The London School of Economics and Political Science

Essays in Financial Economics

Vincent Fardeau

A thesis submitted to the Department of Finance of the London School of Economics for the degree of Doctor of Philosophy, London, December 2011.

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of 54223 words.

Acknowledgments

I am greatly indebted to Dimitri Vayanos and Denis Gromb for their guidance and supervision, for providing inspiration and food for thought on many of the topics presented here, and for dispensing their clarity and intellectual rigor. I also benefited from numerous discussions with permanent or visiting finance faculty at the London School of Economics. In particular, I would like to thank Ulf Axelson, Mike Burkart, Maria-Cecilia Bustamante, Mike Chernov, Amil Dasgupta, Daniel Ferreira, Stéphane Guibaud, Philippe Mueller, Stavros Panageas, Christopher Polk, Rohit Rahi, Andrea Vedolin, Michela Verardo, Paul Woolley and Jean-Pierre Zigrand. I also owe special thanks to David Webb, Kathy Yuan, Yves Nosbusch, Ron Anderson, Amil Dasgupta, Mary Comben and Osmana Raie for their support in various ways throughout the doctoral programme. I was lucky enough to benefit from the hospitality of the department of finance at INSEAD during several short stays. I would like to thank Bernard Dumas, Harald Hau, Pascal Maenhoult, Massimo Massa, Joel Peress, Astrid Schornick for their time and warm welcome. I also benefited from the encouragement of my new colleagues and co-workers at the Federal Reserve Board, where I completed the second chapter of the dissertation, in particular Erik Heitfield and Michael Gordy. I enjoyed the company, discussions, wit and support of many good friends at LSE and elsewhere. To name but a few, let me mention Miguel Anton, Fabian Garavito, Carlos Brando, Alex Castaneda, Claudia Custodio, Olivier Guéant, Vincent Pons, Yuki Sato, Daniel Metzger, Dragana Cvijanovic, Gyuri Venter, Sean Lew, Yu-Min Yen, Giorgia Piacentino, Zhigang Qiu, Nelson Camanho Da Costa, John Kuong, Christian von Drathen, Pierre Chaigneau, Moqi Xu, Christian Mueller, Mark Bohlund, Beatrice Della Torre, Nick Meade, Madda Ghezzi, Sofia Eng, Raul De Novoa, Mayra Vega, Mark Rolland and Tiana Rakoton. My family, in particular my parents and my sister, also deserve the warmest thanks for their continuous and constant support, encouragement and understanding. Several former teachers or co-workers played an inspirational and / or instrumental role in raising my interest for financial economics or academic research in general, including Jean-Pierre Noreck, Jean-Yves Chevallier, Hubert Aupetit, Michel Bouché, Hubert Behles, Roger Guesnerie, Christophe Chamley, Ulrich Hege, Jean-Luc Marion, Eric Maurin, Peter Oppenheimer, Carlos Mejia and Thomas Piketty. Finally, I would like to thank Antoine Faure-Grimaud for giving me the opportunity to study in the programme.

Abstract

In this thesis, I study the effects of market power and financial constraints on arbitrage, liquidity provision, financial stability and welfare. In Chapter 1, I consider a dynamic model of imperfectly competitive arbitrage with time-varying supply. The model can explain the well-documented empirical features that (quasi)-identical assets can trade at significantly different prices; these price differences vanish slowly over time, resulting in apparently slow-moving capital; the price differences can invert over time; market depth is time-varying. I also show that entry does not necessarily correct these effects, although the mere threat of entry may improve liquidity.

In Chapter 2, I introduce in the model the realistic feature that trading requires capital and assume that arbitrageurs' positions must be fully collateralized, which rules out default. I compare liquidity provision, asset prices and welfare in the monopoly case to the perfect competition case studied by Gromb and Vayanos (2002). I show that relative to the competitive case, the monopoly is less efficient but also less capital-intensive, as rents captured over time allow her to build up capital. Consequently, when capital is scarce, financially-constrained competitive arbitrageurs may provide less liquidity at later stages than an unconstrained monopoly. In some cases, this increases aggregate welfare but without being Pareto-improving. I discuss implications for market-making via a specialist.

In Chapter 3, I assume that some arbitrageurs have deeper pockets than others and allow for default. The capital-rich arbitrageurs (predators) either provide liquidity to other market participants (competitive hedgers) or engage in predatory trading against a financially-constrained peer (prey). In this strategy, predators depress the price of the asset to trigger a margin call on the prey's position and gain from her subsequent firesales. I show that the hedgers' reactions to the possibility of predation can make predatory trading cheaper, reducing the prey's staying power. In anticipation of the prey's firesales, hedgers may run on the asset, strengthening and to some extent substituting to the predators' price pressure. Further, their reaction leads to a reduction in the prey's price impact, which decreases her already limited ability to support the price and avoid a margin call. Predatory trading is likely to occur when hedgers are sufficiently risk-averse or the asset sufficiently risky.

Contents

Chapter 1

Dynamic Strategic Arbitrage

Abstract: Real-world arbitrage often involves a limited number of large financial intermediaries (e.g. dealers, hedge funds) with price impact. I study a multi-period model of imperfectly competitive arbitrage, in which supply shocks generate price differences between two identical assets traded in segmented markets. Arbitrageurs seeking to exploit these price differences split up their orders to limit their price impact. I show that order split-up and specific supply shock patterns can explain the empirical evidence that i) identical assets can trade at different prices, ii) these price differences revert slowly over time, as if capital was slow-moving, and iii) the sign of price differences can switch over time. The model also yields new predictions about the determinants and evolution of market depth, about which arbitrage strategies attract arbitrageurs in equilibrium, and how the number of arbitrageurs in a given strategy evolves over time.

1.1 Introduction

In contrast to the textbook case, in which many atomistic investors ensure that the law of one price holds, arbitrage opportunities and mispricings are often chased in the real world by only a limited number of large, highly specialized financial institutions (e.g. broker-dealers, hedge funds, proprietary trading desks). These *arbitrageurs* manage sizable portfolios and recognize that their trades affect asset prices, implying that arbitrage should be studied as a strategic choice.[1] In this chapter, I study how the behaviour of a limited number of strategic arbitrageurs affects the dynamics of asset prices and market liquidity. I also determine which markets attract arbitrageurs and how the level of competition between arbitrageurs changes over time.

[1]For instance, Quantum Fund and Tiger Management closed funds in 2000 because their sizes impaired their ability to take advantage of pricing anomalies (Attari and Mello, 2006). Chen, Stanzl and Watanabe (2002) find evidence of substantial price impact costs in equity markets.

Arbitrageurs who recognize their price impact avoid trading too aggressively against mispricings and split up their orders to preserve the profitability of their strategy. Based on this simple insight and variation in the supply of assets that can be arbitraged, I show that imperfectly competitive arbitrage can explain three well-documented empirical phenomena: i) assets with identical cash-flows can trade at different prices, ii) this price difference vanishes only slowly over time, resulting in apparently slow-moving capital[2]; iii) price differences between identical assets may switch sign over time (*sign inversion*). Sign inversions between pairs of similar assets have been documented during the recent crisis in the interest rate swap spread and the municipal bond market (Bergstresser, Cohen and Shenai, 2011).

In addition to these price dynamics, I determine how the *endogenous* number of arbitrageurs depends on entry costs[3], the risk-return profile of the arbitrage opportunity, or the existing market structure. This allows me to characterize which markets or arbitrage opportunities are likely to be - or *remain* - concentrated, and the implied effects on asset prices and liquidity. First, the model predicts a non-monotonic relationship between the volatility of fundamentals and the number of arbitrageurs, confirming a conjecture made by Shleifer and Vishny (1997): more volatile markets do not necessarily attract more arbitrageurs, in particular if the risk-bearing capacity of other market participants is low. Second, the model generates new predictions about the evolution of competition and its implications for liquidity and the speed of arbitrage. When new arbitrageurs can enter a strategy over time, incumbents attempt to deter them. They are likely to succeed when entry costs are sufficiently high, so that a concentrated structure may persist over time even though concentration means that large rents are available. With lower entry costs, however, the mere *threat* of entry can bring prices closer to fundamental values. With even lower entry costs, the model predicts that liquidity, defined as market depth, improves *ahead* (more precisely, in anticipation) of future entry. Hence a novel prediction of the model is that market depth should be a leading indicator of the number of arbitrageurs active in a given strategy.

I consider a setting with two identical assets (say A and B) traded in segmented markets and a risk-free asset. Risk-averse *local investors* operate in the segmented markets and receive

[2] Price differences between (quasi) identical assets are well-documented for Siamese stocks (Froot and Dabora, 1999, Lamont and Thaler, 2003), on-the-run / off-the-run bonds (Krishnamurthy, 2002). The swap spread and the CDS-bond basis are two other prominent examples. The second phenomenon - slow-moving capital - has been documented at different frequencies and in various markets. See, for instance, Duffie (2010), Mitchell, Pedersen and Pulvino (2007) and Coval and Stafford (2007).

[3] One can imagine that trading across different exchanges, identifying mispricings and / or setting up arbitrage desks require some fixed investment in the first place.

endowment (liquidity) shocks over time. The shocks are correlated with the fundamental of the assets, and thus affect local investors' valuation for their local assets. Since local investors in market A receive opposite shocks to local investors in market B, trading would help investors to fully share risk, but cannot take place because of market segmentation. This pushes the prices of assets A and B apart. Another group of risk-averse investors, the *arbitrageurs*, have the ability to trade freely across markets and can exploit this price difference by intermediating trades between A- and B-investors. There is however only a finite number of arbitrageurs, who understand how their trades affect prices in each market. In particular, arbitrageurs understand that by fully intermediating the trades across markets, they would bring prices in line with fundamentals, and earn zero profits. As a result, they limit the quantities they buy from local investors with low valuation for the asset and sell to investors with high valuation, keeping the spread between the prices of assets A and B open, and earning profits. Of course, as competition among arbitrageurs intensifies, the pressure to intermediate trades increases and the spread decreases. Hence the model predicts that the magnitude of mispricings should increase in the concentration of arbitrageurs. Ruf (2011) finds evidence consistent with this prediction in the commodities options market.

As arbitrageurs' trades bring prices closer to each other in a permanent manner, arbitrageurs are willing to split up orders to limit their price impact over time and exploit mispricings as long as possible. Time-variation in the endowment shocks received by local investors cause changes in the supply of assets that arbitrageurs can arbitrage. Changes in the supply may be known in advance or be uncertain. The first contribution of the paper is to provide an interesting laboratory to understand the effects of known (and potentially time-varying) and risky shocks on the dynamics of asset prices.[4] When shocks are constant over time, corresponding to a constant arbitrage supply, arbitrageurs increase gradually their positions to limit their price impact, which leads to gradual convergence of prices towards fundamentals. Hence the arbitrageurs' strategic considerations can account for the observed slow movement of capital towards buying opportunities documented after supply shocks (Oehmke, 2010, Duffie, 2010). Again, an increase in competition reduces this effect, and increases the speed of convergence of the arbitrage.

When shocks are known to decrease over time, arbitrageurs' activity leads to sign inver-

[4]In papers analyzing price impact in other Cournot-based models (e.g. DeMarzo and Urosevic, 2007, Pritsker, 2009), the supply of the asset is constant. Introducing time-varying shocks is equivalent to consider a time-varying risky asset supply in these models. The model also allows for uncertainty about future shocks / asset supply. See Section 1.3.2 for an analysis of arbitrageurs' risk-management strategies in the presence of uncertain shocks.

sion. Suppose for instance that a large positive shock in market A is followed by a small positive shock (and vice versa in market B).[5] In this case, the equilibrium spread is first positive and then negative, even if the sign of shocks is unchanged, implying that the *same* asset should be more expensive throughout. A-investors, who receive a positive supply shocks, first sell the asset to hedge their risk. As shocks decrease, they find that they have oversold the asset, and seek to buy back. Arbitrageurs limit liquidity, pushing the price of asset A above its expected value. It is then optimal for A-investors to remain excessively short the asset, as the price of asset is expected to revert to its expected value (on average) when the asset pays off.

The prediction of sign inversion may shed light on recent anecdotal and empirical evidence. In 2010, the swap spread turned negative for the first time. Bergstresser, Cohen and Shenai (2011) find that insured municipal bonds became cheaper than similar uninsured bonds issued by the same city. The present model highlights the role of the market structure of dealers / arbitrageurs to explain sign inversion in these markets. Further, one can interpret the decrease in shocks as an easing of supply imbalances or liquidity needs in the market, which seems to correspond to a post-crisis situation.

If the magnitude of future shocks is uncertain, arbitrageurs face a risky trading opportunity. Uncertainty about future shocks increases local investors' willingness to hedge, and consequently the profitability of the arbitrage. However, arbitrageurs do not necessarily react to an increase in uncertainty by an increase in their positions, *even if they tend towards risk neutrality*. In fact, as uncertainty also raises local investors' reluctance to hold their assets, arbitrageurs' price impact increases, and this may prompt them to *decrease* their positions. Further I show that even if arbitrageurs increase their positions, their reaction does not offset local investors' increased liquidity demand, so that in equilibrium, heightened uncertainty about future shocks always leads to less efficient pricing. These predictions are novel relative to the limits of arbitrage and noise trader risk literatures, which focused on the level rather than the uncertainty of supply imbalances / noise trader risk (see e.g. Shleifer and Vishny, 1997, Brunnermeier and Pedersen, 2008).

These predictions obtain with a fixed market structure. However rents available from strategic arbitrage should attract new players over time. Indeed, successful trading strate-

[5] For the sake of exposition, I assume that shocks are known in advance, i.e. the arbitrage is risk-free. I show that similar effects obtain when the arbitrage is risky, i.e. when future shocks are random. In this case, there is an additional effect since arbitrageurs must also adjust their strategies when they learn about the size of the shock.

gies or financial innovations tend to attract copycats and imitators. For instance, LTCM's relative-value and convergence strategies became increasingly popular as the hedge fund produced double-digit returns in the 1990s.[6] What determines the number of arbitrageurs active in a given strategy? How does the market structure evolve over time?

The second contribution of the paper is to formally address these questions by i) analyzing a simultaneous entry game between *risk-averse* arbitrageurs - the previous literature has considered risk-neutral arbitrageurs; and ii) analyzing a strategic deterrence game between incumbent arbitrageurs and new entrants.

My results about simultaneous entry show that allowing for entry does not necessarily correct the strategic rationing of liquidity (measured as the spread) caused by imperfect competition among arbitrageurs. The risk-averse arbitrageurs must decide ex-ante whether to enter (and sink a fixed cost) under uncertainty about the arbitrage profitability. An increase in average shocks makes the arbitrage more profitable, and increases entry. But an increase in the volatility of shocks does not necessarily increase entry, in particular if local investors are sufficiently risk-averse. Indeed, on one hand, a higher volatility makes the arbitrage more risky, which hurts the risk-averse arbitrageurs, and on the other hand, it increases local investors' willingness to hedge their exposure to liquidity shocks (indirect effect), and thus the arbitrage profitability. The first effect dominates and reduces entry if the risk-bearing capacity is small (highly risk-averse investors), and / or the market is concentrated, and / or if volatility is small relative to the average shock (most likely, a large shock will hit the market). Hence entry depends on the "market structure of risk-bearing capacity" (Pritsker, 2009). These results imply that, all else equal, the number of arbitrageurs may first decrease or be stable and then increase along the northeastern direction of the mean-variance frontier of arbitrage opportunities (Figure 1.1).

Once arbitrageurs are in place, their ability to move prices can help them limit future entry, in particular if entry costs are high for new arbitrageurs, and even if the arbitrage is risk-free. Deterring new entrants requires to decrease the profitability of the arbitrage, i.e. by reducing the spread between A- and B-asset prices more quickly. This contradicts arbitrageurs' objective to decrease the spread only gradually. When entry costs are sufficiently large for the entrant, the cost of deterrence is low for incumbents. As entry costs

[6]Similarly, there is anecdotal evidence that Leland and Rubinstein's portfolio insurance strategy became widely imitated in the 1980s: "The LOR principals could do little to prevent rivals from producing similar portfolio insurance products. (...) the competition accelerated and firms like Morgan Stanley, Bankers Trust, Chase Investors Management and Kidder Peabody entered the business." (Kyrillos and Tufano, 1994)

decrease, arbitrageurs must tackle the mispricing more aggressively ex-ante, which increases liquidity and the speed of arbitrage. In this sense, the arbitrage is contestable and the mere threat of entry improves market efficiency. As entry costs decrease further, this deterrence strategy becomes very costly as incumbent arbitrageurs must bear the cost of their adverse price impact. As a result, they engage in less aggressive preemptive buying and let the new arbitrageur enter.

When entry occurs in equilibrium, market liquidity improves along two dimensions. First, arbitrageurs keep trading more aggressively than without entry threat, which decreases the spread. This effect is consistent with evidence presented by Tufano (1989), who shows that intermediaries launching new financial products charge nearly competitive prices even during the early stages where they enjoy a monopolistic position in the product. Second, local investors' current demand becomes more elastic as they rationally anticipate a more favourable market structure in the future. This leads to an increase in market depth and shows that with low entry costs, an improvement in market depth should be a leading indicator of an increase in the number of arbitrageurs. More generally, this result highlights a key feature of the model: market depth is endogenously determined by the current and anticipated market structure, as arbitrageurs' market power and its evolution determine the risk-sharing opportunities available to local investors, and thus the prices at which they are willing to absorb arbitrageurs' trades.

This paper introduces time-varying and uncertain shocks and an endogenous market structure in models where large investors competing à la Cournot trade with a competitive fringe of investors (e.g. DeMarzo and Urosevic, 2007, Pritsker 2009).[7] The time-variation in shocks generates time-variation in the arbitrage profitability and encompasses "gradual arbitrage", as in Oehmke (2010), with the difference that the market depth is endogenous. Time variation in shocks also generates a novel sign inversion effect, in which arbitrageurs prevent prices from converging. Arbitrageurs' destabilizing behaviour arises as an endogenous response to the deterioration of the arbitrage profitability. This is in contrast to the predatory trading literature, where arbitrageurs can be destabilizing as a response to the need of other traders to reduce their positions, or to induce them to do so (Brunnermeier and Pedersen, 2005, Attari, Mello and Ruckes, 2006, Fardeau, 2011a).

The analysis of arbitrageurs' entry in the literature has been either informal (e.g. Shleifer

[7] There is another class of dynamic models of imperfectly competitive trading without competitive fringe: see Vayanos (2001) and Rostek and Weretka(2011). I draw some comparisons between the two types of models in Section 1.3.

and Vishny 1997, Kondor 2009) or based on risk-neutral arbitrageurs (Oehmke, 2010, Zigrand, 2004 and 2006). Allowing for risk-averse arbitrageurs generates new effects such as the non-monotonic relationship between entry and volatility conjectured by Shleifer and Vishny. More generally, the model shows that entry decisions depend on the interaction between the market structure and the risk-bearing capacity of all market participants. Pritsker (2009) highlights the role of the "market structure of risk-bearing capacity" in a related paper about large investors, but does not consider entry.

Sequential entry of arbitrageurs has - to the best of my knowledge - not been studied in an asset pricing context. While the literature has traditionally focused on information asymmetry and traders' risk-aversion (or inventory effects) as determinants of market depth, the model highlights market structure and its potential evolution as a new determinant of market depth. Sequential entry and contestability are the subjects of classic papers in Industrial Organization (e.g. Fudenberg-Tirole, 1987, Baumol, 1982).[8] In a financial market, it is interesting to see that the anticipations of consumers (here local investors) of the product (liquidity) play an important role and make the equilibria self-fulfilling. For instance, the mere anticipation of entry improves market depth, which makes it harder for arbitrageurs to deter new traders from coming in. In other words, while classic IO papers typically assume that there exist a representative consumer with a continuum of asset valuation, here the elasticity of the liquidity demand (price impact) is endogenous. It affects and is affected by the firms' (i.e. arbitrageurs) strategic entry decisions.

Some predictions of the model are observationally equivalent to predictions delivered by limits of arbitrage models.[9] In particular, both types of models predict that assets with identical cash-flows and risks can trade at different prices and that the spread between these assets should decrease over time. The drivers of these effects are imperfect competition on one hand, and capital constraints on the other hand, therefore it should be empirically possible to disentangle these theories. Ruf (2011) shows that both effects matter to explain the skewness risk premium in options market.

Imperfect competition among financial intermediaries (market-makers) is the subject of an extensive literature in market microstructure (e.g. Dennert, 1993, Biais, Martimort and

[8]Note that simultaneous entry is also studied in the literature on non-competitive foundations of general equilibrium. See Zigrand (2004) and references therein.

[9]Some examples of this extensive literature include Shleifer and Vishny (1997), Xiong (2001) , Kyle and Xiong (2001), Gromb and Vayanos (2002, 2009, 2010) and Brunnermeier and Pedersen (2009). A related strand of literature attempts to explain the well-documented slow movement in capital (Duffie, 2010, and Duffie and Strulovici, 2010)

Rochet, 2000). The key difference with these papers is that I assume that all information is public, which allows me to isolate the effect of market power on liquidity. Instead, Dennert considers price competition in a framework were market-makers face adverse selection. Further, in his model, market-makers post quotes first, while here arbitrageurs compete in quantities taking local investors' schedules as given. In Biais et al., market-makers supply liquidity by posting limit order schedules, whereas in my set-up arbitrageurs submit market orders (Cournot competition).

I proceed as follows. In section 1.2, I describe the model. I solve for the equilibrium with a given market structure in Section 1.3. In Section 1.4, I endogenize the number of arbitrageurs. Section 1.5 concludes. All proofs and figures are in the appendix.

1.2 A model of imperfectly competitive arbitrage

The model features two markets for identical assets (A and B). Some investors (arbitrageurs) can trade freely across both markets, while others (local investors) are constrained to trade in only one market, a building block similar to Gromb and Vayanos (2002). A key difference with this paper is that arbitrageurs are imperfectly competitive and are not financially constrained.

1.2.1 Set-up

Assets and timeline. The economy has three periods 0, 1 and 2, and consists of two identical risky assets (A and B) in zero net supply. The risky assets pay off a liquidating dividend at time 2, $D_2 = D + \epsilon_1 + \epsilon_2$, where ϵ_t are iid normal variables with mean 0 and variance σ^2. The fundamental shocks ϵ_1 and ϵ_2 are realized at time 1 and time 2, respectively, and are publicly observed. I denote $D_t = \mathbb{E}_t\left(D_2\right)$, the conditional expected value of the dividend. There is also a risk-free asset in perfectly elastic supply with return r normalized to 0. Trading takes places at time 0 and time 1 and consumption at time 2.

Agents and preferences. The economy is made of two types of traders. First, there are local investors in markets A and B, each represented by a competitive agent with CARA utility and absolute risk-aversion coefficient a: $u\left(C_2^k\right) = -\exp\left(-aC_2^k\right)$, with $k = A, B$. Second, there is a finite number n of arbitrageurs indexed by i ($i = 1, 2, \ldots, n < \infty$)). In the basic version of the model, n is fixed. Later on, I endogenize the number of arbitrageurs and allow for n to change endogenously over time through the entry of new arbitrageurs at time 1. Arbitrageurs have CARA utility with absolute risk-aversion coefficient b: $U\left(C_2^i\right) = -\exp\left(-bC_2^i\right)$, with $i = 1, ..., n$. Arbitrageurs have no endowment in the risky assets. Importantly, local investors are restricted to trade their local risky asset, while arbitrageurs can trade all risky assets. In other words, markets A and B for the risky asset are completely segmented. All investors have access to the risk-free asset.

Liquidity / supply shocks. The local investors in market k receive endowment shocks $s_0^k \epsilon_1$ at time 1 and $s_1^k \epsilon_2$ at time 2. I assume that shocks are opposite across markets: $s_t^A = -s_t^B = s_t$, for $t = 0, 1$. The shocks are correlated to the payoff of the risky asset, and are opposite across markets. Since risky assets are identical and A- and B markets, local investors could achieve perfect risk sharing by trading with each other in the risky asset. However market segmentation prevents direct trading between local investors and creates a trading opportunity for arbitrageurs, who can intermediate trades by buying from investors with low valuation (in market A) and selling to investors with high valuation (in market B). Doing so, arbitrageurs will contribute to integrate markets A and B and provide liquidity to local investors. Thus the endowment shocks create a demand for liquidity and constitute the "supply" of assets available for arbitrage from the point of view of arbitrageurs.

I consider two situations. In the first situation, all traders know in advance the values of s_0 and s_1, i.e. they know the magnitude of the supply. In this case, the trading opportunity corresponds to a textbook arbitrage since it involves no risk. Allowing for s_0 to be different from s_1 helps me understand how changes in supply affect the dynamics of arbitrage. Many results can be derived in this simple *risk-free arbitrage* case, in particular gradual arbitrage and sign inversion. The second situation is closer to a real-life arbitrage opportunity, because it entails some risk. I assume that the magnitude of the second shock, s_1, is random from the point of view of time 0 for all traders: s_1 is normally distributed with mean $\bar{s}_1$ and variance z_1^2, and is independent of ϵ_t. This *risky arbitrage* case allows me to investigate how uncertainty affects the dynamics of strategic arbitrage and arbitrageurs' risk-management strategies.

Discussion. The framework has two different interpretations. In the first one, arbitrageurs stand for market-makers providing immediacy for traders with opposite trading motives who arrive in the market at different times, as in Grossman-Miller (1988). In this interpretation, assets A and B stand for the same asset in two different subperiods of each time and arbitrageurs smooth out the temporary order imbalances by holding the asset in between subperiods. In the second interpretation, arbitrageurs can be thought of as large hedge funds or prop trading desks chasing mispricings between identical or quasi-identical assets, such as on-the-run and off-the-run Treasuries, Siamese stocks (e.g. Royal Dutch and Shell), etc. Long-Term Capital Management (LTCM) is a standard example of this kind of traders.[10] The shocks affecting local investors can stem from institutional or regulatory frictions. For instance, index trackers and mutual fund managers must rebalance their portfolios following index additions or deletions because of benchmarking constraints. Negative shocks in one asset can force portfolio managers of open-ended funds to sell other assets to meet redemptions, etc.[11]

1.2.2 Maximization problems

Local investors. At time 2, local investors consume their entire wealth:

$$C_2^k = W_2^k = Y_1^k D_2 + E_1^k, \text{ k=A,B}$$

In this equation, Y_1^k and E_1^k represent investors k's end-of-period positions in the risky and risk-free assets at time 1. That is, local investors in market k enter period 2 at which they only consume with a position Y_1^k in the risky asset and E_1^k in the risk-free asset.[12] I denote y_t^k the time t trade in risky asset k and p_t^k its price. The law of motion of positions is: $Y_t^k = Y_{t-1}^k + y_t^k$ for the risky asset and $E_t^k = E_{t-1}^k - y_t^k + s_t^k \epsilon_{t+1}$, for the risk-free asset, for $k = A, B$.[13] The local investors' dynamic budget constraint follows:

$$W_{t+1}^k = W_t^k + Y_t^k \left(p_{t+1}^k - p_t^k\right) + s_t^k \epsilon_{t+1}, \text{ k=A,B} \tag{1.1}$$

This equation shows that local investors' wealth changes either because of capital gains, $Y_t^k \left(p_{t+1}^k - p_t^k\right)$, or because of shocks, $s_t^k \epsilon_{t+1}$. The local investors maximize the expected

[10] LTCM also provide a good illustration of the issue of size and market impact. As Pérold (1999) puts it: "The firm had also experienced many instances in which prices moved adversely while LTCM was attempting to exit a position after it had converged, suggesting that the firm's trades were having a larger market impact" (than previously).

[11] Gromb and Vayanos (2010) provide more details and other examples.

[12] Observe also that at time 2, each market is perfectly liquid, so that $p_2^k = D_2$.

[13] Since local investors have CARA preferences, we can set their initial endowment $E_{-1}^k = 0$ without loss of generality.

utility of consumption subject to the dynamic budget constraint:

$$\begin{aligned} \text{for } k = A, B, \quad \max_{\left(Y_t^k\right)_{t=0,1}} \quad & \mathbb{E} \quad \left[u\left(C_2^k\right)\right] \\ s.t.\ W_{t+1}^k \quad &= \quad W_t^k + Y_t\left(p_{t+1}^k - p_t^k\right) + s_t^k \epsilon_{t+1} \end{aligned} \tag{1.2}$$

Arbitrageurs. Arbitrageurs face a different budget constraint because they can trade in both markets. Their final wealth W_2^i is equal to:

$$W_2^i = B_1^i + \sum_{k=A,B} X_1^{i,k} D_2,\ i = 1, \ldots, n \tag{1.3}$$

Note that $X_t^{i,k}$ and $x_t^{i,k}$ represent the arbitrageur i's position and trades at time t in asset k, and are related as follows: $X_t^{i,k} = X_{t-1}^{i,k} + x_t^{i,k}$. The position in the risk-free asset evolves as: $B_t^i = B_{t-1}^i - \sum_{k=A,B} x_t^{i,k} p_t^k$. Therefore the dynamic budget constraint is:

$$W_{t+1}^i = W_t^i + \sum_{k=A,B} X_t^{i,k}\left(p_{t+1}^k - p_t^k\right),\ i = 1, \ldots, n$$

As in Gromb and Vayanos (2002), I will focus on equilibria in which arbitrageurs take opposite positions in each market: for $t = 0, 1$, $x_t^{i,A} = -x_t^{i,B} = x_t^i$. Given that assets A and B are both in zero net supply, this implies that arbitrageurs do not bear any aggregate risk.[14] With opposite positions in markets A and B, the dynamic budget constraint becomes:

$$W_{t+1}^i = W_t^i + X_t\left(p_t^B - p_t^A - \left(p_{t+1}^B - p_{t+1}^A\right)\right) = W_t^i + X_t\left(\Delta_t - \Delta_{t+1}\right),\ i = 1, \ldots, n \tag{1.4}$$

The arbitrageur's dynamic budget constraint shows that their wealth changes via capital gains in the arbitrage. The arbitrageurs' problem is to choose trades x_t^i, $t = 0, 1$, to maximize their expected utility of consumption subject to (1.4) and the price schedules for assets A and B. The price schedules are derived from local investors' inverted demand schedules, and imposing market-clearing:

$$Y_t^k + \sum_{i=1}^{n} X_t^{i,k} = 0,\ k = A, B,\ t = 0, 1 \tag{1.5}$$

[14] In the more general case where the supply is different from zero, an additional risk-sharing motive would emerge along the results presented in the paper.

The price schedules map the effect of arbitrageurs' trades into the price in each market. That is, a price schedule represent the market-clearing price at which the competitive fringe of local investors in each market is ready to trade all possible quantities submitted by arbitrageurs. Hence arbitrageurs will internalize their price impact in each market when choosing their positions in the risky asset. Of course, the specific form of the local investors' demand schedules also depends on the liquidity / supply shocks, and in particular, on whether future shocks are known in advance or are random. In the next section, I derive the equilibrium in the risk-free and risky arbitrage cases.

1.3 Equilibrium with risk-free and risky arbitrage

In this section, I solve for local investors' and arbitrageurs' equilibrium strategies, taking the number of arbitrageurs as given. When the arbitrage is risk-free, the price dynamics depend crucially on whether the supply shocks are constant or not. The risky arbitrage case allows me to analyze in details arbitrageurs' risk-management strategies.

1.3.1 Risk-free arbitrage

Price schedules. Here I assume that s_0 and s_1 are positive shocks and are known in advance by all market participants. As a first step, it is useful to look at the price schedules faced by arbitrageurs at time 1. In our standard CARA-normal framework, local investors' demand in market A is:

$$Y_1^A = \frac{\mathbb{E}\left(D_2\right) - p_1^A}{a\sigma^2} - s_1 \tag{1.6}$$

Local investors in market A experience a positive shock s_1, which reduces their demand for asset A. In market B, local investors have similar demand functions (in p_1^B), except that they experience an opposite shock, increasing their demand for asset B. Using the assumption of opposite positions in markets A and B, and imposing market-clearing (1.5), these demand functions generate the following price schedules $p_1^k\left(Q_1\right)$, where I use as a shorthand $Q_1 = \sum_{i=1}^{n} X_1^i$:

$$\begin{aligned} p_1^A(Q_1) &= \mathbb{E}_1(D_2) - a\sigma^2\left[s_1 - \sum_{i=1}^{n} X_1^i\right] = \mathbb{E}_1(D_2) - a\sigma^2\left[s_1 - \sum_{i=1}^{n} x_0^i - \sum_{i=1}^{n} x_1^i\right] \\ p_1^B(Q_1) &= \mathbb{E}_1(D_2) + a\sigma^2\left[s_1 - \sum_{i=1}^{n} X_1^i\right] = \mathbb{E}_1(D_2) + a\sigma^2\left[s_1 - \sum_{i=1}^{n} x_0^i - \sum_{i=1}^{n} x_1^i\right] \end{aligned}$$

Note that we used the assumption that arbitrageurs have no preexisting position in any of the risky assets, i.e. $x_0^i = X_0^i$. Combining the two schedules, we get the following schedule for the arbitrage spread, $\Delta_1(Q_1) = p_1^B(Q_1) - p_1^A(Q_1)$:

$$\Delta_1(.) = 2a\sigma^2\left[s_1 - \sum_{i=1}^{n} x_0^i - \sum_{i=1}^{n} x_1^i\right] \tag{1.7}$$

The schedule has an intuitive form. The first component, $2a\sigma^2 s_1$ is the price wedge that would prevail between assets A and B in the absence of trading. That is, A-investors, who experience a positive liquidity (supply) shock, would have to hold all the additional supply and would thus value the asset at a discount $a\sigma^2 s_1$ relative to its expected payoff $E_1(D_2)$. B-investors would value the risky asset at exactly the opposite premium, as they experience a negative shock of similar magnitude. Hence in total the price wedge would be $2a\sigma^2 s_1$, increasing with the risk of the asset, σ^2, the risk-aversion of local investors a, and the size of the liquidity shock s_1. The second component of (1.7) represents the impact of arbitrageurs' trades. Arbitrageurs can bring prices of assets A and B closer by setting up a long position in the spread (corresponding to a long position in asset A minus a short position in asset B). Arbitrageurs' price impact, $|\frac{\partial \Delta_1}{\partial x_1^i}| = 2a\sigma^2$, depends on local investors' risk-aversion and the risk of the fundamental.[15] When they are more risk-averse, local investors are more reluctant to hold the risky asset, and thus will require larger price concessions when trading, resulting in a larger price impact.

Equilibrium strategies and spreads. To illustrate the strategic choice faced by arbitrageurs, note that because arbitrageurs set up opposite positions, their objective at time 1 boils down to maximizing the trading profit, $x_1^i \Delta_1(.)$, where $\Delta_1(.)$ is given by (1.7) and depends not only on arbitrageur i's trade, x_1^i, but also all other arbitrageurs' trades $\sum_{-i} x_1^{-i}$, with $\sum_{-i} x_1^{-i} + x_1^i = \sum_{i=1}^{n} x_1^i$, and on the positions established at time 0, $\sum_i x_0^i$. Hence at time 0, arbitrageurs takes into account the *dynamic* impact of their own trades as well as of

[15] I take the absolute value of the derivative as it is more intuitive to compare positive numbers.

other arbitrageurs' on the spread. In the appendix, I work backwards to derive arbitrageurs' optimal trading strategies and obtain the following result:

Proposition 1 *In the risk-free case, there is a unique (symmetric) equilibrium in which arbitrageurs' trades in market A are:*

$$x_0^i = x_0 = \frac{1}{\phi_n} s_0 + \frac{n-1}{(n+1)^2 \phi_n} s_1 \tag{1.8}$$

$$x_1^i = x_1 = -\frac{n}{(n+1)\phi_n} s_0 + \bar{\phi}_n s_1, \tag{1.9}$$

The equilibrium spread is:

$$\Delta_0 = 2a\sigma^2 \left[\psi_n s_0 + \bar{\psi}_n s_1\right] \tag{1.10}$$

$$\Delta_1 = 2a\sigma^2 \left[-\frac{n}{(n+1)\phi_n} s_0 + \bar{\phi}_n s_1\right] \tag{1.11}$$

$$\textit{with } \phi_n = \frac{n^3+4n^2+3n+2}{(n+1)^2}; \ \bar{\phi}_n = \frac{1}{n+1} - \frac{n(n-1)}{(n+1)^3 \phi_n}$$

$$\psi_n = \frac{n^2+n+2}{n^3+4n^2+3n+2}; \ \bar{\psi}_n = \frac{3n^2+5n+2}{n^3+4n^2+3n+2}$$

Gradual arbitrage with constant liquidity shocks ($s_0 = s_1 = s$)

To gain intuition into the equilibrium, it is useful to consider the special case $s_0 = s_1 = s$, with $s > 0$, to fix ideas. Then arbitrageurs' trades are $x_0^i = \kappa_{0,n} s$ and $x_1^i = \kappa_{1,n} s$, with for all $n \geq 1$, $\kappa_{0,n} \equiv \frac{1}{\phi_n}\left(1 + \frac{n-1}{(n+1)^2}\right) \in]0,1[$ and $\kappa_{1,n} \equiv -\frac{n}{(n+1)\phi_n} + \bar{\phi}_n \in]0,1[$. Further, the total purchases are:

$$\sum_{i=1}^{n} x_0^i = n\kappa_{0,n} s < s \text{ and } \sum_{i=1}^{n} x_1^i = n\kappa_{1,n} s < s$$

Hence arbitrageurs never fully absorb the asset supply caused by the liquidity shock in each market. As a result, the spreads between A- and B-asset prices remain strictly positive in equilibrium:

$$\Delta_0 = 2a\sigma^2 \bar{\kappa}_{0,n} s > 0, \ \Delta_1 = 2a\sigma^2 \bar{\kappa}_{1,n} s > 0$$

Why does competition not eliminate the mispricing as soon as $n > 1$? It can be seen from the time-1 objective:

$$\max_{x_1^i} x_1^i \Delta_1(.) = \max_{x_1^i} 2a\sigma^2 x_1^i \left(s - \sum_{i=1}^{n} x_0^i - \sum_{-i} x_1^{-i} - x_1^i \right) \tag{1.12}$$

For a given liquidity shock s, given other arbitrageurs' trades $\sum_{-i} x_1^{-i}$, and initial positions $\sum_i x_0^i$, arbitrageur i has no interest to buy the entire residual supply, $s - \sum_{i=1}^n x_0^i - \sum_{-i} x_1^{-i}$, for he would then close the spread and make a zero profit on his trade. Instead, his best response, from the first-order condition of problem (1.12), is to trade half the residual supply: $x_1^i = \frac{s - \sum_{i=1}^n x_0^i - \sum_{-i} x_1^{-i}}{2}$. Since each arbitrageur has the same impact on the price, all arbitrageurs play a symmetric role, and in the unique (subgame) equilibrium, all arbitrageurs trade the same quantity, $x_1^i = \frac{s - \sum_{i=1}^n x_0^i}{n+1}$.

This quantity is negatively related to the arbitrageurs' first period trades, $\sum_{i=1}^n x_0^i$. Indeed, to keep the spread open, arbitrageurs need to limit their price impact, which is permanent, as shown by equation (1.7). The reason why price impact is permanent is that for local investors in, say market A, who have a low valuation for the asset, selling the asset to arbitrageurs helps insure against the first liquidity shock, but also, to some extent, against the second liquidity shock. Indeed the second shock is also correlated with the asset payoff (and has same constant part s). Hence hedging at time 0 can serve as proxy hedging for time 1. Thus there is some substitutability between insurance (liquidity) received from arbitrageurs at time 0 and that received at time 1. The fact that the liquidity received by local investors at time 0 "durably" reduces their hedging demand at time 1 erodes arbitrageurs' market power. Hence providing liquidity by intermediating trades across markets bears resemblance to the provision of a durable good by a monopolist and is subject to similar Coasian dynamics.[16]

The equilibrium implication of these dynamics is that when the profitability of the arbitrage is constant over time ($s_0 = s_1 = s$), arbitrageurs increase their positions only gradually. This results in gradual convergence of prices towards the fundamental, even more so if the market is particularly concentrated. When competition increases, each arbitrageurs buys

[16]Coase (1972)'s intuitions about the durable goods problem for a monopoly have been formalized by Stockey (1981), Bulow (1982) Gul, Sonnenschein and Wilson (1986) and Kahn (1986), among others. In an asset pricing context, see Vayanos (1999), Kihlstrom (2000), DeMarzo and Urosevic (2007), Pritsker (2009), and Edelstein, Sureda-Gomilla, Urosevic, and Wonder (2010). Here the fixed horizon of the model works as a commitment device for arbitrageurs. The discrete trading periods also limits the substitutability between time 0 and time 1 liquidity. In a full-fledged model, with shrinking time among trading periods and an infinite horizon, the Coase conjecture would apply and the spread would always be zero.

(sells) a smaller amount in market A (B). However the aggregate quantity traded in equilibrium increases, as Figure 1.3 shows. In the limit, arbitrageurs fully intermediate trades between A- and B-investors and the equilibrium spread converges to zero. The following corollary summarizes these results:

Corollary 1 *Suppose $s_0 = s_1 = s$, then*

- *the spread is always positive and decreases with the number of arbitrageurs at time 0 and time 1:* $\frac{\partial \Delta_t}{\partial n} < 0$, $t = 0, 1$,
- *the spread decreases over time:* $\Delta_2 = 0 < \Delta_1 < \Delta_0$,
- *and it decreases faster as n increases:* $\frac{\partial \left[\frac{\Delta_1 - \Delta_0}{\Delta_0}\right]}{\partial n} < 0$.
- *When $n \to \infty$, the arbitrageurs absorb the entire liquidity shock at time 0 and time 1, and the spread converges to zero:* $\lim_{n \to \infty} \Delta_t = 0$, $t = 0, 1$.[17]

These results generalize the idea of "gradual arbitrage" developed in Oehmke (2010) in a setting where the price schedules against which arbitrageurs trade are endogenous. Owing to imperfect competition, arbitrageurs can slow down the speed of arbitrage across markets, resulting in gradual convergence of prices towards the fundamental value.[18] Said differently, as arbitrageurs will set better prices in the future, it is optimal for local investors to hold some of the excess supply created by their liquidity shock. As Oehmke points out, this mechanism can account for the observed slow reversal of prices towards fundamentals following shocks documented, for instance, by Mitchell, Pulvino and Stafford (2002) in the convertible arbitrage market, and Coval and Stafford (2007) in the equity market.

Time-varying price impact. In Oehmke's model, as in other related papers in the literature (e.g. Carlin, Sousa-Lobo and Viswanathan, 2007, Brunnermeier and Pedersen, 2005), arbitrageurs trade against an exogenous price schedule with constant price impact coefficient.[19] With endogenous price schedules, the arbitrageurs' price impact is no longer constant over time. It decreases as time passes and depends on the market structure:[20]

[17] Note that this result does not depend on the assumption $s_0 = s_1 = s$.
[18] In Oehmke's model, the arbitrage is risky but arbitrageurs are risk-neutral.
[19] As some of these models are framed in continuous time, there is also a temporary price impact component that helps pin down the equilibrium speed of trading.
[20] This result is general and does not depend on the assumption that shocks are constant over time, or that shocks are known in advance.

Corollary 2 *Price impact decreases over time, even more so if the market is concentrated (n small):*

- *At time 1, arbitrageurs' price impact is* $|\frac{\partial \Delta_1}{\partial x_1^i}| = 2a\sigma^2$ *(*$i = 1, \ldots, n$*)*
- *At time 0, the equilibrium spread schedule is*

$$\Delta_0\left(Q_0\right) = 2a\sigma^2 \left[s_0 + \frac{s_1}{(n+1)} - \frac{n+2}{n+1} \sum_i x_0^i\right], \tag{1.13}$$

i.e. arbitrageurs' price impact is $|\frac{\partial \Delta_0}{\partial x_0^i}| = 2a\sigma^2 \frac{n+2}{n+1} > |\frac{\partial \Delta_1}{\partial x_1^i}|$.

The spread schedule at time 0 has two components. The first component, $2a\sigma^2\left(s_0 + \frac{s_1}{(n+1)}\right)$, is the spread that would obtain if arbitrageurs did not trade at time 0 in equilibrium. It is increasing in s_0 *and* s_1, because local investors anticipate that risk-sharing at time 1 will be limited due to arbitrageurs' market power. Indeed, an increase in market competitiveness improves risk-sharing and in the limit eliminates s_1. The second component represents the arbitrageurs' price impact, $a\sigma^2 \frac{n+2}{n+1}$. Two opposite effects determine the evolution of price impact over time. First, given that new information accrues over time, the conditional variance of the asset payoff is decreasing over time as uncertainty realizes. This implies that local investors in each market are "more risk-averse" at time 0 than at time 1. Since the variance of each innovation ϵ_t is constant over time, price impact should be twice as large at time 0 than at time 1. This is not the case, however, because a second effect tends to reduce price impact.[21] As local investors anticipate that arbitrageurs will provide further liquidity at time 1, they understand that they will have another trading opportunity to share risk, and this reduces their effective level of risk-aversion ex-ante. Said differently, local investors are less desperate to receive liquidity if they anticipate that more liquidity is coming later on.[22] The more concentrated the market is, however, the more rationed liquidity will be ($a\sigma^2 \frac{n+2}{n+1}$ is maximal for $n = 1$), and therefore price impact is higher at time 0 if the market is concentrated - or, more precisely, expected to remain concentrated.[23] Note that at time 1, arbitrageurs' price impact depends only on risk-aversion and not on the market structure. This is because at time 2, the asset pays off, which is equivalent to

[21] It is easy to see that, indeed for any $n \geq 1$, $2a\sigma^2 \frac{n+2}{n+1} < 4a\sigma^2$.

[22] As noted above, these Coasian dynamics crucially depend on the fixed horizon of the model, and arbitrageurs' inability to commit to trade only once.

