Brownian Excursions in Mathematical Finance

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For my mother.

Declaration

I certify that the thesis I have presented for examination for the Ph.D. degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

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Abstract

The Brownian excursion is defined as a standard Brownian motion conditioned on starting and ending at zero and staying positive in between. The first part of the thesis deals with functionals of the Brownian excursion, including first hitting time, last passage time, maximum and the time it is achieved. Our original contribution to knowledge is the derivation of the joint probability of the maximum and the time it is achieved. We include a financial application of our probabilistic results on Parisian default risk of zero-coupon bonds. In the second part of the thesis the Parisian, occupation and local time of a drifted Brownian motion is considered, using a two-state semi-Markov process. New versions of Parisian options are introduced based on the probabilistic results and explicit formulae for their prices are presented in form of Laplace transforms. The main focus in the last part of the thesis is on the joint probability of Parisian and hitting time of Brownian motion. The difficulty here lies in distinguishing between different scenarios of the sample path. Results are achieved by the use of infinitesimal generators on perturbed Brownian motion and applied to innovative equity exotics as generalizations of the Barrier and Parisian option with the advantage of being highly adaptable to investors' beliefs in the market.

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Chapter 1

Introduction

1.1 Motivation and Literature review

An option is a derivative financial instrument that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a certain price on or before a specific date. A Call option offers the right to buy the underlying asset, whereas a Put option holder has the right to sell the underlying asset. The price at which the underlying transaction will occur, is called the strike price, denoted by K. The payoff is defined as $(S_T - K)^+$ for a Call option, and $(K - S_T)^+$ for a Put option, where S_T is the price of the underlying asset at expiration date T. The simplest options are European options, which can only be exercised at the expiration date and their values depend merely on the underlying asset at that time. In contrast, Barrier options are path-dependent and the underlying asset up to the expiration date has to be considered. The terminal payoff depends on whether the price of the underlying asset reaches a certain barrier before the expiration date. The four main types of these options are Down-and-Out, Down-and-In, Up-and-Out and Up-and-In, i.e. the right to exercise either appears ("In") or disappears ("Out") at some barrier B. This barrier is either set below ("Down") or above ("Up") the underlying asset at the start time $t = 0$. The Down-and-In

Call option, for instance, has final payoff $(S_T - K)^+ \mathbf{1}_{\{\min_{0 \le t \le T} S_t < B\}}$. These Barrier options have major disadvantages concerning risk management. The discontinuity of the payoff at the barrier makes price manipulation by driving the underlying possible. Furthermore, take the Up-and-Out Call option as an illustration. An accidental price jump across the barrier can wipe out a buyer's entire investment, in spite of having the correct view on the overall market trend. Also the discontinuity of the Delta at the barrier for all times and its unboundedness when maturity approaches, makes it hard to find a replicating strategy based on the Black-Scholes theory.

In order to counteract these problems, Parisian options were introduced by Chesney, Jeanblanc-Picqué, and Yor [1997]. They are similar to path-dependent Barrier options, where the contract is defined in terms of staying above or below a certain barrier L for a fixed time period d before maturity date, instead of just touching the barrier. Similar to Barrier options, which are limiting cases of Parisian options, they can have the form of a Down-and-Out, Down-and-In, Up-and-Out, Up-and-In Call or Put. The payoff of the Parisian Down-and-In Call, for instance, can be found to be $(S_T - K)^+ \mathbf{1}_{\{\tau^-_{L,d}(S) < T\}}$, where $\tau^-_{L,d}(S)$ accounts for the first time the excursion below L exceeds option window d. We also call this time the Parisian time below L. One motivation of introducing Parisian options lies in the insensitivity to influential agents; it is a lot more expensive to manipulate these kind of options. Furthermore, this Parisian criterion is to smooth the option value and consequently the Delta and Gamma near the barrier, preventing the option's Gamma to undergo an infinite jump. In contrast to a standard Barrier option, where small price movements around the barrier result in large changes of Delta, the hedging ratio of the Parisian option varies smoothly when crossing the barrier.

Even though Parisian options are not exchange traded, they are used as building blocks in structured products, such as convertible bonds, which offer the holder the right but not the obligation to convert the bond at any time to a pre-specified number of shares. Most convertible bonds contain the call provision, allowing the issuer to buy back the bond at the so-called call price, in order to manage the company's debt-equity ratio. Upon issuer's call, the holder either redeems at call price or converts. Apart from the hard call constraint, which protects the conversion privilege to be called away too early, the soft call constraint requires the stock price to be higher than a certain price level. This is sensitive to market manipulation by the issuer, which can be counteracted with the Parisian feature. The Parisian feature requires the stock price to stay above a level for a certain time. These callable convertible bonds with Parisian feature are commonly traded in the OTC market in Hong Kong , see Avellaneda and Wu [1999], Lau and Kwok [2004].

Pricing derivatives in the Black-Scholes framework rely on the distribution of Brownian functionals. Familiar functionals, such as the first hitting time or the maximum, have been well studied and used for pricing Barrier or Lookback options. The key in pricing Parisian options lies in deriving the distribution of the time above or below a certain barrier. In the case of consecutive Parisian options we are interested in the excursion time, which denotes the time spent between two crossovers of the fixed barrier. In the cumulative Parisian case we are interested in the occupation time, which is the summation of all excursion times above or below the barrier up to a certain time. This has been studied by Chesney, Jeanblanc-Picqué, and Yor [1997], Dassios and Wu [2011d], Cai, Chen, and Wan [2010] and Zhang [2015]. Variations of the Parisian option can be found in the double sided Parisian option by Anderluh and van der Weide [2009] or the double barrier Parisian option by Dassios and Wu [2011b]. American-style

Parisian options have been studied by Haber, Schönbucher, and Wilmott [1999] and Chesney and Gauthier [2006]. If we do not specify the term, we consider consecutive European-style Parisian options. Taking thoughts about Parisian-type questions any further, one can bridge financial application to insurance mathematics. Ruin probabilities of risk models with Parisian delay have been studied by Dassios and Wu [2009b], Dassios and Wu [2008], Dassios and Wu [2011c], Loeffen, Czarna, and Palmowski [2013], Landriault, Renaud, and Zhou [2011] and others. This concept occurs if the surplus process stays below zero for a continuous time period longer than a fixed time.

1.2 Organization and Outline of the thesis

The thesis is structured as follows. In Chapter 3 we explore functionals of the Brownian excursion, including first hitting time, last passage time, maximum and the time it is achieved. The Brownian excursion is defined as a standard Brownian motion, conditioned on starting and ending at zero and staying positive in between. Using conditioned martingales, we relate the excursion to Brownian motion and the three-dimensional Bessel process. Referring to Doob's h-transform we find the dynamics of the Brownian excursion, leading to the derivation of the first hitting time with three different methods. Our proofs use elementary arguments from probability theory and emphasize the nature of excursions. Our main result of Chapter 3, which is new as far as we are concerned, studies the joint probability of the maximum and time it is achieved. We find the financial application of our probabilistic results in Parisian default risk of bonds. The principal difference between stocks and bonds is that the latter reaches a predetermined face value at maturity, whereas a stock's final value is unknown a priori. Brownian excursions possess the features of being non-negative and taking a specific value at maturity, being a suitable model for bond prices.

Chapter 4 deals with the joint probability of Parisian, occupation and local time of drifted Brownian motion. The results are achieved via a two-state semi-Markov model on a perturbed Brownian motion, which has been proposed by Dassios and Wu [2009a]. This perturbed Brownian motion has the same behaviour as a drifted Brownian motion, except it moves toward the other side of the barrier by a jump of size ϵ each time it hits zero, disposing of the difficulty of the origin being regular. The semi-Markov process allows us to define an infinitesimal generator, where the solution of the martingale problem provides us with the triple Laplace transform of Parisian, occupation time and number of downcrossings of the perturbed Brownian motion. The relation between the number of downcrossings by the Brownian motion and the Brownian local time, manifested by Lévy [1948], motivates our study of downcrossings and yields the triple Laplace transform of Parisian, occupation and local time of the drifted Brownian motion. The connection between the local time and the drawdown time stimulates research on the relative drawdown process serving as an indicator of market stability. It has been noted that the relative drawdown process is low in stable periods and shoots up during market recession [see e.g. Vecer et al., 2006]. It can be assumed that a realization of a large drawdown is followed by a default happening at the first instance the relative drawdown exceeds a threshold. We extend this definition to default occurring with Parisian delay, i.e. if the underlying process stays below zero for a pre-specified time period. We consider this to be a more realistic measure of risk, giving regulators more time to react to shortfalls and keeping in mind that relative drawdowns cannot be monitored continuously. To insure against the event of the relative drawdown exceeding some percentage with Parisian delay we introduce the so-called Parisian Crash option with digital payoff and the Parisian Lookback option.

Chapter 5 is devoted to extending the Parisian concept and introducing innovative equity exotics which are generalizations of Barrier and Parisian options and extremely adaptable to investors' needs and beliefs in the market. One version of the so-called ParisianHit option, the MinParisianHit, gets triggered if either the age of the excursion above a level reaches a certain time or another barrier is hit before maturity. The MaxParisianHit on the other hand gets activated when both excursion age exceeds a certain time and a barrier is hit. The advantages of our options are cost benefits for investors, insensitivity to market manipulation and smoothening of the Delta around the barrier in order to find a replicating strategy. The key for pricing these kinds of options lies in deriving the joint law of Parisian and hitting time which we have achieved via a three-state semi-Markov model indicating whether the process is in a positive or negative excursion and above or below a fixed barrier. Results are found in terms of double Laplace transforms.

Chapter 2

Nomenclature

For any stochastic process Y and probability measure $\mathbb{P}$, we use $\mathbb{P}_y$ to denote $\mathbb{P}(.|Y_0 = y)$. With the subscript behind the expected value we denote the starting position of the stochastic process Y, i.e. for any function f

$$\mathbb{E}_y^{\mathbb{P}}(f(Y)) = \mathbb{E}^{\mathbb{P}}(f(Y); Y_0 = y).$$

The superscript announces under which probability measure we take the expectation, i.e.

$$\mathbb{E}_y^{\mathbb{P}}(f(Y)) = \int_{-\infty}^{\infty} f(x)\mathbb{P}_y(Y \in dx).$$

We fix the notation for inverse Laplace transforms. Given a function $F(\beta)$, the inverse Laplace transform of F, denoted by $\mathcal{L}^{-1}\{F(\beta)\}$, is the function f whose Laplace transform is F, i.e.

$$f(t) = \mathcal{L}_\beta^{-1}\{F(\beta)\}|_t \quad \Longleftrightarrow \quad \mathcal{L}_t\{f(t)\}(\beta) := \int_0^\infty e^{-\beta t} f(t)dt = F(\beta).$$

Note, that we consider the inverse Laplace transform with respect to the transformation variable β at the evaluation point t. If not otherwise stated we take from now on $\mathcal{L}_\beta^{-1}\{F(\beta)\}|_t$ as a function of the time variable t.

In the same way we define the triple Laplace transform F for a given function $f(y, x, t)$ of three variables $y, x, t \geq 0$ by

$$F(\gamma, \mu, \beta) = \mathcal{L}_t\mathcal{L}_x\mathcal{L}_y\{f(y, x, t)\}(\gamma, \mu, \beta) := \int_{t=0}^{\infty} e^{-\beta t} \int_{x=0}^{\infty} e^{-\mu x} \int_{y=0}^{\infty} e^{-\gamma y} f(y, x, t) dy\, dx\, dt.$$

Hence, the inverse triple Laplace is

$$\mathcal{L}_\beta^{-1}\mathcal{L}_\mu^{-1}\mathcal{L}_\gamma^{-1}\{F(\gamma, \mu, \beta)\}|_{(y,x,t)} = f(y, x, t),$$

where (y, x, t) are the evaluation points. Analogously, inverse double and quadruple Laplace transforms are defined.

Stochastic Processes		
$(W_t)_{t\geq 0}$		standard Brownian motion
$(W_t^\mu)_{t\geq 0}$	$=$	$(\mu t + W_t)_{t\geq 0}$ - Brownian motion with drift $\mu \geq 0$
$\left(W_i^{br}(t)\right)_{t\geq 0}$		Brownian bridge, $i \in \mathbb{N}$
$(R_t)_{t\geq 0}$		three-dimensional Bessel process
$(m_t)_{0\leq t\leq T}$		Brownian excursion of length T
$(\tilde{m}_t)_{0\leq t\leq T}$		price process of risky zero-coupon bond paying \$1 at time T if no default occurs and \$0 otherwise
$(S_t)_{t\geq 0}$		risk-neutral geometric Brownian motion solving SDE $dS_t = rS_t dt + \sigma S_t dW_t$
$(W_t^{\epsilon,\mu})_{t\geq 0}$		perturbed Brownian motion with drift, see (4.10)

$(X_t)_{t\geq 0}$		two-state semi-Markov process with state space $\{1,-1\}$, see (4.11)
$(\hat{X}_t)_{t\geq 0}$		three-state semi-Markov process with state space $\{2,1,-1\}$, see (5.1)

Random times		
$\bar{Y}_t$	$=$	$\sup_{0\leq s\leq t} Y_s$ - running maximum of a generic stochastic process Y
$DD_t(Y)$	$=$	$\bar{Y}_t - Y_t$ - Drawdown of process Y
$RDD_t(Y)$	$=$	$\frac{\bar{Y}_t - Y_t}{\bar{Y}_t}$ - relative Drawdown of process Y
$H_b(Y)$	$=$	$\inf\{t\geq 0 \mid Y_t = b\}$ - first hitting time of constant b
$H_{a,b}(Y)$	$=$	$\inf\{t\geq 0 \mid Y_t = a \text{ or } Y_t = b\}$ - first exit time of interval (a,b) with $a,b\in\mathbb{R}$ and $a<b$
$G_b(Y)$	$=$	$\sup\{t\geq 0 \mid Y_t = b\}$ - last passage time of constant b
$\theta_t(Y)$	$=$	$\inf\{0\leq s\leq t \mid Y_s = \bar{Y}_t\}$ - first time the maximum is achieved before time t
$\tilde{\theta}_t(Y)$	$=$	$\inf\{0\leq s\leq t \mid Y_s = \min_{0\leq u\leq t} Y_u\}$ - first time the minimum is achieved before time t
$g_t(Y)$	$=$	$\sup\{s\leq t \mid \operatorname{sgn}(Y_s)\neq \operatorname{sgn}(Y_t)\}$ - last crossing time of 0 before time t
$d_t(Y)$	$=$	$\inf\{s\geq t \mid \operatorname{sgn}(Y_s)\neq \operatorname{sgn}(Y_t)\}$ - first crossing time of 0 after time t
$g_{L,t}(Y)$	$=$	$\sup\{s\leq t : Y_s = L\}$ - last crossing time of $L\neq 0$ before time t

$$
\begin{aligned}
d_{L,t}(Y) &= \inf\{s \geq t : Y_s = L\} \text{ - first crossing time of } L \neq 0 \text{ after time } t \\
\tau^+_{d_1}(Y) &= \inf\{t > 0 | (t - g_t(Y))\mathbf{1}_{Y_t > 0} \geq d_1\} \text{ - first time the excursion above 0 exceeds time period } d_1\text{, or alternatively, Parisian time above 0} \\
\tau^-_{d_2}(Y) &= \inf\{t > 0 | (t - g_t(Y))\mathbf{1}_{Y_t < 0} \geq d_2\} \text{ - first time the excursion below 0 exceeds time period } d_2 \\
\tau^+_{L,d}(Y) &= \inf\{t > 0 | (t - g_{L,t}(Y))\mathbf{1}_{Y_t > L} \geq d\} \text{ - first time the excursion above } L \text{ exceeds time period } d \\
\tau^-_{L,d}(Y) &= \inf\{t > 0 | (t - g_{L,t}(Y))\mathbf{1}_{Y_t < L} \geq d\} \text{ - first time the excursion below } L \text{ exceeds time period } d \\
\tau(Y) &= \min\{\tau^+_d(Y), H_b(Y)\} \text{ - minimum of the two stopping times} \\
\bar{\tau}(Y) &= \max\{\tau^+_d(Y), H_b(Y)\} \text{ - maximum of the two stopping times} \\
N_t & \quad \text{number of downcrossings from } \epsilon \text{ to 0 of the process } W^\mu \\
C^1_t(Y) &= \int_0^t \mathbf{1}_{Y_s > 0} ds \text{ - occupation time above 0} \\
C^2_t(Y) &= \int_0^t \mathbf{1}_{Y_s < 0} ds \text{ - occupation time below 0} \\
U_t(Y) &= t - g_t(Y) \text{ - time elapsed in current state, either above or below 0} \\
L^a_t(Y) & \quad \text{local time of } a \text{ for the process } Y
\end{aligned}
$$

Miscellaneous		
$(A-B)^+$	$=$	$\max\{A-B,0\}$ - positive part
$A \wedge B$	$=$	$\min\{A,B\}$ - Minimum
$\bar{\mathbb{Q}}$		equivalent martingale measure
δ_x	$=$	$\begin{cases} 0 & \text{, if } x \neq 0 \\ \infty & \text{, if } x = 0 \end{cases}$ - Dirac function with $\int\limits_{-\infty}^{\infty} \delta_x dx = 1$
$\mathcal{N}(x)$	$=$	$\int\limits_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ - cumulative distribution function of the standard normal distribution
λ_{ij}		transition intensity from state i to j of process X, see (4.12), (4.13)
$p_{ij}(t)$		transition density of X, see (4.15)
$\bar{P}_i(t)$		survival probability in state i, see (4.14)
$\hat{P}_{ij}(\beta)$	$=$	$\int_0^{d_i} e^{-\beta s} p_{ij}(s) ds$, see (4.17)
$\tilde{P}_{ij}(\beta)$	$=$	$\int_0^{\infty} e^{-\beta s} p_{ij}(s) ds$ - Laplace transform, see (4.18)
$\hat{\lambda}_{ij}$		transition intensity from state i to j of process $\hat{X}$, see (5.2), (5.3)
$q_{ij}(t)$		transition density of $\hat{X}$, see (5.5)
$\bar{Q}_i(t)$		survival probability in state i, see (5.4)
$\hat{Q}_{ij}(\beta)$	$=$	$\int_0^{d_i} e^{-\beta s} q_{ij}(s) ds$, see (5.6)
$\tilde{Q}_{ij}(\beta)$	$=$	$\int_0^{\infty} e^{-\beta s} q_{ij}(s) ds$ - Laplace transform, see (5.7)

Chapter 3

Functionals of the Brownian excursion

The relationship between the Brownian excursion and the three-dimensional Bessel bridge is well understood in the literature. We provide three proofs of the result on the first hitting time of the Brownian excursion, use time reversion and derive the density of the last passage time. As a consequence we derive the law of the maximum of the Brownian excursion and extend this result to the joint law of maximum and the time it is achieved. As an application we discuss default probabilities of bonds.

3.1 Introduction

We study functionals of the Brownian excursion, including first hitting time, last passage time, maximum and the time it is achieved. The Brownian excursion is defined as a standard Brownian motion, conditioned on starting and ending at zero and staying positive in between. Using conditioned martingales, as introduced in Perkowski and Ruf [2012], we relate the excursion to the Brownian motion and the three-dimensional Bessel process, abbreviated with $BES(3)$. Since the work of Doob [1957] and McKean [1963], the connection between Brownian motion and Bessel process has been well known. Williams [1974] shows that these two processes are dual and establishes path decomposition theorems, which can even be extended by use of the method of random time substitution. Chung [1975] explores maxima of the Brownian excursion using the last exit decomposition, whereas Durrett, Iglehart, and Miller [1977] and Durrett and Iglehart [1977] develop various relationships between meanders, excursions and bridges by using limit processes of conditional functionals of the Brownian motion. Density factorization makes it possible for Imhof [1984] to derive joint densities concerning the maximum of the Brownian meander and the $BES(3)$ process. McKean [1963] and Williams [1974] derive path decompositions of one-dimensional diffusions by welding together various conditioned processes relying on Doob's h-transforms [Doob, 1957] and the Martin boundary. A representation of the $BES(3)$ process in terms of the Brownian motion was given by Pitman [1975]. Kennedy [1976] derives the distribution of the maximum of the excursion via limiting processes and relates it to the standard Brownian motion. Pitman and Yor [1998] find the maximum of the Bessel process and its Mellin transform through the Gikhman-Kiefer formula. The relationship between the Brownian excursion and the Bessel bridge goes back to McKean [1963] and was formalized by Pitman [1975]. Hernandez-del Valle [2013] shows the relationship of the hitting time of a moving boundary by Brownian motion and excursion by means of the Girsanov theorem leading to Schrödinger's equation with linear potential.

The survey of last passage times play an important role in financial mathematics. Since they look into the future and are not stopping times, the standard theorems in martingale theory cannot be applied and therefore they are much harder to handle. Since Madan, Roynette, and Yor [2008] discovered that European option prices can be written in terms of last passage times, where they allow great flexibility to the local martingale modelling the stock price, they came into focus of interest for option pricing. Last passage times also play an important role in path decomposition of diffusions [see.e.g. Williams, 1970] and enlargement of filtration [Barlow, 1978] [Nikeghbali and Yor, 2006]. Other applications can be found in Imkeller [2002] about insider trading or default risk [see e.g. Elliott, Jeanblanc, and Yor, 2000]. Probabilities of last passage times have been found for drifted Brownian motion yielding European option prices in the Black-Scholes framework [see Madan, Roynette, and Yor [2008], Cheridito, Nikeghbali, and Platen [2012]]. We, on the other hand, focus on the last passage time of the Brownian excursion.

To the best of our knowledge, there has not been any study of the last passage time of the Brownian excursion and the joint density of the maximum and the time it is achieved. We start this chapter by fixing notations in section 3.2 and refer to Doob's h-transform and conditioned martingales in order to find the dynamics of the Brownian excursion in section 3.3. In section 3.4 we derive the density of the first hitting time with three different methods. Our proofs use elementary arguments from probability theory and emphasize the nature of excursions. We show that Brownian excursions are reversible relying on Pitman's Bessel bridge representation [Pitman, 1975, 1999a] in section 3.4.4. Our focus lies in reversing the process and deriving the last passage time. Using the law of the hitting time, we derive the law of the maximum in section 3.4.5, which coincides with known results by Chung [1975], Bertoin, Chaumont, and Pitman [2003], Kobayashi, Izumi, and Katori [2008] and others, relying on a different approach and argument. In section 3.5 we conclude with our main result, studying the joint probability of maximum and time it is achieved, followed by the law of the time of the maximum. Section 3.6 is devoted to an application on default risk of bonds. The principal difference between stocks and bonds is that the latter reach predetermined face values at maturity whereas a stock's final value is unknown a priori. Brownian excursions possess the features of being non-negative and taking a specific value at maturity, being a suitable model for bond prices.

3.2 Definitions

In order to fix notations we define $(W_t)_{t\geq 0}$ to be a standard Brownian motion and $(R_t)_{t\geq 0}$ to be a three-dimensional Bessel process starting at zero with probability measures $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ respectively. For fixed $t > 0$, let

$$g_t(W) = \sup\{s \leq t| \text{ sgn } (W_s) \neq \text{ sgn } (W_t)\},$$
$$d_t(W) = \inf\{s \geq t| \text{ sgn } (W_s) \neq \text{ sgn } (W_t)\}$$

denote the last passage time of zero before time t and first hitting time of zero after t respectively. The time interval $(d_t(W), g_t(W))$ is the Brownian excursion interval straddling time t. Using Brownian scaling we fix the excursion length to be $T > 0$, hence, W restricted to this interval is the excursion process $(m_t)_{0\leq t\leq T}$ with probability measure $\mathbb{P}$. The first hitting times of a constant $a \geq 0$ by the processes are defined as

$$H_a(W) = \inf\{t \geq 0|W_t = a\},$$
$$H_a(R) = \inf\{t \geq 0|R_t = a\},$$
$$H_a(m) = \inf\{t \geq 0|m_t = a\}.$$

The last passage time of the Brownian excursion of length T is denoted by

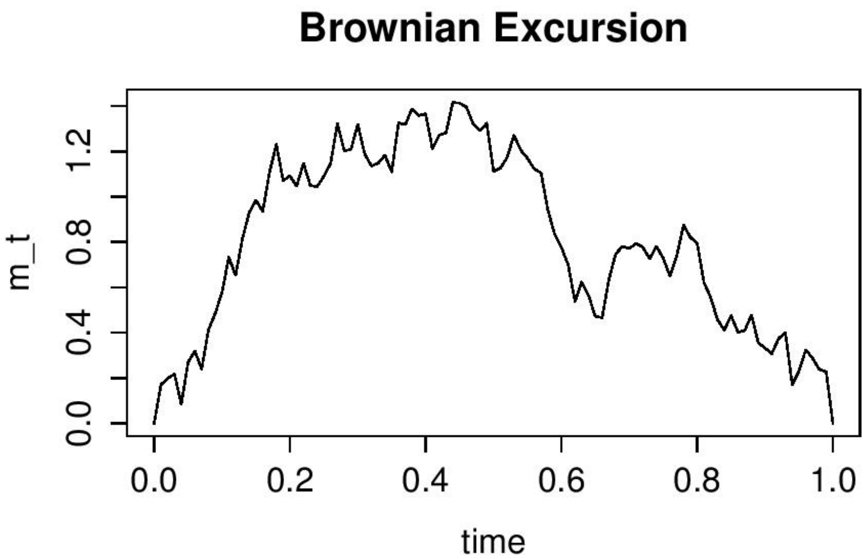
$$G_a(m) = \sup\{t \leq T | m_t = a\}.$$

For the maximum and the time it is achieved, we define

$$\bar{m}_T = \max_{0 \leq s \leq T} m_s,$$
$$\theta_T(m) = \inf\{0 \leq s \leq T | m_s = \bar{m}_T\}.$$

An illustration of a Brownian excursion, where the excursion length is fixed to $T = 1$, is given in Figure 3.1.

Figure 3.1: Path trajectory of an excursion process



3.3 Dynamics of the Brownian excursion

We want to derive the dynamics of the Brownian excursion m by conditioning a standard Brownian motion on starting and ending at zero and staying positive in between. The tool is given through Doob's h-transform [Doob, 1957], where an additional drift arises. H-transforms have the intuitive interpretation of conditioning a Markov process on its behaviour at maturity of the process.

Theorem 3.3.1 (Doob's h-transform)**.** *Let X be a Markov process, i.e. a solution of the martingale problem for the infinitesimal operator $\mathcal{A}$, and let $\hat{h}$ be a harmonic function. Define the measure $\mathbb{P}^{\hat{h}}$ s.t. for any $\mathcal{F}_t$-measurable random variable Y,*

$$\mathbb{E}_x^{\hat{h}}(Y) = \frac{1}{\hat{h}(x)} \mathbb{E}_x(\hat{h}(X_t)Y).$$

Then $\mathbb{P}^{\hat{h}}$ is the law of a solution of the martingale problem for the generator $\mathcal{A}^*$ defined by

$$\mathcal{A}^* f(t,x) = \frac{\mathcal{A}(f\hat{h})(t,x)}{\hat{h}(t,x)}.$$

This means in particular, assuming the original diffusion X having the generator

$$\mathcal{A} f(t,x) = \frac{\partial f}{\partial t} + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2},$$

it follows

$$\mathcal{A}^* f(t,x) = \frac{\mathcal{A}(f\hat{h})(t,x)}{\hat{h}(t,x)} = \frac{1}{\hat{h}(t,x)} \left(\frac{\partial (f\hat{h})}{\partial t} + b(x)\frac{\partial (f\hat{h})}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 (f\hat{h})}{\partial x^2} \right)$$
$$= \frac{\partial f}{\partial t} + \frac{f}{\hat{h}}\frac{\partial \hat{h}}{\partial t} + b(x)\frac{\partial f}{\partial x} + b(x)\frac{f}{\hat{h}}\frac{\partial \hat{h}}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2} + \frac{a(x)}{\hat{h}}\frac{\partial f}{\partial x}\frac{\partial \hat{h}}{\partial x} + \frac{1}{2}a(x)\frac{f}{\hat{h}}\frac{\partial^2 \hat{h}}{\partial x^2}$$
$$= \frac{f}{\hat{h}}\left(\frac{\partial \hat{h}}{\partial t} + b(x)\frac{\partial \hat{h}}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2} \right) + \frac{\partial f}{\partial t} + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2} + \frac{a(x)}{\hat{h}}\frac{\partial f}{\partial x}\frac{\partial \hat{h}}{\partial x}.$$

Assuming that $\hat{h}$ is harmonic, i.e. $\mathcal{A}\hat{h} \equiv 0$, the new generator becomes

$$\mathcal{A}^* f(t,x) = \frac{\partial f}{\partial t} + \left(b(x) + a(x)\frac{1}{\hat{h}}\frac{\partial \hat{h}}{\partial x} \right)\frac{\partial f}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f}{\partial x^2}.$$

We notice that the new process has an additional drift component $a(x)\frac{1}{\hat{h}}\frac{\partial \hat{h}}{\partial x}$. It is shown by Perkowski and Ruf [2012] (Prop. 3.2), that this conditioned process is indeed a diffusion. We refer to Doob [1957], Williams [1974] and Chung and Walsh [2005] for greater detail on Doob's h-transform.

