Study of New Models for Insider Trading and Impulse Control

Pucheng Shi

Department of Mathematics

London School of Economics and Political Science

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

Abstract

This thesis presents the development and study of two stochastic models. The first one is an equilibrium model for a market involving risk-averse insider trading. In particular, the static information model is considered under new assumptions: a) the insider is risk-averse, b) the signal received by the insider is not necessarily Gaussian, and c) the price set by the market maker is a function of a weighted signal that is not necessarily Gaussian either. Conditions on the weighting and pricing functions ensuring the existence of equilibrium are discussed. Equilibrium pricing and weighting functions as well as the insider's optimal trading strategy are derived. Furthermore, the influence of the risk aversion on the equilibrium outcome is investigated.

The second model studied, we derive the explicit solution to an impulse control problem with non-linear penalisation of control expenditure. This solution has several features that are not present in impulse control problems with affine penalisation of control effort. The state dependence of the free-boundaries characterising the optimal strategy is the first one. The possibility for the so-called continuation region to not be an interval and the optimal strategy to involve multiple simultaneous jumps while the problem data is convex are further such aspects.

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Contents

Chapter 1

Introduction

In the first part of the thesis, we consider a new model of insider trading. The phenomenon of insider trading in stock market has attracted significant interest from both economists and mathematicians. The characterisation of the optimal strategy for an insider who possesses superior information than general public has been widely studied in financial mathematics. Especially with the development of enlargement of filtration by Jeulin and Yor [32], there has been quite an interests studying models of insider trading, e.g., Ankirchner, Dereich and Imkeller [2]. These papers have a common assumption: the insider's trading amount will not affect the market pricing dynamic.

On the other hand, assuming the opposite, consider "large investors", Kyle [34] studied the equilibrium model and showed the existence of unique linear equilibrium if the asset value is Gaussian random variable. Under such equilibrium the price process is a Brownian motion in market maker's filtration and Brownian bridge in the insider's filtration. Back [5] assumed the price process to depend only on the cumulative order of the stocks. He considered the model under continuous time trading where the insider can infer the cumulative trading amount of noise traders by observing the price process continuously. Under such assumptions, he extended Kyle's result and proved the existence of equilibrium beyond Gaussian linear framework. Both Kyle and Back studied the risk neutral case where the insider has linear utility function.

Cho [12] followed Back's framework, considered risk averse cases where the insider has exponential utility function. He showed existence of equilibrium under Gaussian linear framework. In addition, he allowed market maker to determine the price process depending not only on the cumulative order of the asset, but also took into account the history of cumulative order. From an economic sense, the more recent trade is better indicator to market maker to determine the price of the asset.

Here, we follow Back's framework, with continuous trading, and extend Cho's results to the risk averse case. In particular, for exponential utility, we characterise all optimal strategies for the insider within a general, non-Gaussian framework. We also establish one inconspicuous equilibrium that allows the insider to trade undetected by the market maker. Moreover, we introduce the weighting function for market maker to determine the price process depending on the paths of the cumulative order.

In the second part of the thesis, we consider a stochastic system whose state dynamics are given by

$$X_t = x - \bar{Z}_t + W_t, \quad \text{for } t \geq 0,$$

where W is a standard one-dimensional Brownian motion and $\bar{Z}$ is an impulsively controlled process. In particular, the process $\bar{Z}$ is given by

$$\bar{Z}_t = \sum_{n=1}^{\infty} Z_n \mathbf{1}_{\{\tau_n < t\}},$$

where (τ_n) is the increasing sequence of stopping times at which impulsive action is applied to the system and the positive real-valued random variables Z_n, $n \geq 1$, are the sizes of the corresponding actions. In this context, the collection

$$\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; Z_1, Z_2, \ldots, Z_n, \ldots)$$

fully characterises any admissible control strategy. The objective of the optimisation problem that we study is to minimise the performance criterion

$$J_x(\mathcal{Z}) = \mathbb{E}\left[\lambda \int_0^\infty e^{-\delta t} X_t^2 \, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^\alpha\big)\right]$$

over all strategies $\mathcal{Z}$, where $\alpha, \delta, \kappa, \lambda > 0$ are given constants. The problem's value function is defined by

$$v(x) = \inf_{\mathcal{Z}} J_x(\mathcal{Z}), \quad \text{for } x \in \mathbb{R}.$$

The theory of stochastic impulse control has attracted considerable interest and has been applied in several fields. In mathematical finance, economics and operations research, important contributions include Richard [51], Harrison, Sellke and Tayor [27], Mundaca and Øksendal [43], Korn [35], Bar-Ilan, Sulem and Zanello [11], Bar-Ilan, Perry and Stadje [10], Ohnishi and Tsujimura [47], Cadenillas and Zapatero [14], Cadenillas, Sarkar and Zapatero [19], Cadenillas, Lakner and Pinedo [18], Feng and Muthuraman [25], Jeanblanc-Picqué and Shiryaev [30], Alvarez and Lempa [4], and several references therein. Models motivated by the optimal management of renewable resources have been studied by Alvarez [1], and Alvarez and Koskela [3]. Also, the general mathematical theory of stochastic impulse control is well-developed: see Lepeltier and Marchal [41], Perthame [48], Djehiche, Hamadène and Hdhiri [21], as well as the books by Bensoussan and Lions [8], Øksendal and Sulem [46], Pham [50], and several references therein.

In view of the general theory of stochastic impulse control, the value function of the optimisation problem that we study identifies with a classical solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$\min \left\{ \frac{1}{2} w''(x) - \delta w(x) + \lambda x^2, \; -w(x) + \inf_{z>0} \big[w(x-z) + 1 + \kappa z^\alpha \big] \right\} = 0.$$

Our objective is to derive and characterise the solution to this quasi-variational inequality. To the best of our knowledge, this is the first impulse control problem with non-linear penalisation of control expenditure that has been explicitly solved in the literature. It turns out that its solution has features that have not been observed in the literature. These include the state dependence of the free-boundaries characterising the optimal strategy as well as the possibility for the so-called continuation region to not be an interval despite the problem data being convex. Furthermore, it may be the case that minimal costs can be achieved only by multiple simultaneous jumps, which implies that an optimal strategy may not exist.

Chapter 2

Insider trading with static information: impact of insider's risk aversion on equilibrium

2.1 Market Model

In this thesis, we model the market affected by private information. In particular, we consider a company which released a risky asset (i.e. a claim on the company value). This asset is assumed to be traded continuously. At some future time, assumed to be time 1 without loss of generality[1], the value of the company, V will become public. As all the agents will agree on the value of the company, they also will agree on the price of the asset being V. For simplicity, we assume no information release directly to the public between the beginning of the market and time 1. If all the agents in the market are risk-neutral, this will imply constant price until the information release and abrupt price adjustment at the moment of information release.

[1]The choice of deterministic time of the information release has no impact on our market model, the generalisation of any other time will be straightforward.

To describe this model in rigorous terms, consider filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{Q}),$$

satisfying the usual conditions. We assume that this probability space is large enough to support a Brownian motions B as well as a normally distributed random variable Z which is independent of B.

We assume there is a risk-less asset on the market and for simplicity we set interest rate to zero. The price of the risky asset is determined by the company's fundamental value at time 1, V, which will be released at time 1. We assume $V = f(Z)$ where the function $f : \mathbb{R} \to \mathbb{R}$ and random variable Z satisfy the following assumption.

Assumption 1 *We assume the fundamental price, V, satisfies:*

1. *f is continuously differentiable and strictly increasing.*

2. *$|f|$ and $|f'|$ are bounded by constant K.*

3. *$\lim_{x\downarrow-\infty} f'(x) = \lim_{x\uparrow\infty} f'(x) = 0$.*

4. *$Z = \mathcal{N}(0, 1)$.*

Remark 1 *The assumption that f is strictly increasing implies that the larger the signal Z the large the asset value. Since f is bounded, we immediately have $\mathbb{E}[f^2(Z)] < \infty$, i.e., the terminal price of the asset is in L^2. Since f is strictly increasing and bounded, we have the limits exist for f when $x \to \pm\infty$. Denote b and d to be the upper and lower limits of f, i.e., $\lim_{x\to-\infty} f(x) = b$ and $\lim_{x\to\infty} f(x) = d$. By assuming $V = f(Z)$ with above conditions, we capture most random variables with smooth distribution functions for V.*

The agents in the market are differentiated by the information they have access to, hence by filtrations their actions are adapted to. In particular, we consider three types of agents populating the market: *noise trader*, *market maker* and *insider*.

Noise Traders trade for reasons other than maximising their utilities, for example for liquidity reasons by Grossman and Stiglitz [26] and we assume that their cumulative demand follows a standard Brownian motion B.

Market Maker observes total cumulated orders, which is the sum of orders from both noise trader and informed trader, i.e., $Y_t = \theta_t + B_t$, where we denote the cumulative order from the insider by time t to be θ_t. The admissible trading strategy of the insider, θ, will be assumed to be an absolutely continuous process, thus Y is a continuous semimartingale in $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$. Then the market maker's filtration at time t, $\mathcal{F}_t^M$, is defined as $\mathcal{F}_t^M := \mathcal{F}_t^Y$ for $t \in [0, 1[$ and $\mathcal{F}_1^M := \mathcal{F}_1^Y \vee \sigma(Z)$.

The market maker sets the asset price, P_t. We define a weighted signal ξ of Y where ξ satisfies the following SDE and initial condition:

$$d\xi_t = w(t, \xi_t)dY_t, \quad \xi_0 = 0 \quad a.s., \tag{2.1.1}$$

where w is called weighting function[2] which satisfies admissibility conditions that we will define shortly. In principle, P depends on the whole path of Y, i.e., the whole path of ξ. For simplicity we assume $P_t = \xi_t + c$ with some constant c for any $t \in [0, 1[$ and $P_1 = f(Z)$. The admissibility conditions imposed on θ and w will ensure that SDE (2.1.1) will admit a unique strong Markov solution. We will denote by $P^{0,z}$ the time 0 law of the process ξ and random variable Z. Now we consider the probability measure $\mathbb{P}$ defined on $(\Omega, \mathcal{F}_1^Y \vee \sigma(Z))$ by

$$\mathbb{P}(E) = \int_{\mathbb{R}} P^{0,z}(E)\mu(dz), \quad \forall E \in \mathcal{F}_1^Y \vee \sigma(Z), \tag{2.1.2}$$

which is the market maker's measure. We will denote $\mathbb{E}$, the expectation taken under market maker's measure and $\mathbb{E}^{0,z}$, the expectation taken under insider's measure.

Insider observes the price process P up to any time t and distribution of Z at $t = 0$, thus her filtration is given by $\mathcal{F}_t^I = \mathcal{F}_t^P \vee \sigma(Z)$. Insider's objective is to maximise the expected utility of final wealth, i.e.:

$$\sup_{\theta \in \mathcal{A}(w)} \mathbb{E}^{0,z} \left[-\frac{1}{\gamma} \exp\{-\gamma W_1^\theta\} \right],$$

where $\mathcal{A}(w)$, which will be specified later, is the set of admissible trading strategies given the pricing rule (P, w). The expectation is taken under the measure $P^{0,z}$, which is the time

[2] We apply the weighting function as the market maker may wish to put price dependency more emphasised on recent trades. This definition is a generalisation of [12] and [15].

0 law of the coupled process ξ and signal Z. In reality, the insider observes the process P and signal Z, we will show later in Remark 6 the equivalence of the filtrations $\mathcal{F}^P \vee \sigma(Z)$ and $\mathcal{F}^Y \vee \sigma(Z)$. We use exponential utility function here with $\gamma > 0$. The ad-hoc reasoning for choosing such utility function is discussed in Remark 7. We denote by W_1^θ an insider's wealth at terminal time if she chooses to follow the admissible trading strategy θ. It is comprised of the continuous gain over the time interval $[0, 1[$ and gain from the possible price discrepancy at terminal time $t = 1$, i.e.

$$W_1^\theta = \int_0^{1^-} \theta_t dP_t + (f(Z) - P_{1^-})\theta_{1^-}. \tag{2.1.3}$$

2.2 Admissibility and Equilibrium

The above market model suggests a feedback mechanism for the insider, as her trading strategy will be reflected upon the asset price which in turn will influence her trading strategy itself. In this thesis, we focus on finding the equilibrium of such market model in the sense:

1. given the pricing rule, insider's trading strategy is optimal;
2. given the trading strategy, there exists a unique strong solution for SDE (2.1.1) over $[0, 1[$ and the pricing rule is rational, i.e., martingale over $[0, 1[$.

To formalise the definition of equilibrium and rational pricing, we need to define the sets of admissible pricing rules and admissible trading strategies.

Definition 1 *Let b and d be the constants defined in Remark 1. An admissible pricing rule is a measurable weighting function w and a constant c such that:*

1. *$w \in \mathcal{C}^{1,2}\left([0,1]\times]\tilde{b}, \tilde{d}[\right)$ is bounded and positive in the interior of its domain where $\tilde{b} = b - c$ and $\tilde{d} = d - c$.*
2. *The weighting function $w \in \mathcal{C}^{1,2} : [0,1]\times]\tilde{b}, \tilde{d}[\to \mathbb{R}^+$ satisfies:*

$$\frac{w_t}{w^2}(t,\xi) + \frac{w_{\xi\xi}(t,\xi)}{2} = -\gamma \tag{2.2.4}$$

   for some positive $\gamma \in \mathbb{R}$.

3. *There exists a unique strong solution* ξ *to the SDE (2.1.1) in* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1[}, \mathbb{P})$ *such that* $\tau > 1$ *where* $\tau = \inf\{t \geq 0 : \xi_t \notin]\tilde{b}, \tilde{d}[\} = \inf\{t \geq 0 : P_t \notin]b, d[\}$.

The first condition is inherited from the boundedness of f, hence the boundedness of the pricing signal ξ. The second condition is a necessary condition for the existence of optimal strategy which we will show later in Lemma 1.

Definition 2 *We will call an admissible pricing rule rational if it satisfies*

$$P_t = \mathbb{E}\left[f(Z)|\mathcal{F}_t^Y\right]$$

for a given admissible trading strategy θ *(we will define shortly in Definition 3 what an admissible trading strategy is). In particular* $P_1 = f(Z)$ *and* $P_0 = \mathbb{E}[f(Z)]$ *where expectation is taken w.r.t. the probability measure* $\mathbb{P}$ *defined in (2.1.2).*

Remark 2 *We can use Bertrand undercutting argument to explain rational pricing. The market maker sets the price to be equal to the expectation of the liquidation value of the asset, conditional on his information set at the time the price is determined. Thus market maker earn on average zero profit. Suppose there are several market makers and one of them is aggressive and makes profit by setting the price higher than the rational price. As a result of competition, other market makers will set price in-between the rational price and the price set by the aggressive market maker. Over the time, prices will converge to rational price.*

Remark 3 *Suppose* P *is rational pricing rule where* $P_t = \xi_t + c$. *Since* ξ *satisfies the SDE (2.1.1), we know it is local martingale. Moreover, since* w *is bounded, we know* ξ *is a true martingale. Therefore*

$$\mathbb{E}[f(Z)] = P_0 = \xi_0 + c = c.$$

We will consider rational pricing P *with* $c = \mathbb{E}[f(Z)]$ *without loss of generality. As discussed in the previous remark, we have* P *is bounded by the range* $]b, d[$. *We immediately have* $\mathbb{E}[\int_0^1 P_t^2 dt] < \infty$. *As we are taking expectation under market maker's measure, this means the process* $P \in L^2$ *from market maker's point of view, i.e., without the existence of the insider.*

Remark 4 *We will show that the rational price process P is bounded with a state space $]b, d[$, where b and d are the constants in Remark 1. In other words, condition 3 of Definition 1 is not restricting our choice of weighting function w. Indeed, as stated in the Remark 3, $P_t \in [b, d]$ a.s. for any $t \in [0, 1]$. Hence, the continuity of the process P implies that its state space is at most $[b, d]$.*

Consider a stopping time $\tau_b := \inf\{t \geq 0 : P_t = b\} \wedge 1$. As process P is a martingale that is closed by the random variable $f(Z)$, optional sampling theorem (e.g. Theorem I.16 in Protter [49]) implies that

$$\mathbb{E}[f(Z)\mathbf{1}_{\tau_b<1}|\mathcal{F}^Y_{\tau_b}] = \mathbb{E}[f(Z)|\mathcal{F}^Y_{\tau_b}]\mathbf{1}_{\tau_b<1} = P_{\tau_b}\mathbf{1}_{\tau_b<1} = b\mathbf{1}_{\tau_b<1}.$$

Therefore,

$$\begin{aligned} b\mathbb{P}[\tau_b < 1] &= \mathbb{E}[f(Z)\mathbf{1}_{\tau_b<1}] = \mathbb{E}[f(Z)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)=b} + f(Z)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b}] \\ &= \mathbb{E}[b\mathbf{1}_{\tau_b<1} + (f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b}] \\ &= b\mathbb{P}[\tau_b < 1] + \mathbb{E}[(f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b}]. \end{aligned}$$

This yields that $\mathbb{E}[(f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b}] = 0$ and, since $(f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b}$ is non-negative random variable, that $(f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b} = 0$ a.s.. Observe that

$$0 = \mathbb{P}[(f(Z) - b)\mathbf{1}_{\tau_b<1}\mathbf{1}_{f(Z)>b} > 0] = \mathbb{P}[\tau_b < 1,\ f(Z) > b] = \mathbb{P}[\tau_b < 1],$$

where the last equality follows from the fact that

$$\mathbb{P}[f(Z) > b] = \mathbb{P}[Z > -\infty] = 1,$$

since by the definition of f (and, in particular, Remark 1) $f(z) = b \Leftrightarrow z = -\infty$, and Z is normally distributed.

Similar consideration applies to $\tau_d := \inf\{t \geq 0 : P_t = d\} \wedge 1$, and therefore $\tau := \inf\{t \geq 0 : P_t \notin]b, d[\} \wedge 1 = 1$ a.s.. This, together with the fact that

$$\mathbb{P}[f(Z) \in]b, d[] = \mathbb{P}[-\infty < Z < +\infty] = 1,$$

yields the desired conclusion.

Remark 5 *We want to show that choosing $P = \xi + \mathbb{E}[f(Z)]$ is in line with the conventional pricing assumption and provide a brief ad-hoc proof of why we choose P to be linear. Assume we have an alternative pricing rule that $P_t = H(t, \zeta_t)$ for $t \in [0, 1[$ where H is a strictly increasing function w.r.t. the space variable. Therefore H^{-1} is well defined. ζ is the solution to the SDE $d\zeta = a(t, \zeta_t)dY_t$ and $\zeta_0 = 0$ a.s. where a is strictly positive in the interior of its domain.*

Apply Ito's formula on process P stopped at τ_n where $\tau_n = \inf\{t \geq 0 : P_t \notin]b + \frac{1}{n}, d - \frac{1}{n}[\}$, denote $P^n_t = P_{t\wedge\tau_n}$ and $\zeta^n_t = \zeta_{t\wedge\tau_n}$, we have:

$$\begin{aligned} P^n_t &= P_0 + \int_0^{t\wedge\tau_n} \left(H_s(s, \zeta^n_s) + \frac{a^2(s, \zeta^n_s)}{2} H_{\zeta\zeta}(s, \zeta^n_s) \right) ds + \int_0^{t\wedge\tau_n} H_\zeta(s, \zeta^n_s) a(s, \zeta^n_s) d\beta_s \\ &= \int_0^{t\wedge\tau_n} \left(H_s(s, H^{-1}(s, P^n_s)) + \frac{a^2(s, H^{-1}(s, P^n_s))}{2} H_{\zeta\zeta}(s, H^{-1}(s, P^n_s)) \right) ds \\ &\quad + \int_0^{t\wedge\tau_n} H_\zeta(s, H^{-1}(s, P^n_s)) a(s, H^{-1}(s, P^n_s)) d\beta_s + H(0, \zeta_0) \end{aligned}$$

Since $\lim_{n\to\infty} \tau_n = \tau > 1$ due to Remark 4, we have the above equality is true for any $t \in [0, 1]$. Suppose P is rational pricing, we have $P_t = H(t, \zeta_t) = \mathbb{E}\left[f(Z)|\mathcal{F}^Y_t\right]$. Therefore, we know P is a martingale in its own filtration. Thus the drift term must be zero, i.e.,

$$H_t(t, y) + \frac{a^2(t, y)}{2} H_{yy}(t, y) = 0, \quad t \in [0, 1]. \tag{2.2.5}$$

Therefore

$$\begin{aligned} P_t &= \int_0^t H_y(s, H^{-1}(t, P_s)) a(s, H^{-1}(t, P_s)) d\beta_s + H(0, \zeta_0) \\ &= \int_0^t H_y(s, H^{-1}(t, P_s)) a(s, H^{-1}(t, P_s)) d\beta_s + \mathbb{E}[f(Z)]. \end{aligned}$$

Since this pricing rule is equivalent to the linear model, we have $w(t, x) = H_y(t, H^{-1}(t, x - \mathbb{E}[f(Z)]))a(t, H^{-1}(t, x - \mathbb{E}[f(Z)]))$ and ξ as the solution of the SDE $d\xi_t = w(t, \xi_t)d\beta_t$ with initial condition $\xi_0 = 0$ a.s. as $P_t = \xi_t + \mathbb{E}[f(Z)]$. In other words, the alternative weighting function a is given by:

$$a(t, y) = \frac{w(t, H(t, y) + \mathbb{E}[f(Z)])}{H_y(t, y)}.$$

We will show that a satisfies a similar PDE w satisfies which is inline with the PDE (2.3.6) we derived in Remark 7. Differentiate a w.r.t. t and y, we have:

$$\begin{aligned} a_t(t,y) &= \frac{w_t(t,H(t,y)+\mathbb{E}[f(Z)])+w_x(t,H(t,y)+\mathbb{E}[f(Z)])H_t(t,y)}{H_y(t,y)} \\ &\quad -\frac{w(t,H(t,y)+\mathbb{E}[f(Z)])H_{ty}(t,y)}{H_y^2(t,y)}, \end{aligned}$$

$$a_y(t,y) = w_x(t,H(t,y)+\mathbb{E}[f(Z)]) - \frac{w(t,H(t,y)+\mathbb{E}[f(Z)])H_{yy}(t,y)}{H_y^2(t,y)},$$

$$\begin{aligned} a_{yy}(t,y) &= w_{xx}(t,H(t,y)+\mathbb{E}[f(Z)])H_y(t,y) + \frac{2w(t,H(t,y)+\mathbb{E}[f(Z)])H_{yy}^2(t,y)}{H_y^3(t,y)} \\ &\quad -\frac{w_x(t,H(t,y)+\mathbb{E}[f(Z)])H_y(t,y)H_{yy}(t,y)+w(t,H(t,y)+\mathbb{E}[f(Z)])H_{yyy}(t,y)}{H_y^2(t,y)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{L}(a) &= \frac{a_t(t,y)}{a^2(t,y)} + \frac{a_{yy}(t,y)}{2} \\ &= \frac{(w_t(t,H(t,y)+\mathbb{E}[f(Z)])+w_x(t,H(t,y)+\mathbb{E}[f(Z)])H_t(t,y))H_y(t,y)}{w^2(t,H(t,y)+\mathbb{E}[f(Z)])} \\ &\quad -\frac{H_{ty}(t,y)}{w(t,H(t,y)+\mathbb{E}[f(Z)])} + \frac{w_{xx}(t,H(t,y)+\mathbb{E}[f(Z)])H_y(t,y)}{2} \\ &\quad -\frac{w_x(t,H(t,y)+\mathbb{E}[f(Z)])H_y(t,y)H_{yy}(t,y)+w(t,H(t,y)+\mathbb{E}[f(Z)])H_{yyy}(t,y)}{2H_y^2(t,y)} \\ &\quad +\frac{w(t,H(t,y)+\mathbb{E}[f(Z)])H_{yy}^2(t,y)}{H_y^3(t,y)} \\ &= H_y(t,y)\left(\frac{w_t(t,H(t,y)+\mathbb{E}[f(Z)])}{w^2(t,H(t,y)+\mathbb{E}[f(Z)])} + \frac{w_{xx}(t,H(t,y)+\mathbb{E}[f(Z)])}{2}\right) \\ &\quad +\frac{w_x(t,H(t,y)+\mathbb{E}[f(Z)])H_y(t,y)}{w^2(t,H(t,y)+\mathbb{E}[f(Z)])}\left(H_t(t,y)+\frac{a^2(t,y)}{2}H_{yy}(t,y)\right) \\ &\quad -\frac{1}{w(t,H(t,y)+\mathbb{E}[f(Z)])}\left(H_t(t,y)+\frac{a^2(t,y)}{2}H_{yy}(t,y)\right)_y \\ &= -\gamma H_y(t,y). \end{aligned}$$

This is the same PDE we derive later in Remark 7. Therefore, we have the equivalence of the following pricing rules:

1. *Pricing rule* $P_t = \xi_t + \mathbb{E}[f(Z)]$ *and weighting function* w *satisfying (2.2.4) where* ξ *is strong solution of SDE* $d\xi_t = w(t, \xi_t)dY_t$, $\xi_0 = 0$ *a.s..*

2. *Pricing rule* $P_t = H(t, \zeta_t)$ *(*H *satisfies PDE (2.2.5),* $H(0,0) = \mathbb{E}[f(Z)]$*) and weighting function* a *satisfying (2.3.6) where* ζ *is strong solution of SDE* $d\zeta_t = w(t, \zeta_t)dY_t$, $\zeta_0 = 0$ *a.s..*

Moreover, if we have w*, and pricing rule* H*, the corresponding weighting function* a *will be given by*

$$a(t,y) = \frac{w(t, H(t,y) + \mathbb{E}[f(Z)])}{H_y(t,y)}.$$

On the other hand if we have w *and weighting function* a*, the corresponding pricing rule will be given by*

$$H_y(t,y) = -\frac{1}{\gamma}\left(\frac{a_t(t,y)}{a^2(t,y)} + \frac{a_{yy}(t,y)}{2}\right),$$
$$H(t,y) = w^{-1}\left(t, a(t,y)H_y(t,y)\right) - \mathbb{E}[f(Z)].$$

Thus without loss of generality, we only need to solve for the linear case where $P_t = \xi_t + \mathbb{E}[f(Z)]$ *and corresponding weighting function* w *satisfying (2.2.4).*

Remark 6 *We also would like to justify by choosing* P *(effectively* ξ*) over* Y *as market maker's pricing signal, we are not losing any information. As straightforwardly by the SDE defined in (2.1.1) we have* $\mathcal{F}^\xi \subseteq \mathcal{F}^Y$*. Since* $\tilde{b} = b - \mathbb{E}[f(Z)] \leq 0$ *with equality when* $f(Z) = b$ $\mathbb{P}$*-a.s.,* $\tilde{d} = d - \mathbb{E}[f(Z)] \geq 0$ *with equality when* $f(Z) = d$ $\mathbb{P}$*-a.s.. The initial condition of* $\xi_0 = 0 \in]\tilde{b}, \tilde{d}[$*. We define for* $\forall x \in]\tilde{b}, \tilde{d}[$*:*

$$A(t,x) = \int_0^x \frac{dy}{w(t,y)} + \int_0^t \frac{1}{2} w_x(s,0)ds,$$

a strictly increasing function w.r.t. x*. Therefore* $A^{-1}(t,y)$ *exists and is well defined. From the definition, we calculate for* $\forall x \in]\tilde{b}, \tilde{d}[$*:*

$$A_x(t,x) = \frac{1}{w(t,x)},$$

$$\begin{aligned} A_t(t,x) + \frac{w^2(t,x)}{2} A_{xx} &= -\int_0^x \frac{w_t(t,y)}{w^2(t,y)} dy + \frac{1}{2} w_x(t,0) - \frac{1}{2} w_x(t,x) \\ &= -\int_0^x \left(\frac{w_t(t,y)}{w^2(t,y)} + \frac{w_{yy}(t,y)}{2}\right) dy. \end{aligned}$$

Denote $\eta_t = A(t, \xi_t)$*. Apply Ito's formula to process* η *stopped at time* τ_n *where* $\tau_n = \inf\{t \geq 0 : P_t \notin]b + \frac{1}{n}, d - \frac{1}{n}[\} = \inf\{t \geq 0 : \xi_t \notin]\tilde{b} + \frac{1}{n}, \tilde{d} - \frac{1}{n}[\}$*, denote* $\eta_t^n = \eta_{t\wedge\tau_n}$ *and* $Y_t^n = Y_{t\wedge\tau_n}$*, we have*

$$\begin{aligned} \eta_t^n &= \int_0^{t\wedge\tau_n} A_x(s, \xi_s^n) d\xi_s^n + \int_0^{t\wedge\tau_n} \left(A_s(s, \xi_s^n) + \frac{w^2(s, \xi_s^n)}{2} A_{xx}(s, \xi_s^n) \right) ds \\ &= Y_t^n - \int_0^{t\wedge\tau_n} \left(\int_0^{A^{-1}(s,\eta_s^n)} \left(\frac{w_s(s, y)}{w^2(s, y)} + \frac{w_{yy}(s, y)}{2} \right) dy \right) ds. \end{aligned}$$

Since $\lim_{n\to\infty} \tau_n = \tau > 1$ *due to Remark 4, we have*

$$Y_t = \eta_t + \int_0^t \left(\int_0^{A^{-1}(s,\eta_s)} \left(\frac{w_s(s, y)}{w^2(s, y)} + \frac{w_{yy}(s, y)}{2} \right) dy \right) ds,$$

i.e., Y_t *solely depends on* $\eta_{[0,t]}$*. Therefore* $\mathcal{F}^Y \subseteq \mathcal{F}^\eta$ *for any* $t \in [0, 1]$*. Moreover since* $A(t, x)$ *is invertible w.r.t. space variable and is continuous, we have* $\mathcal{F}^\eta = \mathcal{F}^\xi$*. Thus* $\mathcal{F}^Y \subseteq \mathcal{F}^\xi$*. Combining with previous result that* $\mathcal{F}^\xi \subseteq \mathcal{F}^Y$*, we have* $\mathcal{F}^Y = \mathcal{F}^\xi$*. Hence the insider's filtration is equivalently generated by processes* Y *and* $\sigma(Z)$*, i.e., insider has full information of the market.*

The definition of admissible strategy θ is based on the set of admissible pricing rule w. Back [5] proved that any strategy as a discontinuous process or with nonzero martingale part is strictly suboptimal. We also limit admissible trading strategies to absolutely continuous set. The formal definition is as follows.

Definition 3 *An admissible trading strategy* $\theta \in \mathcal{A}(w)$ *for insider given any admissible pricing rule is* $\mathcal{F}^\xi \vee \sigma(Z)$ *adapted process satisfying:*

1. θ *is absolutely continuous, i.e.,* $d\theta_t = \alpha_t dt$*.*
2. *There exists a unique strong solution of SDE (2.1.1) in* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1[}, \mathbb{Q})$*.*
3. (ξ, Z) *is a Markov process adapted to* $(\mathcal{F}_t)$ *with measure* $P^{0,z}$*;*

We already have $\mathbb{E}^{0,z}\left[\int_0^1 P_t dt\right] < \infty$ since P is bounded process. Thus doubling strategies are eliminated since the admissible trading strategies are restricted in L^2. (see Duffie and Huang (1985) [20]).

In addition, the insider will prefer not to be detected by the market maker. In this case, she will hide her trading among the noise traders. Therefore, we only consider the inconspicuous strategies for the insider.

Definition 4 *We will call an admissible pricing strategy inconspicuous if*

$$\mathbb{E}[\theta_s|\mathcal{F}_t^{\xi}] = 0$$

for every $0 \leq t \leq s \leq 1$.

Thus the cumulative trading amount $Y_t = B_t + \theta_t$ will appear as a local martingale in market maker's filtration. Moreover, since θ is absolutely continuous, quadratic variation does not depend on the filtration, we have $\langle Y \rangle_t = \langle B \rangle_t = t$. Thus we know Y is a local martingale in $\mathcal{F}^{\xi}$ with $\langle Y \rangle_t = t$. By Levy's characterisation, Y is a $\mathcal{F}^{\xi}$ Brownian motion. Now we can formally define the market equilibrium given the definitions on admissible pricing rules and admissible trading strategies.