[23] In Section 1.4.2, I allow for a new arbitrageur to enter at time 1 upon sinking a fixed cost, therefore local investors' expectations about the future number of arbitrageurs determines price impact at time 0.

restoring perfect liquidity in the market. If the market was perfectly competitive also at time 1, the market structure adjustment of time 0 price impact would disappear, and price impact would be constant$a\sigma^2 \frac{n+2}{n+1} \to a\sigma^2$.

This dynamic "contamination" of illiquidity from period 1 to period 0 is therefore due to imperfect competition and the limited risk-sharing that it implies. The same dynamic effect is present in Rostek and Weretka (2010), who study a setting with n strategic arbitrageurs and no competitive fringe. Their model, however, predicts that price impact should increase over time, because only the second effect, stemming from the opportunity to retrade and diversify risk further in the future, is present. The comparison of our results therefore reveals that the direction of change of market depth over time - whether it increases or decreases over time - should depend not only on the market microstructure but also on the uncertainty surrounding the asset payoff. Here the model predicts that price impact should decrease as the date of the asset payoff approaches, but even more so if only a few large arbitrageurs are active in the trade.

Optimal execution with endogenous market depth. An interesting implication of the time-varying price impact is that a monopolistic arbitrageur does not equally split his trade across periods: for $n = 1$, $\kappa_{0,1} = \frac{2}{5} > \kappa_{1,1} = \frac{3}{10}$, i.e. $x_0 > x_1$. (More generally, for an arbitrary number of arbitrageurs, $x_0 > x_1$) This is a key difference with the literature on optimal execution of large orders (e.g. Bertsimas and Lo, 1998), which shows that with constant price impact, it is optimal for a monopolistic trader to break up orders equally over time. Thus the model highlights that in concentrated markets, optimal order execution and market depth are *jointly determined* and depend on the deep characteristics of the market, such as investors' risk-aversion, asset volatility and the market structure.

Sign inversion with changing supply shocks ($s_0 \geq 0$, $s_1 \geq 0$)

The case where the supply of arbitrage changes over time brings further insight into the mechanisms and generates new predictions. First, proposition 1 shows that the time-0 trade (1.8) depends on both s_0 and s_1, unless there is a single arbitrageur. This shows that when competition increases, the pressure to share risk with local investors increases, so that local investors are able to start hedging their future risk. Note that, independently of the number of arbitrageurs, the time-0 equilibrium spread always depends on both s_0 and s_1, because future certain shocks are immediately reflected in asset prices.

Sign inversion. Changes in shocks, i.e. variation in the arbitrage supply, generates sign inversion: when both shocks are positive, Δ_1 may become negative, even though $s_1 \geq 0$. Remember that a positive shock implies that A-investors should value the asset less than B-investors, suggesting that Δ_1 should be positive. Note from equation (1.10), that Δ_0 is always positive with $s_0 \geq 0$ and $s_1 \geq 0$. The sign of the spread can switch over time if the profitability of the arbitrage deteriorates:

Corollary 3 *Suppose that s_0 and s_1 are positive.*

- *At time 0, the spread is always positive and decreases with the number of arbitrageurs:* $\frac{\partial \Delta_0}{\partial n} < 0$.
- *At time 1:*
  - *The spread is negative if and only if s_1 is small enough relative to s_0:* $\Delta_1 \leq 0 \Leftrightarrow s_1 \leq \alpha_n s_0$, *with* $0 < \alpha_n < 1$.

The condition for sign inversion is that liquidity shocks decrease sufficiently over time, i.e. that the arbitrage profitability decreases sufficiently. An interpretation of the condition is that sign inversion may occur in the aftermath of a large shock, and close to the time where assets mature, or where convergence occurs for exogenous reasons (e.g. when an on-the-run bond is close to becoming off-the-run). Hence the model predicts that sign inversion should occur following periods of low liquidity (or equivalently large price divergence).

To understand the intuition of the mechanism, consider an example in which $s_0 > 0$ and $s_1 = 0$. In this case, local investors in market A initially short the asset, receiving partial insurance against the positive supply (liquidity) shock from arbitrageurs who limit the amount they buy thanks to market power. At time 1, since there is no reason to hedge anymore ($s_1 = 0$), local investors seek to close their hedge by buying back the asset (indeed, $y_1^A > 0$). However, arbitrageurs continue to limit liquidity at this time, so that local investors cannot fully close their short position. This pushes the price of asset A above its expected value. As a consequence, local investors remain short, $Y_1^A < 0$, as one can see by setting $s_1 = 0$ in equation (1.6). This is optimal since the price of asset A will (on average) drop at time 2. (Arbitrageurs are not subject to this effect since they are not exposed to market risk, taking opposite positions across markets.) Since the opposite must occur in market B, the price of asset A trades at a premium relative to the fundamental and the price of asset B at a discount, resulting in a negative spread at time 1. Of course, arbitrageurs earn a profit even if the spread sign inverts, because their profit depends on the fact that prices do not converge and not on the sign. As I show in the proof of Proposition 1, the trading profit at time 1, $x_1\Delta_1$, is equal to $2a\sigma^2\frac{\left(s_1-\sum_{i=1}^n x_0^i\right)^2}{(n+1)^2}$, with $x_1 = \frac{s_1-\sum_{i=1}^n x_0^i}{n+1}$ and $\Delta_1 = 2a\sigma^2\frac{s_1-\sum_{i=1}^n x_0^i}{n+1}$. Hence, arbitrageurs care about the magnitude of the mispricing rather than the sign.

As Lemma 3 shows, s_1 does not have to be zero, but small enough relative to s_0. Intuitively, the need to revert the hedge must simply be large enough for the spread to invert at time 1. Hence, the time-1 spread can turn negative even though all liquidity shocks imply that it should be positive. Interestingly, it is precisely when local investors' demand pressures decrease that arbitrageurs push the spread to invert. Hence it is when asset prices should converge towards their fundamental value that arbitrageurs cause a breakdown of the intuitive relationship between A-and-B asset prices. What causes this breakdown is that arbitrageurs limit liquidity both when local investors need to sell and to buy. Because it is driven by the variation in the arbitrage profitability, this result is not present in Oehmke (2010) in which only the initial shock matters.[24]

This result may shed light on recent puzzling evidence about closely-related assets. Indeed several standard and intuitive relationships broke down in the aftermath of the 2007-2009 financial crisis. For instance, the 7-and 10-year swap spread turned negative for the first time in 2010 (*Business Week*, 23/03/2010). Uninsured municipal bonds became more expensive than similar insured bonds issued by the same city also in 2010 (Bergstresser et al., 2011).[25] The extent of the mispricing, in particular in the municipal bond market, makes standard explanation implausible. For instance, a negative swap spread may be justified by heightened concerns about sovereign risk. Similarly, concerns about monoline insurers may reduce the premium attached to insured bonds to zero. However, it is hard to see how it could generate a negative premium. Although these explanations may be partially correct, the model offers a single complementary mechanism based on market structure and easing of demand pressures.

[24]Specifically, Oehmke considers the time-inconsistent trading strategies of strategic arbitrageurs facing two exogenous demand curves for the same asset. Since the strategy is solved ex-ante, all results are a function of the initial liquidity shock, which eliminates the possibility of time variation in the arbitrage profitability.

[25]Inflation-protected Treasuries also became cheaper than similar nominal bonds (Pflueger and Viceira, 2011). Although our explanation could be appealing, the timing seems less consistent with our mechanism, since the sign inversion occurred in 2009, presumably, in the middle of the crisis instead of at the end. Pflueger and Viceira show that the negative breakeven inflation in the Treasury market can be attributed to a larger liquidity discount for the TIPS and not to a sign of deflation expectations.

1.3.2 Risky arbitrage

In this section, I assume that s_1 is not known at time 0. Investors only know that it is normally distributed with mean $\bar{s}_1$ and variance z_1^2. I also make the following assumption about the parameters:

Assumption 1 $a^2\sigma^2 z_1^2 < \frac{(n+1)^2}{2n+1}$

Since the second shock is random from the point of view of time 0, the arbitrage is no longer risk-free. Therefore, even if arbitrageurs can eliminate all fundamental risk by taking opposite positions in assets A and B, they face uncertainty about the future profitability of the arbitrage. As in standard noise trader risk models, the potential deepening of the mispricing is short-lived, and the prices assets A and B converge at time 2 when the assets pay off. This risky arbitrage case allows me to delve more deeply into the mechanisms and to analyze arbitrageurs' risk-management strategies.

Price schedules and equilibrium

At time 1, the problem is not different from the risk-free case. However at time 0, all investors face uncertainty about the magnitude of the future liquidity shock. I show in the appendix that at time 0, the spread schedule faced by arbitrageurs is the following:

$$\Delta_0(.) = 2a\sigma^2\left[s_0 + \frac{\bar{s}_1}{(n+1)\,r_a} - \frac{n+2}{n+1}(1+\phi_a)\sum_i x_0^i\right], \tag{1.14}$$

$$\text{with } \phi_a = \frac{a^2\sigma^2 z_1^2}{(n+1)^2\, r_a} \text{ and } r_a = 1 - a^2\sigma^2 z_1^2\frac{2n+1}{(n+1)^2} \tag{1.15}$$

There are two key differences with respect to the risk-free case, in which the schedule is given by equation (1.13), which I reproduce here for convenience: $\Delta_0(.) = 2a\sigma^2\left[s_0 + \frac{s_1}{(n+1)} - \frac{n+2}{n+1}\sum_i x_0^i\right]$. The first part of the schedule, $s_0 + \frac{\bar{s}_1}{(n+1)r_a}$, represents the price divergence that would prevail in equilibrium in the absence of trade. Given that $r_a < 1$, we have: $s_0 + \frac{\bar{s}_1}{(n+1)r_a} > \mathbb{E}_0\left[s_0 + \frac{s_1}{(n+1)}\right]$, which captures the effect of convexity, as in Jensen's inequality. The second part represents arbitrageurs' price impact. It increases by a factor $1+\phi_a > 1$ relative to the risk-free case. The increase is larger if the volatility of the liquidity shock z_1^2, fundamental volatility σ^2, or risk aversion a is large. The effect of the market structure, captured by the term $\frac{n+2}{n+1}$, is amplified by the uncertainty about future liquidity shocks. Price impact increases because local investors require larger discounts to hold their risky asset when future shocks are random.

Uncertainty about future liquidity shocks also affects arbitrageurs' strategies at time 0, both through their own risk aversion and through the change in the price schedules. The different channels appear clearly in their value function:

Proposition 2 *At time 0, the arbitrageurs' value function is given by*

$$J_0^i = \max_{x_0^i} -r_b^{-\frac{1}{2}} \exp\left[-2ba\sigma^2\left(x_0^i\hat{\Delta}_0 + (1-\phi_b)\frac{\left(\sum_i x_0^i\right)^2}{(n+1)^2} - \frac{\bar{s}_1}{(n+1)^2 r_b}\left(2\sum_i x_0^i - \bar{s}_1\right)\right)\right]$$

$$\text{where } \hat{\Delta}_0 = \frac{\Delta_0(.)}{2a\sigma^2},\ r_b = 1 + \frac{4ab\sigma^2 z_1^2}{(n+1)^2} \text{ and } \phi_b = \frac{4ab\sigma^2 z_1^2}{(n+1)^2 r_b} \tag{1.16}$$

Arbitrageurs' value function is made of three components:

1. Their time-0 trading profit 0, $x_0^i\hat{\Delta}_0$, i.e. quantity times (normalized) price gap $\hat{\Delta}$.

2. The time-1 continuation profit, in which we can distinguish two parts, depending on their relation to risk aversion:

   (a) The first part, $(1-\phi_b)\frac{\left(\sum_i x_0^i\right)^2}{(n+1)^2}$, is decreasing in arbitrageurs' risk aversion b, and more generally in z_1^2, σ_2, and a. Hence I will refer to it as the precautionary (or hedging) motive. The coefficient ϕ_b measures by how much arbitrageurs reduce their aggressiveness in tackling the arbitrage gap at time 0 for fear of facing too much risk at time 1. Note that ϕ_b depends on the product of a and b because an increase in local investors' risk-aversion makes them more reluctant to hold the risky asset and thus restricts arbitrageurs' risk-sharing opportunities. The hedging motive is, perhaps surprisingly, increasing in the total size of previous trades, $\sum_i x_0^i$. This is because trading aggressiveness at time 0 works as an indirect hedge against large shocks at time 1 by reducing the spread permanently. The strength of the hedging motive also depends on the number of arbitrageurs. A change in market structure has two conflicting effects:

   Corollary 4 *At time 1, when the number of arbitrageurs increases, there is*

   - *a business-stealing effect:* $\frac{\partial \frac{1}{(n+1)^2}}{\partial n} < 0$, *which reduces the coefficient* $\frac{1-\phi_b}{(n+1)^2}$,
   - *a co-insurance effect:* $\frac{\partial(1-\phi_b)}{\partial n} > 0$, *which increases it.*

The business-stealing effect always dominates the co-insurance effect, i.e. $\frac{\partial \frac{1-\phi_b}{(n+1)^2}}{\partial n} < 0$.

The business-stealing effect is the standard consequence of stronger competition in a Cournot setting. The co-insurance effect is positive because as n increases, risk-sharing becomes more effective: more competition means that arbitrageurs supply more liquidity and each arbitrageur benefits from this collective effect.

(b) The second part of the time-1 continuation payoff represents the "strategic motive":

$$-\frac{\bar{s}_1}{(n+1)^2 r_b}\left(2\sum_i x_0^i - \bar{s}_1\right)$$

It is *increasing* in arbitrageurs' risk-aversion b and decreasing in previous trades, as arbitrageurs have an incentive to strategically limit their positions at time 0 to be able to fully exploit the arbitrage opportunity later. The key driver of the strategic motive is the expected level of arbitrage risk $\bar{s}_1$ instead of the risk of the arbitrage risk z_1^2, as explained in more details below.

Proposition 3 *When arbitrage is risky, there is a unique (symmetric) equilibrium characterized by:*

$$x_0^i = \frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a} - \frac{2\bar{s}_1}{(n+1)^2 r_b}}{\phi_n + (n+2)\phi_a + 2n\phi_b} \quad (1.17)$$

$$x_1^i = \frac{s_1 - \sum_i x_0^i}{n+1} \quad (1.18)$$

The equilibrium spread between asset B and asset A is

$$\Delta_0 = 2a\sigma^2\left[\Phi_a\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right) + (1-\Phi_a)\frac{2\bar{s}_1}{(n+1)^2 r_b}\right], \text{ with } \Phi_a \in [0,1] \quad (1.19)$$

$$\Delta_1 = \frac{2a\sigma^2}{n+1}\left[-\frac{n}{d}s_0 + \left(s_1 - \frac{n}{(n+1)^2}\frac{(n+1)r_b - 2r_a}{dr_ar_b}\bar{s}_1\right)\right] \quad (1.20)$$

with $d = \phi_n + (n+2)\phi_a + 2n\phi_b$.

The time 1 subgame is similar to the risk-free case, thus I focus on time 0 where uncertainty about future shocks generates a number of interesting effects. By comparing (1.8) and (1.17), one can see that arbitrageurs equilibrium trades generalize in a very intuitive way.

To understand the mechanisms, it is helpful to decompose x_0 in two terms:

$$x_0 = \frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a}}{\phi_n + (n+2)\,\phi_a + 2n\phi_b} - \frac{\frac{2\bar{s}_1}{(n+1)^2 r_b}}{\phi_n + (n+2)\,\phi_a + 2n\phi_b} \tag{1.21}$$

The first term shows that arbitrageurs buy a fraction $\frac{1}{\phi_n+(n+2)\phi_a+2n\phi_b}$ of the expected spread that would prevail in the absence of liquidity provision (the maximum spread), $s_0 + \frac{\bar{s}_1}{(n+1)r_a}$. This maximum spread represents the demand for liquidity addressed to arbitrageurs at time 0. Only part of this demand is served as arbitrageurs' market power allows them to ration liquidity. Arbitrageurs serve a smaller fraction of the demand as their risk aversion b increases, due to precautionary concerns, and as local investors' risk aversion a increases, because arbitrageurs have a larger price impact at time 0, which prompts them to scale back their trade (as captured by the coefficient ϕ_a). What is interesting is that arbitrageurs provide less than a fraction of the maximum spread, since the second term in (1.21) is negative. The second term captures the effect of the arbitrageurs' strategic motive. It becomes more negative as risk-aversion decreases. In the limit, as arbitrageurs become risk-neutral, the strategic motive is strongest:

$$\text{when } b \to 0,\ x_0^i \to \frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a}}{\phi_n + (n+2)\,\phi_a} - \frac{\frac{2\bar{s}_1}{(n+1)^2}}{\phi_n + (n+2)\,\phi_a} \tag{1.22}$$

In fact, the strategic motive is present even in the absence of uncertainty about liquidity shocks. To eliminate uncertainty, consider the limit case where arbitrageurs are risk-neutral $b \to 0$, and uncertainty vanishes $z_1 \to 0$. Then x_0 converges to (1.8), its equilibrium quantity when the arbitrage is risk-free (assuming $s_1 = \bar{s}_1$):

$$\text{When } z_1 \to 0, b \to 0,\ x_0^i \to \frac{s_0 + \frac{\bar{s}_1}{(n+1)}}{\phi_n} - \frac{\frac{2\bar{s}_1}{(n+1)^2}}{\phi_n} = \frac{1}{\phi_n} s_0 + \frac{n-1}{(n+1)^2\,\phi_n} s_1$$

Hence in hindsight, this decomposition highlights a fact that was hard to identify when the arbitrage was risk-free. Arbitrageurs respond to their commitment problem by buying less than a fraction of the maximum spread, $s_0 + \frac{s_1}{(n+1)r_a}$. This reduction, $-\frac{2\bar{s}_1}{(n+1)^2 r_b}$, arises because arbitrageurs strategically refrain from tackling the spread too aggressively at time 0, in the hope that a large shock will increase the local investors' risk-sharing needs at time 1. Arbitrageurs "speculate" more when the market is more concentrated (n small) and the expected shock $\bar{s}_1$ is large. Given the partial substitutability between liquidity provision at time 0 and time 1, arbitrageurs must reduce liquidity provision, i.e. decrease x_0, to exploit large liquidity needs as much as possible later on.

The strategic motive resembles the standard risk-management mechanism that arises in models where competitive arbitrageurs face financial constraints. Several papers in the limits of arbitrage literature (e.g. Shleifer and Vishny, 1997, Gromb and Vayanos, 2002) show that financially-constrained arbitrageurs refrain from taking on too much risk early on in order to save capital and be able to exploit potentially large price discrepancies at later periods. This mechanism is based on the limited amount of capital available to competitive arbitrageurs in the short-term. Here the effect is related to market power and is a response to the perfect foresight of local investors, which erodes arbitrageurs' market power as in the classic durable goods monopoly problem. Interestingly, the strategic motive is strongest when arbitrageurs are risk-neutral, precisely when they are most likely to aggressively tackle arbitrage opportunities.

Since the precautionary and the strategic motives have opposite dependence on arbitrageurs' risk aversion, an increase in b has an ambiguous effect.

Arbitrageurs' risk aversion and liquidity

According to Friedman (1953), speculators reduce price volatility by smoothing out temporary price fluctuations. Given that this view implies a contrarian behaviour, it may seem desirable to have risk-loving arbitrageurs for markets to be efficient. This is no longer the case when arbitrageurs have price impact: the spread between assets A and B may increase as arbitrageurs become risk-neutral. On one hand, a decrease in risk aversion increases arbitrageurs' trading aggressiveness to tackle the arbitrage. On the other hand, a lower risk-aversion makes them more likely to engage in strategic "speculation", as shown by the following result:

Corollary 5 *An increase in arbitrageurs' risk-aversion may result in them providing more or less liquidity at time 0. There are two opposite effects:*

$$\frac{\partial x_0^i}{\partial b} = \kappa \left[\underbrace{-n\left(n+1\right)^2 \left(s_0 + \frac{\bar{s}_1}{\left(n+1\right) r_a}\right)}_{\textit{precautionary motive} < 0} + \underbrace{\bar{s}_1 \left(d + \frac{2n}{r_b}\right)}_{\textit{reduction in strategic motive} > 0} \right], \ \kappa > 0$$

The reduction in strategic motive dominates iff s_1 is large enough relative to s_0:

$$\frac{\partial x_0^i}{\partial b} \geq 0 \Leftrightarrow \bar{s}_1 \left(d - \frac{n\left(\left(n+1\right) r_b - 2r_a\right)}{r_a r_b} \right) \geq n\left(n+1\right)^2 s_0$$

The following lemma shows a special case in which the strategic motive is so strong that a decrease in risk aversion does lead to an decrease in liquidity provision (and conversely, an increase in risk aversion leads to higher liquidity):

Lemma 1 *Suppose that $s_0 \to 0$. If $n \leq 2$ and local investors' risk-aversion a is small enough (or equivalently, σ^2 or z_1^2 small enough), then, following a small increase in their risk-aversion, arbitrageurs provide more liquidity, which decreases the time 0 spread and increases the expected return of the arbitrage. This effect is stronger if they are not very risk-averse.*

Unsurprisingly, the strategic motive dominates in a very concentrated market, and even more so if arbitrageurs are not too risk-averse. Note that if s_0 is very small, on average, the spread will decrease between time 0 and time 1, implying a negative return. As b increases, arbitrageurs increase their trade at time 0, and this reduces the time 0 spread more than the time 1 spread, leading to a less negative return.

How do arbitrageurs respond to an increase in arbitrage risk?

In the presence of arbitrage risk, it is important to understand whether arbitrageurs' reactions to changes in risk are stabilizing (i.e. leading to smaller spreads), or destabilizing. In the limits of arbitrage literature, it is common to study how positions and prices respond to an increase in "noise trader risk" (Shleifer and Vishny, 1997), or demand pressures / supply imbalances (Gromb and Vayanos, 2010, Brunnermeier and Pedersen, 2009). It is shown that arbitrageurs do not necessarily increase their positions ex-ante when they face larger future shocks, and this may push prices further away from their fundamental values. Here, I analyze arbitrageurs' responses to an increase in the level of the future shock, $\bar{s}_1$, and in the volatility of the shock z_1. Surprisingly, the literature on limits of arbitrage has to the best of my knowledge focused only on the first comparative static (dubbed noise trader risk).

Corollary 6 *Following an increase in the expected shock $\bar{s}_1$, arbitrageurs increase their positions at time 0, but the spread nevertheless increases: $\frac{\partial x_0^i}{\partial \bar{s}_1} \geq 0$ and $\frac{\partial \Delta_0}{\partial \bar{s}_1} \geq 0$.*

The two parts of the result may seem contradictory, as one would expect the increase in arbitrageurs' positions to lead to a smaller spread. It is not the case because an increase in $\bar{s}_1$ also causes an increase in local investors' liquidity demand, and arbitrageurs' response, albeit positive, is not commensurate with local investors' increased need for liquidity. This is in particular due to the fact that an increase in $\bar{s}_1$ increases the profitability of the arbitrage butt also arbitrageurs' strategic motive. This result contrasts with predictions in models of financially-constrained arbitrage, where an increase in positions leads to more efficient prices (e.g. Shleifer and Vishny, 1997, Brunnermeier and Pedersen, 2009).

Next, it is interesting to understand how arbitrageurs respond to increased *uncertainty* about the future profitability of the arbitrage. As one would expect, increased uncertainty reduces arbitrageurs' strategic motives and increases their precautionary motives. However, uncertainty about future profitability matters even when arbitrageurs are risk-neutral, as it affects local investors' liquidity demand, as well as arbitrageurs' price impact.

Corollary 7 *Consider the limit case where arbitrageurs are risk neutral, i.e.* $b \to 0$*. Then arbitrageurs respond to an increase in arbitrage risk* z_1^2 *by taking larger positions if and only if volatility is small enough and the expected shock is large enough relative to the current shock. Otherwise, arbitrageurs decrease their positions.*

$$\frac{\partial x_0^i}{\partial z_1^2} \geq 0 \;\Leftrightarrow\; \begin{cases} a^2\sigma^2 z_1^2 < c_n \text{ with } c_n < \frac{(n+1)^2}{2n+1} \\ \bar{s}_1 \geq \theta_{n,a} s_0 \end{cases}$$

No matter how arbitrageurs respond, the spread always increases following an increase in z_1^2*:* $\frac{\partial \Delta_0}{\partial z_1^2} \geq 0$.

The result shows that even if arbitrageurs are risk-neutral, they may scale down their positions when uncertainty about arbitrage profitability increases. There are two effects: first, local investors are demanding more liquidity, as the convexity effect (i.e. the need to insure against shocks) increases with z_1 (see equation (1.14). This increases the arbitrage supply, which prompts arbitrageurs to increase their positions, but also increases the spread. Second, an increase in uncertainty steepens local investors' demand for the asset in each market, as local investors are more reluctant to hold the asset. This results in larger price impact, which pushes arbitrageurs to decrease their positions. When volatility is already high, this effect dominates, and an increase in volatility leads to a reduction in arbitrageurs' positions. Interestingly, whether arbitrageurs increase their positions or not, the spread always increases, showing that the increase in liquidity demand always outweighs the increase in arbitrageurs' positions.

Market power and spread autocorrelation

Arbitrageurs' activity also implies a number of properties for the behaviour of the spread between assets A and B. First, one can see in Proposition 3 that at time 0, the spread is a weighted average (since $\Phi_a \in [0,1]$) of the maximum spread ($s_0 + \frac{\bar{s}_1}{(n+1)r_a}$) and arbitrageurs' strategic motive. At time 1, the spread is decreasing in the previous shock and reflects the adjustment between the expected shock $\bar{s}_1$ and its realization s_1. Market power also generates autocorrelation of the spreads at one lag:

Corollary 8 *The spread has the following properties:*

- *Comparative statics: the current shock increases the spread.*

$$\frac{\partial \Delta_t}{\partial s_t} > 0, \quad t = 0, 1$$

- *Serial correlation: suppose s_0 is random from the point of view of time -1. Then, when the number of arbitrageurs is finite, the half spread exhibits negative serial correlation between time 0 and time 1:*

$$autocov_{-1}\left(\frac{\Delta_0}{2}, \frac{\Delta_1}{2}\right) < 0$$

When perfect competition obtains, the serial correlation vanishes:

$$\lim_{n\to\infty} autocov_{-1}\left(\frac{\Delta_0}{2}, \frac{\Delta_1}{2}\right) = 0$$

Given that arbitrageurs revert trades in proportion of previous shocks to keep the spread open as long as possible, previous shocks continue to affect the current spread. This generates serial correlation at one lag. As competition increases, arbitrageurs absorb liquidity shocks fully in each period, thus serial correlation disappears.

1.4 Entry

I now turn to endogenize the number of arbitrageurs. I consider ex-ante free entry, as well as gradual entry. I assume that arbitrageurs only know the distribution of future shocks when they decide ex-ante. High ex-ante uncertainty about the profitability may in this regard reduce risk-averse arbitrageurs' incentive to enter if there is enough risk-bearing capacity. When new arbitrageurs can enter gradually, arbitrageurs already active can deter new ones from entering, perpetuating market concentration. The mere threat of future entry, however, can improve liquidity.

1.4.1 Simultaneous (free) entry

I assume that arbitrageurs must sink a set-up cost I at time -1 to enter the market. Throughout this section and the next, one can think of I as the investments required to set up an arbitrage desk, gather information, subscribe to data-providers, etc.

Risk-free arbitrage

For the sake of tractability, I consider the special case $s_0 = s_1 = s$ and assume that from the perspective of time -1, s is random and normally distributed with mean $\bar{s}$ and variance z^2.

Proposition 4 *At time -1, the arbitrageurs' expected utility from entering the market (net of entry cost) is:*

$$\tilde{J}_0^i = -\frac{1}{\sqrt{1+4ab\sigma^2 z^2 \pi_n}} \exp\left[-b\left(\frac{2a\sigma^2\pi_n\bar{s}^2}{1+4ab\sigma^2 z^2\pi_n} - I\right)\right] \tag{1.23}$$

$\tilde{J}$ is decreasing in n. Thus there exists n^*, defined as the first integer such that

$$\begin{cases} \tilde{J}(n^*) \geq -1 \\ \tilde{J}(n^*+1) < -1 \end{cases}$$

Note that as long as I is not too large, arbitrageurs will enter the trade even if there will be no liquidity shock on average, i.e. even if $\bar{s} = 0$. This is because the random nature of local investors' liquidity shocks (from the point of view of time -1) creates "optionality" in the trade, and this induces arbitrageurs to enter. Since $\tilde{J}$ is decreasing in n, there is a single Nash equilibrium in the free-entry game. The equilibrium number of arbitrageurs n^* depends on the characteristics of the segmented markets as follows:

Corollary 9 *There is more entry into markets characterized by large arbitrage risk and / or risk-averse local investors: n^* increases with $\bar{s}$, a, and σ^2 and decreases with I.*

The effect of an increase in the volatility of the arbitrage risk, z, is ambiguous:

- *n^* decreases with z when the market is initially concentrated, or if local investors (or arbitrageurs) are sufficiently risk-averse, or equivalently, if the asset is sufficiently risky, and increases in the opposite situations.*
- *Hence markets that are likely to be illiquid from an ex-ante perspective attract fewer arbitrageurs.*

Arbitrageurs prefer to enter in markets in which the arbitrage gap is large (on average). However, facing an increasing uncertainty about the size of the arbitrage gap may lead to more or less entry depending on the level of risk-aversion of investors and arbitrageurs in the economy. Intuively, uncertainty can be desirable because it increases the "optionality" of the trade, but at same time, it can be costly because arbitrageurs are risk-averse. I show in the proof of the corollary that a key driver of the comparative static is the variable $\theta = 2ab\sigma^2\pi_n$, which measures the risk-bearing capacity of the market. If the market has a small risk-bearing capacity, an increase in the volatility of shocks may discourage arbitrageurs from entering. Because the concentration of the market is also a (negative) determinant of risk-bearing capacity, the same applies if the initial number of arbitrageurs is small.[26] The dependance on n highlights the fact that there are both strategic substitutabilities (business-stealing) and strategic complementarities (co-insurance) between arbitrageurs. Note that this result does not depend on the entry cost: the effect is present even when $I \to 0$.[27] Intuitively, if the market cannot absorb enough risk, arbitrageurs will be reluctant to enter. Thus, the markets that are the most likely ex-ante to be illiquid (high volatility of arbitrage risk, and low risk-bearing capacity) are those in which arbitrageurs enter the least. However, the picture is not entirely bleak because, as shown above, a high level of arbitrage risk attracts arbitrageurs, which then exert a corrective force on market liquidity:

[26] One can imagine a situation in which, at $t = -1^a$, all investors believe that the volatility of the arbitrage risk is z and the number of entrants is determinant. Then at $t = -1^b$, investors learn that the volatility is actually $z' > z$. The corollary states that if the market was concentrated at -1^a, it will be even more at -1^b. Intuitively, the presence of many arbitrageurs has a positive externality in terms of risk-absorption.

[27] As I decreases, there is more and more entry. However, an increase in the uncertainty about the profitability of the arbitrage dampens the effect of a reduction in entry costs.

Corollary 10 *Entry of arbitrageurs counters the effect of an increase in local investors' risk-aversion (or equivalently volatility of the fundamental) on the spread:*

$$\frac{\mathrm{d}\Delta_0}{\mathrm{d}a} = \underbrace{2\sigma^2}_{>0}\left[\psi_{n^*}s_0 + \bar{\psi}_{n^*}s_1\right] + 2a\sigma^2\left[\underbrace{\frac{\partial\psi_{n^*}}{\partial a}}_{<0}s_0 + \underbrace{\frac{\partial\bar{\psi}_{n^*}}{\partial a}}_{<0}s_1\right]$$

Oehmke (2010) derives results that are similar in spirit. He finds, however, that illiquid markets attract less arbitrageurs, taking the coefficient of short-term price impact as a measure of illiquidity. Here, an increase in local investors' risk-aversion makes the market more illiquid but also attracts more arbitrageurs. An increase in a in my model would correspond to an increase in the coefficient of permanent price impact in Oehmke's model.

Combining the two previous results, we can predict what kind of risk-return profile is most attractive to arbitrageurs. The number of arbitrageurs unambiguously increases with the average magnitude of the mispricing $\bar{s}$. However, there is a non-monotonic relation between entry and the volatility of liquidity shocks z. I show in the proof of Corollary 9 that an increase in volatility reduces the equilibrium number of arbitrageurs if

$$z^2 \leq s^2 - \frac{1}{4ab\sigma\pi_n}$$

Hence for z small enough, an increase in volatility reduces entry. The constraint is looser (i.e. one can meet the constraint with larger volatility z), if risk-aversion (a or b) is high or the fundamental is risky (high σ) or the market very concentrated (small n, leading to a large π_n). Figure 1.1 illustrates the analysis. It shows that for a high enough level of volatility ($z \geq z^*$), there should be more arbitrageurs as one goes in the northeastern direction of the risk-return diagram. In this region, entry plays a corrective role against illiquidity (caused by large shocks and large uncertainty). It is the opposite in the left-hand side of the graph. In this region, the number of arbitrageurs should be about stable as one goes towards the northeast, since two forces work in opposite directions: large shocks (on average) attract more arbitrageurs, while an increase in volatility reduces entry. The cutoff z^* increases as the market structure of risk-bearing capacity in the market becomes weaker and converges to $\bar{s}$ as risk-aversion increases.

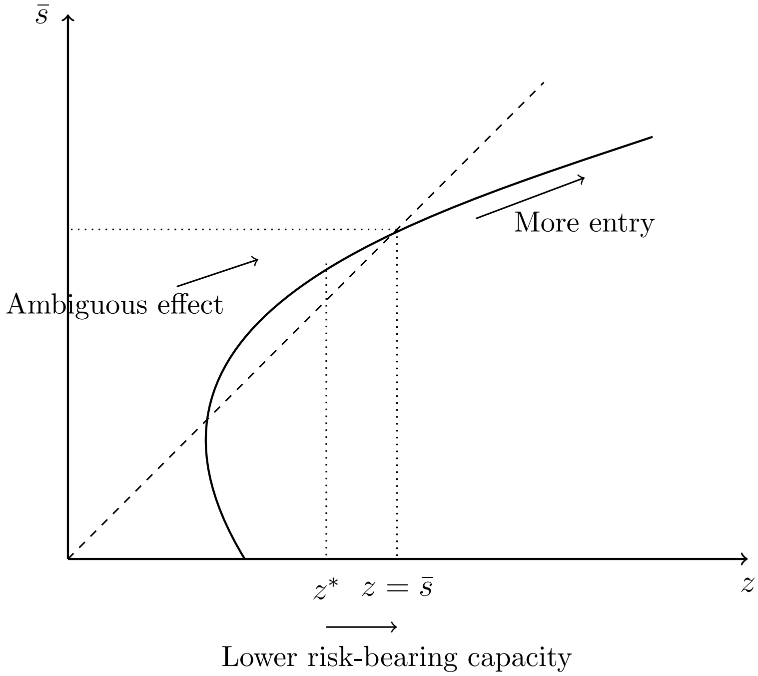


Figure 1.1: **Entry as a function of the risk-return profile of the arbitrage**

Risky arbitrage

In this section, I assume that the first shock, s_0, is unknown at time -1, when arbitrageurs must decide or not to invest and that it is randomly distributed: $s_0 \sim \mathcal{N}(\bar{s}_0, z_0^2)$. The expected utility of entering the market, net of entry cost, is given in the following proposition.

Proposition 5 *At time 0, arbitrageurs' certainty equivalent is given by*

$$CE_0^i = \pi_0 s_0^2 + \pi_1 s_1^2 + \pi_{0,1} s_0 \bar{s}_1 \tag{1.24}$$

At time -1, arbitrageurs' expected utility, net of entry cost, is:

$$\hat{J}_0^i = -\frac{\exp\left[-\eta\left(\frac{\pi_{0,1}\left(\bar{s}_0\bar{s}_1 - \frac{\eta}{2}\pi_{0,1}\bar{s}_1^2 z_0^2\right) + \pi_0\bar{s}_0^2}{1+2\eta\pi_0 z_0^2} + \pi_1\bar{s}_1^2\right)\right]\exp(bI)}{\sqrt{r_b\left(1+2\eta\pi_0 z_0^2\right)}}, \quad \text{with } \eta = 2ab\sigma^2 \tag{1.25}$$

Although the expected utility can be calculated in closed form, it is hard to produce general comparative statics or monotonicity results as the coefficients π_0, π_1 and $\pi_{0,1}$ are complex functions of the parameters. Instead, I investigate some numerical examples.

Numerical examples. Figures 1.5, 1.6 and 1.7 show comparative statics of the endogenous number of arbitrageurs with respect to risk-aversion, and the level and volatility of arbitrage risk. The figures show that $\hat{J}$ is decreasing in the number of arbitrageurs, ensuring that there is a unique equilibrium to the entry game. Arbitrageurs respond positively to an increase in local investors' risk-aversion, or equivalently, in the volatility of the risky assets. The sensitivity to the level and volatility of the arbitrage risk is more complex and seems to be different across periods. For instance an increase in $\bar{s}_0$ seems to increase n^* more than an increase in $\bar{s}_1$. This seems intuitive, because local investors have more opportunities to handle future shocks than the immediate shock. The comparative statics with respect to the volatility of the liquidity shocks at time 0 and time 1 are interesting. An increase in z_0 seems to play the same role as z in the risk-free case: it may decrease the number of entrants, as shown in panel a of Figure 1.7, or increase it (panel b), depending on the risk-bearing capacity of the market.[28] Intuitively, the volatility of the first shock is undiversifiable, since there is no trading at time -1, thus the risk-bearing capacity of the market plays a key role in arbitrageurs' entry decisions. An increase in the volatility of the second shock, z_1, however, seems to have an unambiguous positive effect on n^*. In this case, arbitrageurs can manage risk ex-ante by trading in the risky asset. Figure 1.8 puts this analysis in perspective by comparing the impact on n^* of different parameters in the risk-free and risky arbitrage cases. For the comparison to make sense, I set the level and risk of arbitrage risk to be equal from an ex-ante perspective. Namely, I set $z^2 = z_0^2 + z_1^2$. When the total volatility of the arbitrage risk increases, whether it comes from z_0 or z_1 makes a difference. In the risk-free case, if z^2 increases from 1 to 3, n^* tends to decrease (for other parameter values, there could be an increase, see Corollary 9). In the risky case, if the increase in volatility is matched through an increase in z_0, the effect is similar: n^* tends to decrease (panel a). If it is matched by an increase in z_1, the effect is opposite n^* unambiguously increases (panel b).

1.4.2 Gradual entry and strategic deterrence

I the previous section, I analyzed which markets are likely to attract arbitrageurs ex-ante. In practice, there are many reasons for financial markets to be concentrated at the early stages of their development (e.g. because of financial innovation, learning, uncertainty etc.), but one could expect the degree of competition to increase over time as strategies become copied and

[28]Note however, that it is much harder to obtain the case in which the number of entrants increases: the variance must be multiplied by 10, while risk-aversion of market participants is divided by 3.

rents lure new players in. This is particularly true in the financial industry, where trading strategies are difficult to patent, and personnel mobility, as well as business relationships (e.g. with broker-dealers and other counterparties) make it hard to keep strategies secret. For instance, there is anecotal evidence that LTCM's strategies were widely copied on Wall-Street following the fund's high returns between 1994 and 1997.[29] Similarly, portfolio insurance strategies became widely popular in the 1980s, as they were easy to copy and implement.[30]

To assess whether and when concentration is likely to persist, I now assume that there are n arbitrageurs in place at time 0 (incumbents) and that at time 1, a new arbitrageur may enter upon sinking a fixed cost, I. For simplicity, I focus on the risk-free arbitrage case and assume that the liquidity schocks are identical at time 0 and 1: $s_0 = s_1 = s$. The new arbitrageur (indexed by $n+1$) will enter if her expected payoff at time 1 is larger than the entry cost:

$$2a\sigma^2 \frac{\left(s - \sum_{j=1}^{n} x_0^j\right)^2}{(n+2)^2} \geq I \tag{1.26}$$

However, as equation (1.26) shows, because the new arbitrageur enters with a lag, the expected payoff depends on the previous trade by the n incumbents. It may be in the incumbents' interests to decrease the rents available at time 1 in order to prevent entry (*deter*). When entry costs are low, this can however be significantly costly, as incumbents must alter their trading strategies and tackle the spread more aggressively at time 0 than they would otherwise do. Hence incumbents may as well choose to *accommodate* and let the new arbitrageur enter. The following result describes the equilibrium.

Equilibrium

Proposition 6 *Suppose that* $s_0 = s_1 = s$ *and define* $\rho = \sqrt{\frac{I}{2a\sigma^2}}$.

If there is a monopolist arbitrageur at time 0, the new arbitrageur enters at time 1 if and only if $\rho < \hat{\rho}$. *If* $\rho > \hat{\rho}^{bmk}$, *the incumbent arbitrageur can deter at no cost.*

If there is an oligopoly of $n \geq 2$ *arbitrageurs at time 0:*

- *If* $\rho \in \left]0, \underline{\rho}\right]$, *incumbent arbitrageurs accommodate and the new arbitrageur enters.*

[29] See Perold (1999) and Lowenstein (2000).

[30] Kyrillos and Tufano (1994) note: "the basic ideas underlying the product were well described in the academic literature and could not be patented".

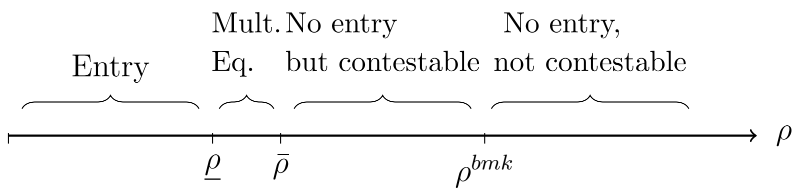


Figure 1.2: **Equilibrium with sequential entry and strategic deterrence**

- *If $\rho \in]\bar{\rho}, \infty)$, incumbents deter the new arbitrageur from entering. If $\rho > \rho^{bmk} > \bar{\rho}$, the incumbents deter the new arbitrageur at no cost.*
- *if $\rho \in \left]\underline{\rho}, \bar{\rho}\right]$, the two equilibria coexist.*

The equilibrium has a very intuitive form, summarized in Figure 1.2. When entry costs are low (or the arbitrage is very profitable because of high local investors' risk-aversion or equivalently large volatility of the fundamental), incumbents accommodate and the new arbitrageur enters at time 1. Given the low entry costs, deterring would carry a large opportunity cost. Indeed note that if entry costs are particularly high ($\rho > \rho^{bmk}$ or $\rho > \bar{\rho}^{bmk}$ in the monopoly case)), there is no need for incumbents to deter.

When there is an oligopoly of incumbents, the two types of equilibria may coexist. The multiplicity of equilibria stems from a coordination problem between arbitrageurs at time 0. For instance, when his current competitors decide to deter the new arbitrageur, an incumbent may decide between altering his optimal strategy to avoid losing market power, or sticking to it and keep the same market structure. Local investors play an interesting role for the outcome of the game, even though they take prices as given: if they anticipate entry, their liquidity demand at time 0 decreases, because they expect the market to be more competitive in the future. One can see this by comparing the price schedule for the spread between both assets faced by arbitrageurs at time 0. I show in the appendix that if local investors anticipate entry, the price impact coefficient decreases from $\frac{n+2}{n+1}$ to $\frac{n+3}{n+2}$. This optimal delaying of liquidity demand adds a self-fulfilling flavour to the equilibrium that generates the multiplicity: the anticipation of entry by local investors makes it harder for an incumbent to decrease the spread and deter the entrant, which, in turn, makes the anticipation more likely to realize.

When does entry occur in equilibrium? Proposition 6, shows that entry costs are the key driver of the equilibrium. But the thresholds $\underline{\rho}$, $\bar{\rho}$ and ρ^{bmk} are functions of the liquidity shocks s and the number of incumbent arbitrageurs n. Hence it is possible to calculate the likelihood that the market remain concentrated as a function of the number of incumbent arbitrageurs. The model delivers an interesting prediction regarding the persistence of market concentration over time:

Corollary 11 *If I, a, σ, s and n are such that $\rho < \underline{\rho}$, i.e. entry occurs in equilibrium, then, all else equal, a decrease in the number of incumbent arbitrageurs n does not modify the equilibrium.*

If parameters are such that $\rho \geq \bar{\rho}$, i.e. the incumbent arbitrageurs deter at no cost in equilibrium, then all else equal, a decrease in the number of incumbent arbitrageurs can shift the equilibrium towards entry, multiple equilibria, or deterrence. The more concentrated the market initially is, the more likely it is that the equilibrium remains deterrence (assuming uniform distribution for ρ).

Intuitively, there are two effects driving this comparative statics. On one hand, a smaller number of incumbents means that there are larger rents available, which for a given entry cost will make entry more profitable for the new arbitrageur. On the other hand, a small number of incumbent arbitrageurs makes coordination on deterrence easier to achieve. Figure 1.11 illustrates the second effect: I plot the likelihood of the accommodate and deter equilibria, assuming that the variable ρ is uniformly distributed between 0 and ρ^{bmk}, i.e. I normalize entry cost to be always sufficiently low, so it is not possible to deter at no cost. The graph shows that concentration benefits coordination among incumbents to deter new entrants: the light-shaded grey area, which represents the deter equilibrium, is largest when n is small.

To take into account the first effect, I construct a different comparative statics. Starting from given entry costs, number of incumbents and determinants of arbitrage profitability, I compare the equilibrium that results from an increase in market concentration for the different regions given in Proposition 6. Since $\underline{\rho}$ is decreasing in n, for a given level of entry cost such that $\rho < \underline{\rho}$, an increase in market concentration can only widen the region in which entry is the equilibrium. In this region, the first effect always dominate. Given that the other thresholds $\bar{\rho}$, ρ^{bmk} are also decreasing in n, it is not necessarily the case when the initial situation is that $\rho \geq \bar{\rho}$ or $\rho \geq \rho^{bmk}$. A new equilibrium may arise following an increase in market concentration.