The Brownian excursion m is a continuous inhomogeneous Markov process (see Knight [1980] for its Feller property), which is distributionally defined as

$$\mathbb{P}_0(m_t \in dx) = \mathbb{Q}_0(W_t \in dx | W_s > 0 \text{ for all } 0 < s < T \text{ and } W_T = 0).$$

The generator of the Brownian motion is $\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}$. We now condition on the event of hitting zero at time T and not before, hence we choose $\hat{h}$ to be the first hitting time density

$$\hat{h}(t,x) = \frac{x}{\sqrt{2\pi(T-t)^3}} e^{-\frac{x^2}{2(T-t)}}\ , \ \ t,x \geq 0.$$

This function can easily be checked to be harmonic with respect to the generator of the standard Brownian motion $\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}$.
Hence, the additional drift component becomes

$$\frac{1}{\hat{h}(t,x)}\frac{d\hat{h}(t,x)}{dx} = \frac{1}{x} - \frac{x}{T-t},$$

and the generator of the Brownian excursion m is therefore

$$\mathcal{A}^* f(t,x) = \frac{\partial f(t,x)}{\partial t} + \left(\frac{1}{x} - \frac{x}{T-t}\right)\frac{\partial f(t,x)}{dx} + \frac{1}{2}\frac{\partial^2 f(t,x)}{dx^2}. \tag{3.1}$$

This yields the dynamics of the Brownian excursion to be

$$dm_t = \left(\frac{1}{m_t} - \frac{m_t}{T-t}\right)dt + dW_t,$$
$$m_0 = 0.$$

We refer to Pitman and Yor [1982] for further reference. It is a remarkable fact that conditioning can be exactly reproduced by applying the right drift.

Remark 3.3.1. Note that conditioning a Brownian motion with non-zero drift on the first hitting time provides us with the same dynamics as a Brownian motion without drift. In particular, a drifted Brownian motion conditioned to hit zero at time T is indistinguishable from a standard Brownian excursion $(m_t)_{0\le t\le T}$.

3.4 Density of the First Hitting Time of the Brownian excursion

Referring to Doob's h-transform the dynamics of the Brownian excursion for $0 < t \leq T$ has been found (see also Bloemendal [2010] and Hernandez-del Valle [2013]),

$$dm_t = \left(\frac{1}{m_t} - \frac{m_t}{T-t}\right) dt + dW_t, \tag{3.2}$$
$$m_0 = 0.$$

This stochastic differential equation has a unique strong solution [see Campi, Cetin, and Danilova, 2013, Prop. 3.5]. As our first result on functionals of the Brownian excursion, we derive the first hitting time of a constant barrier $a > 0$.

Theorem 3.4.1. *The first hitting time of level $a > 0$ by the Brownian excursion m is given by*

$$\mathbb{P}_0(H_a(m) \in dt) = a\sqrt{\frac{2T^3}{\pi t^5 (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \sum_{k=0}^{\infty} \left((2k+1)^2 a^2 - t\right) e^{-\frac{(2k+1)^2 a^2}{2t}} dt. \tag{3.3}$$

We proof this theorem with three different methods. The purpose is to emphasize the probabilistic nature of Brownian excursions and the relationship to similar stochastic processes.

3.4.1 Proof using Bessel process

The next two methods are probabilistically straightforward and accentuate the behaviour of excursions. The connection between Brownian motion and the three-dimensional Bessel process, abbreviated by $BES(3)$, has been well studied. Our motivation for using the $BES(3)$ process lies in Pitman's Bessel bridge representation [Pitman, 1999a,b] of the Brownian excursion. For the $BES(3)$ process, the Laplace transform of first hitting time of a starting in x where $0 < x < a$ is known to be [see e.g. Borodin and Salminen, 2002, formula 2.0.1]

$$\mathbb{E}_x^{\tilde{\mathbb{Q}}}(e^{-\alpha H_a(R)}) = \frac{a \sinh(x\sqrt{2\alpha})}{x \sinh(a\sqrt{2\alpha})}.$$

By using L'Hôpital's rule and letting x approach zero, we find

$$\mathbb{E}_0^{\tilde{\mathbb{Q}}}(e^{-\alpha H_a(R)}) = \frac{a\sqrt{2\alpha}}{\sinh(a\sqrt{2\alpha})}.$$

Using series expansion and inverting term by term, gives us the density

$$\tilde{\mathbb{Q}}_0(H_a(R) \in dt) = \frac{a}{\sqrt{2\pi t^5}} \sum_{k=-\infty}^{\infty} \left((2k+1)^2 a^2 - t\right) e^{-\frac{(2k+1)^2 a^2}{2t}} dt. \tag{3.4}$$

In order to derive the first hitting time $H_a(m)$ of a Brownian excursion, we multiply the first hitting time $H_a(R)$ of the $BES(3)$ with the transition density of getting from a to ϵ from time t to time T and divide by the density of going from zero to ϵ from time zero to time T. Finally we let ϵ go to zero,

$$\mathbb{P}_0(H_a(m) \in dt) = \lim_{\epsilon \to 0} \frac{\tilde{\mathbb{Q}}_0(H_a(R) \in dt)\tilde{p}_{T-t}(a, \epsilon)}{\tilde{p}_T(0, \epsilon)}. \tag{3.5}$$

For the $BES(3)$ process we know, that the transition density is given by [see e.g. Göing, 1997][Göing-Jaeschke and Yor, 2003]

$$\tilde{p}_t(x,y) = \frac{y}{x}\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right), \tag{3.6}$$

$$\tilde{p}_t(0,y) = \sqrt{\frac{2}{\pi t^3}}y^2 e^{-\frac{y^2}{2t}}. \tag{3.7}$$

Using L'Hôpital's rule we can find the limit for $\epsilon \to 0$ to be

$$\begin{aligned}\mathbb{P}_0(H_a(m) \in dt) &= \tilde{\mathbb{Q}}_0(H_a(R) \in dt)\sqrt{\frac{T^3}{(T-t)^3}}e^{-\frac{a^2}{2(T-t)}} \\ &= a\sqrt{\frac{T^3}{2\pi t^5(T-t)^3}}e^{-\frac{a^2}{2(T-t)}}\sum_{k=-\infty}^{\infty}\left((2k+1)^2a^2 - t\right)e^{-\frac{(2k+1)^2a^2}{2t}}dt.\end{aligned}$$

This finalizes the proof of Theorem 3.4.1 by splitting the infinite sum into positive and negative parts.

3.4.2 Proof using Brownian motions

In the case where we want to model the excursion using a conditioned Brownian motion, we have to be more careful, since the Brownian motion might return to zero before time T. Hence, we decompose the hitting time of the Brownian excursion into the probability that the Brownian motion starting at ϵ hits $a > 0$ before hitting zero at time t and the hitting time of zero starting at a at time $T-t$ and divide by the hitting time of zero at time T starting at ϵ, i.e.

$$\mathbb{P}_0(H_a(m) \in dt) = \lim_{\epsilon \to 0}\frac{\mathbb{Q}_\epsilon(H_a(W) \wedge H_0(W) \in dt, H_a(W) < H_0(W))\mathbb{Q}_a(H_0(W) \in d(T-t))}{\mathbb{Q}_\epsilon(H_0(W) \in dT)}. \tag{3.8}$$

We know the hitting time distribution for a Brownian motion absorbed at zero [see e.g. Karatzas and Shreve, 1991, p.100] to be

$$\mathbb{Q}_x(H_a(W) \in du, H_a(W) < H_0(W)) = \frac{1}{\sqrt{2\pi u^3}} \sum_{n=-\infty}^{\infty} (2na + a - x) e^{-\frac{(2na+a-x)^2}{2u}} du.$$

Lemma 3.4.1. *The hitting time of a Brownian excursion starting at* $0 < x < a$ *is*

$$\mathbb{P}_x(H_a(m) \in dt) = \frac{a}{x} \sqrt{\frac{T^3}{2\pi t^3 (T-t)^3}} e^{-\frac{a^2}{2(T-t)} + \frac{x^2}{2T}} \sum_{n=-\infty}^{\infty} (2na + a - x) e^{-\frac{(2na+a-x)^2}{2t}}.$$

Proof. This lemma is a generalization of Theorem 3.4.1 with the starting position of the excursion not being fixed to zero. We will see later that this result yields Theorem 3.4.1 by letting x approach zero. But in the meantime we take it as a separate result.

$$
\begin{aligned}
&\mathbb{P}_x(H_a(m) \in dt) \\
&\quad = \frac{\mathbb{Q}_x(H_a(W) \wedge H_0(W) \in dt, H_a(W) < H_0(W))\mathbb{Q}_a(H_0(W) \in d(T-t))}{\mathbb{Q}_x(H_0(W) \in dT)} \\
&\quad = \frac{\frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na + a - x)e^{-\frac{(2na+a-x)^2}{2t}} \frac{ae^{-\frac{a^2}{2(T-t)}}}{\sqrt{2\pi(T-t)^3}}}{\frac{x}{\sqrt{2\pi T^3}} e^{-\frac{x^2}{2T}}} dt \\
&\quad = \frac{a}{x}\sqrt{\frac{T^3}{2\pi t^3 (T-t)^3}} e^{-\frac{a^2}{2(T-t)} + \frac{x^2}{2T}} \sum_{n=-\infty}^{\infty} (2na + a - x)e^{-\frac{(2na+a-x)^2}{2t}}
\end{aligned}
$$

□

Hence, taking the limit of x approaching zero yields the hitting time of the excursion.

$$
\begin{aligned}
\mathbb{P}_0(H_a(m) \in dt) &= \frac{ae^{-\frac{a^2}{2(T-t)}}}{\sqrt{2\pi(T-t)^3}} \lim_{x \to 0} \frac{\frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na + a - x)e^{-\frac{(2na+a-x)^2}{2t}}}{\frac{x}{\sqrt{2\pi T^3}} e^{-\frac{x^2}{2T}}} dt \\
&= a\sqrt{\frac{T^3}{2\pi t^5 (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \sum_{k=-\infty}^{\infty} \left((2k+1)^2 a^2 - t\right) e^{-\frac{(2k+1)^2 a^2}{2t}} dt.
\end{aligned}
$$

This finalizes the proof of Theorem 3.4.1 by splitting the infinite sum into positive and negative parts.

3.4.3 Proof using conditioned Martingales

The similarity in dynamics suggests the change of measure from Brownian excursion to $BES(3)$. Hernandez-del Valle [2011] (Theorem 2.9) provides us with the relevant formula. Compare the dynamics of the $BES(3)$ process, which is known as

$$dR_t = \frac{1}{R_t}dt + dW_t$$

[see e.g. Borodin and Salminen, 2002, Chapter IV.6. Bessel processes], with dynamics (3.2).

Theorem 3.4.2 (Hernandez-del-Valle)**.** *Let W denote a Brownian motion and let $k^{(1)}$ and $k^{(2)}$ both satisfy the heat equation $-\frac{\partial k^{(1)}}{\partial t} = \frac{1}{2}\frac{\partial^2 k^{(1)}}{\partial x^2}$ and $-\frac{\partial k^{(2)}}{\partial t} = \frac{1}{2}\frac{\partial^2 k^{(2)}}{\partial x^2}$. We assume that we can write the diffusions X, Z and Y with probability measures $\mathbb{P}$, $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ respectively as*

$$\begin{aligned}
dX_t &= \left(\frac{1}{k^{(2)}(t,X_t)}\frac{\partial k^{(2)}(t,X_t)}{\partial x} + \frac{1}{k^{(1)}(t,X_t)}\frac{\partial k^{(1)}(t,X_t)}{\partial x}\right)dt + dW_t,\\
dZ_t &= \frac{1}{k^{(2)}(t,X_t)}\frac{\partial k^{(2)}(t,X_t)}{\partial x}dt + dW_t,\\
dY_t &= dW_t.
\end{aligned}$$

Then assuming $\frac{1}{k^{(2)}(s,z)}\frac{\partial k^{(2)}(s,z)}{\partial z}\frac{1}{k^{(1)}(s,z)}\frac{\partial k^{(1)}(s,z)}{\partial z} \neq 0$ for some strip $\mathbb{R}_+ \times \mathbb{R}$ the following identity holds

$$\begin{aligned}
\mathbb{P}_0(X_t \in A) &= \mathbb{E}_0^{\mathbb{Q}}\left(k^{(1)}(t,Z_t)e^{-\int_0^t \frac{1}{k^{(2)}(s,Z_s)}\frac{\partial k^{(2)}(s,Z_s)}{\partial z}\frac{1}{k^{(1)}(s,Z_s)}\frac{\partial k^{(1)}(s,Z_s)}{\partial z}ds}\mathbf{1}_{Z_t\in A}\right)\\
&= \mathbb{E}_0^{\tilde{\mathbb{Q}}}\left(k^{(1)}(t,Y_t)k^{(2)}(t,Y_t)e^{-\int_0^t \frac{1}{k^{(2)}(s,Y_s)}\frac{\partial k^{(2)}(s,Y_s)}{\partial y}\frac{1}{k^{(1)}(s,Y_s)}\frac{\partial k^{(1)}(s,Y_s)}{\partial y}ds}\mathbf{1}_{Y_t\in A}\right).
\end{aligned}$$

The dynamics of the $BES(3)$ process is given by $dR_t = \frac{1}{R_t}dt + dW_t$ [see e.g. Borodin and Salminen, 2002, Chapter IV.6. Bessel processes].

Take $k^{(1)}(t,x) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}}$ and $k^{(2)}(t,x) = x$. Both functions satisfy the heat equation and we can write

$$dm_t = \left(\frac{1}{k^{(2)}(t,m_t)} \frac{\partial k^{(2)}(t,m_t)}{\partial m} + \frac{1}{k^{(1)}(t,m_t)} \frac{\partial k^{(1)}(t,m_t)}{\partial m} \right) dt + dW_t,$$
$$dR_t = \frac{1}{k^{(2)}(t,R_t)} \frac{\partial k^{(2)}(t,R_t)}{\partial r} dt + dW_t,$$

and with Theorem 3.4.2 the change of measure becomes

$$\begin{aligned} \left.\frac{d\mathbb{P}}{d\tilde{\mathbb{Q}}}\right|_{\mathcal{F}_t} &= k^{(1)}(t,R_t) e^{-\int_0^t \frac{1}{k^{(2)}(s,R_s)} \frac{\partial k^{(2)}(s,R_s)}{\partial r} \frac{1}{k^{(1)}(s,R_s)} \frac{\partial k^{(1)}(s,R_s)}{\partial r} ds} \\ &= \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{R_t^2}{2(T-t)}} e^{\int_0^t \frac{1}{T-s} ds} \\ &= \frac{T}{\sqrt{2\pi(T-t)^3}} e^{-\frac{R_t^2}{2(T-t)}}, \end{aligned}$$

which is indeed a local martingale up to time T.

We now calculate the first hitting time of the Brownian excursion using this change of measure and a normalization factor:

$$
\begin{aligned}
\mathbb{P}_0(H_a(m) < t) &= \frac{\mathbb{E}_0^{\tilde{\mathbb{Q}}}\left(\frac{T}{\sqrt{2\pi(T-t)^3}} e^{-\frac{R_t^2}{2(T-t)}} \mathbf{1}_{H_a(R)<t}\right)}{\mathbb{E}_0^{\tilde{\mathbb{Q}}}\left(\frac{T}{\sqrt{2\pi(T-0)^3}} e^{0}\right)} \\
&= \frac{\sqrt{2\pi T^3}}{T} \mathbb{E}_0^{\tilde{\mathbb{Q}}}\left[\mathbb{E}_0^{\tilde{\mathbb{Q}}}\left(\frac{T}{\sqrt{2\pi(T-t)^3}} e^{-\frac{R_t^2}{2(T-t)}} \middle| \mathcal{F}_{H_a(R)}\right) \mathbf{1}_{H_a(R)<t}\right] \\
&= \frac{\sqrt{2\pi T^3}}{T} \mathbb{E}_0^{\tilde{\mathbb{Q}}}\left[\frac{T}{\sqrt{2\pi(T-H_a(R))^3}} e^{-\frac{a^2}{2(T-H_a(R))}} \mathbf{1}_{H_a(R)<t}\right] \\
&= \int_0^t \sqrt{\frac{T^3}{(T-s)^3}} e^{-\frac{a^2}{2(T-s)}} \tilde{\mathbb{Q}}_0(H_a(R) \in ds).
\end{aligned}
$$

Hence, with (3.4) we derive

$$
\mathbb{P}_0(H_a(m) \in dt) = a\sqrt{\frac{T^3}{2\pi t^5 (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \sum_{k=-\infty}^{\infty} \left((2k+1)^2 a^2 - t\right) e^{-\frac{(2k+1)^2 a^2}{2t}} dt. \tag{3.9}
$$

Splitting the infinite sum yields Theorem 3.4.1. All three proofs explore excursions from different perspectives making use of relations to Brownian motion and $BES(3)$ process.

3.4.4 Density of the Last Passage Time of the Brownian excursion

We now want to use our results on the first hitting time to derive the probability of the last passage time of the Brownian excursion. Since they look into the future and are not stopping times the standard theorems in martingale theory cannot be applied and therefore they are much harder to handle. Relying on the

Bessel bridge representation formalized by Pitman [1999a], we derive the following Corollary.

Corollary 3.4.1. *The distribution of the last passage time of the Brownian excursion is given by*

$$\mathbb{P}_0(G_a(m) \in dt) = a\sqrt{\frac{2T^3}{\pi(T-t)^5 t^3}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \left((2k+1)^2 a^2 - T + t\right) e^{-\frac{(2k+1)^2 a^2}{2(T-t)}} dt$$

Proof. The excursion starts and ends at zero. Pitman [1999a] allows the following representation for $0 \leq t \leq T$

$$m_t = \sqrt{\left(W_1^{br}(t)\right)^2 + \left(W_2^{br}(t)\right)^2 + \left(W_3^{br}(t)\right)^2}, \tag{3.10}$$

where W_i^{br}, $i = 1, 2, 3$, are three independent Brownian bridges. Hence, the Brownian excursion can be identified with the three-dimensional Bessel bridge from zero to zero. Hence, m is reversible. Denote by $\tilde{m}_t := m_{T-t}$ the time-reversed excursion with $\tilde{m}_0 = \tilde{m}_T = 0$ and let $H_a(\tilde{m})$ be the first hitting time of level a of the reversed process. Let $G_a(m) = \sup\{t \leq T | m_t = a\}$ be the last passage time of the original excursion. Then, trivially

$$\mathbb{P}_0(G_a(m) \in dt) = \mathbb{P}_0(H_a(\tilde{m}) \in dt),$$

which completes the proof after applying the result from Theorem 3.4.1 for $T - t$.

□

3.4.5 Density of the Maximum of the Brownian excursion

Having derived the law of first hitting time, we can deduce the law of the maximum of the Brownian excursion. Our outcome coincides with well known results by Chung [1975], Bertoin, Chaumont, and Pitman [2003], Kobayashi, Izumi, and Katori [2008] and others. However, our approach differs in terms of derivation and only uses standard theorems in probability theory.

Corollary 3.4.2. *The distribution of the maximum of the Brownian excursion of length T is given by*

$$\mathbb{P}_0(\max_{0\le s\le T} m_s \ge a) = 2\sum_{k=1}^{\infty}\left(\frac{(2ak)^2}{T}-1\right)e^{-\frac{(2ak)^2}{2T}},$$

and its density is

$$\mathbb{P}_0(\max_{0\le s\le T} m_s \in da) = 8\sum_{k=1}^{\infty}\left(\frac{4a^3k^4}{T^2}-\frac{3ak^2}{T}\right)e^{-\frac{(2ak)^2}{2T}}da. \tag{3.11}$$

Proof. We integrate up the hitting time density of the excursion and derive the distribution of the maximum of the excursion, i.e.

$$\int_0^T \mathbb{P}_0(H_a(m)\in dt) = \mathbb{P}_0(H_a(m)\le T) = \mathbb{P}_0(\max_{0\le s\le T} m_s \ge a).$$

Let $b=(2k+1)a$, then

$$\int_0^T \mathbb{P}_0(H_a(m) \in dt) = \sqrt{2\pi T^3} \sum_{k=-\infty}^{\infty} \int_0^T \frac{a}{\sqrt{2\pi (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \frac{b^2}{\sqrt{2\pi t^5}} e^{-\frac{b^2}{2t}} - \frac{a}{\sqrt{2\pi (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt. \tag{3.12}$$

The r.h.s. of (3.12) can be written as the convolution of the functions

$$f^{(1)}(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}, \qquad f^{(2)}(t) = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}}.$$

The Laplace transform of the convolution

$$f^{(3)}(T) := (f^{(1)} * f^{(2)})(T)$$

is the product $F^{(3)}(z) = F^{(1)}(z)F^{(2)}(z)$. The Laplace transform of the Inverse Gaussian distribution is given by

$$F^{(1)}(z) = \int_0^\infty e^{-zt} f^{(1)}(t) dt = e^{-a\sqrt{2z}}, \qquad F^{(2)}(z) = e^{-b\sqrt{2z}},$$

and

$$F^{(3)}(z) = e^{-(a+b)\sqrt{2z}}$$

Hence, the inverse Laplace transform $f^{(3)}(T)$ becomes

$$(f^{(1)} * f^{(2)})(T) = \frac{a+b}{\sqrt{2\pi T^3}} e^{-\frac{(a+b)^2}{2T}}. \tag{3.13}$$

Differentiating both sides of equation (3.13) with respect to b yields

$$\frac{d}{db}\int_0^T \frac{a}{\sqrt{2\pi(T-t)^3}}e^{-\frac{a^2}{2(T-t)}}\frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}\,dt = \frac{d}{db}\frac{a+b}{\sqrt{2\pi T^3}}e^{-\frac{(a+b)^2}{2T}}$$

$$\int_0^T \frac{ae^{-\frac{a^2}{2(T-t)}}}{\sqrt{2\pi(T-t)^3}}\left(\frac{b^2}{\sqrt{2\pi t^5}}-\frac{1}{\sqrt{2\pi t^3}}\right)e^{-\frac{b^2}{2t}}\,dt = \left(\frac{(a+b)^2}{\sqrt{2\pi T^5}}-\frac{1}{\sqrt{2\pi T^3}}\right)e^{-\frac{(a+b)^2}{2T}}. \tag{3.14}$$

Notice that the l.h.s. of (3.14) is exactly the same as the convolution on the r.h.s. of (3.12). With this trick we can now calculate the probability of the maximum of the excursion.

$$\begin{aligned}\int_0^T \mathbb{P}_0(H_a(m)\in dt) &= \sum_{k=-\infty}^{\infty}\left(\frac{(a+b)^2}{T}-1\right)e^{-\frac{(a+b)^2}{2T}}\\ &= 2\sum_{k=1}^{\infty}\left(\frac{(2ak)^2}{T}-1\right)e^{-\frac{(2ak)^2}{2T}}.\end{aligned}$$

This concludes the first part of the Corollary. For the density we differentiate with respect to a. Straightforward calculation yields

$$\begin{aligned}\mathbb{P}_0(\max_{0\le s\le T} m_s\in da) = \frac{d}{da}\mathbb{P}_0(\max_{0\le s\le T} m_s\le a) &= \frac{d}{da}\left(1+2\sum_{k=1}^{\infty}\left(1-\frac{(2ak)^2}{T}\right)e^{-\frac{(2ak)^2}{2T}}\right)\\ &= 8\sum_{k=1}^{\infty}\left(\frac{4a^3k^4}{T^2}-\frac{3ak^2}{T}\right)e^{-\frac{(2ak)^2}{2T}}.\end{aligned}$$

□

As a generalization we also compute the probability of the maximum for an excursion starting at $x\in(0,a)$. This coincides with Corollary 3.4.2 by letting x approach zero and applying L'Hôpital's rule.

Corollary 3.4.3. *The law of the maximum of a Brownian excursion of length T starting at x, $0 < x < a$, is*

$$\mathbb{P}_x(\max_{0\le s\le T} m_s \ge a) = \sum_{n=0}^{\infty} \left(\frac{4na}{x} \sinh\left(\frac{2nax}{T}\right) - 2\cosh\left(\frac{2nax}{T}\right)\right) e^{-\frac{(2na)^2}{2T}},$$

and its density is

$$\mathbb{P}_x(\max_{0\le s\le T} m_s \in da) = \sum_{n=0}^{\infty} \frac{4n}{Tx} \left((T - 4a^2n^2 - x^2)\sinh\left(\frac{2nax}{T}\right) + 4nax\cosh\left(\frac{2nax}{T}\right)\right) e^{-\frac{(2na)^2}{2T}}.$$

Proof. From Lemma 3.4.1 we derive

$$\begin{aligned}
\mathbb{P}_x(\max_{0\le s\le T} m_s \ge a) &= \int_0^T \mathbb{P}_x(H_a(m) \in dt) \\
&= \int_0^T \frac{a}{x}\sqrt{\frac{T^3}{2\pi t^3 (T-t)^3}} e^{-\frac{a^2}{2(T-t)} + \frac{x^2}{2T}} \sum_{n=-\infty}^{\infty} (2na + a - x) e^{-\frac{(2na+a-x)^2}{2t}} \\
&= \frac{\sqrt{2\pi T^3}}{x} e^{\frac{x^2}{2T}} \sum_{n=-\infty}^{\infty} \int_0^T \frac{a}{\sqrt{2\pi (T-t)^3}} e^{-\frac{a^2}{2(T-t)}} \frac{2na + a - x}{\sqrt{2\pi t^3}} e^{-\frac{(2na+a-x)^2}{2t}} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{2na}{x} - 1\right) e^{-\frac{(2na-x)^2}{2T} + \frac{x^2}{2T}} \\
&= \sum_{n=0}^{\infty} e^{-\frac{(2na)^2}{2T}} \left(\left(\frac{2na}{x} - 1\right) e^{\frac{2nax}{T}} - \left(\frac{2na}{x} + 1\right) e^{-\frac{2nax}{T}} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{4na}{x} \sinh\left(\frac{2nax}{T}\right) - 2\cosh\left(\frac{2nax}{T}\right)\right) e^{-\frac{(2na)^2}{2T}}.
\end{aligned}$$

For the density we differentiate with respect to a

$$
\begin{aligned}
\mathbb{P}_x(\max_{0\le s\le T} m_s \in da) &= \frac{d}{da}\left(1 - \mathbb{P}_0\left(\max_{0\le s\le T} m_s \ge a\right)\right)\\
&= \sum_{n=-\infty}^{\infty} \frac{2n}{Tx}(4a^2n^2 - 4nax - T + x^2)e^{-\frac{(2na)^2-4nax}{2T}}\\
&= \sum_{k=0}^{\infty} \frac{4n}{Tx}\left((T - 4a^2n^2 - x^2)\sinh\left(\frac{2nax}{T}\right) + 4nax\cosh\left(\frac{2nax}{T}\right)\right)e^{-\frac{(2na)^2}{2T}}.
\end{aligned}
$$

□

3.5 Joint law of the Maximum and the Time it is achieved by the Brownian excursion

We recall the definition of the first time the maximum of the Brownian excursion is achieved,

$$\theta_T(m) = \inf\{s \le T | m_s = \bar{m}_T\}.$$

Theorem 3.5.1. *The joint probability of maximum and the time it is achieved for Brownian excursion is given by*

$$
\begin{aligned}
&\mathbb{P}_0\left(\bar{m}_T \in da, \theta_T(m) \in ds\right)\\
&= 4\sqrt{\frac{T^3}{2\pi s^5 (T-s)^5}} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left((2n+1)^2a^2 - s\right) e^{-\frac{(2n+1)^2a^2}{2s}} \times\\
&\qquad \times \left((2k+1)^2a^2 - (T-s)\right) e^{-\frac{(2k+1)^2a^2}{2(T-s)}} da\ ds. \quad (3.15)
\end{aligned}
$$

Before we prove this proposition, we firstly derive the joint law of an excursion starting at x, $0 < x < a$, and let x approach zero in a next step. We emphasize the speed of convergence of the infinite sums, which will be demonstrated in Appendix 7.1, Table (7.2). Numerically, we will not need to compute more than three terms to get precision up to four decimal places.

