Matchings and Covers of Multipartite Hypergraphs

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Abstract

König's theorem is a classic result in combinatorics which states that for every bipartite graph G, the cover number of $\mathcal{G}$ (denoted by $\tau(\mathcal{G})$) is equal to its matching number (denoted by $\nu(\mathcal{G})$). The theorem's importance stems from its many applications in various areas of mathematics, such as optimisation theory and algorithmic analysis.

Ryser's Conjecture for multipartite hypergraphs is a proposed generalisation of König's theorem made in the 1970s. It asserts that for every r-partite hypergraph $\mathcal{H}$, we have the following inequality: $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$. The conjecture is only known to be true for tripartite hypergraphs and a few other special cases.

In the first part of this thesis, we present algorithms that – for a given r – are able to prove or disprove the conjecture in the case of r-partite intersecting hypergraphs. Moreover, for a given r, the algorithms can also be used to enumerate all r-partite extremal hypergraphs to Ryser's Conjecture. Extremal hypergraphs are r-partite hypergraphs for which the cover number is exactly $r-1$ times the matching number.

The second part of this thesis focuses on the case of 4-partite hypergraphs. It is motivated by a recent result on Ryser's Conjecture for tripartite hypergraphs. The result classifies all tripartite extremal hypergraphs, and implies that if $\mathcal{H}$ is a tripartite extremal hypergraph, then it must contain $\nu(\mathcal{H})$ vertex-disjoint tripartite intersecting extremal hypergraphs.

This result leads to the natural question of whether a similar characterisation of r-partite extremal hypergraphs is possible for other values of r? In particular, for the first open case of Ryser's Conjecture, the case of $r = 4$.

We shed some light on this question, by first classifying all 4-partite intersecting extremal hypergraphs. We then present a list of 4-partite extremal hypergraphs with matching number equal to two, such that none of them contain two vertex-disjoint 4-partite extremal hypergraphs.

Our result shows that a straightforward characterisation of 4-partite extremal hypergraphs is not possible in terms of vertex-disjoint intersecting extremal hypergraphs. However, we still conjecture an analogue to the classification of tripartite extremal hypergraphs, which is not contradicted by our extremal examples.

In the final part of the thesis we focus on intersecting extremal hypergraphs to Ryser's Conjecture. Apart from a few sporadic constructions in the literature, there is only one known family of r-partite extremal hypergraphs, which comes from finite projective planes. The family contains an r-partite extremal hypergraph to Ryser's Conjecture, whenever a finite projective plane of order $r-1$ exists.

Our contribution is to first calculates bounds on the sparsest possible extremal hypergraphs for small values of r. We then prove the existence of a new family of extremal hypergraphs to Ryser's Conjecture.

The new family contains an r-partite intersecting extremal hypergraph to Ryser's Conjecture, whenever a finite projective plane of order $r-2$ exists. Moreover we are able to show via a number theoretic argument, that there are infinitely many cases for which our new family contains an extremal hypergraph, when the currently known family of extremal hypergraphs is known not to contain one.

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work, with the following exceptions:

- Chapters 2, 6, and 7 are joint work with Alexey Pokrovskiy and are the subject of a working paper.
- Chapter 8 is joint work with Janos Barát, Alexey Pokrovskiy and Tibor Szabó and is the subject of a working paper.

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CHAPTER 1
INTRODUCTION

"The great importance of the covering problem is supported by the fact that apparently all *combinatorial problems can be reformulated as the determination of the covering number of a certain hypergraph."*

—Z. Füredi *[11]*

The matching number and the cover number of a graph are two of the most well-studied parameters in combinatorics. This is due not only to their theoretical utility, but also because it is often the case that when graph theory is used to solve practical problems – such as analysing network flows in operations research or optimising donated kidney exchanges in economics [21]– the two parameters turn out to model a significant aspect in the context of the problem being solved.

Formally, a *graph* $G = (V, E)$ is given by a set V of vertices, and a set $E \subset \binom{V}{2}$ of edges. A *matching* of a graph G, is a set of pairwise disjoint edges of G. The *matching number* of a graph G, denoted by $\nu(G)$ is the cardinality of a largest matching of G. On the other hand a *cover* of C a graph G is a subset of V such that every edge of G contains one of the vertices in C. The *cover number* of a graph G, denoted by $\tau(G)$ is the cardinality of a smallest possible cover of G.

We call a graph G *bipartite* if its vertex set V can be divided into two disjoint sets A and B such that every edge of G contains one vertex from A and one vertex from B. A classical result in graph theory asserts the equivalence of the matching number and the cover number in the case of bipartite graphs.

König's Theorem. *If G is a bipartite graph, then:*

$$\tau(G) = \nu(G)$$

König's Theorem, proved by Dénes Kőnig [16], has a rich history and is equivalent to numerous other min-max results in combinatorics. Moreover it has been generalised in many different ways (including a generalisation to non-bipartite graphs [23], and a generalisation for infinite bipartite graphs [5]). We refer the reader to the first chapter of [18] for more on the history of this theorem and its interesting variants.

In the 1970's Ryser [22] proposed a generalisation of König's Theorem for multipartite hypergraphs. A *hypergraph* $\mathcal{H}$ consists of a vertex set $V(\mathcal{H})$ and a set $E(\mathcal{H})$ of edges where each edge is a non-empty subset of $V(\mathcal{H})$. We say that $\mathcal{H}$ is r*-uniform* if every edge of $\mathcal{H}$ has cardinality r, and we say that it is r*-partite* if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$, called the *sides*, such that each edge has exactly one vertex from each side.

For an r-partite hypergraph $\mathcal{H}$, we similarly define the matching number $\nu(\mathcal{H})$ of $\mathcal{H}$ to be the size of a largest matching in $\mathcal{H}$, where a matching of $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$. The covering number $\tau(\mathcal{H})$ of $\mathcal{H}$ is the size of a smallest cover of $\mathcal{H}$, where a cover of $\mathcal{H}$ is a subset $W \subset V(\mathcal{H})$ such that every edge of $\mathcal{H}$ contains a vertex of W.

Ryser's Conjecture. *If $\mathcal{H}$ is an r-partite hypergraph then:*

$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$$

The case $r = 2$ in Ryser's Conjecture gives König's Theorem. Apart from this case, the only other general case of that conjecture which has been proved is the case $r = 3$, which has been proved by Aharoni [3] via an elegant topological argument. It has also been shown by Haxell and Scott [14] that there exists $\epsilon > 0$, such that for every 4-partite and 5-partite hypergraphs the inequality $\tau(\mathcal{H}) \leq (r - \epsilon)\nu(\mathcal{H})$ holds.

An important special case of the conjecture is when the hypergraph $\mathcal{H}$ is intersecting. An *intersecting* r-partite hypergraph is one in which every two edges contain at least one vertex in common, or equivalently, its matching number is equal to one.

Intersecting Ryser's Conjecture. *If $\mathcal{H}$ is an intersecting r-partite hypergraph, then:*

$$\tau(\mathcal{H}) \leq (r-1)$$

A bit more is known about the intersecting case of Ryser's Conjecture. It has been proven for $r \leq 5$ by Tuza [24, 25] and has recently been shown to be true for $r \leq 9$ for all intersecting linear hypergraphs by Francetić, Herke, McKay and Wanless [10], where a hypergraph is called *linear* if every pair of its edges contain at most one vertex in common.

1.1 Intersecting extremals

Aside from trying to prove the conjecture, there has also been considerable effort expended recently on trying to understand which hypergraphs are extremal to Ryser's Conjecture. An r-partite hypergraph $\mathcal{H}$ is extremal to Ryser's Conjecture if it satisfies $\tau(\mathcal{H}) = (r-1)\nu(\mathcal{H})$. For the rest of this thesis we shall use the term *extremal hypergraph* to refer exclusively to such hypergraphs, unless otherwise stated.

Studying extremal structure is an important part of graph theory in its own right. In the context of researching a conjecture, understanding extremal cases is particularly important, because if the conjecture is false it provides important insights on how counter-examples to the conjecture can potentially be constructed. On the other hand, if the conjecture is true, then studying extremal cases can provide useful hints on difficult cases that a proof of the conjecture will need to account for.

An example of a tripartite intersecting extremal hypergraph is presented in the following figure. For the rest of the thesis we present r-partite hypergraphs pictorially by vertically aligning vertices in the same sides, and using different colours for edges.

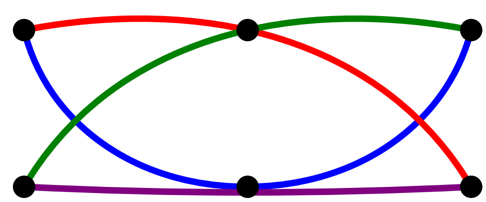

Figure 1.1: A tripartite intersecting extremal hypergraph.

Only one known infinite family of extremal cases to Ryser's Conjecture is known, which come from finite projective planes. We recall that a *finite projective plane of order* n is a collection of $n^2 + n + 1$ lines and $n^2 + n + 1$ points that satisfies the following axioms:

1. Every line contains $n + 1$ points,
2. Every point is on $n + 1$ lines,
3. Any two distinct lines intersect at exactly one point, and
4. Any two distinct points lie on exactly one line.

We will use $\mathcal{P}_r$ to denote a projective plane of order $r - 1$. It follows from the above axioms that $\mathcal{P}_r$ can be thought of as an r-uniform hypergraph, where the lines correspond to edges, and the points correspond to vertices, so that $\mathcal{P}_r$ is also r-regular and contains $r^2 - r + 1$ edges.

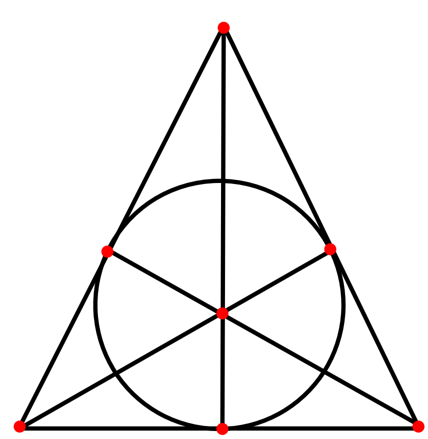

Figure 1.2: A finite projective plane of order 2.

For our purpose we can turn any finite projective plane $\mathcal{P}_r$ to an r-partite hypergraph by a simple transformation to get what is known as *the truncated projective plane*. Denoted by $\mathcal{T}_r$, the truncated projective plane of uniformity r is obtained from $\mathcal{P}_r$ by the removal of a single vertex v of $\mathcal{P}_r$ and the lines containing v. The sides $V_1, \ldots, V_r$ of $\mathcal{T}_r$ are the sets of vertices other than v on the lines containing v. This makes $\mathcal{T}_r$ an r-partite intersecting hypergraph that is also $(r-1)$-regular. Since $\mathcal{T}_r$ also contains $r^2 - r$ vertices, it follows that it requires at least $r-1$ vertices to cover it, and since each side contains $r-1$ vertices we see that $\tau(\mathcal{T}_r) = r-1$, making it also an extremal hypergraph.

As an example, Figure 1.3 turns the finite projective plane of order 2 presented in Figure 1.2, into the tripartite extremal hypergraph presented in Figure 1.1.

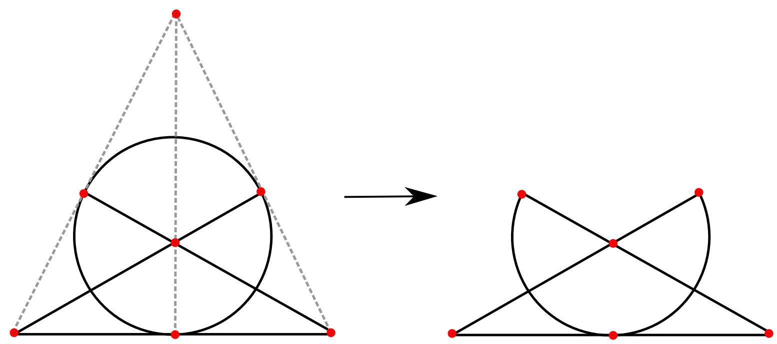

Figure 1.3: Constructing a truncated projective plane.

Thus the truncated projective plane construction provides us with a recipe that allows us to obtain an r-partite extremal hypergraph to Ryser's Conjecture whenever a finite projective plane of order $r-1$ exists. Finite projective planes are only known to exist for orders that are prime powers and a long-standing open problem asks whether there exists any other q for which a finite projective plane of order q exists. However, a few non-existence results are known about projective planes, the main one having been proved by Bruck and Ryser [7].

Bruck-Ryser Theorem. *If a finite projective plane of order q exists and $q \equiv 1$ or 2 (mod 4), then q must be the sum of two squares.*

Apart from the Bruck-Ryser Theorem, the only other non-existence result concerning finite projective planes is due to Lam, Thiel and Swiercz [17] who proved the non-existence of the plane of order 10, famously by the help of a massive computer search. Hence it follows that the first two values of r, for which r-partite truncated extremal hypergraphs are ruled out, are the values $r = 7$ and $r = 11$.

Motivated by the lack of examples attaining Ryser's bound for these values, Aharoni, Barát and Wanless [4] constructed a 7-partite intersecting hypergraph with covering number 6. This was also obtained independently by the author and Pokrovskiy [2], who also constructed an 11-partite intersecting hypergraph with covering number 10. Finally, even though the existence of a finite projective plane of order 12 is still open, a sporadic 13-partite intersecting extremal hypergraph was constructed by Francetić, Herke, McKay, and Wanless [10].

Thus we notice that one of the problems with the truncated projective plane construction is that for infinitely many values it doesn't provide an extremal hypergraph to Ryser's Conjecture, such as the cases that are ruled out by the Bruck-Ryser Theorem.

Another problem with the truncated projective plane construction, is that the truncated extremal hypergraph seems to have many more edges than necessary. This was observed by Mansour, Song and Yuster [19] who defined $f(r)$ as the minimum integer so that there exists an r-partite extremal intersecting hypergraph with $f(r)$ edges. They showed that $f(3) = 3$, $f(4) = 6$, $f(5) = 9$ and $12 \leq f(6) \leq 15$.

Subsequently it was shown by Aharoni, Barát and Wanless [4] that $f(6) = 13$ and $f(7) = 17$. The result $f(6) = 13$ was also proved independently by the author and Pokrovskiy who also showed that $f(7) \leq 22$ and $f(11) \leq 51$ (these results originally appeared in [2], and we present them again in Chapter 7 of this thesis).

1.2 Non-intersecting extremals

It can be seen from the brief survey in the previous section, that in general extremal intersecting hypergraphs to Ryser's Conjecture are still not very well understood, and our knowledge about the properties of these hypergraphs is still very patchy. However, this still compares favourably to what is known about non-intersecting extremal hypergraphs, where knowledge about these hypergraphs in the literature is even more limited.

An exception is the case of tripartite hypergraphs. This case was settled recently by a result of Haxell, Narins and Szabó [13] that characterises all tripartite extremal hypergraphs.

Tripartite Extremals. *Every tripartite extremal hypergraph to Ryser's Conjecture, $\mathcal{H}$, contains $\nu(\mathcal{H})$ vertex-disjoint copies of the unique tripartite intersecting edge-minimal extremal hypergraph.*

An intersecting r-partite hypergraph $\mathcal{H}$ is an edge-minimal extremal hypergraph if $\tau(\mathcal{H}) = r - 1$ and such that for every edge $h \in E(\mathcal{H})$, we have that $\tau(\mathcal{H} - h) = r - 1$.

For $r = 3$, it is easy to see that there is only one unique intersecting edge-minimal tripartite extremal hypergraph, which is obtained by removing any one edge from the truncated extremal hypergraph.

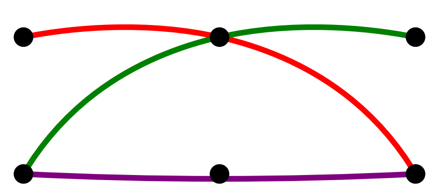

Figure 1.4: The unique tripartite edge-minimal intersecting extremal hypergraph.

Currently the only known infinite family of r-partite extremal non-intersecting hypergraphs with matching number K for a given $K > 1$, is constructed trivially by taking K copies of r-partite extremal intersecting hypergraphs. Thus the characterisation theorem of tripartite extremal hypergraphs asserts that every tripartite extremal hypergraph is essentially a member of the trivial family of extremal hypergraphs.

The successful characterisation of all tripartite extremal hypergraphs raises the natural question if a similar characterisation of r-partite hypergraphs is possible for other values of r, in particular the first open case, the case $r = 4$.

It is easy to see that a necessary first step towards answering the above question, would be to enumerate and classify all 4-partite edge-minimal extremal intersecting hypergraphs.

1.3 The Sunflower Lemma

One of the important tools used in this thesis is a celebrated result in extremal set theory proved by Erdős and Rado, known as the Sunflower Lemma [9]. Even though the original Sunflower Lemma is phrased in terms of sets and families of sets, for our purpose we will rephrase it in terms of edges and r-partite hypergraphs.

An r-partite *sunflower* with a *core* $\mathcal{C}$ and k *petals* is an r-partite hypergraph $\mathcal{S}$ with k edges such that $h_i \cap h_j = \mathcal{C}$ for all $h \in E(\mathcal{S})$ with $i \neq j$. The sets $h_i \setminus C$ are petals, and we require that none of them is empty.

