Two explicitly solvable problems with discretionary stopping

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I would like to dedicate this thesis to my father

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

Abstract

This thesis is concerned with two explicitly solvable stochastic control problems that incorporate discretionary stopping. The first of these problems combines the features of the so-called monotone follower of singular stochastic control theory with optimal stopping. The uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. The aim of the problem is to maximise the utility derived from the system's controlled state at the discretionary time when the system's control is terminated. This objective is reflected by an appropriate performance criterion, which also penalises control expenditure as well as waiting. In the presence of rather general assumptions, the optimal strategy, which can take one of three qualitatively different forms, depending on the problem data, is fully characterised.

The second problem is concerned with the optimal stopping of a diffusion with generalised drift over an infinite horizon. The dynamics of the underlying state process are similar to the ones of a geometric Brownian motion. In particular, the drift of the state process incorporates the process' local time at a given level in an additive way. The objective of this problem is to maximise the expected discounted payoff that stopping the underlying diffusion yields over all stopping times. The associated reward function is the one of a financial call option. The optimal stopping strategy can take six qualitatively different forms, depending on parameter values.

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Contents

Chapter 1

Introduction

In this thesis, we study two explicitly solvable stochastic control problems that incorporate discretionary stopping. First, we consider a stochastic system whose state is modelled by the controlled one-dimensional positive Itô diffusion

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{1.1}$$

where W is a standard one-dimensional Brownian motion, and the controlled process Z is an adapted càglàd increasing process. The objective of the optimisation problem that we solve is to maximise the performance criterion

$$J_x(Z,\tau) = \mathbb{E}\left[\int_0^\tau e^{-\Lambda_t} H(X_t)\,dt - \int_0^\tau e^{-\Lambda_t} K'(X_t) \circ dZ_t + e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right], \tag{1.2}$$

over all admissible choices of Z and all stopping times τ, where

$$\Lambda_t = \int_0^t r(X_u)\,du, \tag{1.3}$$

and

$$\int_0^\tau e^{-\Lambda_t} K'(X_t) \circ dZ_t = \int_0^\tau e^{-\Lambda_t} K'(X_t)\,dZ_t^{\mathrm{c}} + \sum_{0\le t\le\tau} \int_0^{\Delta Z_t} e^{-\Lambda_t} K'(X_t+s)\,ds, \tag{1.4}$$

in which expression, Z^{c} is the continuous part of the increasing process Z. It is worth noting that the integral given by (1.4), which we use to penalise control expenditure, was introduced by Zhu [38] and is now standard in the singular stochastic control literature.

Next, we consider the problem of optimally stopping the process X given by

$$dX_t = bX_t\, dt + \beta\, dL_t^z + \sigma X_t\, dW_t, \quad X_0 = x > 0, \tag{1.5}$$

for some constants $b \in \mathbb{R}$, $\beta \in]-1, 1[\setminus \{0\}$, $z > 0$ and $\sigma \neq 0$. The process L^z appearing here is the symmetric local time of X at level z (see Revuz and Yor [35, Exercise VI.1.25] for the precise definition), while W is a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion that is defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The stochastic differential equation (1.5) has a unique strong solution that is a strictly positive process (see Engelbert and Schmidt [18]). The value function of the optimal stopping problem that we study is defined by

$$v(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}(X_\tau - K)^+\right], \tag{1.6}$$

for some constants $r, K > 0$, where $\mathcal{T}$ is the set of all $(\mathcal{F}_t)$-stopping times.

To avoid repetitions, we discuss these problems and their relevant literature in more detail in the introductions of Chapters 2 and 3.

Chapter 2

A model for optimally advertising and launching a product

2.1 Introduction

In this chapter, we study the stochastic control problem defined by (1.1)–(1.4) in the introduction. This stochastic control problem is motivated by the following application that arises in the context of the so-called goodwill problem. A company considers the timing of launching a new product that they have developed. Prior to launching it in a given market, the company attributes an image to the product based on the market's attitudes to similar products, the new product's quality differences from existing products, and the company's own image in the market. We use X to model the evolution in time of the product's image. In this context, Z represents the effect of costly interventions, such as advertising, that the company can make to raise the product's image. The company's objective is to maximise their utility from launching the product minus their "dis-utility" associated with the cost of intervention and the cost of waiting. In particular, the company aims at maximising the performance index defined by (1.2)–(1.4) over all intervention strategies Z and launching times τ.

Optimal control problems addressing this type of application have attracted significant interest in the literature for about half a century. Most of the models that have been studied in this area involve deterministic control and can be traced back to Nerlove and Arrow [31] (see Buratto and Viscolani [14] and the references therein). More realistic models in which the product's image evolves randomly over time have also been proposed and studied (see Feichtinger, Hartl and Sethi [19] for a review and Marinelli [29] for some more recent references). In particular, Marinelli [29] considers extensions of the classical Nerlove and Arrow model, and studies a class of problems that involve linear dynamics of the state process, absolutely continuous control and linear or quadratic payoff functions. Also, Jack, Johnson and Zervos [21] study a related model involving singular control only, in which, the product is assumed launched at time 0 and the objective is to select an advertising strategy that maximises the expected payoff resulting from its marketing.

The problem that we solve combines the features of the so-called monotone follower of the singular stochastic control theory with optimal stopping. Singular stochastic control, which was introduced by Bather and Chernoff [7] and Beneš, Shepp and Witsenhausen [12], has a well-developed body of theory, and we do not attempt a comprehensive literature survey. Also, we refer the interested reader to Peskir and Shiryaev [32] for a recent exposition of the theory of optimal stopping. Models that combine singular control with discretionary stopping were introduced by Davis and Zervos [15] who assumed that the uncontrolled system dynamics follow a standard Brownian motion and considered quadratic cost functions. In the same context, Karatzas, Ocone, Wang and Zervos [23] solved the problem that arises if an additional finite-fuel constraint is incorporated. A problem combining the singular control of a Brownian motion with drift with optimal stopping was later studied by Ly Vath, Pham and Villeneuve [28]. More recently, Morimoto [30] studied a model similar to the one in Davis and Zervos [15] but with a controlled geometric Brownian motion instead of a controlled standard Brownian motion. Also, Bayraktar and Egami [9], motivated by issues in initial public offerings rather than the goodwill problem, solved a problem that has the same general structure as the one of the problem we consider here. These authors assumed that the uncontrolled state dynamics are given by a Brownian motion with drift added to a compound Poisson process with exponentially distributed jump sizes, and that $H(x) = 0$, $K'(x) = 1$, $r(x) = \varrho$ and $U(x) = \lambda x$ for all x, for some constants $\varrho, \lambda > 0$. It is of interest to observe that the optimal strategy derived in that paper has a qualitatively different form from the one we obtain here. In particular, reflecting the state process at a given level figures among the optimal tactics in Bayraktar and Egami [9] but is never optimal in the problem we study here (see also the discussion of our main results below).

The control of one-dimensional Itô diffusions such as the one we considered here has recently attracted considerable interest in the literature. The optimal stopping of such processes has been studied by Salminen [36], Alvarez [2, 3], Beibel and Lerche [11], Dayanik and Karatzas [17], Dayanik [16] and Lamberton and Zervos [27], among others. Also, Alvarez [1, 4], Bayraktar and Egami [8], and Jack, Johnson and Zervos [21] have studied several singular control problems, Alvarez [5], and Alvarez and Lempa [6] have studied models with impulse control, while Bayraktar and Egami [10], Pham, Ly Vath and Zhou [34] and Johnson and Zervos [22] have analysed models with sequential switching (see also Pham [33]). In the spirit of certain references in this rather incomplete list, we solve the problem we consider by constructing an appropriate solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. To the best of our knowledge, the model that we study here is the first one that combines the singular control of a general one-dimensional Itô diffusion with optimal stopping.

It turns out that the optimal strategy of the problem that we solve here may involve only a single impulse applied to the state process. In particular, the optimal strategy does not involve reflecting the state process in the boundary of a state space's subset, which characterises singular stochastic control problems. Beyond this observation, the optimal strategy can take one of three different possible forms, depending on parameter values. These forms involve combinations of the following three tactics: wait, move (i.e., advertise the product), and stop (i.e., launch the product). Specifically, it is optimal either to move and stop, or to wait and stop, or to wait, move and stop, in which list, we order the sequence of optimal tactics according to small, moderate and large values of the underlying state process X (see Theorem 3, which is our main result).

We illustrate our main result by means of several special cases. Apart from an independent interest that each of these has, they reveal that the form of the optimal strategy is dependent on the functional form of the problem data as well as on parameter values.

Indeed, if the uncontrolled system dynamics are modelled by a geometric Brownian motion, then the move and stop strategy is always optimal if the terminal payoff function U is a power utility function, while the move and stop strategy is never optimal if U is the logarithmic utility function. On the other hand, if the uncontrolled system dynamics are modelled by a mean-reverting square-root process, such as the one appearing in the Cox-Ingersoll-Ross model, then the optimal strategy can take any of the three different possible forms, whether U is a power or the logarithmic utility function.

2.2 Problem formulation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a stochastic system whose uncontrolled dynamics are modelled by the Itô diffusion associated with the stochastic differential equation

$$dX_t^0 = b(X_t^0)\, dt + \sigma(X_t^0)\, dW_t, \quad X_0^0 = x > 0, \tag{2.1}$$

and we make the following assumption.

Assumption 1 *The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are locally Lipschitz, and $\sigma^2(x) > 0$ for all $x > 0$.*

This assumption implies that (2.1) has a unique strong solution. It also implies that, given any $c > 0$, the scale function p_c, given by

$$p_c(c) = 0, \quad p_c'(x) = \exp\left(-2 \int_c^x \frac{b(s)}{\sigma^2(s)}\, ds\right), \tag{2.2}$$

is well-defined, and the speed measure m_c, given by

$$m_c(dx) = \frac{2}{\sigma^2(x) p_c'(x)}\, dx,$$

is a Radon measure. Additionally, we assume that the solution of (2.1) is non-explosive, so that, given any initial condition x, $X_t^0 \in]0, \infty[$ for all $t \geq 0$, with probability 1 (see Karatzas and Shreve [24, Theorem 5.5.29] for appropriate necessary and sufficient analytic conditions).

Assumption 2 *The Itô diffusion X^0 defined by (2.1) is non-explosive.*

Feller's test for explosions (see Theorem 5.5.29 in Karatzas and Shreve [24]) provides a necessary and sufficient condition for this assumption to hold true.

We model the system's controlled dynamics by the SDE (1.1). With each admissible intervention strategy, we associate the performance criterion defined by (1.2)–(1.4).

Definition 1 *The set $\mathcal{A}$ of all* admissible strategies *is the set of all pairs (Z, τ) where τ is an $(\mathcal{F}_t)$-stopping time and Z is an $(\mathcal{F}_t)$-adapted increasing càglàd process such that $Z_0 = 0$,*

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t} K'(X_t) \circ dZ_t\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[e^{-\Lambda_\tau} U^-(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right] < \infty, \tag{2.3}$$

where $U^-(x) = -\min\{0, U(x)\}$.

The objective of our control problem is to maximise J_x over all admissible strategies. Accordingly, we define the problem's value function v by

$$v(x) = \sup_{(Z,\tau)\in\mathcal{A}} J_x(Z, \tau), \quad \text{for } x > 0.$$

For our optimisation problem to be well-posed, we need additional assumptions.

Assumption 3 *The discounting rate function r is absolutely continuous. Also, there exists a constant $r_0 > 0$ such that $r(x) \geq r_0$ for all $x > 0$.*

Assumption 4 *The functions K and U are C^2 with absolutely continuous second derivatives, and the function H is absolutely continuous. There exists a point $\beta > 0$ such that*

$$K'(x) - U'(x) = \begin{cases} \leq 0, & \textit{for } x < \beta, \\ \geq 0, & \textit{for } x > \beta. \end{cases} \tag{2.4}$$

Also, the function H/r is bounded, and $K'(x)$ remains bounded as $x \downarrow 0$.

In the context of the goodwill problem that has motivated this paper, it is worth noting that (2.4) in this assumption has a simple economic interpretation. In view of (1.4), which provides the cost of an intervention strategy Z, $K'(x)\,\varepsilon$ is the cost of raising the product's image from x to $x+\varepsilon$, for small $\varepsilon > 0$. Also, $U'(x)\,\varepsilon$ is the change in the utility that the company derives if the product is launched when its image is $x+\varepsilon$ rather than x, for small $\varepsilon > 0$. In light of these observations, assumption (2.4) captures the idea that the marginal cost of advertising is less (resp., greater) than the marginal utility derived from the product's launch when the product's image is low (resp., high), which is a rather natural one.

In the presence of Assumption 4, we can see that, if we define

$$\Theta(x) = \begin{cases} U(\beta) - \int_x^\beta K'(s)\,ds, & \text{for } x < \beta, \\ U(x), & \text{for } x \geq \beta, \end{cases} \tag{2.5}$$

then the function Θ is C^1 in $]0,\infty[$ and C^2 with absolutely continuous second derivative in $]0,\beta[\,\cup\,]\beta,\infty[$, and it satisfies

$$\max\{\Theta'(x) - K'(x),\ U(x) - \Theta(x)\} = 0. \tag{2.6}$$

In the context of the goodwill problem, Θ would be the value function of the control problem if advertise and launch immediately were the only tactics available to the decision maker, i.e., if waiting for any amount of time were not a possibility.

We need to make additional assumptions. To this end, we consider the operator $\mathcal{L}$ acting on C^1 functions with absolutely continuous first derivatives that is defined by

$$\mathcal{L}w(x) = \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x), \tag{2.7}$$

and the operator $\mathcal{D}_r$ acting on absolutely continuous functions that is defined by

$$\mathcal{D}_r w(x) = \frac{r(x)w'(x) - r'(x)w(x)}{r(x)} \equiv r(x)\left(\frac{w}{r}\right)'(x). \tag{2.8}$$

At first glance, the conditions in the following assumption may appear involved. However, they are quite general, and, apart from a growth and an integrability condition, they have a natural economic interpretation (see the discussion below). Furthermore, they are rather easy to verify in practice, as we will see in Section 2.4.

Assumption 5 *The function* Θ *satisfies*

$$\lim_{x \downarrow 0} \frac{\Theta(x)}{\varphi(x)} = \lim_{x \to \infty} \frac{\Theta(x)}{\psi(x)} = 0, \tag{2.9}$$

where the functions φ *and* ψ *span the solution space of the homogeneous ODE* $\mathcal{L}w(x) = 0$ *and satisfy (2.48)–(2.50) in the Appendix. Furthermore,* Θ *satisfies*

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t} |\mathcal{L}\Theta(X_t^0)| \, dt\right] < \infty. \tag{2.10}$$

There exists a point $x^* \geq 0$ *such that*

$$[\mathcal{L}\Theta + H](x-) = \begin{cases} > 0, & \text{for } x < x^*, \text{ if } x^* > 0, \\ \leq 0, & \text{for } x > x^*. \end{cases} \tag{2.11}$$

Furthermore,

$$[\mathcal{L}\Theta + H](\beta-) \geq [\mathcal{L}\Theta + H](\beta+), \tag{2.12}$$

$$\mathcal{D}_r[\mathcal{L}\Theta + H](x) \leq 0 \quad \text{Lebesgue-a.e. in }]0, \beta[\cup]\beta, \infty[, \tag{2.13}$$

$$\frac{[\mathcal{L}\Theta + H](x)}{r(x)} \text{ remains bounded as } x \downarrow 0, \tag{2.14}$$

$$\liminf_{x \to \infty} [\Theta - R_H](x) > 0, \tag{2.15}$$

where R_H *is defined by (2.57) in the Appendix for* $F = H$*.*

The operator $\mathcal{L}$ is the infinitesimal generator of the uncontrolled diffusion X^0 killed at a rate given by the discounting rate function r. Also, as we have discussed after Assumption 4, Θ is the best value that the company can get from just advertising and launching the product, while H is the running payoff that the company accumulates by delaying the product's launch. Therefore, $[\mathcal{L}\Theta + H](x) \, \Delta t$ is the expected payoff associated with the company's waiting for a small amount of time $\Delta t > 0$ before advertising and launching. In view of this observation, (2.11)–(2.12) capture the following natural idea: if the product's image is low (resp., high), then waiting may be a good (resp., bad) choice because the product's image may improve (resp., deteriorate) due to its stochastic dynamics.