Definition 5 *An equilibrium of the insider is a pair* (w^*, θ^*) *s.t.,* w^**, an admissible pricing rule, and* $\theta^* \in \mathcal{A}(w^*)$*, an admissible strategy satisfying:*

1. w^* *is a rational pricing rule given* θ^*.

2. θ^* *is insider's optimal trading strategy, i.e.,*

$$\mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\left\{-\gamma W_1^{\theta^*}\right\}\right] = \sup_{\theta \in \mathcal{A}(w^*)} \mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\left\{-\gamma W_1^{\theta}\right\}\right].$$

In this thesis, we focus on existence of equilibrium in which the insider trading strategy is inconspicuous. We will call an equilibrium with this property an *inconspicuous* equilibrium.

2.3 Characterisation of Insider's Optimal Strategy

The following Lemma characterises the insider's optimal strategy. For simplicity we denote $\tilde{f}(x) = f(x) - \mathbb{E}[f(Z)]$ where f will satisfy all the conditions in Assumption 1. Moreover, we have $\lim_{x\to\infty} \tilde{f}(x) = \tilde{d} = d - \mathbb{E}[f(Z)]$ and $\lim_{x\to-\infty} \tilde{f}(x) = \tilde{b} = b - \mathbb{E}[f(Z)]$.

Lemma 1 *Suppose the rational pricing rule w satisfies the following condition: $\theta^* \in \mathcal{A}(w)$ satisfies $\xi_1^* = \tilde{f}(z)$ $P^{0,z} - a.s.$ for every $z \in \mathbb{R}$, where ξ^* is the strong solution to the SDE $\xi_t = \int_0^t w(s, \xi_s) dY_s^*$, $\xi_0 = 0$ a.s. with $Y^* = B + \theta^*$. Then θ^* is the optimal strategy, i.e., for any $\theta \in \mathcal{A}(w)$,*

$$\mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\{-\gamma W_1^{\theta^*}\}\right] \geq \mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\{-\gamma W_1^{\theta}\}\right].$$

PROOF.

We will adapt Wu's proof of his Lemma 4.2 in [54].

Due to Remark 4, we have that $\xi_t \in]\tilde{b}, \tilde{d}[$ a.s. for any $t \in [0,1]$, thus we can define the following function for any $\xi \in]\tilde{b}, \tilde{d}[$:

$$\varphi(t, \xi) = \int_{\tilde{f}(z)}^{\xi} \frac{y - \tilde{f}(z)}{w(t, y)} dy + \frac{1}{2}\int_t^1 w(s, \tilde{f}(z)) ds.$$

Since $w \in \mathcal{C}^{1,2}([0,1] \times]\tilde{b}, \tilde{d}[\to \mathbb{R})$ and $w(t, x) > 0$, $\varphi(t, \xi)$ is well defined in $\mathcal{C}^{1,2}([0,1] \times]\tilde{b}, \tilde{d}[\to \mathbb{R})$. The idea of defining such a function comes from the solution of HJB equations and φ will be used to give an upper bound of the insider's expected terminal utility.

First we derive some important properties of φ. Differentiate φ w.r.t. ξ to second order we have

$$\varphi_\xi(t, \xi) = \frac{\xi - \tilde{f}(z)}{w(t, \xi)}, \quad \varphi_{\xi\xi} = \frac{1}{w(t, \xi)} - \frac{[\xi - \tilde{f}(z)] w_\xi(t, \xi)}{w^2(t, \xi)}.$$

Differentiate φ w.r.t. t we have

$$\varphi_t(t, \xi) = \int_{\tilde{f}(z)}^{\xi} -\frac{(y - \tilde{f}(z)) w_t(t, y)}{w^2(t, y)} dy - \frac{1}{2} w(t, \tilde{f}(z)).$$

Note by Leibniz integral rule, we can move derivative inside the integral provided the integrand and derivative of the integrand are continuous functions over the integral intervals. In this case, $-\frac{(y-\tilde{f}(z))w_t(t,y)}{w^2(t,y)}$ is continuous function given the differentiability of w. Therefore,

$$\begin{aligned}
I &= \varphi_t(t, \xi) + \frac{w^2(t, \xi)}{2}\varphi_{\xi\xi}(t, \xi) \\
&= \int_{\tilde{f}(z)}^{\xi} -\frac{(y - \tilde{f}(z)) w_t(t, y)}{w^2(t, y)} dy - \frac{1}{2} w(t, \tilde{f}(z)) + \frac{1}{2} w(t, \xi) - \frac{1}{2}(\xi - \tilde{f}(z)) w_\xi(t, \xi) \\
&= -\int_{\tilde{f}(z)}^{\xi} (y - \tilde{f}(z)) \left(\frac{w_t(t, y)}{w^2(t, y)} + \frac{w_{yy}(t, y)}{2}\right) dy + \frac{1}{2}\int_{\tilde{f}(z)}^{\xi} (y - \tilde{f}(z)) w_{yy}(t, y) dy - \frac{1}{2} w(t, \tilde{f}(z)) \\
&\quad + \frac{1}{2} w(t, \xi) - \frac{1}{2}(\xi - \tilde{f}(z)) w_\xi(t, \xi).
\end{aligned}$$

Integration by parts of the second integral we have

$$\begin{aligned} J &= \frac{1}{2}\int_{\tilde{f}(z)}^{\xi}(y-\tilde{f}(z))w_{yy}(t,y)dy \\ &= \frac{1}{2}(y-\tilde{f}(z))w_y(t,y)|_{\tilde{f}(z)}^{\xi} - \frac{1}{2}\int_{\tilde{f}(z)}^{\xi} w_y(t,y)dy \\ &= \frac{1}{2}(\xi-\tilde{f}(z))w_\xi(t,\xi) - \frac{1}{2}w(t,\xi) + \frac{1}{2}w(t,\tilde{f}(z)). \end{aligned}$$

Substitute into the equation of I we have

$$I = -\int_{\tilde{f}(z)}^{\xi}(y-\tilde{f}(z))\left(\frac{w_t(t,y)}{w^2(t,y)} + \frac{w_{yy}(t,y)}{2}\right)dy.$$

Due to condition (2.2.4), we have

$$\varphi_t(t,\xi) + \frac{w^2(t,\xi)}{2}\varphi_{\xi\xi}(t,\xi) = \gamma\int_{\tilde{f}(z)}^{\xi}(y-\tilde{f}(z))dy = \frac{\gamma}{2}(\xi-\tilde{f}(z))^2.$$

We apply Ito's formula on $\varphi(t,\xi_t)$ stopped at τ_n where $\tau_n = \inf\{t\geq 0 : \xi_t \notin (\tilde{b}+\frac{1}{n}, \tilde{d}-\frac{1}{n})\}$, denote $\xi_t^n = \xi_{t\wedge\tau_n}$:

$$\begin{aligned} \varphi(t,\xi_t^n) - \varphi(0,0) &= \int_0^{t\wedge\tau_n}\varphi_\xi(s,\xi_s^n)d\xi_s^n + \int_0^{t\wedge\tau_n}\left(\varphi_s(s,\xi_s^n) + \frac{w^2(s,\xi_s^n)}{2}\varphi_{\xi\xi}(s,\xi_s^n)\right)ds \\ &= \int_0^{t\wedge\tau_n}\frac{\xi_s^n-\tilde{f}(z)}{w(s,\xi_s^n)}d\xi_s^n + \frac{\gamma}{2}\int_0^{t\wedge\tau_n}(\xi_s^n-\tilde{f}(z))^2ds. \end{aligned}$$

Taking the limit $n\to\infty$, we have $\tau_n \to \tau > 1$ due to the condition 3 of Definition 1, the admissible pricing rule. Therefore we have the above expression holds for any $t\in[0,1]$, i.e.,

$$\varphi(t,\xi_t) - \varphi(0,0) = \int_0^t \frac{\xi_s-\tilde{f}(z)}{w(s,\xi_s)}d\xi_s + \frac{\gamma}{2}\int_0^t(\xi_s-\tilde{f}(z))^2ds.$$

In addition, we have the boundary condition

$$\varphi(1,\xi_1) = \int_{\tilde{f}(z)}^{\xi_1}\frac{y-\tilde{f}(z)}{w(1,y)}dy \geq 0.$$

To see this, if $\xi_1 > \tilde{f}(z)$, then $y > \tilde{f}(z)$ for all $y\in[\tilde{f}(z),\xi_1]$, hence integral is positive On the other hand, if $\xi_1 < \tilde{f}(z)$, then $y < \tilde{f}(z)$ for all $y\in[\xi_1,\tilde{f}(z)]$, hence integral is also positive. Equality holds if and only if $\xi_1 = \tilde{f}(z)$.

The wealth process of the insider at terminal time is W_1. Apply integration by part to W_1^θ defined in (2.1.3), we have

$$\begin{aligned} W_1^\theta &= \int_0^{1-} \theta_t d\xi_t + \left(f(Z) - \left(\xi_{1^-} + \mathbb{E}[f(Z)]\right)\right)\theta_{1^-} \\ &= \int_0^{1-} \theta_t d\xi_t + (\tilde{f}(Z) - \xi_{1^-})\theta_{1^-} \\ &= \theta_{1^-}\xi_{1^-} - [\theta,\xi]_{1^-} + \int_0^{1-} \xi_t d\theta_t + (\tilde{f}(Z) - \xi_{1^-})\theta_{1^-} \\ &= \int_0^{1-} (\tilde{f}(Z) - \xi_t) d\theta_t = \int_0^1 (\tilde{f}(Z) - \xi_t) d\theta_t. \end{aligned}$$

In the above calculation, $[\theta,\xi]_{1^-} = 0$ since θ is absolutely continuous process. The last equality is due to continuity of process ξ.

Therefore the insider's expected wealth:

$$\begin{aligned} R &= \sup_\theta \mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\left\{-\gamma W_1^\theta\right\}\right] \\ &= -\frac{1}{\gamma}\inf_\theta \mathbb{E}^{0,z}\left[\exp\left\{-\gamma\int_0^1 (\tilde{f}(z) - \xi_t)d\theta\right\}\right] \\ &= -\frac{1}{\gamma}\inf_\theta \mathbb{E}^{0,z}\left[\exp\left\{\gamma\int_0^1 \frac{\xi_t - \tilde{f}(z)}{w(t,\xi_t)}d\xi_t - \gamma\int_0^1 (\xi_t - \tilde{f}(z))dB_t\right\}\right]. \end{aligned}$$

The last equality is due to $d\xi = w(t,\xi)(dB_t + d\theta_t)$. Substitute Ito's formula on $\varphi(t,\xi_t)$ into the above equation, we have

$$\begin{aligned} R &= -\frac{1}{\gamma}\inf_\theta \mathbb{E}^{0,z}\left[\exp\left\{-\gamma\varphi(0,0) + \gamma\varphi(1,\xi_1) - \gamma\int_0^1 (\xi_t - \tilde{f}(z))dB_t - \frac{\gamma^2}{2}\int_0^1 (\xi_t - \tilde{f}(z))^2 dt\right\}\right] \\ &\le -\frac{\exp\{-\gamma\varphi(0,0)\}}{\gamma}\inf_\theta \mathbb{E}^{0,z}\left[\mathcal{E}_1^\theta\right] \end{aligned}$$

since $\varphi(1,\xi_1) \ge 0$ with equality if and only if $\xi_1 = \tilde{f}(z)$ a.s..

$$\mathcal{E}_t^\theta = \exp\left\{-\gamma\int_0^t (\xi_s - \tilde{f}(z))dB_s - \frac{\gamma^2}{2}\int_0^t (\xi_s - \tilde{f}(z))^2 ds\right\}$$

is an $\mathcal{F}^{\xi,Z}$ exponential local martingale with $\mathcal{E}_0 = 1$.

Since $|f|$ and $|f'|$ are bounded by K, suppose $|\tilde{f}|$ and $|\tilde{f}'|$ are bounded by $\tilde{K}$. By rational pricing rule Definition 2, we know $\xi_t = \mathbb{E}[\tilde{f}(Z)|\mathcal{F}_t^\xi]$ is bounded by $\tilde{K}$ for any $t \in [0,1]$. The

Novikov's condition is satisfied since

$$\mathbb{E}^{0,z} \exp\left\{\gamma \int_0^1 \left|\xi_t - \tilde{f}(z)\right|^2 dt\right\} \le \exp\{4\gamma \tilde{K}^2\} < \infty.$$

Thus it is a true martingale and in particular $\mathbb{E}^{0,z}\left[\mathcal{E}_1^\theta\right] = \mathbb{E}^{0,z}\left[\mathcal{E}_0\right] = 1$. Therefore the insider's expected utility satisfies

$$\mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\left\{-\gamma W_1^\theta\right\}\right] \le \mathbb{E}^{0,z}\left[-\frac{1}{\gamma}\exp\left\{-\gamma W_1^{\theta^*}\right\}\right] = \frac{\exp\left\{-\gamma\varphi(0,0)\right\}}{\gamma}.$$

Equality holds when $\theta^* \in \mathcal{A}(w)$ satisfies $\xi_1^* = \tilde{f}(z)$ $P^{0,z}$-a.s. ■

We introduce the following ad hoc derivation towards our consideration of exponential utility above. It gives us an insightful, yet not rigorous justification of the reason we choose exponential utility to study.

Remark 7 *In this remark, we consider general pricing rule* (H, a) *where* $P_t = H(t, \zeta)$, ζ *is the strong solution to SDE* $d\zeta_t = a(t, \zeta_t)dY_t$, $\zeta_0 = 0$ *a.s.,* $Y_t = B_t + \theta_t$ *is Brownian motion in its own filtration due to inconspicuous trading. Our aim is the following, given* (H, a) *satisfying (2.2.5) which is due to rational pricing in Remark 5, find out under which utility functions there exists equilibrium and what other conditions* (H, a) *need to satisfy.*

We know the terminal wealth $W_1^\theta = \int_0^1 (f(Z) - H(t, \zeta_t))\, d\theta_t$. *Define process* X *satisfying*

$$dX_t = (f(Z) - H(t, \zeta_t))d\theta_t, \quad X_0 = 0\, a.s..$$

X_t *is an* $\mathcal{F}^\zeta \vee \sigma(Z)$ *adapted process. The insider is trying to maximize the expected utility of terminal wealth, i.e.,* $\sup_\theta \mathbb{E}^{0,z}[W_1^\theta]$. *We can define:*

$$v(t, x, \zeta, \theta) = \mathbb{E}_t^{0,z}\left[u\left(x + \int_t^1 (f(z) - H(s, \zeta_s))d\theta_s\right)\right],$$

where $u \in \mathcal{C}^3$ *is strictly increasing concave utility function. Define the conditional value function* $\phi(t, \zeta, x)$*:*

$$\phi(t, \zeta, x) = \sup_{\theta \in \mathcal{A}(H,a)} v(t, x, \zeta, \theta),$$

where $\mathcal{A}(H, a)$ *is the set of admissible trading strategy given pricing rule* (H, a). *The Bellman's optimality principle, introduced by El Karoui [33] suggests the process* $\phi(t, \zeta, X)$ *be a* *supermartingale for any $\theta \in \mathcal{A}(H,a)$ and a martingale if and only if θ is optimal. By Ito's formula*

$$\begin{aligned} d\phi(t,\zeta_t,X_t) &= \phi_\zeta(t,\zeta_t,X_t)a(t,\zeta_t)dB_t + \phi_x(t,\zeta_t,X_t)(f(z)-H(t,\zeta_t))d\theta_t \\ &\quad + \left(\phi_t(t,\zeta_t,X_t) + \frac{a^2(t,\zeta_t)}{2}\phi_{\zeta\zeta}(t,\zeta_t,X_t)\right)dt. \end{aligned}$$

The drift term is negative for supermartingale and zero for martingale when θ is the optimal strategy, hence

$$\begin{aligned} 0 &= \phi_t(t,\zeta,x) + \frac{a^2(t,\zeta)}{2}\phi_{\zeta\zeta}(t,\zeta,x) \\ &\quad + \sup_{\theta\in\mathcal{A}(H,w)} [\phi_\zeta(t,\zeta,x)a(t,\zeta) + \phi_x(t,\zeta,x)(f(z)-H(t,\zeta))]\frac{d\theta}{dt}. \end{aligned}$$

Notice that the Bellman equation is linear in $\frac{d\theta}{dt}$ and has a solution if and only if

$$\phi_t(t,\zeta,x) + \frac{a^2(t,\zeta)}{2}\phi_{\zeta\zeta}(t,\zeta,x) = 0,$$

$$\phi_\zeta(t,\zeta,x)a(t,\zeta) + \phi_x(t,\zeta,x)(f(z)-H(t,\zeta)) = 0$$

with boundary condition $\phi(1,\zeta,x) = u(x)$. Differentiate the above PDEs w.r.t. ζ and t respectively we have

$$\phi_{t\zeta}(t,\zeta,x) = -a(t,\zeta)a_\zeta(t,\zeta)\phi_{\zeta\zeta}(t,\zeta,x) - \frac{a^2(t,\zeta)}{2}\phi_{\zeta\zeta\zeta}(t,\zeta,x),$$

$$\begin{aligned} \phi_{\zeta t}(t,\zeta,x) &= \frac{\phi_{xt}(t,\zeta,x)(H(t,\zeta)-f(z)) + \phi_x(t,\zeta,x)H_t(t,\zeta)}{a(t,\zeta)} \\ &\quad - \frac{a_t(t,\zeta)\phi_x(t,\zeta,x)(H(t,\zeta)-f(z))}{a^2(t,\zeta)}. \end{aligned}$$

Let $\phi_{xt}(t,\zeta,x) = \phi_{tx}(t,\zeta,x)$, $\phi_{\zeta\zeta x}(t,\zeta,x) = \phi_{x\zeta\zeta}(t,\zeta,x)$ and finally $\phi_{t\zeta}(t,\zeta,x) = \phi_{\zeta t}(t,\zeta,x)$ due to continuity, we can use (2.2.5) to simplify the above equations to satisfy the following condition:

$$\frac{\phi_{xx}(t,\zeta,x)}{\phi_x(t,\zeta,x)} = \frac{1}{H_\zeta(t,\zeta)}\left(\frac{a_t(t,\zeta)}{a^2(t,\zeta)} + \frac{a_{\zeta\zeta}(t,\zeta)}{2}\right) = -\gamma(t,\zeta) \qquad (2.3.6)$$

1. *If* $\gamma = 0$ *we have* $\phi_{xx} = 0$. *Therefore we have* $\phi(t,\zeta,x) = A(t,\zeta)x + B(t,\zeta)$. *Apply boundary condition* $\phi(1,\zeta,x) = u(x) = A(1,\zeta)x + B(1,\zeta)$. *We conclude that only linear utility case (w.l.o.g.* $u(x) = x$*) applies.*

2. *If* $\gamma = \gamma(t,\zeta)$ *we have* $\phi(t,\zeta,x) = A(t,\zeta)\exp\{-\gamma(t,\zeta)x\} + B(t,\zeta)$. *Apply boundary condition* $\phi(1,\zeta,x) = u(x) = A(1,\zeta)\exp\{-\gamma(1,\zeta)x\} + B(1,\zeta)$*, i.e.,* $\gamma(1,\zeta) = const. = \gamma$, $A(1,\zeta)$ *and* $B(1,\zeta)$ *are both constants. This imply the exponential utility case. Without loss of generality* $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$ *where* $\gamma > 0$ *due to concavity of utility function.*

The linear utility case has been widely studied and will not be the focus of this paper. We conclude that given (H,a) *satisfying (2.2.5), Bellman's optimality principle suggests (2.3.6) and* $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$ *are the necessary conditions for the existence of the conditional value function. This ad-hoc derivation inspires us to consider equilibrium under above exponential utility.*

From previous section we obtained the sufficient conditions for insider's strategy to be optimal given suitable conditions on pricing w. Now we provide the following sufficient condition for (w^*,θ^*) to be an inconspicuous equilibrium.

Lemma 2 *A triplet* (P^*,w^*,θ^*) *where* w^* *is an admissible pricing rule and* $\theta^* \in \mathcal{A}(w^*)$*, is an inconspicuous equilibrium if it satisfies the following conditions:*

1. $Y^*_t = B_t + \theta^*_t$ *is a standard Brownian motion in its own filtration.*

2. $\xi^*_1 = \tilde{f}(z)$, $P^{0,z} - a.s.$ *for every* $z \in \mathbb{R}$ *and* ξ^* *is the strong solution to* $\xi_t = \int_0^t w^*(s,\xi_s)dY^*_s$ *on* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,1[},\mathbb{P})$ *with initial condition* $\xi_0 = 0$ *a.s..*

3. $P^* = \xi^* + \mathbb{E}[f(Z)]$ *is an* $(\mathcal{F}^{Y^*})$*-martingale w.r.t.* $\mathbb{P}$.

Proof. Suppose (P^*,w^*,θ^*) is a triplet satisfying conditions 1 to 3 in the statement of the Lemma.

Condition 1 ensures that the insider's trading strategy is inconspicuous since the total order is a Brownian motion in market maker's filtration. Conditions 2 and 3 imply that the pricing rule w^* is rational in the sense that

$$\begin{aligned} P_t^* &= \xi_t^* + \mathbb{E}[f(Z)] = \mathbb{E}[\xi_1^* \mid \mathcal{F}_t^{Y^*}] + \mathbb{E}[f(Z)] \\ &= \mathbb{E}[f(Z) - \mathbb{E}[f(Z)] \mid \mathcal{F}_t^{Y^*}] + \mathbb{E}[f(Z)] = \mathbb{E}[f(Z) \mid \mathcal{F}_t^{Y^*}] \end{aligned}$$

where the second equality is due to martingale property of ξ^* by condition 3, the third equality is due to the convergence of the terminal distribution by condition 2 and last equality is by tower property.

Finally, by Lemma 1, conditions 2 imply that θ^* is optimal.

■

By condition 2 of the above Lemma, ξ^* need to have required terminal distribution of $f(Z)$, this effectively put the condition on w^* as $\xi_t^* = \int_0^t w^*(s, \xi_s) dY_s^*$ where Y^* is a standard Brownian motion by condition 1. The following subsection discuss the existence of such pricing rule.

2.4 Existence of Pricing Rule

Now we discuss the existence of w^* for equilibrium. For brevity we will drop the asterisk in this sub-section. The market maker has to solve PDE (2.2.4) and

$$\xi_1 \stackrel{d}{=} \tilde{f}(Z). \tag{2.4.7}$$

Provided ξ is the strong solution to SDE

$$d\xi_t = w(t, \xi_t) d\beta_t, \quad \xi_0 = 0 \ a.s.$$

and β is a $\mathbb{P}$ standard Brownian motion. The following proposition gives sufficient conditions for existence of $w(t,x) \in \mathcal{C}^{1,2} : [0,1] \times]\tilde{b}, \tilde{d}[\to \mathbb{R}^+$ such that there exists unique strong solution to the above SDE (2.4.12) with required terminal distribution (2.4.7) and w solves the PDE (2.2.4) .

Proposition 1 *Consider function* $\tilde{f} = f - \mathbb{E}[f(Z)]$ *where* f *satisfies Assumption 1 and such that* $\mathbb{E}[\tilde{f}(Z)] = 0$*. Define* $\lim_{x\to\infty} \tilde{f}(x) = \tilde{d}$*, and* $\lim_{x\to-\infty} \tilde{f}(x) = \tilde{b}$*.*

Suppose there exists $\lambda(t,x) \in \mathcal{C}^{1,3} : [0,1] \times \mathbb{R} \to]\tilde{b}, \tilde{d}[$ *such that*

1. λ *is bounded, strictly increasing,* $\lim_{x\to\pm\infty} \lambda_x(t,x) = 0$ *and*

$$\lim_{x\to-\infty} \lambda(t,x) = \tilde{b}, \quad \lim_{x\to\infty} \lambda(t,x) = \tilde{d}.$$

2. λ *satisfies the Burger's equation:*

$$\lambda_t(t,x) + \frac{1}{2}\lambda_{xx}(t,x) = -\gamma\lambda(t,x)\lambda_x(t,x). \tag{2.4.8}$$

3. λ *satisfies the boundary condition:*

$$\lambda(1,x) = \tilde{f} \circ \Phi^{-1} \circ P(1,x) \tag{2.4.9}$$

where Φ *is the cumulative distribution function (CDF) of* $\mathcal{N}(0,1)$ *and* P *is the CDF of* κ_t *with* κ *being the unique strong solution of*

$$d\kappa_t = d\beta_t + \gamma\lambda(t,\kappa_t)dt \tag{2.4.10}$$

with initial condition $\kappa_0 = 0$ *a.s..*

4. λ *satisfies the initial condition* $\lambda(0,0) = 0$*.*

Then the weighting function w *given by:*

$$w(t,y) = \frac{1}{\frac{\partial\lambda^{-1}}{\partial y}(t,y)} \tag{2.4.11}$$

is well defined and $w(t,y) \in \mathcal{C}^{1,2} : [0,1] \times]\tilde{b}, \tilde{d}[\to \mathbb{R}_+$*. It will satisfy the following:*

1. w *is positive in the interior of its domain,* $\lim_{y\downarrow\tilde{b}} w(t,y) = \lim_{y\uparrow\tilde{d}} w(t,y) = 0$ *for any* $t \in [0,1]$*.*

2. *For any* $\xi \in]\tilde{b}, \tilde{d}[$*,* $\xi_t = \lambda(t,\kappa_t)$ *is a unique strong solution for SDE*

$$d\xi_t = w(t,\xi_t)d\beta_t, \tag{2.4.12}$$

with initial condition $\xi_0 = 0$ *a.s.. Moreover, the stopping* $\tau := \inf\{t > 0 : \xi_t \notin]\tilde{b}, \tilde{d}[\}$ *satisfies* $\tau > 1$ *a.s..*

3. $\frac{w_t(t,\xi)}{w(t,\xi)^2} + \frac{w_{\xi\xi}(t,\xi)}{2} = -\gamma$ *with boundary condition* $\xi_1 \overset{d}{=} \tilde{f}(Z)$.

Please refer to Appendix for proof. The sketch of the proof is as follows: we begin by proving w defined by (2.4.11) is well-defined and satisfies the properties in statement 1 for w due to condition 1 for λ. Secondly, due to $\xi_t = \lambda(t, \kappa_t)$ we can apply Ito's formula to obtain the SDE for ξ as in statement 2 for w. Stopping time $\tau > 1$ is due to the process κ being non-explosive since its drift is bounded. Thirdly the SDE for w in statement 3 for w is shown by substituting (2.4.11) into Burger's equation satisfied by λ due to condition 2. Finally the boundary condition of ξ_1 is satisfied due to condition 3 and 4 of λ and the definition $\xi_t = \lambda(t, \kappa_t)$. Therefore to show existence of the rational pricing rule w, it is sufficient to demonstrate existence of solutions for the Burger's equation (2.4.8) and (2.4.9) where P is the CDF of κ_t and κ is the unique strong solution of (2.4.10). We can further relax the sufficient condition to existence of solution of an integral equation.

Lemma 3 *Consider function* $\tilde{f} = f - \mathbb{E}[f(Z)]$ *where* f *satisfies Assumption 1 and such that* $\mathbb{E}[\tilde{f}(Z)] = 0$. *Define* $\lim_{x\to\infty} \tilde{f}(x) = \tilde{d}$, *and* $\lim_{x\to-\infty} \tilde{f}(x) = \tilde{b}$. *Define*

$$\lim_{x\to\infty} \tilde{f}(x) = \tilde{d} > 0, \qquad \lim_{x\to-\infty} \tilde{f}(x) = \tilde{b} < 0.$$

Let $\tilde{P} \in \mathcal{C}^2 : \mathbb{R} \to (0,1)$ *be a function strictly increasing w.r.t.* x, *with* $\tilde{P}(-\infty) = 0$ *and* $\tilde{P}(\infty) = 1$. *It also satisfies the integral equation:*

$$\tilde{P}(x) = \frac{c^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{ \gamma \int_0^u \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s) ds - \frac{u^2}{2} \right\} du.$$

Then

$$\lambda(t,x) := \frac{\int_{\mathbb{R}} \Gamma(t, x-y) \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) e^{\int_0^y \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du} dy}{\gamma \int_{\mathbb{R}} \Gamma(t, x-y) e^{\int_0^y \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du} dy}, \quad \forall t \in [0,1], \tag{2.4.13}$$

where $\Gamma(t,x) = \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{ -\frac{x^2}{2(1-t)} \right\}$, *is well defined, continuously differentiable with respect to the space variable on* $[0,1] \times \mathbb{R}$ *and infinitely continuously differentiable on* $[0,1[\times \mathbb{R}$. *Moreover,* $\lambda_x(t,x)$ *is uniformly bounded and at terminal time we have*

$$\lambda(1,x) = \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x).$$

Such defined λ *satisfies all conditions of Proposition 1. Furthermore,* $P(1,x) = \tilde{P}(x)$ *where* $P(t,x)$ *is the CDF of* κ_t *satisfying the SDE (2.4.10).*

Proof. We will first show that λ given by (2.4.13) is well defined and has the degree of regularity as stated.

Indeed, consider a PDE

$$u_t(t,x) + \frac{u_{xx}(t,x)}{2} = 0 \tag{2.4.14}$$

with the terminal condition

$$u(1,x) = \exp\left\{\int_0^x \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u)du\right\}.$$

As $\tilde{f}$ is bounded, the terminal condition has at most exponential growth, and therefore Theorem 1.12 in [24] yields that there exits a classical solution to this Cauchy problem on $[0,1]$ (note that we can take h as small as needed in this theorem). Moreover, the solution, u, is given by

$$u(t,x) = \int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy \quad \forall t \in [0,1[.$$

Note that $u \in C([0,1] \times \mathbb{R})$ as the solution of the Cauchy problem. Furthermore, Theorem 9.10 in [24] yields that $u \in C^\infty([0,1[\times\mathbb{R})$.

Thus, $u_x(t,x)$ is well-defined and continuous on $[0,1[\times\mathbb{R}$. Moreover, for any $(t,x) \in [0,1[\times\mathbb{R}$ we will have (differentiation under the integral sign is justified as $\left|\frac{\partial}{\partial x}\Gamma(t,x-y)\right| = \frac{|x-y|}{1-t}\Gamma(t,x-y)$ and $u(1,y) < e^{\tilde{K}|y|}$ where $\tilde{K}$ is the upper bound for $|\tilde{f}|$ and $|\tilde{f}'|$)

$$\begin{aligned} u_x(t,x) &= \int_{\mathbb{R}} \frac{\partial}{\partial x}\Gamma(t,x-y)u(1,y)dy = -\int_{\mathbb{R}} \frac{\partial}{\partial y}\Gamma(t,x-y)u(1,y)dy \\ &= \Gamma(t,x-y)u(1,y)\,|_{-\infty}^{\infty} - \int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy. \end{aligned}$$

Note that the last integral is well defined as $|u_y(1,y)| \le \tilde{K}e^{\tilde{K}|y|}$ since $|\tilde{f}|$ is bounded by constant $\tilde{K}$. Moreover,

$$0 \le \lim_{y\to\pm\infty} \Gamma(t,x-y)u(1,y) \le \lim_{y\to\pm\infty} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{-\frac{(x-y)^2}{2(1-t)} + \gamma\tilde{K}|y|\right\} = 0.$$

As $|u_y(1,y)| \le \tilde{K}e^{\tilde{K}|y|}$, Theorem 1.12 in [24] yields that

$$u_x(t,x) = -\int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy \tag{2.4.15}$$

is a solution to the PDE (2.4.14) with the terminal condition

$$u_x(1,x) = \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x) \exp\left\{ \int_0^x \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du \right\}.$$

In particular, $u_x \in C([0,1] \times \mathbb{R})$ and, in view of Theorem 9.10 in [24], $u_x \in C^\infty([0,1[\times\mathbb{R})$. Furthermore,

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy \geq \int_{\mathbb{R}} \Gamma(t,x-y)e^{-\tilde{K}|y|}dy \\ &= \int_{-\infty}^0 \Gamma(t,x-y)e^{-\tilde{K}|y|}dy + \int_0^\infty \Gamma(t,x-y)e^{-\tilde{K}|y|}dy = I_1(t,x) + I_2(t,x), \end{aligned}$$

$$\begin{aligned} I_1(t,x) &= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{ -\frac{(x-y)^2}{2(1-t)} + \tilde{K}y \right\} dy \\ &= \exp\left\{ \tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\} \int_{-\infty}^0 \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{ -\frac{(y-(x+\tilde{K}(1-t)))^2}{2(1-t)} dy \right\} \\ &= \exp\left\{ \tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\} \Phi^{\sqrt{1-t}}(-x - \tilde{K}(1-t)), \end{aligned}$$

$$\begin{aligned} I_2(t,x) &= \int_0^\infty \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{ -\frac{(x-y)^2}{2(1-t)} - \tilde{K}y \right\} dy \\ &= \exp\left\{ -\tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\} \int_0^\infty \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{ -\frac{(y-(x-\tilde{K}(1-t)))^2}{2(1-t)} dy \right\} \\ &= \exp\left\{ -\tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\} \Phi^{\sqrt{1-t}}(-x + \tilde{K}(1-t)). \end{aligned}$$

Therefore, for any $x > \tilde{K}(1-t)$, we have

$$u(t,x) \geq I_2(t,x) > \frac{1}{2} \exp\left\{ -\tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\},$$

and for any $x < \tilde{K}(1-t)$, we have

$$u(t,x) \geq I_1(t,x) > \frac{1}{2} \exp\left\{ \tilde{K}x + \frac{\tilde{K}^2(1-t)}{2} \right\}.$$

Due to continuity of u, the function defined by

$$\lambda(t,x) = \frac{u_x(t,x)}{\gamma u(t,x)}, \quad \forall t \in [0,1],$$

is well-defined, continuous on $[0,1]\times\mathbb{R}$ and infinitely continuously differentiable on $[0,1[\times\mathbb{R}$.

