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# Stochastic modelling and equilibrium in mathematical finance and statistical sequential analysis

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# Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).



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# Abstract

The focus of this thesis are the equilibrium problem under derivative market imbalance, the sequential analysis problems for some time-inhomogeneous diffusions and multidimensional Wiener processes, and the first passage times of certain non-affine jump-diffusions.

First, we investigate the impact of imbalanced derivative markets - markets in which not all agents hedge - on the underlying stock market. The availability of a closed-form representation for the equilibrium stock price in the context of a complete (imbalanced) market with terminal consumption allows us to study how this equilibrium outcome is affected by the risk aversion of agents and the degree of imbalance. In particular, it is shown that the derivative imbalance leads to significant changes in the equilibrium stock price process: volatility changes from constant to local, while risk premia increase or decrease depending on the replicated contingent claim, and become stochastic processes. Moreover, the model produces implied volatility smiles consistent with empirical observations.

Secondly, we study the sequential hypothesis testing and quickest change-point (disorder) detection problem with linear delay penalty costs for certain observable time-inhomogeneous Gaussian diffusions and fractional Brownian motions. The method of proof consists of the reduction of the initial problems into the associated optimal stopping problems for one-dimensional time-inhomogeneous diffusion processes and the analysis of the associated free boundary problems. We derive explicit estimates for the Bayesian risk functions and optimal stopping boundaries for the associated weighted likelihood ratios and obtain their exact rates of convergence under large time values.

Thirdly, we study the quickest change-point detection problems for the correlated components of a multidimensional Wiener process changing their drift rates at certain random times. These problems seek to determine the times of alarm which are as close as possible to the unknown change-point (disorder) times at which some of the components have changed their drift rates. The optimal times of alarm are shown to be the first times at which the appropri-

ate posterior probability processes exit certain regions restricted by the stopping boundaries. We characterize the value functions and optimal boundaries as unique solutions of the associated free boundary problems for partial differential equations. We provide estimates for the value functions and boundaries which are solutions to the appropriately constructed ordinary differential free boundary problems.

Fourthly, we compute the Laplace transforms of the first times at which certain non-affine one-dimensional jump-diffusion processes exit connected regions restricted by two constant boundaries. The method of proof is based on the solution of the associated integro-differential boundary problems for the corresponding value functions. We derive analytic expressions for the Laplace transforms of the first exit times of the jump-diffusion processes driven by compound Poisson processes with multi-exponential jumps. The results are illustrated on the constructed non-affine pure jump analogues of the diffusion processes which represent closed-form solutions of the appropriate stochastic differential equations.

Finally, we obtain closed-form expressions for the values of generalised Laplace transforms of the first times at which two-dimensional jump-diffusion processes exit from regions formed by constant boundaries. It is assumed that the processes form the models of stochastic volatility with independent driving Brownian motions and independent compound Poisson processes with exponentially distributed jumps. The proof is based on the solution to the equivalent boundary-value problems for partial integro-differential operators. We illustrate our results in the examples of Stein and Stein, Heston, and other jump analogues of stochastic volatility models.

# Contents

# Introduction

## I. Description of the subject

The main themes of this thesis are the equilibrium problem in mathematical finance under derivative market imbalance, the sequential analysis problems of mathematical statistics and the first passage times of non-affine jump-diffusions driven by solvable equations.

The question of how the market of financial derivatives impacts the underlying asset prices in equilibrium plays an important role in financial economics and mathematical finance. With the current market of over-the-counter derivatives having outstanding notional amount of more than ten times that of the world stock market, it is crucial to understand the potential impact trading in such contracts can have on the stock prices. In standard frictionless (complete) models of financial markets the introduction of structured financial products does not have an influence on asset prices in equilibrium - this is due to the fact that derivatives are assumed to be in zero net supply and long positions can be offset by taking the corresponding short ones. In reality, however, a lot of the counterparties in such contracts do not hedge them or do so only infrequently. Effectively, the market in the underlying asset becomes imbalanced - an extra supply or demand is created which could potentially impact the dynamics of asset prices.

Apart from the intuitive considerations, there has been number of studies supporting the idea that hedging has an effect on market risk premia and volatility (see e.g. Basak [6] and Grossman and Zhou [49]). The event that triggered investigations into the impact of dynamic hedging strategies was the market crash of 1987. The rise of the so-called portfolio insurance strategies, which guarantee a minimum level of wealth at some horizon, together with automated trading in the years surrounding the crash, led researchers to study them as a possible cause for the high volatility during the crash. Moreover, after the crash the implied volatility started exhibiting the now characteristic smile, suggesting that the Black-Scholes model may not describe the dynamics of the stock prices accurately. There is still no consensus, however, on the magnitude and direction of the market impact and our main motivation here is to provide a general setting which can account for both increasing/decreasing risk premia and market volatilities.

In practice, in order to be able to find the equilibrium stock prices in the above problem, we need to have some externally given quantities (e.g. the dividend growth rate of the underlying asset) that we have estimated through statistical methods. However, no agent has perfect information - the dividends contain noise and the growth rate can change without the agent realizing it. Nevertheless we have to rely on observable data, as it arrives, in order to infer the true value - this is a problem of statistical sequential analysis.

Sequential analysis problems are concerned with the analysis of data that doesn't have a fixed sample size. These problems were initially used in improving industrial quality control but later numerous applications were found in many real-world systems in which the amount of observation data is increasing over time (see, e.g. Carlstein et al. [20] for an overview). Two of the classical problems of this type are the sequential hypothesis testing and quickest change-point (disorder) detection. In the sequential hypothesis testing problem the aim is to determine the true value, among two alternatives, for the parameter of some observable quantity. The problem was first studied for sequences of independent and identically distributed observations by Wald and Wolfowitz [115, 116]. The problem of quickest change-point detection seeks to determine a stopping time which is as close as possible to the time of change-point at which the observable quantity changes its probabilistic properties. Originating from the control charts introduced by Shewhart [100], different variants of the problem were subsequently developed (see Page [84]).

In both of the sequential analysis problems described above one faces a tradeoff between minimizing the observation time and the error due to noise in the observations. The usual method of solving these problems, as developed in Mikhalevich [79] and Shiryaev [101, 102, 103, 104], is to reduce them to optimal stopping problems for Markov processes called sufficient statistics, and then prove verification theorems that characterize the value functions and optimal stopping boundaries as unique solutions to free boundary problems for ordinary or partial (integro-)differential operators. In order to carry out the verification arguments additional conditions are imposed, which guarantee the uniqueness of the solution of the free boundary problem. The smooth-fit condition was seen to hold for the value functions when the underlying sufficient statistics can leave the continuation region determined by the optimal stopping boundaries continuously. An extensive treatment of sequential analysis problems and the associated optimal stopping theory can be found in the books of Shiryaev [105] and Peskir and Shiryaev [90].

The link between optimal stopping and free boundary problems led to the availability of analytic expressions for the solutions of the sequential analysis problems. Nevertheless, even for simple model specifications (e.g. when the observable is one-dimensional Brownian motion with changing/unknown constant drift), finding explicit solutions to the associated free boundary problems is nontrivial and additional relations between the model parameters are often assumed. Thus, one is often lead to search for estimates of the original value functions and optimal stopping times, which are easier to compute. Our aim here is to provide verification theorems and estimates in new and more general models for the observable processes.

Stochastic processes representing solutions to stochastic differential equations are used in modelling phenomena that exhibit random behaviour. Therefore, in the theory of stochastic differential equations, it is important to have analytical tractability of the resulting models. A lot of problems in these models become computationally feasible if probabilistic properties of the related stochastic processes, such as the probability densities or characteristic functions of their marginal distributions, have closed-form expressions. Well-known examples can be found, beginning with the seminal work of Bachélier [5], where he constructed a discrete pre-image of Brownian motion for the description of the stock prices on a financial market, in Ornstein and Uhlenbeck [112], where the authors used a mean-reverting process to study velocity of a massive particle in a fluid under the bombardment by molecules, and in the geometric Brownian motion proposed by Samuelson [97] for modelling the behavior of financial assets. A recently popularized general class of tractable models, for which the form of the characteristic function is known, are the affine processes (see Duffie et al. [33]). An alternative class of continuous processes that can be used in modelling, and which can be non-affine, are those that satisfy *solvable* stochastic differential equations. These equations can be solved explicitly as shown in Gard [45; Chapter IV] or can be reduced to first-order ordinary differential equations as in Øksendal [83; Chapter V], and thus provide tractability of the resulting models. Another form of model tractability comes from the ability to compute the Laplace transforms of the first passage times of a stochastic process - these are the times at which the process crosses given values. Knowledge of the Laplace transform of the first passage times gives rise to numerous applications in engineering (e.g. see Blake and Lindsey [17]) and mathematical finance (see Kou and Wang [68]). Our objective in the final part of the thesis is to obtain analytic expressions and, in certain cases, closed-form solutions for these Laplace transforms for non-affine processes solving stochastic differential equations, which contain jumps and are extensions of the solvable class, as well as for certain jump analogues of stochastic volatility models.

# II. Historical notes and references

We present here historical notes and references to the relevant literature on the problems solved in this thesis, by also pointing out the position of our results.

The problem of finding equilibrium on the market is central in economic theory and has received a lot of attention in mathematical finance recently. The essence of equilibrium is to regard the asset prices as results of the aggregate trading decisions of rational agents on the market, that bring the supply and demand in balance. Starting from microeconomic principles one usually works with agents which have concave preferences, maximize expected consumption and possess exogenously given income streams (i.e. endowments).

The concept of an economy in equilibrium, by looking at prices as a result of supply and demand forces, was introduced in Walras [117]. For the first time existence of equilibrium was proved in a static mathematical framework containing several agents and commodities by Arrow and Debreu [4]. The earlier equilibrium models were in discrete-time and extending them to continuous-time introduced an infinite dimensional problem. This difficulty was overcome in Karatzas et al. [61, 62, 59] in a continuous-time complete market setting. There the authors present the now standard method of finding equilibrium, by using results from portfolio optimization (see Karatzas et al. [60]) together with a finite-dimensional fixed point argument first introduced in Negishi [81]. Numerous extensions to the above classical setting has been considered - see Karatzas and Shreve [64; Chapter 4] for an overview.

The study of equilibrium with agents that are not pure utility maximizers was motivated by the emergence of the volatility smile effect after the market crash of 1987 and the possible influence that dynamic hedging strategies had on the stock price volatility (see Grossman [47], Grossman and Villa [48] ). In Brennan and Schwarz [18] the effect of portfolio insuring on the equilibrium stock prices was investigated. The final wealth of a portfolio insurer was given by a fixed terminal payoff containing an implicit put option on a proportion of the total market wealth. This lead to increase in market risk premium and (implied) volatilities. Portfolio insurers were modelled as final wealth utility maximizers having lower bound on wealth in Grossman and Zhou [49]. Existence of equilibrium prices was proved for logarithmic and power utility with risk aversion coefficient $1/2$. While the main focus of the authors was the magnitude of change in market quantities like volatilities and risk premia in different market states, they provided evidence that market volatility increases. In a related setting Basak [6] proved existence of equilibrium where the portfolio insurers maximized CRRA utility from consumption, and had insurance horizon which ended before the terminal market date. The conclusion was that the market price of risk level stays the same, while the volatility decreases due to the presence of portfolio insurers, which hinted at the importance of the specification of agent's utilities and the market investment horizon (see also Basak [7] for an alternative modelling of the agents' utilities).

In equilibrium literature the completeness of the market is often assumed to hold apriori. However it is more desirable to obtain a complete market as an outcome of equilibrium, which gives rise to the notion *endogenous* completeness. Recently a series of papers concentrated in proving endogenous completeness of equilibrium - see Anderson and Raimondo [2], Hugonnier et al. [52], Riedel and Hirzberg [94] and Kramkov and Predoiu [70]. The key assumptions in the above articles are the Markov property of the model primitives (e.g. dividends or market factors) as well as the real analyticity of the exogenous volatility. In Chapter 1 we prove the existence of equilibrium and its endogenous completeness in a setting where not all agents hedge - i.e. some contingent claims are not in zero net supply and the market for them is imbalanced. We achieve this effect by including a hedging agent in the market that acts as a risk minimizer and wants to perfectly replicate a contingent claim underwritten to another agent that is outside of the market and does not hedge. This is more in line with the definition used in [18] and we have a clear separation of the risk-minimizing and the utility-maximizing effects on the market prices.

The problems of statistical sequential analysis that we are interested in seek to determine the distributional properties of continuously observable stochastic processes with minimal costs. The problem of sequential testing for two simple hypotheses about the drift rate of an observable Gaussian process is to detect the form of its drift rate from one of the two given alternatives. In the Bayesian formulation of this problem, it is assumed that these alternatives have an a priori given distribution. The problem of quickest change-point (disorder) detection for an observable Gaussian process is to find a stopping time of alarm $\tau$ which is as close as possible to the unknown time of change-point $\theta$ at which the local drift rate of the process changes from one form to another. In the classical Bayesian formulation, it is assumed that the random time $\theta$ takes the value 0 with probability $\pi$ and is exponentially distributed given that $\theta > 0$. These problems were originally formulated and solved for sequences of observable independent identically distributed random variables (see, e.g. Shiryaev [105; Chapter IV, Sections 1,3]). The first solutions of the problems in the continuous-time setting were obtained in the case of observable Wiener processes with constant drift rates (see Shiryaev [105; Chapter IV, Sections 2 and 4]). The standard disorder problem for observable Poisson processes with unknown intensities was introduced and solved in Davis [25], under certain restrictions on the model parameters. Peskir and Shiryaev [88, 89] solved both sequential analysis problems for Poisson processes in full generality (see also [90; Chapter VI, Sections 23 and 24]). The case of observable compound Poisson processes, in which the unknown characteristics were the intensity and distribution of jumps, was investigated in Dayanik and Sezer [27, 28]. Other formulations based on the exponential delay penalty setting were studied in Beibel [12] for a Wiener process and in Bayraktar and Dayanik [8] for a Poisson process. These problem settings are suitable when modelling situations in which the costs of delay in disorder detection are not necessarily linear and another measure of the error due to false alarms is preferable (e.g. continuous compounding of interest rate in financial applications). The classical change-point detection problem for Poisson processes for various types of probabilities of false alarm and delay penalty costs was studied in Bayraktar et al. [9]. More general versions of the standard Poisson disorder problem were solved by Bayraktar et al. [10], where the intensities of the observable processes changed to unknown values. These problems for observable jump processes were solved by successive approximations of the value functions of the corresponding optimal stopping problems. This method was also applied in the solution of the disorder problem for observable Wiener process in Sezer [99], in which disorder happens at one of the arrival times of an observable Poisson process. Further extensions of both sequential analysis problems for observable Wiener processes were studied in Gapeev and Peskir [41, 42] in the finite horizon setting, and for certain time-homogeneous diffusions in Gapeev and Shiryaev [43, 44] on infinite time intervals.

In the classical infinite horizon setting for the observable Wiener processes explicit solutions can be obtained, since the corresponding differential operator is an ordinary one. This fails to hold in the finite horizon setting, because the corresponding partial differential operator contains a time derivative. However, in the studies of more realistic models with non-stationary increments, the equivalent free boundary problem becomes parabolic and no explicit solutions exist in general, even in the infinite horizon case (see Chapter 2).

Multidimensional versions of the quickest disorder detection problems naturally arise when one models real-world systems described by several stochastic processes which may have dependent components. Bayraktar and Poor [11] solved the disorder problem for two observable independent Poisson processes, in which stopping times were sought as close as possible to the minimum of the two disorder times. Dayanik et al. [26] solved the disorder problem for observable multidimensional Wiener and Poisson processes with independent components, which change their local characteristics simultaneously. The quickest change-point detection problem for observable multidimensional Wiener process with *correlated* components that change their local drift rates at *different* disorder times is studied in Chapter 3. Possible applications of the solutions of these quickest detection problems include: assembly line breakdown in plant production of an item when we aim to detect the minimum of all disorder times (see [11]); abnormal returns in one of many stocks when we aim to detect just one of the disorder times; total system breakdown when we aim to detect the maximum of all disorder times.

The method of reducing stochastic differential equations to solvable ones was studied in Gard [45; Chapter IV], where closed-form strong solutions to a class of stochastic differential equations with linear coefficients were obtained, by introducing an integrating factor process. The idea is further developed in Øksendal [83; Chapter V], for equations with general drift coefficients, which are reduced to the ordinary differential form. Certain reducibility criteria were provided in Gapeev [38] for diffusions driven by a Wiener process and a Poisson random measure of a finite intensity. Jump analogues of continuous diffusions satisfying solvable equations were constructed and shown to have the same support of marginal distributions as the original processes, making them a suitable modelling alternative. The latter fact was justified by Iyigunler et al. [54], where simulations studies were provided for this model.

An introduction to the topic of financial modelling with jump-diffusions is provided in Runggaldier [96], where asset price and term structure models are studied in the context of pricing and hedging. An extensive overview of Lévy process models with multiple numerical and empirical examples is given in the book of Cont and Tankov [21]. The general class of affine processes, which includes Lévy processes, was introduced in Duffie et al. [33]. The logarithm of the characteristic function of these processes is affine in their initial value and is known in an analytic form through a solution of a family of ordinary differential equations. This leads to tractability of the resulting models and makes them suitable for applications to the term-structure of interest rates (see [33; Chapter 13] and references therein), credit risk (see Duffie [32]), stochastic volatility (see Kallsen [57]) and option pricing by Fourier methods (see e.g. Kallsen et al. [58]). Despite the recent focus on affine processes, there are still models that fall outside this general framework. Some well-known examples are the CEV and SABR models introduced in Cox [23] and Hagan et al. [50], respectively, and for which model-dependent calibration methods are known (see [50]). An overview of both affine and non-affine models for interest rates can be found in Shiryaev [106; Chapter III, Section 4].

The Laplace transform of the first time to a given drawdown of a Brownian motion with linear drift and the running maximum stopped at that time was computed by Taylor [110], and the joint law of those variables was obtained by Lehoczky [74]. Some explicit expressions for other related characteristics such as the expectation and the density of the maximum drawdown of the Brownian motion with linear drift were derived by Douady, Shiryaev and Yor [31] and Magdon-Ismail et al. [76], respectively. More recently, Sepp [98] derived closed-form expressions for the Laplace transforms of the first hitting time of constant boundaries for double-exponential jump-diffusion process. Mijatović and Pistorius [78] obtained the laws of the first-passage times of spectrally positive and negative Lévy processes over constant levels as well as analytically explicit identities for a number of characteristics of drawdowns and drawups in those models.

## III. Contribution of the thesis

Let us now describe the contribution of the thesis to the problems of equilibrium, sequential analysis and stochastic modelling described above.

We prove the existence of endogenous equilibrium in an imbalanced derivative market (Chapter 1). We begin by specifying the financial market, which consists of a (representative) agent that maximizes utility from final wealth and a hedging agent that wants to exactly replicate the payoff of a given contingent claim. There is a bond and a risky stock that represents a claim to a dividend at the final trading date. The dividend is the final value of an exogenously given Markov process. We prove existence of an equilibrium stock price process that makes the market complete, and provide its local volatility form for utilities having index of relative risk aversion less than 1. This is in contrast with the constant volatility resulting from classical equilibrium setting containing only power utility maximizers. By varying the replicated contingent claim we can obtain any volatility smile shape. Thus we can explain the presence of volatility smile by the presence of hedgers on the market, confirming one of the explanations for the Black Monday market crash of 1987. In particular, in comparison to the usual setting with only a representative agent, hedging strategies corresponding to long positions in European options lead to higher implied volatility levels at their associated strike prices, while risk premia increase.

In order to find the equilibrium stock price process we use results from portfolio optimization in complete markets (see [60]), to obtain a guess for the state-price density. Indeed, if equilibrium exists and the resulting market is complete, the hedger can replicate exactly the contingent claim and, assuming zero initial wealth, his final wealth will be equal to the contingent claim minus its arbitrage-free price. By market clearance we obtain the final wealth of the utility-optimizing agent and we use duality results from Kramkov and Schachermayer [71] to find the state-price density process as conditional expectation of the marginal utility at the agent's final wealth. Knowing the state-price density we can obtain the stock price process again as conditional expectation of the terminal dividend. We find the arbitrage-free price of the contingent claim as a solution to a fixed point problem. Finally, we prove that the obtained guess for the stock price process results in complete market by using the recent result on endogenous completeness in [70].

We consider the two classical problems of sequential analysis in their Bayesian formulations for certain Gaussian processes with non-stationary increments (Chapter 2). We begin by providing a unifying optimal stopping problem for the likelihood ratio processes, which are time-inhomogeneous diffusions. This allows us to work with both original problems in a consistent way. We prove a verification theorem and show that the optimal stopping times are the first times at which the associated likelihood ratios exit from certain regions. Such regions are restricted by the curved stopping boundaries, which are solutions to the equivalent parabolic free boundary problems. Since we intend to provide an explicit analysis for the asymptotic rates of the solutions, we introduce an auxiliary ordinary differential free boundary problem in which the time variable is a parameter, by removing the time derivative from the initial parabolic operator. The resulting ordinary differential equation admits an explicit solution, and we can obtain closed-form estimates for the solutions of the original parabolic problem. We derive analytic expressions for the optimal boundaries in the auxiliary problem, and specify their exact asymptotic behaviour under large time values. Combining these results with the estimates of the solutions of the original optimal stopping problem, we can check that the assumption of the main verification theorem, that the optimal stopping time has finite expectation, is indeed satisfied. We demonstrate this in a setting in which the observable process is a fractional Brownian motion with a constant drift rate. In that case we can reduce the sequential analysis problems to the original unifying optimal stopping problem for time-inhomogeneous diffusion processes.

We study the quickest change-point (disorder) detection problem for observable multidimensional Wiener process (Chapter 3). This problem seeks to determine the times of alarm at which some of the components of the process change their local drift rates as soon as possible and with minimal error probabilities. The classical Bayesian formulation of these problems consists of minimization of linear combinations of the probabilities of false alarm and the expected linear penalty costs in detecting the change-points correctly. It is customary assumed that the change-point (disorder) times are independent exponentially distributed random variables. Our setting is closer to the one of [11], since the component disorder times are different, but is more general in the sense that we observe multiple correlated components.

We begin by reducing the original disorder problem to an optimal stopping problem for a multidimensional Markov diffusion. The components of the diffusion form a family of posterior probability processes, corresponding to every subset of disorder times, and play the role of sufficient statistics for the original disorder problem. When doing the reduction, we use the ideas from [40], where the filtering equations for the posterior probabilities are derived for two observable correlated Wiener processes. It is shown that the optimal stopping times are the first times at which one of the posterior probability processes exits from a region restricted by a stochastic boundary surface, determined by the current values of the other sufficient statistics. We formulate the equivalent free boundary problem and prove a verification theorem that identifies its unique solution with the value function of the optimal stopping problem. The main complication in our setting arises from the higher dimensions of the sufficient statistics needed to formulate the optimal stopping problem for a Markov process, due to the presence of several disorder times. Moreover, the correlation structure of the observable processes has to be taken into account when deriving the filtering equations. The proof of the verification theorem uses the change-of-variable formula with local time on surfaces from Peskir [87]. As we do not have explicit solutions to the free boundary problem, we provide lower estimates for the value functions, which inherently construct the upper estimates for the stochastic boundary surfaces, in the case in which we aim to detect the infimum of component disorder times. These estimates are solutions to free boundary problems for ordinary differential equations.

We introduce an analytically tractable framework in which the Laplace transforms of certain exit times for non-affine jump analogues of continuous diffusion models can be computed (Chapter 4). We begin by extending the method of [45; Chapter IV] for finding solvable stochastic differential equations to a general class of jump-diffusions. By applying a smooth invertible transformation, the original equation is reduced to a simpler one with linear diffusion and jump coefficients, and we can choose an appropriate integrating factor process to obtain closed-form solutions. Moreover, we construct jump analogues of certain continuous diffusion models driven by solvable equations, by following the method described in [38]. We provide examples of reducing solvable equations and constructing their non-affine jump-diffusion analogues for several popular models. Finally, we consider the first times at which non-affine jump analogues of continuous diffusion models, with compensator measures correspond to compound Poisson processes, exit from an open interval on the real line. We characterize the integrals of the Laplace transforms of these exit times as solutions to ordinary differential boundary value problems, by reducing the integro-differential equation corresponding to the original jump analogue generator. Explicit solutions are provided for the pure jump analogues of the CIR, CEV and the nonlinear filter models with compensator measures corresponding to a compound Poisson process with one-sided exponentially distributed jumps.

We derive closed-form expressions for the generalised Laplace transforms of the first exit times of the two-dimensional jump-diffusion processes from certain connected regions formed by constant boundaries (Chapter 5). We consider two-dimensional jump-diffusion processes driven by independent standard Brownian motions and independent compound Poisson processes with exponential jumps. We provide closed-form solutions of the partial integro-differential boundary-value problems associated with the values of the generalised Laplace transforms as iterated stopping problems for the two-dimensional jump-diffusion processes forming the models of stochastic volatility. In particular, we derive closed-form expressions for the generalised Laplace transforms in jump analogues of Stein and Stein and Heston as well as in other stochastic volatility models.

## IV. Structure of the thesis

In Section 1.1 we specify our financial market and remark on some useful properties of the exogenous Markov process that models the dividends. In Section 1.2 we prove the existence of endogenously complete equilibrium and provide analytic expressions for the equilibrium stock price drift and diffusion coefficients as well as the optimal portfolio of the representative agent. Moreover we prove the local volatility form of the stock price process for certain utility functions. Finally, in Section 1.3, we illustrate our results when the exogenous Markov process modelling the dividends is of Black-Scholes type, and the representative agent maximizes power utility. In this simple setting, we show the effect of the replicated contingent claim on the implied volatility and the market price of risk of the stock.

In Section 2.1 we formulate a unifying optimal stopping problem for the time-inhomogeneous diffusion likelihood ratio process and show how this problem arises from the Bayesian sequential testing and quickest change-point detection settings. We formulate an equivalent free boundary problem and derive explicit solutions of the auxiliary ordinary free boundary problems which have the time variable as a parameter. In Section 2.2 we study the asymptotic behavior of the resulting stopping boundaries under large time values, by means of deriving their Taylor expansions with respect to the local drift rate of the observable process. In Section 2.3 we apply these results to models with observable fractional Brownian motions by proving that the optimal stopping times have finite expectations and, hence, the verification theorem can be applied to characterize the solutions of the sequential analysis problems.

In Section 3.1 we introduce the setting of the model for the quickest change-point detection problem for observable multidimensional Wiener processes. We derive stochastic differential equations for a family of posterior probability processes corresponding to subsets of the disorder times, by means of generalized Bayes' formula (see [75; Theorem 7.23]). In Section 3.2 we construct the associated optimal stopping problem for the posterior probability processes and formulate the equivalent high-dimensional free boundary problem. The verification theorem is proved providing characterization of the optimal stopping boundary surface as the unique solution to the free boundary problem. Finally, in Section 3.3, we provide estimates for the original solution to the problem of detection of the infimum of all disorder times.

In Section 4.1, we apply the method of [45; Chapter IV] to obtain explicit solutions to jump-diffusion stochastic differential equations with linear coefficients. Then we follow [83; Chapter V, Example 5.16] to reduce the equations with general drift and linear diffusion and jump coefficients to ordinary differential equations that are satisfied pathwise (see also [38]). In Section 4.2, we extend the class of solvable stochastic differential equations via smooth invertible transformations, and provide sufficient conditions for their reducibility. We also construct jump analogues of continuous diffusions and give some examples. In Section 4.3, we show that the Laplace transforms of the first exit times from a region restricted by two constant boundaries for certain finite activity pure jump analogues of continuous diffusions can be obtained by solving ordinary differential equations, and provide explicit solutions for some popular models.

In Section 5.1, we first introduce the setting and notation of the model with a two-dimensional jump-diffusion Markov process which has the price of the risky asset and the volatility rate as the state space components. We define the generalised Laplace transforms of the first times at which the process exits certain regions restricted by constant boundaries. In

Section 5.2, we obtain a closed-form solution to the partial integro-differential boundary-value problem under several additional conditions on the parameters of the model. In Section 5.3, we verify that the resulting solution to the boundary-value problem provides the joint Laplace transform. The main results of the paper are stated in Theorem 5.3.1.

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# Chapter 1

# Equilibrium with imbalance of the derivative market

This chapter is based on joint work with Dr. Albina Danilova.

## 1.1. Financial market and model primitives

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space rich enough to support a Brownian motion $(W_t)_{t\in[0,T]}$ and let $(\mathcal{F}_t)_{t\in[0,T]}$ be its filtration satisfying the usual conditions, where $T \geq 0$ is a terminal time. Consider a financial market consisting of two assets:

- A riskless zero yield bond with maturity $T$ and in total supply of $K \in \mathbb{R}$ units.
- A risky asset, i.e. a stock with an adapted price process $S = (S_t)_{t\in[0,T]}$, which is in total supply of 1 unit and represents a time $T$ claim to an exogenously given random dividend.

Both assets are continuously traded on the time interval $[0, T]$ and we assume that the market terminates after this time. Let the exogenously given log-dividend process $Z = (Z_t)_{t\in[0,T]}$ be the unique strong solution of the stochastic differential equation (SDE)

$$dZ_t = \mu_Z(t, Z_t)\, dt + \sigma_Z(t, Z_t)\, dW_t \quad \text{for} \quad t \in [0, T], \tag{1.1.1}$$

with initial condition $Z_0 = z_0 \in \mathbb{R}$ and some functions $\mu_Z(t, z) : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma_Z(t, z) : [0, T] \times \mathbb{R} \to \mathbb{R}$. Denote by $\mathbf{C}_b(\mathbb{R})$ the space of bounded and continuous real-valued functions on $\mathbb{R}$.

**Assumption 1.1.1.** *The functions $\mu_Z(t,z)$ and $\sigma_Z(t,z)$ satisfy the following conditions:*

**(C1)** Uniform ellipticity*: $\sigma_Z^2(t,z)$ is uniformly bounded away from zero, i.e. there exists $\underline{\sigma}>0$ such that $\sigma_Z^2(t,z)\geq\underline{\sigma}$ on $[0,T]\times\mathbb{R}$.*

**(C2)** Boundedness and analyticity*: $\mu_Z(t,z)$ and $\sigma_Z^2(t,z)$ are bounded on $[0,T]\times\mathbb{R}$. The maps $t\to\mu_Z(t,\cdot)$ and $t\to\sigma_Z(t,\cdot)$ from $[0,T]$ to $\mathbf{C}_b(\mathbb{R})$ are analytic on $(0,T)$, i.e. for all $t\in(0,T)$ there is a constant $\varepsilon(t)>0$ and sequences $(A_n(t))_{n\geq 0}$, $(B_n(t))_{n\geq 0}$ in $\mathbf{C}_b(\mathbb{R})$ such that*

$$\mu_Z(s,\cdot)=\sum_{n=0}^{\infty}A_n(t)(s-t)^n \quad \textit{and} \quad \sigma_Z(s,\cdot)=\sum_{n=0}^{\infty}B_n(t)(s-t)^n,$$

*for any $s\in(0,T)$ with $|s-t|<\varepsilon(t)$.*

**(C3)** Continuity*: $\mu_Z(t,z)$ and $\sigma_Z(t,z)$ are uniformly Hölder-continuous in $t$ for all $z\in\mathbb{R}$, and $\sigma_Z^2(t,z)$ is uniformly Hölder-continuous in $z$ for all $t\in[0,T]$. Moreover, $\mu_Z(t,z)$ and $\sigma_Z(t,z)$ are locally Lipschitz-continuous in $z$ for all $t\in[0,T]$.*

**Remark 1.1.1.** *From Theorems 5.3.11 and 5.3.7 in [35] we can see that* (C2) *and* (C3) *guarantee the existence of a weak solution to* (1.1.1) *that is pathwise unique up to an explosion time. From the boundedness in* (C2) *we get that the explosion time is a.s. infinite (see Chapter IX, Exercise 2.10 in [93]) and therefore the solution is pathwise unique for all $t\in[0,T]$. From Theorem IV.1.1 in [53] it follows that there exists a unique strong solution to* (1.1.1) *with initial condition $Z_0=z_0\in\mathbb{R}$. Moreover, for any $(t,z)\in[0,T]\times\mathbb{R}$, the SDE in* (1.1.1) *has unique strong solution $Z^{(t,z)}$ on $[t,T]$ satisfying $\mathbb{P}[Z_t^{(t,z)}=z]=1$.*

*We use conditions* (C1)-(C3) *to prove some properties of the marginal distributions of $Z$ (see Lemma 1.A.1 in the Appendix) and to obtain unique solutions to certain terminal value (Cauchy) problems with respect to the infinitesimal generator $\mathbb{L}_Z$ of $(t,Z_t)_{t\in[0,T]}$. Moreover, we can apply Theorem 9.2 in [37] to obtain a fundamental solution (see Definition 5.7.9 in [63]) of the partial differential equation (PDE)*

$$\mathbb{L}_Z G(t,z):=\frac{\partial G}{\partial t}(t,z)+\mu_Z(t,z)\frac{\partial G}{\partial z}(t,z)+\frac{\sigma_Z^2(t,z)}{2}\frac{\partial^2 G}{\partial z^2}(t,z)=0, \qquad (1.1.2)$$

*for $(t,z)\in[0,T)\times\mathbb{R}$. We denote this fundamental solution by $p(t_1,z;t_2,v)$ where $0\leq t_1<t_2\leq T$ and $z,v\in\mathbb{R}$.*

*The analyticity condition in* (C2) *allows us to use results from [70] on the analiticity of solutions to Cauchy problems and prove that the volatility of the stock price in our market is* *nonzero a.e. a.s., which will lead to the endogenous completeness of the equilibrium market (see [69]).*

Let us now specify the properties of the stock price processes on the market.

**Definition 1.1.1.** *The stock price process $S$ is admissible if the following conditions are satisfied:*

- *$S$ is a continuous, strictly positive semimartingale with absolutely continuous finite variation part, meaning that it satisfies*

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t) \quad \textit{for} \quad t \in [0,T], \tag{1.1.3}$$

*for some $\mathcal{F}_t$-progressively measurable processes $(\mu_t)_{t\in[0,T]}$ and $(\sigma_t)_{t\in[0,T]}$ such that*

$$\int_0^T |\mu_t| dt < \infty, \quad \int_0^T \sigma_t^2 dt < \infty, \quad a.s..$$

- *The equality $S_T = \exp(Z_T)$ holds.*

- *The market is complete, i.e. we have that*

$$\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt < \infty, \quad a.s.,$$

*the process*

$$\exp\Big( - \int_0^t \frac{\mu_s^2}{\sigma_s^2} dW_s - \frac{1}{2} \int_0^t \frac{\mu_s^2}{\sigma_s^2} ds \Big),$$

*is a martingale and $\sigma_t \neq 0$ a.e. a.s..*

**Remark 1.1.2.** *It is known from Theorem 7.2 in [29] (see [65, 13] for more recent results) that the No Free Lunch with Vanishing Risk (NFLVR) property together with the local boundedness of the stock price process implies its semimartingality. This fact is used in [3] to show that the boundedness of an agent's expected utility implies the NFLVR property, and therefore that the stock price is a semimartingale (see also [15, 73, 65]). The continuity of the stock price process is a consequence of its local martingality under some equivalent measure change and the fact that we work in a Brownian filtration. Therefore, the assumption that $S$ is a continuous*

[1] For a discussion as to why the conditions on the stock price process imply this representation, see [64; Appendix B]

*semimartingale is not too restrictive. Furthermore, the intuitive requirement that the stock price should be equal to the random dividend at time* $T$*, i.e.* $S_T = \exp(Z_T)$*, can be justified by the fact that, otherwise, an obvious arbitrage opportunity exists and NFLVR is not satisfied.*

*It is reasonable to expect that an admissible stock price process* $S$ *leads to a complete financial market, since there is a single source of risk and an asset that allows agents to trade this risk. Our definition of a complete market follows the one of a standard market in Definition 1.5.1 in [64] together with the characterization of a complete market in Theorem 1.6.6 in [64].*

There are two agents trading in the bond and the stock on the financial market – the *hedger* and the *optimizer*. The agents differ in their endowments and portfolio optimization problems. The hedger wants to replicate a nontraded contingent claim $h(S_T)$, where $h(z) : [0,\infty) \to \mathbb{R}$ is a payoff function. The optimizer has utility from final wealth $u(z) : (0,\infty) \to \mathbb{R}$ and wants to maximize its expectation. In the following definition we specify the *admissible* portfolios on the market.

**Definition 1.1.2.** *Let* $S$ *be an admissible stock price process. An* $\mathcal{F}_t$*-progressively measurable process* $\pi = (\pi_t)_{t\in[0,T]}$ *is called a self-financing portfolio process if we have*

$$\int_0^T |\pi_t \mu_t| dt < \infty \quad \textit{and} \quad \int_0^T \pi_t^2 \sigma_t^2 dt < \infty \quad \textit{a.s.}, \tag{1.1.4}$$

*and the corresponding wealth process* $X^\pi = (X_t^\pi)_{t\in[0,T]}$ *satisfies*

$$X_t^\pi = X_0^\pi + \int_0^t \pi_u dS_u \quad \textit{for} \quad t \in [0,T], \tag{1.1.5}$$

*for some initial wealth* $X_0^\pi \in \mathbb{R}$*. We define the set* $\mathcal{A}^b$ *of all (self-financing) portfolios with wealth processes that are bounded from below by a constant* $b \in \mathbb{R}$ *as*

$$\mathcal{A}^b := \big\{\pi \textit{ is a self-financing portfolio process} : X_t^\pi \geq b \quad \textit{a.s. for } t \in [0,T]\big\},$$

*and denote* $\mathcal{A}^B := \bigcup_{b\in\mathbb{R}} \mathcal{A}^b$*. The portfolio process* $\pi$ *will be called admissible if* $\pi \in \mathcal{A}^B$*.*

We set the initial endowments (i.e. wealth) of the agents are zero for the hedger and $S_0 + K$ for the optimizer, respectively. The following conditions on the payoff $h$ will be needed:

**Assumption 1.1.2.**

- $h(z)$ *is a continuous function and there exist* $\underline{k}, \overline{k} > 0$ *such that*

$$h(z) = a_1 z + b_1 \quad \textit{for} \quad z \in [0, \underline{k}] \quad \textit{and} \quad h(z) = a_2 z + b_2 \quad \textit{for} \quad z \geq \overline{k}, \tag{1.1.6}$$

*for some* $a_1, a_2, b_1, b_2 \in \mathbb{R}$*.*

- *$h(z)$ is bounded from below, $h \not\equiv 0$, and the condition*

$$h(z) < z + h_0 \quad \textit{for} \quad z > 0, \tag{1.1.7}$$

*holds for some constant $h_0 \geq 0$.*

- *We have that $h_1 \leq K - h_0$ where*

$$h_1 := \max\Big(0, -\min_{z \geq 0} h(z)\Big). \tag{1.1.8}$$

**Remark 1.1.3.** *The assumption that $h(z)$ is linear for small and large $z$ allows us to prove integrability of certain expressions of the marginal utility (see Lemma 1.A.1 in the Appendix). The boundedness from below of $h(z)$ guarantees that the hedger will be able to replicate the claim with an admissible portfolio.*

*We require that the upper bounds on $h(z)$ and $h_1$ hold, because they guarantee that the optimizer has a strictly positive final wealth (see Theorem 1.2.1 below). One can easily see this in the case when the payoff $h(z)$ is nonnegative, since then we have from* (1.1.8) *that $K \geq h_0$ and, hence, condition* (1.1.7) *leads to $S_T + K > h(S_T)$, i.e., the total endowment on the market, which is initially held by the optimizer, is larger than the replicated claim by the hedger.*

Let us precisely define the solutions to both agents' problems.

**Definition 1.1.3.** *Let $S$ be an admissible stock price process.*

1. *The process $\pi$ is a solution to the hedger's problem if $\pi$ is an admissible portfolio and the corresponding wealth process $X^\pi$, with $X_0^\pi = 0$, satisfies $X_T^\pi = h(S_T) - x^h$, where $x^h \in \mathbb{R}$ is the arbitrage-free price of the contingent claim $h(S_T)$ given by*

$$x^h = \mathbb{E}\Big[h(S_T) \exp\Big( - \int_0^T \frac{\mu_t^2}{\sigma_t^2} dW_t - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} dt\Big)\Big]. \tag{1.1.9}$$

2. *The process $\pi$ is a solution to the optimizer's problem if $\pi$ is an admissible portfolio that solves the final wealth utility maximization problem*

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}[u(X_T^\pi)],$$

*where $\mathcal{A} := \big\{\pi \in \mathcal{A}^0 : \mathbb{E}[\min(0, u(X_T^\pi))] > -\infty\big\}$ and the corresponding wealth process satisfies $X_0^\pi = S_0 + K$.*

Since we want the above utility maximization problem to be well-posed we introduce the following set of assumptions:

**Assumption 1.1.3.**

- *$u(z)$ is a strictly increasing, strictly concave, $C^2((0,\infty))$ function satisfying*

$$\lim_{z\to 0+} u'(z) = \infty, \quad \lim_{z\to\infty} u'(z) = 0 \quad \textit{(Inada conditions)}. \tag{1.1.10}$$

- *The asymptotic elasticity of $u(z)$ is less than $1$, meaning that*

$$\limsup_{z\to\infty} \frac{zu'(z)}{u(z)} < 1. \tag{1.1.11}$$

- *The index of relative risk aversion of $u(z)$ is bounded, i.e.*

$$\frac{-zu''(z)}{u'(z)} \leq R \quad \textit{for} \quad z > 0, \tag{1.1.12}$$

  *for some constant $R > 0$.*

**Remark 1.1.4.** *We need the standard assumptions* (1.1.10)-(1.1.11) *on the utility function $u(z)$ in order to guarantee the existence of a unique solution to the optimizer's problem. The condition* (1.1.12) *was used in [69] to prove the completeness of the financial market in equilibrium. In particular, from* (1.1.12) *we can see that the decreasing function $\log u'(e^z)$ has derivative bounded from below by $-R$ and, hence, there exists a constant $N > 0$ such that $\ln u'(e^z) < N(1+|z|)$. It follows that (see also Lemma 6.1 in [69])*

$$u'(e^z) \leq e^{N(1+|z|)}, \quad -u''(e^z) \leq R\, e^{N+(N+1)|z|} \quad \textit{for} \quad z > 0. \tag{1.1.13}$$

**Example 1.1.5.** *Some payoff functions $h(z)$ that satisfy the above conditions are bounded from below linear combinations of European call and put options, such that the sum of the coefficients in front of the call payoffs is at most 1, i.e.*

$$h(z) = \sum_{i=1}^{n} \alpha_i (z - K_i)^+ + \beta_i (K_i - z)^+,$$

*where $\alpha_i, \beta_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \alpha_i \in [0,1]$ for $n \in \mathbb{N}$. For the utility function $u(z)$ we can take $u(z) = \log(z)$ or $u(z) = z^{1-p}/(1-p)$ for $p \in (0,1) \cup (1,\infty)$.*

Let us define what is equilibrium in our finite-horizon financial market.

**Definition 1.1.4.** *Equilibrium in the finite-horizon financial market is a process triple* $(S, \pi^h, \widehat{\pi})$ *such that the stock price process* $S$ *is admissible, the processes* $\pi^h$ *and* $\widehat{\pi}$ *solve the hedger's and optimizer's problems in Definition 1.1.3, respectively, and the following condition holds:*

- *Clearing of the stock market:*

$$\pi^h + \widehat{\pi} = 1 \qquad \lambda([0,T]) \otimes \mathbb{P} \quad \text{a.e. a.s.}, \tag{1.1.14}$$

*where* $\lambda([0,T])$ *denotes the Lebesgue measure on the interval* $[0,T]$.

Since the wealth processes of both agents are of the form (1.1.5) and their initial wealth is given, from the clearing of stock market condition it follows:

- Clearing of the bond market:

$$X^h - \pi^h S + \widehat{X} - \widehat{\pi} S = K \qquad \lambda([0,T]) \otimes \mathbb{P} \quad \text{a.e. a.s.}, \tag{1.1.15}$$

where we have denoted the hedger's and optimizer's wealth processes by $X^h = (X_t^h)_{t\in[0,T]}$ and $\widehat{X} = (\widehat{X}_t)_{t\in[0,T]}$ respectively.

**Remark 1.1.6.** *Let us comment on the form of condition* (1.1.15). *The quantities* $X^h - \pi^h S$ *and* $\widehat{X} - \widehat{\pi} S$ *on its left hand side correspond to the wealth of each agent that is invested in bonds. However, since the bonds have zero yield, these quantities also represent the number of bonds held by each agent. Since on the right hand side we have the total number of bonds on the market, the condition* (1.1.15) *indeed means that the bond market clears, i.e. the supply and demand of bonds are equal. In combination with* (1.1.14) *this also leads to the clearing of the whole market wealth, i.e.* $X^h + \widehat{X} = S + K$ *a.e. a.s..*

**Remark 1.1.7.** *We have assumed, without loss of generality, that the interest rate on the market is* $0$. *This is due to the fact that the optimizer derives utility only from final wealth at time* $T$ *and, therefore, does not have a time preference for money. This means that the price processes of the bond and the money market account will be constant, and the total amount invested by the equilibrium economy in the money market account will be equal to* $K$. *Actually, by discounting, we could obtain an equilibrium for any integrable interest rate (see e.g. Chapter 1, Definition 1.3 in [64]).*

While our notion of equilibrium is the classical one, our model is nonstandard, as the market contains an agent that does not maximize utility – the hedger. The introduction of a hedging agent in the market allows us to study how equilibrium prices are affected when there are derivatives which are not in zero net supply, as is the case with the contingent claim $h(S_T)$.

## 1.2. Main results

In order to find the equilibrium stock price process $S$ we use ideas from portfolio optimization in complete markets. We describe below the heuristic argument through which we obtain a guess for the state-price density and, subsequently, the stock price process.

Suppose that equilibrium exists and the resulting market is complete. The hedger can replicate exactly the contingent claim with final wealth given by $X_T^h = h(S_T) - x^h$, where the constant $x^h$ is the arbitrage-free price of $h(S_T)$. Since the market clears at time $T$, the final wealth of the optimizer will be $\widehat{X}_T = S_T + K - h(S_T) + x^h$. Now we can use duality results (e.g. see Theorems 2.0 and 2.2 in [71]) to get that the state-price density process $L$ at time $T$ is given by

$$L_T = \frac{u'(S_T + K - h(S_T) + x^h)}{\mathbb{E}[u'(S_T + K - h(S_T) + x^h)]}.$$

If, moreover, $L$ is a martingale, we obtain $L$ at any $t \in [0, T)$ as $L_t = \mathbb{E}[L_T|\mathcal{F}_t]$. Thus we have obtained a guess for the state-price density. Finally, if the process $L\,S$ is a martingale (and not only a local martingale), we can obtain a guess for the stock price process $S_t$ by taking conditional expectation, i.e. $L_t S_t = \mathbb{E}[L_T S_T|\mathcal{F}_t]$ for any $t \in [0, T)$.

After obtaining the guess for the stock price process $S$, what is left is to check that the resulting market is indeed complete and in equilibrium. However, for this line of reasoning to work, we need to apriori specify the arbitrage-free price $x^h$ of the contingent claim $h(S_T)$, which, by looking at the form of $L_T$, should satisfy

$$x^h = \mathbb{E}[h(S_T)L_T] = \frac{\mathbb{E}[h(S_T)u'(S_T + K - h(S_T) + x^h)]}{\mathbb{E}[u'(S_T + K - h(S_T) + x^h)]}.$$

Let us first prove a lemma that gives the existence and uniqueness of a solution to the equation for $x^h$.

**Lemma 1.2.1.** *Let Assumptions 1.1.1, 1.1.2 and 1.1.3 be satisfied. There exists a constant* $x^h \geq -h_1$ *satisfying*

$$\mathbb{E}[(x^h - h(\exp(Z_T)))u'(\exp(Z_T) + K - h(\exp(Z_T)) + x^h)] = 0, \tag{1.2.1}$$

*where* $x^h > -h_1$ *if* $h(z)$ *is not a negative constant, and* $x^h = -h_1$ *otherwise. Moreover, if* $u(z)$ *satisfies*

$$\frac{-zu''(z)}{u'(z)} \leq 1 \quad \textit{for} \quad z > 0, \tag{1.2.2}$$

*then the equation* (1.2.1) *has a unique solution.*

*Proof.* We begin by proving the existence of a solution in the interval $[-h_1, \infty)$ via an application of the intermediate value theorem.

Denote $\xi(z) = (z - h(\overline{Z}))u'(\overline{Z} + K + z - h(\overline{Z}))$ for $z \geq -h_1$, where $\overline{Z} := \exp(Z_T)$. Since $K \geq h_0 + h_1$ and (1.1.7)-(1.1.8) hold, we have that $\overline{Z} + K + z - h(\overline{Z}) > 0$ and $\xi(z)$ is well-defined.

We will first prove that $\mathbb{E}[\xi(z)]$ is a continuous function for $z \geq -h_1$. Choose $z \geq -h_1$ and $\delta > 0$ and let $z' \in [-h_1, z + \delta)$. Since $u'(z)$ is decreasing and the conditions in (1.1.7)-(1.1.8) are satisfied, we obtain

$$\begin{aligned}|\xi(z')| &\leq \left|z' - h(\overline{Z})\right| u'(\overline{Z} + K + z' - h(\overline{Z})) \\ &\leq (z + \delta + \max(h_1, h_0 + \overline{Z}))u'(\overline{Z} + h_0 - h(\overline{Z})).\end{aligned}$$

From Lemma 1.A.1 in the Appendix we conclude that $(z+\delta+\max(h_1, h_0+\overline{Z}))u'(\overline{Z}+h_0-h(\overline{Z}))$ is an integrable random variable and we have by the dominated convergence theorem

$$\lim_{\overline{z}\to z} \mathbb{E}[\xi(\overline{z})] = \mathbb{E}[\lim_{\overline{z}\to z} \xi(\overline{z})] = \mathbb{E}[\xi(z)].$$

Hence $\mathbb{E}[\xi(z)]$ is a continuous function for $z \geq -h_1$.

Let us now find $\overline{z} \geq -h_1$ such that $\mathbb{E}[\xi(\overline{z})] > 0$. Since $h(z)$ satisfies (1.1.6)-(1.1.7) and is bounded from below, we have that

$$h(z) = a_k z + b_k h_0 \quad \text{for} \quad z \geq \overline{k},$$

where $a_k, b_k \in \mathbb{R}$ are such that $a_k \in [0, 1]$ and $a_k\overline{k} + b_k h_0 < \overline{k} + h_0$. In particular, $h(z)$ and $\overline{h}(z) := z + K - h(z)$ are nondecreasing for $z \geq \overline{k}$. Denoting $p_k = \mathbb{P}\left[\overline{Z} \in [\overline{k}, \overline{k}+1]\right]$, from Lemma 1.A.1 we have $p_k > 0$. Since $h(z)$ satisfies (1.1.7) we also have $\mathbb{E}[\max(h(\overline{Z}), 0)] < \infty$. Therefore we can choose $\overline{z} \geq -h_1$ such that

$$\max\left(\sup_{z\in[0,\overline{k}]} h(z)\,,\; h(\overline{k}+1) + \frac{\mathbb{E}[\max(h(\overline{Z}), 0)]}{p_k}\right) < \overline{z} < \infty,$$

and we obtain

$$\begin{aligned}\mathbb{E}\left[\xi(\overline{z})\right] &\geq \mathbb{E}\left[\xi(\overline{z})|\overline{Z}\in\left[\overline{k},\overline{k}+1\right]\right]p_k+\mathbb{E}\left[\xi(\overline{z})|\overline{Z}\geq\overline{k}+1\right]\mathbb{P}\left[\overline{Z}\geq\overline{k}+1\right]\\ &\geq(\overline{z}-h(\overline{k}+1))u'\left(\overline{z}+\overline{h}(\overline{k}+1)\right)p_k-u'(\overline{z}+\overline{h}(\overline{k}+1))\mathbb{E}[\max(h(\overline{Z}),0)]\\ &=u'(\overline{z}+\overline{h}(\overline{k}+1))\left((\overline{z}-h(\overline{k}+1))p_k-\mathbb{E}[\max(h(\overline{Z}),0)]\right)>0.\end{aligned}$$

On the other hand, by using (1.1.8) we have

$$\mathbb{E}\left[\xi(-h_1)\right]=\mathbb{E}[(-h_1-h(\overline{Z}))u'(\overline{Z}+K-h_1-h(\overline{Z}))]\leq 0.$$

Therefore, by the intermediate value theorem, a solution $x^h\geq -h_1$ to (1.2.1) exists. Notice that if $h$ is not a negative constant function then there exists an open set $A\subseteq\mathbb{R}$ such that $h(z)>-h_1$ for $z\in A$ and from Lemma 1.A.1 it follows that

$$\begin{aligned}\mathbb{E}\left[\xi(-h_1)\right]&=\mathbb{E}[(-h_1-h(\overline{Z}))u'(\overline{Z}+K-h_1-h(\overline{Z}))]\\ &\leq\mathbb{E}[(-h_1-h(\overline{Z}))u'(\overline{Z}+K-h_1-h(\overline{Z}))|\overline{Z}\in A]\mathbb{P}[\overline{Z}\in A]<0.\end{aligned}$$

Hence, when $h$ is not a negative constant the solution $x^h$ to (1.2.1) satisfies $x^h>-h_1$. If $h$ is a negative constant then $h\equiv -h_1$ and the solution to (1.2.1) is trivially seen to be $x^h=-h_1$.

We will now show the uniqueness of $x^h$ under the condition (1.2.2). To establish this result, we need to show that $\xi'(z)$ is integrable for $z>-h_1$ and then prove, by differentiating, that $\mathbb{E}[\xi(z)]$ is strictly increasing for $z>-h_1$.

Differentiating $\xi(z)$ gives

$$\xi'(z)=u'(\overline{Z}+K+z-h(\overline{Z}))+(z-h(\overline{Z}))u''(\overline{Z}+K+z-h(\overline{Z})).$$

For the first term, by the strict concavity of $u$ and $z>-h_1$, we have $u'(\overline{Z}+K+z-h(\overline{Z}))<u'(\overline{Z}+h_0-h(\overline{Z}))$. Therefore, from Lemma 1.A.1, we obtain that $u'(\overline{Z}+K+z-h(\overline{Z}))$ is bounded by an integrable random variable, and, hence, it is integrable. For the second term, from the negativity of $u''$ and (1.2.2) we have

$$0>u''(\overline{Z}+K+z-h(\overline{Z}))\geq-\frac{u'(\overline{Z}+K+z-h(\overline{Z}))}{\overline{Z}+K+z-h(\overline{Z})},$$

and therefore

$$|(z-h(\overline{Z}))u''(\overline{Z}+K+z-h(\overline{Z}))|\leq\frac{|\xi(z)|}{\overline{Z}+K+z-h(\overline{Z})}\leq\frac{|\xi(z)|}{z+h_1}.$$

Since $z+h_1>0$ and $\xi(z)$ is integrable for $z>-h_1$, we see that $|(z-h(\overline{Z}))u''(\overline{Z}+K+z-h(\overline{Z}))|$ is bounded by an integrable random variable and is therefore integrable. It follows that the random variable $\xi'(z)$ is integrable for any $z>-h_1$.

Next, we show that $\mathbb{E}[\xi(z)]$ is differentiable and its derivative is strictly positive. Let us fix $z+h_1>\delta>0$ and notice that

$$\begin{aligned}\mathbb{E}\left[\sup_{\overline{z}\in(z-\delta,z+\delta)}|\xi'(\overline{z})|\right]&\leq\mathbb{E}\Bigg[\sup_{\overline{z}\in(z-\delta,z+\delta)}u'(\overline{Z}+K+\overline{z}-h(\overline{Z}))\\&\qquad+\frac{|(\overline{z}-h(\overline{Z}))|u'(\overline{Z}+K+\overline{z}-h(\overline{Z}))}{\overline{Z}+K+\overline{z}-h(\overline{Z})}\Bigg]\\&\leq\mathbb{E}\left[u'(z+h_1-\delta)+u'(z+h_1-\delta)\frac{z+\delta+h(\overline{Z})}{z+h_1-\delta}\right]<\infty.\end{aligned}$$

By the mean value theorem for any $h\in(-\delta,\delta)$ we get for some $\theta\in(0,1)$

$$\left|\frac{\xi(z+h)-\xi(z)}{h}\right|=|\xi'(z+\theta h)|\leq\sup_{\overline{z}\in(z-\delta,z+\delta)}|\xi'(\overline{z})|,$$

and applying the dominated convergence theorem we get

$$\mathbb{E}[\xi'(z)]=\mathbb{E}\left[\lim_{h\to0}\frac{\xi(z+h)-\xi(z)}{h}\right]=\lim_{h\to0}\frac{\mathbb{E}[\xi(z+h)]-\mathbb{E}[\xi(z)]}{h}=\frac{d}{dz}\mathbb{E}[\xi(z)].$$

Additionally, by using (1.2.2) and the strict negativity of $u''$ we get

$$\begin{aligned}\xi'(z)&=u'(\overline{Z}+K+z-h(\overline{Z}))+(z-h(\overline{Z}))u''(\overline{Z}+K+z-h(\overline{Z}))\\&=u'(\overline{Z}+K+z-h(\overline{Z}))\times\\&\quad\times\left(1+\frac{(\overline{Z}+K+z-h(\overline{Z})-\overline{Z}-K)u''(\overline{Z}+K+z-h(\overline{Z}))}{u'(\overline{Z}+K+z-h(\overline{Z}))}\right)>0,\end{aligned}$$

and therefore for any $z>-h_1$ we obtain

$$\frac{d}{dz}\mathbb{E}[\xi(z)]=\mathbb{E}[\xi'(z)]>0.$$

It follows that $\mathbb{E}[\xi(z)]$ is strictly increasing in $z$ for $z>-h_1$ and since $\mathbb{E}[\xi(z)]$ is continuous for $z\geq-h_1$ the solution $x^h$ to (1.2.1) is unique in $[-h_1,\infty)$ under condition (1.2.2). □

We are now ready to prove the following theorem, which is the main result of this paper.

**Theorem 1.2.1.** *Let Assumptions 1.1.1, 1.1.2 and 1.1.3 be satisfied. The stock price process given by*

$$S_t:=\frac{\mathbb{E}[L_T\exp(Z_T)|\mathcal{F}_t]}{L_t}\quad\textit{for}\quad t\in[0,T],\tag{1.2.3}$$

*is an admissible price process. In the above, the (state-price density) process $L$ is defined as*

$$L_t := \frac{\mathbb{E}[u'(\exp(Z_T) + K - h(\exp(Z_T)) + x^h)|\mathcal{F}_t]}{\lambda} \quad \textit{for} \quad t \in [0,T], \tag{1.2.4}$$

*with the constant $\lambda \geq 0$ given by*

$$\lambda := \mathbb{E}\left[u'(\exp(Z_T) + K + x^h - h(\exp(Z_T)))\right], \tag{1.2.5}$$

*and $x^h$ being a solution to (1.2.1). Moreover, there exist processes $\pi^h$ and $\widehat{\pi}$ such that $(S, \pi^h, \widehat{\pi})$ is an equilibrium (in the sense of Definition 1.1.4). Finally, if $u(z)$ satisfies (1.2.2) then for any other equilibrium $(\overline{S}, \pi^{(1)}, \pi^{(2)})$ we have that $(S, \pi^h, \widehat{\pi}) = (\overline{S}, \pi^{(1)}, \pi^{(2)})$ a.e. a.s..*

**Remark 1.2.2.** *The condition in (1.2.2), which is satisfied for $u(z) = \log(z)$ or $u(z) = z^{1-p}/(1-p)$ for $0 < p < 1$, is also used in Chapter 4 in [64] to prove the uniqueness of equilibrium in a standard setting. Moreover, it will be proved in Theorem 1.2.4 below that the stock price $S$ from (1.2.3) follows a local volatility model if we assume that (1.2.2) holds. In particular, from (1.2.3)-(1.2.4) and the fact that $Z$ is a Markov process, we will obtain that $S_t$ is a deterministic function of $t$ and $Z_t$ for any $t \in [0,T]$. The invertibility of that function would follow if $u$ satisfies (1.2.2) and $h(S_T)$ is a linear combination of European call and put option payoffs with nonnegative coefficients.*

**Remark 1.2.3.** *In the case of no hedger on the market (i.e. $h \equiv 0$ and $h_0 = h_1$), we have that $x^h = 0$ and the state-price density process from (1.2.4) is given by*

$$L_t = \frac{\mathbb{E}[u'(S_T)|\mathcal{F}_t]}{\mathbb{E}[u'(S_T)]} \quad \textit{for} \quad t \in [0,T],$$

*which is just the expectation of the marginal utility evaluated at the total market endowment (we have set $K = 0$), and in agreement with the known complete market case (see e.g. Chapter 4.5 in [64]).*

*Proof of Theorem 1.2.1.* Let us outline the steps of the proof. First we will show that the stock price process is admissible. In particular, we will check that the state-price density process $L$, given by (1.2.4), is a martingale and the stock price process $S$ given by (1.2.3) satisfies an SDE of the form

$$dS_t = S_t\left(\mu_t dt + \sigma_t dW_t\right), \tag{1.2.6}$$

for $t \in [0,T]$, where $\mu$ and $\sigma$ are $\mathcal{F}_t$-progressively measurable processes satisfying $\sigma_t \neq 0$ a.e. a.s. and

$$\int_0^T |\mu_t| dt < \infty, \quad \int_0^T \sigma_t^2 dt < \infty, \quad \int_0^T \frac{\mu_t^2}{\sigma_t^2} dt < \infty, \quad \text{a.s..} \tag{1.2.7}$$

Then, after obtaining the solutions $\pi^h$ and $\widehat{\pi}$ to the hedger and the optimizer problems given in Definition 1.1.3, we will check the clearing of the stock market condition from Definition 1.1.4. Finally, we will prove the uniqueness of the equilibrium financial market when (1.2.2) is satisfied.