Sunflower Lemma. *Let $\mathcal{H}$ be an r-partite hypergraph. If $|\mathcal{H}| > r!(k-1)^r$ then $\mathcal{H}$ contains a sunflower with k petals.*

The importance of the Sunflower Lemma to Ryser's Conjecture is that it implies that one can verify if the conjecture is correct for r-partite intersecting hypergraphs for a given r by checking only a finite number of hypergraphs, an assertion that we will prove concretely in the first part of this thesis.

This also implies that one can potentially prove or disprove the intersecting case of the conjecture for a given r using computer search. The lemma also has other implications, such as for a given r there is essentially only a finite number of edge-minimal r-partite intersecting extremal hypergraphs, and other facts which we will also prove in this thesis.

1.4 Contributions of this thesis

Having surveyed what is currently known about related results to our work, we now outline new results that are presented in this thesis.

In the first part of the thesis we utilise the Sunflower Lemma to present an algorithm that for a given r, is able to prove or disprove the conjecture for the case of r-partite intersecting hypergraphs. Roughly speaking the algorithm works by traversing a search tree of r-partite intersecting hypergraphs, and has two possible outputs: either a sequence of hypergraphs that encodes a proof of the intersecting version of the conjecture for the given value of r (if it is true), or else it outputs a counter-example to the conjecture for the given value of r.

In the second part of the thesis we focus on 4-partite extremal hypergraphs. Even though the most interesting use of the aforementioned algorithms is to find counter-examples to Ryser's Conjecture, a variation of the algorithms we present can also be used to enumerate – for a given r – all possible r-partite edge-minimal extremal hypergraphs. This enables us to present a short computer-generated proof that there exist (up to isomorphism) only three 4-partite edge-minimal extremal hypergraphs.

We then present a list of 4-partite extremal hypergraphs with matching number two, that do not contain vertex-disjoint copies of edge-minimal 4-partite extremal hypergraphs. These examples are sufficient to show that – unlike the tripartite case – it is not possible to characterise 4-partite extremal hypergraphs in terms of vertex-disjoint edge-minimal 4-partite extremal hypergraphs. However, we conjecture an analogue of the tripartite characterisation theorem, but for the 4-partite case and which involves a notion of vertex-minimality.

In the final part of the thesis we deal with extremal intersecting hypergraphs. First we present computed bounds on the sparsest possible extremal hypergraph for some low values of r, and then we present a new family of extremal hypergraphs.

The new family of extremal hypergraphs contains an r-partite extremal hypergraph whenever $r = q + 2$, for q a prime power. Moreover, we are able to show – using the Bruck-Ryser Theorem and a number theoretic argument – that for infinitely many values of r, the new family contains an r-partite extremal hypergraph when an r-partite truncated projective plane extremal hypergraph is known not to exist.

1.5 Summary

Knuth [15] defined combinatorics as "…the study of the ways in which discrete objects can be arranged into various kinds of patterns." Adding that "Five basic types of questions typically arise when combinatorial problems are studied, some more difficult than others:

i) Existence: Are there any arrangements X that conform to the pattern?

ii) Construction: If so, can such an X be found quickly?

iii) Enumeration: How many different arrangements X exist?

iv) Generation: Can all arrangements $X_1, X_2, \ldots$ be visited systematically?

v) Optimisation: What arrangement maximise or minimise $f(X)$, given an objective function f."

Since this is a thesis on combinatorics, we find it apt to end this introduction by summarising its content using Knuth's classification of combinatorial questions.

Questions of existence underpin all parts of this thesis, since proving the existence of extremal hypergraphs (and family of extremal hypergraphs) with certain properties is one of the main themes of this thesis.

Questions of construction and generation are the motivation behind presenting the algorithms in the first part of the thesis. These algorithms are designed to construct counter-examples and also systematically visit all arrangements of hypergraphs that are extremal.

Questions on enumeration are the subject of the second part of this thesis, that part deals with enumerating all edge-minimal 4-partite extremal intersecting hypergraphs.

Finally, questions on how sparse an extremal hypergraph can be, are dealt with in the third part of the thesis. These can be seen as combinatorial questions on optimisation, where the objective is to minimise the number of edges an extremal hypergraph can contain.

Part I: Algorithms for Proving Ryser's Conjecture

CHAPTER 2

RYSER-STABILITY & OTHER DEFINITIONS

In this chapter we introduce definitions and notations related to coverings of r-partite intersecting hypergraphs, which we use in the rest of the thesis. In particular, the definitions we introduce in this chapter make it easier to discuss and prove the correctness of the algorithms we present in the next chapter.

2.1 Ryser-stability

Let $\mathcal{H}$ be an r-partite intersecting hypergraph, if there exists a cover $\mathcal{C}$ of $\mathcal{H}$ such that $|\mathcal{C}| \leq c$, and such that every r-partite intersecting hypergraph that contains $\mathcal{H}$ is also covered by $\mathcal{C}$, then we call the pair $(\mathcal{H}, \mathcal{C})$ a *c-Ryser-stable pair*, and $\mathcal{H}$ a *c-Ryser-stable hypergraph.*

Thus if a hypergraph is a c-Ryser-stable hypergraph, then every intersecting hypergraph that contains it is also a c-Ryser-stable hypergraph. In particular, an intersecting counter-example to Ryser's Conjecture can't contain an $(r-1)$-Ryser-stable hypergraph.

A natural class of Ryser-stable hypergraphs comes from sunflowers as shown in the following lemma, where we make use of the notation $SF(r, m, n)$ to denote an r-partite sunflower with a core of size m and n petals.

Lemma 2.1. *For all r, and $c < r$, sunflowers of the form $SF(r, c, r-c+1)$ are c-Ryser-stable.*

Proof. For a given r and $c < r$, let $\mathcal{C}$ be the set of c vertices that form the core of the sunflower $SF(r, c, r-c+1)$. Let $\mathcal{H}'$ be an r-partite intersecting hypergraph that contains $SF(r, c, r-c+1)$, we claim that $\mathcal{C}$ covers $\mathcal{H}'$. Otherwise, assume $\mathcal{C}$ doesn't cover $\mathcal{H}'$, then there must exist an edge $h' \in E(\mathcal{H}')$, such that h' intersects each edge of $SF(r, c, r-c+1)$ but is not covered by $\mathcal{C}$. Thus h' must intersect the $r-c+1$ petals of $SF(r, c, r-c+1)$ in $r-c$ sides, and since petals are disjoint, this leads to a contradiction.

Therefore we conclude that $\mathcal{C}$ must cover every r-partite intersecting hypergraph $\mathcal{H}'$ that contains $SF(r, c, r - c + 1)$, which makes $(SF(r, c, r - c + 1), \mathcal{C})$ a c-Ryser-stable pair and $SF(r, c, r - c + 1)$ a c-Ryser-stable hypergraph.

□

Let $\mathcal{H}$ be an r-partite intersecting hypergraph such that $\tau(\mathcal{H}) = c$, then we call $\mathcal{H}$ a *c-cover edge-minimal* hypergraph, if for every edge $h \in \mathcal{H}$ we have that $\tau(\mathcal{H} - h) = c - 1$. In particular, a counter-example to Ryser's Conjecture for intersecting r-partite hypergraphs must contain an r-cover edge-minimal hypergraph, and an extremal hypergraph to Ryser's Conjecture for intersecting hypergraphs must contain an $(r - 1)$-cover edge-minimal hypergraph.

Given r and c, we will define $f(r, c)$ to be the smallest integer such that all r-partite intersecting c-cover edge-minimal hypergraphs have fewer than $f(r, c)$ edges. Using the Sunflower Lemma we will obtain a crude bound on $f(r, c)$, which also shows that it is finite for all combinations of r and c.

We define the set of sunflowers $\mathcal{U}(r, c)$ to be the set of $r - 1$ hypergraphs with $c + 1$ petals and cores of sizes $1, \ldots, r-1$. In other words, $\mathcal{U}(r, c) = \{SF(r, 1, c+1), \ldots, SF(r, r-1, c+1)\}$.

Lemma 2.2. *If $\mathcal{H}$ is an r-partite intersecting hypergraph that contains any of the hypergraphs in $\mathcal{U}(r, c)$, then $\mathcal{H}$ cannot be a c-cover edge-minimal hypergraph.*

Proof. Let $\mathcal{H}$ be an r-partite intersecting hypergraph that contain a subhypergraph $\mathcal{H}'$, which is isomorphic to a sunflower in $\mathcal{U}(r, c)$, so that $\mathcal{H}'$ has $c + 1$ petals. We will assume that $\mathcal{H}$ is a c-cover edge-minimal hypergraph and show that this leads to a contradiction.

Fix an edge $h' \in \mathcal{H}'$. Since $\mathcal{H}$ is a c-cover edge-minimal hypergraph, then $\tau(\mathcal{H} - h') = c - 1$. Moreover any cover of size $c - 1$ of $\mathcal{H} - h'$ cannot intersect the core of $\mathcal{H}'$, otherwise this cover will also cover the whole of $\mathcal{H}$ contradicting $\tau(\mathcal{H}) = c$. Thus any cover of size $c - 1$ of $\mathcal{H} - h'$ must cover the c petals of $\mathcal{H}' - h'$ using only $c - 1$ vertices, and since petals are disjoint this gives the required contradiction.

□

Lemma 2.3. $f(r, c) \leq r!c^r$

Proof. Assume that $\mathcal{H}$ is an intersecting r-partite, c-cover edge-minimal hypergraph with more than $r!c^r$ edges. It follows from the Sunflower Lemma that $\mathcal{H}$ contains a sunflower with $c+1$ petals, therefore it contains one of the sunflower from the set $\mathcal{U}(r, c)$. Thus by Lemma 2.2 it can't be a c-cover, edge-minimal hypergraph.

□

Corollary 2.4. *Given r and c, there exists a finite number (up to isomorphism) of r-partite intersecting c-cover edge-minimal hypergraphs.*

An important special case of Corollary 2.4 is the case when c is equal to r, as this implies that there are a finite number of edge-minimal counter-examples to Ryser's Conjecture for r-partite intersecting hypergraphs.

Corollary 2.5. *If every r-partite intersecting hypergraph with less than $r!r^r$ edges has a cover of size less than r, then Ryser's Conjecture is true for r-partite intersecting hypergraphs.*

It follows from Corollary 2.5 that a naive algorithm to prove – for a given r – Ryser's Conjecture for r-partite intersecting hypergraphs, would be to enumerate all r-partite intersecting hypergraphs with less than $r!r^r$ edges, and verify that all of them can be covered by $r-1$ vertices.

A similar approach could be used to enumerate (again for a given r) all r-partite extremal hypergraphs. While these approaches are correct, they are also clearly not practical even for small values of r. This motivates a generalisation of the notion of a Ryser-stable hypergraph, which can potentially lead to a more practical algorithm.

2.2 Ryser-stable sequences

Let $\mathcal{H}$ be an r-partite intersecting hypergraph and $\mathcal{C}$ be a cover of $\mathcal{H}$ with $|\mathcal{C}| \leq c$, and let $\mathcal{X}$ be a (possibly empty) set of r-partite intersecting hypergraphs. We say that $(\mathcal{H}, \mathcal{C})$ is a *c-Ryser-stable pair relative to $\mathcal{X}$*, if every r-partite intersecting hypergraph $\mathcal{H}'$ that contains $\mathcal{H}$ is either covered by $\mathcal{C}$, or contains a subhypergraph that is isomorphic to a hypergraph from the set $\mathcal{X}$.

We note that when $(\mathcal{H}, \mathcal{C})$ is c-Ryser-stable pair relative to $\emptyset$, then this is the same as $(\mathcal{H}, \mathcal{C})$ being a c-Ryser-stable pair.

For a given r, let S be a sequence of pairs $(\mathcal{H}_1, \mathcal{C}_1), \ldots, (\mathcal{H}_n, \mathcal{C}_n)$, such that each $\mathcal{H}_i$ is an r-partite intersecting hypergraph, and each corresponding $\mathcal{C}_i$ is a cover of $\mathcal{H}_i$ with $|\mathcal{C}_i| \leq c$, and let $\mathcal{X}$ be a set of r-partite hypergraphs. Then we call S a *c-Ryser-stable sequence of $\mathcal{H}_1$ relative* to $\mathcal{X}$, if every pair in the sequence $(\mathcal{H}_i, \mathcal{C}_i)$ is a c-Ryser-stable pair relative to the set $\{\mathcal{H}_{i+1}, \ldots, \mathcal{H}_n\} \cup \mathcal{X}$. When $\mathcal{X}$ is empty we sometimes omit the relative part in the term and simply say that S is a c-Ryser-stable sequence of $\mathcal{H}_1$.

We note the following properties of Ryser-stable sequences. First, if $S = (\mathcal{H}_1, \mathcal{C}_1), \ldots, (\mathcal{H}_n, \mathcal{C}_n)$ is a c-Ryser-stable relative to a set $\mathcal{X}$, then for all m less than or equal to n, the subsequence $S' = (\mathcal{H}_m, \mathcal{C}_m), \ldots, (\mathcal{H}_n, \mathcal{C}_n)$ is also a c-Ryser-stable sequence relative to $\mathcal{X}$.

Another property of c-Ryser-stable sequences, is that they are closed under concatenation relative to the same set $\mathcal{X}$. More formally, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two r-partite intersecting hypergraphs, and let S_1 be a c-Ryser-stable sequence of $\mathcal{H}_1$ relative to a set $\mathcal{X}$, and let S_2 be a c-Ryser-stable sequence of $\mathcal{H}_2$ relative to the same set $\mathcal{X}$. Then if S_1 and S_2 are concatenated such that the sequence S_2 follows the sequence S_1, the resultant sequence is also a c-Ryser-stable sequence relative to $\mathcal{X}$.

Lemma 2.6. *For a given r and c, let $\mathcal{H}$ be an intersecting r-partite hypergraph, and $\mathcal{X}$ a set of r-partite intersecting hypergraphs.*
If there exists a sequence S, such that S is a c-Ryser-stable sequence of $\mathcal{H}$ relative to $\mathcal{X}$, then it follows that every r-partite intersecting hypergraph that contains $\mathcal{H}$ can be covered with c vertices, or contains one of the hypergraphs in $\mathcal{X}$.

Proof. Let $r, c, \mathcal{H}, \mathcal{X}$ and S be as in the statement of the lemma and let $\mathcal{H}'$ be an r-partite intersecting hypergraph that contains $\mathcal{H}$. We will prove the lemma by induction on the length of the sequence S.

For the base case we will assume that S consist of only one pair $(\mathcal{H}, \mathcal{C})$, in which case this pair is c-Ryser-stable relative to $\mathcal{X}$. It therefore follows that either $\mathcal{H}'$ is covered by $\mathcal{C}$ or contains one of the hypergraphs in $\mathcal{X}$, proving the lemma for this case.

For the induction step, assume that the statement is true for all sequences of length K, and assume that S has $K+1$ elements, so that $S = (\mathcal{H}_1, \mathcal{C}_1), \ldots, (\mathcal{H}_{K+1}, \mathcal{C}_{K+1})$ where $\mathcal{H}' = \mathcal{H}_1$.

In this case it follows from the definition of a Ryser-stable sequence that either $\mathcal{H}'$ is covered by $\mathcal{C}_1$, or it contains one of the hypergraphs in $\mathcal{X} \cup \{\mathcal{H}_2, \ldots, \mathcal{H}_{k+1}\}$. If it is covered by $\mathcal{C}_1$ or contains any of the hypergraphs in $\mathcal{X}$, then we are done. If not, then let m be the smallest integer less than or equal to $K+1$, such that $\mathcal{H}'$ contains $\mathcal{H}_m$.

Let S' be the c-Ryser-stable sequence $S' = (\mathcal{H}_m, \mathcal{C}_m), \ldots (\mathcal{H}_{K+1}, \mathcal{C}_{K+1})$ relative to $\mathcal{X}$. Now S' is a c-Ryser-stable sequence of $\mathcal{H}_m$ relative to $\mathcal{X}$. Moreover, since the length of S' is equal to $K+1-m \leq K$, it follows from the induction hypothesis that since $\mathcal{H}'$ contains $\mathcal{H}_m$ it must be covered by c vertices or contains one of the hypergraphs in $\mathcal{X}$. This allows us to complete the induction step and the proof of the lemma. $\square$

An important special case of Lemma 2.6 is stated in the following corollary.

Corollary 2.7. *For a given r, let $\mathcal{E}$ be the r-partite hypergraph that has only one edge. Then if there exists an $(r-1)$-Ryser-stable sequence of $\mathcal{E}$, it follows that Ryser's Conjecture is true for r-partite intersecting hypergraphs.*

The algorithm we present in the next chapter computes a Ryser-stable sequence for a given hypergraph. In particular, this algorithm can be used to compute – for a given r – a Ryser-stable sequence of the r-partite hypergraph consisting of one edge, from which it follows by Corollary 2.7, that Ryser's Conjecture is true for r-partite intersecting hypergraphs.

In Chapter 3, we will present $(r-1)$-Ryser-stable sequences that are generated by the aforementioned algorithm, and hence that prove Ryser's Conjecture for intersecting tripartite and 4-partite hypergraphs computationally. However, in this section we will present another illustration of using Ryser-stable sequences, by proving an extension of Lemma 2.1.