Building on the above ideas, we can view the function $[\mathcal{L}\Theta + H]/r$ as the expected rate at which the company's payoff from advertising and launching changes by delaying taking action, measured in units of time that are proportional to the discounting rate r. In light of this interpretation and the definition (2.8) of the operator $\mathcal{D}_r$, (2.13) reflects the idea that the expected rate at which the best payoff resulting from "pure" action changes by waiting is decreasing as the product's image increases. Furthermore, (2.14) reflects the idea that waiting cannot be associated with an infinite expected rate of improvement.

In view of (2.56) in the Appendix, the function R_H identifies with the expected payoff that the company face if they exert no advertising effort and they never launch the product. Combining this observation with the interpretation of the function Θ as the optimal payoff that the company can receive if advertising and launching were the only available tactics, we can see that (2.15) is a necessary condition for guaranteeing that waiting forever and never taking any action is not an optimal strategy.

Remark 1 The conditions (2.9)–(2.10) in the previous assumption imply that the function Θ admits the representation

$$\Theta(x) = R_{-\mathcal{L}\Theta}(x) \quad \text{for all } x > 0, \tag{2.16}$$

where $R_{-\mathcal{L}\Theta}$ is defined by (2.56) or (2.57) in the Appendix with $F = -\mathcal{L}\Theta$ (see also the discussion at the end of the Appendix). The boundedness of H/r (see Assumption 4) and the definition (1.3) of Λ imply that

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t} |H(X_t^0)|\, dt\right] = -\mathbb{E}\left[\int_0^\infty \frac{|H(X_t^0)|}{r(X_t^0)}\, de^{-\Lambda_t}\right] \le \sup_{x>0} \frac{|H(x)|}{r(x)} < \infty.$$

This observation and (2.55) in the Appendix imply that the function R_H given by (2.56)–(2.57) with $F = H$ is well-defined and satisfies

$$\mathcal{L}R_H(x) + H(x) = 0 \quad \text{for all } x > 0. \tag{2.17}$$

□

Remark 2 In view of Assumption 4, the function $\mathcal{L}\Theta + H$ is absolutely continuous in $]0, \beta[\cup]\beta, \infty[$ but may have a discontinuity at β. It is for this reason why we have included condition (2.12) in Assumption 5. Also, (2.13), as well as any other such inequality that we may encounter in our analysis, is understood to hold Lebesgue-a.e. if $\mathcal{L}\Theta + H$ is not C^1 in $]0, \beta[\cup]\beta, \infty[$. □

2.3 The solution of the control problem

In light of the general theory of stochastic optimal control and optimal stopping, we expect that the value function v of our control problem identifies with a solution w of the HJB equation

$$\max\left\{\mathcal{L}w(x) + H(x),\ w'(x) - K'(x),\ U(x) - w(x)\right\} = 0. \tag{2.18}$$

A function w is a solution of this equation if it is C^1 with absolutely continuous first derivative, and it satisfies

$$\mathcal{L}w(x) + H(x) \leq 0 \quad \text{Lebesgue-a.e. in }]0,\infty[,$$
$$w'(x) \leq K'(x) \quad \text{and} \quad U(x) \leq w(x) \quad \text{for all } x > 0,$$

and

$$[\mathcal{L}w(x) + H(x)]\,[w'(x) - K'(x)]\,[U(x) - w(x)] = 0 \quad \text{Lebesgue-a.e. in }]0,\infty[.$$

We now solve the control problem by constructing an appropriate solution of this equation. To this end, we have to consider two possibilities. The first one arises when it is optimal to move and stop immediately.

Lemma 1 *In the presence of Assumptions 1–5, the function Θ defined by (2.5) satisfies the HJB equation (2.6) if and only if $x^* = 0$, where x^* is the point in (2.11) of Assumption 5.*

Proof. In view of (2.6), we can see that Θ satisfies the HJB equation of (2.18) if and only if

$$\mathcal{L}\Theta(x) + H(x) \leq 0 \quad \text{Lebesgue-a.e. in }]0,\infty[,$$

which is true if and only if $x^* = 0$, where x^* is the point appearing in (2.11) of Assumption 5. □

The second possibility arises when waiting enters the set of optimal tactics. In this case, we postulate that it is optimal to wait for as long as the state process X takes values below a given threshold level, and move and stop as soon as the state process exceeds the threshold level. If we denote by α this threshold level, then we look for a solution w of the HJB equation (2.18) that satisfies the ODE $\mathcal{L}w(x) + H(x) = 0$ Lebesgue-a.e. in $]0, \alpha[$, and is such that

$$\max\left\{w'(x) - K'(x),\ U(x) - w(x)\right\} = 0 \quad \text{for all } x \geq \alpha.$$

In view of (2.6) and (2.17), we therefore look for a solution of the form

$$w(x) = \begin{cases} A\psi(x) + R_H(x), & \text{for } x < \alpha, \\ \Theta(x), & \text{for } x \geq \alpha, \end{cases} \tag{2.19}$$

where A is an appropriate constant, ψ is as in (2.49)–(2.50), and R_H is defined by (2.56)–(2.57) with $F = H$ (see also Remark 1).

To specify the parameter A and the free-boundary point α, we postulate that w satisfies the so-called "principle of smooth fit". In particular, we assume that w is C^1 at α, which gives rise to the system of equations

$$A\psi(\alpha) + R_H(\alpha) = \Theta(\alpha) \quad \text{and} \quad A\psi'(\alpha) + R_H'(\alpha) = \Theta'(\alpha),$$

which is equivalent to

$$A = \frac{\Theta(\alpha) - R_H(\alpha)}{\psi(\alpha)} = \frac{\Theta'(\alpha) - R_H'(\alpha)}{\psi'(\alpha)}. \tag{2.20}$$

In view of the fact that

$$\Theta - R_H = -R_{\mathcal{L}\Theta + H}, \tag{2.21}$$

which follows from (2.16) and (2.61) with $F = \mathcal{L}\Theta + H$, we can check that the second identity in (2.20) is equivalent to

$$\left(\frac{R_{\mathcal{L}\Theta+H}}{\psi}\right)'(\alpha) = 0.$$

It follows that the free-boundary point α should satisfy the equation

$$q(\alpha) := \int_0^\alpha \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p_c'(s)}\, ds = 0, \tag{2.22}$$

because (2.59) in the Appendix with $F = \mathcal{L}\Theta + H$ implies the expression

$$\left(\frac{R_{\mathcal{L}\Theta+H}}{\psi}\right)'(x) = -\frac{2p_c'(x)}{\psi^2(x)} \int_0^x \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p_c'(s)}\, ds = -\frac{2p_c'(x)}{\psi^2(x)} q(x). \tag{2.23}$$

The following result is concerned with the solvability of this equation and with the associated solution of the HJB equation (2.18).

Lemma 2 *In the presence of Assumptions 1–5, equation (2.22) has a unique solution* $\alpha > 0$ *if and only if* $x^* > 0$*, where* x^* *is the point appearing in (2.11) of Assumption 5. In this case,* $\alpha > x^*$*, and the function* w *defined by (2.19), where* A *is given by (2.20), is* C^1 *with absolutely continuous first derivative and satisfies the HJB equation (2.18).*

Proof. In view of (2.11), we can see that the left-hand derivative $q'(x-)$ of q at $x > 0$ satisfies

$$q'(x-) = \frac{[\mathcal{L}\Theta + H](x-)\psi(x)}{\sigma^2(x)p_c'(x)} \begin{cases} > 0, & \text{for } x < x^*, \text{ if } x^* > 0, \\ \leq 0, & \text{for } x > x^*. \end{cases} \tag{2.24}$$

Combining this observation with the fact that $q(0) = 0$, we can see that the equation $q(\alpha) = 0$ has a unique solution $\alpha > 0$ if and only if $x^* > 0$ and

$$\lim_{x\to\infty} q(x) < 0. \tag{2.25}$$

Furthermore, this solution is such that

$$x^* < \alpha \quad \text{and} \quad q(x) = \begin{cases} > 0, & \text{for } x < \alpha, \\ < 0, & \text{for } x > \alpha. \end{cases} \tag{2.26}$$

To see that the inequality (2.25) is indeed true, we first note that (2.23)–(2.24) imply that the function $R_{\mathcal{L}\Theta+H}/\psi$ is monotone as $x \to \infty$. In particular, this expression implies (2.25) if the function $R_{\mathcal{L}\Theta+H}/\psi$ is actually increasing as $x \to \infty$. To prove that this is indeed the case, we note that (2.15) in Assumption 5 and (2.21) imply that

$$\limsup_{x\to\infty} R_{\mathcal{L}\Theta+H}(x) < 0.$$

This observation, the fact that

$$\lim_{x\to\infty} \frac{R_{\mathcal{L}\Theta+H}(x)}{\psi(x)} = 0,$$

(see (2.58) in the Appendix with $F = \mathcal{L}\Theta + H$) and (2.23) imply (2.25) if and only if

$$R_{\mathcal{L}\Theta+H}(x) \equiv -[\Theta - R_H](x) < 0,$$

for all x sufficient large, which is true thanks to (2.15) in Assumption 5.

In view of the construction of w and the fact that Θ satisfies (2.6), we will prove that w satisfies the HJB equation (2.18) if we show that

$$[\mathcal{L}\Theta + H](x) \le 0 \quad \text{Lebesgue-a.e. in }]\alpha, \infty[, \tag{2.27}$$

$$A\psi(x) + R_H(x) \ge U(x) \quad \text{for all } x \le \alpha, \tag{2.28}$$

$$A\psi'(x) + R_H'(x) \le K'(x) \quad \text{for all } x \le \alpha. \tag{2.29}$$

To this end, we note that (2.27) follows immediately from (2.11) in Assumption 5 and the first inequality in (2.26). To establish (2.28), it suffices to show that

$$A\psi(x) + R_H(x) \ge \Theta(x) \quad \text{for all } x < \alpha,$$

because $\Theta \ge U$ (see (2.6)). In view of (2.20), (2.21) and the fact that $\psi > 0$, we can see that this inequality is equivalent to

$$\frac{R_{\mathcal{L}\Theta+H}(x)}{\psi(x)} \ge \frac{R_{\mathcal{L}\Theta+H}(\alpha)}{\psi(\alpha)} \quad \text{for all } x < \alpha,$$

which is true thanks to (2.23) and (2.26).

Finally, (2.29) will follow if we prove that

$$A\psi'(x) + R_H'(x) \le \Theta'(x) \quad \text{for all } x < \alpha,$$

because $\Theta' \le K'$ (see (2.6)). Combining (2.20) with (2.21) and the strict positivity of ψ', we can see that this inequality is equivalent to

$$\frac{R'_{\mathcal{L}\Theta+H}(x)}{\psi'(x)} \le \frac{R'_{\mathcal{L}\Theta+H}(\alpha)}{\psi'(\alpha)} \quad \text{for all } x < \alpha. \tag{2.30}$$

Using the identity (2.60) in the Appendix with $F = [\mathcal{L}\Theta + H]$ and the definition (2.22) of q, we can see that the left-hand derivative $(R'_{\mathcal{L}\Theta+H}/\psi')'(x-)$ exists for all $x > 0$, and is given by

$$\left(\frac{R'_{\mathcal{L}\Theta+H}}{\psi'}\right)'(x-) = \frac{2r(x)p'_c(x)}{[\sigma(x)\psi'(x)]^2}\left[2q(x) - \frac{[\mathcal{L}\Theta + H](x-)}{r(x)}\frac{\psi'(x)}{p'_c(x)}\right]. \tag{2.31}$$

Furthermore, recalling that the function $\mathcal{L}\Theta+H$ is absolutely continuous in $]0, \beta[\cup]\beta, \infty[$ (see Remark 2), we can use the integration by parts formula, the expression (2.54) in the Appendix and the definition (2.8) of the operator $\mathcal{D}_r$ to calculate

$$\begin{aligned}\frac{[\mathcal{L}\Theta + H](x-)}{r(x)}\frac{\psi'(x)}{p'_c(x)} &= \frac{[\mathcal{L}\Theta + H](x_0-)}{r(x_0)}\frac{\psi'(x_0)}{p'_c(x_0)} + \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)}\frac{\psi'(\beta)}{p'_c(\beta)}\mathbf{1}_{[x_0,x[}(\beta)\\ &\quad + \int_{x_0}^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p'_c(s)}\mathbf{1}_{[x_0,x]\setminus\{\beta\}}(s)\,ds + 2\int_{x_0}^x \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p'_c(s)}\,ds.\end{aligned} \tag{2.32}$$

The limits (2.53) in the Appendix and (2.14) in Assumption 5 imply that

$$\lim_{x_0\downarrow 0}\frac{[\mathcal{L}\Theta + H](x_0)}{r(x_0)}\frac{\psi'(x_0)}{p'_c(x_0)} = 0.$$

In light of (2.11)–(2.13) in Assumption 5, we can use the monotone convergence theorem and this observation to pass to the limit $x_0 \downarrow 0$ in (2.32) to obtain

$$\begin{aligned}\frac{[\mathcal{L}\Theta + H](x-)}{r(x)}\frac{\psi'(x)}{p'_c(x)} = &\int_0^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p'_c(s)}\mathbf{1}_{[0,x]\setminus\{\beta\}}(s)\,ds + 2q(x)\\ &+ \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)}\frac{\psi'(\beta)}{p'_c(\beta)}\mathbf{1}_{]0,x[}(\beta).\end{aligned}$$

This calculation and (2.31) imply that

$$\begin{aligned}\left(\frac{R'_{\mathcal{L}\Theta+H}}{\psi'}\right)'(x-) = &-\frac{2r(x)p'_c(x)}{[\sigma(x)\psi'(x)]^2}\Bigg(\int_0^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p'_c(s)}\mathbf{1}_{[0,x]\setminus\{\beta\}}(s)\,ds\\ &+ \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)}\frac{\psi'(\beta)}{p'_c(\beta)}\mathbf{1}_{]0,x[}(\beta)\Bigg)\\ \geq 0,&\end{aligned}$$

the inequality following thanks to (2.12) and (2.13) in Assumption 5. It follows that the function $R'_{\mathcal{L}\Theta+H}/\psi'$ is increasing, which establishes (2.30). □

We can now prove our main result of the section.

Theorem 3 *Consider the stochastic control problem formulated in Section 2.2 and suppose that Assumptions 1–5 hold true. The optimal strategy takes the form of one of the following mutually exclusive cases, which are characterised by the point $\beta > 0$ appearing in (2.4) of Assumption 4, the point $x^* \geq 0$ appearing in (2.11) of Assumption 5, and the solution $\alpha > x^*$ of equation (2.22):*

1. *If $x^* = 0$, then it is optimal to move and stop, and the optimal strategy is given by*

$$\tau^* = 0 \quad \textit{and} \quad Z_t^* = (\beta - x)^+ \mathbf{1}_{]0,\infty[}(t).$$

2. *If $x^* > 0$ and $\alpha < \beta$, then it is optimal to wait, move and stop, and the optimal strategy is given by*

$$\tau^* = \inf\{t \geq 0 \mid \ X_t^0 \geq \alpha\} \quad \textit{and} \quad Z_t^* = (\beta - \alpha \vee x)^+ \mathbf{1}_{]\tau^*,\infty[}(t).$$

3. *If $x^* > 0$ and $\alpha \geq \beta$, then it is optimal to wait and stop, and the optimal strategy is given by*

$$\tau^* = \inf\{t \geq 0 \mid \ X_t^0 \geq \alpha\} \quad \textit{and} \quad Z^* \equiv 0.$$

In the first case, the value function v identifies with the function Θ defined by (2.5), while, in cases (2) and (3), the value function v identifies with the function w constructed in Lemma 2.