First notice that by the definition in (1.2.3) we obtain $S_T = \exp(Z_T)$. To check that (1.2.6) and (1.2.7) are satisfied, we will obtain martingale representations for the process $L$ and the process $f$ defined by

$$f_t := \mathbb{E}[L_T S_T | \mathcal{F}_t] \quad \text{for } t \in [0,T],$$

and subsequently apply Ito's formula to $f/L$. First, observe that for the constant $\lambda$ defined in (1.2.5) we have $\lambda \in (0,\infty)$. Indeed, by the strict concavity of $u(z)$ on $(0,\infty)$ and Lemma 1.A.1 in the Appendix, we have that

$$\begin{aligned}
&\mathbb{E}\left[u'(S_T + K + x^h - h(S_T))\right] \leq \mathbb{E}\left[u'(S_T + h_0 - h(S_T))\right] < \infty, \\
&\mathbb{E}\left[u'(S_T + K + x^h - h(S_T))\right] > \mathbb{E}\left[u'(S_T + K + x^h + h_1) | S_T < 1\right] \mathbb{P}\left[S_T < 1\right], \\
&\qquad\qquad > u'(1 + K + x^h + h_1)\mathbb{P}\left[S_T < 1\right] > 0.
\end{aligned}$$

Moreover, if $h(z)$ is not a negative constant we have that $x^h > -h_1$ and therefore $u'(z + K + x^h - h(z)) \leq u'(x^h + h_1) < \infty$, while if $h$ is a negative constant we have that $x^h = -h_1 = -K$ and $h_1 > 0$, leading to $u'(z + K + x^h - h(z)) \leq u'(h_1) < \infty$ for $z \geq 0$. Therefore

$$u'(z + K + x^h - h(z)) \leq \overline{u} < \infty, \quad \text{for } z \geq 0,$$

where we have denoted the constant $\overline{u}$ as

$$\overline{u} = \begin{cases} u'(x^h + h_1), & \text{if } x^h > -h_1 \\ u'(h_1), & \text{if } x^h = -h_1. \end{cases}$$

The process $L$ is obviously a nonnegative local martingale that is bounded from above by $\overline{u}/\lambda$ and therefore it is a martingale. Since the constant $\lambda$ defined in (1.2.5) is positive and $u$ is strictly concave on $(0,\infty)$, by using Lemma 1.A.1 in the Appendix, we see that

$$\mathbb{E}[L_t^2] = \mathbb{E}[\mathbb{E}[L_T|\mathcal{F}_t]^2] \le \mathbb{E}[L_T^2] = \frac{\mathbb{E}[(u'(S_T+K+x^h-h(S_T)))^2]}{\lambda^2} < \frac{\overline{u}^2}{\lambda^2} < \infty,$$
$$\mathbb{E}[f_t^2] = \mathbb{E}[\mathbb{E}[f_T|\mathcal{F}_t]^2] \le \mathbb{E}[f_T^2] = \frac{\mathbb{E}[(u'(S_T+K+x^h-h(S_T))S_T)^2]}{\lambda} < \frac{\overline{u}^2\mathbb{E}[S_T^2]}{\lambda^2} < \infty,$$

for any $t \in [0,T]$. Therefore, $L$ and $f$ are square-integrable martingales which we assume, without loss of generality, to be right-continuous (see Theorem 1.3.13 in [63]). Now we can apply Theorem 3.4.15 in [63] to $L$ and $f$ to conclude that they are continuous processes and there exist $\mathcal{F}_t$-progressively measurable processes $(\sigma_t^L)_{t\in[0,T]}$ and $(\sigma_t^f)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[\int_0^T (\sigma_t^L)^2 dt\right] < \infty, \quad \mathbb{E}\left[\int_0^T (\sigma_t^f)^2 dt\right] < \infty, \tag{1.2.8}$$

and

$$dL_t = \sigma_t^L dW_t, \quad df_t = \sigma_t^f dW_t \quad \text{for} \quad t \in [0,T]. \tag{1.2.9}$$

Moreover, this representation is unique in the following sense – for any other $\mathcal{F}_t$-progressively measurable processes $\overline{\sigma}^L$ and $\overline{\sigma}^f$ satisfying (1.2.8)-(1.2.9) we have $\overline{\sigma}^L = \sigma^L$ and $\overline{\sigma}^f = \sigma^f$ a.e. a.s. on $[0,T]\times\Omega$.

Noting that $u'$ is strictly positive and decreasing, the Inada conditions (1.1.10) are satisfied and the process $Z$ does not have a point mass at $\infty$ (see Lemma 1.A.1), it follows that $L, f$ and, consequently, $S = f/L$ are strictly positive processes. We conclude that $S$ is a continuous process, and, by applying Ito's formula, we obtain that it is of the form (1.2.6) where $\mu_t$ and $\sigma_t$ are given by

$$\mu_t = \frac{(\sigma_t^L)^2}{L_t^2} + \frac{-\sigma_t^L\sigma_t^f}{L_t f_t}, \quad \sigma_t = \frac{-\sigma_t^L}{L_t} + \frac{\sigma_t^f}{f_t} \quad \text{for} \quad t \in [0,T].$$

Using the fact that both $L$ and $f$ are continuous and strictly positive processes, the Hölder's inequality and (1.2.8), we obtain

$$\int_0^T \sigma_t^2 dt \le \int_0^T \frac{(\sigma_t^L)^2}{L_t^2} dt + 2\left(\int_0^T \frac{(\sigma_t^L)^2}{L_t^2} dt \int_0^T \frac{(\sigma_t^f)^2}{f_t^2} dt\right)^{\frac{1}{2}} + \int_0^T \frac{(\sigma_t^f)^2}{f_t^2} dt < \infty \quad \text{a.s.},$$
$$\int_0^T |\mu_t| dt = \int_0^T \frac{|\sigma_t \sigma_t^L|}{L_t} dt \le \left(\int_0^T \sigma_t^2 dt \int_0^T \frac{(\sigma_t^L)^2}{L_t^2} dt\right)^{\frac{1}{2}} < \infty \quad \text{a.s.},$$
$$\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt = \int_0^T \frac{(\sigma_t^L)^2}{L_t^2} dt < \infty \quad \text{a.s.}.$$

Let us now prove that $\sigma_t$ is a.e. a.s. nonzero by providing a Markovian form for the processes $L$ and $f$. Since $\mu_Z$ and $\sigma_Z$ satisfy conditions (C1)-(C3), and $u'(e^z+K+x^h-h(e^z))<\overline{u}<\infty$ for $z\in\mathbb{R}$, we can apply Theorem 9.3 in [37] to obtain that there exists a solution $L(t,z)\in C^{1,2}([0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ to the PDE in (1.1.2) with the terminal condition

$$L(T,z)=\frac{1}{\lambda}u'(e^z+K+x^h-h(e^z))\quad\text{for}\quad z\in\mathbb{R}. \tag{1.2.10}$$

Moreover, from Theorem 2.10 in [37], this solution is unique in the class of functions satisfying the growth condition $|L(t,z)|\le c_1\exp(c_2z^2)$ for some positive constants $c_1$ and $c_2$. Furthermore, the solution has the form

$$L(t,z)=\frac{1}{\lambda}\int_{-\infty}^{+\infty}p(t,z;T,v)u'(e^v+K+x^h-h(e^v))dv\quad\text{for}\quad(t,z)\in[0,T)\times\mathbb{R}, \tag{1.2.11}$$

where $p$ is the fundamental solution defined in Remark 1.1.1.

We want to find a Feynman-Kac representation for $L(t,z)$ and, therefore, we need to obtain some bounds on it. From (1.2.11) we obtain the uniform bound $L(t,z)\le\overline{u}/\lambda$ for $(t,z)\in[0,T)\times\mathbb{R}$. Moreover, from (C1)-(C3) the martingale problem for $\mu_Z$ and $\sigma_Z^2$ is well-posed and the corresponding family of measures on the canonical space $\{\mathbb{P}^{t,z}:(t,z)\in[0,T]\times\mathbb{R}\}$ is strongly Markov (see Theorem 7.2.1 in [109]). In particular, from Corollary 5.4.8 in [63] we have that $\mathbb{P}^{t,z}=\mathbb{P}(Z^{(t,z)})^{-1}$ and, therefore, for any nonnegative function $g:\mathbb{R}\to[0,\infty)$ we get

$$\mathbb{E}^{t,z}[g(X(T))]=\mathbb{E}[g(Z_T^{(t,z)})], \tag{1.2.12}$$

where $X$ is the coordinate process on the canonical space. Hence, by (C2)-(C3) and the fact that $L(T,z)>0$, we can apply (1.2.12) and the Feynman-Kac representation of Theorem 5.7.6 in [63], to obtain that $L(t,z)$ has the form

$$\begin{aligned}L(t,z)&=\frac{1}{\lambda}\mathbb{E}\Big[u'\Big(\exp(Z_T^{(t,z)})+K+x^h-h\big(\exp(Z_T^{(t,z)})\big)\Big)\Big]\\&=\frac{1}{\lambda}\mathbb{E}^{t,z}\left[u'\left(\exp(X_T)+K+x^h-h\left(\exp(X_T)\right)\right)\right]\quad\text{for}\quad(t,z)\in[0,T)\times\mathbb{R},\end{aligned}$$

where $X$ is the coordinate process on the canonical space. Since the family of measures on the canonical space $\{\mathbb{P}^{t,z}:(t,z)\in[0,T]\times\mathbb{R}\}$ is Markov, by using Lemma 1.A.2 in the Appendix and (1.2.12), we get

$$\begin{aligned}L(t,Z_t)&=\frac{1}{\lambda}\mathbb{E}\Big[u'\Big(\exp(Z_T^{(0,z_0)})+K+x^h-h\big(\exp(Z_T^{(0,z_0)})\big)\Big)\Big|\mathcal{F}_t\Big]\\&=\frac{1}{\lambda}\mathbb{E}[u'(S_T+K+x^h-h(S_T))|\mathcal{F}_t]=L_t\quad\text{for}\quad t\in[0,T].\end{aligned}$$

[2]Strictly speaking, the solution exists on a strip $[0,T']$ with $T'=\min\{T,c/a_2\}$, where $c$ is a positive constant depending only on $\mu_Z$ and $\sigma_Z$, and $a_1,a_2$ are positive constants such that $L(T,z)\le a_1\exp(a_2z^2)$. Since $L(T,z)$ is bounded we can choose $a_2$ arbitrarily small so that $T'=T$.

By using (C1)-(C3) we can apply Theorem 2.11 in [37] to obtain that $p(t_1, z; t_2, v) > 0$ and, since $u'(e^v + K + x^h - h(e^v)) > 0$ for all $v \in \mathbb{R}$, from (1.2.10) and (1.2.11) we also get that $L(t,z) > 0$ for $(t,z) \in [0,T] \times \mathbb{R}$.

Now we can apply the Ito's formula to the function $L(t,z)$ to obtain

$$d\,L_t = d\,L(t, Z_t) = \sigma_L(t, Z_t) dW_t \quad \text{for} \quad t \in [0,T), \tag{1.2.13}$$

where $\sigma_L(t,z)$ is given by

$$\begin{aligned} \sigma_L(t,z) &= \sigma_Z(t,z)\frac{\partial L}{\partial z}(t,z) \\ &= \frac{\sigma_Z(t,z)}{\lambda}\int_{-\infty}^{+\infty} \frac{\partial p}{\partial z}(t,z;T,v) u'(e^v + K + x^h - h(e^v))dv \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R}. \end{aligned} \tag{1.2.14}$$

The interchange of differentiation and integration in (1.2.14) is justified by using the bounds on the first derivative of the fundamental solution $p$ from Theorem 9.2 in [37] and the dominated convergence theorem.

By using similar arguments as above, since $e^z u'(e^z + K + x^h - h(e^z)) < e^z \overline{u} \leq a_1 \exp(a_2 z^2)$ with $a_1, a_2$ positive constants and $a_2$ arbitrarily small, we obtain that there exists a unique solution $f(t,z) \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ to the PDE in (1.1.2) with the terminal condition

$$f(T,z) = \frac{e^z}{\lambda} u'(e^z + K + x^h - h(e^z)) \quad \text{for} \quad z \in \mathbb{R}, \tag{1.2.15}$$

satisfying the growth condition $|f(t,z)| \leq \overline{c}_1 \exp(\overline{c}_2 z^2)$ for some constants $\overline{c}_1, \overline{c}_2 > 0$, and having the form

$$f(t,z) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} p(t,z;T,v) e^v u'(e^v + K + x^h - h(e^v))dv \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R}. \tag{1.2.16}$$

By using (C1)-(C3) we can apply Theorem 9.2 in [37] to obtain the bound

$$p(t,z;T,v) \leq \frac{C}{\sqrt{T-t}} \exp\left(-c\frac{(v-z)^2}{T-t}\right),$$

for some constants $C, c > 0$. Therefore from (1.2.16), by using change of variables and the fact that for any $(t,v) \in [0,T) \times \mathbb{R}$ and any constant $\overline{c} > 0$

$$\overline{c}\,v^2 - v\sqrt{T-t} + \frac{T}{4\overline{c}} \geq \overline{c}\,v^2 - v\sqrt{T-t} + \frac{T-t}{4\overline{c}} = \left(v\sqrt{\overline{c}} - \frac{\sqrt{T-t}}{2\sqrt{\overline{c}}}\right)^2 \geq 0,$$

it follows that

$$f(t,z) \le \frac{\overline{u}}{\lambda}\int_{-\infty}^{+\infty}\frac{C}{\sqrt{T-t}}\exp\left(v - c\frac{(v-z)^2}{T-t}\right)dv$$
$$= \frac{C\overline{u}e^z}{\lambda}\int_{-\infty}^{+\infty} e^{v\sqrt{T-t}-cv^2}dv \le \frac{C\overline{u}e^{z+\frac{T}{4\overline{c}}}}{\lambda}\int_{-\infty}^{+\infty} e^{(\overline{c}-c)v^2}dv.$$

By choosing the constant $\overline{c}$ such that $\overline{c} < c$ holds we get that $f(t,z) \le const\, e^z \le const\, \exp(\widetilde{c}\, z^2 + 1/(4\widetilde{c}))$ for any constant $\widetilde{c} > 0$. Hence again, by (C2)-(C3) and the fact that $f(T,z) > 0$, we can apply a Feynman-Kac representation (see Theorem 5.7.6 in [63]) together with Problem 5.7.7 in [63], to obtain that $f(t,z)$ has the form

$$f(t,z) = \frac{1}{\lambda}\mathbb{E}\Big[\exp(Z_T^{(t,z)})u'\Big(\exp(Z_T^{(t,z)}) + K + x^h - h\big(\exp(Z_T^{(t,z)})\big)\Big)\Big]$$
$$= \frac{1}{\lambda}\mathbb{E}^{t,z}\left[\exp(X_T)u'\left(\exp(X_T) + K + x^h - h\left(\exp(X_T)\right)\right)\right] \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R}.$$

and, by analogy to the case for $L(t,z)$, we get

$$f(t,Z_t) = \frac{1}{\lambda}\mathbb{E}[S_T u'(S_T + K + x^h - h(S_T))|\mathcal{F}_t] = f_t \quad \text{for} \quad t \in [0,T].$$

From (1.2.15) and (1.2.16), as in the case for $L(t,z)$, we also get $f(t,z) > 0$ for $(t,z) \in [0,T]\times\mathbb{R}$ since $e^v u'(e^v + K + x^h - h(e^v)) > 0$ for all $v \in \mathbb{R}$.

Applying Ito's formula to the function $f(t,z)$ we get

$$d\,f_t = d\,f(t,Z_t) = \sigma_f(t,Z_t)dW_t \quad \text{for} \quad t \in [0,T), \tag{1.2.17}$$

where $\sigma_f(t,z)$ is given by

$$\sigma_f(t,z) = \sigma_Z(t,z)\frac{\partial f}{\partial z}(t,z) \tag{1.2.18}$$
$$= \frac{\sigma_Z(t,z)}{\lambda}\int_{-\infty}^{+\infty}\frac{\partial p}{\partial z}(t,z;T,v)e^v u'(e^v + K + x^h - h(e^v))dv \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R},$$

and the interchange of differentiation and integration is justified as in (1.2.14).

The equations (1.2.13)-(1.2.14) and (1.2.17)-(1.2.18), apart from providing analytic expressions for the SDE coefficients, give us martingale representations for the processes $L_t$ and $f_t$ for $t \in [0,T)$. Comparing (1.2.9) with (1.2.13) and (1.2.17), by using the uniqueness of $\sigma^L$ and $\sigma^f$, we get that $\sigma_t^L = \sigma_L(t,Z_t)$ and $\sigma_t^f = \sigma_f(t,Z_t)$ a.e. a.s. on $[0,T) \times \Omega$. In particular, we have $\mu_t = \mu(t,Z_t)$ and $\sigma_t = \sigma(t,Z_t)$ a.e. a.s. on $[0,T) \times \Omega$, and

$$dS_t = S_t\left(\mu(t,Z_t)dt + \sigma(t,Z_t)dW_t\right), \tag{1.2.19}$$

for $t \in [0,T)$, where $\mu(t,z)$ and $\sigma(t,z)$ are given by

$$\mu(t,z) = \frac{\sigma_L^2(t,z)}{L^2(t,z)} + \frac{-\sigma_L(t,z)\sigma_f(t,z)}{L(t,z)f(t,z)}, \quad \sigma(t,z) = \frac{-\sigma_L(t,z)}{L(t,z)} + \frac{\sigma_f(t,z)}{f(t,z)}, \tag{1.2.20}$$

for $(t,z) \in [0,T) \times \mathbb{R}$.

Note that to prove that $\sigma_t$ is a.e. a.s. nonzero it is enough to show that $\sigma(t, Z_t)$ is a.e. a.s. nonzero because $\sigma_t = \sigma(t, Z_t)$ a.e. a.s. on $[0,T) \times \Omega$. For this purpose, we will check that our setting satisfies the conditions (A1)-(A3) from Section 2 in [70].

From (C3) we have that $\sigma_Z(t,z)$ is continuous and from (C1) it follows that $\sigma_Z(t,z)$ doesn't change sign. Therefore from (C1) we have for $z_1, z_2 \in \mathbb{R}$ and $t \in [0,T]$

$$|\sigma_Z^2(t,z_1) - \sigma_Z^2(t,z_2)| = |\sigma_Z(t,z_1) - \sigma_Z(t,z_2)||\sigma_Z(t,z_1) + \sigma_Z(t,z_2)| \geq 2\underline{\sigma}\, |\sigma_Z(t,z_1) - \sigma_Z(t,z_2)|,$$

and from (C3) it follows that $\sigma_Z(t,z)$ is also uniformly Hölder-continuous in $z$. From this and the conditions (C1)-(C3) we see that condition (A1) in [70] is satisfied. Moreover, by using (1.1.13), the functions $e^z$ and $u'(e^z + K + x^h - h(e^z))$ satisfy condition (A2). In our case condition (A3) is trivially satisfied because all functions in its statement are identically zero in our setting. We also note that the filtration considered in [70] is the (augmented) filtration generated by the exogenously given process $Z$, but from Lemma 1.A.2 in the Appendix this filtration coincides with $(\mathcal{F}_t)_{t\in[0,T]}$. Now we can apply Lemma 4.3 in [70] to obtain that the functions $L(t,z)$ and $f(t,z)$ coincide with the functions that are solutions to the two Cauchy problems from Lemma 4.1 in [70]. Rewriting (1.2.20) as

$$\sigma(t,z) = \frac{-\sigma_L(t,z)}{L(t,z)} + \frac{\sigma_f(t,z)}{f(t,z)} = \frac{\sigma_Z(t,z)}{L(t,z)f(t,z)} \left( L(t,z)\frac{\partial f}{\partial z}(t,z) - f(t,z)\frac{\partial L}{\partial z}(t,z) \right),$$

and using Lemma 4.2 in [70], the continuity of $L(t,z)$ and $f(t,z)$ and the fact that $\sigma_Z(t,z)$ is bounded away from $0$, we get that $\sigma(t,z)$ is a.e. a.s. nonzero with respect to the Lebesgue measure on $[0,T) \times \mathbb{R}$. Since the law of $Z_t$ is equivalent to the Lebesgue measure on $\mathbb{R}$ for $t \in [0,T]$, it follows that $\sigma(t, Z_t)$ is a.e. a.s. nonzero. Therefore we conclude that $S$ is an admissible stock price process.

Since the stock price process $S$ is admissible and $u(z)$ satisfies the asymptotic elasticity condition (1.1.11), we can use the results on portfolio optimization from [71] in order to find the wealth process of the optimizer $\widehat{X}$. Indeed, comparing Definition 1.1.2 with the definitions of complete market in [51], by using that $S$ is a martingale under the equivalent measure $\mathbb{Q}$ with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = L_T$, we obtain from the theorem of [51] that the set of equivalent martingale measures is a singleton. Therefore, denoting the inverse of $u'(z)$ as $I(z)$, we can apply Theorems 2.0 and 2.2 (i) in [71] to get

$$\widehat{X}_T = I\left(\overline{\lambda} L_T\right), \tag{1.2.21}$$

$$L_t \widehat{X}_t = \mathbb{E}[L_T \widehat{X}_T | \mathcal{F}_t] \quad \text{for} \quad t \in [0,T], \tag{1.2.22}$$

where $\overline{\lambda}$ is the unique solution for $z$ to the equation $\widehat{X}_0 = \mathbb{E}[L_T I(z L_T)]$. By using the definition of $L$ and $\lambda$ in (1.2.4) and (1.2.5), and the fact that $x^h$ solves (1.2.1) we get

$$\begin{aligned}
\mathbb{E}[L_T I(\lambda L_T)] =& \frac{1}{\lambda}\mathbb{E}[u'(S_T + K + x^h - h(S_T)) I(u'(S_T + K + x^h - h(S_T)))] \\
=& \frac{1}{\lambda}\mathbb{E}[u'(S_T + K + x^h - h(S_T))(S_T + K + x^h - h(S_T))] \\
=& \frac{1}{\lambda}\mathbb{E}[u'(S_T + K + x^h - h(S_T))(S_T + K)] = \mathbb{E}[L_T(S_T + K)] = S_0 + K = X_0^{\widehat{\pi}},
\end{aligned}$$

and this means that $\overline{\lambda} = \lambda$. Therefore, by evaluating (1.2.4) at time $T$ and substituting in (1.2.21), we get

$$\widehat{X}_T = S_T + K + x^h - h(S_T), \tag{1.2.23}$$

and from (1.2.22) we can obtain $\widehat{X}$ for all $t \in [0,T)$. From the fact that $L$ is bounded we can see that Assumption 3.2.2 in [64] holds. Moreover, since the utility function $u$ satisfies (1.1.10) and (1.1.11) we can apply Theorems 2.0 (iii) and 2.2 (i) in [71] to obtain that $\mathbb{E}[L_T I(y L_T)]$ is continuous for $y > 0$ and, therefore, Assumption 3.7.2 in [64] also holds. Hence, we can apply Theorem 3.7.6 (iii) in [64] to obtain the unique a.e. a.s. solution $\widehat{\pi}$ to the optimizer's problem in Definition 1.1.3 through a martingale representation of the process $L\widehat{X}$.

Consider the portfolio process $\pi^h := 1 - \widehat{\pi}$ and let us check that $\pi^h$ solves the hedger's problem as specified in Definition 1.1.3. From (1.1.5) we obtain

$$X_t^h = \int_0^t (1 - \widehat{\pi}_u) dS_u = S_t - S_0 - \widehat{X}_t + S_0 + K = S_t - X_t^{\widehat{\pi}} + K, \tag{1.2.24}$$

for $t \in [0,T]$ and by using (1.2.23) it follows that

$$X_T^h = h(S_T) - x^h. \tag{1.2.25}$$

Now, since $S$ and $\widehat{X}$ are martingales under the measure $\mathbb{Q}$, from (1.2.24) we have that $X^h$ is also a $\mathbb{Q}$-martingale. Therefore, by using (1.2.25) we get

$$X_t^h = \mathbb{E}^{\mathbb{Q}}[X_T^h | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[h(S_T) - x^h | \mathcal{F}_t] \geq -x^h - h_1 \quad \text{for} \quad t \in [0,T],$$

so that $\pi^h \in \mathcal{A}^{-x^h-h_1}$ and, hence, $\pi^h \in \mathcal{A}^B$. Therefore, since $x^h$ satisfies (1.1.9) by definition, we have that $\pi^h$ solves the hedger's problem.

Now we will prove the uniqueness of the hedger's portfolio $\pi^h$ for the admissible stock price process $S$. Assume there is another process $\widetilde{\pi}^h \in \mathcal{A}^B$ such that the corresponding wealth process $X^{\widetilde{h}}$ satisfies $X^{\widetilde{h}}_T = h(S_T) - x^h$ with $X^{\widetilde{h}}_0 = 0$. Since $\mathbb{E}[X^{\widetilde{h}}_T L_T] = 0 = X^{\widetilde{h}}_0$ it follows that $LX^{\widetilde{h}}$ is a martingale. Therefore

$$X^{\widetilde{h}}_t = \mathbb{E}^{\mathbb{Q}}[X^{\widetilde{h}}_T] = \mathbb{E}^{\mathbb{Q}}[(S_T + K - X^{\widehat{\pi}}_T)L_T] = S_t - X^{\widehat{\pi}}_t + K \quad \text{for} \quad t \in [0,T],$$

and from (1.2.24) we have $X^{\widetilde{h}} \equiv X^h$. In particular we have

$$\int_0^t \pi^h_u dS_u = X^h_t = X^{\widetilde{h}}_t = \int_0^t \widetilde{\pi}^h_u dS_u \quad \text{for} \quad t \in [0,T]. \tag{1.2.26}$$

Notice that $X^h$ is a square-integrable $\mathbb{Q}$-martingale because for any $t \in [0,T]$ we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}[(X^h_t)^2] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[X^h_T|\mathcal{F}_t]^2] \le \mathbb{E}^{\mathbb{Q}}[(X^h_T)^2] = \mathbb{E}[L_T(X^h_T)^2]\\ &= \frac{\mathbb{E}[u'(S_T + K + x^h - h(S_T))(h(S_T) - x^h)^2]}{\lambda} < \frac{\overline{u}\mathbb{E}[(\max(h_1, S_T + h_0) + |x^h|)^2]}{\lambda} < \infty.\end{aligned}$$

Hence from the fact that $\pi^h$ and $\widetilde{\pi}^h$ satisfy (1.2.26), applying Lemma 1.A.3 in the Appendix, we have that

$$\int_0^T (\pi^h_t - \widetilde{\pi}^h_t)^2\, dt = 0 \quad a.s.,$$

and the optimal hedger's portfolio $\pi^h$ is unique a.e. a.s.. By the definition of $\pi^h$ we conclude that the stock market clears and, therefore, the triple $(S, \pi^h, \widehat{\pi})$ is an equilibrium.

Let us now prove the uniqueness of the equilibrium market under condition (1.2.2). Assume that there exists an equilibrium $(\overline{S}, \pi^{(1)}, \pi^{(2)})$ and denote the corresponding wealth processes of the hedger and the optimizer by $X^{(1)}$ and $X^{(2)}$, respectively. Denote the corresponding unique equivalent local martingale measure by $\overline{\mathbb{Q}}$ and its density process with respect to $\mathbb{P}$ by $\overline{L}$. By the definition of admissibility we know that the process $\overline{L}$ is a martingale with $\overline{L}_0 = 1$, and we have $\overline{S}_T = \exp(Z_T) = S_T$.

At time $T$ we know that $X^{(1)}_T = h(S_T) - \overline{x}^h$, where $\overline{x}^h$ is given by $\overline{x}^h = \mathbb{E}[h(S_T)\overline{L}_T]$. By market clearance at time $T$ we have $X^{(2)}_T = S_T + K - h(S_T) + \overline{x}^h$. Applying the duality results from [71], as in (1.2.21), we get that $X^{(2)}_T = I(\overline{\lambda}\overline{L}_T)$ for a constant $\overline{\lambda} > 0$, and we also obtain that $\overline{L}X^{(2)}$ is a martingale. This leads to

$$\overline{\lambda L}_T = u'\left(S_T + K - h(S_T) + \overline{x}^h\right), \tag{1.2.27}$$

where, by using that $\overline{L}$ is a martingale and taking expectations, we have that the constant $\overline{\lambda} > 0$ is given by

$$\overline{\lambda} = \mathbb{E}\left[u'(S_T + K - h(S_T) + \overline{x}^h)\right]. \tag{1.2.28}$$

Since $X^{(1)}$ is the wealth process corresponding to the hedger's portfolio $\pi^{(1)}$, from (1.1.5), we know that $X^{(1)}$ is a local martingale under $\overline{\mathbb{Q}}$, and, since it is bounded from below, it is in fact a $\overline{\mathbb{Q}}$-supermartingale. Therefore, by using that $\mathbb{E}^{\overline{\mathbb{Q}}}[X_T^{(1)}] = \mathbb{E}^{\overline{\mathbb{Q}}}[h(S_T) - \overline{x}^h] = 0 = X_0^{(1)}$, it follows that $\overline{L}X^{(1)}$ is in fact a martingale.

Market clearance implies $\overline{S}_t = X_t^{(1)} + X_t^{(2)} - K$ for all $t \in [0, T]$, and, therefore, $\overline{LS}$ is a martingale. Hence, taking into account that $X_0^{(2)} = \overline{S}_0 + K$ and $\overline{S}_T = S_T$ we obtain

$$\begin{aligned}\mathbb{E}[(S_T + K)u'(S_T + K - h(S_T) + \overline{x}^h)] &= \mathbb{E}[(\overline{S}_T + K)\overline{\lambda L_T}] = \overline{\lambda}(\overline{S}_0 + K) = \overline{\lambda}X_0^{(2)} \\ = \mathbb{E}[X_T^{(2)}\overline{\lambda L_T}] &= \mathbb{E}[X_T^{(2)}u'\left(S_T + K - h(S_T) + \overline{x}^h\right)].\end{aligned} \tag{1.2.29}$$

From (1.2.29) it follows that $\overline{x}$ satisfies (1.2.1). Moreover it is clear that $\overline{x} \geq -h_1$ since otherwise $\mathbb{E}[\xi(\overline{x})] < 0$. Now since (1.2.1) has a unique solution in $[-h_1, \infty)$ under condition (1.2.2) it follows that $\overline{x} = x^h$. Using (1.2.5) and (1.2.28) we get

$$\overline{\lambda} = \mathbb{E}[u'(S_T + K + x^h - h(S_T))] = \lambda. \tag{1.2.30}$$

Finally, taking expectations in (1.2.27) and using (1.2.30) we obtain

$$\overline{L}_t = \frac{\mathbb{E}[u'(S_T + K + x^h - h(S_T))|\mathcal{F}_t]}{\mathbb{E}[u'(S_T + K + x^h - h(S_T))]} = L_t,$$

and

$$\overline{S}_t = \frac{\mathbb{E}[\overline{L}_T S_T|\mathcal{F}_t]}{\overline{L}_t} = \frac{\mathbb{E}[L_T S_T|\mathcal{F}_t]}{L_t} = S_t, \tag{1.2.31}$$

for $t \in [0, T]$ and uniqueness follows. □

In the following corollary, which directly follows from the proof above, we give the Markovian form of the SDE satisfied by the stock price process $S$. The analytic expressions will be used later, when we discuss specific examples.

**Corollary 1.2.1.** *Under the assumptions of Theorem 1.2.1 the equilibrium stock price process $S$ satisfies the SDE*

$$dS_t = S_t\left(\mu(t, Z_t)dt + \sigma(t, Z_t)dW_t\right), \tag{1.2.32}$$

*where the functions* $\mu(t,z)$ *and* $\sigma(t,z)$ *are given by*

$$\mu(t,z) = \frac{\sigma_L^2(t,z)}{L^2(t,z)} + \frac{-\sigma_L(t,z)\sigma_f(t,z)}{L(t,z)f(t,z)}, \quad \sigma(t,z) = \frac{-\sigma_L(t,z)}{L(t,z)} + \frac{\sigma_f(t,z)}{f(t,z)}, \tag{1.2.33}$$

*with*

$$L(t,z) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} p(t,z;T,v)u'(e^v + K + x^h - h(e^v))dv, \tag{1.2.34}$$

$$f(t,z) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} p(t,z;T,v)e^v u'(e^v + K + x^h - h(e^v))dv, \tag{1.2.35}$$

$$\sigma_L(t,z) = \sigma_Z(t,z)\frac{\partial L}{\partial z}(t,z) = \frac{\sigma_Z(t,z)}{\lambda}\int_{-\infty}^{+\infty} \frac{\partial p}{\partial z}(t,z;T,v)u'(e^v + K + x^h - h(e^v))dv, \tag{1.2.36}$$

$$\sigma_f(t,z) = \sigma_Z(t,z)\frac{\partial f}{\partial z}(t,z) = \frac{\sigma_Z(t,z)}{\lambda}\int_{-\infty}^{+\infty} \frac{\partial p}{\partial z}(t,z;T,v)e^v u'(e^v + K + x^h - h(e^v))dv, \tag{1.2.37}$$

*for* $(t,z) \in [0,T) \times \mathbb{R}$. *Moreover* $L_t = L(t,Z_t)$ *and* $L_t S_t = f(t,Z_t)$ *for* $t \in [0,T)$.

We can also obtain analytic expressions for the portfolios of both agents and the corresponding wealth processes.

**Corollary 1.2.2.** *Under the assumptions of Theorem 1.2.1 the wealth process of the optimizer* $\widehat{X}$ *satisfies* $\widehat{X}_t = X(t,Z_t)/L(t,Z_t)$ *where*

$$X(t,z) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} p(t,z;T,v)(e^v + K + x^h - h(e^v))u'(e^v + K + x^h - h(e^v))dv, \tag{1.2.38}$$

*for* $(t,z) \in [0,T) \times \mathbb{R}$. *The portfolio of the optimizer* $\widehat{\pi}$ *satisfies* $\widehat{\pi}_t = \widehat{\pi}(t,Z_t)$ *and the function* $\widehat{\pi}(t,z)$ *is given by*

$$\widehat{\pi}(t,z) = \frac{\widehat{\sigma}(t,z)X(t,z)}{\sigma(t,z)f(t,z)}, \tag{1.2.39}$$

*where*

$$\widehat{\sigma}(t,z) = \frac{-\sigma_L(t,z)}{L(t,z)} + \frac{\sigma_X(t,z)}{X(t,z)}, \tag{1.2.40}$$

$$\begin{aligned}\sigma_X(t,z) =& \frac{\sigma_Z(t,z)}{\lambda}\times \\ &\times \int_{-\infty}^{+\infty} \frac{\partial p}{\partial z}(t,z;T,v)(e^v + K + x^h - h(e^v))u'(e^v + K + x^h - h(e^v))dv,\end{aligned} \tag{1.2.41}$$

*for* $(t,z) \in [0,T) \times \mathbb{R}$.

*Proof.* By using that the process $X := L\widehat{X}$ is a martingale and following the same reasoning as for the process $LS$ in the proof of Theorem 1.2.1, we get that $X_t = X(t, Z_t)$ for $t \in [0,T]$ where the function $X(t,z) \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ is solution to the PDE in (1.1.2) with terminal condition

$$X(T,z) = \frac{e^z + K + x^h - h(e^z)}{\lambda} u'(e^z + K + x^h - h(e^z)),$$

and has the form

$$X(t,z) = \frac{1}{\lambda} \int_{-\infty}^{+\infty} p(t,z;T,v)(e^v + K + x^h - h(e^v))u'(e^v + K + x^h - h(e^v))dv,$$

for $(t,z) \in [0,T) \times \mathbb{R}$. Applying Ito's formula to the function $X(t,z)$ we get

$$d\,X_t = d\,X(t,Z_t) = \sigma_X(t,Z_t)dW_t \quad \text{for} \quad t \in [0,T), \tag{1.2.42}$$

where $\sigma_X(t,z)$ is given by

$$\begin{aligned} \sigma_X(t,z) &= \sigma_Z(t,z)\frac{\partial X}{\partial z}(t,z) \\ &= \frac{\sigma_Z(t,z)}{\lambda} \int_{-\infty}^{+\infty} \frac{\partial p}{\partial z}(t,z;T,v)(e^v + K + x^h - h(e^v))u'(e^v + K + x^h - h(e^v))dv, \end{aligned} \tag{1.2.43}$$

for $(t,z) \in [0,T) \times \mathbb{R}$. Since $\widehat{X} = X/L$ by applying Ito's formula we obtain

$$d\widehat{X}_t = \widehat{X}_t\left(\widehat{\mu}(t,Z_t)dt + \widehat{\sigma}(t,Z_t)dW_t\right) \quad \text{for} \quad t \in [0,T), \tag{1.2.44}$$

where $\widehat{\mu}(t,z)$ and $\widehat{\sigma}(t,z)$ are given by

$$\widehat{\mu}(t,z) = \frac{\sigma_L^2(t,z)}{L^2(t,z)} + \frac{-\sigma_L(t,z)\sigma_X(t,z)}{L(t,z)X(t,z)}, \quad \widehat{\sigma}(t,z) = \frac{-\sigma_L(t,z)}{L(t,z)} + \frac{\sigma_X(t,z)}{X(t,z)}, \tag{1.2.45}$$

for $(t,z) \in [0,T) \times \mathbb{R}$. Comparing (1.2.44) and (1.1.5), by using (1.2.32) and the fact that $X$ is a square-integrable martingale (with a right-continuous modification), we can apply Theorem 3.4.15 in [63] to get that $\widehat{\pi}_t = \widehat{\pi}(t,Z_t)$ a.e. a.s. on $[0,T) \times \Omega$ where the function $\widehat{\pi}(t,z)$ is given by

$$\widehat{\pi}(t,z) = \frac{\widehat{\sigma}(t,z)X(t,z)}{\sigma(t,z)f(t,z)} \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R}.$$

□

Now we will prove the local volatility form of the equilibrium stock price process under the condition (1.2.2). For this purpose we will assume that $\mu_Z(t,z)$ and $\sigma_Z(t,z)$ satisfy the following additional conditions:

**Assumption 1.2.1.**

**(D1)** Differentiability*: $\mu_Z(t,z)$ is once differentiable and $\sigma_Z^2(t,z)$ is twice differentiable in $z$ on $[0,T]\times\mathbb{R}$.*

**(D2)** Boundedness*: $\frac{\partial\mu_Z}{\partial z}(t,z)$, $\frac{\partial\sigma_Z^2}{\partial z}(t,z)$ and $\frac{\partial^2\sigma_Z^2}{\partial z^2}(t,z)$ are bounded on $[0,T]\times\mathbb{R}$.*

**(D3)** Continuity*: $\frac{\partial\mu_Z}{\partial z}(t,z)$, $\frac{\partial\sigma_Z^2}{\partial z}(t,z)$ and $\frac{\partial^2\sigma_Z^2}{\partial z^2}(t,z)$ are continuous in $t$ for $t\in[0,T]$ and locally Hölder-continuous in $z$ on $[0,T]\times\mathbb{R}$.*

**Theorem 1.2.4.** *Let the Assumptions 1.1.1, 1.1.2, 1.1.3 and 1.2.1 hold. Suppose that the optimizer's utility $u(z)$ satisfies* (1.2.2) *and $h(z)$ is of the form*

$$h(z)=\sum_{i=1}^{n}\alpha_i(z-K_i)^+ + \beta_i(K_i-z)^+ \quad \textit{(long puts and calls)},$$

*where $\alpha_i,\beta_i\geq 0$ and $\max(\alpha_i,\beta_i)>0$ for $1\leq i\leq n$ for some $n\in\mathbb{N}$, and $0<K_1<K_2<\cdots<K_n<\infty$. Then the function $f(t,z)/L(t,z)$ has an inverse $g(t,s)$ w.r.t. $z$ and the SDE satisfied by the stock price process $S_t$ is in the local volatility form*

$$dS_t = S_t\left(\mu(t,g(t,S_t))dt+\sigma(t,g(t,S_t))dW_t\right) \quad \textit{for} \quad t\in[0,T).$$

*Proof.* We will prove that for any $t\in[0,T)$ the stock price $S_t$ is strictly increasing function of $Z_t$ and the result will follow from (1.2.32).

Recall from Theorem 1.2.1 and Corollary 1.2.1 that

$$S_t=\frac{\mathbb{E}[L_TS_T|\mathcal{F}_t]}{L_t}=\frac{f(t,Z_t)}{L(t,Z_t)} \quad \text{for} \quad t\in[0,T], \tag{1.2.46}$$

where $L(t,z)$ and $f(t,z)$ are the unique solutions of the PDE

$$\frac{\partial G}{\partial t}(t,z)+\mu_Z(t,z)\frac{\partial G}{\partial z}(t,z)+\frac{\sigma_Z^2(t,z)}{2}\frac{\partial^2 G}{\partial z^2}(t,z)=0 \quad \text{for} \quad (t,z)\in[0,T)\times\mathbb{R}, \tag{1.2.47}$$

in the class of functions satisfying the growth condition $|G(t,z)|\leq c_1\exp(c_2z^2)$ for some constants $c_1,c_2>0$, with the final conditions

$$L(T,z)=\frac{1}{\lambda}u'(e^z+K+x^h-h(e^z)), \quad f(T,z)=\frac{e^z}{\lambda}u'(e^z+K+x^h-h(e^z)), \tag{1.2.48}$$

for $z\in\mathbb{R}$, where the constant $\lambda>0$ is given by (1.2.5). In what follows we will prove that $L(t,z)$ is decreasing function in $z$ and $f(t,z)$ is strictly increasing function in $z$ for $t\in[0,T]$.

Consider $\mathcal{K} = \{\log K_1, \dots, \log K_n\}$ and notice that $L(T,z)$ and $f(T,z)$ belong to the class $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \mathcal{K})$. Using that $h(z)$ is nonnegative we obtain from (1.1.8) that $h_1 = 0$ and therefore $\overline{K} := K - h_0 \geq h_1 = 0$ and $x^h > 0$. Notice that since $h(z) < z + h_0$ for $z > 0$ (from condition (1.1.7)) we have $\sum_{i=1}^n \alpha_i \leq 1$. Hence, denoting

$$\gamma_j = \left(1 - \sum_{i=1}^{j} \alpha_i + \sum_{i=j+1}^{n} \beta_i\right), \quad \delta_j = \sum_{i=1}^{j} (\alpha_i + \beta_i) K_i,$$

for $0 \leq j \leq n$, we have that $\gamma_0 > 1$, $\delta_0 = 0$, $\gamma_j \geq 0$ and $\delta_j > 0$ for $1 \leq j \leq n$. Differentiating the final conditions in (1.2.48) we get

$$\frac{\partial L}{\partial z}(T,z) = \frac{\gamma_j e^z}{\lambda} u''(\gamma_j e^z + \overline{K} + x^h + \delta_j), \tag{1.2.49}$$

$$\frac{\partial f}{\partial z}(T,z) = \frac{e^z}{\lambda} \left(\gamma_j e^z u''(\gamma_j e^z + \overline{K} + x^h + \delta_j) + u'(\gamma_j e^z + \overline{K} + x^h + \delta_j)\right), \tag{1.2.50}$$

for $z \in (\log K_j, \log K_{j+1})$ and $0 \leq j \leq n$, with the convention $K_0 = 0$ and $K_{n+1} = +\infty$. Since $u$ is strictly concave and $\gamma_j \geq 0$ for $0 \leq j \leq n$, from (1.2.49) we have that $L(T,z)$ is decreasing for $z \in \mathbb{R}$, and since $\gamma_0 > 1$ we have that $L(T,z)$ is strictly decreasing for $z \in (-\infty, \log h_1)$. From (1.2.2), the strict concavity of $u$ and the fact that $\overline{K} + x^h + \delta_j > 0$ and $\gamma_j \geq 0$, we have that

$$\begin{aligned}
&\gamma_j e^z u''(\gamma_j e^z + \overline{K} + x^h + \delta_j) + u'(\gamma_j e^z + \overline{K} + x^h + \delta_j) \\
&> (\gamma_j e^z + \overline{K} + x^h + \delta_j) e^z u''(\gamma_j e^z + \overline{K} + x^h + \delta_j) + u'(\gamma_j e^z + \overline{K} + x^h + \delta_j) \geq 0.
\end{aligned}$$

Therefore from (1.2.50) we get that $f(T,z)$ is strictly increasing for $z \in \mathbb{R}$. Moreover, by using (1.1.13), we have that there exist constants $N_1, N_2, N_3, N_4 > 0$ such that

$$\begin{aligned}
&|L(t,z)| \leq e^{N_1(1+|z|)}, \quad |f(t,z)| \leq e^{N_2(1+|z|)} \quad \text{for} \quad z \in \mathbb{R}, \\
&\left|\frac{\partial L}{\partial z}(T,z)\right| \leq e^{N_3(1+|z|)}, \quad \left|\frac{\partial f}{\partial z}(T,z)\right| \leq e^{N_4(1+|z|)} \quad \text{for} \quad z \in \mathbb{R} \setminus \mathcal{K}.
\end{aligned}$$

Therefore, since Assumption 1.2.1 holds, we can apply Lemma 1.A.4 in the Appendix to obtain that $L(t,z)$ is strictly decreasing and $f(t,z)$ is strictly increasing in $z$ for $t \in [0,T)$. So there exists a function $g(t,s)$ which is the inverse of $f(t,z)/L(t,z)$, i.e.

$$\frac{f(t,g(t,s))}{L(t,g(t,s))} = s \quad \text{for} \quad (t,s) \in [0,T) \times \mathbb{R}.$$

From (1.2.46) we see that $Z_t = g(t,S_t)$. Hence, by substituting in (1.2.32), the stock price process SDE can be written in the local volatility form

$$dS_t = S_t \left(\mu(t, g(t,S_t))dt + \sigma(t, g(t,S_t))dW_t\right) \quad \text{for} \quad t \in [0,T).$$

□

**Remark 1.2.5.** *Under the assumptions of Theorem* (1.2.4) *we can deduce that the stock volatility coefficient* $\sigma(t, Z_t)$ *is a.e. a.s. nonzero without referring to the endogenous completeness results in [70]. Indeed, let us assume without loss of generality that* $\sigma_Z(t,z)$ *is strictly positive on* $[0,T)\times\mathbb{R}$*. Then, by using that* $L(t,z)$ *is strictly decreasing and* $f(t,z)$ *is strictly increasing in* $z$*, from* (1.2.36)-(1.2.37) *we can see that* $\sigma_L(t,z)$ *is strictly negative and* $\sigma_f(t,z)$ *is strictly positive on* $[0,T)\times\mathbb{R}$*. Hence, from* (1.2.33) *we can conclude that* $\sigma(t,z)$ *is strictly positive on* $[0,T)\times\mathbb{R}$ *and therefore* $\sigma(t,Z_t)$ *is a.e. a.s. nonzero.*

## 1.3. An example with power utility

In this section we illustrate our results by studying how the imbalance of the derivative market, modelled by the presence of the hedger, impacts the equilibrium stock price $S$ in a simple example. In particular, we will show the changes that occur in the market price of risk, stock volatility and implied volatility as we vary the degree of imbalance and the risk aversion of the optimizer.

Let us specify the primitives of the model. We assume that the hedger's payoff function $h(z)$ is given by the European call payoff

$$h(z) = \alpha(z - \overline{K})^+ \quad \text{for} \quad z \geq 0, \tag{1.3.1}$$

for some weight $\alpha \in (0,1]$ and strike price $\overline{K} > 0$. The European call payoff weight $\alpha$ is the parameter controlling the degree of imbalance on the market. The optimizer has a power utility function $u(z) = z^{1-p}/(1-p)$ with the risk aversion parameter $p \in (0,1)\cup(1,\infty)$. Let the process $Z$ be given by

$$Z_t = \mu t + \sigma W_t \quad \text{for} \quad t \in [0,T],$$

where we have taken $\mu_Z(t,z) \equiv \mu \in \mathbb{R}$ and $\sigma_Z(t,z) \equiv \sigma \in \mathbb{R}$ for $(t,z) \in [0,T]\times\mathbb{R}$. The functions $h(z)$ and $u(z)$, and the process $Z$ clearly satisfy the assumptions from Section 1.1. From (1.1.7)-(1.1.8) we have that $h_1 = 0$ and letting $h_0 = 0$ it follows that the total supply of bonds on the market $K$ can be set to $0$.

In order to compute the stock price SDE coefficients $\mu(t,z)$ and $\sigma(t,z)$ we will use Corollary 1.2.1. Let us obtain analytic expressions for the functions $L(t,z)$, $f(t,z)$, $\sigma_L(t,z)$ and $\sigma_f(t,z)$. Denote

$$d(t,z) = \frac{\log(\overline{K}/e^z) - \mu(T-t)}{\sigma\sqrt{T-t}}, \quad e(t,T,z) = e^{\sigma z\sqrt{T-t}+\mu(T-t)},$$

for $(t,z)\in[0,T)\times\mathbb{R}$. By straightforward computation of the conditional expectations in (1.2.3) and (1.2.4), and using that from Corollary 1.2.1 we have $L_t = L(t,Z_t)$ and $L_tS_t = f(t,Z_t)$ for $t\in[0,T)$, we obtain

$$L(t,z) = \frac{1}{\lambda\sqrt{2\pi}}\bigg(\int_{-\infty}^{d(t,z)} e^{-\frac{v^2}{2}}u'(e^z e(t,T,v)+x^h)dv + \int_{d(t,z)}^{\infty} e^{-\frac{v^2}{2}}u'((1-\alpha)e^z e(t,T,v)+\alpha\overline{K}+x^h)dv\bigg), \tag{1.3.2}$$

$$f(t,z) = \frac{1}{\lambda\sqrt{2\pi}}\bigg(\int_{-\infty}^{d(t,z)} e^z e(t,T,v)e^{-\frac{v^2}{2}}u'(e^z e(t,T,v)+x^h)dv + \int_{d(t,z)}^{\infty} e^z e(t,T,v)e^{-\frac{v^2}{2}}u'((1-\alpha)e^z e(t,T,v)+\alpha\overline{K}+x^h)dv\bigg), \tag{1.3.3}$$

for $(t,z)\in[0,T)\times\mathbb{R}$, where $\lambda$ is defined in (1.2.5). Direct calculations from (1.2.36)-(1.2.37), by differentiating (1.3.2)-(1.3.3), lead to

$$\sigma_L(t,z) = \frac{\sigma e^z}{\lambda\sqrt{2\pi}}\bigg(\int_{-\infty}^{d(t,z)} e(t,T,v)e^{-\frac{v^2}{2}}u''(e^z e(t,T,v)+x^h)dv + \int_{d(t,z)}^{\infty}(1-\alpha)e(t,T,v)e^{-\frac{v^2}{2}}u''((1-\alpha)e^z e(t,T,v)+\alpha\overline{K}+x^h)dv\bigg), \tag{1.3.4}$$

$$\begin{aligned}\sigma_f(t,z) = \frac{\sigma e^z}{\lambda\sqrt{2\pi}}\times &\\ \times\bigg(\int_{-\infty}^{d(t,z)} &(e^z e^2(t,T,v)u''(e^z e(t,T,v)+x^h) + e(t,T,v)u'(e^z e(t,T,v)+x^h))e^{-\frac{v^2}{2}}dv \\ &+\int_{d(t,z)}^{\infty}((1-\alpha)e^z e^2(t,T,v)u''((1-\alpha)e^z e(t,T,v)+\alpha\overline{K}+x^h) \\ &\quad + e(t,T,v)u'((1-\alpha)e^z e(t,T,v)+\alpha\overline{K}+x^h))e^{-\frac{v^2}{2}}dv\bigg),\end{aligned} \tag{1.3.5}$$

for $(t,z)\in[0,T)\times\mathbb{R}$. Now, substituting (1.3.2)-(1.3.5) into (1.2.33), we can compute $\mu(t,z)$ and $\sigma(t,z)$. Similarly, we can also obtain an analytic expression for the optimizer's portfolio $\widehat{\pi}(t,z)$ from (1.2.39).

Our reference case will be a market without a hedger , i.e. $h\equiv 0$ and $x^h=0$. In this case we set $K=0$ and for the equilibrium stock price we have

$$S_t = \frac{\mathbb{E}[L_TS_T|\mathcal{F}_t]}{L_t} = \frac{\mathbb{E}[S_T^{1-p}|\mathcal{F}_t]}{\mathbb{E}[S_T^{-p}|\mathcal{F}_t]} = e^{Z_t}e^{(T-t)(\mu+\frac{(1-2p)\sigma^2}{2})}.$$

This means that $\mu(t,z)\equiv p\sigma^2$ and $\sigma(t,z)\equiv\sigma$ for $(t,z)\in[0,T)\times\mathbb{R}$. In particular, we have a simple Black-Scholes model for the stock price process and the implied volatility is equal to $\sigma$.

We are interested in the dependence of the stock volatility $\sigma(t,z)$ and the market price of risk $\mu(t,z)/\sigma(t,z)$ on $\alpha$ and $p$. Comparison will be made with the case when there is no imbalance (i.e. no hedger) on the market. Below we work with the parameters $T = 1$, $\overline{K} = 0.5$, $\mu = 0.02$ and $\sigma = 0.2$.

**1.3.1. Optimal portfolio and market price of risk** Figure 1.1 shows the optimal portfolio function $\widehat{\pi}(t,z)$ of the optimizer for different levels of the market imbalance $\alpha$, where we have taken $t = 0.1$ and $p = 6$. It can be seen that the hedger holds more of the stock for larger values of the "dividend process" $\exp(Z_t)$, with most of the hedging activity happening near the strike price. We interpret this as the process $\exp(Z)$ serving as a proxy for the stock price $S$, and the hedging activity being similar to a delta hedging strategy but with respect to $\exp(Z)$.

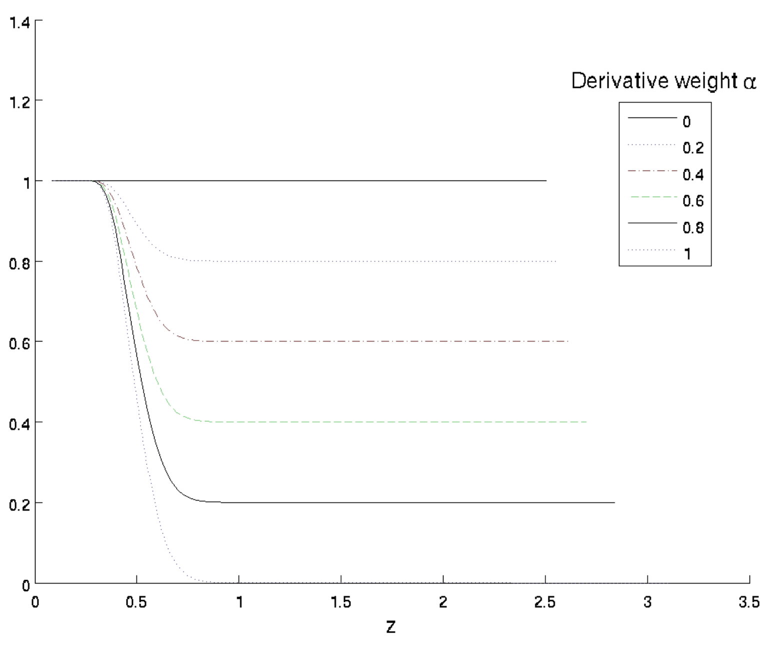



Figure 1.1: Optimizer's portfolio $\widehat{\pi}(t,z)$ for different values of $\alpha$

Figure 1.2 shows the market price of risk $\mu(t,z)/\sigma(t,z)$ for different levels of the market imbalance $\alpha$, where we have taken $t = 0.1$ and $p = 6$. We notice that as $\alpha$ increases, the market price of risk decreases. This is explained by the need of the hedger to hold more of the underlying when $\alpha$ is larger. Since the market price of risk is a measure of the attractiveness of the stock to the risk averse agents, it should decrease as the optimizer should be willing to hold less of the stock.

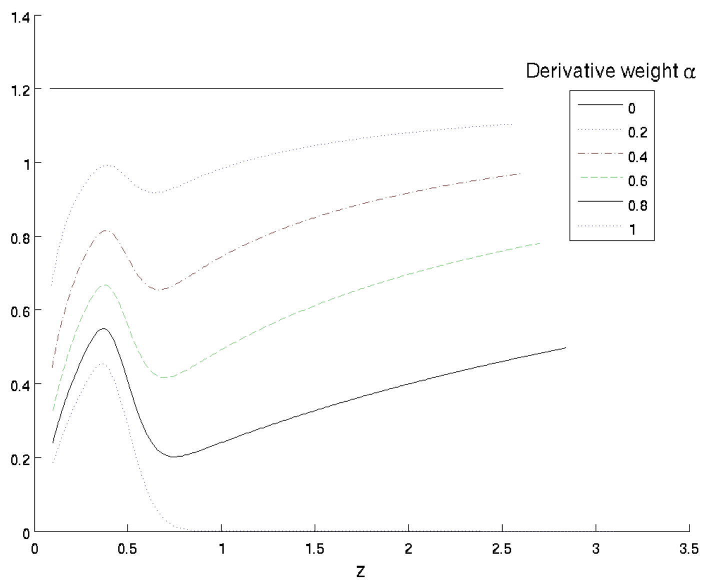



Figure 1.2: Market price of risk $\mu(t,z)/\sigma(t,z)$ for different values of $\alpha$

Figure 1.3 shows the market price of risk $\mu(t,z)/\sigma(t,z)$ for different levels of the optimizer risk aversion parameter $p$, where we have taken $t = 0.1$ and $\alpha = 0.5$. We notice that as $p$ increases, the market price of risk increases. This is due to the fact that when the optimizer is more risk averse, more compensation is required for holding the same amount of risk.

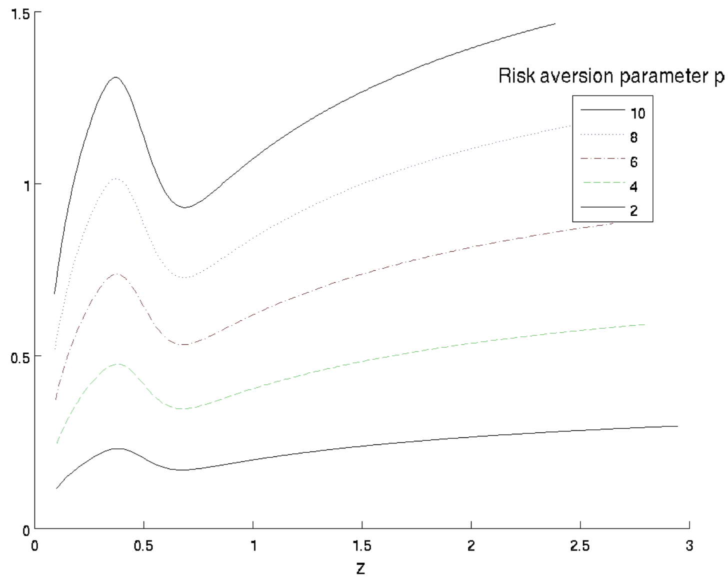



Figure 1.3: Market price of risk $\mu(t, z)/\sigma(t, z)$ for different values of $p$

**1.3.2. Stock volatility** Figure 1.4 shows the stock volatility $\sigma(t, z)$ for different levels of the market imbalance $\alpha$, where we have taken $t = 0.1$ and $p = 6$. The volatility function is exhibiting a spike at the strike price $\overline{K}$ of the European payoff. We notice that as $\alpha$ increases, the volatility spike around the strike price increases. This is explained by the increase in trading volume when the amount of the replicated European call option is higher. Since most of the hedging activity occurs near the strike price, as can be seen from Figure 1.1, this will lead to higher volatility levels.