Lemma 3.5.1. *For the Brownian excursion starting at x, $0 < x < a$, we find the joint distribution to be*

$$\mathbb{P}_x\left(\bar{m}_T \in da, \theta_T(m) \in ds\right) =$$

$$= \sqrt{\frac{2T^3}{\pi s^3 (T-s)^5}} \frac{1}{x e^{-\frac{x^2}{2T}}} \sum_{k=0}^{\infty} \left((2k+1)^2 a^2 - (T-s)\right) e^{-\frac{(2k+1)^2 a^2}{2(T-s)}} \times$$
$$\times \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2 a^2 + x^2}{2s}} \left[(2n+1) a \sinh\left(\frac{(2n+1)ax}{s}\right) - x \cosh\left(\frac{(2n+1)ax}{s}\right)\right] da\ ds. \tag{3.16}$$

Proof of Lemma 3.5.1. We recall Pitman's Bessel bridge representation [Pitman, 1999a],

$$m_t = \sqrt{\left(W_1^{br}(t)\right)^2 + \left(W_2^{br}(t)\right)^2 + \left(W_3^{br}(t)\right)^2}. \tag{3.17}$$

Let R be a $BES(3)$ process with probability measure $\tilde{\mathbb{Q}}$, then the well-known change of measure allows us to represent it as a Brownian motion W,

$$\tilde{\mathbb{Q}}_x(A, R_t \in dz) = \frac{z}{x} \mathbb{Q}_x(A, W_t \in dz, H_0(W) > t) \tag{3.18}$$

for any measurable event A.

Imhof [1984] shows that the joint distribution for a standard Brownian motion is given by

$$\mathbb{Q}_x(\bar{W}_t \in dy, \theta_t(W) \in ds, W_t \in dz) = 2\mathbb{Q}_x(H_y(W) \in ds)\mathbb{Q}_z(H_y(W) \in dt-s)dy\ dz, \tag{3.19}$$

where $H_y(W)$ denotes the first hitting time of y for a Brownian motion and $\bar{W}_t$ its running maximum up to time t. Together with equation (3.18) it becomes

$$\tilde{\mathbb{Q}}_x(\bar{R}_t \in dy, \theta_t(R) \in ds, R_t \in dz) = \frac{z}{x}\mathbb{Q}_x(\bar{W}_t \in dy, \theta_t(W) \in ds, W_t \in dz) =$$
$$\frac{2z}{x}\mathbb{Q}_x(H_y(W) \in ds, H_0(W) > H_y(W))\mathbb{Q}_z(H_y(W) \in dt-s, H_0(W) > H_y(W))dy\ dz, \tag{3.20}$$

where $\bar{R}$ denotes the running maximum of the Bessel process. The joint density of maximum and the time it is achieved for Brownian excursion therefore decomposes into

$$\begin{aligned} \mathbb{P}_x\left(\bar{m}_T \in da, \theta_T(m) \in ds\right) &= \tilde{\mathbb{Q}}_x(\bar{R}_T \in da, \theta_T(R) \in ds | R_T = 0) \\ &= \lim_{\epsilon \to 0} \frac{\tilde{\mathbb{Q}}_x(\bar{R}_T \in da, \theta_T(R) \in ds, R_T \in d\epsilon)}{\tilde{\mathbb{Q}}_x(R_T \in d\epsilon)}. \end{aligned} \tag{3.21}$$

The hitting time distribution for a Brownian motion absorbed at zero is commonly known [see e.g. Karatzas and Shreve, 1991, p.100] to be

$$\mathbb{Q}_x(H_y(W) \in ds, H_y(W) < H_0(W)) = \frac{1}{\sqrt{2\pi s^3}} \sum_{n=-\infty}^{\infty} (2ny + y - x)e^{-\frac{(2ny+y-x)^2}{2s}} ds.$$

Hence, with equation (3.20) the numerator of (3.21) becomes

$$\frac{2\epsilon}{x}\sum_{n=-\infty}^{\infty}\frac{2na+a-x}{\sqrt{2\pi s^3}}e^{-\frac{(2na+a-\delta)^2}{2s}}\sum_{k=-\infty}^{\infty}\frac{2ka+a-\epsilon}{\sqrt{2\pi(T-s)^3}}e^{-\frac{(2ka+a-\epsilon)^2}{2(T-s)}}ds\;da\;d\epsilon.$$

For the denominator of (3.21), we know the density of the $BES(3)$ process from equation (3.6) to be

$$\frac{\epsilon}{x\sqrt{2\pi T}}\left(e^{-\frac{(\epsilon-x)^2}{2t}}-e^{-\frac{(\epsilon+x)^2}{2t}}\right)d\epsilon.$$

Applying L'Hôpital's rule on ϵ leaves us with

$$\begin{aligned}
&\mathbb{P}_x\left(\bar{m}_T\in da,\theta_T(m)\in ds\right)=\sqrt{\frac{T^3}{2\pi s^3(T-s)^5}}\frac{1}{xe^{-\frac{x^2}{2T}}}\times\\
&\quad\times\sum_{n=-\infty}^{\infty}(2na+a-x)e^{-\frac{(2na+a-x)^2}{2s}}\sum_{k=-\infty}^{\infty}\left((2k+1)^2a^2-(T-s)\right)e^{-\frac{(2k+1)^2a^2}{2(T-s)}}ds\;da\\
&=\sqrt{\frac{2T^3}{\pi s^3(T-s)^5}}\frac{2}{xe^{-\frac{x^2}{2T}}}\sum_{k=0}^{\infty}\left((2k+1)^2a^2-(T-s)\right)e^{-\frac{(2k+1)^2a^2}{2(T-s)}}\times\\
&\quad\times\sum_{n=0}^{\infty}e^{-\frac{(2n+1)^2a^2+x^2}{2s}}\left[(2n+1)a\sinh\left(\frac{(2n+1)ax}{s}\right)-x\cosh\left(\frac{(2n+1)ax}{s}\right)\right]
\end{aligned}$$

This yields the claim of the Lemma.

□

Proof of Theorem 3.5.1. For the Brownian excursion pinned at both endpoints to zero, we let the start point x from Lemma 3.5.1 approach zero, i.e. equation (3.21) yields

$$\mathbb{P}_0\left(\bar{m}_T \in da, \theta_T(m) \in ds\right) = \tilde{\mathbb{Q}}_0(\bar{R}_T \in da, \theta_T(R) \in ds | R_T = 0)$$
$$= \lim_{x\to 0}\lim_{\epsilon\to 0} \frac{\tilde{\mathbb{Q}}_x(\bar{R}_T \in da, \theta_T(R) \in ds, R_T \in d\epsilon)}{\tilde{\mathbb{Q}}_x(R_T \in d\epsilon)}. \quad (3.22)$$

Using the result from Lemma 3.5.1 we apply L'Hôpital's rule in order to find the limit of x approaching zero,

$$\mathbb{P}_0\left(\bar{m}_T \in da, \theta_T(m) \in ds\right) = \lim_{x\to 0} \sqrt{\frac{T^3}{2\pi s^3 (T-s)^5}} \frac{1}{xe^{-\frac{\delta^2}{2T}}} \times$$
$$\times \sum_{n=-\infty}^{\infty} (2na + a - x) e^{-\frac{(2na+a-x)^2}{2s}} \sum_{k=-\infty}^{\infty} \left((2k+1)^2 a^2 - (T-s)\right) e^{-\frac{(2k+1)^2 a^2}{2(T-s)}} ds\ da$$
$$= 4\sqrt{\frac{T^3}{2\pi s^5 (T-s)^5}} \sum_{n=0}^{\infty} \left((2n+1)^2 a^2 - s\right) e^{-\frac{(2n+1)^2 a^2}{2s}} \times$$
$$\times \sum_{k=0}^{\infty} \left((2k+1)^2 a^2 - (T-s)\right) e^{-\frac{(2k+1)^2 a^2}{2(T-s)}} ds\ da.$$

□

Remark 3.5.1. Integrating over s using Laplace transforms yields the density of the maximum of the Brownian excursion which coincides with equation (3.11)

$$\mathbb{P}_0\left(\bar{m}_T \in da\right) = \int_{s=0}^{T} \mathbb{P}_0\left(\bar{m}_T \in da, \theta_T(m) \in ds\right)$$

$$= \sqrt{\frac{2\pi T^3}{s^2 (T-s)^2}} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{s(T-s)}{\sqrt{2\pi T^3}} \left(\frac{8(n+k+1)^3 a^3}{T^2} - \right.$$
$$\left. - \frac{6(n+k+1)}{T} \right) e^{-\frac{4(n+k+1)^2 a^2}{2T}} da$$

$$= 8\sum_{k=1}^{\infty}\left(\frac{4a^3k^4}{T^2} - \frac{3ak^2}{T}\right)e^{-\frac{(2ak)^2}{2T}}da.$$

As an immediate result we derive the density of the time the maximum is achieved.

Corollary 3.5.1. *The distribution of the time the maximum is achieved by the Brownian excursion is given by*

$$\mathbb{P}_0\left(\theta_T(m) \in ds\right) = 6\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2n+1)^2(2k+1)^2\sqrt{\frac{s^2T^3}{[(2n+1)^2(T-s)+(2k+1)^2s]^5}}\, ds.$$

Proof. For the distribution of the time of the maximum we integrate (3.15) over a.

$$\mathbb{P}_0\left(\theta_T(m) \in ds\right) = \int_{a=0}^{\infty}\mathbb{P}_0\left(\bar{m}_T \in da, \theta_T(m) \in ds\right)$$

$$= 4\sqrt{\frac{T^3}{2\pi s^5(T-s)^5}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\int_{a=0}^{\infty}\left((2n+1)^2a^2 - s\right)e^{-\frac{(2n+1)^2a^2}{2s}}\times$$
$$\left((2k+1)^2a^2-(T-s)\right)e^{-\frac{(2k+1)^2a^2}{2(T-s)}}da\, ds. \quad (3.23)$$

We will be using moments of Gaussian distribution to evaluate the integral on the r.h.s, which we denote by (A). Let $c^2 := (2n+1)^2$ and $d^2 := (2k+1)^2$.

$$(A) = \int_0^{\infty}\left[c^2d^2a^4 - (c^2(T-s)+d^2s)a^2 + s(T-s)\right]e^{-\frac{(c^2(T-s)+d^2s)a^2}{2s(T-s)}}da$$

Let $b := \frac{s(T-s)}{c^2(T-s)+d^2s}$ and with substitution $a = \sqrt{b}x$ we have

$$(A) = \int_0^\infty \sqrt{b} \left[c^2 d^2 b^2 x^4 - \underbrace{(c^2(T-s) + d^2 s) b}_{s(T-s)} x^2 + s(T-s) \right] e^{-\frac{x^2}{2}} dx =$$

$$\sqrt{\frac{\pi}{2}} \left(c^2 d^2 b^{5/2} \mathbb{E}(N(0,1)^4) - s(T-s) b^{1/2} \mathbb{E}(N(0,1)^2) + s(T-s) b^{1/2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right),$$

where $N(0,1)$ denotes a normal distributed random variable and $\mathbb{E}\left(N(0,1)^m\right)$ describe the moments. We know $\mathbb{E}\left(N(0,1)^2\right) = 1$ and $\mathbb{E}\left(N(0,1)^4\right) = 3$, hence,

$$(A) = 3c^2 d^2 b^{5/2} \sqrt{\frac{\pi}{2}}.$$

Inserting (A) into equation (3.23) yields the proposition.

□

3.6 Application to Default Probability of Zero-Coupon Bonds

Let $(\tilde{m}_t)_{0 \le t \le T}$ denote the price process of a risky zero-coupon bond paying \$1 at maturity T, if no default occurred and \$0 otherwise. Since the final state of a Brownian motion is uncertain, it is not suitable for modelling an asset, where the final value is known a priori. A bond gets redeemed at the par value at maturity, hence, the stochastic process has to be tied down to the final state. This gives us the motivation to use the Brownian excursion to model the bond price process. We notice here that by reflecting and shifting $\tilde{m}$ starting at zero and ending at 1 has the same distribution as the Brownian excursion m starting at 1 and tied down to zero with dynamics

$$dm_t = \left(\frac{1}{m_t} - \frac{m_t}{T-t}\right) dt + dW_t,$$
$$m_0 = 1.$$

We modify the definition of default occurring if the bond price process $\tilde{m}$ goes below a certain barrier $-b \leq 0$ and the minimum is being reached before a specified time $u \leq T$, in which case regulators are advised to take action. This is highly adaptable to the situation where a company does not default, if it is short of reserve for a brief time period of time, but default happens, when the minimum is reached too close to maturity.

The time at which the minimum is reached, is denoted by $\tilde{\theta}_T(\tilde{m}) := \inf\{0 \leq s \leq T | \tilde{m}_s = \min_{0<r\leq T} \tilde{m}_r\}$. This is equivalent to saying that default occurs as soon as the maximum of the Brownian excursion starting at 1 reaches $1+b$ before time u. Here is where the importance of knowing the default probability comes in. The probability of default can now be calculated via Lemma 3.5.1.

$$\begin{aligned}
&\mathbb{P}_0(\min_{0<s\leq T} \tilde{m}_s < -b, \tilde{\theta}_T(\tilde{m}) < u) = \mathbb{P}_1(\bar{m}_T > 1+b, \theta_T(m) < u) \\
&= \int_{a=1+b}^{\infty} \int_{s=0}^{u} \sqrt{\frac{2T^3}{\pi s^3 (T-s)^5}} \frac{2}{e^{-\frac{1}{2T}}} \sum_{k=0}^{\infty} \left((2k+1)^2 a^2 - (T-s)\right) e^{-\frac{(2k+1)^2 a^2}{2(T-s)}} \times \\
&\times \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2 a^2 + 1}{2s}} \left[(2n+1)a \sinh\left(\frac{(2n+1)a}{s}\right) - \cosh\left(\frac{(2n+1)a}{s}\right)\right] ds\, da.
\end{aligned} \tag{3.24}$$

Numerical computations can be found in Appendix 7.1.

Chapter 4

Joint distribution of Parisian, Occupation and Local times of Brownian motion

This chapter studies the joint distribution of Parisian, occupation and local times for Brownian motion. Relying on Lévy's representation of drawdown processes, we find explicit expressions for the Laplace transform of Parisian drawdown times which can be exploited for pricing innovative options. As applications we introduce the Parisian Crash option and the Parisian Lookback option under the Black-Scholes framework.

4.1 Introduction

Pricing derivatives in the Black-Scholes framework rely on the distribution of Brownian functionals. Familiar functionals such as the first hitting time or the maximum have been well studied and used for pricing Barrier or Lookback options. In this chapter we concentrate on Parisian-style options.

The key in pricing Parisian options lies in deriving the distribution of the time spent above or below a certain barrier. In the case of consecutive Parisian options we are interested in the excursion time, whereas in the cumulative Parisian case we are interested in the occupation time, which is the summation of all excursion times above or below the barrier up to a time t. Occupation times are also fundamental for pricing α-Quantile options (see Akahori [1995], Dassios [1995]). As in Chesney, Jeanblanc-Picqué, and Yor [1997], Dassios and Wu [2009a] we reduce the problem to finding the Laplace transform of the first time the excursion exceeds the option window, which we call the Parisian time. Chesney, Jeanblanc-Picqué, and Yor [1997] relied on Brownian meander and the Azéma martingale, which have the restriction of relying heavily on the properties of the Brownian motion, making the results inflexible for extension. We do not rely on excursion theory techniques, but rather derive Laplace transforms using Brownian perturbation. Applying our result to risk management, we consider contracts on drawdown processes which have come into focus a few years ago. Vecer [2006], Cheridito, Nikeghbali, and Platen [2012] and Carr, Zhang, and Hadjiliadis [2011] introduce methods to control the maximum drawdown by proposing Vanilla or Barrier options as hedges. Vecer, Novotny, and Pospisil [2006] discuss techniques for relative drawdowns and coin the term Crash option. Yamamoto, Sato, and Takahashi [2010] find analytical approximation formulae for drawdown options in a stochastic volatility environment. Recently, Zhang [2015] considers both drawdown and drawup times and finds probabilities of one preceding the other. As an important byproduct he proves that in the case of the Brownian motion and the three-dimensional Bessel process the distribution of the occupation time is the same as that of the first passage time of a barrier. Kudryavtsev and Levendorskii [2011] derive general formulae for pricing options with barrier and lookback features.

We, on the other hand, include the Parisian criterion and consider contracts to insure the event the relative drawdown process exceeds a certain percentage and the underlying stays below a level for longer than a fixed period of time. In case of this default event our so-called Parisian Crash option pays off \$1 whereas our so-called Parisian Lookback option pays off the difference of relative drawdown and fixed percentage at maturity.

This chapter is structured as follows. In section 4.2 we introduce a two-state semi-Markov model on a perturbed Brownian motion with drift, which has been proposed by Dassios and Wu [2009a]. This perturbed Brownian motion has the same behaviour as a drifted Brownian motion, except it moves toward the other side of the barrier by a jump of size ϵ each time it hits zero, disposing of the difficulty of the origin being regular. The semi-Markov process allows us to define an infinitesimal generator, where the solution of the martingale problem provides us with the triple Laplace transform of Parisian, occupation times and number of downcrossings of the perturbed Brownian motion. The relation between the number of downcrossings by the Brownian motion and the Brownian local time, manifested by Lévy [1948] (proof can be found in Karatzas and Shreve [1991], Theorem 2.21), motivates our study of downcrossings and yields the triple Laplace transform of Parisian, occupation and local times of the drifted Brownian motion in section 4.3. We extend the result to the quadruple Laplace transform of Parisian, occupation, local times and position of the Brownian motion in section 4.4 by applying the Girsanov theorem. Amongst all times studied, we are most interested in the drawdown time which we will relate to Lookback options. In section 4.5 we provide the tool to connect the distribution of local times and drawdown times, which is given in Lévy [1939]. As an application we suggest two classes of equity exotics in section 4.6: Parisian Crash options and Parisian Lookback options with the advantage of smoothening the Delta around the barrier and being less sensitive to price manipulation. These options are extremely innovative in the sense that they not only take the Parisian time but also the drawdown process into consideration, insuring against the event of a market crash. Parisian Crash options get triggered if the relative drawdown at Parisian time exceeds a certain percentage, whereas Parisian Lookback options payout the drawdown at Parisian time.

4.2 Perturbed Brownian motion and the Martingale problem

For a continuous stochastic process Y we define for fixed $t > 0$ the times

$$g_t(Y) = \sup\{s \le t | \text{sgn}(Y_s) \neq \text{sgn}(Y_t)\}, \tag{4.1}$$

$$d_t(Y) = \inf\{s \ge t | \text{sgn}(Y_s) \neq \text{sgn}(Y_t)\}, \tag{4.2}$$

$$\tau^+_{d_1}(Y) = \inf\{t > 0 | (t - g_t(Y))\mathbf{1}_{Y_t > 0} \ge d_1\}, \tag{4.3}$$

$$\tau^-_{d_2}(Y) = \inf\{t > 0 | (t - g_t(Y))\mathbf{1}_{Y_t < 0} \ge d_2\}, \tag{4.4}$$

$$C^1_t(Y) = \int_0^t \mathbf{1}_{Y_s > 0} ds, \tag{4.5}$$

$$C^2_t(Y) = \int_0^t \mathbf{1}_{Y_s < 0} ds. \tag{4.6}$$

The time interval $(d_t(Y), g_t(Y))$ is the excursion interval straddling time t and the time $g_t(Y) - d_t(Y)$ is called excursion time. $C^1_t(Y)$ denotes the occupation time above zero; obviously we have for the occupation time below zero, $C^2_t(Y) = t - C^1_t(Y)$.

Let W^μ, with $W^\mu_t = W_t + \mu t$, be a Brownian motion with drift $\mu \ge 0$ and

$W_0^\mu = 0$, where W is a standard Brownian motion under the probability measure $\mathbb{Q}$. Let N_t denote the number of times the Brownian motion W^μ crosses down from $\epsilon > 0$ to zero by time t. We notice that the origin zero is a regular point of the process, resulting in the occurrence of infinitely many small excursions. In order to counteract this problem, the perturbed Brownian motion $W^{\epsilon,\mu}$ has been introduced by Dassios and Wu [2009a] as follows. Define the sequence of stopping times for $\epsilon > 0$ and $n \in \mathbb{N}_0$,

$$\delta_0 = 0, \tag{4.7}$$

$$\sigma_n = \inf\{t > \delta_n | W_t^\mu = -\epsilon\}, \tag{4.8}$$

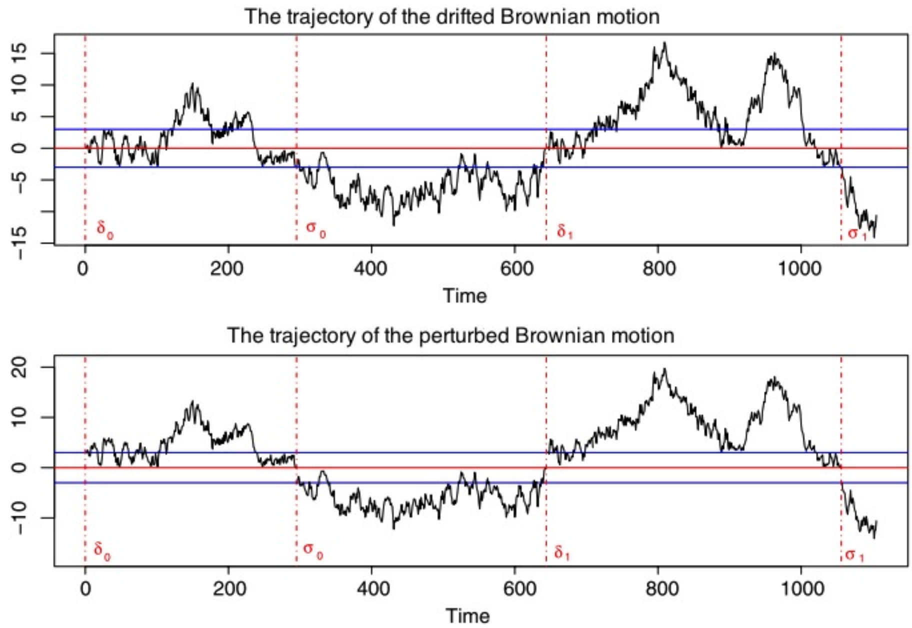
$$\delta_{n+1} = \inf\{t > \sigma_n | W_t^\mu = 0\}. \tag{4.9}$$

Define the perturbed drifted Brownian motion

$$W_t^{\epsilon,\mu} = \begin{cases} W_t^\mu + \epsilon & , \text{if } \delta_n \le t < \sigma_n \\ W_t^\mu & , \text{if } \sigma_n \le t < \delta_{n+1} \end{cases} \tag{4.10}$$

By introducing the jumps of size ϵ towards the other side of zero whenever zero is hit by W^μ we get a process $W^{\epsilon,\mu}$ with a very clear structure of excursions above and below zero, making zero an irregular point. This construction has been introduced by Dassios and Wu [2009a]. See Figure 4.1 for illustration. With the superscript ϵ we denote quantities based on the perturbed process $W^{\epsilon,\mu}$, e.g. $H_b(W^{\epsilon,\mu}) = \inf\{t \ge 0 | W_t^{\epsilon,\mu} = b\}$. By construction we have $W_t^{\epsilon,\mu} \xrightarrow{a.s.} W_t^\mu$ for all $t \ge 0$, as ϵ approaches zero. The quantities defined based on $W_t^{\epsilon,\mu}$ also converge to those of the drifted Brownian motion W_t^μ. This has been proven in Dassios and Wu [2009a], Dassios and Wu [2011a] and Lim [2013].

Figure 4.1: Sample paths of W^{μ} and $W^{\epsilon,\mu}$, see Dassios and Wu [2009a]



It becomes clear that so far we are only concerned about two states, namely above and below zero. We introduce a new process based on $W^{\epsilon,\mu}$ by

$$X_t = \begin{cases} 1 & , \text{ if } W_t^{\epsilon,\mu} > 0 \\ -1 & , \text{ if } W_t^{\epsilon,\mu} < 0. \end{cases} \tag{4.11}$$

Clearly, definitions (4.1), (4.2), (4.3), (4.4)and (4.5) hold similarly for the process X. Define $U_t(X) = t - g_t(X)$ to be the time elapsed in the current state. $(X_t, U_t(X))$ becomes a Markov process. Hence, X is a two state semi-Markov process with state space $\{1, -1\}$, where 1 denotes the process X above zero and

-1 the process below zero. The transition intensities $\lambda_{ij}(u)$ for X satisfy

$$\mathbb{Q}\left(X_{t+\Delta t} = j, i \neq j | X_t = i, U_t(X) = u\right) = \lambda_{ij}(u)\Delta t + o(\Delta t) \tag{4.12}$$

$$\mathbb{Q}\left(X_{t+\Delta t} = i | X_t = i, U_t(X) = u\right) = 1 - \sum_{j \neq i} \lambda_{ij}(u)\Delta t + o(\Delta t) \tag{4.13}$$

for $i, j = 1, -1$. Define the survival probability and transition density by

$$\bar{P}_i(t) = e^{-\int_0^t \sum\limits_{j \neq i} \lambda_{ij}(v)dv}, \tag{4.14}$$

$$p_{ij}(t) = \lambda_{ij}(t)\bar{P}_i(t). \tag{4.15}$$

In particular we have the densities

$$p_{1,-1}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon+\mu t)^2}{2t}}, \qquad p_{-1,1}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon-\mu t)^2}{2t}}. \tag{4.16}$$

In order to simplify notations, we define $\hat{P}_{ij}(\beta)$ and $\tilde{P}_{ij}(\beta)$ to be

$$\hat{P}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} p_{ij}(s)ds, \tag{4.17}$$

$$\tilde{P}_{ij}(\beta) = \int_0^{\infty} e^{-\beta s} p_{ij}(s)ds. \tag{4.18}$$

We consider a bounded function $f : \{1, -1\} \times \mathbb{R}^4 \to \mathbb{R}$. The infinitesimal generator $\mathcal{A}$ is an operator making

$$f(X_t, U_t(X), N_t, t, C_t^1(W^{\epsilon,\mu})) - \int_0^t \mathcal{A}(X_s, U_s(X), N_s, s, C_s^1(W^{\epsilon,\mu}))ds$$

a martingale. Throughout the chapter we shall use the shortcuts $f_i(u,n,t,c) = f(i,u,n,t,c)$ and $\mathcal{A}_{X_t}(U_t(X), N_t, t, C_t^1(W^{\epsilon,\mu})) = \mathcal{A}(X_t, U_t(X), N_t, t, C_t^1(W^{\epsilon,\mu}))$.

Hence, solving $\mathcal{A}f \equiv 0$ will provide us with martingales of the form $f_{X_t}(U_t(X), N_t, t, C_t^1(W^{\epsilon,\mu}))$, to which we can apply the optional sampling theorem in order to obtain the Laplace transforms of interest. We have for the generator

$$\begin{aligned}
\mathcal{A}f_1(u,n,t,c) =& \frac{\partial f_1(u,n,t,c)}{\partial t} + \frac{\partial f_1(u,n,t,c)}{\partial u} + \frac{\partial f_1(u,n,t,c)}{\partial c} + \\
&+ \lambda_{1,-1}(u)\left(f_{-1}(0,n+1,t,c) - f_1(u,n,t,c)\right), \\
\mathcal{A}f_{-1}(u,n,t,c) =& \frac{\partial f_{-1}(u,n,t,c)}{\partial t} + \frac{\partial f_{-1}(u,n,t,c)}{\partial u} + \\
&+ \lambda_{-1,1}(u)\left(f_1(0,n,t,c) - f_{-1}(u,n,t,c)\right).
\end{aligned}$$

We assume f_i having the form

$$f_i(u,n,t,c) = e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_i(u),$$

where $\beta, \gamma, \delta \in \mathbb{R}^+$ are positive constants and h_i a bounded function. We are interested in the stopping times $\tau_{d_1}^+(W^{\epsilon,\mu})$ and $\tau_{d_2}^-(W^{\epsilon,\mu})$, hence, we solve

$$\mathcal{A}f \equiv 0 \text{ subject to } h_1(d_1) = \alpha_1 \text{ and } h_{-1}(d_2) = \alpha_2, \tag{4.19}$$

where α_1 and α_2 are arbitrary constants.