To calculate the probability of the new equilibrium, consider the following example where $\rho \geq \rho^{bmk}(n)$ for a given level entry costs, volatility, liquidity shock and number of incumbents. I assume that ρ is uniformly distributed between $\rho^{bmk}(n)$ and $\rho^{bmk}(n-1)$, with

$\rho^{bmk}(n) < \rho^{bmk}(n-1)$, and calculate the thresholds $\underline{\rho}(n-1)$ and $\bar{\rho}(n-1)$. I find that $\rho^{bmk}(n) < \bar{\rho}(n-1)$ and $\bar{\rho}(n) < \underline{\rho}(n-1)$, which implies that the equilibrium must shift if $\rho \in \left[\rho^{bmk}(n), \bar{\rho}(n-1)\right[$ or $\rho \in \left[\bar{\rho}(n), \underline{\rho}(n-1)\right[$. Using the uniform distribution assumption for ρ, I then calculate the probability of shifting to entry, multiple equilibria or remaining a deterrence equilibrium. My calculations show that the probability of remaining in the deterrence equilibrium increases as the market becomes more concentrated. For instance, when the number of arbitrageurs decreases from 4 to 3, and 3 to 2, the probabilities of the different equilibria are given in Table 1.1. Note that the assumption that ρ's distribution

Table 1.1: Probability of equilibrium change if $\rho > \rho^{bmk}$

Probability	From n=4 to n=3	From n=3 to n=2
Switch to Entry	0.44	0.41
Switch to Multiple	0.14	0.10
Remain Deterrence	0.42	0.49

has an upper bound understates the likelihood that the equilibrium remains deterrence. Overall, the key insight from Corollary 11 is that when entry costs are sufficiently low, the first effect dominates, and a decrease in market concentration makes it (all else equal) even more profitable for a new arbitrageur to enter. Instead, if entry costs are high enough, the second effect dominates and as the market becomes more concentrated, then the probability of remaining in the deterrence equilibrium increases as arbitrageurs are able to coordinate on deterring.

Implications for liquidity and predictions

Since reducing the available rents require to decrease the spread, strategic deterrence leads to an improvement in liquidity along several dimensions, provided entry costs are sufficiently low:

Corollary 12 *Liquidity, measured by the spread, improves at all dates for any $\rho < \rho^{bmk}$. Liquidity measured by the price impact coefficient ("Kyle's lambda") improves at time 0 for any $\rho \leq \underline{\rho}$.*

The speed of convergence towards the fundamental value is higher than in the benchmark case without entrant if $\rho \leq \underline{\rho}$.

Panel (c) of Figure 1.11 plots the time-0 spread as a function of ρ. A similar pattern would emerge at time 1. If there is entry, the spread remains strictly positive as I assumed that there is only one entrant.[31] Corollary 12 shows that liquidity improves in terms of the spread, the price impact and speed of convergence if and only if $\rho < \underline{\rho}$, i.e. when there is entry in equilibrium. Hence the model predicts that faster price convergence and a deeper market should *lead* the increase in the number of traders. If entry is simply a threat, the model predicts only a decrease in the spread. When arbitrageurs deter the entrant, the spread exhibits surprising properties:

Corollary 13 *For $\rho \in \left[\bar{\rho}, \rho^{bmk}\right[$, the spread increases with the entry cost I at all dates and with the number of arbitrageurs at time 0, and is independent of the liquidity shock s.*

A decrease in entry costs can cause a discontinuous increase in liquidity (measured by the spread and the price impact).

Based on Corollary 13, the model predicts that when entry costs are in the middle range (i.e. when the market can be contested), the spread should be higher at time 0 in more competitive markets. The intuition for this result is that more competition exhausts rents available for the entrant, meaning that the incumbents need not decrease the spread as intensively at time 0. Of course, on this interval, there is substitutability between the size of entry costs and the deterrence, meaning that the spread will decrease the lower the entry costs are. Because of the multiplicity of equilibria, there is an interval in which the equilibrium is undetermined. A small decrease in entry costs can then lead to downward, discontinuous jump in the spread, as illustrated by Figure 1.11.

1.5 Conclusion

In this chapter, I study an asset pricing model in which market segmentation and demand pressure effects cause the prices of two identical assets to diverge. Only a small number of strategic arbitrageurs can exploit this price difference and internalize their impact on asset prices. I show that this results in rationing of liquidity, a gradual convergence of prices towards fundamentals, and, when the demand pressures decrease, an inversion of the spread between the two assets. I also highlight the role of the market structure of risk-bearing capacity, i.e. the interaction between the market structure and the risk-bearing capacity of the economy, as a determinant of asset prices dynamics and the arbitrageurs' entry decisions.

[31] With an arbitrary number of potential entrants, the spread would certainly be lower. Discontinuities due to integer issues may however prevent the spread from falling to 0, even when the entry costs converge to 0.

The model shows that arbitrageurs do not necessarily enter more aggressively in markets with more volatile price differences, although this increases potential rents from liquidity provision. This holds in particular if the market structure of risk-bearing capacity is not strong enough. Another interesting conclusion is that the mere threat of entry can improve liquidity (i.e. reduce price differences), and that the prospect of future entry also improves market depth ex-ante.

An important feature of real-world arbitrage this chapters abstracts from is that trading requires capital. In the next chapter, I study how the market structure interacts with financial constraints. I focus on the case of constant profitability and show that relative to a competitive market, a monopolistic arbitrageur provides less liquidity but also operates with less capital in equilibrium. As a result, when capital is scarce in the economy, the monopoly may provide more liquidity than the competitive market at certain dates. These results stress the importance of analyzing arbitrage as a strategic choice for our understanding of capital markets.

1.6 Proofs and Figures

1.6.1 Two useful results

Lemma 2 *Let* $(p,q)\in\mathbb{R}^2$ *with* $p\neq q$ *and consider the* (n,n) *matrix*

$$M_n=\begin{pmatrix} p & q & \cdots & q\\ q & p & \cdots & q\\ \vdots & \vdots & \ddots & \vdots\\ q & q & \cdots & p\end{pmatrix}$$

M_n *is invertible and its inverse is given by:*

$$M_n^{-1}=\frac{1}{(p-q)\,(p+(n-1)\,q)}\begin{pmatrix} p+(n-2)\,q & -q & \cdots & -q\\ -q & p+(n-2)\,q & \cdots & -q\\ \vdots & \vdots & \ddots & \vdots\\ -q & -q & \cdots & p+(n-2)\,q\end{pmatrix}$$

Proof. M_n being a square matrix with independent lines and columns, it is invertible. It is straightforward to check that $M_n.M_n^{-1}=M_n^{-1}.M_n=I$. ■

Lemma 3 *Let* $(A,B)\in\mathbb{R}^2$ *and* $X\sim\mathcal{N}\left(\mu,\sigma^2\right)$, *then*

$$\mathbb{E}\left(\exp\left(-AX^2+BX\right)\right)=\frac{\exp\left(y\right)}{\sqrt{2A\sigma^2+1}},\ \textit{with}\ y=\tfrac{B^2\sigma^2+2\mu(B-A\mu)}{2(2A\sigma^2+1)}$$

.

Proof. Since X is normally distributed:

$$\begin{aligned}\mathbb{E}\left(\exp\left(-AX^2+BX\right)\right) &= \int_{-\infty}^{\infty}\frac{\exp\left(-Ax^2+Bx\right)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\frac{\exp\left(-\frac{2A\sigma^2+1}{2\sigma^2}x^2+\frac{\mu+\sigma^2B}{\sigma^2}x-\frac{\mu}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}\,\mathrm{d}x\end{aligned}$$

Rewrite the exponential in the integrand as $\exp\left(-\frac{(x-m)^2}{2z^2}\right)\exp(y)$. This gives, by identification of the terms:

$$\begin{aligned}\frac{1}{2z^2} &= \frac{2A\sigma^2+1}{2\sigma^2} \Rightarrow z^2 = \frac{\sigma^2}{2A\sigma^2+1}\\ \frac{m}{z^2} &= \frac{\mu+B\sigma^2}{\sigma^2} \Rightarrow m = \frac{\mu+B\sigma^2}{2A\sigma^2+1}\end{aligned}$$

This implies that $\frac{m^2}{2z^2} - y = \frac{\mu^2}{2\sigma^2}$, i.e. $y = \frac{B^2\sigma^2+2\mu(B-A\mu)}{2(2A\sigma^2+1)}$. Thus,

$$\mathbb{E}\left(\exp\left(-AX^2+BX\right)\right) = \frac{\exp(y)}{\sqrt{2A\sigma^2+1}}\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{(x-m)^2}{2z^2}\right)}{z\sqrt{2\pi}}\,\mathrm{d}x = \frac{\exp(y)}{\sqrt{2A\sigma^2+1}}$$

■

1.6.2 Risk-free arbitrage

Proposition 1

Proof. At each date, going backward, I first solve for the demand of local investors in markets A and B, and then solve for the arbitrageurs' optimal trades.

Time 1 - local investors' problem It is enough to solve for the demand of local investors in market A, as market B is the symmetric case, thus I drop the superscript A. At time 1, $W_2 = E_1 + Y_1 D_2 = E_0 - y_1 p_1 + s_1\epsilon_2 + Y_1 D_2$. s_1 is revealed to all agents at time 1.

The local investors' maximization problem is

$$V_1 = \max_{y_1}\mathbb{E}(W_2) = \max_{y_1} -\exp\left(-a\left(E_0 - y_1 p_1 + Y_1 D_1 - \frac{a\sigma^2}{2}(Y_1+s_1)^2\right)\right)$$

$$\begin{aligned}\text{From the FOC, } Y_1 + s_1 &= \frac{D_1 - p_1}{a\sigma^2} && (1.27)\\ \text{Using market-clearing, } p_1^A &= D_1 - a\sigma^2\left(s_1 - \sum_{i=1}^{n} X_1^{i,A}\right) && (1.28)\\ \text{By analogy, } p_1^B &= D_1 - a\sigma^2\left(-s_1 - \sum_{i=1}^{n} X_1^{i,B}\right) && (1.29)\end{aligned}$$

Arbitrageurs' problem at time 1

Starting from arbitrageurs' wealth at time 2 given by equation (1.3), and using the assumption of opposite positions, we can rewrite wealth as:

$$\begin{aligned} W_2^i &= B_0^i + x_1^i \left(p_1^B - p_1^A\right) \\ &= B_0^i + 2a\sigma^2 x_1^i \left(s_1 - \sum_{i=1}^n X_1^i\right), \end{aligned}$$

where the second line of the maximization problem comes from equations (1.28) and (1.29). **Note that by an abuse of notation, I use i both as a counting variable and to refer to arbitrageur i.** Arbitrageurs maximize their expected utility of wealth, thus:

$$J_1^i = \max_{x_1^i} \mathbb{E}\left[u\left(B_0^i + 2a\sigma^2 x_1^i \left(s_1 - \sum_{i=1}^n X_1^i\right)\right)\right]$$

We can write $\sum_{i=1}^n X_1^i = \sum_{i=1}^n X_0^i + \sum_{-i}^n x_1^{-i} + x_1^i = \sum_{i=1}^n x_0^i + \sum_{-i}^n x_1^{-i} + x_1^i$, where $-i$ denote all arbitrageurs but arbitrageur i, and solve for the zero of the first-order condition for each arbitrageur i:

$$\begin{aligned} 2x_1^i + \sum_{-i} x_1^{-i} &= s_1 - \sum_{i=1}^n x_0^i, \ i = 1, \ldots, n \\ \text{thus, } x_1^i + \sum_{i=1}^n x_1^i &= s_1 - \sum_{i=1}^n x_0^i, \ i = 1, \ldots, n \end{aligned}$$

Stacking the n equations together and using matrix notation gives:

$$A_n \tilde{x}_1 = \left(s_1 - \sum_{i=1}^n x_0^i\right) \mathbb{I},$$

where A_n is an (n, n) matrix with 2's on the diagonal and 1's elsewhere, $\tilde{x}_1$ is a $(n, 1)$ vector of trades, $\tilde{x}_1^{-1} = (x_1^1, \ldots, x_1^n)$ and $\mathbb{I}$ is the identity matrix. We can then use Lemma 3 to find A_n^{-1}, invert the system and get the equilibrium trade at time 1,

$$x_1^i = \frac{s_1 - \sum_i x_0^i}{n+1}, i = 1, \ldots, n \tag{1.30}$$

Then, plugging equation (1.30) into (1.28) and arbitrageur i's objective function gives the equilibrium price and value function in the time 1 subgame:

$$\begin{aligned} p_1^A &= D_1 - a\sigma^2 \frac{s_1 - \sum_{i=1}^n x_0^i}{n+1} \\ J_1^i &= u\left(B_0^i + 2a\sigma^2 \frac{\left(s_1 - \sum_{i=1}^n x_0^i\right)^2}{(n+1)^2}\right) \end{aligned}$$

Similarly, plugging the previous expression for p_1^A into (1.27) gives the local investors' equilibrium certainty equivalent in the subgame:

$$\begin{aligned} CE_1 &= E_0 + Y_0 p_1 + Y_1 (D_1 - p_1) - \frac{a\sigma^2}{2}(Y_1 + s_1)^2 \\ &= E_0 + Y_0 p_1 + \frac{a\sigma^2}{2}\frac{\left(s_1 - \sum_{i=1}^n x_0^i\right)^2}{(n+1)^2} - a\sigma^2 s_1 \frac{s_1 - \sum_{i=1}^n x_0^i}{n+1} \end{aligned}$$

with $E_0 = E_{-1} - x_0 p_0 + s_0 \epsilon_1 = -x_0 p_0 + s_0 \epsilon_1$, for simplicity.

Time 0 - local investors

Going backward and using the expression for their certainty equivalent and for E_0, the A investors' problem is:

$$V_0 = \max_{y_0} -\mathbb{E} \exp\left(-a\left(-y_0 p_0 + Y_0 p_1 + \frac{a\sigma^2}{2}\frac{\left(s_1 - \sum_i x_0^i\right)^2}{(n+1)^2} - a\sigma^2 s_1 \frac{s_1 - \sum_i x_0^i}{n+1} + s_0 \epsilon_1\right)\right)$$

Hence, evaluating the expectation,

$$\begin{aligned} V_0 = \max_{y_0} - \exp -a\left(-y_0 p_0 + Y_0\left(D - a\sigma^2 \frac{s_1 - \sum_i x_0^i}{n+1}\right) - \frac{a\sigma^2}{2}(Y_0 + s_0)^2\right) \\ . \exp -a\left(\frac{a\sigma^2}{2}\frac{\left(s_1 - \sum_i x_0^i\right)^2}{(n+1)^2} - \frac{a\sigma^2}{2} s_1 \frac{s_1 - \sum_i x_0^i}{n+1}\right) \end{aligned}$$

From the first-order condition,

$$\begin{aligned}
a\sigma^2 \left(Y_0 + s_0\right) &= D - a\sigma^2 \frac{s_1 - \sum_i x_0^i}{n+1} - p_0 \\
\text{i.e. } p_0 &= D - a\sigma^2 \frac{s_1 - \sum_i x_0^i}{n+1} - a\sigma^2 s_0 + a\sigma^2 \sum_i x_0^i \text{ by market-clearing} \\
\text{i.e. } p_0 &= D - a\sigma^2 \left(s_0 + \frac{s_1}{n+1}\right) + \frac{n+2}{n+1} a\sigma^2 \sum_i x_0^i \\
\text{By analogy, } p_0^B &= D + a\sigma^2 \left(s_0 + \frac{s_1}{n+1}\right) - \frac{n+2}{n+1} a\sigma^2 \sum_i x_0^i
\end{aligned}$$

where I also used the fact that $x_t^i = x_t^{i,A} = -x_t^{i,B}$ for the last equation. The spread between A and B is thus

$$\Delta_0 = p_0^B - p_0^A = 2a\sigma^2 \left(s_0 + \frac{s_1}{n+1} - \frac{n+2}{n+1} \sum_i x_0^i\right)$$

Time 0 - Arbitrageurs

Using this expression for Δ_0 and $B_0^i = B_{-1}^i - \sum_{k=A,B} x_0^{i,k} p_0^k = -x_0 \Delta_0$ (again I assume that $B_{-1}^i = 0$ for simplicity and wlog in this setting), arbitrageur i's problem is:

$$\begin{aligned}
J_0^i &= \max_{x_0^i} -\mathbb{E}_0 \exp -b \left(2a\sigma^2 x_0^i \hat{\Delta}_0 + 2a\sigma^2 \frac{\left(s_1 - \sum_i x_0^i\right)^2}{\left(n+1\right)^2}\right), \text{ with } \hat{\Delta}_0 = \tfrac{\Delta_0}{2a\sigma^2} \\
&= \max_{x_0^i} - \exp -b \left(2a\sigma^2 x_0^i \left(s_0 + \frac{s_1}{n+1} - \frac{n+2}{n+1} \sum_i x_0^i\right) + 2a\sigma^2 \frac{\left(s_1 - \sum_i x_0^i\right)^2}{\left(n+1\right)^2}\right)
\end{aligned}$$

From the first-order condition,

$$\begin{aligned}
s_0 + \frac{s_1}{n+1} - \frac{n+2}{n+1} \left(\sum_{i=1}^n x_0^i + x_0^i\right) &= \frac{2}{\left(n+1\right)^2} \left(s_1 - \sum_{i=1}^n x_0^i\right), i = 1, \ldots, n \\
\text{i.e. } \frac{n+2}{n+1} x_0^i + \frac{n^2+3n}{\left(n+1\right)^2} \sum_{i=1}^n x_0^i &= s_0 + \frac{n-1}{\left(n+1\right)^2} s_1
\end{aligned}$$

Stacking the n equations together and solving for the equilibrium using Lemma 3, I get after some simple algebra:

$$x_0^i = \frac{s_0}{\phi_n} + \frac{n-1}{\left(n+1\right)^2} \frac{s_1}{\phi_n}, \text{ with } \phi_n = \tfrac{n^3+4n^2+3n+2}{\left(n+1\right)^2}$$

The equilibrium quantities for x_1^i, Δ_0 and Δ_1 follow from plugging x_0^i into (1.30) and the price schedules. ■

Corollary 2

This result is proved in the proof of Proposition 1.

Corollary 1

Proof. If $s_0 = s_1 = s$, then after some algebra, we get:

$$\Delta_0 = \underbrace{\frac{4n^2+6n+4}{n^3+4n^2+3n+2}}_{\bar{\kappa}_{0,n}} s$$
$$\Delta_1 = \underbrace{\frac{n+2}{n^3+4n^2+3n+2}}_{\bar{\kappa}_{1,n}} s$$

Clearly, for any $n \geq 1$, $\Delta_0 > \Delta_1 > 0 = \Delta_2$, and $\frac{\partial \Delta_t}{\partial n} < 0$ (t=0,1). Further, $\frac{\Delta_1}{\Delta_0} = \frac{n+2}{4n^2+6n+4}$ is decreasing in n and $lim_{n\to\infty}\Delta_t = 0 (t = 0, 1)$.

In the more general case with s_0, s_1, $\lim_{n\to\infty}\phi_n = 0$ implies that Δ_0 and Δ_1 converge to 0 when n becomes large. ■

Corollary 3

Proof. The first comparative statics is straightforward.

For the second part of the result, note that $\Delta_1 \leq 0$ iff $s_1 \leq \frac{n^3+2n^2+n}{n^3+4n^2+3n+2} s_0 \equiv \alpha_n s_0$. The RHS is smaller than s_0 for any n.

I now calculate the first derivative of Δ_1 with respect to n:

$$\begin{aligned}
\frac{\partial\left(\frac{n}{(n+1)\phi_n}\right)}{\partial n} &= -\frac{n^4+2n^3+n^2-4n-2}{(n^3+4n^2+3n+2)^2} \\
\text{and } \frac{\partial \bar{\phi}_n}{\partial n} &= -\frac{n^6+6n^5+20n^4+46n^3+37n^2+16n+2}{(n+1)^2(n^3+4n^2+3n+2)^2}
\end{aligned}$$

Hence $\frac{\partial \Delta_1}{\partial n} \geq 0 \Leftrightarrow \kappa_n s_0 \geq s_1$, with $\kappa_n = \frac{n^6+4n^5+6n^4-9n^2-12n-2}{n^6+6n^5+20n^4+46n^3+37n^2+16n+2}$. Clearly, $\kappa_1 < 0$ and $\forall n \geq 2,\ 0 < n^6+4n^5+6n^4-9n^2-12n-2 < n^6+6n^5+20n^4+46n^3+37n^2+16n+2$, thus if $n \geq 2$, $\kappa_n \in]0,1[$.

■

1.6.3 Risky arbitrage

Proposition 2

Proof. At time 1, the problem is similar to the risk-free arbitrage case. From the proof of Proposition 1, recall that:

$$\begin{aligned}
p_1^A &= D_1 - a\sigma^2 \frac{s_1 - \sum_{i=1}^n x_0^i}{n+1} \\
J_1^i &= u\left(B_0^i + 2a\sigma^2 \frac{\left(s_1 - \sum_{i=1}^n x_0^i\right)^2}{(n+1)^2}\right) \\
CE_1 &= E_0 + Y_0 p_1 + Y_1\left(D_1 - p_1\right) - \frac{a\sigma^2}{2}\left(Y_1 + s_1\right)^2 \\
&= E_0 + Y_0 p_1 + \frac{a\sigma^2}{2}\frac{\left(s_1 - \sum_{i=1}^n x_0^i\right)^2}{(n+1)^2} - a\sigma^2 s_1 \frac{s_1 - \sum_{i=1}^n x_0^i}{n+1}
\end{aligned}$$

Hence, after rearranging terms,

$$\begin{aligned}
CE_1 = E_0 + Y_0\left(D + a\sigma^2 \frac{\sum_{i=1}^n x_0^i}{n+1}\right) + a\sigma^2 \frac{\left(\sum_{i=1}^n x_0^i\right)^2}{(n+1)^2} \\
-a\sigma^2 s_1\left(\frac{Y_0}{n+1} - \frac{n}{n+1}\sum_i x_0^i\right) - a\sigma^2 \frac{2n+1}{2(n+1)^2} s_1^2
\end{aligned} \tag{1.31}$$

with $E_0 = -y_0 p_0 + s_0 \epsilon_1$.

Time 0 - local investors

At time 0, the local investors in market A solve the following problem:

$$V_0 = \max_{y_0} \mathbb{E}^{s_1,\epsilon_1}\left[-\exp\left(-a\left(CE_1\right)\right)\right]$$

$$\begin{aligned}
\mathbb{E}^{s_1,\epsilon_1}\left[-\exp\left(-a\left(CE_1\right)\right)\right] = -\exp\left[-a\left(-y_0 p_0 + Y_0\left(D + a\sigma^2 \frac{\sum_{i=1}^n x_0^i}{n+1}\right) + a\sigma^2 \frac{\left(\sum_{i=1}^n x_0^i\right)^2}{(n+1)^2}\right)\right] \\
.\mathbb{E}^{s_1,\epsilon_1}\left[\exp\left(-a\left(-\frac{a\sigma^2}{2}\left(Y_0 + s_0\right)^2 - a\sigma^2 s_1\left(\frac{Y_0}{n+1} - \frac{n\sum_i x_0^i}{(n+1)^2}\right) - a\sigma^2 \frac{2n+1}{2(n+1)^2} s_1^2\right)\right)\right]
\end{aligned}$$

By Lemma 3, and setting $-A = a^2\sigma^2 \frac{2n+1}{2(n+1)^2}$, $B = a^2\sigma^2\left(\frac{Y_0}{n+1} - \frac{n\sum_i x_0^i}{(n+1)^2}\right)$, $\mu = \bar{s}_1$ and $\sigma_x^2 = z_1^2$, we have:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-As_1^2 + Bs_1\right)\right] &= r_a^{-\frac{1}{2}} \exp\left[\frac{1}{2r_a} C\right] \\
\text{with } C &= a^4\sigma^4 z_1^2 \left(\frac{Y_0}{n+1} - \frac{n}{(n+1)^2}\sum_i x_0^i\right)^2 + 2\bar{s}_1 a^2\sigma^2 \left(\left(\frac{Y_0}{n+1} - \frac{n}{(n+1)^2}\sum_i x_0^i\right)\right. \\
\text{and } r_a &= 1 - a^2\sigma^2 z_1^2 \frac{2n+1}{(n+1)^2}
\end{aligned}$$

Thus investors in market A solve the following problem:

$$\max_{y_0} -r_a^{-\frac{1}{2}} \exp -a\left(-x_0p_0 + Y_0\left(D + a\sigma^2\frac{\sum_i x_0^i}{n+1}\right) + a\sigma^2\frac{\left(\sum_i x_0^i\right)^2}{(n+1)^2} - \frac{a\sigma^2}{2}\left(Y_0 + s_0\right)^2\right)$$

$$.\exp -a\left(-\frac{a^3\sigma^4 z_1^2}{2r_a}\left(\frac{Y_0}{n+1} - \frac{n\sum_i x_0^i}{(n+1)^2}\right)^2 - \frac{a\sigma^2\bar{s}_1}{r_a}\left(\frac{Y_0}{n+1} - \frac{n\sum_i x_0^i}{(n+1)^2} + \frac{2n+1}{2(n+1)^2}\bar{s}_1\right)\right)$$

The FOC yields

$$D + a\sigma^2\frac{\sum_i x_0^i}{n+1} - a\sigma^2\left(X_0 + s_0\right) - \frac{a^3\sigma^4 z_1^2}{(n+1)\,r_a}\left(\frac{X_0}{n+1} - \frac{n\sum_i x_0^i}{(n+1)^2}\right) - \frac{a\sigma^2\bar{s}_1}{r_a} = p_0$$

By market-clearing: $Y_0 = -\sum_i x_0^i$, since $x_0 = X_0$ by the assumption of zero net supply of the asset and that arbitrageurs do not initially hold the risky asset. After regrouping terms, this gives:

$$\begin{aligned}
p_0 &= D - a\sigma^2\left(s_0 + \frac{\bar{s}_1}{(n+1)\,r_a}\right) + a\sigma^2\frac{n+2}{n+1}\left(1+\phi_a\right)\sum_i x_0^i \qquad (1.33) \\
\text{with } \phi_a &= \frac{a^2\sigma^2 z_1^2}{(n+1)^2\, r_a}
\end{aligned}$$

By symmetry, the price schedule faced by arbitrageurs in market B is:

$$p_0^B = D + a\sigma^2\left(s_0 + \frac{\bar{s}_1}{(n+1)\,r_a}\right) - a\sigma^2\frac{n+2}{n+1}\left(1+\phi_a\right)\sum_i x_0^i \qquad (1.34)$$

Arbitrageurs' problem at time 0

Arbitrageurs' value function:

$$\begin{aligned} J_0^i &= \max_{x_0^i} \mathbb{E}\left[u\left(B_0^i + 2a\sigma^2 \frac{s_1 - \sum_i x_0^i}{(n+1)^2}\right)\right] \\ &= \max_{x_0^i} -\mathbb{E}\left[\exp -b\left(x_0^i \Delta_0 + 2a\sigma^2 \frac{s_1^2 - 2s_1 \sum_i x_0^i + \left(\sum_i x_0^i\right)^2}{(n+1)^2}\right)\right] \end{aligned} \tag{1.35}$$

Using Lemma 3, I get

$$\mathbb{E}\left[\exp\left(-\frac{2ab\sigma^2}{(n+1)^2} s_1^2 + \frac{4ab\sigma^2 \sum_i x_0^i}{(n+1)^2}\right)\right] =$$
$$r_b^{-\frac{1}{2}} \exp -b\left(-\frac{8a^2 b\sigma^4 z_1^2 \left(\sum_i x_0^i\right)^2}{(n+1)^4 r_b} - \frac{\bar{s}_1}{(n+1)^2 r_b}\left(4a\sigma^2 \sum_i x_0^i - \frac{2a\sigma^2}{(n+1)^2}\bar{s}_1\right)\right) \tag{1.36}$$

From equations (1.33) and (1.34), I get $\Delta_0 = 2a\sigma^2\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a} - \frac{n+2}{n+1}(1+\phi_a)\sum_i x_0^i\right)$. Denoting $\hat{\Delta}_0 = \frac{\Delta_0}{2a\sigma^2}$, and using equations (1.35) and (1.36), and rearranging terms gives the value function stated in the proposition. ■

Corollary 4

Proof. Direct from Proposition 2. ■

Proposition 3

Proof. Using Proposition 2, the first-order condition gives, for all $i \in \{1, ..., n\}$:

$$s_0 + \frac{\bar{s}_1}{(n+1) r_a} - \frac{n+2}{n+1}(1+\phi_a)\left(\sum_i x_0^i + x_0^i\right) + 2(1-\phi_b)\frac{\sum_i x_0^i}{(n+1)^2} - 2\frac{\bar{s}_1}{(n+1)^2 r_b} = 0$$

Stacking the n equations together and using Lemma 2 to solve for the equilibrium, gives, after some algebra:

$$x_0^i = \frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a} - \frac{2\bar{s}_1}{(n+1)^2 r_b}}{\phi_n + (n+2)\phi_a + 2n\phi_b} \tag{1.37}$$

Note that the facts that $1-\phi_b < 1$ and $\frac{n+2}{n+1}(1+\phi_a) > \frac{n+2}{n+1} > 1$ ensures that the maximand is concave in x_0^i, which guarantees that the optimum is a maximum.

Using equations (1.33) and (1.37), one can get the equilibrium price of asset and the spread between assets B and A:

$$\begin{aligned}
\sum_i x_0^i &= \frac{n}{d}\left[s_0 + \frac{\bar{s}_1}{(n+1)\,r_a} - \frac{2\bar{s}_1}{(n+1)^2 r_b}\right] \\
p_0^A &= D - a\sigma^2\left(s_0 + \frac{\bar{s}_1}{(n+1)\,r_a}\right) + a\sigma^2\frac{n+2}{n+1}\left(1+\phi_a\right)\sum_i x_0^i \\
\Rightarrow p_0^A &= D - a\sigma^2\Phi_a s_0 - a\sigma^2\Phi_a\frac{\bar{s}_1}{(n+1)\,r_a} - 2a\sigma^2\left(1-\Phi_a\right)\frac{\bar{s}_1}{(n+1)^2 r_b} \\
\text{with } \Phi_a &= 1 - \frac{n\,(n+2)}{(n+1)\,d}\left(1+\phi_a\right) \\
d &= \phi_n + (n+2)\,\phi_a + 2n\phi_b
\end{aligned}$$

$$\begin{aligned}
\text{Note that } \Phi_a &= 1 - \frac{n\,(n+2)\,(1+\phi_a)}{(n+1)\left(\phi_n + (n+2)\,\phi_a + 2n\phi_b\right)} \\
&= \frac{\frac{n^2+n+2}{n+1} + n\,(n+2)\,\phi_a + 2n\,(n+1)\,\phi_b}{(n+1)\left(\phi_n + (n+2)\,\phi_a + 2n\phi_b\right)}
\end{aligned}$$

The second equation follows from the definition of ϕ_n given in Proposition 2. Since $(n+1)\,\phi_n > \frac{n^2+n+2}{n+1}$, $\Phi_a \in [0,1]$. ■

Corollary 5

Proof. From the expression of x_0^i given in Proposition 3,

$$\frac{\partial x_0^i}{\partial b} = \frac{\partial}{\partial b}\left[\frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a}}{\phi_n + (n+2)\,\phi_a + 2n\phi_b}\right] - 2\frac{\partial}{\partial b}\left[\frac{\frac{\bar{s}_1}{(n+1)^2 r_b}}{\phi_n + (n+2)\,\phi_a + 2n\phi_b}\right]$$

I first calculate the second term in brackets.

$$\begin{aligned}
\text{First, note that } (n+1)^2 r_b &= (n+1)^2 + 4ab\sigma^2 z_1^2 \\
\text{Thus } \frac{\bar{s}_1}{(n+1)^2 r_b} &= \frac{\bar{s}_1}{4ab\sigma^2 z_1^2\left[1+f(b)\right]}, \text{ with } f(b) = \frac{(n+1)^2}{4ab\sigma^2 z_1^2} \\
\Rightarrow \frac{\partial}{\partial b}\frac{1}{(n+1)^2 r_b} &= -\frac{1}{\left(4ab\sigma^2 z_1^2\right)^2}\frac{4a\sigma^2 z_1^2}{\left[1+f(b)\right]^2} = -\frac{1}{4ab^2\sigma^2 z_1^2\left[1+f(b)\right]^2} \qquad (1.38)
\end{aligned}$$

$$\begin{aligned}
\text{Second, } \frac{\partial \phi_b}{\partial b} &= \frac{\partial}{\partial b}\left(\frac{1}{1+f(b)}\right) = -\frac{f'(b)}{[1+f(b)]^2} \\
\text{Noting that } f'(b) &= -\frac{(n+1)^2}{b^2 4a\sigma^2 z_1^2} = -\frac{f(b)}{b} \\
\text{gives } \frac{\partial \phi_b}{\partial b} &= \frac{f(b)}{b[1+f(b)]^2} \qquad (1.39)
\end{aligned}$$

Hence, using equations (1.38) and (1.39), and the notation $d = \phi_n + (n+2)\phi_a + 2n\phi_b$, gives:

$$\begin{aligned}
\frac{\partial}{\partial b}\frac{\frac{\bar{s}_1}{(n+1)^2 r_b}}{\phi_n + (n+2)\phi_a + 2n\phi_b} &= \frac{\bar{s}_1}{d^2}\left[-\frac{d}{4ab^2\sigma^2 z_1^2[1+f(b)]^2} - \frac{2nf(b)}{b[1+f(b)]^2(n+1)^2 r_b}\right] \\
&= -\frac{\bar{s}_1}{4ab^2\sigma^2 z_1^2 d^2[1+f(b)]^2}\left(d + \frac{2n}{r_b}\right) \qquad (1.40)
\end{aligned}$$

The second line follows from the fact that $\frac{f(b)}{b(n+1)^2} = \frac{1}{4ab^2\sigma^2 z_1^2}$. I now turn to the first term in brackets:

$$\frac{\partial}{\partial b}\left[\frac{s_0 + \frac{\bar{s}_1}{(n+1)r_a}}{\phi_n + (n+2)\phi_a + 2n\phi_b}\right] = -\frac{2n\phi_b'\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right)}{d^2} = -\frac{2nf(b)\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right)}{bd^2[1+f(b)]^2} \qquad (1.41)$$

Comibining equations (1.41) and (1.40), noting that $\frac{f(b)}{b} = \frac{(n+1)^2}{4ab^2\sigma^2 z_1^2}$ and rearranging terms gives:

$$\frac{\partial x_0^i}{\partial b} = \frac{1}{2ab^2\sigma^2 z_1^2[1+f(b)]^2 d^2}\left[-n(n+1)^2\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right) + \bar{s}_1\left(d + \frac{2n}{r_b}\right)\right]$$

The rest of the corollary follows immediately. ■

Lemma 1

Proof. The result follows as a limit case of two lemmata:

Lemma 4 *It holds that:*

i) The sign of $\frac{\partial x_0^i}{\partial b}$ is independent of b.

ii) If $n \leq 2$, $d + \frac{2n}{r_b} - \frac{n(n+1)}{r_a} > 0$ if $a^2\sigma^2 z_1^2$ is small enough. Thus, in this case, $\frac{\partial x_0^i}{\partial b} \geq 0$ is equivalent to $\frac{\bar{s}_1}{s_0}$ large enough if $s_0 > 0$, and is always satisfied if $s_0 \leq 0$.

iii) If $n > 2$ or $n \leq 2$ and $a^2\sigma^2 z_1^2$ is large enough, $\frac{\partial x_0^i}{\partial b} \geq 0$ is equivalent to $\frac{\bar{s}_1}{|s_0|}$ small enough for $s_0 < 0$, and is never satisfied if $s_0 > 0$.

iv) $\frac{\partial x_0^i}{\partial b} \geq 0 \;\Rightarrow\; \frac{\partial^2 x_0^i}{\partial b^2} \leq 0.$

Proof. Recall from Corollary 5 that

$$\frac{\partial x_0^i}{\partial b} = \kappa \left[-n(n+1)^2 \left(s_0 + \frac{\bar{s}_1}{(n+1) r_a} \right) + \bar{s}_1 \left(d + \frac{2n}{r_b} \right) \right]$$

with $\kappa = \frac{1}{2ab^2\sigma^2 z_1^2 d^2 [1+f(b)]^2}$. Hence the sign of the derivative depends on the expression in parenthesis. Given that $d = \phi_n + (n+2)\phi_a + 2n\phi_b$, the terms in b are given by

$$\begin{aligned} 2n\phi_b + \frac{2n}{r_b} &= 2n \frac{4ab\sigma^2 z_1^2}{(n+1)^2 r_b} + \frac{2n}{r_b} \\ &= \frac{2n}{r_b}\left(1 + \frac{4ab\sigma^2 z_1^2}{(n+1)^2}\right) = \frac{2n}{r_b} r_b = 2n \end{aligned}$$

This proves i).

As a consequence, one can write:

$$d + \frac{2n}{r_b} - \frac{n(n+1)}{r_a} = \phi_n + (n+2)\phi_a - \frac{n(n+1)}{r_a} + 2n = \phi_n + 2n + \frac{(n+2)a^2\sigma^2 z_1^2 - n(n+1)^3}{(n+1)^2 r_a}$$

Developing and rearranging the terms,

$$\begin{aligned} d + \frac{2n}{r_b} - \frac{n(n+1)}{r_a} &= \frac{3n^3 + 8n^2 + 5n + 2}{(n+1)^2} + \frac{(n+2)a^2\sigma^2 z_1^2 - n(n+1)^3}{(n+1)^2 r_a} \\ &= \frac{-n^4 + 5n^2 + 4n + 2 - \frac{n\left(6n^3+18n^2+14n+4\right)}{(n+1)^2} a^2\sigma^2 z_1^2}{(n+1)^2 r_a} \end{aligned}$$

Note that if $n = 1$, the numerator equals $10 - \frac{42}{4} a^2\sigma^2 z_1^2$. From Assumption 1, $a^2\sigma^2 z_1^2 < \frac{4}{3}$. Hence, $d + \frac{2n}{r_b} - \frac{n(n+1)}{r_a} \geq 0$ iff a (or σ^2 or z_1^2) is small enough. The same applies if $n = 2$. If $n > 2$, $-n^4 + 5n^2 + 4n + 2 < 0$, thus $d + \frac{2n}{r_b} - \frac{n(n+1)}{r_a} < 0$. ii) and iii) follow.

Finally, I compute the second derivative of x_0^i with respect to b:

$$\frac{\partial^2 x_0^i}{\partial b^2} = \frac{-\frac{2nr_b'\bar{s}_1}{r_b^2} - \kappa_b'\left(-n(n+1)^2\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right) + \bar{s}_1\left(d + \frac{2n}{r_b}\right)\right)}{\kappa^2}$$

It is easy to see that $r_b' > 0$. Recall that $\kappa = 2ab^2\sigma^2 z_1^2 d^2 [1 + f(b)]^2$, with $f(b) = \frac{(n+1)^2}{4ab\sigma^2 z_1^2}$. $b^2 [1 + f(b)]^2$ increases with b. Further, d^2 and r_b increase with b. Hence, κ increases with

b. Given that $\frac{\partial x_0^i}{\partial b} \geq 0 \Leftrightarrow -n(n+1)^2\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right) + \bar{s}_1\left(d + \frac{2n}{r_b}\right) \geq 0$, $\frac{\partial x_0^i}{\partial b} \geq 0$ implies $\frac{\partial^2 x_0^i}{\partial b^2} \leq 0$. This proves iv). ■

Lemma 5 *Define the expected return of arbitrage* $r_{0,1} \equiv \mathbb{E}\left[\frac{\Delta_1}{\Delta_0} - 1\right]$, *then if*

$$\begin{cases} \frac{\partial x_0^i}{\partial b} \geq 0 \\ \bar{s}_1 \geq \frac{s_0}{\frac{n+2}{n+1}(1+\phi_a) - \frac{1}{(n+1)r_a}} \end{cases}, \text{ then } \frac{\partial r_{0,1}}{\partial b} \geq 0$$

Proof. Recall from the proof of Proposition 3 that

$$\begin{aligned} \Delta_0 &= 2a\sigma^2\left[s_0 + \frac{\bar{s}_1}{(n+1)r_a} - \frac{n+2}{n+1}(1+\phi_a)\sum_i x_0^i\right] \\ \Delta_1 &= \frac{2a\sigma^2}{n+1}\left(s_1 - \sum_i x_0^i\right) \end{aligned}$$

Thus using the definition of $r_{0,1}$:

$$\begin{aligned} \frac{\partial r_{0,1}}{\partial b} \geq 0 \quad &\Leftrightarrow \quad \sum_i \frac{\partial x_0^i}{\partial b}\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right) - \frac{n+2}{n+1}(1+\phi_a)\sum_i x_0^i \\ &+ \quad \frac{n+2}{n+1}(1+\phi_a)\sum_i \frac{\partial x_0^i}{\partial b}\frac{\bar{s}_1 - \sum_i x_0^i}{n+1} \geq 0 \\ &\Leftrightarrow \quad \sum_i \frac{\partial x_0^i}{\partial b}\left(-s_0 - \frac{\bar{s}_1}{(n+1)r_a} + \frac{n+2}{n+1}(1+\phi_a)\bar{s}_1\right) \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Note that } \frac{n+2}{n+1}(1+\phi_a) - \frac{1}{(n+1)r_a} &= \frac{(n+2)(n+1)^2 r_a + a^2\sigma^2 z_1^2(2n^2+4n+3) - (n+1)^3}{(n+1)^3 r_a} \\ &= \frac{n(n+1)^2 - (n-1)a^2\sigma^2 z_1^2}{(n+1)^3 r_a} \end{aligned}$$

Thus $\frac{n+2}{n+1}(1+\phi_a) - \frac{1}{(n+1)r_a} > 0 \Leftrightarrow a^2\sigma^2 z_1^2 < \frac{n(n+1)^2}{n-1}$. Note that $\frac{(n+1)^2}{2n+1} < \frac{n(n+1)^2}{n-1}$, hence Assumption 1 implies that $\frac{n+2}{n+1}(1+\phi_a) - \frac{1}{(n+1)r_a} > 0$. Therefore if $\frac{\partial x_0^i}{\partial b} \geq 0$ and $\bar{s}_1 > \frac{s_0}{\frac{n+2}{n+1}(1+\phi_a) - \frac{1}{(n+1)r_a}}$, then the derivative is positive . ■

The results of Lemma 1 obtain by taking $s_0 \to 0$ in Lemmata 4 and 5.

To plot Figure 1.10, I calculate $\mathbb{E}(\Delta_1) - \Delta_0$. The following lemma gives two useful results:

Lemma 6 *When* $s_0 \to 0$, $\Delta_0 > 0$.

If $n \leq 2$, $\mathbb{E}(\Delta_1)|_{s_0 \to 0} \geq 0$

$$\forall n \geq 1, \ \mathbb{E}(\Delta_1 - \Delta_0)|_{s_0 \to 0} = \frac{2an\sigma^2 \left[(n+2)(1+\phi_a) - 1\right]}{(n+1)^2 d} \left(\frac{1}{r_a} - \frac{2}{(n+1) r_b}\right) \bar{s}_1$$

Proof. The first result follows taking the limit of Δ_0 given in Proposition 3, when $s_0 \to 0$.