In Lemma 2.1 it was implicitly shown that for a given r, the sunflower $SF(r, 1, r)$ cannot be contained in a counter-example to Ryser's Conjecture because it is $(r-1)$-Ryser-stable. In the following lemma, we will show that the sunflower $SF(r, 1, r-1)$ cannot be contained in a counter-example to Ryser's Conjecture, because it has an $(r-1)$-Ryser-stable sequence.

Lemma 2.8. *For a given r, $SF(r,1,r-1)$ cannot be contained in a counter-example to Ryser's Conjecture for r-partite intersecting hypergraphs.*

Proof. For a given r, we will define the set of r-partite hypergraphs $SF^+(r,1,r-1)$ to be a smallest set that contains (up to isomorphism) all r-partite hypergraphs $\mathcal{H}$ that satisfy the following two properties. First, $\mathcal{H}$ contains $SF(r,1,r-1)$ as a subhypergraph $\mathcal{H}'$. Second, $\mathcal{H}$ also contain one extra edge h, such that h contains the core of $\mathcal{H}'$, and at least one other vertex that is also not contained in any of the petals of $\mathcal{H}'$.

Claim 2.9. *For a given r, every r-partite hypergraph in $SF^+(r,1,r-1)$ is $(r-1)$-Ryser-stable.*

Proof. For a given r, let $\mathcal{H}$ be an r-partite hypergraph in $SF^+(r,1,r-1)$. Then it follows from the definition that $\mathcal{H}$ contains the sunflower $SF(r,1,r-1)$ and an edge h. We know that h contains the core of the sunflower, which we will identify as the vertex v. We also know that h contains at least one other vertex not contained in the petals of the sunflower, which we will identify as the vertex w.

We will now show that $(\mathcal{H}, h-w)$ is an $(r-1)$-Ryser-stable pair. It is clear that $h-w$ covers $\mathcal{H}$ since it contains v. Now assume that $\mathcal{H}'$ is an intersecting hypergraph that contains $\mathcal{H}$, and at least one edge h' that is not covered by $h-w$.

If $h-w$ doesn't cover h', then since $\mathcal{H}'$ is intersecting this implies that h' must contain w to intersect with h. Moreover, since it doesn't contain v (otherwise it will be covered by $h-w$), this implies it will have to intersect the remaining $r-1$ edges of $\mathcal{H}$ in $r-2$ sides. However, since these edges are disjoint in these sides, this leads to a contradiction. Therefore $(\mathcal{H}, h-w)$ covers all r-partite intersecting hypergraphs that contain $\mathcal{H}$, which proves the Ryser-stability of hypergraphs in $SF^+(r,1,r-1)$.

□

Claim 2.10. *For a given r, there exists a cover $\mathcal{C}$ of $SF(r,1,r-1)$, such that the pair $\big(SF(r,1,r-1),\mathcal{C}\big)$ is $(r-1)$-Ryser-stable relative to the set $SF^+(r,1,r-1)$.*

Proof. Without loss of generality, assume that the sides of $SF(r,1,r-1)$ are $V_1,\dots,V_r$ and such that the core of $SF(r,1,r-1)$ is in V_1.

Set $\mathcal{C}=V_2$. We claim that $(SF(r,1,r-1),\mathcal{C})$ is $(r-1)$-Ryser-stable relative to the set of hypergraphs in $SF^+(r,1,r-1)$. Note that $\mathcal{C}$ clearly covers $SF(r,1,r-1)$ and contains $r-1$ vertices.

Now assume that $\mathcal{H}'$ is an r-partite intersecting hypergraph that contains $SF(r,1,r-1)$ but is not covered by $\mathcal{C}$. Thus $\mathcal{H}'$ must contain at least one edge h' that intersects $SF(r,1,r-1)$, but doesn't intersect any of the edges in the second side of $\mathcal{H}'$. Thus we see that h' will contain the core of $SF(r,1,r-1)$, and at least one new vertex in the second side of $SF(r,1,r-1)$. This implies that the subhypergraph $SF(r,1,r-1)+h'$ of $\mathcal{H}'$, is isomorphic to one of the hypergraphs in $SF^+(r,1,r-1)$.

Therefore we conclude that any $\mathcal{H}'$ that contains $SF(r,1,r-1)$, is either covered by $\mathcal{C}$, or contains one of the hypergraphs in $SF^+(r,1,r-1)$. In other words, $\big(SF(r,1,r-1),\mathcal{C}\big)$ is $(r-1)$-Ryser-stable relative to $SF^+(r,1,r-1)$. □

Claim 2.11. *For a given r, there exists an $(r-1)$-Ryser-stable sequence for $SF(r,1,r-1)$.*

Proof. For a given r, let $\mathcal{C}$ be a cover of $SF(r,1,r-1)$ such that $\big(SF(r,1,r-1),\mathcal{C}\big)$ is $(r-1)$-Ryser-stable relative to $SF^+(r,1,r-1)$. From Claim 2.10 we know that such a cover exists.

Now let $SF^+(r,1,r-1) = \{\mathcal{H}_1,\ldots,\mathcal{H}_m\}$. For every $\mathcal{H}_i \in SF^+(r,1,r-1)$, let $\mathcal{C}_i$ be a cover of $\mathcal{H}_i$ such that $(\mathcal{H}_i,\mathcal{C}_i)$ are a $(r-1)$-Ryser-stable pair. We know from Claim 2.9 that such a pairing must exists for every $\mathcal{H}_i \in SF^+(r,1,r-1)$.

We will define the sequence S to be the sequence of $m+1$ pairs:

$$\big(SF(r,1,r-1),\mathcal{C}\big),\big(\mathcal{H}_1,\mathcal{C}_1\big),\ldots,\big(\mathcal{H}_m,\mathcal{C}_m\big)$$

We will show that S is a Ryser-stable sequence of $SF(r,1,r-1)$.

The last m pairs $(\mathcal{H}_i,\mathcal{C}_i)$ of S are $(r-1)$-Ryser-stable pairs, so are trivially $(r-1)$-Ryser-stable pair relative to each other. Moreover, we know that the first pair $(SF(r,1,r-1),\mathcal{C})$ is an $(r-1)$-Ryser-stable pair relative to the hypergraphs in the rest of S. This shows that each pair in S, is $(r-1)$-Ryser-stable relative to the hypergraphs that proceed it in the sequence. Thus we conclude that S is also an $(r-1)$-Ryser-stable sequence of $SF(r,1,r-1)$. □

The existence of an $(r-1)$-Ryser-stable sequence for $SF(r,1,r-1)$ combined with Corollary 2.7, imply that for a given r, the sunflower $SF(r,1,r-1)$ cannot be contained in a counter-example to Ryser's Conjecture for intersecting hypergraphs.

□

We conclude this chapter by proving a lemma which roughly say that if an intersecting r-partite hypergraph has a Ryser-stable sequence then all the hypergraphs that contain it must have a Ryser-stable sequence, and the other way around. This lemma will allow us to prove correctness of the recursive mechanism in the algorithm we present in the next chapter.

We will use the following notation in the lemma (and the algorithms). Let $\mathcal{H}$ be an r-partite intersecting hypergraph. We denote by $\mathcal{E}^+(\mathcal{H})$ a smallest set of hypergraphs, such that any r-partite intersecting hypergraph $\mathcal{H}'$ that contains $\mathcal{H}$, and that also satisfies the property that $|E(\mathcal{H}')| = |E(\mathcal{H})|+1$, is isomorphic to one of the hypergraphs in $\mathcal{E}^+(\mathcal{H})$.

Lemma 2.12. *Let $\mathcal{H}$ be an r-partite intersecting hypergraph, such that $\tau(\mathcal{H}) \le c$ for a given c, and let $\mathcal{X}$ be a set of r-partite intersecting hypergraphs.*
Then there exists a c-Ryser-stable sequence of $\mathcal{H}$ relative to $\mathcal{X}$, iff for every $\mathcal{H}'$ in $\mathcal{E}^+(\mathcal{H})$ that doesn't contain any of the hypergraphs in $\mathcal{X}$, there exists a c-Ryser-stable sequence of $\mathcal{H}'$ relative to $\mathcal{X}$.

Proof. To prove the forward direction, let $\mathcal{X}$ and $\mathcal{H}$ be as in the statement of the lemma, and assume that $\mathcal{H}$ has a cover $\mathcal{C}$, such that $|C| \le c$. Further assume that every hypergraph in $\mathcal{E}^+(\mathcal{H})$ that doesn't contain a hypergraph in $\mathcal{X}$, has a c-Ryser-stable sequence relative to $\mathcal{X}$, and let S be the sequence that results from concatenating all the sequences of these hypergraphs.

Since Ryser-stable sequence are closed under concatenation relative to the same set, S is a c-Ryser-stable sequence relative to $\mathcal{X}$. Now by inserting $(\mathcal{H}, \mathcal{C})$ at the beginning of S, we get a c-Ryser-stable sequence of $\mathcal{H}$ relative to $\mathcal{X}$.

For the backward direction, assume that $\mathcal{H}$ has a c-Ryser-stable sequence S relative to $\mathcal{X}$, and let $\mathcal{H}'$ be a hypergraph in $\mathcal{E}^{+}(\mathcal{H})$ that doesn't contain any of the hypergraphs in $\mathcal{X}$. It follows from Lemma 2.6 that there must exist a cover $\mathcal{C}$ of $\mathcal{H}'$ such that $|\mathcal{C}| \leq c$. Let $\mathcal{S}'$ be the sequence that results from inserting the pair $(\mathcal{H}', \mathcal{C})$ at the beginning of S. To see that $\mathcal{S}'$ is a c-Ryser-stable sequence of $\mathcal{H}'$, note that any hypergraph that contains $\mathcal{H}'$ – and is not covered by $\mathcal{C}$ – must contain $\mathcal{H}$, so that the pair $(\mathcal{H}', \mathcal{C})$ is trivially Ryser-stable relative to the graphs in the sequence S.

$\square$

CHAPTER 3

ALGORITHMS

We first present an algorithm `COMPUTE-RYSER-SEQ`, that constructs a Ryser-stable sequence for a given r-partite hypergraph relative to a set of input hypergraphs $\mathcal{X}$.

The algorithm `COMPUTE-RYSER-SEQ` is used heavily by the main algorithm of this chapter `ENUMERATE-EDGE-MIN(r,c)`, which as its name implies, enumerates for a given r and c, all r-partite intersecting c-cover edge-minimal hypergraphs.

In the next chapter we will illustrate the working of the algorithms, by presenting their outputs on small values of r and c.

3.1 Computing Ryser-stable sequences

Informally speaking, the algorithm `COMPUTE-RYSER-SEQ` presented in this section tries to compute a Ryser-stable sequence for a given r-partite intersecting hypergraph $\mathcal{H}$, relative to an input set of r-partite hypergraphs $\mathcal{X}$, and it does so by recursively calling itself on hypergraphs that contain $\mathcal{H}$, until one of two possibilities occur.

The first possibility is that at some point the algorithm calls itself on a hypergraph that is Ryser-stable relative to the given sequence of graphs. In which case it adds it the sequence, and then back-tracks and keeps iterating the process, until it constructs a Ryser-stable sequence to the original hypergraph, at which point it returns the sequence and terminates.

The other possibility, is that at some point during the computation of the sequence, it finds a hypergraph with cover number more than c. If this happens then it returns the found hypergraph, and terminates.

We will also show that the second possibility can only materialise if in fact no sequence can be constructed for $\mathcal{H}$ relative to the inputted set of hypergraphs $\mathcal{X}$.

Algorithm 1: COMPUTE-RYSER-SEQ($\mathcal{H}, S, \mathcal{X}, c$)

Input : An r-partite intersecting hypergraph $\mathcal{H}$, a Ryser-stable sequence S, a set $\mathcal{X}$ of hypergraphs (none of which are contained in $\mathcal{H}$), and an integer c.

Output:

- c-Ryser-stable sequence S' for $\mathcal{H}$ relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$, and s.t S' extends S, or,
- If no such sequence exists, then output $\mathcal{H}'$, s.t $\tau(\mathcal{H}') > c$, and $\mathcal{H}'$ doesn't contain any of the graphs in $\mathcal{X}$.

if *there exists* $\mathcal{C}$*, s.t.* $(\mathcal{H}, \mathcal{C})$ *is c-Ryser-stable relative to graphs in* $\mathcal{X} \cup \mathcal{U}(r, c+1)$ *and to graphs in* S **then**
  Update S by inserting $(\mathcal{H}, \mathcal{C})$ to the beginning of it;
  return the sequence S.
end
if $\tau(\mathcal{H}) > c$ **then**
  return the hypergraph $\mathcal{H}$;
end
for *every* $\mathcal{J} \in \mathcal{E}^+(\mathcal{H})$ *that doesn't contain any of the graphs in* $\mathcal{X} \cup \mathcal{U}(r, c+1)$ **do**
  Set TEMP = COMPUTE-RYSER-SEQ($\mathcal{J}, S, \mathcal{X}, c$);
  if TEMP *is equal to a hypergraph* $\mathcal{H}'$ **then**
    return the hypergraph $\mathcal{H}'$;
  else
    Replace S by the sequence TEMP;
  end
end
Let $\mathcal{C}$ be a cover of $\mathcal{H}$, s.t $|\mathcal{C}| \leq c$;
Update S by appending $(\mathcal{H}, \mathcal{C})$ to the beginning of it;
return the sequence S.

We note that in the algorithm's output, making the sequence S Ryser-stable relative to the hypergraphs in $\mathcal{U}(r, c+1)$, is a technicality designed to prevent the algorithm from entering an "infinite loop" in some situations. This can occur if the algorithm keeps adding edges to sunflowers in $\mathcal{U}(r, c+1)$, without the resultant hypergraph ever becoming Ryser-stable or its cover number increasing.

Lemma 3.1. *Given an r-partite intersecting hypergraph* $\mathcal{H}$*, a set* $\mathcal{X}$ *of hypergraphs that are not contained in* $\mathcal{H}$*, a Ryser-stable sequence* S*, and an integer* c*, then the algorithm* COMPUTE-RYSER-SEQ($\mathcal{H}, S, \mathcal{X}, c$) *returns either:*

- *A c-Ryser-stable sequence* S' *for* $\mathcal{H}$ *relative to* $\mathcal{X} \cup \mathcal{U}(r, c+1)$*, and such that* S' *is an extension of* S*, or,*
- *If no such sequence* S' *exists then, it outputs a hypergraph* $\mathcal{H}'$ *such that* $\tau(\mathcal{H}') > c$*, and* $\mathcal{H}'$ *doesn't contain any of the hypergraphs that are in* $\mathcal{X}$*.*

Proof. Let $\mathcal{H}, \mathcal{X}, S$ and c be as in the statement of the lemma. We will show by induction on the number of edges of $\mathcal{H}$ that the Lemma is true.

For the base case, we'll assume that $|E(\mathcal{H})| \geq r!(c+1)^r$, so that it follows from the Sunflower Lemma that $\mathcal{H}$ contains one of the sunflowers in $\mathcal{U}(r, c+1)$. If there exists a cover $\mathcal{C}$ of $\mathcal{H}$ such that $|\mathcal{C}| \leq c$, then $(\mathcal{H}, \mathcal{C})$ will trivially be c-Ryser-stable relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$, since it contains a sunflower from $\mathcal{U}(r, c+1)$. Hence the algorithm will correctly return a sequence at line 3. If no such $\mathcal{C}$ exists, then it will return the hypergraph $\mathcal{H}$ at line 6, and since by assumption $\mathcal{H}$ doesn't contain any of the hypergraphs in $\mathcal{X}$, this will be a correct output.

For the induction step, assume that the Lemma is correct for all hypergraphs $\mathcal{H}$ with $|E(\mathcal{H})| \geq K$ for some K. Now let the inputted hypergraph $\mathcal{H}'$ to the algorithm, be an r-partite intersecting hypergraph that has $K-1$ edges.

If $\mathcal{H}'$ satisfies the test in line 1 then it will correctly return in line 3, otherwise if it satisfies the test in line 5 then it will correctly return in line 6 (we note again that by assumption the inputted hypergraph $\mathcal{H}'$ doesn't contain any of the hypergraphs in $\mathcal{X}$ as required). Thus we assume that $\mathcal{H}'$ doesn't satisfy either of these tests.

There are now two possibilities, either there exists a c-Ryser-stable sequence of $\mathcal{H}'$ relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$, or no such sequence exists.

If there exists a c-Ryser-stable sequence of $\mathcal{H}'$ relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$, then it follows from Lemma 2.12 that every $\mathcal{H}'' \in \mathcal{E}^+(\mathcal{H}')$ that doesn't contain any of the hypergraphs in $\mathcal{X} \cup \mathcal{U}(r, c+1)$, has a c-Ryser-stable sequence relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$. Since every $\mathcal{H}'' \in \mathcal{E}^+(\mathcal{H}')$ has K edges, it follows by the induction hypothesis that the algorithm will correctly extend the sequence in each iteration of the loop in line 8. In particular, in each iteration of the loop the sequence is extended to include one more of the hypergraphs in $\mathcal{E}^+(\mathcal{H}')$.