Figure 1.5 shows the stock volatility $\sigma(t, z)$ for different levels of the optimizer risk aversion parameter $p$, where we have taken $t = 0.1$ and $\alpha = 0.5$. We notice that as $p$ increases, the volatility spike around the strike price increases. The intuition behind this effect is that, as risk aversion of the optimizer increases, it takes larger moves in the stock price to make the trades with the hedger possible.

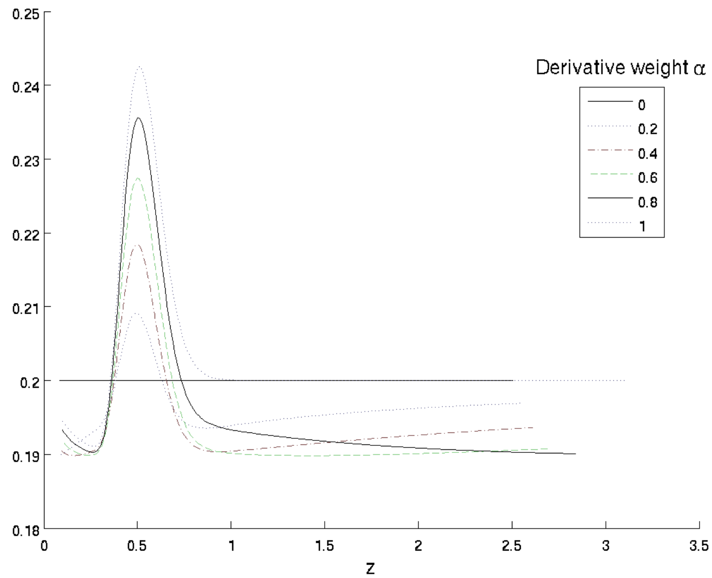



Figure 1.4: Stock volatility $\sigma(t, z)$ for different values of $\alpha$

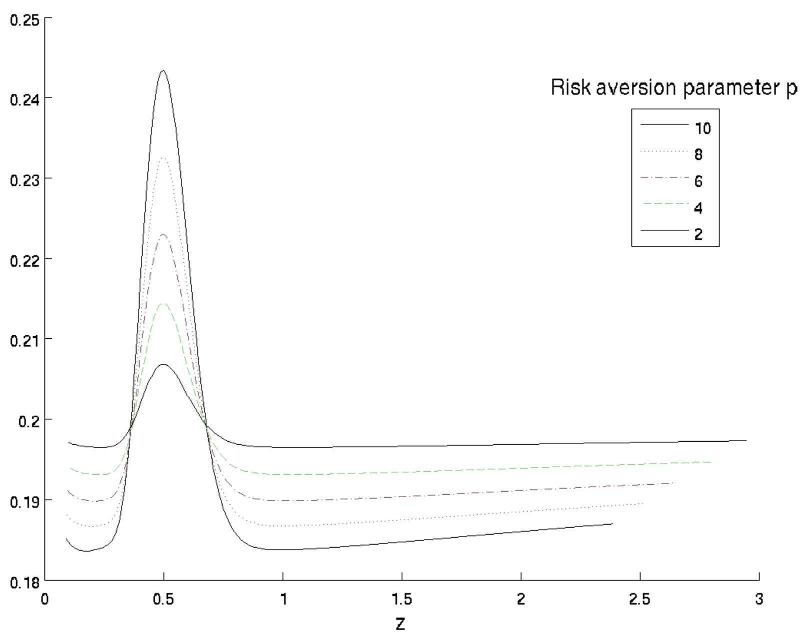



Figure 1.5: Stock volatility $\sigma(t, z)$ for different values of $p$

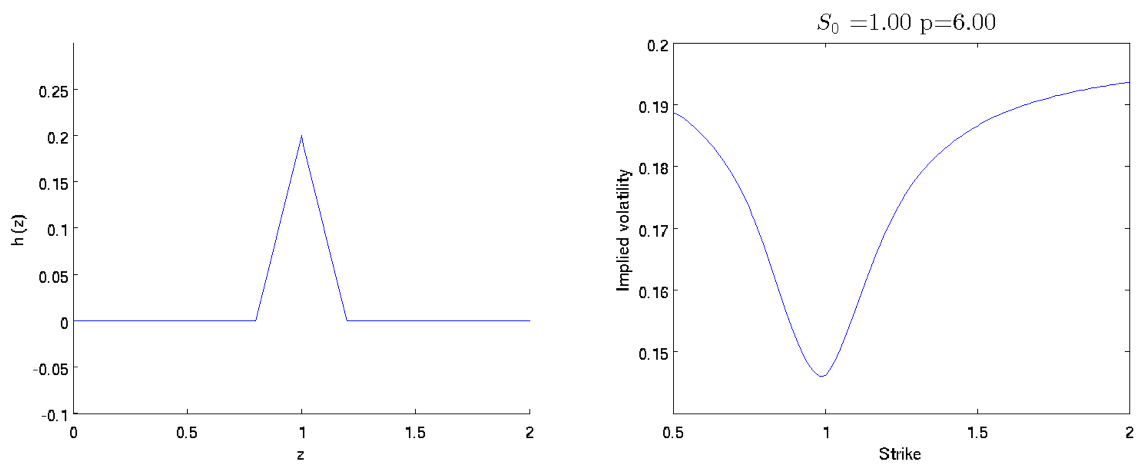
**1.3.3. Implied volatility smile** We illustrate how the implied volatility smile is affected by the imbalance on the derivative market for various payoff functions, i.e. we drop the assumption that $h(z)$ is given by (1.3.1). In each of the Figures 1.6, 1.7 and 1.8 we show the replicated payoff $h(z)$ on the left side together with the implied volatility at the initial time 0 for different strikes at the right side.

The hedger's contingent claims that we illustrate are butterfly spreads, strangles, condors and straddles. Long butterfly and long condor positions correspond to betting on low volatility, while short butterfly and long strangle/straddle positions correspond to betting on high volatility. Keeping in mind that the stock volatility with no imbalance on the market is $\sigma = 0.2$, we can see that the presence of a hedger on the market increases the chance of the hedged payoff $h(z)$ to expire in the money. This is due to the fact that for bets on high/low volatility the whole implied volatility curve shifts above/below the base volatility level of $0.2$.

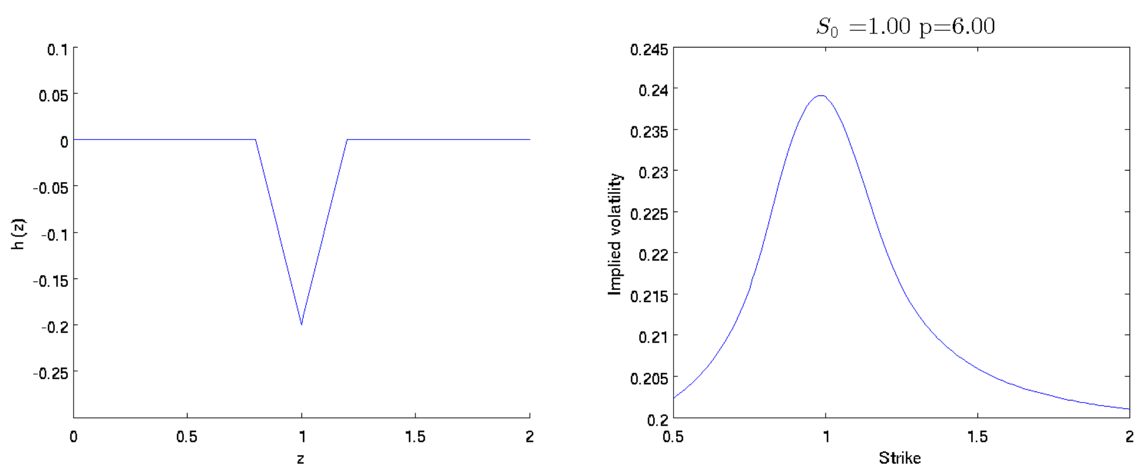
Assume for a moment that the hedged contingent claim was originally underwritten by the hedger to a speculating agent which is not trading on the market. We can conclude that this betting on volatility by the speculator becomes a self-fulfilling prophecy as the hedging activity of the counterparty (i.e. the hedger) affects the equilibrium stock price such that the price of the hedged payoff increases.

In general, the high/low points in the implied volatility coincide with long/short positions in the European options constituting the payoff $h(z)$, and this allows us to obtain any possible shape of the volatility smile, where strikes that correspond to higher/lower implied volatilities are evidence of hedging of long/short positions in European call and put options.

Figure 1.6: Butterfly spread implied volatility

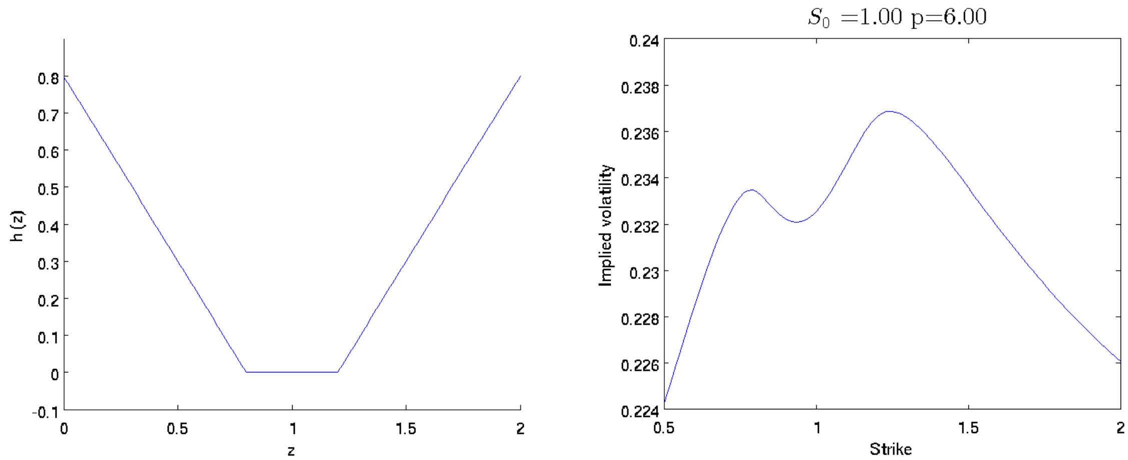



(a) Long butterfly $h(z) = (z - 0.8)^+ - 2(z - 1)^+ + (z - 1.2)^+$

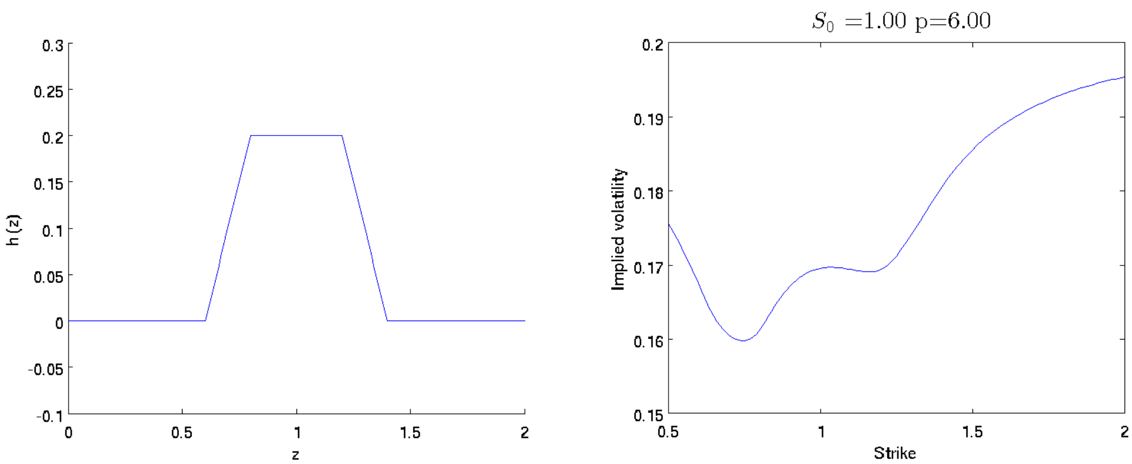



(b) Short butterfly $h(z) = -(z - 0.8)^+ + 2(z - 1)^+ - (z - 1.2)^+$

Figure 1.7: Strangle implied volatility

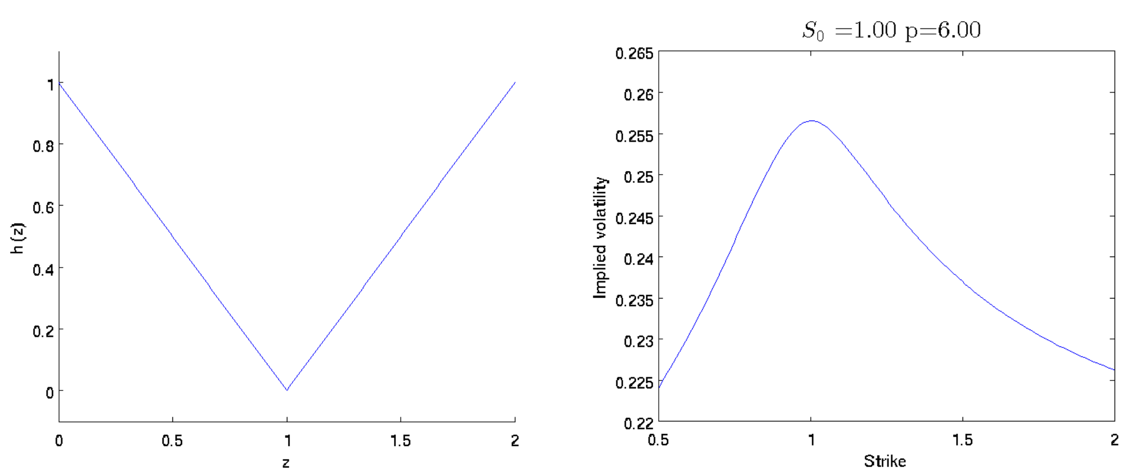



(a) Long strangle $h(z) = (0.8 - z)^{+} + (z - 1.2)^{+}$




(b) Long condor $h(z) = (z - 0.6)^{+} - (z - 0.8)^{+} - (z - 1.2)^{+} + (z - 1.4)^{+}$

Figure 1.8: Straddle implied volatility




(a) Long straddle $h(z) = (1 - z)^{+} + (z - 1)^{+}$

## 1.4. Appendix

**Lemma 1.A.1.** *Let Assumptions 1.1.1, 1.1.2 and 1.1.3 be satisfied. Then we have that* $\mathbb{P}[Z_t = a] = 0$ *and* $\mathbb{P}[Z_T < a] > 0$ *for any* $a \in \mathbb{R}$. *Moreover,* $\mathbb{E}[\exp(nZ_T)] < \infty$ *and* $\mathbb{E}[\exp(n|Z_T|)] < \infty$ *for all* $n \in \mathbb{N}$, *and we also have*

$$\mathbb{E}[\exp(nZ_T)u'(\exp(Z_T) + h_0 - h(\exp(Z_T)))] < \infty \quad \textit{for} \quad n \in \mathbb{N} \cup \{0\}.$$

*Proof.* Since (C1)-(C3) are satisfied, we can apply Theorem 9.1.9 in [109] to obtain that the law of $Z_t$ admits density under the Lebesgue measure on $\mathbb{R}$ and therefore does not have a point mass, i.e. $\mathbb{P}[Z_t = a] = 0$ for any $a \in \mathbb{R}$ and $t \in [0, T]$. Again, from (C1)-(C3), by applying Theorem 3.1 in [108], we have that the measure $\mathbb{P}Z^{-1}$ has full support on the set of real-valued continuous functions $f(t)$ on $[0, T]$ such that $f(0) = z_0$. In particular $\exp(Z_T)$ has full support on $(0, \infty)$. Indeed, for any $a \in \mathbb{R}$ and $\varepsilon > 0$, consider $f(t) \in C([0, T])$ such that $f(0) = z_0$ and $f(T) = a - \varepsilon$, and observe that

$$\mathbb{P}[Z_T < a] \geq \mathbb{P}[|Z_T - a + \varepsilon| < \varepsilon] \geq \mathbb{P}\left[\sup_{t \in [0,T]} |Z_t - f(t)| < \varepsilon\right] > 0,$$

where the last inequality follows from the full support of the measure $\mathbb{P}Z^{-1}$. Note also that due to the boundedness of coefficients in (C2) we can apply Problem 3.4.12 in [63] to the process $Z$ to obtain that it has all exponential moments and, therefore, $\mathbb{E}[\exp(nZ_T)] < \infty$ and $\mathbb{E}[\exp(n|Z_T|)] < \infty$ for all $n \in \mathbb{N}$.

Note that, from the fact that $h(z)$ is continuous and the condition (1.1.7), we have $h_0 \geq h(0)$. Hence, since $h(z)$ also satisfies (1.1.6), we get that

$$\min_{z \geq 0}(z + h_0 - h(z)) > 0, \quad \text{if } h(0) < 0 \text{ or } h_0 > h(0),$$
$$\min_{z \geq \varepsilon}(z + h_0 - h(z)) > 0, \quad \text{if } h(0) \geq 0 \text{ and } h_0 = h(0),$$

for all $\varepsilon > 0$. Moreover, if $h(0) \geq 0$ and $h_0 = h(0)$, from (1.1.6) we also have that

$$h(z) = a_1 z + h_0 \quad \text{for} \quad z \in [0, \underline{k}],$$

where $a_1 < 1$. This means that, if $h(0) \geq 0$ and $h_0 = h(0)$, we obtain

$$z + h_0 - h(z) \geq \mathbf{1}_{\{z < \underline{k}\}}(1 - a_1)z + \mathbf{1}_{\{z \geq \underline{k}\}} \min_{\overline{z} \geq \underline{k}}(\overline{z} + h_0 - h(\overline{z})) \quad \text{for} \quad z > 0.$$

Therefore, from the fact that $u'$ is strictly decreasing, if $h(0) < 0$ or $h_0 > h(0)$ we get that

$$\mathbb{E}[\exp(nZ_T)u'(\exp(Z_T) + h_0 - h(\exp(Z_T)))] < u'\Big(\min_{z\geq 0}(z + h_0 - h(z))\Big)\mathbb{E}[\exp(nZ_T))] < \infty,$$

and if $h(0) \geq 0$ and $h_0 = h(0)$, by using the bounds in (1.1.13), we obtain

$$\begin{aligned}
&\mathbb{E}[\exp(nZ_T)u'(\exp(Z_T) + h_0 - h(\exp(Z_T)))] \\
&\leq \mathbb{P}[\exp(Z_T) < \underline{k}]\mathbb{E}[\exp(nZ_T)u'((1-a_1)\exp(Z_T))|\exp(Z_T) < \underline{k}] \\
&\quad + \mathbb{P}[\exp(Z_T) \geq \underline{k}]u'\Big(\min_{z\geq \underline{k}}(z + h_0 - h(z))\Big)\mathbb{E}[\exp(nZ_T)] \\
&\leq \mathbb{E}[\exp(nZ_T))u'((1-a_1)\exp(Z_T))] + u'\Big(\min_{z\geq \underline{k}}(z + h_0 - h(z))\Big)\mathbb{E}[\exp(nZ_T))] \\
&\leq \mathbb{E}[\exp((N+n)(1 + |\log(1-a_1)| + |Z_T|))] + u'\Big(\min_{z\geq \underline{k}}(z + h_0 - h(z))\Big)\mathbb{E}[\exp(nZ_T))] < \infty,
\end{aligned}$$

for any $n \in \mathbb{N} \cup \{0\}$. □

**Lemma 1.A.2.** *Under Assumption 1.1.1 the filtration generated by the process $Z$ defined in* (1.1.1) *coincides with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian motion $W$.*

*Proof.* Denote by $(\mathcal{F}_t^Z)_{t\in[0,T]}$ the filtration generated by $Z$. Since $Z$ is a strong solution to (1.1.1) it is $\mathcal{F}_t$-adapted and therefore $\mathcal{F}_t^Z \subseteq \mathcal{F}_t$. On the other hand from (C1)-(C2), letting $N_1 > 0$ be a lower bound for $|\sigma_Z|$ and $N_2 > 0$ be an upper bound for $|\mu_Z|$ and $|\sigma_Z|$, we notice that

$$\begin{aligned}
&\left|\frac{\mu_Z(t,z_1)}{\sigma_Z(t,z_1)} - \frac{\mu_Z(t,z_2)}{\sigma_Z(t,z_2)}\right| = \frac{|\mu_Z(t,z_1)\sigma_Z(t,z_2) - \mu_Z(t,z_2)\sigma_Z(t,z_1)|}{|\sigma_Z(t,z_1)\sigma_Z(t,z_2)|} \\
&\leq \frac{|\sigma_Z(t,z_2)||\mu_Z(t,z_1) - \mu_Z(t,z_2)| + |\mu_Z(t,z_2)||\sigma_Z(t,z_2) - \sigma_Z(t,z_1)|}{N_1^2} \\
&\leq N_2\frac{|\mu_Z(t,z_1) - \mu_Z(t,z_2)| + |\sigma_Z(t,z_2) - \sigma_Z(t,z_1)|}{N_1^2},
\end{aligned}$$

and it follows from (C3) that $\mu_Z/\sigma_Z$ is locally Lipschitz. By similar arguments the same holds for $1/\sigma_Z$. Moreover, from (C1)-(C2) it follows that $\mu_Z/\sigma_Z$ and $1/\sigma_Z^2$ are also bounded. Therefore the SDE

$$d\widetilde{W}_t = -\frac{\mu_Z(t,Z_t)}{\sigma_Z(t,Z_t)}\,dt + \frac{1}{\sigma_Z(t,Z_t)}\,dZ_t \quad \text{for} \quad t\in[0,T], \tag{1.A.1}$$

has a unique strong solution $\widetilde{W}$ which is $\mathcal{F}_t^Z$-adapted. But from (1.1.1), by substituting the expression for $Z$ in (1.A.1), we get that $\widetilde{W} = W$ a.e.a.s. and therefore $W$ is also $\mathcal{F}_t^Z$-adapted, which means that $\mathcal{F}_t \subseteq \mathcal{F}_t^Z$. This leads to $\mathcal{F}_t = \mathcal{F}_t^Z$. □

**Lemma 1.A.3.** *In the setting of Theorem 1.2.1 let* $(M_t)_{t\in[0,T]}$ *be a square-integrable martingale under the equivalent martingale measure* $\mathbb{Q}$ *with Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}} = L_T$*. Then there exists an* $\mathcal{F}_t$*-progressively measurable process* $\varphi = (\varphi_t)_{t\in[0,T]}$ *such that*

$$\mathbb{E}\left[\int_0^T \varphi_s^2 ds\right] < \infty, \tag{1.A.2}$$

$$M_t = M_0 + \int_0^t \varphi_s \, d\widetilde{W}_s \quad a.e.a.s., \tag{1.A.3}$$

*where* $\widetilde{W}$ *is a Brownian motion under* $\mathbb{Q}$*. Moreover for any other* $\mathcal{F}_t$*-progressively measurable process* $\widetilde{\varphi} = (\widetilde{\varphi}_t)_{t\in[0,T]}$ *satisfying* (1.A.2)-(1.A.3) *we have*

$$\int_0^T (\varphi_t - \widetilde{\varphi}_t)^2 \, dt = 0 \quad a.s.. \tag{1.A.4}$$

*Proof.* From Lemma 1.6.7 in [64] we know that there exists an $\mathcal{F}_t$-progressively measurable process $\varphi = (\varphi_t)_{t\in[0,T]}$ satisfying condition (1.A.3) such that

$$\int_0^T \varphi_s^2 ds < \infty \quad a.s..$$

However, since $M$ is square-integrable we can use Ito isometry together with (1.A.3) to get

$$\mathbb{E}\left[\int_0^T \varphi_s^2 ds\right] = \mathbb{E}\left[\left(\int_0^T \varphi_s d\widetilde{W}_s\right)^2\right] = \mathbb{E}[(M_T - M_0)^2] < \infty,$$

and therefore $\varphi$ satisfies (1.A.2).

Assume that there exists another $\mathcal{F}_t$-progressively measurable process $\widetilde{\varphi} = (\widetilde{\varphi}_t)_{t\in[0,T]}$ satisfying conditions (1.A.2)-(1.A.3). Then we have that the process $\widetilde{M}$ defined as

$$\widetilde{M}_t := \int_0^t (\varphi_s - \widetilde{\varphi}_s) \, d\widetilde{W}_s \quad \text{for} \quad t \in [0,T],$$

is a square-integrable martingale that is identically zero, and therefore its quadratic variation is also zero. By the Ito isometry we conclude that (1.A.4) holds. □

**Lemma 1.A.4.** *Let Assumptions 1.1.1 and 1.2.1 hold, and the function* $g(z) : \mathbb{R} \to \mathbb{R}$ *belongs to the class* $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus A)$*, where* $A = \{a_1, \ldots, a_m\} \subset \mathbb{R}$ *is the set of points for which* $g(z)$ *is not differentiable and* $a_1 < a_2 < \cdots < a_m$*, for some* $m \in \mathbb{N}$*. Assume also that* $g(z)$ *is decreasing for* $z \in \mathbb{R}$ *and strictly decreasing for* $z < a_1$*, and that* $|g(z)| \le e^{N_1(1+|z|)}$ *for* $z \in \mathbb{R}$ *and* $|g'(z)| \le e^{N_2(1+|z|)}$ *for* $z \in \mathbb{R} \setminus A$*, for some constants* $N_1, N_2 > 0$*. Let* $G(t,z) \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$ *be the unique solution of the PDE*

$$\mathbb{L}_Z G(t,z) = 0 \quad for \quad (t,z) \in [0,T) \times \mathbb{R}, \tag{1.A.5}$$

*with the terminal condition*

$$G(T,z) = g(z) \quad \textit{for} \quad z \in \mathbb{R}, \tag{1.A.6}$$

*in the class of functions satisfying the growth condition* $|G(t,z)| \le c_1 \exp(c_2 z^2)$ *for some constants* $c_1, c_2 > 0$*. Then we have that* $G(t,z)$ *is strictly decreasing function in* $z$ *for all* $t \in [0,T)$*.*

*Proof.* That the solution $G(t,z)$ to the PDE in (1.A.5) exists and is unique follows from the fact that (C1)-(C3) are satisfied by Theorems 9.3 and 2.10 in [37].

Since $G(T,z) = g(z)$ is not differentiable only for $z \in A$, we introduce a class of functions which approximate $G(T,z)$ and smoothen the $m$ discontinuities of $\frac{\partial G}{\partial z}(T,z)$. First let $\overline{n} \in \mathbb{N}$ be defined as

$$\overline{n} = \left[ \frac{1}{\min_{a_i \neq a_j \in A} |a_i - a_j|} \right] + 1,$$

and denote $M = \{1, \dots, m\} \subset \mathbb{N}$. For any $l \in M$ and $n \in \mathbb{N}$, such that $n \ge \overline{n}$, denote $k_{1,l,n} = a_l - 1/n$ and $k_{2,l,n} = a_l + 1/n$, and introduce the constants

$$\varepsilon_{l,n} = \frac{2(G(T,k_{2,l,n}) - G(T,k_{1,l,n}))}{\frac{\partial G}{\partial z}(T,k_{1,l,n}) + \frac{\partial G}{\partial z}(T,k_{2,l,n})}, \tag{1.A.7}$$

$$\delta_{l,n} = n(G(T,k_{2,l,n}) - G(T,k_{1,l,n})) - \frac{1}{2}\left(\frac{\partial G}{\partial z}(T,k_{1,l,n}) + \frac{\partial G}{\partial z}(T,k_{2,l,n})\right). \tag{1.A.8}$$

Notice that, since $G(T,z)$ is decreasing and not differentiable at $a_l$, we have that $G(T,k_{2,l,n}) - G(T,k_{1,l,n}) < 0$. Moreover $G(T,z)$ is differentiable at $z = k_{1,l,n}$ and $z = k_{2,l,n}$ since $n \ge \overline{n}$. Therefore $\varepsilon_{l,n}$ is well-defined and $\varepsilon_{l,n} \in \{-\infty\} \cup (0,\infty)$. Denote $B = \bigcup_{1 \le l \le m}(k_{1,l,n}, k_{2,l,n})$ and define the function $G_n(z)$ as

$$G_n(z) := G(T,z) \quad \text{for} \quad z \in \mathbb{R} \setminus B,$$
$$G_n(z) := \int_{k_{1,l,n}}^{z} \varphi_{l,n}(v)dv + G(T,k_{1,l,n}) \quad \text{for} \quad z \in (k_{1,l,n}, k_{2,l,n}),$$

where the piecewise linear function $\varphi_{l,n}(z)$ defined for $z \in [k_{1,l,n}, k_{2,l,n}]$ is given by

$$\begin{aligned}\varphi_{l,n}(z) = {} & \frac{k_{1,l,n} + \varepsilon_{l,n} - z}{\varepsilon_{l,n}} \frac{\partial G}{\partial z}(T,k_{1,l,n}) \mathbf{1}_{\{z \in [k_{1,l,n}, k_{1,l,n}+\varepsilon_{l,n}]\}} \\ & + \frac{z - (k_{2,l,n} - \varepsilon_{l,n})}{\varepsilon_{l,n}} \frac{\partial G}{\partial z}(T,k_{2,l,n}) \mathbf{1}_{\{z \in [k_{2,l,n}-\varepsilon_{l,n}, k_{2,l,n}]\}} \quad \text{if} \quad \varepsilon_{l,n} \in (0,1/n], \\ \varphi_{l,n}(z) = {} & n\left((a_l - z)\frac{\partial G}{\partial z}(T,k_{1,l,n}) + (z - k_{1,l,n})\delta_{l,n}\right) \mathbf{1}_{\{z \in [k_{1,l,n}, a_l]\}} \\ & + n\left((z - a_l)\frac{\partial G}{\partial z}(T,k_{2,l,n}) + (k_{2,l,n} - z)\delta_{l,n}\right) \mathbf{1}_{\{z \in [a_l, k_{2,l,n}]\}} \quad \text{if} \quad \varepsilon_{l,n} \in \{-\infty\} \cup (1/n, +\infty).\end{aligned}$$

Since $G(T,z)$ is decreasing, and noticing from (1.A.7)-(1.A.8) that $\delta_{l,n}<0$ when $\varepsilon_{l,n}\in\{-\infty\}\cup(1/n,+\infty)$, we get that $\varphi_{l,n}(z)$ is nonpositive and continuous, and moreover it satisfies

$$\int_{k_{1,l,n}}^{k_{2,l,n}}\varphi_{l,n}(v)dv=G(T,k_{2,l,n})-G(T,k_{1,l,n}), \tag{1.A.9}$$

$$\varphi_{l,n}(k_{1,l,n})=\frac{\partial G}{\partial z}(T,k_{1,l,n}),\quad \varphi_{l,n}(k_{2,l,n})=\frac{\partial G}{\partial z}(T,k_{2,l,n}). \tag{1.A.10}$$

By the fact that $G(T,z)\in C^1(\mathbb{R}\setminus A)$ and the continuity of $\varphi_{l,n}(z)$ for $l\in M$, we have that $G_n\in C^1(\mathbb{R}\setminus B)\cap C^1(B)$. By using the continuity of $\varphi_{l,n}(z)$ and (1.A.9)-(1.A.10) we obtain

$$\begin{aligned}
\lim_{z\downarrow k_{1,l,n}}G_n(z)&=\lim_{z\downarrow k_{1,l,n}}\int_{k_{1,l,n}}^{z}\varphi_{l,n}(v)dv+G(T,k_{1,l,n})=G(T,k_{1,l,n})=\lim_{z\uparrow k_{1,l,n}}G_n(z),\\
\lim_{z\uparrow k_{2,l,n}}G_n(z)&=\lim_{z\uparrow k_{2,l,n}}\int_{k_{1,l,n}}^{z}\varphi_{l,n}(v)dv+G(T,k_{1,l,n})=G(T,k_{2,l,n})=\lim_{z\downarrow k_{2,l,n}}G_n(z),\\
\lim_{z\downarrow k_{1,l,n}}G_n'(z)&=\lim_{z\downarrow k_{1,l,n}}\varphi_{l,n}(z)=\frac{\partial G}{\partial z}(T,k_{1,l,n})=\lim_{z\uparrow k_{1,l,n}}G_n'(z),\\
\lim_{z\uparrow k_{2,l,n}}G_n'(z)&=\lim_{z\uparrow k_{2,l,n}}\varphi_{l,n}(z)=\frac{\partial G}{\partial z}(T,k_{2,l,n})=\lim_{z\downarrow k_{2,l,n}}G_n'(z),
\end{aligned}$$

and it follows that $G_n(z)\in C^1(\mathbb{R})$. Since $G(T,z)$ is nonincreasing and $\varphi_{l,n}$ is nonpositive for $l\in M$ it follows that $G_n(z)$ is nonincreasing. Therefore we have

$$|G_n(z)-G(T,z)|\le G(T,k_{1,l,n})-G(T,k_{2,l,n})\quad\text{for}\quad z\in(k_{1,l,n},k_{2,l,n}),$$

and from the fact that $G_n(z)=G(T,z)$ for $z\in\mathbb{R}\setminus B$ and the continuity of $G(T,z)$ for $z\in A$, it follows that $G_n(z)$ converge uniformly to $G(T,z)$ as $n\to\infty$. In particular, since $|G(T,z)|\le e^{N_1(1+|z|)}$ and $\sup_{z\in\overline{B}}G_n(z)=G_n(z^*)<\infty$ for some $z^*\in\overline{B}$, where $\overline{B}$ denotes the closure of the set $B$, it follows that $|G_n(z)|\le e^{g_n(1+|z|)}$ for some constant $g_n>0$. Moreover we have

$$\begin{aligned}
|G_n'(z)|&=\left|\frac{\partial G}{\partial z}(T,z)\right|=|g'(z)|\le e^{N_2(1+|z|)}\quad\text{for}\quad z\in\mathbb{R}\setminus B,\\
|G_n'(z)|&=|\varphi_{l,n}(z)|\quad\text{for}\quad z\in(k_{1,l,n},k_{2,l,n}),
\end{aligned}$$

and, similarly, since $\varphi_{l,n}(z)$ achieves its maximum in the closed interval $[k_{1,l,n},k_{2,l,n}]$, we obtain that $|G_n'(z)|\le e^{\overline{g}_n(1+|z|)}$ for some constant $\overline{g}_n>0$ and for all $z\in\mathbb{R}$.

By using conditions (C1)-(C3) and the fact that $|G_n(z)|\le e^{g_n(1+|z|)}$, from Theorem 9.3 in [37] we have that there exists a solution $G_n(t,z)\in C^{1,2}([0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$ to the PDE in (1.A.5) with the final condition $G_n(T,z)=G_n(z)$. The solution is of the form

$$G_n(t,z)=\int_{-\infty}^{+\infty}p(t,z;T,v)G_n(v)dv\quad\text{for}\quad(t,z)\in[0,T)\times\mathbb{R}, \tag{1.A.11}$$

where we recall that $p(t_1, z; t_2, v)$ is the fundamental solution of (1.A.5). Moreover, from Theorem 9.2 in [37] we have the bound

$$\left|\frac{\partial p}{\partial z}(t, z; T, v)\right| \le \frac{C}{T-t}\exp\left(-c\frac{(v-z)^2}{T-t}\right), \tag{1.A.12}$$

for some positive constants $C$ and $c$. Therefore, by differentiating (1.A.11) and using a change-of-variable formula and the fact that $\exp(g_n v\sqrt{T-t}) \le \exp(\bar{c}\, v^2 + Tg_n^2/(4\bar{c}))$ for any $(t, v) \in [0, T) \times \mathbb{R}$ and any constant $\bar{c} > 0$, it follows that

$$\begin{aligned}\left|\frac{\partial G_n}{\partial z}(t, z)\right| &\le e^{g_n}\int_{-\infty}^{+\infty}\frac{C}{T-t}\exp\left(g_n|v| - c\frac{(v-z)^2}{T-t}\right)dv \le \frac{e^{g_n+|z|}C}{\sqrt{T-t}}\int_{-\infty}^{+\infty} e^{g_n\sqrt{T-t}|v|-cv^2}dv \\ &= \frac{2e^{g_n+|z|}C}{\sqrt{T-t}}\int_0^{+\infty} e^{g_n\sqrt{T-t}v-cv^2}dv \le \frac{e^{g_n+|z|+\frac{Tg_n^2}{4\bar{c}}}C}{\sqrt{T-t}}\int_{-\infty}^{+\infty} e^{(\bar{c}-c)v^2}dv.\end{aligned}$$

for $(t, z) \in [0, T) \times \mathbb{R}$. Hence, if we choose $\bar{c}$ such that $\bar{c} < c$ we get

$$\left|\frac{\partial G_n}{\partial z}(t, z)\right| \le \frac{e^{g_n+|z|+\frac{Tg_n^2}{4\bar{c}}}C\sqrt{\pi}}{\sqrt{(c-\bar{c})(T-t)}},$$

for $(t, z) \in [0, T) \times \mathbb{R}$. Hence for any $\widetilde{c} > 0$ we obtain

$$\int_0^T\int_{-\infty}^{+\infty}\left|\frac{\partial G_n}{\partial z}(t, z)\right| e^{-\widetilde{c}z^2}dzdt \le \frac{e^{g_n+\frac{Tg_n^2}{4\bar{c}}}C\sqrt{T\pi}}{\sqrt{c-\bar{c}}}\int_{-\infty}^{+\infty} e^{|z|-\widetilde{c}z^2}dz < \infty. \tag{1.A.13}$$

From condition (D1) we can differentiate (1.A.5) once with respect to $z$ to get that $\frac{\partial G_n}{\partial z}(t, z)$ solves the PDE

$$\frac{\partial K}{\partial t}(t, z) + \frac{\partial \mu_Z}{\partial z}(t, z)K(t, z) + \left(\mu_Z(t, z) + \frac{1}{2}\frac{\partial \sigma_Z^2}{\partial z}(t, z)\right)\frac{\partial K}{\partial z}(t, z) + \frac{\sigma_Z^2(t, z)}{2}\frac{\partial^2 K}{\partial z^2}(t, z) = 0 \tag{1.A.14}$$

$$\text{for } (t, z) \in [0, T) \times \mathbb{R},$$

with the final condition

$$\frac{\partial G_n}{\partial z}(T, z) = G_n'(z). \tag{1.A.15}$$

Moreover, since (C1)-(C3) and (D1)-(D3) are satisfied, and since we know that $|G_n'(z)| \le e^{\bar{g}_n(1+|z|)}$ and $\frac{\partial G_n}{\partial z}(t, z)$ satisfies (1.A.13), we can apply Theorems 9.3 and 9.6 in [37] to get that $\frac{\partial G_n}{\partial z}(t, z)$ is the unique solution to (1.A.14)-(1.A.15) and has the form

$$\frac{\partial G_n}{\partial z}(t, z) = \int_{-\infty}^{+\infty}\widetilde{p}(t, z; T, v)G_n'(v)dv \quad \text{for} \quad (t, z) \in [0, T) \times \mathbb{R}, \tag{1.A.16}$$

where $\widetilde{p}(t_1, z; t_2, v)$ is the fundamental solution of (1.A.14). From Theorem 2.11 in [37] we have that $\widetilde{p}(t, z; T, v) > 0$, and using that $G_n'(z) \leq 0$ together with the fact that $G(T, z)$ is strictly decreasing for $z < a_1$, we have that for any $n \geq \overline{n}$

$$\begin{aligned}\frac{\partial G_n}{\partial z}(t,z) &= \int_{-\infty}^{+\infty} \widetilde{p}(t,z;T,v) G_n'(v) dv \leq \int_{-\infty}^{k_{1,1,\overline{n}}} \widetilde{p}(t,z;T,v) G_n'(v) dv \\ &= \int_{-\infty}^{k_{1,1,\overline{n}}} \widetilde{p}(t,z;T,v) \frac{\partial G}{\partial z}(T,v) dv < 0 \quad \text{for} \quad (t,z) \in [0,T) \times \mathbb{R}.\end{aligned}$$

This means that $G_n(t, z)$ is strictly decreasing in $z$ for $(t, z) \in [0, T) \times \mathbb{R}$. Moreover, by the mean value theorem, for any $z_1, z_2 \in \mathbb{R}$ such that $z_1 < z_2$ and for any $n \geq \overline{n}$, there is $z^\theta \in [z_1, z_2]$ such that

$$\begin{aligned}G_n(t,z_1) - G_n(t,z_2) &= (z_1 - z_2) \frac{\partial G_n}{\partial z}(t, z^\theta) \geq (z_1 - z_2) \int_{-\infty}^{k_{1,1,\overline{n}}} \widetilde{p}(t, z^\theta; T, v) \frac{\partial G}{\partial z}(T, v) dv \quad (1.A.17) \\ &\geq (z_1 - z_2) \sup_{z \in [z_1, z_2]} \int_{-\infty}^{k_{1,1,\overline{n}}} \widetilde{p}(t,z;T,v) \frac{\partial G}{\partial z}(T,v) dv > 0 \quad \text{for} \quad t \in [0,T).\end{aligned}$$

Notice that from (1.A.11) we get

$$|G_n(t,z) - G(t,z)| \leq \int_{-\infty}^{+\infty} p(t,z;T,v) |G_n(v) - G(T,v)| dv,$$

and, since $G_n(z)$ converge uniformly to $G(T, z)$, we get that $G_n(t, z)$ converge uniformly to $G(t, z)$ with respect to $z$ for $(t, z) \in [0, T) \times \mathbb{R}$ as $n \to \infty$. Now taking $z_1, z_2 \in \mathbb{R}$ such that $z_1 < z_2$ we have from (1.A.17)

$$\begin{aligned}G(t,z_1) - G(t,z_2) &= \lim_{n \to \infty} (G_n(t,z_1) - G_n(t,z_2)) \\ &\geq (z_1 - z_2) \sup_{z \in [z_1, z_2]} \int_{-\infty}^{k_{1,1,\overline{n}}} \widetilde{p}(t,z;T,v) \frac{\partial G}{\partial z}(T,v) dv > 0 \quad \text{for} \quad t \in [0,T),\end{aligned}$$

and it follows that $G(t, z)$ is strictly decreasing in $z$ for $t \in [0, T)$. □

# Chapter 2

# On the sequential testing and quickest change-point detection problems for Gaussian processes

This chapter is based on joint work with Dr. Pavel V. Gapeev.

## 2.1. Preliminaries

In this section, we give a formulation of the unifying optimal stopping problem for a one-dimensional time-inhomogeneous regular diffusion process and consider the associated partial and ordinary differential free-boundary problems.

**2.1.1.** For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{G}, P)$ with a standard Brownian motion $\overline{B} = (\overline{B}_t)_{t \geq 0}$. Let $\Phi = (\Phi_t)_{t \geq 0}$ be a one-dimensional time-inhomogeneous diffusion process with the state space $[0, \infty)$, which is a pathwise (strong) solution of the stochastic differential equation

$$d\Phi_t = \eta(t, \Phi_t)\, dt + \zeta(t, \Phi_t)\, d\overline{B}_t \quad (\Phi_0 = \phi), \tag{2.1.1}$$

where $\eta(t, \phi)$ and $\zeta(t, \phi) > 0$ are some continuously differentiable functions of at most linear growth in $\phi$ on $[0, \infty)$. Let us consider an optimal stopping problem with the value function

$$V_*(t, \phi) = \inf_{\tau} E_{t,\phi} \left[ G(\Phi_{t+\tau}) + \int_0^{\tau} F(\Phi_{t+s})\, ds \right], \tag{2.1.2}$$

where $E_{t,\phi}$ denotes the expectation under the assumption that $\Phi_t = \phi$, for some $\phi \in [0,\infty)$. Here, the gain function $G(\phi)$ and the cost function $F(\phi)$ are assumed to be non-negative, continuous and bounded, $G(\phi)$ is concave and continuously differentiable on $((0,c') \cup (c',\infty))$ for some $c' \in [0,\infty]$, and the infimum in (2.1.2) is taken over all stopping times $\tau$ such that the integral above has a finite expectation, so that $E_{t,\phi}\tau < \infty$ holds. Such time-inhomogeneous optimal stopping problems for diffusion processes within a finite horizon setting have been considered in McKean [77], van Moerbeke [113], Jacka [55], Broadie and Detemple [19], Myneni [80], Peskir [87, 86], and [41, 42] among others (see also Peskir and Shiryaev [90; Chapter VII] and Detemple [30] for an overview and further references). Other time-inhomogeneous optimal stopping problems with infinite time horizon were recently considered in [39].

**Example 2.1.1 (Sequential testing problem.).** Suppose that we observe a continuous process $X = (X_t)_{t\geq 0}$ of the form $X_t = \theta\mu(t) + B_t$, where $\mu(t) > 0$ is increasing and two times continuously differentiable function for $t > 0$, $\mu(0) = 0$, and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion which is independent of the random variable $\theta$. We assume that $P(\theta = 1) = \pi$ and $P(\theta = 0) = 1 - \pi$ holds for some $\pi \in (0,1)$ fixed. The problem of sequential testing of two simple hypotheses about the values of the parameter $\theta$ can be embedded into the optimal stopping problem of (2.1.2) with $G(\phi) = ((a\phi) \wedge b)/(1+\phi)$ and $F(\phi) = 1$, where $a, b > 0$ are some given constants (see, e.g. [105; Chapter IV, Section 2] and [90; Chapter VI, Section 21]). In this case, the *likelihood ratio* process $\Phi$ takes the form

$$\Phi_t = \frac{\pi}{1-\pi}\, L_t \quad \text{with} \quad L_t = \exp\left( \int_0^t \mu'(s)\, dX_s - \frac{1}{2}\int_0^t (\mu'(s))^2\, ds \right), \tag{2.1.3}$$

and thus solves the stochastic differential equation of (2.1.1) with the coefficients $\eta(t,\phi) = (\mu'(t)\phi)^2/(1+\phi)$ and $\zeta(t,\phi) = \mu'(t)\phi$, where the process $\overline{B} = (\overline{B}_t)_{t\geq 0}$ defined by

$$\overline{B}_t = X_t - \int_0^t \frac{\mu'(s)\Phi_s}{1+\Phi_s}\, ds \tag{2.1.4}$$

is the *innovation* standard Brownian motion generating the same filtration $(\mathcal{F}_t)_{t\geq 0}$ as the process $X$.

**Example 2.1.2 (Quickest change-point detection problem.).** Suppose that we observe a continuous process $X = (X_t)_{t\geq 0}$ of the form $X_t = (\mu(t) - \mu(\theta))^+ + B_t$, where $\mu(t) > 0$ is increasing and two times continuously differentiable function for $t > 0$, $\mu(0) = 0$, and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion which is independent of the random variable $\theta$. We assume that $P(\theta = 0) = \pi$ and $P(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ holds for all $t \geq 0$, and some

$\pi \in (0,1)$ and $\lambda > 0$ fixed. The problem of quickest detection of the change-point parameter $\theta$ can be embedded into the optimal stopping problem of (2.1.2) with $G(\phi) = 1/(1+\phi)$ and $F(\phi) = c\phi/(1+\phi)$, where $c > 0$ is a given constant (see, e.g. [105; Chapter IV, Section 4] and [90; Chapter VI, Section 22]). In this case, the *likelihood ratio* process $\Phi$ takes the form

$$\Phi_t = \frac{L_t}{e^{-\lambda t}} \left( \frac{\pi}{1-\pi} + \int_0^t \frac{\lambda e^{-\lambda s}}{L_s} \, ds \right) \tag{2.1.5}$$

with $L = (L_t)_{t \geq 0}$ given by (2.1.3), and thus solves the stochastic differential equation (2.1.1) with the coefficients $\eta(t,\phi) = \lambda(1+\phi) + (\mu'(t)\phi)^2/(1+\phi)$ and $\zeta(t,\phi) = \mu'(t)\phi$, where the innovation standard Brownian motion $\overline{B} = (\overline{B}_t)_{t \geq 0}$ is given by (2.1.4).

**2.1.2.** It follows from the general theory of optimal stopping for Markov processes (see, e.g. [90; Chapter I, Section 2.2]) that the optimal stopping time in the problem of (2.1.2) is given by

$$\tau_* = \inf\{ s \geq 0 \,|\, V_*(t+s, \Phi_{t+s}) = G(\Phi_{t+s}) \} \tag{2.1.6}$$

whenever it exists. We further search for an optimal stopping time of the form

$$\tau_* = \inf\{ s \geq 0 \,|\, \Phi_{t+s} \notin (g_*(t+s), h_*(t+s)) \} \tag{2.1.7}$$

for some functions $0 \leq g_*(t) < h_*(t) \leq \infty$ to be determined (see, e.g. [90; Chapter IV, Section 14] for a time-inhomogeneous finite-horizon setting).

**2.1.3.** By means of standard arguments (see, e.g. [63; Chapter V, Section 5.1]), it can be shown that the infinitesimal generator $\mathbb{L}$ of the process $(t, \Phi) = (t, \Phi_t)_{t \geq 0}$ is given by the expression

$$\mathbb{L} = \partial_t + \eta(t,\phi)\, \partial_\phi + \frac{\zeta^2(t,\phi)}{2}\, \partial^2_{\phi\phi} \tag{2.1.8}$$

for all $(t,\phi) \in (0,\infty)^2$. In order to find analytic expressions for the unknown value function $V_*(t,\phi)$ from (2.1.2) and the unknown boundaries $g_*(t)$ and $h_*(t)$ from (2.1.7), we use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [105; Chapter III, Section 8] and [90; Chapter IV, Section 8]). We formulate the associated free boundary problem

$$(\mathbb{L}V)(t,\phi) = -F(\phi) \quad \text{for} \quad g(t) < \phi < h(t) \tag{2.1.9}$$

$$V(t, g(t)+) = G(g(t)) \quad \text{and} \quad V(t, h(t)-) = G(h(t)) \quad (\textit{instantaneous stopping}) \tag{2.1.10}$$

$$V(t,\phi) = G(\phi) \quad \text{for} \quad \phi < g(t) \quad \text{and} \quad \phi > h(t) \tag{2.1.11}$$

$$V(t,\phi) < G(\phi) \quad \text{for} \quad g(t) < \phi < h(t) \tag{2.1.12}$$

$$(\mathbb{L}G)(\phi) > -F(\phi) \quad \text{for} \quad \phi < g(t) \quad \text{and} \quad \phi > h(t) \tag{2.1.13}$$

for some $0 \leq g(t) < c' < h(t) \leq \infty$ and all $t \geq 0$. Note that the superharmonic characterization of the value function (see, e.g. [105; Chapter III, Section 8] and [90; Chapter IV, Section 9]) implies that $V_*(t,\phi)$ from (2.1.2) is the largest function satisfying (2.1.9)-(2.1.13) with the boundaries $g_*(t)$ and $h_*(t)$. Moreover, since the system in (2.1.9)-(2.1.13) may admit multiple solutions, we need to use some additional conditions which would uniquely determine the value function and the optimal stopping boundaries for the initial problem of (2.1.2). For this reason, we will need to assume that the *smooth-fit* conditions

$$\partial_\phi V(t, g(t)+) = \partial_\phi G(g(t)) \quad \text{and} \quad \partial_\phi V(t, h(t)-) = \partial_\phi G(h(t)) \quad (\textit{smooth fit}) \tag{2.1.14}$$

hold for all $t > 0$.

We further provide an analysis of the parabolic free boundary problem of (2.1.9)-(2.1.13), satisfying the conditions of (2.1.14), and such that the resulting boundaries are continuous and of bounded variation. Since such free-boundary problems cannot normally be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the variational inequalities, arising in the context of optimal stopping problems, have been extensively studied in the literature (see, e.g. Friedman [36], Bensoussan and Lions [14], Krylov [72], or Øksendal [83]). Although the necessary conditions for existence and uniqueness of such solutions in [36; Chapter XVI, Theorem 11.1], [72; Chapter V, Section 3, Theorem 14] with [72; Chapter VI, Section 4, Theorem 12], and [83; Chapter X, Theorem 10.4.1] can be verified by virtue of the regularity of the coefficients of the diffusion process in (2.1.1), the application of these classical results would still have rather inexplicit character. We therefore continue with the following verification assertion related to the free boundary problem formulated above, which is proved in the Appendix.

**Theorem 2.1.3.** *Let the process $\Phi$ be a pathwise unique solution of the stochastic differential equation in (2.1.1). Suppose that the functions $G(\phi)$ and $F(\phi)$ are bounded and continuous,* *and $G$ is concave and continuously differentiable on $((0,c') \cup (c',\infty))$ for some $c' \in [0,\infty]$. Assume that the couple $g_*(t)$ and $h_*(t)$, such that $0 \le g_*(t) < c' < h_*(t) \le \infty$, together with $V(t,\phi;g_*(t),h_*(t))$ form a solution of the free boundary problem of (2.1.9)-(2.1.14), while the boundaries $g_*(t)$ and $h_*(t)$ are continuous and of bounded variation. Define the stopping time $\tau_*$ as the first exit time of the process $\Phi$ from the interval $(g_*(t),h_*(t))$ as in (2.1.7), and assume that $E_{t,\phi}\tau_* < \infty$ holds. Then, the value function $V_*(t,\phi)$ takes the form*

$$V_*(t,\phi) = \begin{cases} V(t,\phi;g_*(t),h_*(t)), & \text{if } g_*(t) < \phi < h_*(t) \\ G(\phi), & \text{if } \phi \le g_*(t) \text{ or } \phi \ge h_*(t) \end{cases} \tag{2.1.15}$$

*with*

$$V(t,\phi;g_*(t),h_*(t)) = E_{t,\phi}\Big[G(\Phi_{t+\tau_*}) + \int_0^{\tau_*} F(\Phi_{t+s})\,ds\Big], \tag{2.1.16}$$

*and the boundaries $g_*(t)$ and $h_*(t)$ are uniquely determined by the smooth-fit conditions of (2.1.14).*

**2.1.4.** Note that the solution of the free boundary problem in (2.1.9)-(2.1.14) cannot be found in an explicit form for the sequential testing and quickest change-point detection problems formulated in Examples 2.1 and 2.2 above. In this respect, let us introduce the function $\widehat{V}(t,\phi)$ and the boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ satisfy the second-order ordinary differential equation

$$(\mathbb{L}V)(t,\phi) = -F(\phi) + \partial_t V(t,\phi) \;\text{ for }\; g(t) < \phi < h(t), \tag{2.1.17}$$

and the conditions of (2.1.10)-(2.1.14), where the variable $t$ plays the role of a parameter. We further provide a connection of the original and the auxiliary free boundary problems associated with the differential equations in (2.1.9) and (2.1.17), respectively. In particular, we will show that, under certain conditions, the lower and upper optimal stopping boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ of the auxiliary problem provide lower and upper estimates of the optimal stopping boundaries $g_*(t)$ and $h_*(t)$ of the original problem.

Let us first state the corresponding verification assertion for the modified free boundary problem which directly follows from Theorem 2.1.3.

**Corollary 2.1.1.** *Let the process $\Phi$ be a pathwise unique solution of the stochastic differential equation in (2.1.1). Suppose that the functions $G(\phi)$ and $F(\phi)$ are bounded and continuous, and $G$ is concave and continuously differentiable on $((0,c') \cup (c',\infty))$ for some $c' \in [0,\infty]$. Assume that the couple $\widehat{g}(t)$ and $\widehat{h}(t)$, such that $0 \le \widehat{g}(t) < c' < \widehat{h}(t) \le \infty$, together with*

*$V(t,\phi;\widehat{g}(t),\widehat{h}(t))$ form a unique solution of the ordinary differential free boundary problem of (2.1.17)+(2.1.10)-(2.1.14), the derivative $\partial_t V(t,\phi;\widehat{g}(t),\widehat{h}(t))$ exists and is continuous, and the boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ are continuous and of bounded variation. Then, the function $\widehat{V}(t,\phi)$ defined by*

$$\widehat{V}(t,\phi) = \begin{cases} V(t,\phi;\widehat{g}(t),\widehat{h}(t)), & \text{if } \widehat{g}(t) < \phi < \widehat{h}(t) \\ G(\phi), & \text{if } \phi \le \widehat{g}(t) \text{ or } \phi \ge \widehat{h}(t) \end{cases} \tag{2.1.18}$$

*is the value function for the optimal stopping problem*

$$\widehat{V}(t,\phi) = \inf_{\tau} E_{t,\phi}\bigg[G(\Phi_{t+\tau}) + \int_0^{\tau} \Big(F(\Phi_{t+s}) - \partial_t\widehat{V}(t+s,\Phi_{t+s})\, I\big(\Phi_{t+s} \in (\widehat{g}(t+s),\widehat{h}(t+s))\big)\Big)\, ds\bigg] \tag{2.1.19}$$

*where $I(\cdot)$ denotes the indicator function and the stopping time $\widehat{\tau}$ of the form*

$$\widehat{\tau} = \inf\{s \ge 0 \,|\, \Phi_{t+s} \notin (\widehat{g}(t+s),\widehat{h}(t+s))\} \tag{2.1.20}$$

*is optimal in (2.1.19), whenever the integral above is of finite expectation, and $\widehat{\tau} = 0$ otherwise.*

**Remark 2.1.4.** Let us fix some $t \ge 0$ and assume that $\partial_t\widehat{V}(t+s,\phi) \ge 0$ holds for all $s \ge 0$ and $\phi \in (\widehat{g}(t+s),\widehat{h}(t+s))$. Then, the value function $\widehat{V}(t+s,\phi)$ of the auxiliary optimal stopping problem in (2.1.19) represents a lower estimate for the value function $V_*(t+s,\phi)$ of (2.1.2), i.e. $\widehat{V}(t+s,\phi) \le V_*(t+s,\phi)$ for all $s \ge 0$ and $\phi > 0$. Indeed, it follows from the fact that $\partial_t\widehat{V}(t+s,\phi) \ge 0$ for all $s \ge 0$ and $\phi \in (\widehat{g}(t+s),\widehat{h}(t+s))$ that the stopping times $\tau$ over which the infimum is taken in (2.1.19) include those for which $E_{t,\phi}\tau < \infty$ holds. Hence, comparing the right-hand sides of (2.1.2) and (2.1.19), and using again the property $\partial_t\widehat{V}(t+s,\phi) \ge 0$, we obtain $\widehat{V}(t+s,\phi) \le V_*(t+s,\phi)$ for all $s \ge 0$ and $\phi > 0$. It thus follows from the structure of the optimal stopping times $\tau_*$ and $\widehat{\tau}$ in (2.1.7) and (2.1.20) that the inequality $\tau_* \le \widehat{\tau}$ should hold ($P_{t,\phi}$-a.s.). In this case, the optimal stopping boundaries $\widehat{g}(t+s)$ and $\widehat{h}(t+s)$ from (2.1.20) are lower and upper estimates for the original optimal stopping boundaries $g_*(t+s)$ and $h_*(t+s)$ in (2.1.7), that is $\widehat{g}(t+s) \le g_*(t+s)$ and $h_*(t+s) \le \widehat{h}(t+s)$ for all $s \ge 0$.