For the second result, let's start from equation (1.20):

$$\mathbb{E}(\Delta_1) = \frac{2a\sigma^2}{n+1} \left[-\frac{n}{d} s_0 + \left(1 - \frac{n}{(n+1)^2 d} \left(\frac{n+1}{r_a} - \frac{2}{r_b}\right)\right) \bar{s}_1\right]$$

We can prove that $1 - \frac{n}{(n+1)^2 d}\left(\frac{n+1}{r_a} - \frac{2}{r_b}\right) \in [0,1]$ when $n \leq 2$. Let's show first that 1 is an upper bound. Skipping some lines of algebra, I get:

$$1 - \frac{n}{(n+1) dr_a} + \frac{2n}{(n+1)^2 dr_b} < 1 \ \Leftrightarrow -\frac{4ab\sigma^2 z_1^2}{n+1} - \frac{2(2n+1)}{(n+1)^2} < n-1$$

The second inequality is always verified. Second, $1 - \frac{n}{(n+1)dr_a} + \frac{2n}{(n+1)^2 dr_b} > 0$ is equivalent to $\frac{(n+1)^2}{n} d > \frac{n+1}{r_a} - \frac{2}{r_b}$. Given that $d = \phi_n + (n+2)\phi_a + 2n\phi_b = \phi_n + (n+2)\frac{a^2\sigma^2 z_1^2}{(n+1)^2 r_a} + 2n\frac{4ab\sigma^2 z_1^2}{(n+1)^2 r_b}$, the inequality is equivalent to

$$\frac{(n+1)^2 \phi_n}{n} + \frac{(n+2) a^2\sigma^2 z_1^2 - n(n+1)}{(n+1)^2 r_a} + \frac{2(1 + 4ab\sigma^2 z_1^2)}{(n+1)^2 r_b} > 0$$

The third term is clearly positive. Thus it is sufficient that the sum of the first two terms is positive. Using the definition of ϕ_n, the sum of the first two terms is equal to:

$$\frac{(n+1)^2 \phi_n r_a + (n+2) a^2\sigma^2 z_1^2 - n(n+1)}{nr_a} =$$

$$\frac{(n^3 + 4n^2 + 3n + 2)\left(1 - \frac{2n+1}{(n+1)^2} a^2\sigma^2 z_1^2\right) + (n+2) a^2\sigma^2 z_1^2 - n(n+1)}{nr_a}$$

$$\text{The numerator is} \quad n^3+3n^2+2n+2+a^2\sigma^2 z_1^2\left(n+2-\frac{(2n+1)\left(n^3+4n^2+3n+2\right)}{(n+1)^2}\right)>0$$

$$\Leftrightarrow \qquad a^2\sigma^2 z_1^2<\frac{(n+1)^2\left(n^3+3n^2+2n+2\right)}{2n\left(n^3+4n^2+3n+1\right)}$$

Note that $\frac{(n+1)^2\left(n^3+3n^2+2n+2\right)}{2n(n^3+4n^2+3n+1)}-\frac{(n+1)^2}{2n+1}=\frac{(n+1)^2\left(-n^3+n^2+3n+2\right)}{2n(2n+1)(n^3+4n^2+3n+1)}$. This expression is strictly positive for $n=1$, and $n=2$, and negative for $n>2$. Hence for $n\leq 2$, Assumption 1 implies that $1-\frac{n}{(n+1)dr_a}+\frac{2n}{(n+1)^2 dr_b}>0$, which implies that $\mathbb{E}\left(\Delta_1\right)|_{s_0\to 0}\geq 0$. ■

Finally, I calculate the expected change of the spread $\mathbb{E}\left(\Delta_1-\Delta_0\right)$ using equations (1.19) and (1.20):

$$\begin{aligned}\mathbb{E}\left(\Delta_1-\Delta_0\right) &= 2a\sigma^2\left[\frac{1}{n+1}\left(1-\frac{n}{(n+1)\,dr_a}+\frac{2n}{(n+1)^2\,dr_b}\right)-\left(\frac{\Phi_a}{(n+1)\,r_a}+\frac{2\left(1-\Phi_a\right)}{(n+1)^2\,r_b}\right)\right]\bar{s}_1\\ &- 2a\sigma^2\left(\frac{n}{(n+1)\,d}+\Phi_a\right)s_0\end{aligned}$$

Recall that $\Phi_a=1-\frac{n(n+2)(1+\phi_a)}{(n+1)d}$, we have:

$$\mathbb{E}\left(\Delta_1-\Delta_0\right)|_{s_0\to 0}=\frac{2an\sigma^2\left[(n+2)\left(1+\phi_a\right)-1\right]}{(n+1)^2\,d}\left(\frac{1}{r_a}-\frac{2}{(n+1)\,r_b}\right)\bar{s}_1$$

■

Corollary 6

Proof. From the expression of the equilibrium spread (1.19), $\frac{\partial\Delta_0}{\partial\bar{s}_1}=2a\sigma^2\left[\frac{\Phi_a}{(n+1)r_a}+\frac{2(1-\Phi_a)}{(n+1)^2 r_b}\right]>0$ since $\Phi_a\in\left]0,1\right[$. Similarly, from equation (1.17),

$$\frac{\partial x_0^i}{\partial\bar{s}_1}=\frac{1}{d}\left(\frac{1}{(n+1)\,r_a}-\frac{2}{(n+1)^2\,r_b}\right),\text{with } d=\phi_n+(n+2)\,\phi_a+2n\phi_b$$

Replacing r_a and r_b by their expressions, and rearranging terms, this simplifies into:

$$\frac{\partial x_0^i}{\partial\bar{s}_1}=\frac{n-1+\frac{4ab\sigma^2 z_1^2}{n+1}+2a\sigma^2 z_1^2\frac{2n+1}{(n+1)^2}}{d}>0,\text{for any } n\geq 1$$

■

Corollary 7

Proof. Since both the denominator and the numerator of x_0^i depend on z_1^2, I first calculate the derivative of each part. Starting from the expression of the equilibrium trade

x_0^i given by (1.17), and the definition of r_a, ϕ_a, r_b and ϕ_b given by equations (1.15) and (1.16), we get:

$$\frac{\partial r_a}{\partial z_1^2} = -a^2\sigma^2\frac{2n+1}{(n+1)^2} < 0; \quad \frac{\partial \phi_a}{\partial z_1^2} = \frac{a^2\sigma^2(n+1)^2 r_1 + a^2\sigma^2\frac{2n+1}{(n+1)^2}a^2\sigma^2 z_1^2}{(n+1)^2 r_a^2} = \frac{a^2\sigma^2}{(n+1)^2 r_a^2} > 0$$

$$\frac{\partial r_b}{\partial z_1^2} = \frac{4ab\sigma^2}{(n+1)^2} > 0, \quad \frac{\partial \phi_b}{\partial z_1^2} = \frac{4ab\sigma^2(n+1)^2 r_b - \frac{4ab\sigma^2}{(n+1)^2}4ab\sigma^2 z_1^2}{(n+1)^4 r_b^2} = \frac{4ab\sigma^2}{(n+1)^2 r_b^2} > 0$$

This implies that the derivative of the numerator of x_0^i is:

$$\begin{aligned}
\partial\frac{\frac{\bar{s}_1}{(n+1)r_a} - \frac{2\bar{s}_1}{(n+1)^2 r_b}}{\partial z_1^2} &= \frac{\bar{s}_1}{n+1}\left[\frac{\partial r_a^{-1}}{\partial z_1^2} - \frac{2}{n+1}\frac{\partial r_b^{-1}}{\partial z_1^2}\right] \\
&= \frac{\bar{s}_1}{n+1}\left[\frac{(2n+1)a^2\sigma^2}{(n+1)^2 r_a^2} + \frac{8ab\sigma^2}{(n+1)^3 r_b^2}\right]
\end{aligned}$$

Similarly, the derivative of the denominator of x_0^i is:

$$\frac{\partial(\phi_n + (n+2)\phi_a + 2n\phi_b)}{\partial z_1^2} = \frac{(n+2)a^2\sigma^2}{(n+1)^2 r_a^2} + 2n\frac{4ab\sigma^2}{(n+1)^2 r_b^2}$$

Thus combining both derivatives and using the notation $d = \phi_n + (n+2)\phi_a + 2n\phi_b$, I get:

$$\frac{\partial x_0^i}{\partial z_1^2} = \frac{\frac{d\bar{s}_1}{n+1}\left[\frac{(2n+1)a^2\sigma^2}{(n+1)^2 r_a^2} + \frac{8ab\sigma^2}{(n+1)^3 r_b^2}\right] - \left(s_0 + \frac{\bar{s}_1}{(n+1)r_a} - \frac{2\bar{s}_1}{(n+1)^2 r_b}\right)\left(\frac{(n+2)a^2\sigma^2}{(n+1)^2 r_a^2} + 2n\frac{4ab\sigma^2}{(n+1)^2 r_b^2}\right)}{d^2}$$

Thus $\frac{\partial x_0^i}{\partial z_1^2} \geq 0$ iff

$$\frac{d\bar{s}_1}{n+1}\left[\frac{(2n+1)a^2\sigma^2}{(n+1)^2 r_a^2} + \frac{8ab\sigma^2}{(n+1)^3 r_b^2}\right] \geq \left(s_0 + \frac{\bar{s}_1((n+1)r_b - 2r_a)}{(n+1)^2 r_a r_b}\right)\left(\frac{(n+2)a^2\sigma^2}{(n+1)^2 r_a^2} + 2n\frac{4ab\sigma^2}{(n+1)^2 r_b^2}\right)$$

Now let's consider the limit case where arbitrageurs become risk-neutral, $b \to 0$. The previous condition becomes:

$$\frac{\partial x_0^i}{\partial z_1^2} \geq 0 \iff \frac{d^{b\to 0}\bar{s}_1}{n+1}\frac{(2n+1)a^2\sigma^2}{(n+1)^2 r_a^2} \geq \left(s_0 + \frac{\bar{s}_1(n+1-2r_a)}{(n+1)^2 r_a}\right)\frac{(n+2)a^2\sigma^2}{(n+1)^2 r_a^2}$$

Since $d^{b\to 0} = \phi_n + (n+2)\,\phi_a$, we can rearrange terms and get:

$$\text{If } b \to 0,\ \frac{\partial x_0^i}{\partial z_1^2} \geq 0 \ \Leftrightarrow\ \underbrace{\left(\frac{(2n+1)\,(\phi_n + (n+2)\,\phi_a)}{(n+1)\,(n+2)} - \frac{n+1-2r_a}{(n+1)^2\,r_a}\right)}_{\Theta}\bar{s}_1 \geq s_0$$

After a simple calculation, I get $\Theta > 0$ iff $a^2\sigma^2 z_1^2 < c_n \equiv \frac{(n+1)^2}{2n+1}\frac{(n+1)(2n+1)\phi_n+n+3-(n+1)^2}{(n+1)(2n+1)\phi_n+n+3}$. Clearly, $\frac{(n+1)(2n+1)\phi_n+n+3-(n+1)^2}{(n+1)(2n+1)\phi_n+n+3} < 1$, thus $c_n < \frac{(n+1)^2}{2n+1}$. Hence there are two cases:

- If $c_n \leq a^2\sigma^2 z_1^2 < \frac{(n+1)^2}{2n+1}$, then $\Theta < 0$ and $\frac{\partial x_0^i}{\partial z_1^2} < 0$

- If $0 < a^2\sigma^2 z_1^2 < c_n$, then $\Theta > 0$ and $\frac{\partial x_0^i}{\partial z_1^2} \geq 0$ iff $s_1 > \Theta^{-1} s_0$

In the result, I use the notation $\theta_{n,a} = \Theta^{-1}$.

To derive the comparative statics of the spread, I start from the expression of the spread schedule (1.14), and get:

$$\frac{\partial \Delta_0}{\partial z_1^2} = 2a\sigma^2\left[\frac{\partial\left(s_0 + \frac{\bar{s}_1}{(n+1)r_a}\right)}{\partial z_1^2} - \frac{n+2}{n+1}\,(1+\phi_a)\,\frac{\partial\left(\sum_{i=1}^n x_0^i\right)}{\partial z_1^2} - \frac{n+2}{n+1}\sum_{i=1}^n x_0^i\frac{\partial \phi_a}{\partial z_1^2}\right]$$

When $b \to 0$, this gives:

$$\frac{\partial \Delta_0}{\partial z_1^2} = \frac{2n+1}{(n+1)^3\,r_a^2}a^2\sigma^2\bar{s}_1 - \frac{n+2}{n+1}\,(1+\phi_a)\,\frac{\frac{d\bar{s}_1}{n+1}a^2\sigma^2\frac{2n+1}{(n+1)^2 r_a^2} - \left(s_0 + \frac{\bar{s}_1(n+1-2r_a)}{(n+1)^2 r_a}\right)\frac{(n+2)a^2\sigma^2}{(n+1)^2 r_a^2}}{d^2}$$
$$-\frac{n+2}{n+1}\frac{a^2\sigma^2}{(n+1)^2\,r_a^2}\frac{n\left(s_0 + \frac{\bar{s}_1(n+1-2r_a)}{(n+1)^2 r_a}\right)}{d}$$

After rearranging terms, I get:

$$\frac{\partial \Delta_0}{\partial z_1^2} = \alpha a^2\sigma^2\bar{s}_1 + \beta a^2\sigma^2\left(s_0 + \frac{\bar{s}_1\,(n+1-2r_a)}{(n+1)^2\,r_a}\right), \text{with}$$

$$\begin{aligned}
\alpha &= \frac{2n+1}{(n+1)^3 r_a^2} - \frac{(n+2)(2n+1)}{(n+1)^4 r_a^2 d}(1+\phi_a) = \frac{2n+1}{(n+1)^4 r_a^2 d}\left[\frac{n^3+2n^2+n}{n+1} + (n+2)(n-1)\phi_a\right] > 0 \\
\beta &= \frac{(n+2)^2(1+\phi_a)}{(n+1)^3 r_a^2} - \frac{n(n+2)}{(n+1)^3 r_a^2 d} \\
&= \frac{n+2}{(n+1)^3 r_a^3 d}\left[\underbrace{(n+2)\phi_n - n}_{>0} + \phi_a (n+2)(\phi_n + (n+2)\phi_a + n + 2)\right]
\end{aligned}$$

Hence with α and β strictly positive, the derivative is positive. ■

Corollary 8

Proof. Follows from Proposition 3. ■

1.6.4 Entry

Simultaneous (free) entry

Risk-free arbitrage **Proposition 4**

Proof. From Proposition 1, I calculate the arbitrageurs' certainty equivalent in equilibrium. Skipping a few lines of algebra, this gives, assuming w.l.o.g. that $B_0^i = 0$:

$$\begin{aligned}
CE_0^i &= 2a\sigma^2\left[\omega_0 s_0^2 + \omega_1 s_1^2 + \omega_{0,1} s_0 s_1\right] \\
\text{with } \omega_0 &= \frac{n^4+3n^3+6n^2+5n+2}{(n^3+4n^2+3n+2)^2} \\
\omega_1 &= \frac{4n^6+14n^5+21n^4+22n^3+21n^2+14n+4}{(n+1)^2(n^3+4n^2+3n+2)^2} \\
\omega_{0,1} &= \frac{2n^4+5n^3+8n^2+3n+2}{(n^3+4n^2+3n+2)^2}
\end{aligned}$$

Note that $\omega_0+\omega_1+\omega_{0,1} = \frac{7n^6+28n^5+54n^4+66n^3+55n^2+30n+8}{(n+1)^2(n^3+4n^2+3n+2)^2} \equiv \pi_n$. Thus assuming that $s_0 = s_1 = s$ and $s \sim \mathcal{N}(\bar{s}, z^2)$, we have by Lemma 3 (with "B"$= 0$):

$$-\mathbb{E}\left(\exp -2ab\sigma^2\pi_n s^2\right) = -\frac{\exp\left[-\frac{2ab\sigma^2\bar{s}^2\pi_n}{1+4ab\sigma^2 z^2\pi_n}\right]}{(1+4ab\sigma^2 z^2\pi_n)^{\frac{1}{2}}}$$

Let us denote $\theta = -2ab\sigma^2\pi_n$.

$$\tilde{J}_0^i = -\frac{\exp\left(\frac{\theta\bar{s}^2}{1-2\theta z^2} + bI\right)}{\sqrt{1-2\theta z^2}} \tag{1.42}$$

$\tilde{J}$ must be compared to the payoff from the alternative, which consists in doing nothing and brings the agent a utility of $-\exp(0) = -1$. To see that $\tilde{J}$ is decreasing in n, note that $\tilde{J}$ is decreasing in θ. Since θ decreases with π_n and π_n decreases with n, θ increases with n, i.e. $\tilde{J}$ decreases with n. ∎

Corollary 9

Proof. $\tilde{J}$ decreases with I, hence $\tilde{J}^0$ decreases with I. Since $\theta = -2ab\sigma^2\pi_n$ increases with n, n^* decreases with I. Since $\tilde{J}$ increases with $\bar{s}$, the same logic applies and n^* increases with $\bar{s}$. Further, θ decreases with a and σ^2. Since $\tilde{J}$ increases in both a and σ^2, $\tilde{J}^{-1}$ also does and therefore n^* increases in a and σ^2.

The comparative static with respect to z requires to calculate the derivative. Using equation (1.42), and noting that

$$\frac{\partial \frac{\theta\bar{s}^2}{1-2\theta z^2}}{\partial z^2} = \frac{2\theta^2\bar{s}^2}{\left(1-2\theta z^2\right)^2}$$

$$\begin{aligned}
\frac{\partial \tilde{J}_0^i}{\partial z^2} &= -\frac{\frac{2\theta^2\bar{s}^2\sqrt{1-2\theta z^2}}{\left(1-2\theta z^2\right)^2}\exp\left(\frac{\theta\bar{s}^2}{1-2\theta z^2}\right) + \frac{\theta}{\sqrt{1-2\theta z^2}}\exp\left(\frac{\theta\bar{s}^2}{1-2\theta z^2}\right)}{1-2\theta z^2} \\
&= -\frac{\exp\left(\frac{\theta\bar{s}^2}{1-2\theta z^2}\right)}{\left(1-2\theta z^2\right)^{\frac{3}{2}}}\left[\frac{2\theta^2\bar{s}^2}{1-2\theta z^2} + \theta\right]
\end{aligned}$$

Hence $\frac{\partial \tilde{J}_0^i}{\partial z^2} \leq 0 \;\Leftrightarrow\; \frac{2\theta^2\bar{s}^2}{1-2\theta z^2} + \theta \geq 0 \;\Leftrightarrow\; \frac{2\theta\bar{s}^2}{1-2\theta z^2} + 1 \leq 0$, because $\theta < 0$. This is equivalent to

$$\bar{s}^2 - z^2 \geq \frac{-1}{2\theta} = \frac{1}{4ab\sigma^2\pi_n}$$

The condition is satisfied if a or b or σ^2 are large, or if π_n is large, i.e. if n is small. Since $\tilde{J}$ decreases with n, n^* decreases with z if θ is large enough. θ is large when a or b or σ^2 are large or n is small. ∎

Corollary 10

Proof. ϕ_n and $\bar{\psi}_n$ are given in Proposition 1. It is easy to see that they are decreasing in n. Since n^* is increasing in a, ψ_{n^*} and $\bar{\psi}_{n^*}$ are decreasing in a. ∎

Risky arbitrage

Proposition 5

Proof. To calculate arbitrageurs' certainty equivalent in equilibrium, recall from the proof of Proposition 3 that

$$\begin{aligned}
x_0^i &= \frac{s_0}{d} + \frac{(n+1)\,r_b - 2r_a}{(n+1)^2\,r_a r_b d}\bar{s}_1 \\
\sum_i x_0^i &= \frac{n}{d}s_0 + \frac{n\,(n+1)\,r_b - 2nr_a}{(n+1)^2\,r_a r_b d}\bar{s}_1 \\
\hat{\Delta}_0 &= \Phi_a s_0 + \frac{(n+1)\,r_b\Phi_a + (1-\Phi_a)\,2r_a}{(n+1)^2\,r_a r_b}\bar{s}_1 \\
\text{with } \Phi_a &= 1 - \frac{n\,(n+2)}{n+1}\frac{1+\phi_a}{d}
\end{aligned}$$

Thus, skipping some tedious algebra, arbitrageurs' certainty equivalent is given by

$$\begin{aligned}
CE_0^i &= x_0^i\hat{\Delta}_0 + (1-\phi_b)\,\frac{\left(\sum_i x_0^i\right)^2}{(n+1)^2} - \frac{\bar{s}_1}{(n+1)^2\,r_b}\left(2\sum_i x_0^i - \bar{s}_1\right) \\
&= \pi_0 s_0^2 + \pi_1 s_1^2 + \pi_{0,1} s_0 \bar{s}_1
\end{aligned}$$

$$\begin{aligned}
\text{with } \pi_0 &= \frac{\Phi_a}{d} + \frac{1-\phi_b}{(n+1)^2}\frac{n^2}{d^2} && (1.43) \\
&= \frac{(n+1)^2\,\phi_n - n\,(n^2+2n+2) + (n+1)\,(n+2)\,\phi_a + n\,(2n^2+2n+2)\,\phi_b}{(n+1)^2\,d^2} \\
\pi_1 &= \frac{\left[(n+1)\,r_b - 2r_a\right]^2}{(n+1)^4\,r_a^2 r_b^2 d^2}\left[\Phi_a d + \frac{n^2\,(1-\phi_b)}{(n+1)^2}\right] \\
&- 2\,(n-1)\,\frac{(n+1)\,r_b - 2r_a}{(n+1)^4\,r_a r_b^2 d} + \frac{1}{(n+1)^2\,r_b} && (1.44) \\
\pi_{0,1} &= \frac{2\left[(n+1)\,r_b - 2r_a\right]}{(n+1)^2\,r_a r_b d}\left[\Phi_a + \frac{n}{(n+1)^2\,d}\right] - \frac{2\,(n-1)}{(n+1)^2\,r_b d} && (1.45)
\end{aligned}$$

Therefore at time 0 arbtrageurs' equilibrium utility is

$$\begin{aligned} J_0^i &= -r_b^{-\frac{1}{2}} \exp\left(-2ab\sigma^2\left(\pi_0 s_0^2 + \pi_1 s_1^2 + \pi_{0,1} s_0 \bar{s}_1\right)\right) \\ &= -r_b^{-\frac{1}{2}} \exp\left(-2ab\sigma^2 \pi_1 \bar{s}_1\right) \exp\left(-2ab\sigma^2 \pi_0 s_0^2 - 2ab\sigma^2 \pi_{0,1} s_0 \bar{s}_1\right) \end{aligned}$$

Hence assuming that $s_0 \sim \mathcal{N}\left(\bar{s}_0, z_0^2\right)$, and using Lemma 3:

$$\begin{aligned} \mathbb{E}_{-1}\left[\exp\left(-2ab\sigma^2\pi_0 s_0^2 - 2ab\sigma^2\pi_{0,1}s_0\bar{s}_1\right)\right] &= \frac{\exp\left(\frac{2a^2b^2\sigma^4\pi_{0,1}^2\bar{s}_1^2 z_0^2 + \bar{s}_0\left(-2ab\sigma^2\pi_{0,1}\bar{s}_1 - 2ab\sigma^2\pi_0\bar{s}_0\right)}{1+4ab\sigma^2\pi_0 z_0^2}\right)}{\sqrt{1+4ab\sigma^2\pi_0 z_0^2}} \\ &= \frac{\exp\left(-2ab\sigma^2\frac{\bar{s}_0(\pi_{0,1}\bar{s}_1 + \pi_0\bar{s}_0) - ab\sigma^2\pi_{0,1}^2\bar{s}_1^2 z_0^2}{1+4ab\sigma^2\pi_0 z_0^2}\right)}{\sqrt{1+4ab\sigma^2\pi_0 z_0^2}} \end{aligned}$$

This implies that arbitrageurs' expected utility at time -1 (net of entry cost) is:

$$\hat{J}_0^i = -\frac{\exp\left[-\eta\left(\frac{\pi_{0,1}\left(\bar{s}_0\bar{s}_1 - \frac{\eta}{2}\pi_{0,1}\bar{s}_1^2 z_0^2\right) + \pi_0\bar{s}_0^2}{1+2\eta\pi_0 z_0^2} + \pi_1\bar{s}_1^2\right)\right]\exp(bI)}{\sqrt{r_b\left(1+2\eta\pi_0 z_0^2\right)}}, \quad \text{with } \eta = 2ab\sigma^2 \tag{1.46}$$

■

1.6.5 Gradual entry

Proposition 6

Proof.

I start with the oligopoly case ($n \geq 2$). The proposition is based on the following three results.

Proposition 7 *Arbitrageurs accommodate entry iff $\rho < \bar{\rho}$. There is no accommodate equilibrium if $\rho > \rho^{acc} > \bar{\rho}$. The equilibrium trades are:*

$$\begin{aligned} \forall i = 1, \ldots, n,\ x_0^i &= \frac{(n+1)(n+4)}{n^3+6n^2+9n+6}s \\ \forall i = 1, \ldots, n+1,\ x_1^i &= \frac{s - \sum_i x_0^i}{n+2} \end{aligned}$$

Proof. At time 1, following arguments given in the proof of Proposition 1, the payoff (certainty equivalent) of the subgame is given by $2a\sigma^2\frac{\left(s-\sum_j x_0^j\right)^2}{(n+2)^2}$. Hence, the new arbitrageur enters iff

$$\frac{\left(s-\sum_j x_0^j\right)^2}{\left(n+2\right)^2} \geq \frac{I}{2a\sigma^2} = \rho^2 \tag{1.47}$$

In the benchmark case with no gradual entry, the equilibrium trades are (from Proposition 1):

$$x_0^i = \frac{n^2+3n}{n^3+4n^2+3n+2}s$$

This implies that $s - \sum_j x_0^j = \frac{(n+1)(n+2)}{n^3+4n^2+3n+2}s$. Plugging this expression into condition 1.47 shows that at time 1, if $\rho > \rho^{bmk} \equiv \frac{n+1}{n^3+4n^2+3n+2}s$, it is not profitable for the new arbitrageur to enter. Thus incumbent arbitrageurs can deter the entrant without altering their trading strategy, i.e. at no cost.

Let's now assume that local investors believe at time 0 that entry will occur at time 1 and that incumbents trade accordingly, i.e. assume that $n+1$ traders will be active at time 1. Following the same steps as in the benchmark case of Proposition 1, the spread at time 0 is:

$$\Delta_0 = 2a\sigma^2\left[s_0 + \frac{s_1}{n+2} - \frac{n+3}{n+2}\sum_j x_0^j\right] = 2a\sigma^2\frac{n+3}{n+2}\left[s - \sum_j x_0^j\right], \text{ with } s_0 = s_1 = s$$

Hence incumbents solve the following problem:

$$\frac{CE_0^{i,acc}}{2a\sigma^2} = \max_{x_0^i}\frac{n+3}{n+2}x_0^i\left(s-\sum_j x_0^j\right) + \frac{\left(s-\sum_i x_0^i\right)^2}{\left(n+2\right)^2}$$

Solving for the first-order condition:

$$\forall i = 1,\ldots,n,\ \frac{n+3}{n+2}\left(s-\sum_j x_0^j - x_0^i\right) - \frac{2}{\left(n+2\right)^2}\left(s-\sum_j x_0^j\right) = 0$$

Collecting the n equations and stacking them together in vectors, and applying Lemma 3 gives the equilibrium trade:

$$x_0^i = \frac{(n+1)(n+4)}{n^3+6n^2+9n+6}s$$

As a result, $s - \sum_j x_0^j = \frac{(n+2)(n+3)}{n^3+6n^2+9n+6}s$, and the anticipation that the new arbitrageur enters is verified iff $\frac{\left(s-\sum_{j=1}^n x_0^j\right)^2}{(n+2)^2} \geq \rho^2 \Leftrightarrow \rho \leq \rho^{acc} \equiv \frac{n+3}{n^3+6n^2+9n+6}s$. Of course, the above strategy is a Nash equilibrium only if the entrant can commit ex-ante to enter no matter what. Since it is not possible here, we must take into account the possibility for arbitrageurs to deter entry by deviating at time 0. When $\rho \leq \rho^{acc}$, the incumbents' certainty equivalent is

$$CE_0^{i,acc} = \frac{(n+3)^2(n^2+5n+5)}{(n^3+6n^2+9n+6)^2}s^2 \equiv Ms^2 \tag{1.48}$$

Since each incumbent is pivotal, a deviation by one incumbent arbitrageur can deter the new arbitrageur from entering by reducing the available rent at time 1. In this case, the time 1 payoff changes to $2a\sigma^2\frac{\left(s-\sum_j x_0^j\right)^2}{(n+1)^2}$. To analyze the deviaton, let's assume that $n-1$ incumbents (indexed by $-i$) trade $x_0^{-i} = \frac{(n+1)(n+4)}{n^3+6n^2+9n+6}s$ ("accommodate"). This implies that $s - \sum_{-i} x_0^{-i} = \frac{2\left(n^2+5n+5\right)}{n^3+6n^2+9n+6}s$. A deviating incumbent thus solves (holding local investors' beliefs fixed):[32]

$$\begin{aligned} \frac{CE_0^{i,dev,acc}}{2a\sigma^2} &= \max_{x_0^i} \frac{n+3}{n+2}x_0^i\left(s - \sum_{-i} x_0^{-i} - x_0^i\right) + \frac{\left(s-\sum_{-i} x_0^{-i} - x_0^i\right)^2}{(n+1)^2} \\ & s.t. \quad \frac{\left(s-\sum_i x_0^i\right)^2}{(n+2)^2} < \rho^2 \end{aligned} \tag{1.49}$$

Ignoring the constraint first, and solving the maximisation problem gives:

$$x_0^{i,dev} = \frac{n^3+5n^2+5n-1}{2(n^3+5n^2+6n+1)}\left(s - \sum_{-i} x_0^{-i}\right) = \frac{(n^3+5n^2+5n-1)(n^2+5n+5)}{(n^3+5n^2+6n+1)(n^3+6n^2+9n+6)}s$$

One can check whether by deviating to this quantity, an incumbent deters entry: in this case, $s - \sum_{-i} x_0^{-i} - x_0^i = \frac{\left(n^3+5n^2+7n+3\right)\left(n^2+5n+5\right)}{(n^3+5n^2+6n+1)(n^3+6n^2+9n+6)}s$ and therefore the no-entry constraint is satisfied iff $\rho > \frac{\left(n^3+5n^2+7n+3\right)\left(n^2+5n+5\right)}{(n+2)(n^3+5n^2+6n+1)(n^3+6n^2+9n+6)}s \equiv \hat{\rho}$. However, we are considering the interval $[0, \rho^{acc}[$, and $\rho^{acc} - \hat{\rho} = -\frac{n^2+9n+9}{(n+2)(n^3+5n^2+6n+1)(n^3+6n^2+9n+6)} < 0$. This means that to deter entry,

[32] Since I assumed that $\rho \leq \rho^{acc}$, the only potentially optimal deviation is to prevent entry. Without the constraint in the maximisation problem, the solution would simply be $x_0^i = \frac{2\left(n^2+5n+5\right)}{n^3+6n^2+9n+6}s$, by definition.

the deviating incumbent must trade $x_0^{i,dev}$ to meet the no-entry constraint (1.49), i.e. such that:

$$\frac{\left(s - \sum_{-i} x_0^{-i} - x_0^{i,dev}\right)^2}{(n+2)^2} < \rho^2$$

Assuming that $s - \sum_{-i} x_0^{-i} - x_0^{i,dev} \geq 0$ (which will be true in equilibrium), this amounts to $\frac{x_0^{i,dev}}{n+2} > \frac{s-\sum_{-i} x_0^{-i}}{n+2} - \rho$. Since it is suboptimal to increase his position too much, the deviating arbitrageur chooses $x_0^{i,dev} = s - \sum_{-i} x_0^{-i} - (n+2)(\rho - \epsilon)$, with $\epsilon > 0$ and small, i.e.

$$x_0^{i,dev} = \frac{2\left(n^2+5n+5\right)}{n^3+6n^2+9n+6} - (n+2)(\rho - \epsilon)$$

We can thus compute the payoff from deviating. Skipping a few lines of algebra, I get:

$$\begin{aligned} \frac{CE_0^{i,dev,acc}}{2a\sigma^2} = & - \frac{(n+2)\left(n^3+5n^2+6n+1\right)}{(n+1)^2}\rho^2 \\ & + \left(\frac{2(n+3)\left(n^2+5n+5\right)}{n^3+6n^2+9n+6}s + \frac{2(n+2)\left(n^3+5n^2+6n+1\right)}{(n+1)^2}\epsilon\right)\rho \\ & - \frac{(n+2)\left(n^3+5n^2+6n+1\right)}{(n+1)^2}\epsilon^2 - \frac{2(n+3)\left(n^2+5n+5\right)}{n^3+6n^2+9n+6}s\epsilon \end{aligned}$$

We can derive the condition under which accommodate is a Nash equilibrium in rational expectations by comparing payoffs:

$$\begin{aligned} \frac{1}{2a\sigma^2}\left(CE_0^{acc} - CE_0^{dev,acc}\right)|_{\epsilon\to 0} & \geq 0 \Leftrightarrow a_1\rho^2 + a_2\rho + a_3 \geq 0 \\ \text{with } a_1 & = \frac{(n+2)\left(n^3+5n^2+6n+1\right)}{(n+1)^2} \\ a_2 & = -\frac{2(n+3)\left(n^2+5n+5\right)}{n^3+6n^2+9n+6}s \\ a_3 & = \frac{(n+3)^2\left(n^2+5n+5\right)}{\left(n^3+6n^2+9n+6\right)^2}s^2 \end{aligned}$$

The discriminant of $a_1\rho^2 + a_2\rho + a_3$ is $\Delta = \frac{(n+3)^2\left(n^2+5n+5\right)}{\left(n^3+6n^2+9n+6\right)^2}\tilde{\Delta}s^2$, with

$$\tilde{\Delta} = \left(n^2+5n+5\right) - \frac{(n+2)\left(n^3+5n^2+6n+1\right)}{(n+1)^2} = \frac{2n+3}{(n+1)^2} \tag{1.50}$$

There are two positive roots ρ_1 and ρ_2, with $\rho_1 < \rho_2$, given by $\rho_1 = -\frac{a_2+\sqrt{\Delta}}{2a_1} \equiv \bar{\rho}$ and $\rho_2 = \frac{-a_2+\sqrt{\Delta}}{2a_1}$. After some simplifications, I get:

$$\bar{\rho} = \frac{n+3}{n^3+6n^2+9n+6} \frac{(n+1)^2\left(u_n - \sqrt{\tilde{\Delta} u_n}\right)}{(n+2)\,v_n} s = \rho^{acc} \frac{(n+1)^2\left(u_n - \sqrt{\tilde{\Delta} u_n}\right)}{(n+2)\,v_n} \tag{1.51}$$

$$\rho_2 = \frac{n+3}{n^3+6n^2+9n+6} \frac{(n+1)^2\left(u_n + \sqrt{\tilde{\Delta} u_n}\right)}{(n+2)\,v_n} s = \rho^{acc} \frac{(n+1)^2\left(u_n + \sqrt{\tilde{\Delta} u_n}\right)}{(n+2)\,v_n} \tag{1.52}$$

$$\begin{aligned} \text{with } u_n &= n^2+5n+5 \\ v_n &= n^3+5n^2+6n+1 \end{aligned}$$

It is clear that $\rho_2 > \underline{\rho}_2 = \rho^{acc}\frac{(n+1)^2 u_n}{(n+2)v_n}$. Further, the fact that for all $n \geq 2$, $\frac{(n+1)^2 u_n}{(n+2)v_n} > 1$, implies that $\rho_2 > \rho^{acc}$.

Regarding the position of $\bar{\rho}$ relative to ρ^{acc}, note that $\forall\,(a,b) \in (\mathbb{R}^+)^2$, with $a > b$, $a - b - \sqrt{a\,(a-b)} = \sqrt{a-b}\left[\sqrt{a-b} - \sqrt{a}\right] < 0$, as the square root is increasing. Rewrite $\tilde{\Delta} = u_n - \frac{n+2}{(n+1)^2} v_n$, and apply the previous result to the thresholds:

$$\begin{aligned} \bar{\rho} < \rho^{acc} \quad &\Leftrightarrow \quad \frac{(n+1)^2\left(u_n - \sqrt{\tilde{\Delta} u_n}\right)}{(n+2)\,v_n} < 1 \\ &\Leftrightarrow \quad u_n - \frac{n+2}{(n+1)^2} v_n < \sqrt{u_n\left(u_n - \frac{n+2}{(n+1)^2} v_n\right)} \end{aligned}$$

The previous inequality holds. Thus incuments accommodate iff $\rho < \bar{\rho} < \rho^{acc}$. Note that for $\rho > \rho_{acc}$, the payoff from deviating will always dominate the payoff from accommodating, as can be seen from Figure 1.11. The constrained part of $CE_0^{i,dev,acc}$ is above $CE_0^{i,acc}$ when $\rho > \rho^{acc}$ because the function is still increasing after $\bar{\rho}$ for some time: the peak of $CE_0^{i,dev,acc}$ is $\rho^m > \rho^{acc} > \bar{\rho}$. This can be verified by direct calculation. ■

Proposition 8 *If $\rho > \rho^{bmk}$, arbitrageurs deter without altering their optimal strategy. Equilibrium trades are as in Proposition 1, with $s_0 = s_1 = s$.*

If $\rho \leq \rho^{bmk}$, arbitrageurs deter iff $\rho > \underline{\rho}$, with $\rho^{bmk} > \underline{\rho}$. Arbitrageurs must alter their trading strategy as follows:

$$\begin{aligned} \forall i = 1,\ldots,n,\ x_0^i &= \frac{1}{n}\left[s - (n+2)\,\rho + \epsilon\right],\ \epsilon > 0 \textit{ and } \epsilon \approx 0 \\ \forall i = 1,\ldots,n,\ x_1^i &= \frac{s - \sum_i x_0^i}{n+1} \end{aligned}$$

Proof. The case $\rho > \rho^{bmk}$ was treated above. I focus instead on $\rho \leq \rho^{bmk}$. I assume that local investors believe at time 0 that the new arbitrageur will not enter the market in the next period, meaning that the incumbent arbitrageurs face the following schedule for the spread: $\Delta_0 = 2a\sigma^2 \frac{n+2}{n+1}\left(s - \sum_j x_0^j\right)$. When $\rho \leq \rho^{bmk}$, incumbents must alter their benchmark trading strategies to decrease the time 1 payoff and prevent entry. x_0 satisfies: $\frac{\left(s-\sum_j x_0^j\right)^2}{(n+2)^2} < \rho^2$, i.e. assuming that $s - \sum_j x_0^j \geq 0$ (which will be true in equilibrium), $\sum_j x_0^j > s - (n+2)\,\rho$. Since incumbents have an interest in minimizing the deviation from their optimal strategy, they choose the smallest quantity such that the previous inequality is satisfied. Imposing symmetry across incumbents, this gives the following candidate equilibrium strategy:

$$\forall i = 1, \ldots, n,\ x_0^i = \frac{1}{n}\left[s - (n+2)\,\rho + \epsilon\right],\ \text{with } \epsilon \text{ positive and small} \tag{1.53}$$

The objective function of arbitrageurs is $\frac{CE_0^{i,det}}{2a\sigma^2} = \max_{x_0^i} \frac{n+2}{n+1} x_0^i \left(s - \sum_j x_0^j\right) + \frac{\left(s-\sum_j x_0^j\right)^2}{(n+1)^2}$. Plugging the strategy (1.53) into the objective function, and skipping a few lines of calculation, I get:

$$\begin{aligned}\frac{CE_0^{i,det}}{2a\sigma^2} = & - \frac{\left(n^2+2n+2\right)(n+2)^2}{n\,(n+1)^2}\rho^2 + \left[\frac{(n+2)^2}{n\,(n+1)}s + \frac{2\left(n^2 2n+2\right)(n+2)}{n\,(n+1)^2}\epsilon\right]\rho \\ & + \frac{n+2}{n\,(n+1)}s\epsilon + \frac{n^2+4n+2}{n\,(n+1)^2}\epsilon^2\end{aligned} \tag{1.54}$$

This is the payoff from deterring entrance, when local investors believe that there will be no entry in equilibrium. Since deterring requires to alter the time-0 trading strategy, it may be too costly for incumbent arbitrageurs. I now analyze under which conditions (1.53) forms a Nash equilibrium in rational expectations.

Suppose $n-1$ incumbents, indexed by $-i$, follow the deterrence strategy and trade $x_0^{-i} = \frac{1}{n}\left[s - (n+2)\,\rho + \epsilon\right]$. This leads to $s - \sum_{-i} x_0^{-i} = \frac{s}{n} + \frac{(n-1)(n+2)}{n}\rho - \frac{n-1}{n}\epsilon$. Then for incumbent arbitrageur i, A deviating incumbent solves the following problem:

$$\begin{aligned}\frac{CE_0^{i,dev,det}}{2a\sigma^2} &= \max_{x_0^i} \frac{n+2}{n+1} x_0^i \left(s - \sum_{-i} x_0^{-i} - x_0^i \right) + \frac{\left(s - \sum_{-i} x_0^{-i} - x_0^i\right)^2}{(n+2)^2} \\ s.t.\ x_0^{-i} &= \frac{1}{n} \left[s - (n+2)\rho + \epsilon \right] \\ 2a\sigma^2 \frac{\left(s - \sum_{i=1}^n x_0^i\right)^2}{(n+2)^2} &\geq I \end{aligned} \tag{1.55}$$

Solving the first-order condition (assuming the entry constraint is satisfied) gives

$$x_0^i = \frac{n^3 + 6n^2 + 10n + 6}{2\left(n^3 + 6n^2 + 11n + 7\right)} \left[s + \frac{(n-1)(n+2)}{n} \rho - \frac{n-1}{n} \epsilon \right] \tag{1.56}$$

Note $x_0^i > 0$. Further, $x_0^{-i} \geq 0 \ \Leftrightarrow \ s > (n+2)\rho - \epsilon$, which is verified for any $\rho < \rho^{bmk}$.

We can check under which condition the deviation leads to the new arbitrageur's entry at time 1. Given the construction of the deterrence strategy x_0^{-i}, it is sufficient to compare x_0^i and x_0^{-i}. In particular if $x_0^i < x_0^{-i}$, then the deviation leads to entry. Skipping a few lines of algebra,

$$\begin{aligned} x_0^{-i} - x_0^i &= \frac{n^3 + 6n^2 + 12n + 8}{2\left(n^3 + 6n^2 + 11n + 7\right)} \frac{s}{n} - \frac{n^4 + 7n^3 + 16n^2 + 18n + 8}{2\left(n^3 + 6n^2 + 11n + 7\right)} \left[\frac{n+2}{n} \rho - \frac{\epsilon}{n} \right] \\ \text{Hence, } x_0^{-i} &> x_0^i \Leftrightarrow \rho - \frac{\epsilon}{n+2} < \rho^{dev} \equiv \frac{(n+2)^2 s}{n^4 + 7n^3 + 16n^2 + 18n + 8} \end{aligned} \tag{1.57}$$

Comparing ρ^{dev} with ρ^{bmk} shows that $\forall n \geq 1, \rho^{dev} < \rho^{bmk}$. Hence if $\rho \in \left[\rho^{dev}, \rho^{bmk}\right[$, the deviation does not lead to entry. Since on this interval $x_0^i \geq x_0^{-i}$, the deviating arbitrageur would have to buy $x_0^{-i} - \eta$, with $eta > 0$ and small. This strategy must be dominated, since it implies a strictly lower time-1 continuation payoff (for the entrant steals some business), but only a very small gain at time 0, relative to the deterrence strategy. Thus if $\rho \in \left[\rho^{dev}, \rho^{bmk}\right[$, then deterring is optimal.

If $\rho < \rho^{dev}$, there is a trade-off between deviating and letting the new arbitrageur enter and deterring for arbitrageur i. I plug (1.56) into the objective function and, after rearranging terms, I obtain the payoff from deviating for arbitrageur i:

$$\frac{CE_0^{i,dev,det}}{2a\sigma^2} = \frac{(n+2)^4}{4(n+1)\left(n^3 + 6n^2 + 11n + 7\right)} \left[s + \frac{(n-1)(n+2)}{n} \rho \frac{n-1}{n} \epsilon \right]^2 \tag{1.58}$$

I now derive under which condition the strategy is a Nash equilibrium in rational expectations by comparing payoffs. Recalling equation (1.54), and rearranging terms, I take $\epsilon \to 0$, and get:

$$\begin{aligned}
\frac{CE_0^{i,det} - CE_0^{i,dev,det}}{2a\sigma^2} &\geq 0 \Leftrightarrow b_1\rho^2 + b_2\rho + b_3 \geq 0 \\
\text{with } b_1 &= -\frac{(n+2)^2}{n}\left[\frac{n^2+2n+2}{(n+1)^2} + \frac{(n-1)^2 v_n}{n} < 0\right] \\
b_2 &= \left[\frac{(n+2)^2}{n(n+1)} - \frac{2(n-1)(n+2)v_n}{n^2}\right]s \\
b_3 &= -\frac{v_n}{n^2}s^2 < 0 \\
\text{and } v_n &\equiv \frac{(n+2)^4}{4(n+1)(n^3+6n^2+11n+7)}
\end{aligned}$$

After some algebra, I get the following expressions for b_1, b_2 and the discriminant:

$$\begin{aligned}
b_1 &= -\frac{(n+2)^2 w_n}{4n^2(n+1)^2(n^3+6n^2+11n+7)} \\
b_2 &= \frac{(n+2)^2(n^4+7n^3+16n^2+18n+8)}{2n^2(n+1)(n^3+6n^2+11n+7)}s \\
\Delta &= \frac{(n+2)^4\bar{\Delta}}{4n^4(n+1)^2(n^3+6n^2+11n+7)^2} \\
\text{with } w_n &= n^7+11n^6+47n^5+101n^4+132n^3+120n^2+72n+16 \\
\bar{\Delta} &= \left(n^4+7n^3+16n^2+18n+8\right)^2 - \frac{(n+2)^2}{n+1}w_n \\
&= \frac{n^2(7n^4+60n^3+160n^2+188n+84)}{n+1}
\end{aligned}$$

The discriminant being positive, there are two roots $\rho_3 = \frac{-b_2+\sqrt{\Delta}}{2b_1} \equiv \underline{\rho}$ and $\rho_4 = \frac{-b_2-\sqrt{\Delta}}{2b_1}$ with $\rho_4 > \underline{\rho} > 0$. It is possible to calculate these roots explicitly. Skipping a few lines of algebra, I get:

$$\rho_{3+k} = \frac{(n+1)\left(n^4+7n^3+16n^2+18n+8-(-1)^k\sqrt{\bar{\Delta}}\right)}{w_n}s,\ k=0,1$$

Finally, I compare the roots to the threshold ρ^{bmk}. Note that $\rho_4 > \frac{(n+1)\left(n^4+7n^3+16n^2+18n+8\right)}{w_n} \equiv \underline{\rho}_4$. Further, $\underline{\rho}_4 - \rho^{bmk} > 0 \Leftrightarrow \frac{n(n+1)\left(4n^3+10n^2-2n-12\right)}{(n^3+4n^2+3n+2)w_n} > 0$, which is true $\forall n \geq 2$. Hence

$\rho_4 > \rho_{bmk}$.