To establish that λ_x is continuous on $[0,1]\times\mathbb{R}$, observe that

$$\lambda_x(t,x) = \frac{u_{xx}(t,x)u(t,x) - (u_x(t,x))^2}{\gamma u^2(t,x)}$$

is well defined on $[0,1]\times\mathbb{R}$. Moreover, since both u and u_x are continuous on $[0,1]\times\mathbb{R}$ and u is strictly positive, the continuity of λ_x will follow from the continuity of u_{xx}. Note that from (2.4.15) we have (differentiation under the integral sign is justified as $\left|\frac{\partial}{\partial x}\Gamma(t,x-y)\right| = \frac{|x-y|}{1-t}\Gamma(t,x-y)$ and $|u_y(1,y)| < \tilde{K}e^{\tilde{K}|y|}$)

$$\begin{aligned} u_{xx}(t,x) &= -\int_{\mathbb{R}} \frac{\partial}{\partial x}\Gamma(t,x-y)u_y(1,y)dy = \int_{\mathbb{R}} \frac{\partial}{\partial y}\Gamma(t,x-y)u_y(1,y)dy \\ &= -\Gamma(t,x-y)u_y(1,y)\ |_{-\infty}^{\infty} + \int_{\mathbb{R}} \Gamma(t,x-y)u_{yy}(1,y)dy \\ &= \int_{\mathbb{R}} \Gamma(t,x-y)u_{yy}(1,y)dy. \end{aligned} \tag{2.4.16}$$

The last equality is due to

$$0 \le \lim_{y\to\pm\infty} \Gamma(t,x-y)|u_y(1,y)| \le \lim_{y\to\pm\infty} \frac{\tilde{K}}{\sqrt{2\pi(1-t)}} \exp\left\{-\frac{(x-y)^2}{2(1-t)} + \gamma\tilde{K}|y|\right\} = 0,$$

And the $\int_{\mathbb{R}}\Gamma(t,x-y)u_{yy}(1,y)dy$ is well defined as we have

$$\begin{aligned} |u_{xx}(1,x)| &= \left|\left[\left(\gamma\tilde{f}(\Phi^{-1}(\tilde{P}(x)))\right)^2 + \gamma\tilde{f}'(\Phi^{-1}(\tilde{P}(x)))\frac{\tilde{P}_x(x)}{\Phi'(\Phi^{-1}(\tilde{P}(x)))}\right] e^{\int_0^x \gamma\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)du}\right| \\ &\le \gamma\tilde{K}\left[\gamma\tilde{K} + \left|\frac{\tilde{P}_x(x)}{\Phi'(\Phi^{-1}(\tilde{P}(x)))}\right|\right] e^{\gamma\tilde{K}|x|} \\ &= \gamma\tilde{K}\left[\gamma\tilde{K} + c^*\exp\left\{\int_0^x \gamma\tilde{f}(\Phi^{-1}(\tilde{P}(u)))du + \frac{1}{2}\left(\Phi^{-1}(\tilde{P}(x)) - x^2\right)\right\}\right] e^{\gamma\tilde{K}|x|} \\ &\le \gamma\tilde{K}\left[\gamma\tilde{K} + c^* e^{\gamma\tilde{K}|x| + \frac{1}{2}\left((\Phi^{-1}(\tilde{P}(x)))^2 - x^2\right)}\right] e^{\gamma\tilde{K}|x|} \\ &\le \bar{K}e^{2\gamma\tilde{K}|x|} \end{aligned} \tag{2.4.17}$$

where the first and second inequalities are due to the boundedness of $\tilde{f}$ and $\tilde{f}'$, and $\bar{K} = (\gamma\tilde{K})^2 + \gamma\tilde{K}c^* \sup_x e^{\frac{1}{2}\left((\Phi^{-1}(\tilde{P}(x)))^2 - x^2\right)}$. We have $\bar{K} < \infty$ since, due to the Lemma 6 in Appendix, $\lim_{x\to\pm\infty} e^{\frac{1}{2}\left((\Phi^{-1}(\tilde{P}(x)))^2 - x^2\right)} \leq 1$ and $\Phi^{-1} \circ \tilde{P}$ being a continuous function. The bound (2.4.17) together with the representation (2.4.16) yield, via application of the Theorem 1.12 in [24], that u_{xx} is a solution to the PDE (2.4.14) with the terminal condition $u_{xx}(1, x)$.

Next, we show that this λ satisfies the conditions of Proposition 1.

Direct calculation yield that it solves the equation (2.4.8). Indeed,

$$\begin{aligned}
\lambda_t(t,x) &= \frac{1}{\gamma}\left(\frac{u_{tx}(t,x)}{u(t,x)} - \frac{u_x(t,x)u_t(t,x)}{u^2(t,x)}\right), \\
\lambda_x(t,x) &= \frac{1}{\gamma}\left(\frac{u_{xx}(t,x)}{u(t,x)} - \frac{u_x^2(t,x)}{u^2(t,x)}\right), \\
\lambda_{xx}(t,x) &= \frac{1}{\gamma}\left(\frac{u_{xxx}(t,x)}{u(t,x)} - \frac{3u_{xx}(t,x)u_x(t,x)}{u^2(t,x)} + \frac{2u_x^3(t,x)}{u^3(t,x)}\right),
\end{aligned}$$

and therefore

$$\begin{aligned}
I &= \gamma\left(\lambda_t(t,x) + \frac{1}{2}\lambda_{xx}(t,x) + \gamma\lambda_x(t,x)\lambda(t,x)\right) \\
&= \frac{1}{u(t,x)}\left(u_t(t,x) + \frac{u_{xx}(t,x)}{2}\right)_x - \frac{u_x(t,x)}{u(t,x)^2}\left(u_t(t,x) + \frac{u_{xx}(t,x)}{2}\right) = 0
\end{aligned}$$

Next, we demonstrate that condition 1 of Proposition 1 is satisfied. Indeed, λ is bounded since

$$\begin{aligned}
|\lambda(t,x)| &\leq \frac{\int_{\mathbb{R}} \Gamma(t, x-y)\,|u_y(1,y)|\,dy}{\gamma\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\
&= \frac{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)\left|\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(y)\right|dy}{\gamma\int_{\mathbb{R}}\Gamma(t,x-y)u(1,y)dy} \leq \frac{\tilde{K}}{\gamma}
\end{aligned}$$

due to the boundedness of $\tilde{f}$.

To show that λ is strictly increasing, we observe that

$$\lambda_x(1,x) = \frac{d}{dx}\left[\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(x)\right] > 0$$

due to $\tilde{f}$, Φ and $\tilde{P}$ being strictly increasing functions. Moreover,

$$\lambda_x(t,x) = \frac{u_{xx}(t,x)u(t,x) - u_x(t,x)^2}{\gamma u(t,x)^2}.$$

Integration by parts (and using bounds on $\tilde{f}$) will yield

$$\begin{aligned}
I &= u_{xx}(t,x)u(t,x) - u_x(t,x)^2 \\
&= \left(\int_{\mathbb{R}} \Gamma_{xx}(t,x-y)u(1,y)dy\right)\left(\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy\right) - \left(\int_{\mathbb{R}} \Gamma_x(t,x-y)u(1,y)dy\right)^2 \\
&= \left(\int_{\mathbb{R}} \Gamma(t,x-y)u_{yy}(1,y)dy\right)\left(\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy\right) - \left(\int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy\right)^2 \\
&\geq \left(\int_{\mathbb{R}} \Gamma(t,x-y)\sqrt{u_{yy}(1,y)u(1,y)}dy\right)^2 - \left(\int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy\right)^2 > 0
\end{aligned}$$

where the inequality before last is just an application of Cauchy-Schwarz inequality ($u_{xx} > 0$ as we will see shortly) and the last inequality holds since

$$u_{xx}(1,x)u(1,x) - u_x^2(1,x) = \gamma u^2(1,x)\lambda_x(1,x) > 0.$$

Therefore $\lambda_x(t,x) > 0$ for all $(t,x) \in [0,1] \times \mathbb{R}$, i.e. λ is strictly increasing.

Next, we need to establish that $\lim_{x\to\pm\infty} \lambda_x(t,x) = 0$. First we show this is true for $t = 1$. We have

$$\lim_{x\to\pm\infty} \lambda_x(1,x) = \lim_{x\to\pm\infty} \tilde{f}' \circ \Phi^{-1} \circ \tilde{P}(x) \frac{\tilde{P}'(x)}{\Phi'(\Phi^{-1} \circ \tilde{P}(x))}$$

Observe that since $\lim_{x\to\pm\infty} \tilde{f}'(x) = 0$, we only need to show that

$$\lim_{x\to\pm\infty} \frac{\tilde{P}'(x)}{\Phi'(\Phi^{-1} \circ \tilde{P}(x))} = \sqrt{2\pi} \lim_{x\to\pm\infty} \frac{\tilde{P}'(x)}{e^{-\frac{1}{2}(\Phi^{-1}\circ\tilde{P}(x))^2}}$$

is finite. Observe that

$$\begin{aligned}
\sqrt{2\pi} \lim_{x\to\pm\infty} \frac{\tilde{P}'(x)}{e^{-\frac{1}{2}(\Phi^{-1}\circ\tilde{P}(x))^2}} &= \sqrt{2\pi} \lim_{x\to\pm\infty} \frac{\tilde{P}''(x)}{e^{-\frac{1}{2}(\Phi^{-1}\circ\tilde{P}(x))^2}(-\Phi^{-1} \circ \tilde{P}(x))\frac{d}{dx}\left(\Phi^{-1} \circ \tilde{P}(x)\right)} \\
&= \sqrt{2\pi} \lim_{x\to\pm\infty} \frac{\tilde{P}'(x)(\gamma\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x) - x)}{e^{-\frac{1}{2}(\Phi^{-1}\circ\tilde{P}(x))^2}(-\Phi^{-1} \circ \tilde{P}(x))\frac{\tilde{P}'(x)}{\Phi'\circ\Phi^{-1}\circ\tilde{P}(x)}} \\
&= \lim_{x\to\pm\infty} \frac{x - \gamma\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x)}{\Phi^{-1} \circ \tilde{P}(x)} \\
&= \lim_{x\to\pm\infty} \frac{x}{\Phi^{-1} \circ \tilde{P}(x)} = 1 < \infty.
\end{aligned}$$

where the one to the last equality is due to the boundedness of $\tilde{f}$, the last equality is due to the Lemma 6 in Appendix and the first one is the application of L'Hopital rule. Note that the rule is applicable since $\lim_{x\to+\infty} \Phi^{-1} \circ \tilde{P}(x) = +\infty$ and

$$0 < \lim_{x\to\pm\infty} \tilde{P}'(x) = \frac{c^*}{\sqrt{2\pi}} \lim_{x\to\pm\infty} e^{\gamma \int_0^x \tilde{f}\circ\Phi^{-1}\circ\tilde{P}(s)ds - \frac{x^2}{2}} \leq \frac{c^*}{\sqrt{2\pi}} \lim_{x\to\pm\infty} e^{\gamma\tilde{K}|x| - \frac{x^2}{2}} = 0.$$

Therefore

$$\lim_{x\to\pm\infty} \frac{\tilde{P}'(x)}{\Phi'(\Phi^{-1} \circ \tilde{P}(x))}$$

is finite. Therefore, the previous considerations yield

$$\lim_{x\to\pm\infty} \lambda_x(1,x) = \lim_{x\to\pm\infty} \frac{d}{dx} \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x) = 0. \tag{2.4.18}$$

Next, we need to show that $\lim_{x\to\pm\infty} \lambda_x(t,x) = 0$ for any $t \in [0,1[$. We prove this statement only for $x \to +\infty$ as the case $x \to -\infty$ is done similarly. First, observe that

$$\begin{aligned} u_{yy}(1,y) &= \frac{d}{dy}\left(\gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) u(1,y)\right) \\ &= \gamma^2 \left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right)^2 u(1,y) + \gamma \frac{d}{dy}\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right) u(1,y), \end{aligned}$$

and therefore

$$\begin{aligned} \lambda_x(t,x) &= \frac{u_{xx}(t,x)u(t,x) - (u_x(t,x))^2}{\gamma u^2(t,x)} \\ &= \frac{\int_{\mathbb{R}} \Gamma(t,x-y)u_{yy}(1,y)dy}{\gamma \int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} - \frac{1}{\gamma}\left(\frac{\int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}\right)^2 \\ &= \frac{\gamma \int_{\mathbb{R}} \Gamma(t,x-y)\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right)^2 u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\ &\quad + \frac{\int_{\mathbb{R}} \Gamma(t,x-y)\frac{d}{dy}\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right) u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\ &\quad -\gamma \left(\frac{\int_{\mathbb{R}} \Gamma(t,x-y)\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right) u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}\right)^2 \\ &= \gamma I_1(x) + I_2(x) - \gamma I_3^2(x). \end{aligned}$$

We will show that $\lim_{x\to\infty} I_1(x) = \lim_{x\to\infty} I_3^2(x) = \tilde{d}^2$ and $\lim_{x\to\infty} I_2(x) = 0$, which will yield the required result. Due to the fact that $\tilde{f}$ is increasing and $\lim_{x\to\infty} \tilde{f}(x) = \tilde{d} > 0$, we have $\tilde{f}(x) < \tilde{d}$ for all x and therefore $I_1(x) < \tilde{d}$ and $I_3(x) < \tilde{d}$ for all x.

Moreover, fix an $\epsilon > 0$ and let N be such that $\tilde{f}(N) > \tilde{d} - \epsilon$, then we will have

$$\begin{aligned}
\lim_{x\to+\infty} I_3(x) &= \lim_{x\to+\infty} \frac{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} \\
&\geq \lim_{x\to+\infty} \frac{\tilde{b} \int_{-\infty}^{N} \Gamma(t, x-y) u(1,y) dy + (\tilde{d} - \epsilon) \int_{N}^{\infty} \Gamma(t, x-y) u(1,y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} \\
&= \tilde{d} - \epsilon + \lim_{x\to+\infty} \frac{(\tilde{b} - \tilde{d} + \epsilon) \int_{-\infty}^{N} \Gamma(t, x-y) u(1,y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} = \tilde{d} - \epsilon,
\end{aligned}$$

where the last equality is due to the Lemma 5 in Appendix. Due to the arbitrariness on ϵ, and the previous bound on I_3 we have $\lim_{x\to\infty} I_3(x) = \tilde{d}$ as claimed.

Similarly, (N is the same as before)

$$\begin{aligned}
\lim_{x\to+\infty} I_1(x) &= \lim_{x\to+\infty} \frac{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) (\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y))^2 dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} \\
&\geq \lim_{x\to+\infty} \frac{(\tilde{d} - \epsilon)^2 \int_{N}^{\infty} \Gamma(t, x-y) u(1,y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} \\
&= (\tilde{d} - \epsilon)^2 - \lim_{x\to+\infty} \frac{(\tilde{d} - \epsilon)^2 \int_{-\infty}^{N} \Gamma(t, x-y) u(1,y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} = (\tilde{d} - \epsilon)^2,
\end{aligned}$$

and therefore $\lim_{x\to\infty} I_1(x) = \tilde{d}$ in the same way as before.

Due to (2.4.18) and the fact that $\frac{d}{dy}\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right)$ is continuous, for any $\epsilon > 0$ there exist constants M and N such that

$$\left| \frac{d}{dy} \left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right) \right| < \epsilon, \ \ \forall y > N$$

and

$$\left| \frac{d}{dy} \left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right) \right| < M, \ \ \forall y \leq N$$

Thus, we will have

$$\begin{aligned}
\lim_{x\to+\infty} |I_2(x)| &\leq \lim_{x\to+\infty} \frac{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) \left| \frac{d}{dy} \left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right) \right| dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} \\
&\leq \lim_{x\to+\infty} (M - \epsilon) \frac{\int_{-\infty}^{N} \Gamma(t, x-y) u(1,y) dy}{\int_{\mathbb{R}} \Gamma(t, x-y) u(1,y) dy} + \epsilon = \epsilon,
\end{aligned}$$

where the last equality is due to the Lemma 5 in Appendix.

We also notice that, since $|\tilde{f}|$ is bounded by $\tilde{K}$, we have for any $t \in [0,1]$:

$$|I_1(x)| = \frac{\int_{\mathbb{R}} \Gamma(t, x-y) \left| \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right|^2 u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \leq \tilde{K}^2,$$

$$|I_3(x)| \leq \frac{\int_{\mathbb{R}} \Gamma(t, x-y) \left| \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right| u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \leq \tilde{K},$$

$$|I_2(x)| = \frac{\int_{\mathbb{R}} \Gamma(t, x-y) \left| \frac{d}{dy} \left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \right) \right| u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}.$$

To show $\lambda_x(t,x)$ is uniformly bounded for $(t,x) \in [0,1] \times \mathbb{R}$, it suffices to show $\frac{d}{dy}\left(\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)\right)$ is uniformly bounded in $\mathbb{R}$, which is true due to 2.4.18.

To conclude that condition 1 of Proposition 1 holds, we need to demonstrate that $\lim_{x\to\infty} \lambda(t,x) = \tilde{d}$, and $\lim_{x\to-\infty} \lambda(t,x) = \tilde{b}$. Notice that by the definition of $\lambda(t,x)$,

$$\begin{aligned}
\lim_{x\to\infty} \lambda(t,x) &= \lim_{x\to\infty} \frac{u_x(t,x)}{\gamma u(t,x)} = \lim_{x\to\infty} \frac{\int_{\mathbb{R}} \Gamma(t,x-y)u_y(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\
&= \lim_{x\to\infty} \frac{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}.
\end{aligned}$$

Since $\tilde{f} \circ \Phi^{-1} \circ \tilde{P}$ is bounded by $\tilde{d}$, we have

$$\lim_{x\to\infty} \lambda(t,x) \leq \lim_{x\to\infty} \tilde{d} \frac{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} = \tilde{d}.$$

Since $\tilde{f} \circ \Phi^{-1} \circ \tilde{P}$ is strictly increasing and converge to $\tilde{d}$ when $x \to +\infty$. We have for any $\epsilon > 0$, there exists $N > 0$ s.t. for any $x > N$, we have $\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x) > \tilde{d} - \epsilon$. We have,

$$\begin{aligned}
\lambda(t,x) &= \frac{\int_{-\infty}^{N} \Gamma(t,x-y)u(1,y)\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\
&\quad + \frac{\int_{N}^{\infty} \Gamma(t,x-y)u(1,y)\tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} = I_4(x) + I_5(x).
\end{aligned}$$

Due to boundedness of $\tilde{f}$ and Lemma 5 in Appendix, we have

$$\lim_{x\to\infty} |I_4(x)| \leq \tilde{K} \lim_{x\to\infty} \frac{\int_{-\infty}^{N} \Gamma(t,x-y)u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} = 0$$

and therefore $\lim_{x\to\infty} I_4(x) = 0$. Moreover, due to the same lemma

$$\begin{aligned}\lim_{x\to\infty} I_5(x) &\geq (\tilde{d}-\epsilon)\frac{\int_N^\infty \Gamma(t,x-y)u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy} \\ &= (\tilde{d}-\epsilon)\left[1 - \lim_{x\to\infty}\frac{\int_{-\infty}^N \Gamma(t,x-y)u(1,y)dy}{\int_{\mathbb{R}} \Gamma(t,x-y)u(1,y)dy}\right] \\ &= \tilde{d}-\epsilon.\end{aligned}$$

Since ϵ is arbitrarily chosen, we have $\lim_{x\to\infty} I_5 = \tilde{d}$. Therefore, $\lim_{x\to\infty} \lambda(t,x) = \tilde{d}$. Similarly we can show $\lim_{x\to-\infty} \lambda(t,x) = \tilde{b}$.

Next we will show the connection between P and $\tilde{P}$ that would imply that the condition 3 of Proposition 1 holds.

Since λ and λ_x are uniformly bounded due to Lemma 3, Proposition 5.2.9 and Theorem 5.2.5 in [38] yield that for any fixed $(t,x) \in [0,1] \times \mathbb{R}$ there exists unique strong solution to SDE (2.4.10) with initial condition $\kappa_t = x$. Denote $P(t,x)$ the CDF of κ_t.

Our goal is to derive $P(t,x)$ via an application of Girsanov theorem. Consider a local martingale L given by:

$$L_t = \exp\left\{-\int_0^t \gamma\lambda(s,\kappa_s)d\beta_s - \frac{1}{2}\int_0^t \gamma^2\lambda^2(s,\kappa_s)ds\right\}.$$

Since $\lambda(t,x)$ is bounded, L is a true martingale, and therefore a measure defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \mid_{\mathcal{F}_t} = L_t$$

is equivalent to $\mathbb{P}$. Moreover, under $\tilde{\mathbb{P}}$ the process κ satisfies

$$\kappa_t = \tilde{\beta}_t$$

by Girsanov Theorem and $\mathbb{P}[\kappa_t < x] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{\mathbf{1}_{\kappa_t<x}}{L_t}\right]$.

Observe that

$$\frac{1}{L_t} = \exp\left\{\int_0^t \gamma\lambda(s,\kappa_s)d\tilde{\beta}_s - \frac{1}{2}\int_0^t \gamma^2\lambda^2(s,\kappa_s)ds\right\} = \exp\left\{I(t,\kappa_t) - N_t\right\},$$

where $I(t,x) = \int_0^x \gamma\lambda(t,u)du$, and

$$N_t = \int_0^t \left[I_t(s,\kappa_s) + \frac{1}{2}\gamma\lambda_x(s,\kappa_s) + \frac{1}{2}\gamma^2\lambda^2(s,\kappa_s)\right] ds.$$

Indeed, application of Ito's formula gives

$$\begin{aligned} I(t,\kappa_t) &= \int_0^t I_t(s,\kappa_s)ds + \int_0^t I_x(s,\kappa_s)d\kappa_s + \frac{1}{2}\int_0^t I_{xx}(s,\kappa_s)d\langle\kappa\rangle_s \\ &= \int_0^t I_t(s,\kappa_s)ds + \int_0^t \gamma\lambda(s,\kappa_s)d\tilde{\beta}_s + \int_0^t \frac{1}{2}\gamma\lambda_x(s,\kappa_s)ds, \end{aligned}$$

which yields the required representation for $1/L$.

Moreover,

$$\begin{aligned} N_t &= \int_0^t \left\{\gamma\int_0^{\kappa_s}\lambda_t(s,x)dx + \frac{\gamma}{2}\lambda_x(s,\kappa_s) + \frac{\gamma^2}{2}\lambda^2(s,\kappa_s)\right\}ds \\ &= \int_0^t \gamma\left\{\int_0^{\kappa_s}\left(\lambda_t(s,x) + \frac{1}{2}\lambda_{xx}(s,x) + \gamma\lambda(s,x)\lambda_x(s,x)\right)dx + \frac{1}{2}\lambda_x(s,0) + \frac{\gamma}{2}\lambda^2(s,0)\right\}ds \\ &= \frac{\gamma}{2}\int_0^t \left\{\lambda_x(s,0) + \gamma\lambda^2(s,0)\right\}ds = c(t), \end{aligned}$$

where the last equality is due to the fact that λ satisfies (2.4.8) and c is a deterministic function.

Due to the above considerations, we have

$$\begin{aligned} P(t,x) &= \mathbb{P}[\kappa_t < x] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\kappa_t<x}e^{I(t,\kappa_t)}\right]e^{-c(t)} \\ &= e^{-c(t)}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^x e^{I(t,y)}e^{-\frac{y^2}{2t}}dy. \end{aligned}$$

The last equality is because κ_t is a $\tilde{\mathbb{P}}$ Brownian motion with normal distribution $\mathcal{N}(0,t)$. Thus we have

$$\begin{aligned} P_x(1,x) &= \frac{e^{-c(1)}}{\sqrt{2\pi}}\exp\left\{\gamma\int_0^x \lambda(1,u)du - \frac{x^2}{2}\right\} \\ &= \frac{e^{-c(1)}}{\sqrt{2\pi}}\exp\left\{\gamma\int_0^x \tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)du - \frac{x^2}{2}\right\} \\ &= \frac{e^{-c(1)}}{c^*}\tilde{P}_x(x). \end{aligned}$$

Since $P(1,x)$ is the CDF of κ_1, we have

$$1 = \int_{-\infty}^{\infty} P_x(1,x)dx = \frac{e^{-c(1)}}{c^*}\int_{-\infty}^{\infty}\tilde{P}_x(x)dx = \frac{e^{-c(1)}}{c^*}$$

due to the definition of $\tilde{P}$. Therefore we have $c^* = e^{-c(1)}$ and $P_x(1,x) = \tilde{P}_x(x)$, integrate both sides we have $P(1,x) = \tilde{P}(x)$ since $P(1,\infty) = \tilde{P}(\infty) = 1$.

Finally we will show condition 4 of Proposition 1, i.e., $\lambda(0,0) = 0$. By definition of λ, we have $\lambda(0,0) = \frac{u_x(0,0)}{\gamma u(0,0)}$. Since

$$u(0,0) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2} + \gamma \int_0^y \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du} dy = \frac{\tilde{P}(\infty)}{c^*} = \frac{1}{c^*}.$$

Due to (2.4.15), we also have

$$u_x(0,0) = \int_{\mathbb{R}} \Gamma(0,-y) u_y(1,y) dy = \int_{\mathbb{R}} \frac{\gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y)}{\sqrt{2\pi}} \exp\left\{ -\frac{y^2}{2} + \gamma \int_0^y \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du \right\} dy$$

Therefore

$$\begin{aligned} \lambda(0,0) &= \frac{c^*}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) \exp\left\{ -\frac{y^2}{2} + \gamma \int_0^y \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du \right\} dy \\ &= \int_{\mathbb{R}} \tilde{P}_y(y) \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) dy = \int_0^1 \tilde{f} \circ \Phi^{-1}(u) du \\ &= \int_{\mathbb{R}} \tilde{f}(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \mathbb{E}[\tilde{f}(Z)] = 0, \end{aligned}$$

where the second equality is due to the definition of $\tilde{P}$, the third equality is by change of variable $u = \tilde{P}(y)$ and fourth equality is by change of variable $z = \Phi^{-1}(u)$. ■

To collect the results we have so far before moving onto the final Lemma for existence of equilibrium pricing, we conclude that if given the existence of $\tilde{P}$ solving the integral equation (2.4.19) defined in Lemma 4 with boundary conditions $\tilde{P}(-\infty) = 0$ and $\tilde{P}(\infty) = 1$, we can define function λ as (2.4.13), a strictly increasing function satisfying (2.4.8). Moreover, $\tilde{P}$ is the terminal distribution of process κ which is the unique strong solution of (2.4.10). By Proposition 1, we can define $w(t,y) = \frac{1}{\frac{d}{dy}\lambda^{-1}(t,y)} \in \mathcal{C}^{1,2} : [0,1] \times]\tilde{b}, \tilde{d}[\to \mathbb{R}^+$ with $\lim_{y \downarrow \tilde{b}} w(t,y) = \lim_{y \uparrow \tilde{d}} w(t,y) = 0$ for any $t \in [0,1]$ and $\xi_t = \lambda(t, \kappa_t)$ such that ξ is the unique strong solution to (2.4.12) and w solves (2.2.4) and (2.4.7). The following Lemma completes the existence of such pricing rule.

Lemma 4 *Consider function $\tilde{f} = f - \mathbb{E}[f(Z)]$ where f satisfies Assumption 1 and such that $\mathbb{E}[\tilde{f}(Z)] = 0$. Define $\lim_{x \to \infty} \tilde{f}(x) = \tilde{d}$, and $\lim_{x \to -\infty} \tilde{f}(x) = \tilde{b}$. Then there exists*

$\tilde{P} \in \mathcal{C}^2 : \mathbb{R} \to \mathbb{R}$, *a function strictly increasing w.r.t.* x, *with* $\tilde{P}(-\infty) = 0$ *and* $\tilde{P}(\infty) = 1$ *satisfying the integral equation*

$$\tilde{P}(x) = \frac{c^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{\gamma \int_0^u \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds - \frac{u^2}{2}\right\} du \tag{2.4.19}$$

where c^* *is chosen such that* $\tilde{P}(\infty) = 1$.

PROOF. First the integral equation (2.4.19) will make sense because $\tilde{f}$ is bounded, therefore $\int_0^u \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds$ is at most linear. Thus $-\frac{u^2}{2}$ will be the dominating term in the integral, i.e.,

$$\int_{-\infty}^{\infty} \exp\left\{\gamma \int_0^u \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds - \frac{u^2}{2}\right\} du < \infty.$$

Thus c^* is well-defined. Denote $g(x) = \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(x)$ and $G(x) = \int_0^x g(u)du$, we have the integral expression for $\tilde{P}$:

$$\tilde{P}(x) = \frac{c^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{\gamma G(u) - \frac{u^2}{2}\right\} du.$$

where c^* is constant to normalise the integral such that:

$$1 = \frac{c^*}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\gamma G(u) - \frac{u^2}{2}\right\} du.$$

Here we have a recursive relation to obtain a sequence of $g^n(x)$, $G^n(x)$, $\tilde{P}^n(x)$ and c_n^* such that

$$g^n(x) = \tilde{f} \circ \Phi^{-1} \circ \tilde{P}^n(x), \quad G^n(x) = \int_0^x g^n(u)du,$$

$$1 = \frac{c_n^*}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\gamma G^n(u) - \frac{u^2}{2}\right\} du, \quad \tilde{P}^{n+1}(x) = \frac{c_n^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{\gamma G^n(u) - \frac{u^2}{2}\right\} du.$$

Denote transformation T, the mapping such that $\tilde{P}^{n+1} = T\tilde{P}^n$. Define

$$\mathcal{D} = \left\{\tilde{P} \in \mathcal{C}_b(\mathbb{R}) : \tilde{P}\, nondecreasing, \tilde{P}(x) = \int_{-\infty}^{x} \tilde{P}_x(u)du; \tilde{P}(-\infty) = 0; \tilde{P}(\infty) = 1; \right.$$
$$\left. 0 \le \tilde{P}_x(x) \le \frac{c}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}\right\}$$

where we choose any $\sigma^2 > 1$ and $c = \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{\frac{\gamma^2\tilde{K}^2}{2(\sigma^2-1)}\right\}$.