**Example 2.1.5 (Sequential testing problem.).** Let us first solve the free-boundary problem in (2.1.17)+(2.1.10)–(2.1.14) with $G(\phi) = (a\phi \wedge b)/(1+\phi)$ and $F(\phi) = 1$ as in Example 2.1.1 above. For this, we follow the arguments of [105; Chapter IV, Section 2] and [90; Chapter VI, Section 21] and integrate the second-order ordinary differential equation in (2.1.17) twice with respect to the variable $\phi/(1+\phi)$ as well as use the conditions of (2.1.10) and (2.1.14) at the upper boundary $\widehat{h}(t)$ to obtain

$$V(t,\phi;\widehat{g}(t),\widehat{h}(t)) = \frac{b}{1+\phi} - \frac{2}{(\mu'(t))^2}\Big(\Big(\frac{\widehat{h}(t)}{1+\widehat{h}(t)} - \frac{\phi}{1+\phi}\Big)\Upsilon(\widehat{h}(t)) - \Psi(\widehat{h}(t)) + \Psi(\phi)\Big), \tag{2.1.21}$$

where we denote

$$\Psi(\phi) = -\frac{1-\phi}{1+\phi}\ln\phi \quad \text{and} \quad \Upsilon(\phi) = \phi - \frac{1}{\phi} + 2\ln\phi, \tag{2.1.22}$$

for all $\phi > 0$. Then, applying the conditions of (2.1.10) and (2.1.14) at the lower boundary $\widehat{g}(t)$, we obtain that the functions $\widehat{g}(t)$ and $\widehat{h}(t)$ solve the system of arithmetic equations

$$\frac{a(\mu'(t))^2 g(t)}{2(1+g(t))} = \frac{b(\mu'(t))^2}{2(1+g(t))} - \Upsilon(h(t))\left(\frac{h(t)}{1+h(t)} - \frac{g(t)}{1+g(t)}\right) + \Psi(h(t)) - \Psi(g(t)), \tag{2.1.23}$$

$$\frac{(b+a)(\mu'(t))^2}{2} = \Upsilon(h(t)) - \Upsilon(g(t)), \tag{2.1.24}$$

which is equivalent to the system

$$\frac{(b-a)(\mu'(t))^2}{2} = h(t) + \frac{1}{h(t)} - g(t) - \frac{1}{g(t)}, \tag{2.1.25}$$

$$\frac{b(\mu'(t))^2}{2} = h(t) + \ln h(t) - g(t) - \ln g(t), \tag{2.1.26}$$

for all $t > 0$. It is shown in [105; Chapter IV, Section 2] and [90; Chapter VI, Section 21] that the system in (2.1.25)-(2.1.26) admits the unique solution $0 < \widehat{g}(t) < b/a < \widehat{h}(t) < \infty$, for any $\mu'(t)$ and $t \geq 0$ fixed. Moreover, by using the implicit function theorem, we can differentiate (2.1.25)-(2.1.26) to get

$$(b-a)\,\mu'(t)\mu''(t) = h'(t) - \frac{h'(t)}{h^2(t)} - g'(t) + \frac{g'(t)}{g^2(t)}, \tag{2.1.27}$$

$$b\,\mu'(t)\mu''(t) = h'(t) + \frac{h'(t)}{h(t)} - g'(t) - \frac{g'(t)}{g(t)}, \tag{2.1.28}$$

from which we deduce that

$$g'(t) = \frac{\mu'(t)\mu''(t)(b-ah(t))g^2(t)}{(g(t)+1)(h(t)-g(t))} \quad \text{and} \quad h'(t) = \frac{\mu'(t)\mu''(t)(b-ag(t))h^2(t)}{2(h(t)+1)(h(t)-g(t))} \tag{2.1.29}$$

holds for all $t > 0$. In particular, we also obtain that the partial derivative $\partial_t \widehat{V}(t,\phi)$ exists and is continuous.

**Example 2.1.6 (Quickest change-point detection problem.).** Let us now solve the free-boundary problem in (2.1.17)+(2.1.10)–(2.1.14) with $G(\phi) = 1/(1+\phi)$ and $F(\phi) = c\phi/(1+\phi)$ as in Example 2.1.2 above, where we set $\widehat{g}(t) = 0$ for all $t \geq 0$. For this, we follow the arguments of [105; Chapter IV, Section 4] or [90; Chapter VI, Section 22] and integrate the second-order ordinary differential equation in (2.1.17) twice with respect to the variable $\phi/(1+\phi)$ as well as use the conditions of (2.1.10) and (2.1.14) at the upper boundary $\widehat{h}(t)$ to obtain

$$V(t, \phi; \widehat{h}(t)) = \frac{1}{1+\widehat{h}(t)} + \int_{\phi}^{\widehat{h}(t)} \frac{C(t)}{(1+y)^2} \int_0^y \exp\Big( - \Lambda(t)(H(y) - H(x))\Big) \frac{1+x}{x} dx dy, \quad (2.1.30)$$

where we denote

$$C(t) = \frac{2c}{(\mu'(t))^2}, \quad \Lambda(t) = \frac{2\lambda}{(\mu'(t))^2}, \quad \text{and} \quad H(x) = \ln x - \frac{1+x}{x}, \qquad (2.1.31)$$

for all $t \geq 0$ and $\phi > 0$. It thus follows from the condition of (2.1.14) that the boundary $\widehat{h}(t)$ solves the arithmetic equation

$$C(t) \int_0^{h(t)} \exp\Big( - \Lambda(t)\ (H(h(t)) - H(x))\Big)\, \frac{1+x}{x}\, dx = 1, \qquad (2.1.32)$$

for all $t \geq 0$. It is shown in [105; Chapter IV, Section 4] and [90; Chapter VI, Section 22] that the equation in (2.1.32) admits the unique solution $\lambda/c \leq \widehat{h}(t)$, for any $\mu'(t)$ and $t \geq 0$ fixed. Moreover, by using the implicit function theorem, we can also obtain that $\widehat{h}(t)$ is continuosly differentiable, as well as the partial derivative $\partial_t \widehat{V}(t, \phi)$ exists and is continuous.

## 2.2. Asymptotic behaviour of the stopping boundaries

In this section, we are interested in how the optimal stopping boundaries $\widehat{g}(t)$ and $\widehat{h}(t)$ in the modified problem behave asymptotically with respect to the derivative $\mu'(t)$ of the drift function $\mu(t)$ in Example 2.1.1 and Example 2.1.2, as $t \to \infty$. More precisely, we will obtain the limits and the asymptotic expansions of $\widehat{g}(t)$ and $\widehat{h}(t)$ with respect to $\mu'(t)$ in some particular cases, when either $\mu'(t) \to 0$ or $\mu'(t) \to \infty$ holds as $t \to \infty$.

**Example 2.2.1 (Sequential testing problem.).** Let us introduce the function $W(x)$ which is the inverse of $e^x x$, and thus, solves the equation

$$e^{W(x)} W(x) = x \quad \text{for} \quad x \geq 0 \qquad (2.2.1)$$

(see, e.g. [22; Formula (1.5)]). Note that $W(x)$ is strictly increasing and satisfy the properties $W(0) = 0$, and $W(x) \to \infty$ as $x \to \infty$, and it has the asymptotic series expansion

$$W(x) \sim \ln(x) - \ln(\ln(x)) \quad \text{as} \quad x \to \infty \tag{2.2.2}$$

(see, e.g. [22; Formula (4.19)]). Then, by solving the quadratic equation in (2.1.25) for $h(t)$, we obtain that $\widehat{g}(t)$ and $\widehat{h}(t)$ satisfy

$$h_{\pm}(t) = \frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4} \pm \sqrt{\left(\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4}\right)^2 - 1}, \tag{2.2.3}$$

where $\widehat{h}(t) = \widehat{h}_{-}(t)$ or $\widehat{h}(t) = \widehat{h}_{+}(t)$, for all $t \geq 0$. Hence, by substituting the expression of (2.2.3) into the formula of (2.1.26) and taking exponentials on both sides, we have that $\widehat{g}(t)$ satisfies the following equation

$$\begin{aligned} &\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4} \pm \sqrt{\left(\frac{g(t)}{2} + \frac{1}{2g(t)} + \frac{(b-a)(\mu'(t))^2}{4}\right)^2 - 1} \\ &= W\big(e^{g(t)+b(\mu'(t))^2/2} g(t)\big), \end{aligned} \tag{2.2.4}$$

which contains both the positive and negative branch of the function on the left-hand side, depending on the root which we have chosen for $\widehat{h}(t)$ in (2.2.3). If we rearrange the terms and square both sides of the expression in (2.2.4), we get that $\widehat{g}(t)$ should satisfy

$$1 + W^2\big(e^{g(t)+b(\mu'(t))^2/2} g(t)\big) = \left(g(t) + \frac{1}{g(t)} + \frac{(b-a)(\mu'(t))^2}{2}\right) W\big(e^{g(t)+b(\mu'(t))^2/2} g(t)\big), \tag{2.2.5}$$

for all $t \geq 0$.

Let us first consider the case in which $b > a$ and $\mu'(t) \to \infty$ holds as $t \to \infty$. If we assume that $\widehat{h}(t) = \widehat{h}_{-}(t)$, by using the assumption that $b > a$ and $0 < \widehat{g}(t) < b/a$, we obtain that $\widehat{h}_{-}(t) \to 0$, which contradicts the fact that $b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$. It follows that $\widehat{h}(t) = \widehat{h}_{+}(t)$ and $\widehat{g}(t)$ should solve the equation in (2.2.4) with the positive branch of the function taken on the left-hand side. Hence, the left-hand side of the expression in (2.2.4) converges to $\infty$ as $t \to \infty$, so that $e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t) \to \infty$ holds by virtue of the properties of the function $W(x)$ defined in (2.2.1). In particular, the functions on both sides of (2.2.5) converge to $\infty$ with the same speed, and thus, the following expression holds

$$W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \sim \frac{(b-a)(\mu'(t))^2}{2} + \widehat{g}(t) + \frac{1}{\widehat{g}(t)} \quad \text{as} \quad t \to \infty. \tag{2.2.6}$$

Furthermore, taking into account the asymptotic series expansion of (2.2.2), we see that

$$W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \sim \frac{b(\mu'(t))^2}{2} + \widehat{g}(t) + \ln(\widehat{g}(t)) \quad \text{as} \quad t \to \infty. \tag{2.2.7}$$

Since $\widehat{g}(t)$ is bounded from above by $b/a$ for all $t \geq 0$ and using the equation of (2.2.3) for $\widehat{h}(t)$, we therefore conclude that

$$\widehat{g}(t) \sim \frac{2}{a(\mu'(t))^2} \quad \text{and} \quad \widehat{h}(t) \sim \frac{b(\mu'(t))^2}{2} \quad \text{as} \quad t \to \infty. \tag{2.2.8}$$

Let us now consider the case in which $b < a$ and $\mu'(t) \to \infty$ holds as $t \to \infty$. Since the function on the left-hand side of (2.1.25) converges to $-\infty$ as $t \to \infty$, taking into account the fact that $\widehat{g}(t) < b/a < \widehat{h}(t)$ holds for $t \geq 0$, we obtain that $\widehat{g}(t) \to 0$ as $t \to \infty$. Assuming that $W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t))$ does not converge to $\infty$ implies that there exists a sequence $(t_n)_{n\in\mathbb{N}}$, such that $t_n \to \infty$ and $\widehat{g}(t_n) = O(e^{-b(\mu'(t_n))^2/2})$ as $n \to \infty$. Now if $\widehat{h}(t) = \widehat{h}_+(t)$, we obtain that $\widehat{h}(t_n) \to \infty$ as $n \to \infty$, while the assumption that the right-hand side of (2.2.4) does not converge to $\infty$ leads to contradiction. On the other hand, if $\widehat{h}(t) = \widehat{h}_-(t)$, we obtain that $\widehat{h}(t) \to 0$, which contradicts the assumption that $b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$. We therefore obtain that $W(e^{\widehat{g}(t)+b(\mu'(t))^2/2}\widehat{g}(t)) \to \infty$, and by the same considerations as in the case $b > a$ above, regarding the asymptotic behaviour of the both sides of (2.2.5), we obtain (2.2.8).

Let us finally consider the case in which $\mu'(t) \to 0$ holds as $t \to \infty$. Since the left-hand side of (2.1.26) converges to $0$ in this case, by using the fact that the function $x + \ln(x)$ is strictly increasing for $x > 0$, and $0 < \widehat{g}(t) < b/a < \widehat{h}(t) < \infty$ holds for all $t \geq 0$, we may conclude that $\widehat{g}(t) \to b/a$ and $\widehat{h}(t) \to b/a$ holds as $t \to \infty$.

**Example 2.2.2 (Quickest change-point detection problem.).** Integrating by parts and using the notations of (2.1.31), we obtain

$$C(t) \int_0^y \frac{(1+x)}{x} \exp\Big( - \Lambda(t)\,(H(y) - H(x))\Big)\, dx = \frac{cy}{\lambda} \left(1 - \frac{Q(-\Lambda(t)-1, \Lambda(t)/y)}{\Lambda(t)+1}\right), \tag{2.2.9}$$

where we denote

$$Q(z,y) = -zy^{-z}e^{y}\Gamma(z,y) \quad \text{with} \quad \Gamma(z,y) = \int_y^\infty e^{-u}u^{z-1}\, du, \tag{2.2.10}$$

for all $z \leq 0$ and $y \geq 0$. In this case, the expression in (2.1.32) takes the form

$$h(t) \left(1 - \frac{Q(-\Lambda(t)-1, \Lambda(t)/h(t))}{\Lambda(t)+1}\right) = \frac{\lambda}{c}, \tag{2.2.11}$$

for all $t \geq 0$. We also recall the properties of the function $Q(z, y)$ in [111; Section 9] (see also [46; Section 2.5]) and note that $0 \leq Q(z, y) \leq 1$ as well as $Q(z, 0) = 1$ holds for all $z \leq 0$.

Let us first consider the case in which $\mu'(t) \to \infty$, and thus $\Lambda(t) \to 0$ as $t \to \infty$. Since $\lambda/c \leq \widehat{h}(t)$ holds, we have $\Lambda(t)/\widehat{h}(t) \to 0$, so that $Q(-\Lambda(t) - 1, \Lambda(t)/\widehat{h}(t)) \to 1$ as $t \to \infty$. Therefore, by using the fact that $\widehat{h}(t)$ satisfies the equation in (2.2.11), we get that $\widehat{h}(t) \to \infty$ holds as $t \to \infty$.

Suppose that $\mu'(t) \to 0$, so that $\Lambda(t) \to \infty$ holds as $t \to \infty$. Then, using the property $0 \leq Q(z, y) \leq 1$, it follows from (2.2.11) that

$$\widehat{h}(t) \sim \frac{\lambda}{c} \quad \text{as} \quad t \to \infty. \tag{2.2.12}$$

Let us now determine the exact rate of increase for $\widehat{h}(t)$ in the case in which $\mu'(t) \to \infty$ as $t \geq \infty$. In this case, the expression in (2.1.32) can be written as

$$\Lambda(t) \int_0^{h(t)} \exp\Big(\Lambda(t)\, H(x)\Big) \frac{1+x}{x}\, dx = \frac{\lambda}{c} \exp\Big(\Lambda(t)\, H(h(t))\Big), \tag{2.2.13}$$

for $t \geq 0$. Then, using the definition of the function $H(x)$ in (2.1.31), we obtain the expansion on the right-hand side of (2.2.13) in the form

$$\frac{\lambda}{c} \exp\Big(\Lambda(t)\, H(\widehat{h}(t))\Big) \sim \frac{\lambda\, \widehat{h}(t)^{\Lambda(t)}}{c}, \tag{2.2.14}$$

under $\mu'(t) \to \infty$. Note that the assumption of

$$\limsup_{t \to \infty} \widehat{h}(t)^{\Lambda(t)} = \infty \tag{2.2.15}$$

implies that there exists a sequence $(t_n)_{n \in \mathbb{N}}$, such that $t_n \to \infty$ and $\exp(\Lambda(t_n) H(\widehat{h}(t_n))) \to \infty$ as $n \to \infty$. Since we have $\widehat{h}(t) \to \infty$, there exists $t' \geq 0$ such that $2\lambda/c < \widehat{h}(t)$ holds for all $t \geq t'$. Moreover, since the function $H(x)$ is strictly increasing for $x > 0$, by evaluating the left-hand side of (2.2.13) at $\widehat{h}(t)$, we obtain that

$$\begin{aligned} &\int_0^{\widehat{h}(t)} \Lambda(t) \exp\Big(\Lambda(t)\, H(x)\Big) \frac{1+x}{x}\, dx = \int_0^{\widehat{h}(t)} x\, d \exp\Big(\Lambda(t) H(x)\Big) \\ &> \int_{\frac{2\lambda}{c}}^{\widehat{h}(t)} x\, d \exp\Big(\Lambda(t)\, H(x)\Big) > \frac{2\lambda}{c} \bigg( \exp\Big(\Lambda(t)\, H(\widehat{h}(t))\Big) - \exp\Big(\Lambda(t) H\Big(\frac{2\lambda}{c}\Big)\Big) \bigg) \end{aligned} \tag{2.2.16}$$

holds for all $t \geq t'$. This fact means that the leading term of the left-hand side of (2.2.13) is larger than the leading term on the right-hand side of (2.2.13) along the sequence $t_n$ as $n \to \infty$, and thus, the assumption of (2.2.15) cannot be satisfied. Since $\widehat{h}(t) \to \infty$ and $\Lambda(t) \to 0$, we have $\widehat{h}(t)^{\Lambda(t)} \gtrsim 1$ as $t \to \infty$. The latter fact implies that $\widehat{h}(t)^{\Lambda(t)}$ is bounded, so that $\ln \widehat{h}(t) = O((\mu'(t))^2)$ as $t \to \infty$.

## 2.3. The fractional Brownian motion setting

In this section, we apply the asymptotic results obtained above to demonstrate the existence of solutions in the problems of sequential analysis for an observable fractional Brownian motion with linear drift. In particular, we will prove that the optimal stopping time $\tau_*$ has a finite expectation.

**Example 2.3.1 (Sequential testing problem.).** Suppose that in the setting of Example 2.1.1 the observable continuous process $X \equiv Y^H = (Y_t^H)_{t \geq 0}$ is given by $Y_t^H = \theta \rho t + B_t^H$, where $B^H = (B_t^H)_{t \geq 0}$ is a fractional Brownian motion with parameter $H \in (1/2, 1)$ independent of $\theta$, and $\rho > 0$ is a constant. Introduce the process $\overline{M}^H = (\overline{M}_t^H)_{t \geq 0}$ by

$$\overline{M}_t^H = Z_t^H - c_1 \int_0^t \rho \, \frac{s^{1-2H} \Phi_s}{1 + \Phi_s} \, ds \quad \text{with} \quad \langle \overline{M}^H \rangle_t = \langle Z^H \rangle_t = \frac{c_1 t^{2-2H}}{2 - 2H}, \tag{2.3.1}$$

where the process $Z^H = (Z_t^H)_{t \geq 0}$ is defined by

$$Z_t^H = \int_0^t \frac{s^{1/2-H} (t-s)^{1/2-H}}{2H \Gamma(3/2 - H) \Gamma(H + 1/2)} \, dY_s^H \quad \text{and} \quad c_1 = \frac{\Gamma(3/2 - H)}{2H \Gamma(H + 1/2) \Gamma(2 - 2H)}, \tag{2.3.2}$$

with $\Phi$ being the likelihood ratio process as in (2.1.3).

It follows from the result of [82; Theorem 3.1] that the process $\overline{M}^H$ is a fundamental martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and thus admits the following representation with respect to the innovation standard Brownian motion

$$\overline{M}_t^H = \sqrt{c_1} \int_0^t s^{1/2-H} \, d\overline{B}_s \quad \text{so that} \quad \overline{B}_t = \frac{1}{\sqrt{c_1}} \int_0^t s^{H-1/2} \, d\overline{M}_s^H. \tag{2.3.3}$$

for all $t \geq 0$ (see, e.g. [82; Section 5.2]). In this case, the process $L$ from (2.1.3) is given by

$$L_t = \exp\left( \rho Z_t^H - \frac{\rho^2}{2} \langle Z^H \rangle_t \right), \tag{2.3.4}$$

so that the process $\Phi$ satisfies the stochastic differential equation in (2.1.1) with $\eta$ and $\zeta$ as in Example 2.1.1 with $\mu'(t) = \rho \sqrt{c_1} t^{1/2-H}$, for all $t \geq 0$. Hence, the analysis from the previous section can be applied for the drift rate $\mu'(t) \to 0$ when $1/2 < H < 1$ as $t \to \infty$.

Let us fix a starting time $t \geq 0$ and introduce the deterministic time change $\beta(t,s)$ with the rate $(\mu'(s))^2$ defined as

$$\beta(t,s) = \int_t^{t+s} (\mu'(u))^2 \, du \equiv \frac{c_1 \rho^2 ((t+s)^{2-2H} - t^{2-2H})}{2 - 2H}, \tag{2.3.5}$$

and its inverse $\gamma(t,s)$ shifted by $t$, such that $\beta(t,\gamma(t,s)-t)=s$ for all $s\geq 0$. Since the process $\Phi$ satisfies the stochastic differential equation of (2.1.1), by applying the time-change formula for Itô integrals in [83; Theorems 8.5.1 and 8.5.7], we obtain

$$\Phi_{\gamma(t,s)} = \Phi_t \exp\left(\widetilde{B}_s - \frac{s}{2} + \int_0^s \frac{\Phi_{\gamma(t,u)}}{1+\Phi_{\gamma(t,u)}}\,du\right) \quad \text{with} \quad \widetilde{B}_s = \int_t^{\gamma(t,s)} \mu'(u)\,d\overline{B}_u, \tag{2.3.6}$$

where $\widetilde{B}=(\widetilde{B}_s)_{s\geq 0}$ is a standard Brownian motion with respect to the filtration $(\mathcal{F}_{\gamma(t,s)})_{s\geq 0}$. Therefore, by using the definition of $\widehat{\tau}$ in (2.1.20) and taking into consideration the time change $\beta(t,s)$ from (2.3.5), we conclude that the stopping time $\beta(t,\widehat{\tau})$ with respect to the filtration $(\mathcal{F}_{\gamma(t,s)})_{s\geq 0}$ can be represented as

$$\beta(t,\widehat{\tau}) = \inf\left\{ s\geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} + \int_0^s \frac{\Phi_{\gamma(t,u)}}{1+\Phi_{\gamma(t,u)}}du + \ln\Phi_t \notin \big(\ln\widehat{g}(\gamma(t,s)), \ln\widehat{h}(\gamma(t,s))\big)\right\}, \tag{2.3.7}$$

for all $t\geq 0$.

Assume that $b\neq a$ in Example 2.1.1. In this case, noticing from (2.3.5) that $\gamma(t,s)\to\infty$ and using the fact that $\widehat{g}(t)\to b/a$ and $\widehat{h}(t)\to b/a$ as $t\to\infty$, it follows that for any $\varepsilon>0$ there exists $t_*>0$ large enough such that the inequalities

$$\frac{b}{a}-\varepsilon < \widehat{g}(\gamma(t,s)) < \frac{b}{a} < \widehat{h}(\gamma(t,s)) < \frac{b}{a}+\varepsilon \tag{2.3.8}$$

hold for all $t>t_*$ and $s\geq 0$. Let us now fix an arbitrary $\varepsilon>0$ such that $\varepsilon<b/a$, and assume from now on that $t>t_*$. Then, introducing the sets of sample paths $A_0=\{\omega\in\Omega \,|\, \widehat{g}(t) < \Phi_t < \widehat{h}(t)\}$,

$$A_s = \big\{\omega\in A_0 \,\big|\, \widehat{g}(\gamma(t,s)) < \Phi_{\gamma(t,s)} < \widehat{h}(\gamma(t,s))\big\}, \;\; C_s = \big\{\omega\in\Omega \,\big|\, |\Phi_{\gamma(t,s)} - b/a| < \varepsilon\big\}, \tag{2.3.9}$$

and using the inequalities in (2.3.8), we get the inclusion $A_s\subseteq C_s$ for any $s\geq 0$. Therefore, by the definition of the event $C_s$, for the upper bounds $c_1(\varepsilon)$ and $c_2(\varepsilon)$ defined below, we have

$$c_1(\varepsilon) \equiv \frac{b-a\varepsilon}{a+b-a\varepsilon} < \frac{\Phi_{\gamma(t,s)}}{1+\Phi_{\gamma(t,s)}} < \frac{b+a\varepsilon}{a+b+a\varepsilon} \equiv c_2(\varepsilon), \quad \text{for} \quad \omega\in A_s, \tag{2.3.10}$$

for any $\varepsilon>0$. It follows from the notations in (2.3.6) and the structure of the event $A_0$ that $A_s\subseteq D_s$ holds, where we define

$$D_s = \left\{\omega\in\Omega \,\middle|\, \widetilde{B}_s - \frac{s}{2} \in \left(\ln\left(\frac{\widehat{g}(\gamma(t,s))}{\widehat{h}(t)}\right) - c_2(\varepsilon)\,s, \ln\left(\frac{\widehat{h}(\gamma(t,s))}{\widehat{g}(t)}\right) - c_1(\varepsilon)\,s\right)\right\}, \tag{2.3.11}$$

for all $s\geq 0$. Define the stopping time $\overline{\tau}$ as

$$\overline{\tau} = \inf\left\{s\geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} \notin \left(\ln\left(\frac{\widehat{g}(\gamma(t,s))}{\widehat{h}(t)}\right) - c_2(\varepsilon)\,s, \ln\left(\frac{\widehat{h}(\gamma(t,s))}{\widehat{g}(t)}\right) - c_1(\varepsilon)\,s\right)\right\}, \tag{2.3.12}$$

and notice that the stopping times $\beta(t,\widehat{\tau}) = \beta(t,\widehat{\tau}(\omega))$ and $\overline{\tau} = \overline{\tau}(\omega)$ admit the representations

$$\beta(t,\widehat{\tau}(\omega)) = \sup\left\{ s \geq 0 \,\middle|\, \omega \in \bigcap_{0\leq u\leq s} A_u \right\} \quad \text{and} \quad \overline{\tau}(\omega) = \sup\left\{ s \geq 0 \,\middle|\, \omega \in \bigcap_{0\leq u\leq s} D_u \right\}, \tag{2.3.13}$$

for any $\omega \in \Omega$. Then, it follows from the inclusion $A_s \subseteq D_s$ for $s \geq 0$ that $\beta(t,\widehat{\tau}) \leq \overline{\tau}$ holds. Because of the assumption $b \neq a$, we can choose $\varepsilon < b/a$ such that either $1-\varepsilon > b/a$ holds when $b < a$ or $1+\varepsilon < b/a$ holds when $b > a$. Hence, assuming that $b < a$, we have $1/2 - c_2(\varepsilon) > 0$. Thus, it follows from the expressions in (2.3.8) and (2.3.12) that $\overline{\tau} \leq \tau'$ holds, where we set

$$\tau' = \inf\big\{ s \geq 0 \,|\, \widetilde{B}_s \leq \ln(b - a\varepsilon) - \ln(a\widehat{h}(t)) + (1/2 - c_2(\varepsilon))s \big\}, \tag{2.3.14}$$

which is a stopping time with polynomial moments of all orders (see, e.g. [107; Chapter IV]). Therefore, it follows from the fact that $\beta(t,\widehat{\tau}) \leq \overline{\tau} \leq \tau'$ holds and the structure of the time change in (2.3.5) that $E_{t,\phi}\widehat{\tau} \leq E_{t,\phi}\gamma(t,\tau') - t < \infty$ is satisfied, and we get the same inequalities in the case of $b > a$, similarly.

Let us now prove that $\partial_t V(t,\phi;\widehat{g}(t),\widehat{h}(t)) > 0$ holds for all $\phi \in (\widehat{g}(t),\widehat{h}(t))$ and $t > 0$ large enough. For this purpose, by differentiating the expression in (2.1.21) and using the expressions in (2.1.22) and (2.1.29), we get

$$\begin{aligned} &\partial_t V(t,\phi;\widehat{g}(t),\widehat{h}(t)) = 2(2H-1)(\Psi(\widehat{h}(t)) - \Psi(\phi))/(t(\mu'(t))^2) \\ &- \frac{2}{(\mu'(t))^2}\left(\frac{\widehat{h}(t)}{1+\widehat{h}(t)} - \frac{\phi}{1+\phi}\right)\left(\frac{(2H-1)\xi(\widehat{h}(t))}{t} + \frac{\widehat{h}'(t)(\widehat{h}(t)+1)^2}{\widehat{h}(t)^2}\right) = \frac{2(2H-1)\Xi(t,\phi)}{t(\mu'(t))^2}, \end{aligned} \tag{2.3.15}$$

where we denote

$$\Xi(t,\phi) = \phi + \ln\phi - \widehat{h}(t) - \ln\widehat{h}(t) + \frac{\phi}{1+\phi}\big(\Upsilon(\widehat{h}(t)) - \Upsilon(\phi)\big) + \frac{(\widehat{h}(t)-\phi)(b-a\widehat{g}(t))}{2(\widehat{h}(t)-\widehat{g}(t))(1+\phi)}, \tag{2.3.16}$$

for all $t > 0$ and $\phi > 0$. It is clear that $\Xi(t,\widehat{h}(t)) = 0$ holds and, thus, we obtain from the expressions in (2.1.24) and (2.1.26) that

$$\Xi(t,\widehat{g}(t)) = \frac{(\mu'(t))^2}{2}\left(\frac{\widehat{g}(t)(a+b)}{1+\widehat{g}(t)} - b\right) + \frac{(b-a\widehat{g}(t))}{2(1+\widehat{g}(t))} = \frac{(b-a\widehat{g}(t))}{2(1+\widehat{g}(t))}\left(1 - \frac{(\mu'(t))^2}{2}\right), \tag{2.3.17}$$

holds for $t > 0$. Since $b/a > \widehat{g}(t) > 0$ is satisfied, and there exists $t' > 0$ such that $\mu'(t) < \sqrt{2}$ holds for all $t \geq t'$, we have $\Xi(t,\widehat{g}(t)) > 0$ for $t \geq t'$. Then, by differentiating the expression in (2.3.16), we get

$$\partial_\phi \Xi(t,\phi) = \frac{1}{(1+\phi)^2}\left(\Upsilon(\widehat{h}(t)) - \Upsilon(\phi) - \frac{(b-a\widehat{g}(t))(1+\widehat{h}(t))}{2(\widehat{h}(t)-\widehat{g}(t))}\right), \tag{2.3.18}$$

for all $t > 0$ and $\phi > 0$. Observe that, since $\Upsilon(\phi)$ is an increasing function, it follows that $\partial_\phi \Xi(t,\phi)$ changes its sign at most once in the region $\phi \in (\widehat{g}(t), \widehat{h}(t))$ for all $t \geq t'$. It is easily seen that the inequality $\partial_\phi \Xi(t, \widehat{h}(t)) < 0$ holds, which means that either $\Xi(t,\phi)$ is decreasing for $\phi \in (\widehat{g}(t), \widehat{h}(t))$ or there exists some $\phi_* \in (\widehat{g}(t), \widehat{h}(t))$ such that $\Xi(t,\phi)$ is increasing for $\phi \in (\widehat{g}(t), \phi_*]$ and decreasing for $\phi \in (\phi_*, \widehat{h}(t))$. Hence, since $\Xi(t, \widehat{g}(t)) > 0$ and $\Xi(t, \widehat{h}(t)) = 0$ holds, we get that $\Xi(t,\phi) > 0$ is satisfied in both cases for $\phi \in (\widehat{g}(t), \widehat{h}(t))$ and $t \geq t'$. For $1/2 < H < 1$, it follows from the expressions in (2.3.15) that the inequality $\partial_t V(t,\phi; \widehat{g}(t), \widehat{h}(t)) > 0$ holds for all $\phi \in (\widehat{g}(t), \widehat{h}(t))$ and $t \geq t'$. We can therefore apply the assertions of Remark 2.1.4 and use the fact that $E_{t,\phi}\widehat{\tau} < \infty$ to obtain that $E_{t,\phi}\tau_* \leq E_{t,\phi}\widehat{\tau} < \infty$ holds when the starting time $t$ satisfies $t > t' \vee t_*$.

**Example 2.3.2 (Quickest disorder detection problem.).** Suppose that in the setting of Example 2.1.2 the observable continuous process $X \equiv Y^H = (Y_t^H)_{t \geq 0}$ is given by $Y_t^H = (t-\theta)^+\rho + B_t^H$, where $B^H = (B_t^H)_{t\geq 0}$ is a fractional Brownian motion with parameter $H \in (1/2, 1)$ independent of $\theta$, and $\rho > 0$ is a constant. Let the likelihood ratio process $\Phi$ be defined as in (2.1.5), where the process $L$ is given by (2.3.4). Therefore, by using the same reasoning as in Example 2.3.1, we obtain that the process $\Phi$ satisfies the stochastic differential equation in (2.1.1) with $\eta(t,\phi)$ and $\zeta(t,\phi)$ as in Example 2.1.2, where $\mu'(t) = \rho\sqrt{c_1}t^{1/2-H}$ for all $t \geq 0$. Hence, the analysis from the previous section can be applied for the drift rate $\mu'(t) \to 0$ when $1/2 < H < 1$ as $t \to \infty$.

Let us fix a starting time $t \geq 0$ and define the deterministic time change $\beta(t,s)$ and its inverse $\gamma(t,s)$ as in (2.3.5) for all $s \geq 0$. By using the expression in (2.1.5) we get that $\Phi_s \geq \Phi_0 e^{\lambda s} L_s$ holds for all $s \geq 0$. Therefore, if we define the stopping time $\widetilde{\tau}$ as

$$\widetilde{\tau} = \inf\{s \geq 0 \,|\, \Phi_0\, e^{\lambda(t+s)} L_{t+s} \geq \widehat{h}(t+s)\}, \tag{2.3.19}$$

we have that $\widehat{\tau} \leq \widetilde{\tau}$ holds, where $\widehat{\tau}$ is defined in (2.1.20). In order to simplify further notations, we define the process $\widetilde{\Phi} = (\widetilde{\Phi}_s)_{s\geq 0}$ by $\widetilde{\Phi}_s = \Phi_0 e^{\lambda\gamma(t,s)} L_{\gamma(t,s)}$ for $s \geq 0$. Since $L$ has the form of (2.3.4), by applying the time-change formula for Itô integrals in [83; Theorems 8.5.1 and 8.5.7], we obtain

$$\widetilde{\Phi}_s = \widetilde{\Phi}_0 \, \exp\left( \widetilde{B}_s - \frac{s}{2} + \lambda(\gamma(t,s) - t) + \int_0^s \frac{\widetilde{\Phi}_u}{1 + \widetilde{\Phi}_u}\, du \right), \tag{2.3.20}$$

where the process $\widetilde{B} = (\widetilde{B}_s)_{s\geq 0}$ defined in (2.3.6) is a standard Brownian motion. Therefore, by using the definition of $\widetilde{\tau}$ in (2.3.19) and taking into consideration the time change, the stopping time $\beta(t,\widetilde{\tau})$ can be represented as

$$\beta(t,\widetilde{\tau}) = \inf\left\{ s \geq 0 \,\middle|\, \widetilde{B}_s - \frac{s}{2} + \lambda(\gamma(t,s)-t) + \int_0^s \frac{\widetilde{\Phi}_u}{1+\widetilde{\Phi}_u} du + \ln \widetilde{\Phi}_0 \geq \ln \widehat{h}(\gamma(t,s)) \right\}. \quad (2.3.21)$$

Since $\gamma(t,s) \to \infty$ as $t \to \infty$, it follows from (2.2.12) that for any $\varepsilon > 0$ there exists $t^* > 0$ large enough such that the inequalities

$$\frac{\lambda}{c} < \widehat{h}(\gamma(t,s)) < \frac{\lambda}{c} + \varepsilon \quad (2.3.22)$$

hold for all $t > t^*$ and $s \geq 0$.

Let us now fix an arbitrary $\varepsilon > 0$ and assume from now on that $t > t^*$. By using the fact that $\widetilde{\Phi}$ is a nonnegative process, we obtain from (2.3.5) that the inequalities

$$\lambda(\gamma(t,s)-t) + \int_0^s \frac{\widetilde{\Phi}_u}{1+\widetilde{\Phi}_u} du \geq \lambda(\gamma(t,s)-t) \geq \lambda \left( \frac{s(2-2H)}{c_1 \rho^2} \right)^{1/(2-2H)} \quad (2.3.23)$$

hold for all $s \geq 0$. Define the random variable $\Delta_t$ as

$$\Delta_t = \sup_{s \geq 0} \left( s + \ln \left( \frac{\lambda + c\varepsilon}{c\, \widetilde{\Phi}_0} \right) - \lambda \left( \frac{s(2-2H)}{c_1 \rho^2} \right)^{1/(2-2H)} + \frac{s}{2} \right), \quad (2.3.24)$$

and notice that it follows from the inequalities in (2.3.22) and (2.3.23) that

$$\ln \left( \frac{\widehat{h}(\gamma(t,s))}{\widetilde{\Phi}_0} \right) - \lambda(\gamma(t,s)-t) - \int_0^s \frac{\widetilde{\Phi}_u}{1+\widetilde{\Phi}_u} du + \frac{s}{2} < \Delta_t - s \quad (2.3.25)$$

holds for all $s \geq 0$. Subsequently, we obtain from (2.3.21) that $\beta(t,\widetilde{\tau}) \leq \tau''$, where we set

$$\tau'' = \inf\{ s \geq 0 \,|\, \widetilde{B}_s \leq \Delta_t - s \}, \quad (2.3.26)$$

for any $t > t^*$. Moreover, by introducing the event $A = \{\omega \in \Omega \,|\, \widetilde{\Phi}_0 < \widehat{h}(t)\}$, we also obtain that $\beta(t,\widetilde{\tau}) = 0$ on $\Omega \setminus A$, and hence, we conclude that $\beta(t,\widetilde{\tau}) \leq \tau'' I(A)$ holds. Since we have that $\Delta_t > 0$ on the event $A$ and $\Delta_t < \infty$ ($P_{t,\phi}$-a.s.), for $1/2 < H < 1$, we get that $\tau'' I(A)$ has polynomial moments of all orders (see, e.g. [107; Chapter IV]). Therefore, it follows from the fact that $\beta(t,\widehat{\tau}) \leq \beta(t,\widetilde{\tau}) \leq \tau'' I(A)$ holds and the structure of the time change in (2.3.5) that $E_{t,\phi} \widehat{\tau} \leq E_{t,\phi} \gamma(t,\tau'') - t < \infty$ is satisfied.

Let us finally show that $\partial_t V(t,\phi;\widehat{h}(t)) > 0$ holds for all $\phi \in (0,\widehat{h}(t))$ and $t > 0$. For this purpose, differentiating the expression in (2.1.30) and using the expressions in (2.1.32) and (2.2.9), we get

$$\partial_t V(t,\phi;\widehat{h}(t)) = \int_\phi^{\widehat{h}(t)} \frac{\partial}{\partial t} \left( \frac{C(t)}{(y+1)^2} \int_0^y \frac{(1+x)}{x} \exp\Big( -\Lambda(t)(H(y)-H(x)) \Big) dx \right) dy \quad (2.3.27)$$

$$= \int_\phi^{\widehat{h}(t)} \frac{cy}{\lambda(y+1)^2} \frac{\partial}{\partial t} \left( 1 - \frac{Q(-\Lambda(t)-1, \Lambda(t)/y)}{\Lambda(t)+1} \right) dy$$

for all $\phi < \widehat{h}(t)$ and $t > 0$. Note that we also have

$$x^{-a}e^{x}\Gamma(a,x) = x^{-a}e^{x}\int_{x}^{\infty} e^{-u}u^{a-1}du = \int_{0}^{\infty} e^{-xu}(u+1)^{a-1}du, \tag{2.3.28}$$

for $a < 0$ and $x > 0$. It is shown by differentiation of the expressions in (2.3.28) that the function $x^{-a}e^{x}\Gamma(a,x)$ is decreasing in $x$ and increasing in $a$ (see, e.g. [46; Section 2.5] for similar results). Hence, the function

$$\frac{y}{x+1}Q\Big(-x-1, \frac{x}{y}\Big) = y\Big(\frac{x}{y}\Big)^{x+1}e^{x/y}\Gamma\Big(-x-1, \frac{x}{y}\Big) \tag{2.3.29}$$

is decreasing in $x$ for $x, y > 0$, where the functions $Q(z,y)$ and $\Gamma(z,y)$ are defined in (2.2.10). Recall that, for $1/2 < H < 1$, the function $\mu'(t)$ is decreasing, so that $\Lambda(t)$ is increasing in $t$. Hence, by using the formulas from (2.3.29), we obtain from the expressions in (2.3.27) that $V(t,\phi;\widehat{h}(t))$ is increasing in $t$, which leads to $\partial_t V(t,\phi;\widehat{h}(t)) > 0$ for all $\phi \in (0,\widehat{h}(t))$ and $t > 0$. We can therefore apply the assertions of Remark 2.1.4 and use the fact that $E_{t,\phi}\widehat{\tau} < \infty$ to conclude that $E_{t,\phi}\tau_* \le E_{t,\phi}\widehat{\tau} < \infty$, when the starting time $t$ satisfies $t > t^*$.

## 2.4. Appendix

Let us now prove the verification assertion stated in Theorem 2.1.3 above.

*Proof.* In order to verify the assertions stated above, let us denote by $V(t,\phi)$ the right-hand side of the expression in (2.1.15). Then, using the fact that the function $V(t,\phi)$ satisfies the conditions of (2.1.11)-(2.1.13) by construction, we can apply the local time-space formula from Peskir [85] (see also [90; Chapter II, Section 3.5] for a summary of the related results and further references) to obtain

$$\begin{aligned} V(t+u,\Phi_{t+u}) + \int_0^u F(\Phi_{t+s})\,ds &= V(t,\phi) + M_u + K_u \\ &+ \int_0^u (\mathbb{L}V + F)(t+s,\Phi_{t+s})\, I\big(\Phi_{t+s} \neq g_*(t+s), \Phi_{t+s} \neq h_*(t+s)\big)\,ds \end{aligned} \tag{2.4.1}$$

for all $t \ge 0$, where the process $M = (M_u)_{u \ge 0}$ defined by

$$M_u = \int_0^u V_\phi(t+s,\Phi_{t+s})\,\zeta(t+s,\Phi_{t+s})\,I\big(\Phi_{t+s} \neq g_*(t+s), \Phi_{t+s} \neq h_*(t+s)\big)\,d\overline{B}_s \tag{2.4.2}$$

is a continuous local martingale with respect to the probability measure $P_{t,\phi}$. Here, the process $K = (K_u)_{u \geq 0}$ is given by

$$K_u = \frac{1}{2} \int_0^u \Delta_\phi V(t+s, g_*(t+s))\, I\big(\Phi_{t+s} = g_*(t+s)\big)\, d\ell_s^{g_*} \\ + \frac{1}{2} \int_0^u \Delta_\phi V(t+s, h_*(t+s))\, I\big(\Phi_{t+s} = h_*(t+s)\big)\, d\ell_s^{h_*} \tag{2.4.3}$$

where $\Delta_\phi V(t+s, g_*(t+s)) = V_\phi(t+s, g_*(t+s)+) - V_\phi(t+s, g_*(t+s)-)$, $\Delta_\phi V(t+s, h_*(t+s)) = V_\phi(t+s, h_*(t+s)+) - V_\phi(t+s, h_*(t+s)-)$, and the processes $\ell^{g_*} = (\ell_u^{g_*})_{u \geq 0}$ and $\ell^{h_*} = (\ell_u^{h_*})_{u \geq 0}$ defined by

$$\ell_u^{g_*} = P_{t,\phi} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(g_*(t+s) - \varepsilon < \Phi_{t+s} < g_*(t+s) + \varepsilon\big)\, \zeta^2(t+s, \Phi_{t+s})\, ds \tag{2.4.4}$$

and

$$\ell_u^{h_*} = P_{t,\phi} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(h_*(t+s) - \varepsilon < \Phi_{t+v} < h_*(t+s) + \varepsilon\big)\, \zeta^2(t+s, \Phi_{t+s})\, ds \tag{2.4.5}$$

are the local times of $\Phi$ at the curves $g_*(t)$ and $h_*(t)$, at which $V_\phi(t, \phi)$ may not exist. It follows from the concavity and continuous differentiability of the gain function $G(\phi)$ in (2.1.2), and the stopping time $\tau_*$ in (2.1.7), that the inequalities $\Delta_\phi V(t, g_*(t)) \leq 0$ and $\Delta_\phi V(t, h_*(t)) \leq 0$ should hold for all $t \geq 0$, so that the continuous process $K$ defined in (2.4.3) is non-increasing. We may therefore conclude that $K_u = 0$ can hold for all $u \geq 0$ if and only if the smooth-fit conditions of (2.1.14) are satisfied.

Using the assumption that the inequality in (2.1.13) holds for the function $G(\phi)$ with the boundaries $g_*(t)$ and $h_*(t)$, we conclude that $(\mathbb{L}V + F)(t, \phi) \geq 0$ holds for any $\phi \neq g_*(t)$ and $\phi \neq h_*(t)$. Moreover from the conditions in (2.1.10)-(2.1.12) the inequality $V(t, \phi) \leq G(\phi)$ holds for all $(t, \phi) \in [0, \infty)^2$. Thus, for any stopping time $\tau$ such that $E_{t,\phi}\tau < \infty$, the expression in (2.4.1) yields the inequalities

$$G(\Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s})ds - K_\tau \geq V(t+\tau, \Phi_{t+\tau}) + \int_0^\tau F(\Phi_{t+s})ds - K_\tau \\ \geq V(t, \phi) + M_\tau. \tag{2.4.6}$$

Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence of stopping times for the process $M$ such that $\tau_n = \inf\{s \geq 0 \mid |M_s| \geq n\}$. Taking the expectations with respect to the probability measure $P_{t,\phi}$ in (2.4.6), by means of the optional sampling theorem (see, e.g. [75; Chapter III, Theorem 3.6] or [63;

Chapter I, Theorem 3.22]), we get the inequalities

$$
\begin{aligned}
&E_{t,\phi}\Big[G(\Phi_{t+\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} F(\Phi_{t+s})\,ds - K_{\tau\wedge\tau_n}\Big] \\
&\geq E_{t,\phi}\Big[V(t+\tau\wedge\tau_n, \Phi_{t+\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} F(\Phi_{t+s})\,ds - K_{\tau\wedge\tau_n}\Big] \geq V(t,\phi) + E_{t,\phi}\,M_{\tau\wedge\tau_n} = V(t,\phi).
\end{aligned} \tag{2.4.7}
$$

Hence, letting $n$ go to infinity and using Fatou's lemma, we obtain

$$
\begin{aligned}
&E_{t,\phi}\Big[G(\Phi_{t+\tau}) + \int_0^{\tau} F(\Phi_{t+s})\,ds - K_{\tau}\Big] \\
&\geq E_{t,\phi}\Big[V(t+\tau, \Phi_{t+\tau}) + \int_0^{\tau} F(\Phi_{t+s})\,ds - K_{\tau}\Big] \geq V(t,\phi)
\end{aligned} \tag{2.4.8}
$$

for any stopping time $\tau$ such that $E_{t,\phi}\tau < \infty$ and $E_{t,\phi}K_\tau > -\infty$, and all $(t,\phi) \in [0,\infty)^2$, where $K_\tau = 0$ holds whenever the conditions of (2.1.14) are satisfied. By virtue of the structure of the stopping time in (2.1.7) and the conditions of (2.1.11), it is readily seen that the equalities in (2.4.6) hold with $\tau_*$ instead of $\tau$ when either $\phi \leq g_*(t)$ or $\phi \geq h_*(t)$, respectively.

Let us now show that the equalities are attained in (2.4.8) when $\tau_*$ replaces $\tau$ and the smooth-fit conditions of (2.1.14) hold for $g_*(t) < \phi < h_*(t)$. By virtue of the fact that the function $V(t,\phi)$ and the boundaries $g_*(t)$ and $h_*(t)$ solve the partial differential equation in (2.1.9) and satisfy the conditions in (2.1.10) and (2.1.14), it follows from the expression in (2.4.1) and the structure of the stopping time in (2.1.7) that

$$
\begin{aligned}
&G(\Phi_{t+\tau_*\wedge\tau_n}) + \int_0^{\tau_*\wedge\tau_n} F(\Phi_{t+s})\,ds \\
&\geq V(t+\tau_*\wedge\tau_n, \Phi_{t+\tau_*\wedge\tau_n}) + \int_0^{\tau_*\wedge\tau_n} F(\Phi_{t+s})\,ds = V(t,\phi) + M_{\tau_*\wedge\tau_n}
\end{aligned} \tag{2.4.9}
$$

holds for $g_*(t) < \phi < h_*(t)$. Hence, taking expectations and letting $n$ go to infinity in (2.4.9), using the assumptions that $G(\phi)$ is bounded and the integral in (2.1.16) is of finite expectation, we apply the Lebesgue dominated convergence theorem to obtain the equality

$$
E_{t,\phi}\Big[G(\Phi_{t+\tau_*}) + \int_0^{\tau_*} F(\Phi_{t+s})\,ds\Big] = V(t,\phi) \tag{2.4.10}
$$

for all $(t,\phi) \in [0,\infty)^2$. We may therefore conclude that the function $V(t,\phi)$ coincides with the value function $V_*(t,\phi)$ of the optimal stopping problem in (2.1.2) whenever the smooth-fit conditions of (2.1.14) hold.

In order to prove the uniqueness of the value function $V_*(t,\phi)$ and the boundaries $g_*(t)$ and $h_*(t)$ as solutions to the free-boundary problem in (2.1.9)-(2.1.13) with the smooth-fit conditions of (2.1.14), let us assume that there exist other continuous boundaries of bounded variation $\widetilde{g}(t)$ and $\widetilde{h}(t)$ such that $0 \le \widetilde{g}(t) < c' < \widetilde{h}(t) \le \infty$ holds. Then, define the function $\widetilde{V}(t,\phi)$ as in (2.1.15) with $\widetilde{V}(t,\phi;\widetilde{g}(t),\widetilde{h}(t))$ satisfying (2.1.9)-(2.1.14) and the stopping time $\widetilde{\tau}$ as in (2.1.7) with $\widetilde{g}(t)$ and $\widetilde{h}(t)$ instead of $g_*(t)$ and $h_*(t)$, respectively, such that $E_{t,\phi}\widetilde{\tau} < \infty$. Following the arguments from the previous part of the proof and using the fact that the function $\widetilde{V}(t,\phi)$ solves the partial differential equation in (2.1.9) and satisfies the conditions of (2.1.10) and (2.1.14) with $\widetilde{g}(t)$ and $\widetilde{h}(t)$ instead of $g(t)$ and $h(t)$ by construction, we apply the change-of-variable formula from [85] to get

$$\begin{aligned} &\widetilde{V}(t+u,\Phi_{t+u}) + \int_0^u F(\Phi_{t+s})\,ds = \widetilde{V}(t,\phi) + \widetilde{M}_u \\ &+ \int_0^u (\mathbb{L}\widetilde{V} + F)(t+s,\Phi_{t+s})\, I\big(\Phi_{t+s} \notin (\widetilde{g}(t+s),\widetilde{h}(t+s))\big)\,ds \end{aligned} \tag{2.4.11}$$

where the process $\widetilde{M} = (\widetilde{M}_u)_{u\ge 0}$ defined as in (2.4.2) with $\widetilde{V}_\phi(t,\phi)$ instead of $V_\phi(t,\phi)$ is a continuous local martingale with respect to the probability measure $P_{t,\phi}$. Thus, taking into account the structure of the stopping time $\widetilde{\tau}$, from (2.4.11) we obtain that

$$\begin{aligned} &G(\Phi_{t+\widetilde{\tau}\wedge\widetilde{\tau}_n}) + \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n} F(\Phi_{t+s})\,ds \\ &\ge \widetilde{V}(t+\widetilde{\tau}\wedge\widetilde{\tau}_n,\Phi_{t+\widetilde{\tau}\wedge\widetilde{\tau}_n}) + \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n} F(\Phi_{t+s})\,ds = \widetilde{V}(t,\phi) + \widetilde{M}_{\widetilde{\tau}\wedge\widetilde{\tau}_n} \end{aligned} \tag{2.4.12}$$

holds for $\widetilde{g}(t) < \phi < \widetilde{h}(t)$ and any localizing sequence $(\widetilde{\tau}_n)_{n\in\mathbb{N}}$ of $\widetilde{M}$. Hence, taking expectations and letting $n$ go to infinity in (2.4.12), using the assumptions that $G(\phi)$ and $F(\phi)$ are bounded and the integral in (2.1.16) taken up to $\widetilde{\tau}$ is of finite expectation, by means of the Lebesgue dominated convergence theorem, we have that the equality

$$E_{t,\phi}\bigg[G(\Phi_{t+\widetilde{\tau}}) + \int_0^{\widetilde{\tau}} F(\Phi_{t+s})\,ds\bigg] = \widetilde{V}(t,\phi) \tag{2.4.13}$$

is satisfied. Therefore, recalling the fact that $\tau_*$ is the optimal stopping time in (2.1.2) and comparing the expressions in (2.4.10) and (2.4.13), we see that the inequality $\widetilde{V}(t,\phi) \ge V(t,\phi)$ should hold for all $(t,\phi) \in [0,\infty)^2$.

We finally show that $\widetilde{g}(t)$ and $\widetilde{h}(t)$ should coincide with $g_*(t)$ and $h_*(t)$. By using the fact that $\widetilde{V}(t,\phi)$ and $V(t,\phi)$ satisfy (2.1.10)-(2.1.12), and $\widetilde{V}(t,\phi) \ge V(t,\phi)$ holds for all $(t,\phi) \in [0,\infty)^2$ we get that $g_*(t) \le \widetilde{g}(t)$ and $\widetilde{h}(t) \le h_*(t)$. Inserting $\tau_* \wedge \widetilde{\tau}_n$ into (2.4.11) in place of $u$ and using the assumptions that $G(\phi)$ is bounded and the appropriate integrals are of finite expectation, by means of the arguments similar to the ones above, we obtain

$$E_{t,\phi}\left[\widetilde{V}(t+\tau_*,\Phi_{t+\tau_*})+\int_0^{\tau_*}F(\Phi_{t+s})\,ds\right]=\widetilde{V}(t,\phi) \tag{2.4.14}$$
$$+\,E_{t,\phi}\int_0^{\tau_*}(\mathbb{L}\widetilde{V}+F)(t+s,\Phi_{t+s})\,I\big(\Phi_{t+s}\notin(\widetilde{g}(t+s),\widetilde{h}(t+s))\big)\,ds.$$

for all $(t,\phi)\in[0,\infty)^2$. Thus, since we have $\widetilde{V}(t,\phi)=V(t,\phi)=G(\phi)$ for $\phi=g_*(t)$ and $\phi=h_*(t)$, and $\widetilde{V}(t,\phi)\geq V(t,\phi)$, we see from the expressions in (2.4.10) and (2.4.14) that the inequality

$$E_{t,\phi}\int_0^{\tau_*}(\mathbb{L}\widetilde{V}+F)(t+s,\Phi_{t+s})\,I\big(\Phi_{t+s}\notin(\widetilde{g}(t+s),\widetilde{h}(t+s))\big)\,ds\leq 0, \tag{2.4.15}$$

should hold. Due to the assumption of continuity of $\widetilde{g}(t)$ and $\widetilde{h}(t)$ we may therefore conclude that $g_*(t)=\widetilde{g}(t)$ and $h_*(t)=\widetilde{h}(t)$, so that $\widetilde{V}(t,\phi)$ coincides with $V(t,\phi)$ for all $(t,\phi)\in[0,\infty)^2$.

□

# Chapter 3

# Quickest change-point detection problems for multidimensional Wiener processes

This chapter is based on joint work with Dr. Pavel V. Gapeev.

## 3.1. The problem formulation

Let $(\Omega, \mathcal{G}, P_{\vec{\pi}})$ be a probability space, $B = (B^1, \ldots, B^n)$ is an $n$-dimensional Wiener process with constantly correlated components, where $\vec{\pi}$ is an $n$-dimensional vector such that $\vec{\pi} = (\pi_1, \ldots, \pi_n) \in [0,1]^n$ and $n \in \mathbb{N}$. Denote $N := \{1, \ldots, n\}$ and let, for any $i \in N$, the nonnegative random variable $\theta_i$ be such that $P_{\vec{\pi}}(\theta_i = 0) = \pi_i$ and $P_{\vec{\pi}}(\theta_i > t \,|\, \theta_i > 0) = e^{-\lambda_i t}$ with $\lambda_i > 0$, for all $t \geq 0$. Let also $\theta_i$ be independent of $B^j$ for all $i, j \in N$, and $\theta_i$ be independent of $\theta_j$ for all $i \neq j \in N$. Assume that we observe the processes $X^i = (X^i_t)_{t \geq 0}$ satisfying the stochastic differential equation

$$dX^i_t = \mu_i \, I(\theta_i \leq t) \, dt + \nu_i \, dB^i_t \quad (X^i_0 = 0), \tag{3.1.1}$$

where $\mu_i, \nu_i > 0$ for $i \in N$. Let the functions $f_i : [0, \infty)^n \mapsto [0, \infty)$ be given for $i = 1, \ldots, m$, $m \in \mathbb{N}$, and denote $\vec{\theta} := (\theta_1, \ldots, \theta_n)$. Our aim is to find a stopping time of alarm $\tau_*$ with respect to the (observable) filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by all $X^i$ for $i \in N$, that is $\mathcal{F}_t = \sigma(X^i_s, i \in N \,|\, 0 \leq s \leq t)$, which is as close as possible to every function $f_j(\vec{\theta})$ for $j = 1, \ldots, m$. Specifically, the quickest change-point detection problem for a multidimensional Wiener process is to compute the Bayesian risk function

$$V_*(\vec{\pi}) = \inf_\tau \left( \sum_{i=1}^m \Big( b_i\, P_{\vec{\pi}}\big(\tau < f_i(\vec{\theta})\big) + c_i\, E_{\vec{\pi}}\big[(\tau - f_i(\vec{\theta}))^+\big] \Big) \right), \tag{3.1.2}$$

and find the optimal stopping time $\tau_*$ at which the infimum is attained in (3.1.2), where $b_i, c_i > 0$ are given constants for $i = 1, \ldots, m$. Here $P_{\vec{\pi}}(\tau < f_i(\vec{\theta}))$ represents the probability of false alarm and $E_{\vec{\pi}}[(\tau - f_i(\vec{\theta}))^+]$ represents the average delay of detecting the function $f_i(\vec{\theta})$ for $i = 1, \ldots, m$.

By using standard arguments (see [105; pages 195-197]) we get that

$$\begin{aligned} P_{\vec{\pi}}\big(\tau < f_i(\vec{\theta})\big) = E_{\vec{\pi}}\big[I(\tau < f_i(\vec{\theta}))\big] &= E_{\vec{\pi}}\big[E_{\vec{\pi}}\big[I(\tau < f_i(\vec{\theta}))\,\big|\,\mathcal{F}_\tau\big]\big] \\ &= E_{\vec{\pi}}\big[P_{\vec{\pi}}\big(\tau < f_i(\vec{\theta})\,\big|\,\mathcal{F}_\tau\big)\big], \end{aligned} \tag{3.1.3}$$

and

$$\begin{aligned} E_{\vec{\pi}}\big[(\tau - f_i(\vec{\theta}))^+\big] &= E_{\vec{\pi}} \int_0^\tau I(f_i(\vec{\theta}) \le t)\, dt = E_{\vec{\pi}} \int_0^\infty I(f_i(\vec{\theta}) \le t, t \le \tau)\, dt \\ &= E_{\vec{\pi}} \int_0^\infty E_{\vec{\pi}}\big[I(f_i(\vec{\theta}) \le t, t \le \tau)\,\big|\,\mathcal{F}_t\big]\, dt = E_{\vec{\pi}} \int_0^\tau P_{\vec{\pi}}\big(f_i(\vec{\theta}) \le t\,\big|\,\mathcal{F}_t\big)\, dt, \end{aligned} \tag{3.1.4}$$

holds for $i = 1, \ldots, m$, where $I(\cdot)$ denotes the indicator function.

**3.1.1. Sufficient statistics and filtering equations** Let us now reduce the original problem of (3.1.2) to an optimal stopping problem for a multidimensional (strong) Markov process. We define the posterior probability processes $(\Pi_t^{*,i})_{t\ge 0}$ as $\Pi_t^{*,i} = P_{\vec{\pi}}(f_i(\vec{\theta}) \le t | \mathcal{F}_t)$ for $t \ge 0$ and $i = 1, \ldots, m$, and observe that it follows from (3.1.3)-(3.1.4) that the Bayesian risk function in (3.1.2) can be represented as

$$V_*(\vec{\pi}) = \inf_\tau E_{\vec{\pi}} \left[ \sum_{i=1}^m b_i\, (1 - \Pi_\tau^{*,i}) + c_i \int_0^\tau \Pi_t^{*,i}\, dt \right]. \tag{3.1.5}$$

For each $J \subseteq N$, we define the posterior probability process $(\Pi_t^J)_{t\ge 0}$ as $\Pi_t^J := P_{\vec{\pi}}(\bigcap_{i\in J}\{\theta_i \le t\}|\mathcal{F}_t)$. In order to simplify the notation, we will order the processes $\Pi^J$ by choosing an arbitrary integer-valued bijection $O : \{1, \ldots, 2^n\} \mapsto 2^N$ from the set of integers $\{1, \ldots, 2^n\}$ to the power set (i.e. the set of all subsets) of $N$ and denoting by $\vec{\Pi} = (\Pi^1, \ldots, \Pi^{2^n})$ the $2^n$-dimensional process with components given by $\Pi^j = \Pi^{O(j)}$ for $j = 1, \ldots, 2^n$. Let us now assume that the functions $f_i$ are such that $\Pi^{*,i}$ is of the form

$$\Pi_t^{*,i} \equiv P_{\vec{\pi}}(f_i(\vec{\theta}) \le t\,|\,\mathcal{F}_t) = \sum_{j=1}^{2^n} a_{ji}\, \Pi_t^j, \tag{3.1.6}$$

for some constants $a_{ji}$, for all $t \geq 0$ and every $i = 1, \ldots, m$ and $j = 1, \ldots, 2^n$ (examples of such functions $f_i$ will be provided in Section 3.3). In what follows, we prove that the process $\vec{\Pi}$ has the strong Markov property.