Further, note that

$$\begin{aligned}
\frac{\underline{\rho} - \rho^{dev}}{s} &= \frac{(n+1)\left(n^4 + 7n^3 + 16n^2 + 18n + 8 - \sqrt{\bar{\Delta}}\right)}{w_n} - \frac{(n+2)^2}{n^4 + 7n^3 + 16n^2 + 18n + 8} \\
&= \frac{(n+1)\left[\bar{\Delta} - \left(n^4 + 7n^3 + 16n^2 + 18n + 8\right)\sqrt{\bar{\Delta}}\right]}{w_n\left(n^4 + 7n^3 + 16n^2 + 18n + 8\right)}
\end{aligned}$$

Then write $\bar{\Delta} = a - b$, with $a = -\left(n^4 + 7n^3 + 16n^2 + 18n + 8\right)^2$ and $b = \frac{(n+2)^2}{n+1} w_n$. $\bar{\Delta} > 0$ $\Rightarrow a > b$, hence $a - b - \sqrt{a(a-b)} = \sqrt{a-b}\left[\sqrt{a-b} - \sqrt{a}\right] < 0$ (since $a - b < a$ and $\sqrt{}$ is increasing). Thus, $\underline{\rho} < \rho^{dev}$. Since $\rho^{dev} < \rho^{bmk}$, $\underline{\rho} < \rho^{bmk}$. Summing up, deterring is a Nash equilibrium in rational expectations on $\left[\underline{\rho}, \rho^{bmk}\right]$. ■

Proposition 9 $\forall n \geq 2$, $\underline{\rho} < \bar{\rho}$.

Proof. By direct calculation. Recall from the proofs of Propositions 7 and 8 that

$$\begin{aligned}
\bar{\rho} &= \rho^{acc} \frac{(n+1)^2\left[u_n - \sqrt{\tilde{\Delta} u_n}\right]}{(n+2)\, v_n} \\
\text{with } v_n &= n^3 + 5n^2 + 6n + 1,\ \tilde{\Delta} = \frac{2n+3}{(n+1)^2},\ \rho^{acc} = \frac{n+3}{n^3 + 6n^2 + 9n + 6} \\
\text{and } \underline{\rho} &= \frac{(n+1)\left[z_n - \sqrt{\bar{\Delta}}\right]}{w_n} \\
\text{with } w_n &= n^7 + 11n^6 + 47n^5 + 101n^4 + 132n^3 + 120n^2 + 72n + 16 \\
z_n &= n^4 + 7n^3 + 16n^2 + 18n + 8 \\
\bar{\Delta} &= \frac{n^2\left(7n^4 + 60n^3 + 160n^2 + 188n + 84\right)}{n+1}
\end{aligned}$$

Hence $\bar{\rho} > \underline{\rho} \Leftrightarrow (n+1)\rho^{acc} w_n \left[u_n - \sqrt{\tilde{\Delta} u_n}\right] - v_n z_n + v_n \sqrt{\bar{\Delta}} > 0 \Leftrightarrow (n+1)\rho^{acc} w_n - v_n z_n > (n+1)\rho^{acc} w_n \sqrt{\tilde{\Delta} u_n} - v_n \sqrt{\bar{\Delta}}$. Skipping several lines of algebra, I find that the numerator of the LHS is $q_n = n^{11} + 19n^{10} + 156n^9 + 729n^8 + 2160n^7 + 4134n^6 + 6107n^5 + 6454n^4 + 5254n^3 + 3142n^2 + 1172n + 192$. Since $u_n < (n+3)^2$, $\sqrt{\tilde{\Delta} u_n} < \frac{(n+3)\sqrt{2n+3}}{n+1} < \frac{(n+3)(2n+3)}{n+1}$. Hence, it is sufficient to prove that $q_n > (n+3)^2 (2n+3) w_n$. After some algebra, I find that the LHS is equal to $2n^{10} + 37n^9 + 295n^8 + 1330n^7 + 3768n^6 + 7215n^5 + 9423n^4 + 8996n^3 + 6072n^2 + 2520n + 432 < q_n$ for all $n \geq 2$. Hence $\bar{\rho} > \underline{\rho}$. ■

Monopoly case.

Last, I consider the case $n = 1$. If there is a monopolistic arbitrageur at time 0, the equilibrium at time 0 is not a Nash equilibrium in rational expectations, but simply a rational expectations equilibrium.

The thresholds ρ^{bmk} and ρ^{acc} remain the same, with $n = 1$. From the results of Proposition 7, I get: $\rho^{bmk} = 0.2$ and $\rho^{acc} = \frac{2}{11}$. For $\rho > \rho^{bmk}$, the monopolist has no interest to let the new arbitrageur enter. For $\rho < \rho^{bmk}$, the monopolist compares the payoff from accommodating and deterring. If the monopolist accommodates, his payoff is $CE_0^{acc} = 2a\sigma^2 \frac{4}{11} s^2$ (from equation (1.48). If he deters, since $\rho \leq \rho^{bmk}$, he must set x_0^1 such that the time 1 payoff is smaller than the entry cost for the new arbitrageur, i.e. $x_0^1 = s - 3\rho - \epsilon$, with ϵ small and strictly positive. The payoff, for ϵ very small, is $CE_0^{det} = 2a\sigma^2 \left(-\frac{9}{2}\rho^2 + \frac{9}{2}s\rho\right)$.

Given that there is no coordination problem, I compare directly the payoffs CE_0^{acc} and CE_0^{det}, while in the $n \geq 2$ case, I was holding the local investors' beliefs fixed and checking each arbitrageur's incentives to deter or accommodate. After a straightforward calculation, I find that $CE_0^{acc} \geq CE_0^{det} \Leftrightarrow \rho \leq \hat{\rho}$ or $\rho \geq \hat{\rho}'$, with $\hat{\rho} = 1 - \frac{2}{9}\sqrt{\delta}$ and $\hat{\rho}' = 1 + \frac{2}{9}\sqrt{\delta}$, where $\delta = \frac{\sqrt{747}}{2\sqrt{11}}$. Clearly, $\hat{\rho}' > 1 > \rho^{bmk} > \rho^{acc} > \hat{\rho}$. Thus, in equilibrium, the monopolist accommodates if $\rho \leq \hat{\rho}$ and the new arbitrageur enters, and the monopolist deters if $\rho > \hat{\rho}$, with no entry. The deterrence is not "costly" if $\rho > \rho^{bmk}$. ■

Corollary 12

Proof. Using the results of Propositions 7 and 8, I get, after straightforward calculations:

$$\text{If } \rho > \rho^{bmk},\ \Delta_0^{bmk} = 2a\sigma^2 \frac{(n+2)^2}{n^3+4n^2+3n+2} s \tag{1.59}$$

$$\Delta_1^{bmk} = 2a\sigma^2 \frac{n+2}{n^3+4n^2+3n+2} s = \frac{\Delta_0^{bmk}}{n+2}$$

$$\text{If } \rho \leq \underline{\rho},\ \Delta_0^{acc} = 2a\sigma^2 \frac{(n+3)(n+2)}{n^3+6n^2+9n+6} s \tag{1.60}$$

$$\Delta_1^{acc} = 2a\sigma^2 \frac{n+2}{n^3+6n^2+9n+6} s = \frac{\Delta_0^{acc}}{n+3}$$

$$\text{If } \rho \in \left[\bar{\rho}, \rho^{bmk}\right], \text{ and } \epsilon \text{ small } \Delta_0^{det} = 2a\sigma^2 \frac{(n+2)^2}{n+1} \rho \tag{1.61}$$

$$\Delta_1^{det} = 2a\sigma^2 \frac{n+2}{n+1} \rho = \frac{\Delta_0^{det}}{n+2}$$

Comparison of spreads across the three regimes

First, a simple calculation shows that $\forall n \geq 2$, $\Delta_0^{bmk} > \Delta_0^{acc}$, which implies that $\Delta_1^{bmk} > \Delta_1^{acc}$.

Second, $\Delta_0^{det}(\bar{\rho}) > \Delta_0^{acc}$ iff $\frac{(n+2)^2}{n+1}\bar{\rho} > \frac{(n+1)(n+3)}{n^3+6n^2+9n+6}s = \frac{n+1}{n+2}\rho^{acc}$. Recall from equation (1.51) in the proof of Proposition 7 that $\bar{\rho} = \rho^{acc}\frac{(n+1)^2\left(u_n-\sqrt{\tilde{\Delta}u_n}\right)}{(n+2)v_n}$. Hence $\Delta_0^{det}(\bar{\rho}) > \Delta_0^{acc}$ iff $v_n - (n+1)u_n + (n+1)\sqrt{\tilde{\Delta}u_n} < 0$, which is equivalent to $\sqrt{\tilde{\Delta}u_n} < \frac{(n+2)^2}{n+1}$. Using the definition of $\tilde{\Delta}$, this simplifies to $(2n+3)u_n = (2n+3)(n^2+5n+5) < (n+2)^4$, i.e. $2n^3 + 13n^2 + 25n + 15 < (n+2)^4$, which holds for any n. Hence $\Delta_0^{det}(\bar{\rho}) > \Delta_0^{acc}$. This implies that $\Delta_1^{det}(\bar{\rho}) > \Delta_1^{acc}$.

Third, consider $\Delta_0^{det}\left(\underline{\rho}\right) - \Delta_0^{acc} = 2a\sigma^2(n+2)s\left(\frac{(n+2)\left(z_n-\sqrt{\bar{\Delta}}\right)}{w_n} - \frac{(n+3)}{n^3+6n^2+9n+6}\right)$. Hence $\Delta_0^{det}\left(\underline{\rho}\right) > \Delta_0^{acc}$ iff $(n+2)(n^3+6n^2+9n+6)z_n - (n+3)w_n - (n^3+6n^2+9n+6)\sqrt{\bar{\Delta}} > 0$. By the definition of $\bar{\Delta}$, and given that $(n+2)(n^3+6n^2+9n+6)z_n - (n+3)w_n = n^7 + 13n^6 + 75n^5 + 233n^4 + 394n^3 + 360n^2 + 176n + 48$, this is equivalent to $\sqrt{\frac{7n^4+60n^3+160n^2+188n+84}{n+1}} < \frac{n^7+13n^6+75n^5+233n^4+394n^3+360n^2+176n+48}{n(n+2)(n^3+6n^2+9n+6)}$. Taking the square on each side and developing, one can check that this inequality is always satisfied after some tedious algebra. Hence $\Delta_0^{det}\left(\underline{\rho}\right) - \Delta_0^{acc}$, and thus $\Delta_1^{det}\left(\underline{\rho}\right) - \Delta_1^{acc}$

Fourth, it is immediate that $\Delta_0^{det}\left(\rho^{bmk}\right) - \Delta_0^{bmk} = 0$. This implies that the inequality is also verified at time 1.

Note that $\Delta_t^{det}(\rho)$ is increasing in ρ on its interval, so that it was sufficient to compare the spread at the thresholds $\underline{\rho}$ and $\bar{\rho}$, ρ^{bmk}. Figure 1.11 represents Δ_0 as a function of ρ, and summarizes all the previous results.

Speed of convergence

Clearly, $\frac{\Delta_1^{bmk}}{\Delta_0^{bmk}} - 1 = \frac{\Delta_1^{det}}{\Delta_0^{det}} - 1 = -\frac{n+1}{n+2} > \frac{\Delta_1^{acc}}{\Delta_0^{acc}} - 1 = -\frac{n+2}{n+3}$.

The price impact coefficient is $\frac{n+2}{n+1}$ when $\rho > \bar{\rho}$ and $\frac{n+3}{n+2}$ when $\rho \leq \underline{\rho}$. Thus arbitrageurs have a lower price impact when $\rho \leq \underline{\rho}$. ■

Corollary 13

Proof. The first part of the corollary follows from the definition of (1.61). The second part follows from the facts that $\Delta_0^{acc} < \Delta_0^{det}\left(\underline{\rho}\right) < \Delta_0^{det}(\bar{\rho})$ and the multiplicity of equilibria on $\left[\underline{\rho}, \bar{\rho}\right]$. ■

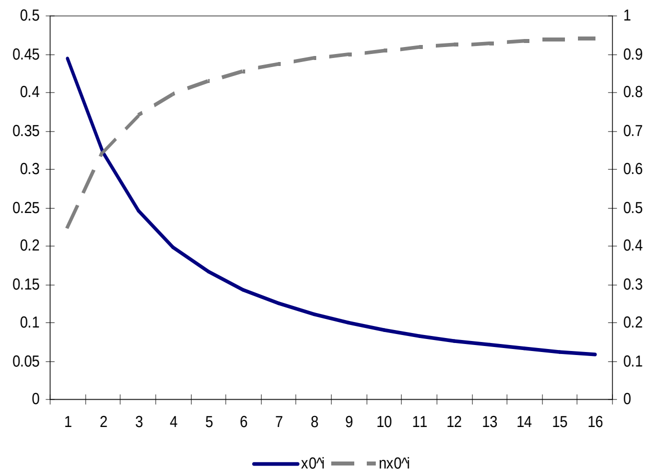
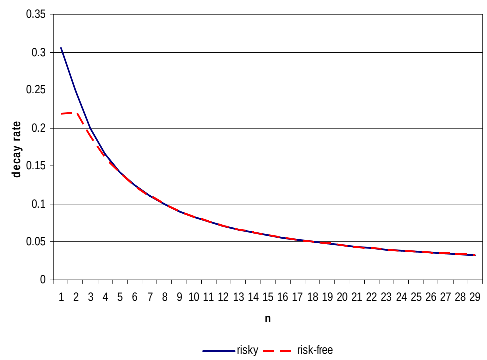
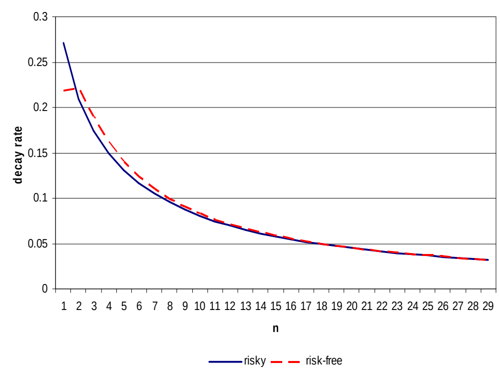


Figure 1.3: **Trades and race effect in the risk-free case.** The parameters are: $s_0 = s_1 = 1$.

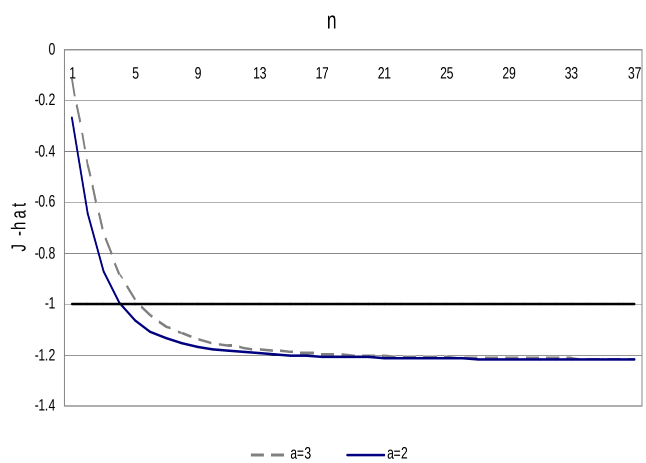


(a) Local investors more risk-averse than arbitrageurs ($a = 3.5$, $b = 0.5$)

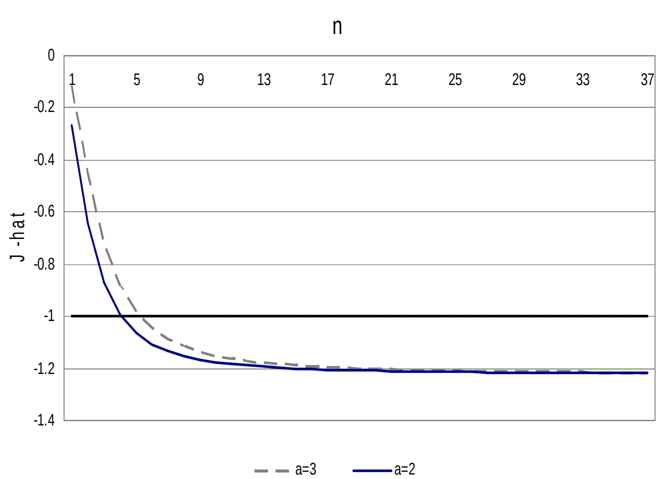


(b) Local investors as risk-averse than arbitrageurs ($a = b = 2$)

Figure 1.4: **Trades (panel a) and decay rate (panel b and c) of arbitrageurs' position at time 0 as a function of the number of arbitrageurs.** In all cases, the parameters are: $s_0 = 1$, $s_1 = \bar{s}_1 = 1$, $z_1^2 = 1$, $\sigma^2 = 0.1$.

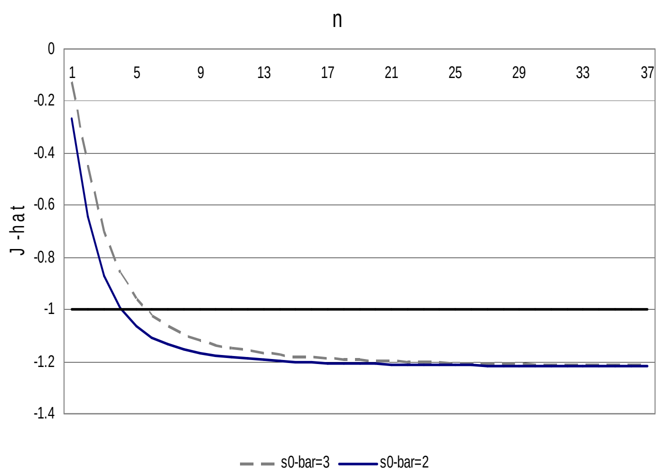


(a) Increase in local investors' risk-aversion (a)

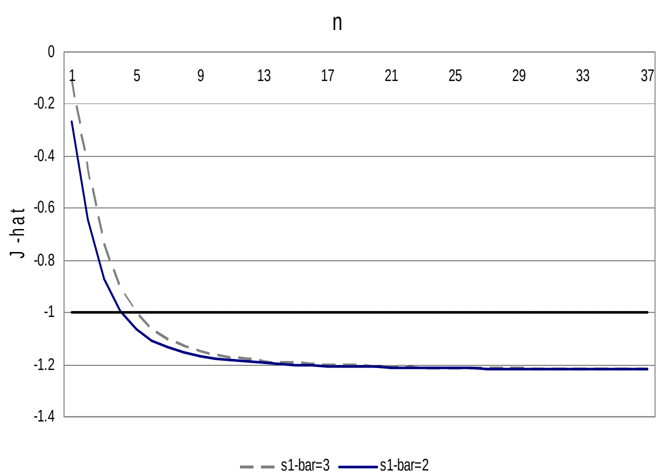


(b) Increase in arbitrageurs' risk-aversion (b)

Figure 1.5: **Comparative statics of the equilibrium number of arbitrageurs n^* in the risky case.** The baseline parameters are: $a = 2$, $b = 2$, $\bar{s}_0 = \bar{s}_1 = 2$, $z_0^2 = z_1^2 = 0.5$, $\sigma^2 = 0.1$, $I = 0.1$.

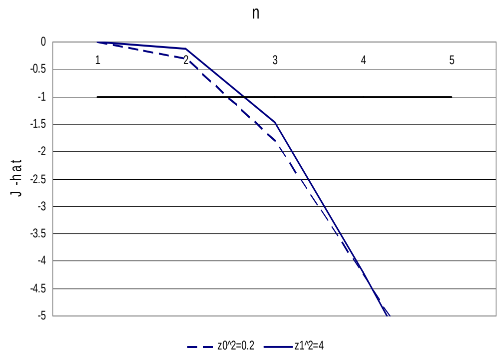


(a) Increase in $\bar{s}_0$

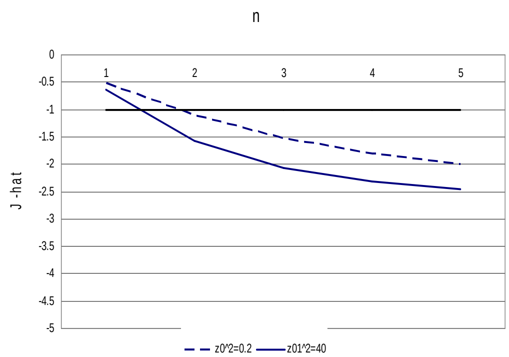


(b) Increase in $\bar{s}_1$

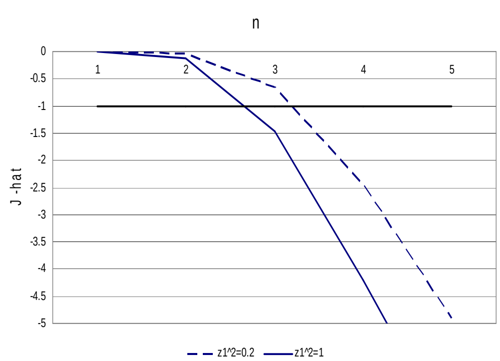
Figure 1.6: **Comparative statics of the equilibrium number of arbitrageurs n^* in the risky arbitrage case.** The baseline parameters are: $a = 2$, $b = 2$, $\bar{s}_0 = \bar{s}_1 = 2$, $z_0^2 = z_1^2 = 0.5$, $\sigma^2 = 0.1$, $I = 0.1$.



(a) Increase in z_0 (high risk aversion $a = b = 3$)

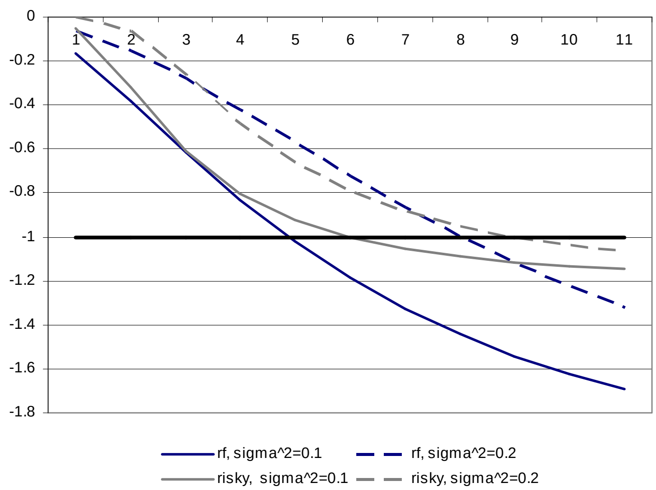


(b) Increase in z_0 (low risk aversion $a = b = 1$)

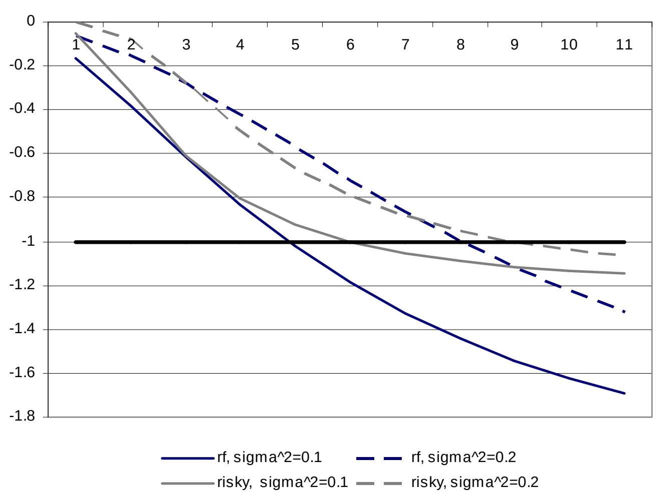


(c) Increase in z_1

Figure 1.7: **Comparative statics of the equilibrium number of arbitrageurs n^* in the risky arbitrage case.** The baseline parameters are: $a = b = 3$, $\bar{s}_0 = \bar{s}_1 = 4$, $z_0^2 = z_1^2 = 0.2$, $\sigma^2 = 0.1$, $I = 1$.

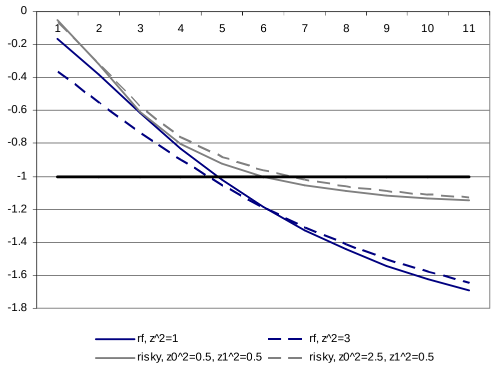


(a) Increase in a

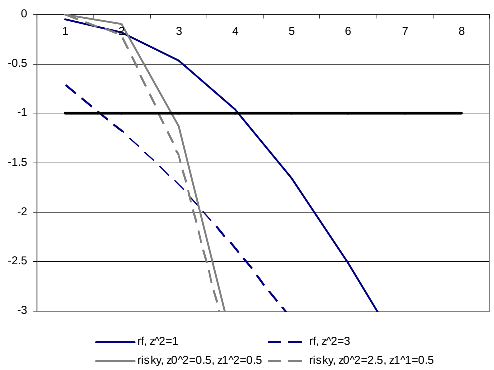


(b) Increase in σ^2

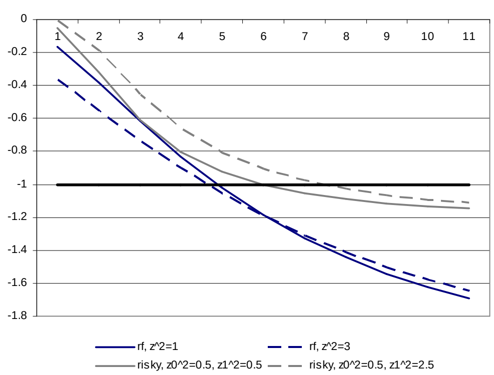
Figure 1.8: **Comparing entry in the riskfree and risky arbitrage cases. Effect of a change in investors' risk-aversion and volatility of the fundamental.** The baseline parameters are: $a = 2$, $b = 2$, $\bar{s}_0 = \bar{s}_1 = \bar{s} = 2$, $z_0^2 = z_1^2 = .5$, $z^2 = 1$, $\sigma^2 = 0.1$, $I = 1$.



(a) Increase in z_0

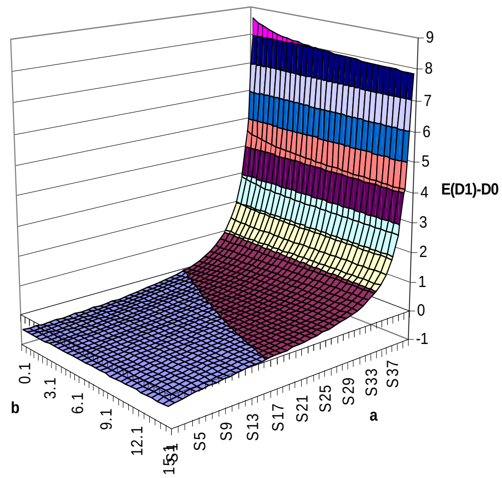


(b) Increase in z_0 with other parameters

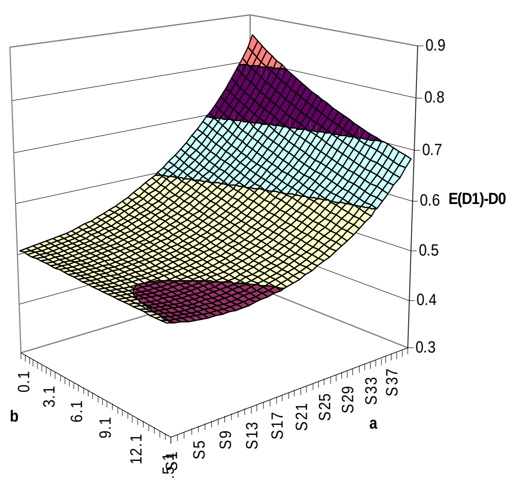


(c) Increase in z_1

Figure 1.9: **Comparing entry in the riskfree and risky arbitrage cases. Effect of a change in the volatility of the arbitrage risk at time 0 or time 1.** The baseline parameters for panel b are: $a = b = 3$, $\bar{s}_0 = \bar{s}_1 = \bar{s} = 4$, $\sigma^2 = 0.1$, $I = 1$. In panel a and c, the parameters are $a = 2$, $b = 2$, $\bar{s}_0 = \bar{s}_1 = \bar{s} = 2$, $z_0^2 = z_1^2 = .5$, $z^2 = 1$, $\sigma^2 = 0.1$, $I = 1$

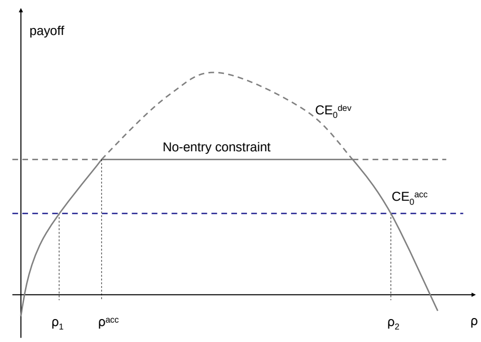


(a) Concentrated market ($n = 2$)

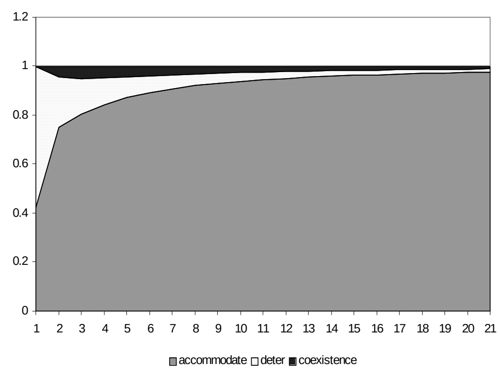


(b) More competitive market ($n = 10$)

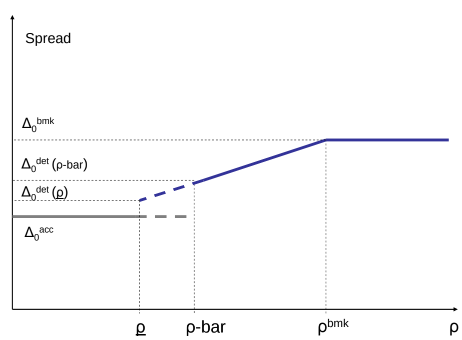
Figure 1.10: **Expected return of the arbitrage at time 0.** The baseline parameters are $z^2 = 1$, $\sigma^2 = 0.1$. a varies from 0.1 to 10.



(a) Accommodate equilibrium - constrained case.



(b) Coexistence of equilibria.



(c) Spread as a function of ρ.

Figure 1.11: **Gradual Entry**

Chapter 2

Market Structure and the Limits of Arbitrage

Abstract: How does imperfect competition among financially-constrained arbitrageurs affect liquidity and asset prices? I study a multi-period model in which arbitrageurs (e.g. market-makers, dealers, hedge funds) provide market liquidity and face capital constraints, and compare monopolistic competition to the competitive case studied by Gromb and Vayanos (2002). I show that the monopoly is both less efficient *and* less capital-intensive, as rents captured over time allow her to build up capital. Consequently, when capital is scarce, arbitrageurs may provide more liquidity at later stages under monopolistic competition than under perfect competition. In some cases, this increases aggregate welfare but without being Pareto-improving. Further, when a monopolistic arbitrageur has an intermediate level of capital, she may tackle the arbitrage in a way that leaves her constrained in the future. Surprisingly, this may improve liquidity relative to a situation without financial constraints. I discuss implications for market-making via a monopolistic specialist.

2.1 Introduction

In line with the theoretical predictions of the limits of arbitrage literature, there is a growing body of empirical evidence showing that the amount of capital held by financial institutions affects market liquidity and asset prices.[1] The theory is based on the assumption that arbitrageurs are competitive, and predicts that the *aggregate* amount of arbitrageurs' capital matters. In practice, however, arbitrage is carried out by large, highly specialized financial institutions (hedge funds, proprietary trading desks, broker-dealers, etc.), who recognize their

[1] For recent empirical evidence, see for example Hu, Pan and Wang (2011), Mitchell and Pulvino (2011) and Jylha and Suominen (2009).

price impact.[2][3] Hence in reality capital is often concentrated in the hands of a few large, strategic players instead of being distributed across a large number of small competitive investors. This implies that the market structure and the distribution of capital should be an important determinant of asset prices and liquidity.

To understand how the size of arbitrageurs and the distribution of capital affect market efficiency, I extend Gromb and Vayanos' (2002) model of financially constrained *competitive* arbitrage to the non-competitive case. In Gromb and Vayanos' model, competitive arbitrageurs exploit price differences between two identical risky assets traded in segmented markets. Market liquidity, defined as the spread between the prices of the risky assets, depends on arbitrageurs' aggregate capital: with abundant capital, competitive arbitrageurs eliminate the spread immediately. With a smaller amount of capital, financial constraints bind and the spread decreases gradually over time, as capital gains captured by arbitrageurs progressively relax their financial constraint. By contrast, I study a situation in which all the capital is deployed through a single, monopolistic arbitrageur who recognizes her price impact. This allows me to compare liquidity and welfare across the two market structures. I show that switching from perfect to monopolistic competition (or equivalently from a largely disaggregated capital distribution to an extremely concentrated one) has substantial implications for liquidity provision, asset prices and welfare that go *beyond* the simple efficiency loss associated with market power.

First, I show that when capital is concentrated in the hands of a single arbitrageur, a trade-off between market efficiency and capital intensity appears. A monopolistic arbitrageur ("‘the arbitrageur", or "the monopoly") internalizes her impact on the spread. Thus, facing a positive spread between the two identical assets, the arbitrageur trades in a way that keeps the spread open in equilibrium, even if her financial constraint is not binding. This allows her to reap profits over time and therefore to increase her capital. Understanding that the price of each asset will not converge to its fundamental value for this precise reason (even in the

[2] See Chen, Stanzl and Watanabe (2002) and references therein for empirical evidence of institutional investors' price impact. Numerous firms propose algorithms or programmes to help institutional investors minimize their price impact. In extreme cases, one trader may become the dominant player in one market or in a specific strategy, such as LTCM with relative-value and convergence trades or Amaranth and Enron with energy derivatives.

[3] There is a large theoretical literature on limits of arbitrage. See for example, Shleifer and Vishny (1997), Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009). In effect, the assumption of perfect competition also means that entry has already occurred in the market, so that an increase in capital cannot stem from new players. Attari and Mello (2006) do analyze the trading strategy of a monopolistic financially-constrained arbitrageur but do not draw comparison across market structures. I discuss their paper in more details when I review the literature below.

absence of risk), financiers demand less collateral to fund the arbitrageur's position. Hence in equilibrium the monopoly prevents the arbitrage to close while competitive arbitrageurs would eliminate all mispricings. At the same time, the monopoly requires less initial capital to do so than a competitive market.

The key driver of this result is the modeling of the arbitrageur's financial constraint which implies that the arbitrageur's financiers are sophisticated enough to understand the impact of the market structure on the equilibrium price. Indeed, in the model, the arbitrageur is required to post collateral in each leg of the arbitrage only to cover the risk associated with the position, which depends on the volatility of the fundamental. Predictable discounts/ premia arising from the monopoly's rationing lower the collateral requirement, because financiers realize that they are not facing additional risks. In other words, margins have a countercyclical effect relative to liquidity.[4] Although arguably strong, this assumption constitutes an interesting and unavoidable benchmark: according to Brunnermeier and Pedersen (2009), countercyclical, stabilizing margins are "hard to escape in a theoretical model". Moreover, if the market is concentrated, the dominant arbitrageur is likely to be visible to other market participants and financiers may understand how she affects liquidity. There is anecdotal evidence that LTCM, which held very large positions in fixed-income markets, had access to cheaper funding than its competitors.[5]

An interesting consequence of the efficiency - capital intensity trade-off is that, contrary to the competitive case, an increase in the volatility of the fundamental does not necessarily tighten the arbitrageur's financial constraint. There are two opposite effects at work. On one hand, an increase in volatility increases the maximum potential loss on the arbitrageur's position and makes it more risky for financiers to fund it, which tightens the constraint. This effect is the same as in the competitive case (Gromb and Vayanos, 2002, 2010). On the other hand, an increase in volatility increases the profitability of the arbitrage opportunity, which lowers collateral requirement. The reason is that the arbitrage opportunity arises in the first place because of demand pressure stemming from local investors who trade in segmented markets. These investors receive liquidity shocks that are correlated with the payoff of the risky asset. Local investors in each market (say A and B) receive opposite shocks and would thus benefit from trading with each other to share risk but are prevented

[4]I keep Gromb and Vayanos (2002)'s modeling of the financial constraint in order to facilitate comparisons. Brunnermeier and Pedersen (2009) consider closely-related constraints but allow for the financiers to be imperfectly informed about sources of illiquidity in the market, and show that this can generate procyclical margins.

[5]See Pérold (1999).

to do so by market segmentation. The arbitrageur instead is the only investor who can trade across both markets. As volatility increases, local investors are ready to accept larger price concessions to share their risk, which benefits the arbitrageur. Financiers recognize this second effect, which loosens the financial constraint.

Second, and quite surprisingly, I show that the arbitrageur may *choose* to be constrained at certain dates in equilibrium. Specifically, when the amount of capital is intermediate, and volatility sufficiently low, the monopoly trades in a way that makes her financial constraint binding in the future, even though with a different trading strategy, the amount of capital she holds would allow her to remain unconstrained until the assets pay off. The reason for this behaviour is that to maximize profits, the arbitrageur seeks not only to keep the arbitrage spread open while buying, but also to keep it open as long as possible. Since the positions already established continue to affect the spread over time (permanent price impact), the arbitrageur split up her order and increases her position progressively over time. Local investors anticipate that this will gradually reduce the arbitrage spread. This reduces their willingness to accept large price concessions at present dates, which erodes the arbitrageur's current market power.[6] In this context, the financial constraint can work as a *commitment device* for the arbitrageur to keep the spread open over time: if the constraint binds, future spreads will be large, which increases the local investors' willingness to share risk and accept a large price concession early on.

This equilibrium arises only for an intermediate level of capital. Indeed, if the arbitrageur had a large amount of capital, her trading strategy would not be dynamically consistent: she would be able to reoptimize in the future, which would be optimal from her viewpoint. Since local investors are smart, they can foresee this behaviour and therefore the commitment power of the financial constraints unravels if the arbitrageur is too well-capitalized. At the same time, if capital is too scarce, the arbitrageur cannot respond to the increase in local investors' demand for liquidity at earlier dates that result from the anticipation that her constraint will be binding in the future.

Third, I compare liquidity provision and welfare across market structures. When there is abundant capital, the market is perfectly liquid under perfect competition and imperfectly

[6] Hence providing liquidity is subject to the same dynamics as a durable good provided by a monopolist. In the model, trading happens at discrete dates and the spread closes as the asset matures. These features represent "frictions" that limit the substitutability of one trade with another. With an infinite horizon, or with a shrinking time between two trading dates, the Coase conjecture would apply, i.e. the spread would immediately reach zero. See Vayanos (2001), DeMarzo and Urosevic (2007, and Pritsker (2009).

liquid under monopolistic competition, because the monopolistic arbitrageur limits liquidity to extract rents. Since the monopoly is less capital intensive than the competitive market, a more interesting case is where there is not enough capital for the competitive market to be unconstrained but enough for a monopoly to be unconstrained (or voluntarily constrained). I show that in this case the monopoly may provide more liquidity, resulting in a tighter spread, before the asset matures. For some parameter cases, I can show analytically that the improvement occurs only close to the date at which the asset matures, and is associated with a reduction in liquidity before. Suppose for instance that the monopoly is voluntarily constrained. The monopoly is able to decrease the spread in the future precisely because she limits liquidity further at the beginning. Doing so, she maximizes intermediate capital gains, which relaxes the financial constraint and results in lower future spreads. I show that this change can hurt local investors' welfare, as the deterioration of current liquidity offsets the benefit of the future improvement. Arbitrageurs, however, benefit from this change in market structure. In an example, I show that aggregate welfare may increase as a result.

The analysis has implications for the debate about the size of financial intermediaries and how much capital they should hold. The debate about the size of intermediaries follows from the failure of large institutions during the 2007-2009 crisis. In the media or among economists, it often revolves around the lack of competition in the financial industry, and the implicit government protection on large systemically important institutions. This paper abstracts from government protection in case of default but shows that market power has consequences not only for market efficiency and the law of one price but also for margins and access to funding liquidity. The model predicts that large institutions with market power can operate with less capital and that this may be beneficial for market liquidity in situations where capital is scarce. However, the benefits in terms of liquidity seems small (in numerical examples) and involve some wealth transfers to arbitrageurs.

I show that, when dividend volatility is not too large, voluntarily-constrained monopolistic arbitrageurs provide more liquidity than unconstrained monopolistic arbitrageurs. This has implications for the level of capital monopolistic market-makers, such as NYSE specialists, should hold, depending on the underlying characteristics of the market they make. In assets with limited dividend volatility, it is preferable, surprisingly, that specialists are *not* too much capitalized if one cares about market liquidity. This however leads to transfers relative a market with many competitive market-makers. Thus opening specialists businesses to competition would redistribute gains from trade to customers and liquidity consumers but may reduce aggregate welfare and affect the provision of liquidity through time (better liquidity ahead of dividend payoff, lower close to payoff).

This paper departs from the literature of limits of arbitrage by studying the effects of the concentration of capital into a large arbitrageur. Consistent with the idea that the capital is concentrated, I relax the assumption of price-taking behaviour that prevails in models of financially constrained arbitrage.[7]

Attari and Mello (2006) also study the trading strategy of a monopolistic arbitrageur. There are two key differences with my analysis. First, in Attari and Mello, the arbitrageur faces a constraint based only on current prices, which generates an immediate feedback effect from capital to asset prices. This effect is absent in my model, as I assume that the financial constraint is forward-looking and is based not only on current but also future prices.[8] Second, all investors are rational in my model. In particular, local investors' demand for the asset is endogenous. By contrast, Attari and Mello assume that local investors have an exogenous downward-sloping demand curve. This assumption has two important consequences: i) it allows me to carry out a welfare analysis under different market structures; ii) it plays a central role in the dynamics of market depth and asset prices. It generates Coasian dynamics: as local investors rationally anticipate the price path, a monopolistic arbitrageur competes with herself over time. Vayanos (1999, 2001), Kihlstrom (2000) and DeMarzo and Urosevic (2007) have emphasized the analogy between the durable goods problem studied by Coase (1972) and asset pricing with non-competitive investors. Basak (1997) studies a Lucas economy with a monopolistic trader able to make future commitments. To the best of my knowledge, this paper is the first to solve the dynamic problem of a monopolistic investor under realistic financial constraints when all investors are rational and the monopolist cannot commit ex-ante. My contribution in this context is to show that financial constraints may alleviate the arbitrageur's commitment problem by providing a credible commitment device without impairing market liquidity.[9] Finally, this paper builds on a companion paper (Fardeau, 2011), where I derive the trading strategies and entry decisions of an *oligopoly* of unconstrained arbitrageurs. There I focus on price effects, and derive new predictions about the dynamics of entry.

[7]Gromb and Vayanos (2010) survey this extensive theoretical literature. My model is similar to a two-period version of Gromb and Vayanos (2002) with risk-free arbitrage and mean-variance preferences.

[8]Both constraints are plausible, but may stem from different contractual frictions. However, financial constraints of financial firms is a largely unexplored area. As an exception, He and Krishnamurthy (2011) embed a static contractual problem à la Holmstrom Tirole (1997), which generates an equity constraint, into an asset pricing model. The mechanisms generated by this constraint seems consistent with our results.

[9]There is a loose analogy between this effect and the use of leverage as a strategic bargaining tool by shareholders against unions (Perotti and Spier, 1993).

I proceed as follows. I present the model in the next section. In section 2.3, I review the competitive equilibrium and its properties. In Section 2.4, I study the monopolistic equilibrium. I compare liquidity across market structures in Section 2.5. Section 2.6 concludes. The appendix contains the proofs.

2.2 Model

The model extends Gromb and Vayanos (2010)'s setting to imperfectly competitive arbitrageurs. It has three periods, indexed by $t = 0, 1, 2$. The financial market is open at time 0 and time 1, and consumption takes place at time 2. There are two identical risky assets, A and B, and a risk-free asset with return r_f normalized to 0. Assets A and B are in zero net supply and pay a dividend D_2 at time 2, with $D_2 = D + \epsilon_1 + \epsilon_2$, where ϵ_t is a random variable with a symmetric bounded support $[-\bar{e}, \bar{e}]$, a mean of 0 and volatility σ. The distribution need not be further specified, but to facilitate the interpretation of the results, I will sometimes use a particular distribution described below. The information ϵ_t is revealed to all investors at time t before trading. The price of asset k at time t is denoted p_t^k. Each asset k is traded on its own, segmented market.

There are two types of investors. First, in each market, there are risk-averse local investors with mean-variance preferences: for $k = A, B$, $U\left(W_2^k\right) = \mathbb{E}\left(W_2^k\right) - \frac{a}{2}\mathbb{V}\left(W_2^k\right)$. Local investors experience liquidity shocks $s\epsilon_t$ that are correlated with the dividend of the risky asset. That is, at time $t = 0, 1$, local investors in market A receive a shock $s\epsilon_{t+1}$, where the magnitude of the shock, $s > 0$, is deterministic. B-investors receive opposite shocks, $-s\epsilon_{t+1}$.[10] Since k-investors have only access to asset k (market segmentation) and the risk-free security, they cannot share risk, although they could perfectly insure each other. The shocks and market segmentation imply potential price differences between assets A and B, although their cash-flows are identical. In particular, A-investors have a low valuation for the asset, and B-investors a high valuation.

At time 2, local investors consume their wealth W_2^k. Let E_t^k and Y_t^k denote their end-of-period positions in the risk-free and risky asset k, respectively. Then we can write local

[10] The results do not depend on the mean-variance framework. I use these preferences because they offer greater tractability when liquidity shocks are stochastic, an extension that I am planning to consider in future work.

investors' final wealth as follows:

$$\text{for } k = A, B,\ W_2^k = E_1^k + Y_1^k D_2,$$

The dynamic budget constraint follows from the dynamics of asset holdings: $Y_t^k = Y_{t-1}^k + y_t^k$ and $E_t^k = E_{t-1}^k - y_t^k p_t^k + s\epsilon_{t+1}$, where y_t^k denotes the time-t trade of investors k.