Proof. Throughout the proof, we consider the solution w of the HJB equation (2.18) that is as in Lemma 1 or in Lemma 2, depending on whether $x^* = 0$ or not, and we fix any initial condition $x > 0$ and any admissible strategy $(Z, \tau) \in \mathcal{A}$. Also, we consider the local martingale defined by

$$M_T = \int_0^T e^{-\Lambda_t} \sigma(X_t) w'(X_t) \, dW_t,$$

and we let (τ_n) be any localising sequence of $(\mathcal{F}_t)$-stopping times such that $\tau_n \leq n$, for all $n \geq 1$. Using Itô's formula and the fact that $\Delta X_t = \Delta Z_t$, we calculate

$$\begin{aligned}
e^{-\Lambda_{\tau\wedge\tau_n}} w(X_{\tau\wedge\tau_n+}) &= w(x) + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} \mathcal{L}w(X_t)\, dt + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} w'(X_t)\, dZ_t \\
&\quad + \sum_{0\leq t\leq\tau\wedge\tau_n} e^{-\Lambda_t} \left[w(X_{t+}) - w(X_t) - w'(X_t)\Delta X_t\right] + M_{\tau\wedge\tau_n} \\
&= w(x) + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} \mathcal{L}w(X_t)\, dt + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} w'(X_t)\, dZ_t^{\mathrm{c}} \\
&\quad + \sum_{0\leq t\leq\tau\wedge\tau_n} e^{-\Lambda_t} \left[w(X_t + \Delta Z_t) - w(X_t)\right] + M_{\tau\wedge\tau_n},
\end{aligned}$$

where the operator $\mathcal{L}$ is defined by (2.7) and Z^{c} is the continuous part of the process Z. In view of (1.4) and the fact that w satisfies the HJB equation (2.18), we can therefore see that

$$\begin{aligned}
&\int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} H(X_t)\, dt - \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t + e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau\leq\tau_n\}} \\
&\qquad = w(x) + e^{-\Lambda_\tau} \left[U(X_{\tau+}) - w(X_{\tau+})\right] \mathbf{1}_{\{\tau\leq\tau_n\}} - e^{-\Lambda_{\tau_n}} w(X_{\tau_n+}) \mathbf{1}_{\{\tau_n<\tau\}} \\
&\qquad\quad + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} \left[\mathcal{L}w(X_t) + H(X_t)\right] dt \\
&\qquad\quad + \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} \left[w'(X_t) - K'(X_t)\right] dZ_t^{\mathrm{c}} \\
&\qquad\quad + \sum_{0\leq t\leq\tau\wedge\tau_n} e^{-\Lambda_t} \int_0^{\Delta Z_t} \left[w'(X_t+s) - K'(X_t+s)\right] ds + M_{\tau\wedge\tau_n} \\
&\qquad \leq w(x) + e^{-\Lambda_{\tau_n}} w^-(X_{\tau_n+}) \mathbf{1}_{\{\tau_n<\tau\}} + M_{\tau\wedge\tau_n},
\end{aligned} \tag{2.33}$$

where $w^-(x) = -\min\{0, w(x)\}$. Taking expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} H(X_t)\, dt - \int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t + e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau\leq\tau_n\}}\right] \\
&\qquad \leq w(x) + \mathbb{E}\left[e^{-\Lambda_{\tau_n}} w^-(X_{\tau_n+}) \mathbf{1}_{\{\tau_n<\tau\}}\right].
\end{aligned} \tag{2.34}$$

The assumption that H/r is bounded, the fact that the process E defined by $E_t = -\exp(-\Lambda_t)$ is increasing and the dominated convergence theorem imply that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} H(X_t)\, dt\right] &= \lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau} \mathbf{1}_{\{t\leq\tau_n\}} \frac{H(X_t)}{r(X_t)}\, dE_t\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau\wedge\tau_n} \frac{H(X_t)}{r(X_t)}\, dE_t\right] \\
&= \mathbb{E}\left[\int_0^{\tau} e^{-\Lambda_t} H(X_t)\, dt\right],
\end{aligned} \tag{2.35}$$

while the monotone convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t\right] = \mathbb{E}\left[\int_0^{\tau} e^{-\Lambda_t} K'(X_t) \circ dZ_t\right].$$

The admissibility condition (2.3) and the monotone convergence theorem imply that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau\le\tau_n\}}\right] &= \lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_\tau} U^+(X_{\tau+}) \mathbf{1}_{\{\tau\le\tau_n\}}\right] \\ &\quad - \lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_\tau} U^-(X_{\tau+}) \mathbf{1}_{\{\tau\le\tau_n\}}\right] \\ &= \mathbb{E}\left[e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right]. \end{aligned} \tag{2.36}$$

Also, since w^- is bounded, which follows from the inequality $w \ge \Theta$ and the fact that Θ is bounded from below (see (2.5) and the last claim in Assumption 4), we can use the dominated convergence theorem and Assumption 3 to obtain

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_{\tau_n}} w^-(X_{\tau_n+}) \mathbf{1}_{\{\tau_n<\tau\}}\right] = 0.$$

In view of these observations, we can pass to the limit as $n \to \infty$ in (2.34) to obtain $J_x(Z,\tau) \le w(x)$, which implies that $v(x) \le w(x)$.

In each of the cases (1)–(3) in the theorem's statement, we can check that the strategy (Z^*, τ^*) is admissible in the sense of Definition 1 because the process Z^* has at most one jump and because $U(X^*_{\tau^*}) = U(\alpha) \in \mathbb{R}$. Furthermore, we can check that (2.33) and (2.34) both hold with equality, which, combined with (2.35)–(2.36), implies that $J_x(Z^*, \tau^*) = w(x)$. This conclusion and the inequality $v(x) \le w(x)$, which we have established above, imply that $v(x) = w(x)$ and that (Z^*, τ^*) is optimal. □

2.4 Special cases

We now consider a number of special cases that arise when the uncontrolled system's dynamics are modelled by a geometric Brownian motion (Section 2.4.1) or by a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model (Section 2.4.2). In these special cases, we assume that

$$H(x) = -\gamma, \quad K'(x) = \kappa \quad \text{and} \quad r(x) = \varrho \quad \text{for all } x > 0,$$

where $\gamma \geq 0$ and $\kappa, \varrho > 0$ are constants. Also, we assume that the terminal payoff function U is a power utility function, given by

$$U(x) = \frac{x^p}{p} \quad \text{for all } x > 0, \tag{2.37}$$

for some $p \in]0, 1[$, in which case, the function Θ defined by (2.5) takes the form

$$\Theta(x) = \begin{cases} \frac{1-p}{p} \kappa^{-\frac{p}{1-p}} + \kappa x, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta, \\ \frac{x^p}{p}, & \text{for } x \geq \kappa^{-\frac{1}{1-p}} \equiv \beta, \end{cases} \tag{2.38}$$

or the logarithmic utility function, namely

$$U(x) = \ln x \quad \text{for all } x > 0, \tag{2.39}$$

in which case,

$$\Theta(x) = \begin{cases} \kappa x - 1 - \ln \kappa, & \text{for } x < \kappa^{-1} \equiv \beta, \\ \ln x, & \text{for } x > \kappa^{-1} \equiv \beta. \end{cases} \tag{2.40}$$

It is straightforward to verify that these choices satisfy all of the conditions appearing in Assumptions 3 and 4.

2.4.1 Geometric Brownian motion

Suppose that X^0 is a geometric Brownian motion, so that

$$dX_t^0 = bX_t^0 \, dt + \sigma X_t^0 \, dW_t, \quad X_0^0 = x > 0,$$

for some constants b and $\sigma \neq 0$, and assume that $\varrho > b$. In this case, Assumptions 1 and 2 both hold true, and it is a standard exercise to verify that, if we choose $c = 1$, then

$$\varphi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p_c'(x) = x^{n+m-1}, \tag{2.41}$$

where the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\tfrac{1}{2}\sigma^2 k^2 + \left(b - \tfrac{1}{2}\sigma^2\right) k - \varrho = 0.$$

Also, it is well-known that

$$\varrho > b \quad \Leftrightarrow \quad n > 1, \tag{2.42}$$

in which case,

$$\mathbb{E}\left[\int_0^\infty e^{-\varrho t} X_t^0 \, dt\right] = \frac{x}{\varrho - b} < \infty. \tag{2.43}$$

Since there exists a constant $C_1 > 0$ such that $|\mathcal{L}\Theta(x)| \leq C_1(1+x)$ for all $x > 0$, whether Θ is given by (2.38) or (2.40), it follows from (2.41)–(2.43) that conditions (2.9) and (2.10) in Assumption 5 hold true. Also, we can use (2.56) in the Appendix with $F = H \equiv -\gamma$ to calculate $R_H = -\gamma/\varrho$, which implies that (2.15) in Assumption 5 is satisfied, whether Θ is given by (2.38) or (2.40).

In the following two subsections, we show that the choices for the problem data that we have made satisfy the remaining conditions (2.11)–(2.14) in Assumption 5, and we discuss the possible forms that the optimal strategy takes.

Power utility function U

If the terminal payoff function U is the power utility function given by (2.37), then we can check that the function Θ defined by (2.38) satisfies

$$\begin{aligned}[\mathcal{L}\Theta + H](x) &= \begin{cases} -(\varrho - b)\kappa x - \varrho\frac{1-p}{p}\kappa^{-\frac{p}{1-p}} - \gamma, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta, \\ -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right] x^p - \gamma, & \text{for } x > \kappa^{-\frac{1}{1-p}} \equiv \beta, \end{cases} \\ &< 0,\end{aligned}$$

where the inequality follows from the assumption that $\varrho > b$ and the fact that $p \in]0, 1[$. It follows that (2.11) is satisfied with $x^* = 0$ and that (2.14) holds true. We can also calculate

$$\begin{aligned}[\mathcal{L}\Theta + H](\beta-) &= -\left[\frac{\varrho}{p} - b\right]\kappa^{-\frac{p}{1-p}} - \gamma \\ &> -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right]\kappa^{-\frac{p}{1-p}} - \gamma \\ &= [\mathcal{L}\Theta + H](\beta+),\end{aligned}$$

which establishes (2.12), and

$$\begin{aligned}\mathcal{D}_r[\mathcal{L}\Theta + H](x) &= \frac{d}{dx}[\mathcal{L}\Theta + H](x) \\ &= \begin{cases} -(\varrho - b)\kappa, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta, \\ -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right] p x^{-(1-p)}, & \text{for } x > \kappa^{-\frac{1}{1-p}} \equiv \beta, \end{cases} \\ &< 0,\end{aligned}$$

which implies that (2.13) is also true.

Finally, the fact that $x^* = 0$ puts us in the context of case (1) of Theorem 3, so the move-and-stop strategy is the optimal strategy.

Logarithmic utility function U

If the terminal payoff function U is the logarithmic utility function given by (2.39), then we can check that the function Θ defined by (2.40) satisfies

$$[\mathcal{L}\Theta + H](x) = \begin{cases} -(\varrho - b)\kappa x + \varrho \ln \kappa + \varrho - \gamma, & \text{for } x < \kappa^{-1} \equiv \beta, \\ -\varrho \ln x - \frac{1}{2}\sigma^2 + b - \gamma, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases}$$

$$[\mathcal{L}\Theta + H](\beta-) = \varrho \ln \kappa + b - \gamma > \varrho \ln \kappa - \tfrac{1}{2}\sigma^2 + b - \gamma = [\mathcal{L}\Theta + H](\beta+),$$

as well as

$$\begin{aligned}\mathcal{D}_r[\mathcal{L}\Theta + H](x) &= \frac{d}{dx}[\mathcal{L}\Theta + H](x) \\ &= \begin{cases} -(\varrho - b)\kappa, & \text{for } x < \kappa^{-1} \equiv \beta, \\ -\varrho x^{-1}, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases} \\ &< 0.\end{aligned}$$

These calculations imply that (2.13)–(2.14) hold true, and that (2.11) is satisfied with

$$x^* = \begin{cases} \frac{\varrho + \varrho \ln \kappa - \gamma}{(\varrho - b)\kappa}, & \text{if } \varrho \ln \kappa < -b + \gamma, \\ \beta, & \text{if } -b + \gamma \le \varrho \ln \kappa \le \frac{1}{2}\sigma^2 - b + \gamma, \\ \exp\left(\frac{-\frac{1}{2}\sigma^2 + b - \gamma}{\varrho}\right), & \text{if } \frac{1}{2}\sigma^2 - b + \gamma < \varrho \ln \kappa. \end{cases}$$

In this case $x^* > 0$, so "waiting" belongs to the set of optimal tactics. To obtain the free-boundary point $\alpha > 0$ that determines the waiting region, we use (2.22) and (2.41) to calculate

$$q(\alpha) = \begin{cases} -\frac{(\varrho-b)\kappa}{\sigma^2(1-m)\alpha^m}\left[\alpha - \frac{(1-m)(\varrho+\varrho\ln\kappa-\gamma)}{-m(\varrho-b)\kappa}\right], & \text{if } \alpha \leq \kappa^{-1} \equiv \beta, \\ \frac{\varrho}{\sigma^2 m\alpha^m}\left[\ln\alpha + \frac{1}{m} - \frac{\varrho+\varrho\ln\kappa-\gamma}{\varrho} + (\alpha\kappa)^m\left(\ln\kappa - \frac{1}{m} - \frac{m(\varrho-b)}{(1-m)\varrho}\right)\right], & \text{if } \alpha > \kappa^{-1} \equiv \beta. \end{cases}$$

From these calculations, it follows that the unique solution $\alpha > 0$ of the equation $q(\alpha) = 0$ is strictly less than $\beta \equiv \kappa^{-1}$ if and only if

$$\varrho\ln\kappa < \frac{-m}{-m+1}(\varrho - b) - \varrho + \gamma. \tag{2.44}$$

In light of this analysis, we can see that the optimal strategy takes one of the following forms. If the parameter values are such that (2.44) is true, then we are in the context of case (2) of Theorem 3, and the wait-move-and-stop strategy is optimal. Otherwise, we are in the context of case (3) of Theorem 3, and the wait-and-stop strategy is optimal.

2.4.2 Mean-reverting square-root process

Suppose that X^0 is a mean-reverting square-root process, so that

$$dX_t^0 = \zeta(\vartheta - X_t^0)\,dt + \sigma\sqrt{X_t^0}\,dW_t, \quad X_0^0 = x > 0,$$

for some constants $\zeta, \vartheta, \sigma > 0$, and assume that

$$\zeta\vartheta - \tfrac{1}{2}\sigma^2 > 0, \tag{2.45}$$

which is a necessary and sufficient condition for X^0 to be non-explosive. In this context, Assumptions 1 and 2 are plainly satisfied. With reference to Jack, Johnson and Zervos [21, Section 5.2], we can deduce that, if we choose $c = 1$, then

$$\phi(x) = \frac{\mathrm{U}\left(\frac{\varrho}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}x\right)}{\mathrm{U}\left(\frac{\varrho}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}\right)}, \quad \psi(x) = \frac{{}_1\mathrm{F}_1\left(\frac{\varrho}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}x\right)}{{}_1\mathrm{F}_1\left(\frac{\varrho}{\zeta}, \frac{2\zeta\vartheta}{\sigma^2}; \frac{2\zeta}{\sigma^2}\right)},$$

where U and ${}_1\mathrm{F}_1$ are confluent hypergeometric functions, and

$$p_c'(x) = x^{-2\zeta\vartheta/\sigma^2}e^{2\zeta(x-1)/\sigma^2}.$$

The functions φ and ψ identify with confluent hypergeometric functions and ψ has exponential growth as x tends to ∞. Also, the calculation

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\varrho t} X_t^0 \, dt\right] &= \int_0^\infty e^{-\varrho t} \mathbb{E}\left[X_t^0\right] \, dt \\ &= \int_0^\infty e^{-\varrho t}\left[\vartheta + (x-\vartheta)e^{-\zeta t}\right] \, dt \\ &= \frac{\zeta\vartheta + \varrho x}{\varrho(\zeta+\varrho)} \\ &< \infty\end{aligned}$$

is a standard exercise in financial mathematics. This calculation implies that Θ satisfies (2.10) in Assumption 5 because there exists a constant $C_2 > 0$ such that $|\mathcal{L}\Theta(x)| \leq C_2(1+x)$ for all $x > 0$, whether Θ is given by (2.38) or (2.40) (see also (2.46) and (2.47) below). Such a bound of Θ also implies that (2.9) in Assumption 5 holds true because $\lim_{x\downarrow 0}\varphi(x) = \infty$ and $\psi(x)$ has exponential growth as x tends to ∞. Furthermore, the fact that $R_H \equiv -\gamma/\varrho$, which follows from (2.56), implies that (2.15) in Assumption 5 holds true, whether Θ is given by (2.38) or (2.40).