We introduce the probability measure $P^J(\cdot) := P_{\vec{\pi}}(\cdot \mid \bigcap_{i\in J}\{\theta_i = 0\} \bigcap \bigcap_{j \in N\setminus J}\{\theta_j = \infty\})$ and the (weighted) density process $(Z_t^J)_{t\geq 0}$ as

$$Z_t^J := \exp\left(t \sum_{i\in J} \lambda_i\right) \frac{d(P^J|\mathcal{F}_t)}{d(P^{\varnothing}|\mathcal{F}_t)}, \tag{3.1.7}$$

for $J \subseteq N$, where $P^J \,|\, \mathcal{F}_t$ denotes the restriction of the measure $P^J$ to $\mathcal{F}_t$. Let the correlation matrix $\Sigma = (\sigma_{ij})_{i,j\in N}$ of the $n$-dimensional process $X = (X^1, \ldots, X^n)$ be given by

$$\sigma_{ij} = \frac{\langle X^i, X^j \rangle_1}{\nu_i \nu_j}, \tag{3.1.8}$$

for $i, j \in N$, and denote the entries of the inverse correlation matrix as $\Sigma^{-1} = (\nu_{ij})_{i,j\in N}$, which exists because $\Sigma$ is a symmetric and positive definite matrix. We can express the density process from (3.1.7) in terms of processes adapted to the observable filtration, and these processes will be linear combinations of the observed processes $X^i$ for $i \in N$, as the following lemma shows. The arguments are essentialy based on the application of the Girsanov theorem for a multidimensional Wiener process.

**Lemma 3.1.1.** *We have*

$$Z_t^J = \exp\left(\sum_{i\in J} \lambda_i t + \sum_{i \in J} Y_t^i - \frac{1}{2}\left(\sum_{i,j\in J} \frac{\mu_i}{\nu_i}\frac{\mu_j}{\nu_j}\nu_{lj}\right) t\right), \tag{3.1.9}$$

*for $J \subseteq N$, where we have defined*

$$Y_t^i := \frac{\mu_i}{\nu_i} \sum_{j=1}^{n} \frac{\nu_{ij}}{\nu_j} X_t^j, \tag{3.1.10}$$

*for $i \in N$ and $t \geq 0$.*

*Proof.* See Appendix. □

Let us now define the process $(\Phi_t^{\alpha,L})_{t\geq 0}$ recursively as

$$\Phi_t^{\alpha,L} := \lambda_{\alpha_k} \int_0^t \Phi_u^{[\alpha_1,\ldots,\alpha_{k-1}],L} \frac{Z_t^{K\cup L}}{Z_u^{K\cup L}} du, \quad \Phi_t^{\varnothing,L} := \pi^L Z_t^L, \quad \Phi^{\varnothing,\varnothing} \equiv 1, \tag{3.1.11}$$

for $K, L \subseteq N$ such that $K \neq \varnothing, K \cap L = \varnothing$, and any permutation $\alpha := [\alpha_1, \ldots, \alpha_k] \in \mathrm{Perm}(K)$, where $\mathrm{Perm}(K)$ denotes the set of all permutations of $K$, and $\pi^L := \prod_{l\in L} \pi_l$. The process $\Phi^{\alpha,L}$

can be regarded as a (weighted) likelihood ratio process corresponding to the event $\bigcap_{l\in L}\{\theta_l = 0\}$ $\bigcap\{0 < \theta_{\alpha_1} \leq \cdots \leq \theta_{\alpha_k} \leq t\} \bigcap \bigcap_{i\in N\setminus(K\cup L)}\{t < \theta_i\}$ since it can be written in the form

$$\Phi_t^{\alpha,L} = \pi^L \exp\left(t\sum_{i\in N}\lambda_i\right)\int_{A_t}\frac{d(P^{u,L}|\mathcal{F}_t)}{d(P^{\varnothing}|\mathcal{F}_t)}\prod_{i=1}^{k+r}\lambda_{\alpha_i}e^{-u_i\lambda_{\alpha_i}}d^{k+r}\vec{u}, \tag{3.1.12}$$

where $r$ is the number of elements of the set $N\setminus(K\cup L)$ and

$$\{\alpha_{k+1},\ldots,\alpha_{k+r}\} = N\setminus(K\cup L), \tag{3.1.13}$$

$$A_t = \{x\in\mathbb{R}^{k+r} \mid 0 < x_1 \leq \cdots \leq x_k \leq t \text{ and } t < x_{k+i} \text{ for } i = 1,\ldots,r\}, \tag{3.1.14}$$

$$P^{u,L}(\cdot) = P_{\vec{\pi}}(\cdot \mid \textstyle\bigcap_{i\in L}\{\theta_i = 0\}\bigcap\bigcap_{j=1,\ldots,k+r}\{\theta_{\alpha_j} = u_j\}), \tag{3.1.15}$$

for $\vec{u} = (u_1,\ldots,u_{k+r}) \in \mathbb{R}^{k+r}$ and $t \geq 0$. Therefore, the processes $(\Psi_t^{J,L})_{t\geq 0}$ and $(\Psi_t^J)_{t\geq 0}$ defined as

$$\Psi_t^{J,L} := \sum_{J\subseteq K\subseteq N\setminus L}\ \sum_{\alpha\in\mathrm{Perm}(K)}\Phi_t^{\alpha,L} \quad\text{and}\quad \Psi_t^J := \sum_{L_1\subseteq N\setminus J, L_2\subseteq J}\Psi_t^{J\setminus L_2, L_1\cup L_2}, \tag{3.1.16}$$

for $J, L \subseteq N$ such that $J\cap L = \varnothing$, can be regarded as a (weighted) likelihood ratio processes corresponding to the events $\{(\theta_l = 0)_{l\in L}\}\bigcap\{(0 < \theta_i \leq t)_{i\in J}\}\bigcap\{(0 < \theta_i)_{i\in N\setminus(J\cup L)}\}$ and $\{(\theta_i \leq t)_{i\in J}\}$, respectively. Hence, by using the generalized Bayes formula from [75; Theorem 7.23], we obtain that the posterior probability process $(\Pi_t^J)_{t\geq 0}$ takes the form

$$\Pi_t^J = \frac{\Psi_t^J}{\Psi_t^{\varnothing}}, \tag{3.1.17}$$

for $J\subseteq N$.

It follows from the expression in (3.1.9) that $Z^J$ satisfies the following stochastic differential equation

$$dZ_t^J = Z_t^J\left(\sum_{i\in J}\lambda_i\,dt + \sum_{i\in J}dY_t^i\right), \tag{3.1.18}$$

for $J\subseteq N$. By using Itô's formula, from (3.1.18) and (3.1.11) we get

$$d\Phi_t^{\alpha,L} = \left(\lambda_{\alpha_k}\Phi_t^{[\alpha_1,\ldots,\alpha_{k-1}],L} + \sum_{i\in K\cup L}\lambda_i\Phi_t^{\alpha,L}\right)dt + \sum_{i\in K\cup L}\Phi_t^{\alpha,L}\,dY_t^i, \tag{3.1.19}$$

$$d\Phi_t^{\varnothing,L} = \sum_{i\in L}\lambda_i\Phi_t^{\varnothing,L}\,dt + \sum_{i\in L}\Phi_t^{\varnothing,L}\,dY_t^i, \tag{3.1.20}$$

for $K, L \subseteq N$ such that $K \neq \varnothing$, $K \cap L = \varnothing$ and any $\alpha := [\alpha_1, \dots, \alpha_k] \in \mathrm{Perm}(K)$. Therefore, by using (3.1.16), we further obtain

$$d\Psi_t^{J,L} = \left( \sum_{i \in J} \lambda_i \Psi_t^{J\setminus\{i\},L} + \sum_{i \notin J} \lambda_i \Psi_t^{J,L} \right) dt + \sum_{i \in J\cup L} \Psi_t^{J,L} \, dY_t^i + \sum_{i \notin J \cup L} \Psi_t^{J\cup\{i\},L} \, dY_t^i, \tag{3.1.21}$$

and, by aggregating, we get

$$d\Psi_t^{J} = \left( \sum_{i \in J} \lambda_i \Psi_t^{J\setminus\{i\}} + \sum_{i \notin J} \lambda_i \Psi_t^{J} \right) dt + \sum_{i \in J} \Psi_t^{J} \, dY_t^i + \sum_{i \notin J} \Psi_t^{J\cup\{i\}} \, dY_t^i, \tag{3.1.22}$$

for $J, L \subseteq N$ such that $J \cap L = \varnothing$. Hence, by applying Itô's formula to (3.1.17), we conclude that

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \Big( \Pi_t^{J\setminus\{i\}} - \Pi_t^J \Big) \, dt + \sum_{i \in N} \Big( \Pi_t^{J\cup\{i\}} - \Pi_t^J \Pi_t^{\{i\}} \Big) \Big( dY_t^i - \sum_{j=1}^n \Pi_t^{\{j\}} \, d\langle Y^i, Y^j \rangle_t \Big), \tag{3.1.23}$$

for $J \subseteq N$.

Furthermore, we get from (3.1.10) that

$$\langle Y^i, Y^j \rangle_t = \frac{\mu_i \mu_j}{\nu_i \nu_j} t \sum_{k,l=1}^n \nu_{ik} \nu_{jl} \sigma_{kl} = \frac{\mu_i \mu_j}{\nu_i \nu_j} \nu_{ji} t, \tag{3.1.24}$$

and, therefore, we can write the equation in (3.1.23) as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \Big( \Pi_t^{J\setminus\{i\}} - \Pi_t^J \Big) dt + \sum_{i \in N} \Big( \Pi_t^{J\cup\{i\}} - \Pi_t^J \Pi_t^{\{i\}} \Big) \sum_{j=1}^n \frac{\mu_i}{\nu_i} \frac{\nu_{ji}}{\nu_j} \Big( dX_t^j - \mu_j \Pi_t^{\{j\}} dt \Big). \tag{3.1.25}$$

Defining the innovation processes $\overline{B}^i = (\overline{B}_t^i)_{t\geq 0}$, $i \in N$, by

$$\overline{B}_t^i := \frac{X_t^i}{\nu_i} - \frac{\mu_i}{\nu_i} \int_0^t \Pi_s^{\{i\}} \, ds, \tag{3.1.26}$$

and using the Lévy's characterization theorem (see, e.g. [75; Chapter IV, Theorem 4.1]), we see that $\overline{B}^i$ is a standard Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ under the probability measure $P_{\overline{\pi}}$. Moreover, we have $\langle \overline{B}^i, \overline{B}^j \rangle_t = \sigma_{ij} t$ for all $t \geq 0$ and every $i, j \in N$, and we can rewrite (3.1.25) as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \Big( \Pi_t^{J\setminus\{i\}} - \Pi_t^J \Big) \, dt + \sum_{i \in N} \Big( \Pi_t^{J\cup\{i\}} - \Pi_t^J \Pi_t^{\{i\}} \Big) \sum_{j=1}^n \frac{\mu_i}{\nu_i} \nu_{ji} \, d\overline{B}_t^j. \tag{3.1.27}$$

Alternatively, by defining the processes $\widehat{B}^i = (\widehat{B}_t^i)_{t\geq 0}$, $i \in N$, as

$$\widehat{B}_t^i := \frac{Y_t^i - \sum_{j=1}^n \int_0^t \Pi_s^{\{j\}} \, d\langle Y^i, Y^j \rangle_s}{\sqrt{\langle Y^i, Y^i \rangle_t}} \sqrt{t} = \left( Y_t^i - \sum_{j=1}^n \int_0^t \Pi_s^{\{j\}} \frac{\mu_i \mu_j}{\nu_i \nu_j} \nu_{ji} \, ds \right) \frac{\nu_i}{\mu_i \sqrt{\nu_{ii}}}, \tag{3.1.28}$$

and using the Lévy's characterization theorem we see that $\widehat{B}^i$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ under the probability measure $P_{\vec{\pi}}$. Moreover, by (3.1.24), we have

$$\langle \widehat{B}^i, \widehat{B}^j \rangle_t = \frac{\nu_{ji}}{\sqrt{\nu_{ii}\nu_{jj}}}\, t, \tag{3.1.29}$$

for all $i, j \in N$ and $t \geq 0$, and we can rewrite (3.1.23) as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \left( \Pi_t^{J \setminus \{i\}} - \Pi_t^J \right) dt + \sum_{i \in N} \left( \Pi_t^{J \cup \{i\}} - \Pi_t^J \Pi_t^{\{i\}} \right) \frac{\mu_i \sqrt{\nu_{ii}}}{\nu_i}\, d\widehat{B}_t^i. \tag{3.1.30}$$

Therefore, by using either (3.1.27) or (3.1.30), we obtain that the process $\vec{\Pi}$ satisfies the conditions of [83; Chapter V, Theorem 5.2.1]) about the existence and uniqueness of strong solutions of stochastic differential equations, and thus, by virtue of [83; Chapter VII, Theorem 7.2.4], it has the strong Markov property with respect to its natural filtration which coincides with $(\mathcal{F}_t)_{t\geq 0}$. Moreover, since we have the representations

$$\Pi_t^J \equiv P_{\vec{\pi}}(\textstyle\bigcap_{i\in J}\{\theta_i \leq t\} \,|\, \mathcal{F}_t) = \displaystyle\sum_{J \subseteq K \subseteq N} P_{\vec{\pi}}(\textstyle\bigcap_{i\in K}\{\theta_i \leq t\} \bigcap \bigcap_{i \in N\setminus K}\{t < \theta_i\} \,|\, \mathcal{F}_t), \tag{3.1.31}$$

$$P_{\vec{\pi}}(\textstyle\bigcap_{i\in K}\{\theta_i \leq t\} \bigcap \bigcap_{i \in N\setminus K}\{t < \theta_i\} \,|\, \mathcal{F}_t) = \Pi_t^K - \displaystyle\sum_{i \in N\setminus K} \Pi_t^{K\cup\{i\}} + \sum_{i\neq j \in N\setminus K} \Pi_t^{K\cup\{i,j\}} + \tag{3.1.32}$$
$$\cdots + (-1)^{n-k-1} \sum_{i\in N\setminus K} \Pi_t^{N\cup\{i\}} + (-1)^{n-k}\Pi_t^N,$$

for $J, K \subseteq N$, where $k$ is the number of elements of $K$ and

$$\sum_{K\subseteq N} P_{\vec{\pi}}(\{(\theta_i \leq t)_{i\in K}\} \bigcap \{(t < \theta_i)_{i\in N\setminus K}\} | \mathcal{F}_t) = 1, \tag{3.1.33}$$

holds, it follows that the state space of the process $\vec{\Pi}$ is given by

$$\mathcal{D} := \Bigg\{ \vec{p} \in [0,1]^{2^n} \,\Bigg|\, \text{for some } \vec{q} \in [0,1]^{2^n} \text{ with } \sum_{j=1}^{2^n} q_j = 1 \tag{3.1.34}$$
$$\text{we have that } p_i = \sum_{O(i)\subseteq O(j)\subseteq N} q_j \quad \text{for } i = 1, \ldots, 2^n \Bigg\}.$$

Finally, by using (3.1.5)-(3.1.6) and the strong Markov property of the process $\vec{\Pi}$, we can reduce the problem of (3.1.2) to the Markovian optimal stopping problem

$$V_*(\vec{p}) = \inf_\tau E_{\vec{p}} \Bigg[ \sum_{j=1}^m b_j \Bigg( 1 - \sum_{i=1}^{2^n} a_{ij}\Pi_\tau^i \Bigg) + c_j \int_0^\tau \sum_{i=1}^{2^n} a_{ij}\, \Pi_t^i \, dt \Bigg], \tag{3.1.35}$$

where the infimum is taken over all stopping times $\tau$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ such that the integrals above have finite expectation, so that $E_{\vec{p}}\tau < \infty$ (see, e.g. [105; Chapter IV, Section 4] and [90; Chapter VI, Section 22]). Here, the process $\vec{\Pi}$ starts at some $\vec{p} \in \mathcal{D}$ under the probability measure $P_{\vec{p}}$. Notice that from the linearity of the representations in (3.1.31)-(3.1.32) it follows that the value function $V_*(\vec{p})$ is concave.

## 3.2. Main results

The main results of the paper are presented in this section. We obtain certain properties of the optimal stopping time and the optimal boundaries in the problem of (3.1.35). We also provide characterization of the optimal stopping boundary surface and value function $V_*$ as the unique solution to a multidimensional free boundary problem.

Let us first introduce some further notations. For any $j = 1, \ldots, 2^n$, we denote by $J$ the subset of $N$ corresponding to the index $j$, that is $J := O(j) \subseteq N$. For any set $K \subseteq N$, we denote the number of its elements by $|K|$, and $\lambda(K) := \sum_{k\in K} \lambda_k$.

**3.2.1. The structure of the optimal stopping time** Define the linear function $F^j(\vec{p})$ as

$$F^j(\vec{p}) = \sum_{i=1}^{2^n} f_{ji} p_i, \tag{3.2.1}$$

where the constants $f_{ji}$ are given by

$$f_{jj} = -\frac{1}{\lambda(J)}, \quad \text{if } J \neq \varnothing, \tag{3.2.2}$$

$$f_{ji} = -\frac{\prod_{k\in(J\setminus O(i))} \lambda_k}{\lambda(O(i))} \sum_{\alpha\in\mathrm{Perm}(J\setminus O(i))} \prod_{q=1}^{|J\setminus O(i)|} \frac{1}{\lambda(O(i)) + \sum_{r=1}^{q} \lambda_{\alpha_r}}, \quad \text{if } \varnothing \neq O(i) \subset J, \tag{3.2.3}$$

$$f_{ji} = 0, \quad \text{otherwise}, \tag{3.2.4}$$

for any $\vec{p} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. Applying Itô's formula to $F^j(\vec{\Pi}_\tau)$ and the optional sampling theorem (see, e.g. [75; Chapter III, Theorem 3.6] or [63; Chapter I, Theorem 3.22]), by using (3.1.30), we can see that

$$E_{\vec{p}}\big[F^j(\vec{\Pi}_\tau)\big] = F^j(\vec{p}) + E_{\vec{p}}\bigg[\int_0^\tau \Pi_t^j \, dt - \tau\bigg], \tag{3.2.5}$$

for any $\vec{p} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$, and for any stopping time $\tau$ such that $E_{\vec{p}}\tau < \infty$. Therefore, the optimal stopping problem (3.1.35) can be rewritten as

$$\overline{V}_*(\vec{p}) := V_*(\vec{p}) + \sum_{k=1}^{m} \left( \sum_{i=1}^{2^n} c_k a_{ik} F^i(\vec{p}) \right) - b_k = \inf_{\tau} E_{\vec{p}}\big[G(\vec{\Pi}_\tau) + c\tau\big], \tag{3.2.6}$$

where we have defined

$$G(\vec{p}) := \sum_{k=1}^{m} \left( \sum_{i=1}^{2^n} c_k a_{ik} F^i(\vec{p}) \right) - b_k a_{ik} p_i \quad \text{and} \quad c := \sum_{k=1}^{m} \sum_{i=1}^{2^n} c_k a_{ik}, \tag{3.2.7}$$

for $\vec{p} \in \mathcal{D}$. Note that we can conclude from (3.1.6) that the constants $a_{ji}$ satisfy

$$0 \leq \sum_{j=1}^{2^n} a_{ji} p_j \leq 1, \tag{3.2.8}$$

for $i = 1, \ldots, m$ and $\vec{p} \in \mathcal{D}$, and we obtain that $c \geq 0$, so that the optimal stopping problem in (3.2.6) is well-posed. Moreover, by using (3.2.1), we can rewrite $G$ as

$$G(\vec{p}) = \sum_{i=1}^{2^n} g_i p_i \quad \text{with} \quad g_i = \sum_{k=1}^{m} \left( \sum_{j=1}^{2^n} c_k a_{jk} f_{ji} \right) - b_k a_{ik}, \tag{3.2.9}$$

and from the concavity of $V_*(\vec{p})$ and the linearity of $F^j(\vec{p})$, $j = 1, \ldots, 2^n$, we also get that the value function $\overline{V}_*(\vec{p})$ is concave.

From the general optimal stopping theory for Markov processes (see, e.g. [90; Chapter I, Section 2.2]) and the form of the value function in (3.2.6), we know that the optimal stopping time in (3.1.35) is given by

$$\tau_* = \inf\big\{ s \geq 0 \,\big|\, \overline{V}_*(\vec{\Pi}_s) = G(\vec{\Pi}_s) \big\}, \tag{3.2.10}$$

whenever it exists.

Let us choose an integer $l$ such that $1 \leq l \leq 2^n$ and denote by $\vec{\Pi}^{-l}$ the process $\vec{\Pi}$ without its $l$-th component, and by $\vec{p_l}$ the vector $\vec{p} \in \mathcal{D}$ without its $l$-th component $p_l$. Assume that $g_l < 0$ (the case $g_l > 0$ can be considered similarly) and $G(\vec{p})$ achieves its minimum for all $\vec{p} \in \mathcal{D}$ such that $p_l = 1$. We see from (3.2.9) that the linear function $G(\vec{p})$ is decreasing in $p_l$, and by the concavity of $\overline{V}_*(\vec{p})$ and the fact that $\overline{V}_*(\vec{p}) = G(\vec{p})$ for all $\vec{p} \in \mathcal{D}$ such that $p_l = 1$, we get that the optimal stopping time from (3.2.10) is of the form

$$\tau_* = \inf\big\{ s \geq 0 \,|\, \Pi_s^l \geq b_*(\vec{\Pi}_s^{-l}) \big\}, \tag{3.2.11}$$

for some function $0 \leq b_*(\vec{p}_l) \leq 1$ and all $\vec{p} \in \mathcal{D}$. Finally, we may conclude from the fact that $G(\vec{p})$ is linear and $\overline{V}_*(\vec{p})$ is concave that the boundary $b_*(\vec{p}_l)$ is continuous and of bounded variation.

Summarising the facts proved above, we are now in a position to state the following result.

**Lemma 3.2.1.** *Let the posterior probability processes $\Pi^{*,i}$ be such that the expression in* (3.1.6) *holds. Assume there exists an integer $l$ such that $g_l < 0$ and $G(\vec{p})$ achieves its minimum for all $\vec{p} \in \mathcal{D}$ with $p_l = 1$, for some $l = 1, \ldots, 2^n$. Then, the optimal stopping time $\tau_*$ in the problems* (3.1.35) *and* (3.2.6) *is of the form* (3.2.11)*, whenever it exists, and the optimal stopping boundary $b_*(\vec{p}_l)$ is continuous and of bounded variation for $\vec{p} \in \mathcal{D}$.*

In what follows, we work under the assumptions of Lemma 3.2.1.

**3.2.2. The free-boundary problem** By means of standard arguments (see, e.g. [63; Chapter V, Section 5.1]), it can be seen from (3.1.30) that the infinitesimal operator $\mathbb{L}$ of the process $\vec{\Pi}$ is given by the expression

$$\mathbb{L} = \sum_{j=1}^{2^n} \sum_{i \in J} \lambda_i \left(p_{O^{-1}(J \setminus \{i\})} - p_j\right) \partial_{p_j} \tag{3.2.12}$$

$$+ \frac{1}{2} \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \sum_{k,l \in N} \frac{\mu_k \mu_l \nu_{kl}}{\nu_k \nu_l} (p_{O^{-1}(J \cup \{k\})} - p_j p_{O^{-1}(\{k\})})(p_{O^{-1}(O(i) \cup \{l\})} - p_i p_{O^{-1}(\{l\})}) \partial^2_{p_j p_i},$$

for all $\vec{p} \in \mathcal{D}$. In order to find analytic expressions for the unknown value function $\overline{V}_*(\vec{p})$ from (3.2.6) and the unknown boundary $b_*(\vec{p}_l)$ from (3.2.11), we will use results from the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [105; Chapter III, Section 8] and [90; Chapter IV, Section 8]). Specifically, we formulate the associated free boundary problem

$$(\mathbb{L}V)(\vec{p}) = -c \quad \text{for} \quad p_l < b(\vec{p}_l), \tag{3.2.13}$$

$$V(p_1, \ldots, p_{l-1}, b(\vec{p}_l)-, p_{l+1}, \ldots, p_{2^n}) = G(p_1, \ldots, p_{l-1}, b(\vec{p}_l), p_{l+1}, \ldots, p_{2^n}), \tag{3.2.14}$$

$$V(\vec{p}) = G(\vec{p}) \quad \text{for} \quad p_l > b(\vec{p}_l), \tag{3.2.15}$$

$$V(\vec{p}) < G(\vec{p}) \quad \text{for} \quad p_l < b(\vec{p}_l), \tag{3.2.16}$$

$$(\mathbb{L}V)(\vec{p}) > -c \quad \text{for} \quad p_l > b(\vec{p}_l), \tag{3.2.17}$$

for some $0 \leq b(\vec{p}_l) \leq 1$, where the *instantaneous stopping* condition of (3.2.14) is satisfied at $b(\vec{p}_l)$ for all $\vec{p}_l \in [0,1]^{2^n-1}$ such that $\vec{p} \in \mathcal{D}$. Since the problem in (3.2.13)-(3.2.17) may admit multiple solutions, we need to use some additional conditions which would specify the appropriate solution, and thus provide the value function and the optimal stopping boundary for the initial problem of (3.2.6) (and (3.1.35)). Therefore, we will assume that

$$\partial_{p_l} V(p_1, \ldots, p_{l-1}, b(\vec{p}_l)-, p_{l+1}, \ldots, p_{2^n}) = g_l \quad (\textit{smooth fit}), \tag{3.2.18}$$

holds for all $\vec{p} \in \mathcal{D}$. Note that the *smooth-fit* conditions of (3.2.18) are naturally used for the value function at the optimal stopping boundary, whenever the general payoff function $G(\vec{p})$ is continuously differentiable in $p_l$ at the boundary $b(\vec{p}_l)$ (see [90; Chapter IV, Section 9] for an extensive overview).

We further search for analytic solutions of the elliptic-type free boundary problem in (3.2.13)-(3.2.16) satisfying the conditions of (3.2.17)-(3.2.18) and such that the resulting boundary is continuous and of bounded variation. Since such free boundary problems cannot normally be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the variational inequalities arising in the context of optimal stopping problems have been extensively studied in the literature (see, e.g. Friedman [36], Bensoussan and Lions [14], Krylov [72], or Øksendal [83]). Although the necessary conditions for existence and uniqueness of such solutions in [36; Chapter XVI, Theorem 11.1], [72; Chapter V, Section 3, Theorem 14] with [72; Chapter VI, Section 4, Theorem 12], and [83; Chapter X, Theorem 10.4.1] can be verified by virtue of the properties of the coefficients of the process $\vec{\Pi}$, the application of these classical results would still have a rather inexplicit character.

We therefore continue with the following verification assertion related to the free boundary problem formulated above.

**Theorem 3.2.1.** *Assume that $V(\vec{p}; b_*(\vec{p}_l))$ together with $0 \leq b_*(\vec{p}_l) \leq 1$ form a solution of the free boundary problem of* (3.2.13)-(3.2.17)*, and the boundary $b_*(\vec{p}_l)$ is continuous and of bounded variation. Define the stopping time $\tau_*$ as the first exit time of the process $\Pi^l$ from the interval $[0, b_*(\vec{\Pi}^{-l}))$ as in* (3.2.11)*, and assume that $E_{\vec{p}}\tau_* < \infty$ holds for $\vec{p} \in \mathcal{D}$. Then, the value function $\overline{V}_*(\vec{p})$ takes the form*

$$\overline{V}_*(\vec{p}) = \begin{cases} V(\vec{p}; b_*(\vec{p}_l)), & \text{if } p_l < b_*(\vec{p}_l) \\ G(\vec{p}), & \text{if } p_l \geq b_*(\vec{p}_l) \end{cases} \tag{3.2.19}$$

*with*

$$V(\vec{p}; b_*(\vec{p}_l)) = E_{\vec{p}}\big[G(\vec{\Pi}_{\tau_*}) + c\tau_*\big], \tag{3.2.20}$$

*and the boundary $b_*(\vec{p}_l)$ is uniquely determined by the smooth-fit condition of (3.2.18).*

*Proof.* In order to verify the assertions stated above, let us denote by $V(\vec{p})$ the right-hand side of the expression in (3.2.19). Then, using the fact that the function $V(\vec{p})$ satisfies the conditions of (3.2.15)-(3.2.16) by construction, we can apply the local time-space formula from [87] (see also [90; Chapter II, Section 3.5] for a summary of the related results and further references) to obtain

$$V(\vec{\Pi}_t) + c\,t = V(\vec{p}) + M_t + L_t + \int_0^t \big((\mathbb{L}V)(\vec{\Pi}_s) + c\big)\, I\big(\Pi_s^l \geq b_*(\vec{\Pi}_s^{-l})\big)\, ds, \tag{3.2.21}$$

where the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t = \sum_{i=1}^{2^n} \sum_{k \in N} \int_0^t V_{p_i}(\vec{\Pi}_s) \frac{\mu_k \sqrt{\nu_{kk}}}{\nu_k} \left(\Pi_s^{O^{-1}(O(i)\cup\{k\})} - \Pi_s^i \Pi_s^{O^{-1}(\{k\})}\right) I\big(\Pi_s^l \neq b_*(\vec{\Pi}_s^{-l})\big) d\widehat{B}_s^k, \tag{3.2.22}$$

is a continuous local martingale under the probability measure $P_{\vec{p}}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Here, the process $L = (L_t)_{t \geq 0}$ is given by

$$L_t = \frac{1}{2} \int_0^t \Delta_{p_l} V(\vec{\Pi}_s)\, I(\Pi_s^l = b_*(\vec{\Pi}_s^{-l}))\, d\ell_s^i, \tag{3.2.23}$$

where the function $\Delta_{p_l} V(\vec{p})$ is given by

$$\Delta_{p_l} V(\vec{p}) = V_{p_l}(p_1, \ldots, p_{l-1}, p_l+, p_{l+1}, \ldots, p_{2^n}) - V_{p_l}(p_1, \ldots, p_{l-1}, p_l-, p_{l+1}, \ldots, p_{2^n}), \tag{3.2.24}$$

and the process $\ell^i = (\ell_t^i)_{t \geq 0}$ defined by

$$\ell_t = P_{\vec{p}} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\left(b_*(\vec{\Pi}_s^{-l}) - \varepsilon < \Pi_s^l < b_*(\vec{\Pi}_s^{-l}) + \varepsilon\right) d\langle \Pi^l - b_*(\vec{\Pi}^{-l}) \rangle_s, \tag{3.2.25}$$

is the local time of $\Pi^l$ at the surface $b_*(\vec{p}_l)$, at which the partial derivative $V_{p_l}(\vec{p})$ may not exist. It follows from the fact that the gain function $G(\vec{p})$ in (3.2.6) is decreasing in $p_l$ and the conditions (3.2.15)-(3.2.16) that the inequality $\Delta_{p_l} V(\vec{p}) \leq 0$ should hold for all $\vec{p} \in \mathcal{D}$, so that the continuous process $L$ defined in (3.2.23) is non-increasing. We may therefore conclude that $L_t = 0$ can hold for all $t \geq 0$ if and only if the smooth-fit condition of (3.2.18) is satisfied.

Using the assumption that the inequality in (3.2.17) holds with the boundary $b_*(\vec{p}_l)$, we conclude from the condition in (3.2.15) that $(\mathbb{L}V)(\vec{p}) + c \geq 0$ holds for any $\vec{p} \in \mathcal{D}$ such that

$p_l \neq b_*(\vec{p}_l)$. Moreover, it follows from the conditions of (3.2.14)-(3.2.16) that the inequality $V(\vec{p}) \leq G(\vec{p})$ holds for all $\vec{p} \in \mathcal{D}$. Thus, the expression in (3.2.21) yields that the inequalities

$$G(\vec{\Pi}_\tau) + c\,\tau - L_\tau \geq V(\vec{\Pi}_\tau) + c\,\tau - L_\tau \geq V(\vec{p}) + M_\tau, \tag{3.2.26}$$

hold for any finite stopping time $\tau$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times for the process $M$ such that $\tau_n = \inf\{t \geq 0 \mid |M_t| \geq n\}$. Taking the expectations with respect to the probability measure $P_{\vec{p}}$ in (3.2.26), by means of the optional sampling theorem, we get the inequalities

$$\begin{aligned} E_{\vec{p}}\big[G(\vec{\Pi}_{\tau\wedge\tau_n}) + c\,(\tau\wedge\tau_n) - L_{\tau\wedge\tau_n}\big] &\geq E_{\vec{p}}\big[V(\vec{\Pi}_{\tau\wedge\tau_n}) + c\,(\tau\wedge\tau_n) - L_{\tau\wedge\tau_n}\big] \\ &\geq V(\vec{p}) + E_{\vec{p}}\, M_{\tau\wedge\tau_n} = V(\vec{p}). \end{aligned} \tag{3.2.27}$$

Hence, letting $n$ go to infinity and using Fatou's lemma, we obtain

$$E_{\vec{p}}\big[G(\vec{\Pi}_\tau) + c\,\tau - L_\tau\big] \geq E_{\vec{p}}\big[V(\vec{\Pi}_\tau) + c\,\tau - L_\tau\big] \geq V(\vec{p}), \tag{3.2.28}$$

for any stopping time $\tau$ such that $E_{\vec{p}}\tau < \infty$ and $E_{\vec{p}}L_\tau > -\infty$, and all $\vec{p} \in \mathcal{D}$, where $L_\tau = 0$ holds whenever the condition of (3.2.18) is satisfied. By virtue of the structure of the stopping time in (3.2.11) and the condition (3.2.15), it is readily seen that the equalities in (3.2.26) hold with $\tau_*$ instead of $\tau$ when $p_l \geq b_*(\vec{p}_l)$.

Let us now show that the equalities are attained in (3.2.28) for $p_l < b_*(\vec{p}_l)$, when $\tau_*$ replaces $\tau$ and the smooth-fit condition of (3.2.18) hold. By virtue of the fact that the function $V(\vec{p})$ and the continuous boundary of bounded variation $b_*(\vec{p}_l)$ solve the partial differential equation in (3.2.13) and satisfy the conditions in (3.2.14) and (3.2.18), it follows from the expression in (3.2.21) and the structure of the stopping time in (3.2.11) that

$$G(\vec{\Pi}_{\tau_*\wedge\tau_n}) + c\,(\tau_*\wedge\tau_n) = V(\vec{\Pi}_{\tau_*\wedge\tau_n}) + c\,(\tau_*\wedge\tau_n) = V(\vec{p}) + M_{\tau_*\wedge\tau_n}, \tag{3.2.29}$$

holds for $p_l < b_*(\vec{p}_l)$. Hence, taking expectations and letting $n$ go to infinity in (3.2.29), using the facts that $G(\vec{p})$ is bounded and $E_{\vec{p}}\tau_* < \infty$, we apply the Lebesgue dominated convergence theorem to obtain the equality

$$E_{\vec{p}}\big[G(\vec{\Pi}_{\tau_*}) + c\,\tau_*\big] = V(\vec{p}), \tag{3.2.30}$$

for all $\vec{p} \in \mathcal{D}$. We may therefore conclude that the function $V(\vec{p})$ coincides with the value function $\overline{V}_*(\vec{p})$ of the optimal stopping problem in (3.2.6) whenever the smooth-fit condition of (3.2.18) holds.

In order to prove the uniqueness of the value function $\overline{V}_*(\vec{p})$ and the boundary $b_*(\vec{p}_l)$ as solutions of the free-boundary problem in (3.2.13)-(3.2.17) with the smooth-fit condition of (3.2.18), let us assume that there exists another continuous boundary of bounded variation $b'(\vec{p}_l)$ such that $0 \le b'(\vec{p}_l) \le 1$ holds. Then, define the function $V'(\vec{p})$ as in (3.2.19) with $V'(\vec{p}; b'(\vec{p}_l))$ satisfying (3.2.13)-(3.2.17) and the stopping time $\tau'$ as in (3.2.11) with $b'(\vec{p}_l)$ instead of $b_*(\vec{p}_l)$, such that $E_{\vec{p}}\, \tau' < \infty$. Following the arguments from the previous part of the proof and using the fact that the function $V'(\vec{p})$ solves the partial differential equation in (3.2.13) and satisfies the conditions of (3.2.14) and (3.2.18) with $b'(\vec{p}_l)$ instead of $b_*(\vec{p}_l)$ by construction, we apply the change-of-variable formula from [87] to get

$$V'(\vec{\Pi}_t) + c\, t = V'(\vec{p}) + M'_t + \int_0^t \big( (\mathbb{L}V')(\vec{\Pi}_s) + c \big)\, I\big(\Pi^l_s \ge b'(\vec{\Pi}^{-l}_s)\big)\, ds, \tag{3.2.31}$$

where the process $M' = (M'_t)_{t \ge 0}$ defined as in (3.2.22) with $V'_{p_i}(\vec{p})$ instead of $V_{p_i}(\vec{p})$ is a continuous local martingale with respect to the probability measure $P_{\vec{p}}$. Thus, taking into account the structure of the stopping time $\tau'$, we obtain from (3.2.31) that

$$G(\vec{\Pi}_{\tau' \wedge \tau'_n}) + c\, (\tau' \wedge \tau'_n) \ge V'(\vec{\Pi}_{\tau' \wedge \tau'_n}) + c\, (\tau' \wedge \tau'_n) = V'(\vec{p}) + M'_{\tau' \wedge \tau'_n}, \tag{3.2.32}$$

holds for $p_l < b'(\vec{p}_l)$ and any localising sequence $(\tau'_n)_{n \in \mathbb{N}}$ of $M'$. Hence, taking expectations and letting $n$ go to infinity in (3.2.32), using the fact that $G(\vec{p})$ is bounded and $E_{\vec{p}}\, \tau' < \infty$, by means of the Lebesgue dominated convergence theorem, we have that the equality

$$E_{\vec{p}}\big[ G(\vec{\Pi}_{\tau'}) + c\, \tau' \big] = V'(\vec{p}), \tag{3.2.33}$$

is satisfied. Therefore, recalling the fact that $\tau_*$ is the optimal stopping time in (3.2.6) and comparing the expressions in (3.2.30) and (3.2.33), we see that the inequality $V'(\vec{p}) \ge V(\vec{p})$ should hold for all $\vec{p} \in \mathcal{D}$.

Finally, we show that $b'(\vec{p}_l)$ should coincide with $b_*(\vec{p}_l)$. By using the fact that $V'(\vec{p})$ and $V(\vec{p})$ satisfy (3.2.14)-(3.2.16), and $V'(\vec{p}) \ge V(\vec{p})$ holds for all $\vec{p} \in \mathcal{D}$ we get that $b'(\vec{p}_l) \le b_*(\vec{p}_l)$. Inserting $\tau_* \wedge \tau'_n$ into (3.2.31) in place of $t$ and using arguments similar to the ones above, we obtain

$$E_{\vec{p}}\big[ V'(\vec{\Pi}_{\tau_*}) + c\tau_* \big] = V'(\vec{p}) + E_{\vec{p}} \int_0^{\tau_*} \big( (\mathbb{L}V')(\vec{\Pi}_s) + c \big)\, I\big(\Pi^l_s \ge b'(\vec{\Pi}^{-l}_s)\big)\, ds, \tag{3.2.34}$$

for all $\vec{p} \in \mathcal{D}$. Thus, since we have $V'(\vec{p}) = V(\vec{p}) = G(\vec{p})$ for $p_l = b_*(\vec{p}_l)$, and $V'(\vec{p}) \ge V(\vec{p})$, we see from the expressions in (3.2.30) and (3.2.34) that the inequality

$$E_{\vec{p}} \int_0^{\tau_*} \big( (\mathbb{L}V')(\vec{\Pi}_s) + c \big)\, I\big(\Pi^l_s \ge b'(\vec{\Pi}^{-l}_s)\big)\, ds \le 0 \tag{3.2.35}$$

should hold. Due to the assumption of continuity of $b'(\vec{p}_l)$ we may therefore conclude that $b_*(\vec{p}_l) = b'(\vec{p}_l)$, so that $V'(\vec{p})$ coincides with $V(\vec{p})$ for all $\vec{p} \in \mathcal{D}$. □

**3.2.3. The location and structure of the optimal stopping boundary** Let us define the linear function $H^j(\vec{p})$ as

$$H^j(\vec{p}) = \sum_{i \in J} \lambda_i \left( p_{O^{-1}(J \setminus \{i\})} - p_j \right) = \sum_{i=1}^{2^n} h_{ji} p_i, \tag{3.2.36}$$

where the constants $h_{ji}$ are given by

$$h_{jj} = -\lambda(J), \quad \text{for } J \neq \varnothing, \tag{3.2.37}$$

$$h_{ji} = \lambda_k, \quad \text{if } O(i) = J \setminus \{k\} \text{ with } k \in J, \tag{3.2.38}$$

$$h_{ji} = 0, \quad \text{otherwise.} \tag{3.2.39}$$

for any $\vec{p} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. By using (3.1.30) and the optional sampling theorem, we get

$$E_{\vec{p}} \int_0^\tau H^j(\vec{\Pi}_t)\, dt + p_j = E_{\vec{p}}\, \Pi^j_\tau, \tag{3.2.40}$$

for any $\vec{p} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$, and for any stopping time $\tau$ such that $E_{\vec{p}} \tau < \infty$. Therefore, the optimal stopping problem of (3.1.35) is equivalent to

$$\widetilde{V}_*(\vec{p}) := V_*(\vec{p}) + \sum_{k=1}^{m} \Big( \sum_{i=1}^{2^n} b_k a_{ik} \Big) - b_k = \inf_\tau E_{\vec{p}} \int_0^\tau H(\vec{\Pi}_t)\, dt, \tag{3.2.41}$$

where we denote

$$H(\vec{p}) = \sum_{k=1}^{m} \Big( \sum_{i=1}^{2^n} c_k a_{ik} p_i \Big) - b_k a_{ik} H^i(\vec{p}), \tag{3.2.42}$$

for $\vec{p} \in \mathcal{D}$. By using (3.2.36), we can rewrite $H$ as

$$H(\vec{p}) = \sum_{i=1}^{2^n} h_i p_i \quad \text{with} \quad h_i = \sum_{k=1}^{m} \Big( c_k a_{ik} - \sum_{j=1}^{2^n} b_k a_{jk} h_{ji} \Big). \tag{3.2.43}$$

It is seen from (3.2.41) that, whenever $H(\vec{\Pi}_t) < 0$, it is not optimal to stop, or equivalently

$$H(\vec{p}) \geq 0 \quad \text{for} \quad \vec{p} \in S, \tag{3.2.44}$$

where the stopping region $S$ is defined as (compare with (3.2.11))

$$S := \big\{ \vec{p} \in \mathcal{D} \,\big|\, p_l \geq b_*(\vec{p}_l) \big\}. \tag{3.2.45}$$

By using (3.2.43), this means that the set

$$\left\{ \vec{p} \in \mathcal{D} \,\middle|\, \sum_{i=1}^{2^n} h_i p_i < 0 \right\} \tag{3.2.46}$$

belongs to the continuation region $C$ defined by

$$C := \left\{ \vec{p} \in \mathcal{D} \,\middle|\, p_l < b_*(\vec{p}_l) \right\}. \tag{3.2.47}$$

If we assume $h_l > 0$, the above leads to

$$b_*(\vec{p}_l) \geq \overline{b}_*(\vec{p}_l) \equiv \frac{h_l p_l - \sum_{i=1}^{2^n} h_i p_i}{h_l}, \tag{3.2.48}$$

so that $\overline{b}_*(\vec{p}_l) \leq b_*(\vec{p}_l)$ holds for all $\vec{p}_l \in [0,1]^{2^n-1}$ such that $\vec{p} \in \mathcal{D}$. Therefore we call *admissible* the parameters of the model that satisfy (3.2.48), because otherwise the optimal stopping time is not of the form (3.2.11), whenever it exists.

Let us take $\vec{p}, \vec{q} \in \mathcal{D}$ such that $p_l < b_*(\vec{p}_l)$, $q_k \leq p_k$, and $\vec{q}_k = \vec{p}_k$, for some $k \neq l$, and assume $h_k > 0$. Using the fact that $\vec{\Pi}$ is a time-homogeneous strong Markov process and taking into account the comparison resuts for solutions of stochastic differential equations in [114] we get

$$\begin{aligned} \overline{V}_*(\vec{q}) - G(\vec{q}) \equiv \widetilde{V}_*(\vec{q}) \leq E_{\vec{q}} \int_0^{\tau_*(\vec{p})} H(\vec{\Pi}_t)\, dt \leq E_{\vec{p}} \int_0^{\tau_*(\vec{p})} H(\vec{\Pi}_t)\, dt \\ = \widetilde{V}_*(\vec{p}) \equiv \overline{V}_*(\vec{p}) - G(\vec{p}) < 0, \end{aligned} \tag{3.2.49}$$

which leads to $p_l \equiv q_l < b_*(\vec{q}_l)$. Since we can choose $p_l$ arbitrarily close to $b_*(\vec{p}_l)$, it follows that $b_*(\vec{p}_l) \leq b_*(\vec{q}_l)$ and therefore the boundary $b_*(\vec{p}_l)$ is decreasing in $p_k$. The case when $h_k < 0$ leads by analogy to the fact that $b_*(\vec{p}_l)$ is increasing in $p_k$.

Let us summarise the results proved above in the following assertion.

**Proposition 3.2.1.** *Under the assumption that $h_l > 0$ the inequality* (3.2.48) *holds and the parameters of the model are admissible. Moreover, if for some $k \neq l$ we have that $h_k > 0$ ($h_k < 0$), the boundary $b_*(\vec{p}_l)$ is decreasing (increasing) in $p_k$ for $\vec{p} \in \mathcal{D}$.*

## 3.3. Examples and estimates

In the previos sections we characterized the Bayesian risk function of (3.1.2) as the solution to the Markovian optimal stopping problem in (3.1.35) and, under certain assumptions, to the free boundary problem in (3.2.13)-(3.2.18). However, explicit solutions to this multidimensional free boundary problem are not available in general. Therefore, in what follows, we first study specific examples that satisfy the assumptions in Lemma 3.2.1 and Proposition 3.2.1, and then provide estimates for the value function and optimal boundaries in (3.1.35) which are easier to compute. We assume for notational convenience that the bijection $O$ satisfies $O(1) = \varnothing$, so that we have $\Pi^1 = \Pi^\varnothing \equiv 1$.

**3.3.1. The case of infimum and supremum** Let us now present an example, in which we can can indeed find $l = 1, \ldots, 2^n$ such that $g_l < 0$ and $h_l > 0$, and $G(\vec{p})$ achieves its minimum for all $\vec{p} \in \mathcal{D}$ with $p_l = 1$. Let $m = 2$ and the functions $f_1(\vec{\theta})$ and $f_2(\vec{\theta})$ in (3.1.2) be given by $f_1(\vec{\theta}) = \bigwedge_{i\in N} \theta_i$ and $f_2(\vec{\theta}) = \bigvee_{i\in K} \theta_i$ for some $\varnothing \neq K \subseteq N$. This means that the posterior probability processes $\Pi^{*,1}$ and $\Pi^{*,2}$ from (3.1.5) are of the form (3.1.6) with

$$a_{11} = 0, \quad a_{j1} = (-1)^{|O(j)|-1} \quad \text{for } j = 2, 3, \ldots, 2^n, \tag{3.3.1}$$

$$a_{k2} = 1, \quad a_{j2} = 0 \quad \text{for } j = 1, \ldots, k-1, k+1, \ldots, 2^n, \tag{3.3.2}$$

where we have taken $1 < k \leq 2^n$ to be such that $O(k) = K$. Notice that from (3.2.2)-(3.2.4) we have

$$\sum_{K\subseteq O(j)\subseteq N} (-1)^{|O(j)\setminus K|} f_{jk} = \frac{1}{\lambda(N)}, \tag{3.3.3}$$

and by using (3.2.9), (3.2.43) and (3.3.1)-(3.3.2) we get

$$g_j = -a_{j1}\Big(b_1 + \frac{c_1}{\lambda(N)}\Big) - b_2 a_{j2} + c_2 f_{kj} \quad \text{if } O(j) \subseteq K, \tag{3.3.4}$$

$$g_j = -a_{j1}\Big(b_1 + \frac{c_1}{\lambda(N)}\Big) \qquad \text{otherwise}, \tag{3.3.5}$$

and

$$h_k = a_{k1}(b_1\lambda(N) + c_1) + b_2\lambda(K) + c_2, \tag{3.3.6}$$

$$h_j = a_{j1}(b_1\lambda(N) + c_1) - b_2\lambda_i \quad \text{if } \varnothing \neq O(j) = K \setminus \{i\} \text{ with } i \in K, \tag{3.3.7}$$

$$h_1 = -b_1\lambda(N) - b_2\lambda_i \qquad \text{if } K \equiv \{i\}, \tag{3.3.8}$$

$$h_j = a_{j1}(b_1\lambda(N) + c_1) \qquad \text{if } \varnothing \neq O(j) \neq K \setminus \{i\} \text{ with } i \in K, \tag{3.3.9}$$

$$h_1 = -b_1\lambda(N) \qquad \text{if } K \not\equiv \{i\}. \tag{3.3.10}$$

If $|K|$ is odd number we can choose $l \equiv k$ and from (3.3.4)-(3.3.10) and the fact that $a_{l1} \equiv a_{k1} = 1$, it follows that $g_l < 0$ and $h_l > 0$. If $|K|$ is even number and $K \neq N$ we can choose $l$

such that $O(l) = K \cup \{\overline{k}\}$ with $\overline{k} \in N \setminus K$, and from (3.3.4)-(3.3.10) and the fact that $a_{l1} = 1$, it follows that $g_l < 0$ and $h_l > 0$. If $K \equiv N$ and $|K|$ is even number we additionally assume that

$$b_1 - b_2 + \frac{c_1 - c_2}{\lambda(N)} < 0. \tag{3.3.11}$$

Therefore we can again choose $l \equiv k$ and from (3.3.4)-(3.3.10) with (3.3.11) and the fact that $a_{l1} \equiv a_{k1} = -1$ it follows that $g_l < 0$ and $h_l > 0$.

By using the definition of $\mathcal{D}$ in (3.1.34), we obtain that

$$p_j = 1 \quad \text{if } O(j) \subseteq O(l), \tag{3.3.12}$$

$$p_j = p_i \quad \text{if } O(j) = O(i) \cup \{r\} \text{ with } r \in O(l), \tag{3.3.13}$$

holds for all $\vec{p} \in \mathcal{D}$ with $p_l = 1$. Therefore, by using that $a_{j1} = -a_{i1}$ for $O(i) = O(j) \setminus \{r\}$ with $r \in O(j)$, we get that $\sum_{j=1}^{2^n} a_{j1} p_j = 1$. If we choose $j$ such that $O(j) \subseteq K$, it follows that $f_{kj}$ is negative and $K \subseteq O(l)$ implies $p_j = 1$. Hence, we conclude from (3.2.8), (3.2.9) and (3.3.4)-(3.3.5) that $G(\vec{p})$ achieves its minimum for all $\vec{p} \in \mathcal{D}$ with $p_l = 1$.

Let us finally note that, in the case when $m = 1$ and the function $f_1(\vec{\theta})$ is defined as above, we can choose $l = 2, 3 \ldots, 2^n$, such that $|O(l)| = 1$, and we will have that $g_l < 0$ and $h_l > 0$, and $G(\vec{p})$ achieves its minimum for all $\vec{p} \in \mathcal{D}$ with $p_l = 1$.

**3.3.2. Estimates in the infimum case** In order to find estimates for the value function $V_*(\vec{p})$ from (3.1.35) and the boundary $b_*(\vec{p}_l)$ from (3.2.11) we will use the solution to the ordinary free boundary problem from [105; pages 203-204] (see also [90; Chapter VI, Section 22.1]). We assume that $m = 1$, the function $f_1(\vec{\theta})$ is given as in Section 3.3.1. and $b_1 = 1$ in (3.1.2). Therefore, the problem in (3.1.2) reduces to finding a stopping time of alarm $\tau_*$, with respect to the observable filtration $(\mathcal{F}_t)_{t \geq 0}$, which is as close as possible to the infimum of all disorder times.

Denote $k_i = \mu_i \sqrt{\nu_{ii}} / \nu_i$ for $i \in N$ and define the ordinary differential operator $\mathbb{L}_*$ as

$$\mathbb{L}_* := \frac{\pi_*^2 (1 - \pi_*)^2}{2} \sum_{i,j \in N} |k_i k_j| \frac{d^2}{d\pi_*^2} + \lambda(N)(1 - \pi_*) \frac{d}{d\pi_*}. \tag{3.3.14}$$

Let us formulate the ordinary free boundary problem

$$(\mathbb{L}_* V_1)(\pi_*) = -c_1\pi_* \quad \text{for} \quad \pi_* \in [0,h), \tag{3.3.15}$$

$$V_1(h-) = 1-h \quad (\textit{continuous fit}), \tag{3.3.16}$$

$$V_1'(h-) = -1 \quad (\textit{smooth fit}), \tag{3.3.17}$$

$$V_1(\pi_*) < 1-\pi_* \quad \text{for} \quad \pi_* \in [0,h), \tag{3.3.18}$$

$$V_1(\pi_*) = 1-\pi_* \quad \text{for} \quad \pi_* \in (h,1], \tag{3.3.19}$$

for some $0 \le h \le 1$. It is shown in [105; pages 203-204] that there exist a unique concave solution $V_1(\pi_*)$ to the problem in (3.3.15)-(3.3.19) with the property that $V_1'(0+) = 0$. In particular, the solution is given by

$$V_1(\pi_*) = \begin{cases} (1-h) - \int_{\pi_*}^h \psi(x)dx & \text{if } \pi_* \in [0,h), \\ 1-\pi_* & \text{if } \pi_* \in [h,1], \end{cases} \tag{3.3.20}$$

and the constant $h$ is the unique root of the equation

$$\psi(h) = -1, \tag{3.3.21}$$

and satisfies $h \ge \lambda(N)/(\lambda(N)+c_1)$, where

$$\psi(\pi_*) := -\frac{c_1}{\gamma} e^{-\lambda(N)\delta(\pi_*)/\gamma} \int_0^{\pi_*} \frac{e^{\delta(x)}}{x(1-x)^2} dx, \tag{3.3.22}$$

$$\delta(\pi_*) := \log \frac{\pi_*}{1-\pi_*} - \frac{1}{\pi_*}, \quad \gamma := \frac{\sum_{i,j\in N} |k_i k_j|}{2}, \tag{3.3.23}$$

for $\pi_* \in (0,1)$. By using the fact that $V_1(\pi_*)$ satisfies (3.3.19), we obtain

$$(\mathbb{L}_* V_1)(\pi_*) \ge -c_1\pi_*, \tag{3.3.24}$$

for $\pi_* \in (\lambda(N)/(\lambda(N)+c_1), 1]$ and, hence, for all $\pi_* \in [0,h) \cup (h,1]$ since $V_1(\pi_*)$ satisfies (3.3.15) and $h \ge \lambda(N)/(\lambda(N)+c_1)$.

Denoting $\Pi^* \equiv \Pi^{*,1}$, we obtain from (3.1.6) and (3.3.1) that

$$\Pi_t^* \equiv P_{\vec{\pi}}(\theta_1 \wedge \theta_2 \cdots \wedge \theta_n \le t \,|\, \mathcal{F}_t^X) = \sum_{i\in N} \Pi_t^{\{i\}} - \sum_{i\ne j\in N} \Pi_t^{\{i,j\}} + \sum_{i\ne j\ne k\in N} \Pi_t^{\{i,j,k\}} - \dots \tag{3.3.25}$$

$$+ (-1)^{n-2} \sum_{i\in N} \Pi_t^{N\setminus\{i\}} + (-1)^{n-1}\Pi_t^N,$$

and applying Itô's formula, by using (3.1.30) and (3.3.1)-(3.3.2), we can see that the process $\Pi^*$ satisfies

$$d\Pi_t^* = \sum_{i\in N} \lambda_i (1-\Pi_t^*)\, dt + \sum_{i\in N} k_i \Pi_t^{\{i\}} (1-\Pi_t^*)\, d\widehat{B}_t^i, \tag{3.3.26}$$

for all $t \geq 0$. Therefore, using the fact that the function $V_1(\pi_*)$ satisfies the smooth-fit condition (3.3.17) and (3.3.19), we can apply the local time-space formula from [85] to obtain

$$\begin{aligned} V_1(\Pi_t^*) = V_1(\Pi_0^*) &+ \int_0^t V_1'(\Pi_s^*)\, \lambda(N)(1-\Pi_s^*)\, ds + \sum_{i\in N} \int_0^t V_1'(\Pi_s^*)\, k_i\, \Pi_s^{\{i\}}(1-\Pi_s^*)\, d\widehat{B}_s^i \\ &+ \frac{1}{2}\int_0^t V_1''(\Pi_s^*) \sum_{i,j\in N} \left( \frac{k_i k_j \nu_{ji}}{\sqrt{\nu_{ii}\nu_{jj}}} \Pi_s^{\{i\}} \Pi_s^{\{j\}} \right) (1-\Pi_s^*)^2\, I(\Pi_t^* \neq h)\, ds. \end{aligned} \tag{3.3.27}$$

From (3.3.18)-(3.3.19), by means of the optional sampling theorem, we get that

$$\begin{aligned} &E_{\vec{p}}\left[ 1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt \right] \geq E_{\vec{p}}\left[ V_1(\Pi_\tau^*) + c_1 \int_0^\tau \Pi_t^*\, dt \right] \\ &= V_1(\Pi_0^*) + E_{\vec{p}} \int_0^\tau \Big( V_1'(\Pi_t^*)\, \lambda(N)\, (1-\Pi_t^*) + c_1 \Pi_t^* \Big)\, dt \\ &\quad + \frac{1}{2} E_{\vec{p}} \int_0^\tau V_1''(\Pi_t^*) \sum_{i,j\in N} \left( \frac{k_i k_j \nu_{ji}}{\sqrt{\nu_{ii}\nu_{jj}}} \Pi_t^{\{i\}} \Pi_t^{\{j\}} \right) (1-\Pi_t^*)^2\, I(\Pi_t^* \neq h)\, dt, \end{aligned} \tag{3.3.28}$$

for any stopping time $\tau$ such that $E_{\vec{p}}\,\tau < \infty$ for $\vec{p} \in \mathcal{D}$. Since $V_1(\pi_*)$ is two times differentiable and concave we have that $V_1''(\pi_*) \leq 0$ for $\pi_* \in [0,h) \cup (h,1]$. From (3.3.28) and the fact that $-1 \leq \nu_{ji}/\sqrt{\nu_{ii}\nu_{jj}} \leq 1$, we therefore have

$$\begin{aligned} E_{\vec{p}}\left[ 1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt \right] &\geq V_1(\Pi_0^*) + E_{\vec{p}} \int_0^\tau \Big( V_1'(\Pi_t^*)\, \lambda(N)(1-\Pi_t^*) + c_1 \Pi_t^* \Big)\, dt \\ &\quad + \frac{1}{2} E_{\vec{p}} \int_0^\tau V_1''(\Pi_t^*) \sum_{i,j\in N} \Big( |k_i k_j| \Pi_t^{\{i\}} \Pi_t^{\{j\}} \Big)\, (1-\Pi_t^*)^2\, I(\Pi_t^* \neq h)\, dt. \end{aligned} \tag{3.3.29}$$

By using that

$$\Pi_t^{\{i\}} \equiv P_{\vec{\pi}}\, (\theta_i \leq t | \mathcal{F}_t) \leq P_{\vec{\pi}} (\theta_1 \wedge \theta_2 \cdots \wedge \theta_n \leq t \,|\, \mathcal{F}_t^X) \equiv \Pi_t^* \tag{3.3.30}$$

holds for any $i \in N$ and $t \geq 0$, and (3.3.24) is satisfied, we obtain

$$\begin{aligned} E_{\vec{p}}\left[ 1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt \right] &\geq V_1(\Pi_0^*) + E_{\vec{p}} \int_0^\tau \Big( (\mathbb{L}_* V_1)(\Pi_t^*) + c_1 \Pi_t^* \Big)\, I(\Pi_t^* \neq h)\, dt \\ &\geq V_1(\Pi_0^*), \end{aligned} \tag{3.3.31}$$

for any stopping time $\tau$ such that $E_{\vec{p}}\tau < \infty$ for $\vec{p} \in \mathcal{D}$. Since $\Pi_0^* = \sum_{j=1}^{2^n} a_{j1}p_j$ under the measure $P_{\vec{p}}$, by using (3.1.35), we have

$$V_*(\vec{p}) \equiv \inf_{\tau} E_{\vec{p}}\Big[1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt\Big] \geq V_1\Big(\sum_{j=1}^{2^n} a_{j1}p_j\Big), \tag{3.3.32}$$

for $\vec{p} \in \mathcal{D}$.