There is an additional investor, the arbitrageur, who can trade all assets without restriction. The arbitrageur is also endowed with mean-variance preferences over wealth: $u(W_2) = \mathbb{E}(W_2) - \frac{b}{2}\mathbb{V}(W_2)$, albeit with a potentially different risk-aversion b. Given that she has access to all securities, the arbitrageur's final wealth is

$$W_2 = \sum_{k=A,B} X_1^k D_2 + B_1$$

with for each asset k, $X_t^k = X_{t-1}^k + x_t^k$ denotes the end-of-period position at time t in asset k, x_t^k the corresponding trade, and $B_t = B_{t-1} - \sum_{k=A,B} x_t^k p_t^k$, the arbitrageur's risk-free asset holdings at the end of period t. I assume that the arbitrageur has no endowment in the risky assets $X_{-1}^k = 0$, $k = A, B$ and starts with an initial wealth $W_{-1} = B_{-1}$. Apart from Section 2.3, where I assume that the arbitrageur stands for a continuum of competitive investors, I assume that the arbitrageur is a price-setter in both A and B markets. Specifically, I assume that the arbitrageur chooses positions, knowing the local investors' demand in each market, and imposing market-clearing.

Whether the arbitrageur is price-taker or price-setter, she needs capital to trade the risky assets. I model the financial constraint in the same fashion as Gromb and Vayanos (2002, 2010). Arbitrageurs have a margin account V_t^k in each market, and their positions must be fully collateralized. That is, the arbitrageur's wealth in this account must cover the maximum possible loss on the position over the next period:

$$V_{t-1}^k \geq \max_{p_{t+1}^k} X_t^k \left(p_t^k - p_{t+1}^k\right)$$

Hence, in total, the arbitrageur's wealth must cover the total maximum loss on each account:[11]

$$W_{t-1} \geq \sum_{k=A,B} \max_{p_{t+1}^k} X_t^k \left(p_t^k - p_{t+1}^k\right) \tag{2.1}$$

The presence of the financial constraint implies that arbitrageurs may not be able to fully eliminate the price differences between A and B assets. The modeling of the constraint also implies that asset A cannot be used as collateral for asset B (and vice-versa). In other words, cross-collateralization is not allowed, which can be viewed as a consequence of the assumption of market segmentation. In practice, cross-collateralization is often limited by financiers who are concerned about imperfect correlation between assets (although this would not be an issue here). Sometimes traders also voluntarily avoid it in order to avoid revealing their trading strategies.[12] The full-collateralization of each separate account rules out default in equilibrium.

The constraint corresponds to one-periods VaR constraint at the 100 percent level (as implied by the assumption of full collateralization). The 100 percent level is for simplicity only, as it rules out default in equilibrium and thus makes welfare comparisons simpler[13], but the constraint is motivated by real-world margin setting.[14] An important feature of the constraint is that it is forward-looking, in the sense that it is based on both current and future prices. This is in contrast to Attari and Mello (2006), who also study the trading strategy of a monopolistic arbitrageur but consider a constraint based only on current prices.

Following Gromb and Vayanos (2002), I will focus on equilibria in which the arbitrageur holds symmetric positions in both assets, i.e. $X_t^A = -X_t^B = X_t$. Given that the arbitrageur starts with no endowment in the risky assets, this implies that $x_t^A = -x_t^B = x_t$, for $t = 0, 1$. Using the symmetry assumption, we can rewrite the arbitrageur's budget constraint as

[11] I define W_{t-1} as the end-of-period wealth, while Gromb and Vayanos (2002) use $\tilde{W}_t$ as the beginning-of-period wealth. Given this difference in notation: $W_{t-1} = \tilde{W}_t$. The same applies to the definition of margin accounts.

[12] For instance, Pérold (1999) reports: "LTCM inernalized most of the back-office functions associated with contractual arrangements, due to the complexity and and advanced nature of many of the firm's trades. This also helped maintain the confidentiality of its positions. LTCM chose Bear Stearns as a clearing agent partly because Bear Stearns was committed to customer business rather than being focused on proprietary trading, and thus there were fewer conflicts of interest."

[13] There is no need to compute the welfare of financiers on the other side of the constraint.

[14] See Brunnermeier and Pedersen (2009), Appendix A, for additional institutional details.

follows:

$$W_2 = W_{-1} + \sum_{t=0,1} x_t \Delta_t, \text{ with } \Delta_t = p_t^B - p_t^A$$

The equation shows that by setting up opposite position in each leg of the arbitrage, the arbitrageur eliminates all fundamental risk and derives all her profits from exploiting the price difference Δ between the two markets. The symmetry assumption also simplifies the financial constraint, because it implies that the risk premia on asset A and B are opposite. That is, $\phi_t^A = D_t - p_t^A = \frac{\Delta_t}{2} = -\phi_t^B$, where D_t is the conditional expected value of the asset at time t: $D_t = D_{t-1} + \epsilon_t$. This implies that $p_t^k - p_{t+1}^k = \phi_{t+1}^k - \phi_t^k - \epsilon_{t+1} = \frac{\Delta_{t+1} - \Delta_t}{2} - \epsilon_{t+1}$. As a result, we can rewrite the financial constraint (2.1) as follows:

$$\begin{aligned} W_{t-1} &\geq \sum_{k=A,B} \max_{p_t^k} X_t^k \left(p_t^k - p_{t+1}^k\right) \\ &\geq \max_{\epsilon_{t+1}} X_t \left(\frac{\Delta_{t+1} - \Delta_t}{2} - \epsilon_{t+1}\right) + \max_{\epsilon_{t+1}} -X_t \left(-\frac{\Delta_{t+1} - \Delta_t}{2} - \epsilon_{t+1}\right) \\ &\geq 2X_t \left(\frac{\Delta_{t+1} - \Delta_t}{2}\right) + \max_{\epsilon_{t+1}} X_t \left(-\epsilon_{t+1}\right) + \max_{\epsilon_{t+1}} -X_t \left(-\epsilon_{t+1}\right) \\ &\geq 2|X_t|\bar{e} - X_t \left(\Delta_t - \Delta_{t+1}\right) \end{aligned}$$

The last step follows from the symmetric support of the distribution. Since the arbitrage is risk-free, the arbitrageur will hold a long position in equilibrium, $X_t \geq 0$. Thus, we can rewrite the right-hand side as $m_t X_t$, where the margin m_t is

$$m_t = 2\bar{e} - \left(\Delta_t - \Delta_{t+1}\right)$$

The properties of the margin are key for the dynamics of the model. Clearly, margins increase with the dispersion (and consequently, volatility) of the fundamental $\bar{e}$. A more volatile asset leads to a larger potential loss on the position, which induces financiers to ask for more collateral. Margins also depend on the mispricing between asset A and B. More specifically, they depend on the *change* in the mispricing, $\Delta_t - \Delta_{t+1}$. If financiers expect market liquidity to improve, i.e. $\Delta_{t+1} \leq \Delta_t$, they reduce current margins. Hence, the financiers' behaviour assumed here leads to countercyclical margins relative to mispricings (illiquidity). Said differently, margins play a stabilizing role for asset prices. If a drop in liquidity (i.e. large Δ_t) is temporary, then financiers do not necessarily ask for more capital. This stands in sharp contrast to the financial constraint considered in Attari and Mello (2006), which is based only on current prices. It also differs from the uninformed financier case in Brunnermeier and Pedersen (2009), in which uncertainty about whether the mispricing will decrease or not in the future can lead to procyclical, destabilizing margins.[15]

Remark: volatility, tail risk, and dispersion of the fundamental. I show in section 2.4 that the ratio $\frac{\bar{e}}{\sigma^2}$ plays an important role for the equilibrium. Since the dispersion of the fundamental $\bar{e}$ and its volatility are related, it is useful to specify a simple distribution of fundamental shocks. The following coarse four-point symmetric distribution is enough to gain intuition:

Lemma 7 *Let $(\mu, p) \in \left]0, \infty\right[\times \left]0, 1\right[$, and $\epsilon_t \sim \mathcal{E}\left[-\bar{e}, \bar{e}\right]$, where the random variable $\mathcal{E}$ takes the following values:*

$$\mathcal{E} = \begin{cases} -\bar{e} \text{ with probability } \frac{1}{2} - p \\ -\frac{\bar{e}}{\mu} \text{ with probability } p \\ \frac{\bar{e}}{\mu} \text{ with probability } p \\ \bar{e} \text{ with probability } \frac{1}{2} - p \end{cases}$$

Then $\mathbb{E}\left(\mathcal{E}\right) = 0$ *and* $\sigma^2 = \mathbb{V}\left(\mathcal{E}\right) = \bar{e}^2 \left[1 + 2p\left(\frac{1}{\mu^2} - 1\right)\right]$.

This example shows how the variance of fundamental shocks relates to the support boundary, $\bar{e}$. Although this distribution is just meant to fix ideas, the relation between σ^2 and $\bar{e}^2$ is more general. Further, this example can help us clarify how the volatility in a symmetric distribution can relate to the shape of the tails. The parameter μ measures how far the median values are from the mean 0, while p measures the weight of the tails: a small p means that tail events in which ϵ_t takes the extreme values $\bar{e}$ or $-\bar{e}$ are likely. Clearly the variance decreases with μ and with p (since $\frac{1}{\mu^2} - 1 < 0$). Hence, when extreme events are likely (small p), the variance is large. More generally, this example shows that while an increase in the boundaries of the distribution of fundamentals $\bar{e}$ always increases volatility, volatility may also increase because of a change in the shape of the distribution, without changing the dispersion $\bar{e}$.

[15] Brunnermeier and Pedersen show that a margin spiral, in which low liquidity leads to higher margins, which further limits the ability of arbitrageurs to provide liquidity, can result from the uninformed case. This margin spiral complements and amplifies the loss spiral created by the financial constraint ("a decrease in arbitrageurs' capital impairs their ability to provide liquidity and eliminate the mispricing, which in turn reduces their capital"). Under our assumptions, there can be a loss spiral, but no margin spiral.

2.3 Competitive equilibrium benchmark

In this section, I briefly recall the competitive benchmark derived in Gromb and Vayanos (2002). The model illustrates how liquidity (given by the spread between assets A and B) depends on arbitrageurs' capital.

Proposition 10 *(Gromb and Vayanos, 2002) There exists a unique symmetric competitive equilibrium given by:*

- *If $W_{-1} \geq \omega^* \equiv 2s\bar{e}$, the financial constraint never binds, the arbitrageurs absorb the liquidity shock s, i.e. $X_t = s$ at $t = 0, 1$, and the spread between assets A and B is always 0: $\Delta_0 = \Delta_1 = \Delta_2 = 0$*

- *If $0 \leq W_{-1} < \omega^*$, the financial constraint binds at $t = 0$ and $t = 1$ and the spread between assets A and B narrows over time and is closed only at $t = 2$, i.e. $\Delta_0 > \Delta_1 > \Delta_2 = 0$. The arbitrageur position in asset A is given by:*

$$x_0 - x_0 \frac{a\sigma^2 (s - x_0)}{\bar{e}} = \frac{W_{-1}}{2\bar{e}} \tag{2.2}$$

$$X_1 - X_1 \frac{a\sigma^2 s - X_1}{\bar{e}} = x_0 \tag{2.3}$$

The equilibrium links liquidity (via the spread) to arbitrageurs' initial capital and has a simple form: if arbitrageurs' capital is large enough, then the market is perfectly liquid, as reflected by the absence of spread between assets A and B; if instead arbitrageurs start with less capital, then the financial constraints are binding, and assets A and B trade at a positive spread, which decreases over time. An increase in the liquidity shock s affecting local investors or in the dispersion of the fundamental (increase in $\bar{e}$) tightens (proportionately) the financial constraint: the financiers anticipate that the price divergence between assets A and B is potentially larger and demand more collateral.

To facilitate comparison with the monopolistic case and gain further insight, I derive the equilibrium positions and spread as a function of arbitrageurs' capital:

Corollary 14 *If $0 \leq W_{-1} < \omega^*$, the arbitrageurs' positions in asset A are:*

$$x_0 = \frac{a\sigma^2 s - \bar{e} + \sqrt{Q}}{2a\sigma^2}; \; X_1 = \frac{a\sigma^2 s - \bar{e} + \sqrt{U}}{2a\sigma^2}$$

with $Q = (\bar{e} - a\sigma^2 s)^2 + 2a\sigma^2 W_{-1}$ *and* $U = (\bar{e} - a\sigma^2 s)^2 + 4a\sigma^2 x_0 \bar{e}$.

In equilibrium, the spreads are:

$$\Delta_0 = 2\left(a\sigma^2 s + \bar{e}\right) - \sqrt{Q} - \sqrt{U};\ \Delta_1 = a\sigma^2 s + \bar{e} - \sqrt{U}$$

This result shows how the positions and the spreads depend on arbitrageurs' capital: clearly the spread at time 0 and 1 decreases with capital W_{-1} (i.e. $\frac{\partial \Delta_t}{\partial W_{-1}} < 0$, and even more so if capital is low, a non-linear effect (i.e. $\frac{\partial^2 \Delta_t}{\partial W_{-1}^2} < 0$).[16]

2.4 Equilibrium with a monopolistic arbitrageur

In this section I derive the trading strategy of a monopolistic arbitrageur and compare it to the competitive case. In the monopoly case, market power allows the arbitrageur to limit liquidity but also to operate with a much lower level of initial capital thanks to rent capture. However, the arbitrageur also faces a commitment problem as local investors recognize that, even though the arbitrageur can limit liquidity at the current stage, she always has an interest to provide further liquidity at a later stage. Liquidity provision by a single arbitrageur thus resembles the provision of a durable good by a monopolist. In this context, financial constraints can work as a commitment device to limit liquidity at a later stage. This device is credible, however, only if capital is not too abundant. If instead the arbitrageur's capital is sufficiently large, this device does not work as local investors recognize that the arbitrageur has an incentive to deviate at a later stage (time inconsistency).

2.4.1 Liquidity provision and time consistency

I start by introducing some useful notation and presenting the solution method. Since the arbitrageur may be constrained (superscript c) or not (superscript u) at each date, there are two payoffs associated with the different combinations at time 1: $J_1^{c_1}$ and $J_1^{u_1}$, and four at time 0: $J_0^{u_1,u_0}$, $J_0^{u_1,c_0}$, $J_0^{c_1,u_0}$, $J_0^{c_1,c_0}$.

At time 1, the arbitrageur enters with a position x_0 in the asset A (and the opposite in asset B). The local investors' first-order conditions and market-clearing imply that $\Delta_1(X_1) =$

[16] See also Brunnermeier and Pedersen (2008).

$2a\sigma^2 (s - X_1)$, where $X_1 = X_0 + x_1 = x_0 + x_1$. The arbitrageur's maximization problem is thus:

$$\begin{aligned} J_1 &= \max_{x_1} B_0 + x_1 \Delta_1 (X_1) \\ s.t. \quad & W_0 \geq 2X_1 [\bar{e} - (\Delta_1 - \Delta_2)] \end{aligned}$$

Therefore, denoting $x_1^u = \frac{s - x_0}{2}$ the first-best solution, and x_1^c the constrained solution (which saturates the constraint: $W_0 = 2X_1^c [\bar{e} - (\Delta_1 - \Delta_2)]$), the two possible payoffs at time 1 are given by:

$$\begin{aligned} J_1^{u_1}(x_0) &= B_0 + 2a\sigma^2 x_1^u (s - X_1^u) = B_0 + \frac{a\sigma^2}{2} (s - x_0)^2 \\ J_1^{c_1}(x_0) &= B_0 + 2a\sigma^2 x_1^c (s - X_1^c), \text{ with } X_1^k = x_0 + x_1^k, \; k = u, c \end{aligned}$$

Of course, by construction $J_1^{c_1}(x_0) \leq J_1^{u_1}(x_0)$. Similarly, the time-1 equilibrium spread depends on the state and the arbitrageur's beginning-of-period position x_0: $\Delta_1 = \Delta_1^{u_1}(x_0)$ if the constraint is slack, and $\Delta_1 = \Delta_1^{c_1}(x_0)$ otherwise.

At time 0, the relation between the four payoffs is more complicated: local investors being forward-looking, the price schedule $\Delta_0 (x_0)$ depends on their beliefs about the state of the market in the next period (constrained or unconstrained arbitrageur). In particular, their first-order conditions and market-clearing always imply the following price schedule (shown in the appendix):

$$\Delta_0 (x_0) = \Delta_1 (x_0) + 2a\sigma^2 (s - x_0)$$

That is, we can define $\Delta_0^{u_1}(x_0)$, and $\Delta_0^{c_1}(x_0)$ the time-0 price schedule implied by the corresponding beliefs about the state at time 1. Depending on the anticipated time-1 state, the arbitrageur's maximization problem is:

$$\begin{aligned} J_0^{u_1} &= \max_{x_0} W_{-1} + x_0 \Delta_0^{u_1}(x_0) + 2a\sigma^2 x_1^u (s - X_1^u) \\ s.t. \quad & W_{-1} \geq 2x_0 [\bar{e} - (\Delta_0 - \Delta_1)] \\ \text{or } J_0^{c_1} &= \max_{x_0} W_{-1} + x_0 \Delta_0^{c_1}(x_0) + 2a\sigma^2 x_1^c (s - X_1^c) \\ s.t. \quad & W_{-1} \geq 2x_0 [\bar{e} - (\Delta_0 - \Delta_1)] \end{aligned}$$

Local investors have rational expectations and can anticipate the price path, and thus whether the arbitrageur's constraint will be binding or not. Hence a necessary condition for the maximization problems to make sense is that in equilibrium, the time-0 trade does satisfy the time-1 constraint if local investors expect the arbitrageur's constraint to be slack, and vice-versa if they expect the constraint to be binding. The condition is:

Lemma 8 *Suppose that in equilibrium the arbitrageur chooses to trade a quantity x_0 at time 0. It is consistent with being unconstrained at time 1 if and only if*

$$W_{-1} - s\bar{e} + a\sigma^2 \frac{s^2}{2} + \left(2a\sigma^2 s - \bar{e}\right) x_0 - \frac{5}{2} a\sigma^2 x_0^2 \geq 0 \tag{2.4}$$

Depending on whether the constraint binds or not at time 0, the arbitrageur can trade the first-best quantity x_0^u or the constrained quantity x_0^c which "maxes out" her financial constraint.[17] The corresponding payoffs are $J_0^{u_1,u_0}$, $J_0^{u_1,c_0}$, $J_0^{c_1,u_0}$, $J_0^{c_1,c_0}$, which are functions of the base parameters.

The arbitrageur chooses $\max\left(J_0^{u_1,u_0}, J_0^{u_1,c_0}, J_0^{c_1,u_0}, J_0^{c_1,c_0}\right)$. By definition: $J_0^{u_1,u_0} \geq J_0^{u_1,c_0}$ and $J_0^{c_1,u_0} \geq J_0^{c_1,c_0}$, i.e. conditional on the state at time 1, it is better to be unconstrained at time 0. However, because the price schedule is different at time 0 depending on what local investors believe about time 1, it is not guaranteed that $J_0^{u_1,u_0}$ or $J_0^{u_1,c_0}$ are greater than $J_0^{c_1,u_0}$. In other words, from the point of view of time 0, being unconstrained at time 1 may not be more profitable than being constrained. In particular, when capital is large enough, the first-best solution is likely to be feasible, and so are other trading strategies. In the absence of competitive pressure, the arbitrageur can in principle deviate from, say, the unconstrained strategy to a constrained strategy if this raises her profit. This yields a time consistency issue. Suppose, for instance, that $J_0^{c_1,u_0} \geq J_0^{u_1,u_0}$ for some parameter region. Then it is in the arbitrageur's interest to trade in a way that leaves her constrained at time 1. However, if the arbitrageur has abundant capital, she may have enough dry powder left when time 1 comes to be unconstrained. She may therefore be able to re-optimize and trade her first-best quantity, a time-inconsistent behaviour. This will be the case if inequality (2.4) in Lemma 8 is not satisfied. This should not occur in equilibrium, however, because local investors are rational: their expectations must therefore be correct in equilibrium.[18] The definition of the equilibrium in the monopoly case summarizes this discussion:

[17] This is optimal, since the arbitrage is riskfree.

[18] This rules out any exogenous commitment device that the arbitrageur could credibly use to "tie her hands" in such cases.

Definition 1 *An equilibrium is a collection of arbitrageur's trades* $(x_t)_{t=0,1}$ *(or equivalently positions* x_0, X_1*) in asset A and opposite trades (positions) in asset B, such that*

- *given prices, the local investors maximize their expected utility of final consumption,*
- *the arbitrageur maximizes her expected payoff subject to financial constraints, local investors' demands and market-clearing,*
- *local investors have rational expectations.*

I now describe the strategies, under which conditions they are feasible and / or credible and the equilibrium.

2.4.2 Equilibrium

Scarce capital

I start with the case where the arbitrageur's capital is low. Then the arbitrageur has no financial flexibility and there is only one feasible strategy, so that the equilibrium is easy to determine:

Proposition 11 *If* $W_{-1} < \omega^c \equiv \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2 - \frac{\bar{e}^2}{10a\sigma^2}$*, then the financial constraint binds at* $t = 0$ *and* $t = 1$*, and the arbitrageur's trades and the equilibrium spreads are the same as in Proposition 10 and Corollary 14.*

If $W_{-1} \geq \omega^c$*, it is always possible for the arbitrageur to be unconstrained at time 1 and trade* $x_1^u = \frac{s-x_0}{2}$*. Other strategies in which the arbitrageur is constrained in one or two periods are also feasible.*

It is already clear from this result that the condition under which the arbitrageur is fully or partially constrained is very different from the competitive case. In particular, the threshold ω^c is no longer linear in s and $\bar{e}$ and also depends on volatility of the fundamental σ. I proceed with the analysis of the more complicated abundant capital case and relegate comments and intuitions to the end of this section.

Strategies with more abundant capital

If capital is more abundant ($W_{-1} \geq \omega^c$), the arbitrageur has more financial flexibility and can choose from a larger set of strategies. I first describe the strategies available to the arbitrageur, starting with the case where she chooses to be unconstrained at time 1.

Lemma 9 *Denote* $\omega_0^m = \frac{4}{5}s\bar{e} - \frac{12}{25}a\sigma^2 s^2$, $\omega_1^m = \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2$, *and note that* $\omega^c = \omega_1^m - \frac{\bar{e}^2}{10a\sigma^2} < w_1^m$.

The following holds:

- *If* $W_{-1} \geq \max(\omega_0^m, \omega_1^m)$, *the arbitrageur can be unconstrained at time 0 and time 1. The unconstrained strategy* (u_1, u_0) *consists of the following trades*

$$x_0 = \frac{2}{5}s,\ x_1 = \frac{3}{10}s$$

and yields a payoff $J_0^{u_1,u_0} = W_{-1} + \frac{9}{10}a\sigma^2 s^2$.

- *If* $W_{-1} \in [\omega^c, \max(\omega_0^m, \omega_1^m)[$, *then the arbitrageur must reduce her time-0 position in order to remain unconstrained at time 1 (if* $\omega_0^m \leq W_{-1} < \omega_1^m$*) or to satisfy the time-0 constraint (if* $\omega_1^m \leq W_{-1} < \omega_0^m$*), or to satisfy both constraints (if* $\omega^c \leq W_{-1} < \min(\omega_0^m, \omega_1^m)$*). The arbitrageur's constraints are:*

$$\begin{aligned} &\textit{at time 0:} \quad W_{-1} - 2\left(\bar{e} - a\sigma^2 s\right)x_0 - 2a\sigma^2 x_0^2 \geq 0 \\ &\textit{at time 1:} \quad W_{-1} - s\bar{e} + a\sigma^2\frac{s^2}{2} + \left(2a\sigma^2 s - \bar{e}\right)x_0 - \frac{5}{2}a\sigma^2 x_0^2 \geq 0 \end{aligned}$$

The arbitrageur's strategy to remain unconstrained at time 1, if feasible, is the maximum of four quantities, saturating the time-0 and time-1 constraints:

$$\begin{aligned} x_0^0 &= \frac{a\sigma^2 s - \bar{e} + \sqrt{Q}}{2a\sigma^2} > 0,\ x_0^{0'} = \frac{a\sigma^2 s - \bar{e} - \sqrt{Q}}{2a\sigma^2} < 0 \\ x_0^1 &= \frac{2a\sigma^2 s - \bar{e} - \sqrt{R}}{5a\sigma^2},\ x_0^{1'} = \frac{2a\sigma^2 s - \bar{e} + \sqrt{R}}{5a\sigma^2} \end{aligned}$$

where $R = 10a\sigma^2 W_{-1} + \bar{e}^2 - 14a\sigma^2 s\bar{e} + 9a^2\sigma^4 s^2$. *The sign of* x_0^1 *and* $x_0^{1'}$ *depends on the ratio* $\frac{\bar{e}}{a\sigma^2 s}$ *and the position of arbitrageur capital relative to the threshold* $\omega^p = s\bar{e} - \frac{1}{2}a\sigma^2 s^2$.

The payoff of this strategy, as a function of the time-0 trade, is $J_0^{u_1,c_0}(x_0) = W_{-1} + \frac{a\sigma^2 s^2}{2} + 2a\sigma^2 s x_0 - \frac{5}{2}a\sigma^2 x_0^2$.

The intuition for this result is straightforward. With abundant capital, $W_{-1} \geq \max(\omega_0^m, \omega_1^m)$, the arbitrageur is unconstrained. With a somewhat lower level of capital, the arbitrageur must alter her trading strategy if she wants to remain unconstrained at time 1. Since capital is relatively abundant ($W_{-1} > \omega^c$), other strategies in which the arbitrageur chooses to be constrained at time 1 or at both time 0 and time 1 are feasible too, but may not be time consistent. (I refer to a time consistent strategy as "credible".)

Lemma 10 *Suppose $W_{-1} \geq \omega^c$. Then strategies (c_1, u_0) and (c_1, c_0) in which the arbitrageur voluntarily chooses to be constrained at time 1, or at time 0 and time 1, are available under the following conditions.*

- *The (c_1, u_0) strategy consists of the following positions:*

$$x_0 = \frac{s}{2}, \; X_1 = \frac{s}{2} - \frac{\bar{e} - \sqrt{U^m}}{2a\sigma^2}, \; \text{with } U^m = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2 W_{-1} + a^2\sigma^4 s^2$$

It is feasible and credible (i.e. time consistent) if and only if $W_{-1} \geq \omega^p \equiv s\bar{e} - \frac{1}{2}a\sigma^2 s^2$ and $W_{-1} < \bar{\omega}^p \equiv \frac{3}{2}s\bar{e} - \frac{7}{8}a\sigma^2 s^2$, with $\omega^p > \omega^c$. Its payoff is $J_0^{c_1,u_0} = \frac{\bar{e}}{a\sigma^2}\left[\sqrt{U^m} - \left(\bar{e} - a\sigma^2 s\right)\right]$.

- *The (c_1, c_0) strategy is given in Corollary 14 and its payoff is*

$$J_0^{c_1,c_0} = \frac{\bar{e}}{a\sigma^2}\left[a\sigma^2 s - \bar{e} + \sqrt{a^2\sigma^4 s^2 - \bar{e}^2 + 2\bar{e}\sqrt{Q}}\right]$$

It is credible if and only if $h\left(W_{-1}\right) < 0$, with

$$h\left(W_{-1}\right) = -\frac{1}{4}W_{-1} + \frac{1}{4}a^2\sigma^4 s^2 - \frac{s\bar{e}}{4} + \frac{\bar{e}^2}{2a\sigma^2} + \frac{3\bar{e} - a\sigma^2 s}{4a\sigma^2}\sqrt{Q} \tag{2.5}$$

Given the feasibility and credibility conditions, determining the equilibrium requires to compare $J_0^{u_1,u_0}$ to $J_0^{c_1,u_0}$ or $J_0^{c_1,c_0}$ if the unconstrained strategy is feasible at time 1, and $J_0^{u_1,c_0}$ to $J_0^{c_1,u_0}$ or $J_0^{c_1,c_0}$ if it is not. This is no easy task given that the order of the thresholds ω_0^m, ω_1^m, ω^c, ω^p and $\bar{\omega}^p$ changes with the ratio $\frac{\bar{e}}{a\sigma^2 s}$. Further, note that the last credibility condition is not explicit since Q depends on W_{-1}. In spite of the large number of cases, it is possible to derive some general results. There are three main regions for the equilibrium. I now state the main results of this section:

Abundant capital: trading off efficiency and capital intensity

Proposition 12 *The following holds:*

- *If* $W_{-1} \geq \omega^m = \Lambda s\bar{e} - \Gamma a\sigma^2 s^2$, *there exists a unique equilibrium in which the financial constraint is slack at* $t = 0$ *and* $t = 1$, *the arbitrageur's trades in asset A are* $x_0 = \frac{2}{5}s$ *and* $x_1 = \frac{3}{10}s$ *and the equilibrium spreads are* $\Delta_0^m = \frac{9}{5}a\sigma^2 s$ *and* $\Delta_1^m = \frac{3}{5}a\sigma^2 s$.

- Λ *and* Γ *are such that* $0 < \Lambda < 2$ *and* $0 < \Gamma < 1$, *and depend on the ratio* $\frac{\bar{e}}{a\sigma^2 s}$ *as follows:*

$$\begin{cases} \Lambda = \frac{4}{5} \text{ and } \Gamma = \frac{12}{25} \text{ if } \frac{\bar{e}}{a\sigma^2 s} \in \left[0, \frac{21}{10\left(1+\sqrt{5}\right)}\right[\\ \Lambda = 1 + \frac{1}{\sqrt{5}} \text{ and } \Gamma = \frac{9}{10} \text{ if } \frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)} \end{cases}$$

- *Relative to the benchmark competitive case, there is a trade-off between liquidity provision and capital intensity:*
  - *For all* $\bar{e}$, a, σ, s, $\omega^m < \omega^*$, *i.e. the monopolistic arbitrageur can remain unconstrained with a lower initial capital.*
  - *However, the monopoly provides less liquidity:* $\Delta_t^m > \Delta_t^*$, $t = 0, 1$.

This region is comparable to the first region of the competitive case ($W_{-1} \geq \omega^*$). There are three noticeable differences. First, although the arbitrageur is unconstrained, assets A and B trade at a spread, i.e. liquidity is imperfect in the economy. This is simply due to the arbitrageur's market power. Given the absence of competition, the arbitrageur limits the amount she buys from local investors with low valuation for the asset and sells to those with high valuation. This keeps the spread open in equilibrium, which allows the arbitrageur to make a profit.[19] Second, and consequently, the financial constraint is no longer linear in the dispersion of the fundamental $\bar{e}$ and the liquidity shock s. In fact, it is now quadratic in s[20], so that the following comparative statics obtains:

Corollary 15 *The threshold* $\bar{\omega}^m$ *features the following comparative statics:*

- *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[0, \frac{21}{10\left(1+\sqrt{5}\right)}\right[$, *i.e. if volatility is high enough, then a small increase in the liquidity shock* s *loosens the financial constraint,*

- *If* $\frac{\bar{e}}{a\sigma^2 s} > \frac{21}{10\left(1+\sqrt{5}\right)}$, *i.e. if volatility is low enough, a small increase in* s *tightens the financial constraint.*

[19] Indeed in equilibrium the arbitrageur's position is $X_t < s$. (note that $x_0 = X_0$)

[20] It is also quadratic in $\bar{e}$ since σ^2 is a function of $\bar{e}^2$.

The intuition for this result is simple. On one hand, an increase in the dispersion of the fundamental $\bar{e}$ increases its volatility σ, which in turn increases the magnitude of the potential divergence from fundamental in the next period and makes a default by the arbitrageur more likely from the viewpoint of financiers. This tightens the financial constraint, an effect akin to the competitive case. On the other hand, under our modeling assumptions, the financiers (implicitly) recognize that an increase in volatility is equivalent to an increase in the willingness of local investors to share their risk and to accept large price concessions. This increases the profitability of the arbitrage strategy and allows the arbitrageur to capture larger rents. The arbitrageur reaps larger capital gains, which relaxes the financial constraint. This second effect relies on the assumption that financiers understand the sources of illiquidity and the dynamics of liquidity provision, which generates countercyclical (stabilizing) margins.[21]

Bearing this simple trade-off in mind, it is easy to interpret $\bar{\omega}^m$ as the sum of two terms: the first term, $\Lambda s\bar{e}$, represents the maximum loss on the position caused by a change in the fundamental and is therefore a multiple of $\bar{e}$, which measures the largest possible change in the fundamental. Note that it depends on the arbitrageur's position , which is less than $2s$. By contrast, in the competitive case, $\omega^* = 2s\bar{e}$, because arbitrageurs fully absorb the liquidity shock affecting market A and B, which is $2s$ in total (corresponding to a position of size s in each leg of the arbitrage). The second term in $\bar{\omega}^m$, $-\Gamma a\sigma^2 s^2$, is an adjustment measuring how much past or future profits due to rent extraction lower the capital requirement. It is thus specific to the monopoly case, as financiers anticipate that perfect competition drives profits to zero.

The third noticeable difference is that there are two different regions for the threshold ω^m, while there is a unique threshold ω^* in the competitive benchmark. These regions can be expressed in terms of low or high volatility since by using the four-point distribution given in Section 1.2, the ratio $\frac{\bar{e}}{a\sigma^2 s}$ becomes $\frac{\bar{e}}{a\bar{e}^2\left(1+2p\left(\frac{1}{\mu^2}-1\right)\right)s} = \frac{1}{a\bar{e}\left(1+2p\left(\frac{1}{\mu^2}-1\right)\right)s}$. For simplicity, I will refer to a situation with large fundamental dispersion $\bar{e}$ as high volatility and small fundamental dispersion as low volatility.[22] Note that $\bar{\omega}^m = \omega_0^m$ in the high volatility region, i.e. the time-0 feasibility constraint is binding in this case. In the region of low volatility, $\bar{\omega}^m = v_1$, where v_1 is the threshold such that $J_0^{u_1,u_0} \geq J_0^{c_1,u_0}$. The interpretation is in terms

[21] Brunnermeier and Pedersen (2008)'s model nests both stabilizing and destabilizing margins. With destabilizing margins, the second effect would remain but would bite less.

[22] Equivalently, low volatility can stem from low risk in the tail (large μ or low p), so that we could rephrase the analysis in terms of large or small tail risk. Note that because the distribution is symmetric, tail risk equally includes good and bad events.

of which effects of the maximum position loss and the profit adjustment dominates. When volatility is high, the profit adjustment is large and therefore $v_1 < \omega_0^m$, because v_1 takes into account all expected profits, while ω_0^m reflects only one period expected profits. It is the opposite in the low volatility region, where the profit adjustment is small, meaning that the feasibility constraint is not the binding constraint. The uniqueness of the threshold in the competitive benchmark is due to the absence of profit adjustment, since competition drives profit to zero.

Intermediate level of capital: voluntarily-constrained trading

When the arbitrageur has less capital, she may credibly choose to be constrained at time 1.

Proposition 13 *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{2\sqrt{5}}{5}, 1\right[$ *and* $W_{-1} \in [\omega^p, v_1[$ *or if* $\frac{\bar{e}}{a\sigma^2 s} > 1$ *and* $W_{-1} \in [\omega_1^m, v_1[$, *then the unconstrained strategy is feasible. However, in the unique equilibrium, the financial constraint is slack at* $t = 0$ *and binding at* $t = 1$, *i.e. the arbitrageur's positions are:* $x_0 = \frac{s}{2}$ *and* $X_1 = \frac{s}{2} - \frac{\bar{e}-\sqrt{U^m}}{2a\sigma^2}$. *The equilibrium spreads are* $\Delta_0^{c_1,u_0} = 2a\sigma^2 s + \bar{e} - \sqrt{U^m}$ *and* $\Delta_1^{c_1,u_0} = a\sigma^2 s + \bar{e} - \sqrt{U^m}$.

The result shows that the arbitrageur voluntarily chooses to be constrained when volatility is low enough and her level of capital is intermediate. The reason why being constrained at time 1 might be optimal is related to the Coasian dynamics of the model. Intuitively, the arbitrageur chooses her trading strategy to keep the spread open as long as possible. Since the asset matures only at time 2, local investors have some freedom to chose the date at which they consume liquidity. They rationally anticipate that after providing liquidity (i.e. tackling the arbitrage opportunity) at time 0, the arbitrageur will provide further liquidity at time 1, further decreasing the spread. Hence providing liquidity early, at time 0, reduces the profitability of later liquidity provision for the arbitrageur, unless she is able to credibly commit to keep the spread large in the future. To this extent, the financial constraint works as a commitment device for the arbitrageur, who can then extract larger rents at time 0. Indeed, in equilibrium, local investors anticipate that the arbitrageur's constraint binds time 1, which increases their willingness to accept large price concessions at time 0, increasing the potential price gap and thus the arbitrageur's capital gain.

When does this occur in equilibrium? The conditions on capital and volatility given in Proposition 13 have an intuitive interpretation. First, the arbitrageur's capital must be below some threshold $v_1 < \bar{\omega}^p$, which guarantees that the arbitrageur cannot re-optimize (by Lemma 10). Intuitively, if the arbitrageur has a very abundant capital, local investors anticipate that she will not be actually constrained at time 1, and the equilibrium unravels. If capital is too low, however, the arbitrageur cannot serve the additional liquidity demand at time 0 and cannot benefit from committing to be constrained. These conditions on arbitrageur's capital are combined with the requirement that the volatility be low enough, i.e. that $\frac{\bar{e}}{a\sigma^2 s}$ is low enough. Intuitively, if this was not the case, the unconstrained strategy would be so profitable that the arbitrageur would not be tempted to boost her trading profit by organizing a liquidity shortage. In particular, I show in the appendix that for a high enough volatility, the feasibility of the c_1, u_0 strategy always implies that it is dominated by the unconstrained strategy.

At a deeper level, one may wonder how it is possible for the constraint to be binding only at one date in equilibrium, while in the competitive case, either the constraint binds at all dates or never. This point is related to the trade-off between position funding and profit adjustment. The position funding effect depends on the size of the arbitrageur's position. The profit adjustment depends on expected profits. As time passes, the position increases, and therefore the constraint should tighten. But at the same time, the profit adjustment also increases, so that the constraint at time 1 may be less severe than the constraint at time 0. When arbitrageurs are competitive, they collectively fully absorb the liquidity shock in each period, so that their total position is always $2s$. Further, perfect competition eliminates the arbitrageurs' profits, hence there is no profit adjustment, and the constraint either binds all the time or never.

Similarly, one can wonder why the arbitrageur chooses to make her constraint binding at time 1 and not at time 0. Intuitively, from the viewpoint of time 0, there are larger rents to collect since local investors are aware that they will face two liquidity shocks and therefore have a larger willingness to share risk. At time 1, only one shock remains and it is too late to hedge the first one. Hence the arbitrageur prefers to be constrained at time 1.

When the arbitrageur chooses to be constrained at time 1, she has just enough wealth to respond to the increased willingness of local investors to diversify risk at time 0. The arbitrageur responds to this additional liquidity demand by tackling the arbitrage gap more aggressively at time 0 than in the unconstrained case: she sets up a trade $x_0 = \frac{s}{2}$ instead of $\frac{2}{5}s$. Conversely, her time-1 trade is lower than if she were unconstrained.

Corollary 16 *In the (c_1, u_0) equilibrium given in Proposition 13, the arbitrageur trades a larger quantity at time 0 and a smaller at time 1, than if she were using the (feasible)* *unconstrained strategy, i.e. in the relevant parameter space,* $x_0^{c_1,u_0} > x_0^{u_1,u_0}$ *and* $x_1^{c_1,u_0} < x_1^{u_1,u_0}$.

The overall effect is that the arbitrageur builds a larger position in the (c_1, u_0) *equilibrium than if she chose to remain unconstrained if* $W_{-1} \in [\underline{\omega}, v_1[$*, with* $\underline{\omega} \equiv \frac{7}{5}s\bar{e} - \frac{23}{25}a\sigma^2 s^2$.

It is interesting to see that the arbitrageur's increased trading aggressiveness at time 0 may be so strong that she may be able to build a larger position than if she was unconstrained, even though she is financially constrained at time 1. The condition that the arbitrageur must hold enough capital to build a large position is intuitive, since in this equilibrium trade size is increasing in the arbitrageur's capital.

Implications for market liquidity and empirical predictions. The arbitrageur trades more aggressively in this equilibrium than when she is unconstrained. But her behaviour is motivated by the fact that local investors shift liquidity demand towards the first period, pushing the prices of assets A and B further apart. Given these conflicting effects, it is natural to analyze the overall impact on the equilibrium spread. The following result shows that the increased trading aggressiveness always dominates:

Corollary 17 *The spread is lower at all dates when the arbitrageur chooses to be constrained in equilibrium, i.e.* $\Delta_t^{c_1,u_0} < \Delta_t^m$, $t = 0, 1$.

This improvement in liquidity means either that the arbitrageur more than compensates the additional liquidity demand at time 0, or that her trades have a larger price impact than in the unconstrained case. Even more surprising is the result that the liquidity improves at all dates: this is related to the fact that trades have a permanent impact on the price. Moreover, the arbitrageur may acquire a larger position than if she was unconstrained.

More generally, this result implies that a drop in arbitrageur's capital may not have a monotonically decreasing effect on market liquidity. If volatility is low enough, a reduction in the arbitrageur's capital may first leave market liquidity unchanged (if $W_{-1} \geq \omega^m$), then improve it and decrease it again later. This is in contrast to the competitive case.

NYSE specialists can be seen as real-world counterparts to our monopolistic arbitrageur, and one can use effective spreads as proxy for market liquidity as in Comerton-Forde, Jones, Hendershott, Moulton, Seasholes (2010).[23] The model prediction is that for firms with

[23] To strengthen the analogy with market-making, consider that market A and B at time t may represent two subperiods t^A and t^B of date t. Thus A-investors may come to the market at time t^A to share the risk of

low enough dividend volatility, effective spreads should increase in the amount of capital available to the specialists running the stock, for capital large enough. For capital low enough, effective spreads should be decreasing in the amount of specialists' capital, and even more so if specialist capital is low, for any level of dividend volatility. Comerton-Forde et al. find evidence for the latter, but base their tests on the assumption that there are two regions for market-maker capital (as in the competitive case), and not three as in the monopolistic case. Instead of the cross-section, the test may be run in the time-series, i.e. by comparing spreads in times where specialists appear to be more constrained than others.

Another interesting implication of Proposition 13 is that when the arbitrageur is constrained at time 1, the amount of capital she owns affects her price impact at time 0.[24] Further, an increase in capital has an ambiguous effect on the arbitrageur's price impact (Kyle's lambda):

Corollary 18 *Under the conditions of Proposition 13 or if $W_{-1} < \omega^c$, then the arbitrageur's price impact at t=0 (Kyle's lambda) depends on her initial capital and following an increase in capital,*

- *the arbitrageur's price impact increases for small trades ($x_0 < \frac{s}{2}$)*
- *the arbitrageur's price impact decreases for large trades ($x_0 \geq \frac{s}{2}$)*

This result is based on a substitution effect between the arbitrageur's initial capital and her intermediate capital gain. If the arbitrageur's capital increases, it must be that her capital gain between time 0 and time 1 decreases for her to remain constrained. The difference between the effect of small and large trades on the price comes from whether the trade will increase or decrease the capital gain between $t = 0$ and $t = 1$. If $x_0 \leq \frac{s}{2}$, a small increase in the trade following from an injection of capital would raise the intermediate capital gain, which is equal to $x_0 2a\sigma^2 (s - x_0)$. Therefore, for the arbitrageur to remain constrained at time 1, it must be that her capital gain is small, i.e. that her price impact increases. This result has an interesting implication. Suppose capital is injected into the arbitrageur with the view to improve market liquidity. If capital injections are too limited to push the arbitrageur

their liquidity shocks, while B-investors with whom there would be gains from trade arrive only later at time t^B. Arbitrageurs fill the gap between the two subperiods, providing immediacy as market-makers smoothing out order imbalances. See Gromb and Vayanos (2010) for further details on this alternative interpretation of the model.

[24] This point holds for the c_1, u_0 equilibrium, as well as the c_1, c_0 equilibrium. It also holds more generally in the competitive case, if one think of the arbitrageurs' *collective* price impact.

out of the constrained region, using the spread and the arbitrageur's price impact to assess the effect of the policy on liquidity may produce conflicting results. The model predicts that the spread will decrease, but the price impact may either increase or decrease.

