a *greater* risk-free rate than exponential preferences as well as a *greater* return on equity. This implies that it would be easier to reconcile the size of the equity premium by simply explaining

[22]We are grateful to a referee for pointing out this aspect.

the size of the risk-free rate. In other words, under hyperbolic discounting there may be less pressure to explain the size of returns on stocks and more pressure to explain the observed low size of the risk-free rate than under a constant rate of time preference. In this sense, some of the potential solutions to the risk-free rate puzzle posited in the literature may perhaps be sufficient to explain the size of the equity premium if discounting is hyperbolic rather than exponential.[23]

iii. THE INSTANTANEOUS GRATIFICATION CASE

The Instantaneous Gratification (IG) case put forward in Harris and Laibson (2008), in which $\lambda \to \infty$, is of particular relevance for it has dealt with a number of problems inherent in the discrete-time approximation of hyperbolic discount functions, such as a kinked discount factor and the necessity to define the expected duration of the present term. Importantly, the IG model has resolved the pathologies of multiplicity of equilibria and non-monotonicity of the consumption function that have flawed previous theoretical advances in the literature of time-inconsistent preferences. In particular, the properties of the IG model include the existence and uniqueness of equilibrium as well as the continuity and monotonicity of the consumption function.[24]

For these fundamental reasons, in the sections that follow we focus on this particular case of interest.

iv. COMPARATIVE STATICS OF CONSUMPTION WITH RESPECT TO RISK: THE MAGNIFICATION EFFECT OF CHANGES IN RISK

Despite the fact that there are no implications for the risk premium other than quantitative implications for consumption-saving allocations and the level of asset returns, an important difference arises with regard to how consumption is related to risk.

Risk has a linear effect on consumption and depends on the degree of risk aversion. Merton (1969) calculated the elasticities of consumption with respect to expected return

[23] See, for instance, Kocherlakota (1996), Campbell (2000) and Mehra (2008) for reviews of the literature.

[24] See Harris and Laibson (2008, Section 5).

$\bar{\mu}$ and to variance $\bar{\sigma}^2$ for the exponential case, and noted that their sign depend on the parameter of risk aversion b :

$$\epsilon_{c,\bar{\mu}}\big|_M = \bar{\mu}\frac{b-1}{b}\frac{1}{\alpha_M}$$

and

$$\epsilon_{c,\bar{\sigma}^2}\big|_M = -\bar{\sigma}^2\frac{b-1}{2}\frac{1}{\alpha_M}$$

In the hyperbolic IG case, the sign of the corresponding elasticities also depend on b being greater or lower than 1. However, for a given MPC, the absolute value is unambiguously greater than in the Merton case for $b \neq 1$:[25]

$$\epsilon_{c,\bar{\mu}}\big|_{IG} = \bar{\mu}\frac{b-1}{b-(1-\beta)}\frac{1}{\alpha_H}$$

and

$$\epsilon_{c,\bar{\sigma}^2}\big|_{IG} = -\bar{\sigma}^2\frac{b}{b-(1-\beta)}\frac{b-1}{2}\frac{1}{\alpha_H}$$

This result arises from the fact that the sensitivity of the hyperbolic MPC to changes in $\bar{\sigma}^2$ and $\bar{\mu}$ is greater than in the exponential case. In particular, in the exponential case the derivative of the MPC with respect to risk is

$$\frac{\partial \alpha_M}{\partial \bar{\sigma}^2} = \frac{1-b}{2}. \tag{3.21}$$

while in the IG model, the effect of risk on the marginal propensity to consume is

$$\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2} = \frac{b}{b-(1-\beta)}\frac{1-b}{2}. \tag{3.22}$$

We note that in both cases risk has a linear effect on the marginal propensity to consume, and that the derivatives are decreasing in risk for $b > 1$ and increasing in risk

[25] See Appendix for the derivation of the elasticities of consumption with respect to $\bar{\mu}$ and $\bar{\sigma}^2$.

for $b < 1$.[26] However, the absolute value of this relationship is greater in the hyperbolic case than in the exponential discounting case:

$$\left|\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2}\right| > \left|\frac{\partial \alpha_M}{\partial \bar{\sigma}^2}\right|, \qquad \forall b \neq 1.$$

This implies that for any $b \neq 1$, hyperbolic discounting generates a greater response of the propensity to consume to changes in risk.[27]

One implication is that although the hyperbolic discounting's MPC may well be observationally equivalent to the exponential case for given $\bar{\sigma}^2$, $\bar{\mu}$ and risk aversion parameter b (i.e. there would be a suitable subjective rate γ_M that would make $\alpha_{IG} = \alpha_M$), the MPC would react more to changes in the risk parameter. In models where risk is allowed to change (e.g. models allowing for stochastic volatility), this result provides a magnification mechanism to explain the excess price volatility puzzle.

v. IMPLICATIONS FOR ASSET PRICES IN A LUCAS TREE MODEL

We have shown that the relationship between the marginal propensity to consume and portfolio volatility is *magnified* through a greater absolute value of its slope for coefficients of relative risk aversion different than one. However, asset prices are exogenous in the Merton model. Hence, to determine the quantitative importance of hyperbolic discounting as a driving force behind stock market volatility, the introduction of a model with endogenous asset prices is in order.