In the following two subsections, we verify that conditions (2.11)–(2.14) of Assumption 5 are satisfied as well, and we discuss the possible forms that the optimal strategy takes.

Power utility function U

If the terminal payoff function U is the power utility function given by (2.37), then we can check that the function Θ defined by (2.38) satisfies

$$[\mathcal{L}\Theta + H](x) = \begin{cases} -(\varrho+\zeta)\kappa x + \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}} - \gamma, & \text{for } x < \kappa^{\frac{-1}{1-p}} \equiv \beta, \\ \left[\zeta\vartheta - \frac{1}{2}\sigma^2(1-p)\right] x^{-(1-p)} - \left(\zeta + \frac{\varrho}{p}\right) x^p - \gamma, & \text{for } x > \kappa^{\frac{-1}{1-p}} \equiv \beta, \end{cases} \tag{2.46}$$

$$\begin{aligned}[\mathcal{L}\Theta + H](\beta-) &= -\left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} + \zeta\vartheta\kappa - \gamma \\ &> -\left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} + \left[\zeta\vartheta - \tfrac{1}{2}(1-p)\sigma^2\right]\kappa - \gamma \\ &= [\mathcal{L}\Theta + H](\beta+)\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{D}_r[\mathcal{L}\Theta + H](x) \\
&\quad = \frac{d}{dx}[\mathcal{L}\Theta + H](x) \\
&\quad = \begin{cases} -(\varrho + \zeta)\kappa, & \text{for } x < \kappa^{\frac{-1}{1-p}} \equiv \beta, \\ -\left[\zeta\vartheta - \frac{1}{2}\sigma^2(1-p)\right](1-p)x^{-(2-p)} - \left(\zeta + \frac{\varrho}{p}\right)px^{-(1-p)}, & \text{for } x > \kappa^{\frac{-1}{1-p}} \equiv \beta, \end{cases} \\
&\quad < 0,
\end{aligned}
$$

where the inequality follows from the assumption (2.45) and the fact that $p \in]0,1[$. These calculations imply immediately that (2.13)–(2.14) hold true. Also, these calculations imply that there exists a unique point x^* such that (2.11) in Assumption 5 is true. In particular,

$$
\begin{aligned}
x^* &= 0, \quad \text{if } \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}} \le \gamma, \\
x^* &\in]0,\beta[, \quad \text{if } \zeta\vartheta\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} < \gamma < \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}}, \\
x^* &= \beta, \quad \text{if } \left[\zeta\vartheta - \tfrac{1}{2}(1-p)\sigma^2\right]\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} \le \gamma \le \zeta\vartheta\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}},
\end{aligned}
$$

and

$$
x^* > \beta, \quad \text{if } \gamma < \left[\zeta\vartheta - \tfrac{1}{2}(1-p)\sigma^2\right]\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}}.
$$

In view of Lemmas 1 and 2, we conclude that, in the special case of the general problem that we consider here, the optimal strategy can take the form of any of the cases (1)–(3) of Theorem 3, depending on parameter values.

Logarithmic utility function U

If the terminal payoff function U is the logarithmic utility function given by (2.39), then we can calculate

$$
[\mathcal{L}\Theta + H](x) = \begin{cases} -(\zeta + \varrho)\kappa x + \zeta\vartheta\kappa + \varrho\ln\kappa + \varrho - \gamma, & \text{for } x < \kappa^{-1} \equiv \beta, \\ \left[\zeta\vartheta - \frac{1}{2}\sigma^2\right]x^{-1} - \zeta - \varrho\ln x - \gamma, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases} \tag{2.47}
$$

$$
\begin{aligned}
[\mathcal{L}\Theta + H](\beta-) &= -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa - \gamma \\
&> -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa - \tfrac{1}{2}\sigma^2\kappa - \gamma \\
&= [\mathcal{L}\Theta + H](\beta+)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_r[\mathcal{L}\Theta + H](x) &= \frac{d}{dx}[\mathcal{L}\Theta + H](x) \\
&= \begin{cases} -(\zeta + \varrho)\kappa, & \text{for } x < \kappa^{-1} \equiv \beta, \\ -\left[\zeta\vartheta - \tfrac{1}{2}\sigma^2\right] x^{-2} - \varrho x^{-1}, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases} \\
&< 0,
\end{aligned}
$$

where the inequality follows from the assumption (2.45). These calculations imply immediately that (2.13)–(2.14) are satisfied and that there exists a unique point x^* such that (2.11) is true. In particular,

$$
\begin{aligned}
x^* &= 0, \quad \text{if } \zeta\vartheta\kappa + \varrho \ln \kappa + \varrho \le \gamma, \\
x^* &\in]0, \beta[, \quad \text{if } -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa < \gamma < \zeta\vartheta\kappa + \varrho \ln \kappa + \varrho, \\
x^* &= \beta, \quad \text{if } -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa - \tfrac{1}{2}\sigma^2\kappa \le \gamma \le -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa,
\end{aligned}
$$

and

$$
x^* > \beta, \quad \text{if } \gamma < -\zeta + \zeta\vartheta\kappa + \varrho \ln \kappa - \tfrac{1}{2}\sigma^2\kappa.
$$

As in the previous case, the optimal strategy can be as in any of the cases (1)–(3) of Theorem 3, depending on parameter values.

2.5 Appendix: A second order linear ODE

In this section, we review a range of results regarding the solvability of a second order linear ODE on which part of our analysis has been based. All of the claims that we do not prove here are standard, and can be found in various forms in several references (e.g., see Borodin and Salminen [13, Chapter II]).

In the presence of Assumptions 1, 2 and 3, the general solution of the second-order linear homogeneous ODE

$$\mathcal{L}w(x) \equiv \tfrac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x > 0,$$

is given by

$$w(x) = A\varphi(x) + B\psi(x),$$

for some constants $A, B \in \mathbb{R}$. The functions φ and ψ are C^2,

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{2.48}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0, \tag{2.49}$$

and

$$\lim_{x \downarrow 0} \varphi(x) = \lim_{x \to \infty} \psi(x) = \infty. \tag{2.50}$$

In this context, φ and ψ are unique, modulo multiplicative constants. To simplify the notation we assume, without loss of generality, that $\varphi(c) = \psi(c) = 1$, where $c > 0$ is the same constant as the one that we used in the definition (2.2) of the scale function p_c. Also, these functions satisfy

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp_c'(x), \tag{2.51}$$

where $C := [\psi'(c) - \varphi'(c)] > 0$. Furthermore, the identity

$$\varphi''(x)\psi'(x) - \varphi'(x)\psi''(x) = \frac{2Cr(x)}{\sigma^2(x)} p_c'(x), \tag{2.52}$$

follows immediately from the fact that φ and ψ satisfy the ODE $\mathcal{L}f(x) = 0$ and (2.51).

Combining the inequalities

$$0 < \frac{\varphi(x)\psi'(x)}{Cp_c'(x)} < 1 \quad \text{and} \quad 0 < -\frac{\varphi'(x)\psi(x)}{Cp_c'(x)} < 1,$$

which follow from (2.48)–(2.49) and (2.51), with (2.50), we can see that

$$\lim_{x \downarrow 0} \frac{\psi'(x)}{p_c'(x)} = \lim_{x \to \infty} \frac{\varphi'(x)}{p_c'(x)} = 0. \tag{2.53}$$

Also, the calculation

$$\frac{d}{dx}\left(\frac{1}{p_c'(x)}\right) = \frac{2b(x)}{\sigma^2(x)p_c'(x)},$$

and the fact that ψ satisfies the ODE $\mathcal{L}w(x) = 0$, imply that

$$\frac{d}{dx}\left(\frac{\psi'(x)}{p_c'(x)}\right) = \frac{2}{\sigma^2(x)p_c'(x)}\left[\tfrac{1}{2}\sigma^2(x)\psi''(x) + b(x)\psi'(x)\right] = \frac{2r(x)\psi(x)}{\sigma^2(x)p_c'(x)}. \tag{2.54}$$

Now, we consider any Borel measurable function F such that

$$\int_0^x \frac{|F(s)|\psi(s)}{\sigma^2(s)p_c'(s)}\,ds + \int_x^\infty \frac{|F(s)|\varphi(s)}{\sigma^2(s)p_c'(s)}\,ds < \infty \quad \text{for all } x > 0.$$

A function F satisfies this integrability condition if and only if

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}|F(X_t^0)|\,dt\right] < \infty \tag{2.55}$$

for every initial condition $x > 0$ of the SDE (2.1). Given such F, the function R_F defined by

$$R_F(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}F(X_t^0)\,dt\right], \quad \text{for } x > 0, \tag{2.56}$$

admits the analytic representation

$$R_F(x) = \frac{2}{C}\varphi(x)\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds + \frac{2}{C}\psi(x)\int_x^\infty \frac{F(s)\varphi(s)}{\sigma^2(s)p_c'(s)}\,ds, \tag{2.57}$$

and satisfies the ODE $\mathcal{L}R_F(x) + F(x) = 0$, Lebesgue-a.e., as well as

$$\lim_{x\downarrow 0}\frac{|R_F(x)|}{\varphi(x)} = \lim_{x\to\infty}\frac{|R_F(x)|}{\psi(x)} = 0. \tag{2.58}$$

In view of (2.51)–(2.52) and (2.57), we can calculate

$$\left(\frac{R_F}{\psi}\right)'(x) = \frac{R_F'(x)\psi(x) - R_F(x)\psi'(x)}{\psi^2(x)} = -\frac{2p_c'(x)}{\psi^2(x)}\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds, \tag{2.59}$$

and we can check that the function R_F'/ψ' is absolutely continuous with derivative

$$\left(\frac{R_F'}{\psi'}\right)'(x) = \frac{4r(x)p_c'(x)}{[\sigma(x)\psi'(x)]^2}\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds - \frac{2F(x)}{\sigma^2(x)\psi'(x)}. \tag{2.60}$$

Noting that $-\mathcal{L}R_F = F$, we can see that, if $R_{-\mathcal{L}F}$ (resp., $R_{\mathcal{L}F}$) is defined as in (2.56)–(2.57) with $-\mathcal{L}F$ (resp., $\mathcal{L}F$) in the place of F, then

$$R_F = R_{-\mathcal{L}R_F} = -R_{\mathcal{L}R_F}. \tag{2.61}$$

Also, if Θ is a C^1 function with absolutely continuous first derivative that satisfies (2.9) and (2.10) then Θ satisfies (2.16).

Chapter 3

An explicitly solvable problem of optimally stopping a diffusion with generalised drift

3.1 Introduction

We consider the problem of optimally stopping the process X given by

$$dX_t = bX_t\,dt + \beta\,dL_t^z + \sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{3.1}$$

for some constants $b \in \mathbb{R}$, $\beta \in]-1, 1[\setminus \{0\}$, $z > 0$ and $\sigma \neq 0$. The process W is a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion that is defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Also, L^z is the symmetric local time of X at level z, which is defined by

$$L_t^z = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \sigma^2 X_s^2 \mathbf{1}_{]z-\varepsilon, z+\varepsilon[}(X_s)\,ds$$

(see Revuz and Yor [35, Exercise VI.1.25]). The stochastic differential equation (3.1) has a unique strong solution that is a strictly positive process (see Engelbert and Schmidt [18]).

The process X behaves like a usual geometric Brownian motion inside $]0, z[\cup]z, \infty[$. The difference is that the direction of each excursion of X away from z is determined by an "independent" Bernouli random variable with parameter $p = (1+\beta)/2$. In other words, X is reflected in z in the positive direction with probability p and in the negative direstion with probability $1-p$.

The value function of the optimal stopping problem that we study is defined by

$$v(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}(X_\tau - K)^+\right], \tag{3.2}$$

for some constants $r, K > 0$, where $\mathcal{T}$ is the set of all $(\mathcal{F}_t)$-stopping times. We make the following assumption.

Assumption 6 *$b \in \mathbb{R}$, $\beta \in \,]-1,1[\,\setminus \{0\}$, $z > 0$, $\sigma \neq 0$, $r, K > 0$ and $r > b$.*

The theory of optimal stopping has a well-developed body of theory that has been documented in several references, including the monographs by El Karoui [25], Friedman [20], Krylov [26], Peskir and Shiryaev [32], and Shiryayev [37]. Apart from results of a general nature, there are several problems involving the optimal stopping of diffusions that have been explicitly solved. To the best of our knowledge, the only examples involving the optimal stopping of diffusions with generalised drift such as the one given by (3.1) can be found in Peskir and Shiryaev [32, Section IV.9.3] who are motivated by the range of validity of the so-called "principle of smooth fit".

3.2 Preliminary considerations

It is well-known that every solution to the Euler ODE

$$\tfrac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0 \tag{3.3}$$

is given by

$$w(x) = Ax^n + Bx^m,$$

for some constants $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are the solutions to the quadratic equation

$$\tfrac{1}{2}\sigma^2 k^2 + \left(b - \tfrac{1}{2}\sigma^2\right) k - r = 0,$$

given by

$$m, n = \frac{-\left(b - \frac{1}{2}\sigma^2\right) \mp \sqrt{\left(b - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

It is straightforward to verify that

$$r > b \quad \Leftrightarrow \quad n > 1, \tag{3.4}$$

$$n + m - 1 = -\frac{2b}{\sigma^2}, \quad nm = -\frac{2r}{\sigma^2}, \tag{3.5}$$

and

$$\frac{r}{r-b} = \frac{nm}{(n-1)(m-1)} < \frac{n}{n-1}. \tag{3.6}$$

We will need the fact that these identities and the assumption $r > b$ imply that

$$bx - r(x - K) \leq 0 \quad \text{for all } x \geq \frac{rK}{r-b}. \tag{3.7}$$

The solution to the ODE (3.3) that satisfies

$$(1+\beta)w'_+(z) = (1-\beta)w'_-(z) \tag{3.8}$$

and identifies with the function ψ defined by

$$\psi(x) = \begin{cases} x^n, & \text{if } x < z, \\ Ax^n + Bx^m, & \text{if } x \geq z, \end{cases} \tag{3.9}$$

for

$$A = \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1+\beta)} \begin{cases} > 1, & \text{if } \beta < 0, \\ \in \,]0, 1[, & \text{if } \beta > 0, \end{cases} \tag{3.10}$$

and

$$B = \frac{2n\beta}{(n-m)(1+\beta)} z^{n-m} \begin{cases} < 0, & \text{if } \beta < 0, \\ > 0, & \text{if } \beta > 0. \end{cases} \tag{3.11}$$

will play a fundamental role in our analysis. The following result is concerned with properties of this function.

Lemma 4 *In the presence of Assumption 6, the function* ψ *defined by (3.9)–(3.11) satisfies*

$$\psi'_-(z) = nz^{n-1} < nAz^{n-1} + mBz^{m-1} = \psi'_+(z) \quad \Leftrightarrow \quad \beta < 0 \tag{3.12}$$

and is convex if and only if

$$\beta < 0 \quad \text{and} \quad (n-1)(1-\beta) - 2m\beta \geq 0. \tag{3.13}$$

Furthermore,

$$\psi(x) = \psi(y)\mathbb{E}\left[e^{-rT_y}\right] \quad \text{for all } y > x, \tag{3.14}$$

where $T_y = \inf\{t \geq 0 \mid \ X_t = y\}$.