Using the results from Section 3.3.1. in the case $m = 1$, we can choose $l = 1, \dots, 2^n$, where $O(l) = \{r\}$ for some $r \in N$, and apply Lemma 3.2.1 to obtain that the optimal stopping time $\tau_*$ is of the form (3.2.11). Therefore, by using the fact that $\Pi^*$ is of the form (3.3.25), we have that $a_{l1} = 1$ and, hence, $\tau_*$ is of the form

$$\tau_* = \inf\big\{t \geq 0 \,\big|\, \Pi_t^* \geq g_1^*(\vec{\Pi}_t)\big\}, \tag{3.3.33}$$

with $g_1^*(\vec{p})$ given by

$$g_1^*(\vec{p}) = b_*(\vec{p}_l) + \sum_{j=1}^{2^n} a_{j1}p_j - p_l, \tag{3.3.34}$$

for $\vec{p} \in \mathcal{D}$. Moreover from (3.2.48) and (3.3.6)-(3.3.10) we obtain that

$$b_*(\vec{p}_l) \geq \overline{b}_*(\vec{p}_l) = p_l - \sum_{j=1}^{2^n} a_{j1}p_j + \frac{\lambda(N)}{\lambda(N) + c_1}, \tag{3.3.35}$$

and it follows that $0 < \lambda(N)/(\lambda(N) + c_1) \leq g_1^*(\vec{p})$ for $\vec{p} \in \mathcal{D}$.

We can deduce from Theorem 3.2.1 that the function $\overline{V}_*(\vec{p})$ defined in (3.2.6) satisfies (3.2.15)-(3.2.16) and therefore, by using (3.3.34), we have that $V_*(\vec{p}) < 1 - \sum_{j=1}^{2^n} a_{j1}p_j$ holds for all $\vec{p} \in \mathcal{D}$ such that $0 \leq \sum_{j=1}^{2^n} a_{j1}p_j < g_1^*(\vec{p})$. Since $V_1(\pi_*)$ satisfies (3.3.18)-(3.3.19), it follows from (3.3.32) that $g_1^*(\vec{p}) \leq h$ and we also get from (3.3.34) that

$$b_*(\vec{p}_l) \leq h + p_l - \sum_{j=1}^{2^n} a_{j1}p_j, \tag{3.3.36}$$

for $\vec{p} \in \mathcal{D}$.

Summarising the facts proved above, we are now ready to state the main result of this section.

**Theorem 3.3.1.** *Suppose that the function $V_1(\pi_*)$ is concave and, together with the constant $h \in [0, 1]$, solves the ordinary free boundary problem in* (3.3.15)-(3.3.19). *Then we have that the* *lower bound in* (3.3.32) *holds for the value function* $V_*(\vec{p})$ *from* (3.1.35) *and the upper bound in* (3.3.36) *holds for the boundary* $b_*(\vec{p}_l)$ *from* (3.2.11). *Moreover, the optimal stopping time in* (3.1.35) *can be written in the form of* (3.3.33), *where the optimal boundary* $g_1^*(\vec{p})$ *is such that* $0 < \lambda(N)/(\lambda(N) + c_1) \leq g_1^*(\vec{p}) \leq h \leq 1$ *for* $\vec{p} \in \mathcal{D}$.

## 3.4. Appendix

**3.A.1. Proof of Lemma 3.1.1** Define the $n$-dimensional row vector $\mu^J = (\mu_1^J, \ldots, \mu_n^J)$ and the row process $\overline{X} = (\overline{X}^1, \ldots, \overline{X}^n)$ as

$$\mu_i^J = \frac{\mu_i}{\nu_i} \quad \text{for} \quad i \in J, \quad \mu_i^J = 0 \quad \text{for} \quad i \in N \setminus J, \quad \overline{X}_t^i = \frac{X_t^i}{\nu_i} \quad \text{for} \quad i \in N, \tag{3.A.1}$$

for $t \geq 0$. From the definition of $X$ in (3.1.1), under the measure $P^\varnothing$ we have

$$\frac{X_t^i}{\nu_i} = B_t^i \quad \text{for} \quad i \in N, \tag{3.A.2}$$

and under the measure $P^J$ we have

$$\frac{X_t^i}{\nu_i} = \frac{\mu_i}{\nu_i} t + B_t^i \quad \text{for} \quad i \in J, \quad \frac{X_t^i}{\nu_i} = B_t^i \quad \text{for} \quad i \in N \setminus J, \tag{3.A.3}$$

for $t \geq 0$. Therefore, by the Girsanov theorem for an $n$-dimensional Brownian motion (see, e.g. [75; Chapter VI, Theorem 6.4]), we conclude that the weighted density process $Z^J$ satisfies

$$\begin{aligned} Z_t^J &= \exp\left(t \sum_{i \in J} \lambda_i\right) \frac{d(P^J | \mathcal{F}_t)}{d(P^\varnothing | \mathcal{F}_t)} = \exp\left(\sum_{i \in J} \lambda_i t + \mu^J \Sigma^{-1} (\overline{X}_t)^T - \frac{1}{2} \mu^J \Sigma^{-1} (\mu^J)^T t\right) \\ &= \exp\left(\sum_{i \in J} \lambda_i t + \sum_{i \in J} \frac{\mu_i}{\nu_i} \sum_{j=1}^n \frac{\nu_{ij}}{\nu_j} X_t^j - \frac{1}{2} \sum_{i,j \in J} \frac{\mu_i}{\nu_i} \frac{\mu_j}{\nu_j} \nu_{lj} t\right) \\ &= \exp\left(\sum_{i \in J} \lambda_i t + \sum_{i \in J} Y_t^i - \frac{1}{2} \sum_{i,j \in J} \frac{\mu_i}{\nu_i} \frac{\mu_j}{\nu_j} \nu_{lj} t\right), \end{aligned} \tag{3.A.4}$$

for $t \geq 0$, where the processes $Y^i$ are defined as in (3.1.10) for $i \in N$ and $(\cdot)^T$ denotes the vector transpose. □

**3.A.2. Sufficient statistics in the case of an exponential delay penalty costs** We describe here the sufficient statistics and their corresponding stochastic differential (filtering) equations in the case of exponential delay penalty costs. We are interested in detecting the so-called $k^{th}$-to-default event, which is a generalization of the infimum and the supremum of all disorder times. Specifically, keeping the notation from Section 3.1, let $m = 1$ and let the Bayesian risk function from (3.1.2) be of the form

$$V_*(\vec{\pi}) = \inf_{\tau} \Big( b_1 \, P_{\vec{\pi}}\big(\tau < f_1(\vec{\theta})\big) + c_1 \, E_{\vec{\pi}}\big[e^{\beta(\tau - f_1(\vec{\theta}))^+} - 1\big] \Big), \tag{3.A.5}$$

where $\beta > 0$ and the function $f_1(\vec{\theta})$ is equal to the $k$-th element $\theta_{i_k}$ in the ordering $\theta_{i_1} \le \theta_{i_2} \le \cdots \le \theta_{i_n}$ of the elements of $\vec{\theta}$, that is, it is given by

$$f_1(\vec{\theta}) = \bigwedge_{J \subseteq N, |J| = k} \bigvee_{j \in J} \theta_j, \tag{3.A.6}$$

for some $k \in N$. The term $E_{\vec{\pi}}[e^{\beta(\tau - f_1(\vec{\theta}))^+} - 1]$ represents the average *exponential* delay of detecting the function $f_1(\vec{\theta})$. We also notice that

$$\begin{aligned} E_{\vec{\pi}}\big[e^{\beta(\tau - f_1(\vec{\theta}))^+} - 1\big] &= E_{\vec{\pi}} \int_0^\infty I(f_1(\vec{\theta}) \le t, t \le \tau) \beta e^{\beta(t - f_1(\vec{\theta}))} \, dt \\ &= E_{\vec{\pi}} \int_0^\infty E_{\vec{\pi}}\big[I(f_1(\vec{\theta}) \le t, t \le \tau) \beta e^{\beta(t - f_1(\vec{\theta}))} \,\big|\, \mathcal{F}_t\big] \, dt \\ &= E_{\vec{\pi}} \int_0^\tau \beta E_{\vec{\pi}}\big[I(f_1(\vec{\theta}) \le t) e^{\beta(t - f_1(\vec{\theta}))} \,\big|\, \mathcal{F}_t\big] \, dt. \end{aligned} \tag{3.A.7}$$

In order to reduce the problem in (3.A.5) to an optimal stopping problem for a multidimensional Markov process we define the process $(\widetilde{\Pi}_t^{*,1})_{t \ge 0}$ as $\widetilde{\Pi}_t^{*,1} = E_{\vec{\pi}}[I(f_1(\vec{\theta}) \le t) e^{\beta(t - f_1(\vec{\theta}))} \,|\, \mathcal{F}_t]$ for $t \ge 0$. Hence, from (3.1.3) and (3.A.7), it follows that the Bayesian risk function in (3.A.5) can be written as

$$V_*(\vec{\pi}) = \inf_{\tau} E_{\vec{\pi}} \Big[ b_1 \, (1 - \Pi_\tau^{*,1}) + c_1 \int_0^\tau \beta \, \widetilde{\Pi}_t^{*,1} \, dt \Big]. \tag{3.A.8}$$

Define the posterior probability process $(\widetilde{\Pi}_t^J)_{t \ge 0}$ as $\widetilde{\Pi}_t^J := E_{\vec{\pi}}[I(\bigcap_{i \in J} \{\theta_i \le t\}) e^{\beta(t - f_1(\vec{\theta}))^+} \,|\, \mathcal{F}_t]$, for $J \subseteq N$, and denote by $\widetilde{\Pi} = (\widetilde{\Pi}^1, \ldots, \widetilde{\Pi}^{2^n})$ the $2^n$-dimensional process with components given by $\widetilde{\Pi}^j = \widetilde{\Pi}^{O(j)}$ for $j \in \{1, \ldots, 2^n\}$. Notice that, by the inclusion-exclusion principle, we have that

$$I(f_1(\vec{\theta}) \le t) = \sum_{i=k}^n (-1)^{i-k} \, \frac{(i-1)!}{(k-1)!(i-k)!} \sum_{J \subseteq N, |J| = i} I(\textstyle\bigcap_{j \in J} \{\theta_j \le t\}), \tag{3.A.9}$$

and, therefore, the representation in (3.1.6) is satisfied and $\widetilde{\Pi}^{*,1}$ is of the form

$$\widetilde{\Pi}_t^{*,1} \equiv E_{\vec{\pi}}\big[I(f_1(\vec{\theta}) \le t) e^{\beta(t - f_1(\vec{\theta}))} \,\big|\, \mathcal{F}_t\big] = \sum_{j=1}^{2^n} a_{j1} \, \widetilde{\Pi}_t^j, \tag{3.A.10}$$

where

$$a_{j1} = (-1)^{i-k} \frac{(i-1)!}{(k-1)!(i-k)!} \quad \text{for} \quad k = 1, \ldots, |O(j)| = i, \quad a_{j1} = 0 \quad \text{otherwise}, \tag{3.A.11}$$

for $j = 1, \ldots, 2^n$. Moreover, by using the fact that

$$\begin{aligned} & I(\textstyle\bigcap_{i \in J}\{\theta_i \le t\} \bigcap \{f_1(\vec{\theta}) \le t\}) \\ & = \sum_{i=k}^{n} (-1)^{i-k} \frac{(i-1)!}{(k-1)!(i-k)!} \sum_{L \subseteq N, |L| = i} I(\textstyle\bigcap_{j \in L \cup J}\{\theta_j \le t\}), \end{aligned} \tag{3.A.12}$$

$$\begin{aligned} I(\textstyle\bigcap_{i \in J}\{\theta_i \le t\}) e^{\beta(t - f_1(\vec{\theta}))^+} &= I(\textstyle\bigcap_{i \in J}\{\theta_i \le t\} \bigcap \{f_1(\vec{\theta}) \le t\}) e^{\beta(t - f_1(\vec{\theta}))} \\ &\quad + (1 - I(f_1(\vec{\theta}) \le t))\, I(\textstyle\bigcap_{i \in J}\{\theta_i \le t\}), \end{aligned} \tag{3.A.13}$$

we get that

$$\widetilde{\Pi}_t^J = \Pi_t^J + \sum_{i=k}^{n} (-1)^{i-k} \frac{(i-1)!}{(k-1)!(i-k)!} \sum_{L \subseteq N, |L| = i} (\widetilde{\Pi}_t^{J \cup L} - \Pi_t^{J \cup L}), \tag{3.A.14}$$

for $J \subseteq N$ and $t \ge 0$. It follows that, for any $J \subseteq N$ such that $|J| < k$, the process $\widetilde{\Pi}^J$ can be written as a linear combination of the processes $\Pi^J$, $\Pi^{J \cup L}$ and $\widetilde{\Pi}^{J \cup L}$ where $L \subseteq N$ and $|J \cup L| \ge k$. Therefore, we only need to obtain the stochastic differential equations satisfied by the processes $\widetilde{\Pi}^J$ for all $J \subseteq N$ such that $|J| \ge k$.

For any $R, L \subseteq N$ such that $R \ne \varnothing$, $R \cap L = \varnothing$ and any permutation $\alpha := [\alpha_1, \ldots, \alpha_r] \in Perm(R)$ we define the process $(\widetilde{\Phi}_t^{\alpha, L})_{t \ge 0}$ recursively as

$$\widetilde{\Phi}_t^{\alpha, L} := \lambda_{\alpha_r} \int_0^t \widetilde{\Phi}_u^{[\alpha_1, \ldots, \alpha_{r-1}], L} \frac{Z_t^{R \cup L} e^{\beta t}}{Z_u^{R \cup L} e^{\beta u}}\, du \quad \text{for} \quad |R \cup L| \ge k, \tag{3.A.15}$$

$$\widetilde{\Phi}_t^{\alpha, L} := \Phi_t^{\alpha, L} \quad \text{for} \quad |R \cup L| < k, \quad \widetilde{\Phi}_t^{\varnothing, L} := \pi^L e^{\beta t} Z_t^L \quad \text{for} \quad |L| \ge k, \tag{3.A.16}$$

where $Z^L$ and $\Phi^{\alpha, L}$ are given by (3.1.7) and (3.1.11). By analogy to Section 2, from the generalized Bayes formula in [75; Theorem 7.23], we obtain that the posterior probability process $(\widetilde{\Pi}_t^J)_{t \ge 0}$ takes the form

$$\widetilde{\Pi}_t^J = \frac{\widetilde{\Psi}_t^J}{\Psi_t^{\varnothing}}, \tag{3.A.17}$$

where

$$\widetilde{\Psi}_t^J := \sum_{\substack{L_1 \subseteq N \setminus J \\ L_2 \subseteq J}} \sum_{\substack{R \supseteq J \setminus L_2 \\ R \subseteq N \setminus (L_1 \cup L_2)}} \sum_{\alpha \in \mathrm{Perm}(R)} \widetilde{\Phi}_t^{\alpha, L_1 \cup L_2}, \tag{3.A.18}$$

for $J \subseteq N$ and $\Psi^{\varnothing}$ as in (3.1.16). By using Itô's formula, from (3.1.18) and (3.A.15) we get

$$d\widetilde{\Phi}_t^{\alpha,L} = \Big(\lambda_{\alpha_r}\widetilde{\Phi}_t^{[\alpha_1,\dots,\alpha_{r-1}],L} + \Big(\beta + \sum_{i\in R\cup L}\lambda_i\Big)\widetilde{\Phi}_t^{\alpha,L}\Big)\,dt + \sum_{i\in R\cup L}\widetilde{\Phi}_t^{\alpha,L}\,dY_t^i, \tag{3.A.19}$$

for $R, L \subseteq N$ such that $R \neq \varnothing$, $R \cap L = \varnothing$ and $|R \cup L| \geq k$, and any $\alpha := [\alpha_1, \dots, \alpha_r] \in \mathrm{Perm}(R)$. We also obtain from (3.A.16) that

$$d\widetilde{\Phi}_t^{\varnothing,L} = \Big(\beta + \sum_{i\in L}\lambda_i\Big)\,\widetilde{\Phi}_t^{\varnothing,L}\,dt + \widetilde{\Phi}_t^{\varnothing,L}\sum_{i\in L}dY_t^i \tag{3.A.20}$$

holds for $L \subseteq N$ such that $|L| \geq k$. Therefore, by using (3.A.18) and aggregating, we further obtain

$$d\widetilde{\Psi}_t^J = \Big(\sum_{i\in J}\lambda_i\widetilde{\Psi}_t^{J\setminus\{i\}} + \Big(\beta + \sum_{i\notin J}\lambda_i\Big)\,\widetilde{\Psi}_t^J\Big)\,dt + \sum_{i\in J}\widetilde{\Psi}_t^J\,dY_t^i + \sum_{i\notin J}\widetilde{\Psi}_t^{J\cup\{i\}}\,dY_t^i. \tag{3.A.21}$$

Hence, by applying Itô's formula to (3.A.17) and using the same reasoning as in Section 3.1, we conclude that

$$d\widetilde{\Pi}_t^J = \Big(\sum_{i\in J}\lambda_i\widetilde{\Pi}_t^{J\setminus\{i\}} + \Big(\beta - \sum_{i\in J}\lambda_i\Big)\widetilde{\Pi}_t^J\Big)\,dt + \sum_{i\in N}\Big(\widetilde{\Pi}_t^{J\cup\{i\}} - \widetilde{\Pi}_t^J\Pi_t^{\{i\}}\Big)\,\frac{\mu_i\sqrt{\nu_{ii}}}{\nu_i}\,d\widehat{B}_t^i, \tag{3.A.22}$$

for $J \subseteq N$ such that $|J| \geq k$. It follows that $(\vec{\Pi}, \widetilde{\Pi})$ is a (time-homogeneous strong) Markov process, even after removing all components $\widetilde{\Pi}^J$, where $J \subseteq N$ and $|J| < k$.

Finally, by using (3.A.8), (3.1.6) and (3.A.10), we can reduce the problem of (3.A.5) to the optimal stopping problem

$$V_*(\vec{p}) = \inf_\tau E_{\vec{p}}\Big[b_1\Big(1 - \sum_{i=1}^{2^n}a_{i1}\Pi_\tau^i\Big) + c_1\int_0^\tau\sum_{i=1}^{2^n}a_{i1}\widetilde{\Pi}_t^i\,dt\Big]. \tag{3.A.23}$$

Here, the processes $\vec{\Pi}$ and $\widetilde{\Pi}$ start at the same $\vec{p} \in \mathcal{D}$ under the probability measure $P_{\vec{p}}$.

**3.A.3. Filtering equations in the case of a two-dimensional Poisson process** Our aim in this section is to describe the sufficient statistics in a setting with dependent observable Poisson processes and for that purpose we will obtain the corresponding filtering equations. Let in the setting of Section 3.1 we have that $n = 2$ and $\pi_i = 0$ for $i = 1, 2$ and for ease of notation let $P \equiv P_{\vec{\pi}}$. Let $N^i = (N_t^i)_{t\geq 0}$ for $i = 0, 1, 2$, be pure jump processes, and assume that they are independent of the disorder times $\theta_j$, and also independent of one another. In particular, we assume that $N_t^i$ $,i = 0,1,2$, are Poisson processes with intensities

$$\varkappa_{1,0}\lambda_{1,0}, \quad (1-\varkappa_{1,0})\lambda_{1,0}, \quad (1-\varkappa_{2,0})\lambda_{2,0}, \quad \text{for} \quad 0 \le t < \theta_1 \wedge \theta_2, \tag{3.A.24}$$

$$\varkappa_{1,1}\lambda_{1,1}, \quad (1-\varkappa_{1,1})\lambda_{1,1}, \quad (1-\varkappa_{2,1})\lambda_{2,0}, \quad \text{for} \quad \theta_1 \le t < \theta_2, \tag{3.A.25}$$

$$\varkappa_{1,3}\lambda_{1,0}, \quad (1-\varkappa_{1,3})\lambda_{1,0}, \quad (1-\varkappa_{2,3})\lambda_{2,1}, \quad \text{for} \quad \theta_2 \le t < \theta_1, \tag{3.A.26}$$

$$\varkappa_{1,2}\lambda_{1,1}, \quad (1-\varkappa_{1,2})\lambda_{1,1}, \quad (1-\varkappa_{2,2})\lambda_{2,1}, \quad \text{for} \quad \theta_1 \vee \theta_2 \le t, \tag{3.A.27}$$

respectively, for some constants $0 < \varkappa_{i,j} < 1$ , $i = 1,2$, $j = 0,1,2,3$, and $\lambda_{i,j} > 0$, $i = 1,2$, $j = 0,1$, which satisfy

$$\varkappa_{1,0}\lambda_{1,0} = \varkappa_{2,0}\lambda_{2,0}, \quad \varkappa_{1,1}\lambda_{1,1} = \varkappa_{2,1}\lambda_{2,0}, \quad \varkappa_{1,3}\lambda_{1,0} = \varkappa_{2,3}\lambda_{2,1}, \quad \varkappa_{1,2}\lambda_{1,1} = \varkappa_{2,2}\lambda_{2,1}. \tag{3.A.28}$$

Let the pure jump (observable) processes $X^1$ and $X^2$ be given as $X_t^i = N_t^i + N_t^0$ for $i = 1,2$. Specifically, from (3.A.24)-(3.A.27)+(3.A.28), we conclude that $X^i$ has the form

$$dX_t^i = I(t \le \theta_i)dX_t^{i,0} + I(t > \theta_i)dX_t^{i,1}, \tag{3.A.29}$$

where $X_t^{i,j}$ is Poisson process with intensity $\lambda_{i,j}$ for $i = 1,2$, $j = 0,1$ and $t \ge 0$. Note that the dependence between the observable processes $X^1$ and $X^2$ is realised through the common pure jump process $N^0$.

Let us introduce the processes $\Phi^i = (\Phi_t^i)_{t\ge0}$ and $\Psi^i = (\Psi_t^i)_{t\ge0}$ defined as

$$\Phi_t^i = \lambda_i \int_0^t \frac{Z_t^{i,0}}{Z_v^{i,0}}\,dv \quad \text{and} \quad \Psi_t^i = \lambda_{3-i}\int_0^t \Phi_u^i \frac{Z_t^{i,0}}{Z_u^{i,0}}\frac{Z_t^{3-i,1}}{Z_u^{3-i,1}}\,du, \tag{3.A.30}$$

where the (weighted) density process $Z^{i,j} = (Z_t^{i,j})_{t\ge0}$ is given by

$$Z_t^{i,0} = e^{\lambda_i t}\frac{d(P(\cdot \mid \{\theta_i = 0\}\bigcap\{\theta_{3-i} = \infty\})|\mathcal{F}_t)}{d(P(\cdot \mid \{\theta_i = \theta_{3-i} = \infty\})|\mathcal{F}_t)}, \tag{3.A.31}$$

$$Z_t^{i,1} = e^{\lambda_i t}\frac{d(P(\cdot \mid \{\theta_i = 0\}\bigcap\{\theta_{3-i} = 0\})|\mathcal{F}_t)}{d(P(\cdot \mid \{\theta_i = \theta_{3-i} = 0\})|\mathcal{F}_t)} \tag{3.A.32}$$

for $i = 1,2$. The process $Z^{i,j}$ satisfies (see [75; Theorem 19.7])

$$Z_t^{i,j} = \exp\bigg(\sum_{l=0}^{2}\alpha_{i,j,l}N_t^l - \delta_{i,j}t\bigg), \tag{3.A.33}$$

for $t \ge 0$, where we have defined the constants

$$\alpha_{i,j,i} = \ln\frac{(1-\varkappa_{i,2i+3j-2ij-1})\lambda_{i,1}}{(1-\varkappa_{i,j(5j-2i)})\lambda_{i,0}}, \quad \alpha_{i,j,3-i} = \ln\frac{(1-\varkappa_{3-i,2i+3j-2ij-1})\lambda_{3-i,j}}{(1-\varkappa_{3-i,j(5j-2i)})\lambda_{3-i,j}} \tag{3.A.34}$$

$$\alpha_{i,j,0} = \ln\frac{\varkappa_{i,2i+3j-2ij-1}\lambda_{i,1}}{\varkappa_{i,j(5j-2i)}\lambda_{i,0}} \tag{3.A.35}$$

$$\delta_{i,j} = -\lambda_i + \lambda_{i,1} - \lambda_{i,0} + \lambda_{3-i,j}(\varkappa_{3-i,j(5j-2i)} - \varkappa_{3-i,2i+3j-2ij-1}) \tag{3.A.36}$$

for $i = 1, 2$ and $j = 0, 1$. Here, the processes $\Phi_t^i$ and $\Psi_t^i$ can be regarded as the (weighted) likelihood ratio processes corresponding to the events $\{\theta_i \le t < \theta_{3-i}\}$ and $\{\theta_i < \theta_{3-i} \le t\}$, respectively, for all $t \ge 0$ and $i = 1, 2$.

By means of standard arguments, resulting from the application of the generalized Bayes formula from [75; Theorem 7.23], it is shown that the posterior probability processes $\Pi = (\Pi_t)_{t \ge 0}$ and $\Pi^i = (\Pi_t^i)_{t \ge 0}$ defined by $\Pi_t = P(\theta_1 \le t, \theta_2 \le t \,|\, \mathcal{F}_t^X)$ and $\Pi_t^i = P(\theta_i \le t \,|\, \mathcal{F}_t^X)$, $i = 1, 2$, respectively, take the form

$$\Pi_t = \frac{\Psi_t}{1 + \Xi_t} \quad \text{and} \quad \Pi_t^i = \frac{\Upsilon_t^i}{1 + \Xi_t}, \tag{3.A.37}$$

where the processes $\Psi = (\Psi_t)_{t \ge 0}$, $\Upsilon^i = (\Upsilon_t^i)_{t \ge 0}$ and $\Xi = (\Xi_t)_{t \ge 0}$ are given by

$$\Psi_t = \Psi_t^i + \Psi_t^{3-i}, \quad \Upsilon_t^i = \Phi_t^i + \Psi_t \quad \text{and} \quad \Xi_t = \Phi_t^i + \Phi_t^{3-i} + \Psi_t \tag{3.A.38}$$

for all $t \ge 0$ and $i = 1, 2$.

Applying Itô's formula, we get that the process $Z^{i,j}$ from (3.A.33) admits the representation

$$dZ_t^{i,j} = Z_{t-}^{i,j} \lambda_i dt + Z_{t-}^{i,j} \sum_{l=0}^{2} \int (e^{\alpha_{i,j,l} v} - 1) \, (\mu_l(dt, dv) - e^{j\, \alpha_{3-i,0,l} v} \nu_l^\infty(dt, dv)), \tag{3.A.39}$$

with $Z_0^{i,j} = 1$ for $i = 1, 2$ and $j = 0, 1$. Here the measures $\nu_l^\infty(dt, dv)$ are given by

$$\nu_0^\infty(dt, dv) = \varepsilon_1(dv) \varkappa_{1,0} \lambda_{1,0} dt, \quad \nu_i^\infty(dt, dv) = \varepsilon_1(dv)(1 - \varkappa_{i,0}) \lambda_{i,0} dt, \quad \text{for} \quad i = 1, 2, \tag{3.A.40}$$

and represent the compensators, conditional on $\{\theta_1 > t, \theta_2 > t\}$ and with respect to the observable filtration $\mathcal{F}_t = \sigma(X_s^1, X_s^2 \,|\, 0 \le s \le t)$, of the jump measures $\mu_l(dt, dv)$ of $N^l$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ for $l = 0, 1, 2$, where $\varepsilon_1$ is the Dirac measure at the point $1$. Then, defining the process $U^i = (U_t^i)_{t \ge 0}$ by $U_t^i = Z_t^{i,0} Z_t^{3-i,1}$ we see that the following expression holds

$$dU_t^i = U_{t-}^i (\lambda_i + \lambda_{3-i}) dt + U_{t-}^i \sum_{l=0}^{2} \int \left(e^{k_l v} - 1\right) (\mu_l - \nu_l^\infty)(dt, dv), \tag{3.A.41}$$

for all $t \ge 0$, where $U_0^i = 1$, $k_i = \alpha_{i,0,i} + \alpha_{3-i,1,i}$ and $k_0 = \alpha_{i,0,0} + \alpha_{3-i,1,0}$, for $i = 1, 2$. Let us introduce the notation

$$dR_t^i = \frac{dZ_t^{i,0}}{Z_{t-}^{i,0}}, \quad dS_t = \frac{dU_t^i}{U_{t-}^i} = \frac{dU_t^{3-i}}{U_{t-}^{3-i}}, \quad \widetilde{Z}_t^{i,j} = e^{-\lambda_i t} Z_t^{i,j}, \quad d\widetilde{R}_t^i = \frac{d\widetilde{Z}_t^{i,0}}{\widetilde{Z}_{t-}^{i,0}} = dR_t^i - \lambda_i dt \tag{3.A.42}$$

$$\widetilde{U}_t^i = \widetilde{Z}_t^{i,0} \widetilde{Z}_t^{3-i,1}, \quad d\widetilde{S}_t = \frac{d\widetilde{U}_t^i}{\widetilde{U}_{t-}^i} = \frac{d\widetilde{U}_t^{3-i}}{\widetilde{U}_{t-}^{3-i}} = dS_t - (\lambda_1 + \lambda_2) dt, \quad \widetilde{\delta}_{i,j} = \delta_{i,j} + \lambda_i. \tag{3.A.43}$$

Hence, using again the Itô's formula, we obtain that the processes $\Phi^i$ and $\Psi^i$ from (3.A.30) solve the stochastic differential equations

$$d\Phi_t^i = \lambda_i(1 + \Phi_{t-}^i)dt + \Phi_{t-}^i d\widetilde{R}_t^i \tag{3.A.44}$$

with $\Phi_0^i = 0$, and

$$d\Psi_t^i = \big(\lambda_{3-i}\Phi_t^i + (\lambda_1 + \lambda_2)\Psi_{t-}^i\big)dt + \Psi_{t-}^i d\widetilde{S}_t \tag{3.A.45}$$

with $\Psi_0^i = 0$, for $i = 1, 2$. Thus, the processes defined in (3.A.38) admit the representations

$$d\Psi_t = \big(\lambda_{3-i}\Phi_t^i + \lambda_i\Phi_t^{3-i} + (\lambda_1 + \lambda_2)\Psi_{t-}\big)dt + \Psi_{t-}d\widetilde{S}_t \tag{3.A.46}$$

with $\Psi_0^i = 0$,

$$d\Upsilon_t^i = \big(\lambda_i(1 + \Xi_t) + \lambda_{3-i}\Upsilon_t^i\big)dt + \Upsilon_{t-}^i d\widetilde{R}_t^i + \Psi_{t-}d(\widetilde{S}_t - \widetilde{R}_t^i) \tag{3.A.47}$$

with $\Upsilon_0^i = 0$, and

$$d\Xi_t = (\lambda_i + \lambda_{3-i})(1 + \Xi_t)dt + \Upsilon_{t-}^i d\widetilde{R}_t^i + \Upsilon_{t-}^{3-i} d\widetilde{R}_t^{3-i} + \Psi_{t-}d(\widetilde{S}_t - \widetilde{R}_t^i - \widetilde{R}_t^{3-i}) \tag{3.A.48}$$

with $\Xi_0 = 0$, for $i = 1, 2$. We therefore conclude, due to the Itô's formula, that the processes defined in (3.A.37) solve the stochastic differential equations

$$d\Pi_t = \big((\Pi_t^1 - \Pi_t)\lambda_2 + (\Pi_t^2 - \Pi_t)\lambda_1\big)\,dt + \sum_{i=0}^{2}\int f_i(\Pi_{t-}, \Pi_{t-}^1, \Pi_{t-}^2)(\mu_i - \nu_i)(dt, dv), \tag{3.A.49}$$

with $\Pi_0 = 0$, where

$$f_i(\pi, \pi^1, \pi^2) = \frac{\pi g_i(\pi, \pi^1, \pi^2)}{e^{k_i v} - g_i(\pi, \pi^1, \pi^2)} \tag{3.A.50}$$

$$g_j(\pi, \pi^1, \pi^2) = (1 - \pi)(e^{k_j v} - 1) + (\pi - \pi^j)(e^{\alpha_{j,0,j} v} - 1) + (\pi - \pi^{3-j})(e^{\alpha_{3-j,1,j} v} - 1) \tag{3.A.51}$$

$$g_0(\pi, \pi^1, \pi^2) = (1 - \pi)(e^{k_0 v} - 1) + (\pi - \pi^1)(e^{\alpha_{1,0,0} v} - 1) + (\pi - \pi^2)(e^{\alpha_{2,0,0} v} - 1), \tag{3.A.52}$$

for $i = 0, 1, 2$ and $j = 1, 2$, and

$$d\Pi_t^i = \lambda_i(1 - \Pi_t^i)dt + \sum_{j=0}^{2}\int f_j^i(\Pi_{t-}, \Pi_{t-}^1, \Pi_{t-}^2)(\mu_j - \nu_j)(dt, dv), \tag{3.A.53}$$

with $\Pi_0^i = 0$, where

$$f_j^i(\pi,\pi^1,\pi^2) = \frac{g_j^i(\pi,\pi^1,\pi^2)}{e^{k_j v} - g_j(\pi,\pi^1,\pi^2)}, \quad \text{for} \quad j = 0,1,2, \tag{3.A.54}$$

$$g_j^i(\pi,\pi^1,\pi^2) = \pi(1-\pi^i)(e^{k_j v}-1) + (1-\pi^i)(\pi^i-\pi)(e^{\alpha_{j,0,j}v}-1) \tag{3.A.55}$$
$$+ \pi^i(\pi-\pi^{3-i})(e^{\alpha_{3-j,1,j}v}-1), \quad \text{for} \quad j = 1,2,$$

$$g_0^i(\pi,\pi^1,\pi^2) = \pi(1-\pi^i)(e^{k_0 v}-1) + (1-\pi^i)(\pi^i-\pi)(e^{\alpha_{1,0,0}v}-1) \tag{3.A.56}$$
$$+ \pi^i(\pi-\pi^{3-i})(e^{\alpha_{2,0,0}v}-1),$$

for $i = 1,2$ and $(\pi,\pi^1,\pi^2) \in [0,1]^3$. In the equations (3.A.49)+(3.A.53) the measures $\nu_l(dt,dv)$ are given by

$$\nu_i(dt,dv) = \varepsilon_1(v)\big(1 + \Pi_{t-}(e^{k_i v}-1) + (\Pi_{t-}-\Pi_{t-}^i)(e^{\alpha_{i,0,i}v}-1) \tag{3.A.57}$$
$$+ (\Pi_{t-}-\Pi_{t-}^{3-i})(e^{\alpha_{3-i,1,i}v}-1)\big)(1-\varkappa_{i,0})\lambda_{i,0}dt, \quad \text{for} \quad i = 1,2,$$

$$\nu_0(dt,dv) = \varepsilon_1(v)\big(1 + \Pi_{t-}(e^{k_0 v}-1) + (\Pi_{t-}-\Pi_{t-}^1)(e^{\alpha_{1,0,0}v}-1) \tag{3.A.58}$$
$$+ (\Pi_{t-}-\Pi_{t-}^2)(e^{\alpha_{2,0,0}v}-1)\big)\varkappa_{1,0}\lambda_{1,0}dt,$$

and represent the compensators of the jump measures $\mu_l(dt,dv)$ for $l = 0,1,2$ with respect to the observable filtration $\mathcal{F}_t$ for $(t,v) \in \mathbb{R}_+ \times \mathbb{R}$.

# Chapter 4

# On the Laplace transforms of the first exit times in one-dimensional non-affine jump-diffusion models

This chapter is based on joint work with Dr. Pavel V. Gapeev.

## 4.1. Solvable stochastic jump differential equations

In this section, we suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ there exists a standard Wiener process $W = (W_t)_{t\geq 0}$ and a homogeneous Poisson random measure $\mu(dt, dv)$ on $(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}))$ with the intensity (compensator) measure $\nu(dt, dv) = dtF(dv)$ (see [56; Definition II.1.20]), where $F$ is a positive $\sigma$-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(\{0\}) = 0$ and $W$ is assumed to be independent of $\mu(dt, dv)$.

**4.1.1.** Let us consider the stochastic differential equation

$$dX_t = \beta(t, X_t)\, dt + \gamma(t, X_t)\, dW_t + \int h(\delta(t, X_{t-}, v))\, (\mu - \nu)(dt, dv) + \int \overline{h}(\delta(t, X_{t-}, v))\, \mu(dt, dv), \tag{4.1.1}$$

where $h(x) = xI_{\{|x|\leq 1\}}$ with $I_{\{\cdot\}}$ as the indicator function, $\overline{h}(x) = x - h(x)$, and $\beta(t, x)$, $\gamma(t, x) > 0$ and $\delta(t, x, v)$ are continuous functions on $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+ \times \mathbb{R}^2$, respectively. Assume that, for any $n \in \mathbb{N}$, there exist a constant $C_n > 0$ and a function $\rho_n(v)$ with

$\int \rho_n^2(v)F(dv) < \infty$ such that

$$|\beta(t,x) - \beta(t,y)| + |\gamma(t,x) - \gamma(t,y)| \le C_n\,|x-y|, \tag{4.1.2}$$

$$|\beta(t,x)| + |\gamma(t,x)| \le C_n\,(1+|x|), \tag{4.1.3}$$

$$|h(\delta(t,x,v)) - h(\delta(t,y,v))| \le \rho_n(v)\,|x-y|, \tag{4.1.4}$$

$$|h(\delta(t,x,v))| \le \rho_n(v)\,(1+|x|), \tag{4.1.5}$$

$$|\overline{h}(\delta(t,x,v)) - \overline{h}(\delta(t,y,v))| \le \rho_n^2(v)\,|x-y|, \tag{4.1.6}$$

$$|\overline{h}(\delta(t,x,v))| \le (\rho_n^2(v) \wedge \rho_n^4(v))\,(1+|x|), \tag{4.1.7}$$

for all $0 \le t \le n$ and $x, y, v \in \mathbb{R}$. These conditions guarantee the existence of a unique strong solution $X = (X_t)_{t\ge 0}$ to (4.1.1) for a given $X_0 \in \mathbb{R}$ (see [56; Chapter III, Theorem 2.32]). We additionally assume that

$$\gamma(t,x) = \gamma_0(t) + \gamma_1(t)\,x \quad \text{and} \quad \delta(t,x,v) = \delta_0(t,v) + \delta_1(t,v)\,x, \tag{4.1.8}$$

where $\gamma_i(t)$ and $\delta_i(t,v)$ for $i = 0,1$ are continuous functions such that $\delta_1(t,v) > -1$, for all $t \ge 0$ and $x, v \in \mathbb{R}$. Finally, the equation in (4.1.1) takes the form

$$\begin{aligned} dX_t &= \beta(t,X_t)\,dt + (\gamma_0(t) + \gamma_1(t)X_t)\,dW_t \\ &+ \int h(\delta_0(t,v) + \delta_1(t,v)X_{t-})\,(\mu-\nu)(dt,dv) + \int \overline{h}(\delta_0(t,v) + \delta_1(t,v)X_{t-})\,\mu(dt,dv). \end{aligned} \tag{4.1.9}$$

**4.1.2.** Following the arguments in [45; Chapter IV], we see that if we have

$$\beta(t,x) = \beta_0(t) + \beta_1(t)\,x, \tag{4.1.10}$$

for all $t \ge 0$ and $x \in \mathbb{R}$, then the stochastic differential equation (4.1.9) can be solved explicitly. For this, we assume that the condition

$$\int_0^t \int \bigg( \frac{\delta_1^2(s,v) I_{\{|\delta(s,x,v)| \le 1\}}}{1 + |\delta_1(s,v)|} + |\log(1+\delta_1(s,v)) - \delta_1(s,v) I_{\{|\delta(s,x,v)|\le 1\}}| \bigg) F(dv)ds < \infty, \tag{4.1.11}$$

holds for all $t \ge 0$ and $x \in \mathbb{R}$. Therefore, the integrating factor process $Z = (Z_t)_{t\ge 0}$ given by

$$\begin{aligned} Z_t = \exp\bigg( &\int_0^t \frac{\gamma_1^2(s)}{2}\,ds - \int_0^t \gamma_1(s)\,dW_s - \int_0^t \int \delta_1(s,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\,(\mu-\nu)(ds,dv) \\ &- \int_0^t \int \big(\log(1+\delta_1(s,v)) - \delta_1(s,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\big)\;\mu(ds,dv) \bigg), \end{aligned} \tag{4.1.12}$$

is well-defined according to [106; Chapter VII, §3a, Theorem 2]. Hence, applying Itô's formula to (4.1.12), we get that the process $Z$ satisfies the equation

$$\begin{aligned} dZ_t = Z_{t-}\bigg(&\gamma_1^2(t)\,dt - \gamma_1(t)\,dW_t - \int \delta_1(t,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\,(\mu-\nu)(dt,dv) \\ &- \int \frac{\delta_1(t,v) I_{\{|\delta(s,X_{s-},v)|>1\}} - \delta_1^2(t,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}}{1+\delta_1(t,v)}\,\mu(dt,dv)\bigg). \end{aligned} \tag{4.1.13}$$

It follows from the expressions in (4.1.9) and (4.1.10) that the process $F = (F_t)_{t\ge 0}$ defined by

$$F_t = R_t^{-1} Z_t X_t \quad \text{with} \quad R_t = \exp\bigg(\int_0^t \beta_1(s)\,ds\bigg), \tag{4.1.14}$$

admits the representation

$$\begin{aligned} dF_t &= R_t^{-1}\big(Z_{t-}\,dX_t + X_{t-}\,dZ_t + d\langle Z^c, X^c\rangle_t + \Delta Z_t \Delta X_t - Z_{t-}X_{t-}\beta_1(t)\,dt\big) \\ &= R_t^{-1} Z_{t-}\bigg(\big(\beta_0(t) - \gamma_0(t)\gamma_1(t)\big)\,dt + \gamma_0(t)\,dW_t + \int \delta_0(t,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\,(\mu-\nu)(dt,dv) \\ &\qquad + \int \bigg(\frac{\delta_0(t,v)}{1+\delta_1(t,v)} - \delta_0(t,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\bigg)\,\mu(dt,dv)\bigg). \end{aligned} \tag{4.1.15}$$

Therefore, we may conclude from the expressions in (4.1.14) and (4.1.15) that the process $X = (X_t)_{t\ge 0}$ given by

$$\begin{aligned} X_t =& Z_t^{-1} R_t \bigg(X_0 + \int_0^t R_s^{-1} Z_s \big(\beta_0(s) - \gamma_0(s)\gamma_1(s)\big)\,ds + \int_0^t R_s^{-1} Z_s \gamma_0(s)\,dW_s \\ &+ \int_0^t R_{s-}^{-1} Z_{s-}\bigg(\int \delta_0(s,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\,(\mu-\nu)(ds,dv) \\ &\qquad + \int \bigg(\frac{\delta_0(s,v)}{1+\delta_1(s,v)} - \delta_0(s,v) I_{\{|\delta(s,X_{s-},v)|\le 1\}}\bigg)\,\mu(ds,dv)\bigg)\bigg), \end{aligned} \tag{4.1.16}$$

provides a (unique strong) solution of the equation in (4.1.9) under the condition of (4.1.10) for a given $X_0 \in \mathbb{R}$.

**4.1.3.** Following the arguments in [83; Chapter V, Example 5.16], we now show that the stochastic differential equation in (4.1.9) can be reduced to an ordinary differential equation if we assume that $\gamma_0(t) = \delta_0(t,v) = 0$ in (4.1.8), for all $t \ge 0$ and $v \in \mathbb{R}$. By applying the integration-by-parts formula to the process $G = (G_t)_{t\ge 0}$ given by $G_t = Z_t X_t$, and using the form of the functions $h$ and $\overline{h}$, and the expressions in (4.1.9) and (4.1.13), we obtain

$$\begin{aligned} dG_t &= Z_{t-}\,dX_t + X_{t-}\,dZ_t + d\langle Z^c, X^c\rangle_t + \Delta Z_t\,\Delta X_t \qquad (4.1.17)\\ &= Z_{t-}\bigg(\beta(t,X_{t-})\,dt + \gamma_1(t)X_{t-}\,dW_t \\ &\qquad + \int h(\delta_1(t,v)X_{t-})\,(\mu-\nu)(dt,dv) + \int \overline{h}(\delta_1(t,v)X_{t-})\,\mu(dt,dv)\bigg) \\ &\quad + Z_{t-}X_{t-}\bigg(\gamma_1^2(t)\,dt - \gamma_1(t)\,dW_t - \int \frac{h(\delta_1(t,v)X_{t-})}{X_{t-}}\,(\mu-\nu)(dt,dv) \\ &\qquad - \int \frac{\overline{h}(\delta_1(t,v)X_{t-}) - \delta_1(t,v)h(\delta_1(t,v)X_{t-})}{(1+\delta_1(t,v))X_{t-}}\,\mu(dt,dv)\bigg) \\ &\quad - Z_{t-}X_{t-}\gamma_1^2(t)\,dt - Z_{t-}X_{t-}\int \frac{\delta_1^2(t,v)}{1+\delta_1(t,v)}\,\mu(dt,dv). \end{aligned}$$

Therefore, if $\beta(t,x)$ satisfies the conditions in (4.1.2)-(4.1.3), then the (unique strong) solution $X$ of (4.1.9) is given by $X_t = G_t Z_t^{-1}$, where for all $\omega\in\Omega$ the process $G(\omega) = (G_t(\omega))_{t\geq 0}$ is the unique solution of the ordinary differential equation

$$dG_t(\omega) = Z_t(\omega)\,\beta(t, Z_t^{-1}(\omega)G_t(\omega))\,dt. \qquad (4.1.18)$$

**4.1.4.** Let us finally consider the stochastic differential equation of (4.1.1) with the truncation function $h(x)=x$, so that it takes the form

$$dX_t = \beta(t,X_t)\,dt + \gamma(t,X_t)\,dW_t + \int \delta(t,X_{t-},v)\,(\mu-\nu)(dt,dv). \qquad (4.1.19)$$

Now the conditions in (4.1.4)-(4.1.7) can be written as

$$|\delta(t,x,v)-\delta(t,y,v)| \leq \rho_n(v)\,|x-y| \quad \text{and} \quad |\delta(t,x,v)| \leq \rho_n(v)\,(1+|x|), \qquad (4.1.20)$$

for all $0\leq t\leq n$, $n\in\mathbb{N}$, and $x,y,v\in\mathbb{R}$. In this case, the equation of (4.1.9) takes the form

$$dX_t = \beta(t,X_t)\,dt + (\gamma_0(t)+\gamma_1(t)X_t)\,dW_t + \int \big(\delta_0(t,v)+\delta_1(t,v)X_{t-}\big)\,(\mu-\nu)(dt,dv). \quad (4.1.21)$$

The condition in (4.1.11) can then be simplified to

$$\int_0^t \int \bigg(\frac{\delta_1^2(s,v)}{1+|\delta_1(s,v)|} + \big|\log(1+\delta_1(s,v)) - \delta_1(s,v)\big|\bigg)\,F(dv)\,ds < \infty. \qquad (4.1.22)$$

Then, the integrating factor process $Z$ from (4.1.12) admits the representation

$$\begin{aligned} Z_t = \exp\bigg(&\int_0^t \frac{\gamma_1^2(s)}{2}\,ds - \int_0^t \gamma_1(s)\,dW_s - \int_0^t\int \delta_1(s,v)\,(\mu-\nu)(ds,dv) \qquad (4.1.23)\\ &- \int_0^t\int \big(\log(1+\delta_1(s,v)) - \delta_1(s,v)\big)\,\mu(ds,dv)\bigg). \end{aligned}$$

Hence, the application of Itô's formula to the expression in (4.1.23) yields

$$dZ_t = Z_{t-}\bigg(\gamma_1^2(t)dt - \gamma_1(t)dW_t - \int \delta_1(t,v)(\mu-\nu)(dt,dv) + \int \frac{\delta_1^2(t,v)}{1+\delta_1(t,v)}\mu(dt,dv)\bigg). \quad (4.1.24)$$

In a way similar to the one presented above, by using the expressions from (4.1.21) and (4.1.24), we apply the Itô's formula to the processes $F$ and $G$ defined as in (4.1.14) and Section 4.1.3., respectively, and obtain the equations of (4.1.15) and (4.1.17). We again conclude that if $\beta(t,x)$ satisfies conditions (4.1.2)-(4.1.3), then the (unique strong) solution $X$ of equation (4.1.19) is given by (4.1.16) in the setting of Section 4.1.2. and given by $X_t = Z_t^{-1}G_t$ in the setting of Section 4.1.3.. Note that in this case, however, the indicator functions appearing in (4.1.15)-(4.1.16) are equal to one and $\overline{h}(x) \equiv 0$ holds in (4.1.17).

## 4.2. Reducibility to solvable equations

**4.2.1.** Let us consider the stochastic differential equation

$$\begin{aligned} dY_t &= \eta(t,Y_t)\,dt + \sigma(t,Y_t)\,dW_t \\ &\quad + \int h(\theta(t,Y_{t-},v))\,(\mu(dt,dv)-\nu(dt,dv)) + \int \overline{h}(\theta(t,Y_{t-},v))\,\mu(dt,dv), \end{aligned} \quad (4.2.1)$$

where $\eta(t,y)$, $\sigma(t,y)>0$ and $\theta(t,y,v)$ are continuous functions on $\mathbb{R}_+\times\mathcal{D}_Y$ and $\mathbb{R}_+\times\mathcal{D}_Y\times\mathbb{R}$, respectively, for some open set $\mathcal{D}_Y\subseteq\mathbb{R}$. Suppose that $f(t,y)$ is an *invertible* function from the class $C^{1,2}(\mathbb{R}_+,\mathcal{D}_Y)$ in the sense that there exists a function $g(t,x)$ such that $f(t,g(t,x))=x$ and $g(t,f(t,y))=y$ for all $t\ge 0$, $x\in\mathcal{D}_X$ and $y\in\mathcal{D}_Y$, where $\mathcal{D}_X$ denotes the range of $f(t,y)$. Let the functions $\beta(t,x)$, $\gamma(t,x)$, and $\delta(t,x,v)$ be given by

$$\beta(t,x) = \partial_t f(t,g(t,x)) + \eta(t,g(t,x))\partial_y f(t,g(t,x)) + \frac{\sigma^2(t,g(t,x))}{2}\partial_{yy}^2 f(t,g(t,x)), \quad (4.2.2)$$

$$\gamma(t,x) = \sigma(t,g(t,x))\,\partial_y f(t,g(t,x)), \quad (4.2.3)$$

$$h(\delta(t,x,v)) = h(\theta(t,g(t,x),v))\,\partial_y f(t,g(t,x)), \quad (4.2.4)$$

$$\begin{aligned} \overline{h}(\delta(t,x,v)) &= f(t,g(t,x)+\theta(t,g(t,x),v)) - f(t,g(t,x)) \\ &\quad - h(\theta(t,g(t,x),v))\,\partial_y f(t,g(t,x)), \end{aligned} \quad (4.2.5)$$

for $t\ge 0$, $x\in\mathcal{D}_X$, and $v\in\mathbb{R}$, and assume that they satisfy the conditions (4.1.2)-(4.1.7), so that the equation in (4.1.1) has a (unique strong) solution $X$ with a state space $\mathcal{D}_X$ and $X_0\in\mathcal{D}_X$. By virtue of the invertibility of the function $f(t,y)$ and an application of Itô's formula, we conclude that $Y$ defined as $Y_t = g(t, X_t)$ is a (unique strong) solution to the equation (4.2.1) with a state space $\mathcal{D}_Y$ and $Y_0 = g(0, X_0) \in \mathcal{D}_Y$. Moreover, if the functions $\gamma(t,x)$ and $\delta(t,x,v)$ satisfy (4.1.8), the equation (4.2.1) is reduced to the equation (4.1.9), which is solvable in a closed form under one of the conditions (4.1.10) or $\gamma_0(t) = \delta_0(t,v) = 0$.

On the other hand, if the equation in (4.2.1) has a (unique strong) solution $Y$ with a state space $\mathcal{D}_Y$, by means of Itô's formula applied to the process $X_t = f(t, Y_t)$, we get

$$\begin{aligned} dX_t = &\Big(\partial_t f(t,Y_t) + \eta(t,Y_t)\,\partial_y f(t,Y_t) + \frac{\sigma^2(t,Y_t)}{2}\,\partial^2_{yy} f(t,Y_t)\Big)\,dt \\ &+ \sigma(t,Y_t)\,\partial_y f(t,Y_t)\,dW_t + \int h(\theta(t,Y_{t-},v))\,\partial_y f(t,Y_{t-})\,(\mu(dt,dv)-\nu(dt,dv)) \\ &+ \int \Big(f(t,Y_{t-}+\theta(t,Y_{t-},v)) - f(t,Y_{t-}) - h(\theta(t,Y_{t-},v))\,\partial_y f(t,Y_{t-})\Big)\,\mu(dt,dv). \end{aligned} \tag{4.2.6}$$

Therefore, if $f(t,y)$ solves the equations

$$\partial_t f(t,y) + \eta(t,y)\,\partial_y f(t,y) + \frac{\sigma^2(t,y)}{2}\,\partial^2_{yy} f(t,y) = \beta(t, f(t,y)), \tag{4.2.7}$$

$$\sigma(t,y)\,\partial_y f(t,y) = \gamma_0(t) + \gamma_1(t) f(t,y), \tag{4.2.8}$$

$$h(\theta(t,y,v))\,\partial_y f(t,y) = h(\delta_0(t,v) + \delta_1(t,v) f(t,y)), \tag{4.2.9}$$

$$f(t,y+\theta(t,y,v)) - f(t,y) - h(\theta(t,y,v))\,\partial_y f(t,y) = \overline{h}(\delta_0(t,v) + \delta_1(t,v) f(t,y)), \tag{4.2.10}$$

for some continuous functions $\beta(t,x)$, $\gamma_i(t)$, and $\delta_i(t,v)$, $i = 0,1$, $t \geq 0$, $x \in \mathcal{D}_X$, $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$, we obtain that the equation in (4.2.1) is reduced to the one of (4.1.9), which is solvable in a closed form under one of the conditions of either (4.1.10) or $\gamma_0(t) = \delta_0(t,v) = 0$.

**Example 4.2.1.** *(Black-Karasinski model [16].)* Suppose that in (4.2.1) we have $\eta(t,y) = y(\eta_0(t) + \eta_1(t)\log y)$, $\sigma(t,y) = \sigma_0(t) y$ and $\theta(t,y,v) = 0$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t,y) = \log y$, $y > 0$, with the inverse $g(t,x) = e^x$, $x \in \mathbb{R}$, reduces the equation in (4.2.1) to the equation of (4.1.9) with (4.1.10), where $\beta_0(t) = \eta_0(t) - \sigma_0^2(t)/2$, $\beta_1(t) = \eta_1(t)$, $\gamma_0(t) = \sigma_0(t)$, $\gamma_1(t) = \delta_i(t,v) = 0$, $i = 0,1$, for all $t \geq 0$ and $v \in \mathbb{R}$ .

**Example 4.2.2.** *(Stochastic population model [83; Chapter V, Example 5.15].)* Suppose that in (4.2.1) we have $\eta(t,y) = \eta_0(t) y(\eta_1(t) - y)$, $\eta_0(t) > 0$, $\eta_1(t) > 0$, $\sigma(t,y) = \sigma_0(t) y$ and $\theta(t,y,v) = 0$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t,y) = 1/y$, $y > 0$, with the inverse $g(t,x) = 1/x$, $x > 0$, reduces (4.2.1) to the equation (4.1.9) with (4.1.10), where $\beta_0(t) = \eta_0(t)$, $\beta_1(t) = \sigma_0^2(t) - \eta_0(t)\eta_1(t)$, $\gamma_1(t) = -\sigma_0(t)$, $\gamma_0(t) = \delta_i(t,v) = 0$, $i = 0,1$, for all $t \geq 0$ and $v \in \mathbb{R}$.

**Remark 4.2.3.** Observe that in Examples 4.2.1 and 4.2.2 the function $\eta(t,y)$ does not satisfy the condition (4.1.3), but we see that the equation in (4.2.1) has a unique solution, since it is reducible to the linear equation of (4.1.9) with (4.1.10).

**4.2.2.** Let us now describe the invertible transformations $f(t,y)$ that reduce the equation in (4.2.1) to the equation in (4.1.9), and thus, to a solvable equation, in the *time-homogeneous case*. Suppose that (4.2.1) has a (unique strong) solution $Y$, where $\eta(t,y)=\eta(y)$, $\sigma(t,y)=\sigma(y)$, $\theta(t,y,v)=\theta(y,v)$ and $f(t,y)=f(y)$, $g(t,x)=g(x)$ for all $t\geq 0$, $x\in\mathcal{D}_X$, $y\in\mathcal{D}_Y$ and $v\in\mathbb{R}$. Assume that $\eta(y)$, $\sigma(y)$, and $\theta(y,v)$ are twice continuously differentiable functions, $\sigma(y)>0$, and denote

$$r(y)=\int^y\frac{dz}{\sigma(z)},\quad p(y)=\frac{\eta(y)}{\sigma(y)}-\frac{1}{2}\sigma'(y),\ \text{ and }\ q(y,v)=\exp\big(r(y+\theta(y,v))-r(y)\big),\tag{4.2.11}$$

for all $y\in\mathcal{D}_Y$ and $v\in\mathbb{R}$. Let us introduce the following set of conditions:

(C1) either the equality

$$(q\partial_y q+\sigma(\partial_y q)^2)(y,v)=(q\partial_y\sigma\partial_y q-\sigma(\partial_y q)^2+\sigma\partial_{yy}^2 q)(y,v)=0,\tag{4.2.12}$$

or the equality

$$\left(\frac{q\partial_y\sigma\partial_y q-\sigma(\partial_y q)^2+\sigma\partial_{yy}^2 q}{q\partial_y q+\sigma(\partial_y q)^2}\right)(y,v)=c_1,\tag{4.2.13}$$

is satisfied for some constant $c_1\in\mathbb{R}$ and all $y\in\mathcal{D}_Y$ and $v\in\mathbb{R}$;

(C2) either the equality $p'(y)=0$ or the condition

$$\left(\frac{(\sigma p')'}{p'}\right)(y)=c_2,\ \text{ and }\ \frac{(\sigma p')'}{p'}=\frac{q\partial_y\sigma\partial_y q-\sigma(\partial_y q)^2+\sigma\partial_{yy}^2 q}{q\partial_y q+\sigma(\partial_y q)^2}\ \text{ with (4.2.13)},\tag{4.2.14}$$

is satisfied for some constant $c_2\in\mathbb{R}$ and all $y\in\mathcal{D}_Y$ and $v\in\mathbb{R}$;

(C3) the equality

$$\left(\frac{\sigma\partial_y q}{q}\right)(y,v)=c_3(v)\tag{4.2.15}$$

is satisfied for some function $c_3(v)$ and all $y\in\mathcal{D}_Y$ and $v\in\mathbb{R}$.

We are now ready to state the reducibility criterion for jump-diffusion processes solving the equation (4.2.1).