Consider a simple continuous-time version of Lucas' (1978) representative-agent fruit-tree model of asset pricing. A tree (stock) yields fruit (dividends) D_t according to a geometric Brownian motion:

$$\frac{dD_t}{D_t} = v\,dt + \sigma\,dz_t.$$

[26]See Merton (1969), Section 7, for a discussion on the effect of changes in $\bar{\mu}$ and $\bar{\sigma}^2$ on consumption.
[27]Note that for logarithmic utility both slopes are equal to zero, i.e. $\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2}\big|_{b=1} = \frac{\partial \alpha_M}{\partial \bar{\sigma}^2}\big|_{b=1} = 0$.

Investors can buy shares in the stock at the ex-dividend price P_t. The supply of shares is normalized to 1 and we assume zero net supply of the risk-free asset. In equilibrium, the representative agent follows the optimal policy $c_t^* = \alpha^* x_t$, where α^* is the optimal MPC out of wealth. Ignoring bubble solutions, we conjecture that the equilibrium price is proportional to dividends. Since in equilibrium all dividends are consumed and wealth is equal to the value of the stock:

$$P_t = \frac{1}{\alpha^*} D_t.$$

The variability of prices will be directly linked to the variability of dividends and to the variablity of the MPC in specifications where, for example, the parameter σ were stochastic. As we indicated earlier, it can be shown that

$$\frac{\partial \alpha^*}{\partial \sigma^2} < 0 \text{ for } b > 1, \quad \frac{\partial \alpha^*}{\partial \sigma^2} > 0, \quad \text{for } b < 1.$$

The intuition for this is the same as in the exponential case as established in Merton (1969).The slope of the schedule will depend on the relative strenght of the substitution and income effects of the volatility parameter on consumption, which is determined by the risk aversion parameter b.[28] However, in the IG model the MPC and consequently prices are more responsive to changes in risk.

Next we discuss a number of theoretical and empirical implications of these findings:

1. FROM A THEORETICAL PERSPECTIVE, these drastic differences in the comparative statics of consumption and asset prices with respect to risk mean that hyperbolic discounting offers a novel mechanism whereby changes in risk may affect consumption and stock prices.

2. FROM AN EMPIRICAL PERSPECTIVE, in order to get a sense of the possible

[28] See Merton (1969), Section 7, for a detailed discussion.

quantitative size of the volatility effects, it is useful to study a calibrated model in the region of the parameter space that is empirically plausible. Laibson et al. (2007) use a structural model and field data to estimate an unrestricted discount function that allows the discount rate to differ in the short-run and in the long-run. Theirs are, to the best of our knowledge, the best available estimates obtained in field data. Their structural procedure yields estimates for their benchmark case (which sets the relative risk aversion parameter at a value of 2) of $\beta = 0.7031$ and $\delta = 0.9580$, with standard errors of 0.1093 and 0.0068 respectively. Letting, for instance, the relative risk aversion parameter be 3, yields an estimate of $\beta = 0.5776$ with a standard error of 0.1339, leaving basically unchanged the estimate of δ.

Since they consider a rich consumption model that includes stochastic labor income, liquid and illiquid assets, revolving credit and other ingredients, they can also perform several robustness tests to changes in the different parameters of the model. These tests include compound cases where parameter changes are allowed to reinforce each other. Given the estimates they obtain, reasonable ranges for the parameters we are interested in are β from0.40 to 0.80 and b from 1.4 to 3.[29]

In the Table below we report the results of the calibrations for different parameter values of the ratio:

$$\Omega = \left.\frac{\partial \alpha_{IG}}{\partial \sigma^2}\right|_{\beta<1} / \left.\frac{\partial \alpha_M}{\partial \sigma^2}\right|_{\beta=1} = \frac{b}{b-(1-\beta)}$$

which captures the *magnification* in the response of the MPC to changes in risk relative to the standard, exponential-discounting case. We explore different combinations of the parameter of relative risk aversion b and the short-run discount factor β.