Proof. The equivalence stated in (3.12) is straightforward to see. Also, the claim that ψ is convex if and only if (3.13) is true follows immediately from the inequality $n > 1$ (see also (3.4)) and the calculations

$$\begin{aligned}\psi''(x) &= x^{m-2}\big[n(n-1)Ax^{n-m} + m(m-1)B\big] \\ &= \frac{nx^{m-2}}{(n-m)(1+\beta)}\Big\{(n-1)\big[n(1-\beta) - m(1+\beta)\big]x^{n-m} + 2m(m-1)\beta z^{n-m}\Big\}\end{aligned}$$

and

$$\psi''_+(z) = \frac{n\big[(n-1)(1-\beta) - 2m\beta\big]z^{n-2}}{1+\beta}.$$

To establish (3.14), we first note that the second distributional derivative $\psi''(dy)$ of the function ψ has Lebesgue decomposition that is given by

$$\psi''(dy) = \mathbf{1}_{]0,z[\cup]z,\infty[}(y)\psi''(y)\, dy + \left[\psi'_+(z) - \psi'_-(z)\right]\delta_z(dy),$$

where $\delta_z(dy)$ is the Dirac measure that assigns mass 1 on z. Combining this observation with the Itô-Tanaka-Meyer and the occupation times formulae (see Revuz and Yor [35, Exercise VI.1.25]) we can calculate

$$\begin{aligned}
&\psi(X_t) - r\int_0^t \psi(X_s)\,ds \\
&\quad = \psi(x) + \tfrac{1}{2}\int_0^t \big(\psi'_+ + \psi'_-\big)(X_s)\,dX_s + \tfrac{1}{2}\int_0^\infty L_t^y\,\psi''(dy) - r\int_0^t \psi(X_s)\,ds \\
&\quad = \psi(x) + \tfrac{1}{2}\int_0^t \big(\psi'_+ + \psi'_-\big)(X_s)\,dX_s - r\int_0^t \psi(X_s)\,ds \\
&\qquad + \tfrac{1}{2}\int_0^\infty L_t^y\Big(\mathbf{1}_{]0,z[\cup]z,\infty[}(y)\psi''(y)\,dy + \big[\psi'_+(z) - \psi'_-(z)\big]\,\delta_z(dy)\Big) \\
&\quad = \psi(x) + \tfrac{1}{2}\int_0^t \big(\psi'_+ + \psi'_-\big)(X_s)\,dX_s - r\int_0^t \psi(X_s)\,ds \\
&\qquad + \tfrac{1}{2}\int_0^t \sigma^2 X_s^2\psi''(X_s)\mathbf{1}_{\{X_s\neq z\}}\,ds + \tfrac{1}{2}\big[\psi'_+(z) - \psi'_-(z)\big]L_t^z \\
&\quad = \psi(x) + \int_0^t \Big[\tfrac{1}{2}\sigma^2 X_s^2\psi''(X_s) + bX_s\psi'(X_s) - r\psi(X_s)\Big]\mathbf{1}_{\{X_s\neq z\}}\,ds \\
&\qquad + \tfrac{1}{2}\int_0^t \big(\psi'_+ + \psi'_-\big)(X_s)\beta\,dL_s^z + \tfrac{1}{2}\big[\psi'_+(z) - \psi'_-(z)\big]L_t^z + \sigma\int_0^t X_s\psi'_-(X_s)\,dW_s.
\end{aligned}$$

In view of the facts that the measure dL_t^z is supported on the set $\{X_t = z\}$ and the function ψ satisfies (3.3) and (3.8), we can see that

$$\begin{aligned}
&\psi(X_t) - r\int_0^t \psi(X_s)\,ds \\
&\qquad = \psi(x) + \int_0^t \Big[\tfrac{1}{2}\sigma^2 X_s^2\psi''(X_s) + bX_s\psi'(X_s) - r\psi(X_s)\Big]\mathbf{1}_{\{X_s\neq z\}}\,ds \\
&\qquad\quad + \tfrac{1}{2}\big[(1+\beta)\psi'_+(z) - (1-\beta)\psi'_-(z)\big]L_t^z + \sigma\int_0^t X_s\psi'_-(X_s)\,dW_s \\
&\qquad = \psi(x) + \sigma\int_0^t X_s\psi'_-(X_s)\,dW_s.
\end{aligned}$$

Using the integration by parts formula, we obtain

$$e^{-rt}\psi(X_t) = \psi(x) + \sigma\int_0^t e^{-rs}X_s\psi'_-(X_s)\,dW_s. \tag{3.15}$$

In view of the fact that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^{t\wedge T_y} e^{-rs}X_s\psi'_-(X_s)\,dW_s\right)^2\right] &= \mathbb{E}\left[\int_0^t \big[\mathbf{1}_{\{s\leq T_y\}}e^{-rs}X_s\psi'_-(X_s)\big]^2 ds\right] \\
&\leq y^2\sup_{u\leq y}\big[\psi'_-(u)\big]^2 t \\
&< \infty \quad \text{for all } t\geq 0,
\end{aligned}$$

which follows from Itô's isometry, we can see that the stochastic integral in (3.15) is a square integrable martingale if stopped at T_y. It follows that

$$\psi(x) = \psi(y)\mathbb{E}\left[e^{-rT_y}\mathbf{1}_{\{T_y \le t\}}\right] + \mathbb{E}\left[e^{-rt}\psi(X_t)\mathbf{1}_{\{t<T_y\}}\right].$$

Using the monotone and the dominated convergence theorems, we can pass to the limit as $t \to \infty$ to obtain (3.14). □

Remark 3 The second inequality (3.13) is true and ψ is convex for all $\beta \in \,]-1,0[$ if and only if it is true for $\beta = -1$, namely, if and only if

$$n + m - 1 \overset{(3.5)}{=} -\frac{2b}{\sigma^2} \ge 0.$$

In view of this observation, we can see that,

if $b \le 0$, then the second inequity (3.13) is true for all $\beta \in \,]-1,0[$,

and

if $b > 0$, then the second inequity (3.13) is true for all $\beta \in \,]-1,0[$ such that

$$\beta \ge \frac{n-1}{n+2m-1} > -1.$$

□

3.3 A verification theorem

We now establish sufficient conditions under which the value function v identifies with a function $w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$(1+\beta)w'_+(z) \le (1-\beta)w'_-(z), \tag{3.16}$$

and

$$\max\Big\{\tfrac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x), \\ (x-K)^+ - w(x)\Big\} = 0 \quad \text{inside }]0,z[\,\cup\,]z,\infty[. \tag{3.17}$$

Proposition 5 *Consider the optimal stopping problem defined by (3.1)–(3.2) and suppose that Assumption 6 holds true. Let* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *be a positive function satisfying the variational inequality (3.16)–(3.17) in the sense that*
(I) w *is* C^1 *inside* $]0, z[\,\cup\,]z, \infty[$ *and* C^2 *inside* $]0, z[\,\cup\,]z, \infty[\,\setminus S$*, where* S *is a finite set,*
(II) w *satisfies (3.16),*
(III) w *satisfies (3.17) inside* $]0, z[\,\cup\,]z, \infty[\,\setminus S$*,*
(IV) w *satisfies (3.16) with equality if* $z \in \{x > 0 \mid \; w(x) > (x-K)^+\}$*, and*
(V) w *satisfies*

$$\sup_{y \in]0,n[} |w'_-(y)| < \infty \text{ for all } n \geq 1 \quad \text{and} \quad \lim_{y \to \infty} \frac{w(y)}{\psi(y)} = 0, \tag{3.18}$$

where w_- *is the left-hand derivative of* w*. Then* $w(x) = v(x)$ *for all* $x > 0$ *and*

$$\tau_\star = \inf\{t \geq 0 \mid \; w(X_t) = (X_t - K)^+\} \tag{3.19}$$

defines an optimal stopping time.

Proof. Using the Itô-Tanaka-Meyer and the occupation times formula (see Revuz and Yor [35, Exercise VI.1.25]) as in the proof of Lemma 4, we can calculate

$$\begin{aligned} w(X_t) &- r\int_0^t w(X_s)\, ds \\ &= w(x) + \tfrac{1}{2}\int_0^t (w'_+ + w'_-)(X_s)\, dX_s + \tfrac{1}{2}\int_0^\infty L_t^y\, w''(dy) - r\int_0^t w(X_s)\, ds \\ &= w(x) + \int_0^t \Big[\tfrac{1}{2}\sigma^2 X_s^2 w''(X_s) + bX_s w'(X_s) - rw(X_s)\Big] \mathbf{1}_{\{X_s \neq z\}}\, ds \\ &\quad + \tfrac{1}{2}\left[(1+\beta)w'_+(z) - (1-\beta)w'_-(z)\right] L_t^z + \sigma\int_0^t X_s w'_-(X_s)\, dW_s. \end{aligned}$$

Using the integration by parts formula, we obtain

$$\begin{aligned} e^{-rt}w(X_t) = w(x) &+ \int_0^t e^{-rs}\left[\tfrac{1}{2}\sigma^2 X_s^2 w''(X_s) + bX_s w'(X_s) - rw(X_s)\right] \mathbf{1}_{\{X_s \neq z\}}\, ds \\ &+ \tfrac{1}{2}\left[(1+\beta)w'_+(z) - (1-\beta)w'_-(z)\right]\int_0^t e^{-rs}\, dL_s^z + M_t, \end{aligned} \tag{3.20}$$

where

$$M_t = \sigma\int_0^t e^{-rs} X_s w'_-(X_s)\, dW_s.$$

If we define

$$T_n = \inf\bigl\{t \geq 0 \mid \ X_t \geq n\bigr\},$$

then we can see that Itô's isometry implies that

$$\begin{aligned}\mathbb{E}\bigl[M^2_{t\wedge T_n}\bigr] &= \sigma^2 \mathbb{E}\left[\int_0^t \bigl[\mathbf{1}_{\{s\leq T_n\}} e^{-rs} X_s w'_-(X_s)\bigr]^2\, ds\right] \\ &\leq \sigma^2 n^2 \sup_{y\leq n}\bigl[w'_-(y)\bigr]^2 t \\ &< \infty,\end{aligned}$$

which proves that the stopped process M^{T_n} is a square integrable martingale. This observation and (3.20) imply that, given any $(\mathcal{F}_t)$-stopping time τ,

$$\begin{aligned}&\mathbb{E}\bigl[e^{-r\tau}(X_\tau - K)^+ \mathbf{1}_{\{\tau\leq T_n\}}\bigr] + \mathbb{E}\left[e^{-rT_n} w(X_{T_n}) \mathbf{1}_{\{\tau > T_n\}}\right] \\ &\quad = w(x) + \mathbb{E}\left[e^{-r\tau}\bigl\{(X_\tau - K)^+ - w(X_\tau)\bigr\}\mathbf{1}_{\{\tau\leq T_n\}}\right] \\ &\qquad + \mathbb{E}\left[\int_0^{\tau\wedge T_n} e^{-rs}\left[\tfrac{1}{2}\sigma^2 X_s^2 w''(X_s) + bX_s w'(X_s) - rw(X_s)\right]\mathbf{1}_{\{X_s\neq z\}}\, ds\right] \\ &\qquad + \tfrac{1}{2}\left[(1+\beta)w'_+(z) - (1-\beta)w'_-(z)\right]\mathbb{E}\left[\int_0^{\tau\wedge T_n} e^{-rs}\, dL^z_s\right].\end{aligned} \tag{3.21}$$

If τ is any $(\mathcal{F}_t)$-stopping time, then (3.21) and the fact that w satisfies the variational inequality (3.16)–(3.17) imply that

$$\mathbb{E}\bigl[e^{-r\tau}(X_\tau - K)^+ \mathbf{1}_{\{\tau\leq T_n\}}\bigr] + w(n)\mathbb{E}\left[e^{-rT_n}\mathbf{1}_{\{\tau > T_n\}}\right] \leq w(x).$$

Similarly, we can see that, if $\tau_\star$ is defined by (3.19), then

$$\mathbb{E}\bigl[e^{-r\tau_\star}(X_{\tau_\star} - K)^+ \mathbf{1}_{\{\tau_\star\leq T_n\}}\bigr] + w(n)\mathbb{E}\left[e^{-rT_n}\mathbf{1}_{\{\tau_\star > T_n\}}\right] = w(x).$$

Combining these observations with the identity

$$\lim_{n\to\infty} w(n)\mathbb{E}\left[e^{-rT_n}\mathbf{1}_{\{\tau_\star > T_n\}}\right] \overset{(3.14)}{=} \lim_{n\to\infty} \frac{w(n)\psi(x)}{\psi(n)} \frac{\mathbb{E}\left[e^{-rT_n}\mathbf{1}_{\{\tau_\star > T_n\}}\right]}{\mathbb{E}\left[e^{-rT_n}\right]} \overset{(3.18)}{=} 0,$$

we can see that $v(x) = w(x)$ and that $\tau_\star$ is optimal. □

3.4 The case when $-1 < \beta < 0$

We will solve the optimal stopping problem that arises when $\beta \in \,]-1, 0[$ under the assumption that the problem data is such that

$$(n-1)(1-\beta) - 2m\beta \geq 0 \quad \Leftrightarrow \quad \frac{r}{r-b} \leq \frac{n}{n - \frac{1+\beta}{1-\beta}}, \tag{3.22}$$

where the equivalence follows from (3.6). In view of Lemma 4, (3.22) is equivalent to the convexity of ψ (see also Remark 3).

In this case, we are going to show that the function w defined by

$$w(x) = \begin{cases} \Gamma\psi(x), & \text{if } x \leq a, \\ x - K, & \text{if } x > a, \end{cases} \tag{3.23}$$

satisfies the requirements of Proposition 5 and identifies with the value function v for appropriate choices for the constant Γ and the free-boundary point $a > 0$. To this end, we are guided by the intuition that we can get from a careful inspection of Figures 1–3. For sufficiently small values of z, we expect that $a > z$, while, for sufficiently large values of z, we expect that $a < z$. In both of these cases, we use the so-called "principle of smooth fit", namely, the requirement that the value function should be C^1 along the free-boundary point a, to determine Γ, a (see Figures 1 and 3), which yields the system of equations

$$\begin{cases} \Gamma\psi(a) = a - K, \\ \Gamma\psi'(a) = 1, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \Gamma = (a-K)/\psi(a) = 1/\psi'(a), \\ a\psi'(a) - \psi(a) - K\psi'(a) = 0. \end{cases} \tag{3.24}$$

In view of the definition (3.9) of ψ, we can see that the possibility that $a < z$ implies that

$$a = \frac{nK}{n-1} > 0 \tag{3.25}$$

solves the equation for a in (3.24). On the other hand, we can check that, if $z < a$, then the equation for a in (3.24) is equivalent to

$$F(a) = 0, \tag{3.26}$$

where F is defined by

$$F(a) = \big[(n-1)a - nK\big] A a^{n-m} + \big[(m-1)a - mK\big] B, \tag{3.27}$$

because

$$F(a) = a^{-m+1}\big[(a-K)\psi'(a) - \psi(a)\big] \quad \text{for all } a > z.$$

For intermediate values of z, Figure 2 suggests that $a = z$ and the function w given by (3.23) is not C^1 at a.