Lemma 4.2.1. *With the above definitions* $\left(e^{-\beta t}e^{-\gamma N_t}e^{-\delta C_t}h_{X_t}(U_t(X))\right)_{t\geq 0}$ *is a martingale with*

$$h_1(u) = \alpha_1 e^{-\int_u^{d_1}\beta+\delta+\lambda_{1,-1}(v)dv} + e^{-\gamma}h_{-1}(0)\int_u^{d_1}\lambda_{1,-1}(w)e^{-\int_u^w \beta+\delta+\lambda_{1,-1}(v)dv}dw,$$
$$0 \leq u \leq d_1$$
$$h_{-1}(u) = \alpha_2 e^{-\int_u^{d_2}\beta+\lambda_{-1,1}(v)dv} + h_1(0)\int_u^{d_2}\lambda_{-1,1}(w)e^{-\int_u^w \beta+\lambda_{-1,1}(v)dv}dw,$$
$$0 \leq u \leq d_2$$

and initial values

$$h_1(0) = \frac{\alpha_1 e^{-(\delta+\beta)d_1}e^{-\int_0^{d_1}\lambda_{1,-1}(v)dv} + \alpha_2 e^{-\beta d_2-\gamma}e^{-\int_0^{d_2}\lambda_{-1,1}(v)dv}\int_0^{d_1}e^{-(\beta+\delta)w}\lambda_{1,-1}(w)e^{-\int_0^w\lambda_{1,-1}(v)dv}dw}{1-e^{-\gamma}\int_0^{d_2}e^{-\beta t}\lambda_{-1,1}(t)e^{-\int_0^t\lambda_{-1,1}(v)dv}dt\int_0^{d_1}e^{-(\beta+\delta)w}\lambda_{1,-1}(w)e^{-\int_0^w\lambda_{1,-1}(v)dv}dw},$$
$$h_{-1}(0) = \frac{\alpha_2 e^{-\beta d_2}e^{-\int_0^{d_2}\lambda_{-1,1}(v)dv} + \alpha_1 e^{-(\beta+\delta)d_1}e^{-\int_0^{d_1}\lambda_{1,-1}(v)dv}\int_0^{d_2}e^{-\beta w}\lambda_{-1,1}(w)e^{-\int_0^w\lambda_{-1,1}(v)dv}dw}{1-e^{-\gamma}\int_0^{d_1}e^{-(\beta+\delta)t}\lambda_{1,-1}(t)e^{-\int_0^t\lambda_{1,-1}(v)dv}dt\int_0^{d_2}e^{-\beta w}\lambda_{-1,1}(w)e^{-\int_0^w\lambda_{-1,1}(v)dv}dw}.$$

Proof. Solving $\mathcal{A}f \equiv 0$, where f has the form $f_i(u,n,t,c) = e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_i(u)$ becomes

$$-\beta e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_1(u) + e^{-\beta t}e^{-\gamma n}e^{-\delta c}\frac{\partial h_1(u)}{\partial u} + \lambda_{1,-1}(u)e^{-\beta t}e^{-\gamma(n+1)}e^{-\delta c}h_{-1}(0) -$$
$$-\lambda_{1,-1}(u)e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_1(u) - \delta e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_1(u) = 0,$$
$$-\beta e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_{-1}(u) + e^{-\beta t}e^{-\gamma n}e^{-\delta c}\frac{\partial h_{-1}(u)}{\partial u} + \lambda_{-1,1}(u)e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_1(0) -$$
$$-\lambda_{1,-1}(u)e^{-\beta t}e^{-\gamma n}e^{-\delta c}h_{-1}(u) = 0,$$

simplified to

$$\frac{dh_1(u)}{\partial u} - (\beta + \lambda_{1,-1}(u) + \delta)h_1(u) = -\lambda_{1,-1}(u)e^{-\gamma}h_{-1}(0),$$
$$\frac{dh_{-1}(u)}{\partial u} - (\beta + \lambda_{-1,1}(u))h_{-1}(u) = -\lambda_{1,-1}(u)h_1(0).$$

Using the integrating factor method to solve ordinary differential equations and the constraints $h_1(d_1) = \alpha_1$ and $h_{-1}(d_2) = \alpha_2$, we find

$$\begin{aligned} h_1(u) &= \alpha_1 e^{-\int_u^{d_1} \beta+\delta+\lambda_{1,-1}(v)dv} + e^{-\gamma}h_{-1}(0)\int_u^{d_1} \lambda_{1,-1}(w)e^{-\int_u^w \beta+\delta+\lambda_{1,-1}(v)dv}dw, \\ &\qquad 0 \le u \le d_1, \\ h_{-1}(u) &= \alpha_2 e^{-\int_u^{d_2} \beta+\lambda_{-1,1}(v)dv} + h_1(0)\int_u^{d_2} \lambda_{-1,1}(w)e^{-\int_u^w \beta+\lambda_{-1,1}(v)dv}dw, \\ &\qquad 0 \le u \le d_2. \end{aligned} \tag{4.20}$$

Setting $u = 0$ and solving the system of equations

$$h_1(0) = \alpha_1 e^{-\int_0^{d_1} \beta+\delta+\lambda_{1,-1}(v)dv} + e^{-\gamma}h_{-1}(0)\int_0^{d_1} \lambda_{1,-1}(w)e^{-\int_0^w \beta+\delta+\lambda_{1,-1}(v)dv}dw,$$
$$h_{-1}(0) = \alpha_2 e^{-\int_0^{d_2} \beta+\lambda_{-1,1}(v)dv} + h_1(0)\int_0^{d_2} \lambda_{-1,1}(w)e^{-\int_0^w \beta+\lambda_{-1,1}(v)dv}dw$$

provides us with the initial values of $h_1(0)$ and $h_{-1}(0)$:

$$h_1(0) = \frac{\alpha_1 e^{-(\delta+\beta)d_1} e^{-\int_0^{d_1} \lambda_{1,-1}(v)dv} + \alpha_2 e^{-\beta d_2 - \gamma} e^{-\int_0^{d_2} \lambda_{-1,1}(v)dv} \int_0^{d_1} e^{-(\beta+\delta)w} \lambda_{1,-1}(w) e^{-\int_0^w \lambda_{1,-1}(v)dv} dw}{1 - e^{-\gamma} \int_0^{d_2} e^{-\beta t} \lambda_{-1,1}(t) e^{-\int_0^t \lambda_{-1,1}(v)dv} dt \int_0^{d_1} e^{-(\beta+\delta)w} \lambda_{1,-1}(w) e^{-\int_0^w \lambda_{1,-1}(v)dv} dw}$$

$$h_{-1}(0) = \frac{\alpha_2 e^{-\beta d_2} e^{-\int_0^{d_2} \lambda_{-1,1}(v)dv} + \alpha_1 e^{-(\beta+\delta)d_1} e^{-\int_0^{d_1} \lambda_{1,-1}(v)dv} \int_0^{d_2} e^{-\beta w} \lambda_{-1,1}(w) e^{-\int_0^w \lambda_{-1,1}(v)dv} dw}{1 - e^{-\gamma} \int_0^{d_1} e^{-(\beta+\delta)t} \lambda_{1,-1}(t) e^{-\int_0^t \lambda_{1,-1}(v)dv} dt \int_0^{d_2} e^{-\beta w} \lambda_{-1,1}(w) e^{-\int_0^w \lambda_{-1,1}(v)dv} dw} \quad (4.21)$$

Finally, plugging (4.21) into (4.20) we derive the solution $h_i(u)$ of the ordinary differential equation. As a result, we have obtained the martingale

$$\hat{M}_t := f_{X_t}(U_t(X), N_t, t, C_t(W^{\epsilon,\mu})) = e^{-\beta t} e^{-\gamma N_t} e^{-\delta C_t(W^{\epsilon,\mu})} h_{X_t}(U_t(X)).$$

□

Lemma 4.2.2. *The triple Laplace transform of Parisian, occupation times and number of crossings of the perturbed Brownian motion $W^{\epsilon,\mu}$ is*

$$\alpha_1 \mathbb{E}_\epsilon^{\mathbb{Q}} \left(e^{-\beta\tau_{d_1}^+(W^{\epsilon,\mu}) - \gamma N_{\tau_{d_1}^+(W^{\epsilon,\mu})} - \delta C_{\tau_{d_1}^+(W^{\epsilon,\mu})}} \mathbf{1}_{\tau_{d_1}^+(W^{\epsilon,\mu}) < \tau_{d_2}^-(W^{\epsilon,\mu})} \right) +$$
$$+ \alpha_2 \mathbb{E}_\epsilon^{\mathbb{Q}} \left(e^{-\beta\tau_{d_2}^-(W^{\epsilon,\mu}) - \gamma N_{\tau_{d_2}^-(W^{\epsilon,\mu})} - \delta C_{\tau_{d_2}^-(W^{\epsilon,\mu})}} \mathbf{1}_{\tau_{d_2}^-(W^{\epsilon,\mu}) < \tau_{d_1}^+(W^{\epsilon,\mu})} \right)$$
$$= \Bigg\{ \alpha_1 e^{-(\delta+\beta)d_1} \left(1 - e^{-2\mu\epsilon} \mathcal{N}\left(\frac{\mu d_1 - \epsilon}{\sqrt{d_1}} \right) - \mathcal{N}\left(\frac{-\mu d_1 - \epsilon}{\sqrt{d_1}} \right) \right) + \alpha_2 e^{-\beta d_2 - \gamma} \times$$
$$\times \left[1 - \mathcal{N}\left(\frac{\mu d_2 - \epsilon}{\sqrt{d_2}} \right) - e^{2\mu\epsilon} \mathcal{N}\left(\frac{-\mu d_2 - \epsilon}{\sqrt{d_2}} \right) \right] \Bigg[e^{-(\sqrt{2(\beta+\delta)+\mu^2}+\mu)\epsilon} \times$$
$$\times \mathcal{N}\left(\sqrt{(2(\beta+\delta)+\mu^2)d_1} - \frac{\epsilon}{\sqrt{d_1}} \right) + e^{(\sqrt{2(\beta+\delta)+\mu^2}-\mu)\epsilon} \times$$
$$\times \mathcal{N}\left(-\sqrt{(2(\beta+\delta)+\mu^2)d_1} - \frac{\epsilon}{\sqrt{d_1}} \right) \Bigg] \Bigg\} \times \Bigg\{ 1 - e^{-\gamma} \Bigg[e^{(\mu - \sqrt{2\beta+\mu^2})\epsilon} \times$$
$$\times \mathcal{N}\left(\sqrt{(2\beta+\mu^2)d_2} - \frac{\epsilon}{\sqrt{d_2}} \right) + e^{(\mu+\sqrt{2\beta+\mu^2})\epsilon} \mathcal{N}\left(-\sqrt{(2\beta+\mu^2)d_2} - \frac{\epsilon}{\sqrt{d_2}} \right) \Bigg] \times$$
$$\times \Bigg[e^{-(\sqrt{2(\beta+\delta)+\mu^2}+\mu)\epsilon} \mathcal{N}\left(\sqrt{(2(\beta+\delta)+\mu^2)d_1} - \frac{\epsilon}{\sqrt{d_1}} \right) + e^{(\sqrt{2(\beta+\delta)+\mu^2}-\mu)\epsilon} \times$$
$$\times \mathcal{N}\left(-\sqrt{(2(\beta+\delta)+\mu^2)d_1} - \frac{\epsilon}{\sqrt{d_1}} \right) \Bigg] \Bigg\}^{-1},$$

where α_1 and α_2 are arbitrary constants.

Proof. Lemma 4.2.1 provides us with a martingale of the form

$$\hat{M}_t := f_{X_t}(U_t(X), N_t, t, C_t(W^{\epsilon,\mu})) = e^{-\beta t} e^{-\gamma N_t} e^{-\delta C_t(W^{\epsilon,\mu})} h_{X_t}(U_t(X)).$$

The optional sampling theorem on martingale $\hat{M}_t$ with stopping time $\tau_{d_1}^+(W^{\epsilon,\mu}) \wedge \tau_{d_2}^-(W^{\epsilon,\mu}) \wedge t$ yields

$$\mathbb{E}_\epsilon^{\mathbb{Q}}\left(\hat{M}_{\tau^+_{d_1}(W^{\epsilon,\mu})\wedge\tau^-_{d_2}(W^{\epsilon,\mu})\wedge t}\right) = \mathbb{E}_\epsilon^{\mathbb{Q}}(\hat{M}_0).$$

$h_i(u)$, $i = 1, -1$, are continuos functions and therefore bounded on the compact intervals $[0, d_1]$ or $[0, d_2]$ respectively. Hence, there exists a constant K, such that $|h_i(U_t(X))| \leq K$ for all $U_t(X)1 \in [0, d_i]$, $i = 1, -1$. Hence, Lebesgue's Dominated Convergence Theorem is applicable, yielding

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_\epsilon^{\mathbb{Q}}\left(\hat{M}_{\tau^+_{d_1}(W^{\epsilon,\mu})\wedge\tau^-_{d_2}(W^{\epsilon,\mu})\wedge t}\right) &= \mathbb{E}_\epsilon^{\mathbb{Q}}\left(\hat{M}_{\tau^+_{d_1}(W^{\epsilon,\mu})\wedge\tau^-_{d_2}(W^{\epsilon,\mu})}\right) \\
&= \alpha_1 \mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta\tau^+_{d_1}(W^{\epsilon,\mu})-\gamma N_{\tau^+_{d_1}(W^{\epsilon,\mu})}-\delta C_{\tau^+_{d_1}(W^{\epsilon,\mu})}}\mathbf{1}_{\tau^+_{d_1}(W^{\epsilon,\mu})<\tau^-_{d_2}(W^{\epsilon,\mu})}\right) \\
&\quad + \alpha_2 \mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta\tau^-_{d_2}(W^{\epsilon,\mu})-\gamma N_{\tau^-_{d_2}(W^{\epsilon,\mu})}-\delta C_{\tau^-_{d_2}(W^{\epsilon,\mu})}}\mathbf{1}_{\tau^-_{d_2}(W^{\epsilon,\mu})<\tau^+_{d_1}(W^{\epsilon,\mu})}\right);
\end{aligned}$$

notice, that $h_{X_{\tau^+_{d_1}}}(U_{\tau^+_{d_1}}(X)) = h_1(d_1) = \alpha_1$ and $h_{X_{\tau^-_{d_2}}}(U_{\tau^-_{d_2}}(X)) = h_2(d_2) = \alpha_2$. Also, $\mathbb{E}_\epsilon^{\mathbb{Q}}(\hat{M}_0) = h_{X_0}(0)$, where $h_{X_0}(0)$ is (4.21) respectively, depending on the state it starts.

We are starting in state 1 by definition, hence from equation (4.21), we follow

$$h_1(0) = \frac{\alpha_1 e^{-(\delta+\beta)d_1}\bar{P}_1(d_1) + \alpha_2 e^{-\beta d_2 - \gamma}\bar{P}_{-1}(d_2)\hat{P}_{1,-1}(\beta+\delta)}{1 - e^{-\gamma}\hat{P}_{-1,1}(\beta)\hat{P}_{1,-1}(\beta+\delta)}. \tag{4.22}$$

Straightforward calculation yield for (4.14), (4.15), (4.17) and (4.18):

$$
\begin{aligned}
\hat{P}_{1,-1}(\beta+\delta) &= \int_0^t e^{-(\beta+\delta)w}\lambda_{1,-1}(w)e^{-\int_0^w \lambda_{1,-1}(v)dv}dw = \int_0^t e^{-(\beta+\delta)w}p_{1,-1}(w)dw \\
&= e^{-(\sqrt{2(\beta+\delta)+\mu^2}+\mu)\epsilon}\mathcal{N}\left(\sqrt{(2(\beta+\delta)+\mu^2)t}-\frac{\epsilon}{\sqrt{t}}\right) \\
&\quad + e^{(\sqrt{2(\beta+\delta)+\mu^2}-\mu)\epsilon}\mathcal{N}\left(-\sqrt{(2(\beta+\delta)+\mu^2)t}-\frac{\epsilon}{\sqrt{t}}\right),
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&\hat{P}_{-1,1}(\beta) = \int_0^t e^{-\beta w}\lambda_{-1,1}(w)e^{-\int_0^w \lambda_{-1,1}(v)dv}dw = \int_0^t e^{-\beta w}p_{-1,1}(w)dw \\
&= e^{(\mu-\sqrt{2\beta+\mu^2})\epsilon}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)t}-\frac{\epsilon}{\sqrt{t}}\right)+e^{(\mu+\sqrt{2\beta+\mu^2})\epsilon}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)t}-\frac{\epsilon}{\sqrt{t}}\right),
\end{aligned}
\tag{4.24}
$$

$$
\bar{P}_1(t) = e^{-\int_0^t \lambda_{1,-1}(v)dv} = 1 - e^{-2\mu\epsilon}\mathcal{N}\left(\frac{\mu t-\epsilon}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-\mu t-\epsilon}{\sqrt{t}}\right), \tag{4.25}
$$

$$
\bar{P}_{-1}(t) = e^{-\int_0^t \lambda_{-1,1}(v)dv} = 1 - \mathcal{N}\left(\frac{\mu t-\epsilon}{\sqrt{t}}\right) - e^{2\mu\epsilon}\mathcal{N}\left(\frac{-\mu t-\epsilon}{\sqrt{t}}\right). \tag{4.26}
$$

Inserting calculations (4.23), (4.24), (4.25) and (4.26) into equation (4.22) yields the result of the proposition.

□

4.3 Laplace transform of Parisian, Occupation and Local times

We now relate the number of downcrossings to the local time via Lévy's Downcrossing Theorem (see [Revuz and Yor, 1999, pp.227-228]). For a continuous local martingale M the local time for every a and t is

$$L_t^a(M) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(a-\epsilon, a+\epsilon)}(M_s) d\langle M, M \rangle_s \quad \text{a.s.}$$

Lévy's Downcrossing theorem (Chung and Durrett [1976], Revuz and Yor [1999]) states that

$$\mathbb{Q}_0 \left(\lim_{\epsilon \to 0} \epsilon N_t = L_t^0(M) \right) = 1. \tag{4.27}$$

We will use this result to derive the triple Laplace transform of Parisian, local and occupation times for a drifted Brownian motion W^μ. This is done by firstly replacing the number of downcrossings by the local time yielding results for the perturbed Brownian motion. Finally we examine the limiting behaviour of the perturbed process, which by construction is the drifted Brownian motion.

Proposition 4.3.1. *The triple Laplace transform of Parisian, local and occupation times of the drifted Brownian motion W^μ is*

$$\alpha_1 \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta\tau_{d_1}^+(W^\mu) - \gamma L^0_{\tau_{d_1}^+}(W^\mu) - \delta C_{\tau_{d_1}^+(W^\mu)}} \mathbf{1}_{\tau_{d_1}^+(W^\mu) < \tau_{d_2}^-(W^\mu)} \right) +$$
$$+ \alpha_2 \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta\tau_{d_2}^-(W^\mu) - \gamma L^0_{\tau_{d_2}^-}(W^\mu) - \delta C_{\tau_{d_2}^-(W^\mu)}} \mathbf{1}_{\tau_{d_2}^-(W^\mu) < \tau_{d_1}^+(W^\mu)} \right)$$
$$= \left\{ \alpha_1 e^{-(\delta+\beta)d_1} \left(2\mu \mathcal{N}\left(\mu\sqrt{d_1}\right) + \sqrt{\frac{2}{\pi d_1}} e^{-\frac{\mu^2 d_1}{2}} \right) + \alpha_2 e^{-\beta d_2} \left(\sqrt{\frac{2}{\pi d_2}} e^{-\frac{\mu^2 d_2}{2}} - \right.\right.$$
$$\left.\left. - 2\mu \mathcal{N}\left(-\mu\sqrt{d_2}\right) \right) \right\} \times \left\{ \sqrt{\frac{2}{\pi d_2}} e^{-\frac{(2\beta+\mu^2)d_2}{2}} + \sqrt{\frac{2}{\pi d_1}} e^{-\frac{(2(\beta+\delta)+\mu^2)d_1}{2}} + \right.$$
$$+ 2\sqrt{2(\beta+\delta)+\mu^2}\,\mathcal{N}\left(\sqrt{(2(\beta+\delta)+\mu^2)d_1}\right) + 2\sqrt{2\beta+\mu^2}\,\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d_2}\right) +$$
$$\left. + \gamma - \sqrt{2\beta+\mu^2} - \sqrt{2(\beta+\delta)+\mu^2} \right\}^{-1},$$

where α_1, α_2 are arbitrary constants.

Proof. Equation (4.27) suggests replacing γ with $\epsilon\gamma$ in Lemma 4.2.2. Lemma 4.2.2 can then be transformed into the triple Laplace transform of Parisian, local and occupation times of the perturbed Brownian motion. The next task is to let ϵ approach zero. By construction of our semi-Markov model, we have for all $t \geq 0$

$$W_t^{\epsilon,\mu} \xrightarrow{\epsilon \downarrow 0} W_t^\mu \quad \text{a.s.}$$

The stopping times based on $W_t^{\epsilon,\mu}$ converge almost surely to those of the drifted Brownian motion W_t^μ. Furthermore,

$$e^{-\beta\tau_{d_1}^+(W^{\epsilon,\mu}) - \gamma N_{\tau_{d_1}^+(W^{\epsilon,\mu})} - \delta C_{\tau_{d_1}^+(W^{\epsilon,\mu})}} < 1$$

and

$$e^{-\beta\tau^-_{d_2}(W^{\epsilon,\mu})-\gamma N_{\tau^-_{d_2}(W^{\epsilon,\mu})}-\delta C_{\tau^-_{d_2}(W^{\epsilon,\mu})}} < 1,$$

thus dominated convergence applies to get the result for W^μ,

$$\begin{aligned}
&\alpha_1\mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta\tau^+_{d_1}(W^\mu)-\gamma L^0_{\tau^+_{d_1}}(W^\mu)-\delta C_{\tau^+_{d_1}(W^\mu)}}\mathbf{1}_{\tau^+_{d_1}(W^\mu)<\tau^-_{d_2}(W^\mu)}\right)+\\
&+\alpha_2\mathbb{E}^{\mathbb{Q}}_0\left(e^{-\beta\tau^-_{d_2}(W^\mu)-\gamma L^0_{\tau^-_{d_2}}(W^\mu)-\delta C_{\tau^-_{d_2}(W^\mu)}}\mathbf{1}_{\tau^-_{d_2}(W^\mu)<\tau^+_{d_1}(W^\mu)}\right)\\
=&\lim_{\epsilon\to0}\alpha_1\mathbb{E}^{\mathbb{Q}}_\epsilon\left(e^{-\beta\tau^+_{d_1}(W^{\epsilon,\mu})-\gamma N_{\tau^+_{d_1}(W^{\epsilon,\mu})}-\delta C_{\tau^+_{d_1}(W^{\epsilon,\mu})}}\mathbf{1}_{\tau^+_{d_1}(W^{\epsilon,\mu})<\tau^-_{d_2}(W^{\epsilon,\mu})}\right)+\\
&+\alpha_2\mathbb{E}^{\mathbb{Q}}_\epsilon\left(e^{-\beta\tau^-_{d_2}(W^{\epsilon,\mu})-\gamma N_{\tau^-_{d_2}(W^{\epsilon,\mu})}-\delta C_{\tau^-_{d_2}(W^{\epsilon,\mu})}}\mathbf{1}_{\tau^-_{d_2}(W^{\epsilon,\mu})<\tau^+_{d_1}(W^{\epsilon,\mu})}\right)\\
=&\lim_{\epsilon\to0}\frac{\alpha_1e^{-(\delta+\beta)d_1}\bar{P}_1(d_1)+\alpha_2e^{-\beta d_2-\gamma}\bar{P}_{-1}(d_2)\hat{P}_{1,-1}(\beta+\delta)}{1-e^{-\gamma}\hat{P}_{-1,1}(\beta)\hat{P}_{1,-1}(\beta+\delta)}.
\end{aligned}\tag{4.28}$$

See Dassios and Wu [2009a] for further reference. Therefore, letting ϵ go to zero in the result of Lemma 4.2.2 will provide us with the triple Laplace transform for the drifted Brownian motion. In particular, plugging in calculations (4.23), (4.24), (4.25) and (4.26) into equation (4.28) and applying L'Hôpital's rule, we obtain the result in Proposition 4.3.1.

□

4.4 Laplace transform of Parisian, Occupation, Local times and Position of Brownian motion

Proposition 4.3.1 gives us the triple Laplace transform of Parisian, occupation and local times for both scenarios, $\tau_{d_1}^{+}(W^{\mu}) < \tau_{d_2}^{-}(W^{\mu})$ and $\tau_{d_2}^{-}(W^{\mu}) < \tau_{d_1}^{+}(W^{\mu})$, indicated by α_1 and α_2 respectively. Now we also want to include the distribution of the position of the Brownian motion at Parisian time. This will be achieved by applying Girsanov theorem on results for drifted Brownian motion in Proposition 4.3.1. For reasons of clarity we distinguish between the cases $\tau_{d_1}^{+}(W) < \tau_{d_2}^{-}(W)$ and $\tau_{d_2}^{-}(W) < \tau_{d_1}^{+}(W)$.

4.4.1 Case $\tau_{d_1}^{+}(W) < \tau_{d_2}^{-}(W)$

Proposition 4.4.1. *The joint moment generating function and Laplace transforms of Parisian, local, occupation times and position of the standard Brownian motion W when the excursion above zero exceeds d_1 before the excursion below zero exceeds d_2 is*

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{d_1}^{+}(W)-\gamma L^0_{\tau_{d_1}^{+}}(W)-\delta C_{\tau_{d_1}^{+}(W)}+\mu W_{\tau_{d_1}^{+}(W)}}\right)\\
&\quad=\left\{\sqrt{\frac{2}{\pi d_1}}e^{-(\beta+\delta)d_1}+2\mu e^{-(\beta+\delta)d_1+\frac{\mu^2 d_1}{2}}\mathcal{N}\left(\mu\sqrt{d_1}\right)\right\}\times\left\{\sqrt{\frac{2}{\pi d_2}}e^{-\beta d_2}+\right.\\
&\quad+\sqrt{\frac{2}{\pi d_1}}e^{-(\beta+\delta)d_1}+2\sqrt{2(\beta+\delta)}\mathcal{N}\left(\sqrt{2(\beta+\delta)d_1}\right)+2\sqrt{2\beta}\mathcal{N}\left(\sqrt{2\beta d_2}\right)+\\
&\quad\left.+\gamma-\sqrt{2\beta}-\sqrt{2(\beta+\delta)}\right\}^{-1}
\end{aligned}$$

Proof. Under probability measure $\mathbb{Q}$, W^{μ} is a Brownian motion with drift μ.

Girsanov theorem provides us with a measure $\hat{\mathbb{Q}}$ under which W^μ is a standard Brownian motion. The proposition follows after applying the change of measure to Proposition 4.3.1 with $\beta = \beta + \frac{\mu^2}{2}$ and Radon-Nikodym derivative

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = e^{\mu W_t - \frac{1}{2}\mu^2 t}.$$

□

4.4.2 Case $\tau_{d_2}^-(W) < \tau_{d_1}^+(W)$

Proposition 4.4.2. *The joint moment generating function and Laplace transforms of Parisian, local, occupation times and position of the standard Brownian motion W when the excursion below zero exceeds d_2 before the excursion above zero exceeds d_1 is*

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{d_2}^-(W) - \gamma L^0_{\tau_{d_2}^-}(W) - \delta C_{\tau_{d_2}^-(W)} + \mu W_{\tau_{d_2}^-(W)}}\right)$$
$$= \left\{\sqrt{\frac{2}{\pi d_2}}e^{-\beta d_2} - 2\mu e^{-\beta d_2 + \frac{\mu^2 d_2}{2}}\mathcal{N}\left(-\mu\sqrt{d_2}\right)\right\} \times \left\{\sqrt{\frac{2}{\pi d_2}}e^{-\beta d_2} + \sqrt{\frac{2}{\pi d_1}}e^{-(\beta+\delta)d_1} + \right.$$
$$\left. + 2\sqrt{2(\beta+\delta)}\mathcal{N}\left(\sqrt{2(\beta+\delta)d_1}\right) + 2\sqrt{2\beta}\mathcal{N}\left(\sqrt{2\beta d_2}\right) + \gamma - \sqrt{2\beta} - \sqrt{2(\beta+\delta)}\right\}^{-1}$$

Proof. This proposition follows in the same way as in Proposition 4.4.1.

□

4.5 Lévy's Theorem and Drawdown processes

The reflection principle implies, that the maximum of a Brownian motion at a certain time t has the same distribution as the absolute value of a Brownian motion at time t. This results does not extend to the maximum process $\left(\bar{W}_t\right)_{t\geq 0}$, where $\bar{W}_t = \sup_{0\leq s\leq t} W_s$, and the reflected Brownian motion $(|W_t|)_{t\geq 0}$. However, Lévy [1948] described a similar relationship.