This implies that when the algorithm is at line 16, the sequence S is a c-Ryser-stable sequence that contains the original inputted sequence of the algorithm, and such that every hypergraph $\mathcal{H}'' \in \mathcal{E}^+(\mathcal{H}')$ is either contained in the sequence or in $\mathcal{X} \cup \mathcal{U}(r, c+1)$. It follows that by inserting $(\mathcal{H}', \mathcal{C})$ at the beginning of this sequence (where $\mathcal{C}$ is the cover computed at line 16), the resultant sequence is a c-Ryser-stable sequence of $\mathcal{H}'$ relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$. The algorithm correctly returns this sequence at line 18. Thus we see the algorithm returns the correct output when $\mathcal{H}'$ does have a c-Ryser-stable sequence relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$.

We now assume that $\mathcal{H}'$ doesn't have such a sequence. Then it will again follow from Lemma 2.12, that there exists at least one hypergraph in $\mathcal{E}^+(\mathcal{H}')$ – which we will denote by $\mathcal{H}''$ – such that $\mathcal{H}''$ has the following two properties. The first property is that $\mathcal{H}''$ doesn't contain any of the hypergraphs in $\mathcal{X} \cup \mathcal{U}(r, c+1)$. The second property is that there exists no c-Ryser-stable sequence of $\mathcal{H}''$ relative to $\mathcal{X} \cup \mathcal{U}(r, c+1)$.

Thus we know that at some point, the algorithm will set $\mathcal{J}$ in line 8, to $\mathcal{H}''$. Since $\mathcal{H}''$ contains K edges, it follows from the induction hypothesis, that when $\mathcal{J}$ is set to $\mathcal{H}''$, then at line 9 of the algorithm, the variable `TEMP` will be set to a hypergraph. Moreover, this hypergraph will have cover number more than c (which also follows from the induction hypothesis), and it will not contain any of the hypergraphs in $\mathcal{X}$. Hence the algorithm will correctly return this hypergraph in line 11 and terminate.

$\square$

3.2 Enumerating edge-minimal extremals

An important aspect of the algorithm that we present in this section, is that – aside from enumerating all r-partite c-cover intersecting edge-minimal hypergraphs – the algorithm also outputs a sequence which can be thought of as a certificate, proving that the hypergraphs it outputs, are the only possible r-partite c-cover edge-minimal hypergraphs (for a given combination of r and c).

Algorithm 2: `ENUMERATE-EDGE-MINS`(r, c)

Input : Integers r and c.
Output:
- A sequence S of r-partite hypergraphs, and,
- A set $\mathcal{X}$ of hypergraphs, such that S is a $(c-1)$-Ryser-stable sequence of $\mathcal{E}$ relative to $\mathcal{X} \cup \mathcal{U}(r, c)$, where $\mathcal{E}$ is the r-partite hypergraph that contains only one edge.

1 Set S as an empty sequence;
2 Set $\mathcal{E}$ as an r-partite hypergraph with one edge;
3 Set $\mathcal{X}$ as an empty set;
4 Set `TEMP = COMPUTE-RYSER-SEQ(`$\mathcal{E}, S, \mathcal{X}, c-1$`)`
5 **while** `TEMP` *equal to some hypergraph* $\mathcal{E}'$ **do**
6   Add to $\mathcal{X}$ all c-cover edge-minimal subhypergraphs of `TEMP`;
7   Set `TEMP = COMPUTE-RYSER-SEQ(`$\mathcal{E}, S, \mathcal{X}, c-1$`);`
8 **end**
9 **return** Sequence `TEMP` and the set $\mathcal{X}$.

Lemma 3.2. *For a given r and c, let $\mathcal{E}$ be the r-partite hypergraph consisting of only one edge. The algorithm* `ENUMERATE-EDGE-MINS(`r, c`)` *returns a sequence S and a set of r-partite hypergraphs $\mathcal{X}$, such that S is a $(c-1)$-Ryser-stable sequence S of $\mathcal{E}$ relative to $\mathcal{X} \cup \mathcal{U}(r, c)$. Moreover, the set $\mathcal{X}$ contains a list of all r-partite c-cover edge-minimal intersecting hypergraph.*

Proof. We know from Lemma 3.1 that every time `COMPUTE-RYSER-SEQ(`$\mathcal{E}, S, \mathcal{X}, c-1$`)` returns a hypergraph, it must not be contained in any of the hypergraphs already in $\mathcal{X}$.

For the rest of the proof, fix the algorithm's input parameters r and c. Now each time the algorithm executes the loop that starts at line 5, the number of (non-isomorphic) c-cover edge-minimal r-partite hypergraphs in $\mathcal{X}$ increases by at least 1. However, we also know from Corollary 2.4 that there are a finite number (up to isomorphism) of c-cover edge-minimal r-partite intersecting hypergraphs.

Thus, eventually the algorithm will find that at some point when it carries out the test at line 5, the variable `TEMP` will be equal to a sequence rather than a hypergraph. This implies that the algorithm will always halt in a finite amount of time.

This eventuality also implies that when the algorithm halts, and by using Lemma 3.1, the returned sequence will be a $(c-1)$-Ryser-stable sequence of $\mathcal{E}$ relative to $\mathcal{X} \cup \mathcal{U}(r,c)$, and $\mathcal{X}$ will only contain edge-minimal c-cover hypergraphs, as required by the first part of the lemma.

It remains to show that the returned set $\mathcal{X}$ will also contain all r-partite c-cover edge-minimal hypergraphs. For the sake of contradiction assume this is not the case, and that there exists an r-partite hypergraph $\mathcal{H}$ that is not contained in $\mathcal{X}$, and such that $\mathcal{H}$ is also a c-cover edge-minimal hypergraphs.

Then it follows from Lemma 2.6, and the sequence generated by the algorithm, that $\mathcal{H}$ must contain one of the hypergraphs in $\mathcal{X} \cup \mathcal{U}(r,c)$. But it also follows from Lemma 2.2 that it can't contain any of the hypergraphs in $\mathcal{U}(r,c)$.

This leaves the possibility of $\mathcal{H}$ containing one of the hypergraphs in $\mathcal{X}$. Since all the hypergraphs in $\mathcal{X}$ are also c-cover edge-minimal hypergraphs, $\mathcal{H}$ must therefore be isomorphic to one of them, a contradiction.

□

3.3 Verifying Ryser's Conjecture

An important special case of `ENUMERATE-EDGE-MINS` is the case when the parameter c is set to r, in which case the returned set $\mathcal{X}$ of the algorithm will contain all edge-minimal counter-example to Ryser's conjecture. Due to the importance of this case we present it as a separate algorithm.

Algorithm 3: `VERIFY-RYSER(`r`)`

Input : Integer r.
Output: `TRUE` if Ryser's conjecture is true for intersecting r-partite hypergraphs, `FALSE` otherwise.

Set $\mathcal{X}$ and S to the set of hypergraphs and sequence returned by `ENUMERATE-EDGE-MINS(`r, r`)`;
if *the set $\mathcal{X}$ is empty* **then**
  Print S;
  return `TRUE`;
else
  Print $\mathcal{X}$;
  return `FALSE`;
end

We see from the description of `VERIFY-RYSER(`r`)`, that when it verifies the conjecture for a given r, it prints a Ryser-stable sequence that proves that the conjecture is true for r-partite intersecting hypergraphs.

On the other hand, if it finds that the conjecture is false for the given value of r, then it prints out a list of r-partite counter-examples disproving the conjecture.

CHAPTER 4

EXAMPLE OUTPUT

The focus of this chapter is to illustrate the working of the algorithms introduced in this thesis. This will be done by presenting computer generated proofs of some special cases of Ryser's Conjecture. Namely, we present a computer generated proof that Ryser's Conjecture is true for intersecting tripartite hypergraphs, and for intersecting 4-partite hypergraphs.

One of the motivations behind investigating algorithms for proving Ryser's Conjecture was to settle more special cases of the conjecture via a computer generated proof. In particular, it was hoped that the algorithms would facilitate a proof of the first open case of the intersecting version of the conjecture, that of 6-partite intersecting hypergraphs. This case has remained open since the 1970s, and seems to be a difficult problem. Settling this case would also potentially enable more theoretical results, such as extending Haxell and Scott's result on 4-partite and 5-partite hypergraphs, to 6-partite hypergraphs.

However, even though our implementation of the algorithm `VERIFY-RYSER` terminated in few seconds for all inputs $r \leq 5$, this was not the case for $r = 6$, and in fact we experimented running the algorithm on this case for a few weeks without it terminating.

We therefore find it apt to dedicate parts of this chapter to discuss ways in which the algorithm `VERIFY-RYSER` could potentially be improved to be more practically efficient, and possibly yield a proof (or a counter-example) to Ryser's Conjecture for 6-partite intersecting hypergraphs.

To further illustrate the use of the algorithms, we will also demonstrate their capability in enumerating edge-minimal extremal hypergraphs. We will use the algorithms to enumerate c-cover edge-minimal hypergraphs for the case $r = 3$ and $r = 4$.

The case of enumerating all 3-cover 4-partite edge-minimal hypergraphs (i.e. edge-minimal extremals) is particularly interesting, since this is the first instance in which the algorithm provides non-obvious theoretical results that were not previously known in the literature. However, we present this result in the next part of this thesis, which is focused on investigating 4-partite edge-minimal extremal hypergraphs.

4.1 Computer generated proofs

In this section we present computer generated proofs of Ryser's Conjecture for tripartite and for 4-partite hypergraphs, which are returned by invoking `VERIFY-RYSER(3)` and `VERIFY-RYSER(4)` respectively.

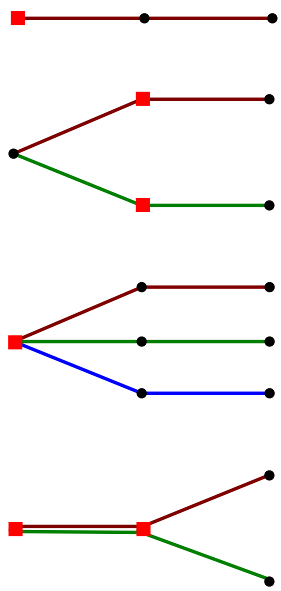

Figure 4.1: A tripartite 2-Ryser-stable sequence.

We recall that the algorithm `VERIFY-RYSER` returns `TRUE` or `FALSE` depending on whether the intersecting case of the conjecture is true or false for the inputted value of r. It also prints an $(r-1)$-Ryser-stable sequence when it returns `TRUE`, which acts as a certificate or a proof that the conjecture is true. On the other hand when it returns `FALSE`, it also prints a set of edge-minimal r-partite counter-examples to the conjecture.

The sequence of hypergraphs in Figure 4.1 is a 2-Ryser-stable sequence of tripartite hypergraphs (relative to the empty set), which is printed by the algorithm when invoked with the input $r = 3$.

Hence by Corollary 2.7, the sequence in Figure 4.1 shows that Ryser's Conjecture is true for all tripartite intersecting hypergraphs.

In the figure, we identified the corresponding cover of each hypergraph in the sequence by placing red squares over vertices that are in the cover.

We note that the sequence doesn't contain any hypergraph with more than four edges, which also implies that the algorithm didn't have to consider any hypergraph with more than four edges when computing the sequence. This compares very favourably with the bound given by the naive algorithm discussed in Chapter 2, and also the bound given in Corollary 2.5.

The naive algorithm proposed in chapter 2, suggests proving Ryser's Conjecture for intersecting tripartite hypergraphs by checking all intersecting hypergraphs with less than $3!3^3 = 162$ edges.

This comparison also holds for the case $r = 4$ which can be seen in the sequence of hypergraphs in Figure 4.2. This figure shows a 4-partite 3-Ryser-stable sequence, proving that Ryser's Conjecture is true for 4-partite intersecting hypergraphs.

The algorithm doesn't have to consider hypergraphs with more than 5 edges for this case, while the bound suggested by the naive algorithm consists in checking all 4-partite hypergraphs with less than $4!4^4 = 6,144$ edges.

4.2 More efficient heuristics

Two sources of inefficiency in the algorithm `VERIFY-RYSER` come from its use of the algorithm `COMPUTE-RYSER-SEQ` which in-turn contains two known NP-hard problems. The first such problem is computing the cover number of a hypergraph, and the second is deciding if one hypergraph is isomorphic to a subhypergraph of another hypergraph (for a proof of these problems intractability, see for example [12]).

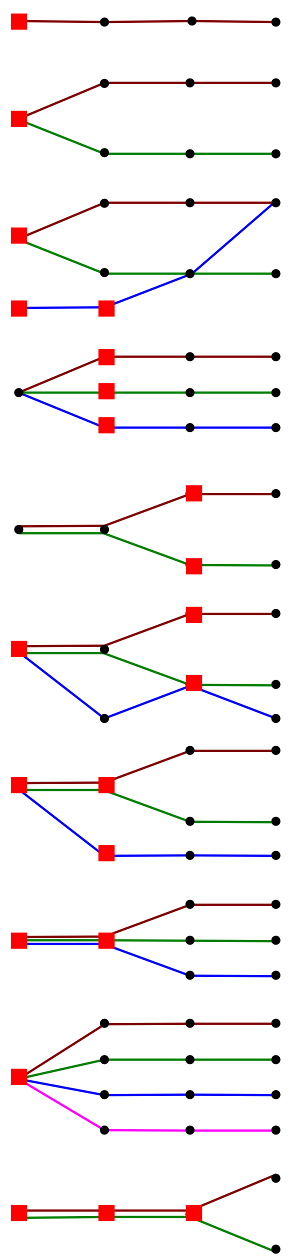

Figure 4.2: A 4-partite 3-Ryser-stable sequence.

Computing cover numbers is required by `COMPUTE-RYSER-SEQ` in three places: line 1, line 5 and line 16. While deciding if one hypergraph is a subhypergraph of another is used in line 1, when deciding if a hypergraph $\mathcal{H}$ is Ryser-stable relative to a set $\mathcal{X}$ of hypergraphs. This is because relative Ryser-stability implicitly involves checking if $\mathcal{H}$ contains any of the hypergraphs in $\mathcal{X}$ as a subhypergraph.

When running `VERIFY-RYSER(6)`, it turns out that the efficiency cost incurred due to computing cover numbers of the hypergraphs the algorithm encounters, is much less than the costs incurred due to subhypergraph isomorphism computations, even though both computations are theoretically intractable problems.

Informally, one way to limit the practical inefficiency caused by these two sources of hard computation, is to limit the size of the hypergraphs considered by the algorithm. This involves limiting the depth of the search tree traversed by the algorithm, or equivalently limiting the number of times the algorithm `COMPUTE-RYSER-SEQ` recursively calls itself in line 9; since each time it recursively calls this line it adds one edge to the inputted hypergraph.

One way in which this might be done is by designing better sorting rules (or heuristics). This takes place in line 8 of the `COMPUTE-RYSER-SEQ`. In our description, we didn't specify a sorting rule since it doesn't effect the correctness of the algorithm. However, in practice we observed that different sorting rules have a considerable impact on the size of the generated sequence.

Intuitively, it is clearly more efficient if the algorithm sorts hypergraphs in line 8 by starting with hypergraphs that are most likely to become relative Ryser-stable first.

This raises an interesting question of what are the properties of a hypergraph that makes it more likely that it will become Ryser-stable relative to a set of hypergraphs? We were unable to come up with a satisfactory answer, though we propose it as an interesting question to pursue if one intends to make our algorithms more practically efficient.

Another potential way to improve the practical running time of the algorithm `COMPUTE-RYSER-SEQ` is to implement it in a more efficient manner. Our implementation was run on a standard desktop environment, and we made basic use the *nauty* [20] package for computing subgraph isomorphism.

However, a considerable practical speed up might be attained by utilising *nauty* in a more sophisticated manner, or by using more advanced computing technology than a standard desktop computer. This can involve capabilities offered by cloud computing services designed for scientific computing.

Thus we conclude that there are various ways in which the algorithms we presented can be enhanced – or re-implemented in a more sophisticated manner – for the purpose of making them more practically efficient. These enhancement might potentially be sufficient to at least enable the algorithm to resolve the case of 6-partite intersecting hypergraphs in a reasonable amount of time.

4.3 Example enumerations

We now turn to utilising the algorithms developed in the previous chapter for the purpose of enumerating edge-minimal extremal hypergraphs. We recall that the algorithm `ENUMERATE-EDGE-MIN` takes as input r and c, and returns two objects: a c-Ryser-stable sequence S, and a set $\mathcal{X}$ that contains r-partite c-cover edge-minimals.

The outputted sequence S is a c-Ryser-stable sequence relative to $\mathcal{X} \cup \mathcal{U}(r, c)$, which can also be thought of as a certificate (or proof) that the set $\mathcal{X}$ contains all possible r-partite c-cover edge-minimal hypergraphs.

In figure 4.3 we present the output for invoking `ENUMERATE-EDGE-MIN` on the inputs $r = 3$ and $c = 2$, which is equivalent to enumerating all tripartite intersecting extremal hypergraphs.

For reference, the hypergraphs in the first column are the hypergraphs in $\mathcal{U}(3, 2)$. While the hypergraph in the middle column is the unique tripartite 2-cover edge-minimal hypergraphs found by the algorithm. Finally, the last column is the 1-Ryser-stable sequence returned by the algorithm which proves there is only one tripartite intersecting extremal hypergraph.