[29]As Laibson et al. (2007) discuss, the picture with respect to the relative risk aversion coefficient is not entirely clear. They consider a value of 2 for their benchmark case, and also the values 1 and 3. The usual view in the asset pricing and consumption-savings literatures is that it is in the range of 0.5 to 5. Gourinchas and Parker (2002) find values between 0.1 and 5.3. Barro (2006) notes that savings rates (excluding human capital) fall markedly as a country develops if b is much below 2, and are counterfactually low it is much above 4.

β	0.4		0.5		0.6		0.7		0.8		1	
b	α	Ω	α	Ω	α	Ω	α	Ω	α	Ω	α	Ω
1.4	0.149	1.750	0.133	1.556	0.119	1.400	0.108	1.273	0.099	1.167	0.085	1.000
1.6	0.126	1.600	0.115	1.455	0.105	1.333	0.097	1.231	0.090	1.143	0.079	1.000
1.8	0.110	1.500	0.101	1.385	0.094	1.286	0.088	1.200	0.082	1.125	0.073	1.000
2	0.098	1.429	0.091	1.333	0.086	1.250	0.080	1.176	0.076	1.111	0.068	1.000
2.2	0.088	1.375	0.083	1.294	0.078	1.222	0.074	1.158	0.071	1.100	0.064	1.000
2.4	0.081	1.333	0.076	1.263	0.073	1.200	0.069	1.143	0.066	1.091	0.061	1.000
2.6	0.075	1.300	0.071	1.238	0.068	1.182	0.065	1.130	0.062	1.083	0.057	1.000
2.8	0.069	1.273	0.066	1.217	0.064	1.167	0.061	1.120	0.059	1.077	0.054	1.000
3	0.065	1.250	0.062	1.200	0.060	1.154	0.058	1.111	0.056	1.071	0.052	1.000

Figure 3.1: Magnification effect of changes in the risk parameter σ on the MPC. Calibration: $\gamma = 10\%$, $\delta = 0.95$, $\mu = 8.7\%$; $r = 1\%$ and $\sigma = 23.5\%$.

We consider values for the relative risk aversion from 1.4 to 3. For each combination, we report the value of Ω and α_{IG}. For the preferred estimate in Laibson et al (2007) of $\beta = 0.70$, we find that hyperbolic discounting generates between 11.1 percent to 27.3 percent greater responsiveness of the MPC to changes in risk than in the standard formulation while keeping α_{IG} in the 5.8 - 10.8% range. As β gets lower, both the ratio Ω for a given value of α_{IG} and the range of the values that Ω takes increase.

Taking the case where $\beta = 0.40$, a value which should not be considered unrealistically low given the findings in Laibson et al (2007), we find that hyperbolic discounting generates a magnification of the response of the MPC to changes in risk that is between 25 to 75 percent greater than in the exponential case. Even in the case of $\beta = 0.80$, which is a value that may seem perhaps unlikely high, hyperbolic discounting generates between a 7.1 to 16.7 percent greater response of prices to changes in risk than exponential discounting.

Summing up, calibrations of the Lucas model with hyperbolic discounting in the empirically plausible region of the parameter space reveal that the MPC, and therefore prices, are much more responsive to changes in risk than in the standard case. Most of the calibrations, e.g., for $\beta \in [0.5, 0.7]$, suggest that this greater responsiveness is in the range of 14 to 55 percent. These results arise for low values of the relative risk aversion coefficient (between 1.4 to 2.4).

We conclude that in light of the difficulties in the literature for explaining stock market volatility, also known as the excess-volatility puzzle for stocks, hyperbolic discounting offers a great deal of promise for contributing to explaining an important and challenging aspect of asset market data.

3.4 Concluding Comments

The analysis has introduced dynamically inconsistent preferences in the standard setting where capital markets are perfect and frictionless, and where wealth is generated by stochastic returns on assets. Introducing labor income jointly with the constraint that consumers may not borrow against future labor income, incomplete markets, and other market frictions (e.g., taxes, transaction costs) are directions that merit future research, even though it is typically not possible to obtain closed-form solutions in these settings.

Over the last couple of decades a large literature has significantly departed from the tractable framework of a representative-agent, time-additive isoelastic preferences, and complete frictionless markets in an attempt to explain asset pricing puzzles. This paper maintains the assumption of time and state-separable preferences defined only over consumption.

Since our paper considers a frictionless economy with no liquidity constraints, it becomes readily comparable to Merton (1969, 1971) and differentiates our work from Laibson (1994, 1997) and Harris and Laibson (2003, 2008). As in the Merton's setup with constant relative risk aversion, the porfolio-selection decision is independent of the consumption decision. We provide with closed-form solutions for a hyperbolic agent's optimal consumption and the optimal porfolio-selection problems and show that the latter is identical to the Merton's exponential case.