Proposition 6 *Consider the optimal stopping problem defined by (3.1)–(3.2) and suppose that Assumption 6 holds true. Also, suppose that $\beta \in \,]-1, 0[$ and that (3.22) is satisfied. If the problem data is such that $z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}$, then equation (3.26) has a unique solution a such that*

$$\frac{nK}{n - \frac{1+\beta}{1-\beta}} < a < \frac{nK}{n-1}. \tag{3.28}$$

If the rest of the problem data is kept fixed, then this solution $a = a(z)$ defines a strictly decreasing C^1 function such that

$$\lim_{z \downarrow 0} a(z) = \frac{nK}{n-1} \quad \text{and} \quad \lim_{z \uparrow \frac{nK}{n - \frac{1+\beta}{1-\beta}}} a(z) = \frac{nK}{n - \frac{1+\beta}{1-\beta}}. \tag{3.29}$$

Furthermore,

(I) *if $\frac{nK}{n-1} < z$, then let a be given by (3.25),*
(II) *if $\frac{nK}{n - \frac{1+\beta}{1-\beta}} \leq z \leq \frac{nK}{n-1}$, then let $a = z$, and*
(III) *if $z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}$, then let a be the unique solution to (3.26).*

Given such choices for a, the function w, which is defined by (3.23) for $\Gamma > 0$ being given by (3.24) in cases (I), (III), and $\Gamma = (z-K)z^{-n} > 0$ in case (II), identifies with the value function v of the discretionary stopping problem. Furthermore, the $(\mathcal{F}_t)$-stopping time defined by

$$\tau_\star = \inf\big\{t \geq 0 \mid \; X_t \geq a\big\},$$

is optimal.

Proof. To study the solvability of equation (3.26), we first use the definitions (3.10), (3.11) of A, B to calculate

$$F(K) = -K^{n-m+1}\big[A + BK^{m-n}\big], \quad F'(K) = (m-1)K^{n-m}\big[A + BK^{m-n}\big], \tag{3.30}$$

and

$$\begin{aligned} A + BK^{m-n} &= \frac{1}{(1+\beta)(n-m)}\left[n(1-\beta) - m(1+\beta) + 2n\beta\left(\frac{z}{K}\right)^{n-m}\right] \\ &> 0 \qquad \text{for all } z < \left[-\frac{n(1-\beta)-m(1+\beta)}{2n\beta}\right]^{\frac{1}{n-m}} K. \end{aligned} \tag{3.31}$$

In view of the calculation

$$F''(a) = (n-m)\big[(n-1)(n-m+1)a - n(n-m-1)K\big]Aa^{n-m-2},$$

and the fact that $A > 0$, we can see that

$$F' \begin{cases} \text{is strictly decreasing in }]0, a_\ddagger[, \\ \text{is strictly increasing in }]a_\ddagger, \infty[, \end{cases} \qquad \text{where } a_\ddagger = \frac{n(n-m-1)K}{(n-1)(n-m+1)}. \tag{3.32}$$

If the problem data is such that

$$z < \min\left\{\frac{nK}{n - \frac{1+\beta}{1-\beta}},\ \left[-\frac{n(1-\beta)-m(1+\beta)}{2n\beta}\right]^{\frac{1}{n-m}} K\right\},$$

then (3.30) and (3.31) imply that $F(K) < 0$ and $F'(K) < 0$. Combining these inequalities with (3.32) and the calculations

$$\begin{aligned} F\left(\frac{nK}{n-\frac{1+\beta}{1-\beta}}\right) &= \frac{K}{(1-\beta)\left(n - \frac{1+\beta}{1-\beta}\right)} \\ &\quad \times \left[2n\beta A\left(\frac{nK}{n-\frac{1+\beta}{1-\beta}}\right)^{n-m} - \big[n(1-\beta)-m(1+\beta)\big]B\right] \\ &= \frac{2nK\beta\big[n(1-\beta)-m(1+\beta)\big]}{(n-m)(1+\beta)(1-\beta)\left(n-\frac{1+\beta}{1-\beta}\right)}\left[\left(\frac{nK}{n-\frac{1+\beta}{1-\beta}}\right)^{n-m} - z^{n-m}\right] \\ &< 0 \qquad \text{for all } z < \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \end{aligned} \tag{3.33}$$

and

$$F\left(\frac{nK}{n-1}\right) = -\frac{(n-m)K}{n-1}B > 0, \tag{3.34}$$

we can see that equation (3.26) has a unique solution such that (3.28) is true. In particular, this solution is such that

$$F'(a) > 0 \quad \text{and} \quad F(x) < 0 \quad \text{for all } x \in \,]K, a[. \tag{3.35}$$

On the other hand, if the problem data is such that

$$\left[-\frac{n(1-\beta) - m(1+\beta)}{2n\beta}\right]^{\frac{1}{n-m}} K \le z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}, \tag{3.36}$$

then (3.30) and (3.31) imply that $F(K) \ge 0$. This inequality and the calculation

$$F(z) = \frac{1-\beta}{1+\beta}\left[\left(n - \frac{1+\beta}{1-\beta}\right)z - nK\right] z^{n-m} < 0 \quad \text{for all } z < \frac{nK}{n - \frac{1+\beta}{1-\beta}} \tag{3.37}$$

imply that there exists a subset of $]K, z]$ in which F' is strictly negative because F is continuous in $]k, z[$. Combining this observation with (3.32) and (3.33)–(3.34), we can see that equation (3.26) has a unique solution such that (3.28) is true. In particular, this solution is such that

$$F'(a) > 0 \quad \text{and} \quad F(x) < 0 \quad \text{for all } x \in \,]z, a[. \tag{3.38}$$

To show that the solution to (3.26) is a strictly decreasing function of z that satisfies (3.29) when the rest of the problem data is kept constant, we note that $a(z)$ satisfies the equation $\tilde{F}\big(a(z), z\big) = 0$, where

$$\tilde{F}(a, z) = \big[(n-1)a - nK\big]Aa^{n-m} + \big[(m-1)a - mK\big]B(z), \tag{3.39}$$

and A, B are given by (3.10), (3.11) (see also (3.27) defining F). In view of the identification of $\tilde{F}$ with F if z is considered to be constant, we can see that (3.35) and (3.38) imply that $\tilde{F}_a\big(a(z), z\big) > 0$ for all $z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}$. Therefore, differentiating the identity $\tilde{F}\big(a(z), z\big) = 0$ with respect to z, we obtain

$$a'(z) = -\frac{\tilde{F}_z\big(a(z), z\big)}{\tilde{F}_a\big(a(z), z\big)} = -\frac{2n\big[(m-1)a(z) - mK\big]\beta}{1+\beta} \frac{z^{n-m-1}}{\tilde{F}_a\big(a(z), z\big)} < 0, \tag{3.40}$$

the inequality following because

$$(m-1)a(z) - mK \leq (m-1)\frac{nK}{n - \frac{1+\beta}{1-\beta}} - mK = -\frac{K}{n - \frac{1+\beta}{1-\beta}} \frac{n(1-\beta) - m(1+\beta)}{1-\beta} < 0.$$

The first of the limits in (3.29) follows from the observation that

$$0 = \lim_{z \downarrow 0} \tilde{F}\big(a(z), z\big) \overset{(3.11),(3.27)}{=} A \lim_{z \downarrow 0} \big[(n-1)a(z) - nK\big] a^{n-m}(z),$$

while, the second limit in (3.29) follows from (3.33) and (3.35)–(3.38).

In view of its construction, we will prove that the function w that is defined as in the statement of this result satisfies (3.16)–(3.17) if we show that

$$(1+\beta)w'_+(z) \leq (1-\beta)w'_-(z), \quad \text{if } z \geq a, \tag{3.41}$$

$$(x-K)^+ \leq w(x) \quad \text{for all } x < a, \tag{3.42}$$

and

$$\tfrac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) \leq 0 \quad \text{inside }]a, \infty[. \tag{3.43}$$

The inequality (3.41) is equivalent to $\beta \leq 0$ if $z > a$, which is true by assumption, while, if $z = a$, then it is equivalent to

$$nK \leq \left(n - \frac{1+\beta}{1-\beta}\right) z,$$

which also holds true.

In light of (3.23) and the first expression for $\Gamma > 0$ in (3.24), we can see that (3.42) is equivalent to

$$\frac{x-K}{\psi(x)} \leq \frac{a-K}{\psi(a)} \quad \text{for all } x < a. \tag{3.44}$$

The continuity of ψ and the calculation

$$\frac{d}{dx}\frac{x-K}{\psi(x)} = \begin{Bmatrix} -\big[(n-1)x - nK\big]x^{-1-n}, & \text{if } x < z \\ -x^{m-1}F(x)/\psi^2(x), & \text{if } x > z \end{Bmatrix} > 0 \quad \text{for all } x < a,$$

where the inequality follows from the fact that $a < nK/(n-1)$ and (3.35)–(3.38), imply that (3.44) is indeed true. The inequality (3.43) is equivalent to

$$bx - r(x-K) \leq 0 \quad \text{for all } x > a, \tag{3.45}$$

which is true thanks to (3.6), (3.7), (3.22) and the fact that, in all cases, $a \geq nK/\left(n - \frac{1+\beta}{1-\beta}\right)$.

Finally, the identification of w with the discretionary stopping's value function v and the optimality of $\tau_\star$ follow immediately from Proposition 5. □

3.5 The case when $0 < \beta < 1$

We first show that the function w defined by (3.23) may identify with the value function v for appropriate values of the constant Γ and the free-boundary point $a > 0$, depending on parameter values (see Figures 4 and 5). To this end, we appeal to the "principle of smooth fit", which yields the system of equations (3.24) and equation (3.26) for a. The next result is concerned with the solvability of (3.26) in the context that we consider here as well as with necessary and sufficient conditions on the problem data for this case to be optimal.

Proposition 7 *Consider the optimal stopping problem defined by (3.1)–(3.2) and suppose that Assumption 6 holds true and that $\beta \in \,]0,1[$. Equation (3.26) has a unique solution $a > 0$. This solution is such that*

$$a \begin{cases} \in \left]\frac{nK}{n-1} \vee z, \infty\right[, & \text{if } n \leq \frac{1+\beta}{1-\beta}, \\ \in \left]\frac{nK}{n-1} \vee z, \frac{nK}{n-\frac{1+\beta}{1-\beta}}\right[, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z < \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \\ = \frac{nK}{n-\frac{1+\beta}{1-\beta}}, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z = \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \\ \in \left]\frac{nK}{n-\frac{1+\beta}{1-\beta}}, z\right[, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z > \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \end{cases} \tag{3.46}$$

and

$$F(\zeta) \begin{cases} < 0, & \text{for all } \zeta \in \left]z \wedge \frac{nK}{n-1}, a\right[, \\ > 0, & \text{for all } \zeta > a. \end{cases} \tag{3.47}$$

If the rest of the problem data is kept fixed, then this solution $a = a(z)$ defines a strictly increasing C^1 function. If we further assume that $z < \frac{nK}{n-\frac{1+\beta}{1-\beta}}$ if $n > \frac{1+\beta}{1-\beta}$, then there *exists a unique*

$$z^* \in \begin{cases} \left]\frac{nK}{n-1}, \infty\right[, & \text{if } n \leq \frac{1+\beta}{1-\beta}, \\ \left]\frac{nK}{n-1}, \frac{nK}{n-\frac{1+\beta}{1-\beta}}\right[, & \text{if } n > \frac{1+\beta}{1-\beta}, \end{cases} \tag{3.48}$$

such that

$$a(z) - K - \frac{1}{n}\left(\frac{nK}{n-1}\right)^{-n+1} \psi\big(a(z)\big) \begin{cases} > 0, & \text{if } z < z^*, \\ < 0, & \text{if } z > z^*. \end{cases} \tag{3.49}$$

Furthermore, the function w that is defined by (3.23) for $\Gamma > 0$ being given by (3.24) and $a > 0$ being the unique solution of (3.26) satisfies the variational inequality (3.16)–(3.17) and identifies with the value function v of the discretionary stopping problem if and only if $z \leq z^*$. In particular, the $(\mathcal{F}_t)$-stopping time defined by

$$\tau_\star = \inf\big\{t \geq 0 \mid \; X_t \geq a\big\},$$

is optimal.

Proof. Recalling that $A > 0$, $B > 0$ (see (3.10), (3.11)), $m < 0$ and $1 < n$, we can see that the calculation

$$F'(a) = (n-1)(n-m+1)\left[a - \frac{(n-m)nK}{(n-1)(n-m+1)}\right] Aa^{n-m-1} + (m-1)B$$

implies that

$$\lim_{a \downarrow 0} F'(a) = (m-1)B < 0 \quad \text{and} \quad \lim_{a \to \infty} F'(a) = \infty.$$

Combining this observation with (3.32), we can see that there exists a unique $a_\dagger > 0$ such that

$$F'(a) \begin{cases} < 0, & \text{if } a < a_\dagger, \\ > 0, & \text{if } a > a_\dagger. \end{cases} \tag{3.50}$$

In view of these inequalities and the calculations

$$F\left(\frac{nK}{n-1}\right) = -\frac{(n-m)K}{n-1}B < 0 \quad \text{and} \quad \lim_{a \to \infty} F(a) = \infty, \tag{3.51}$$

we can conclude that equation (3.26) has a unique solution $a > 0$. In particular, we can see that this solution is such that $a > \frac{nK}{n-1}$. Furthermore, we can obtain (3.46) and (3.47) by considering (3.50) and (3.51) in connection with the facts that

$$\begin{aligned}
&\text{if } n \le \frac{1+\beta}{1-\beta}, \ \text{then } F(z) < 0 \text{ for all } z > 0,\\
&\text{if } n > \frac{1+\beta}{1-\beta}, \ \text{then } F(z) < 0 \text{ for all } z \in \Big]0, \frac{nK}{n - \frac{1+\beta}{1-\beta}}\Big[,\\
&\text{if } n > \frac{1+\beta}{1-\beta}, \ \text{then } F(z) > 0 \text{ for all } z > \frac{nK}{n - \frac{1+\beta}{1-\beta}},
\end{aligned}$$

which follow from the calculation in (3.37), and the inequalities

$$F\left(\frac{nK}{n - \frac{1+\beta}{1-\beta}}\right) \begin{cases} > 0, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z < \frac{nK}{n-\frac{1+\beta}{1-\beta}},\\ = 0, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z = \frac{nK}{n-\frac{1+\beta}{1-\beta}},\\ < 0, & \text{if } n > \frac{1+\beta}{1-\beta} \text{ and } z > \frac{nK}{n-\frac{1+\beta}{1-\beta}},\end{cases}$$

which follow from (3.33) because $\beta > 0$ here.