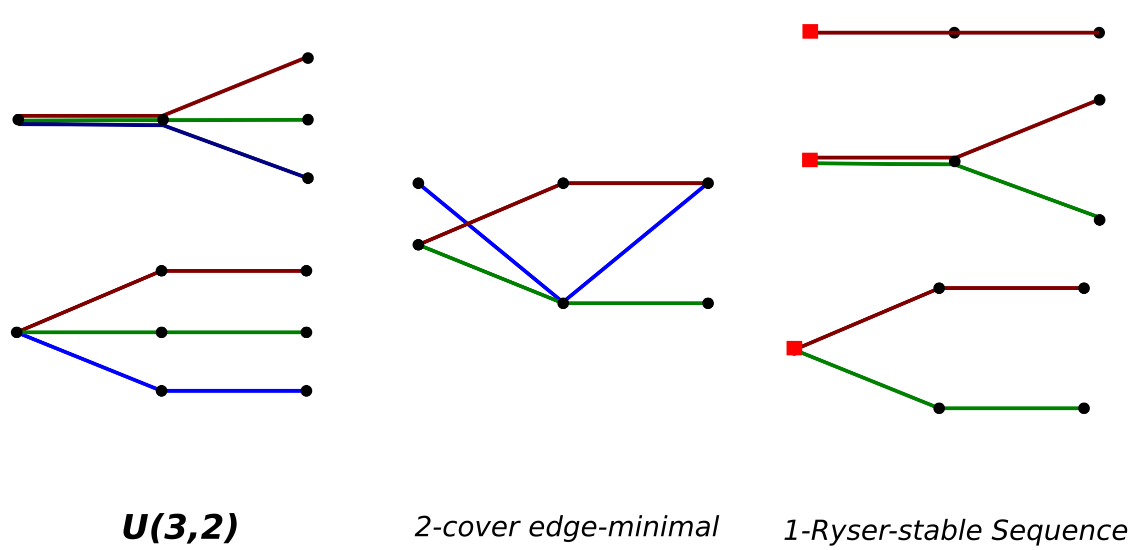

Figure 4.3: Enumeration of tripartite 2-cover edge-minimal hypergraphs.

In the same vein Figure 4.4 presents a proof that there are only three 4-partite 2-cover edge-minimal intersecting hypergraphs.

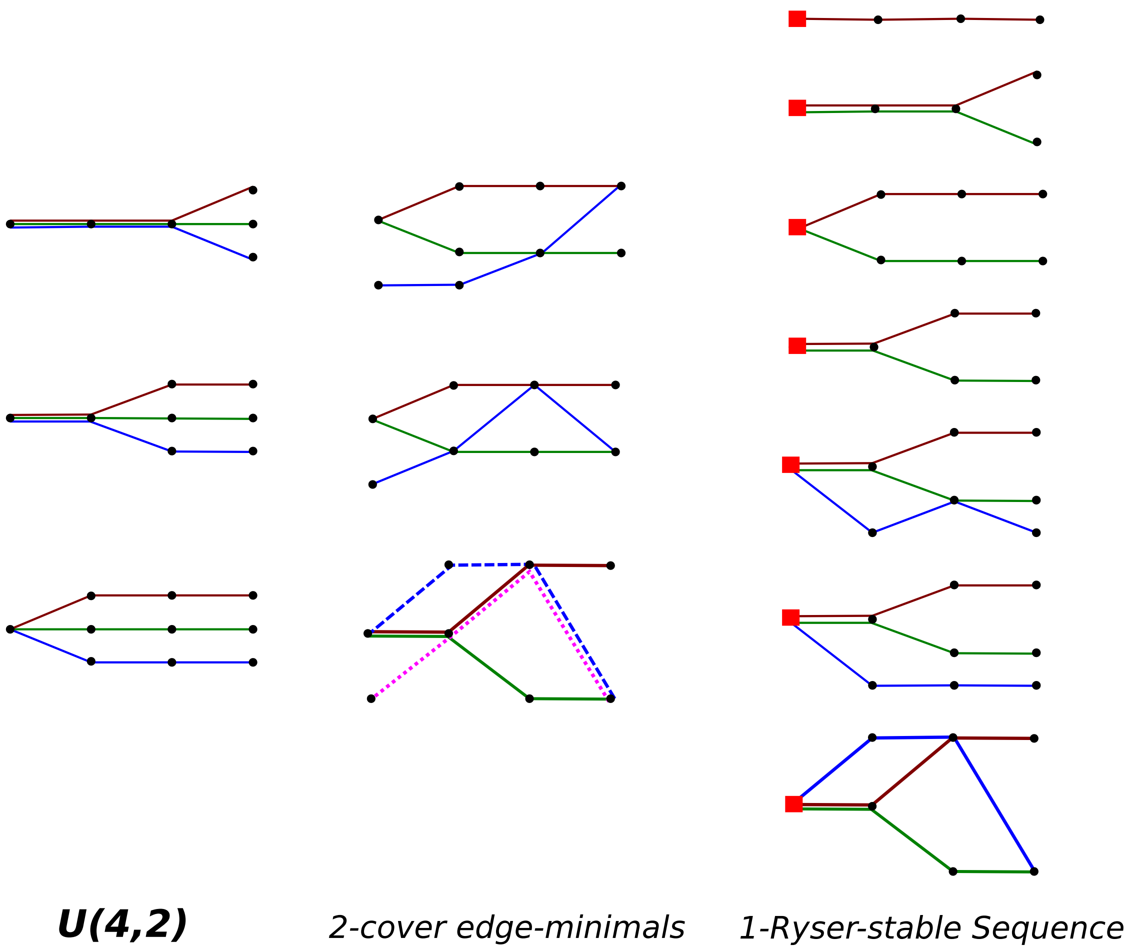

Figure 4.4: Enumeration of 4-partite 2-cover edge-minimal hypergraphs.

This leaves the case of 4-partite 3-cover edge-minimal extremal hypergraphs, or in other words, 4-partite edge-minimal extremal hypergraphs. We present this case in the next chapter, where we commence the second part of this thesis, which is dedicated to investigating 4-partite extremal hypergraphs and their structural properties.

Part II: The Case of 4-partite Hypergraphs

CHAPTER 5

EDGE-MINIMAL EXTREMALS

As mentioned in the introduction of this thesis, a necessary first step for characterising 4-partite extremal hypergraphs, is to first enumerate all edge-minimal 4-partite intersecting extremal hypergraphs. We accomplish this step in the beginning of this chapter, which paves the way for its main finding. We show that 4-partite extremal hypergraphs cannot be characterised the way tripartite extremal hypergraphs were characterised in terms of intersecting extremal hypergraphs.

5.1 The case of $\nu = 1$

It is easy to check that the following three 4-partite intersecting hypergraphs – which we denote as $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ – are edge-minimal extremal hypergraphs.

1111	1111	1111
1222	1222	1222
1333	1233	1233
2123	1332	1332
3132	2132	2132
3213	3231	3231
	4212	2212
$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$

4-partite intersecting extremals

In particular, we note that $\mathcal{F}_1$ is the smallest subhypergraph of the 4-partite truncated projective plane extremal hypergraph (and hence is also a linear hypergraph). While $\mathcal{F}_2$ and $\mathcal{F}_3$ differ in only one edge, which nevertheless is sufficient to make them non-isomorphic (since $\mathcal{F}_2$ contains four vertices in its first side, while all sides of $\mathcal{F}_3$ has three vertices).

Theorem 5.1. *The set $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ contains (up to isomorphism) all possible 4-partite intersecting edge-minimal extremal hypergraphs to Ryser's Conjecture.*

Proof. We will present a computer-generated proof that consists of a 2-Ryser-stable sequence of $\mathcal{E}$ relative to $\mathcal{F} \cup \mathcal{U}(4,3)$, where $\mathcal{E}$ is the 4-partite hypergraph consisting of one edge.

The sequence presented in the following table was computed by running an implementation of the algorithm `ENUMERATE-EDGE-MIN(4,3)`. Thus by Lemma 3.2, we know that all 4-partite edge-minimals extremal hypergraphs are contained in $\mathcal{F}$, proving the lemma.

1111	$\{(1,1),(2,1)\}$
1111 1122	$\{(1,1),(2,1)\}$
1111 1222	$\{(1,1)\}$
1111 1222 2112	$\{(2,1),(2,2)\}$
1111 1222 2112 1132	$\{(2,1),(2,2)\}$
1111 1222 1113	$\{(1,1),(4,2)\}$
1111 1222 1133	$\{(1,1),(2,1)\}$
1111 1222 2123	$\{(1,1),(1,2)\}$
1111 1222 2123 1131	$\{(1,1),(3,2)\}$
1111 1222 2123 1132	$\{(1,1),(2,1)\}$
1111 1222 2123 1233	$\{(1,1),(1,2)\}$
1111 1222 2123 1233 3123	$\{(1,1),(2,1)\}$
1111 1222 2123 2231	$\{(1,1),(1,2)\}$
1111 1222 2123 2231 1321	$\{(1,1),(1,2)\}$
1111 1222 2123 2231 1321 3121	$\{(4,1),(3,2)\}$
1111 1222 2123 2214	$\{(1,1),(1,2)\}$
1111 1222 2123 1134	$\{(1,1),(1,2)\}$
1111 1222 2123 1134 1321	$\{(1,1),(1,2)\}$
1111 1222 2123 1134 1321 3121	$\{(1,1),(2,1)\}$
1111 1222 2123 1134 1213	$\{(1,1),(1,2)\}$
1111 1222 2123 1134 1243	$\{(1,1),(1,2)\}$
1111 1222 2123 1134 1322	$\{(1,1),(2,1)\}$
1111 1222 2123 1134 1323	$\{(1,1),(2,1)\}$
1111 1222 2123 1134 1325	$\{(1,1),(2,1)\}$
1111 1222 1333	$\{(1,1)\}$
1111 1222 1333 2123	$\{(1,1),(1,2)\}$
1111 1222 1333 2123 3123	$\{(1,1),(2,1)\}$
1111 1222 1333 2123 3132	$\{(1,1),(2,1)\}$
1111 1222 1333 1144	$\{(1,1),(2,1)\}$
1111 1222 1333 1444	$\{(1,1)\}$
1111 1122 1133	$\{(1,1),(2,1)\}$

3-Ryser-stable sequence of $\mathcal{E}$ relative to $\mathcal{F} \cup \mathcal{U}(4,3)$

Each row i of the above table contains the pair $(\mathcal{H}_i, \mathcal{C}_i)$. In each ith row of the table, the first column displays the hypergraph $\mathcal{H}_i$ of the sequence, while the second column displays the cover $\mathcal{C}_i$ of the sequence. To specify the vertices in the cover we use the notation (j,k) to represent the kth vertex in the jth partition. Thus the vertex $(3,2)$ covers the edge 2123 because it contains 2 in its third entry.

□

5.2 The case of $\nu = 2$

We recall the characterisation theorem of 3-partite edge-minimal extremal hypergraphs:

Tripartite Extremals (Haxell, Narins, Szabó)**.** *Every tripartite extremal hypergraph to Ryser's Conjecture, $\mathcal{H}$, contains $\nu(\mathcal{H})$ vertex-disjoint copies of the unique tripartite intersecting edge-minimal extremal hypergraph.*

In light of Theorem 5.1, we can now ask concrete questions to test whether 4-partite extremal hypergraphs can be characterised in the same way as tripartite extremal hypergraphs. The simplest such question is the following: does every 4-partite edge-minimal extremal hypergraph with matching number 2, contain two disjoint hypergraphs (not necessarily different) from the set $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$?

The following table presents three hypergraphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, each with matching number two, and each sufficient to answer the question in the negative.

$(\nu = 2)$, **4-partite Edge-Minimal Extremal Hypergraphs**				$(\nu = 1)$, **4-partite Edge-Minimal Extremal Subhypergraphs**				
$\mathcal{G}_1 = \{$	1111	2222		$\mathcal{F}_1 \simeq \{$	1111	1333	1444	
	1333	2155			5314	6341	6413	$\}$
	1444	2562		$\mathcal{F}_1 \simeq \{$	2222	2155	3162	
	3162	4652			4652	4265	2666	$\}$
	5314	4265		$\mathcal{F}_1 \simeq \{$	1333	1444	5314	
	6341	2666			6341	6413	1211	$\}$
	6413	1211	$\}$					
$\mathcal{G}_2 = \{$	1111	2222		$\mathcal{F}_1 \simeq \{$	1111	1333	1555	
	1333	2444			3135	5531	5153	$\}$
	1555	2266		$\mathcal{F}_1 \simeq \{$	1141	1333	1555	
	3135	4426			3135	5531	5153	$\}$
	5531	6224		$\mathcal{F}_2 \simeq \{$	2222	2444	2266	
	5153	2624			2624	4426	6224	
	1141	4214	$\}$		4214			$\}$
$\mathcal{G}_3 = \{$	1111	2222		$\mathcal{F}_1 \simeq \{$	1111	1333	1555	
	1333	2444			3135	5531	5153	$\}$
	1555	2266		$\mathcal{F}_1 \simeq \{$	1141	1333	1555	
	3135	4426			3135	5531	5153	$\}$
	5531	6224		$\mathcal{F}_3 \simeq \{$	2222	2444	2266	
	5153	2624			2624	4426	6224	
	1141	4264	$\}$		4264			$\}$

4-partite edge-minimal extremal hypergraphs,
and their intersecting extremal subhypergraphs

For each row i of the table, the first column presents one of the hypergraphs $\mathcal{G}_i$, while the second column shows all possible subhypergraphs of $\mathcal{G}_i$ that are isomorphic to any of the hypergraphs in the set $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. For example, the table shows that $\mathcal{G}_1$ contains only three different subhypergraphs that are isomorphic to $\mathcal{F}_1$, and no subhypergraphs that are isomorphic to either $\mathcal{F}_2$ or $\mathcal{F}_3$.

We note that the hypergraphs $\mathcal{G}_2$ and $\mathcal{G}_3$ are different only in the last edge, in that the last edge in G_2 contains the first vertex in the third partition, while the last edge in G_3 contains the sixth vertex in the third partition. This is however sufficient to make them non-isomorphic, since as can be seen from the table, G_2 contains a copy of $\mathcal{F}_2$ as a subhypergraph, while $\mathcal{G}_3$ contains a copy of $\mathcal{F}_3$.

5.3 Characterising 4-partite extremals

By looking at the subhypergraph structure of $\mathcal{G}_1$ presented in the previous table, we note that none of the intersecting 4-partite edge-minimal extremal subhypergraphs it contains are vertex disjoint.

For example, the first two subhypergraphs it contains (the one in row 1 and row 2) overlap in the first vertex of the second partition. This can be seen as the first subhypergraph contains the edge 1111, while the second subhypergraph contains the edges 2155 and 3162.

We also note a similar situation occurs with each of the hypergraphs presented in the table, in that each of their subhypergraphs that are isomorphic to one of the hypergraphs from the set $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$, overlap in at least one vertex.

Lemma 5.2. *There exist 4-partite edge-minimal extremal hypergraphs $\mathcal{H}$, such that $\mathcal{H}$ doesn't contain $\tau(\mathcal{H})$ vertex-disjoint copies of 4-partite minimal intersecting extremal hypergraphs.*

Proof. The existence of any of the hypergraphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ suffice to prove the lemma. $\square$

The case of 4-partite extremal hypergraphs with matching number two, is the simplest case possible after tripartite extremal hypergraphs, for which a characterisation of all extremal hypergraphs is still open. However, from the above discussion we see that even for this case the picture is more complicated than what has proven to be the case for tripartite hypergraphs.

Nevertheless, in the next chapter we will take a closer look at the hypergraphs $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$, to investigate if other characterisations are still possible for 4-partite extremals in terms of intersecting extremals. In particular, ones that use a more relaxed condition than vertex-disjointedness.

CHAPTER 6

VERTEX-MINIMAL EXTREMALS

We begin this chapter by motivating the definition of a *vertex-minimal* r-partite intersecting hypergraph. To apply this definition in the context of Ryser's Conjecture we will have to relax the condition of multipartiteness, and instead use the setting of what we call *weakly r-partite* hypergraphs.

We then use the lens of vertex-minimality to take a closer look at the 4-partite edge-minimal extremal hypergraphs presented in the previous chapter. We will show that they do not contradict a vertex-minimal analogue to the tripartite characterisation of extremal hypergraphs.

6.1 Weakly r-partite hypergraphs

When we look closely at the hypergraphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ presented in the previous chapter, we notice that the reason each can't be described in terms of disjoint copies of edge-minimal intersecting extremal hypergraph, is due to only one vertex from each of their respective set of vertices.

For example, looking at the hypergraph $\mathcal{G}_2$, we see that it has a subhypergraph isomorphic to $\mathcal{F}_1$, and another that is isomorphic to $\mathcal{F}_2$, and they both overlap in one vertex. In the table, these are respectively the first and third entry in G_2's subhypergraphs entries. For the rest of the discussion, we will denote these subhypergraphs by $\mathcal{F}_1'$ and $\mathcal{F}_2'$.

The subhypergraphs $\mathcal{F}_1'$ and $\mathcal{F}_2'$ overlap in the first vertex of the third side, since $\mathcal{F}_1'$ contains the edge 1111, and $\mathcal{F}_2'$ contains the edge 4214.

Thus we can say that $\mathcal{G}_2$ includes "almost disjoint" copies of hypergraphs from the set of 4-partite edge-minimal extremal intersecting hypergraphs. It can also be seen that $\mathcal{G}_1$ and $\mathcal{G}_3$ can be characterised in the same way with respect to their extremal subhypergraphs.