As for consumption, we provided a closed form solution that shows that the marginal propensity to consume **can be pinned down** from a system of ordinary differential equations. We show that the hyperbolic MPC is unambiguously greater compared to the classic exponential case. In addition, we showed how the MPC is more sensitive to changes in the risk and expected return parameters, and suggested that our model of time inconsistent preferences could help explain aspects of asset market data, particularly of stock market volatility. We leave this as a recommendation for further research.

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3.5 Appendix

3.5.1 Derivation of Bellman equation (3.7)

The objective function can be written as in (3.6)

$$w(x) = u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[e^{-\gamma dt} \beta v(x+dx)]$$

Multiply both sides by $e^{\gamma dt}$ to get

$$e^{\gamma dt} w(x) = e^{\gamma dt} u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[\beta v(x+dx)]$$

For small dt, we can approximate

$$\begin{aligned} e^{-\lambda dt} &\approx 1 - \lambda dt \\ e^{-\gamma dt} &\approx 1 - \gamma dt \\ e^{\gamma dt} &\approx 1 + \gamma dt \end{aligned}$$

Therefore, the above equation can be written as

$$(1 + \gamma dt)\, w(x) \approx (1 + \gamma dt)\, u(c(x))\, dt + (1 - \lambda dt)\, E[w(x+dx)] + \lambda dt E[\beta v(x+dx)]$$

Substracting $w(x)$ from both sides we get

$$\gamma dt w(x) \approx (1 + \gamma dt)\, u(c(x))\, dt + (1 - \lambda dt)\, E[dw(x)] + \lambda dt E[\beta v(x+dx) - \beta v(x) - w(x) + \beta v($$

where

$$\begin{aligned} dw(x) &= w(x+dx) - w(x), \\ dv(x) &= v(x+dx) - v(x) \end{aligned}$$

Dividing by dt

$$\gamma w(x) \approx (1 + \gamma dt)\, u(c(x)) + (1 - \lambda dt)\, \frac{E[dw(x)]}{dt} + \lambda E[\beta dv(x) - w(x) + \beta v(x)]$$

and letting $dt \to 0$, we obtain

$$\gamma w(x) = u(c(x)) + \frac{E[dw(x)]}{dt} + \lambda E[\beta v(x) - w(x)]. \tag{3.23}$$

Applying Ito's Lemma and taking expectations we find

$$E[dw(x)] = w' E[dx] + \frac{1}{2} w'' E\left[(dx)^2\right]$$

where, from (3.2),

$$E[dx] = (\mu\theta x + (1-\theta) r\theta x - c)\, dt$$

and

$$E\left[(dx)^2\right] = \sigma^2\theta^2 x^2 dt$$

Thus, the Bellman equation (3.23) can be written as

$$\gamma w(x) = u(c(x)) + (\mu\theta x + (1-\theta) r\theta x - c) w'(x) + \frac{1}{2}\sigma^2\theta^2 x^2 w''(x) + \lambda(\beta v(x) - w(x)).$$

Q.E.D.

3.5.2 Derivation of (3.16)

Recall the definitions

$$w(x_t) = E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds + \int_{t+\tau_t}^{\infty} \beta e^{-\gamma(s-t)} u(c(x_s))\, ds\right]$$

and

$$\begin{aligned} v(x_t) &\equiv E_t \int_t^{\infty} e^{-\gamma(s-t)} u(c(x_s))\, ds \\ &= E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds + \int_{t+\tau_t}^{\infty} e^{-\gamma(s-t)} u(c(x_s))\, ds\right] \end{aligned}$$

or

$$\beta v(x_t) = E_t \left[\int_t^{t+\tau_t} \beta e^{-\gamma(s-t)} u(c(x_s))\, ds + \int_{t+\tau_t}^{\infty} \beta e^{-\gamma(s-t)} u(c(x_s))\, ds \right]$$

Therefore,

$$\begin{aligned}
\beta v(x_t) - w(x_t) &= E_t \left[\int_t^{t+\tau_t} \beta e^{-\gamma(s-t)} u(c(x_s))\, ds - \int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds \right] \\
&= -(1-\beta) E_t \left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds \right] \\
&= -(1-\beta) \int_t^{\infty} e^{-\lambda(s-t)} e^{-\gamma(s-t)} E_t [u(c(x_s))]\, ds \\
&= -(1-\beta) \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t [u(c(x_s))]\, ds
\end{aligned}$$

So,

$$\lambda(\beta v(x_t) - w(x_t)) = -\lambda(1-\beta) \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t [u(c(x_s))]\, ds.$$

or, using (3.14),

$$\lambda(\beta v(x_t) - w(x_t)) = -\lambda(1-\beta)\alpha_H \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t [w(x_s)]\, ds$$

Q.E.D.