We will show that the set $\mathcal{D}$ has the following properties:

1. $\mathcal{D}$ is a convex set: Suppose $\tilde{P}^1, \tilde{P}^2 \in \mathcal{D}$ and $\lambda \in [0,1]$, then

$$\tilde{P} := \lambda \tilde{P}^1 + (1-\lambda)\tilde{P}^2 \in \mathcal{C}_b,\ \textit{nondecreasing},$$

$$\tilde{P}(-\infty) = \lambda \tilde{P}^1(-\infty) + (1-\lambda)\tilde{P}^2(-\infty) = 0,$$

$$\tilde{P}(\infty) = \lambda \tilde{P}^1(\infty) + (1-\lambda)\tilde{P}^2(\infty) = 1.$$

   Moreover

$$\begin{aligned}\tilde{P}(x) &= \lambda \tilde{P}^1(x) + (1-\lambda)\tilde{P}^2(x) \\ &= \lambda \int^x \tilde{P}^1_x(s)ds + (1-\lambda)\int^x \tilde{P}^2_x(s)ds \\ &= \int^x \left(\lambda \tilde{P}^1_x(s) + (1-\lambda)\tilde{P}^2_x(s)\right) ds,\end{aligned}$$

$$0 \leq \tilde{P}_x(x) = \lambda \tilde{P}^1_x(x) + (1-\lambda)\tilde{P}^2(x) \leq \frac{c}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}.$$

   Thus $\tilde{P} \in \mathcal{D}$.

2. $\mathcal{D}$ is closed: Since $\mathcal{C}_b(\mathbb{R})$ is a Banach space, given $\{\tilde{P}^n\}$, a sequence of elements in $\mathcal{D}$ converging to some element, $\tilde{P} \in \mathcal{C}_b$ in the sup norm, i.e., for any $\epsilon > 0$ there exists N s.t. for any $n > N$, we have

$$\sup_{x \in \mathbb{R}} | \tilde{P}^n(x) - \tilde{P}(x) | \leq \epsilon.$$

   Therefore for any $y > x$, $\epsilon > 0$ and $n > N$, we have

$$\tilde{P}(y) - \tilde{P}(x) = \tilde{P}(y) - \tilde{P}^n(y) + \tilde{P}^n(y) - \tilde{P}^n(x) + \tilde{P}^n(x) - \tilde{P}(x) \geq -2\epsilon$$

   as $\tilde{P}^n(y) - \tilde{P}^n(x) \geq 0$ for each n and $y > x$. Since ϵ can be arbitrarily small, we have $\tilde{P}$ is a non-decreasing function.
   Moreover, since $\tilde{P}^n(-\infty) = 0$ for any n, choose $n > N$. Then for $\epsilon > 0$ there exists $-L < 0$ s.t. for any $x < -L$ we have $\tilde{P}^n(x) < \epsilon$. Therefore for any $x < -L$

$$0 \leq \tilde{P}(x) = \tilde{P}(x) - \tilde{P}^n(x) + \tilde{P}^n(x) < 2\epsilon.$$

Thus we have $\tilde{P}(-\infty) = 0$ due to the arbitrary choice of ϵ.

Similarly, since $\tilde{P}^n(\infty) = 1$ for any n, choose $n > N$. Then for $\epsilon > 0$ there exists $L > 0$ s.t. for any $x > L$ we have $1 - \tilde{P}^n(x) < \epsilon$. Therefore for any $x > L$

$$0 \leq 1 - \tilde{P}(x) = 1 - \tilde{P}^n(x) + \tilde{P}^n(x) - \tilde{P}(x) < 2\epsilon.$$

Thus we have $\tilde{P}(\infty) = 1$ due to the arbitrary choice of ϵ.

In addition, for any $x \leq y$ in $\mathbb{R}$, it follows from Fatou's lemma that

$$0 \leq \tilde{P}(y) - \tilde{P}(x) = \lim_{n\to\infty} \int_x^y \tilde{P}_x^n(u)du \leq \int_x^y \limsup_{n\to\infty} \tilde{P}_x^n(u)du.$$

Since each $\tilde{P}_x^n$ is bounded from above by the same integrable functions, so will be $\limsup_{n\to\infty} \tilde{P}_x^n(u)$ for every $u \in [x, y]$. This implies that $\tilde{P}$ is absolutely continuous and, in particular, there exists a function $\tilde{P}_x$ with $0 \leq \tilde{P}_x(x) \leq \limsup_{n\to\infty} \tilde{P}_x^n(1, x) \leq \frac{c}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2\sigma^2}\}$ for all $x \in \mathbb{R}$. Hence, $\mathcal{D}$ is closed.

3. $T\tilde{P} \in \mathcal{D}$. We start from

$$T\tilde{P}(x) = \frac{c^*_{T\tilde{P}}}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{\gamma \int_0^u \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds - \frac{u^2}{2}\right\} du,$$

From the definition of $T\tilde{P}$ we know it is increasing function in $\mathcal{C}_b(\mathbb{R})$ with $T\tilde{P}(-\infty) = 0$ and $c^*_{T\tilde{P}}$ is to normalise the integral such that $T\tilde{P}(\infty) = 1$. Moreover, it is absolutely continuous and can be written as $T\tilde{P}(x) = \int^x (T\tilde{P})_x(s)ds$ with

$$(T\tilde{P})_x = \frac{c^*_{T\tilde{P}}}{\sqrt{2\pi}} \exp\left\{\gamma \int_0^x \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds - \frac{x^2}{2}\right\}.$$

First we obtain an estimate on $c^*_{T\tilde{P}}$. By definition of $c^*_{T\tilde{P}}$, we have

$$\begin{aligned}
c^*_{T\tilde{P}} &= \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\left\{\gamma G(u) - \frac{u^2}{2}\right\} du} \leq \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\left\{-\gamma |G(u)| - \frac{u^2}{2}\right\} du} \\
&\leq \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\{-\gamma \tilde{K}|u| - \frac{u^2}{2}\}du} = \frac{\sqrt{2\pi}}{\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_{-\infty}^{\infty} \exp\{-\frac{(|u|+\gamma\tilde{K})^2}{2}\}du} \\
&= \frac{\sqrt{2\pi}}{2\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_0^{\infty} \exp\{-\frac{(u+\gamma\tilde{K})^2}{2}\}du} = \frac{\sqrt{2\pi}}{2\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_{\gamma\tilde{K}}^{\infty} \exp\{-\frac{u^2}{2}\}du} \\
&= \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{-\frac{\gamma^2\tilde{K}^2}{2}\right\}, \qquad (2.4.20)
\end{aligned}$$

where $G(x) = \int_0^x \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(s)ds$, $\tilde{K}$ is the upper bound of $|\tilde{f}|$ and $|\tilde{f}'|$. Therefore, with $\sigma^2 > 1$, we have

$$\begin{aligned}
(T\tilde{P})_x(x)e^{\frac{x^2}{2\sigma^2}} &= \frac{c^*_{T\tilde{P}}}{\sqrt{2\pi}} \exp\left\{\gamma G(x) - \left(\frac{1}{2} - \frac{1}{2\sigma^2}\right) x^2\right\} \\
&\leq \frac{c^*_{T\tilde{P}}}{\sqrt{2\pi}} \exp\left\{\gamma \tilde{K}|x| - \left(\frac{1}{2} - \frac{1}{2\sigma^2}\right) x^2\right\} \\
&\leq \frac{c^*_{T\tilde{P}}}{\sqrt{2\pi}} \exp\left\{\frac{\gamma^2 \tilde{K}^2 \sigma^2}{2(\sigma^2 - 1)}\right\} \\
&\leq \frac{1}{\sqrt{2\pi}} \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{-\frac{\gamma^2\tilde{K}^2}{2} + \frac{\gamma^2 \tilde{K}^2 \sigma^2}{2(\sigma^2 - 1)}\right\} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{\frac{\gamma^2\tilde{K}^2}{2(\sigma^2 - 1)}\right\} = \frac{c}{\sqrt{2\pi}},
\end{aligned}$$

where $c = \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{\frac{\gamma^2\tilde{K}^2}{2(\sigma^2-1)}\right\}$ from definition of $\mathcal{D}$. Thus we have $\tilde{P}_x(x) \leq \frac{c}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2\sigma^2}\}$. Therefore, $T\tilde{P} \in \mathcal{D}$.

Concluding from above, we have $\mathcal{D}$ is a closed convex subset of Banach space, transformation T maps from $\mathcal{D}$ to $\mathcal{D}$. Thus $T\mathcal{D}$ is an equicontinuous family of functions. By Ascoli-Arzela Theorem (Corollary III.3.3 Lang [39]), if $\tilde{P}^n$ is a sequence in $T\mathcal{D}$, then there is a subsequence which converges not only point-wise to $\tilde{P} \in \mathcal{C}_b(\mathbb{R})$ but also uniform on every compact interval of $\mathbb{R}$. We will show the convergence is uniform for any $x \in \mathbb{R}$.

Assume $\tilde{P}^{n_k}$ is the convergent subsequence. By the definition of $\mathcal{D}$, since $\tilde{P}^{n_k}_x(x) \leq \frac{c}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$ we have:

$$\tilde{P}^{n_k}(x) \leq \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2\sigma^2}\right\} du, \quad 1 - \tilde{P}^{n_k}(x) \leq \frac{c}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left\{-\frac{u^2}{2\sigma^2}\right\} du.$$

Therefore, there exist $x^* < 0$ and $x_* > 0$ such that for any $\epsilon > 0$,

$$\tilde{P}^{n_k}(x) \leq \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{x_*} \exp\left\{-\frac{u^2}{2\sigma^2}\right\} du = c\sigma\Phi^\sigma(x_*) \leq \epsilon,$$

$$1 - \tilde{P}^{n_k}(x) \leq \frac{c}{\sqrt{2\pi}} \int_{x^*}^{\infty} \exp\left\{-\frac{u^2}{2\sigma^2}\right\} du = c\sigma\Phi^\sigma(-x^*) \leq \epsilon,$$

where Φ^σ is the CDF of $\mathcal{N}(0, \sigma)$.

Since $\tilde{P}^{n_k}$ converges to $\tilde{P}$ point-wise, we also have that for any $x \le x_*$, $\tilde{P}(x) \le \epsilon$ and for any $x \ge x^*$, $1 - \tilde{P}(x) \le \epsilon$.

We also have the convergence is uniform on the compact $[x_*, x^*]$. Thus there exist $N \in \mathbb{N}$ such that for all $k \ge N$

$$\sup_{x \in [x_*, x^*]} |\tilde{P}^{n_k}(x) - \tilde{P}(x)| \le \epsilon.$$

Thus for any $k \ge N$ we have

$$\begin{aligned} \sup_{x \in \mathbb{R}} |\tilde{P}^{n_k}(x) - \tilde{P}(x)| &\le \sup_{x \in [x_*, x^*]} |\tilde{P}^{n_k}(x) - \tilde{P}(x)| + \sup_{x \in [-\infty, x_*]} (\tilde{P}^{n_k}(x) + \tilde{P}(x)) \\ &\quad + \sup_{x \in [x^*, \infty]} (1 - \tilde{P}^{n_k}(x) + 1 - \tilde{P}(x)) \le 5\epsilon. \end{aligned}$$

Therefore the convergence of $\tilde{P}^{n_k}$ to $\tilde{P}$ is uniform for all $x \in \mathbb{R}$. Thus $T\mathcal{D}$ is pre-compact in $\mathcal{C}_b(\mathbb{R})$.

Next we show the transformation T is continuous. Assume without loss of generality $\tilde{P}^n$ converge to $\tilde{P} \in \mathcal{D}$ in sup norm. It suffices to show the point-wise convergence of $T\tilde{P}^n$ to $T\tilde{P}$ (as we have shown above, since $T\tilde{P}^n \in \mathcal{D}$, if $T\tilde{P}^n$ converge to $T\tilde{P}$ point-wise, we have $T\tilde{P}^n$ converge to $T\tilde{P}$ uniformly in $\mathbb{R}$ under sup-norm). Since $T\tilde{P}^n(x) = \frac{c_n^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{\gamma G^n(u) - \frac{u^2}{2}\} du$ and denote

$$G(x) = \int_0^x g(u)du = \int_0^x f \circ \Phi^{-1} \circ \tilde{P}(u)du.$$

We already know from (2.4.20) that c_n^* has an upper bound $\frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{-\frac{\gamma^2\tilde{K}^2}{2}\right\}$. Similarly we can achieve a lower bound for c_n^* as well:

$$\begin{aligned} c_n^* &= \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\left\{\gamma G(u) - \frac{u^2}{2}\right\} du} \ge \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\left\{\gamma |G(u)| - \frac{u^2}{2}\right\} du} \\ &\ge \frac{\sqrt{2\pi}}{\int_{-\infty}^{\infty} \exp\{\gamma\tilde{K}|u| - \frac{u^2}{2}\} du} = \frac{\sqrt{2\pi}}{\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_{-\infty}^{\infty} \exp\{-\frac{(|u|-\gamma\tilde{K})^2}{2}\} du} \\ &= \frac{\sqrt{2\pi}}{2\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_0^{\infty} \exp\{-\frac{(u-\gamma\tilde{K})^2}{2}\} du} = \frac{\sqrt{2\pi}}{2\exp\{\frac{\gamma^2\tilde{K}^2}{2}\} \int_{-\gamma\tilde{K}}^{\infty} \exp\{-\frac{u^2}{2}\} du} \\ &= \frac{1}{2\Phi(\gamma\tilde{K})} \exp\left\{-\frac{\gamma^2\tilde{K}^2}{2}\right\}, \end{aligned}$$

where $\tilde{K}$ is the upper bound of $|\tilde{f}|$ and $|\tilde{f}'|$. Therefore c_n^* has a converging subsequence $c_{n_k}^* \to c^*$. To show the point-wise convergence of $T\tilde{P}^n$ to $T\tilde{P}$, it suffices to show point-wise convergence of $T\tilde{P}^{n_k}$ to $T\tilde{P}$.

To do so, it suffices to show point-wise convergence of $c_{n_k}^* \exp\{\gamma G^{n_k}\}$ to $c^* \exp\{\gamma G\}$ by Scheffe's theorem (Theorem 3.16.12 by Billingsley [6]). We have

$$\begin{aligned} & \lim_{k\to\infty} \left|c_{n_k}^* \exp\{\gamma G^{n_k}(x)\} - c^* \exp\{\gamma G(x)\}\right| \\ \leq & \lim_{k\to\infty} \left|c_{n_k}^* \exp\{\gamma G^{n_k}(x)\} - c_{n_k}^* \exp\{\gamma G(x)\}\right| + \lim_{k\to\infty} \left|c_{n_k}^* \exp\{\gamma G(x)\} - c^* \exp\{\gamma G(x)\}\right| \\ = & I_1 + I_2. \end{aligned}$$

To investigate the first term, we notice

$$\begin{aligned} I_1 &= \lim_{k\to\infty} c_{n_k}^* \exp\{\gamma G(x)\} \left|\exp\left\{\gamma \int_0^x (\tilde{f}\circ\Phi^{-1}\circ\tilde{P}^{n_k} - \tilde{f}\circ\Phi^{-1}\circ\tilde{P})(u)du\right\} - 1\right| \\ &\leq \lim_{k\to\infty} c_{n_k}^* \exp\{\gamma\tilde{K}|x|\} \left|\exp\left\{\gamma|x| \sup_{[0,x]} \left|\tilde{f}\circ\Phi^{-1}\circ\tilde{P}^{n_k}(u) - \tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)\right|\right\} - 1\right|. \end{aligned}$$

Since $\tilde{P}^{n_k}$ converges to $\tilde{P}$ uniformly, we have for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for any $k > N$, we have

$$\sup_{x\in\mathbb{R}} |\tilde{P}^{n_k}(x) - \tilde{P}(x)| < \epsilon.$$

Therefore for any $x \in \mathbb{R}$, when $k > N$, $\tilde{P}(x) - \epsilon < \tilde{P}^{n_k}(x) < \tilde{P}(x) + \epsilon$. Thus, we have for any $0 < \epsilon < \frac{1}{2}(\tilde{P}(x) - \tilde{P}(0))$,

$$\begin{aligned} J &= \sup_{[0,x]} \left|\tilde{f}\circ\Phi^{-1}\circ\tilde{P}^{n_k}(u) - \tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)\right| \\ &\leq \sqrt{2\pi} \sup_{[\tilde{P}(0)-\epsilon, \tilde{P}(x)+\epsilon]} \tilde{f}'(\Phi^{-1}(y)) \exp\left\{\frac{(\Phi^{-1}(y))^2}{2}\right\} \sup_{[0,x]} |\tilde{P}^{n_k}(u) - \tilde{P}(u)| \\ &< \sqrt{2\pi}\tilde{K}\epsilon \sup_{[\tilde{P}(0)-\epsilon, \tilde{P}(x)+\epsilon]} \exp\left\{\frac{(\Phi^{-1}(y))^2}{2}\right\} = \sqrt{2\pi}\tilde{K}\epsilon M(x,\epsilon) \end{aligned}$$

where $M(x,\epsilon)$ is an increasing function of ϵ. Moreover,

$$\lim_{\epsilon\downarrow 0} M(x,\epsilon) = \sup_{[\tilde{P}(0), \tilde{P}(x)]} \exp\left\{\frac{(\Phi^{-1}(y))^2}{2}\right\} = M(x).$$

Therefore we have

$$\begin{aligned}
I_1 &< \lim_{k\to\infty} c^*_{n_k} \exp\{\gamma\tilde{K}|x|\} \left|\exp\left\{\sqrt{2\pi}\gamma|x|\tilde{K}\epsilon M(x,\epsilon)\right\} - 1\right| \\
&\leq c^* \exp\{\gamma\tilde{K}|x|\} \lim_{\epsilon\downarrow 0} \left|\exp\left\{\sqrt{2\pi}\gamma|x|\tilde{K}\epsilon M(x,\epsilon)\right\} - 1\right| \\
&= \frac{1}{2\Phi(-\gamma\tilde{K})} \exp\left\{-\frac{\gamma^2\tilde{K}^2}{2} + \gamma\tilde{K}|x|\right\} \lim_{\epsilon\downarrow 0} \left|\exp\left\{\sqrt{2\pi}\gamma|x|\tilde{K}M(x)\epsilon\right\} - 1\right| = 0.
\end{aligned}$$

Note in the above inequality, the choice of k and ϵ are independent.

Since $c^*_{n_k} \to c^*$, there exists $\tilde{N} \in \mathbb{N}$ such that for any$\epsilon > 0$ and $k > \tilde{N}$ we have

$$|c^*_{n_k} - c^*| \leq \epsilon.$$

Thus

$$\begin{aligned}
I_2 &= \lim_{k\to\infty} |c^*_{n_k} - c^*| \exp\{\gamma G(x)\} \\
&\leq \exp\{\gamma\tilde{K}|x|\} \lim_{k\to\infty} |c^*_{n_k} - c^*| = 0.
\end{aligned}$$

Therefore

$$\lim_{k\to\infty} \left|c^*_{n_k} \exp\{\gamma G^n(x)\} - c^* \exp\{\gamma G(x)\}\right| = I_1 + I_2 = 0.$$

Hence $T\tilde{P}^n$ converges to $T\tilde{P}$ point-wise. This implies that $T\tilde{P}^n$ converge to $T\tilde{P}$ uniformly in $\mathbb{R}$ under sup-norm. As $\tilde{P}^n$ converge to $\tilde{P}$ uniformly in $\mathbb{R}$ under sup-norm, we have T is a continuous operator. $\mathcal{D}$ is a closed and convex subset of a Banach space and $T\mathcal{D}$ is pre-compact. Therefore, by Schauder's fixed point theorem(Theorem 7.1.2 by Friedman [24]), T has a fixed point P, i.e. $T\tilde{P} = \tilde{P}$. Moreover $\tilde{P} \in \mathcal{C}^1$ due to the definition of operator $T\tilde{P}$, $\tilde{P}$ is differentiable with continuous derivatives. We know $\tilde{P} \in \mathcal{C}^2$ since directly differentiate (2.4.19) we have $\tilde{P}_{xx} = \tilde{P}_x \left(\gamma f \circ \Phi^{-1} \circ \tilde{P}(x) - x\right)$.

In addition, $\tilde{P}$ is strictly increasing as the derivative is strictly positive. ■

Now we have the existence of $\tilde{P}$ solving the integral equation, as discussed before the Lemma, we have the equilibrium pricing rule $w(t,\xi)$ where w is defined in Proposition 1. Moreover, we will show in the following Corollary that ρ, the transition density of process ξ, and p, the transition density of process κ, exist and can be derived one another through the connection between w and λ.

Corollary 2 *Let $\tilde{P}$ be the cumulative distribution function given by Lemma 4 and λ given by*

$$\lambda(t,x) := \frac{\int_{\mathbb{R}} \Gamma(t,x-y)\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(y)e^{\int_0^y \gamma\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)du}dy}{\gamma\int_{\mathbb{R}} \Gamma(t,x-y)e^{\int_0^y \gamma\tilde{f}\circ\Phi^{-1}\circ\tilde{P}(u)du}dy}, \quad \forall t\in[0,1].$$

Then the unique strong solution of SDE (2.4.10) with initial condition $\kappa_0 = 0$ a.s., κ, admits a transition density, denoted by p. Moreover, this transition density will satisfy Chapman-Kolmogorov equation

$$p(t,y;s,z) = \int_{\mathbb{R}} p(t,y;u,x)p(u,x;s,z)dx,$$

and have the following smoothness properties:

1. *(continuity in the forward space variable) $p(s,x;t,.) : \mathbb{R} \to [0,\infty[$ is continuous for $s\in[0,t[$ and $x\in\mathbb{R}$;*

2. *(smoothness in backward variables) $(s,x)\to p(s,x;t,y)$ belongs to $\mathcal{C}^{1,2}([0,t[\times\mathbb{R})$ for every $y\in\mathbb{R}$.*

Furthermore, κ is a Feller process.

Moreover, consider the weighting function $w(t,x) = \frac{1}{\lambda_x^{-1}(t,x)}$ and ξ, the unique strong solution of SDE (2.4.12) with initial condition $\xi_0 = 0$ a.s.. Then ξ admits a transition density, denoted by ρ which is given by

$$\rho(t,y;s,z) = \frac{p(t,\lambda^{-1}(t,y);s,\lambda^{-1}(s,z))}{w(s,z)}.$$

Proof. The existence and uniqueness of a strong solution of SDE (2.4.10) with initial condition $\kappa_0 = 0$ a.s., was established in Lemma 3. Moreover, since λ and λ_x are uniformly bounded due to Lemma 3, Proposition 5.2.9 and Theorem 5.2.5 in [38] yield that for any fixed $(t,x)\in[0,1]\times\mathbb{R}$ there exists unique strong solution to SDE (2.4.10) with initial condition $\kappa_t = x$.

Existence, uniqueness, and smoothness in backward variables (s,x) of the transition density function p follows from pp. 368-369 of [38]. Indeed, those considerations apply due to the uniform (on $[0,1]\times\mathbb{R}$) boundedness of λ and λ_x and the fact that there exists unique strong solution to (2.4.10) with initial condition $\kappa_t = x$ for any $(t,x)\in[0,1]\times\mathbb{R}$.

In particular, there exists a unique fundamental solution of

$$v_t(t,x) + \frac{1}{2}v_{xx}(t,x) + \gamma\lambda(t,x)v_x(t,x) = 0, \tag{2.4.21}$$

which is the transition density p.

To show that p satisfies Chapman-Kolmogorov equation, fix any $(s,z) \in [0,1] \times \mathbb{R}$ and $0 \le u < s$. Observe that $p(t,y;s,z)$ satisfies (2.4.21) on $[0,u] \times \mathbb{R}$ with terminal condition $v(u,x) = p(u,x;s,z)$. Then, due to Theorem 5.7.6 in [38], we have the required representation

$$p(t,y;s,z) = \int_{\mathbb{R}} p(t,y;u,x)p(u,x;s,z)dx.$$

The above considerations, together with the definition of the fundamental solution and the Theorem 11 in Chapter 1, Section 6 in [24] implies that κ is a Feller process.

The continuity in the forward space variable follows from Theorem 3.2.1 in [53] and the fact that p is the unique fundamental solution of (2.4.21).

Finally, we turn to the transition density of ξ. Due to the Proposition 1, $\xi_t = \lambda(t,\kappa_t)$. Moreover, considerations similar to the ones in the Proposition 1 yield that for any fixed $(t,y) \in [0,1]\times]\tilde{b},\tilde{d}[$, we have $\xi_s^{(t,y)} = \lambda(s, \kappa_s^{(t,\lambda^{-1}(t,y))})$ for all $s \in [t,1]$, where $\xi^{(t,y)}$ is the unique strong solution of SDE (2.4.12) with initial condition $\xi_t^{(t,y)} = y$ and $\kappa^{(t,\lambda^{-1}(t,y))}$ is the unique strong solution of SDE (2.4.10) with initial condition $\kappa_t^{(t,\lambda^{-1}(t,y))} = \lambda^{-1}(t,y)$. The proof is similar to the consideration in Proposition 1. We have shown in the first paragraph that SDE (2.4.10) with initial condition $\kappa_t = x$ where $(t,x) \in [0,1] \times \mathbb{R}$ has unique strong solution. Since $y \in]\tilde{b},\tilde{d}[$, we have $\lambda^{-1}(t,y) \in \mathbb{R}$. Denote κ as the unique strong solution to SDE (2.4.10) with initial condition $\kappa_t = \lambda^{-1}(t,y)$. Define $\xi_s = \lambda(s,\kappa_s)$, which is consistent with the initial condition

$$\xi_t = \lambda(t,\kappa_t) = \lambda(t,\lambda^{-1}(t,y)) = y.$$

Application of Ito's formula will yield

$$\begin{aligned}
\xi_s &= \xi_t + \int_t^s \lambda_x(u,\kappa_u)d\kappa_u + \int_t^s \left(\lambda_u(u,\kappa_u) + \frac{\lambda_{xx}(u,\kappa_u)}{2}\right) du \\
&= y + \int_t^s \lambda_x(u,\kappa_u)\left(d\beta_u + \gamma\lambda(u,\kappa_u)du\right) - \gamma\int_t^s \lambda(u,\kappa_u)\lambda_x(u,\kappa_u)du \\
&= y + \int_t^s \lambda_x(u,\kappa_u)d\beta_u = \int_t^s \lambda_x(u,\lambda^{-1}(u,\xi_u))d\beta_u = \int_t^s w(u,\xi_u)d\beta_u.
\end{aligned}$$

Therefore $\xi_s = \lambda(s, \kappa_s)$ is a strong solution to SDE (2.4.12) with initial condition $\xi_t = y$ for $(t, y) \in [0, 1] \times]\tilde{b}, \tilde{d}[$. Denote that $\tau = \inf\{s > t : \xi_s \notin]\tilde{b}, \tilde{d}[\} = \inf\{s > t : \kappa_s \notin] - \infty, +\infty[\}$. Since κ is non-explosive due to the boundedness of the drift term $\gamma\lambda(s, \kappa_s)$, we have $\tau > 1$.

To show the uniqueness of the solution to the SDE (2.4.12), suppose there is another strong solution $\tilde{\xi}$. Denote a sequence of open sets $V_n =]\tilde{b} + \frac{1}{n}, \tilde{d} - \frac{1}{n}[$, $n = 1, 2, ...$ and define a sequence of stopping times ν_n by $\nu_n = \inf\{s \geq t : \tilde{\xi}_s \notin V_n\}$. Then the process $\tilde{\kappa}_s := \lambda^{-1}(s, \tilde{\xi}_s)$ is well defined on $[t, \nu_n]$ for all n. Note that since λ is increasing function, $\nu_n = \inf\{s \geq t : \tilde{\kappa}_s \notin U_n\}$, where $U_n = \Big]\lambda^{-1}(s, \tilde{b} + \frac{1}{n}), \lambda^{-1}(s, \tilde{d} - \frac{1}{n})\Big[$.

Application of Ito's formula to $\tilde{\kappa}$ stopped at ν_n will yield

$$\begin{aligned}
\tilde{\kappa}_{s\wedge\nu_n} &= \lambda^{-1}(t, y) + \int_t^{s\wedge\nu_n} \lambda_x^{-1}(u, \tilde{\xi}_u) d\tilde{\xi}_u + \int_t^{s\wedge\nu_n} \left(\lambda_u^{-1}(u, \tilde{\xi}_u) + \frac{\lambda_{xx}^{-1}(u, \tilde{\xi}_u) w^2(u, \tilde{\xi}_u)}{2} \right) du \\
&= \lambda^{-1}(t, y) + \int_t^{s\wedge\nu_n} \lambda_x^{-1}(u, \tilde{\xi}_u) w(u, \tilde{\xi}_u) d\beta_u + \int_t^{s\wedge\nu_n} \left(\lambda_u^{-1}(u, \tilde{\xi}_u) + \frac{\lambda_{xx}^{-1}(u, \tilde{\xi}_u)}{2\left(\lambda_x^{-1}(u, \tilde{\xi}_u)\right)^2} \right) du \\
&= \lambda^{-1}(t, y) + \int_t^{s\wedge\nu_n} \lambda_x^{-1}(u, \tilde{\xi}_u) w(u, \tilde{\xi}_u) d\beta_u + \int_t^{s\wedge\nu_n} \gamma\tilde{\xi}_u du \\
&= \lambda^{-1}(t, y) + \int_t^{s\wedge\nu_n} d\beta_u + \int_t^{s\wedge\nu_n} \gamma\lambda(u, \tilde{\kappa}_u) du,
\end{aligned}$$

and therefore $\tilde{\kappa}$ is a strong solution of (2.4.10) in $[t, s \wedge \nu_n]$ for each $n \in \mathbb{N}$. Since solution to (2.4.10) with initial condition $\kappa_t = \lambda^{-1}(t, y)$ is unique, we have $\tilde{\kappa}_{s\wedge\nu_n} = \kappa_{s\wedge\nu_n}$ for all $s \in [t, 1]$. Taking the limit, in view of continuity of κ, we have $\tilde{\kappa}_{s\wedge\nu} = \kappa_{s\wedge\nu}$ for all $s \in [t, 1]$, where $\nu = \lim_{n\to\infty} \nu_n$. In particular, $\tilde{\kappa}_{1\wedge\nu} = \kappa_{1\wedge\nu}$ and therefore $\nu < 1$ is equivalent to $\tau < 1$ which has probability zero due to the arguments similar to Remark 4. Thus,

$$\tilde{\kappa}_s = \tilde{\kappa}_{s\wedge\nu} = \kappa_{s\wedge\nu} = \kappa_s \qquad t \in [t, 1].$$

Due to the connection between $\tilde{\kappa}$ and $\tilde{\xi}$ and continuity of $\tilde{\kappa}$ as well as κ, the above implies that

$$\mathbb{P}[\tilde{\xi}_s = \xi_s, s \in [0, 1]] = 1.$$

Therefore $\xi_s^{(t,y)} = \lambda(s, \kappa_s^{(t,\lambda^{-1}(t,y))})$ for all $s \in [t, 1]$ is the unique strong solution of SDE (2.4.12) with initial condition $\xi_t^{(t,y)} = y$.

Let ϕ be any bounded continuous function on $\mathbb{R}$. For any $(t,y) \in [0,1] \times]\tilde{b}, \tilde{d}[$ and $s \in [t,1]$ we have:

$$\begin{aligned}
\mathbb{E}[\phi(s, \xi_s^{(t,y)})] &= \mathbb{E}[\phi(s, \lambda(s, \kappa_s^{(t,\lambda^{-1}(t,y))}))] = \int_{\mathbb{R}} \phi(s, \lambda(s,z)) p(t, \lambda^{-1}(t,y); s, z) dz \\
&= \int_{\tilde{b}}^{\tilde{d}} \phi(s,u) p(t, \lambda^{-1}(t,y); s, \lambda^{-1}(s,u)) d\lambda^{-1}(s,u) \\
&= \int_{\tilde{b}}^{\tilde{d}} \phi(s,u) \frac{p(t, \lambda^{-1}(t,y); s, \lambda^{-1}(s,u))}{w(s,u)} du \\
&= \int_{\tilde{b}}^{\tilde{d}} \phi(s,u) \rho(t,y;s,u) du
\end{aligned}$$

which implies that the density of ξ exists and is as stated. The second equality is by applying the transition density of κ, the third equality is by change of variable $u = \lambda(s,z)$, the fourth equality is by connection between w and λ. ■

In the next Subsection, we complete by giving the optimal trading strategy of the insider under equilibrium described by the transition density $\rho(t,\xi)$.