If the rest of the problem data is kept fixed, then we can see that the same arguments and calculations as the ones in (3.40) imply that the function $z \mapsto a'(z)$ is continuous and strictly positive because $\beta > 0$ and

$$(m-1)a(z) - mK \le (m-1)\frac{nK}{n-1} - mK = -\frac{(n-m)K}{n-1} < 0.$$

In the rest of the proof, we assume that $z < \frac{nK}{n-\frac{1+\beta}{1-\beta}}$ if $n > \frac{1+\beta}{1-\beta}$. To establish the existence and uniqueness of z^* such that (3.48) and (3.49) hold true, we consider the inequality

$$\frac{x-K}{\tilde{\psi}(x,z)} \le \frac{a(z)-K}{\tilde{\psi}\big(a(z),z\big)} \quad \text{for } x < a(z), \tag{3.52}$$

where $a(z)$ is the solution to the equation $\tilde{F}\big(a(z),z\big) = 0$ that identifies with (3.26) for $\tilde{F}$ given by (3.39) and we write $\tilde{\psi}(\cdot,z)$ instead of ψ to stress the explicit dependence of this function on z. The calculation

$$\frac{\partial}{\partial x}\frac{x-K}{\tilde{\psi}(x,z)} = \begin{cases} -\big[(n-1)x - nK\big]x^{-1-n}, & \text{if } x < z,\\ -x^{m-1}\tilde{F}(x,z)/\tilde{\psi}^2(x,z), & \text{if } x > z,\end{cases} \tag{3.53}$$

implies that, if $z \leq \frac{nK}{n-1}$, then (3.52) holds with strict inequality for all $x < a(z)$. On the other hand, if $z > \frac{nK}{n-1}$, then (3.47) and (3.53) imply that

$$\frac{\partial}{\partial x} \frac{x-K}{\tilde{\psi}(x,z)} \begin{cases} < 0, & \text{if } x \in \left]\frac{nK}{n-1}, z\right[, \\ > 0, & \text{if } x \in \left]0, \frac{nK}{n-1}\right[\cup \left]z, a(z)\right[. \end{cases}$$

Therefore, if $z > \frac{nK}{n-1}$, then (3.52) is true for all $x < a$ if and only if the inequality

$$\left.\frac{x-K}{\tilde{\psi}(x,z)}\right|_{x=\frac{nK}{n-1}} = \left.\frac{x-K}{x^n}\right|_{x=\frac{nK}{n-1}} = \frac{1}{n}\left(\frac{nK}{n-1}\right)^{-n+1} \leq \frac{a(z)-K}{\tilde{\psi}\big(a(z),z\big)} \tag{3.54}$$

holds true. If $n > \frac{1+\beta}{1-\beta}$, then (3.46) and (3.53) imply that

$$\left.\frac{x-K}{x^n}\right|_{x=\frac{nK}{n-1}} > \left.\frac{x-K}{x^n}\right|_{x=\frac{nK}{n-\frac{1+\beta}{1-\beta}}} = \lim_{z \uparrow \frac{nK}{n-\frac{1+\beta}{1-\beta}}} \frac{a(z)-K}{\psi\big(a(z),z\big)}.$$

Also, if $n \leq \frac{1+\beta}{1-\beta}$, then (3.10), (3.11) and the fact that $a(z) > z$ for all $z > 0$ imply that

$$\lim_{z\to\infty} \frac{a(z)-K}{\psi\big(a(z),z\big)} = 0.$$

Combining these observations with the calculation

$$\begin{aligned} \frac{d}{dz}\frac{a(z)-K}{\tilde{\psi}\big(a(z),z\big)} &= -\frac{\tilde{F}\big(a(z),z\big)a^{m-1}(z)}{\tilde{\psi}^2\big(a(z),z\big)}a'(z) - \frac{2n\beta\big[a(z)-K\big]a^m(z)}{(1+\beta)\tilde{\psi}^2\big(a(z),z\big)}z^{n-m-1} \\ &= -\frac{2n\beta\big[a(z)-K\big]a^m(z)}{(1+\beta)\tilde{\psi}^2\big(a(z),z\big)}z^{n-m-1} \\ &< 0, \end{aligned}$$

and the fact that (3.52) holds with strict inequality for all $x < a(z)$ if $z \leq \frac{nK}{n-1}$, we can see that there exists a unique z^* satisfying (3.48) and such that (3.54) holds true if $z \leq z^*$ and is false if $z > z^*$. In particular,

$$\text{(3.52) holds true for all } x < a(z) \text{ if and only if } z \leq z^*, \tag{3.55}$$

and that the inequalities in (3.49) are all true.

In light of its construction, we will prove that the function w that is as in the statement of this result satisfies the variational inequality (3.16)–(3.17) if we show that

$$(x-K)^+ \leq w(x) \quad \text{for all } x < a, \tag{3.56}$$

and

$$\tfrac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) \leq 0 \quad \text{inside }]a, \infty[. \tag{3.57}$$

The definition (3.23) of w and the first expression for $\Gamma > 0$ in (3.24) imply that (3.56) is equivalent to (3.52), which is true if and only if $z \leq z^*$ (see (3.55)). The inequality (3.57) is equivalent to

$$bx - r(x - K) \leq 0 \quad \text{for all } x > a, \tag{3.58}$$

which is true thanks to (3.6), (3.7) and the fact that $a \geq \frac{nK}{n-1}$.

Finally, the identification of w with the discretionary stopping's value function v and the optimality of $\tau_\star$ follow from Proposition 5. □

The function w defined by (3.23) with $\Gamma, a > 0$ being determined by the requirement that it is C^1 at a is depicted by Figure 4 if $z < z^*$ and by Figure 5 if $z = z^*$. When the problem data is such that $z > z^*$, we will show that there exist constants C_ℓ, D_ℓ, C_r, D_r and free boundary points $\gamma \in \left]\frac{nK}{n-1}, z\right[$, $\zeta > z$ such that the function w defined by

$$w(x) = \begin{cases} \frac{1}{n}\left(\frac{nK}{n-1}\right)^{-n+1} x^n, & \text{if } x \leq \frac{nK}{n-1}, \\ C_\ell x^n + D_\ell x^m, & \text{if } x \in [\gamma, z], \\ C_r x^n + D_r x^m, & \text{if } x \in]z, \zeta], \\ x - K, & \text{if } x \in \left]\frac{nK}{n-1}, \gamma\right[\cup]\zeta, \infty[, \end{cases} \tag{3.59}$$

identifies with the value function v. This function satisfies the ODE (3.3) inside the set $\left]0, \frac{nK}{n-1}\right[\cup]\gamma, z[\cup]z, \zeta[$ and is C^1 at $\frac{nK}{n-1}$. The requirements that w is continuous at z and

$$(1+\beta)w'_+(z) = (1-\beta)w'_-(z) \tag{3.60}$$

yield the identities

$$C_r = \frac{n(1-\beta) - m(1+\beta)}{(n-m)(1+\beta)} C_\ell - \frac{2m\beta}{(n-m)(1+\beta)} D_\ell z^{-(n-m)} \tag{3.61}$$

and

$$D_r = \frac{2n\beta}{(n-m)(1+\beta)} C_\ell z^{n-m} + \frac{n(1+\beta) - m(1-\beta)}{(n-m)(1+\beta)} D_\ell, \tag{3.62}$$

while, C^1 fit at γ, ζ yields

$$C_\ell = -\frac{1}{n-m}\big[(m-1)\gamma - mK\big]\gamma^{-n}, \quad C_r = -\frac{1}{n-m}\big[(m-1)\zeta - mK\big]\zeta^{-n}, \tag{3.63}$$

$$D_\ell = \frac{1}{n-m}\big[(n-1)\gamma - nK\big]\gamma^{-m} \quad \text{and} \quad D_r = \frac{1}{n-m}\big[(n-1)\zeta - nK\big]\zeta^{-m}. \tag{3.64}$$

Substituting the expressions given by (3.61)–(3.62) for the constants C_ℓ, D_ℓ, C_r, D_r into (3.63)–(3.64), we obtain the system of equations

$$\begin{aligned}\big[(n-1)\zeta - nK\big]z^m\zeta^{-m} + \frac{2n\beta}{(1+\beta)(n-m)}\big[(m-1)\gamma - mK\big]z^n\gamma^{-n}& \\ -\frac{n(1+\beta)-m(1-\beta)}{(1+\beta)(n-m)}\big[(n-1)\gamma - nK\big]z^m\gamma^{-m} &= 0, \end{aligned}\tag{3.65}$$

$$\begin{aligned}\big[(m-1)\zeta - mK\big]z^n\zeta^{-n} - \frac{n(1-\beta)-m(1+\beta)}{(1+\beta)(n-m)}\big[(m-1)\gamma - mK\big]z^n\gamma^{-n}& \\ -\frac{2m\beta}{(1+\beta)(n-m)}\big[(n-1)\gamma - nK\big]z^m\gamma^{-m} &= 0. \end{aligned}\tag{3.66}$$

Subtracting (3.65) from (3.66), we obtain

$$\begin{aligned}G(\gamma,\zeta) := \big[(n-1)\zeta - nK\big]z^m\zeta^{-m} - \big[(m-1)\zeta - mK\big]z^n\zeta^{-n}& \\ - \big[(n-1)\gamma - nK\big]z^m\gamma^{-m} + \big[(m-1)\gamma - mK\big]z^n\gamma^{-n} &= 0\end{aligned}\tag{3.67}$$

On the other hand, solving (3.65) for $\big[(m-1)\gamma - mK\big]z^n\gamma^{-n}$ and substituting the resulting expression in (3.66), we obtain

$$H(\gamma,\zeta) := \zeta^{-n}F(\zeta) - \frac{1-\beta}{1+\beta}\big[(n-1)\gamma - nK\big]\gamma^{-m} = 0, \tag{3.68}$$

where F is defined by (3.27).

Proposition 8 *Consider the optimal stopping problem defined by (3.1)–(3.2) and suppose that Assumption 6 holds true and that $\beta \in \,]0,1[$. The system of equations (3.67)–(3.68) has a unique solution (γ,ζ) such that $\frac{nk}{n-1} \le \gamma < z < \zeta$ if and only if*

$$\textit{either} \quad z \ge \frac{nK}{n-\frac{1+\beta}{1-\beta}}, \ \textit{if } n > \frac{1+\beta}{1-\beta}, \quad \textit{or} \quad z \ge z^*, \ \textit{otherwise}, \tag{3.69}$$

where z^* is as in Proposition 7. If the problem data is such that $z < \frac{nK}{n-\frac{1+\beta}{1-\beta}}$ if $n > \frac{1+\beta}{1-\beta}$, then this solution satisfies

$$(\gamma,\zeta) = \left(\frac{nK}{n-1}, a\right) \ \textit{if } z = z^* \quad \textit{and} \quad \frac{nK}{n-1} < \gamma < z < a < \zeta \ \textit{if } z > z^*, \tag{3.70}$$

where a is the unique solution to the equation $F(a) = 0$. In either case, the function w defined by (3.59) for $C_\ell, D_\ell, C_r, D_r > 0$ being given by (3.63)–(3.64) and $\gamma < \zeta$ being the solution considered above satisfies the variational inequality (3.16)–(3.17) and identifies with the value function v of the discretionary stopping problem. In particular, the $(\mathcal{F}_t)$-stopping time defined by

$$\tau_\star = \inf \left\{ t \geq 0 \mid \ X_t \in \left] \frac{nK}{n-1}, \gamma \right[\cup]\zeta, \infty[\right\},$$

is optimal.

Proof. In view of the inequality

$$\frac{nmK}{(n-1)(m-1)} < \frac{nK}{n-1},$$

which follows from the fact that $m < 0 < n$, and the calculations

$$G(\gamma, \gamma) = 0, \qquad \lim_{\zeta \to \infty} G(\gamma, \zeta) = \infty,$$

and

$$\frac{\partial}{\partial \zeta} G(\gamma, \zeta) = -(n-1)(m-1) \left[\zeta - \frac{nmK}{(n-1)(m-1)} \right] z^n \zeta^{-n-1} \left[\left(\frac{\zeta}{z} \right)^{n-m} - 1 \right]$$
$$\begin{cases} < 0, & \text{for all } \zeta \in \left] \frac{nK}{n-1}, z \right[, \\ > 0, & \text{for all } \zeta > z, \end{cases}$$

we can see that the equation $G(\gamma, \zeta) = 0$ defines uniquely a function $L : \left[\frac{nK}{n-1}, z \right[\to]z, \infty[$ such that

$$z < L(\gamma) \quad \text{and} \quad G\big(\gamma, L(\gamma)\big) = 0 \quad \text{for all } \gamma \in \left[\frac{nK}{n-1}, z \right[. \tag{3.71}$$

In particular, we can see that

$$G(\gamma, \zeta) \begin{cases} < 0, & \text{for all } \frac{nK}{n-1} \leq \gamma < \zeta < L(\gamma), \\ > 0, & \text{for all } \frac{nK}{n-1} \leq \gamma < L(\gamma) < \zeta, \end{cases} \quad \text{and} \quad \lim_{\gamma \uparrow z} L(\gamma) = z. \tag{3.72}$$

Also, differentiating the identity $G\big(\gamma, L(\gamma)\big) = 0$ with respect to γ, we obtain

$$L'(\gamma) = \frac{\left[\gamma - \frac{nmK}{(n-1)(m-1)} \right] \gamma^{-n-1} \left[\left(\frac{\gamma}{z} \right)^{n-m} - 1 \right]}{\left[L(\gamma) - \frac{nmK}{(n-1)(m-1)} \right] L^{-n-1}(\gamma) \left[\left(\frac{L(\gamma)}{z} \right)^{n-m} - 1 \right]} < 0 \quad \text{for all } \gamma \in \left] \frac{nK}{n-1}, z \right[\tag{3.73}$$

To resolve the solvability of the system of equations (3.67)–(3.68), we need to establish conditions under which the equation $H\big(\gamma, L(\gamma)\big) = 0$ has a unique solution $\gamma \in \big[\frac{nK}{n-1}, z\big[$. To this end, we note that the definition (3.68) of H and the limit in (3.72) imply that

$$H(z,z) = -\frac{2\beta z^{-m+1}}{1+\beta} < 0,$$

and we use (3.71), (3.73) to calculate

$$\begin{aligned}
&\frac{d}{d\gamma} H\big(\gamma, L(\gamma)\big) \\
&\quad = -(n-1)(m-1)\frac{1-\beta}{1+\beta}\left[\gamma - \frac{nmK}{(n-1)(m-1)}\right]\gamma^{-m-1} \\
&\qquad \times \left\{ \frac{\big[n(1-\beta) - m(1+\beta)\big]L^{n-m}(\gamma) + 2n\beta z^{n-m}}{(n-m)(1-\beta)} \frac{\left(\frac{\gamma}{z}\right)^{n-m} - 1}{\left(\frac{L(\gamma)}{z}\right)^{n-m} - 1}\gamma^{-n+m} - 1 \right\} \\
&\quad < 0 \qquad \text{for all } \gamma \in \left]\frac{nK}{n-1}, z\right[.
\end{aligned}$$

In light of these calculations, we can see that

$$\text{there exists } \gamma^* \in \left[\frac{nK}{n-1}, z\right[\text{ such that } H\big(\gamma^*, L(\gamma^*)\big) = 0 \tag{3.74}$$

if and only if

$$H\big(\gamma, L(\gamma)\big)\Big|_{\gamma = \frac{nK}{n-1}} = L^{-n}(\gamma) F\big(L(\gamma)\big)\Big|_{\gamma = \frac{nK}{n-1}} \geq 0. \tag{3.75}$$

If the problem data is such that $n > \frac{1+\beta}{1-\beta}$ and $z \geq \frac{nK}{n - \frac{1+\beta}{1-\beta}}$, then (3.46) and (3.71) imply that $a \leq z < L\big(\frac{nK}{n-1}\big)$. Therefore, (3.75) holds with strict inequality thanks to (3.47). On the other hand, if the problem data is such that $z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}$ if $n > \frac{1+\beta}{1-\beta}$, then (3.47) implies that the inequality (3.75) is true if and only if $L\big(\frac{nK}{n-1}\big) \geq a$, where $a > z \vee \frac{nK}{n-1}$ is the unique solution to the equation $F(a) = 0$. Furthermore, the inequality $L\big(\frac{nK}{n-1}\big) \geq a$ is equivalent to

$$\begin{aligned}
G\left(\frac{nK}{n-1}, a\right) &= \big[(n-1)a - nK\big] z^m a^{-m} \\
&\qquad - \big[(m-1)a - mK\big] z^n a^{-n} - \frac{n-m}{n}\left(\frac{nK}{n-1}\right)^{-n+1} z^n \\
&\leq 0,
\end{aligned} \tag{3.76}$$

thanks to (3.72). Using the identities $F(a) = 0$ and $Bz^m + Az^n = z^n$ to eliminate the term $\big[(m-1)a - mK\big]$ in (3.76), we can calculate

$$Ba^m G\left(\frac{nK}{n-1}, a\right) = \big[(n-1)a - nK\big] z^n - \frac{n-m}{n}\left(\frac{nK}{n-1}\right)^{-n+1} z^n B a^m.$$

Similarly, we can eliminate the term $\big[(n-1)a - nK\big]$ in (3.76) to obtain

$$Aa^n G\left(\frac{nK}{n-1}, a\right) = -\big[(m-1)a - mK\big] z^n - \frac{n-m}{n}\left(\frac{nK}{n-1}\right)^{-n+1} z^n A a^n.$$

Adding up these identities yields

$$z^{-n}\psi(a) G\left(\frac{nK}{n-1}, a\right) = (n-m)\left[a - K - \frac{1}{n}\left(\frac{nK}{n-1}\right)^{-n+1} \psi(a)\right]$$

because $\psi(a) = Aa^n + Ba^m$ when the problem data is such that $z < \frac{nK}{n - \frac{1+\beta}{1-\beta}}$ if $n > \frac{1+\beta}{1-\beta}$. In view of (3.49), it follows that (3.76) is true if and only if $z \geq z^*$. Therefore, the system of equations (3.67)–(3.68) has a unique solution, which identifies with the pair $\big(\gamma^*, L(\gamma^*)\big)$ considered in (3.74), if and only if the problem data satisfy (3.69). In particular, the arguments that we have developed reveal that (3.70) holds true.