Theorem 4.5.1 (Lévy 1948). *The pairs of processes* $\left(\bar{W}_t - W_t, \bar{W}_t\right)_{t\geq 0}$ *and* $(|W_t|, 2L_t^0(W))_{t\geq 0}$ *have the same law. In particular,* $\left(\bar{W}_t - W_t\right)_{t\geq 0}$ *is a Markov process.*

It is clear that $2\mathbb{Q}_0(W_t > a) = \mathbb{Q}_0(|W_t| > a)$ for all $a \geq 0$, hence it follows directly

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma L^0_{\tau^+_{d_1}}(W)+\mu W_{\tau^+_{d_1}(W)}}\right) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\frac{\gamma}{2}\bar{W}_{\tau^+_{d_1}(W)}+\frac{\mu}{2}\left(\bar{W}_{\tau^+_{d_1}(W)}-W_{\tau^+_{d_1}(W)}\right)}\right)$$

and similiarly

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma L^0_{\tau^-_{d_2}}(W)+\mu W_{\tau^-_{d_2}(W)}}\right) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\frac{\gamma}{2}\bar{W}_{\tau^-_{d_2}(W)}+\frac{\mu}{2}\left(\bar{W}_{\tau^-_{d_2}(W)}-W_{\tau^-_{d_2}(W)}\right)}\right). \tag{4.29}$$

We distinguish between the case where we are only concerned about the excursion above and the case below zero. In the latter case, the time the excursion above zero reaches d_1, which we exchangeably call the Parisian time above zero, vanishes by letting d_1 approach infinity and hence $\tau^+_{d_1}(W) \to \infty$.

Corollary 4.5.1. *The joint moment generating function and Laplace transform of Parisian time below zero and drawdown and maximum at Parisian time is given by*

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{d_2}^-(W)-\gamma\bar{W}_{\tau_{d_2}^-(W)}+\mu\left(\bar{W}_{\tau_{d_2}^-(W)}-W_{\tau_{d_2}^-(W)}\right)}\right) = \frac{\sqrt{\frac{2}{\pi d_2}}e^{-\beta d_2} - 4\mu e^{-\beta d_2+2\mu^2 d_2}\mathcal{N}\left(-2\mu\sqrt{d_2}\right)}{\sqrt{\frac{2}{\pi d_2}}e^{-\beta d_2} + 2\sqrt{2\beta}\mathcal{N}\left(\sqrt{2\beta d_2}\right) + 2\gamma}.$$

Proof. Apply equation (4.29) into Proposition 4.4.2 and set $\delta = 0$ in order to dispose of the occupation time. Letting $\tau_{d_1}^+(W)$ approach infinity yields the Corollary.

□

Similarly, we treat the case where we are only concerned about the Parisian time above zero. Here we suggest $\tau_{d_2}^-(W) \to \infty$ and yield the following Corollary.

Corollary 4.5.2. *The joint moment generating function and Laplace transform of Parisian time below zero and drawdown and maximum at Parisian time is given by*

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{d_1}^+(W)-\gamma\bar{W}_{\tau_{d_1}^+(W)}+\mu\left(\bar{W}_{\tau_{d_1}^+(W)}-W_{\tau_{d_1}^+(W)}\right)}\right) = \frac{\sqrt{\frac{2}{\pi d_1}}e^{-\beta d_1} + 4\mu e^{-\beta d_1+2\mu^2 d_1}\mathcal{N}\left(2\mu\sqrt{d_1}\right)}{\sqrt{\frac{2}{\pi d_1}}e^{-\beta d_1} + 2\sqrt{2\beta}\mathcal{N}\left(\sqrt{2\beta d_1}\right) + 2\gamma}.$$

4.6 Application to Risk Management

The relative drawdown RDD of an underlying process S is defined as the percentage drop of the underlying price from its running maximum. This concept serves as an alternative measure of risk, which has the advantage of capturing the path property of the price process in contrast to the commonly used Value-at-Risk measure. We make the definitions

$$\begin{aligned}\bar{S}_t &= \sup_{0\leq s\leq t} S_s,\\ DD_t(S) &= \bar{S}_t - S_t,\\ RDD_t(S) &= \frac{\bar{S}_t - S_t}{\bar{S}_t}.\end{aligned}$$

We notice that in the context of risk management, the relative drawdown RDD serves as an indicator of market stability, where the relative drawdown process shoots up during market recession and is low in stable periods. It can be assumed, that a realization of a large drawdown is followed by a default, motivating a new definition of a market crash introduced by Vecer, Novotny, and Pospisil [2006],

$$T_a = \inf\left\{ t \geq 0 \middle| \left(1 - \frac{S_t}{\bar{S}_t}\right) \geq a \right\}. \tag{4.30}$$

In order to insure the event the maximum relative drawdown $\max_{0\leq t\leq T} RDD_t(S)$ exceeds a certain threshold, Vecer, Novotny, and Pospisil [2006] introduce the crash option with digital payoff. This contract pays \$1 at the time the relative drop of the asset from its maximum exceeds a percentage a before maturity and expires worthlessly elsewise. The value of this binary option becomes

$$\mathbb{E}\left(e^{-r(T_a-t)}\mathbf{1}_{\max\limits_{0\leq s\leq T} RDD_s(S)>a}\Big|\mathcal{F}_t\right)=\mathbb{E}\left(e^{-r(T_a-t)}\mathbf{1}_{T_a<T}\Big|\mathcal{F}_t\right).$$

Vecer, Novotny, and Pospisil [2006] find the price and the Delta hedging strategy for this crash option.

Our contribution lies in extending the definition of default, see (4.30), to default occurring with Parisian delay, i.e. if the underlying process stays below zero for a pre-specified time period $d \geq 0$. We consider this to be a more realistic measure of risk, giving regulators more time to react to shortfalls and keeping in mind that relative drawdowns can not be monitored continuously. To insure against the event of the relative drawdown exceeding some percentage with Parisian delay, we introduce two related contract: The Parisian Crash and the Parisian Lookback option.

4.6.1 Parisian Crash Options

In this section, we suggest a new class of equity exotics: Crash option triggered at Parisian time. This so-called Parisian Crash option pays \$1 at the time the underlying price process stays underneath a barrier, which we without loss of generality assume to be 1, for a consecutive time longer than option window $d \in [0, T]$, if the relative drawdown exceeds a certain percentage $a \in (0, 1]$. Otherwise, the contract expires worthless. The payoff at Parisian time becomes

$$\mathbf{1}_{RDD_{\tau_d^-}(S)>a}\mathbf{1}_{\tau_{1,d}^-(S)<T}.$$

We assume the Black-Scholes model and let our underlying process be generated by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The measure $\bar{\mathbb{Q}}$, under which the discounted process is a martingale, is called the equivalent martingale measure. Under the equivalent martingale measure $\bar{\mathbb{Q}}$ and risk-free interest rate $r \geq 0$ and no dividends, the underlying asset and its standardized log function $Z_t = \frac{1}{\sigma}\log S_t$ have the following dynamics respectively

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t, \\ dZ_t &= \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)dt + dW_t, \end{aligned}$$

with $Z_0 = 0$ and $S_0 = 1$. By the standard pricing formula under the equivalent martingale measure $\bar{\mathbb{Q}}$, we find the value at time t of the Parisian Crash option, denoted by $PCO(t, T, r, d, a)$, to be

$$PCO(t, T, r, d, a) = \mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left(e^{-r(\tau_{1,d}^-(S)-t)}\mathbf{1}_{RDD_{\tau_{1,d}^-}(S)>a}\mathbf{1}_{\tau_{1,d}^-(S)<T}\middle|\mathcal{F}_t\right).$$

The fair price can be expressed in terms of the drawdown and maximum of a standard Brownian motion and the Parisian time in the following way:

$$
\begin{aligned}
PCO(0,T,r,d,a) &= \mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left(e^{-r\tau_d^-(S)}\mathbf{1}_{RDD_{\tau_{1,d}^-}(S)>a}\mathbf{1}_{\tau_{1,d}^-(S)<T}\right)\\
&= \mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left(e^{-r\tau_{1,d}^-(S)}\mathbf{1}_{\{1-S_{\tau_{1,d}^-}/\bar{S}_{\tau_{1,d}^-}>a,\tau_{1,d}^-(S)<T\}}\right)\\
&= \mathbb{E}_0^{\mathbb{Q}}\left(e^{-r\tau_d^-(Z)}e^{mZ_{\tau_d^-}-\frac{m^2}{2}\tau_d^-}\mathbf{1}_{\{\bar{Z}_{\tau_d^-}-Z_{\tau_d^-}>\frac{-\ln(1-a)}{\sigma},\tau_d^-(Z)<T\}}\right)\\
&= \int\limits_{y=0}^{\infty}\int\limits_{x=\frac{-\ln(1-a)}{\sigma}}^{y}\int\limits_{t=0}^{T} e^{-(r+\frac{m^2}{2})t+m(y-x)}\mathbb{Q}_0\left(\bar{Z}_{\tau_d^-}\in dy,\bar{Z}_{\tau_d^-}-Z_{\tau_d^-}\in dx,\tau_d^-(Z)\in dt\right),
\end{aligned}
$$

with the following definitions

$$m=\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right), \tag{4.31}$$

$$\tau_{1,d}^-(S)=\inf\{t>0|\mathbf{1}_{S_t\le 1}(t-g_t)\ge d\}, \tag{4.32}$$

$$\tau_d^-(Z)=\inf\{t>0|\mathbf{1}_{Z_t\le 0}(t-g_t)\ge d\}, \tag{4.33}$$

and Girsanov theorem, where $\mathbb{Q}$ is a new measure under which $Z_t=mt+W_t$ is a standard Brownian motion. The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}}\Big|_{\mathcal{F}_T}=e^{mZ_T-\frac{m^2}{2}T}. \tag{4.34}$$

The pricing of the Parisian Crash option has now been reduced to finding the joint distribution of the drawdown and maximum of a standard Brownian motion and the Parisian time.

Applying Corollary 4.5.1, the fair price of the Parisian Crash option can be written in terms of the triple Laplace transform in the following way.

$$PCO(0,T,r,d,a) =$$
$$= \int\limits_{y=0}^{\infty} \int\limits_{x=\frac{-\ln(1-a)}{\sigma}}^{y} \int\limits_{t=0}^{T} e^{-(r+\frac{m^2}{2})t+m(y-x)} \mathcal{L}_{\beta}^{-1} \mathcal{L}_{\mu}^{-1} \mathcal{L}_{\gamma}^{-1} \{F(\gamma, -\mu, \beta)\}|_{(y,x,t)} dt\ dx\ dy,$$

where

$$\begin{aligned} F(\gamma, -\mu, \beta) &= \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_d^-(Z) - \gamma \bar{Z}_{\tau_d^-(Z)} + \mu \left(\bar{Z}_{\tau_d^-(Z)} - Z_{\tau_d^-(Z)} \right)} \right) \\ &= \frac{\sqrt{\frac{2}{\pi d}} e^{-\beta d} - 4\mu e^{-\beta d + 2\mu^2 d} \mathcal{N}\left(-2\mu\sqrt{d}\right)}{\sqrt{\frac{2}{\pi d}} e^{-\beta d_2} - 2\sqrt{2\beta}\mathcal{N}\left(-\sqrt{2\beta d}\right) + 2\gamma} \end{aligned} \tag{4.35}$$

and

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right).$$

4.6.2 Parisian Lookback Options

The second hybrid exotic option we are introducing, is the so-called Parisian Lookback option, which can be regarded as a combination of a Parisian option and a Lookback Put option with floating strike. Lookback options with floating strike have payoffs being the drawdown DD with the disadvantage of possibly having a final drawdown far below the maximum drawdown. Our proposed Parisian Lookback option expires worthless if the stock's Parisian time underneath the barrier 1 exceeds option window d. Otherwise, the option has the lookback's payoff at Parisian time $\tau_{1,d}^{-}(S)$. Hence, the payoff of the Parisian Lookback option becomes

$$\left(\bar{S}_{\tau_{1,d}^{-}(S)} - S_{\tau_{1,d}^{-}(S)}\right) \mathbf{1}_{\tau_{1,d}^{-}(S)\leq T}.$$

Using risk-neutral valuation the fair price can be written as

$$\begin{aligned}
&PLP(0,T,r,d,a) = \mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left(e^{-r\tau_{1,d}^{-}(S)}\left(\bar{S}_{\tau_{1,d}^{-}(S)} - S_{\tau_{1,d}^{-}(S)}\right)\mathbf{1}_{\tau_{1,d}^{-}(S)\leq T}\right)\\
&= \mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left(e^{-r\tau_{1,d}^{-}(S)}\bar{S}_{\tau_{1,d}^{-}(S)}\left(1-\frac{S_{\tau_{1,d}^{-}(S)}}{\bar{S}_{\tau_{1,d}^{-}(S)}}\right)\mathbf{1}_{\tau_{1,d}^{-}(S)\leq T}\right)\\
&= \mathbb{E}_{0}^{\mathbb{Q}}\left(e^{-r\tau_{d}^{-}(Z)+\sigma\bar{Z}_{\tau_{d}^{-}(Z)}+mZ_{\tau_{d}^{-}(Z)}-\frac{m^2}{2}\tau_{d}^{-}(Z)}\left(1-\frac{1}{e^{\sigma\left(\bar{Z}_{\tau_{d}^{-}(Z)}-Z_{\tau_{d}^{-}(Z)}\right)}}\right)\mathbf{1}_{\tau_{d}^{-}(Z)\leq T}\right)\\
&= \int\limits_{y=0}^{\infty}\int\limits_{x=0}^{y}\int\limits_{t=0}^{T} e^{-(r+\frac{m^2}{2})t+\sigma y+m(y-x)}\left(1-\frac{1}{e^{\sigma x}}\right)\times\\
&\qquad\qquad \times\mathbb{Q}_0\left(\bar{Z}_{\tau_{d}^{-}(Z)}\in dy, \bar{Z}_{\tau_{d}^{-}(Z)} - Z_{\tau_{d}^{-}(Z)}\in dx, \tau_{d}^{-}(Z)\in dt\right),
\end{aligned}$$

with $m = \frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)$ and change of measure $\frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}}\Big|_{\mathcal{F}_T} = e^{mZ_T-\frac{m^2}{2}T}$.

As in section 4.6.1, this joint density can be found by inverting our results on the triple Laplace transform of maximum, drawdown and Parisian times. The fair price becomes

$$PLP(0,T,r,d,a) =$$

$$\int_{y=0}^{\infty}\int_{x=0}^{y}\int_{t=0}^{T} e^{-(r+\frac{m^2}{2})t+\sigma y+m(y-x)}\left(1-\frac{1}{e^{\sigma x}}\right)\mathcal{L}_{\beta}^{-1}\mathcal{L}_{\mu}^{-1}\mathcal{L}_{\gamma}^{-1}\{F(\gamma,-\mu,\beta)\}|_{(y,x,t)}dt\ dx\ dy,$$

where the function F is defined in equation (4.35).

Chapter 5

Joint distribution of Parisian and Hitting times of Brownian motion

We study the joint law of Parisian time and hitting time of a drifted Brownian motion by using a three-state semi-Markov model, obtained through perturbation. We obtain a martingale, to which we can apply the optional sampling theorem and derive the double Laplace transform. This general result is applied to address problems in option pricing. We introduce a new option related to Parisian options, being triggered when the age of an excursion exceeds a certain time or/and a barrier is hit. We obtain an explicit expression for the Laplace transform of its fair price.

5.1 Introduction

In this chapter we introduce a new type of option, the so-called ParisianHit option, which in contrast to the Parisian option takes both the Parisian time and the hitting time of a pre-specified barrier into account. One version of this modification, called MinParisianHit option, is triggered if either the age of an excursion above a level reaches a certain time or another barrier is hit before maturity. The Max-ParisianHit, on the other hand, gets activated when both excursion age exceeds a certain time and a barrier is hit, making market manipulation extremely difficult.

The key for pricing these kind of options lies in deriving the joint law of Parisian and hitting times. Here, we study the Parisian and hitting times using a three-state semi-Markov model, indicating whether the process is in a positive or negative excursion and above or below a fixed barrier. This will allow us to compute the double Laplace transform of these two times, which can be inverted numerically using techniques as in Labart and Lelong [2009]. The study of this combined element of Parisian and hitting times is very difficult due to the amount of possible scenarios that can happen. Each case has to be considered individually.

The chapter is structured as follows. In section 5.2 we introduce a three-state semi-Markov model on a perturbed Brownian motion with drift, which has been introduced by Dassios and Wu [2009a]. This perturbed Brownian motion has the same behaviour as a drifted Brownian motion, except it moves toward the other side of the barrier by a jump of size ϵ each time it hits 0, disposing of the difficulty of the origin being regular. The semi-Markov process allows us to define an infinitesimal generator where the solution of the martingale problem provides us with the single Laplace transform of Parisian and hitting times in section 5.3. Dividing up into the two possible cases in section 5.3.1 and 5.3.2 we derive an explicit form of the double Laplace transform of hitting and Parisian times for drifted Brownian motion. Section 5.4 is devoted to the application to option pricing and introduces the MinParisianHit and the MaxParisianHit option. Using results about the double Laplace transform, we will be able to price

ParisianHit options.

5.2 Perturbed Brownian motion and the Martingale problem

Let W^μ be a drifted Brownian motion under the probability measure $\mathbb{Q}$ and define for fixed time t the endpoints for the excursion interval $(g_t(W^\mu), d_t(W^\mu))$ as in section 4.2. $H_b(W^\mu)$ denotes the first hitting of b and $H_{a,b}(W^\mu)$ denotes the first exit time of interval (a, b) where $a < b$ and $a, b \in \mathbb{R}_0^+$. $\tau_d^+(W^\mu)$ is the Parisian time above zero.

$$
\begin{aligned}
H_b(W^\mu) &= \inf\{t > 0 | W_t^\mu = b\},\\
H_{a,b}(W^\mu) &= \inf\{t > 0 | W_t^\mu = a \text{ or } W_t^\mu = b\},\\
\tau_d^+(W^\mu) &= \inf\{t > 0 | (t - g_t(W^\mu))\mathbf{1}_{W_t^\mu > 0} \geq d\}.
\end{aligned}
$$

We define the perturbed Brownian motion $W^{\epsilon,\mu}$ as in section 4.2, equation (4.10). Recall that this is necessary to escape the occurrence of infinitely many small excursions around the origin. So far, we are only concerned about two states, namely above and below zero. Since we will be working with the hitting time $H_b(W^\mu)$, we construct an artificial absorbing state for the time the process $W^{\epsilon,\mu}$ spends above a specified barrier $b > 0$. We introduce a new process based on

$W^{\epsilon,\mu}$ by

$$\hat{X}_t = \begin{cases} 2 & \text{, if } W_t^{\epsilon,\mu} \geq b \\ 1 & \text{, if } 0 < W_t^{\epsilon,\mu} < b \\ -1 & \text{, if } W_t^{\epsilon,\mu} < 0. \end{cases} \tag{5.1}$$

We will see later, that we are only concerned about the minimum of hitting and Parisian times of the perturbed process, even though we will also derive the joint law of the maximum of both times. Hence, we define state 2 to be an absorbing state, i.e. once b is hit, the process does not return to state 1 anymore. Define $U_t(\hat{X}) := t - g_t(\hat{X})$ to be the time elapsed in the current state, either state -1 or state 1 and 2 combined. Note, that $U_t(\hat{X})$ only distinguishes between above or below zero and converges to $U_t(W^\mu) = t - g_t(W^\mu)$, the time elapsed above or below zero in the current excursion of the drifted Brownian motion W^μ. If the notation is unambiguous, we will abbreviate the definition of the time elapsed for the Brownian motion, $U_t = U_t(W^\mu)$. $(\hat{X}_t, U_t(\hat{X}))$ becomes a Markov process. Hence, $\hat{X}$ is a three-state semi-Markov process with state space $\{2, 1, -1\}$. The transition intensity $\hat{\lambda}_{ij}(u)$ for $\hat{X}$ is defined similarly as in (4.12), (4.13):

$$\mathbb{Q}\left(\hat{X}_{t+\Delta t} = j, i \neq j | \hat{X}_t = i, U_t(\hat{X}) = u\right) = \hat{\lambda}_{ij}(u)\Delta t + o(\Delta t) \tag{5.2}$$

$$\mathbb{Q}\left(\hat{X}_{t+\Delta t} = i | \hat{X}_t = i, U_t(\hat{X}) = u\right) = 1 - \sum_{j \neq i} \hat{\lambda}_{ij}(u)\Delta t + o(\Delta t) \tag{5.3}$$

for $i, j = 2, 1, -1$. Define the survival probability and transition density by

$$\bar{Q}_i(t) = e^{-\int_0^t \sum_{j \neq i} \hat{\lambda}_{ij}(v)dv}, \tag{5.4}$$

$$q_{ij}(t) = \hat{\lambda}_{ij}(t)\bar{Q}_i(t). \tag{5.5}$$

In order to simplify notations we define $\hat{Q}_{ij}(\beta)$ and $\tilde{Q}_{ij}(\beta)$ to be

$$\hat{Q}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} q_{ij}(s)ds, \tag{5.6}$$

$$\tilde{Q}_{ij}(\beta) = \int_0^{\infty} e^{-\beta s} q_{ij}(s)ds. \tag{5.7}$$

We consider a bounded function $f : \{2, 1, -1\} \times \mathbb{R}^2 \to \mathbb{R}$. The infinitesimal generator $\mathcal{A}$ is an operator making

$$f(\hat{X}_t, U_t(\hat{X}), t) - \int_0^t \mathcal{A}f(\hat{X}_s, U_s(\hat{X}), s)ds$$

a martingale. We shall use the shortcut $f_i(z, u) = f(i, z, u)$ and $\mathcal{A}f_{\hat{X}_t}(U_t(\hat{X}), t) = \mathcal{A}f(\hat{X}_t, U_t(\hat{X}), t)$.

Hence, solving $\mathcal{A}f = 0$, subject to certain conditions, will provide us with martingales of the form $f_{\hat{X}_t}(U_t(\hat{X}), t)$, to which we can apply the optional sampling theorem to obtain the Laplace transforms of interest. We have for the generator

$$\mathcal{A}f_1(u, t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \hat{\lambda}_{1,1}(u)\left(f_{-1}(0, t) - f_1(u, t)\right) + \\ + \hat{\lambda}_{12}(u)\left(f_2(u, t) - f_1(u, t)\right),$$

$$\mathcal{A}f_{-1}(u, t) = \frac{\partial f_{-1}}{\partial t} + \frac{\partial f_{-1}}{\partial u} + \hat{\lambda}_{-11}(u)\left(f_1(0, t) - f_{-1}(u, t)\right).$$

Since we are not interested in what happens after the absorbing state 2 has been reached, we do not define $\mathcal{A}f_2$, the generator starting from state 2.

We assume the function f having the form $f_i(u,t) = e^{-\beta t}h_i(u)$, where $\beta \in \mathbb{R}^+$ is a positive constant, and solve $\mathcal{A}f \equiv 0$ with the constraints $h_1(d) = B$ and $h_{-1}(\infty) = 0$ with constant B. Since state 2 is an absorbing state, we may assign any bounded function at will. We choose $h_2(u) = A\tilde{h}(u)$, where A is an arbitrary constant. The function $\tilde{h}$ will be motivated and defined in the proof of Proposition 5.3.2. The intuition behind choosing the constraint $h_{-1}(\infty) = 0$ is, that in this chapter we are not concerned with the time elapsed below zero, hence, we let the excursion window below zero approach infinity. A and B on the other hand are constants, indicating different scenarios and clarified in Lemma 5.2.2.

Lemma 5.2.1. *Using the conditions above, the initial value of the function* $f_1(0,0) = h_1(0)$ *is given by*

$$h_1(0) = \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{12}(w)dw}{1 - \tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}. \tag{5.8}$$

Proof. $\mathcal{A}f \equiv 0$ transforms into

$$\frac{dh_1(u)}{du} - (\beta + \hat{\lambda}_{1,-1}(u) + \hat{\lambda}_{12}(u))h_1(u) + \hat{\lambda}_{1,-1}(u)h_{-1}(0) + A\hat{\lambda}_{12}(u)\tilde{h}(u) = 0,$$
$$\frac{dh_{-1}(u)}{du} - (\beta + \hat{\lambda}_{-1,1}(u))h_{-1}(u) + \hat{\lambda}_{-1,1}(u)h_1(0) = 0.$$

Using the integrating factor method for ordinary differential equations and the constraints we find

$$
\begin{aligned}
h_1(u) = Be^{-\int_u^d \beta\hat{\lambda}_{1,-1}(v)+\hat{\lambda}_{12}(v)dv} + \int_u^d \big(\hat{\lambda}_{1,-1}(w)h_{-1}(0)+ \\
+ A\hat{\lambda}_{12}(w)\tilde{h}(w)\big)e^{-\int_u^w \beta\hat{\lambda}_{1,-1}(v)+\hat{\lambda}_{12}(v)dv}dw\ , \quad 0 \le u \le d \\
h_{-1}(u) = h_1(0)\int_u^\infty \hat{\lambda}_{-1,1}(w)e^{-\int_u^w \beta+\hat{\lambda}_{-1,1}(v)dv}dw\ , \qquad u \ge 0.
\end{aligned}
$$

Setting $u = 0$ and solving the system of equations gives us

$$
\begin{aligned}
h_1(0) &= \frac{Be^{-\int_0^d \beta+\hat{\lambda}_{1,-1}(v)+\hat{\lambda}_{12}(v)dv} + A\int_0^d \hat{\lambda}_{12}(w)\tilde{h}(w)e^{-\int_0^w \beta+\hat{\lambda}_{1,-1}(v)+\hat{\lambda}_{12}(v)dv}dw}{1-\int_0^\infty \hat{\lambda}_{-1,1}(w)e^{-\int_0^w \beta+\hat{\lambda}_{-1,1}(v)dv}dw\int_o^d \hat{\lambda}_{1,-1}e^{-\int_0^w \beta+\hat{\lambda}_{1,-1}(v)+\hat{\lambda}_{12}(v)dv}dw} \\
&= \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{12}(w)dw}{1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)},
\end{aligned}
$$

where $\bar{Q}_i(t)$, $q_{12}(t)$, $\hat{\lambda}_{ij}(u)$, $\hat{Q}_{ij}(\beta)$ and $\tilde{Q}_{ij}(\beta)$ have been defined in (5.4), (5.5), (5.2), (5.6) and (5.7).

□

For the transition densities we use results from Borodin and Salminen [2002] (formula (2.0.2) and formulae (3.0.2), (3.0.6)). Without loss of generality we assume $b > \epsilon > 0$. Therefore, it is not possible to go straight from state -1 to state 2 and vice versa, i.e. $q_{-1,2}(t) = q_{2,-1}(t) = 0$.