Low capital: semi-constrained trading

When the arbitrageur's capital is below $\max(\omega^p, \omega_1^m)$ or if volatility is high enough, it is in general not possible to determine the equilibrium analytically, because equilibrium conditions are implicit. The following result summarizes the equilibrium conditions:

Proposition 14 *Suppose that* $W_{-1} \in [\omega^c, \min(\omega_0^m, \omega_1^m)[\cup [\min(\omega_0^m, \omega_1^m), \max(\omega_0^m, \omega_1^m)[$, *then at least one constraint is binding in equilibrium.*

- *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[0, \frac{7}{10}\right[$, *then remaining unconstrained at time 1 involves saturating the* $t = 0$ *constraint, which is the most tightly binding, i.e. the* u_1, c_0 *strategy involves trading* x_0^0 *at time 0. Further, the* c_1, u_0 *strategy is never feasible / credible.*
  - *If* $W_{-1} \geq \omega^p$, *an equilibrium always exists. If* $h(W_{-1}) \geq 0$, *or if* $h(W_{-1}) > 0$ *and* $\tilde{g}_c(W_{-1}) \geq 0$ *the equilibrium is* u_1, c_0. *If* $h(W_{-1}) > 0$ *and* $\tilde{g}_c(W_{-1}) < 0$, *then the equilibrium is* c_1, c_0.
  - *If* $W_{-1} < \omega^p$, *then if* $h(W_{-1}) \geq 0$, *it is not possible to satisfy the financial constraints and there is no equilibrium. Otherwise, the equilibrium is the same as for* $W_{-1} \geq \omega^p$.
- *If* $\frac{\bar{e}}{a\sigma^2 s} > \frac{7}{10}$, *then remaining unconstrained at time 1 involves saturating the* $t = 1$ *constraint, which is the most tightly binding, i.e. the* u_1, c_0 *strategy involves trading* x_0^1 *at time 0. There are two cases:*
  - *If* $\omega_1^m \geq \omega^p$ *the equilibrium is as in Proposition 13,*
  - *If* $\omega_1^m < \omega^p$, *the equilibrium is* u_1, c_0 *with* $x_0 = x_0^1$ *if* $h(W_{-1}) \geq 0$. *If* $h(W_{-1}) < 0$, *the equilibrium is* u_1, c_0 *if* $g(W_{-1}) \geq 0$, *and* c_1, c_0 *otherwise.*

The functions g *and* $\tilde{g}_c$ *are defined in Lemma 14 and Lemma 15 in the appendix.*

Contrary to the previous case, the unconstrained strategy is not feasible anymore, so that it is the lack of capital and not strategic considerations that dictates the arbitrageur's trading strategy in equilibrium. Even in this case, however, the arbitrageur may keep some financial flexibility by reducing her time-0 trade in order to remain unconstrained at time 1.[25] Note

[25] Consistent with the previous section, I have not been able to find numerical examples where this strategy is an equilibrium.

that the equilibrium may not exist if capital is low and volatility large enough. The reason is that for such parameters, the thresholds, ω^c, ω_0^m, etc. may be negative. As a consequence, the arbitrageur's capital is in some cases negative as well, and this explains why trading may not be feasible.

2.5 Market structure, liquidity provision and welfare

When the competitive economy is unconstrained, because capital is abundant, it is clear that the market is more liquid than in the monopolistic economy. However, given that the monopoly is less capital-intensive than a competitive market, it is natural to ask whether the monopoly can provide more liquidity than a constrained competitive market when capital is relatively scarce. In this section, I show that a monopoly - whether it is unconstrained or voluntarily constrained - may provide more liquidity than a constrained competitive market but only at time 1, just before the asset matures.

2.5.1 Constrained perfect competition vs unconstrained monopoly

Given that the thresholds ω^m and ω_1^m associated with the monopoly are lower than the threshold ω^* of the competitive market, there is a parameter region in which the competitive market is constrained but the monopoly is unconstrained in equilibrium. I denote Δ_t^* and Δ_t^m the spreads at time t in the competitive and monopoly cases, respectively.

Proposition 15 *At time 0: Suppose that* $W_{-1} \in [\omega^m, \omega^*[$. *The constrained competitive market features more liquidity than an unconstrained monopoly at time 0 if* $\frac{\bar{e}}{a\sigma^2 s} < \frac{21}{10\left(1+\sqrt{5}\right)}$ *or if* $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)}$ *and* $W_{-1} \in [\tilde{\omega}, \omega^*[$ *with* $\tilde{\omega} > \omega^m$.

At time 1: Suppose that $W_{-1} \in [\omega_1^m, \omega^*[$ *and* $h\left(W_{-1}\right) \geq 0$ *or* $h\left(W_{-1}\right) < 0$ *and* $f\left(W_{-1}\right) \geq 0$, *then*

- *If* $\frac{\bar{e}}{a\sigma^2 s} \geq 7+q$, *then if* $W_{-1}\, [\omega_1^m, \hat{\omega}[$, *the unconstrained monopoly provides more liquidity than the constrained competitive market, and less if* $W_{-1} \in [\hat{\omega}, \omega^*[$,
- *If* $\frac{\bar{e}}{a\sigma^2 s} < 7+q$, *then constrained competitive market always provide more liquidity than the unconstrained monopoly.*

Since the ratio $\frac{\bar{e}}{a\sigma^2 s}$ can be rewritten as a function of $\bar{e}$, a and s only, we can discuss the result in terms of high and low volatility regions (or dispersion of fundamental $\bar{e}$). To understand the result, note that volatility has different effects in the constrained competitive case and the unconstrained monopoly. For the unconstrained monopoly, volatility has an unequivocal positive effect. It increases local investors' demand for liquidity, making the arbitrage opportunity more profitable. Thus the spread increases with volatility σ^2 (and thus with $\bar{e}$). For the constrained competitive market, volatility has two opposite effects: first, it increases local investors' demand for liquidity, as in the monopoly case, thus pushing asset prices apart. Second, by making the arbitrage more profitable, it can increase the intermediate capital gain and relax the financial constraint.

As a consequence, in the high volatility region (low $\frac{\bar{e}}{a\sigma^2 s}$), the unconstrained monopoly is less liquid than the constrained competitive economy. This is because the constrained competitive economy benefits from the softening effect of volatility on the financial constraint. In the low volatility region, this effect is reduced, and thus there is less liquidity in the monopoly case only if arbitrageurs' capital is large enough at time 0, and there can be more liquidity in the monopoly case at time 1. Intuitively, the intermediate capital gain is small in this case, thus the constrained competitive economy remains severely constrained at time 1. The condition on capital is intuitive, since when competitive arbitrageurs are constrained, the spread decreases in the amount of capital they hold. Note that this result obtains only when considering ω_1^m instead of v_1 as a lower threshold for an unconstrained monopoly. This requires that the conditions $h(W_{-1}) \geq 0$ or $h(W_{-1}) < 0$ and $f(W_{-1}) \geq 0$ be satisfied. In numerical examples, these conditions seem easy to meet.

My results do not rule out the possibility of having more liquidity in the monopoly case also at time 0. However, even if I do not have an analytical proof, I have not been able to generate this case numerically, suggesting that liquidity improvement may occur only at time 1.

2.5.2 Constrained perfect competition vs voluntarily constrained monopoly

When capital is relatively abundant but close to the constrained region, the monopolistic arbitrageur may find it optimal to be constrained at time 1. I showed that this decreases the spread relative to the unconstrained case in Section 2.4. Hence from Proposition 15, one would expect that the monopoly provides more liquidity than the constrained competitive market at least at time 1. The following result confirms this conjecture.

Proposition 16 *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{2\sqrt{5}}{5}, 1\right[$ *and* $W_{-1} \in [\omega^p, v_1[$ *or* $\frac{\bar{e}}{a\sigma^2 s} > 1$ *and* $W_{-1} \in [\omega_1^m, v_1[$ *then the voluntarily-constrained monopoly provides more liquidity at time 1 than the constrained competitive market.*

Given Corollary 17, it is not surprising to see that the condition for $\Delta_1^* \geq \Delta_1^{c_1,u_0}$ is easier to satisfy than in the unconstrained monopoly case. In particular, there is no condition on arbitrageurs' capital, although we are looking at the same region with $W_{-1} \geq \omega_1^m$. The result, however, holds only at time 1. At time 0, I show in the proof that for the monopoly to provide more liquidity at time 0, a non-trivial condition on parameters must be satisfied. In numerical examples, I have always found a larger spread in the monopoly case than in the constrained competitive case. This is confirmed by the following result:

Corollary 19 *Under the conditions of Proposition 16, the voluntarily constrained monopoly captures the largest possible intermediate capital gain,* $x_0 (\Delta_0 - \Delta_1)$*, by rationing liquidity more than the constrained competitive market:* $x_0^{c_1,u_0} \leq x_0^{c_1,c_0}$*.*

This implies that under these conditions, $\Delta_0 - \Delta_1$ *is larger in the voluntarily constrained case than in the constrained competitive case, and thus* $\Delta_0^{c_1,u_0} \geq \Delta_0^{c_1,c_0}$*.*

It may be surprising that the competitive market yields a tighter spread at time 0 and a larger one at time 1, all the more than the model features permanent price impact, implying that a large spread at time 0 should translate into a large spread at time 1. However, the intuition is simple. The monopoly improves liquidity at time 1 relative to competitive arbitrageurs because she captures a larger intermediate capital gain. The larger capital gain, $x_0 (\Delta_0 - \Delta_1) = 2a\sigma^2 x_0 (s - x_0)$, follows precisely from the fact that the monopoly limits liquidity more at time 0 by buying a smaller amount, which causes the spread to be larger at time 0 than in the competitive case. In particular, I show in the proof of the Corollary that the intermediate capital gain in the monopoly case, U^m, is greater than that of the competitive case, U, and that this implies $\Delta_1^* \geq \Delta_1^{c_1,u_0}$.

2.5.3 Welfare

Given that liquidity may improve at time 1 when the market is monopolistic, it is natural to study whether investors' welfare improves. As a first step, I calculate the expression of local investors' welfare as a function of the spreads.

Lemma 11 *Let χ_0^A denote A-local investors' welfare, and let autarky ($\chi_0^{A,a}$) define a situation without arbitrageur ($n = 0$), i.e. where there is no trade across markets, and full insurance (χ_0^{A*}) the situation where a continuum of unconstrained competitive arbitrageurs trade across markets.*

Then we have:

$$\begin{aligned}\chi_0^A &= \frac{(\Delta_0 - \Delta_1)^2 + \Delta_1^2}{8a\sigma^2} - \frac{s}{2}\Delta_0,\\ \chi_0^{A,a} &= -a\sigma^2 s^2 < \chi_0^{A,m} \leq \chi_0^{A,*} = 0\end{aligned}$$

The arbitrageur's profit is larger in the monopoly case than in autarky or full insurance cases.

As expected, for local investors, autarky and full insurance form two polar cases, and the monopolistic case is somewhere in the middle. In autarky, local investors have no options to hedge, and their certainty equivalent is minimal. When there is a continuum of unconstrained competitive arbitrageurs, local investors can trade the asset at its fair value, and can access a perfect hedge thanks to arbitrageurs' intermediation to market B, resulting in perfect insurance. When there is a monopolistic arbitrageur (whether she is constrained or not), local investors receive some imperfect insurance as the market is imperfectly liquid. To understand how the investors' welfare with an unconstrained monopolist ($\chi_0^{A,m}$) compares to a constrained competitive market ($\chi_0^{A,c}$) when we place ourselves under the conditions of Propositions 15 and 16, we could directly compare welfare. However, it is difficult to derive analytical results. Thus I use an indirect approach based on comparative statics.

Corollary 20 *The following holds:*

- *Local investors' welfare decreases with Δ_0, $\frac{\partial \chi_0^A}{\partial \Delta_0} < 0$*
- *If $\Delta_1 < \frac{1}{2}\Delta_0$, then local investors' welfare decreases with the time-1 spread and with a decrease in liquidity: $\frac{\partial \chi_0^A}{\partial \Delta_1} < 0$, $\frac{\partial \chi_0^A}{\partial (\Delta_0 - \Delta_1)} < 0$, and $\frac{\partial \chi_0^A}{\partial (\Delta_1 - \Delta_2)} < 0$*

An immediate implication of this result is that, if Δ_1 is small enough relative to Δ_0, the improvement in liquidity at time 1 - because it is due to a larger difference $\Delta_0 - \Delta_1$ as numerical results and analysis suggest - may not be Pareto improving. Put differently, switching from a constrained competitive market to a monopolistic market unambiguously increases arbitrageurs' welfare but may decrease local investors' if the improvement in liquidity at time 1 does not offset the worsening at time 0:

Corollary 21 *If* $\Delta_1 < \frac{1}{2}\Delta_0$*, and the conditions of Proposition 16 are satisfied, switching from a competitive market to a monopolistic market increases the arbitrageur's welfare but can decrease local investors' welfare. Aggregate welfare may rise.*

Intuitively, the improvement in liquidity at time 1 is associated with a decrease in the improvement of liquidity between time 0 and time 1, measured by $\Delta_0 - \Delta_1$, and a quicker improvement between time 1 and time 2, $\Delta_1 - \Delta_2$.[26] Under the conditions of Corollary 21, the first effect can outweigh the second. There are two reasons why it can be the case. i) Since local investors experience two shocks, they face higher risks at time 0 (conditionally), thus receiving liquidity at time 0 matters more than receiving liquidity at time 1. ii) The improvement in liquidity at time 1 may require a large worsening at time 0, implying that $\frac{\mathrm{d}\chi_0^A}{\mathrm{d}\Delta} = \frac{\partial \chi_0^A}{\partial \Delta_0}\mathrm{d}\Delta_0 + \frac{\partial \chi_0^A}{\partial \Delta_1}\mathrm{d}\Delta_1 \leq 0$ (see numerical example below). The condition for the result of Corollary 20 is that $\Delta_1 < \frac{1}{2}\Delta_0$. In practice it seems verified. In many numerical examples, including the one reported below, the spread at time 1 is somewhere between a third and a half of the spread at time 0. I prove the result on aggregate welfare on a numerical example:

Aggregate welfare. It is interesting to study the aggregate effects of the change in market structure. Although some redistribution effects may be negative, aggregate welfare may increase with a change in market structure (This, of course, takes into account both A and B local investors). For instance, assume that ϵ_t follows the example distribution described in Section 1.2 and consider the following parameter values: $a = 9$, $s = 0.1$, $\bar{e} = 1$, $p = 0.48$, $\mu = 150$, and set the arbitrageur's capital W_{-1} to $\omega_1^m \approx 0.1368$. We are then under the conditions of Proposition 16, and if the market is competitive, the arbitrageurs would be constrained, since $\omega^* = 0.2$. The equilibrium spreads in the monopolistic (voluntarily-constrained) structure are $\Delta_0^{c_1,u_0} \approx 0.058$, and $\Delta_1^{c_1,u_0} \approx 0.0216$. The spreads in the constrained competitive case are $\Delta_0^{c_1,c_0} \approx 0.044$ and $\Delta_1^{c_1,c_0} \approx 0.0217$, implying that the spread at time 1 is less than a half of that of time 0. Comparing the market structure shows that the improvement in liquidity at time 1 is moderate relative to the deterioration at time 0. (observe that we assumed a value of the arbitrageur's capital at the low end of the possible range) This implies that local investors' welfare decreases from -0.0019 to -0.0023. Instead arbitrageur's profit increases 0.1398 to 0.1401. The total effect (taking into account both markets A and B) is positive: 0.013.

[26] Recall that $\Delta_2 = 0$ and that $\Delta_0 > \Delta_1$.

2.6 Conclusion

In this chapter, I studied a model of financially constrained arbitrage and relaxed the assumption that arbitrageurs are price-takers. I show that a monopolistic arbitrageur reduces market efficiency but is less capital-intensive. I also emphasized the role of financial constraints as a commitment device for the monopolistic arbitrageur. Since the arbitrageurs' trading counterparties understand that she will push prices further to their fundamental value in the future, the arbitrageur loses market power today. Thus, in some cases, the arbitrageur may choose to be constrained in order to solve this commitment problem. Relative to the competitive case, a monopolistic arbitrageur does necessarily provide less liquidity. If capital is scarce in the economy, liquidity may improve relative to a constrained competitive market when the asset is close to maturity, since the single arbitrageur's superior financial flexibility loosens her financial constraint.

I have studied a textbook situation in which the arbitrage is risk-free. In practice, arbitrage strategies such as relative-value and convergence trading entail risk. Gromb and Vayanos (2002) show that in this case competitive arbitrageurs may not take the efficient level of risk, as they fail to internalize the effects of their strategies on others' financial constraints. With imperfect competition, one can expect that arbitrageurs would to some extent internalize the impact of their decisions, even though this would also decrease efficiency. Hence, extending the model to risky arbitrage is an interesting research avenue.

Similarly, considering the effects of entry is a promising topic and would help disentangle the effects of an increase in capital for existing arbitrageurs from an increase in capital resulting from the entry of new arbitrageurs.

2.7 Proofs

2.7.1 Competitive equilibrium

Proposition 10

Proof. The result is a special case of Proposition 1 in Gromb and Vayanos (2002) with $T = 2$, and $f'(x) = a\sigma^2 x$. ■

Corollary 14

Proof. I solve the system of equations (2.2)-(2.3). Rearranging terms in equation (2.2) gives:

$$2a\sigma^2 x_0^2 + 2\left(\bar{e} - a\sigma^2 s\right) x_0 - W_{-1} = 0$$

Since $2a\sigma^2 s > 0$ and $-W_{-1} \leq 0$, the unique positive root is

$$x_0 = \frac{a\sigma^2 s - \bar{e} + \sqrt{Q}}{2a\sigma^2}, \text{ with } Q = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2 W_{-1}. \tag{2.6}$$

Similarly, reshuffling terms in equation (2.3) gives:

$$a\sigma^2 X_1 + \bar{e} - a\sigma 2s - x_0 \bar{e} = 0 \tag{2.7}$$

$a\sigma^2$ and $-x_0\bar{e}$ have opposite signs and $\bar{e} - a\sigma 2s > 0$, hence the unique positive root is

$$X_1 = \frac{a\sigma^2 s - \bar{e} + \sqrt{U}}{2a\sigma^2}, \text{ with } U = \left(\bar{e} - a\sigma^2 s\right)^2 + 4a\sigma^2 x_0 \bar{e}.$$

To derive equilibrium spreads, I use the first-order condition of local investors' maximization problems. At time 0, $a\sigma^2\left(x_0^A + s\right) = \mathbb{E}_0\left(p_1^A\right) - p_0^A = \frac{\Delta_0 - \Delta_1}{2}$. Similarly at time 1, $a\sigma^2\left(X_1^A + s\right) = \mathbb{E}_1\left(p_2^A\right) - p_1^A = \frac{\Delta_1 - \Delta_2}{2} = \frac{\Delta_1}{2}$. Market-clearing for asset A requires $x_0^a + x_0 = 0$, and $X_1^A + X_1 = 0$, hence

$$\begin{aligned}
\Delta_1 &= 2a\sigma^2\left(s - X_1\right) \\
\frac{\Delta_0}{2} &= a\sigma^2\left(s - x_0\right) + a\sigma^2\left(s - X_1\right)
\end{aligned}$$

I get the equilibrium spreads by plugging equations (2.6) and (2.7) into the above equations. ■

2.7.2 Monopoly equilibrium

Lemma 8

Proof. At time 1, local investors in market A solve the following problem:

$$\begin{aligned} \chi_1^A &= \max_{y_1^A} U_1 = \mathbb{E}_1\left(W_2^A\right) - \frac{a}{2}\mathbb{V}_1\left(W_2^A\right) \\ &= \max_{y_1^A} E_0^A + Y_1^A D_1 - y_1^A p_1^A - \frac{a\sigma^2}{2}\left(Y_1^A + s_1\right)^2 \end{aligned}$$

From the first-order condition, $a\sigma^2\left(Y_1^A + s_1\right) = D_1 - p_1^A$, and market-clearing, $Y_t^A + X_t = 0$, I obtain the price schedule faced by the arbitrageur in market A:

$$p_1^A\left(X_1\right) = D_1 - a\sigma^2 s_1 + a\sigma^2 X_1$$

By symmetry, in market B: $p_1^B\left(X_1\right) = D_1 + a\sigma^2 s_1 - a\sigma^2 X_1$. With $\Delta_1 = p_1^B - p_1^A$, this gives:

$$\Delta_1\left(X_1\right) = 2a\sigma^2\left(s - X_1\right) \tag{2.8}$$

The arbitrageur takes the price schedule as given and solve the following maximization problem:

$$\begin{aligned} J_1^i &= \max_{x_1} \mathbb{E}_1\left(W_2^i\right) - \frac{b}{2}\mathbb{V}\left(W_2^i\right) \\ s.t.\ W_0 &\geq 2X_1\left[\bar{e} - a\sigma^2\left(s - X_1\right)\right] \\ W_2 &= W_1 = B_0 - x_1 p_1^A + x_1 p_1^B = B_0 + x_1\Delta_1\Delta_1\left(X_1\right) = 2a\sigma^2\left(s - X_1\right) \end{aligned}$$

where x_1 is the arbitrageur's trade in asset A. Given that opposite positions in assets A and B eliminate all fundamental risk, the problem can be rewritten as:

$$\begin{aligned} J_1^i &= \max_{x_1} B_0 + 2a\sigma^2 x_1\left(s - X_1\right) \\ s.t.\ W_0 &\geq 2X_1\left[\bar{e} - a\sigma^2\left(s - X_1\right)\right] \end{aligned}$$

From the first-order condition, and using $X_1 = x_0 + x_1$, the unconstrained solution is

$$x_1 = \frac{s - x_0}{2} \tag{2.9}$$

This trade satisfies the $t = 1$ financial constraint if

$$W_0 = B_{-1} + x_0 \left(\Delta_0 - \Delta_1\right) \geq 2X_1 \left[\bar{e} - a\sigma^2 \left(s - X_1\right)\right] \tag{2.10}$$

To express this inequality as a function of the time-0 trade x_0, it is necessary to derive the price schedule Δ_0, which is a function of x_0. Coming back to the local investors' problem, plugging their demand into the value function χ_1^A, we get:

$$\chi_1^A = E_0^A + Y_0^A p_1^A + \frac{\left(D_1 - p_1^A\right)^2}{2a\sigma^2} - s_1 \left(D_1 - p_1^A\right)$$

At time 0, the local investors choose their holdings (trades) in the risky asset y_0^A. We can rewrite their final wealth as

$$E_{-1}^A - y_0^A p_0^A + s_0 \epsilon_1 + Y_0^a p_1^A + X_1^A \left(D_2 - p_1\right) + s_1 \epsilon_2$$

In equilibrium, the price p_1^A is the sum of the expected conditional value of the asset, D_1 and the liquidity discount, -ϕ_1^A. Hence $D_1 - p_1$ is independent of ϵ_1, which implies that $D_2 - p_1$ depends only on ϵ_2. This means that at time 0,

$$\begin{aligned}
\mathbb{E}_0\left(W_2^A\right) &= E_{-1}^A - y_0^A p_0^A + Y_0^A \mathbb{E}_0\left(p_1^A\right) + X_1^A \left(D_1 - p_1\right) \\
\mathbb{V}_0\left(W_2^A\right) &= \sigma^2 \left(Y_0^A + s_0\right)^2 + \sigma^2 \left(Y_1^A + s_1\right)^2
\end{aligned}$$

Therefore the local investors' time-0 problem is:[27]

$$\begin{aligned}
\chi_0^A &= \max_{y_0^A} \mathbb{E}_0\left(W_2^A\right) - \frac{a}{2}\mathbb{V}_0\left(W_2^A\right) \\
&= \max_{y_0^A} E_{-1}^A - y_0^A p_0^A + Y_0^A \mathbb{E}_0\left(p_1^A\right) + Y_1^A \left(D_1 - p_1\right) - \frac{a}{2}\sigma^2 \left(Y_0^A + s_0\right)^2 - \frac{a}{2}\sigma^2 \left(Y_1^A + s_1\right)^2
\end{aligned} \tag{2.11}$$

The first-order condition is:

$$\mathbb{E}_0\left(p_1^a\right) - p_0^A = a\sigma^2 \left(Y_0^A + s_0\right)$$

[27] Note that following Basak and Chabakauri (2009) we could write a recursive representation for the local investors' problem, using the law of the conditional variance. This would yield of course the same solution.

Using the symmetry of the B-market, and market-clearing in both markets, gives:

$$\Delta_0 - \Delta_1 = 2a\sigma^2 (s - x_0)$$

Using this result, equation (2.9), the fact that $X_1 = x_0 + x_1$, the notation $B_{-1} = W_{-1}$ and the financial constraint (2.10), I can rewrite the condition under which the constraint is slack as

$$W_{-1} \geq \bar{e}(s + x_0) - a\sigma^2 (s - x_0) \frac{s + 5x_0}{2},$$

which gives $W_{-1} - s\bar{e} + a\sigma^2 \frac{s^2}{2} + \left(2a\sigma^2 - \bar{e}\right) x_0 - \frac{5}{2} a\sigma^2 x_0^2 \geq 0$

■

Proposition 11

Proof. Building on Lemma 8, we know that the unconstrained trade $x_1 = \frac{s-x_0}{2}$ is feasible as long as the left-hand side of inequality (2.4) has a solution, i.e. as long as the discriminant is positive:

$$R = \left(2a\sigma^2 s - \bar{e}\right)^2 + 10a\sigma^2 \left(W_{-1} - s\bar{e} + a\sigma^2 \frac{s^2}{2}\right)$$

Hence, rearranging terms, I get that $R \geq 0$ if and only if

$$W_{-1} \geq \frac{7}{5} s\bar{e} - \frac{9}{10} a\sigma^2 s^2 - \frac{e^2}{a\sigma^2} \equiv \omega^c \tag{2.12}$$

Let us assume that $W_{-1} < \omega^c$ so that the arbitrageur is necessarily constrained at $t = 1$, i.e. there is no position x_0 such that $x_1 = \frac{s-x_0}{2}$ is feasible. The arbitrageur's position must therefore saturate the financial constraint, i.e.:

$$W_0 = W_{-1} + 2a\sigma^2 x_0 (s - x_0) = 2X_1 \left(\bar{e} - a\sigma^2 s\right) + 2a\sigma^2 X_1^2$$

$$\text{i.e. } -2a\sigma^2 X_1^2 - 2\left(\bar{e} - a\sigma^2 s\right) X_1 + W_{-1} + 2a\sigma^2 x_0 (s - x_0) = 0$$

At time 1, the arbitrageur's position, x_0, is given, and we can view this equation as a second-order equation in X_1 . The constant term, $W_{-1} + 2a\sigma^2 x_0 (s - x_0)$, is positive, since it represents the wealth accumulated so far, which will be positive in equilibrium. Hence there is a unique solution:

$$X_1^{c_1} = \frac{a\sigma^2 s - \bar{e} + \sqrt{U^m}}{2a\sigma^2}, \text{ with } U^m = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2 \left(W_{-1} + 2a\sigma^2 x_0 \left(s - x_0\right)\right) \quad (2.13)$$

The time-1 payoff in this constrained case is $2a\sigma^2 X_1 \left(s - X_1\right)$. Plugging the quantity (2.13) into this expression gives after some algebra:

$$2a\sigma^2 X_1^{c_1} \left(s - X_1^{c_1}\right) = \frac{\bar{e}}{a\sigma^2}\left[a\sigma^2 s - \bar{e} + \sqrt{U^m}\right] - \left(W_{-1} + 2a\sigma^2 x_0 \left(s - x_0\right)\right) \quad (2.14)$$

At time 0, as we proved in the previous result, the local investors' first-order condition and market-clearing imply that

$$\begin{aligned} \mathbb{E}_0\left(p_1^A - p_0^A\right) &= a\sigma^2\left(s - x_0\right) \\ \text{Similarly, in market B, } \mathbb{E}_0\left(p_1^B - p_0^B\right) &= a\sigma^2\left(-s + x_0\right) \\ \Rightarrow \Delta_0 - \Delta_1 &= 2a\sigma^2\left(s - x_0\right) \end{aligned} \quad (2.15)$$

As a consequence, $\Delta_0 = \Delta_1 + 2a\sigma^2\left(s - x_0\right)$. Plugging equation (2.13) into $\Delta_1^{c_1} = 2a\sigma^2\left(s - X_1^{c_1}\right)$ gives $\Delta_1^{c_1} = a\sigma^2 s + \bar{e} - \sqrt{U^m}$. Using (2.15), this implies that

$$\Delta_0^{c_1}\left(x_0\right) = a\sigma^2 s + \bar{e} - \sqrt{U^m} + 2a\sigma^2\left(s - x_0\right) \quad (2.16)$$

I can now solve the arbitrageur's problem at time 0:

$$\begin{aligned} J_0^{c_1} &= \max_{x_0} W_{-1} + x_0 \Delta_0^{c_1}\left(x_0\right) + 2a\sigma^2 X_1^{c_1}\left(s - X_1^{c_1}\right) \\ & s.t.\ W_{-1} \geq 2x_0\left(\bar{e} - a\sigma^2\left(s - x_0\right)\right) \\ \Delta_0^{c_1}\left(x_0\right) &= a\sigma^2 s + \bar{e} - \sqrt{U^m} + 2a\sigma^2\left(s - x_0\right) \end{aligned}$$

Using equation (2.14) and plugging the spread schedule into the maximand, the problem boils down:

$$\begin{aligned} & J_0^{c_1} = \max_{x_0} \frac{\bar{e}}{a\sigma^2}\left[\sqrt{U^m} - \bar{e} + a\sigma^2 s\right] \\ & s.t.\ W_{-1} \geq 2x_0\left(\bar{e} - a\sigma^2\left(s - x_0\right)\right) \end{aligned}$$

Since $U^m = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2\left(W_{-1} + 2a\sigma^2 x_0\left(s - x_0\right)\right)$ is concave in x_0, the maximization problem admits an interior solution (ignoring the constraint for now):

$$\text{FOC:}\quad \frac{\bar{e}}{a\sigma^2}\frac{\partial\sqrt{U^m}}{\partial x_0} = 0$$

Since $\frac{\partial\sqrt{U^m}}{\partial x_0} = \frac{\frac{\partial U^m}{\partial x_0}}{\sqrt{U^m}} = \frac{4a^2\sigma^4(s-2x_0)}{\sqrt{U^m}}$, the unconstrained optimum is

$$x_0 = \frac{s}{2}$$

Let us now check under which condition this strategy satisfies the time-0 financial constraint. The constraint is satisfied if

$$W_{-1} \geq 2x_0\left(\bar{e} - a\sigma^2\left(s - x_0\right)\right) = s\bar{e} - \frac{1}{2}a\sigma^2 s^2 \equiv \omega^p$$

To determine whether this condition is compatible with the initial assumption that $W_{-1} < \omega^c$, I compare the two thresholds:

$$\omega^c \geq \omega^p \Leftrightarrow \frac{2}{5}s\bar{e} - \frac{2}{5}a\sigma^2 s^2 - \frac{\bar{e}^2}{10a\sigma^2} \Leftrightarrow -\frac{\left(\bar{e} - 2a\sigma^2 s\right)^2}{10a\sigma^2} \geq 0$$

Since the last inequality is never satisfied, $\omega^c \leq \omega^p$, implying that for any arbitrageur capital W_{-1} strictly below ω^c, the arbitrageur is constrained at both $t = 0$ and $t = 1$. The time-0 position and the equilibrium are thus the same as in Proposition 10.

Conversely, for $W_{-1} \geq \omega^c$, not only the fully constrained strategy is financially feasible, but for $W_{-1} \geq \omega^p$, but also the c_1, u_0 strategy. ■

Lemma 9

Proof. Let's assume that $W_{-1} \geq \omega^c$, so that there is always a time-0 trade such that $x_1 = \frac{s-x_0}{2}$ satisfies the financial constraint at $t = 1$.

The u_1, u_0 strategy. I first derive the conditions under which the unconstrained strategy is feasible. Plugging $x_1 = \frac{s-x_0}{2}$ into the arbitrageur's objective function yields her value function in the unconstrained state of the world:

$$J_1^{u_1} = B_0 + \frac{a\sigma^2}{2}\left(s - x_0\right)^2 \tag{2.17}$$

From local investors' first-order conditions and market-clearing, and the symmetry assumption $x_0^A = -x_0^B = x_0$, I get that

$$\begin{aligned}
\mathbb{E}_0\left(p_1^A - p_0^A\right) &= a\sigma^2\left(s - x_0\right) \\
\mathbb{E}_0\left(p_1^B - p_0^B\right) &= a\sigma^2\left(x_0 - s\right)
\end{aligned}$$

Subtracting the first line from the second gives:

$$\Delta_1 - \Delta_0 = 2a\sigma^2\left(x_0 - s\right) \Rightarrow \ \Delta_0^{u_1}\left(x_0\right) = 3a\sigma^2\left(s - x_0\right) \tag{2.18}$$

where the second equation follows from equations (2.8) and (2.9). Hence, using equation (2.17), the arbitrageur's problem at time 0 (if he is unconstrained at time 1) is:

$$\begin{aligned}
J_0^{u_1} = \max_{x_0} W_{-1} + x_0\Delta_0^{u_1}\left(x_0\right) + \frac{a\sigma^2}{2}\left(s - x_0\right)^2 \\
s.t.\ \Delta_0^{u_1}\left(x_0\right) = 3a\sigma^2\left(s - x_0\right) \\
W_{-1} \geq 2x_0\left[\bar{e} - a\sigma^2\left(s - x_0\right)\right]
\end{aligned}$$

Taking the first-order condition and solving for its zero (ignoring the financial constraint), the unconstrained optimal strategy is

$$x_0^{u_1,u_0} = \frac{2}{5}s \tag{2.19}$$

$$\text{This implies: } x_1^{u_1,u_0} = \frac{3}{10}s \tag{2.20}$$

Plugging this quantities into (2.8) and (2.18), this trades translates into the following equilibrium spreads:

$$\Delta_0 = \frac{9}{5}a\sigma^2 s;\ \Delta_1 = \frac{3}{5}a\sigma^2 s \tag{2.21}$$

Further, from equations (2.19) and (2.20), I get the payoff

$$J_0^{u_1,u_0} = W_{-1} + \frac{9}{10}a\sigma^2 s^2 \tag{2.22}$$

I now derive the conditions under which these trades are feasible. Plugging equations

(2.19)-(2.20) into the financial constraints, I get:

$$\text{At } t=0,\ W_{-1} \geq 2x_0\left[\bar{e} - a\sigma^2\left(s - x_0\right)\right] \Leftrightarrow\ W_{-1} \geq \omega_0^m \equiv \frac{4}{5}s\bar{e} - \frac{12}{25}a\sigma^2 s^2 \tag{2.23}$$

$$\text{At } t=1,\ W_0 \geq 2X_1\left[\bar{e} - a\sigma^2\left(s - X_1\right)\right] \Leftrightarrow\ W_{-1} \geq \omega_1^m \equiv \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2 \tag{2.24}$$

From the expressions of ω_0^m and ω_1^m, one can see that a slack constraint at $t=0$ does not necessarily imply the same at $t=1$, in particular, $\omega_1^m \geq \omega_0^m \ \Leftrightarrow\ \bar{e} \geq \frac{7}{10}a\sigma^2 s$. Moreover, by comparing ω_1^m and ω^c, one can see that $\omega^c = \omega_1^m - \frac{\bar{e}}{10a\sigma^2} < \omega_1^m$.

I show in Lemma 16 below that $\omega^c \geq \omega_0^m \Leftrightarrow \left[3 - 2\sqrt{\frac{6}{5}}, 3 + 2\sqrt{\frac{6}{5}}\right]$.

The u_1, c_0 strategy. Since there are two thresholds ω_0^m, ω_1^m, and given that their relative order changes, one constraint may be saturated at a time. To remain unconstrained at time 1, the arbitrageur must trade a quantity that jointly satisfies the following inequalities:

$$\begin{aligned} t=0\text{: } W_{-1} &\geq 2x_0\left[\bar{e} - a\sigma^2\left(s - x_0\right)\right] \\ t=1\text{: } W_{-1} &\geq \bar{e}\left(s + x_0\right) - a\sigma^2\left(s - x_0\right)\frac{s + 5x_0}{2} \end{aligned}$$

I showed in the proof of Proposition 10 that the time-0 constraint has two roots, one positive (denoted x_0^0) and one negative (denoted $x_0^{0'}$). The time-1 constraint has at least one root if its discriminant $R \geq 0$ (defined in the proof of Lemma 8). Let x_0^1 and $x_0^{1'}$ denote the roots if $R > 0$. Since the arbitrageur's problem and the constraints are second-order equations in x_0 and have an inverted U shape, if one constraint is binding, it means that its roots are smaller than the peak of the arbitrageur's the value function at $x_0^{u_1,u_0} = \frac{2}{5}s$, and at least one root of the non-binding constraint is greater than $\frac{2}{5}s$. If both constraints are binding, then all four roots are smaller than $\frac{2}{5}s$. Then to remain unconstrained, the arbitrageur must trade the largest quantity such that both inequalities are weakly satisfied. This may not always be feasible, for instance if $x_0^1 < x_0^{1'} < 0$ and $x_0^{0'} > x_0^{1'}$. The time-1 constraint has two negative roots if $W_{-1} \leq \omega^p = s\bar{e} - \frac{1}{2}a\sigma^2 s^2$, and $\frac{\bar{e}}{a\sigma^2 s} > 2$. Similarly, if $W_{-1} < \omega_1^m < \omega_0^m$, and x$_0^1 > x_0^{1'} > x_0^0 > 0$.

■

Lemma 10

Proof.

The c_1, u_0 strategy was derived, in part, in the proof of Proposition 11. I recall the main equations here for convenience. The positions are:

$$\begin{aligned}
x_0^{c_1,u_0} &= \frac{s}{2} \\
X_1^{c_1,u_0} &= \frac{a\sigma^2 s - \bar{e} + \sqrt{U^m}}{2a\sigma^2} \\
\text{where } U^m &= \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2\left[W_{-1} + 2a\sigma^2 x_0^{c_1,u_0}\left(s - x_0^{c_1,u_0}\right)\right] \\
&= 2a\sigma^2 W_{-1} + \left(\bar{e} - a\sigma^2 s\right)^2 + a^2\sigma^4 s^2 \qquad (2.25)
\end{aligned}$$

The payoff is:

$$J_0^{c_1,u_0} = \frac{\bar{e}}{a\sigma^2}\left[\sqrt{U^m} - \bar{e} + a\sigma^2 s\right] \qquad (2.26)$$

From Proposition 11, the time-0 trade is feasible if and only if $W_{-1} \geq \omega^p$. I now check under which condition the arbitrageur is indeed constrained at time 1 after trading $x_0 = \frac{s}{2}$. It must be that if the arbitrageur re-optimizes at time 1 with a starting position $\frac{s}{2}$, her financial constraint binds. The unconstrained trade at time 1 is $\frac{s-x_0}{2} = \frac{s}{4}$, resulting $X_1 = \frac{s}{2} + \frac{s}{4} = \frac{3}{4}s$. Plugging this quantity into the time-1 financial constraint gives:

$$W_{-1} + 2a\sigma^2 x_0\left(s - x_0\right) \geq 2X_1\left[\bar{e} - a\sigma^2\left(s - X_1\right)\right] \Leftrightarrow W_{-1} \geq \frac{3}{2}s\bar{e} - \frac{7}{8}a\sigma^2 s^2 \equiv \bar{\omega}^p$$

To sum up: the c_1, u_0 strategy is

- feasible if and only if $W_{-1} \geq \omega^p$,
- credible if and only if $W_{-1} \geq \bar{\omega}^p$.

The c_1, c_0 strategy is described in Corollary 14. Plugging the positions into the arbitrageur's profit function, $W_{-1} + 2a\sigma^2 x_0\left(s - x_0\right) + 2a\sigma^2 X_1\left(s - X_1\right)$ gives the payoff:

$$J_0^{c_1,c_0} = \frac{\bar{e}}{a\sigma^2}\left[a\sigma^2 s - \bar{e} + \sqrt{\left(\bar{e} - a\sigma^2 s\right)^2 + 2\bar{e}\left(a\sigma^2 s - \bar{e} + \sqrt{Q}\right)}\right]$$

As shown above, a necessary and sufficient condition for the arbitrageur to be constrained at time 0 is $W_{-1} < \omega^p$, with $\omega^p \geq \omega^c$. When $W_{-1} \in [\omega^c, \omega^p[$, one must therefore check under which condition the strategy is credible. The arbitrage cannot re-optimize at time 1 if the time-1 constraint is binding when it is evaluated based on $x_1 = \frac{s-x_0}{2}$ and $x_0 = \frac{a\sigma^2 s - \bar{e} - \sqrt{Q}}{2a\sigma^2}$. From Lemma 8, the time-1 financial constraint based on $x_1 = \frac{s-x_0}{2}$ is $W_{-1} - s\bar{e} + a\sigma^2\frac{s^2}{2} + \left(2a\sigma^2 s - \bar{e}\right)x_0 - \frac{5}{2}a\sigma^2 x_0^2 \geq 0$. Plugging $x_0 = \frac{a\sigma^2 s - \bar{e} + \sqrt{Q}}{2a\sigma^2}$ into this equation and rearranging terms gives inequality (2.5). ■

The following result will be useful to determine the equilibrium:

Corollary 22 *suppose that $W_{-1} < \omega^p$, then if $W_{-1} \geq \bar{\omega}^p$, the c_1, u_0 strategy is not credible and it implies that the c_1, c_0 is not credible either.*

Proof. This follows from the relative position of time-0 trades:

$$x_0^{c_1,c_0} \leq x_0^{c_1,u_0} = \frac{s}{2} \quad \Leftrightarrow \quad \bar{e} - \sqrt{Q} \geq 0$$
$$\Rightarrow \quad e^2 \geq Q \ \Leftrightarrow W_{-1} \leq s\bar{e} - a\sigma^2\frac{s^2}{2} = \omega^p$$

Then, note that the time-1 constraint, based on $x_1 = \frac{s-x_0}{2}$ is an inverted-U shaped parabola in x_0: $g\left(x_0\right) = W_{-1} - s\bar{e} + a\sigma^2\frac{s^2}{2} + \left(2a\sigma^2 s - \bar{e}\right)x_0 - \frac{5}{2}a\sigma^2 x_0^2$. By Lemma 10, $W_{-1} \geq \bar{\omega}^p$ implies that $g\left(x_0^{c_1,u_0}\right) \geq 0$. Note that g is decreasing for all $x_0 < x_0^1$, where x_0^1 is the largest root of $g\left(x_0\right) = 0$. Further, if $W_{-1} \geq \omega^p$, $x_0^{c_1,c_0} \leq x_0^{c_1,u_0} \leq x_0^1$. Thus on this interval, $g\left(x_0^{c_1,u_0}\right) \geq 0 \ \Rightarrow \ g\left(x_0^{c_1,c_0}\right) \geq 0$, i.e. the fact that c_1, u_0 is not credible implies that c_1, c_0 is not credible. ■

Equilibrium

Propositions 12 and 13 are based on a number of intermediate results that I present here. I first compare the payoffs of the different strategies, then derive parameter conditions to order the capital thresholds, and finally determine the equilibrium in each region.