**Theorem 4.2.4.** *(i) Let the condition of (C1) be satisfied and the assumptions*

$$|\theta(y,v)| > 1 \quad \textit{if and only if} \quad \left|c\left(e^{\gamma_1 r(y+\theta(y,v))} - e^{\gamma_1 r(y)}\right)\right| > 1, \tag{4.2.16}$$

$$0 < |\theta(y,v)| \le 1 \quad \textit{if and only if} \quad e^{\gamma_1 r(y+\theta(y,v))} - e^{\gamma_1 r(y)} = \gamma_1 \frac{\theta(y,v)}{\sigma(y)} e^{\gamma_1 r(y)}, \tag{4.2.17}$$

*hold for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$ and some constants $c \in \mathbb{R}$ and $\gamma_1 \neq 0$. Then the equation in* (4.2.1) *is reducible to the one of* (4.1.9)*, where the appropriate invertible transformation $f(y)$ is given by*

$$f(y) = c e^{\gamma_1 r(y)} - \frac{\gamma_0}{\gamma_1}, \tag{4.2.18}$$

*for all $y \in \mathcal{D}_Y$ and some constant $\gamma_0 \in \mathbb{R}$. Moreover, if the condition of (C2) is also satisfied, we can choose $\gamma_0$ and $\gamma_1$ such that the expression in* (4.1.16) *holds. On the other hand, if the equality $(\partial_y q)(y,v) = 0$ holds for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$, we can choose $\gamma_0 = 0$ and reduce the equation in* (4.1.9) *to the ordinary differential equation of* (4.1.18)*.*

*(ii) Let (C3) be satisfied and the assumptions*

$$|\theta(y,v)| > 1 \quad \textit{if and only if} \quad |\gamma_0(r(y+\theta(y,v)) - r(y))| > 1, \tag{4.2.19}$$

$$0 < |\theta(y,v)| \le 1 \quad \textit{if and only if} \quad r(y+\theta(y,v)) - r(y) = \frac{\theta(y,v)}{\sigma(y)}, \tag{4.2.20}$$

*for some $\gamma_0 \neq 0$, $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. Then, the equation in* (4.2.1) *is reducible to the one of* (4.1.9) *with $\gamma_1 = 0$, where the appropriate invertible transformation $f(y)$ is given by*

$$f(y) = \gamma_0 r(y) + c, \tag{4.2.21}$$

*for all $y \in \mathcal{D}_Y$ and some constant $c \in \mathbb{R}$. Moreover, if the equality $(\sigma p')'(y) = 0$ also holds for all $y \in \mathcal{D}_Y$, we can choose $\gamma_0$ such that the expression in* (4.1.16) *holds.*

*Proof.* In order to prove the reducibility of the equation in (4.2.1) to the one of (4.1.9), we need to check whether the equalities in (4.2.7)-(4.2.10) are satisfied for some $\beta(t,x) = \beta(x)$, $\gamma_i(t) = \gamma_i$, $\delta_i(t,v) = \delta_i(v)$, $i = 0, 1$, and $f(t,y) = f(y)$ for all $t \ge 0$, $y \in \mathcal{D}_Y$, and $v \in \mathbb{R}$.

(i) By using the notations of (4.2.11) and the fact that $\sigma(y) > 0$ for $y \in \mathcal{D}_Y$, we obtain that the function $f(y)$ given by (4.2.18) is invertible. It can be shown by means of direct calculations that the equality in (4.2.8) is satisfied. Then, by summing up the equations in (4.2.9) and (4.2.10), instead of checking the equality in (4.2.10), we can verify whether

$$f(y+\theta(y,v)) - f(y) = \delta_0(v) + \delta_1(v) f(y) \tag{4.2.22}$$

holds. It follows by substituting the expressions of (4.2.18) with (4.2.11) for $f(y)$ that the equation in (4.2.22) is equivalent to

$$\big(q^{\gamma_1}(y,v) - (1+\delta_1(v))\big)e^{\gamma_1 r(y)} = \frac{\gamma_1\delta_0(v) - \gamma_0\delta_1(v)}{c\gamma_1}. \tag{4.2.23}$$

Then, differentiating the expression in (4.2.23), we see that we can verify whether

$$\left(q^{\gamma_1}(y,v) - (1+\delta_1(v)) + \frac{\sigma(y)}{\gamma_1}\partial_y q^{\gamma_1}(y,v)\right)\frac{\gamma_1 e^{\gamma_1 r(y)}}{\sigma(y)} = 0 \tag{4.2.24}$$

holds, while after multiplying both parts of (4.2.24) by $e^{-\gamma_1 r(y)}\sigma(y)/\gamma_1$ and differentiating again, we see that the expression

$$\gamma_1\partial_y q^{\gamma_1}(y,v) + \partial_y(\sigma\partial_y q^{\gamma_1})(y,v) = 0 \tag{4.2.25}$$

needs to be verified. It follows from the direct calculations that the equality of (4.2.25) is equivalent to

$$\gamma_1(q\partial_y q + \sigma(\partial_y q)^2)(y,v) + (q\partial_y\sigma\partial_y q - \sigma(\partial_y q)^2 + \sigma\partial^2_{yy}q)(y,v) = 0. \tag{4.2.26}$$

Hence, the equality in (4.2.25) can be verified by means of either the equality in (4.2.12) or

$$\gamma_0 = 0 \quad\text{and}\quad \gamma_1 = -\left(\frac{q\partial_y\sigma\partial_y q - \sigma(\partial_y q)^2 + \sigma\partial^2_{yy}q}{q\partial_y q + \sigma(\partial_y q)^2}\right)(y,v), \tag{4.2.27}$$

combined with the one of (4.2.13). By choosing

$$\delta_1(v) = q^{\gamma_1}(y,v) - 1 + \frac{\sigma(y)}{\gamma_1}\partial_y q^{\gamma_1}(y,v), \tag{4.2.28}$$

we get that (4.2.24) is also verified. Thus, if we set $\gamma_0 = 0$ and

$$\delta_0(v) = \big(q^{\gamma_1}(y,v) - (1+\delta_1(v))\big)ce^{\gamma_1 r(y)}, \tag{4.2.29}$$

we have that (4.2.22) holds.

Let us now check whether (4.2.9) is satisfied. For this, we define the auxiliary sets

$$\Theta_0 = \{(y,v)\in\mathcal{D}_Y\times\mathbb{R}\colon |\theta(y,v)| = 0\}, \quad \Theta_1 = \{(y,v)\in\mathcal{D}_Y\times\mathbb{R}\colon |\theta(y,v)| > 1\}, \tag{4.2.30}$$

$$\Delta_0 = \{(y,v)\in\mathcal{D}_Y\times\mathbb{R}\colon |\delta(f(y),v)| = 0\}, \quad \Delta_1 = \{(y,v)\in\mathcal{D}_Y\times\mathbb{R}\colon |\delta(f(y),v)| > 1\}, \tag{4.2.31}$$

and note that from the invertibility of $f(y)$ and (4.2.22) we have $\Theta_0 = \Delta_0$. It follows from (4.2.9) that we should verify that $\Theta_1 \subseteq \Delta_1$, but on $\Delta_1\setminus\Theta_1$ we get $f'(y) = 0$, which contradicts invertibility. Therefore, we need to verify that $\Theta_1 = \Delta_1$, but by means of the equality in (4.2.22), the former is just the condition of (4.2.16). Then, substituting (4.2.22) into (4.2.9), on $(\mathcal{D}_Y \times \mathbb{R}) \setminus (\Delta_0 \cup \Delta_1)$ we also need to verify that

$$f(y + \theta(y, v)) - f(y) = \theta(y, v) f'(y) \tag{4.2.32}$$

holds, but the latter equality is equivalent to the condition of (4.2.17). Thus, the conditions of (4.2.16)-(4.2.17) are equivalent to the one in (4.2.9). Finally, the equality (4.2.7) is satisfied if we choose $\beta(x)$ as in (4.2.2) for $x \in \mathcal{D}_X$.

Assuming additionally that the condition of (C2) holds, let us now check that the equality in (4.2.7) is satisfied with $\beta(x)$ of the form (4.1.10), for some $\beta_0, \beta_1 \in \mathbb{R}$. If the expressions in (4.2.14) are satisfied, we can set

$$\gamma_0 = 0 \quad \text{and} \quad \gamma_1 = -\left(\frac{(\sigma p')'}{p'}\right)(y), \tag{4.2.33}$$

and notice that if the expression in (4.2.13) hold then $\gamma_0$ and $\gamma_1$ agree with the ones from (4.2.27). Substituting the expression (4.2.18) with (4.2.11) for $f(y)$ into (4.2.7) and using (4.1.10), we need to check whether

$$\left(\gamma_1 p(y) + \frac{\gamma_1^2}{2} - \beta_1\right) e^{\gamma_1 r(y)} = \frac{\gamma_1 \beta_0 - \gamma_0 \beta_1}{c \gamma_1} \tag{4.2.34}$$

holds. It follows by differentiating the expression in (4.2.34) and using (4.2.11) that

$$\left(\gamma_1 p(y) + \frac{\gamma_1^2}{2} - \beta_1 + \sigma(y) p'(y)\right) \frac{\gamma_1 e^{\gamma_1 r(y)}}{\sigma(y)} = 0 \tag{4.2.35}$$

needs to be verified, and multiplying both parts of (4.2.35) by $e^{-\gamma_1 r(y)} \sigma(y) / \gamma_1$ and differentiating again, we see that

$$\gamma_1 p'(y) + (\sigma p')'(y) = 0, \tag{4.2.36}$$

should also hold. Hence, the equality in (4.2.36) is satisfied under the condition of $p'(y) = 0$ or (4.2.14) with (4.2.33). It follows that the equality in (4.2.35) holds if we set

$$\beta_1 = \gamma_1 p(y) + \frac{\gamma_1^2}{2} + \sigma(y) p'(y). \tag{4.2.37}$$

Thus, the equality in (4.2.34) is verified if we set $\gamma_0 = 0$ and

$$\beta_0 = \left(\gamma_1 p(y) + \frac{\gamma_1^2}{2} - \beta_1\right) c e^{\gamma_1 r(y)}. \tag{4.2.38}$$

We may therefore conclude that the equality in (4.2.7) holds with $\beta(x)$ of the form (4.1.10) and we can solve the equation in (4.1.9) by the expression of (4.1.16).

On the other hand, if the equality $(\partial_y q)(y,v) = 0$ holds for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$, it follows from (4.2.28)-(4.2.29) that $\delta_0(v) = 0$ holds, so that we can set $\gamma_0 = 0$ and reduce the equation in (4.1.9) to the ordinary differential equation of (4.1.18).

(ii) By using the notations of (4.2.11) and the fact that $\sigma(y) > 0$ for $y \in \mathcal{D}_Y$, we obtain that the function $f(y)$ given by (4.2.21) is invertible. Direct calculations show that $f(y)$ satisfies the equality in (4.2.8). It follows by substituting the expression of (4.2.21) with (4.2.11) for $f(y)$ into (4.2.22) that we can equivalently check whether

$$\big(\log q(y,v) - \delta_1(v) r(y)\big)\gamma_0 = \delta_0(v) + \delta_1(v) c \tag{4.2.39}$$

holds for some constant $c \in \mathbb{R}$. Then, differentiating the equality in (4.2.39) and multiplying both parts of the resulting expression by $\sigma(y)$, we see that we can verify whether

$$\left(\frac{\sigma \partial_y q}{q}\right)(y,v) - \delta_1(v) = 0 \tag{4.2.40}$$

holds. It follows from the expression in (4.2.15) that the equation above is satisfied if we set

$$\delta_1(v) = \left(\frac{\sigma \partial_y q}{q}\right)(y,v) \tag{4.2.41}$$

for all $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. Hence, the equality in (4.2.39) is verified if we choose

$$\delta_0(v) = \big(\log q(y,v) - \delta_1(v) r(y)\big)\gamma_0 - \delta_1(v) c \tag{4.2.42}$$

for some $c \in \mathbb{R}$. By means of the arguments similar to the ones used in case (i), the conditions in (4.2.19)-(4.2.20) are equivalent to the ones of (4.2.9). Again, the equality in (4.2.7) holds if we choose $\beta(x)$ as in (4.2.2) for $x \in \mathcal{D}_X$.

Finally, assuming additionally that the equality $(\sigma p')'(y) = 0$ holds for all $y \in \mathcal{D}_Y$, let us check whether the equality in (4.2.7) is satisfied with $\beta(x)$ of the form (4.1.10), for some $\beta_0, \beta_1 \in \mathbb{R}$. It follows by substituting the expression of (4.2.21) with (4.2.11) for $f(y)$ into the one of (4.2.7) with (4.1.10) that we can equivalently check whether

$$\big(p(y) - \beta_1 r(y)\big)\gamma_0 = \beta_0 + c\beta_1 \tag{4.2.43}$$

holds for some constant $c \in \mathbb{R}$. Then, by differentiating the equality in (4.2.43), applying the notations of (4.2.11), and multiplying both parts of the resulting expression by $\sigma(y)$, we can verify whether

$$\sigma(y) p'(y) - \beta_1 = 0 \tag{4.2.44}$$

holds. Hence, by using the equality $(\sigma p')'(y) = 0$, we get that the equality in (4.2.44) is satisfied if we set

$$\beta_1 = \sigma(y)p'(y) \tag{4.2.45}$$

for all $y \in \mathcal{D}_Y$. Thus, the equality in (4.2.43) is verified if we set

$$\beta_0 = c\beta_1 - \big(p(y) - \beta_1 r(y)\big)\gamma_0. \tag{4.2.46}$$

We may therefore conclude that the equality in (4.2.7) holds with $\beta(x)$ of the form (4.1.10) and any $\gamma_0 \neq 0$, so that we can solve the equation in (4.1.9) by the expression of (4.1.16).

□

**Remark 4.2.5.** It follows from the proof presented above that if the truncation function $h(x)$ is non-zero, that is, if the equation in (4.2.9) is not trivially satisfied, the process $Y$ should have the diffusion coefficient $\sigma(y)$ which satisfies either the condition of (4.2.17) or (4.2.20). This is relevant only in the case of infinite jump activity, because the condition of (4.2.9) is always satisfied by putting $h(x) \equiv 0$ for finite jump activity.

**Example 4.2.6.** *(Cox-Ingersoll-Ross model I [24].)* Suppose that in (4.2.1) we have $\eta(y) = \eta_0 + \eta_1 y$, $\sigma(y) = \sigma_0\sqrt{y}$, $\eta_0 \geq \sigma_0^2/2$, $\eta_1 \neq 0$ and $\theta(y,v) = 0$ for all $y > 0$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(2\sqrt{y})$, $y > 0$, with the inverse $g(x) = (\log x/2)^2$, $x > 1$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = x(2\eta_0 + \eta_1 \log^2 x/2 + \sigma_0^2(\log x - 1)/2)/\log x$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x > 1$ and $v \in \mathbb{R}$.

**Example 4.2.7.** *(Cox-Ingersoll-Ross model II [24].)* Suppose that in (4.2.1) we have $\eta(y) = \eta_0 y(\eta_1 - y)$, $\sigma(y) = \sigma_0\sqrt{y^3}$ and $\theta(y,v) = 0$ for all $y > 0$ and $v \in \mathbb{R}$, where $\eta_0$, $\eta_1 \in \mathbb{R}$ and $\sigma_0 > 0$. The function $f(y) = \exp(-2/\sqrt{y})$, $y > 0$, with the inverse $g(x) = 4/\log^2 x$, $x \in (0,1)$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = -\eta_0 x(\eta_1 \log x - 4/\log x)/2 + \sigma_0^2 x(1 + 3/\log x)/2$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x \in (0,1)$ and $v \in \mathbb{R}$.

**Example 4.2.8.** *(Constant elasticity of variance model [23] and [50].)* Suppose that in (4.2.1) we have $\eta(y) = \eta_1 y$, $\sigma(y) = \sigma_0 y^\alpha$ and $\theta(y,v) = 0$ for all $y > 0$ and $v \in \mathbb{R}$, where $\eta_1 \in \mathbb{R}$ and $\sigma_0$, $\alpha > 0$. In the case when $\alpha = 1$, the function $f(y) = y$, $y > 0$, with the inverse $g(x) = x$, $x > 0$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = x\eta_1$, $\gamma_1 = \sigma_0$ and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x > 0$ and $v \in \mathbb{R}$. In the case when $\alpha \in (0,1)$, the function $f(y) = \exp(y^{1-\alpha}/(1-\alpha))$, $y > 0$, with the inverse $g(x) = (\log(x)(1-\alpha))^{1/(1-\alpha)}$,

$x > 1$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = \eta_1(1-\alpha)x\log x + \sigma_0^2 x(1-\alpha/((1-\alpha)\log x))/2$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x > 1$ and $v \in \mathbb{R}$. The case $\alpha > 1$ yields the same reduced equation as the case $\alpha \in (0,1)$ does, but with $\beta(x)$ defined for $x \in (0,1)$.

**Example 4.2.9.** *(Shiryaev filtering model [75; Chapter IX].)* Suppose that in (4.2.1) we have $\eta(y) = \eta_0(1-y)$, $\sigma(y) = \sigma_0 y(1-y)$ and $\theta(y,v) = 0$ for all $y \in (0,1)$ and $v \in \mathbb{R}$. Then the function $f(y) = y/(1-y)$, $y \in (0,1)$, with the inverse $g(x) = x/(1+x)$, $x > 0$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = \eta_0(1+x) + \sigma_0^2 x^2/(1+x)$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x > 0$ and $v \in \mathbb{R}$.

**Example 4.2.10.** *(Jacobi diffusion model [66; p. 335].)* Suppose that in (4.2.1) we have $\eta(y) = \sigma_0^2(\eta_0(1-y) - \eta_1 y)/2$, $\sigma(y) = \sigma_0\sqrt{y(1-y)}$, $\eta_0 \geq 1$, $\eta_1 \geq 1$, and $\theta(y,v) = 0$ for all $y \in (0,1)$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(2\arcsin\sqrt{y})$, $y \in (0,1)$, with the inverse $g(x) = \sin^2(\log\sqrt{x})$, $x \in (1, e^{\pi})$, reduces the equation in (4.2.1) to the one of (4.1.9), where $\beta(x) = \sigma_0^2 x(\eta_0\cos^2(\log\sqrt{x}) - \eta_1\sin^2(\log\sqrt{x}) + (\sin(\log x) - \cos(\log x))/2)/\sin(\log x)$, $\gamma_1 = \sigma_0$, and $\gamma_0 = \delta_0(v) = \delta_1(v) = 0$ for all $x \in (1, e^{\pi})$ and $v \in \mathbb{R}$.

**4.2.3.** In the rest of this section we will construct jump analogues of some diffusions. For this, we will use the Wiener process $W = (W_t)_{t\geq 0}$ and the Poisson random measure $\mu(dt,dv)$ with the compensator $\nu(dt,dv) = dtF(dv)$ existing on the probability space $(\Omega, \mathcal{F}, P)$.

Let $Y = (Y_t)_{t\geq 0}$ be a continuous process with a state space $\mathcal{D}_Y$ solving the stochastic differential equation (4.2.1) with $\theta(t,y,v) = 0$ for $t \geq 0$, $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. Suppose that there exists an invertible transformation $f(t,y) \in C^{1,2}(\mathbb{R}_+, \mathcal{D}_Y)$ satisfying (4.2.7)-(4.2.10) and such that the process $X = (X_t)_{t\geq 0}$, $X_t = f(t,Y_t)$, solves the equation (4.1.9) with $\delta_i(t,v) = 0$ for $i = 0,1$, $t \geq 0$, $v \in \mathbb{R}$. Let us take a continuous function $\widehat{\delta}(t,x,v) = \widehat{\delta}_0(t,v) + x\widehat{\delta}_1(t,v)$ such that $\widehat{\delta}_1(t,v) > -1$ holds and the expression in (4.1.11) is satisfied with $\delta(t,x,v)$ replaced by $\widehat{\delta}(t,x,v)$. Assume also that

$$\widehat{\delta}_i(t,v) \neq 0 \quad \text{if and only if} \quad \gamma_i(t) \neq 0, \tag{4.2.47}$$

for $i = 0,1$ and all $t \geq 0$, $v \in \mathbb{R}$. Consider the stochastic differential equation

$$\begin{aligned} d\widehat{X}_t &= \beta(t,\widehat{X}_t)\,dt + (\gamma_0(t) + \gamma_1(t)\widehat{X}_t)\,dW_t \qquad (4.2.48) \\ &+ \int h(\widehat{\delta}_0(t,v) + \widehat{\delta}_1(t,v)\widehat{X}_{t-})\,(\mu(dt,dv) - \nu(dt,dv)) + \int \overline{h}(\widehat{\delta}_0(t,v) + \widehat{\delta}_1(t,v)\widehat{X}_{t-})\,\mu(dt,dv), \end{aligned}$$

where $\beta(t,x)$ satisfies (4.1.10) or the condition

$$\gamma_0(t) = \widehat{\delta}_0(t,v) = 0 \tag{4.2.49}$$

holds for all $t \geq 0$ and $v \in \mathbb{R}$, and assume that its (unique strong) solution $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$ has the state space $\mathcal{D}_X$. Then, according to the arguments in Section 2, we conclude that equation (4.2.48) is solvable in a closed form, and applying to the solution $\widehat{X}$ the inverse transformation $g(t,x)$ for $t \geq 0$, $x \in \mathcal{D}_X$, we obtain that the process $\widehat{Y}_t = g(t, \widehat{X}_t)$ solves the equation

$$\begin{aligned} d\widehat{Y}_t = {} & \eta(t, \widehat{Y}_t)\, dt + \sigma(t, \widehat{Y}_t)\, dW_t \\ & + \int \widehat{\theta}_0(t, \widehat{Y}_{t-}, v)\, (\mu(dt,dv) - \nu(dt,dv)) + \int \widehat{\theta}_1(t, \widehat{Y}_{t-}, v)\, \mu(dt,dv), \end{aligned} \tag{4.2.50}$$

with

$$\widehat{\theta}_0(t,y,v) = h(\widehat{\delta}_0(t,v) + \widehat{\delta}_1(t,v) f(t,y)) \partial_x g(t, f(t,y)), \tag{4.2.51}$$

$$\widehat{\theta}_1(t,y,v) = g(t, \widehat{\delta}_0(t,v) + (1 + \widehat{\delta}_1(t,v)) f(t,y)) - g(t, f(t,y)) - \widehat{\theta}_0(t,y,v), \tag{4.2.52}$$

for $t \geq 0$, $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. We will call such process $\widehat{Y} = (\widehat{Y}_t)_{t \geq 0}$ a *jump analogue* of the diffusion process $Y = (Y_t)_{t \geq 0}$ (see [38; Section 4]). Note that when $h \equiv 0$ the jump analogue $\widehat{Y}$ also solves equation of the form (4.2.1).

**Remark 4.2.11.** Let us now introduce the *pure jump analogue* $\widetilde{Y} = (\widetilde{Y}_t)_{t \geq 0}$ of the given $Y = (Y_t)_{t \geq 0}$ by setting $\sigma(t,y) = 0$ in (4.2.50) for all $t \geq 0$ and $y \in \mathcal{D}_Y$. Such a process $\widetilde{Y}$ can be defined as a (unique strong) solution of the stochastic differential equation

$$d\widetilde{Y}_t = \eta(t, \widetilde{Y}_t)\, dt + \int \widehat{\theta}_0(t, \widetilde{Y}_{t-}, v)\, (\mu(dt,dv) - \nu(dt,dv)) + \int \widehat{\theta}_1(t, \widetilde{Y}_{t-}, v)\, \mu(dt,dv), \tag{4.2.53}$$

with $\widehat{\theta}_i(t,y,v)$, $i = 0, 1$, given by (4.2.51)-(4.2.52).

Let us now give some examples of jump analogues of diffusion processes presented in this section. We assume throughout that the truncation function $h(x)$ satisfies $h(x) \equiv 0$, and therefore $\widehat{\theta}_0(t,y,v) \equiv 0$.

**Example 4.2.12.** *(Extended Black-Karasinski model.)* Suppose that in (4.2.50) we have the same $\eta(t,y)$ and $\sigma(t,y)$ as in Example 4.2.1. Then for a jump analogue in (4.2.52) we can take $\widehat{\delta}_1(t,v) = 0$, and thus $\widehat{\theta}_1(t,y,v) = y(\exp(\widehat{\delta}_0(t,v)) - 1)$ for all $t \geq 0$, $y > 0$, and $v \in \mathbb{R}$.

**Example 4.2.13.** *(Extended stochastic population model.)* Suppose that in (4.2.50) we have the same $\eta(t,y)$ and $\sigma(t,y)$ as in Example 4.2.2. Then for a jump analogue in (4.2.52) we can take $\widehat{\delta}_0(t,v) = 0$, and thus $\widehat{\theta}_1(t,y,v) = -y(\widehat{\delta}_1(t,v)/(1+\widehat{\delta}_1(t,v)))$ for all $t \geq 0$, $y > 0$, and $v \in \mathbb{R}$.

**Example 4.2.14.** *(Extended Cox-Ingersoll-Ross model I.)* Suppose that in (4.2.50) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 4.2.6. Then for a jump analogue in (4.2.52) we can take $\widehat{\theta}_1(y,v) = \sqrt{y}\log(1+\widehat{\delta}_1(v)) + \log^2(1+\widehat{\delta}_1(v))/4$ for all $y > 0$, and $v \in \mathbb{R}$.

**Example 4.2.15.** *(Extended Cox-Ingersoll-Ross model II.)* Suppose that in (4.2.50) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 4.2.7. Then for a jump analogue in (4.2.52) we can take $\widehat{\theta}_1(y,v) = y\sqrt{y}\log\sqrt{1+\widehat{\delta}_1(v)}(2-\sqrt{y}\log\sqrt{1+\widehat{\delta}_1(v)})/(\sqrt{y}\log\sqrt{1+\widehat{\delta}_1(v)}-1)^2$ for all $y > 0$, and $v \in \mathbb{R}$.

**Example 4.2.16.** *(Extended constant elasticity of variance model.)* Suppose that in (4.2.50) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 4.2.8. In the case when $\alpha = 1$ for a jump analogue in (4.2.52) we can take $\widehat{\theta}_1(y,v) = \widehat{\delta}_0(v) + \widehat{\delta}_1(v)y$ for all $y > 0$ and $v \in \mathbb{R}$. In the cases when $\alpha \in (0,1)$ or $\alpha > 1$, for a jump analogue in (4.2.52) we can put $\widehat{\delta}_0(v) = 0$ and $\widehat{\theta}_1(y,v) = (y^{1-\alpha} + (1-\alpha)\log^{1-\alpha}(1+\widehat{\delta}_1(v)))^{1/(1-\alpha)} - y$ for all $y > 0$ and $v \in \mathbb{R}$.

**Example 4.2.17.** *(Extended Shiryaev filtering model.)* Suppose that in (4.2.50) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 4.2.9. Then for a jump analogue in (4.2.52) we can take $\widehat{\theta}_1(y,v) = y(1-y)\widehat{\delta}_1(v)/(1+y\widehat{\delta}_1(v))$ for all $y \in (0,1)$ and $v \in \mathbb{R}$ (see, e.g. [75; Chapter XIX]).

**Example 4.2.18.** *(Extended Jacobi diffusion model.)* Suppose that in (4.2.50) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 4.2.10. Then for a jump analogue in (4.2.52) we can take $\widehat{\theta}_1(y,v) = \sin(2\arcsin\sqrt{y} + \log\sqrt{1+\widehat{\delta}_1(v)})\ \sin(\log\sqrt{1+\widehat{\delta}_1(v)})$ for all $y \in (0,1)$ and $v \in \mathbb{R}$.

## 4.3. The Laplace transforms of first passage times

In this section, we derive closed-form expressions for the Laplace transforms of first passage times on constant boundaries for some of the jump-diffusion processes constructed above.

**4.3.1. The setting.** Let the continuous process $Y = (Y_t)_{t\geq 0}$, with the state space $\mathcal{D}_Y \subseteq \mathbb{R}$, solve the time-homogeneous stochastic differential equation in (4.2.1) with $\eta(t,y) = \eta(y)$, $\sigma(t,y) = \sigma(y)$, $\theta(t,y,v) = 0$ for all $t \geq 0$, $y \in \mathcal{D}_Y$ and $v \in \mathbb{R}$. Suppose that there exists a strictly increasing function $f(y) \in C^2(\mathcal{D}_Y)$ such that the process $X = (X_t)_{t\geq 0}$ given by $X_t = f(Y_t)$ has a state space $\mathcal{D}_X = (d_1, d_2)$ with $0 \leq d_1 < d_2 \leq \infty$. Moreover, assume that $f(y)$ satisfies the equalities (4.2.7)-(4.2.10), and hence, $X$ solves the equation in (4.1.9), with $\beta(t,x) = \beta(x)$, $\gamma_i(t) = \gamma_i$, $\delta_i(t,v) = 0$, $i = 0, 1$ for all $t \geq 0$, $x \in \mathcal{D}_X$, and $v \in \mathbb{R}$. Consider a jump analogue $\widehat{Y} = (\widehat{Y}_t)_{t\geq 0}$ of the process $Y$, such that $\widehat{Y}_t = g(\widehat{X}_t)$, where the process $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ solves the equation of the form (4.2.48) and has the state space $\mathcal{D}_X$.

For some $a, b \in \mathcal{D}_X$, $a < b$, fixed, let us define the first passage times $\tau_a$ and $\zeta_b$ as

$$\tau_a = \inf\{t \geq 0 \,|\, \widehat{Y}_t \leq g(a)\} \equiv \inf\{t \geq 0 \,|\, \widehat{X}_t \leq a\}, \tag{4.3.1}$$

$$\zeta_b = \inf\{t \geq 0 \,|\, \widehat{Y}_t \geq g(b)\} \equiv \inf\{t \geq 0 \,|\, \widehat{X}_t \geq b\}, \tag{4.3.2}$$

so that $g(a) < g(b)$ holds. Our aim is to find analytic expressions for the Laplace transform of $\tau_a \wedge \zeta_b$. For this purpose, we will compute the value function $V_*(x)$ given by

$$V_*(x) = E_x\big[e^{-\varkappa(\tau_a\wedge\zeta_b)}\, I_{\{\tau_a<\zeta_b\}}\big] \equiv E_x\big[e^{-\varkappa\tau_a}\, I_{\{\tau_a<\zeta_b\}}\big], \tag{4.3.3}$$

for any $x \in \mathcal{D}_X$ and some $\varkappa > 0$ fixed. Here $E_x$ denotes the expectation with respect to the probability measure $P_x$ under which the one-dimensional time-homogeneous (strong) Markov process $\widehat{X}$ starts at $x \in \mathcal{D}_X$.

We consider the case in which the process $\widehat{X}$ satisfies

$$d\widehat{X}_t = (\beta(\widehat{X}_t) - K\widehat{X}_t)\, dt + \gamma_1 \widehat{X}_t\, dW_t + \widehat{X}_{t-}\Big(\exp\Big(\sum_{i=1}^m \Delta Z_t^{i,+} - \sum_{j=1}^n \Delta Z_t^{j,-}\Big) - 1\Big), \tag{4.3.4}$$

where $Z^{i,+} = (Z_t^{i,+})_{t\geq 0}$ and $Z^{j,-} = (Z_t^{j,-})_{t\geq 0}$ are independent compound Poisson processes with intensities $\lambda_{i,+}, \lambda_{j,-} > 0$ and exponentially distributed jump sizes with parameters $\alpha_i, \beta_j > 0$, $\alpha_i \neq 1$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $m, n \in \mathbb{N}$, and

$$K = \sum_{i=1}^m \frac{\lambda_{i,+}}{\alpha_i - 1} - \sum_{j=1}^n \frac{\lambda_{j,-}}{\beta_j + 1}. \tag{4.3.5}$$

In this case, the compensator measure $\nu(dt, dv)$ in the equation of (4.2.48) is given by

$$\nu(dt, dv) = dt\left(I_{\{v>0\}} \sum_{i=1}^m \lambda_{i,+}\alpha_i\, e^{-\alpha_i v} + I_{\{v<0\}} \sum_{j=1}^n \lambda_{j,-}\beta_j\, e^{\beta_j v}\right) dv, \tag{4.3.6}$$

and $\widehat{\delta}(x, v) = (e^v - 1)x$ and $\gamma(x) = \gamma_1 x$ holds for all $x \in \mathcal{D}_X$, $v \in \mathbb{R}$, where the truncation function is $h(v) = v$, for $v \in \mathbb{R}$.

**4.3.2. The boundary value problem.** By means of standard arguments based on the application of Itô's formula for semimartingales, it is shown that the infinitesimal generator $\mathbb{L}$ of the process $\widehat{X}$ acts on a function $V(x) \in C^2(\mathcal{D}_X)$ according to the rule

$$(\mathbb{L}V)(x) = \frac{\gamma_1^2 x^2}{2} V''(x) + (\beta(x) - Kx)\, V'(x) - \left( \sum_{i=1}^m \lambda_{i,+} + \sum_{j=1}^n \lambda_{j,-} \right) V(x) \tag{4.3.7}$$
$$+ \left( \sum_{i=1}^m \lambda_{i,+} \alpha_i \int_0^\infty V(xe^y)\, e^{-\alpha_i y}\, dy + \sum_{j=1}^n \lambda_{j,-} \beta_j \int_{-\infty}^0 V(xe^y)\, e^{\beta_j y}\, dy \right),$$

for all $x \in \mathcal{D}_X$. In order to find analytic expressions for the unknown value function $V_*(x)$ in (4.3.3), let us build on the results of the general theory of Markov processes (see, e.g. [34; Chapter V]). We reduce the problem of computing $V_*(x)$ to the problem of finding a solution $V(x)$ to the boundary value problem

$$(\mathbb{L}V)(x) = \varkappa V(x), \quad \text{for} \quad a < x < b, \tag{4.3.8}$$

$$V(x) = 1, \quad \text{for} \quad x \le a, \quad \text{and} \quad V(x) = 0, \quad \text{for} \quad x \ge b, \tag{4.3.9}$$

$$V(a+) = V(a) \equiv 1 \quad \text{and} \quad V(b-) = V(b) \equiv 0, \tag{4.3.10}$$

where the *continuous fit* conditions of (4.3.10) hold in the cases in which the process $\widehat{X}$ can pass continuously through the boundaries $a$ and $b$, respectively. On the other hand, if $\gamma_1 = 0$ holds, the equation of (4.3.4) for $\widehat{X}$ does not contain a diffusion part, so that the function $V_*(x)$ may be discontinuous at the points $a$ or $b$, depending on the sign of the local drift rate $\beta(x) - Kx$ in (4.3.4), since $\widehat{X}$ may pass through either of them only by jumping. Therefore, in order to determine which of the continuous fit conditions in (4.3.10) should hold for $V(x)$, we will assume that one of the following four cases is satisfied.

(ia) There exists some constant $c \in \mathcal{D}_X$ such that

$$\beta(x) - Kx < 0 \quad \text{for} \quad x > c, \quad \beta(x) - Kx > 0 \quad \text{for} \quad x < c, \quad \text{and} \quad \beta(c) - Kc = 0 \tag{4.3.11}$$

holds, so that the process $\widehat{X}$ is reverting continuously to the level $c$. If $a < c < b$ then the continuous fit condition does not holds at either $a$ or $b$. On the other hand, if either $a > c$ or $b < c$ holds, the process $\widehat{X}$ can pass continuously through $a$ or $b$, respectively, and thus, we assume that $V(x)$ satisfies the left-hand condition of (4.3.10) if $a > c$, and the right-hand condition of (4.3.10) if $b < c$.

(iia) There exists some constant $c \in \mathcal{D}_X$ such that

$$\beta(x) - Kx > 0 \quad \text{for} \quad x > c, \quad \beta(x) - Kx < 0 \quad \text{for} \quad x < c, \quad \text{and} \quad \beta(c) - Kc = 0 \tag{4.3.12}$$

holds, so that the process $\widehat{X}$ moves away from the level $c$ continuously. If $a < c < b$ then the function $V$ solves the equation in (4.3.8) not on the whole interval $(a, b)$, but on the parts $(a, c)$ and $(c, b)$, separately. Moreover, the process $\widehat{X}$ can pass through $a$ or $b$ continuously, and thus, we assume that $V(x)$ satisfies the conditions of (4.3.10). On the other hand, if either $a > c$ or $b < c$ holds, the process $\widehat{X}$ can pass continuously through $a$ or $b$, respectively, and thus, we assume that $V(x)$ satisfies the right-hand part of (4.3.10) if $a > c$, and the left-hand part of (4.3.10) if $b < c$.
(iiia) If $\beta(x) - Kx > 0$ holds for all $x \in \mathcal{D}_X$, then the process $\widehat{X}$ can pass through $b$ continuously, and thus, we assume that $V(x)$ satisfies the right-hand part of (4.3.10).
(iva) If $\beta(x) - Kx < 0$ holds for all $x \in \mathcal{D}_X$, then the process $\widehat{X}$ can pass through $a$ continuously, and thus, we assume that $V(x)$ satisfies the left-hand part of (4.3.10).

When $\gamma_1 = 0$, we will additionally assume that the solution $V(x)$ is bounded. Note that, in the case when $\gamma_1 \neq 0$, this fact follows directly from the condition of (4.3.10).

We now describe a procedure which reduces the integro-differential boundary value problem of (4.3.8)-(4.3.10) to an ordinary differential one based on the exponential distribution of the jump sizes of the compound Poisson processes $Z^{i,+}$ and $Z^{j,-}$. For this purpose, by applying the conditions in (4.3.9), we obtain that the equation in (4.3.8) with (4.3.7) takes the form

$$a_{2,0}(x)\, V''(x) + a_{1,0}(x)\, V'(x) + a_{0,0}(x)\, V(x) + b_0(x) \tag{4.3.13}$$
$$+ \left( \sum_{i=1}^{m} \lambda_{i,+} \alpha_i x^{\alpha_i} \int_x^b V(y)\, y^{-\alpha_i - 1}\, dy + \sum_{j=1}^{n} \lambda_{j,-} \beta_j\, x^{-\beta_j} \int_a^x V(y)\, y^{\beta_j - 1}\, dy \right) = 0, \quad \text{for} \quad a < x < b,$$

where we set

$$a_{2,0}(x) = \frac{\gamma_1^2 x^2}{2}, \quad a_{1,0}(x) = \beta(x) - Kx, \tag{4.3.14}$$
$$a_{0,0}(x) = -\sum_{i=1}^{m} \lambda_{i,+} - \sum_{j=1}^{n} \lambda_{j,-} - \varkappa, \quad \text{and} \quad b_0(x) = \sum_{j=1}^{n} \lambda_{j,-} a^{\beta_j}\, x^{-\beta_j}. \tag{4.3.15}$$

The idea is to get rid of the integrals in (4.3.13), by successively making an appropriate Ansatz and applying integration by parts. Indeed, let us define recursively the functions

$$G_{0,0}(x) = V(x), \quad G_{i,0}(x) = \int_x^b \frac{G_{i-1,0}(y)}{y^{1+\alpha_i - \alpha_{i-1}}}\, dy, \quad \text{and} \quad G_{m,j}(x) = \int_a^x \frac{G_{m,j-1}(y)}{y^{1-\beta_j+\beta_{j-1}}}\, dy, \tag{4.3.16}$$

for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and all $a \le x \le b$, where we have denoted $\alpha_0 = 0$ and $\beta_0 = -\alpha_m$. Define the differential operators

$$\mathbb{L}_i = -x^{\alpha_i - \alpha_{i-1} + 1}\, \frac{d}{dx} \quad \text{and} \quad \mathbb{L}_{m+j} = x^{\beta_{j-1} - \beta_j + 1}\, \frac{d}{dx} \tag{4.3.17}$$

and introduce the notation $\mathbb{L}_{k,k'} = \mathbb{L}_k \circ \mathbb{L}_{k+1} \circ \cdots \circ \mathbb{L}_{k'}$, where $\mathbb{L}_{k,k'}$ is the identity operator if $k > k'$, and notice that the expressions

$$G_{i,0}(x) = (\mathbb{L}_{i+1,i'} G_{i',0})(x), \quad \text{for} \quad i' = i, \ldots, m, \tag{4.3.18}$$

$$G_{m,j}(x) = (\mathbb{L}_{m+j+1,m+j'} G_{m,j'})(x), \quad \text{for} \quad j' = j, \ldots, n, \tag{4.3.19}$$

$$G_{i,0}(x) = (\mathbb{L}_{i+1,m+j} G_{m,j})(x), \tag{4.3.20}$$

hold by definition, as well as $G_{i,0}(b) = 0$ and $G_{m,j}(a) = 0$, for $i = 0, \ldots, m$ and $j = 1, \ldots, n$. Therefore, substituting the expressions of (4.3.18)-(4.3.20) into (4.3.13) and using the integration-by-parts formula, we get that (4.3.13) is equivalent to each of the following boundary value problems

$$\sum_{k=0}^{i+2} a_{k,i}(x) G_{i,0}^{(k)}(x) + b_i(x) + (-1)^i \Bigg( \sum_{k=1}^{m} \lambda_{k,+} \alpha_k x^{\alpha_k} \int_x^b G_{i,0}(y) y^{\alpha_i - \alpha_k - 1} dy \prod_{k'=1}^{i} (\alpha_{k'} - \alpha_k) \tag{4.3.21}$$
$$+ \sum_{l=1}^{n} \lambda_{l,-} \beta_l \, x^{-\beta_l} \int_a^x G_{i,0}(y) \, y^{\alpha_i + \beta_l - 1} \, dy \prod_{k'=1}^{i} (\alpha_{k'} + \beta_l) \Bigg) = 0,$$

$$(\mathbb{L}_{k+1,i} G_{i,0})(b) = 0, \quad \text{for} \quad k = 1, \ldots, i, \tag{4.3.22}$$

for $i = 1, \ldots, m$, and

$$\sum_{l=0}^{m+j+2} a_{l,m+j}(x) \, G_{m,j}^{(l)}(x) + b_{m+j}(x) \tag{4.3.23}$$
$$+ (-1)^m \sum_{l=1}^{n} \lambda_{l,-} \beta_l \, x^{-\beta_l} \int_a^x G_{m,j}(y) \, y^{\beta_l - \beta_j - 1} \, dy \prod_{k=1}^{m} (\alpha_k + \beta_l) \prod_{l'=1}^{j} (\beta_{l'} - \beta_l) = 0,$$

$$(\mathbb{L}_{m+l+1,m+j} G_{m,j})(a) = 0, \quad \text{for} \quad l = 1, \ldots, j, \tag{4.3.24}$$

$$(\mathbb{L}_{i+1,m+j} G_{m,j})(b) = 0, \quad \text{for} \quad i = 1, \ldots, m, \tag{4.3.25}$$

for $j = 1, \ldots, n$ and all $x \in (a, b)$, where the coefficients are given by

$$a_{k,i}(x) = \sum_{k'=k}^{i+2} a_{k'-1,i-1}(x) \, (x^{\alpha_i - \alpha_{i-1} + 1})^{(k'-k)} \, \frac{(k'-1)!}{(k'-k)!(k-1)!}, \tag{4.3.26}$$

$$a_{0,i}(x) = (-1)^{i-1} \, x^{\alpha_i} \Bigg( \sum_{l=1}^{n} \lambda_{l,-} \beta_l \prod_{k=1}^{i-1} (\alpha_k + \beta_l) - \sum_{k=1}^{m} \lambda_{k,+} \alpha_k \prod_{k'=1}^{i-1} (\alpha_{k'} - \alpha_k) \Bigg), \tag{4.3.27}$$

$$b_i(x) = (-1)^i \sum_{l=1}^{n} \lambda_{l,-} a^{\beta_l} \, x^{-\beta_l} \Bigg( 1 + \beta_l \sum_{k=1}^{i} a^{\alpha_k} G_{k,0}(a) \prod_{k'=1}^{k-1} (\alpha_{k'} + \beta_l) \Bigg), \tag{4.3.28}$$

for $k = 1, \dots, i+2$ and $i = 1, \dots, m$, and

$$a_{l,m+j}(x) = \sum_{l'=l}^{m+j+2} a_{l'-1,m+j-1}(x)(x^{\beta_{j-1}-\beta_j+1})^{(l'-l)} \frac{(l'-1)!}{(l'-l)!(l-1)!}, \tag{4.3.29}$$

$$a_{0,m+j}(x) = (-1)^m x^{-\beta_j} \sum_{l=1}^{n} \lambda_{l,-} \beta_l \prod_{k=1}^{m} (\alpha_k + \beta_l) \prod_{l'=1}^{j-1} (\beta_{l'} - \beta_l), \tag{4.3.30}$$

$$b_{m+j}(x) = b_m(x) = (-1)^m \sum_{l=1}^{n} \lambda_{l,-} a^{\beta_l} x^{-\beta_l} \left(1 + \beta_l \sum_{k=1}^{m} a^{\alpha_k} G_{k,0}(a) \prod_{k'=1}^{k-1} (\alpha_{k'} + \beta_l)\right), \tag{4.3.31}$$

for $l = 1, \dots, m+j+2$ and $j = 1, \dots, n$.

In particular, the integro-differential equation (4.3.13) is equivalent to

$$\sum_{k=0}^{m+n+2} a_{k,m+n}(x)\, G_{m,n}^{(k)}(x) + b_{m+n}(x) = 0, \quad \text{for} \quad a < x < b, \tag{4.3.32}$$

$$(\mathbb{L}_{m+j+1,m+n} G_{m,n})(a) = 0, \quad \text{for} \quad j = 1 \dots, n, \tag{4.3.33}$$

$$(\mathbb{L}_{i+1,m+n} G_{m,n})(b) = 0, \quad \text{for} \quad i = 1, \dots, m, \tag{4.3.34}$$

which is an ordinary differential boundary problem. Moreover, by using that $V(x) = G_{0,0}(x) = (\mathbb{L}_{1,m+n} G_{m,n})(x) = 0$, we can rewrite conditions (4.3.9) and (4.3.10) as

$$(\mathbb{L}_{1,m+n} G_{m,n})(a) = 1, \quad (\mathbb{L}_{1,m+n} G_{m,n})(b) = 0, \tag{4.3.35}$$

$$(\mathbb{L}_{1,m+n} G_{m,n})(a+) = (\mathbb{L}_{1,m+n} G_{m,n})(a), \quad (\mathbb{L}_{1,m+n} G_{m,n})(b-) = (\mathbb{L}_{1,m+n} G_{m,n})(b). \tag{4.3.36}$$

Therefore we have transformed the integro-differential boundary problem (4.3.8)-(4.3.10) for the function $V(x)$ to the ordinary differential boundary problem (4.3.32)-(4.3.36) for the function $G_{m,n}(x)$.

The general solution of the ordinary (nonhomogeneous) differential equation in (4.3.32) has the form

$$G_{m,n}(x) = \overline{G}_{m,n}(x) + \sum_{k=1}^{m+n+2} C_k\, U_k(x), \quad \text{for} \quad a < x < b, \tag{4.3.37}$$

where $C_k$, $k = 1, \dots, m+n+2$, are some arbitrary constants, $U_k(x)$, $k = 1, \dots, m+n+2$, constitute the fundamental system of solutions (i.e. nontrivial linearly independent particular solutions) of the homogeneous version of (4.3.32) and $\overline{G}_{m,n}(x)$ is a particular solution of (4.3.32) (see [91; Chapter III, Section 18]). Therefore, we further look for a solution of the equation (4.3.13) in the form

$$V(x; a, b) = (\mathbb{L}_{1,m+n} \overline{G}_{m,n})(x) + \sum_{k=1}^{m+n+2} C_k(a,b)\, (\mathbb{L}_{1,m+n} U_k)(x), \quad \text{for} \quad a < x < b, \tag{4.3.38}$$

where the constants $C_k(a,b)$, $k=1,\ldots,m+n+2$, are specified from the appropriate boundary conditions. It follows from the expressions in (4.3.33)-(4.3.34) that the constants $C_k(a,b)$, $k=1,\ldots,m+n+2$, solve the equations

$$(\mathbb{L}_{m+j+1,m+n}\overline{G}_{m,n})(a)+\sum_{k=1}^{m+n+2}C_k\,(\mathbb{L}_{m+j+1,m+n}U_k)(a)=0, \quad \text{for} \quad j=1,\ldots,n, \tag{4.3.39}$$

$$(\mathbb{L}_{i+1,m+n}\overline{G}_{m,n})(b)+\sum_{k=1}^{m+n+2}C_k\,(\mathbb{L}_{i+1,m+n}U_k)(b)=0, \quad \text{for} \quad i=1,\ldots,m, \tag{4.3.40}$$

If $\gamma_1\neq 0$, then from the conditions of (4.3.10), we get that $C_k(a,b)$, $k=1,\ldots,m+n+2$, also satisfy

$$(\mathbb{L}_{1,m+n}\overline{G}_{m,n})(a)+\sum_{k=1}^{m+n+2}C_k\,(\mathbb{L}_{1,m+n}U_k)(a)=1, \tag{4.3.41}$$

$$(\mathbb{L}_{1,m+n}\overline{G}_{m,n})(b)+\sum_{k=1}^{m+n+2}C_k\,(\mathbb{L}_{1,m+n}U_k)(b)=0. \tag{4.3.42}$$

The existence and uniqueness of solutions for $C_k(a,b)$, $k=1,\ldots,m+n+2$, follows from the linear independence of the fundamental solutions $U_k(x)$, $k=1,\ldots,m+n+2$, of the ordinary differential equation in (4.3.32).

On the other hand, if $\gamma_1=0$ holds, the ordinary differential equation (4.3.32) is of order $m+n+1$ and the general solution of (4.3.13) has the form of (4.3.38) with $C_{m+n+2}=0$. In order to find the constants $C_k$, $k=1,\ldots,m+n+1$, we will revisit each of cases (ia)-(iva) above:

(ib) Assume that the conditions in case (ia) are satisfied. If $a<c<b$ neither of the conditions (4.3.41)-(4.3.42) is satisfied. In this case we assume, without loss of generality, that

$$|(\mathbb{L}_{1,m+n}U_{m+n+1})(c-)|=|(\mathbb{L}_{1,m+n}U_{m+n+1})U_{m+n+1}(c+)|=\infty, \tag{4.3.43}$$

and, hence, that $C_{m+n+1}=0$ holds. Therefore, we have that $V(x)$ is of the form (4.3.38) with $C_k$, $k=1,\ldots,m+n$, solving the equations (4.3.39)-(4.3.40).

If either $a>c$ or $b<c$ holds we have that $V(x)$ is of the form (4.3.38) with $C_k$, $k=1,\ldots,m+n+1$, solving the equations (4.3.39)-(4.3.40)+(4.3.42) if $b<c$, and (4.3.39)-(4.3.40)+(4.3.41) if $a>c$.

(iib) Assume that the conditions in case (iia) are satisfied. If $a<c<b$ the function $V(x)$ is

of the form

$$V(x;a) = (\mathbb{L}_{1,m+n}\overline{G}_{m,n})(x) + \sum_{k=1}^{m+n+1} C_k(a)\,(\mathbb{L}_{1,m+n}U_k)(x), \quad \text{for} \quad a < x < c, \tag{4.3.44}$$

$$V(x;b) = (\mathbb{L}_{1,m+n}\overline{G}_{m,n})(x) + \sum_{k=1}^{m+n+1} C_k(b)\,(\mathbb{L}_{1,m+n}U_k)(x), \quad \text{for} \quad c < x < b, \tag{4.3.45}$$

for some constants $C_k(a)$ and $C_k(b)$ for $k = 1, \ldots, m+n+1$. By similar considerations, these constants solve the equations (4.3.39)-(4.3.40) together with (4.3.41) or (4.3.42), respectively.

On the other hand, if either $a > c$ or $b < c$ holds, the function $V(x)$ is of the form (4.3.38) with $C_k$, $k = 1, \ldots, m+n+1$, solving (4.3.39)-(4.3.40)+(4.3.42) if $a > c$, and (4.3.39)-(4.3.40)+(4.3.41) if $b < c$.

(iiib) Assume that the conditions of the case (iiia) are satisfied. Then $V(x)$ is of the form of (4.3.38) with $C_k$, $k = 1, \ldots, m+n+1$, solving the equations in (4.3.39)-(4.3.40)+(4.3.42).

(ivb) Assume that the conditions of the case (iva) are satisfied. Then $V(x)$ is of the form of (4.3.38) with $C_k$, $k = 1, \ldots, m+n+1$, solving the equations in (4.3.39)-(4.3.40)+(4.3.41).

Summarising the facts exposed above, we now state and prove the corresponding verification assertion relating the solution of the boundary-value problem to the original value function.

**Theorem 4.3.1.** *Suppose that the process $\widehat{X}$ provides a (unique strong) solution of the stochastic differential equation in (4.3.4). Then, the Laplace transform $V_*(x)$ from (4.3.3) of the associated with $\widehat{X}$ random variable $\tau_a$, given that $\tau_a < \zeta_b$ from (4.3.1)-(4.3.2), admits the representation*

$$V_*(x) = V(x;a,b), \quad \textit{for} \quad a < x < b, \tag{4.3.46}$$

*for any fixed $a, b \in \mathcal{D}_X$ with $a < b$, where the function $V(x;a,b)$ is specified as follows:*

*(i) if $\gamma_1 \neq 0$ then the function $V(x;a,b)$ admits the representation of* (4.3.38) *with the coefficients $C_k(a,b)$, $k = 1, \ldots, m+n+2$, which provide a unique solution to the system in* (4.3.39)-(4.3.42)*;*

*(ii) if $\gamma_1 = 0$ then the function $V(x;a,b)$ is bounded and takes the form of either $V(x;a)$ in (4.3.44) or $V(x;b)$ in (4.3.45), respectively, with the coefficients $C_k(a)$ or $C_k(b)$, $k = 1, \ldots, m+n+1$, which provide a unique solution to the systems in the case (iia)-(iib), while if $\beta(x)$ satisfies one of the conditions from the cases (ia), (iiia), or (iva), then $V(x;a,b)$ is bounded and of the form* (4.3.38) *with $C_k(a,b)$, $k = 1, \ldots, m+n+1$, satisfying the corresponding conditions from the cases (ib), (iiib), or (ivb).*

*Proof.* In order to verify the assertion formulated above, it remains us to show that the function defined in (4.3.46) coincides with the value function in (4.3.3). For this, let us denote by $V(x)$ the right-hand side of the expression in (4.3.46).

(i) Let us first consider the case $\gamma_1 \neq 0$. Then, applying the change-of-variable formula for semimartingales with jumps of bounded variation from [87; Theorem 3.1] to the stopped process $e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)} V(\widehat{X}_{t \wedge \tau_a \wedge \zeta_b})$ we get that

$$e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)} \, V(\widehat{X}_{t \wedge \tau_a \wedge \zeta_b}) = V(x) + \int_0^{t \wedge \tau_a \wedge \zeta_b} e^{-\varkappa s} \, (\mathbb{L}V - \varkappa V)(\widehat{X}_s) \, ds + M_t \tag{4.3.47}$$

holds for all $a < x < b$, where the process $M = (M_t)_{t \geq 0}$ defined by

$$\begin{aligned} M_t = & \int_0^{t \wedge \tau_a \wedge \zeta_b} e^{-\varkappa s} \, V'(\widehat{X}_s) \, I_{\{\widehat{X}_s \neq a, \widehat{X}_s \neq b\}} \, \gamma_1 \widehat{X}_s \, dW_s \\ & + \int_0^{t \wedge \tau_a \wedge \zeta_b} \int e^{-\varkappa s} \Big( V(\widehat{X}_{s-} e^y) - V(\widehat{X}_{s-}) \Big) \, (\mu - \nu)(ds, dy) \end{aligned} \tag{4.3.48}$$

is a local martingale under $P_x$.

By virtue of straightforward calculations and the arguments of the previous section, it is verified that the function $V(x)$ solves the ordinary (integro-)differential equation in (4.3.7)+(4.3.8), so that the expression in (4.3.47) takes the form

$$e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)} \, V(\widehat{X}_{t \wedge \tau_a \wedge \zeta_b}) = V(x) + M_t \tag{4.3.49}$$

for $a < x < b$. Since the function $V(x)$ satisfies the boundary conditions of (4.3.9)-(4.3.10), it is continuous and bounded for all $x \in \mathcal{D}_X$. Thus, it follows from the expression in (4.3.49) that the process $M$ is a uniformly integrable martingale. Hence, taking the expectation with respect to $P_x$ in both sides of (4.3.49), by means of the optional sampling theorem (see, e.g. [56; Chapter I, Theorem 1.39]), we get

$$E_x\big[e^{-\varkappa(t \wedge \tau_a \wedge \zeta_b)} \, V(\widehat{X}_{t \wedge \tau_a \wedge \zeta_b})\big] = V(x) + E_x\big[M_{t \wedge \tau_a \wedge \zeta_b}\big] = V(x) \tag{4.3.50}$$

for all $x \in \mathcal{D}_X$ and $t \geq 0$. Therefore, letting $t$ go to infinity and using the conditions in (4.3.9)-(4.3.10) as well as the fact that $V(\widehat{X}_{\tau_a \wedge \zeta_b}) = I_{\{\tau_a < \zeta_b\}}$ on the set $\{\tau_a \wedge \zeta_b < \infty\}$, we can apply the Lebesgue dominated convergence theorem for (4.3.50) to obtain the equalities

$$E_x\big[e^{-\varkappa(\tau_a \wedge \zeta_b)} \, I_{\{\tau_a < \zeta_b\}}\big] = E_x\big[e^{-\varkappa(\tau_a \wedge \zeta_b)} \, V(\widehat{X}_{\tau_a \wedge \zeta_b}) \, I_{\{\tau_a \wedge \zeta_b < \infty\}}\big] = V(x) \tag{4.3.51}$$

for all $x \in \mathcal{D}_X$, that completes the proof in the case $\gamma_1 \neq 0$.