This motivates looking more closely at the vertices at which the "almost disjoint" subhypergraphs of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ intersect, to see if they have any interesting properties. In the case of $\mathcal{G}_2$, we will show in the following discussion, that the overlapping vertex in the subhypergraphs $\mathcal{F}'_1$ and $\mathcal{F}'_2$, does indeed seem to have a distinguishing property from other vertices in the subhypergraphs of $\mathcal{G}_2$.

Restricting our view to the subhypergraph $\mathcal{F}'_1$, we note that if we imagine removing the overlapping vertex we end up with the following hypergraph:

11*1	1333	1555
3135	5531	5153

$\mathcal{F}'_1$ with a vertex removed

In the hypergraph presented above, we used the * to denote that the first edge of the hypergraph no longer contains a vertex from the third side. In particular, the above hypergraph is not 4-partite since not all of its edges contain exactly one vertex from each side. However, the above hypergraph does satisfy all other properties of $\mathcal{F}'_1$, in that it is an extremal hypergraph, and that it is also intersecting and edge-minimal with respect to its cover number.

We also note that we can remove other vertices from the above hypergraph without also changing the properties of being an edge-minimal extremal intersecting hypergraph:

11*1	1*33	1555
*135	5531	5153

Removing more vertices

In the above hypergraph we observe that if we remove any other vertex, the resultant hypergraph will no longer be intersecting.

We also observe that even though the above hypergraph is not 4-partite, it is almost r-partite in that its vertices admit partitioning into four sides, such that each edge contains *at most* one vertex from each side.

The above discussion on $\mathcal{F}_1$ motivate the following definitions. We call a hypergraph $\mathcal{H}$ a *weakly r-partite* hypergraph, if its vertex set can be partitioned into r sides $V_1, \ldots, V_r$, such that each edge has at most one vertex from each side.

Moreover, given two weakly r-partite hypergraphs $\mathcal{H}'$ and $\mathcal{H}$, we say that $\mathcal{H}'$ is *weakly contained* in $\mathcal{H}$, if for every $h' \in E(\mathcal{H}')$ there exists $h \in E(\mathcal{H})$ such that $h' \subseteq h$.

Finally, we say that a weakly r-partite extremal intersecting hypergraph $\mathcal{H}$ is *vertex-minimal* hypergraph, if for every weakly r-partite extremal intersecting hypergraph $\mathcal{H}'$ that is weakly contained in $\mathcal{H}$, we have that $\mathcal{H} = \mathcal{H}'$. Informally speaking, a vertex-minimal extremal hypergraph is one where we can't shorten any edge – by removing vertices from it – without making the hypergraph non-intersecting.

We also note that the notion of a vertex-minimal extremal hypergraph can be thought of as the dual of the notion of an edge-minimal extremal hypergraph, in that removing an edge from an edge-minimal extremal hypergraph decreases the cover number of the resultant hypergraph, while removing a vertex from a vertex-minimal extremal increases the matching number of the resultant hypergraph.

In light of these definitions, we can now characterise the last resultant hypergraph we obtained above (by removing vertices from the subhypergraph $\mathcal{F}_1'$) as a weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraph.

For a given r, we can enumerate all possible weakly r-partite edge-minimal vertex-minimal extremal intersecting hypergraphs as follow. First, we construct the set S_r of all possible r-partite edge-minimal extremal intersecting hypergraphs. As we saw in the first part of this thesis, this set is finite for all r, and can be generated using the algorithms we presented. We then extract all unique (up to isomorphism) weakly r-partite vertex-minimal edge-minimal extremal intersecting hypergraph that are weakly contained in hypergraphs in the set S_r.

In the case of weakly 3-partite intersecting hypergraphs, it is easy to see that there is only one unique edge-minimal vertex-minimal 3-partite extremal intersecting hypergraph, which can be thought of as a triangle.

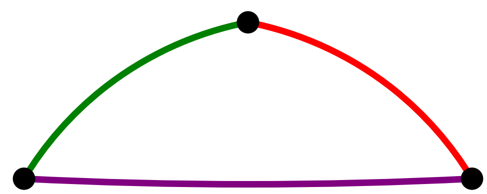

Figure 6.1: The unique weakly tripartite edge-minimal vertex-minimal intersecting extremal hypergraph

As for the case of $r = 4$, we can generate all weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraphs, by enumerating all such non-isomorphic hypergraphs that are weakly contained in the hypergraphs $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. By doing so it is easy to see that we end up with two weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraphs, which we denote as $\mathcal{J}_1$ and $\mathcal{J}_2$.

111*	1111
1222	12*2
1*33	123*
*123	1*32
3132	*132
3213	*231
	*212
$\mathcal{J}_1$	$\mathcal{J}_2$

All weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraphs

Thus we see that $\mathcal{F}_1$ weakly contains one unique weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraph, which is $\mathcal{J}_1$. While each of $\mathcal{F}_2$ and $\mathcal{F}_2$ turn out to weakly contain the same weakly 4-partite hypergraph $\mathcal{J}_2$.

6.2 A conjecture

Equipped with the definitions developed in the previous section, and our enumeration of all weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraphs, we now take another look at the hypergraphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. The purpose this time is to interpret them in terms of the hypergraphs $\mathcal{J}_1$ and $\mathcal{J}_2$.

($\nu = 2$), **4-partite Edge-minimal Extremal Hypergraphs**			($\nu = 1$), **Weakly 4-partite Edge-minimal, Vertex-minimal Extremal Subhypergraphs**			
$\mathcal{G}_1 = \{$	1111	2222	$\mathcal{J}_1 \simeq \{$	1*11	13*3	1444
	1333	2155		*314	6341	6413 $\}$
	1444	2562	$\mathcal{J}_1 \simeq \{$	22*2	2155	*162
	3162	4652		4652	4265	266* $\}$
	5314	4265	$\mathcal{J}_1 \simeq \{$	13*3	1444	*314
	6341	2666		6341	641*	1211 $\}$
	6413	1211 $\}$				
$\mathcal{G}_2 = \{$	1111	2222	$\mathcal{J}_1 \simeq \{$	11*1	1*33	1555
	1333	2444		*135	5531	5153 $\}$
	1555	2266	$\mathcal{J}_1 \simeq \{$	11*1	1*33	1555
	3135	4426		3135	55*1	5153 $\}$
	5531	6224	$\mathcal{J}_2 \simeq \{$	222*	24*4	22*6
	5153	2624		2*24	4426	*224
	1141	4214 $\}$		42*4		$\}$
$\mathcal{G}_3 = \{$	1111	2222	$\mathcal{J}_1 \simeq \{$	11*1	1*33	1555
	1333	2444		*135	5531	5153 $\}$
	1555	2266	$\mathcal{J}_1 \simeq \{$	11*1	1*33	1555
	3135	4426		3135	55*1	5153 $\}$
	5531	6224	$\mathcal{J}_2 \simeq \{$	222*	24*4	22*6
	5153	2624		2*24	4426	*224
	1141	4264 $\}$		42*4		$\}$

$\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and their edge-minimal vertex-minimal extremal intersecting subhypergraphs

By inspecting the above table, we see that they hypergraph $\mathcal{G}_1$ weakly contains two disjoint copies of $\mathcal{J}_1$, which are the entries in its first and second row in the subhypergraph column.

While each of the hypergraphs $\mathcal{G}_2$ and $\mathcal{G}_3$, weakly contain one copy of $\mathcal{J}_1$ alongside a vertex-disjoint copy of $\mathcal{J}_2$.

Thus even though the hypergraphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ contradict a characterisation of 4-partite extremal hypergraphs in terms of copies of edge-minimal extremal hypergraphs, they do not contradict a characterisation in terms of weakly 4-partite edge-minimal vertex-minimal extremal intersecting hypergraphs.

Moreover, even though we attempted computationally to generate a single 4-partite extremal hypergraph with matching number two, which doesn't weakly contain two vertex-disjoint vertex-minimal intersecting extremals, we were unable to do so.

This compels us to conclude our investigation of 4-partite extremal hypergraphs by speculating the following:

Conjecture 6.1. *If $\mathcal{H}$ is a 4-partite extremal hypergraph, then it contains $\nu(\mathcal{H})$ vertex-disjoint weakly 4-partite intersecting extremals.*

Part III: A Family of Extremals

CHAPTER 7

SPARSE EXTREMALS

So far in this thesis, we have seen that a crucial step towards understanding extremal hypergraphs to Ryser's Conjecture in the general setting, seems to be to first understand extremal hypergraphs in the intersecting setting. Thus in the final part of this thesis, we will focus on r-partite extremal intersecting hypergraphs.

We begin by studying how sparse such objects can be, by investigating the values of $f(r)$ for low values of r. We recall from the introduction that the function $f(r)$ was defined as the minimum integer so that there exists an r-partite intersecting extremal with $f(r)$ edges. The current known values of $f(r)$ in the literature are $f(3) = 3$, $f(4) = 6$, $f(5) = 9$ and $12 \le f(6) \le 15$.

We also present a simple heuristic that attempts to randomly construct an $(r+1)$-partite extremal hypergraph from an r-partite extremal hypergraph. Implementing and running an algorithm that makes use of this heuristic, allows us to obtain a bound on $f(7)$ and $f(11)$, by constructing the first known extremal hypergraphs for these cases.

In the next chapter we show that one can remove the random component from the above method of generating extremal hypergraphs. In particular, we show that for a class of extremal hypergraphs there is a deterministic way to always transform an r-partite extremal hypergraph from this class to a new $(r+1)$-partite extremal hypergraph. This class of extremal hypergraphs include truncated projective plane extremal hypergraphs, and this allows us to prove the existence of a new family of extremal hypergraphs.

In this chapter, we prove the following theorem:

Theorem 7.1. *$f(6) = 13$, $f(7) \le 22$ and $f(11) \le 51$.*

As mentioned in the Introduction, the results $f(6) = 13$ and $f(7) = 17$ were also obtained independently in [4]. Moreover, the material in this chapter appeared previously in a version uploaded to the arXiv repository [1], and also in a version submitted for publication [2].

The difference between the arXiv version and this chapter, is that the arXiv version doesn't include the bound for $f(11)$, and also doesn't include the section on generating extremal hypergraphs. While the difference between the submitted version and this chapter, is that they differ slightly on the way the heuristic in the last section of this chapter is presented.

7.1 The value of $f(6)$

To settle the case of $f(6)$ we will first show that $f(6) > 12$, by proving that $f(6) \neq 12$ and then combine it with the result $f(6) > 11$ established in [19]. We will then present a 6-partite intersecting extremal hypergraph with 13 edges, which shows that $f(6) = 13$.

For a given hypergraph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, we let $E(v)$ denote the set $\{e \in E(\mathcal{H}) : v \in e\}$, and we denote the degree of v by $d(v) = |E(v)|$. We also use the notation $\Delta(\mathcal{H})$ to denote the maximum degree over all vertices of $\mathcal{H}$. Finally, for two distinct vertices v and w in $\mathcal{H}$, the *co-degree* of v and w, denoted by $c(v, w)$, is defined as $|E(v) \cap E(w)|$.

In the rest of this chapter we will make use of the following trivial bound on the covering number of an intersecting hypergraph: if $\mathcal{H}$ is an intersecting hypergraph then $\tau(\mathcal{H}) \leq \lceil \frac{|E(\mathcal{H})|}{2} \rceil$. This bound follows since a cover of size $\lceil \frac{|E(\mathcal{H})|}{2} \rceil$ can be established via the greedy algorithm given that every two edges in an intersecting hypergraph intersect in at least one vertex. We will call any cover obtained this way a *greedy* cover of the hypergraph.

The strategy we adopt to prove that $f(6) \neq 12$, is first to assume that $\mathcal{H}$ is a 6-partite extremal hypergraph that contains exactly 12 edges and then showing via a case-by-case analysis that all possible values of $\Delta(\mathcal{H})$ lead to a contradiction. When $\Delta(\mathcal{H})$ is large it can be shown that a cover $\mathcal{C}$ of $\mathcal{H}$ can be formed such that $|\mathcal{C}| < 5$, contradicting the extremality of $\mathcal{H}$. When $\Delta(\mathcal{H})$ is small it can be shown that some of the edges of $\mathcal{H}$ don't intersect each other contradicting the fact that $\mathcal{H}$ is intersecting.

The case $\Delta(\mathcal{H}) = 4$ turns out to be more difficult to deal with than the other cases, and to settle it we will require some facts concerning the degree structure of intersecting 6-partite hypergraphs with 8 edges and a covering number equal to 4. We will start by proving these facts before presenting the proof of $f(6) > 12$.

Lemma 7.2. *If $\mathcal{H}'$ is an intersecting 6-partite hypergraph with 8 edges and $\tau(\mathcal{H}') = 4$, then $\mathcal{H}'$ contains exactly 6 vertices of degree 3, one in each partition, and there exists two edges in $\mathcal{H}'$ such that they share at least two vertices of degree 3 in common.*

Proof. For the rest of proof let $\mathcal{H}'$ be as in the statement of the Lemma. We can assume $\Delta(\mathcal{H}') \leq 3$, otherwise we can find a cover $\mathcal{C}$ of $\mathcal{H}'$ with $|\mathcal{C}| \leq 3$ by including in $\mathcal{C}$ a vertex of degree more than 3, and greedily covering the remaining uncovered edges. We will proceed via a series of claims.

Claim 7.3. *Every 6-partite, intersecting hypergraph $\mathcal{G}$ with 7 edges and satisfying $\Delta(\mathcal{G}) \leq 3$ has at least 2 vertices of degree 3.*

Proof. Suppose, for the sake of contradiction that $\mathcal{G}$ contains at most one vertex of degree 3. Let v be this vertex (if it exists). Since $\mathcal{G}$ is intersecting, there are $\binom{7}{2} = 21$ intersections between the edges. Three of these intersections can occur at v, and the rest must all occur at distinct vertices of degree 2. Therefore there must be at least 19 vertices in $\mathcal{G}'$ of degree ≥ 2. By the Pigeonhole Principle some partition of $\mathcal{G}$ has at least 4 vertices of degree at least 2. Since $\mathcal{G}$ has 7 edges, some edge must pass through two vertices in this partition contradicting $\mathcal{G}$ being 6-partite. ☐

Claim 7.4. *Every edge in $\mathcal{H}'$ contains a vertex of degree 3.*

Proof. If E is an edge of $\mathcal{H}'$, then it has 6 vertices and must intersect the 7 other edges of $\mathcal{H}'$. By the Pigeonhole Principle, one of the vertices of E must have degree 3. ☐

Claim 7.5. *For any pair of vertices u and v of degree 3 in $\mathcal{H}'$, $c(u,v) \geq 1$.*

Proof. Suppose that there are two vertices $u, v \in V(\mathcal{H}')$ of degree 3 which are not contained in a common edge. Then, since $|E(\mathcal{H}')| = 8$, there are only two edges in $\mathcal{H}'$ which do not contain either u or v. These two edges must intersect in some vertex w. This gives a cover $\{u, v, w\}$ of $\mathcal{H}'$ of order 3, contradicting our assumption that $\tau(\mathcal{H}') > 3$. ☐

Let $\mathcal{K}$ be the non-uniform hypergraph formed from $\mathcal{H}'$ by deleting the vertices with degree less than 3. Formally $V(\mathcal{K})$ is the set of vertices of $\mathcal{H}'$ with degree 3, and the edges of $\mathcal{K}$ are defined as $E(\mathcal{K}) = \{A \cap V(\mathcal{K}) : A \in \mathcal{H}'\}$. We allow $\mathcal{K}$ to have repeated edges in the case when $A \cap V(\mathcal{K}) = A' \cap V(\mathcal{K})$ for distinct edges $A, A' \in \mathcal{H}'$.

Notice that by Claim 7.4, we have that $|\mathcal{K}| = |\mathcal{H}'| = 8$, and the edges in $\mathcal{H}$ have order at least 1. Moreover, from the definition of $\mathcal{K}$, we have that $\mathcal{K}$ satisfies the conclusion of Claim 7.5 and $\mathcal{K}$ is 3-regular.

Claim 7.6. *Let A be an edge of $\mathcal{K}$. We have that $|A| \leq |V(\mathcal{K})| - 2$.*

Proof. By the definition of $\mathcal{K}$, there is an edge $A' \in \mathcal{H}'$ satisfying $A = A' \cap V(\mathcal{K})$. Let $\mathcal{H}''$ be the hypergraph formed from $\mathcal{H}'$ by removing the edge A'. It is easy to check that $\mathcal{H}''$ satisfies all the conditions of Claim 7.3, and hence contains two vertices u and v with degree 3. Since $\Delta(\mathcal{H}') \leq 3$, the vertices u and v could not be contained in A' (or A) giving the result. ☐

Claim 7.7. $|V(\mathcal{K})| = 6$.

Proof. All edges in $\mathcal{K}$ contain at least one vertex of degree 3 by Claim 7.4, which by combining with Claim 7.6 implies that $|V(\mathcal{K})| \geq 3$.

Suppose that $|V(\mathcal{K})| = 3$. By Claim 7.6, we have that $|E| \leq 1$ for every edge $E \in \mathcal{K}$. This contradicts $\mathcal{K}$ satisfying Claim 7.5.