With the following definition

$$
H_{a,b}(Y) = \inf\{t \ge 0 | Y_t = a \text{ or } Y_t = b\}
$$

for the first exit time of interval (a, b) with $a, b \in \mathbb{R}$ and $a < b$ by a general stochastic process Y, and the function

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)y-x)^2}{2t}},$$

[see e.g. Borodin and Salminen, 2002, Appendix 2, 9. Theta functions of imaginary argument and related functions], the quantities $q_{ij}(t)$, $\hat{Q}_{ij}(\beta)$, $\tilde{Q}_{ij}(\beta)$ and $\bar{Q}_i(d)$ can be calculated:

$$\begin{aligned}
q_{1,-1}(t) &= \frac{1}{dt}\mathbb{P}_\epsilon(H_{0,b}(W^{\epsilon,\mu}) \in dt, W^{\epsilon,\mu}_{H_{0,b}} = 0) = e^{-\mu\epsilon - \frac{\mu^2 t}{2}} ss_t(b-\epsilon, b) \\
&= e^{-\mu\epsilon - \frac{\mu^2 t}{2}} \sum_{k=-\infty}^{\infty} \frac{\epsilon + 2kb}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon+2kb)^2}{2t}} \\
&= e^{-\mu\epsilon - \frac{\mu^2 t}{2}} \sum_{k=0}^{\infty} \left[\frac{2kb+\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2kb+\epsilon)^2}{2t}} - \frac{2kb-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2kb-\epsilon)^2}{2t}} \right] - \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon+\mu t)^2}{2t}}
\end{aligned}$$

$$q_{-1,1}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon-\mu t)^2}{2t}}$$

$$\begin{aligned}
q_{12}(t) &= \frac{1}{dt}\mathbb{P}_\epsilon(H_{0,b}(W^{\epsilon,\mu}) \in dt, W^{\epsilon,\mu}_{H_{0,b}} = b) = e^{\mu(b-\epsilon) - \frac{\mu^2 t}{2}} ss_t(\epsilon, b) \\
&= e^{\mu(b-\epsilon) - \frac{\mu^2 t}{2}} \sum_{k=-\infty}^{\infty} \frac{b - \epsilon + 2kb}{\sqrt{2\pi t^3}} e^{-\frac{(b-\epsilon+2kb)^2}{2t}} \\
&= e^{\mu(b-\epsilon) - \frac{\mu^2 t}{2}} \sum_{k=0}^{\infty} \frac{(2k+1)b - \epsilon}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)b-\epsilon)^2}{2t}} - \frac{(2k+1)b + \epsilon}{\sqrt{2\pi t^3}} e^{-\frac{((2k+1)b+\epsilon)^2}{2t}}
\end{aligned}$$

$$\hat{Q}_{12}(\beta)=\sum_{k=0}^{\infty}e^{(\mu-(2k+1)\sqrt{2\beta+\mu^2})b}\Bigg[e^{\epsilon(\sqrt{2\beta+\mu^2}-\mu)}\mathcal{N}\left(-\frac{(2k+1)b-\epsilon}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)-$$
$$-e^{-\epsilon(\sqrt{2\beta+\mu^2}+\mu)}\mathcal{N}\left(-\frac{(2k+1)b+\epsilon}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)\Bigg]+$$
$$+e^{(\mu+(2k+1)\sqrt{2\beta+\mu^2})b}\Bigg[e^{-\epsilon(\sqrt{2\beta+\mu^2}+\mu)}\mathcal{N}\left(-\frac{(2k+1)b-\epsilon}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)-$$
$$-e^{\epsilon(\sqrt{2\beta+\mu^2}-\mu)}\mathcal{N}\left(-\frac{(2k+1)b+\epsilon}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]$$

$$\tilde{Q}_{-1,1}(\beta)=e^{(\mu-\sqrt{2\beta+\mu^2})\epsilon}$$

$$\hat{Q}_{1,-1}(\beta)=\int_{s=0}^{d}e^{-\beta s}e^{-\mu\epsilon-\frac{\mu^2 s}{2}}\sum_{k=0}^{\infty}\left[\frac{2kb+\epsilon}{\sqrt{2\pi s^3}}e^{-\frac{(2kb+\epsilon)^2}{2s}}-\frac{2kb-\epsilon}{\sqrt{2\pi s^3}}e^{-\frac{(2kb-\epsilon)^2}{2s}}\right]-$$
$$-e^{-\beta s}\frac{\epsilon}{\sqrt{2\pi s^3}}e^{-\frac{(\epsilon+\mu t)^2}{2s}}ds$$
$$=e^{-\mu\epsilon}\Bigg\{\sum_{k=0}^{\infty}\Bigg[e^{-\sqrt{2\beta+\mu^2}(2kb+\epsilon)}\mathcal{N}\left(-\frac{2kb+\epsilon}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)+$$
$$+e^{\sqrt{2\beta+\mu^2}(2kb+\epsilon)}\mathcal{N}\left(-\frac{2kb+\epsilon}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)-e^{-\sqrt{2\beta+\mu^2}(2kb-\epsilon)}\times$$
$$\times\mathcal{N}\left(-\frac{2kb-\epsilon}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)-e^{\sqrt{2\beta+\mu^2}(2kb-\epsilon)}\mathcal{N}\left(-\frac{2kb-\epsilon}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]-$$
$$-e^{-\sqrt{2\beta+\mu^2}\epsilon}\mathcal{N}\left(-\frac{\epsilon}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)-e^{\sqrt{2\beta+\mu^2}\epsilon}\mathcal{N}\left(-\frac{\epsilon}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg\}$$

$$
\begin{aligned}
\bar{Q}_1(d) &= \mathbb{P}_\epsilon(H_0(W^{\epsilon,\mu}) > d, H_b(W^{\epsilon,\mu}) > d)\\
&= \int_d^\infty e^{-\frac{\mu^2 t}{2}} \left(e^{-\mu\epsilon} ss_t(b-\epsilon, b) + e^{\mu(b-\epsilon)} ss_t(\epsilon, b)\right) dt\\
&= \sum_{k=0}^{\infty} \Bigg\{ e^{-\mu(2kb+2\epsilon)} \mathcal{N}\left(\frac{2kb+\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) - e^{2kb\mu}\mathcal{N}\left(-\frac{2kb+\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) -\\
&\quad - e^{-2kb\mu}\mathcal{N}\left(\frac{2kb-\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) + e^{\mu(2kb-2\epsilon)}\mathcal{N}\left(-\frac{2kb-\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) +\\
&\quad + e^{-2kb\mu}\mathcal{N}\left(\frac{(2k+1)b-\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) - e^{2kb\mu+2\mu(b-\epsilon)}\mathcal{N}\left(-\frac{(2k+1)b-\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) -\\
&\quad - e^{-2kb\mu-2\mu\epsilon}\mathcal{N}\left(\frac{(2k+1)b+\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) + e^{2kb\mu+2\mu b}\mathcal{N}\left(-\frac{(2k+1)b+\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) \Bigg\} -\\
&\quad - e^{-2\mu\epsilon}\mathcal{N}\left(\frac{\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right) + \mathcal{N}\left(-\frac{\epsilon}{\sqrt{d}} - \mu\sqrt{d}\right)
\end{aligned}
$$

Lemma 5.2.2. *For the perturbed Brownian motion with drift, we find the Laplace transform to be*

$$
A\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\epsilon,\mu})}\tilde{h}(U_{H_b}(W^{\epsilon,\mu}))\mathbf{1}_{H_b(W^{\epsilon,\mu})<\tau_d^+(W^{\epsilon,\mu})}\right) + B\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right)
= \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{12}(w)dw}{1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}, \quad (5.9)
$$

where A and B are arbitrary constants.

Proof. Solving $\mathcal{A}f \equiv 0$ with constraints $h_1(d) = B$ and $h_{-1}(\infty) = 0$, provides us with a martingale of the form $\hat{\hat{M}}_t := f_{\hat{X}_t}(U_t(\hat{X}), t) = e^{-\beta t}h_{\hat{X}_t}(U_t(\hat{X}))$. Recall that state 2, which denotes the perturbed Brownian motion above barrier b, is an absorbing state. Hence, we may choose h_2 to be any arbitrary bounded function. We assign h_2 to be $h_2(u) = A\tilde{h}(u)$, where A is a constant and $\tilde{h}$ is a bounded function, which will be specified in the proof of Proposition 5.3.2.

Let $\tau(W^{\epsilon,\mu}) = \min\{H_b(W^{\epsilon,\mu}), \tau_d^+(W^{\epsilon,\mu})\}$, then optional sampling theorem applied to martingale $\hat{\hat{M}}_t$ with stopping time $\tau(W^{\epsilon,\mu}) \wedge t$ yields

$$\mathbb{E}_\epsilon^\mathbb{Q}\left(\hat{\hat{M}}_{\tau(W^{\epsilon,\mu})\wedge t}\right) = \mathbb{E}_\epsilon^\mathbb{Q}(\hat{\hat{M}}_0). \tag{5.10}$$

$h_1(u)$ is a continuous function and therefore bounded on the compact interval $[0, d]$. Hence, there exists a constant K, such that $|h_1(U_t(\hat{X}))| \leq K$ for all $U_t(\hat{X}) \in [0, d]$. Furthermore, we have assumed that $h_2(u)$ is a bounded function. Therefore Lebesgue's Dominated Convergence Theorem applies, yielding for the l.h.s. of (5.10):

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_\epsilon^\mathbb{Q}\left(\hat{\hat{M}}_{\tau(W^{\epsilon,\mu})\wedge t}\right) &= \mathbb{E}_\epsilon^\mathbb{Q}\left(\hat{\hat{M}}_{\tau(W^{\epsilon,\mu})}\right) \\
&= \mathbb{E}_\epsilon^\mathbb{Q}\left(e^{-\beta H_b(W^{\epsilon,\mu})} h_2(U_{H_b(W^{\epsilon,\mu})}(W^{\epsilon,\mu}))\mathbf{1}_{H_b(W^{\epsilon,\mu})<\tau_d^+(W^{\epsilon,\mu})}\right) + \\
&\quad + \mathbb{E}_\epsilon^\mathbb{Q}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})} h_1(U_{\tau_d^+(W^{\epsilon,\mu})}(W^{\epsilon,\mu}))\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right) \\
&= A\mathbb{E}_\epsilon^\mathbb{Q}\left(e^{-\beta H_b(W^{\epsilon,\mu})} \tilde{h}(U_{H_b(W^{\epsilon,\mu})}(W^{\epsilon,\mu}))\mathbf{1}_{H_b(W^{\epsilon,\mu})<\tau_d^+(W^{\epsilon,\mu})}\right) + \\
&\quad + B\mathbb{E}_\epsilon^\mathbb{Q}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right).
\end{aligned}$$

For the r.h.s. of (5.10) we have $\mathbb{E}_\epsilon^\mathbb{Q}(\hat{\hat{M}}_0) = h_1(0)$ and the claim follows from Lemma 5.2.1.

□

5.3 Laplace transform of Parisian and Hitting times

This section is the main part of this chapter and devoted to finding the double Laplace transform of Parisian and hitting times. We firstly derive the limiting Laplace transform through results on the perturbed process and distinguish between the two possible scenarios $H_b(W^\mu) < \tau_d^+(W^\mu)$ and $\tau_d^+(W^\mu) < H_b(W^\mu)$.

Proposition 5.3.1. *The Laplace transform of the hitting and Parisian times for drifted Brownian motion W^μ is given by*

$$
\begin{aligned}
&A\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right) + B\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right) = \\
&\quad = \left\{Be^{-\beta d}\left(\sum_{k=0}^{\infty} 2\left[z(k,0,\mu) - e^{\mu b}z(k+\frac{1}{2},0,\mu)\right] - z(0,0,\mu)\right) + \right.\\
&\quad\left. + A\int_0^d e^{-\beta w}\tilde{h}(w)\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw\right\}\times \\
&\quad \times\left\{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right] - z(0,\beta,\mu) - 2\sqrt{2\beta+\mu^2}\right\}^{-1},
\end{aligned}
$$

where the function z is defined as

$$
\begin{aligned}
&z(k,\beta,\mu) = \\
&\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}} - \sqrt{2\beta+\mu^2}\left(e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)+\right. \\
&\qquad\qquad\left. + e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\right). \quad (5.11)
\end{aligned}
$$

Proof. In order to find the Laplace transform for the drifted Brownian motion, we take the limit from results about $W^{\epsilon,\mu}$ and therefore we let ϵ approach zero in equation (5.9). In particular, notice that by construction we have $W_t^{\epsilon,\mu} \xrightarrow{a.s.} W_t^{\mu}$ for all $t \geq 0$ as ϵ approaches zero. The quantities defined based on $W_t^{\epsilon,\mu}$ also converge to those of the drifted Brownian motion W_t^{μ}. Furthermore, $e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})$ and $e^{-\beta \tau_d^+(W^\mu)}$ are both bounded functions. Recall, that U_{H_b} is the abbreviation for $U_{H_b(W^\mu)}(W^\mu)$. Thus dominated convergence applies to get the result for W_t^μ,

$$
\begin{aligned}
&A\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right) + B\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right) \\
&\quad= \lim_{\epsilon\to 0} A\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\epsilon,\mu})}\tilde{h}(U_{H_b(W^{\epsilon,\mu})}(W^{\epsilon,\mu}))\mathbf{1}_{H_b(W^{\epsilon,\mu})<\tau_d^+(W^{\epsilon,\mu})}\right) + \\
&\qquad + B\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right) \\
&\quad= \lim_{\epsilon\to 0} \frac{Be^{-\beta d}\bar{Q}_1(d) + A\int_0^d e^{-\beta w}\tilde{h}(w)q_{12}(w)dw}{1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)}
\end{aligned} \tag{5.12}
$$

We refer to Dassios and Wu [2009a], Dassios and Wu [2011a] and Lim [2013] for further details. Therefore, letting ϵ go to zero in the result of Lemma 5.2.2 will provide us with the Laplace transform for the drifted Brownian motion. In order to apply L'Hôpital's rule, we take the derivative with respect to ϵ and find for the denominator of (5.9):

$$\frac{\partial}{\partial\epsilon}\left(1-\tilde{Q}_{-1,1}(\beta)\hat{Q}_{1,-1}(\beta)\right)\underset{\epsilon\to 0}{\longrightarrow}$$

$$\sum_{k=0}^{\infty}\Bigg(2\sqrt{2\beta+\mu^2}\Bigg[e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}+\sqrt{(2\beta+\mu^2)d}\right)-$$

$$-e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]+2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}\Bigg)-$$

$$-2\sqrt{2\beta+\mu^2}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d}\right)-\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}}$$

$$=\sum_{k=0}^{\infty}\Bigg(2\sqrt{2\beta+\mu^2}\Bigg[e^{-\sqrt{2\beta+\mu^2}2kb}-e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)-$$

$$-e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\Bigg]+2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}}\Bigg)-$$

$$-2\sqrt{2\beta+\mu^2}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d}\right)-\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}}\qquad(5.13)$$

For the numerator we find

$$\frac{\partial}{\partial\epsilon}\bar{Q}_1(d)\underset{\epsilon\to 0}{\longrightarrow}\sum_{k=0}^{\infty}\Bigg\{2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2kb)^2}{2d}-\frac{\mu^2 d}{2}}-2\sqrt{\frac{2}{\pi d}}e^{-\frac{(2k+1)^2b^2}{2d}-\frac{\mu^2 d}{2}+\mu b}+$$

$$+2\mu\Bigg[e^{(2k+1)\mu b+\mu b}\mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}}-\mu\sqrt{d}\right)+e^{-(2k+1)\mu b+\mu b}\mathcal{N}\left(\frac{(2k+1)b}{\sqrt{d}}-\mu\sqrt{d}\right)-$$

$$-e^{2k\mu b}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\mu\sqrt{d}\right)-e^{-2k\mu b}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\mu\sqrt{d}\right)\Bigg]\Bigg\}-\sqrt{\frac{2}{\pi d}}e^{-\frac{\mu^2 d}{d}}+2\mu\mathcal{N}\left(-\mu\sqrt{d}\right)$$

$$(5.14)$$

and

$$\frac{\partial}{\partial\epsilon}q_{12}(t)\underset{\epsilon\to 0}{\longrightarrow}\sqrt{\frac{2}{\pi t^3}}e^{\mu b-\frac{\mu^2 t}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{t}-1\right)e^{-\frac{(2k+1)^2b^2}{2t}}\qquad(5.15)$$

Inserting calculations (5.13), (5.14) and (5.15) into equation (5.12) yields the proposition.

□

5.3.1 Case $H_b(W^\mu) < \tau_d^+(W^\mu)$

In the case where the barrier b is hit before the excursion above zero of length d is completed, we have found the single Laplace transform of the hitting time of the drifted Brownian motion in Proposition 5.3.1.

Lemma 5.3.1.

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)$$
$$= \frac{\int_0^d e^{-\beta w}\tilde{h}(w)\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw}{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right]-z(0,\beta,\mu)-2\sqrt{2\beta+\mu^2}},$$

where z is defined as in (5.11)

$$z(k,\beta,\mu) =$$
$$\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\beta+\mu^2)d}{2}-\frac{2(kb)^2}{d}} - \sqrt{2\beta+\mu^2}\left(e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)+\right.$$
$$\left.+\,e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\frac{2kb}{\sqrt{d}}-\sqrt{(2\beta+\mu^2)d}\right)\right). \quad (5.16)$$

We are now interested in finding the double Laplace transform of hitting and Parisian times in the case that b is hit before excursion exceeds d. We will now make an appropriate choice of the bounded function $\tilde{h}$, where the intuition will become clear in the proof of the following Proposition.

Proposition 5.3.2. *The double Laplace transform of hitting and Parisian times of a drifted Brownian motion W^μ, where $H_b(W^\mu) < \tau_d^+(W^\mu)$, is*

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right) = \\
&\int_0^d e^{-\beta w}\left[e^{-\gamma d}\left(1-e^{-2\mu b}\mathcal{N}\left(\frac{\mu(d-w)-b}{\sqrt{d-w}}\right)-\mathcal{N}\left(\frac{-\mu(d-w)-b}{\sqrt{d-w}}\right)\right)+\right.\\
&\quad+\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+})\left(e^{-(\sqrt{2\gamma+\mu^2}+\mu)b}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)(d-w)}-\frac{b}{\sqrt{d-w}}\right)+\right.\\
&\quad\left.\left.+e^{\sqrt{2\gamma+\mu^2}-\mu)b}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)(d-w)}-\frac{b}{\sqrt{d-w}}\right)\right)\right]\times\\
&\quad\times\sqrt{\frac{2}{\pi w^3}}e^{\mu b-\frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2b^2}{w}-1\right)e^{-\frac{(2k+1)^2b^2}{2w}}dw\times\\
&\times\left\{\sum_{k=0}^{\infty}2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right]-z(0,\beta,\mu)-2\sqrt{2\beta+\mu^2}\right\}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+(W^\mu)}) &= \frac{2\mu e^{-\gamma d}\mathcal{N}\left(\mu\sqrt{d}\right)+\sqrt{\frac{2}{\pi d}}e^{-\gamma d-\frac{\mu^2 d}{2}}}{2\sqrt{2\gamma+\mu^2}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}\right)+\sqrt{\frac{2}{\pi d}}e^{-\frac{(2\gamma+\mu^2)d}{2}}}\\
&= \frac{e^{-\gamma d}(z(0,0,\mu)+2\mu)}{z(0,\gamma,\mu)+2\sqrt{2\gamma+\mu^2}},
\end{aligned}$$

and the function z is defined in equation (5.11).

Proof. In order to find the double Laplace transform

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}e^{-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)$$

in the case where $H_b(W^\mu) < \tau_d^+(W^\mu)$, we define our previously generic function $\tilde{h}$ to be

$$\tilde{h}(U_{H_b}) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\big|\mathcal{F}_{H_b(W^\mu)}\right),$$

where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the standard filtration associated with the Brownian motion. Hence, the l.h.s. of Lemma 5.3.1 becomes

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\tilde{h}(U_{H_b})\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)\\
&\quad= \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\big|\mathcal{F}_{H_b(W^\mu)}\right)\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)\\
&\quad= \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_b(W^\mu)}e^{-\gamma\tau_d^+(W^\mu)}\mathbf{1}_{H_b(W^\mu)<\tau_d^+(W^\mu)}\right)
\end{aligned}$$

with our choice of $\tilde{h}$. On the other hand, we have

$$\begin{aligned}
\tilde{h}(U_{H_b}) &= \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma(H_b(W^\mu)+d-U_{H_b})}\mathbf{1}_{\tilde{H}_0(W^\mu)>d-U_{H_b}}\big|\mathcal{F}_{H_b(W^\mu)}\right)+\\
&\quad+\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma(H_b(W^\mu)+\tilde{H}_0(W^\mu)+\hat{\tau}_d^+(W^\mu))}\mathbf{1}_{\tilde{H}_0<d-U_{H_b}}\big|\mathcal{F}_{H_b(W^\mu)}\right)\\
&= e^{-\gamma H_b(W^\mu)}\Bigg[e^{-\gamma(d-U_{H_b})}\mathbb{P}_b(\tilde{H}_0(W^\mu)>d-U_{H_b})+\\
&\quad+\mathbb{E}_b^{\mathbb{Q}}\left(e^{-\gamma\tilde{H}_0(W^\mu)}\mathbf{1}_{\tilde{H}_0(W^\mu)<d-U_{H_b}}\right)\mathbb{E}_0^{\mathbb{Q}}(e^{-\gamma\hat{\tau}_d^+(W^\mu)})\Bigg],
\end{aligned}$$

where $\tilde{H}_0(W^\mu)$ is the first hitting time of zero restarted at time $H_b(W^\mu)$ and hence independent of $H_b(W^\mu)$ and $\hat{\tau}_d^+(W^\mu)$ is the first time the excursion lasts time d above zero restarted at time $\tilde{H}_0(W^\mu)$ and therefore also independent of

$H_b(W^\mu)$. For the derivation of the Laplace transform of $\hat{\tau}_d^+(W^\mu)$, we refer to Appendix 7.2. It is shown there, that

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\right) = \frac{2\mu e^{-\gamma d}\mathcal{N}\left(\mu\sqrt{d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\gamma d - \frac{\mu^2 d}{2}}}{2\sqrt{2\gamma+\mu^2}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\frac{(2\gamma+\mu^2)d}{2}}}.$$

For the other quantities, straightforward calculation yields

$$\begin{aligned}
\mathbb{P}_b(\tilde{H}_0(W^\mu) > d - U_{H_b}) &= \int_{d-U_{H_b}}^{\infty} \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{(b+\mu t)^2}{2t}} dt \\
&= 1 - e^{-2\mu b}\mathcal{N}\left(\frac{\mu(d-U_{H_b}) - b}{\sqrt{d-U_{H_b}}}\right) - \mathcal{N}\left(\frac{-\mu(d-U_{H_b}) - b}{\sqrt{d-U_{H_b}}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_b^{\mathbb{Q}}\left(e^{-\gamma\tilde{H}_0(W^\mu)}\mathbf{1}_{\tilde{H}_0 < d - U_{H_b}}\right) &= e^{-(\sqrt{2\gamma+\mu^2}+\mu)b}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)(d-U_{H_b})} - \frac{b}{\sqrt{d-U_{H_b}}}\right) \\
&\quad + e^{\sqrt{2\gamma+\mu^2}-\mu)b}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)(d-U_{H_b})} - \frac{b}{\sqrt{d-U_{H_b}}}\right)
\end{aligned}$$

Inserting these calculations into Lemma 5.3.1 yields the proposition.

□

Remark 5.3.1. The single Laplace transform of $\hat{\tau}_d^+(W^\mu)$ can be derived by setting $A = 0$, $B = 1$ and letting b approach infinity in Proposition 5.3.1. Notice that $\hat{\tau}_d^+(W^\mu)$ and $\tau_d^+(W^\mu)$ are identically distributed, due to the strong Markov property of the Brownian motion. It immediately yields

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\right) = \frac{e^{-\gamma d}(z(0,0,\mu)+2\mu)}{z(0,\gamma,\mu)+2\sqrt{2\gamma+\mu^2}},$$

where the 2μ in the numerator comes in from the odd case in equation (5.14). However, we find it easier and more intuitive to use a two-state semi-Markov model. This will be demonstrated in the Appendix 7.2. Certainly, with both methods the results coincide.

5.3.2 Case $\tau_d^+(W^\mu) < H_b(W^\mu)$

In the case where the excursion has exceeded length d before hitting the barrier $b > 0$, we conclude from Proposition 5.3.1

Lemma 5.3.2.

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right)$$
$$= \frac{e^{-\beta d}\left\{\sum_{k=0}^{\infty} 2\left[z(k,0,\mu) - e^{\mu b}z(k+\frac{1}{2},0,\mu)\right] - z(0,0,\mu)\right\}}{\sum_{k=0}^{\infty} 2\left[z(k,\beta,\mu)+\sqrt{2\beta+\mu^2}e^{-\sqrt{2\beta+\mu^2}2kb}\right] - z(0,\beta,\mu) - 2\sqrt{2\beta+\mu^2}}$$

where the function z is defined in equation (5.11).

This lemma allows us to compute the probability, that the Parisian time happens before the hitting time of b by setting $\beta = \mu = 0$, as outlined in the following corollary.

Corollary 5.3.1. *For the standard Brownian motion W the probability that the excursion exceeds time d before hitting barrier b is given by*

$$\mathbb{Q}\left(\tau_d^+(W) < H_b(W)\right) = 1 - \frac{2\sum_{k=0}^{\infty} e^{-\frac{(2k+1)^2 b^2}{2d}} - 1}{2\sum_{k=0}^{\infty} e^{-\frac{(2kb)^2}{2d}} - 1}$$

Now, the double Laplace transform of hitting and Parisian times in the case where the excursion has exceeded length d before hitting b, can be derived.

Proposition 5.3.3. *The double Laplace transform of hitting and Parisian times for the drifted Brownian motion W^μ in the case where $\tau_d^+(W^\mu) < H_b(W^\mu)$ is given by*

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^\mu)-\gamma H_b(W^\mu)}\mathbf{1}_{\tau_d^+(W^\mu)<H_b(W^\mu)}\right) = \\
&\qquad\left\{e^{-\beta d}\left[e^{-b(\sqrt{2\gamma+\mu^2}-\mu)}\mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{b(\sqrt{2\gamma+\mu^2}-\mu)}\times\right.\right. \\
&\left.\left.\times\mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\right]\sum_{k=0}^{\infty} 2\left[z(k,0,\mu) - e^{\mu b}z(k+\frac{1}{2},0,\mu)\right] - z(0,0,\mu)\right\}\times \\
&\qquad\times\left\{\left[\sum_{k=0}^{\infty} 2\left[z(k,\beta+\gamma,\mu) + \sqrt{2(\beta+\gamma)+\mu^2}e^{-\sqrt{2(\beta+\gamma)+\mu^2}2kb}\right] - \right.\right. \\
&\left.\left. -z(0,\beta+\gamma,\mu) - 2\sqrt{2(\beta+\gamma)+\mu^2}\right]\left[1 - \mathcal{N}\left(\frac{\mu d - b}{\sqrt{d}}\right) - e^{2\mu b}\mathcal{N}\left(\frac{-\mu d - b}{\sqrt{d}}\right)\right]\right\}^{-1},
\end{aligned}$$

where the function z is defined by (5.11).

Proof. In order to find the double Laplace transform in this case, we define a new infinitesimal generator for the perturbed Brownian motion $W^{\epsilon,\mu}$ starting at time

$\tau_d^+(W^{\epsilon,\mu})$. We can do this due to the strong Markov property of the Brownian motion. State 2, which stands for $W^{\epsilon,\mu}$ above barrier b, is an absorbing state, hence nothing comes back from there. Also, we are not concerned with state -1, which denotes $W^{\epsilon,\mu}$ below zero, because our excursion has already exceeded time d and we are now only interested in hitting b. With this motivation the generator becomes

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{12}(u)\left(f_2(u,t) - f_1(u,t)\right),$$

where we choose f_2 to be $f_2(u,t) = e^{-\gamma t}$. Since state 2 is absorbing, the function f_2 can be assigned arbitrarily. Note, that our choice of f_2 is a bounded function.

Furthermore, at time $\tau_d^+(W^{\epsilon,\mu})$ we are in state 1. Similar to the proof of Lemma 5.2.2, we solve $\mathcal{A}f \equiv 0$ in order to derive a martingale of the form $\hat{M}_t := f_{\hat{X}_t}(U_t(\hat{X}), t) = e^{-\beta t} h_{\hat{X}_t}(U_t(\hat{X}))$. However, notice that we have $f_1(d,0) = h_1(d)$, because by definition our time elapsed at starting time $\tau_d^+(W^{\epsilon,\mu})$ is d. Since we have already achieved an excursion above zero of length d, we are not concerned about any excursions any longer, hence we choose the constraint $h_1(\infty) = 0$. Solving $\mathcal{A}f \equiv 0$ yields

$$h_1(u) = \int_u^\infty \lambda_{12}(w) e^{-\int_u^w \gamma + \lambda_{12}(v)dv} dw \ , \qquad 0 \le u \le \infty,$$

where

$$\lambda_{12}(t) e^{-\int_0^t \lambda_{12}(v)dv} = p_{12}(t) = \mathbb{P}_\epsilon(H_b(W^\mu) \in dt) = \frac{b-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(b-\epsilon-\mu t)^2}{2t}}.$$

Hence,

$$
\begin{aligned}
h_1(d) &= \frac{e^{\gamma d}\int_d^{\infty} e^{-\gamma w} p_{12}(w)dw}{1-\int_0^d p_{12}(s)ds} \\
&= \Bigg\{ e^{\gamma d}\Bigg[e^{-(b-\epsilon)(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(\frac{b-\epsilon}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{(b-\epsilon)(\sqrt{2\gamma+\mu^2}-\mu)} \times \\
&\quad \times \mathcal{N}\left(-\frac{b-\epsilon}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\Bigg]\Bigg\} \times \Bigg\{ 1 - \mathcal{N}\left(\frac{\mu d-(b-\epsilon)}{\sqrt{d}}\right) - \\
&\quad - e^{2\mu(b-\epsilon)} \mathcal{N}\left(\frac{-\mu d-(b-\epsilon)}{\sqrt{d}}\right)\Bigg\}^{-1} \\
&\xrightarrow{\epsilon\to 0} \Bigg\{ e^{\gamma d}\Bigg[e^{-b(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right) - e^{b(\sqrt{2\gamma+\mu^2}-\mu)} \times \\
&\quad \times \mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d}\right)\Bigg]\Bigg\} \times \Bigg\{ 1 - \mathcal{N}\left(\frac{\mu d-b}{\sqrt{d}}\right) - \\
&\quad - e^{2\mu b} \mathcal{N}\left(\frac{-\mu d-b}{\sqrt{d}}\right)\Bigg\}^{-1}.
\end{aligned}
$$

As a result, we have found a martingale $\hat{M}_t := f_{\hat{X}_t}(U_t(\hat{X}), t)$ with $\hat{M}_0 = f_1(d, 0) = h_1(d)$. Also, with $\hat{\hat{H}}_b(W^{\epsilon,\mu})$ being the first hitting time of b of our process restarted at $\tau_d^+(W^{\epsilon,\mu})$ and hence $H_b(W^{\epsilon,\mu}) = \tau_d^+(W^{\epsilon,\mu}) + \hat{\hat{H}}_b(W^{\epsilon,\mu})$. Furthermore, note the following:

$$
\hat{M}_{\hat{\hat{H}}_b(W^{\epsilon,\mu})} = f_2(U_{\hat{\hat{H}}_b(W^{\epsilon,\mu})}(\hat{X}), \hat{\hat{H}}_b(W^{\epsilon,\mu})) = e^{-\gamma \hat{\hat{H}}_b(W^{\epsilon,\mu})}.
$$

Notice that at hitting time of b, the process $W^{\epsilon,\mu}$ is in state 2.