3.5.3 Derivation of (3.17)

This equation can be derived making use of the candidate solution (3.11) and the policy rules. First, recall the candidate value function (3.11) and apply Ito's Lemma to find $dw = w'dx + \frac{1}{2}w''(dx)^2$. Recall that the variation dx is given by (3.2), and note that $(dx)^2 = \sigma^2\theta^2 x^2 dt$. Thus,

$$dw = w'dx + \frac{1}{2}w''(dx)^2$$

$$dw = \left([rx + (\mu - r)\theta x - c] w' + \frac{1}{2}\sigma^2\theta^2 x^2 w'' \right) dt + \sigma\theta x w' dz$$

so the expected variation of the value function is

$$E\left[dw\right] = \left(\left[rx + \left(\mu - r\right)\theta x - c\right]w' + \frac{1}{2}\sigma^2\theta^2x^2w''\right)dt$$

where the partial derivatives are $w' = \left(1-b\right)\frac{w}{x}$, and $w'' = -b\left(1-b\right)\frac{w}{x^2}$. It follows from (3.12) and (3.13) that the variation of the *optimal* value function is

$$dw = \left(1-b\right) \quad -\alpha_H + r + \frac{\left(\mu - r\right)^2}{2b\sigma^2}\Bigg) wdt + \frac{\left(1-b\right)}{b}\frac{\left(\mu - r\right)}{\sigma}wdz,$$

with expectation

$$E\left[dw\right] = \left(1-b\right) \quad -\alpha_H + r + \frac{\left(\mu - r\right)^2}{2b\sigma^2}\Bigg) wdt$$

which implies

$$E_t\left[w_s\right] = w\left(x_t\right)\exp\left\{\left(1-b\right) \quad -\alpha_H + r + \frac{\left(\mu - r\right)^2}{2b\sigma^2}\Bigg)\left(s-t\right)\right\}.$$

Q.E.D.

Alternatively we could recall the candidate solution (3.11)

$$w\left(x_s\right) = \alpha_H^{-b}\frac{x_s^{1-b}}{1-b}, \ \forall s \geq t$$

and that $w' = \left(1-b\right)\frac{w}{x}$, and $w'' = -b\left(1-b\right)\frac{w}{x^2}$. And from Ito's Lemma:

$$dw = w'dx + \frac{1}{2}w''\left(dx\right)^2$$

Q.E.D.

3.5.4 Derivation of the elasticities of consumption with respect to the portfolio's expected return and variance

Recall the MPC of the hyperbolic IG model (3.20)

$$\alpha^{IG} \equiv \alpha_H|_{\lambda\to\infty} = \frac{1}{b-(1-\beta)}\left(\gamma-(1-b)\left(\bar{\mu}-b\frac{\bar{\sigma}^2}{2}\right)\right).$$

The elasticity of consumption with respect to expected returns would be

$$\epsilon_{c,\bar{\mu}}|_{IG} \equiv \bar{\mu}\frac{\partial c}{\partial\bar{\mu}}\frac{1}{c}$$

Taking the derivative

$$\frac{\partial c}{\partial\bar{\mu}} = \frac{\partial\alpha}{\partial\bar{\mu}}x + \alpha\frac{\partial x}{\partial\bar{\mu}}$$

we note that $\frac{\partial x_0}{\partial\bar{\mu}} = 0$, and

$$\frac{\partial c}{\partial\bar{\mu}} = \frac{\partial\alpha}{\partial\bar{\mu}}x = \frac{b-1}{b-(1-\beta)}x$$

therefore

$$\epsilon_{c,\bar{\mu}}|_{IG} = \bar{\mu}\frac{b-1}{b-(1-\beta)}\frac{x}{c}$$

Recalling (3.13) , we can write

$$\epsilon_{c,\bar{\mu}}|_{IG} = \bar{\mu}\frac{b-1}{b-(1-\beta)}\frac{1}{\alpha}$$

In similar way we can find the elasticity of consumption with respect to portfolio volatility

$$\epsilon_{c,\bar{\sigma}^2}|_{IG} = -\bar{\sigma}^2\frac{b}{b-(1-\beta)}\frac{b-1}{2}\frac{1}{\alpha_H}.$$

Q.E.D.