The strict positivity of the constants C_ℓ, D_ℓ, C_r, D_r follows from their definition in (3.63)–(3.64) and the inequalities

$$\frac{mK}{m-1} < \frac{nK}{n-1} < \gamma < \zeta.$$

By construction, we will prove that the function w that is as in the statement of this result satisfies the variational inequality (3.16)–(3.17) if we show that

$$(x-K)^+ \leq w(x) \quad \text{for all } x \in \left]0, \frac{nK}{n-1}\right[\cup \,]\gamma, \zeta[, \tag{3.77}$$

and

$$\tfrac{1}{2}\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) \leq 0 \quad \text{inside } \left]\frac{nK}{n-1}, \gamma\right[\cup \,]\zeta, \infty[. \tag{3.78}$$

In view of the definition (3.59) of w and the calculation

$$\frac{d}{dx}\left[\frac{1}{n}\left(\frac{nK}{n-1}\right)^{-n+1} x^n - (x-K)\right] = \left(\frac{nK}{n-1}\right)^{-n+1} x^{n-1} - 1,$$

we can see that the function $x \mapsto w(x) - (x - K)$ is strictly decreasing in the interval $\left]0, \frac{nK}{n-1}\right[$. Combining this observation with the positivity of w, we can see that (3.77) is true for all $x \in \left]0, \frac{nK}{n-1}\right[$. On the other hand, the inequality (3.77) for $x \in]\gamma, \zeta[$ follows from the observation that the restrictions of the function $x \mapsto w(x) - (x - K)$ in the intervals $]\gamma, z[$ and $]z, \zeta[$ both are convex (thanks to the strict positivity of C_ℓ, D_ℓ, C_r, D_r) and the facts that

$$\lim_{x \downarrow \gamma} \bigl[w(x) - (x - K)\bigr] = \lim_{x \downarrow \gamma} \frac{d}{dx} \bigl[w(x) - (x - K)\bigr] = 0$$

and

$$\lim_{x \uparrow \zeta} \bigl[w(x) - (x - K)\bigr] = \lim_{x \uparrow \zeta} \frac{d}{dx} \bigl[w(x) - (x - K)\bigr] = 0.$$

The inequality (3.78) is equivalent to

$$bx - r(x - K) \leq 0 \text{ for all } x \in \left]\frac{nK}{n-1}, \gamma\right[\cup]\zeta, \infty[,$$

which is true thanks to (3.6) and (3.7).

Finally, the identification of w with the discretionary stopping's value function v and the optimality of $\tau_\star$ follow from Proposition 5. □

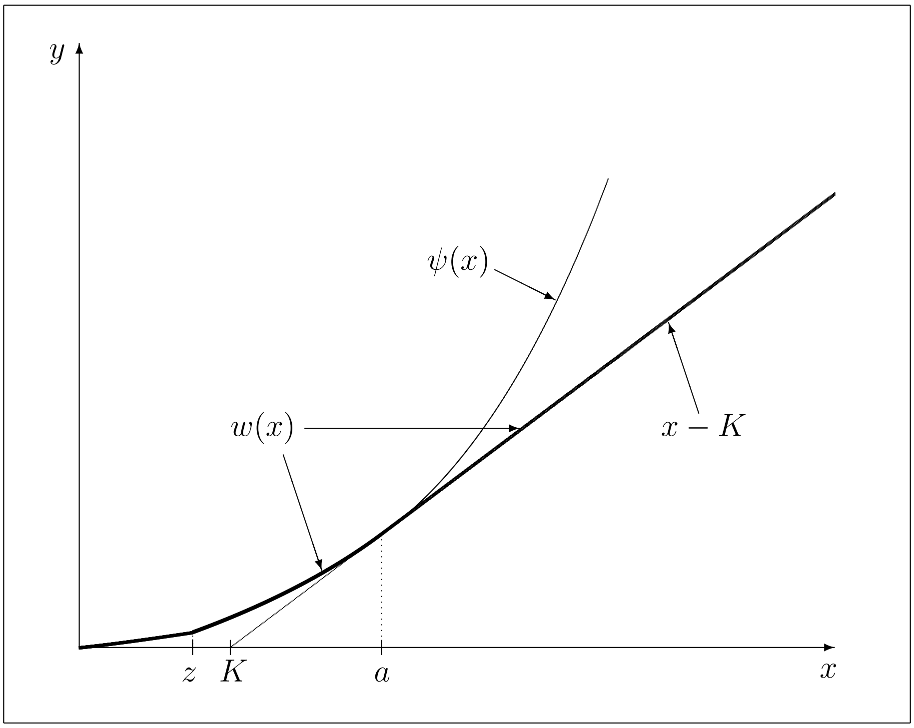

Figure 1

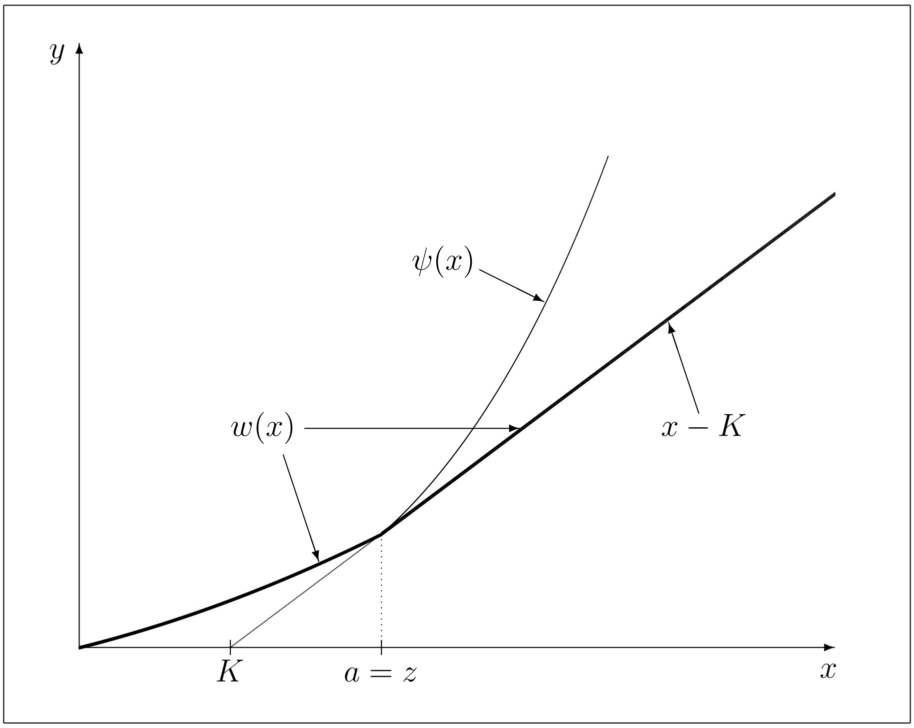

Figure 2

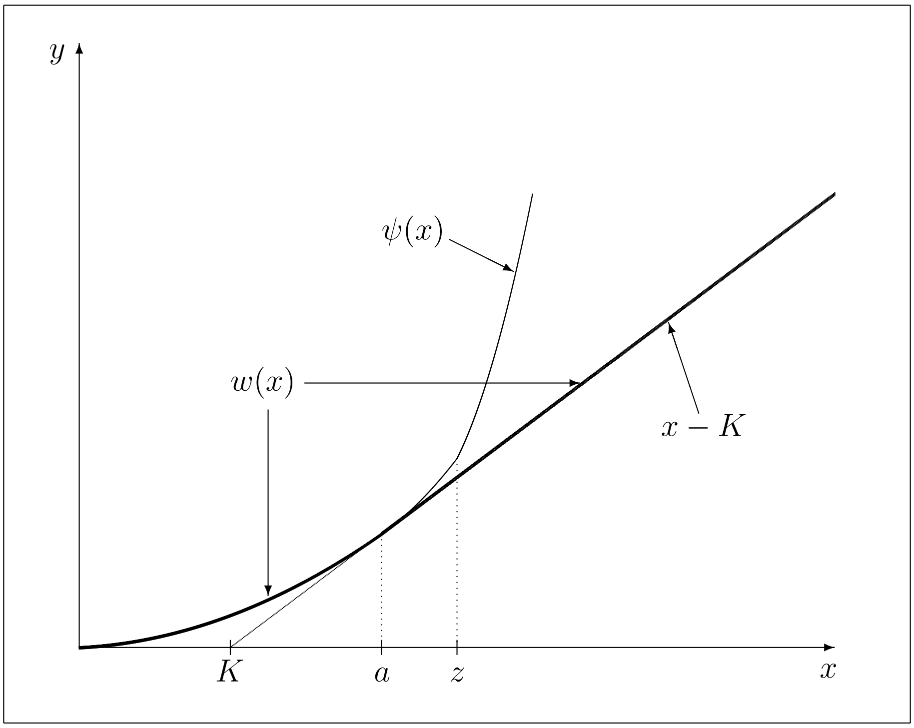

Figure 3

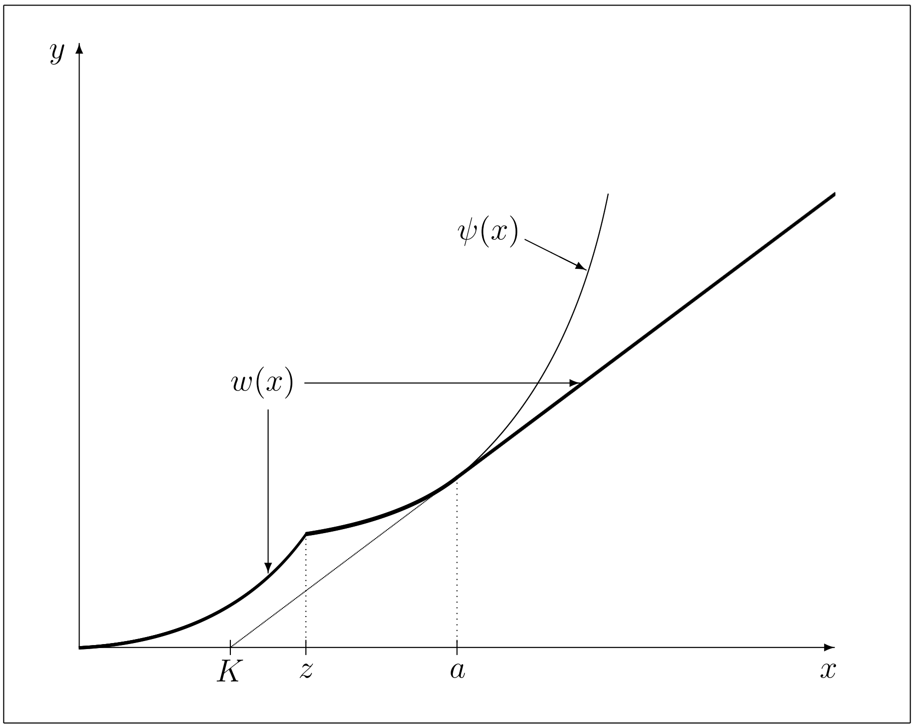

Figure 4

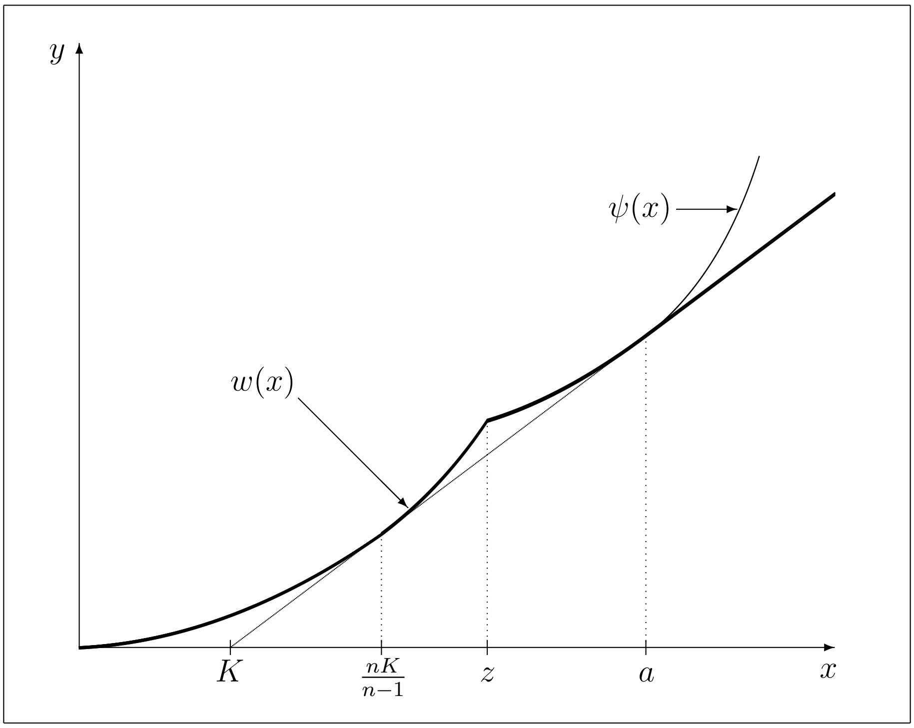

Figure 5

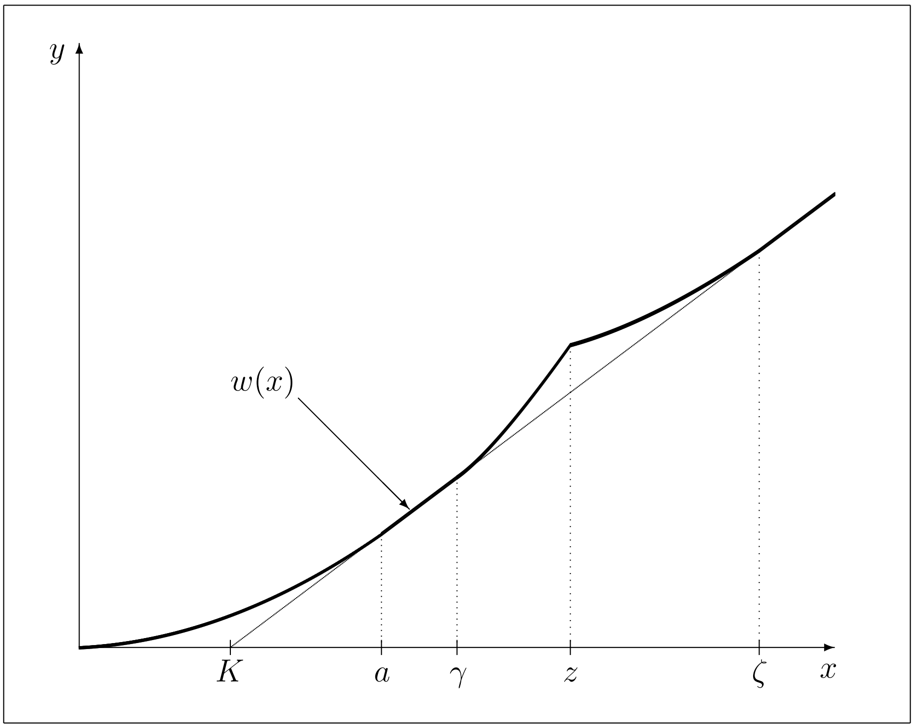

Figure 6

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