Hence, the optional sampling theorem on martingale $\hat{M}_t$ with stopping time $\hat{\hat{H}}_b(W^{\epsilon,\mu}) \wedge t$ yields

$$
\mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\epsilon,\mu})\wedge t}\right) = \mathbb{E}_{\epsilon}^{\mathbb{Q}}(\hat{M}_0).
$$

Notice, that by construction

$$\mathbb{E}_{\epsilon}^{\mathbb{Q}}(\hat{M}_0) = h_1(d).$$

Furthermore, $h_1(u)$ is continuous and decreasing due to the integral limit. Hence, there exists a constant K, such that $|h_1(U_t(\hat{X}))| \leq K$ for all $U_t(\hat{X})$. Therefore, Lebesgue's Dominated Convergence Theorem applies and we derive

$$\lim_{t\to\infty} \mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\epsilon,\mu})\wedge t}\right) = \mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(\hat{M}_{\hat{\hat{H}}_b(W^{\epsilon,\mu})}\right) = \mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\gamma\hat{\hat{H}}_b(W^{\epsilon,\mu})}\right).$$

Hence, $h_1(d) = \mathbb{E}_{\epsilon}^{\mathbb{Q}}(e^{-\gamma\hat{\hat{H}}_b(W^{\epsilon,\mu})})$ and the double Laplace becomes

$$\begin{aligned}
&\mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}e^{-\gamma H_b(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right)\\
&\quad= \mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\mathbb{E}_{\epsilon}^{\mathbb{Q}}(e^{-\gamma H_b(W^{\epsilon,\mu})}\big|\tau_d^+(W^{\epsilon,\mu}))\right)\\
&\quad= \mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\mathbb{E}_{\epsilon}^{\mathbb{Q}}(e^{-\gamma(\tau_d^+(W^{\epsilon,\mu})+\hat{\hat{H}}_b(W^{\epsilon,\mu}))}\big|\tau_d^+(W^{\epsilon,\mu}))\right)\\
&\quad= h_1(d)\mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-(\beta+\gamma)\tau_d^+(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^+(W^{\epsilon,\mu})<H_b(W^{\epsilon,\mu})}\right).
\end{aligned}$$

Together with Lemma 5.3.2 we conclude the proposition.

□

Remark 5.3.2. Until now we have only discussed the case, where the excursion above zero and level $b > 0$ is considered. In the case, where we are interested in excursions below zero and $b > 0$ we define the first time the excursion below zero exceeds time period d to be

$$\tau_d^-(W^\mu) = \inf\{t > 0 | \mathbf{1}_{W_t^\mu < 0}(t - g_t(W^\mu)) > d\}.$$

The infinitesimal generator naturally becomes

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \hat{\lambda}_{1,-1}(u)\left(f_{-1}(0,t) - f_1(u,t)\right) + \hat{\lambda}_{12}(u)\left(Ae^{-\beta t} - f_1(u,t)\right),$$
$$\mathcal{A}f_{-1}(u,t) = \frac{\partial f_{-1}}{\partial t} + \frac{\partial f_{-1}}{\partial u} + \hat{\lambda}_{-1,1}(u)\left(f_1(0,t) - f_{-1}(u,t)\right).$$

We solve $\mathcal{A}f = 0$ subject to $h_1(\infty) = 0$ and $h_{-1}(d) = B$ and find

$$A\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta H_b(W^{\epsilon,\mu})}\mathbf{1}_{H_b(W^{\epsilon,\mu}) < \tau_d^-(W^{\epsilon,\mu})}\right) + B\mathbb{E}_\epsilon^{\mathbb{Q}}\left(e^{-\beta \tau_d^-(W^{\epsilon,\mu})}\mathbf{1}_{\tau_d^-(W^{\epsilon,\mu}) < H_b(W^{\epsilon,\mu})}\right)$$
$$= \frac{A\tilde{Q}_{12}(\beta) + Be^{-\beta d}\tilde{Q}_{1,-1}(\beta)\bar{Q}_{-1}(d)}{1 - \tilde{Q}_{1,-1}(\beta)\hat{Q}_{-1,1}(\beta)}.$$

In both cases $H_b(W^{\epsilon,\mu}) < \tau_d^-(W^{\epsilon,\mu})$ and $\tau_d^-(W^{\epsilon,\mu}) < H_b(W^{\epsilon,\mu})$ we start a new Markov model and derive the double Laplace transform. The other two cases where $b < 0$ can be treated similarly.

5.4 Pricing ParisianHit Options

In the Black-Scholes framework, let $(S_t)_{t\geq 0}$ be the stock price process following a geometric Brownian motion, i.e. solving the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and call L the level. We define the times

$$g_{L,t}(S) = \sup\{s \leq t : S_s = L\},$$
$$d_{L,t}(S) = \inf\{s \geq t : S_s = L\}.$$

The trajectory of S between $g_{L,t}(S)$ and $d_{L,t}(S)$ is the excursion of S at level L, which straddles time t. The variables $g_{L,t}(S)$ and $d_{L,t}(S)$ are called the left and right ends of the excursion. Assuming that the interest rate r is constant, the process representing the risk neutral asset price is given by

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t},$$

solving the stochastic differential equation $dS_t = rS_t dt + \sigma S_t dW_t$. We denote the equivalent martingale measure by $\bar{\mathbb{Q}}$.

We define $\tau_d^+(S)$ as the first time the age of an excursion above L for the price process is greater or equal to d and $H_B(S)$ as the first hitting time of a barrier $B > L$, i.e.

$$\tau_{L,d}^+(S) = \inf\{t \geq 0 | \mathbf{1}_{S_t > L}(t - g_{L,t}^S) \geq d\},$$
$$H_B(S) = \inf\{t \geq 0 | S_t = B\}.$$

We introduce the notation

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right),$$
$$l = \frac{1}{\sigma}\ln\frac{L}{S_0},$$
$$b = \frac{1}{\sigma}\ln\frac{B}{S_0}$$

and define the process $(Z_t)_{t\geq 0} = (W_t + mt)_{t\geq 0}$. We write $S_t = S_0 e^{\sigma Z_t}$ with $Z_t = W_t + mt$. The condition $S_t \leq L$ becomes $Z_t \leq l$. Using Girsanov's theorem we introduce a new probability measure $\mathbb{Q}$, which makes Z a $\mathbb{Q}$-Brownian motion. The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}}\bigg|_{\mathcal{F}_T} = e^{mZ_T - \frac{m^2}{2}T}. \tag{5.17}$$

We define the first time at which the age of an excursion above the level l for the process $(Z_t)_{t\geq 0}$ is greater than or equal to d:

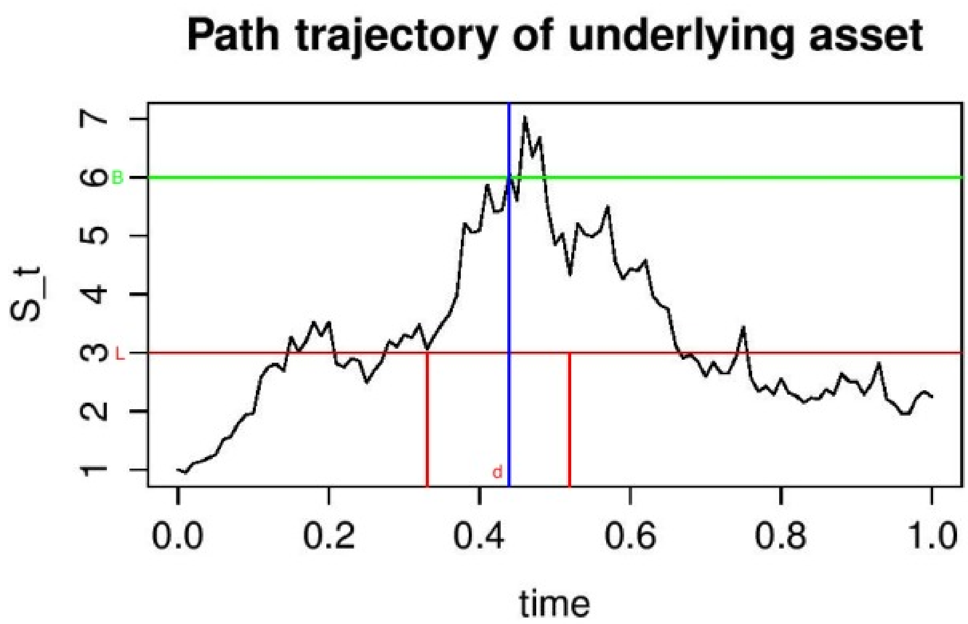
$$\tau^+_{l,d}(Z) = \inf\{t \geq 0 | \mathbf{1}_{Z_t > l}(t - g_{l,t}) \geq d\}$$
$$g_{l,t}(Z) = \sup\{u \leq t | Z_u = l\}$$

In the case where $l = 0$, we shall use the shortcut $\tau^+_d(Z)$ and $g_t(Z)$.

5.4.1 Option triggered at Minimum of Parisian and Hitting times

Our so-called MinParisianHit Option is triggered either when the age of an excursion above L reaches time d or a barrier $B > L$ is hit by the underlying price process S. More precisely, a MinParisianHit Up-and-In is activated at the minimum of both stopping times, i.e. $\min\{\tau^+_{L,d}(S), H_B(S)\}$. This time is illustrated by the blue line in Figure 5.1.

Figure 5.1: Minimum of Parisian and hitting times



To simplify calculations we assume from now on that the underlying process starts at the barrier, i.e. $S_0 = L$ or equivalently $l = 0$, hence we can use results from our three states Semi-Markov model. The more general case, where $S_0 \neq L$ and the strong Markov property of the Brownian motion applies, will be discussed in Appendix 7.3.

The MinParisianHit Up-and-In Call option has payoff

$$(S_T - K)^+ \mathbf{1}_{\min\{\tau^+_{L,d}(S), H_B(S)\} \leq T},$$

where K denotes the strike price.

Using risk-neutral valuation and Girsanov's change of measure (5.17), the price of this option can be written in the following way.

$$\begin{aligned}
&minPHC_i^u(S_0,T,K,L,d,r) = e^{-rT}\mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left((S_T-K)^+\mathbf{1}_{\min\{\tau_{L,d}^+(S),H_B(S)\}\le T}\right)\\
&\quad = e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_0^{\mathbb{Q}}\left((S_0e^{\sigma Z_T}-K)^+e^{mZ_T}\mathbf{1}_{\min\{\tau_d^+(Z),H_b(Z)\}\le T}\right)\\
&\quad = e^{-(r+\frac{1}{2}m^2)T}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0e^{\sigma z}-K)e^{mz}\mathbb{Q}_0\left(Z_T\in dz,\min\{\tau_d^+(Z),H_b(Z)\}\le T\right)
\end{aligned} \tag{5.18}$$

Hence, finding the fair price for a MinParisianHit option reduces to finding the joint probability of position at maturity and minimum of Parisian and hitting times.

Proposition 5.4.1. *The joint density of position at maturity and minimum of hitting and Parisian times for standard Brownian motion is*

$$\mathbb{Q}_0(Z_T\in dz,\min\{\tau_d^+(Z),H_b(Z)\}\le T) = \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}\times$$

$$\times\left[\frac{\sum\limits_{k=-\infty}^{\infty}\frac{w+2kb}{d}e^{-\frac{(w+2kb)^2}{2d}}}{2\sum\limits_{k=0}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}}-e^{-\frac{(2k+1)^2b^2}{2d}}\right)}\mathcal{L}_\beta^{-1}\{H_1(\beta)\}|_t+\delta_{(w-b)}\mathcal{L}_\beta^{-1}\{H_2(\beta)\}|_t\right]dw\ dt$$

with

$$H_1(\beta) = \frac{e^{-\beta d}\left(2\sum\limits_{k=0}^{\infty}\left[z(k,0,0) - z(k+\frac{1}{2},0,0)\right] - z(0,0,0)\right)}{2\sum\limits_{k=0}^{\infty}\left[z(k,\beta,0) + \sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}$$

$$H_2(\beta) = \frac{2\sum\limits_{k=0}^{\infty} z(k+\frac{1}{2},\beta,0) + \sqrt{2\beta}e^{-(2k+1)\sqrt{2\beta}b}}{2\sum\limits_{k=0}^{\infty}\left[z(k,\beta,0) + \sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}$$

and z defined by (5.11) and δ_x being the Dirac delta function.

Proof. Let Z denote a standard Brownian motion and $\tau(Z) := \min\{\tau_d^+(Z), H_b(Z)\}$. The joint probability of position at maturity and minimum of Parisian and hitting times can be decomposed in the following way:

$$\mathbb{Q}_0(Z_T \in dz, \min\{\tau_d^+(Z), H_b(Z)\} \leq T) = \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b} \mathbb{Q}_0(Z_T \in dz, \tau(Z) \in dt, Z_\tau \in dw)$$

$$= \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b} \mathbb{Q}_0(Z_T \in dz | \tau(Z) = t, Z_\tau \in dw)\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw)$$

$$= \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}dz\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw)$$

$$= \int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}dz\Big[\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | H_b(Z) < \tau_d^+(Z))\times$$

$$\times\, \mathbb{Q}_0(H_b(Z) < \tau_d^+(Z)) + \mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | \tau_d^+(Z) < H_b(Z))\times$$

$$\times\, \mathbb{Q}_0(\tau_d^+(Z) < H_b(Z))\Big]$$

We find

$$\begin{aligned}
&\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | \tau_d^+(Z) < H_b(Z)) \mathbb{Q}_0(\tau_d^+(Z) < H_b(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z)) \mathbb{Q}_0(\tau(Z) \in dt | \tau_d^+(Z) < H_b(Z)) \times \\
&\quad\ \times \mathbb{Q}_0(\tau_d^+(Z) < H_b(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z)) \mathbb{Q}_0(\tau(Z) \in dt, \tau_d^+(Z) < H_b(Z)).
\end{aligned} \tag{5.19}$$

For the first term on the r.h.s. we notice

$$\begin{aligned}
&\mathbb{Q}_0(Z_{\tau_d^+} \in dw | \tau(Z) = t, \tau_d^+(Z) < H_b(Z)) \\
&\quad = \lim_{\epsilon \to 0} \mathbb{Q}_\epsilon(Z_d \in dw | \inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b) \\
&\quad = \lim_{\epsilon \to 0} \frac{\mathbb{Q}_\epsilon(Z_d \in dw, \inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b)}{\mathbb{Q}_\epsilon(\inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b)} \\
&\quad = \lim_{\epsilon \to 0} \frac{\sum_{k=-\infty}^{\infty} e^{-\frac{(w-\epsilon+2kb)^2}{2d}} - e^{-\frac{(w+\epsilon+2kb)^2}{2d}}}{\sum_{k=-\infty}^{\infty} \int_0^b e^{-\frac{(z-\epsilon+2kb)^2}{2d}} - e^{-\frac{(z+\epsilon+2kb)^2}{2d}} dz} dw \\
&\quad = \frac{\sum_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}}}{2 \sum_{k=0}^{\infty} \left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2 b^2}{2d}} \right)} dw.
\end{aligned} \tag{5.20}$$

Notice that the first equality results from the position at Parisian time, $Z_{\tau_d^+}$, being independent of time $\tau_d^+(Z) = t$. See Chesney, Jeanblanc-Picqué, and Yor [1997], section 8.3.1, for further details. Formulae for the third line can be found in Borodin and Salminen [2002], Chapter 1. Brownian motion, formulae (1.15.4) and (1.15.8). The second term on the r.h.s. of equation (5.19) can be calculated via inverting the Laplace transform of the minimum of hitting and Parisian times. The Laplace transform has been found in Lemma 5.3.2. With $\mu = 0$ we derive

$$\mathbb{Q}_0(\tau(Z) \in dt, \tau_d^+(Z) < H_b(Z)) = \mathcal{L}_\beta^{-1} \left\{ \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta \tau_d^+(Z)} \mathbf{1}_{\tau_d^+(Z) < H_b(Z)} \right) \right\} \Big|_t \, dt$$
$$= \mathcal{L}_\beta^{-1} \left\{ \frac{e^{-\beta d} \left(\sum_{k=0}^{\infty} 2 \left[z(k,0,0) - z(k+\frac{1}{2},0,0) \right] - z(0,0,0) \right)}{\sum_{k=0}^{\infty} 2 \left[z(k,\beta,0) + \sqrt{2\beta} e^{-\sqrt{2\beta} 2kb} \right] - z(0,\beta,0) - 2\sqrt{2\beta}} \right\} \Bigg|_t \, dt,$$

where $z(k,\beta,\mu)$ is defined as in (5.11) to be

$$z(k,\beta,\mu) =$$
$$\sqrt{\frac{2}{\pi d}} e^{-\frac{(2\beta+\mu^2)d}{2} - \frac{2(kb)^2}{d}} - \sqrt{2\beta+\mu^2} \Bigg(e^{\sqrt{2\beta+\mu^2} 2kb} \mathcal{N}\left(-\frac{2kb}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) +$$
$$+ e^{-\sqrt{2\beta+\mu^2} 2kb} \mathcal{N}\left(\frac{2kb}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \Bigg).$$

We also have in the case that $H_b(Z) < \tau_d^+(Z)$,

$$\mathbb{Q}_0(\tau(Z) \in dt, Z_\tau \in dw | H_b(Z) < \tau_d^+(Z)) \mathbb{Q}_0(H_b(Z) < \tau_d^+(Z))$$
$$= \mathbb{Q}_0(Z_{H_b} \in dw | \tau(Z) = t, H_b(Z) < \tau_d^+(Z)) \mathbb{Q}_0(\tau(Z) \in dt, H_b(Z) < \tau_d^+(Z)).$$

Since Z_{H_b} conditionally on $H_b(Z)$ is deterministic the probability becomes the

Dirac delta function at point b, hence

$$\mathbb{Q}_0(Z_{H_b} \in dw | \tau(Z) = t, H_b(Z) < \tau_d^+(Z)) = \delta_{(w-b)} dw,$$

where the Dirac delta function is defined for all $x \in \mathbb{R}$ as

$$\delta_x = \begin{cases} 0 & , \text{ if } x \neq 0 \\ \infty & , \text{ if } x = 0, \end{cases}$$

and also satisfying the identity

$$\int_{-\infty}^{\infty} \delta_x \; dx = 1.$$

By inversion of the Laplace transform in Lemma 5.3.1 with $h \equiv 1$, we firstly derive for the numerator

$$\frac{\partial}{\partial \epsilon} \hat{Q}_{12}(\beta) \longrightarrow \sum_{k=0}^{\infty} 2\sqrt{\frac{2}{\pi d}} e^{\mu b - \frac{(2k+1)^2 b^2}{2d} - \frac{(2\beta+\mu^2)d}{2}} + 2\sqrt{2\beta + \mu^2} e^{\mu b} \Bigg[e^{-(2k+1)\sqrt{2\beta+\mu^2} b} \times$$
$$\times \mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}} + \sqrt{(2\beta+\mu^2)d} \right) - e^{(2k+1)\sqrt{2\beta+\mu^2} b} \mathcal{N}\left(-\frac{(2k+1)b}{\sqrt{d}} - \sqrt{(2\beta+\mu^2)d} \right) \Bigg]$$
$$= 2e^{\mu b} \sum_{k=0}^{\infty} z(k + \frac{1}{2}, \beta, \mu) + \sqrt{2\beta + \mu^2} e^{-(2k+1)\sqrt{2\beta+\mu^2} b}.$$

Setting $\mu = 0$, we yield

$$\mathbb{Q}_0(\tau(Z) \in dt, H_b(Z) < \tau_d^+(Z)) = \mathcal{L}_\beta^{-1} \left\{ \mathbb{E}_0^{\mathbb{Q}} \left(e^{-\beta H_b(Z)} \mathbf{1}_{H_b(Z) < \tau_d^+(Z)} \right) \right\} \Big|_t \; dt$$
$$= \mathcal{L}_\beta^{-1} \left\{ \frac{2\sum_{k=0}^{\infty} z(k + \frac{1}{2}, \beta, 0) + \sqrt{2\beta} e^{-(2k+1)\sqrt{2\beta} b}}{2\sum_{k=0}^{\infty} \left[z(k, \beta, 0) + \sqrt{2\beta} e^{-\sqrt{2\beta} 2kb} \right] - z(0, \beta, 0) - 2\sqrt{2\beta}} \right\} \Bigg|_t \; dt.$$

Putting things together the proposition follows.

□

We are now able to price a MinParisianHit option by combining Proposition 5.4.1 and equation (5.18), in particular the fair price of a MinParisianHit Up-and-In Call option can be calculated via evaluating the integral

$$\begin{aligned} &minPHC_i^u(S_0, T, K, L, d, r) \\ &= e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0 \left(Z_T \in dz, \min\{\tau_d^+(Z), H_b(Z)\} \leq T \right), \end{aligned} \tag{5.21}$$

where the joint probability has been derived in Proposition 5.4.1.

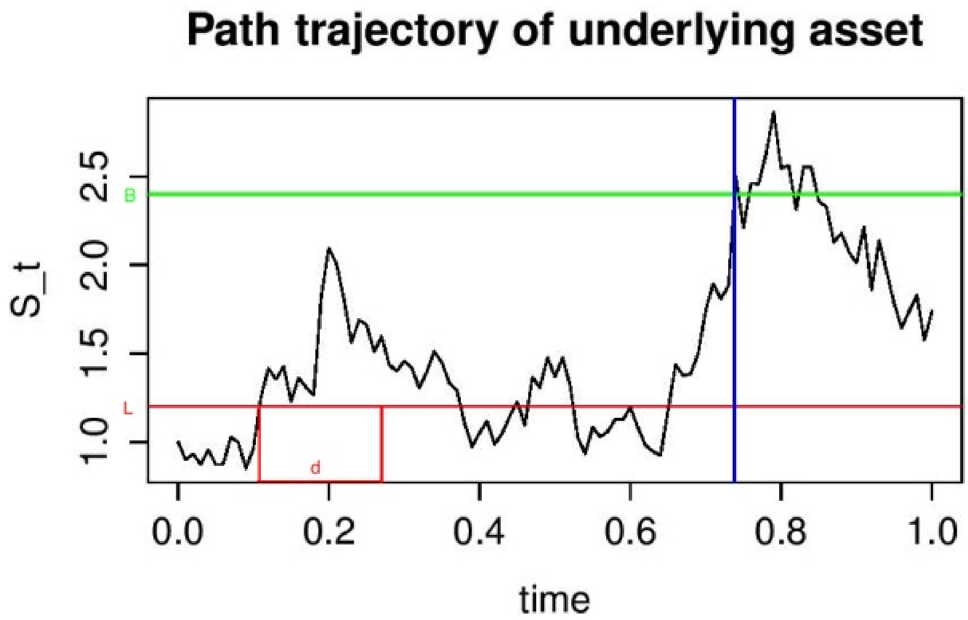
5.4.2 Option triggered at Maximum of Parisian and Hitting times

Our so-called MaxParisianHit Option is triggered, when both the barrier B is hit and the excursion age exceeds duration d above L. Hence, the payoff of a Call option with strike K becomes

$$(S_T - K)^+ \mathbf{1}_{\{\tau_{L,d}^+(S) \leq T, H_B(S) \leq T\}} = (S_T - K)^+ \mathbf{1}_{\{\max\{\tau_{L,d}^+(S), H_B(S)\} \leq T\}}.$$

The maximum of Parisian and hitting times is illustrated by the blue line in Figure 5.2.

Figure 5.2: Maximum of Parisian and hitting times



As in the previous case the problem reduces to finding the joint density of hitting and Parisian times and position for a drifted Brownian motion which then can be related to the joint density of hitting and Parisian time for standard Brownian motion due to Girsanov. We also assume $S_0 = L$, thus $\tau^+_{l,d}(Z) = \tau^+_d(Z)$, and discuss the more general case $S_0 \neq L$ in Appendix 7.3. The fair price becomes

$$
\begin{aligned}
maxPHC_i^u(S_0,T,K,L,d,r) &= e^{-rT}\mathbb{E}_{S_0}^{\bar{\mathbb{Q}}}\left((S_T-K)^+\mathbf{1}_{\{\tau_{L,d}^+(S)\le T, H_B(S)\le T\}}\right)\\
&= e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_0^{\mathbb{Q}}\left((S_0e^{\sigma Z_T}-K)^+e^{mZ_T}\mathbf{1}_{\{\tau_d^+(Z)\le T, H_b(Z)\le T\}}\right)\\
&= e^{-(r+\frac{1}{2}m^2)T}\int\limits_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0e^{\sigma z}-K)e^{mz}\mathbb{Q}_0(Z_T\in dz, \max\{\tau_d^+(Z),H_b(Z)\}\le T).
\end{aligned}
\tag{5.22}
$$

Hence, finding the fair price of a MaxParisianHit option reduces to finding the joint probability of position at maturity and maximum of Parisian and hitting times.

Proposition 5.4.2. *The joint probability of position at maturity and maximum of hitting and Parisian times of standard Brownian motion is*

$$
\begin{aligned}
&\mathbb{Q}_0(Z_T\in dz, \max\{\tau_d^+(Z),H_b(Z)\}\le T)\\
&\quad=\int\limits_{t=0}^{T}\int\limits_{w=-\infty}^{\infty}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-w)^2}{2(T-t)}}\Bigg\{\frac{|w|}{\pi\sqrt{(t-d)d^3}}e^{-\frac{w^2}{2d}}-\\
&-\frac{\sum\limits_{k=-\infty}^{\infty}\frac{w+2kb}{d}e^{-\frac{(w+2kb)^2}{2d}}}{\sum\limits_{k=-\infty}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}}-e^{-\frac{(2k+1)^2b^2}{2d}}\right)}\mathcal{L}_\beta^{-1}\{H_1(\beta)\}|_t\ dt+\delta_{(w-b)}\mathcal{L}_\gamma^{-1}\{H_3(\gamma)\}|_t\Bigg\}dw\ dt\ dz,
\end{aligned}
$$

where

$$
H_1(\beta)=\frac{e^{-\beta d}\left(2\sum\limits_{k=0}^{\infty}\left[z(k,0,0)-z(k+\frac{1}{2},0,0)\right]-z(0,0,0)\right)}{2\sum\limits_{k=0}^{\infty}\left[z(k,\beta,0)+\sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right]-z(0,\beta,0)-2\sqrt{2\beta}},
$$

$$H_3(\gamma) = \left\{ \left[e^{-\sqrt{2\gamma}b} \mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{2\gamma d} \right) - e^{\sqrt{2\gamma}b} \mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{2\gamma d} \right) \right] \times \right.$$
$$\left. \times \sum_{k=0}^{\infty} 2\left[z(k,0,0) - z(k+\frac{1}{2},0,0) \right] - z(0,0,0) \right\} \times$$
$$\times \left\{ \left[\sum_{k=0}^{\infty} 2\left[z(k,\gamma,0) + \sqrt{2\gamma} e^{-\sqrt{2\gamma}2kb} \right] - z(0,\gamma,0) - 2\sqrt{2\gamma} \right] \times \left[1 - 2\mathcal{N}\left(-\frac{b}{\sqrt{d}} \right) \right] \right\}^{-1},$$

with z defined by (5.11) and δ_x denoting the Dirac delta function.