(ii) Assume now that $\gamma_1 = 0$ and $V(x)$ satisfies the right-hand condition in (4.3.10), so that $V(b-) = V(b) \equiv 0$ holds, while it does not satisfy the left-hand condition there, that is $V(a+) \neq V(a) \equiv 1$ holds (the other cases can be dealt with similarly). This feature corresponds to the case in which the process $\widehat{X}$ can pass through the boundary $a$ only by jumping and we particularly have that $P_x(\widehat{X}_{\tau_a} = a) = 0$ holds for $x \in \mathcal{D}_X \setminus \{a\}$. Following the idea of the proof in [67; Theorem 3.1], by using the assumption that $V$ is bounded, we can introduce a sequence of bounded functions $(V_k)_{k\in\mathbb{N}}$ from the class $C^1(\mathcal{D}_X)$ such that $V_k(a) = V(a+)$, $|V_k(x) - V(x)| \leq |V_k(a) - V(a)|$ for all $x \in \mathcal{D}_X$, and $V_k(x) = V(x)$ for $x \in \mathcal{D}_X \setminus \big((a-1/k, a] \cup (b, b+1/k)\big)$. Clearly, we have $V_k(x) \to V(x)$ for all $x \in \mathcal{D}_X \setminus \{a\}$ as $k \to \infty$. By applying the change-of-variable formula for finite variation processes from [92; Chapter II, Theorem 31] to the stopped process $e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)} V_k(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b})$, we get that

$$e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)}\, V_k(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b}) = V_k(x) + \int_0^{t\wedge\tau_a\wedge\zeta_b} e^{-\varkappa s}\, (\mathbb{L}V_k - \varkappa V_k)(\widehat{X}_s)\, ds + M_t^k \tag{4.3.52}$$

holds for $a < x < b$, where the process $M^k = (M_t^k)_{t\geq0}$, $k \in \mathbb{N}$, defined by

$$M_t^k = \int_0^{t\wedge\tau_a\wedge\zeta_b} \int e^{-\varkappa s}\, \big(V_k(\widehat{X}_{s-}e^y) - V_k(\widehat{X}_{s-})\big)\, (\mu - \nu)(ds, dy), \tag{4.3.53}$$

is a local martingale. It follows from the construction of the functions $V_k(x)$ above that the inequality $|V_k(x) - V(x)| \leq |V_k(a) - V(a)|$ holds for all $x \in \mathcal{D}_X$, so that, we have

$$\begin{aligned} \big|(\mathbb{L}V_k - \varkappa V_k)(x)\big| &\leq \lambda\bigg(\sum_{i=1}^m \alpha_i \int_{\log b - \log x}^{\log(b+1/k) - \log x} |V_k(xe^y) - V(xe^y)| dy \\ &\quad + \sum_{j=1}^n \beta_j \int_{\log(a-1/k) - \log x}^{\log a - \log x} \big|V_k(xe^y) - V(xe^y)\big| dy\bigg) \\ &\leq \lambda |V_k(a) - V(a)| \bigg(\log\Big(\frac{b+1/k}{b}\Big) \sum_{i=1}^m \alpha_i + \log\Big(\frac{a}{a-1/k}\Big) \sum_{i=1}^n \beta_i\bigg) \to 0, \end{aligned} \tag{4.3.54}$$

for $a < x < b$ uniformly in $x$ as $k \to \infty$. Hence, we obtain from the expression in (4.3.52) and the fact that $V_k(x)$ is bounded that the inequality

$$|M_t^k| \leq C + \lambda\, \big|V_k(a) - V(a)\big| \bigg(\log\Big(\frac{b+1/k}{b}\Big) \sum_{i=1}^m \alpha_i + \log\Big(\frac{a}{a-1/k}\Big) \sum_{i=1}^n \beta_i\bigg) t \tag{4.3.55}$$

holds for some constant $C > 0$ and all $t \geq 0$, so that the process $M^k$ is a martingale. Thus, taking the expectation with respect to $P_x$ in (4.3.52), we get

$$E_x\bigg[e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)}\, V_k(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b}) - \int_0^{t\wedge\tau_a\wedge\zeta_b} e^{-\varkappa s}\, (\mathbb{L}V_k - \varkappa V_k)(\widehat{X}_s)\, ds\bigg] = V_k(x), \tag{4.3.56}$$

for all $a < x < b$ and $t \geq 0$. Note that, by virtue of the facts that $P_x(\widehat{X}_{\tau_a} = a) = 0$ and $V_k(x) \to V(x)$ holds for all $x \in \mathcal{D}_X \setminus \{a\}$, we get that $V_k(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b}) \to V(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b})$ ($P_x$-a.s.). Therefore, we have by the dominated convergence that

$$\lim_{k\to\infty} E_x\big[e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)}\, V_k(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b})\big] = E_x\big[e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)}\, V(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b})\big], \tag{4.3.57}$$

and by the uniform convergence in (4.3.54), we obtain

$$\lim_{k\to\infty} E_x\bigg[\int_0^{t\wedge\tau_a\wedge\zeta_b} e^{-\varkappa s}\,(\mathbb{L}V_k - \varkappa V_k)(\widehat{X}_s)\, ds\bigg] = 0, \tag{4.3.58}$$

for $a < x < b$. Hence, we conclude that

$$E_x\big[e^{-\varkappa(t\wedge\tau_a\wedge\zeta_b)}\, V(\widehat{X}_{t\wedge\tau_a\wedge\zeta_b})\big] = \lim_{k\to\infty} V_k(x) = V(x) \tag{4.3.59}$$

holds for all $a < x < b$ and $t \geq 0$. Therefore, the same dominated convergence arguments which were used above complete the proof for the case $\gamma_1 = 0$ as well. □

**4.3.3. The case of a single compound Poisson process** We now show how to find the solution $V(x)$ of the boundary value problem (4.3.7)-(4.3.10) in a single compound Poisson process setting. In particular, we let $m = 1$ and $n = 0$ in (4.3.4) and notice that from (4.3.6) the compensator measure $\nu(dt, dv)$ in (4.2.48) is given by

$$\nu(dt, dv) = \lambda\, dt\, \alpha_1 e^{-\alpha_1 v} I_{\{v>0\}}\, dv, \tag{4.3.60}$$

for some $\lambda, \alpha_1 > 0$, and $\alpha_1 \neq 1$. For notational convenience, we set $G(x) = G_{1,0}(x)$ for $a \leq x \leq b$. Note that the equations in (4.3.32)-(4.3.35) read as

$$\frac{\gamma_1^2 x^3}{2}\, G'''(x) + \big(x\,\beta(x) + x^2\,(\gamma_1^2(\alpha_1+1) - K)\big)\, G''(x) - \lambda\alpha_1\, G(x) \tag{4.3.61}$$

$$+ \bigg((\alpha_1+1)\,(\beta(x) - xK) + \frac{\gamma_1^2(\alpha_1+1)\alpha_1 x}{2} - (\lambda+\varkappa)\, x\bigg)\, G'(x) = 0, \quad \text{for} \quad a < x < b,$$

$$G'(a) = -a^{-\alpha_1-1}, \quad G'(b) = 0, \quad G(b-) = G(b) \equiv 0. \tag{4.3.62}$$

The general solution of (4.3.61) has the form

$$G(x) = C_1\, U_1(x) + C_2\, U_2(x) + C_3\, U_3(x), \quad \text{for} \quad a < x < b, \tag{4.3.63}$$

where $U_1(x)$, $U_2(x)$, and $U_3(x)$ constitute the fundamental system of solutions (i.e. nontrivial linearly independent particular solutions) of (4.3.61), which we assume to be continuously differentiable at $x = a$ and $x = b$. By the definition of $G_{1,0}$ in (4.3.16), we obtain that $V(x) = -x^{\alpha_1+1}G'(x)$ holds, so that $V(x)$ has the form

$$V(x) = -x^{\alpha_1+1}\left(C_1\,U_1'(x) + C_2\,U_2'(x) + C_3\,U_3'(x)\right), \quad \text{for} \quad a < x < b. \tag{4.3.64}$$

It follows from (4.3.62) that the constants $C_1$, $C_2$ and $C_3$ satisfy the equality

$$C_1\,U_1(b) + C_2\,U_2(b) + C_3\,U_3(b) = 0. \tag{4.3.65}$$

Note that when $\gamma_1 \neq 0$, by using (4.3.36) we get

$$G'(a+) = G'(a) \equiv -a^{-\alpha_1-1} \quad \text{and} \quad G'(b-) = G'(b) \equiv 0, \tag{4.3.66}$$

and we obtain from the expression of (4.3.63) that

$$C_1\,U_1'(b) + C_2\,U_2'(b) + C_3\,U_3'(b) = G'(b) \equiv 0, \tag{4.3.67}$$

$$C_1\,U_1'(a) + C_2\,U_2'(a) + C_3\,U_3'(a) = G'(a) \equiv -a^{-\alpha_1-1}, \tag{4.3.68}$$

while when $\gamma_1 = 0$, we again follow the case-by-case analysis as in the previous subsection to find the constants $C_1$, $C_2$, and $C_3$.

**4.3.4. Some examples** In the setting of the latter subsection, let us finally find explicit solutions for the functions $G(x)$, and thus, for the Laplace transform $V_*(x)$ of the first exit time $\tau_a \wedge \zeta_b$ for the process $\widehat{X}$, in several examples considered above.

In the examples considered below, we assume that $\gamma_1 = 0$. In this case, the first derivative $G'(x)$ of every solution below has a right-hand limit $G'(a+)$ at $x = a$ and a left-hand limit $G'(b-)$ at $x = b$, so that the function $V(x)$ is bounded and we can apply Theorem 4.3.1. Moreover, in every example, one of the conditions in case (ia)-(iva) is satisfied and we can determine the constants $C_1$ and $C_2$ in the expression of (4.3.64) from the corresponding conditions in cases (ib)-(ivb), where we put $C_3 = 0$.

**Example 4.3.2.** *(Extended Cox-Ingersoll-Ross model I.)* Let the drift coefficient $\beta(x)$ of the process $X$ be given as in Example 4.2.6 and note that $\mathcal{D}_X = (1, \infty)$. However, we still do not have explicit solutions for the equation in (4.3.61) when $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, we assume that $\eta_0 = 0$, so that we have $\beta(x) = x\overline{\eta}\log(x)$ for $x \in \mathcal{D}_X$, where $\overline{\eta} = \eta_1/2$. By making the Ansatz of $H(y) = G(e^y)$, we get from the equation in (4.3.61) that $H(y)$ solves the second-order ordinary differential equation

$$\left(\overline{\eta}y - \frac{\lambda}{\alpha-1}\right)H''(y) + \left(\overline{\eta}\alpha y - \frac{\alpha\lambda}{\alpha-1} - \lambda - \varkappa\right)H'(y) - \lambda\alpha\,H(y) = 0, \tag{4.3.69}$$

for $y \in (\log a, \log b)$. In particular, it follows from [118; Formulas 2.1.2.108 and 2.1.2.70] that the equation in (4.3.69) admits a general solution $H(y) = G(e^y)$ with $G(x)$ being of the form of (4.3.63) and $C_3 = 0$, where we have

$$U_1(x) = x^r \, \Phi(p, q; z(x)), \quad U_2(x) = x^r \, \Psi(p, q; z(x)), \tag{4.3.70}$$

when $q \neq 0, -1, -2, \ldots$, and

$$U_1(x) = x^r \, z(x)^{1-q} \, \Phi(p - q + 1, 2 - q; z(x)), \tag{4.3.71}$$

$$U_2(x) = x^r \, z(x)^{1-q} \, \Psi(p - q + 1, 2 - q; z(x)), \tag{4.3.72}$$

when $q = 0, -1, -2, \ldots$. Here, we denote

$$p = \begin{cases} -\varkappa/\overline{\eta}, \text{ if } \overline{\eta} < 0, \\ -\lambda/\overline{\eta}, \text{ if } \overline{\eta} > 0, \end{cases}, \quad q = -\frac{(\lambda + \varkappa)}{\overline{\eta}}, \quad r = \begin{cases} -\alpha, \text{ if } \overline{\eta} < 0, \\ 0, \quad \text{if } \overline{\eta} > 0, \end{cases}, \tag{4.3.73}$$

$$z(x) = -\operatorname{sign}(\overline{\eta})\alpha\Big(\log(x) - \frac{\lambda}{\overline{\eta}(\alpha - 1)}\Big), \tag{4.3.74}$$

and the functions $\Phi(x, y; z)$ and $\Psi(x, y; z)$ are the Kummer's and Tricomi's confluent hypergeometric functions (see, e.g. [1; Chapter XIII]), respectively. In particular, we have

$$\Phi(x, y; z) = \frac{\Gamma(y)}{\Gamma(x)\Gamma(y - x)} \int_0^1 e^{zv} \, v^{x-1} (1 - v)^{y-x-1} \, dv, \tag{4.3.75}$$

$$\Psi(x, y; z) = \frac{1}{\Gamma(x)} \int_0^\infty e^{-zv} \, v^{x-1} (1 + v)^{y-x-1} \, dv, \tag{4.3.76}$$

for $y > x > 0$ and $z > 0$, where $\Gamma$ is the gamma function.

**Example 4.3.3.** *(Extended Cox-Ingersoll-Ross model II.)* Let the drift coefficient $\beta(x)$ of the process $X$ be given as in Example 4.2.7 and recall that $\mathcal{D}_X = (0, 1)$. However, we still do not have explicit solutions for the equation in (4.3.61) when $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, we assume that $\eta_1 = 0$ holds, so that we have $\beta(x) = 2\eta_0 x / \log x$ for $x \in \mathcal{D}_X$. By making the Ansatz of $H(y) = G(e^y)$, it follows from the equation in (4.3.61) that the function $H(y)$ satisfies the ordinary differential equation

$$\Big(2\eta_0 - \frac{\lambda}{\alpha_1 - 1} \, y\Big) \, H''(y) + \Big(2\alpha_1\eta_0 - \Big(\frac{\alpha_1\lambda}{\alpha_1 - 1} + \lambda + \varkappa\Big) \, y\Big) \, H'(y) - \lambda\alpha_1 \, y \, H(y) = 0, \tag{4.3.77}$$

for $\log a < y < \log b$. Therefore, by analogy to Example 4.3.2, we get that $G(x)$ is of the form of (4.3.63) with $C_3 = 0$ and the functions $U_1(x)$ and $U_2(x)$ satisfy (4.3.71)-(4.3.72) when $q$ is a nonpositive integer, and (4.3.70) otherwise, where we set

$$D = \frac{((\alpha_1 - 1)\varkappa - \lambda)^2}{(\alpha_1 - 1)^2} + \frac{4\varkappa\lambda\alpha_1}{\alpha_1 - 1}, \quad r = -\frac{(\alpha_1 - 1)\sqrt{D} + \alpha_1\lambda + (\alpha_1 - 1)(\varkappa + \lambda)}{2\lambda}, \tag{4.3.78}$$

$$p = \frac{2\eta_0 r(r + \alpha_1)}{\sqrt{D}}, \quad q = \frac{2\eta_0(\alpha_1 - 1)^2(\lambda + \varkappa)}{\lambda^2}, \tag{4.3.79}$$

$$z(x) = \frac{\sqrt{D}(\alpha_1 - 1)}{\lambda}\Big(\log x - \frac{2\eta_0(\alpha_1 - 1)}{\lambda}\Big). \tag{4.3.80}$$

**Example 4.3.4.** *(Extended constant elasticity of variance model.)* Let the drift coefficient $\beta(x)$ of the process $X$ be given as in Example 4.2.8 with the elasticity parameter $\overline{\alpha}$, where $\mathcal{D}_X = (1, \infty)$ if $\overline{\alpha} \in (0, 1)$ and $\mathcal{D}_X = (0, 1)$ if $\overline{\alpha} \in (1, \infty)$. Notice that, by definition, we have $\beta(x) = \overline{\eta} x \log x$ for $x \in \mathcal{D}_X$, where $\overline{\eta} = \eta_1(1 - \overline{\alpha})$. By making the Ansatz of $H(y) = G(e^y)$, we get from (4.3.61) that $H(y)$ solves the ordinary differential equation in (4.3.69), for $y \in (\log a, \log b)$. Therefore, we get that $G(x)$ is of the form (4.3.63) with $C_3 = 0$, and the functions $U_1(x)$ and $U_2(x)$ satisfy (4.3.71)-(4.3.72) if $q$ is a nonpositive integer, and (4.3.70) otherwise, where $p, q, r$ and $z(x)$ are defined as in (4.3.73)-(4.3.74).

**Example 4.3.5.** *(Extended Shiryaev filtering model.)* Let the drift coefficient $\beta(x)$ of the process $X$ be given as in Example 4.2.9 and note that $\mathcal{D}_X = (0, \infty)$. Notice that, by definition, we have $\beta(x) = \eta_0(1 + x)$ for $x \in \mathcal{D}_X$, where $\overline{\eta} = \eta_0 - \lambda/(\alpha - 1)$.

If we assume that $\overline{\eta} \neq 0$ holds, it follows from (4.3.61) that $G(x)$ satisfies the ordinary differential equation of

$$\Big(\frac{\eta_0}{\overline{\eta}} + x\Big)\, x\, G''(x) + \bigg(\Big(\alpha + 1 - \frac{\lambda + \varkappa}{\overline{\eta}}\Big)\, x + \frac{\eta_0(\alpha + 1)}{\overline{\eta}}\bigg)\, G'(x) - \frac{\lambda\alpha}{\overline{\eta}}\, G(x) = 0, \tag{4.3.81}$$

for $a < x < b$. It follows from [118; Formulas 2.1.2.172 and 2.1.2.171] that the general solution of the second-order ordinary differential equation in (4.3.81) is of the form (4.3.64) with $C_3 = 0$ and

$$U_1(x) = z(x)^{1-q}\, {}_2F_1(p_1 - q + 1, p_2 - q + 1, 2 - q; z(x)), \tag{4.3.82}$$

$$U_2(x) = \begin{cases} U_1(x) \int^x \frac{Z(y)}{U_1(y)^2}\, dy, & \text{if } q \text{ is a (negative) integer}, \\ {}_2F_1\left(p_1, p_2, q; z(x)\right), & \text{otherwise}, \end{cases} \tag{4.3.83}$$

where ${}_2F_1(p, q, r; z)$ is the Gauss hypergeometric function (see, e.g. [1; Chapter XV]) and we denote

$$p_{1,2} = \frac{\alpha\overline{\eta} - (\lambda + \varkappa) \pm \sqrt{(\lambda + \varkappa - \alpha\overline{\eta})^2 - 4\lambda\alpha\overline{\eta}}}{2\overline{\eta}}, \quad q = -\alpha - 1, \quad z(x) = -\frac{\overline{\eta}}{\eta_0}x, \tag{4.3.84}$$

$$Z(x) = \exp\bigg( - \int^x \frac{y(\overline{\eta}(\alpha+1) - \lambda - \varkappa) + (\eta_0(\alpha+1))}{y(\overline{\eta}y + \eta_0)}\, dy\bigg). \tag{4.3.85}$$

On the other hand, if we assume that $\overline{\eta} = 0$ holds, it follows from (4.3.61) that $G(x)$ satisfies the ordinary differential equation

$$\eta_0\, x\, G''(x) + \big(\eta_0(\alpha+1) - (\lambda+\varkappa)\, x\big)\, G'(x) - \lambda\alpha\, G(x) = 0, \tag{4.3.86}$$

for $a < x < b$. Therefore, by analogy to Example 4.3.2, we get that $G(x)$ is of the form of (4.3.63) with $C_3 = 0$ and

$$U_1(x) = e^{xr}\, \Phi(p, q; z(x)), \quad U_2(x) = e^{xr}\, \Psi(p, q; z(x)), \tag{4.3.87}$$

where we denote

$$p = 1 + \frac{\alpha\varkappa}{\lambda+\varkappa}, \quad q = \alpha + 1, \quad r = \frac{\lambda+\varkappa}{\eta_0}, \quad \text{and} \quad z(x) = -\frac{x(\lambda+\varkappa)}{\eta_0}, \tag{4.3.88}$$

and the functions $\Phi(x, y; z)$ and $\Psi(x, y; z)$ are defined as in (4.3.75)-(4.3.76).

# Chapter 5

# On the generalised Laplace transforms of the first exit times in jump-diffusion models of stochastic volatility

This chapter is based on joint work with Dr. Pavel V. Gapeev.

## 5.1. Preliminaries

In this section, we introduce the setting and notation in the problem of computation of the Laplace transforms of the first exit times in (two-dimensional) jump-diffusion models of financial markets with stochastic volatility and formulate the associated boundary value problem.

**5.1.1. The model.** Let us consider a probability space $(\Omega, \mathcal{F}, P)$ supporting two independent standard Brownian motions $B^j = (B^j_t)_{t \geq 0}$, $j = 1, 2$. The processes $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are defined by $X_t = \sum_{k=1}^{N^1_t} J^1_k$ and $Y_t = \sum_{k=1}^{N^2_t} J^2_k$, where $N^i = (N^i_t)_{t \geq 0}$, $i = 1, 2$, are independent Poisson processes of intensity $\lambda_i$, $i = 1, 2$, and $(J^i_k)_{k \in \mathbb{N}}$ are independent exponentially distributed random variables with probability density $e^{-\eta_i^+ x} I(x > 0) + e^{-\eta_i^- x} I(x < 0)$ where $\eta_i^+ \geq 0$ and $\eta_i^- \leq 0$, for $i = 1, 2$, and $I(\cdot)$ denotes the indicator function. Suppose that there exists a process $(S, Q) = (S_t, Q_t)_{t \geq 0}$ which is a (pathwise) unique solution of the system of stochastic differential equations

$$dS_t = \delta\,\sigma^2(Q_t)\,S_t\,dt + \varepsilon\,\sigma(Q_t)\,S_t\,dB_t^1 \qquad (5.1.1)$$
$$+ S_{t-}\int \Big(e^x - 1\Big)\sigma^2(Q_{t-})\,(\mu^X - \nu^X)(dt,dx) \quad (S_0 = s)$$

and

$$dQ_t = \phi(Q_t)\,dt + \psi(Q_t)\,dB_t^2 \qquad (5.1.2)$$
$$+ Q_{t-}\int \Big(e^y - 1\Big)\,(\mu^Y - \nu^Y)(dt,dy) \quad (Q_0 = q)$$

for some $s, q > 0$ fixed, where $\delta$ and $\varepsilon$ are some constants, and $\sigma(q) > 0$ and $\phi(q)$ are continuously differentiable functions of at most linear growth on $(0,\infty)$ (see, e.g. [75; Chapter IV, Theorem 4.6] and [56; Chapter III, Theorem 2.32] for the existence and uniqueness of solutions of stochastic differential equations). Here, $\mu^X(dt,dx)$ and $\mu^Y(dt,dy)$ are the measures of jumps of the processes $X$ and $Y$, and $\nu^X(dt,dx)$ and $\nu^Y(dt,dy)$ are their compensators with respect to the probability measure $P$, respectively. It follows from the structure of the processes $X$ and $Y$ that the compensators have the form $\nu^X(dt,dx) = dtF_1(x;\lambda_1,\eta_1^+,\eta_1^-)dx$ and $\nu^Y(dt,dy) = dtF_2(y;\lambda_2,\eta_2^+,\eta_2^-)\,dy$ with

$$F_i(x;\lambda_i,\eta_i^+,\eta_i^-) = \lambda_i\left(e^{-\eta_i^+ x}\,I(x>0) + e^{-\eta_i^- x}\,I(x<0)\right)dx. \qquad (5.1.3)$$

Without loss of generality and because of the nature of the problems as well as the examples considered below, we can further assume that the state space of the process $S$ is $(0,\infty)$. Observe that the process $Q$ forms a one-dimensional (strong) Markov jump-diffusion process, while $(S,Q)$ provides a two-dimensional jump-diffusion Markov process. We further assume that the state space of $Q$ is $(0,\infty)$, so that the state space of $(S,Q)$ is $(0,\infty)^2$. Let us also define the associated with the processes $S$ and $Q$ first hitting (stopping) times

$$\tau_a^- = \inf\{t \ge 0 \,|\, S_t \le a\} \quad \text{and} \quad \tau_b^+ = \inf\{t \ge 0 \,|\, S_t \ge b\} \qquad (5.1.4)$$

and

$$\zeta_g^- = \inf\{t \ge 0 \,|\, Q_t \le g\} \quad \text{and} \quad \zeta_h^+ = \inf\{t \ge 0 \,|\, Q_t \ge h\} \qquad (5.1.5)$$

for some $0 \le a < b \le \infty$ and $0 \le g < h \le \infty$ fixed.

**5.1.2. Formulation of the problem.** The main aim in the present paper is to derive closed form expressions for the generalised Laplace transforms and other related functionals of the random variables $\tau_a^-$, $\tau_b^+$, $\zeta_g^-$, and $\zeta_h^+$. In this respect, let us introduce the value functions $V_-^*(s,q;a)$ and $V_+^*(s,q;b)$ given by

$$V_-^*(s,q;a) = E_{s,q}\Big[e^{-2\varkappa A_{\tau_a^-}}\, U^*(Q_{\tau_a^-})\, I(\tau_a^- \le \zeta_g^- \wedge \zeta_h^+) \\ + e^{-2\varkappa A_{\zeta_g^-}}\, W_-^*(S_{\zeta_g^-}, Q_{\zeta_g^-}; a)\, I(\zeta_g^- < \tau_a^- \wedge \zeta_h^+)\Big] \tag{5.1.6}$$

and

$$V_+^*(s,q;b) = E_{s,q}\Big[e^{-2\varkappa A_{\tau_b^+}}\, U^*(Q_{\tau_b^+})\, I(\tau_b^+ \le \zeta_g^- \wedge \zeta_h^+) \\ + e^{-2\varkappa A_{\zeta_g^-}}\, W_+^*(S_{\zeta_g^-}, Q_{\zeta_g^-}; b)\, I(\zeta_g^- < \tau_b^+ \wedge \zeta_h^+)\Big] \tag{5.1.7}$$

where the function $U^*(q)$ is defined as

$$U^*(q) = E_q\Big[e^{-\varkappa A_{\zeta_g^-}}\, I(\zeta_g^- < \zeta_h^+)\Big] \tag{5.1.8}$$

the functions $W_-^*(s,q;a)$ and $W_+^*(s,q;b)$ are given by

$$W_-^*(s,q;a) = E_{s,q}\Big[e^{-\varkappa A_{\tau_a^-}}\Big] \quad \text{and} \quad W_+^*(s,q;b) = E_{s,q}\Big[e^{-\varkappa A_{\tau_b^+}}\Big] \tag{5.1.9}$$

for $\varkappa > 0$ and all $(s,q) \in (0,\infty)^2$. Here, the process $A = (A_t)_{t \ge 0}$ is defined by

$$A_t = \int_0^t \sigma^2(Q_u)\, du. \tag{5.1.10}$$

**5.1.3. The boundary-value problems.** By means of standard arguments based on Itô's formula, it can be shown that the infinitesimal operator $\mathbb{L}_{(S,Q)}$ of the process $(S,Q)$ from (5.1.1)-(5.1.2) under the probability measure $P$ acts on a function $V(s,q)$ from the class $C^{2,2}$ on $(0,\infty)^2$ according to the rule:

$$(\mathbb{L}_{(S,Q)}V)(s,q) = (\delta - K_1)\,\sigma^2(q)\, s\, \partial_s V(s,q) + \frac{\varepsilon^2\sigma^2(q)}{2}\, s^2\, \partial_{ss}V(s,q) \\ + (\phi(q) - qK_2)\, \partial_q V(s,q) + \frac{\psi^2(q)}{2}\, \partial_{qq}V(s,q) + \int \Big(V\big(se^x, q\big) - V(s,q)\Big)\sigma^2(q)\, F_1(x;\lambda_1,\eta_1^+,\eta_1^-)\, dx \\ + \int \Big(V\big(s, qe^y\big) - V(s,q)\Big)\, F_2(y;\lambda_2,\eta_2^+,\eta_2^-)\, dy \tag{5.1.11}$$

for all $(s,q) \in (0,\infty)^2$, while the infinitesimal operator $\mathbb{L}_Q$ of the process $Q$ under the probability measure $P$ acts on a function $U$ from the class $C^2$ on $(0,\infty)$ like

$$(\mathbb{L}_Q U)(q) = (\phi(q) - qK_2)U'(q) + \frac{\psi^2(q)}{2}U''(q) \\ + \int \Big(U\big(qe^y\big) - U(q)\Big)F_2(y;\lambda_2,\eta_2^+,\eta_2^-)dy \tag{5.1.12}$$

for all $q > 0$, where

$$K_i = \lambda_i \left( \frac{1}{\eta_i^+ - 1} + \frac{1}{\eta_i^- - 1} \right) \quad \text{for} \quad i = 1, 2. \tag{5.1.13}$$

In order to find analytic expressions for the unknown value functions from (5.1.6)-(5.1.7), let us use the results of general theory of Markov processes (see, e.g. [34; Chapter V]). We reduce the problems of (5.1.6)-(5.1.7) for the functions $V_-^*(s, q; a)$ and $V_+^*(s, q; b)$ to the equivalent boundary value problem

$$(\mathbb{L}_{(S,Q)} V - 2\varkappa \sigma^2(q) V)(s, q) = 0 \quad \text{for} \quad a < s \;\text{ or }\; s < b \quad \text{and} \quad g < q < h \tag{5.1.14}$$

$$V(s, q) = U(q) \quad \text{for} \quad s \le a \quad \text{or} \quad V(s, q) = U(q) \quad \text{for} \quad s \ge b \tag{5.1.15}$$

$$V(s, q)\big|_{s=a+} = U(q) \quad \text{or} \quad V(s, q)\big|_{s=b-} = U(q) \tag{5.1.16}$$

$$V(s, q) = W_\pm(s, q) \quad \text{for} \quad q \le g \quad \text{and} \quad V(s, q) = 0 \quad \text{for} \quad q \ge h \tag{5.1.17}$$

$$V(s, q)\big|_{q=g+} = W_\pm(s, q) \quad \text{and} \quad V(s, q)\big|_{q=h-} = 0 \tag{5.1.18}$$

where the conditions in (5.1.16) and (5.1.18) are satisfied for each $s > 0$ and $q > 0$, respectively. Here, the functions $U(q)$ solve the boundary-value problem

$$(\mathbb{L}_Q U - \varkappa \sigma^2(q) U)(q) = 0 \quad \text{for} \quad g < q < h \tag{5.1.19}$$

$$U(q) = 1 \quad \text{for} \quad q \le g \quad \text{and} \quad U(q) = 0 \quad \text{for} \quad q \ge h \tag{5.1.20}$$

$$U(q)\big|_{q=g+} = 1 \quad \text{and} \quad U(q)\big|_{q=h-} = 0 \tag{5.1.21}$$

while the functions $W_-^*(s, q; a)$ and $W_+^*(s, q; b)$ solve the problem

$$(\mathbb{L}_{(S,Q)} W - \varkappa \sigma^2(q) W)(s, q) = 0 \quad \text{for} \quad a < s < b \tag{5.1.22}$$

$$W(s, q) = 1 \quad \text{for} \quad s \le a \quad \text{or} \quad W(s, q) = 1 \quad \text{for} \quad s \ge b \tag{5.1.23}$$

$$W(s, q)\big|_{s=a+} = 1 \quad \text{or} \quad W(s, q)\big|_{s=b-} = 1 \tag{5.1.24}$$

for all $q > 0$.

## 5.2. Solutions of the boundary value problems

In this section, we derive the solutions of the boundary value problems associated with the value functions in (5.1.6)-(5.1.7).

**5.2.1. Solutions of the system (5.1.19)-(5.1.21).** **(i)** Let us assume that $\lambda_2 = 0$ holds. In this case, it is known that the general solution of the second-order differential equation of (5.1.19) has the form

$$U(q) = D_+ \, U_+(q) + D_- \, U_-(q) \tag{5.2.1}$$

where $D_\pm$ are some arbitrary constants, and the two functions $U_+(q)$ and $U_-(q)$ represent the two fundamental positive solutions (i.e. nontrivial linearly independent particular solutions) of the second-order ordinary differential equation in (5.1.12)+(5.1.19) (see [91; Chapter III, Section 18]). Without loss of generality, we may assume that $U_+(q)$ and $U_-(q)$ are (strictly) increasing and decreasing (convex) functions satisfying the properties $U_+(\infty) = \infty$ and $U_+(0+) = +0$ as well as $U_-(0+) = \infty$ and $U_-(\infty) = +0$, respectively (see, e.g. [95; Chapter V, Section 50] for further details in the purely continuous case). Then, by applying the instantaneous-stopping conditions of (5.1.21) to the function in (5.2.1), we obtain that the equalities

$$D_+ \, U_+(g) + D_- \, U_-(g) = 1 \quad \text{and} \quad D_+ \, U_+(h) + D_- \, U_-(h) = 0 \tag{5.2.2}$$

hold. Solving the system of linear equations in (5.2.2), we obtain the function

$$U(q; g, h) = \frac{U_-(h)U_+(q) - U_+(h)U_-(q)}{U_+(g)U_-(h) - U_+(h)U_-(g)} \tag{5.2.3}$$

which satisfies the system in (5.1.19)-(5.1.21). Note that, in the cases $g = 0$ and $h = \infty$, we see that $D_- \equiv D_-(0, h) = 0$ and $D_+ \equiv D_+(g, \infty) = 0$ should hold in (5.2.3), since otherwise $U(q) \to \pm\infty$ as $q \downarrow 0$ and $q \uparrow \infty$, respectively, which must be excluded, by virtue of the obvious fact that the function $U^*(q)$ in (5.1.8) is bounded. Therefore, using arguments similar to the ones above, we conclude that the functions $U(q; 0, h)$ and $U(q; g, \infty)$ have the form

$$U(q; 0, h) = \frac{U_+(q)}{U_+(h)} \quad \text{and} \quad U(q; g, \infty) = \frac{U_-(q)}{U_-(g)} \tag{5.2.4}$$

for all $q < h$ and $q > g$, respectively.

**(ii)** Let us now assume that $\lambda_2 > 0$, $\eta_2^+ > 0$, and $\eta_2^- = 0$ holds. In this case, by using the boundary conditions in (5.1.20), we can rewrite the equation in (5.1.19) as

$$\begin{aligned} &\frac{\psi^2(q)}{2} \, U''(q) + \big(\phi(q) - qK_2\big) \, U'(q) - \big(\varkappa \, \sigma^2(q) + \lambda_2\big) \, U(q) \\ &+ \lambda_2 \eta_2^+ q^{\eta_2^+} \int_q^h U(y) \, y^{-\eta_2^+ - 1} \, dy = 0. \end{aligned} \tag{5.2.5}$$

We define the function $G(q)$ as

$$G(q) = \int_q^h U(y)\, y^{-\eta_2^+ - 1}\, dy, \tag{5.2.6}$$

for $g \le q \le h$, and notice that solving the ordinary integro-differential boundary problem in (5.2.5)+(5.1.20)-(5.1.21) is equivalent to solving the third-order ordinary differential boundary problem

$$\begin{aligned}
&\frac{\psi^2(q) q^2}{2}\, G'''(q) + \big((\phi(q) - qK_2)q + (\eta_2^+ + 1)\psi^2(q)\big)\, q\, G''(q) && (5.2.7)\\
&+ \bigg(\big( - \varkappa\, \sigma^2(q) - \lambda_2\big)\, q^2 + (\eta_2^+ + 1)\, \big(\phi(q) - qK_2\big)\, q + (\eta_2^+ + 1)\eta_2^+ \frac{\psi^2(q)}{2}\bigg)\, G'(q) \\
&- \lambda_2 \eta_2^+\, q\, G(q) = 0 \quad \text{for} \quad g < q < h, \\
&G(q)\big|_{q=h-} = G(h) \equiv 0, && (5.2.8)\\
&G'(q)\big|_{q=g+} = G'(g) \equiv -g^{-\eta_2^+ - 1} \quad \text{and} \quad G'(q)\big|_{q=h-} = G'(h) \equiv 0. && (5.2.9)
\end{aligned}$$

It is known that the general solution of the third-order differential equation of (5.2.7) has the form

$$G(q) = D_1\, G_1(q) + D_2\, G_2(q) + D_3\, G_3(q), \tag{5.2.10}$$

where $D_i$, $i = 1, 2, 3$, are some arbitrary constants, and the functions $G_i(q)$ represent the three fundamental positive solutions (i.e. nontrivial linearly independent particular solutions) of the third-order ordinary differential equation in (5.2.7) for $i = 1, 2, 3$ (see [91; Chapter III, Section 18]). Hence the solution of (5.1.19)-(5.1.21) is of the form

$$U(q; g, h) = D_1\, U_1(q) + D_2\, U_2(q) + D_3\, U_3(q), \tag{5.2.11}$$

where we set $U_i(q) = -q^{\eta_2^+ + 1} G_i'(q)$ for $g \le q \le h$ and $i = 1, 2$, and the constants $D_i$, $i = 1, 2, 3$, satisfy the equations

$$\begin{aligned}
&D_1\, G_1(h) + D_2\, G_2(h) + D_3\, G_3(h) = 0, && (5.2.12)\\
&D_1\, G_1'(g) + D_2\, G_2'(g) + D_3\, G_3'(g) = -g^{-\eta_2^+ - 1}, && (5.2.13)\\
&D_1\, G_1'(h) + D_2\, G_2'(h) + D_3\, G_3'(h) = 0. && (5.2.14)
\end{aligned}$$

**5.2.2. Solutions of the system (5.1.14)-(5.1.18).** **(i)** Let us assume that $\lambda_1 = \lambda_2 = 0$ holds. In this case, let us now look for a solution of the partial differential equation in (5.1.14) in the form

$$V(s,q) = s^{\theta_+}\big(C_{1,+}\,U_1(q) + C_{2,+}\,U_2(q)\big) + s^{\theta_-}\big(C_{1,-}\,U_1(q) + C_{2,-}\,U_2(q)\big) \tag{5.2.15}$$

where $\theta_\pm$ are given by the roots of the equation

$$\frac{\varepsilon^2}{2}\,\theta(\theta-1) + \delta\,\theta = \varkappa, \tag{5.2.16}$$

so that $\theta_- < 0 < \theta_+$ holds, and we have

$$\theta_\pm = \frac{1}{2} - \frac{\delta}{\varepsilon^2} \pm \sqrt{\left(\frac{\delta}{\varepsilon^2} - \frac{1}{2}\right)^2 + \frac{2\varkappa}{\varepsilon^2}}. \tag{5.2.17}$$

Here, $C_{i,\pm}$ are arbitrary constants, and $U_i(q)$ are the appropriate fundamental positive solutions of the second-order ordinary differential equation in (5.1.12)+(5.1.19) considered above, for $i = 1, 2$. Note that the equalities $C_{1,+} = C_{2,+} = 0$ should hold in (5.2.15) if we consider a solution for $V_-^*(s,q;a)$ in (5.1.6), while the equalities $C_{1,-} = C_{2,-} = 0$ should hold there if we consider a solution for $V_+^*(s,q;b)$ in (5.1.7). These properties occur since otherwise $V(s,q) \to \pm\infty$ as $s \downarrow 0$ or $s \uparrow \infty$, which must be excluded, by virtue of the obvious fact that the functions $V_-^*(s,q;a)$ in (5.1.6) and $V_+^*(s,q;b)$ in (5.1.7) are bounded. Then, by applying the instantaneous-stopping conditions of (5.1.16)-(5.1.18) to the function in (5.2.15), we obtain that the equalities

$$a^{\theta_-}\big(C_{1,-}\,U_1(q) + C_{2,-}\,U_2(q)\big) = U(q;g,h) \tag{5.2.18}$$

and

$$b^{\theta_+}\big(C_{1,+}\,U_1(q) + C_{2,+}\,U_2(q)\big) = U(q;g,h) \tag{5.2.19}$$

hold. Solving the equations in (5.2.18)-(5.2.19), we obtain the functions

$$V_-(s,q;a;g,h) = \left(\frac{s}{a}\right)^{\theta_-} U(q;g,h) \quad \text{and} \quad V_+(s,q;b;g,h) = \left(\frac{s}{b}\right)^{\theta_+} U(q;g,h) \tag{5.2.20}$$

which satisfy the system in (5.1.14)-(5.1.18), where the functions $U(q;g,h)$ is given by (5.2.3).

**(ii)** Let us now assume that $\lambda_1 > 0$, $\lambda_2 = 0$, $\eta_1^+ > 0$, and $\eta_1^- = 0$ holds. In this case, let us look for a solution of the partial differential equation in (5.1.14) in the form

$$V(s,q) = s^{\theta_+}\big(C_{1,+}\,U_1(q) + C_{2,+}\,U_2(q)\big) + s^{\theta_-}\big(C_{1,-}\,U_1(q) + C_{2,-}\,U_2(q)\big) \tag{5.2.21}$$

where $\theta_\pm$ are given by two roots of the equations

$$(\delta - K_1)\,\theta^2 - (\eta_1^+(\delta - K_1) + \lambda_1)\,\theta = \varkappa \quad \text{if} \quad \lambda_1, \eta_1^+ > 0, \text{ and } \varepsilon = 0, \tag{5.2.22}$$

$$(\theta - \eta_1^+)\left(\frac{\varepsilon^2}{2}\,\theta(\theta-1) + (\delta - K_1)\,\theta - \lambda_1\right) - \eta_1^+\lambda_1 = \varkappa, \quad \text{if} \quad \lambda_1, \eta_1^+, \varepsilon > 0, \tag{5.2.23}$$

so that $\theta_- < 0 < \theta_+$ holds, and we set

$$\theta_\pm = \frac{\eta_1^+}{2} + \frac{\lambda_1}{2(\delta - K_1)} \pm \sqrt{\Big(\frac{\eta_1^+}{2} + \frac{\lambda_1}{2(\delta - K_1)}\Big)^2 + \frac{\varkappa}{\delta - K_1}} \text{ if } \lambda_1, \eta_1^+ > 0, \text{ and } \varepsilon = 0. \quad (5.2.24)$$

Here $C_{i,\pm}$ are arbitrary constants and $U_i(q)$ are the appropriate fundamental positive solutions from (5.2.11) of the ordinary integro-differential equation (5.2.5) considered above for $i = 1, 2$. Moreover, in order for (5.1.14)+(5.1.15) to be satisfied, we choose $\eta_1^+ > 0$ such that

$$\eta_1^+ b^{\theta_\pm} + \theta_\pm - \eta_1^+ = 0. \quad (5.2.25)$$

Note that the equalities $C_{i,+} = 0$ should hold in (5.2.21) if we consider a solution for $V_-^*(s, q; a)$ in (5.1.6), while the equalities $C_{i,-} = 0$ should hold there if we consider a solution for $V_+^*(s, q; b)$ in (5.1.7), for $i = 1, 2$. These properties occur since otherwise $V(s, q) \to \pm\infty$ as $s \downarrow 0$ or $s \uparrow \infty$, which must be excluded, by virtue of the obvious fact that the functions $V_-^*(s, q; a)$ in (5.1.6) and $V_+^*(s, q; b)$ in (5.1.7) are bounded. Then, by applying the instantaneous-stopping conditions of (5.1.16)-(5.1.18) to the function in (5.2.21), we obtain that the equalities

$$a^{\theta_-} \big(C_{1,-} \, U_1(q) + C_{2,-} \, U_2(q)\big) = U(q; g, h) \quad (5.2.26)$$

and

$$b^{\theta_+} \big(C_{1,+} \, U_1(q) + C_{2,+} \, U_2(q)\big) = U(q; g, h) \quad (5.2.27)$$

should hold. Solving the equations in (5.2.26)-(5.2.27), we obtain the functions

$$V_-(s, q; a; g, h) = \Big(\frac{s}{a}\Big)^{\theta_-} U(q; g, h) \quad \text{and} \quad V_+(s, q; b; g, h) = \Big(\frac{s}{b}\Big)^{\theta_+} U(q; g, h) \quad (5.2.28)$$

which satisfy the system in (5.1.14)-(5.1.18), where the function $U(q; g, h)$ is given by (5.2.11).

**3.3. Solutions of the system (5.1.22)-(5.1.24). (i)** Let us assume that $\lambda_1 = \lambda_2 = 0$ holds and look for a solution of the partial differential equation in (5.1.22) in the form

$$W(s, q) = C_+ \, s^{\theta_+} + C_- \, s^{\theta_-} \quad (5.2.29)$$

where $C_\pm$ are some arbitrary constants, and $\theta_\pm$ are given by (5.2.17). Then, by applying the instantaneous-stopping conditions of (5.1.24) to the function in (5.2.29), we obtain that the equalities

$$C_+ \, a^{\theta_+} + C_- \, a^{\theta_-} = 1 \quad \text{or} \quad C_+ \, b^{\theta_+} + C_- \, b^{\theta_-} = 1 \quad (5.2.30)$$

hold. Note that, in the cases $s \downarrow 0$ or $s \uparrow \infty$, we see that $C_- = 0$ or $C_+ = 0$ should hold in (5.2.29), since otherwise $W(s, q) \to \pm\infty$ as $s \downarrow 0$ and $s \uparrow \infty$, respectively, which must be excluded, by virtue of the obvious fact that the functions $W_-^*(s,q;b)$ in (5.1.9) and $W_+^*(s,q;a)$ in (5.1.10) are bounded. Therefore, using arguments similar to the ones above, we conclude that the functions $W_+^*(s,q;b)$ and $W_-^*(s,q;a)$ have the form

$$W_+^*(s,q;b) = (s/b)^{\theta_+} \quad \text{and} \quad W_-^*(s,q;a) = (s/a)^{\theta_-} \tag{5.2.31}$$

for all $s<b$ and $s>a$, respectively.

**(ii)** Let us now assume that $\lambda_1>0$, $\lambda_2=0$, $\eta_1^+>0$, and $\eta_1^-=0$ holds. We look for solution of the form (5.2.29), where $\theta_\pm$ are given by (5.2.24) or by a positive and a negative root of the cubic equation (5.2.23) when $\varepsilon=0$ or $\varepsilon>0$, respectively, and $\eta_1^+$ satisfies the equality in (5.2.25). Then, similarly as in the previous case, by applying the instantaneous-stopping conditions of (5.1.24) to the function in (5.2.29), we obtain the equalities in (5.2.30) and conclude that $W_+^*(s,q;b)$ and $W_-^*(s,q;a)$ have the form (5.2.31).

**5.2.3. Some examples.** Let us now consider several examples in which we can obtain explicit solutions of the boundary value problems formulated above.

**Example 5.2.1.** Let us consider the case of the (mean-reverting) exponential Stein-Stein model of stochastic volatility for $(S,Q)$. In this case, we set $\sigma(q)=\ln q$, $\phi(q)=q(\alpha-\beta\ln q+\gamma^2/2)$, and $\psi(q)=\gamma q$, for some constants $\alpha\ge 0$, $\beta$, and $\gamma>0$, so that $Q$ is an exponential Ornstein-Uhlenbeck process (Black-Karasinski model) with the state space $E=(0,\infty)$.

Let us assume that $\lambda_2=0$ and denote $\overline{U}(\overline{q})=U(e^{\overline{q}})$ for $\overline{q}\in\mathbb{R}$. Then, the equation in (5.2.5) can be written as

$$\frac{\gamma^2}{2}\,\overline{U}''(\overline{q}) + \big(\alpha-\beta\overline{q}\big)\,\overline{U}'(\overline{q}) - \varkappa\overline{q}^2\,\overline{U}(\overline{q}) = 0. \tag{5.2.32}$$

It follows from [118; Formulas 2.1.31 and 2.1.108] that second-order ordinary differential equation in (5.2.32) admits a general solution $U(q)$ of the form (5.2.1) with

$$U_+(e^{\overline{q}}) = e^{k\overline{q}^2+r\overline{q}}\,\Phi(p,1/2,z(\overline{q})) \quad \text{and} \quad U_-(e^{\overline{q}}) = e^{k\overline{q}^2+r\overline{q}}\,\Psi(p,1/2,z(\overline{q})), \tag{5.2.33}$$

where the constant $k$ solves the quadratic equation $4\gamma^2k^2-4\beta k-2\varkappa\gamma=0$. Here, we denote

$$p=\frac{\gamma^2r^2+2\alpha r+2\gamma^2k}{4(2\gamma^2k-\beta)}, \quad z(\overline{q})=\frac{\beta-2\gamma^2k}{\gamma^2}\Big(\overline{q}-\frac{\alpha\beta}{2\gamma^2k-\beta}\Big)^2, \tag{5.2.34}$$

where

$$r=-\frac{2\alpha k}{2\gamma^2k-\beta}, \tag{5.2.35}$$

and the functions $\Phi$ and $\Psi$ are the Kummer's and Tricomi's confluent hypergeometric functions (see, e.g. [1; Chapter XIII]), respectively. In particular, we have

$$\Phi(x,y,z) = \frac{\Gamma(y)}{\Gamma(x)\Gamma(y-x)} \int_0^1 e^{zv}\, v^{x-1}(1-v)^{y-x-1}\, dv, \tag{5.2.36}$$

$$\Psi(x,y,z) = \frac{1}{\Gamma(x)} \int_0^\infty e^{-zv}\, v^{x-1}(1+v)^{y-x-1}\, dv, \tag{5.2.37}$$

for $y > x > 0$ and $z > 0$, where $\Gamma$ is the gamma function.

**Example 5.2.2.** Let us consider the case of the (mean-reverting) Heston model of stochastic volatility for $(S,Q)$. In this case, we set $\sigma(q) = \sqrt{q}$, $\phi(q) = \alpha - \beta q$, and $\psi(q) = \gamma\sqrt{q}$, for some constants $\alpha \geq 0$, $\beta$, and $\gamma > 0$ such that $\alpha > \gamma^2/2$, so that $Q$ is a Feller square root process (Cox-Ingersoll-Ross model) with the state space $E = (0,\infty)$.

Let us assume that $\lambda_2 = 0$. Then, the equation in (5.2.5) is given by

$$\frac{\gamma^2 q}{2}\, U''(q) + (\alpha - \beta q)\, U'(q) - \varkappa q\, U(q) = 0. \tag{5.2.38}$$

It follows from [118; Formula 2.1.108] that second-order ordinary differential equation in (5.2.38) admits a general solution $U(q)$ of the form (5.2.1) with

$$U_+(q) = e^{rq}\, \Phi(p_1, p_2, z(q)), \quad U_-(q) = e^{rq}\, \Psi(p_1, p_2, z(q)), \tag{5.2.39}$$

when $p_2 \neq 0, -1, -2, \dots$, and

$$U_+(q) = e^{rq}\, z(q)^{1-p_2}\, \Phi(p_1 - p_2 + 1, 2 - p_2, z(q)), \tag{5.2.40}$$

$$U_-(q) = e^{rq}\, z(q)^{1-p_2}\, \Psi(p_1 - p_2 + 1, 2 - p_2, z(q)), \tag{5.2.41}$$

when $p_2 = 0, -1, -2, \dots$. Here, we have denoted

$$p_1 = \frac{\alpha r}{\sqrt{D}}, \quad p_2 = \frac{2\alpha}{\gamma^2}, \quad \text{and} \quad z(q) = -\frac{2\sqrt{D}q}{\gamma^2}, \tag{5.2.42}$$

where

$$r = \frac{\sqrt{D} + \beta}{\gamma^2}, \quad D = \beta^2 + 2\varkappa\gamma^2, \tag{5.2.43}$$

and the functions $\Phi$ and $\Psi$ are defined as in (5.2.36)-(5.2.37).

Let us now assume that $\lambda_2 > 0$, $\eta_2^+ > 0$, and $\eta_2^- = 0$ holds. Moreover, let us assume that $\psi(q) \equiv 0$ holds and denote $\overline{\beta} = \beta + K_2$. Then, the equation in (5.2.7) takes the form

$$q(\alpha - \overline{\beta}q)\, G''(q) + \big((\eta_2^+ + 1)\,(\alpha - \overline{\beta}q) - \varkappa\, q^2 - \lambda_2\, q\big)\, G'(q) - \lambda_2\eta_2^+\, G(q) = 0. \tag{5.2.44}$$

If we additionally assume that $\alpha = 0$, it follows from [118; Formulas 2.1.139 and 2.1.108] that the second-order ordinary equation admits the general solution $G(q)$ of the form (5.2.10) with $D_3 = 0$, where we have

$$G_1(q) = q^k\, e^{rq}\, \Phi(p_1, p_2, z(q)), \quad G_2(q) = q^k\, e^{rq}\, \Psi(p_1, p_2, z(q)), \tag{5.2.45}$$

when $p_2 \neq 0, -1, -2, \dots$, and

$$G_1(q) = q^k\, e^{rq}\, z(q)^{1-p_2}\, \Phi(p_1 - p_2 + 1, 2 - p_2, z(q)), \tag{5.2.46}$$

$$G_2(q) = q^k\, e^{rq}\, z(q)^{1-p_2}\, \Psi(p_1 - p_2 + 1, 2 - p_2, z(q)), \tag{5.2.47}$$

when $p_2 = 0, -1, -2, \dots$. Here, we denote

$$p_1 = \frac{(-2\overline{\beta}k - \lambda_2 - \overline{\beta}(\eta_2^+ + 1))r - \varkappa k}{\varkappa}, \quad p_2 = \frac{2\overline{\beta}k + \lambda_2 + \overline{\beta}(\eta_2 + 1)}{\overline{\beta}}, \quad z(q) \; = \frac{\varkappa q}{\overline{\beta}}, \tag{5.2.48}$$

where

$$r = \frac{-\varkappa}{2\overline{\beta}}, \quad -\overline{\beta}k^2 + (-\lambda_2 - \overline{\beta}\eta_2^+)k - \lambda_2\eta_2^+ = 0, \tag{5.2.49}$$

and the functions $\Phi$ and $\Psi$ are defined as in (5.2.36)-(5.2.37).

**Example 5.2.3.** Let us consider the case of the Dothan model of stochastic volatility for $(S, Q)$. In this case, we set $\sigma(q) = q$, $\phi(q) = \beta q$, and $\psi(q) = \gamma q$, for some constants $\beta$ and $\gamma > 0$, so that $Q$ is a geometric Brownian motion with the state space $E = (0, \infty)$.

Let us assume that $\lambda_2 = 0$ holds. Then, the equation in (5.2.5) takes the form

$$\frac{\gamma^2 q}{2}\, U''(q) + \beta\, U'(q) - \varkappa q\, U(q) = 0. \tag{5.2.50}$$

It follows from [118; Formula 2.1.108] that the general solution of the second-order ordinary equation in (5.2.50) is of the form of (5.2.1) with the functions $U_+(q)$ and $U_-(q)$ satisfying (5.2.40)-(5.2.41) if $p_2$ is a nonpositive integer, and (5.2.39) otherwise, where we have denoted

$$p_1 = \frac{\beta r}{\sqrt{D}}, \quad p_2 = \frac{2\beta}{\gamma^2}, \quad z(q) = -\frac{2\sqrt{D}q}{\gamma^2}, \tag{5.2.51}$$

with

$$r = \frac{\sqrt{D}}{\gamma^2}, \quad D = 2\varkappa\gamma^2. \tag{5.2.52}$$

Let us now assume that $\lambda_2 > 0$, $\eta_2^+ > 0$, and $\eta_2^- = 0$ holds Moreover, let us assume that $\psi(q) \equiv 0$ holds and denote $\overline{\beta} = \beta - K_2$. Then, the equation in (5.2.7) takes the form

$$\overline{\beta} q^2 \, G''(q) + \big( (\eta_2^+ + 1)\overline{\beta} - \lambda_2 - \varkappa \, q^2 \big) \, q \, G'(q) - \lambda_2 \eta_2^+ \, G(q) = 0. \tag{5.2.53}$$

If we additionally assume that $\overline{\beta} = 0$, the equation in (5.2.53) admits an explicit solution which is of the form of (5.2.10) with $D_2 = D_3 = 0$, and the function $G_1(q)$ is given by

$$G_1(q) = \Big( \frac{\lambda_2 + \varkappa q^2}{q^2} \Big)^{\eta_2^+/2}. \tag{5.2.54}$$

**Example 5.2.4.** Let us finally consider the case of the (two-dimensional) Black-Merton-Scholes model for $(S, Q)$. In this case, we set $\sigma(q) = 1$, $\phi(q) = \beta q$, and $\psi(q) = \gamma q$, for some constants $\beta$ and $\gamma > 0$, so that $Q$ is a geometric Brownian motion with the state space $E = (0, \infty)$.

Let us assume that $\lambda_2 = 0$. Then, the equation in (5.2.5) takes the form

$$\frac{\gamma^2 q^2}{2} \, U''(q) + \beta \, q \, U'(q) - \varkappa \, U(q) = 0. \tag{5.2.55}$$

It follows from [118; Formula 2.1.123] that the general solution of the second-order ordinary differential equation in (5.2.55) is of the form (5.2.1) with

$$U_+(q) = q^{(1-p_1+2p_2)/2}, \quad U_-(q) = q^{(1-p_1-2p_2)/2}, \tag{5.2.56}$$

where we denote

$$p_1 = \frac{2\beta}{\gamma^2}, \quad p_2 = \frac{1}{2} \sqrt{(1-p_1)^2 + 4\varkappa}. \tag{5.2.57}$$

Let us now assume that $\lambda_2 > 0$, $\eta_2^+ > 0$, and $\eta_2^- = 0$ holds. Moreover, let us assume that $\psi(q) \equiv 0$ holds and denote $\overline{\beta} = \beta - K_2$. Then, the equation in (5.2.7) takes the form

$$\overline{\beta} q^2 \, G''(q) + (\eta_2^+ + 1 - \varkappa - \lambda_2) \, q \, G'(q) - \lambda_2 \eta_2^+ \, G(q) = 0. \tag{5.2.58}$$

It follows from [118; Formula 2.1.123] that the second-order ordinary differential equation admits the general solution $G(q)$ of the form (5.2.10) with $D_3 = 0$, and the functions $G_1(q)$ and $G_2(q)$ satisfying the same equation (5.2.56) as the functions $U_+(q)$ and $U_-(q)$, respectively, where we set

$$p_1 = \frac{\eta_2^+ + 1 - \varkappa - \lambda_2}{\overline{\beta}}, \quad p_2 = -\lambda_2 \eta_2^+. \tag{5.2.59}$$

## 5.3. Main result and proof

In this section, taking into account the facts proved above, we formulate and prove the main results of the paper.

**Theorem 5.3.1.** *Suppose that the coefficients $\sigma(q) > 0$, $\phi(q)$, and $\psi(q) > 0$ of the jump-diffusion process $(S, Q)$ defined by (5.1.1)-(5.1.2) are continuously differentiable functions of at most linear growth. Then, the generalised transforms $V_-^*(s, q; \theta_- \rho; a)$ and $V_+^*(s, q; \theta_+ \rho; b)$ in (5.1.6)-(5.1.7) of the associated with $(S, Q)$ random times $\tau_a^-$, $\tau_b^+$, $\zeta_g^-$, and $\zeta_h^+$ from (5.1.4)-(5.1.5) admit the representations*

$$V_-^*(s, q; a) = V_-(s, q; a; g, h) = W_-^*(s, q; a)\, U(q; g, h) \tag{5.3.1}$$

*and*

$$V_+^*(s, q; b) = V_+(s, q; b; g, h) = W_+^*(s, q; b)\, U(q; g, h) \tag{5.3.2}$$

*for all $a < s < b$ and $g < q < h$, and any $0 < a < b < \infty$ and $0 < g < h < \infty$ fixed. Here, the function $U(q; g, h)$ takes the form of (5.2.3)-(5.2.4) and (5.2.11), and the functions $W_-^*(s, q; a)$ and $W_+^*(s, q; b)$ take the form of (5.2.31).*

Since the assertions stated above are proved using essentially similar arguments for all the cases of $\varepsilon > 0$ and $\psi(q) > 0$, $\varepsilon > 0$ and $\psi(q) = 0$, $\varepsilon = 0$ and $\psi(q) > 0$, and $\varepsilon = 0$ and $\psi(q) = 0$, we only give a proof for the case $\varepsilon > 0$ and $\psi(q) > 0$ in which both processes $S$ and $Q$ have continuous diffusion parts. Note that the corresponding verification assertions for the value functions $U^*(q) = U(q; g, h)$ for $g < q < h$ in (5.1.8) and (5.2.3)-(5.2.4), and $W_-^*(s, q; a)$ and $W_+^*(s, q; b)$ for $a < s < b$ in (5.1.9) and (5.2.31) can be proved using the arguments similar to the ones presented below.

*Proof.* In order to verify the assertion stated above, it remains to show that the functions defined in (5.3.1)-(5.3.2) coincides with the value functions in (5.1.6)-(5.1.7). For this purpose, let us denote by $V(s, q)$ the right-hand side of the expression in (5.3.1) (the case (5.3.2) is analogical). Then, taking into account the fact that the function $V(s, q)$ is continuous on $(0, \infty)^2$ and $C^2$ on $(a, \infty) \times (g, h)$, by applying the change-of-variable formula for semimartingales with jumps of bounded variation from [87; Theorem 3.1] to $e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t})$, we obtain

$$e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}) \tag{5.3.3}$$
$$= V(s,q) + \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} e^{-2\varkappa A_u} \left( \mathbb{L}_{(S,Q)} V(S_u, Q_u) - 2\varkappa\, \sigma^2(Q_u)\, V(S_u, Q_u) \right) du + M_t$$

for all $a < s$ and $g < q < h$, where the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t = \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} e^{-2\varkappa A_u}\, \partial_s V(S_u, Q_u)\, \sigma(Q_u)\, S_u\, dB_u^1 \tag{5.3.4}$$
$$+ \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} e^{-2\varkappa A_u}\, \partial_q V(S_u, Q_u)\, \psi(Q_u)\, dB_u^2$$
$$+ \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} \int e^{-2\varkappa A_u} \Big( V(S_{u-} e^x, Q_{u-}) - V(S_{u-}, Q_{u-}) \Big) (\mu^X - \nu^X)(du, dx)$$
$$+ \int_0^{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t} \int e^{-2\varkappa A_u} \Big( V(S_{u-}, Q_{u-} e^y) - V(S_{u-}, Q_{u-}) \Big) (\mu^Y - \nu^Y)(du, dy)$$

is a local martingale under $P_{s,q}$.

By virtue of straightforward calculations and the arguments of the previous section, it is verified that the function $V(s,q)$ solves the partial (integro-)differential equation in (5.1.11)+(5.1.14), so that the expression in (5.3.3) takes the form

$$e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}) = V(s,q) + M_t \tag{5.3.5}$$

for all $a < s$ and $g < q < h$. Since the function $V(s,q)$ satisfies the boundary conditions in (5.1.15)-(5.1.18) and is therefore bounded, it follows from the representation of (5.3.5) that the process $M$ is a uniformly integrable martingale. Then, taking the expectation with respect to $P_{s,q}$ in both sides of the expression in (5.3.5), by means of the optional sampling theorem (see, e.g. [56; Chapter I, Theorem 1.39]), we get

$$E_{s,q}\big[e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t})\big] = V(s,q) + E_{s,q} M_t = V(s,q) \tag{5.3.6}$$

for all $a < s$ and $g < q < h$. Therefore, letting $t$ go to infinity and using the boundary conditions in (5.1.15)-(5.1.18) as well as the fact that $e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}) = 0$ on $\{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ = \infty\}$ ($P_{s,q}$-a.s.), we can apply the Lebesgue dominated convergence theorem for the expression in (5.3.6) to obtain the equalities

$$E_{s,q}\Big[e^{-2\varkappa A_{\tau_a^-}}\, U^*(Q_{\tau_a^-})\, I(\tau_a^- \leq \zeta_g^- \wedge \zeta_h^+) + e^{-2\varkappa A_{\zeta_g^-}}\, W_-^*(S_{\zeta_g^-}, Q_{\zeta_g^-}; a)\, I(\zeta_g^- < \tau_a^- \wedge \zeta_h^+)\Big] \tag{5.3.7}$$
$$= E_{s,q}\big[e^{-2\varkappa A_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}} V(S_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t}, Q_{\tau_a^- \wedge \zeta_g^- \wedge \zeta_h^+ \wedge t})\big] = V(s,q)$$

or for all $a < s$ and $g < q < h$, which directly implies the desired assertion. □

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