2.5 Equilibrium

In this Subsection, we give an inconspicuous equilibrium consists of the rational pricing rule (P^*, w^*) and optimal trading strategy θ^* for the insider.

Theorem 3 *Under Assumption 1, a triplet (P^*, w^*, θ^*) given by the following is an inconspicuous equilibrium.*

1. *The weighting function $w^*(t,x) = \frac{1}{\lambda_x^{-1}(t,x)}$ where λ is given by*

$$\lambda(t,x) := \frac{\int_{\mathbb{R}} \Gamma(t, x-y) \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(y) e^{\int_0^y \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du} dy}{\gamma \int_{\mathbb{R}} \Gamma(t, x-y) e^{\int_0^y \gamma \tilde{f} \circ \Phi^{-1} \circ \tilde{P}(u) du} dy}, \quad \forall t \in [0,1].$$

   The existence of $\tilde{P}$ is given by Lemma 4.

2. The insider's trading strategy $\theta^*_t = \int_0^t \alpha^*(s, \xi^*_s)ds$ *where*

$$\alpha^*(t, x) = w^*(t, x)\frac{\rho_x(t, x; 1, \tilde{f}(Z))}{\rho(t, x; 1, \tilde{f}(Z))}.$$

ρ is defined in Corollary 2 as the transition density of process ξ where ξ satisfies the SDE $\xi_t = \int_0^t w^*(s, \xi_s)d\beta_s$.

3. The market maker's pricing function $P^*_t = \xi^*_t + \mathbb{E}[f(Z)]$, *where ξ^* satisfies the SDE*

$$\xi^*_t = \int_0^t w^*(s, \xi^*_s)dY^*_s.$$

The cumulative order $Y^*_t = B_t + \theta^*_t$.

PROOF. We will first show that (P^*, w^*) is admissible in the sense of Definition 1. We know by Proposition 1, Lemma 3 and Lemma 4, we have the weighting function w^* is well-defined, positive in the interior of its domain and satisfies the PDE

$$\frac{w^*_t(t, x)}{w^*(t, x)^2} + \frac{w^*_{xx}(t, x)}{2} = -\gamma.$$

In addition, there exists unique strong solution ξ to the SDE

$$\xi_t = \int_0^t w^*(s, \xi_s)d\beta_s.$$

By Proposition 1, it can be written as $\xi_t = \lambda(t, \kappa_t)$ where κ is the unique strong solution of SDE (2.4.10), i.e.,

$$\kappa_t = \beta_t + \int_0^t \gamma\lambda(s, \kappa_s)ds.$$

From Remark 4, we have $\tau > 1$ a.s. Thus w^* is admissible pricing rule.

Next we show θ^* is admissible, i.e., $\theta^* \in \mathcal{A}(w^*)$. By construction θ^* is absolutely continuous. We need to show there exists unique strong solution for the SDE

$$d\xi^*_t = w^*(t, \xi^*_t)dB_t + w^*(t, \xi^*_t)^2\frac{\rho_\xi(t, \xi^*_t; 1, \tilde{f}(Z))}{\rho(t, \xi^*_t; 1, \tilde{f}(Z))}dt, \tag{2.5.22}$$

with initial condition $\xi^*_0 = 0$ a.s.. We will show under the connection $\xi^*_t = \lambda(t, \kappa^*_t)$, it is equivalent to show there exists unique strong solution for the SDE

$$d\kappa^*_t = dB_t + \gamma\lambda(t, \kappa^*_t)dt + \frac{p_x(t, \kappa^*_t; 1, \lambda^{-1}(1, \tilde{f}(Z)))}{p(t, \kappa^*_t; 1, \lambda^{-1}(1, \tilde{f}(Z)))}dt, \tag{2.5.23}$$

with initial condition $\kappa_0^* = 0$ a.s.. This is consistent with our result from Lemma 3 that $\lambda(0,0) = 0$. To show that $\xi_t^* = \lambda(t, \kappa_t^*)$ is a strong solution, apply Ito's formula, we have

$$\begin{aligned} d\xi_t^* &= \left(\lambda_t(t,\kappa_t^*) + \frac{1}{2}\lambda_{xx}(t,\kappa_t^*)\right)dt + \lambda_x(t,\kappa_t^*)d\kappa_t^* \\ &= -\gamma\lambda(t,\kappa_t^*)\lambda_x(t,\kappa_t^*)dt + \lambda_x(t,\kappa_t^*)dB_t + \gamma\lambda(t,\kappa_t^*)\lambda_x(t,\kappa_t^*)dt \\ &\quad +\lambda_x(t,\kappa_t^*)\frac{p_x(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}{p(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}dt \\ &= w^*(t,\xi_t^*)dB_t + w^*(t,\xi_t^*)\frac{p_x(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}{p(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}dt \end{aligned}$$

where the second equality is due to Burger's equation (2.4.8) and the third equality is due to the equality

$$\lambda_x(t,x) = \frac{1}{\lambda_y^{-1}(t,\lambda(t,x))} = w^*(t,\lambda(t,x)).$$

Due to Corollary 2, we have the connection between ρ and p given as follows

$$p(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z))) = w^*(1,\tilde{f}(Z))\rho(t,\xi_t^*;1,\tilde{f}(Z)).$$

Therefore differentiate w.r.t. the first space variable, we have

$$p_x(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z))) = w^*(1,\tilde{f}(Z))w^*(t,\xi_t^*)\rho_x(t,\xi_t^*;1,\tilde{f}(Z)).$$

Substitute into the Ito's formula we have

$$d\xi_t^* = w^*(t,\xi_t^*)dB_t + w^*(t,\xi_t^*)^2\frac{\rho_x(t,\xi_t^*;1,\tilde{f}(Z))}{\rho(t,\xi_t^*;1,\tilde{f}(Z))}dt.$$

Therefore, $\xi_t^* = \lambda(t,\kappa_t^*)$ is a strong solution to SDE (2.5.22). It remains to show the uniqueness of the solution of the SDE (2.5.22). Suppose there is another strong solution $\tilde{\xi}$. Denote a sequence of open sets $V_n =]\tilde{b} + \frac{1}{n}, \tilde{d} - \frac{1}{n}[$, $n = 1, 2, ...$ and define a sequence of stopping times ν_n by $\nu_n = \inf\{t \geq 0 : \tilde{\xi}_t \notin V_n\}$. Then the process $\tilde{\kappa}_t := \lambda^{-1}(t, \tilde{\xi}_t)$ is well defined on $[0, \nu_n]$ for all n. Note that since λ is increasing function, $\nu_n = \inf\{t \geq 0 : \tilde{\kappa}_t \notin U_n\}$, where $U_n = \left]\lambda^{-1}(t, \tilde{b} + \frac{1}{n}), \lambda^{-1}(t, \tilde{d} - \frac{1}{n})\right[$.

Application of Ito's formula to $\tilde{\kappa}$ stopped at ν_n will yield

$$\begin{aligned}\tilde{\kappa}_{t\wedge\nu_n} &= \int_0^{t\wedge\nu_n} \lambda_x^{-1}(s,\tilde{\xi}_s)d\tilde{\xi}_s + \int_0^{t\wedge\nu_n}\left(\lambda_s^{-1}(s,\tilde{\xi}_s) + \frac{\lambda_{xx}^{-1}(s,\tilde{\xi}_s)w^2(t,\tilde{\xi}_s)}{2}\right)ds \\ &= \int_0^{t\wedge\nu_n} \lambda_x^{-1}(s,\tilde{\xi}_s)\left(w^*(t,\tilde{\xi}_t)dB_t + w^*(t,\tilde{\xi}_t)^2\frac{\rho_x(t,\tilde{\xi}_t;1,\tilde{f}(Z))}{\rho(t,\tilde{\xi}_t;1,\tilde{f}(Z))}dt\right) + \int_0^{t\wedge\nu_n}\gamma\tilde{\xi}_s ds \\ &= \int_0^{t\wedge\nu_n} dB_s + \int_0^{t\wedge\nu_n}\gamma\lambda(s,\tilde{\kappa}_s)ds + \int_0^{t\wedge\nu_n} w^*(t,\tilde{\xi}_t)\frac{\rho_x(t,\tilde{\xi}_t;1,\tilde{f}(Z))}{\rho(t,\tilde{\xi}_t;1,\tilde{f}(Z))}dt\end{aligned}$$

where the second equality is due to a derivation of Burger's equation (2.6.25), the third equation is due to the equality

$$\lambda_y^{-1}(t,y) = \frac{1}{w^*(t,y)}.$$

Similar as before, due to Corollary 2, we have the connection between ρ and p given as follows

$$\rho(t,\tilde{\xi}_t;1,\tilde{f}(Z)) = \frac{p(t,\tilde{\kappa}_t;1,\lambda^{-1}(1,\tilde{f}(Z)))}{w^*(1,\tilde{f}(Z))}.$$

Therefore differentiate w.r.t. the first space variable, we have

$$\rho_x(t,\tilde{\xi}_t;1,\tilde{f}(Z)) = \frac{p_x(t,\tilde{\kappa}_t;1,\lambda^{-1}(1,\tilde{f}(Z)))}{w^*(1,\tilde{f}(Z))w^*(t,\tilde{\xi}_t)}.$$

Substitute into the Ito's formula we have

$$\tilde{\kappa}_{t\wedge\nu_n} = \int_0^{t\wedge\nu_n} dB_s + \int_0^{t\wedge\nu_n}\gamma\lambda(s,\tilde{\kappa}_s)ds + \int_0^{t\wedge\nu_n}\frac{p_x(t,\tilde{\kappa}_t;1,\lambda(1,\tilde{f}(Z)))}{p(t,\tilde{\kappa}_t;1,\lambda(1,\tilde{f}(Z)))}dt.$$

Therefore $\tilde{\kappa}$ is a strong solution of (2.5.23). Since solution to (2.5.23) is unique, we have $\tilde{\kappa}_{t\wedge\nu_n} = \kappa^*_{t\wedge\nu_n}$ for all $t\in[0,1]$. Taking the limit, in view of continuity of κ^*, we have $\tilde{\kappa}_{t\wedge\nu} = \kappa^*_{t\wedge\nu}$ for all $t\in[0,1]$, where $\nu = \lim_{n\to\infty}\nu_n$. In particular, $\tilde{\kappa}_{1\wedge\nu} = \kappa^*_{1\wedge\nu}$ and therefore $\nu < 1$ is equivalent to $\tau < 1$ which has probability zero due to the arguments above. Thus,

$$\tilde{\kappa}_t = \tilde{\kappa}_{t\wedge\nu} = \kappa^*_{t\wedge\nu} = \kappa^*_t \qquad t\in[0,1].$$

We apply Theorem 2.4 by Cetin and Danilova [13] to show that unique strong solution exists for (2.5.23). Denote $E = \mathbb{R}$ and $[0,1]\times E$ be the set containing the range of the process (t,κ^*_t). Conditions to apply the theorem need to be checked are as follows:

1. Assumption 2.1 [13] in one dimensional case where $a = 1$ and $b = \gamma\lambda(t, x)$ is uniformly bounded by $]\tilde{b}, \tilde{d}[$. The local martingale problem for A is well-posed whereas A is defined as

$$A_t = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \gamma\lambda(t, x)\frac{\partial}{\partial x}.$$

   To see that, by Corollary 5.4.8 and 5.4.9 of Karatzas and Shreve [38], the well-posedness of martingale problem is equivalently the existence of weak solution and uniqueness in law of the solution of the induced SDE by the martingale problem. Due to Theorem 5.2.9 [38], since the drift term is global Lipschitz and bounded by Lemma 3, we have the SDE (2.4.10) admits unique strong solution for any initial condition $\kappa_0 = \kappa$ a.s.. Therefore the well-posedness of martingale problem is satisfied.

2. Assumption 2.2 [13]: we know κ as the unique strong solution of SDE (2.4.10) is a Feller process. Moreover, since p satisfies Chapman-Kolmogorov equation, κ admits the semigroup property.

   We have shown in Corollary 2 that p has desired smoothness properties which is condition (H) in Assumption 2.2 [13].

3. We have shown in Lemma 3 that λ_x is uniformly bounded, i.e., there exists constant M s.t. for all $t \in [0, 1]$, $x, y \in \mathbb{R}$,

$$|\lambda(t, x) - \lambda(t, y)| \leq M|x - y|.$$

4. We want to show $p(t, y; 1, y') > 0$ for all $t \in [0, 1[$, $y, y' \in \mathbb{R}$. p is the fundamental solution of the parabolic partial differential equation (2.4.21) with coefficient λ uniformly bounded, hence at most linear growth. We have by Theorem 1.1 [45] that the fundamental solution has a Gaussian lower bound with coefficients depending on uniform boundedness of growth condition on $\gamma\lambda$. Therefore $p(t, y; 1, y') > 0$ for all $t \in [0, 1[$, $y, y' \in \mathbb{R}$.

5. We also need to show $P^{s,x}(\inf\{t > s : \kappa_t \notin \mathbb{R}\} < 1) = 0$ for any $(t, x) \in [0, 1] \times \mathbb{R}$ where κ satisfies SDE (2.4.10). It suffices to show that the escape time of process κ is

a.s. ∞, i.e., the process κ is non-explosive, which it is as the unique strong solution of the bounded drift one dimensional Levy process (2.4.10).

With all of the above conditions being satisfied, we have the SDE (2.5.23) has a unique strong solution. Moreover, $\kappa_1^* = \lambda^{-1}(1, \tilde{f}(Z))$. Therefore we have unique strong solution ξ^* for SDE (2.5.22) and $\xi_1^* = \tilde{f}(Z)$ a.s. which satisfies the second condition of the equilibrium Lemma 2.

Next we need to show (P^*, w^*, θ^*) given satisfy the remaining two conditions of the equilibrium Lemma 2.

From above construction of ξ^* which is the unique strong solution of SDE $\xi_t = \int_0^t w^*(s, \xi_s) dY_s^*$ with terminal distribution $\xi_1 = \tilde{f}(Z)$, we have constructed a Markov bridge adapted to its own filtration. Moreover, we have

$$\begin{aligned}
\mathbb{E}\left[\alpha^*(t,\xi_t^*)|\mathcal{F}_t^{\xi^*}\right] &= \mathbb{E}\left[w^*(t,\xi_t^*)\frac{\rho_x(t,\xi_t^*;1,\tilde{f}(Z))}{\rho(t,\xi_t^*;1,\tilde{f}(Z))}|\mathcal{F}_t^{\xi^*}\right] \\
&= \mathbb{E}\left[\frac{p_x(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}{p(t,\kappa_t^*;1,\lambda^{-1}(1,\tilde{f}(Z)))}|\mathcal{F}_t^{\kappa^*}\right] \\
&= \int_{\mathbb{R}} p_x(t,\kappa_t^*;1,u)du
\end{aligned}$$

where the last equality is because p can be also viewed as the conditional density of $\lambda^{-1}(1, \tilde{f}(Z))$. Since $\int_{\mathbb{R}} p(t, x; 1, u)du = 1$ as p is transition density of κ^*. Moreover, from Theorem 11 of Chapter I [24], we have estimate

$$|p_x(t,x;1,u)| \leq \frac{C(c)}{1-t}\exp\left[-c\frac{(u-x)^2}{2(1-t)}\right]$$

for some C as function of c and any $c < 1$. Thus for any bounded x, we have $|p_x| \leq \tilde{C}\exp(-\tilde{c}u^2)$ where $\tilde{C}$ and $\tilde{c}$ depend on (t, u). Then by Leibniz rule, we can exchange the order of integration and differentiation and obtain

$$\int_{\mathbb{R}} p_x(t,x;1,u)du = 0 = \mathbb{E}\left[\alpha^*(t,\xi_t^*)|\mathcal{F}_t^{\xi^*}\right].$$

Therefore we have $Y_t^* = B_t + \int_0^t \alpha^*(s, \xi_t^*)ds$ is an $\mathcal{F}_t^{\xi^*}$-local martingale. Moreover, since $[Y^*]_t = [B]_t = t$, by Levy's characterisation, we know Y^* is a standard Brownian motion in its own filtration. To show P^* is a $\mathcal{F}^{Y^*}$ martingale, it suffices to show ξ^* is a $\mathcal{F}^{Y^*}$ martingale. We know $\xi^*_t = \int_0^t w(s, \xi^*_s) dY^*_s$ is a local martingale in its own filtration due to continuity of w^*. Moreover, w^* is bounded due to Proposition 1. Thus ξ^* is a true martingale.

Finally we show P^* is rational, equivalently we want to show $P^*_t = \xi^*_t + \mathbb{E}[f(Z)] = \mathbb{E}[f(Z)|\mathcal{F}^{Y^*}_t]$. Since ξ^* is a martingale, we have $\xi^*_t = \mathbb{E}[\tilde{f}(Z)|\mathcal{F}^{\xi^*}_t] = \mathbb{E}[\tilde{f}(Z)|\mathcal{F}^{Y^*}_t]$. Therefore we have

$$P^*_t = \xi^*_t + \mathbb{E}[f(Z)] = \mathbb{E}[\tilde{f}(Z)|\mathcal{F}^{Y^*}_t] + \mathbb{E}[f(Z)] = \mathbb{E}[f(Z)|\mathcal{F}^{Y^*}_t].$$

Thus P^* is rational.

■

The importance of the above equilibrium is as follows. Cho [12] considered exponential utility with weighting function w depending only on time variable. He concluded that there was no equilibrium unless the asset value is normally distributed. Now we see by relaxing the condition on weighting function to depend on the path of cumulative order as well, linear equilibrium exists under general, non-Gaussian framework. In comparison with risk-neutral insider case, the equilibrium is not necessarily unique.

Future research could be made in the following directions. One straightforward extension of the result is to consider unbounded, at most linear valuation function f. During our attempt the difficulty lies in proving the integral equation (2.4.19) has a smooth solution.

Another new angle is to consider the dynamic information case where the insider observes the information over time, instead of observing the full information at time 0. On the other hand, the model for noise traders can be generalised to fit reality of the market, e.g., to have time-varying volatility or to be modelled as Poisson process. We could also consider the case where market makers are made risk-averse.

2.6 Appendix: proof of results in Section 2.4.1

Proof of Proposition 1. We start the proof by showing w is well-defined. Suppose there exist λ solving (2.4.8) and (2.4.9). Since λ is strictly increasing w.r.t. x, $\lambda^{-1}(t, y)$ exists, well-defined and strictly increasing for any $y \in]\tilde{b}, \tilde{d}[$. As $\lambda(t,x) \in \mathcal{C}^{1,3}$, we have

$$\frac{d}{dy}\lambda^{-1}(t,y) = \frac{1}{\lambda_x(t, \lambda^{-1}(t,y))}. \tag{2.6.24}$$

$$\frac{d^2}{dy^2}\lambda^{-1}(t,y) = \frac{-\lambda_{xx}(t,\lambda^{-1}(t,y))}{\lambda_x^2(t,\lambda^{-1}(t,y))}\frac{d}{dy}\lambda^{-1}(t,y) = \frac{-\lambda_{xx}(t,\lambda^{-1}(t,y))}{\lambda_x^3(t,\lambda^{-1}(t,y))}.$$

$$\frac{d^3}{dy^3}\lambda^{-1}(t,y) = \frac{-\lambda_{xxx}(t,\lambda^{-1}(t,y))}{\lambda_x^4(t,\lambda^{-1}(t,y))} + \frac{3\lambda_{xx}^2(t,\lambda^{-1}(t,y))}{\lambda_x^5(t,\lambda^{-1}(t,y))}.$$

We also have

$$\lambda_t^{-1}(t,y) = -\frac{\lambda_t(t,\lambda^{-1}(t,y))}{\lambda_x(t,\lambda^{-1}(t,y))}.$$

Therefore $\lambda^{-1}(t,y) \in \mathcal{C}^{1,3}([0,1]\times]\tilde{b},\tilde{d}[) \to \mathbb{R}$. In particular, since $\lambda(t,x)$ is strictly increasing, differentiable and bounded, we have $\lambda_x(t,x) > 0$ for any $(t,x) \in [0,1]\times]\tilde{b},\tilde{d}[$. Thus by (2.6.24), we have $\frac{\partial\lambda^{-1}}{\partial y}(t,y) > 0$ and $w(t,y) = \frac{1}{\lambda_y^{-1}(t,y)} \in \mathcal{C}^{1,2}([0,1]\times]\tilde{b},\tilde{d}[\to \mathbb{R})$ is well defined. To see the behaviour of w on the boundaries $\tilde{b}$ and $\tilde{d}$, we notice $\lim_{x\to\pm\infty}\lambda_x(t,x) = 0$. Therefore by (2.6.24), we have

$$\lim_{y\downarrow\tilde{b}} w(t,y) = \lim_{y\downarrow\tilde{b}} \frac{1}{\lambda_y^{-1}(t,y)} = \lim_{y\downarrow\tilde{b}} \lambda_x(t,\lambda^{-1}(t,y)).$$

Since we have $\lim_{x\to-\infty}\lambda(t,x) = \tilde{b}$ and $\lambda(t,x)$ is strictly increasing function w.r.t. x, we have

$$\lim_{y\downarrow\tilde{b}} w(t,y) = \lim_{x\to-\infty}\lambda_x(t,x) = 0.$$

Similarly we have

$$\lim_{y\uparrow\tilde{d}} w(t,y) = \lim_{y\uparrow\tilde{d}} \frac{1}{\lambda_y^{-1}(t,y)} = \lim_{y\uparrow\tilde{d}} \lambda_x(t,\lambda^{-1}(t,y)).$$

Since we have $\lim_{x\to\infty}\lambda(t,x) = \tilde{d}$ and $\lambda(t,x)$ is strictly increasing function w.r.t. x, we have

$$\lim_{y\uparrow\tilde{d}} w(t,y) = \lim_{x\to\infty}\lambda_x(t,x) = 0.$$

Moreover, since $\lambda_x(t,x)$ is strictly positive for any $x \in \mathbb{R}$, thus $\lambda_y^{-1}(t,y)$ is strictly positive for any $y \in]\tilde{b},\tilde{d}[$. Therefore $w(t,y)$ is strictly positive for any $(t,y) \in [0,1]\times]\tilde{b},\tilde{d}[$.

To derive PDE of w, we first derive PDE of λ^{-1} from (2.4.8). For simplicity in the rest of the proof, unless specified, λ and its derivatives will be function of $(t, \lambda^{-1}(t,y))$. Since $\lambda(t, \lambda^{-1}(t,y)) = y$, we differentiate w.r.t. t and y

$$\lambda_t + \lambda_x \lambda_t^{-1} = 0, \quad \lambda_x \lambda_y^{-1} = 1, \quad \lambda_{xx}(\lambda_y^{-1})^2 + \lambda_x \lambda_{yy}^{-1} = 0.$$

Thus we have the following

$$\lambda_t^{-1} = -\frac{\lambda_t}{\lambda_x}, \quad \lambda_y^{-1} = \frac{1}{\lambda_x}, \quad \lambda_{yy}^{-1} = -\frac{\lambda_{xx}}{\lambda_x^3}.$$

Therefore

$$\lambda_t^{-1} + \frac{\lambda_{yy}^{-1}}{2(\lambda_y^{-1})^2} = -\frac{1}{\lambda_x}\left(\lambda_t + \frac{1}{2}\lambda_{xx}\right) = \gamma\lambda(t, \lambda^{-1}(t,y)) = \gamma y. \tag{2.6.25}$$

Note $w(t,y) = \frac{1}{\lambda_y^{-1}(t,y)}$, thus $\frac{w_t(t,y)}{w^2(t,y)} = -\lambda_{ty}^{-1}(t,y)$ and $w_y(t,y) = -\frac{\lambda_{yy}^{-1}(t,y)}{(\lambda_y^{-1}(t,y))^2}$. Therefore

$$\frac{w_t(t,y)}{w^2(t,y)} + \frac{w_{yy}(t,y)}{2} = -\left(\lambda_t^{-1} + \frac{\lambda_{yy}^{-1}}{2(\lambda_y^{-1})^2}\right)_y = -\gamma.$$

Thus $w(t,y) = \frac{1}{\lambda_y^{-1}(t,y)}$ satisfies (2.2.4).

Define $\xi_t = \lambda(t, \kappa_t)$, which is consistent with the initial condition

$$0 = \xi_0 = \lambda(0, \kappa_0) = \lambda(0,0).$$

Since λ is strictly increasing w.r.t. space variable, λ^{-1} exists and is well defined. Application of Ito's formula will yield

$$\begin{aligned} \xi_t &= \xi_0 + \int_0^t \lambda_x(s,\kappa_s)d\kappa_s + \int_0^t \left(\lambda_s(s,\kappa_s) + \frac{\lambda_{xx}(s,\kappa_s)}{2}\right)ds \\ &= \int_0^t \lambda_x(s,\kappa_s)\left(d\beta_s + \gamma\lambda(s,\kappa_s)ds\right) - \gamma\int_0^t \lambda(s,\kappa_s)\lambda_x(s,\kappa_s)ds \\ &= \int_0^t \lambda_x(s,\kappa_s)d\beta_s = \int_0^t \lambda_x(s, \lambda^{-1}(s,\xi_s))d\beta_s = \int_0^t w(s,\xi_s)d\beta_s. \end{aligned}$$

Therefore $\xi_t = \lambda(t,\kappa_t)$ is a strong solution to SDE (2.4.12). Observe that $\inf\{t > 0 : \xi_t \notin]\tilde{b}, \tilde{d}[\} = \inf\{t > 0 : \kappa_t \notin]-\infty, +\infty[\}$. Since κ is non-explosive due to the boundedness of the drift term $\gamma\lambda(t,\kappa_t)$, we have $\tau > 1$. Equivalently,

$$\begin{aligned} \mathbb{P}(\kappa_t > \infty) &= \mathbb{P}(\beta_t + \int_0^t \gamma\lambda(s,\kappa_s)ds > \infty) \\ &\leq \mathbb{P}(\beta_t - \max\{-\tilde{b}, \tilde{d}\}t > \infty) = \mathbb{P}(\beta_t > \infty) = 0 \end{aligned}$$

together with

$$\begin{aligned}\mathbb{P}(\kappa_t < -\infty) &= \mathbb{P}(\beta_t + \int_0^t \gamma\lambda(s,\kappa_s)ds < -\infty) \\ &\leq \mathbb{P}(\beta_t + \max\{-\tilde{b},\tilde{d}\}t < -\infty) = \mathbb{P}(\beta_t < -\infty) = 0\end{aligned}$$

yield the non-explosiveness of κ. Therefore $\tau > 1$.

To conclude that statement 2 of the Proposition hold, it remains to show the uniqueness of the solution of the SDE

$$d\xi_t = w(t,\xi_t)d\beta_t.$$

Suppose there is another strong solution $\tilde{\xi}$. Denote a sequence of open sets $V_n =]\tilde{b}+\frac{1}{n},\tilde{d}-\frac{1}{n}[$, $n = 1,2,...$ and define a sequence of stopping times ν_n by $\nu_n = \inf\{t \geq 0 : \tilde{\xi}_t \notin V_n\}$. Then the process $\tilde{\kappa}_t := \lambda^{-1}(t,\tilde{\xi}_t)$ is well defined on $[0,\nu_n]$ for all n. Note that since λ is increasing function, $\nu_n = \inf\{t \geq 0 : \tilde{\kappa}_t \notin U_n\}$, where $U_n = \Big]\lambda^{-1}(t,\tilde{b}+\frac{1}{n}),\lambda^{-1}(t,\tilde{d}-\frac{1}{n})\Big[$.

Application of Ito's formula to $\tilde{\kappa}$ stopped at ν_n will yield

$$\begin{aligned}\tilde{\kappa}_{t\wedge\nu_n} &= \int_0^{t\wedge\nu_n} \lambda_x^{-1}(s,\tilde{\xi}_s)d\tilde{\xi}_s + \int_0^{t\wedge\nu_n}\left(\lambda_s^{-1}(s,\tilde{\xi}_s) + \frac{\lambda_{xx}^{-1}(s,\tilde{\xi}_s)w^2(s,\tilde{\xi}_s)}{2}\right)ds \\ &= \int_0^{t\wedge\nu_n} \lambda_x^{-1}(s,\tilde{\xi}_s)w(s,\tilde{\xi}_s)d\beta_s + \int_0^{t\wedge\nu_n}\left(\lambda_s^{-1}(s,\tilde{\xi}_s) + \frac{\lambda_{xx}^{-1}(s,\tilde{\xi}_s)}{2\left(\lambda_x^{-1}(s,\tilde{\xi}_s)\right)^2}\right)ds \\ &= \int_0^{t\wedge\nu_n} \lambda_x^{-1}(s,\tilde{\xi}_s)w(s,\tilde{\xi}_s)d\beta_s + \int_0^{t\wedge\nu_n}\gamma\tilde{\xi}_s ds \\ &= \int_0^{t\wedge\nu_n} d\beta_s + \int_0^{t\wedge\nu_n}\gamma\lambda(s,\tilde{\kappa}_s)ds,\end{aligned}$$

and therefore $\tilde{\kappa}$ is a strong solution of (2.4.10) in $[0,t\wedge\nu_n]$ for each $n\in\mathbb{N}$. Since solution to (2.4.10) is unique, we have $\tilde{\kappa}_{t\wedge\nu_n} = \kappa_{t\wedge\nu_n}$ for all $t\in[0,1]$. Taking the limit, in view of continuity of κ, we have $\tilde{\kappa}_{t\wedge\nu} = \kappa_{t\wedge\nu}$ for all $t\in[0,1]$, where $\nu = \lim_{n\to\infty}\nu_n$. In particular, $\tilde{\kappa}_{1\wedge\nu} = \kappa_{1\wedge\nu}$ and therefore $\nu < 1$ is equivalent to $\tau < 1$ which has probability zero due to the arguments in Remark 4. Thus,

$$\tilde{\kappa}_t = \tilde{\kappa}_{t\wedge\nu} = \kappa_{t\wedge\nu} = \kappa_t \qquad t\in[0,1].$$

Due to the connection between $\tilde{\kappa}$ and $\tilde{\xi}$ and continuity of $\tilde{\kappa}$ as well as κ, the above implies that

$$\mathbb{P}[\tilde{\xi}_t = \xi_t, t \in [0,1]] = 1.$$

Finally, the distributional equality of the condition 3 of the Proposition holds since (2.4.9) is equivalent to $\lambda(1,\kappa_1) \stackrel{d}{=} \tilde{f}(Z)$ where $Z = \mathcal{N}(0,1)$ since $P(1,x)$ is the cumulative distribution function of κ_1. Therefore

$$\xi_1 = \lambda(1,\kappa_1) \stackrel{d}{=} \tilde{f}(Z).$$

■

Lemma 5 *Consider a function g satisfying*

$$0 < g(x) \le e^{M|x|},\ \forall x \in \mathbb{R}$$

for some constant M. Then for any $z \in \mathbb{R}$ and $t \in [0,1[$ we have

$$\lim_{x \to +\infty} \int_{-\infty}^{z} \Gamma(t, x-y) g(y) dy = 0,$$

where $\Gamma(t,x) = \frac{1}{\sqrt{2\pi(1-t)}} \exp\left\{-\frac{x^2}{2(1-t)}\right\}$.

Proof. As $\int_{-\infty}^{z} \Gamma(t,x-y)g(y)dy$ is increasing in z, we can assume, without loss of generality, that $z > 0$.