Proof. Let $\bar{\tau}(Z) = \max\{\tau_d^+(Z), H_b(Z)\}$, we again have the following decomposition:

$$\mathbb{Q}_0(Z_T \in dz, \max\{\tau_d^+(Z), H_b(Z)\} \le T) \tag{5.23}$$
$$= \int_{t=0}^{T} \int_{w=-\infty}^{\infty} \mathbb{Q}_0(Z_T \in dz, \bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw)$$
$$= \int_{t=0}^{T} \int_{w=-\infty}^{\infty} \mathbb{Q}_0(Z_T \in dz | \bar{\tau}(Z) = t, Z_{\bar{\tau}} \in dw) \mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw)$$
$$= \int_{t=0}^{T} \int_{w=-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} \mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw) dz$$
$$= \int_{t=0}^{T} \int_{w=-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} \Big[\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, H_b(Z) < \tau_d^+(Z)) +$$
$$+ \mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, \tau_d^+(Z) < H_b(Z)) \Big] dz. \tag{5.24}$$

For the second part of the r.h.s. of equation (5.24) we have

$$\begin{aligned}
&\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, \tau_d^+(Z) < H_b(Z)) \\
&\quad = \mathbb{Q}_0(Z_{H_b} \in dw | H_b(Z) = t, \tau_d^+(Z) < H_b(Z)) \mathbb{Q}_0(H_b(Z) \in dt, \tau_d^+(Z) < H_b(Z)) \\
&\quad = \delta_{(w-b)} \ \mathcal{L}_\gamma^{-1}\{H_3(\gamma)\}|_t dw,
\end{aligned}$$

where we know from Proposition 5.3.3 with $\mu = 0$ and $\beta = 0$

$$\begin{aligned}
&H_3(\gamma) = \mathbb{E}(e^{-\gamma H_b(Z)} \mathbf{1}_{\tau_d^+(Z) < H_b(Z)}) = \left\{ \left[e^{-\sqrt{2\gamma} b} \mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{2\gamma d}\right) - \right.\right. \\
&\left. - e^{\sqrt{2\gamma} b} \mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{2\gamma d}\right) \right] \sum_{k=0}^{\infty} 2 \left[z(k,0,0) - z(k + \frac{1}{2}, 0, 0) \right] - z(0,0,0) \Bigg\} \times \\
&\times \left\{ \left[\sum_{k=0}^{\infty} 2 \left[z(k,\gamma,0) + \sqrt{2\gamma} e^{-\sqrt{2\gamma} 2kb} \right] - z(0,\gamma,0) - 2\sqrt{2\gamma} \right] \times \left[1 - 2\mathcal{N}\left(-\frac{b}{\sqrt{d}}\right) \right] \right\}^{-1}.
\end{aligned}$$

Notice the Dirac delta function which is motivated by the deterministic behaviour of Z_{H_b} conditioned on $H_b(Z) = t$.

For the first part of the r.h.s of equation (5.24) we have

$$\begin{aligned}
&\mathbb{Q}_0(\bar{\tau}(Z) \in dt, Z_{\bar{\tau}} \in dw, H_b(Z) < \tau_d^+(Z)) \\
&\quad = \mathbb{Q}_0(\tau_d^+(Z) \in dt, Z_{\tau_d^+} \in dw, H_b(Z) < \tau_d^+(Z)) \\
&\quad = \mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt) - \mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt, \tau_d^+(Z) < H_b(Z)).
\end{aligned}$$

We have found in section 5.4.1, that with equation (5.19) and (5.20) combined we derive

$$\mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt, \tau_d^+(Z) < H_b(Z)) \\ = \frac{\sum\limits_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}}}{2\sum\limits_{k=0}^{\infty}\left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2b^2}{2d}}\right)} \mathcal{L}_\beta^{-1}\{H_1(\beta)\}|_t \, dw \, dt.$$

Also, Chung [1976] provides us with

$$\mathbb{Q}_0(Z_{\tau_d^+} \in dw, \tau_d^+(Z) \in dt) = \frac{|w|}{\pi\sqrt{(t-d)d^3}} e^{-\frac{w^2}{2d}} dw \, dt.$$

Hence, putting terms together we derive the proposition.

□

Proposition 5.4.2 allows us to derive the price of a MaxParisianHit option, in particular with equation (5.22) we find the fair price of a MaxParisianHit Up-and-In Call option

$$maxPHC_i^u(S_0, T, K, L, d, r) \\ = e^{-(r+\frac{1}{2}m^2)T} \int\limits_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0(Z_T \in dz, \max\{\tau_d^+(Z), H_b(Z)\} \leq T), \tag{5.25}$$

where the joint probability has been found in Proposition 5.4.2.

In Proposition 5.3.2 and 5.3.3 we have derived the double Laplace transform of hitting and Parisian times for drifted Brownian motion. This main result leads to finding the joint distribution of the final position of Brownian motion and the minimum or maximum of hitting and Parisian time. We have established pricing formulae for MinParisianHit and MaxParisianHit options. These fair prices contain single Laplace transforms which need to be inverted numerically using techniques as in Labart and Lelong [2009], Abate and Whitt [1995] and Bernard, Courtois, and Quittard-Pinon [2005].

Chapter 6

Conclusion

This thesis investigates Parisian-style questions in risk management and option pricing. The main focus is put on Brownian excursion theory.

We have derived the distribution of functionals of the Brownian excursion, such as the first hitting time, the last passage time, the maximum and the time it is achieved. Our results rely mainly on conditioned martingales and reversibility, making use of the relationship to similar stochastic processes. We present analytically closed-form solutions and apply our results to the calculation of default probabilities of bonds.

Furthermore, the joint probability of Parisian, occupation and local times has been studied. We use the method of Brownian perturbation and a piecewise deterministic semi-Markov model to achieve results in form of explicit triple Laplace transforms with respect to the maturity time. Relating the local time to down-crossings allows us to introduce the so-called Parisian Crash options and Parisian Lookback options.

In the field of option pricing under the Black-Scholes assumptions, we extend the Parisian concept and introduce the so-called ParisianHit option, a generalization of Parisian and Barrier options, with the advantage of being highly adaptable to investors' beliefs in the market. Fair prices are found through a piecewise deterministic semi-Markov framework yielding results in terms of double Laplace transforms of Parisian and hitting time.

The models used in this thesis do not rely heavily on the properties of the Brownian motion and can be extended to more general Markov processes, such as Bessel processes, Ornstein-Uhlenbeck processes, just to name a few. Strong relationships to financial processes, such as geometric Brownian motion or Cox-Ingersoll-Ross processes suggest consideration with Bessel processes as a direction of future research. Cox-Ingersoll-Ross families of diffusions have been proposed to model short term interest rates [see e.g. Cox, Ingersoll, and Ross, 1985] and stochastic volatility [see e.g. Heston, 1993]. Not only this family of processes, but also Ornstein-Uhlenbeck processes and geometric Brownian motion can be represented in terms of Bessel processes, suggesting the extension of our results concerning the joint probability of hitting, excursion, occupation and local times to Bessel processes.

Numerical inversion of single Laplace transforms with respect to maturity time has been well studied [see e.g. Labart and Lelong, 2009]. Recovering the function from its Laplace transform has been done using a contour integral represented by a series, a method proposed by Abate and Whitt [1995]. The accuracy and efficiency of this methods has been tested with Monte Carlo. Bernard, Courtois, and Quittard-Pinon [2005] provide an algorithm to invert the Laplace transform of the Parisian densities by approximating with a linear combination of fractional functions, for which Laplace inverse functions are known. However, new challenges arise when dealing with the inversion of triple Laplace transforms. New discretisation errors demand careful use of theory and open up directions of future research.

Taking into account the recursive formulae for the density of the Parisian stopping time by Dassios and Lim [2013] and Dassios and Lim [2015], it remains an open question whether ParisianHit, Parisian Crash and Parisian Lookback option prices can be found recursively, avoiding numerical inversion of Laplace transforms.

Another direction of future research might be the investigation of the Delta and Gamma of these innovative options, as well as hedging strategies. The Delta and Gamma of Parisian options have been studied in Bernard et al. [2005] by giving closed formulae of the Laplace transforms. The strategy of Delta hedging is a standard technique used in practice, where portfolio weights are adjusted on a continuous basis. However, this has several drawbacks. Firstly, the value of the Delta is very high and changes rapidly near the barrier when the time is close to maturity and the Gamma gets very large near the barrier. Also, continuous weight adjustment is not possible, therefore the adjustments made in discrete time cause small errors, which accumulate over the lifetime of an option and result in big accuracy problems. Thirdly, enormous transaction costs will be generated through frequent trading. Avellaneda and Wu [1999] test the Delta hedging performance for Parisian options and the effect of the option window d on the option values and Deltas. The alternative to Delta hedging is the Static hedging method. Given a target option, e.g. a ParisianHit option, one constructs a portfolio of standard options with different maturities, strikes and weights, which will exactly replicate our target option and just needs very few rebalancing. Work has been done by Carr and Bowie [1994] as well as Carr and Chou [1996] for Barrier option within the Black-Scholes framework making use of the Put-Call symmetry [Carr, Ellis, and Gupta, 1998]. Carr and Chou [1996] use a continuum of Arrow-Debreu securities to decompose a Barrier option. They utilise a modification of the Taylor expansion to derive the adjusted payoff, which is not path-dependent any longer. This is used to derive the initial purchase for the static hedge only consisting of liquid derivatives, such as Puts, Calls, Bonds and Stock. Carr and Nadtochiy [2011] derive exact static hedges via Laplace transformation assuming that the underlying asset is a time-homogeneous diffusion. We believe that the literature is broad enough to find efficient hedging strategies for Parisian options, and even ParisianHit options, in the future.

In conclusion, we realise that the study of Parisian-type questions exerts immense fascination. It can be applied to many different areas in Financial Mathematics with the motivation of being a better measure of risk in the case of ruin probabilities with Parisian delay and of being insensitive to price manipulation and highly adaptable to investors' beliefs in the market in their practical use as derivatives.

Chapter 7

Appendix

7.1 Numerical results on the Default Probability of Bonds

This is the `R` script for deriving the default probability of our newly defined risky zero-coupon bond using the parameters maturity $T = 10$ years, barrier $-b = -0,7$ and minimum being reached by time $u = 6$ years. It remains important to mention, that the infinite sums in equation (3.24) converge extremely fast. To compute the semi-infinite integral we perform the change of variable $a = y + \frac{t}{1-t}$,

$$\int_y^\infty f(a)da = \int_0^1 f\left(y + \frac{t}{1-t}\right) \frac{1}{(1-t)^2} dt.$$

The default probability can be computed in the following way using the `R` package *cubature*.

Table 7.1: R Script for default probability with $T = 10$, $u = 6$, $b = 0.7$, $n = 100$

```r
# Parameters
T <- 10
u <- 6
b <- 0.7
sum1 <- 0
sum2 <- 0

# x[1] = a, x[2] = s
# first infinite sum

summe1 <- function(x){
  for(n in 0:100 ){
    t1 = (2*n+1)*((1+b)+(x[1]/(1-x[1])))*(exp(-((2*n+1)*((1+b)+(x[1]/(1-x[1])))-1)^2/(2*x[2]))-exp(-((2*n+1)*((1+b)+(x[1]/(1-x[1])))+1)^2/(2*x[2])))-(exp(-((2*n+1)*((1+b)+(x[1]/(1-x[1])))-1)^2/(2*x[2]))+exp(-((2*n+1)*((1+b)+(x[1]/(1-x[1])))+1)^2/(2*x[2])))
    sum1 = sum1 + t1
  }
  return(sum1)
}

# second infinite sum

summe2 <- function(x){
  for (k in 0:100){
    t2 = ((2*k+1)^2*((1+b)+(x[1]/(1-x[1])))^2-(T-x[2]))*exp(-((2*k+1)^2*((1+b)+(x[1]/(1-x[1])))^2)/(2*(T-x[2])))
    sum2 = sum2 + t2
  }
  return(sum2)
}

# integrand

f <- function(x){
  sqrt((2*T^3)/(pi*x[2]^3*(T-x[2])^5))*exp(1/(2*T))*summe1(x)*summe2(x)*(1/(1-x[1])^2)
}

# double integration

adaptIntegrate(f, lowerLimit = c(0,0), upperLimit = c(1,u), maxEval=10000, tol = 1e-05)
```

The numerical result for the parameters $T = 10$, $u = 6$ and $-b = -0.7$ is 0.6998403. In order to demonstrate the speed of convergence for the infinite sums, we calculate the default probability with respect to the number of summands $n = 1, 2, 3, 4...$ in the two infinite sums.

Table 7.2: Default probability with $T = 10$, $u = 6$, $b = 0.7$

n	Default Prob.
1	0.6581729
2	0.6987304
3	0.6998313
4	0.6998403
5	0.6998403
100	0.6998403

The following table gives the default probability of a zero-coupon bond with maturity $T = 10$ and barrier $b = 0.7$ for different values of u.

Table 7.3: Default probability with $T = 10$, $b = 0.7$, $n = 100$

u	Default Prob.
2	0.0774475
4	0.3570686
6	0.6998403
8	0.9516138
10	0.9999969

On the other hand, we provide a table of default probabilities for different values of the barrier b. Maturity and u are fixed to be $T = 10$, $u = 6$.

Table 7.4: Default probability with $T = 10$, $u = 6$, $n = 100$

b	Default Prob.
0	0.6998427
2	0.6321395
3	0.3450696
4	0.1061302

7.2 Laplace transform of the Parisian time

This section is devoted to an alternative derivation of the distribution of the Parisian time, which appears in Proposition 5.3.2. We want to derive the Laplace transform of the time the excursion exceeds time d in the one barrier case by introducing a two-state semi-Markov process for an intuitive approach.

Lemma 7.2.1. *The Laplace transform of the age of an excursion above zero reaching time period d for the drifted Brownian motion W^{μ} is*

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\right) = \frac{2\mu e^{-\gamma d}\mathcal{N}\left(\mu\sqrt{d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\gamma d-\frac{\mu^2 d}{2}}}{2\sqrt{2\gamma+\mu^2}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\frac{(2\gamma+\mu^2)d}{2}}}$$

Proof. With the same definition of the perturbed Brownian motion with drift as in (4.7), (4.8), (4.9) and (4.10), we define a two-state semi-Markov process as in (4.11):

$$X_t = \begin{cases} 1 & \text{if } W_t^{\epsilon,\mu} > 0 \\ -1 & \text{if } W_t^{\epsilon,\mu} < 0 \end{cases}$$

The transition densities of X have been found in (4.16) to be

$$p_{1,-1}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}}e^{-\frac{(\epsilon+\mu t)^2}{2t}} \qquad p_{-1,1}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}}e^{-\frac{(\epsilon-\mu t)^2}{2t}}$$

With $U_t(X) = t - g_t(X)$ to denote the time elapsed in the current state, $(X_t, U_t(X))$ is a Markov process. Hence, X_t is a two state semi-Markov process with state space $\{1,-1\}$. We consider a bounded function $f : \{1,-1\} \times \mathbb{R}^2 \to \mathbb{R}$. The generator $\mathcal{A}$ is an operator making

$$f(X_t, U_t(X), t) - \int_0^t \mathcal{A}f(X_s, U_s(X), s)ds$$

a martingale. We shall us the shortcut $f_i(z,u) = f(i,z,u)$ and $\mathcal{A}f_{X_t}(U_t(X),t) = \mathcal{A}f(X_t, U_t(X), t)$. Hence, solving $\mathcal{A}f = 0$ provides us with martingales of the form $f_{X_t}(U_t(X), t)$. We can find for our purposes that

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1(u,t)}{\partial t} + \frac{\partial f_1(u,t)}{\partial u} + \lambda_{1,-1}(u)\left(f_{-1}(0,t) - f_1(u,t)\right),$$
$$\mathcal{A}f_{-1}(u,t) = \frac{\partial f_{-1}(u,t)}{\partial t} + \frac{\partial f_{-1}(u,t)}{\partial u} + \lambda_{-1,1}(u)\left(f_1(0,t) - f_{-1}(u,t)\right).$$

We assume f_i having the form

$$f_i(u,t) = e^{-\beta t} h_i(u),$$

where β is a positive constant. We solve $\mathcal{A}f \equiv 0$ with constraints $\lim_{q\to\infty} h_{-1}(q) = 0$ and $h_1(d) = 1$. This construction ensures $\tau_q^-(W^{\epsilon,\mu}) \to \infty$ as $q \to \infty$ and hence

$$\min\{\tau_d^+(W^{\epsilon,\mu}), \tau_d^-(W^{\epsilon,\mu})\} \longrightarrow \tau_d^+(W^{\epsilon,\mu}) \qquad \text{a.s.}$$

Recall that we are only interested in the excursion above zero reaching time d in Proposition 5.3.2. Proceeding just as in section 4.2, we solve the martingale problem and apply optional sampling theorem in oder to derive the following expression for the perturbed process $W^{\epsilon,\mu}$:

$$\mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^{\epsilon,\mu})}\right) = \frac{e^{-\int_0^d \gamma+\lambda_{1,-1}(v)dv}}{1-\int_0^\infty \lambda_{-1,1}(t)e^{-\int_0^t \gamma+\lambda_{-1,1}(v)dv}dt\int_0^d \lambda_{1,-1}(w)e^{-\int_0^w \gamma+\lambda_{1,-1}(v)dv}dw} \tag{7.1}$$

In Chapter 4 we have already derived in equations (4.25), (4.23) and (4.24), that the following holds:

$$e^{-\int_0^d \lambda_{1,-1}(v)dv} = 1 - e^{-2\mu\epsilon}\mathcal{N}\left(\frac{\mu d-\epsilon}{\sqrt{d}}\right) - \mathcal{N}\left(\frac{-\mu d-\epsilon}{\sqrt{d}}\right),$$

$$\int_0^d e^{-\gamma w}\lambda_{1,-1}(w)e^{-\int_0^w \lambda_{1,-1}(v)dv}dw = \int_0^d e^{-\gamma w}p_{1,-1}(w)dw$$
$$= e^{-(\sqrt{2\gamma+\mu^2}+\mu)\epsilon}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}-\frac{\epsilon}{\sqrt{d}}\right) + e^{(\sqrt{2\gamma+\mu^2}-\mu)\epsilon}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)d}-\frac{\epsilon}{\sqrt{d}}\right),$$

$$\int_0^\infty e^{-\gamma w}\lambda_{-1,1}(w)e^{-\int_0^w \lambda_{-1,1}(v)dv}dw \;=\; \int_0^\infty e^{-\gamma w}p_{-1,1}(w)dw \;=\; e^{(\mu-\sqrt{2\gamma+\mu^2})\epsilon}.$$

Inserting these calculations into equation (7.1) yields

$$\mathbb{E}_{\epsilon}^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^{\epsilon,\mu})}\right)$$

$$= \frac{e^{-\gamma d}\left(1 - e^{-2\mu\epsilon}\mathcal{N}\left(\frac{\mu d-\epsilon}{\sqrt{d}}\right) - \mathcal{N}\left(\frac{-\mu d-\epsilon}{\sqrt{d}}\right)\right)}{1 - e^{-2\sqrt{2\gamma+\mu^2}\epsilon}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d} - \frac{\epsilon}{\sqrt{d}}\right) - \mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)d} - \frac{\epsilon}{\sqrt{d}}\right)}.$$

Applying L'Hôpital's rule and taking the limit distribution for $\epsilon \to 0$ gives us the Laplace transform of $\tau_d^+(W^\mu)$, as proposed in Proposition 5.3.2.

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\gamma\tau_d^+(W^\mu)}\right) = \frac{2\mu e^{-\gamma d}\mathcal{N}\left(\mu\sqrt{d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\gamma d - \frac{\mu^2 d}{2}}}{2\sqrt{2\gamma+\mu^2}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)d}\right) + \sqrt{\frac{2}{\pi d}}e^{-\frac{(2\gamma+\mu^2)d}{2}}}$$

Note that $\hat{\tau}_d^+(W^\mu)$ (from Proposition 5.3.2) and $\tau_d^+(W^\mu)$ are equally distributed relying on the strong Markov property of the Brownian motion.

□

7.3 Laplace transform of Parisian and Hitting times with $S_0 \neq L$

In the case where the underlying asset does not start at the level L, i.e. $S_0 \neq L$, we want to make use of the strong Markov property of the Brownian motion. We distinguish between two possible scenarios, $S_0 < L$ and $S_0 > L$. From a financial point of view, we are only concerned with $L < B$, and therefore $l < b$.

The price of the MinParisianHit Up-and-In Call option (5.21) can be rewritten in the following form,

$$
\begin{aligned}
&minPHC_i^u(S_0, T, K, L, d, r) \\
&= e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0 \left(Z_T \in dz, \min\{\tau_{l,d}^+(Z), H_b(Z)\} \leq T \right),
\end{aligned}
$$

whereas the MaxParisianHit Up-and-In Call option (5.25) becomes

$$
\begin{aligned}
&maxPHC_i^u(S_0, T, K, L, d, r) \\
&\quad = e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty} (S_0 e^{\sigma z} - K) e^{mz} \mathbb{Q}_0 (Z_T \in dz, \max\{\tau_{l,d}^+(Z), H_b(Z)\} \leq T).
\end{aligned}
$$

The proofs of Propositions 5.4.2 and 5.4.2 suggest, that the pricing reduces to finding the Laplace transforms of hitting and Parisian time. This can be achieved by decomposing the stopping times and using known results for $S_0 = L$.

We look at the case $S_0 < L$ first. By definition it follows $l > 0$. Define the first hitting time of l for the $\mathbb{Q}$ - Brownian motion Z, with $Z_0 = 0$, to be $H_l(Z) = \inf\{t \geq 0 | Z_t = l\}$. By definition, we have

$$\tau_{l,d}^{+}(Z) = H_l(Z) + \tau_{l,d}^{+}(\tilde{Z}),$$

where $\tilde{Z}$ stands for a restarted Brownian motion at time $H_l(Z)$, i.e. $\tilde{Z}_0 = l$. Hence, we have equality in distribution of $\tau_{l,d}^{+}(\tilde{Z})$ and $\tau_d^{+}(Z)$. By the strong Markov property of the Brownian motion, we therefore have

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(Z)}\mathbf{1}_{\tau_{l,d}^{+}(Z)<H_b(Z)}\right) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_l(Z)}\right)\mathbb{E}_l^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(\tilde{Z})}\mathbf{1}_{\tau_{l,d}^{+}(\tilde{Z})<H_b(\tilde{Z})}\right).$$

Clearly, $\mathbb{Q}_0\left(H_l(Z) < H_b(Z)\right) = 1$ due to $l < b$. Notice, that $\mathbb{Q}_0\left(\tau_{l,d}^{+}(Z) < H_b(Z)\right) = \mathbb{Q}_l\left(\tau_{l,d}^{+}(\tilde{Z}) < H_b(\tilde{Z})\right)$, since $l < b$ and $\tau_{l,d}^{+}$ is concerned with the Parisian time above l. It is not difficult to see that

$$\mathbb{E}_l^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(\tilde{Z})}\mathbf{1}_{\tau_{l,d}^{+}(\tilde{Z})<H_b(\tilde{Z})}\right) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^{+}(Z)}\mathbf{1}_{\tau_d^{+}(Z)<H_b(Z)}\right),$$

which has been calculated in Lemma 5.3.2 with $\mu = 0$. Also, according to Borodin and Salminen [2002], Chapter 1. Brownian motion, formula (2.0.1), we have

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_l(Z)}\right) = e^{-l\sqrt{2\beta}},$$

yielding

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(Z)}\mathbf{1}_{\tau_{l,d}^{+}(Z)<H_b(Z)}\right)$$
$$= \frac{e^{-l\sqrt{2\beta}-\beta d}\left\{\sum_{k=0}^{\infty} 2\left[z(k,0,0)-z(k+\frac{1}{2},0,0)\right]-z(0,0,0)\right\}}{\sum_{k=0}^{\infty} 2\left[z(k,\beta,0)+\sqrt{2\beta}e^{-\sqrt{2\beta}2kb}\right]-z(0,\beta,0)-2\sqrt{2\beta}}.$$

In the second case where $S_0 > L$, we have by definition $l < 0 < b$. Then $\tau_{l,d}^{+}(Z)$ can be decomposed into

$$\tau_{l,d}^{+}(Z) = \begin{cases} d & \text{, if } H_l(Z) \geq d \\ H_l(Z) + \tau_{l,d}^{+}(\tilde{Z}) & \text{, if } H_l(Z) < d \end{cases}$$

where $\tilde{Z}$ is a restarted Brownian motion at l. Hence,

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(Z)}\mathbf{1}_{\tau_{l,d}^{+}(Z)<H_b(Z)}\right) \\
&= \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta d}\mathbf{1}_{\tau_{l,d}^{+}(Z)<H_b(Z)}\mathbf{1}_{H_l(Z)>d}\right) + \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_l(Z)-\beta\tau_{l,d}^{+}(\tilde{Z})}\mathbf{1}_{\tau_{l,d}^{+}(Z)<H_b(Z)}\mathbf{1}_{H_l(Z)<d}\right) \\
&= e^{-\beta d}\mathbb{Q}_0\left(H_b(Z) > d, H_l(Z) > d\right) + \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_l(Z)}\mathbf{1}_{H_l(Z)<d}\right)\mathbb{E}_l^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^{+}(\tilde{Z})}\mathbf{1}_{\tau_{l,d}^{+}(\tilde{Z})<H_b(\tilde{Z})}\right)
\end{aligned}$$

According to Borodin and Salminen [2002], Chapter 1. Brownian motion, formula (1.15.4),

$$\mathbb{Q}_0\left(H_b(Z) > d, H_l(Z) > d\right) = \mathbb{Q}_0\left(l < \inf_{0\le s\le d} Z_s, \sup_{0\le s\le d} Z_s < b\right)$$

$$= \frac{1}{\sqrt{2\pi d}} \sum_{k=-\infty}^{\infty} \int_a^b \left(e^{-\frac{(z+2k(b-l))^2}{2d}} - e^{-\frac{(z-2l+2k(b-l))^2}{2d}} \right) dz.$$

Also, we can calculate

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta H_l(Z)} \mathbf{1}_{H_l(Z)<d}\right) = \int_0^d e^{-\beta t} \frac{|l|}{\sqrt{2\pi t^3}} e^{-\frac{l^2}{2t}} dt$$

$$= e^{-\sqrt{2\beta}|l|} \mathcal{N}\left(\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right) + e^{\sqrt{2\beta}|l|} \mathcal{N}\left(-\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right).$$

Again, we have the equality in distribution

$$\mathbb{E}_l^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^+(\tilde{Z})} \mathbf{1}_{\tau_{l,d}^+(\tilde{Z})<H_b(\tilde{Z})}\right) = \mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_d^+(Z)} \mathbf{1}_{\tau_d^+(Z)<H_b(Z)}\right),$$

which has been calculated in Lemma 5.3.2 with $\mu = 0$. Altogether, it becomes

$$\mathbb{E}_0^{\mathbb{Q}}\left(e^{-\beta\tau_{l,d}^+(Z)} \mathbf{1}_{\tau_{l,d}^+(Z)<H_b(Z)}\right)$$

$$= \frac{e^{-\beta d}}{\sqrt{2\pi d}} \sum_{k=-\infty}^{\infty} \int_a^b \left(e^{-\frac{(z+2k(b-l))^2}{2d}} - e^{-\frac{(z-2l+2k(b-l))^2}{2d}} \right) dz +$$

$$+ \left[e^{-\sqrt{2\beta}|l|} \mathcal{N}\left(\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right) + e^{\sqrt{2\beta}|l|} \mathcal{N}\left(-\sqrt{2\beta d} - \frac{|l|}{\sqrt{d}}\right) \right] \times$$

$$\times \frac{e^{-\beta d}\left(\sum_{k=0}^{\infty} 2\left[z(k,0,0) - z(k+\frac{1}{2},0,0)\right] - z(0,0,0) \right)}{\sum_{k=0}^{\infty} 2\left[z(k,\beta,0) + \sqrt{2\beta} e^{-\sqrt{2\beta}2kb}\right] - z(0,\beta,0) - 2\sqrt{2\beta}}.$$

Analogously, similar results when $H_b(Z) < \tau_{l,d}^+(Z)$, $l < b$, can be achieved.

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