Suppose that $|V(\mathcal{K})| = 4$. As in the previous case, Claim 7.6 implies that we have $|E| \leq 2$ for every edge $E \in \mathcal{K}$. Then, Claim 7.5 implies that for every pair of distinct vertices $u, v \in V(\mathcal{K})$ the edge $\{u, v\}$ is in $\mathcal{K}$. Since $\mathcal{K}$ is 3-regular, there cannot be any other edges in $\mathcal{K}$, which contradicts $|E(\mathcal{K})| = 8$.

Suppose that $|V(\mathcal{K})| = 5$. Claim 7.6 implies that we have $|E| \leq 3$ for every edge $E \in (K)$. Let e_i be the number of edges $E \in \mathcal{K}$ satisfying $|E| = i$. Notice that since $|E| \leq 3$ for every edge $E \in \mathcal{K}$, we have that $e_i = 0$ for $i > 3$. We also note that an edge of order one cannot be repeated, because this implies there is a vertex $v \in V(\mathcal{K})$ that is contained in two edges of order one. However, since we have $\Delta(\mathcal{K}) \leq 3$, this implies by Claim 7.5 that $\mathcal{K}$ contains an edge that contains v and passes through the other 4 vertices of $\mathcal{K}$, which is a not possible since $e_5 = 0$.

Since $\mathcal{K}$ has 5 vertices and 8 edges and is 3-regular, we have the following.

$$e_1 + e_2 + e_3 = |\mathcal{K}| = 8, \tag{7.1}$$
$$3e_3 + 2e_2 + e_1 = 3|V(\mathcal{K})| = 15. \tag{7.2}$$

Combining (7.1) and (7.2), we obtain the following

$$e_3 = e_1 - 1, \tag{7.3}$$
$$e_2 = 9 - 2e_1, \tag{7.4}$$

There are five cases, depending on the value of e_1.

- Suppose that $e_1 \leq 1$. Then (7.3), together with $e_3 \geq 0$ implies that in fact $e_1 = 1$ and hence from (7.3) and (7.4) we obtain $e_2 = 7$ and $e_3 = 0$. This contradicts Claim 7.5 which implies that $e_2 + 3e_3 \geq \binom{5}{2} = 10$.
- Suppose that $e_1 = 2$. Then we have $e_3 = 1$ and $e_2 = 5$. Again, this contradicts $e_2 + 3e_3 \geq \binom{5}{2} = 10$.
- Suppose that $e_1 = 3$. Then we have $e_3 = 2$ and $e_2 = 3$. Let $\{v_1\}$, $\{v_2\}$, and $\{v_3\}$ be the three edges of $\mathcal{K}$ of order 1. Notice that by Claim 7.5 and $\Delta(\mathcal{K}) \leq 3$, for each i, the vertex v_i must be contained in two edges E, F of order 3 satisfying $E \cap F = \{v_i\}$. This leads to a contradiction since there are only two edges in $\mathcal{K}$ of order 3.
- Suppose that $e_1 = 4$. Then we have $e_3 = 3$ and $e_2 = 1$. Let $\{v_1, v_2\}$ be the edge of order 2 in $\mathcal{K}$. Since $|V(\mathcal{K})| = 5$ and there are four edges of $\mathcal{K}$ of order 1, either $\{v_1\}$ or $\{v_2\}$ must be an edge of $\mathcal{K}$. There can only be one more edge going through this vertex, and by Claim 7.5, it would also have to pass through the remaining three vertices v_3, v_4, and v_5. This contradicts $|E| \leq 3$ holding for every edge in $\mathcal{K}$.
- Suppose that $e_1 > 4$. In this case (7.4) gives $|e_2| < 0$ which is impossible.

□

Claim 7.8. *The hypergraph $\mathcal{K}$ contains two edges E and F such that $E \cap F \geq 2$.*

Proof. Claim 7.6 implies that we have $|E| \leq 4$ for any edge $E \in \mathcal{K}$. Suppose that we have an edge E of order 4 in $\mathcal{K}$. Let $E = \{v_1, v_2, v_3, v_4\}$. Since $\mathcal{K}$ is 3-regular each vertex v_i is contained in two edges F_i^1 and F_i^2 other than E. Since $\mathcal{K}$ has 8 edges, $F_i^a = F_j^b$ for some $i \neq j$. Therefore we have $\{v_i, v_j\} \subseteq F_i^a \cap E$ implying the claim.

Suppose that all edges $E \in \mathcal{K}$ satisfy $|E| \leq 3$. If a vertex $v \in V(\mathcal{K})$ is contained in three edges of order 3, then two of these edges have intersection of size greater than 2, proving the claim. Therefore we have that any $v \in V(\mathcal{K})$ is contained in at most two edges of order 3. By Claim 7.5, every vertex $v \in V(\mathcal{K})$ is then contained in exactly two edges of order 3 and one edge of order 2. The number of edges of order 2 in $\mathcal{K}$ must therefore be $|V(\mathcal{K})|/2 = 3$ and the number of edges of order 3 in $\mathcal{K}$ must be $2|V(\mathcal{K})|/3 = 4$. This contradicts $\mathcal{K}$ having 8 edges. □

Now Claim 7.7 proves that $\mathcal{H}'$ contains six vertices of degree 3, and Claim 7.5 shows that these vertices are all in different partitions of $\mathcal{H}$. Claim 7.8 shows that there exist at least two edges in $\mathcal{H}'$ such that they share at least two vertices of degree 3 in common. Together these facts prove Lemma 7.2. □

Using Lemma 7.2 we are able to determine precisely all possible degree structures of intersecting 6-partite hypergraphs with 8 edges and a covering number equal to 4.

Lemma 7.9. *If $\mathcal{H}'$ is an intersecting 6-partite hypergraph with 8 edges and $\tau(\mathcal{H}') = 4$, then $\mathcal{H}'$ has one of the following degree structure:*

- *In all 6 partitions of $\mathcal{H}'$, each partition contains one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1, or*
- *In 5 partitions of $\mathcal{H}'$ it contains one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1, and in the 6th partition it contains one vertex of degree 3, one vertex of degree 2, and four vertices of degree 1.*

Proof. Since $\mathcal{H}'$ is an intersecting hypergraph that contains 8 edges, the number of intersections between the edges of $\mathcal{H}'$ is at least $\binom{8}{2} = 28$. From Lemma 7.2 we also know that $\Delta(\mathcal{H}') = 3$ and that $\mathcal{H}'$ contains six vertices of degree 3. Since each vertex of degree 3 contributes 3 intersections between the edges of $\mathcal{H}'$, the maximum number of intersections contributed by the vertices of degree 3 is 18.

However, by Lemma 7.2, we know that at least one pair of edges have in common at least two vertices of degree 3, therefore we can reduce the previous bound by 1 to account for this duplication, which makes the maximum number of intersection contributed by the vertices of degree 3 equal to 17. Hence, the vertices of degree 2 in $\mathcal{H}'$ need to account for at least $28 - 17 = 11$ of the intersections in $\mathcal{H}'$.

Since $|E(\mathcal{H}')| = 8$, and each partition of $\mathcal{H}'$ contains a vertex of degree 3, the maximum number of degree 2 vertices that $\mathcal{H}'$ can contain in each partition is two. Therefore if $\mathcal{H}'$ contains 11 vertices of degree 2 then by the Pigeonhole Principle in at least five partitions of $\mathcal{H}'$ it will contain two vertices of degree 2, and in the remaining partition we must have either one vertex of degree 2 or two vertices of degree 2.

If one of the partitions of $\mathcal{H}'$ contains exactly one vertex of degree 2, then apart from the vertex of degree 3 the remaining vertices in that partition will all have degree 1. These two possibilities prove the degree scheme stated in Lemma 7.9. □

Lemma 7.10. $f(6) \neq 12$

Proof. Let $\mathcal{H}$ be a 6-partite intersecting hypergraph containing 12 edges and assume that $\tau(\mathcal{H}) = 5$. We will proceed by showing that all possible values of $\Delta(\mathcal{H})$ lead to a contradiction.

Case $\Delta(\mathcal{H}) \geq 6$: Assume that $\Delta(\mathcal{H}) \geq 6$, and let $v \in V(\mathcal{H})$ be vertex such that $d(v) \geq 6$, finally denote by $\mathcal{H}' \subset E(\mathcal{H})$ the set of edges that don't contain v, which forms an intersecting 6-partite sub-hypergraph of $\mathcal{H}$.

We have $|E(\mathcal{H}')| \leq 6$ and therefore we can greedily cover $\mathcal{H}'$ with a cover $\mathcal{C}$ such that $|\mathcal{C}| \leq 3$. Therefore the set $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers $\mathcal{H}$, and $|\mathcal{C}'| < 5$ which contradicts $\mathcal{H}$ being extremal.

Case $\Delta(\mathcal{H}) = 5$: Assume that $\Delta(\mathcal{H}) = 5$ and let $v \in V(\mathcal{H})$ such that $d(v) = 5$, and define the intersecting 6-partite sub-hypergraph $\mathcal{H}' \subset E(\mathcal{H})$ to consist of the 7 edges in $E(\mathcal{H})$ that don't contain v.

If $\mathcal{H}'$ has a cover C such that $|C| \leq 3$, then the cover $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers $\mathcal{H}$ and $|\mathcal{C}'| < 5$ which contradicts $\mathcal{H}$ being extremal. We can therefore assume that $\tau(\mathcal{H}') = 4$.

If any 3 or more edges of $\mathcal{H}'$ intersect in a vertex v', then we can greedily cover the remaining edges of $\mathcal{H}'$ by 2 vertices or less, contradicting $\tau(\mathcal{H}') = 4$. Therefore, we can suppose that $\Delta(\mathcal{H}') \leq 2$.

However, if $\Delta(\mathcal{H}') \leq 2$ then the maximum number of intersections that can occur in a partition of $\mathcal{H}'$ is 3 intersections, which occurs when a partition of $\mathcal{H}'$ contains three vertices of degree 2. It follows that the maximum number of intersections in all of $\mathcal{H}'$ is equal to 18. However, we require at least $\binom{7}{2} = 21$ intersections between edges of $\mathcal{H}'$, for $\mathcal{H}'$ to be an intersecting hypergraph, which leads to a contradiction.

Case $\Delta(\mathcal{H}) = 4$: Assume that $\Delta(\mathcal{H}) = 4$ and let $v \in V(\mathcal{H})$ be a vertex such that $d(v) = 4$. Let $\mathcal{H}'$ be the intersecting 6-partite sub-hypergraph $\mathcal{H}' \subset E(\mathcal{H})$ consisting of the 8 edges in $E(\mathcal{H})$ that don't contain v.

If we can cover $\mathcal{H}'$ by a cover $\mathcal{C}$ such that $|\mathcal{C}| \leq 3$, then the set $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers the whole of $\mathcal{H}$, and since $|\mathcal{C}'| < 5$ this will contradict $\mathcal{H}$ being extremal. Therefore we can assume that $\tau(\mathcal{H}') = 4$.

Since $\tau(\mathcal{H}') = 4$, then as in the proof of Lemma 7.2, we must have $\Delta(\mathcal{H}') \leq 3$ (since otherwise, we could cover 4 edges by one vertex, and the remaining edges greedily by 2 vertices.).

Denote by $\mathcal{H}''$ the set of four edges that contain the vertex v' of degree 4 (i.e. the edges not in $\mathcal{H}'$). Since $\mathcal{H}$ is an intersecting hypergraph, the number of intersections in $\mathcal{H}$ between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ is equal to $4 \cdot 8 = 32$, and these intersections need to occur in 5 partitions of $\mathcal{H}$; since in the partition that contain v' the edges in $\mathcal{H}''$ are disjoint from the edges in $\mathcal{H}'$.

From Lemma 7.9 we know that $\mathcal{H}'$ can have two types of degree schemes in its partitions, which we will refer to as *Type A* and *Type B*:

Type A: Partitions that have *Type A* contain one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1,

Type B: Partitions that have *Type B* contain one vertex of degree 3, one vertex of degree 2 and four vertices of degree 1.

We will now establish the maximum number of intersections possible that can occur between the edges of $\mathcal{H}''$ and the edges of $\mathcal{H}'$ in each of the two types of degree schemes and show that this is less the minimum required for $\mathcal{H}$ to be intersecting.

Claim 7.11. *Let S be a partition of $\mathcal{H}'$ of Type A. Then the maximum number of intersections in $\mathcal{H}$ that can occur between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ within S is at most* 6.

Proof. If all the edges in $\mathcal{H}''$ contained a vertex from S, then S would cover all of $\mathcal{H}$ and $|S| = 4$, contradicting the fact that $\mathcal{H}$ is extremal. Thus at most three edges in $\mathcal{H}''$ can contain a vertex from S.

Let w' be the vertex in S that has degree 3 in $\mathcal{H}'$. We note that if more than one edge from $\mathcal{H}''$ contained w', then w' will have a degree in $\mathcal{H}$ that exceeds 4, which contradicts $\Delta(\mathcal{H}) = 4$. Therefore at most one edge of $\mathcal{H}''$ can contain w'.

Suppose that at most two edges of $\mathcal{H}''$ contain vertices from S, since at most one of them can contain a vertex of degree 3, this case trivially satisfies the claim.

Thus the only remaining case that needs to be checked is when three edges of $\mathcal{H}''$ contain a vertex from S.

Let e_i be the number of edges in $\mathcal{H}''$ that contain a vertex in S of degree i in $\mathcal{H}'$. From the above we have:

$$e_1 + e_2 + e_3 \leq 3 \tag{7.1}$$
$$e_1 \leq 3 \tag{7.2}$$
$$e_2 \leq 3 \tag{7.3}$$
$$e_3 \leq 1 \tag{7.4}$$

Suppose that exactly three edges of $\mathcal{H}''$ contain vertices from S, and one of the edges in $\mathcal{H}''$ contains w'. Let e and e' denote the remaining two edges of $\mathcal{H}'$ that contain a vertex in S. It can be seen that e and e' contain the vertices in S of degree 2 in $\mathcal{H}'$ in three possible ways, and we first show two of these possibilities lead to a contradiction:

- Each of e and e' contain a different vertex in S of degree 2 in $\mathcal{H}'$. In this case, w' and the two vertices in S of degree 2 will cover $4 + 3 + 3 = 10$ edges of $\mathcal{H}$, and since we can cover the remaining two edges of $\mathcal{H}$ by a vertex, this will contradict $\tau(\mathcal{H}) = 5$.
- Edges e and e' contain the same vertex in S of degree 2 in $\mathcal{H}'$. In this case the aforementioned vertex and w' cover $4 + 4 = 8$ edges of $\mathcal{H}$. Since, we can greedily cover the remaining 4 edges of $\mathcal{H}$ with 2 vertices, this will allow us to cover $\mathcal{H}$ with 4 vertices, contradicting the fact that $\tau(\mathcal{H}) = 5$.

- At most one of the edges e and e' contains a vertex in S of degree 2 in $\mathcal{H}'$.

From the above case analysis, it follows that if one of the edges in $\mathcal{H}''$ contained the vertex in S of degree 3 in $\mathcal{H}'$, then at most one edge from $\mathcal{H}''$ contains a vertex in S of degree 2 in $\mathcal{H}'$. We represent this as the inequality:

$$e_2 + 2e_3 \leq 3 \tag{7.5}$$

The number of intersections between edges in $\mathcal{H}''$ and vertices in S, can be represented as the inequality $e_1 + 2e_2 + 3e_3$. By combining the inequalities (7.1) and (7.5) we obtain the following bound on the number of intersections:

$$e_1 + 2e_2 + 3e_3 \leq 6 \tag{7.6}$$

Which proves that the maximum number of intersections between the set $\mathcal{H}''$ and partitions with degree scheme of *Type A* is equal to 6.

□

Claim 7.12. *Let S be a partition of $\mathcal{H}$ of Type B. Then the maximum number of intersections in $\mathcal{H}$ that can occur between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ within S is at most 7.*

Proof. Let w' be the vertex in S of degree 3 in $\mathcal{H}'$, and let w'' be the vertex in S of degree 2 in $\mathcal{H}'$. We note that no more than one edge of $\mathcal{H}''$ can contain w', otherwise w' will have a degree that exceeds 4 in $\mathcal{H}$ which contradicts $\Delta(\mathcal{H}) = 4$. Similarly, $\Delta(\mathcal{H}) = 4$ implies that the maximum number of edges in $\mathcal{H}''$ that can contain w'' in S is equal to 2.

Let e_i be the number of edges in $\mathcal{H}''$ that contain a vertex in S of degree i in $\mathcal{H}'$. From the above we have:

$$\begin{aligned} e_1 + e_2 + e_3 &\leq 4 && (7.1)\\ e_1 &\leq 4 && (7.2)\\ e_2 &\leq 2 && (7.3)\\ e_3 &\leq 1 && (7.4) \end{aligned}$$

If a edge in $\mathcal{H}''$ contains w', and more than one edge in $\mathcal{H}''$ contain w'', then w' and w'' cover 8 or more edges of $\mathcal{H}$, and therefore the remaining edges can be greedily covered by two vertices or less, contradicting $\tau(\mathcal{H}) = 5$. Thus if one of the edges in $\mathcal{H}''$ contains w', then at most one other edge of E' can contain w'', or in inequality form:

$$e_2 + 2e_3 \leq 3 \tag{7.5}$$

We have that the expression $e_1 + 2e_2 + 3e_3$ represents number of intersections between $\mathcal{H}''$ and $\mathcal{H}'$, which we can bound by combining the inequalities (7.1) and (7.5) we obtain:

$$e_1 + 2e_2 + 3e_3 \leq 7 \tag{7.6}$$

Which proves that the maximum number of intersections between the set of vertices $\mathcal{H}''$ and $\mathcal{H}'$ in a partition with degree scheme of Type B is equal to 7.