We have

$$\begin{aligned} 0 &< \int_{-\infty}^{z} \Gamma(t,x-y)g(y)dy \\ &< \int_{0}^{z} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(x-y)^2}{2(1-t)}+My} dy + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(x-y)^2}{2(1-t)}-My} dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

To complete the squares for each integral, we have

$$\begin{aligned} I_1(x) &= \int_{0}^{z} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{1}{2(1-t)}[x-y+(1-t)M]^2 + \frac{(1-t)M^2}{2} + Mx} dy \\ &= e^{\frac{(1-t)M^2}{2}+Mx} \int_{-x-(1-t)M}^{z-x-(1-t)M} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{u^2}{2(1-t)}} du \\ &= e^{\frac{(1-t)M^2}{2}+Mx} \left[\Phi\left(\frac{z-x-(1-t)M}{\sqrt{1-t}}\right) - \Phi\left(\frac{-x-(1-t)M}{\sqrt{1-t}}\right)\right]. \end{aligned}$$

Similarly, we have

$$I_2(x) = e^{\frac{(1-t)M^2}{2} - Mx}\Phi\left(\frac{-x + (1-t)M}{\sqrt{1-t}}\right).$$

Take limits when $x \to +\infty$, we have $\lim_{x\to\infty} I_2(x) = 0$ and

$$\lim_{x\to\infty} I_1(x) = \lim_{x\to\infty} \frac{\Phi\left(\frac{z-x-(1-t)M}{\sqrt{1-t}}\right)}{e^{-Mx-\frac{(1-t)M^2}{2}}} + \lim_{x\to\infty} \frac{\Phi\left(\frac{-x-(1-t)M}{\sqrt{1-t}}\right)}{e^{-Mx-\frac{(1-t)M^2}{2}}} = I_3(x) + I_4(x).$$

Since

$$\lim_{x\to\infty} \Phi\left(\frac{z - x - (1-t)M}{\sqrt{1-t}}\right) = \lim_{x\to\infty} \Phi\left(\frac{-x - (1-t)M}{\sqrt{1-t}}\right) = \lim_{x\to\infty} e^{-Mx-\frac{(1-t)M^2}{2}} = 0,$$

are continuously differentiable functions, we can apply L'Hopital rule,

$$\begin{aligned}
\lim_{x\to\infty} I_3(x) &= \lim_{x\to\infty} \frac{-\frac{1}{\sqrt{2\pi(1-t)}}e^{-\frac{[z-x-(1-t)M]^2}{2(1-t)}}}{-Me^{-Mx-\frac{(1-t)M^2}{2}}} \\
&= \frac{1}{\sqrt{2\pi(1-t)}M} \lim_{x\to\infty} e^{-\frac{[z-x-(1-t)M]^2}{2(1-t)}+Mx+\frac{(1-t)M^2}{2}} = 0,
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} I_4(x) &= \lim_{x\to\infty} \frac{-\frac{1}{\sqrt{2\pi(1-t)}}e^{-\frac{[-x-(1-t)M]^2}{2(1-t)}}}{-Me^{-Mx-\frac{(1-t)M^2}{2}}} \\
&= \frac{1}{\sqrt{2\pi(1-t)}M} \lim_{x\to\infty} e^{-\frac{[-x+(1-t)M]^2}{2(1-t)}+Mx+\frac{(1-t)M^2}{2}} = 0.
\end{aligned}$$

Therefore we have

$$\lim_{x\to\infty} \int_{-\infty}^{z} \Gamma(t, x-y)g(y)dy = \lim_{x\to\infty} (I_2(x) + I_3(x) + I_4(x)) = 0.$$

■

Lemma 6 *Suppose f satisfies Assumption 1 and $\lim_{x\to\infty} f(x) = d > 0$, $\lim_{x\to-\infty} f(x) = b < 0$. Let $\tilde{P} \in \mathcal{C}^2 : \mathbb{R} \to \mathbb{R}$, a function strictly increasing w.r.t. x, with $\tilde{P}(-\infty) = 0$ and $\tilde{P}(\infty) = 1$ satisfying the integral equation:*

$$\tilde{P}(x) = \frac{c^*}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{\gamma \int_0^u f \circ \Phi^{-1} \circ \tilde{P}(s)ds - \frac{u^2}{2}\right\} du.$$

Then, for any $\varepsilon > 0$ *there exists* $\hat{x}$ *such that for all* $x > \hat{x}$

$$-\gamma(d-\varepsilon) < \Phi^{-1} \circ \tilde{P}(x) - x < 0, \tag{2.6.26}$$

and for any $x < -\hat{x}$

$$0 < \Phi^{-1} \circ \tilde{P}(x) - x < \gamma(b-\varepsilon). \tag{2.6.27}$$

This implies that

$$\lim_{x \to \pm\infty} \frac{\Phi^{-1} \circ \tilde{P}(x)}{x} = 1. \tag{2.6.28}$$

Proof. First we will show bounds in (2.6.26). By L'Hopital's rule, we have

$$\lim_{x\to+\infty} \frac{1-\tilde{P}(x)}{1-\Phi(x)} = \lim_{x\to+\infty} \frac{\tilde{P}'(x)}{\Phi'(x)} = \lim_{x\to+\infty} c^* e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds}.$$

Since f, Φ and $\tilde{P}$ are strictly increasing functions, we know $f \circ \Phi^{-1} \circ \tilde{P}$ is a strictly increasing function. As $\lim_{x\to\infty} \tilde{P}(x) = 1$, we know

$$\lim_{x\to+\infty} f \circ \Phi^{-1} \circ \tilde{P}(x) = \lim_{x\to+\infty} f(x) = d > 0.$$

Therefore, there exists $x_* > 0$ s.t. for all $x > x_*$, we have $f \circ \Phi^{-1} \circ \tilde{P}(x) > \frac{d}{2}$. Thus

$$\int_0^x f \circ \Phi^{-1} \circ \tilde{P}(s)ds > \int_0^{x_*} f \circ \Phi^{-1} \circ \tilde{P}(s)ds + (x - x_*)\frac{d}{2}.$$

Hence we have

$$\lim_{x\to+\infty} \frac{1-\tilde{P}(x)}{1-\Phi(x)} = \lim_{x\to+\infty} c^* e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds} = \infty > 1.$$

Therefore there exists $y^* > 0$ such that $\Phi^{-1}(\tilde{P}(x)) < x$ for all $x > y^*$.

Since Φ is strictly increasing function, the first inequality in (2.6.26) is equivalent to

$$\tilde{P}(x) > \Phi(x - \gamma(d+\epsilon))$$

for all $x > \hat{x}$. Note that

$$\begin{aligned}
\lim_{x\to+\infty} \frac{1-\tilde{P}(x)}{1-\Phi(x-\gamma(d+\epsilon))} &= \lim_{x\to+\infty} \frac{\tilde{P}'(x)}{\Phi'(x-\gamma(d+\epsilon))} \\
&= c^* \lim_{x\to+\infty} e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds - \frac{x^2}{2} + \frac{1}{2}(x-\gamma(d+\epsilon))^2} \\
&= c^* e^{\frac{\gamma^2(d+\epsilon)^2}{2}} \lim_{x\to+\infty} e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds - \gamma(d+\epsilon)x} \\
&< c^* e^{\frac{\gamma^2(d+\epsilon)^2}{2}} \lim_{x\to+\infty} e^{-\gamma\epsilon x} = 0 < 1.
\end{aligned}$$

Therefore, there exists $\tilde{y}^* > 0$ such that for any $x > \tilde{y}^*$, we have $\tilde{P}(x) > \Phi(x - \gamma(d + \epsilon))$.

The inequality (2.6.27) is proved similarly. Indeed,

$$\lim_{x\to-\infty} \frac{\tilde{P}(x)}{\Phi(x)} = \lim_{x\to-\infty} \frac{\tilde{P}'(x)}{\Phi'(x)} = \lim_{x\to-\infty} c^* e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds}.$$

Since $b < 0$, we have

$$\lim_{x\to-\infty} f \circ \Phi^{-1} \circ \tilde{P}(x) = \lim_{x\to-\infty} f(x) = b < 0.$$

Therefore, there exists $x^* < 0$ s.t. for all $x < x^*$, we have $f \circ \Phi^{-1} \circ \tilde{P}(x) < \frac{b}{2}$. Thus

$$\int_0^x f \circ \Phi^{-1} \circ \tilde{P}(s)ds > \int_0^{x_*} f \circ \Phi^{-1} \circ \tilde{P}(s)ds + (x - x_*)\frac{b}{2}.$$

Hence we have

$$\lim_{x\to-\infty} c^* e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds} = \infty > 1.$$

Therefore there exists $y_* < 0$ such that $\Phi^{-1}(\tilde{P}(x)) < x$ for $x < y_*$.

To show the second inequality, as before, we need to show

$$\tilde{P}(x) < \Phi(x - \gamma(b - \epsilon))$$

for x small enough. Notice

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\tilde{P}(x)}{\Phi(x - \gamma(b-\epsilon))} &= \lim_{x\to-\infty} \frac{\tilde{P}'(x)}{\Phi'(x - \gamma(b-\epsilon))} \\
&= c^* \lim_{x\to-\infty} e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds - \frac{x^2}{2} + \frac{1}{2}(x-\gamma(b-\epsilon))^2} \\
&= c^* e^{\frac{\gamma^2(b-\epsilon)^2}{2}} \lim_{x\to\infty} e^{\gamma \int_0^x f\circ\Phi^{-1}\circ\tilde{P}(s)ds - \gamma(b-\epsilon)x} \\
&< c^* e^{\frac{\gamma^2(b-\epsilon)^2}{2}} \lim_{x\to-\infty} e^{\gamma\epsilon x} = 0 < 1.
\end{aligned}$$

Therefore, there exists $\tilde{y}_* < 0$ such that for any $x < \tilde{y}_*$, we have $\tilde{P}(x) < \Phi(x - \gamma(b - \epsilon))$. Thus, the statement of the Lemma holds with $\hat{x} = \max\{y^*, \tilde{y}^*, -y_*, -\tilde{y}_*\}$.

■

Chapter 3

The solution to an impulse control problem with non-linear penalisation of control effort

3.1 Problem formulation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions and carrying a standard $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{T}$ the family of all $(\mathcal{F}_t)$-stopping times. An impulse control is a collection

$$\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; Z_1, Z_2, \ldots, Z_n, \ldots),$$

where $(\tau_n) \subset \mathcal{T}$ is the increasing sequence of the $(\mathcal{F}_t)$-stopping times at which impulsive action is applied to a system and the positive real-valued random variables Z_n, $n \geq 1$, are the sizes of the corresponding jumps of the underlying state process. In particular, we assume that

$$\tau_n < \tau_{n+1} \mathbf{1}_{\{\tau_n < \infty\}} \quad \text{for all } n \geq 1.$$

We denote by $\mathcal{A}$ the family of all impulse controls. Given such a control $\mathcal{Z} \in \mathcal{A}$, we define the cáglád process

$$\bar{Z}_t = \sum_{n=1}^{\infty} Z_n \mathbf{1}_{\{\tau_n < t\}}. \tag{3.1.1}$$

In this context, we model the stochastic dynamics of the controlled system by

$$X_t = x - \bar{Z}_t + W_t. \tag{3.1.2}$$

The objective of the optimisation problem that we study is to minimise the performance criterion given by

$$J_x(\mathcal{Z}) = \mathbb{E}\left[\lambda \int_0^\infty e^{-\delta t} X_t^2 \, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n}\big(1 + \kappa Z_n^\alpha\big)\right] \tag{3.1.3}$$

over all strategies $\mathcal{Z}$, where $\alpha, \delta, \kappa, \lambda > 0$ are given constants. The value function of this optimisation problem is defined by

$$v(x) = \inf_{\mathcal{Z} \in \mathcal{A}} J_x(\mathcal{Z}), \quad \text{for } x \in \mathbb{R}. \tag{3.1.4}$$

In view of the general theory of stochastic impulse control, the value function of the optimisation problem that we study identifies with a classical solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$\min\left\{\frac{1}{2}w''(x) - \delta w(x) + \lambda x^2,\ -w(x) + \inf_{z>0}\big[w(x-z) + 1 + \kappa z^\alpha\big]\right\} = 0. \tag{3.1.5}$$

3.2 The eventual nature of the optimal control

The structure of the problem we consider suggests that it should never be optimal to exercise any control effort if the state process takes negative values and that it should never be optimal to make the state process jump across the origin. The first of these observations suggests that the value function v should satisfy the boundary condition

$$\lim_{x \to -\infty} v(x)\left(\lambda\,\mathbb{E}\left[\int_0^\infty e^{-\delta t}(x + W_t)^2 \, dt\right]\right)^{-1} = \lim_{x \to -\infty} v(x)\left(\frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2}\right)^{-1} = 1 \tag{3.2.6}$$

because the probability of the uncontrolled process hitting $\mathbb{R}_+$ tends to 0 as $x \to -\infty$. Furthermore, these observations suggest that the optimal strategy should be partially characterised by two strictly positive points $a_0 < b_0$. Whenever the state space process X reaches the level b_0, control should be exercised to "push" it instantaneously down to the level a_0. On the other hand, the controller should take no action as long as the state process is inside the interval $]-\infty, b_0[$. Accordingly, the restriction of the value function v in $]-\infty, b_0]$ should identify with a solution to the ODE

$$\frac{1}{2}w''(x) - \delta w(x) + \lambda x^2 = 0, \quad \text{for } x \in \,]-\infty, b_0[, \tag{3.2.7}$$

that satisfies the boundary condition

$$w(b_0) = w(a_0) + 1 + \kappa(b_0 - a_0)^\alpha. \tag{3.2.8}$$

Every solution to the ODE (3.2.7) is given by

$$w(x) = Ae^{\sqrt{2\delta}x} + Be^{-\sqrt{2\delta}x} + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2},$$

for some constants $A, B \in \mathbb{R}$. In view of the boundary condition (3.2.6), we choose $B = 0$ and we look for a solution w to the HJB equation (3.1.5) whose restriction in $]-\infty, b_0]$ takes the form

$$w(x) = \begin{Bmatrix} Ae^{\sqrt{2\delta}x} + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2}, & \text{if } x < b_0 \\ w(a_0) + 1 + \kappa(x - a_0)^\alpha, & \text{if } x = b_0 \end{Bmatrix}, \tag{3.2.9}$$

for some constant $A \in \mathbb{R}$.

To derive a system of appropriate equations to determine the free-boundary points a_0, b_0 and the constant A, we argue as follows. First, we note that the fact that b_0 separates the "wait" region $]-\infty, b_0[$ from the "action" region to which b_0 itself belongs implies that the marginal cost of "waiting" should tend to the marginal cost of optimal "acting" as the state process increases to b_0. This observation suggests the free-boundary condition

$$w'(b_0-) \equiv \sqrt{2\delta}Ae^{\sqrt{2\delta}b_0} + \frac{2\lambda}{\delta}b_0 = \alpha\kappa(b_0 - a_0)^{\alpha-1}, \tag{3.2.10}$$

which is consistent with the C^1 regularity associated with the so-called "principle of smooth fit". Furthermore, we note that the HJB equation (3.1.5) can be satisfied for $x \in \,]-\infty, b_0]$

only if

$$-w(b_0) + w(y) + 1 + \kappa(b_0 - y)^\alpha \begin{Bmatrix} = 0, & \text{for } y = a_0 \\ \geq 0, & \text{for all } y \leq b_0 \end{Bmatrix},$$

which implies that the function $y \mapsto w(y) + 1 + \kappa(b_0 - y)^\alpha$ has a local minimum at $y = a_0$ and yields

$$w'(a_0) \equiv \sqrt{2\delta} A e^{\sqrt{2\delta} a_0} + \frac{2\lambda}{\delta} a_0 = \alpha\kappa(b_0 - a_0)^{\alpha-1}. \tag{3.2.11}$$

We are therefore faced with the system of equations (3.2.8), (3.2.10) and (3.2.11) for the unknowns a_0, b_0 and A.

Subtracting (3.2.11) from (3.2.10), we derive the expression

$$A = -\frac{2\lambda(b_0 - a_0)}{\delta\sqrt{2\delta}\left(e^{\sqrt{2\delta} b_0} - e^{\sqrt{2\delta} a_0}\right)} < 0. \tag{3.2.12}$$

On the other hand, adding (3.2.10) and (3.2.11) side by side and using (3.2.12), we obtain

$$f_1(b_0 - a_0) = f_2(b_0 - a_0) + \frac{\lambda}{\delta}\left[\frac{\delta + \delta\kappa(b_0 - a_0)^\alpha}{\lambda(b_0 - a_0)} + \frac{2}{\sqrt{2\delta}} - (a_0 + b_0)\right], \tag{3.2.13}$$

where

$$f_1(y) = \frac{1}{y} - \kappa(\alpha - 1)y^{\alpha-1} \quad \text{and} \quad f_2(y) = \frac{2\lambda}{\delta\sqrt{2\delta}}\left[\frac{\sqrt{2\delta}y}{2}\coth\frac{\sqrt{2\delta}y}{2} - 1\right]. \tag{3.2.14}$$

Furthermore, we can use (3.2.12) to observe that

$$q(a_0, b_0)\begin{Bmatrix} = 0 \\ < 0 \end{Bmatrix} \quad \Leftrightarrow \quad \frac{\delta + \delta\kappa(b_0 - a_0)^\alpha}{\lambda(b_0 - a_0)} + \frac{2}{\sqrt{2\delta}}\begin{Bmatrix} = a_0 + b_0 \\ < a_0 + b_0 \end{Bmatrix}, \tag{3.2.15}$$

where

$$q(s, x) = -w(x) + w(s) + 1 + \kappa(x - s)^\alpha. \tag{3.2.16}$$

It follows that the system of equations (3.2.8), (3.2.10) and (3.2.11) for the unknowns a_0, b_0 and A is equivalent to the system of equations

$$(3.2.12), \quad f_1(b_0 - a_0) = f_2(b_0 - a_0) \quad \text{and} \quad a_0 + b_0 = \frac{\delta + \delta\kappa(b_0 - a_0)^\alpha}{\lambda(b_0 - a_0)} + \frac{2}{\sqrt{2\delta}}. \tag{3.2.17}$$

At this point, we should note that we have derived the system of equations (3.2.17) in a way that is more involved than necessary because this will facilitate some of our analysis below.

The next result, which we prove in Appendix I, is concerned with the solvability of this system of equations as well as with other issues that we will need in the following sections.

Lemma 7 *Given any values of the constants $\lambda, \delta, \kappa, \alpha > 0$, the following statements are true:*

(I) *The equation $f_1(y) = f_2(y)$, where f_1, f_2 are defined by (3.2.14), has a unique solution $y_* > 0$.*

(II) *The points*

$$a_0 = \frac{\delta + \delta\kappa y_*^\alpha}{2\lambda y_*} + \frac{1}{\sqrt{2\delta}} - \frac{y_*}{2} > 0 \quad \text{and} \quad b_0 = a_0 + y_* > a_0 \tag{3.2.18}$$

provide the unique solution to the system of equations (3.2.17), which is equivalent to the system of equations (3.2.8), (3.2.10) and (3.2.11) for the unknowns a_0, b_0 and A.

(III) *There exist points $y_\dagger \in \,]0, a_0[$ and $y^\dagger \in \,]a_0, b_0[$ such that the concave function ℓ defined by*

$$\ell(y) = w'(y) = \sqrt{2\delta} A e^{\sqrt{2\delta} y} + \frac{2\lambda}{\delta} y \tag{3.2.19}$$

satisfies

$$\ell(y) \begin{cases} < 0, & \text{if } y < y_\dagger \\ > 0, & \text{if } y \in \,]y_\dagger, b_0] \end{cases} \quad \text{and} \quad \ell'(y) \begin{cases} > 0, & \text{if } y < y^\dagger \\ < 0, & \text{if } y \in \,]y^\dagger, b_0] \end{cases}. \tag{3.2.20}$$

Furthermore,

$$\text{if } \alpha \in \,]0,1[, \quad \text{then} \quad g(s, b_0) < 0 \text{ for all } s < a_0 \quad \text{and} \quad g_s(s, b_0) > 0 \text{ for all } s \leq a_0, \tag{3.2.21}$$

$$\text{while, } \quad \text{if } \alpha \geq 1, \quad \text{then} \quad g(s, b_0) \begin{cases} < 0, & \text{if } s < a_0, \\ > 0, & \text{if } s \in \,]a_0, b_0] \end{cases}, \tag{3.2.22}$$

where g is the function defined by

$$g(s, x) = w'(s) - \alpha\kappa(x - s)^{\alpha - 1} \equiv \ell(s) - \alpha\kappa(x - s)^{\alpha - 1}, \quad \text{for } s \leq x. \tag{3.2.23}$$

(IV) *The function q defined by (3.2.16) satisfies*

$$q(s, x) > 0 \quad \text{for all } s < x \leq b_0 \text{ such that } (s, x) \neq (a_0, b_0). \tag{3.2.24}$$

3.3 The solution to the case with concave penalisation of control effort ($\alpha \in]0,1]$)

It turns out that, if $\alpha \in]0,1]$, then it is never optimal to wait for any amount of time if the state process takes values greater than b_0. In particular, given any initial condition $x \geq b_0$, it is optimal to jump immediately to a point $a(x) < b_0$ that we specify below. Otherwise, the optimal strategy takes the form we studied in the previous section. In view of these observations, we look for a solution to the HJB equation (3.1.5) of the form

$$w(x) = \begin{cases} Ae^{\sqrt{2\delta}x} + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2}, & \text{if } x < b_0 \\ w\big(a(x)\big) + 1 + \kappa\big(x - a(x)\big)^\alpha, & \text{if } x \geq b_0 \end{cases}, \tag{3.3.25}$$

for some C^1 function $a : [b_0, \infty[\to]0, b_0[$ such that $a(b_0) = a_0$, where $A < 0$ is the constant given by (3.2.12).

To determine the free-boundary function a, we first note that w can satisfy the HJB equation (3.1.5) only if

$$w\big(a(x)\big) + \kappa\big(x - a(x)\big)^\alpha = \min_{s<x}\big\{w(s) + \kappa(x-s)^\alpha\big\} \quad \text{for all } x \geq b_0.$$

This identity will be true only if $a(x) \in]0, b_0[$ satisfies

$$g\big(a(x), x\big) = 0 \quad \text{for all } x \geq b_0, \tag{3.3.26}$$

where g is defined by (3.2.23).

The next result, which we prove in Appendix II, is concerned with the solution to the HJB equation (3.1.5) when $\alpha \in]0,1]$.

Lemma 8 *Fix any values of the constants $\lambda, \delta, \kappa, \alpha > 0$ such that $\alpha \in]0,1]$. Also, let $A < 0$ and $0 < a_0 < b_0$ be defined by (3.2.12) and (3.2.18), and let $y_\dagger \in]0, a_0[$ and $y^\dagger \in]a_0, b_0[$ be as in Lemma 7.(III). The following statements are true:*

(I) *If $\alpha \in]0,1[$, then there exist a constant $\varepsilon_a \in]0, (b_0 - a_0)/2[$ and a C^∞ function $a :]b_0 - \varepsilon_a, \infty[\to]y_\dagger, a_0 + \varepsilon_a[$ such that (3.3.26) holds true. Furthermore,*

$$a(b_0) = a_0, \quad \lim_{x\to\infty} a(x) = y_\dagger \quad \textit{and} \quad a'(x) < 0 \quad \textit{for all } x \in]b_0 - \varepsilon_a, \infty[. \tag{3.3.27}$$

(II) *If* $\alpha \in \,]0, 1[$, *then let the function* a *be as in part (I). On the other hand, if* $\alpha = 1$, *define* $a(x) = a_0$ *for all* $x \geq b_0$. *The function* w *defined by (3.3.25) is* C^1 *in* $\mathbb{R}$ *as well as* C^∞ *in* $\mathbb{R} \setminus \{b_0\}$ *and satisfies the HJB equation (3.1.5).*

We conclude this section with the following result, which we prove in Appendix III.

Theorem 4 *Consider the stochastic impulse control formulated in Section 3.1 and suppose that* $\alpha \in \,]0, 1]$. *The problem's value function is given by*

$$v(x) = w(x) \quad \textit{for all } x \in \mathbb{R},$$

where $w > 0$ *is as in Lemma 8.(II). Furthermore, the optimal impulse control strategy* $\mathcal{Z}^\star \in \mathcal{A}$ *takes the qualitative form discussed at the beginning of the sections and is defined sequentially by*

$$\tau_1^\star = \inf\{t \geq 0 \mid \ x + W_t \geq b_0\}, \quad Z_1^\star = x + W_{\tau_1^\star} - a(x + W_{\tau_1^\star}), \quad (3.3.28)$$

$$\tau_2^\star = \inf\bigl\{t \geq \tau_1^\star \mid \ a(x + W_{\tau_1^\star}) + W_t - W_{\tau_1^\star} \geq b_0\bigr\}, \quad Z_2^\star = b_0 - a_0, \quad (3.3.29)$$

and

$$\tau_{n+1}^\star = \inf\{t \geq \tau_n^\star \mid \ a_0 + W_t - W_{\tau_n^\star} \geq b_0\}, \quad Z_{n+1}^\star = b_0 - a_0, \quad (3.3.30)$$

for $n \geq 2$.

3.4 The case with strictly convex penalisation of control effort ($\alpha > 1$)

The situation arising when $\alpha > 1$ is fundamentally different from the one we studied in the previous section. Indeed, the following result, which we prove in Appendix IV, reveals that the cost of a single jump can be strictly larger than the total cost incurred by multiple simultaneous jumps of the same total size.

Lemma 9 *Consider any $\alpha > 1$ fixed. The functions $K_n, K : [0,\infty[\to [1,\infty[$ defined by*

$$K_n(z) = n + n\kappa\left(\frac{z}{n}\right)^\alpha \quad \text{and} \quad K(z) = K_n(z), \quad \text{if } z \in [z_{n-1}, z_n[,$$

for $n = 1, 2, \ldots$, where

$$z_0 = 0 \quad \text{and} \quad z_n = \kappa^{-1/\alpha}\left[\left(\frac{1}{n}\right)^{\alpha-1} - \left(\frac{1}{n+1}\right)^{\alpha-1}\right]^{-1/\alpha}, \text{ if } n \geq 1, \tag{3.4.31}$$

are continuous and satisfy

$$K_n(z) = \inf\left\{\sum_{j=1}^n \left(1 + \kappa u_j^\alpha\right) \;\middle|\; u_1, \ldots, u_n \geq 0 \text{ and } u_1 + \cdots + u_n = z\right\}, \tag{3.4.32}$$

$$K(z) = \inf\left\{\sum_{j=1}^n \left(1 + \kappa u_j^\alpha\right) \;\middle|\; n \in \mathbb{N} \setminus \{0\},\ u_1, \ldots, u_n \geq 0 \text{ and } u_1 + \cdots + u_n = z\right\}. \tag{3.4.33}$$

Furthermore,

$$\lim_{n\to\infty}(z_{n+1} - z_n) = \kappa^{-1/\alpha}(\alpha - 1)^{-1/\alpha} \tag{3.4.34}$$

and

$$\lim_{z\to\infty}\frac{K(z)}{z} = \kappa^{1/\alpha}\alpha(\alpha - 1)^{-(\alpha-1)/\alpha}. \tag{3.4.35}$$

This result suggests that we should look for a solution to the HJB equation (3.1.5) of the form

$$w(x) = \left\{\begin{array}{ll} Ae^{\sqrt{2\delta}x} + \frac{\lambda}{\delta}x^2 + \frac{\lambda}{\delta^2}, & \text{if } x < b_0 \\ w\big(a(x)\big) + K\big(x - a(x)\big)^\alpha, & \text{if } x \geq b_0 \end{array}\right\},$$

for some function $a : [b_0, \infty[\to]0, b_0[$ such that $a(b_0) = a_0$, where $A < 0$ is the constant given by (3.2.12). However, such a function would not satisfy the HJB equation even in the sense of distributions because K' is not continuous and $K'(z_n+) < K'(z_n-)$ for all $n \geq 1$. This observation suggests that the waiting region should involve intervals in $]b_0, \infty[$ beyond the interval $]-\infty, b_0[$. On the other hand, Lemma 9 suggests that minimal costs can be achieved only by multiple simultaneous jumps, which implies that an optimal strategy may not exist. Despite most substantial effort in several directions, we have not managed to derive an explicit construction of the value function and the optimal strategy incorporating these features.

The following result, which we prove in Appendix V, identifies the restriction of the value function v in $]-\infty, b_0[$ with the function w that we studied in Section 3.2.

Lemma 10 *Consider the stochastic impulse control problem formulated in Section 3.1 and suppose that $\alpha > 1$. The problem's value function satisfies*

$$v(x) = \inf_{\mathcal{Z} \in \mathcal{A}} \mathbb{E}\left[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2 \, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^\alpha\big) \mathbf{1}_{\{\tau_n \le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w(X_{T_{\mathcal{Z}}+})\right], \quad (3.4.36)$$

where w is defined by (3.2.9) with a_0, b_0 and $A < 0$ being as in Lemma 8.(II).

$$T_{\mathcal{Z}} = \inf\big\{t \ge 0 \mid \ X_t \in \,]-\infty, b_0]\big\}. \quad (3.4.37)$$

In particular,

$$v(x) = w(x) > 0 \quad \text{for all } x \in \,]-\infty, b_0].$$

We conclude with the following result, which we prove in Appendix VI. In this theorem, we establish an iterative procedure for deriving the value function v. This procedure also yields a sequence of ε-optimal strategies, which arise by solving sequentially (3.4.39) and (3.4.40).

Theorem 5 *Consider the stochastic impulse control problem formulated in Section 3.1, suppose that $\alpha > 1$, and define*

$$\bar{w}_0(x) = \min\Big\{w_{\text{ext}}(x), \ \inf_{z>0}\big[w_{\text{ext}}(x - z) + 1 + \kappa z^\alpha\big]\Big\}, \quad (3.4.38)$$

where w_{ext} is the extension of the function $w > 0$ defined by (3.2.9), which is given by

$$w_{\text{ext}}(x) = A e^{\sqrt{2\delta} x} + \frac{\lambda}{\delta} x^2 + \frac{\lambda}{\delta^2}, \quad \text{for } x \in \mathbb{R},$$

with a_0, b_0 and $A < 0$ being as in Lemma 8.(II). Also, define

$$w_j(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\lambda \int_0^\tau e^{-\delta t} \mathring{X}_t^2 \, dt + e^{-\delta \tau} \bar{w}_j(\mathring{X}_\tau)\right], \quad (3.4.39)$$

$$\bar{w}_{j+1}(x) = \min\Big\{w_j(x), \ \inf_{z>0}\big[w_j(x - z) + 1 + \kappa z^\alpha\big]\Big\}, \quad (3.4.40)$$

for $j \geq 0$, where $\mathring{X}_t = x + W_t$. For each $j \geq 0$, the function w_j is the difference of two convex functions and satisfies the variational inequality

$$\min\left\{\frac{1}{2}w_j''(x) - \delta w_j(x) + \lambda x^2,\ \bar{w}_j(x) - w_j(x)\right\} = 0 \tag{3.4.41}$$

in the sense of distributions. Furthermore,

$$\bar{w}_j(x) \geq w_j(x) \geq \bar{w}_{j+1}(x) \text{ for all } j \geq 0 \quad \text{and} \quad \lim_{j\to\infty} w_j(x) = v(x) \quad \text{for all } x \in \mathbb{R}. \tag{3.4.42}$$

3.5 Appendix

3.5.1 Appendix I: proof of Lemma 7

Proof of (I). The calculations

$$f_2'(y) = \frac{2\lambda e^{\sqrt{2\delta}y}}{\delta(e^{\sqrt{2\delta}y} - 1)^2}\left[\sinh\sqrt{2\delta}y - \sqrt{2\delta}y\right] > 0,$$

$$\lim_{y\downarrow 0} f_2(y) = 0 \quad \text{and} \quad \lim_{y\to\infty} f_2(y) = \infty,$$

reveal that f_2 is strictly increasing from 0 to ∞ as y increases from 0 to ∞. Combining this observation with the calculations

$$f_1'(y) = -\frac{1}{y^2} - \kappa(\alpha - 1)^2 y^{\alpha-2} < 0,$$

$$\lim_{y\to\infty} f_1(y) = \left\{\begin{matrix} 0, & \text{if } \alpha \leq 1 \\ -\infty, & \text{if } \alpha > 1 \end{matrix}\right\} \quad \text{and} \quad \lim_{y\downarrow 0} f_1(y) = \infty,$$

we can see that there exists a unique $y_* > 0$ such that

$$f_1(y) - f_2(y) \left\{\begin{matrix} < 0, & \text{if } y > y_* \\ = 0, & \text{if } y = y_* \\ > 0, & \text{if } y \in \left]0, y_*\right[\end{matrix}\right\}. \tag{3.5.43}$$

Proof of (II). The points a_0, b_0 given by (3.2.18) are the unique solution to the corresponding system of equations appearing in (3.2.17) with $b_0 > a_0$. Therefore, they give rise to the unique solution to the system of equations (3.2.8), (3.2.10) and (3.2.11) for the unknowns a_0, b_0 and A. To complete the proof of this part, we need to show that $a_0 > 0$. To this end, it suffices to show that

$$b_0 + a_0 = \frac{\delta + \delta\kappa y_*^\alpha}{\lambda y_*} + \frac{2}{\sqrt{2\delta}} > y_* = b_0 - a_0. \tag{3.5.44}$$

In view of the identity

$$\frac{\delta + \delta\kappa y_*^\alpha}{\lambda y_*} + \frac{2}{\sqrt{2\delta}} = y_* \coth\frac{\sqrt{2\delta}y_*}{2} + \frac{\delta\kappa\alpha y_*^{\alpha-1}}{\lambda},$$

which follows from the equation $f_1(y_*) = f_2(y_*)$, we can see that (3.5.44) is equivalent to

$$y_* \coth\frac{\sqrt{2\delta}y_*}{2} + \frac{\delta\kappa\alpha y_*^{\alpha-1}}{\lambda} > y_*,$$

which is true because $\coth\frac{\sqrt{2\delta}y_*}{2} > 1$.

Proof of (III). The function ℓ defined by (3.2.19) is plainly concave because $A < 0$. Combining this observation with the inequalities

$$\ell(0) = \sqrt{2\delta}A < 0 \quad \text{and} \quad \ell(a_0) = \ell(b_0) \overset{(3.2.10)}{=} \alpha\kappa(b_0 - a_0)^{\alpha-1} > 0,$$

we obtain (3.2.20). We prove (3.2.21) later (see "Proof of (3.5.46) if $\alpha \in\,]0,1[$" further below).

Proof of (IV). In view of (I) and the observations that

$$\lim_{s\to-\infty} q(s,x) = \infty \quad \text{and} \quad \lim_{s\uparrow x} q(s,x) = 1,$$

we can see that (3.2.24) will follow if we show that

$$q \text{ has no strictly negative minimum in } \big\{(s,x) \in \mathbb{R}^2 \mid \ s < x < b_0\big\}, \tag{3.5.45}$$

$$\text{and} \quad q(s,b_0) > 0 \quad \text{for all } s \in\,]-\infty, b_0[\setminus \{a_0\}. \tag{3.5.46}$$

To prove (3.5.45), we argue by contradiction and we assume that there exists $(\bar{s}, \bar{x})$ with $\bar{s} < \bar{x} < b_0$ such that $q(\bar{s}, \bar{x}) < 0$ is a local minimum of q. The first order conditions

$$\begin{Bmatrix} q_s(\bar{s}, \bar{x}) = 0 \\ q_x(\bar{s}, \bar{x}) = 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} w'(\bar{s}) \equiv \ell(\bar{s}) = \alpha\kappa(\bar{x} - \bar{s})^{\alpha-1} \\ w'(\bar{x}) \equiv \ell(\bar{x}) = \alpha\kappa(\bar{x} - \bar{s})^{\alpha-1} \end{Bmatrix}, \tag{3.5.47}$$

the inequality $q(\bar{s}, \bar{x}) < 0$ and the same analysis as the one leading to (3.2.17) gives rise to the inequalities

$$\bar{s} + \bar{x} > \frac{\delta + \delta\kappa(\bar{x} - \bar{s})^\alpha}{\lambda(\bar{x} - \bar{s})} + \frac{2}{\sqrt{2\delta}} \quad \text{and} \quad f_1(\bar{x} - \bar{s}) < f_2(\bar{x} - \bar{s}).$$

The second of these inequalities and (3.5.43) imply that $\bar{x} - \bar{s} > y_* = b_0 - a_0$. On the other hand, the first order conditions (3.5.47) and (3.2.20) in (III) imply that

$$\bar{s} \in [a_0, y^\dagger[\quad \text{and} \quad \bar{x} \in]y^\dagger, b_0] \quad \Rightarrow \quad \bar{x} - \bar{s} \leq b_0 - a_0 = y_*,$$

which is a contradiction.

To prove (3.5.46), we define

$$\bar{q}(s) = q(s, b_0) = -w(b_0) + \underbrace{Ae^{\sqrt{2\delta}s} + \frac{\lambda}{\delta}s^2 + \frac{\lambda}{\delta^2}}_{w(s)} + 1 + \kappa(b_0 - s)^\alpha, \quad \text{for } s \leq b_0,$$

and we observe that

$$\lim_{s \to -\infty} \bar{q}(s) = \infty, \quad \bar{q}(a_0) = 0 \quad \text{and} \quad \bar{q}(b_0) = 1. \tag{3.5.48}$$

Also, we calculate

$$\bar{q}'(s) = g(s, b_0) = \sqrt{2\delta}Ae^{\sqrt{2\delta}s} + \frac{2\lambda}{\delta}s - \alpha\kappa(b_0 - s)^{\alpha-1}, \tag{3.5.49}$$

$$\bar{q}''(s) = g_s(s, b_0) = 2\delta Ae^{\sqrt{2\delta}s} + \frac{2\lambda}{\delta} + \alpha(\alpha - 1)\kappa(b_0 - s)^{\alpha-2}, \tag{3.5.50}$$

$$\bar{q}'''(s) = (2\delta)^{\frac{3}{2}}Ae^{\sqrt{2\delta}s} - \alpha(\alpha - 1)(\alpha - 2)\kappa(b_0 - s)^{\alpha-3}, \tag{3.5.51}$$

and we note that

$$\lim_{s \to -\infty} \bar{q}'(s) = -\infty \quad \text{and} \quad \bar{q}'(a_0) \overset{(3.2.11)}{=} 0. \tag{3.5.52}$$

To complete the proof, we need to distinguish between two different cases.

Proof of (3.5.46) if $\alpha \in \,]0,1[$. Combining the concavity of $\bar{q}' = g(\cdot, b_0)$, which follows from (3.5.51) and the fact that $A < 0$, with (3.5.52), the observation that $\lim_{s \uparrow b_0} \bar{q}'(s) = -\infty$, and the fact that $\bar{q}(a_0) < \bar{q}(b_0)$ (see (3.5.48)), we can conclude that there exists a unique $s_\dagger \in \,]a_0, b_0[$ such that

$$\bar{q}'(s) = g(s, b_0) \begin{Bmatrix} > 0, & \text{if } s \in \,]a_0, s_\dagger[\\ < 0, & \text{if } s \in \,]-\infty, a_0[\cup]s_\dagger, b_0[\end{Bmatrix}.$$

These inequalities and (3.5.48) imply (3.5.46). On the other hand, these inequalities and the concavity of $\bar{q}' = g(\cdot, b_0)$ imply (3.2.21).

Proof of (3.5.46) if $\alpha \geq 1$. In this case, (3.2.20) and (3.5.50) reveal that $\bar{q}'$ is strictly increasing in $]-\infty, a_0[$, which, combined with (3.5.52), implies that

$$\bar{q}'(s) = g(s, b_0) < 0 \quad \text{for all } s \in \,]-\infty, a_0[.$$

On the other hand, we can use (3.2.20) to calculate

$$\bar{q}'(s) = \ell(s) - \alpha\kappa(b_0 - s)^{\alpha-1} > \ell(a_0) - \alpha\kappa(b_0 - a_0)^{\alpha-1} = \bar{q}'(a_0) = 0 \quad \text{for all } s \in \,]a_0, b_0[.$$

These inequalities and (3.5.48) imply (3.5.46) as well as (3.2.22). □

3.5.2 Appendix II: proof of Lemma 8

Proof of (I). Suppose that $\alpha \in \,]0,1[$. The calculations

$$g_x(s, x) = -\alpha(\alpha - 1)\kappa(x - s)^{\alpha-2} > 0 \quad \text{for all } s < x, \tag{3.5.53}$$

$$\text{and} \quad \lim_{x \downarrow s} g(s, x) = -\infty, \quad \lim_{x \to \infty} g(s, x) = \ell(s), \quad \text{for all } s \in \mathbb{R} \tag{3.5.54}$$

imply that, given any s,

there exists a unique $\mathfrak{a}(s) \in \,]s, \infty[$ such that $g\big(s, \mathfrak{a}(s)\big) = 0$ if and only if $\ell(s) > 0$.

This observation and (3.2.20) implies that the equation $g(s, x) = 0$ for $x > s$ defines uniquely a continuous function $\mathfrak{a} : \,]y_\dagger, b_0[\,\to \mathbb{R}_+$ such that $\mathfrak{a}(s) > s$. In particular,

$$\lim_{s \downarrow y_\dagger} \mathfrak{a}(s) = \infty \quad \text{and} \quad \mathfrak{a}(a_0) = b_0, \tag{3.5.55}$$

thanks to (3.2.20) and (3.2.11), respectively. Furthermore, these considerations and the first inequality in (3.2.21) imply that

$$\mathfrak{a}(s) > b_0 \quad \text{for all } s \in \left]y_\dagger, a_0\right[. \tag{3.5.56}$$

In view of second inequality in (3.2.21) and the calculation

$$g_{sx}(s,x) = \alpha(\alpha-1)(\alpha-2)\kappa(x-s)^{\alpha-3} > 0 \quad \text{for all } s < x,$$

we can see that $g_s(s,x) > 0$ for all $s \leq a_0$ and $x \geq b_0$. Combining this observation with (3.5.56) and the continuity of the functions g_s and $\mathfrak{a}$, we can see that there exists $\varepsilon_{\mathfrak{a}} \in \left]0, b_0 - a_0\right[$ such that

$$g_s\big(s, \mathfrak{a}(s)\big) > 0 \quad \text{for all } s \in \left]y_\dagger, a_0 + \varepsilon_{\mathfrak{a}}\right[. \tag{3.5.57}$$

Differentiating the identity $g\big(s, \mathfrak{a}(s)\big) = 0$ with respect to s and using this inequality and (3.5.53), we obtain

$$\mathfrak{a}'(s) = -\frac{g_s\big(s, \mathfrak{a}(s)\big)}{g_x\big(s, \mathfrak{a}(s)\big)} < 0 \quad \text{for all } s \in \left]y_\dagger, a_0 + \varepsilon_{\mathfrak{a}}\right[. \tag{3.5.58}$$

These considerations imply that, given any $\varepsilon_a \in \big]0, b_0 - \mathfrak{a}(a_0 + \varepsilon_{\mathfrak{a}})\big[$, the function a defined by

$$a(x) = \mathfrak{a}^{-1}(x), \quad \text{for } x > b_0 - \varepsilon_a,$$

has all of the properties claimed in the statement of part (I) of the proposition.

For future reference, we also note that differentiation of (3.3.26) yields the expression

$$a'(x) = -\frac{g_x\big(a(x), x\big)}{g_s\big(a(x), x\big)} = \frac{\alpha(\alpha-1)\kappa\big(x - a(x)\big)^{\alpha-2}}{\ell'\big(a(x)\big) + \alpha(\alpha-1)\kappa\big(x - a(x)\big)^{\alpha-2}} < 0 \quad \text{for all } x > b_0. \tag{3.5.59}$$

Proof of (II). Consider the function w defined by (3.3.25) for A, a_0, b_0 and a as in the statement of the proposition. Before addressing the main issues of the proof, we make some preliminary calculations that we will need in several places. First, we note that

$$\begin{aligned} w'(x) &= w'\big(a(x)\big)a'(x) + \alpha\kappa\big(x - a(x)\big)^{\alpha-1}\big(1 - a'(x)\big) \\ &= g\big(a(x), x\big)a'(x) + \alpha\kappa\big(x - a(x)\big)^{\alpha-1} \\ &\overset{(3.3.26)}{=} \alpha\kappa\big(x - a(x)\big)^{\alpha-1} \quad \text{for all } x > b_0. \end{aligned} \tag{3.5.60}$$

Combining these identities with the definition (3.2.23) of g and (3.3.26), we can see that

$$w'(x) = w'\big(a(x)\big) \equiv \ell\big(a(x)\big) \quad \text{for all } x > b_0. \tag{3.5.61}$$

On the other hand, differentiating (3.5.60), we obtain

$$w''(x) = \alpha(\alpha-1)\kappa\big(x - a(x)\big)^{\alpha-2}\big[1 - a'(x)\big] \quad \text{for all } x > b_0. \tag{3.5.62}$$

The function w is plainly C^∞ in $\mathbb{R} \setminus \{b_0\}$. Also, it is C^1 because

$$\lim_{x \downarrow b_0} w(x) = \lim_{x \downarrow b_0} \Big[w\big(a(x)\big) + 1 + \kappa\big(x - a(x)\big)^\alpha\Big] = w(a_0) + 1 + \kappa(b_0 - a_0)^\alpha \overset{(3.2.8)}{=} \lim_{x \uparrow b_0} w(x)$$

and

$$\lim_{x \downarrow b_0} w'(x) \overset{(3.5.60)}{=} \alpha\kappa(b_0 - a_0)^{\alpha-1} \overset{(3.2.10)}{=} \lim_{x \uparrow b_0} w'(x).$$

In view of its definition and Lemma 7.(IV), we will prove that the function w defined by (3.3.25) satisfies the HJB equation (3.1.5) if we show that

$$q(s,x) \overset{(3.2.16)}{\equiv} -w(x) + w(s) + 1 + \kappa(x-s)^\alpha \geq 0 \quad \text{for all } s < x \text{ and } x > b_0, \tag{3.5.63}$$

$$\text{and} \quad \frac{1}{2}w''(x) - \delta w(x) + \lambda x^2 \geq 0 \quad \text{for all } x > b_0. \tag{3.5.64}$$

Proof of (3.5.63) if $\alpha \in \,]0,1[$. Fix any $x \geq b_0$. Combining the concavity of $g(\cdot, x)$, which follows from the calculation

$$g_{ss}(s,x) = (2\delta)^{\frac{3}{2}} A e^{\sqrt{2\delta} s} - \alpha(\alpha-1)(\alpha-2)\kappa(x-s)^{\alpha-3}$$

and the fact that $A < 0$, with the observations that

$$\lim_{s \to -\infty} g(s,x) = -\infty, \quad g\big(a(x), x\big) = 0 \quad \text{and} \quad \lim_{s \uparrow x} g(s,x) = -\infty$$

and (3.5.57) (recall that $a = \mathfrak{a}^{-1}$), we can conclude that there exists a unique $s_\dagger(s) \in \,]a(x), x[$ such that

$$q_s(s,x) = g(s,x) \begin{Bmatrix} > 0, & \text{if } s \in \,]a_0, s_\dagger[\\ < 0, & \text{if } s \in \,]-\infty, a_0[\,\cup\,]s_\dagger, x[\end{Bmatrix}.$$

Combining these inequalities with the observations that

$$\lim_{s\to-\infty} q(s,x) = \infty, \quad q\big(a(x),x\big) = 0 \quad \text{and} \quad q(x,x) = 1,$$

we can see that (3.5.63) is indeed true.

Proof of (3.5.63) if $\alpha = 1$. In this case, the required result follows immediately once we combine Lemma 7.(IV) with the observation that

$$\begin{aligned} q(s,x) &= -w(x) + w(s\vee b_0) + \kappa\big(x - (s\vee b_0)\big) + q(s, s\vee b_0) \\ &= q(s, s\vee b_0) \quad \text{for all } s < x \text{ and } x > b_0. \end{aligned}$$

Proof of (3.5.64) if $\alpha \in \,]0,1[$. We first use (3.5.61) to calculate

$$w''(x) = \ell'\big(a(x)\big)a'(x) \overset{(3.5.59)}{=} \frac{\alpha(\alpha-1)\kappa\big(x-a(x)\big)^{\alpha-2}\ell'\big(a(x)\big)}{\ell\big(a(x)\big) + \alpha(\alpha-1)\kappa\big(x-a(x)\big)^{\alpha-2}}, \quad \text{for all } x > b_0,$$

and

$$\begin{aligned} &\Big[\ell\big(a(x)\big) + \alpha(\alpha-1)\kappa\big(x-a(x)\big)^{\alpha-2}\Big]^2 w'''(x) \\ &\qquad = \alpha^2(\alpha-1)^2\kappa^2\big(x-a(x)\big)^{2\alpha-4}\ell''\big(a(x)\big)a'(x) \\ &\qquad\quad + \alpha(\alpha-1)(\alpha-2)\kappa\big(x-a(x)\big)^{\alpha-3}\big[\ell'\big(a(x)\big)\big]^2\big[1-a'(x)\big]^2 \\ &\qquad > 0 \quad \text{for all } x > b_0, \end{aligned}$$

the inequality following thanks to the concavity of ℓ (see Lemma 7.(III)) and the fact that a is strictly decreasing (see part (I) of this proposition). In view of these calculations and the fact that $A < 0$, we can see that

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{2}w''(x) - \delta w(x) + \lambda x^2\right] &\overset{(3.5.61)}{=} \frac{1}{2}w'''(x) - \delta\ell\big(a(x)\big) + 2\lambda x \\ &\overset{(3.2.19)}{=} \frac{1}{2}w'''(x) - \delta\sqrt{2\delta}Ae^{\sqrt{2\delta}a(x)} + 2\lambda\big[x - a(x)\big] \\ &> 0 \quad \text{for all } x > b_0. \end{aligned} \tag{3.5.65}$$

It follows that (3.5.64) is true if and only if

$$\frac{1}{2}w''(b_0+) - \delta w(b_0) + \lambda b_0^2 \geq 0 \quad \Leftrightarrow \quad w''(b_0+) \geq w''(b_0-). \tag{3.5.66}$$

To show that this inequality is indeed true, we note that (3.2.24) in Lemma 7.(IV) and the definition (3.3.25) of w imply that

$$q\big(a(x),x\big)\begin{cases} >0, & \text{if } x\in\,]b_0-\varepsilon_a,b_0[\\ =0, & \text{if } x\geq b_0\end{cases}\Bigg\}.$$

Since q is $C^{1,1}$ and a is C^{∞}, these inequalities imply that

$$\lim_{x\uparrow b_0}\frac{d^2q\big(a(x),x\big)}{dx^2}\geq 0.$$

Combining this observation with the calculation

$$\begin{aligned}\lim_{x\uparrow b_0}\frac{d^2q\big(a(x),x\big)}{dx^2} &= \lim_{x\uparrow b_0}\frac{d\big[g\big(a(x),x\big)a'(x)+q_x\big(a(x),x\big)\big]}{dx}\\ &= \lim_{x\uparrow b_0}\Big[-w''(x)+\alpha(\alpha-1)\kappa\big(x-a(x)\big)^{\alpha-2}\big[1-a'(x)\big]\Big]\\ &= -w''(b_0-)+\alpha(\alpha-1)\kappa(b_0-a_0)^{\alpha-2}\big[1-a'(b_0)\big]\\ &\overset{(3.5.62)}{=} -w''(b_0-)+w''(b_0+),\end{aligned}$$

where we have used the identities

$$q_s\big(a(x),x\big)=g\big(a(x),x\big)=0,$$

we obtain (3.5.66).

Proof of (3.5.64) if $\alpha=1$. In this case, $w''(x)=0$ for all $x>b_0$ and (3.5.65) follows immediately. On the other hand, (3.5.66) is plainly true because $w''(b_0-)=\ell'(b_0)<0$ (see (3.2.20) in Lemma 7.(III)).

3.5.3 Appendix III: proof of Theorem 4

Throughout the proof, we consider any initial condition $x\in\mathbb{R}$ fixed. Let $\mathcal{Z}\in\mathcal{A}$ be any admissible impulse control strategy such that $J_x(\mathcal{Z})<\infty$. The finiteness of such a strategy's performance implies that

$$\mathbb{E}\left[\int_0^{\infty}e^{-\delta t}X_t^2\,dt\right]<\infty. \tag{3.5.67}$$

Using Itô's formula, we obtain

$$
\begin{aligned}
e^{-\delta T} w(X_{T+}) &= w(x) + \int_0^T e^{-\delta t} \left[\frac{1}{2} w''(X_t) - \delta w(X_t) \right] dt + \int_{[0,T]} e^{-\delta t} w'(X_t)\, d\bar{Z}_t \\
&\quad + \int_0^T e^{-\delta t} w'(X_t)\, dW_t + \sum_{t \in [0,T]} e^{-\delta t} \big[w(X_{t+}) - w(X_t) - w'(X_t) \Delta X_t \big] \\
&= w(x) + \int_0^T e^{-\delta t} \left[\frac{1}{2} w''(X_t) - \delta w(X_t) \right] dt \\
&\quad + \int_0^T e^{-\delta t} w'(X_t)\, dW_t + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big[w(X_{\tau_n} - Z_n) - w(X_{\tau_n}) \big] \mathbf{1}_{\{\tau_n \le T\}},
\end{aligned}
$$

the second equality following from the fact that $\Delta X_s \equiv X_{s+} - X_s = \Delta \bar{Z}_s$ and (3.1.1). These identities imply that

$$
\begin{aligned}
&\lambda \int_0^T e^{-\delta t} X_t^2\, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^{\alpha} \big) \mathbf{1}_{\{\tau_n \le T\}} \\
&\quad = w(x) - e^{-\delta T} w(X_{T+}) + \int_0^T e^{-\delta t} \left[\frac{1}{2} w''(X_t) - \delta w(X_t) + \lambda X_t^2 \right] dt + \int_0^T e^{-\delta t} w'(X_t)\, dW_t \\
&\qquad + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big[w(X_{\tau_n} - Z_n) - w(X_{\tau_n}) + 1 + \kappa Z_n^{\alpha} \big] \mathbf{1}_{\{\tau_n \le T\}}. \qquad (3.5.68)
\end{aligned}
$$

Since w satisfies the HJB equation (3.1.5),

$$
\begin{aligned}
&\lambda \int_0^T e^{-\delta t} X_t^2\, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^{\alpha} \big) \mathbf{1}_{\{\tau_n \le T\}} \\
&\qquad \ge w(x) - e^{-\delta T} w(X_{T+}) + \int_0^T e^{-\delta t} w'(X_t)\, dW_t. \qquad (3.5.69)
\end{aligned}
$$

In view of the definition (3.3.25) of w, we can see that there exists a constant $C > 0$ such that

$$
\big[w'(x) \big]^2 + \big| w(x) \big| \le C \big(1 + |x|^2 \big) \quad \text{for all } x \in \mathbb{R}.
$$

In view of these estimates, Itô's isometry and (3.5.67) imply that

$$
\begin{aligned}
\mathbb{E} \left[\left(\int_0^T e^{-\delta t} w'(X_t)\, dW_t \right)^2 \right] &= \mathbb{E} \left[\int_0^T e^{-2\delta t} \big| w'(X_t) \big|^2\, dt \right] \\
&\le C\, \mathbb{E} \left[\int_0^T e^{-\delta t} \big(1 + X_t^2 \big)\, dt \right] \\
&< \infty
\end{aligned}
$$

and

$$\lim_{T\to\infty} e^{-\delta T}\mathbb{E}\Big[\big|w(X_{T+})\big|\Big] \leq \lim_{T\to\infty} C\Big(e^{-\delta T} + \mathbb{E}\big[e^{-\delta T}X_T^2\big]\Big) = 0. \tag{3.5.70}$$

The first of these observations implies that the stochastic integral in (3.5.69) is a square integrable martingale. Taking expectations in (3.5.69), we therefore obtain

$$\mathbb{E}\left[\lambda\int_0^T e^{-\delta t}X_t^2\,dt + \sum_{n=1}^{\infty} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\alpha}\big)\mathbf{1}_{\{\tau_n\leq T\}}\right] \geq w(x) - \mathbb{E}\big[e^{-\delta T}w(X_{T+})\big].$$

Passing to the limit using the monotone convergence theorem and (3.5.70), we derive

$$J_x(\mathcal{Z}) = \mathbb{E}\left[\lambda\int_0^{\infty} e^{-\delta t}X_t^2\,dt + \sum_{n=1}^{\infty} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\alpha}\big)\right] \geq w(x),$$

which establishes the inequality

$$v(x) \geq w(x). \tag{3.5.71}$$

To prove the reverse inequality and establish the optimality of the impulse control strategy $\mathcal{Z}^*$ defined by (3.3.28)–(3.3.30), we first note that $(\tau_{n+1}^* - \tau_n^*, n\geq 2)$ is a sequence of independent and identically distributed random variables, each having the same distribution as the first hitting time

$$T_{b_0-a_0}(B) = \inf\{t\geq 0 \mid \; B_t \geq b_0 - a_0\},$$

where B is a standard one-dimensional Brownian motion starting from 0. In particular,

$$\mathbb{E}\big[e^{-\delta(\tau_{n+1}^*-\tau_n^*)}\big] = \mathbb{E}\big[e^{-\delta T_{b_0-a_0}(B)}\big] = e^{-\sqrt{2\delta}(b_0-a_0)}$$

and

$$\mathbb{E}\left[e^{-\delta\tau_n^*}\right] = \mathbb{E}\left[e^{-\delta\tau_2^*}\right]\prod_{j=1}^{n-2}\mathbb{E}\big[e^{-\delta(\tau_{j+2}^*-\tau_{j+1}^*)}\big] = \mathbb{E}\left[e^{-\delta\tau_2^*}\right]e^{-(n-2)\sqrt{2\delta}(b_0-a_0)}$$

for all $n\geq 2$. Furthermore, the state process X^* associated with $\mathcal{Z}^*$ satisfies

$$\begin{aligned}\mathbb{E}\left[\int_{\tau_n^*}^{\tau_{n+1}^*} e^{-\delta t}X_t^{*2}\,dt\right] &= \mathbb{E}\left[e^{-\delta\tau_n^*}\int_{\tau_n^*}^{\tau_{n+1}^*} e^{-\delta(t-\tau_n^*)}(a_0+W_t-W_{\tau_n^*})^2\,dt\right]\\ &= \mathbb{E}\big[e^{-\delta\tau_n^*}\big]\mathbb{E}\left[\int_0^{T_{b_0-a_0}(B)} e^{-\delta t}(a_0+B_t)^2\,dt\right] \quad \text{for all } n\geq 2.\end{aligned}$$

In view of these observations and the monotone convergence theorem, we can see that

$$\mathbb{E}\left[\sum_{n=1}^{\infty} e^{-\delta\tau_n^*}\big(1+\kappa Z_n^{*\alpha}\big)\right] = \mathbb{E}\left[\sum_{n=1}^{2} e^{-\delta\tau_n^*}\big(1+\kappa Z_n^{*\alpha}\big)\right] + \big[1+\kappa(b_0-a_0)^\alpha\big]\sum_{n=3}^{\infty}\mathbb{E}\left[e^{-\delta\tau_n^*}\right]$$
$$< \infty$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}X_t^{*2}\,dt\right] = \mathbb{E}\left[\int_0^{\tau_2^*} e^{-\delta t}X_t^{*2}\,dt\right] + \mathbb{E}\left[\int_0^{T_{b_0-a_0}(B)} e^{-\delta t}(a_0+B_t)^2\,dt\right]\sum_{n=3}^{\infty}\mathbb{E}\big[e^{-\delta\tau_n^*}\big]$$
$$< \infty.$$

It follows that $J_x(\mathcal{Z}^*) < \infty$.

To proceed further, we note that the impulse control strategy $\mathcal{Z}^\star$ is such that

$$\frac{1}{2}w''(X_t^\star) - \delta w(X_t^\star) + \lambda X_t^{\star 2} = 0 \quad \text{for all } t \in \mathbb{R}_+ \setminus \{\tau_n^\star,\ n \geq 1\},$$
$$\text{and} \quad w(X_{\tau_n}^\star - Z_n^\star) - w(X_{\tau_n}^\star) + 1 + \kappa Z_n^{\star\alpha} = 0 \quad \text{for all } n \geq 1.$$

In view of these observations, we can see that (3.5.68) implies that

$$\lambda\int_0^T e^{-\delta t}X_t^{\star 2}\,dt + \sum_{n=1}^{\infty} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\star\alpha}\big)\mathbf{1}_{\{\tau_n^\star\leq T\}}$$
$$= w(x) - e^{-\delta T}w(X_{T+}^\star) + \int_0^T e^{-\delta t}w'(X_t^\star)\,dW_t.$$

Using this identity instead of the inequality (3.5.69) and following the same steps as the ones leading to (3.5.71), we can see that $J_x(\mathcal{Z}^\star) = w(x)$. It follows that

$$v(x) \leq w(x), \tag{3.5.72}$$

which, combined with (3.5.71), implies that $v(x) = w(x)$ as well as the optimality of $\mathcal{Z}^\star$.

3.5.4 Appendix IV: proof of Lemma 9

Proof of Lemma 9. The identity (3.4.32) follows from the observation that, given any $\alpha > 1$, $z \geq 0$ and $n \geq 2$,

$$\min_{\substack{u_1,\ldots,u_n\geq 0\\ u_1+\cdots+u_n=z}} \sum_{j=1}^{n}(1+\kappa u_j^\alpha) = n + n\kappa\left(\frac{z}{n}\right)^\alpha. \tag{3.5.73}$$

For $n = 2$, this result follows immediately from the equivalence

$$\frac{d}{du}\big[\big(1+\kappa u^{\alpha}\big)+\big(1+\kappa(z-u)^{\alpha}\big)\big]=0 \quad \Leftrightarrow \quad u=\frac{z}{2}.$$

Given any $n > 2$, the first order conditions

$$\frac{\partial}{\partial u_i}\left(\sum_{j=1}^{n-1}\big(1+\kappa u_j^{\alpha}\big)+\left[1+\kappa\left(z-\sum_{j=1}^{n-1}u_j\right)^{\alpha}\right]\right)=0, \quad \text{for } i=1,2,\dots,n-1, \tag{3.5.74}$$

are equivalent to $u_i+\sum_{j=1}^{n-1}u_j = z$, for $i = 1, 2, \dots, n-1$. Therefore, the first order conditions (3.5.74) are equivalent to $u_1 = u_2 = \cdots = u_{n-1}$. Combining this result with the observation that $u_n = z - \sum_{j=1}^{n-1} u_j = u_1$, we can see that the minimum on the left-hand side of (3.5.73) is achieved by the choice $u_1 = \cdots = u_n = \frac{z}{n}$.

We can see that (3.4.33) is indeed true by combining (3.4.32) with the fact that the inequality

$$n+n\kappa\left(\frac{z}{n}\right)^{\alpha}>(n+1)+(n+1)\kappa\left(\frac{z}{n+1}\right)^{\alpha}$$

is equivalent to the inequality $z > z_n$.