□

Since there is only one partition with degree scheme of *Type B*, and all intersections between $\mathcal{H}''$ and $\mathcal{H}'$ occur in five partitions of $\mathcal{H}'$ then the maximum number of intersection that can occur between $\mathcal{H}''$ and $\mathcal{H}'$ is equal to $7 + 6 \cdot 4 = 31$, which is one short of the 32 intersections required to make $\mathcal{H}$ intersecting, a contradiction.

Case $\Delta(\mathcal{H}) \leq 3$: Since $\mathcal{H}$ is extremal each partition needs to have at least 5 vertices (otherwise the vertices of partition with less than 5 vertices will form a cover of $\mathcal{H}$ contradicting $\tau(H) = 5$), therefore each partition can have at most three vertices with degree 3.

Hence the maximum number of intersections between the edges that can occur in a particular partition of $\mathcal{H}$ is when the partition consists of three vertices with degree 3, along with another vertex of degree 2 and another vertex of degree 1, in which case the maximum number of intersections per partition would be equal to 10. It follows that the maximum total number of intersections that can occur in all the partitions of $\mathcal{H}$ is 60.

However, if $\mathcal{H}$ is an intersecting hypergraph with 12 edges then it will need to have $\binom{12}{2} = 66$ intersections. Therefore a hypergraph with $\Delta(\mathcal{H}) \leq 3$ can't be extremal.

□

We now present a 6-partite intersecting hypergraph $\mathcal{H}$ such that $\tau(\mathcal{H}) = 5$. All sides of $\mathcal{H}$ except the first one contain 5 vertices, while the first side contains 6 vertices.

144535	252553	345343	415455	454244
525514	551325	543252	534433	624351
642424	655132	633545		

6-partite extremal hypergraph, $\mathcal{H}$

Lemma 7.13. $\tau(\mathcal{H}) = 5$

Lemma 7.13 (which can be easily verified using a computer) allows us to complete the first part of Lemma 7.1. From [19] we know that $f(6) > 11$, and by Theorem 7.10 we know that $f(6) \neq 12$. Therefore, by Lemma 7.13 we have that $f(6) = 13$.

7.2 Bounding $f(7)$ and $f(11)$

In this section we prove the second part of Theorem 7.1 by presenting two intersecting hypergraphs: the first is 7-partite with covering number equal to 6, and the second is 11-partite with covering number equal to 10.

Both hypergraphs were generated by the aid of a computer search and we we briefly outline the idea used in the search after we present the graphs.

The following is an intersecting 7-partite hypergraph with 22 edges and a covering number of size 6.

1111111	1222222	1313333	1333434	1444545	1555653
2123563	2341626	2645231	3115524	3126635	3362551
3543132	3543217	3631243	4142443	4251537	4313255
4521531	5314233	5325147	6213641		

7-partite extremal hypergraph, $\mathcal{H}'$

An example of a cover of size 6 of $\mathcal{H}'$, is the cover consisting of all vertices in the first side of $\mathcal{H}'$. While the following hypergraph, $\mathcal{H}''$ is an intersecting 11-partite hypergraph with 51 edges, and a covering number of size 10.

00000000000	00000000111	00000001012	00000010013
00000100014	00001000015	00010000016	00100000017
01222222228	02333333338	03444444448	04555555558
05000000019	13366065264	14273470672	16085636427
21570783465	22465277081	23637590722	24728034894
27342608653	30378296549	32704885626	33299739051
37583042784	38650674235	43823687510	44682863041
47954230962	49305724776	51384579810	57876354021
58235088944	59693205493	61938645071	62287904563
63502376996	64046289737	68329850482	71606438586
72852099474	77027775347	79488380252	82596620842
83780258375	84834706283	86357482091	87208567436
88472935710	89949073524	90000000018	

11-partite extremal hypergraph, $\mathcal{H}''$

An example of a cover of size 10 of hypergraph $\mathcal{H}''$, is the cover consisting of all vertices in the first side of $\mathcal{H}''$.

The existence of $\mathcal{H}'$ and $\mathcal{H}''$ allows us to prove Theorem 7.1.

7.3 Generating extremals

It would have been impractical to find the hypergraphs $\mathcal{H}'$ and $\mathcal{H}''$ presented in the previous section via an exhaustive search (total enumeration), because of the size of the search space. Therefore to find the extremal hypergraphs presented in the previous section, we had to use a randomised method that used an appropriate heuristic that sufficiently limits the search space. For our purpose, we only needed the search space to be limited to the point where it was feasible to construct a 7-partite and 11-partite extremal intersecting hypergraph in a reasonable amount of time.

Our heuristic for finding an r-partite extremal intersecting hypergraph consisted in first starting with an $(r-1)$-partite extremal intersecting hypergraph $\mathcal{K}$. The second step then consisted in adding one vertex to each edge of $\mathcal{K}$, and designating all the new vertices as a new partition, which turns the resultant hypergraph into an r-partite intersecting hypergraph with cover number $r-2$. The third step was to randomly add intersecting edges to the resultant r-partite hypergraph, until its cover number increases by 1, at which point an r-partite extremal hypergraph is constructed. The final step, was to search within the constructed extremal hypergraph, for the sparsest possible extremal subhypergraph.

By experimenting we found that another element of randomness can be added to the above process, namely in the second step. We observed that the edges of the starting hypergraph don't need to be disjoint when extended to the new partition for the process to work. In fact, the edges can be extended in the new partition to intersect each other in various ways – and the exact configuration of the intersections can be randomised – subject to two constraints.

First, each edge should gain only one new vertex in the new partition (to maintain the correct partiteness of the final resultant graph). Second, none of the new intersections should reduce the cover number of the initial set of edges below $r-2$, assuming the original hypergraph was $(r-1)$-partite and had cover number $r-2$.

By implementing the above random process and running it for a relatively short duration (no more than hours), we were able to generate the 7-partite extremal hypergraph $\mathcal{H}'$ and the 11-partite extremal hypergraph $\mathcal{H}''$.

To find the 7-partite extremal hypergraph our starting hypergraph was the 6-partite extremal $\mathcal{H}$ presented in this chapter. While to find the 11-partite extremal hypergraph, our starting hypergraph was built from a subhypergraph of the 10-partite truncated projective plan extremal hypergraph.

Moreover, by experimenting with other low values of r, we observed that the above method succeeded frequently in turning subhypergraphs of the r-partite truncated projective plane extremal hypergraph into new $(r+1)$-partite extremal hypergraphs.

This leads us to the question of whether there are any structural properties that causes this process of randomly generating extremal hypergraphs to succeed? As we show in the next chapter, there does indeed turn out to be an affirmative answer to this question. In particular, we provide a deterministic recipe that always guarantees turning an r-partite truncated projective plane extremal hypergraph, into a new $(r+1)$-partite extremal hypergraph.

CHAPTER 8

NEW EXTREMALS FROM OLD

In this chapter we prove the following theorem:

Theorem 8.1. *Let $\mathcal{T}$ be an r-partite intersecting hypergraph, and S an edge of $\mathcal{T}$.*

If $\mathcal{T}$ and S satisfy the following two conditions:

- *The edge S intersects every other edge of $\mathcal{T}$ in exactly one vertex, and*
- *We have that $\tau(\mathcal{T} - S) = r - 1$, and the only covers of $\mathcal{T} - S$ of size $r - 1$ are sides.*

Then there exists an $(r+1)$-partite intersecting hypergraph that is extremal to Ryser's Conjecture.

We note that the truncated projective plane hypergraph of uniformity r satisfies the conditions in Theorem 8.1 for all $r \geq 4$, where the edge S can be any of its edges. We also recall that truncated projective planes of uniformity r are known to exist for all $r = q + 1$, such that q is a prime power. This allows us to prove the existence of a new infinite family of extremal intersecting hypergraphs.

Corollary 8.2. *For any prime power q there exists an $(q+2)$-partite extremal hypergraph to Ryser's Conjecture.*

In particular, Corollary 8.2 implies the existence of all the sporadic extremal existence results that are currently known in the literature, and which were mentioned in the Introduction of this thesis. Namely it implies the existence of a 7-partite, 11-partite and a 13-partite extremal hypergraph to Ryser's Conjecture.

Moreover, in Section 2 of this chapter we will show by using a number theoretic argument, that for infinitely many values of r, Corollary 8.2 implies the existence of an r-partite extremal hypergraph to Ryser's Conjecture, when an r-partite truncated projective plane extremal hypergraph is ruled out by the Bruck-Ryser Theorem.

8.1 The construction

In this section, we prove Theorem 8.1.

We define an $\{r-1, r\}$-uniform hypergraph to be a family of sets of size $r-1$ and r.

Notice that in order to find an r-uniform hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) = r-1$, it suffices to find an $\{r-1, r\}$-uniform $\mathcal{H}'$ with $\tau(\mathcal{H}') = r-1$. Once we have such a hypergraph we can construct an r-uniform hypergraph from $\mathcal{H}'$ by adding a separate new vertex to each edge of size $r-1$.

Let $\mathcal{T}$ be an r-partite r-uniform intersecting hypergraph with sides $V_1, \ldots, V_r$. Let $S = \{s_1, \ldots, s_r\}$ be an edge of $\mathcal{T}$, with $s_i \in V_i$, that satisfies the conditions in Theorem 8.1.

Let $F_1, \ldots, F_r$ be r edges of $\mathcal{T}$ with $s_i \in F_i \cap S$ for each i, and also $(F_i - s_i) \cap (F_j - s_j) \neq \emptyset$ for all i, j. The edges $F_1, \ldots, F_s$ do not have to be distinct — one possibility is to take $F_1 = \cdots = F_r = S$.

We define an $\{r, r+1\}$-uniform, intersecting hypergraph $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$, which has cover number r.

- The vertex set of $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ consists of the vertex set of $\mathcal{T}$ together with r vertices $v_1, \ldots, v_r$ in side V_{r+1}.
- For an edge $E \neq S$ of $\mathcal{T}$ satisfying $E \cap S = s_i$, we define $\hat{E} = E + v_i$. That is, $\hat{E}$ is an $(r+1)$-edge built from E by adding the vertex v_i corresponding to the vertex of S which E contains. Notice that $\hat{E}$ is well-defined since S intersects any other edge of $\mathcal{T}$ in exactly one point.

  Define

$$\begin{aligned}
\mathcal{E}_1 &= \{\hat{E} : E \in \mathcal{T} - S\} \\
\mathcal{E}_2 &= \{F_i : i = 1, \ldots, r\} \\
\mathcal{E}_3 &= \{F_i - s_i + v_i : i = 1, \ldots, r\}.
\end{aligned}$$

  We let $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r) = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$.

In other words $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ has three parts: The first part consists of taking the $(r+1)$-edges $\hat{E}$ for all $E \in \mathcal{T}$ other than S. The second part consists of the r-edges F_i. The third part consists of the r-edges formed from $F_1, \ldots, F_r$ by deleting the vertex at which they intersect S, and then adding the corresponding vertex v_i.

First we show that these hypergraphs are intersecting.

Lemma 8.3. *$\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ is intersecting.*

Proof. The hypergraph $\mathcal{E}_1 \cup \mathcal{E}_2$ is intersecting simply because restricting it to the first r sides gives $\mathcal{T}$ which is an intersecting hypergraph. Furthermore for any i and j, we have $(F_i - s_i + v_i) \cap (F_j - s_j + v_j) \supseteq (F_i - s_i) \cap (F_j - s_j) \neq \emptyset$ by assumption.

It remains to show that edges in $\mathcal{E}_1$ intersect those in $\mathcal{E}_3$ i.e. that $\hat{E} \cap (F_i - s_i + v_i) \neq \emptyset$ for any $E \neq S$ and $i = 1, \ldots, r$. Since $\mathcal{T}$ is intersecting, there is some vertex $x \in E \cap F_i$. If $x \neq s_i$, then we have $x \in \hat{E} \cap (F_i - s_i + v_i)$. Otherwise, we have $v_i \in \hat{E} \cap (F_i - s_i + v_i)$.

□

We show that the covers of the hypergraphs $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ have a very specific structure.

Lemma 8.4. *If C is a cover of $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$, then $C' = (C \cup \{s_i : v_i \in C\}) \setminus \{v_1, \ldots, v_r\}$ is a cover of $\mathcal{T} - S$.*

Proof. Let E be an arbitrary edge of $\mathcal{T} - S$. We show that $E \cap C' \neq \emptyset$. We know that $C \cap \hat{E} \neq \emptyset$, since C is a cover of $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$. Let y be a vertex in $C \cap \hat{E}$. If $y \notin \{v_1, \ldots, v_r\}$, then $y \in C'$ which implies $C' \cap E \neq \emptyset$. Otherwise $y = v_i$ for some i, which implies that $s_i \in C' \cap E$. □

We now prove that $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ has covering number $r + 1$. This immediately implies Theorem 8.1 (by taking $\mathcal{H}$ to be $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ with a new vertex added to each of its r-edges).

Theorem 8.5. *The hypergraph $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ is an $(r + 1)$-partite, $\{r, r + 1\}$-uniform, intersecting hypergraph with $\tau(H(\mathcal{T}, S, F_1, \ldots, F_r)) = r$.*

Proof. It is immediate that $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ is $(r+1)$-partite and $\{r, r+1\}$-uniform — the r-edges $E \in \mathcal{T} - S$ just gained a vertex v_i in V_{r+1} in order to become $\hat{E}$, whereas the r-edges F_i had vertex s_i deleted in V_i, and v_i added in V_{r+1}. From Lemma 8.3, $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ is intersecting. It remains to prove that $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ has covering number r.

Suppose to the contrary that there is a cover C of $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$ with $|C| \leq r - 1$. By Lemma 8.4, C' is a cover of $\mathcal{T} - S$ and for its size we have $|C'| \leq |C| \leq r - 1$. By the assumption of Theorem 8.1, C' must be one of the sides V_i for $i = 1, \ldots, r$. Then the definition of C' implies that the cover C is either V_i or $V_i - s_i + v_i$. In the first case C does not cover the edge $F_i - s_i + v_i$, while in the second case C does not cover the edge F_i, both contradicting the assumption that C is a cover of $\mathcal{H}(\mathcal{T}, S, F_1, \ldots, F_r)$. □

8.2 Further discussion

We recall the Bruck-Ryser Theorem:

Bruck-Ryser Theorem. *If a finite projective plane of order q exists and $q \equiv 1$ or 2 (mod 4), then q must be the sum of two squares.*

Thus the Bruck-Ryser Theorem allows us to rule out the existence of an r-partite truncated projective plane extremal hypergraph for infinitely many values of r, the first two being 7 and 15. However, we know from Corollary 8.2, that there does exist a 7-partite and 15-partite extremal hypergraph, which can constructed from respectively a 6-partite, and a 14-partite truncated projective plane extremal hypergraph.

This raises the question of how often does this situation occur? Or in other words, for how many values of r does Corollary 8.2 provide an r-partite extremal hypergraph, when a truncated projective plane extremal hypergraph is ruled out by the Bruck-Ryser Theorem?

The following lemma proves that this situation occurs infinitely many times:

Lemma 8.6. *There exists infinitely many values of r, such that an r-partite extremal hypergraph to Ryser's Conjecture exists, when an r-partite truncated projective plane extremal is known not to exist.*

Proof. To be able to use the Bruck-Ryser Theorem, we will first show that a certain class of integers cannot be represented as a sum of two squares.

Claim 8.7. *If M is an integer of the form $8m+5$ for $m \in \mathbb{N}$, then $M+1$ cannot be represented as the sum of two squares.*

Proof. We note that numbers of the form $4m'+3$ have an odd number of prime factors of the same form. Thus if M is an integer of the form $8m+5$, then $M+1 = 8m+6 = 2(4m'+3)$ has also an odd number of prime factors of the form $4m'+3$.

A well-known result in number theory (and a consequence of Fermat's Theorem) asserts the integers that can be represented as the sum of two squares, are precisely the integers whose prime factorisation has an even exponent for every prime factor of the form $4n+3$ (see for example [6] for a simple proof). Since we saw that $M+1$ doesn't satisfy this condition with its prime factors, we conclude that it cannot be represented as the sum of two squares. □

Claim 8.8. *There are infinitely many primes P, such that $P+1 \equiv 2 \bmod 4$, and such that $P+1$ cannot be represented as the sum of two squares.*

Proof. From Dirichlet's Prime Number Theorem [8], we know that there are infinitely many primes of the form $8m+5$. Moreover, it follows from Claim 8.7 that for every prime P of the form $8m+5$, $P+1$ cannot be represented as the sum of two squares, and since $P+1 \equiv 2 \bmod 4$ this proves the claim. □

Thus we see from Claim 8.8 that there are infinitely many primes P, such that $P+1$ doesn't satisfy the conditions of the Bruck-Ryser Theorem. This implies there are infinitely many primes P such that an $(P+1)$-partite truncated projective plane extremal hypergraph exists, but no $(P+2)$-partite truncated projective plane extremal hypergraph exists. However for every such such prime P, Theorem 8.1 guarantees the existence of a $(P+2)$-partite extremal intersecting hypergraph, which can be constructed from the $(P+1)$-partite truncated projective plane extremal hypergraph. □

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