To show (3.4.34), we use the Maclaurin series expansion

$$(1+y)^{\zeta}=1+\zeta y+\frac{\zeta(\zeta-1)}{2}y^2+o(y^2), \quad \text{for } |y|<1,$$

where ζ is a constant, to calculate

$$\begin{aligned} z_n &= \kappa^{-1/\alpha}n^{(\alpha-1)/\alpha}\left[1-\left(1+\frac{1}{n}\right)^{-(\alpha-1)}\right]^{-1/\alpha} \\ &= \kappa^{-1/\alpha}n^{(\alpha-1)/\alpha}\left[\frac{\alpha-1}{n}-\frac{\alpha(\alpha-1)}{2n^2}+o(n^{-2})\right]^{-1/\alpha} \end{aligned} \tag{3.5.75}$$

as well as

$$\begin{aligned} z_{n-1} &= \kappa^{-1/\alpha}n^{(\alpha-1)/\alpha}\left[\left(1-\frac{1}{n}\right)^{-(\alpha-1)}-1\right]^{-1/\alpha} \\ &= \kappa^{-1/\alpha}n^{(\alpha-1)/\alpha}\left[\frac{\alpha-1}{n}+\frac{\alpha(\alpha-1)}{2n^2}+o(n^{-2})\right]^{-1/\alpha}. \end{aligned} \tag{3.5.76}$$

In view of these calculations, we can see that

$$\begin{aligned}
&\lim_{n\to\infty}(z_n - z_{n-1})\\
&= \kappa^{-1/\alpha}(\alpha-1)^{-1/\alpha}\lim_{n\to\infty} n^{(\alpha-1)/\alpha}n^{1/\alpha}\left[\left(1-\frac{\alpha}{2n}+o(n^{-1})\right)^{-1/\alpha} - \left(1+\frac{\alpha}{2n}+o(n^{-1})\right)^{-1/\alpha}\right]\\
&= \kappa^{-1/\alpha}(\alpha-1)^{-1/\alpha}\lim_{n\to\infty} n\left[\left(1+\frac{1}{2n}+o(n^{-1})\right) - \left(1-\frac{1}{2n}+o(n^{-1})\right)\right],
\end{aligned}$$

and (3.4.34) follows.

Given any $z > 0$, there exists $n \geq 1$ such that $z_{n-1} \leq z < z_n$. For any such pair of z and n,

$$\frac{K(z_{n-1})}{z_n} < \frac{K(z)}{z} < \frac{K(z_n)}{z_{n-1}}$$

because K is strictly increasing. In view of this observation, we can see that (3.4.35) will follow if we show that

$$\lim_{n\to\infty}\frac{K(z_{n-1})}{z_n} = \lim_{n\to\infty}\frac{K(z_n)}{z_{n-1}} = \kappa^{1/\alpha}\alpha(\alpha-1)^{-(\alpha-1)/\alpha},$$

namely, if we prove that

$$\lim_{n\to\infty}\frac{n}{z_n}\left[1+\kappa\left(\frac{z_{n-1}}{n}\right)^{\alpha}\right] = \lim_{n\to\infty}\frac{n}{z_{n-1}}\left[1+\kappa\left(\frac{z_n}{n}\right)^{\alpha}\right] = \kappa^{1/\alpha}\alpha(\alpha-1)^{-(\alpha-1)/\alpha}. \tag{3.5.77}$$

To this end, we calculate

$$\begin{aligned}
\lim_{n\to\infty}\frac{n}{z_n} &\overset{(3.5.75)}{=} \lim_{n\to\infty}\kappa^{1/\alpha}n^{1/\alpha}\left[\frac{\alpha-1}{n} - \frac{\alpha(\alpha-1)}{2n^2} + o(n^{-2})\right]^{1/\alpha} = \kappa^{1/\alpha}(\alpha-1)^{1/\alpha},\\
\lim_{n\to\infty}\frac{n}{z_{n-1}} &\overset{(3.5.76)}{=} \lim_{n\to\infty}\kappa^{1/\alpha}n^{1/\alpha}\left[\frac{\alpha-1}{n} + \frac{\alpha(\alpha-1)}{2n^2} + o(n^{-2})\right]^{1/\alpha} = \kappa^{1/\alpha}(\alpha-1)^{1/\alpha},
\end{aligned}$$

and (3.5.77) follows.

3.5.5 Appendix V: proof of Lemma 10

Proof of (I). First, we consider the function u defined by

$$u(x) = \begin{Bmatrix} w(x), & \text{if } x \leq b_0 \\ w(b_0) + \ell(b_0)(x - b_0), & \text{if } x > b_0 \end{Bmatrix},$$

where w is given by (3.2.9). This function is C^1 in $\mathbb{R}$ as well as C^∞ in $\mathbb{R} \setminus \{b_0\}$, and satisfies the inequality

$$\min\left\{\frac{1}{2}u''(x) - \delta u(x) + \lambda x^2,\ -u(x) + \inf_{z>0}\big[u(x-z) + 1 + \kappa z^\alpha\big]\right\} \geq 0. \qquad (3.5.78)$$

In view of its construction and Lemma 7.(IV), we will establish (3.5.78) if we prove that

$$\tilde{q}(s,x) = -u(x) + u(s) + 1 + \kappa(x-s)^\alpha \geq 0 \quad \text{for all } s < x \text{ and } x > b_0, \qquad (3.5.79)$$

$$\text{and} \quad \frac{1}{2}u''(x) - \delta u(x) + \lambda x^2 \geq 0 \quad \text{for all } x > b_0. \qquad (3.5.80)$$

To show (3.5.79), we consider any $x > b_0$ and any $s < x$ and we calculate

$$\begin{aligned}\tilde{q}_x(s,x) &= -\ell(b_0) + \alpha\kappa(x-s)^{\alpha-1} \\ &= -g\big(b_0 - (x-s), b_0\big) \\ &\overset{(3.2.22)}{=} \left\{\begin{array}{ll} > 0, & \text{if } b_0 - (x-s) < a_0 \ \Leftrightarrow\ x > b_0 - a_0 + s \\ < 0, & \text{if } b_0 - (x-s) \in \,]a_0, b_0] \ \Leftrightarrow\ x \in [s, b_0 - a_0 + s[\end{array}\right\}.\end{aligned}$$

These inequalities imply that

$$\text{if } x > b_0 \vee (b_0 - a_0 + s), \quad \text{then } \tilde{q}(s,x) > \left\{\begin{array}{ll} q(s, b_0), & \text{if } s \leq a_0 \\ \tilde{q}(s, b_0 - a_0 + s), & \text{if } s > a_0 \end{array}\right\},$$

$$\text{and,} \quad \text{if } x \in [s, b_0 - a_0 + s[\, \cap\, [b_0, \infty[, \quad \text{then } \tilde{q}(s,x) > \tilde{q}(s, b_0 - a_0 + s),$$

where q is defined by (3.2.16). In view of these implications and Lemma 7.(IV), we can see that (3.5.79) will follow if we prove that

$$\tilde{q}(s, b_0 - a_0 + s) \geq 0 \quad \text{for all } s > a_0. \qquad (3.5.81)$$

To this end, we distinguish between two cases. If $b_0 < s \leq b_0 - a_0 + s$, then

$$\begin{aligned}\tilde{q}(s, b_0 - a_0 + s) &= -\ell(b_0)(b_0 - a_0) + 1 + \kappa(b_0 - a_0)^\alpha \\ &\overset{(3.2.8)}{=} -\ell(b_0)(b_0 - a_0) + w(b_0) - w(a_0) \\ &= \int_{a_0}^{b_0} \big[\ell(s) - \ell(b_0)\big]\, ds \\ &\overset{(3.2.20)}{>} 0.\end{aligned}$$

On the other hand, if $a_0 < s < b_0 < b_0 - a_0 + s$, then the calculation

$$\begin{aligned} \frac{d\tilde{q}(s, b_0 - a_0 + s)}{ds} &= \frac{d\big(-w(b_0) - \ell(b_0)(s - a_0) + w(s) + 1 + \kappa(b_0 - a_0)^\alpha\big)}{ds} \\ &= -\ell(b_0) + \ell(s) \\ &\overset{(3.2.20)}{>} 0 \end{aligned}$$

implies that

$$\tilde{q}(s, b_0 - a_0 + s) > \tilde{q}(a_0, b_0) = q(a_0, b_0) \overset{(3.2.8)}{=} 0.$$

In either case, (3.5.81) holds true, and (3.5.79) has been proved.

In view of the calculations

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{2}u''(x) - \delta u(x) + \lambda x^2\right] &= -\delta\ell(b_0) + 2\lambda x \\ &\overset{(3.2.19)}{=} -\delta\sqrt{2\delta}Ae^{\sqrt{2\delta}b_0} + 2\lambda\big[x - b_0\big] \\ &> 0 \quad \text{for all } x > b_0, \end{aligned}$$

we can see that (3.5.80) is true if and only if

$$\frac{1}{2}u''(b_0+) - \delta u(b_0) + \lambda b_0^2 \geq 0 \quad \Leftrightarrow \quad 0 \geq w''(b_0-) = \ell'(b_0),$$

which is true thanks to (3.2.20) in Lemma 7.(III).

To proceed further, we consider any initial condition $x \in \mathbb{R}$ and any impulse control strategy $\mathcal{Z} \in \mathcal{A}$. Using Itô's formula and (3.5.78), we can follow the same steps as the ones we used to derive (3.5.69) in Appendix III to obtain

$$\begin{aligned} \lambda\int_0^T e^{-\delta t}X_t^2\,dt + &\sum_{n=1}^\infty e^{-\delta\tau_n}\big(1 + \kappa Z_n^\alpha\big)\mathbf{1}_{\{\tau_n \leq T\}} \\ &\geq \lambda\int_0^{T_\mathcal{Z}\wedge T} e^{-\delta t}X_t^2\,dt + \sum_{n=1}^\infty e^{-\delta\tau_n}\big(1 + \kappa Z_n^\alpha\big)\mathbf{1}_{\{\tau_n \leq T_\mathcal{Z}\wedge T\}} \\ &\quad + e^{-\delta(T_\mathcal{Z}\wedge T)}u(X_{(T_\mathcal{Z}\wedge T)+}) - e^{-\delta T}u\big(X_{T+}\big) + \int_{T_\mathcal{Z}\wedge T}^T e^{-\delta t}u'(X_t)\,dW_t, \end{aligned}$$

where $T_\mathcal{Z}$ is the stopping time defined by (3.4.37). Following exactly the same arguments as in the proof of (3.5.71) in Appendix III, we can see that this inequality implies that

$$J_x(\mathcal{Z}) \geq \mathbb{E}\left[\lambda\int_0^{T_\mathcal{Z}} e^{-\delta t}X_t^2\,dt + \sum_{n=1}^\infty e^{-\delta\tau_n}\big(1 + \kappa Z_n^\alpha\big)\mathbf{1}_{\{\tau_n \leq T_\mathcal{Z}\}} + e^{-\delta T_\mathcal{Z}}w(X_{T_\mathcal{Z}+})\right],$$

where we have also used the fact that $w(X_{T_{\mathcal{Z}}+}) = u(X_{T_{\mathcal{Z}}+})$. It follows that

$$v(x) \geq \inf_{\mathcal{Z} \in \mathcal{A}} \mathbb{E}\left[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2 \, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^{\alpha}\big) \mathbf{1}_{\{\tau_n \leq T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w(X_{T_{\mathcal{Z}}+})\right]. \quad (3.5.82)$$

To derive the reverse inequality, we consider any initial condition $x \in \mathbb{R}$ and any impulse control strategy $\mathcal{Z} \in \mathcal{A}$, and we denote by $\tilde{\mathcal{Z}} \in \mathcal{A}$ the impulse control strategy that is identical to $\mathcal{Z}$ up to the stopping time $T_{\mathcal{Z}}$ and then repositions the state process X down to the level a_0 whenever this hits the level b_0 after time $T_{\mathcal{Z}}$. This strategy can be constructed as follows. First, we define

$$\bar{Z}_t^{(i)} = \sum_{n=1}^{\infty} Z_n \mathbf{1}_{\{\tau_n < t\}} \mathbf{1}_{\{\tau_n \leq T_{\mathcal{Z}}\}},$$

$$\tau_1^{(ii)} = \inf\big\{t \geq T_{\mathcal{Z}} \mid \ X_{T_{\mathcal{Z}}+} + W_t - W_{T_{\mathcal{Z}}+} \geq b_0\big\}, \quad Z_1^{(ii)} = b_0 - a_0,$$

$$\tau_{n+1}^{(ii)} = \inf\{t \geq \tau_n^{(ii)} \mid \ a_0 + W_t - W_{\tau_n^{(ii)}} \geq b_0\}, \quad Z_{n+1}^{(ii)} = b_0 - a_0, \quad \text{for } n \geq 1,$$

and

$$\bar{Z}_t^{(ii)} = \sum_{n=1}^{\infty} Z_n^{(ii)} \mathbf{1}_{\{\tau_n^{(ii)} < t\}}.$$

We then define $\tilde{\mathcal{Z}} = \big(\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_n, \ldots; \tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_n, \ldots\big)$, where $\tilde{\tau}_n$ are the stopping times at which the jumps of the process $\tilde{\bar{Z}} = \bar{Z}^{(i)} + \bar{Z}^{(ii)}$ occur and $\tilde{Z}_n$ are the corresponding jump sizes. Furthermore, we denote by $\tilde{\mathcal{A}}$ the family of all such impulse control strategies, and we note that $\tilde{\mathcal{A}} \subseteq \mathcal{A}$. In particular, we note that

$$T_{\tilde{\mathcal{Z}}} = T_{\mathcal{Z}}, \quad \tilde{X}_t \mathbf{1}_{\{t \leq T_{\tilde{\mathcal{Z}}}\}} = X_t \mathbf{1}_{\{t \leq T_{\mathcal{Z}}\}} \quad \text{and} \quad \tilde{Z}_n \mathbf{1}_{\{\tilde{\tau}_n \leq T_{\tilde{\mathcal{Z}}}\}} = Z_n \mathbf{1}_{\{\tau_n \leq T_{\mathcal{Z}}\}}. \quad (3.5.83)$$

Using the fact that the restriction of u in $]-\infty, b_0]$ identifies with w and satisfies (3.5.78) with equality, and following exactly the same arguments as in the proof of (3.5.72) in Appendix III with u in place of w, we can see that

$$J_x(\tilde{\mathcal{Z}}) = \mathbb{E}\left[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2 \, dt + \sum_{n=1}^{\infty} e^{-\delta \tau_n} \big(1 + \kappa Z_n^{\alpha}\big) \mathbf{1}_{\{\tau_n \leq T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w(X_{T_{\mathcal{Z}}+})\right].$$

It follows that

$$
\begin{aligned}
v(x) &= \inf_{\mathcal{Z}\in\mathcal{A}} J_x(\mathcal{Z}) \le \inf_{\tilde{\mathcal{Z}}\in\tilde{\mathcal{A}}} J_x(\tilde{\mathcal{Z}}) \\
&= \inf_{\mathcal{Z}\in\mathcal{A}} \mathbb{E}\left[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt + \sum_{n=1}^{\infty} e^{-\delta\tau_n}\big(1+\kappa Z_n^\alpha\big)\mathbf{1}_{\{\tau_n\le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w(X_{T_{\mathcal{Z}}+})\right],
\end{aligned}
$$

where we have also used (3.5.83). Combining this inequality with (3.5.82), we obtain the required result.

3.5.6 Appendix VI: proof of Theorem 5

The inequalities $\bar{w}_j \ge w_j \ge \bar{w}_{j+1}$ for all $j \ge 0$ are straightforward to see. On the other hand, the regularity of the functions w_j and the fact that they satisfy the variational inequality (3.4.41) follow from Theorem 6.3 of Lamberton and Zervos [40]. In particular, w_j satisfies

$$
w_j(x) = \mathbb{E}\left[\lambda \int_0^{T_j^\star} e^{-\delta t} \mathring{X}_t^2\, dt + e^{-\delta T_j^\star} \bar{w}_j(\mathring{X}_{T_j^\star})\right], \tag{3.5.84}
$$

where

$$
\mathring{X}_t = x + W_t \quad \text{and} \quad T_j^\star = \inf\big\{t\ge 0 \mid \ w_j(\mathring{X}_t) = \bar{w}_j(\mathring{X}_t)\big\}, \tag{3.5.85}
$$

Furthermore, the continuity of the function $z \mapsto w_j(x-z) + 1 + \kappa z^\alpha$ and the fact that this function tends to ∞ as z increases to ∞ imply that there exists a function $\mathfrak{z}_j : [b_0, \infty[\to [0, \infty[$ such that

$$
\mathfrak{z}_j(x) = \operatorname*{arg\,min}_{z\ge 0}\big[w_j(x-z) + 1 + \kappa z^\alpha\big]. \tag{3.5.86}
$$

Also, we note that the definitions of $\bar{w}_j$ in (3.4.38), (3.4.40) and Lemma 7.(IV) imply that

$$
\bar{w}_j(x) = w_{\mathrm{ext}}(x) \quad \text{for all } x \le b_0 \text{ and } j \ge 0. \tag{3.5.87}
$$

We now show by induction that, given any $j \ge 0$,

$$
\begin{aligned}
w_j(x) = \inf_{\mathcal{Z}\in\mathcal{A}_j} \mathbb{E}\Bigg[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt \\
+ \sum_{n=1}^{j} e^{-\delta\tau_n}\big(1+\kappa Z_n^\alpha\big)\mathbf{1}_{\{\tau_n\le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{\mathrm{ext}}(X_{T_{\mathcal{Z}}+})\Bigg],
\end{aligned} \tag{3.5.88}
$$

where

$$T_{\mathcal{Z}} = \inf\bigl\{t \geq 0 \mid \ X_t \leq b_0\bigr\}$$

and $\mathcal{A}_j \subseteq \mathcal{A}$ is the family of all admissible impulse control strategies $\mathcal{Z}$ such that $\tau_{j+2} = \infty$, namely, the class of all strategies that involve a maximum of $j+1$ jumps. To establish (3.5.88) for $j = 0$, we first note that, given any $(\mathcal{F}_t)$-stopping times $\underline{T} \leq \bar{T}$,

$$\mathbb{E}\left[\lambda \int_0^{\bar{T}} e^{-\delta t} X_t^2 \, dt + e^{-\delta \bar{T}} w_{\text{ext}}(\mathring{X}_{\bar{T}})\right]$$
$$= \mathbb{E}\left[\lambda \int_0^{\underline{T} \wedge \bar{T}} e^{-\delta t} X_t^2 \, dt + e^{-\delta(\underline{T} \wedge \bar{T})} w_{\text{ext}}(\mathring{X}_{\underline{T} \wedge \bar{T}})\right] \tag{3.5.89}$$

because the process $\left(\lambda \int_0^t e^{-\delta s} X_s^2 \, ds + e^{-\delta t} w_{\text{ext}}(\mathring{X}_t)\right)$ is a square-integrable martingale. In view of this observation, the definitions (3.4.38), (3.4.39) of $\bar{w}_0$, w_0, we can see that, given any $\mathcal{Z} \in \mathcal{A}_0$,

$$\begin{aligned}
&\mathbb{E}\left[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2 \, dt + e^{-\delta \tau_1}\bigl(1 + \kappa Z_1^\alpha\bigr)\mathbf{1}_{\{\tau_1 \leq T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{\text{ext}}(X_{T_{\mathcal{Z}}+})\right] \\
&\quad = \mathbb{E}\left[\lambda \int_0^{\tau_1 \wedge T_{\mathcal{Z}}} e^{-\delta t} X_t^2 \, dt + e^{-\delta \tau_1}\bigl(1 + \kappa Z_1^\alpha\bigr)\mathbf{1}_{\{\tau_1 \leq T_{\mathcal{Z}}\}} + e^{-\delta(\tau_1 \wedge T_{\mathcal{Z}})} w_{\text{ext}}(X_{(\tau_1 \wedge T_{\mathcal{Z}})+})\right] \\
&\quad = \mathbb{E}\Biggl[\lambda \int_0^{\tau_1 \wedge T_{\mathcal{Z}}} e^{-\delta t} \mathring{X}_t^2 \, dt \\
&\qquad\qquad + e^{-\delta \tau_1}\left[w_{\text{ext}}(\mathring{X}_{\tau_1} - Z_1) + 1 + \kappa Z_1^\alpha\right]\mathbf{1}_{\{\tau_1 \leq T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{\text{ext}}(\mathring{X}_{T_{\mathcal{Z}}})\mathbf{1}_{\{T_{\mathcal{Z}} < \tau_1\}}\Biggr] \\
&\quad \geq \inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\lambda \int_0^\tau e^{-\delta t} \mathring{X}_t^2 \, dt + e^{-\delta \tau} \bar{w}_0(\mathring{X}_\tau)\right] \\
&\quad = w_0(x).
\end{aligned} \tag{3.5.90}$$

To derive the reverse inequality, we consider the strategy $\mathcal{Z}^\star \in \mathcal{A}_0$ given by

$$\tau_1^\star = \inf\bigl\{t \geq 0 \mid \ w_0(\mathring{X}_t) = \bar{w}_0(\mathring{X}_t) \text{ and } \bar{w}_0(\mathring{X}_t) > w_{\text{ext}}(X_t)\bigr\} \quad \text{and} \quad Z_1^\star = \mathfrak{z}_0(\mathring{X}_{\tau_1^\star}),$$

where $\mathfrak{z}_0$ is defined by (3.5.86). The definition (3.4.38) of $\bar{w}_0$ and the identity (3.5.87) imply that the stopping time $T_0^\star$ defined by (3.5.85) satisfies

$$T_0^\star \leq \tau_1^\star \wedge T_{\mathcal{Z}^\star} \quad \text{and} \quad T_0^\star \mathbf{1}_{\{\tau_1^\star \leq T_{\mathcal{Z}^\star}\}} = \tau_1^\star \mathbf{1}_{\{\tau_1^\star \leq T_{\mathcal{Z}^\star}\}}.$$

In view of this observation, (3.5.87) and (3.5.89) we can see that

$$\begin{aligned}
&\mathbb{E}\Bigg[\lambda \int_0^{T_{\mathcal{Z}^\star}} e^{-\delta t} X_t^2\,dt + e^{-\delta\tau_1^\star}\big(1+\kappa Z_1^{\star\alpha}\big)\mathbf{1}_{\{\tau_1^\star\le T_{\mathcal{Z}^\star}\}} + e^{-\delta T_{\mathcal{Z}^\star}} w_{\mathrm{ext}}(X_{T_{\mathcal{Z}^\star}+})\Bigg] \\
&\quad = \mathbb{E}\Bigg[\lambda \int_0^{\tau_1^\star\wedge T_{\mathcal{Z}^\star}} e^{-\delta t} X_t^2\,dt + e^{-\delta\tau_1^\star}\big(1+\kappa Z_1^{\star\alpha}\big)\mathbf{1}_{\{\tau_1^\star\le T_{\mathcal{Z}^\star}\}} + e^{-\delta(\tau_1^\star\wedge T_{\mathcal{Z}^\star})} w_{\mathrm{ext}}(X_{(\tau_1^\star\wedge T_{\mathcal{Z}^\star})+})\Bigg] \\
&\quad = \mathbb{E}\Bigg[\lambda \int_0^{\tau_1^\star\wedge T_{\mathcal{Z}^\star}} e^{-\delta t} \mathring{X}_t^2\,dt \\
&\qquad\qquad + e^{-\delta\tau_1^\star}\Big[w_{\mathrm{ext}}(\mathring{X}_{\tau_1^\star}-Z_1^\star)+1+\kappa Z_1^{\star\alpha}\Big]\mathbf{1}_{\{\tau_1^\star\le T_{\mathcal{Z}^\star}\}} + e^{-\delta T_{\mathcal{Z}^\star}} w_{\mathrm{ext}}(\mathring{X}_{T_{\mathcal{Z}^\star}})\mathbf{1}_{\{T_{\mathcal{Z}^\star}<\tau_1^\star\}}\Bigg] \\
&\quad = \mathbb{E}\Bigg[\lambda \int_0^{T_0^\star} e^{-\delta t} \mathring{X}_t^2\,dt \\
&\qquad\qquad + e^{-\delta T_0^\star}\Big[w_{\mathrm{ext}}(\mathring{X}_{T_0^\star}-Z_1^\star)+1+\kappa Z_1^{\star\alpha}\Big]\mathbf{1}_{\{\tau_1^\star\le T_{\mathcal{Z}^\star}\}} + e^{-\delta T_0^\star} w_{\mathrm{ext}}(\mathring{X}_{T_0^\star})\mathbf{1}_{\{T_{\mathcal{Z}^\star}<\tau_1^\star\}}\Bigg] \\
&\quad = \mathbb{E}\left[\lambda \int_0^{T_0^\star} e^{-\delta t} \mathring{X}_t^2\,dt + e^{-\delta T_0^\star}\bar{w}_0(\mathring{X}_{T_0^\star})\right] \\
&\quad = w_0(x).
\end{aligned} \tag{3.5.91}$$

Combining these identities with (3.5.90), we obtain (3.5.88) for $j = 0$.

To proceed further, we assume that (3.5.88) holds true for $j = k-1$, for some $k \ge 1$. An impulse control strategy $\mathcal{Z} \in \mathcal{A}_k$ involves a maximum of $k+1$ jumps. The evolution of the driving Brownian motion after time τ_1 at which the first jump occurs is independent of the evolution of the state process prior to time τ_1. Therefore, we may assume in what follows that τ_n and Z_n are measurable with respect to the information flow obtained by X_{τ_1+} and the process $\big((W_{\tau_1+t} - W_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}\big)$. In view of this observation

$$
\begin{aligned}
&\mathbb{E}\bigg[\lambda \int_0^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt + \sum_{n=1}^{k+1} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\alpha}\big)\mathbf{1}_{\{\tau_n \le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{\text{ext}}(X_{T_{\mathcal{Z}}+})\bigg] \\
&\quad = \mathbb{E}\bigg[\lambda \int_0^{\tau_1\wedge T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt + e^{-\delta\tau_1}\big(1+\kappa Z_1^{\alpha}\big)\mathbf{1}_{\{\tau_1 \le T_{\mathcal{Z}}\}} \\
&\qquad\quad + \mathbb{E}\bigg[\lambda \int_{\tau_1\wedge T_{\mathcal{Z}}}^{T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt \\
&\qquad\qquad + \sum_{n=2}^{k+1} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\alpha}\big)\mathbf{1}_{\{\tau_n \le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{\text{ext}}(X_{T_{\mathcal{Z}}+}) \,\bigg|\, \mathcal{F}_{\tau_1\wedge T_{\mathcal{Z}}}\bigg]\bigg] \\
&\quad \ge \mathbb{E}\bigg[\lambda \int_0^{\tau_1\wedge T_{\mathcal{Z}}} e^{-\delta t} X_t^2\, dt + e^{-\delta\tau_1}\big(1+\kappa Z_1^{\alpha}\big)\mathbf{1}_{\{\tau_1 \le T_{\mathcal{Z}}\}} + e^{-\delta(\tau_1\wedge T_{\mathcal{Z}})} w_{k-1}(X_{(\tau_1\wedge T_{\mathcal{Z}})+})\bigg] \\
&\quad = \mathbb{E}\bigg[\lambda \int_0^{\tau_1\wedge T_{\mathcal{Z}}} e^{-\delta t} \mathring{X}_t^2\, dt \\
&\qquad\quad + e^{-\delta\tau_1}\big[w_{k-1}(\mathring{X}_{\tau_1} - Z_1) + 1 + \kappa Z_1^{\alpha}\big]\mathbf{1}_{\{\tau_1 \le T_{\mathcal{Z}}\}} + e^{-\delta T_{\mathcal{Z}}} w_{k-1}(\mathring{X}_{T_{\mathcal{Z}}})\mathbf{1}_{\{T_{\mathcal{Z}} < \tau_1\}}\bigg] \\
&\quad \ge \inf_{\tau\in\mathcal{T}} \mathbb{E}\bigg[\lambda \int_0^{\tau} e^{-\delta t} \mathring{X}_t^2\, dt + e^{-\delta\tau} \bar{w}_k(\mathring{X}_\tau)\bigg] \\
&\quad = w_k(x).
\end{aligned} \tag{3.5.92}
$$

Combining the arguments we have used in (3.5.91) and (3.5.92), we can see that the strategy $\mathcal{Z}^\star \in \mathcal{A}_k$ given by

$$
\begin{aligned}
&\tau_n^\star = \inf\big\{t \ge 0 \;\big|\; w_{k+1-n}(\mathring{X}_t) = \bar{w}_{k+1-n}(\mathring{X}_t) \text{ and } \bar{w}_{k+1-n}(\mathring{X}_t) > w_{k-n}(X_t)\big\}, \\
&Z_n^\star = \mathfrak{z}_{k+1-n}(\mathring{X}_{\tau_n^\star}),
\end{aligned}
$$

for $n = 1, \ldots, k+1$, satisfies

$$
\mathbb{E}\bigg[\lambda \int_0^{T_{\mathcal{Z}^\star}} e^{-\delta t} X_t^2\, dt + \sum_{n=1}^{k+1} e^{-\delta\tau_n}\big(1+\kappa Z_n^{\star\alpha}\big)\mathbf{1}_{\{\tau_n \le T_{\mathcal{Z}^\star}\}} + e^{-\delta T_{\mathcal{Z}^\star}} w_{\text{ext}}(X_{T_{\mathcal{Z}^\star}+})\bigg] = w_k(x).
$$

This identity and (3.5.92) imply that (3.5.88) holds true for $j = k$. It follows that (3.5.88) is true for all $j \ge 0$.

To establish the fact that $\lim_{j\to\infty} w_j(x) = v(x)$ for all $x \in \mathbb{R}$ and complete the proof, we first note that $\lim_{j\to\infty} w_j(x)$ exists because the sequence $\big(w_j(x)\big)$ is decreasing. The inequality $\lim_{j\to\infty} w_j(x) \geq v(x)$ follows immediately from (3.5.88) and the fact that $\mathcal{A}_j \subseteq \mathcal{A}_{j+1}$ for all $j \geq 0$. To prove the reverse inequality, we consider any ε-optimal strategy

$$\mathcal{Z}^\varepsilon = (\tau_1^\varepsilon, \ldots, \tau_n^\varepsilon, \ldots; Z_1^\varepsilon, \ldots, Z_n^\varepsilon, \ldots) \in \mathcal{A},$$

namely, any strategy such that $J_x(\mathcal{Z}^\varepsilon) \leq v(x) + \varepsilon$. If we denote by $\mathcal{Z}^{\varepsilon,j} \in \mathcal{A}_j$ the strategy obtained by $\mathcal{Z}^\varepsilon$ by setting $\tau_{j+1+k}^\varepsilon = \infty$ for all $k \geq 1$, then

$$\begin{aligned} v(x) + \varepsilon &\geq \mathbb{E}\left[\lambda \int_0^\infty e^{-\delta t} (X_t^\varepsilon)^2 \, dt + \sum_{n=1}^\infty e^{-\delta \tau_n^\varepsilon} \big(1 + \kappa (Z_n^\varepsilon)^\alpha\big)\right] \\ &= J_x(\mathcal{Z}^{\varepsilon,j}) + \mathbb{E}\left[\lambda \int_{\tau_j^\varepsilon}^\infty e^{-\delta t} \big[(X_t^\varepsilon)^2 - (X_{\tau_j^\varepsilon} + W_t)^2\big] \, dt + \sum_{n=j+2}^\infty e^{-\delta \tau_n^\varepsilon} \big(1 + \kappa (Z_n^\varepsilon)^\alpha\big)\right]. \end{aligned}$$

Combining this observation with the limits

$$\begin{aligned} 0 &\leq \lim_{j\to\infty} \mathbb{E}\left[\lambda \int_{\tau_j^\varepsilon}^\infty e^{-\delta t} (X_t^\varepsilon)^2 \, dt + \sum_{n=j+2}^\infty e^{-\delta \tau_n^\varepsilon} \big(1 + \kappa (Z_n^\varepsilon)^\alpha\big)\right] \\ &\leq \lim_{j\to\infty} \mathbb{E}\left[\lambda \int_{\tau_j^\varepsilon}^\infty e^{-\delta t} (X_{\tau_j^\varepsilon} + W_t)^2 \, dt\right] = \lim_{j\to\infty} \mathbb{E}\left[e^{-\delta \tau_j^\varepsilon} \left(\frac{\lambda}{\delta} (X_{\tau_j^\varepsilon})^2 + \frac{\lambda}{\delta^2}\right)\right] = 0, \end{aligned}$$

which follow from the fact that $\lim_{j\to\infty} \tau_j^\varepsilon = \infty$, we can see that $\lim_{j\to\infty} w_j(x) \leq v(x) + \varepsilon$. It follows that $\lim_{j\to\infty} w_j(x) \leq v(x)$ because ε has been arbitrary.

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