Lemma 12 *The u_1, u_0 strategy (weakly) dominates the c_1, u_0 strategy if and only if $W_{-1} \leq v_2$ or $W_{-1} \geq v_1$, with $v_2 = \left(1 - \frac{1}{\sqrt{5}}\right)s\bar{e} - \frac{9}{10}a\sigma^2 s^2$ and $v_1 = \left(1 + \frac{1}{\sqrt{5}}\right)s\bar{e} - \frac{9}{10}a\sigma^2 s^2$.*

Proof. I recall the expressions of the respective payoffs for convenience: $J_0^{u_1,u_0} = W_{-1} + \frac{9}{10}a\sigma^2 s^2$, and $J_0^{c_1,u_0} = \frac{\bar{e}}{a\sigma^2}\left[\sqrt{U^m} - \left(\bar{e} - a\sigma^2 s\right)\right]$. This gives:

$$J_0^{u_1,u_0} \geq J_0^{c_1,u_0} \ \Leftrightarrow \ a\sigma^2 W_{-1} + \bar{e}^2 + \frac{9}{10}a^2\sigma^4 s^2 - a\sigma^2 s\bar{e} \geq \bar{e}\sqrt{U^m}$$

From Proposition 11, a necessary condition for consider the u_1, u_0 strategy is $W_{-1} \geq \omega^c$, which implies that $a\sigma^2 W_{-1} + \bar{e}^2 + \frac{9}{10}a^2\sigma^4 s^2 - a\sigma^2 s\bar{e} \geq 0$. Hence elevating to the square each side will not change the order of the previous inequality. I get:

$$\left[a\sigma^2 W_{-1} - a\sigma^2\omega^l\right]^2 \geq \bar{e}^2 U^m, \text{ with } \omega^l = s\bar{e} - \tfrac{9}{10}a\sigma^2 s^2 - \tfrac{\bar{e}^2}{a\sigma^2}$$

Using $U^m = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2 W_{-1} + a^2\sigma^4 s^2$ and developing and regrouping terms gives :

$$a^2\sigma^4 W_{-1} - 2a\sigma^2\left(a\sigma^2\omega^l + \bar{e}^2\right)W_{-1} + a^2\sigma^4\left(\omega^l\right)^2 - \bar{e}^2\left(\bar{e}^2 - 2a\sigma^2 s\bar{e} + 2a^2\sigma^4 s^2\right) \geq 0$$

The left-hand side is a second-order equation in W_{-1}. Its discriminant is:

$$\delta = 4a^2\sigma^4\left[a\sigma^2\omega^l + \bar{e}^2\right]^2 - 4a^2\sigma^4\left[a^2\sigma^4\left(\omega^l\right)^2 - \bar{e}^2\left(\bar{e}^2 - 2a\sigma^2 s\bar{e} + 2a^2\sigma^4 s^2\right)\right]$$

Using the definition of ω^l and regrouping terms yields:

$$\delta = \frac{4}{5}a^4\sigma^8 s^2\bar{e}^2 > 0$$

This proves that there are always two roots $v_1 = \frac{2a\sigma^2\left(a\sigma^2\omega^l + \bar{e}^2\right) + \sqrt{\delta}}{2a^2\sigma^4}$ and $v_2 = \frac{2a\sigma^2\left(a\sigma^2\omega^l - \bar{e}^2\right) + \sqrt{\delta}}{2a^2\sigma^4}$, with $v_2 \leq v_1$, so that the inequality is satisfied for $W_{-1} \leq v_2$ or $W_{-1} \geq v_1$. v_1 and v_2 can be simplified to the expressions given in the lemma by replacing δ and ω^l by their expressions. It is easy to check that the conditions are both necessary and sufficient. ■

Lemma 13 *The u_1, u_0 strategy weakly dominates the c_1, c_0 strategy if*

$$f\left(W_{-1}\right) = -a^2\sigma^4 W_{-1}^2 + 2a^2\sigma^4\bar{\omega}^l W_{-1} - a^2\sigma^4\left(\bar{\omega}^l\right)^2 + 2\bar{e}^3\sqrt{Q} + \bar{e}^4 - a^2\sigma^4 s^2\bar{e}^2 \leq 0 \quad (2.27)$$

Proof. From the expressions of $J_0^{u_1,u_0}$ and $J_0^{c_1,c_0}$, I get:

$$J_0^{u_1,u_0} \geq J_0^{c_1,c_0} \Leftrightarrow a\sigma^2 W_{-1} - a\sigma^2 s\bar{e} + \frac{9}{10}a^2\sigma^4 s^2 \leq \bar{e}\sqrt{a^2\sigma^4 s^2 - \bar{e}^2 + 2\bar{e}\sqrt{Q}}$$

The assumption that $W_{-1} \geq \omega_1^m$ implies that the left-hand side is positive. I take the square on each side, which gives:

$$a^2\sigma^4\left[W_{-1} - \bar{\omega}^l\right]^2 \geq a^2\sigma^4 s^2\bar{e}^2 - \bar{e}^4 + 2\bar{e}^3\sqrt{Q}, \text{ with } \bar{\omega}^l = s\bar{e} - \tfrac{9}{10}a\sigma^2 s^2$$

Developing and rearranging terms yields condition (2.27). ■

Lemma 14 *If $\omega_0^m \leq W_{-1} < \omega_1^m$, or $W_{-1} < \omega_1^m \leq \omega_0^m$, the u_1, c_0 strategy involves trading a quantity x_0^1, given in Lemma 9, at time 0. In this case,*

- *the u_1, c_0 strategy is always dominated by c_1, u_0,*
- *the u_1, c_0 strategy dominates c_1, c_0 if*

$$g\left(W_{-1}\right) = \frac{2}{5}a\sigma^2 W_{-1} - \frac{12}{25}a^2\sigma^4 s^2 + \frac{42}{25}e^2 + \frac{2}{25}a\sigma^2 s\bar{e} + \frac{4}{25}\left(a\sigma^2 s + 2\bar{e}\right)\sqrt{R} \geq 2\bar{e}\sqrt{Q} \quad (2.28)$$

Proof. First, I plug the expression of x_0^1 into $J_0^{u_1,c_0}\left(x_0\right)$ to the payoff of the strategy using the results of Lemma 9. This gives:

$$J_0^{u_1,c_0}\left(x_0^1\right) = W_{-1} + \frac{9a^2\sigma^4 s^2 - \left(\bar{e} - \sqrt{R}\right)^2}{10a\sigma^2} = \frac{\bar{e}}{a\sigma^2}\left[\frac{7}{5}a\sigma^2 s + \frac{\sqrt{R} - \bar{e}}{5}\right]$$

Then using the expression for $J_0^{c_1,u_0}$ given in Lemma 10, I find that

$$J_0^{u_1,c_0}\left(x_0^1\right) \geq J_0^{c_1,u_0} \Leftrightarrow \frac{2}{5}\left(a\sigma^2 s + 2\bar{e}\right) + \frac{\sqrt{R}}{5} \geq \sqrt{U^m} > 0$$

Taking the square on each side yields, after developing and rearranging the terms,

$$\left(a\sigma^2 s + 2\bar{e}\right)\sqrt{R} \geq 10a\sigma^2\left(W_{-1} - \underline{\omega}^c\right), \text{ with } \underline{\omega}^c = \tfrac{13}{10}s\bar{e} - \tfrac{23}{20}a\sigma^2 s^2 - \tfrac{e^2}{5a\sigma^2}$$

Given that $W_{-1} \geq \omega^c$ by assumption, and that $\omega^c > \underline{\omega}^c$, the right-hand side is positive and taking the square on each side does not change the sign. This yields the following second-order equation in W_{-1}:

$$-100a^2\sigma^4 W_{-1}^2 + 10a\sigma^2\left[20a\sigma^2\underline{\omega}^c + \left(a\sigma^2 s + 2\bar{e}\right)^2\right]W_{-1} - 10a\sigma^2\left[\left(a\sigma^2 s + 2\bar{e}\right)^2\omega^c + 10a\sigma^2\left(\underline{\omega}^c\right)^2\right] \geq 0$$

The discriminant of the equation is:

$$d = 100a^2\sigma^4\left[20a\sigma^2\underline{\omega}^c + \left(a\sigma^2 s + 2\bar{e}\right)^2\right]^2 - 400a^2\sigma^4\left[10a\sigma^2\left[\left(a\sigma^2 s + 2\bar{e}\right)^2\omega^c + 10a\sigma^2\left(\underline{\omega}^c\right)^2\right]\right]$$

Developing and regrouping the terms gives

$$d = 100a^2\sigma^4 \left(a\sigma^2 s + 2\bar{e}\right)^2 \left[\left(a\sigma^2 s + 2\bar{e}\right)^2 - 40a\sigma^2 \left(\omega^c - \underline{\omega}^c\right)\right]$$

Since $\omega^c - \underline{\omega}^c = \frac{1}{10}s\bar{e} + \frac{1}{4}a\sigma^2 s^2 + \frac{\bar{e}^2}{10a\sigma^2}$, the discriminant boils down to

$$d = -900a^4\sigma^8 s^2 \left(a\sigma^2 s + 2\bar{e}\right)^2 < 0$$

Therefore, given that the coefficient of the second-order term is negative, the inequality is never satisfied.

I now turn to the second point:

$$J_0^{u_1,c_0} \geq J_0^{c_1,c_0} \;\Leftrightarrow\; \frac{2}{5}\left(a\sigma^2 s + 2\bar{e}\right) + \frac{\sqrt{R}}{5} \geq \sqrt{a^2\sigma^4 s^2 - \bar{e}^2 + 2\bar{e}\sqrt{Q}}$$

This yields condition (2.28), after elevating both sides to the square and rearranging terms. ■

Lemma 15 *If $\omega_0^m \leq W_{-1} < \omega_1^m$, or $W_{-1} < \omega_1^m \leq \omega_0^m$, the u_1, c_0 strategy to trade x_0^0, given in Lemma 9. Then*

- *the u_1, c_0 strategy dominates c_1, u_0 if*

$$\tilde{g}_u\left(W_{-1}\right) = -\frac{1}{4}a\sigma^2 W_{-1} + \frac{\left(5\bar{e} - a\sigma^2 s\right)\sqrt{Q} + 2a\sigma^2 s\bar{e} - \bar{e}^2 + a^2\sigma^4 s^2}{4} \geq \bar{e}\sqrt{U^m}$$

- *the u_1, c_0 strategy dominates c_1, c_0 if*

$$\tilde{g}_c\left(W_{-1}\right) = -\frac{1}{4}a\sigma^2 W_{-1} + \frac{\left(5\bar{e} - a\sigma^2 s\right)\sqrt{Q} + 2a\sigma^2 s\bar{e} - \bar{e}^2 + a^2\sigma^4 s^2}{4} \geq \bar{e}\sqrt{a^2\sigma^4 s^2 - \bar{e}^2 + 2\bar{e}\sqrt{Q}}$$

Proof. By plugging x_0^0 into $J_0^{u_1,c_0}\left(x_0\right)$, both given in Lemma 9, I get the payoff of the strategy

$$J_0^{u_1,c_0}\left(x_0^0\right) = -\frac{1}{4}W_{-1} + \frac{a^2\sigma^4 s^2 + 6a\sigma^2 s\bar{e} - 5\bar{e}^2 + \left(5\bar{e} - a\sigma^2 s\right)\sqrt{Q}}{4a\sigma^2}$$

Comparing this payoff to $J_0^{c_1,u_0}$ and $J_0^{c_1,c_0}$ and rearranging terms gives the conditions given in the result. ■

Lemma 16 *The thresholds are ranked in the following order:*

1. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[0, \frac{1}{10}\right[$, *then* $\omega^c \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p \leq \omega_0^m$,
2. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{1}{10}, \frac{79}{140}\right[$, *then* $\omega^c \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega_0^m \leq \omega^p$,
3. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{79}{140}, \frac{21}{10\left(1+\sqrt{5}\right)}\right[$, *then* $\omega^c \leq \omega_1^m \leq v_1 \leq \omega_0^m \leq \bar{\omega}^p \leq \omega^p$,
4. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{21}{10\left(1+\sqrt{5}\right)}, \frac{7}{10}\right[$, *then* $\omega^c \leq \omega_1^m \leq \omega_0^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p$,
5. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{7}{10}, \frac{3}{4}\right[$, *then* $\omega^c \leq \omega_0^m \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p$,
6. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{3}{4}, 3 - 2\sqrt{\frac{6}{5}}\right[$, *then* $\omega^c \leq \omega_0^m \leq \omega_1^m \leq v_1 \leq \omega^p \leq \bar{\omega}^p$,
7. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[3 - 2\sqrt{\frac{6}{5}}, \frac{2\sqrt{5}}{5}\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega_1^m \leq v_1 \leq \omega^p \leq \bar{\omega}^p$,
8. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{2\sqrt{5}}{5}, 1\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega_1^m \leq \omega^p \leq v_1 \leq \bar{\omega}^p$,
9. *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[1, 3 + 2\sqrt{\frac{6}{5}}\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega^p \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p$,
10. *If* $\frac{\bar{e}}{a\sigma^2 s} > 3 + 2\sqrt{\frac{6}{5}}$, *then* $\omega^c \leq \omega_0^m \leq \omega^p \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p$.

Proof. I start by recalling the expressions of the different thresholds:

$$\begin{aligned}
\omega^c &= \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2 - \frac{\bar{e}^2}{10a\sigma^2} \\
\omega_0^m &= \frac{4}{5}s\bar{e} - \frac{12}{25}a\sigma^2 s^2 \\
\omega_1^m &= \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2 \\
\omega^p &= s\bar{e} - \frac{1}{2}a\sigma^2 s^2 \\
\bar{\omega}^p &= \frac{3}{2}s\bar{e} - \frac{7}{8}a\sigma^2 s^2 \\
v_1 &= \left(1 + \frac{\sqrt{5}}{5}\right) s\bar{e} - \frac{9}{10}a\sigma^2 s^2 \\
v_2 &= \left(1 - \frac{\sqrt{5}}{5}\right) s\bar{e} - \frac{9}{10}a\sigma^2 s^2
\end{aligned}$$

I now determine relative positions.

ω_1^m **vs** ω^c**:** $\omega^c = \omega_1^m - \frac{\bar{e}}{10a\sigma^2} < \omega_1^m$.

ω_0^m **vs** ω^c**:**

$$\begin{aligned}
\omega^c \geq \omega_0^m \quad &\Leftrightarrow \quad \frac{7}{5}s\bar{e} - \frac{9}{10}a\sigma^2 s^2 - \frac{\bar{e}^2}{10a\sigma^2} \geq \frac{4}{5}s\bar{e} - \frac{12}{25}a\sigma^2 s^2 \\
&\Leftrightarrow \quad 6a\sigma^2 s\bar{e} > \frac{42}{10}a^2\sigma^4 s^2 + \bar{e}^2 \\
&\Leftrightarrow \quad -\bar{e}^2 + 6a\sigma^2 s\bar{e} - \frac{42}{10}a^2\sigma^4 s^2 > 0 \qquad (2.29)
\end{aligned}$$

One can view the left-hand side as a second-order equation in $\bar{e}$. The discriminant of the left-hand side is $D = \frac{96}{5}a^2\sigma^4 s^2$, and given that the coefficient of the second-order term and the constant have the same sign, and that the coefficient of the first-order term is positive, there are two positive roots, given by $\left(3 - 2\sqrt{\frac{6}{5}}\right)a\sigma^2 s \approx 0.81a\sigma^2 s$ and $\left(3 + 2\sqrt{\frac{6}{5}}\right)a\sigma^2 s \approx 5.2a\sigma^2 s$. Hence $\omega^c \geq \omega_0^m$ if and only if $\frac{\bar{e}}{a\sigma^2 s} \in \left[3 - 2\sqrt{\frac{6}{5}}, 3 + 2\sqrt{\frac{6}{5}}\right]$.

ω^p **vs** ω_1^m **and** ω_0^m**:**

$$\begin{aligned}
\omega^p \geq \omega_0^m \quad &\Leftrightarrow \quad \bar{e} \geq \frac{1}{10}a\sigma^2 s \\
\omega^p \geq \omega_1^m \quad &\Leftrightarrow \quad \bar{e} \leq a\sigma^2 s
\end{aligned}$$

ω^p **vs** ω^c**:**

$$\begin{aligned}
\omega^p \leq \omega^c \quad &\Leftrightarrow \quad \frac{2}{5}s\bar{e} - \frac{2}{5}a\sigma^2 s^2 - \frac{\bar{e}^2}{10a\sigma^2} \geq 0 \\
&\Leftrightarrow \quad -\frac{\left(\bar{e} - 2a\sigma^2 s\right)^2}{10a\sigma^2} \geq 0 \\
&\Rightarrow \quad \text{impossible, hence } \omega^p > \omega^c
\end{aligned}$$

ω^p **vs** $\bar{\omega}^p$**:** $\omega^p \leq \bar{\omega}^p \Leftrightarrow \bar{e} \geq \frac{3}{4}a\sigma^2 s$.

$\bar{\omega}^p$ **vs** ω_0^m **and** ω_1^m**:**

$$\begin{aligned}
\bar{\omega}^p \geq \omega_0^m \quad &\Leftrightarrow \quad \bar{e} \geq \frac{79}{140}a\sigma^2 s \\
\bar{\omega}^p \geq \omega_1^m \quad &\Leftrightarrow \quad \bar{e} \geq -4a\sigma^2 s \text{ which always holds}
\end{aligned}$$

ω_0^m **vs** ω_1^m**:** $\omega_0^m \leq \omega_1^m \Leftrightarrow \bar{e} \geq \frac{7}{10}a\sigma^2 s$.

v_1 vs ω_0^m, ω_1^m, ω^p and $\bar{\omega}^p$:

$$\begin{aligned} v_1 &> \omega_1^m \text{ since } 1+\tfrac{\sqrt{5}}{5} > \tfrac{7}{5} \\ v_1 &\geq \omega_0^m \Leftrightarrow \bar{e} \geq \frac{21}{10\left(1+\sqrt{5}\right)} a\sigma^2 s \\ v_1 &\geq \omega^p \Leftrightarrow \bar{e} \geq \frac{2\sqrt{5}}{5} a\sigma^2 s \\ v_1 &< \bar{\omega}^p \text{since } \tfrac{3}{2} > 1+\tfrac{\sqrt{5}}{5} \text{ and } \tfrac{9}{10} > \tfrac{7}{8} \end{aligned}$$

Note that $v_2 < \omega_1^m$ hence the condition $W_{-1} < v_2$ is not going to bind, and therefore it is not useful to study the relative position of v_2.

Overall, without condition on the parameters, we have: $\bar{\omega}^p \geq \omega_1^m$, $\omega^c < \omega_1^m$, $\omega^c < \omega^p$, $v_1 > \omega_1^m$, $\bar{\omega}^p > v_1$ and $v_2 < \omega_1^m$. For the other relationships, there are 9 thresholds, in ascending order: $\frac{1}{10}$, $\frac{79}{140}$, $\frac{21}{10\left(1+\sqrt{5}\right)}$, $\frac{7}{10}$, $\frac{3}{4}$, $3-2\sqrt{\frac{6}{5}}$, $\frac{2\sqrt{5}}{5}$, 1, $3+2\sqrt{\frac{6}{5}}$. ■

Proposition 12 **Proof.** For u_1, u_0 to be an equilibrium, it must be i) feasible, ii) weakly dominating c_1, u_0 and c_1, c_0. Note that if a strategy dominates c_1, u_0, it also dominates c_1, c_0 (whether c_1, u_0 and c_1, c_0 are credible or not), but the converse is not necessarily true. A necessary and sufficient condition for i) to hold is that $W_{-1} \geq \max\left(\omega_0^m, \omega_1^m\right)$ by Lemma 9. A necessary and sufficient condition for ii) to hold is that $W_{-1} \geq v_1$ or $W_{-1} \leq v_2$. Given that $v_2 \leq \omega_1^m$ and $v_1 > \omega_1^m$ for all parameter values, one can eliminate $W_{-1} \leq v_2$ from the equilibrium conditions. Hence a necessary condition is $W_{-1} \geq \max\left(v_1, \omega_0^m\right)$. Lemma 16 shows that $v_1 \geq \omega_0^m$ if and only if $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)}$, hence the result.

Note that u_1, u_0 may be an equilibrium for a larger parameter interval, because the previous points are based on the weak dominance versus c_1, u_0, which implies weak dominance of c_1, c_0, and do not take into account credibility and feasibility conditions. There are cases (see Lemmata 22-25 below) where c_1, u_0 dominates u_1, u_0 but is either not credible or not feasible. This is the case in particular if $v_1 \leq \omega^p$ and / or $v_1 \geq \bar{\omega}^p$. If at the same time, u_1, u_0 dominates c_1, c_0, and / or c_1, c_0 is not credible, then u_1, u_0 is an equilibrium, but these cases are not accounted for by the proposition. ■

Corollary 15 **Proof.** Building on Proposition 12, if $\frac{\bar{e}}{a\sigma^2 s} \leq \frac{9}{10\left(1+\sqrt{5}\right)} \approx 0.65$, the equilibrium condition is $W_{-1} \geq \omega_0^m = \frac{4}{5}s\bar{e} - \frac{12}{25}a\sigma^2 s^2$. Since ω_0^m is highest for $\bar{e} \geq \frac{6}{5}a\sigma^2 s > \frac{21}{10\left(1+\sqrt{5}\right)}$, an increase in s decreases ω_0^m if $\frac{\bar{e}}{a\sigma^2 s} \leq \frac{21}{10\left(1+\sqrt{5}\right)}$. Conversely, if $\frac{\bar{e}}{a\sigma^2 s} > \frac{21}{10\left(1+\sqrt{5}\right)}$, the threshold is $v_1 = \left(1+\frac{1}{\sqrt{5}}\right) s\bar{e} - \frac{9}{10}a\sigma^2 s^2$. Taking the first-order condition shows that v_1 is increasing in s if $\bar{e} \geq \frac{9}{10\left(1+\frac{1}{\sqrt{5}}\right)} \approx 0.62$. Given that $\frac{9}{10\left(1+\frac{1}{\sqrt{5}}\right)} < \frac{21}{10\left(1+\sqrt{5}\right)}$, an increase in s tightens the constraint if $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)}$. ■

Proposition 13 Proof. The result requires that the u_1, u_0 and c_1, u_0 are both feasible and that c_1, u_0 is the dominant strategy. The first point requires $W_{-1} \geq \max\left(\omega_0^m, \omega_1^m\right)$ and $W_{-1} \in [\omega^p, \bar{\omega}^p[$ (by Lemmata 9 and 10). The second point requires $W_{-1} < v_1$ (Lemma 12). Hence we must have $W_{-1} \geq \max\left(\omega_0^m, \omega_1^m, \omega^p\right)$ and $W_{-1} < \min\left(v_1, \bar{\omega}^p\right)$. From Lemma 16, this interval is non-empty only in the two instances stated in the result. ■

Corollary 17 Proof. Recall the expressions of $x_0^{c_1,u_0}$, $x_1^{c_1,u_0}$ and $X_1^{c_1,u_0}$ from Lemma 10:

$$x_0^{c_1,u_0} = \frac{s}{2};\ x_1^{c_1,u_0} = \frac{\sqrt{U^m}-\bar{e}}{2a\sigma^2};\ X_1^{c_1,u_0} = \frac{a\sigma^2 s - \bar{e} + \sqrt{U^m}}{2a\sigma^2}$$

Using the results of Lemma 9, it is immediate that $x_0^{c_1,u_0} > x_0^{u_1,u_0}$.

$$x_1^{c_1,u_0} - x_1^{u_1,u_0} = \frac{\sqrt{U^m}-\bar{e}}{2a\sigma^2} - \frac{3}{10}s \geq \ \Leftrightarrow\ \sqrt{U^m} \geq \bar{e} + \frac{3}{5}a\sigma^2 s$$

Taking squares on both sides and rearranging terms yields:

$$W_{-1} \geq \frac{8}{5}s\bar{e} - \frac{41}{5}a\sigma^2 s > \omega_1^m$$

Further $v_1 \geq \frac{8}{5}s\bar{e} - \frac{41}{5}a\sigma^2 s \ \Leftrightarrow\ \frac{\sqrt{5}-3}{5}\bar{e} \geq \frac{2}{25}a\sigma^2 s$, which never holds since $\sqrt{5}-3 < 0$. Thus the equilibrium condition $W_{-1} < v_1$ can be satisfied at the same time as $W_{-1} \geq \frac{8}{5}s\bar{e} - \frac{41}{5}a\sigma^2 s$, which implies that for all parameter values satisfying the conditions of Proposition 13, $x_1^{c_1,u_0} < x_1^{u_1,u_0}$.

Turning to the total position:

$$X_1^{c_1,u_0} - X_1^{u_1,u_0} = \frac{a\sigma^2 s - \bar{e} + \sqrt{U^m}}{2a\sigma^2} - \frac{7}{10}s \ \Rightarrow W_{-1} \geq \frac{7}{5}s\bar{e} - \frac{23}{25}a\sigma^2 s^2$$

Since $v_1 = \left(1+\frac{1}{\sqrt{5}}\right) s\bar{e} - \frac{9}{10}a\sigma^2 s^2$, $\frac{7}{5}s\bar{e} - \frac{23}{25}a\sigma^2 s^2 < v_1$. ■

Proposition 14 **Proof.** The result can be derived from Lemmata 18-27 below or directly by using Lemmata 9, 10, 16, 14 and 15. ∎

For the sake of exhaustiveness, I now derive the equilibrium for each of the parameter regions of Lemma 16. Since the algorithm for the equilibrium determination is always the same, I write the proof only in the first case. Note that:

Lemma 17 *If* $\frac{\bar{e}}{a\sigma^2 s} < 7 - 2\sqrt{10}$ *or* $\frac{\bar{e}}{a\sigma^2 s} > 7 + 2\sqrt{10}$, $\omega^c < 0$. *Thus some parameter regions may imply a negative capital (i.e. debt)* $W_{-1} < 0$.

Proof. Rewriting ω^c as $\frac{14a\sigma^2 s\bar{e} - 9a^2\sigma^4 s^2 - \bar{e}^2}{10a\sigma^2}$, one can view the numerator as a second-order equation in $\bar{e}$ and calculate its discriminant, $\delta = 160a^2\sigma^4 s^2$. There are two roots, $\left(7 - 2\sqrt{10}\right) a\sigma^2 s \approx 0.67a\sigma^2 s$ and $\left(7 - 2\sqrt{10}\right) a\sigma^2 s \approx 13.3a\sigma^2 s$ ∎

This result matters as it can explain why in some regions the equilibrium with may not exist. Capital may be so negative that no trade may be feasible. I recall some notation.

Notation 17 *The following notation is used as a shorthand:*

- *Weakly dominant strategy:*
  - u_1, u_0 *dominates* c_1, c_0 *if* $f(W_{-1}) \geq 0$ *where* f *is defined in Lemma 13,*
  - u_1, c_0*, with the time-0 constraint most severely binding, dominates* c_1, c_0 *if* $g(W_{-1}) \geq 0$*, with* g *defined in Lemma 14,*
  - u_1, c_0*, with the time-1 constraint most severely binding, dominates* c_1, u_0 *if* $\tilde{g}_u(W_{-1}) \geq 0$*, with* $\tilde{g}_u$ *defined in Lemma 15,*
  - u_1, c_0*, with the time-1 constraint most severely binding, dominates* c_1, c_0 *if* $\tilde{g}_u(W_{-1}) \geq 0$*, with* $\tilde{g}_c$ *defined in Lemma 15.*
- *Credibility and feasibility conditions:*
  - c_1, c_0 *is credible if and only if* $h(W_{-1}) < 0$*, where* h *is defined in Lemma 10.*

I now present the equilibrium for each parameter region:

Lemma 18 ***Case A:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[0, \frac{1}{10}\right[$*, then* $\omega^c \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p \leq \omega_0^m$*,*

1. *If* $W_{-1} \geq \omega_0^m$, *the equilibrium is* u_1, u_0,

2. *If* $W_{-1} \in [\omega^p, \omega_0^m[$, *then if* W_{-1} *is such that* $h(W_{-1}) \geq 0$, *the equilibrium is* u_1, c_0 *with* $x_0 = x_0^0$, *otherwise, a necessary condition for this equilibrium to hold is* $\tilde{g}_c(W_{-1}) \geq 0$. *If* $\tilde{g}_c(W_{-1}) < 0$, *the equilibrium is* c_1, c_0.

3. *If* $W_{-1} \in [\bar{\omega}^p, \omega^p[$ *then the equilibrium is* c_1, u_0 *with* $x_0 = x_0^0$ *if* $x_0^0 > x_0^{1'}$ *and may not exist otherwise.*

4. *If* $W_{-1} \in [v_1, \bar{\omega}^p[$ *or* $[\omega_1^m, v_1[$, *or* $W_{-1} \in [\omega^c, \omega_1^m[$, *then the equilibrium is* c_1, u_0 *with* $x_0 = x_0^0$ *if* $x_0^0 > x_0^{1'}$ *and* $h(W_{-1}) \geq 0$. *If* $h(W_{-1}) < 0$, *the equilibrium is* u_1, c_0 *if* $\tilde{g}_c(W_{-1}) \geq 0$ *and* $x_0^0 > x_0^{1'}$. $\tilde{g}_c(W_{-1}) < 0$, *then the equilibrium is* c_1, c_0. *If* $x_0 < x_0^{1'}$, *it is not possible to satisfy the financial constraints.*

Proof. If $W_{-1} \geq \omega_0^m$, u_1, u_0 is feasible, and u_1, c_0 is feasible but not credible since $W_{-1} \geq \bar{\omega}^p$. Given that $W_{-1} \geq v_1$, u_1, u_0 would dominate c_1, u_0, hence dominates c_1, c_0. Hence the equilibrium is u_1, u_0.

If $W_{-1} \in [\omega^p, \omega_0^m[$, then u_1, u_0 is not feasible, with the time-0 constraint binding, hence one must compare u_1, c_0 with $x_0 = x_0^0$ to c_1, u_0 or c_1, c_0. c_1, u_0 is still not credible, hence u_1, c_0 must be compared to c_1, c_0 if the latter is credible, i.e. if $h(W_{-1}) < 0$ according to Lemma 10. By Lemma 15, if c_1, c_0 is credible, it is dominated if $\tilde{g}_c(W_{-1}) \geq 0$.

If $W_{-1} \in [\bar{\omega}^p, \omega^p[$, then the time-0 constraint is still the most severely binding. However, since $W_{-1} < \omega^p$, there are two positive roots to equation (2.4). One (x_0^1) is greater than $\frac{2}{5}s$ since the time-1 constraint is not binding, and one is smaller ($x_0^{1'}$). From Corollary 14, the time-0 constraint has always a positive and a negative root: $x_0^0 > 0$ and $x_0^{0'} < 0$. If $x_0^0 < x)^{1'}$ then the two constraints do not cross in the upper-quadrant, meaning that no trade ensures that the time-0 constraint is feasible and $x_1 = \frac{s - x_0}{2}$ is feasible. Given that $W_{-1} \in [\bar{\omega}^p, \omega^p[$, the fact that c_1, u_0 is not credible implies that c_1, c_0 is not credible by Corollary 22. Hence if $x_0^0 < x_0^{1'}$, there may be no equilibrium. If $x_0^0 \geq x_0^{1'}$, there is always an equilibrium, and the equilibrium is u_1, c_0 with $x_0 = x_0^0$.

If $W_{-1} \in [v_1, \bar{\omega}^p[$ or $[\omega_1^m, v_1[$, then the analysis is similar, with the exception that Corollary 22 cannot be used anymore. Hence if $h(W_{-1}) < 0$, c_1, c_0 is a credible strategy and it is played in equilibrium if $g(W_{-1}) < 0$.

If $W_{-1} \in [\omega^c, \omega_1^m[$, the analysis is similar: given that $W_{-1} < \omega_1^m$, the time-1 constraint is also binding but since $\omega_0^m > \omega_1^m$, $x_0^0 < x_0^1$, the time-0 constraint remains the most severely binding constraint. ■

Lemma 19 ***Case B:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{1}{10}, \frac{79}{140}\right[$, *then* $\omega^c \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega_0^m \leq \omega^p$,

1. *If* $W_{-1} \geq \omega^p$ *or* $W_{-1} \in [\omega_0^m, \omega^p[$, *the equilibrium is* u_1, u_0,
2. *If* $W_{-1} \in [\bar{\omega}^p, \omega_0^m[$, *the equilibrium is* u_1, c_0 *with* $x_0 = x_0^0$ *if* $x_0^0 > x_0^{1'}$. *Otherwise, it is not possible to satisfy the financial constraints (A.3.).*
3. *If* $W_{-1} \in [v_1, \bar{\omega}^p[$ *or* $[\omega_1^m, v_1[$, *the equilibrium is the same as A.4.*

Lemma 20 ***Case C:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{79}{140}, \frac{21}{10\left(1+\sqrt{5}\right)}\right[$, *then* $\omega^c \leq \omega_1^m \leq v_1 \leq \omega_0^m \leq \bar{\omega}^p \leq \omega^p$,

1. *If* $W_{-1} \geq \omega^p$ *or* $W_{-1} \in [\bar{\omega}^p, \omega^p[$, *or* $[\omega_0^m, \bar{\omega}^p[$, *the equilibrium is* u_1, u_0.
2. *If* $W_{-1} \in [v_1, \omega_0^m[\cup [\omega_1^m, v_1[\cup [\omega^c, \omega_1^m[$, *the equilibrium is the same as A.4.*

Lemma 21 ***Case D:*** $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{21}{10\left(1+\sqrt{5}\right)}, \frac{7}{10}\right[$, *then* $\omega^c \leq \omega_1^m \leq \omega_0^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p$,

1. *If* $W_{-1} \geq \omega^p$, *or* $W_{-1} \in [\bar{\omega}^p, \omega^p[$ *or* $[v_1, \bar{\omega}^p[$, *the equilibrium is* u_1, u_0.
2. *If* $W_{-1} \in [\omega_0^m, v_1[$, *then the equilibrium is* u_1, u_0 *if* $h\left(W_{-1} \geq 0\right)$. *If* $h\left(W_{-1}\right) < 0$, *then the equilibrium is* u_1, u_0 *if* $f\left(W_{-1}\right) \geq 0$ *and* c_1, c_0 *otherwise.*
3. *If* $W_{-1} \in [\omega_1^m, \omega_0^m[$ *or* $[\omega^c, \omega_1^m[$, *then the equilibrium is as in A.4.*

Proof. The first case is immediate: one must compare the strategies u_1, u_0 and c_1, u_0, and the latter is either not feasible / credible, or dominated in these three intervals, since $W_{-1} \geq v_1$. (this implies that c_1, c_0 is also dominated)

The second case is new and follows from Lemma 10 and Lemma 13. It differs from A.4. because one mus compare u_1, u_0 to c_1, c_0, while in A.4. one compares u_1, c_0 to c_1, c_0. This case arises because for $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)}$, $w_0^m < v_1$. The third case has already been analyzed. ■

Lemma 22 ***Case E:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{7}{10}, \frac{3}{4}\right[$, *then* $\omega^c \leq \omega_0^m \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p \leq \omega^p$,

1. *If* $W_{-1} \geq \omega^p$, *or* $W_{-1} \in [\bar{\omega}^p, \omega^p[$ *or* $[v_1, \bar{\omega}^p[$, *the equilibrium is* u_1, u_0.
2. *If* $W_{-1} \in [\omega_1^m, v_1[$, *the equilibrium is as D.2.*
3. *If* $W_{-1} \in [\omega_0^m, \omega_1^m[$ *or* $[\omega^c, \omega_0^m[$, *then the equilibrium is* u_1, c_0 *with* $x_0 = x_0^1$ *if* $h(W_{-1}) \geq 0$. *If* $h(W_{-1}) < 0$, *the equilibrium is* u_1, c_0 *if* $g(W_{-1}) \geq 0$, *and* c_1, c_0 *otherwise.*

Lemma 23 ***Case F:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{3}{4}, 3 - 2\sqrt{\frac{6}{5}}\right[$, *then* $\omega^c \leq \omega_0^m \leq \omega_1^m \leq v_1 \leq \omega^p \leq \bar{\omega}^p$,

1. *If* $W_{-1} \geq \bar{\omega}^p$ *or* $W_{-1} \in [\omega^p, \bar{\omega}^p[$ *or* $[v_1, \omega^p[$, *then the equilibrium is* u_1, u_0,
2. *If* $W_{-1} \in [\omega_1^m, v_1[$, *then the equilibrium is as D.2.*
3. *If* $W_{-1} \in [\omega_0^m, \omega_1^m[$ *or* $[\omega^c, \omega_0^m[$, *the equilibrium is as E.3.*

Lemma 24 ***Case G:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[3 - 2\sqrt{\frac{6}{5}}, \frac{2\sqrt{5}}{5}\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega_1^m \leq v_1 \leq \omega^p \leq \bar{\omega}^p$,

1. *If* $W_{-1} \geq \bar{\omega}^p$ *or* $W_{-1} \in [\omega^p, \bar{\omega}^p[$ *or* $[v_1, \omega^p[$, *then the equilibrium is* u_1, u_0,
2. *If* $W_{-1} \in [\omega_1^m, v_1[$, *then the equilibrium is as D.2.*
3. *If* $W_{-1} \in [\omega^c, \omega_1^m[$, *the equilibrium is as E.3.*

Lemma 25 ***Case H:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[\frac{2\sqrt{5}}{5}, 1\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega_1^m \leq \omega^p \leq v_1 \leq \bar{\omega}^p$,

1. *If* $W_{-1} \geq \bar{\omega}^p$ *or* $W_{-1} \in [v_1, \bar{\omega}^p[$, *then the equilibrium is* u_1, u_0,
2. *If* $W_{-1} \in [\omega^p, v_1[$, *then the equilibrium is* c_1, u_0,
3. *If* $W_{-1} \in [\omega_1^m, \omega^p[$, *then the equilibrium is as D.2.*
4. *If* $W_{-1} \in [\omega^c, \omega_1^m[$, *the equilibrium is as E.3.*

Lemma 26 ***Case I:*** *If* $\frac{\bar{e}}{a\sigma^2 s} \in \left[1, 3 + 2\sqrt{\frac{6}{5}}\right[$, *then* $\omega_0^m \leq \omega^c \leq \omega^p \leq \omega_1^m \leq v_1 \leq \bar{\omega}^p$,

1. *If* $W_{-1} \geq \bar{\omega}^p$ *or* $W_{-1} \in [v_1, \bar{\omega}^p[$, *then the equilibrium is* u_1, u_0,
2. *If* $W_{-1} \in [\omega_1^m, v_1[$ *or* $[\omega^p, \omega_1^m[$, *then the equilibrium is* c_1, u_0,
3. *If* $W_{-1} \in [\omega^c, \omega_1^m[$, *then the equilibrium is as E.3.*

Lemma 27 ***Case J:*** *If $\frac{\bar{e}}{a\sigma^2 s} > 3 + 2\sqrt{\frac{6}{5}}$, then $\omega^c \le \omega_0^m \le \omega^p \le \omega_1^m \le v_1 \le \bar{\omega}^p$,*

1. *If $W_{-1} \ge \bar{\omega}^p$ or $W_{-1} \in [v_1, \bar{\omega}^p[$, then the equilibrium is u_1, u_0,*

2. *If $W_{-1} \in [\omega_1^m, v_1[$ or $[\omega^p, \omega_1^m[$, then the equilibrium is c_1, u_0,*

3. *If $W_{-1} \in [\omega_0^m, \omega^p[$ or $[\omega^c, \omega_0^m[$, then the equilibrium is as E.3.*

2.7.3 Liquidity and welfare comparisons

Proposition 15

Proof. First I recall the expressions of the spread at time 0 in both cases:

$$\Delta_0^m = \frac{9}{5} a\sigma^2 s;\ \Delta_0^* = 2\left(a\sigma^2 s + \bar{e}\right) - \sqrt{Q} - \sqrt{U}$$

with $Q = \left(\bar{e} - a\sigma^2 s\right)^2 + 2a\sigma^2 W_{-1}$ and $U = \left(\bar{e} - a\sigma^2 s\right)^2 + 4a\sigma^2 x_0 \bar{e}$. It is convenient to rewrite U by plugging the expression for the time-0 constrained trade given in Corollary 14:

$$\begin{aligned} U &= \left(\bar{e} - a\sigma^2 s\right)^2 + 2\bar{e}\left(a\sigma^2 s - \bar{e}\right) + 2\bar{e}\sqrt{Q} \\ &= a^2\sigma^4 s^2 - \bar{e}^2 + 2\bar{e}\sqrt{Q} \end{aligned}$$

$$\Delta_0^* \ge \Delta_0^m \iff \frac{1}{5} a\sigma^2 s + 2\bar{e} \ge \sqrt{Q} + \sqrt{U}$$

This implies that

$$\frac{1}{5} a\sigma^2 s + 2\bar{e} \ge Q + U + 2\sqrt{QU}$$

Since $Q + U = 2a^2\sigma^4 s^2 - 2a\sigma^2 s\bar{e} + 2\bar{e}\sqrt{Q} + 2a\sigma^2 W_{-1}$, the condition becomes, after regrouping terms:

$$-2a\sigma^2 W_{-1} + \frac{14}{5} a\sigma^2 s\bar{e} - \frac{49}{25} a^2\sigma^4 s^2 + 4\bar{e}^2 \ge 2\sqrt{Q}\left(\bar{e} + \sqrt{U}\right)$$

A necessary condition for this inequality to be satisfied is that the LHS is positive, i.e.

$$W_{-1} \le \tilde{\omega} = \frac{7}{5} s\bar{e} - \frac{49}{50} a\sigma^2 s^2 + \frac{2\bar{e}^2}{a\sigma^2} \tag{2.30}$$

The interval of interest to compare $\tilde{\omega}$ to is $[\omega_0^m, \omega^*[$ if $\frac{\bar{e}}{a\sigma^2 s} < \frac{21}{10\left(1+\sqrt{5}\right)}$, $[v_1, \omega^*[$ if $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)}$ and $[v_1, \omega^*[$ if $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{7}{10}$ and $h\left(W_{-1}\right) \geq 0$ or $h\left(W_{-1}\right) < 0$ and $f\left(W_{-1}\right) \geq 0$. Note that the latter case is not given in Proposition 12, which gives only a necessary condition and not a sufficient one. I compare $\tilde{\omega}$ to the different thresholds:

$$v_1 \geq \tilde{\omega} \;\Leftrightarrow\; \frac{5\left(\sqrt{5}-2\right)a\sigma^2 s\bar{e} - 2a^2\sigma^4 s^2 - 50\bar{e}^2}{25a\sigma^2} \geq 0$$

We can consider the numerator of the LHS as a second-order equation in $\bar{e}$. Calculating the discriminant $d = \left(25\left(\sqrt{5}-2\right)^2 - 400\right)a^2\sigma^4 s^2 < 0$ shows that the LHS is always negative (since the second-order term has a negative coefficient), i.e. $v_1 < \tilde{\omega}$.

Next, I compare $\tilde{\omega}$ and ω_0^m:

$$\omega_0^m \leq \tilde{\omega} \;\Leftrightarrow\; \frac{-6a\sigma^2 s\bar{e} + 5a^2\sigma^4 s^2 - 20\bar{e}^2}{10a\sigma^2}$$

The LHS in the numerator can be seen as a second-order equation in $\bar{e}$, I calculate its discriminant: $d = 436a^2\sigma^4 s^2 > 0$. Since the constant and the second-order term have opposite signs, there is a positive and a negative root. The positive root is equal to $\frac{3+\sqrt{109}}{20}a\sigma^2 s \approx 0.68a\sigma^2 s > \frac{21}{10\left(1+\sqrt{5}\right)}a\sigma^2 s \approx 0.65a\sigma^2 s$. Hence if $\frac{\bar{e}}{a\sigma^2 s} < \frac{21}{10\left(1+\sqrt{5}\right)}$, (2.30) does not hold. If

To complete the time-0 case, I assume that $h\left(W_{-1}\right) > 0$ or $h\left(W_{-1}\right) \leq 0$ and $f\left(W_{-1}\right) \geq 0$. The relevant threshold for the monopoly is then ω_1^m:

$$\omega_1^m \leq \tilde{\omega} \;\Leftrightarrow\; \frac{a^2\sigma^4 s^2}{25} \leq \bar{e}^2 \;\Leftrightarrow \bar{e} \geq \frac{a\sigma^2 s}{5} \;\; (\bar{e} > 0)$$

Next, I compare the time-1 spreads, $\Delta_1^* = a\sigma^2 s + \bar{e} - \sqrt{U}$ and $\Delta_1^m = \frac{3}{5}a\sigma^2 s$:

$$\Delta_1^* \geq \Delta_1^m \;\Leftrightarrow\; \frac{4}{5}a\sigma^2 s\bar{e} + 2\bar{e}^2 - \frac{21}{25}a^2\sigma^2 s^4 \geq 2\bar{e}\sqrt{Q} \tag{2.31}$$

I study the sign of the LHS, taking it as a second-order equation in $\bar{e}$. The discriminant is $d = \frac{184}{25}a^2\sigma^4 s^2$ and given that the constant and the second-order term have opposite signs, there is a positive and a negative root. The positive root is $\frac{\sqrt{46}-2}{10}a\sigma^2 s \approx 0.48a\sigma^2 s$. Hence for $\frac{\bar{e}}{a\sigma^2 s} \leq \frac{\sqrt{46}-2}{10}$, $\Delta_1^* \leq \Delta_1^m$. Otherwise, one can take the square in each side of inequality

(2.31), which gives, after a few lines of algebra:

$$\Delta_1^* \geq \Delta_1^m \;\Rightarrow\; W_{-1} \leq \hat{\omega} \equiv \frac{7}{5}s\bar{e} - \frac{21}{25}a\sigma^2 s^2 + \frac{441}{5000}\frac{a^3\sigma^6 s^4}{\bar{e}^2} - \frac{21}{125}\frac{a^2\sigma^4 s^3}{\bar{e}}$$

To assess the existence of this case, I compare $\hat{\omega}$ to the thresholds v_1, ω_0^m and ω_1^m.

$$\hat{\omega} \geq v_1 \;\Leftrightarrow\; \frac{2-\sqrt{5}}{5}s\bar{e} + \frac{3}{50}a\sigma^2 s^2 + \frac{21}{125}\frac{a^2\sigma^4 s^2}{\bar{e}}\left[\frac{21}{40}\frac{a\sigma^2 s}{\bar{e}} - 1\right] \geq 0$$

It is not possible to derive the roots of this equation analytically. However, one can look at sufficient conditions. There are two options and both are not satisfied. i) It is enough that:

$$\left\{\begin{array}{l} \frac{21}{40}\frac{a\sigma^2 s}{\bar{e}} - 1 \geq 0 \\ \frac{2-\sqrt{5}}{5}s\bar{e} + \frac{3}{50}a\sigma^2 s^2 \geq 0 \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{l} \frac{\bar{e}}{a\sigma^2 s} \leq 0.525 \\ \frac{\bar{e}}{a\sigma^2 s} \leq \frac{3}{10\left(\sqrt{5}-2\right)} \approx 1.27 \end{array}\right. \;\Leftrightarrow\; \frac{\bar{e}}{a\sigma^2 s} \leq 0.525$$

This is not compatible with the initial assumption that $\frac{\bar{e}}{a\sigma^2 s} \geq \frac{21}{10\left(1+\sqrt{5}\right)} \approx 0.65$. ii) Another sufficient condition is:

$$\left\{\begin{array}{l} \frac{3}{50}a\sigma^2 s^2 \geq \frac{21}{125}\frac{a^2\sigma^4 s^3}{\bar{e}} \\ \frac{441}{5000\bar{e}^2}a^3\sigma^6 s^4 \geq \frac{\sqrt{5}-2}{5}s\bar{e} \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{l} \frac{\bar{e}}{a\sigma^2 s} \geq \frac{14}{5} \\ \frac{\bar{e}}{a\sigma^2 s} \leq \frac{(441)^{\frac{1}{3}}}{10\left(\sqrt{5}-2\right)^{\frac{1}{3}}} \approx 1.23 \end{array}\right.$$

These two conditions are contradictory.

Now compare $\hat{\omega}$ and ω_0^m:

$$\hat{\omega} \geq \omega_0^m \;\Leftrightarrow\; \frac{3}{5}s\bar{e} - \frac{9}{25}a\sigma^2 s^2 + \frac{441}{5000}\frac{a^3\sigma^6 s^3}{\bar{e}^2} - \frac{21}{125}\frac{a^2\sigma^4 s^3}{\bar{e}} \geq 0$$

Again, there are two types of sufficient conditions and both are not satisfied. i) It is enough that:

$$\left\{\begin{array}{l} \frac{3}{5}s\bar{e} \geq \frac{9}{25}a\sigma^2 s^2 \\ \frac{441}{5000}\frac{a^3\sigma^6 s^3}{\bar{e}^2} \geq \frac{21}{125}\frac{a^2\sigma^4 s^3}{\bar{e}} \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{l} \frac{\bar{e}}{a\sigma^2 s} \geq \frac{3}{5} \\ \frac{\bar{e}}{a\sigma^2 s} \leq \frac{21}{40} \end{array}\right.$$

These two conditions contradict each other. ii) Another sufficient set of sufficient conditions is:

$$\left\{\begin{array}{l} \frac{3}{5}s\bar{e} \geq \frac{21}{125}\frac{a^2\sigma^4 s^3}{\bar{e}} \\ \frac{441}{5000}\frac{a^3\sigma^6 s^3}{\bar{e}^2} \geq \frac{9}{25}a\sigma^2 s^2 \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{l} \frac{\bar{e}}{a\sigma^2 s} \geq \frac{\sqrt{7}}{5} \approx 0.53 \\ \frac{\bar{e}}{a\sigma^2 s} \leq \frac{21}{15\sqrt{8} \approx 0.49} \end{array}\right.$$

Again these two conditions contradict each other. Last I compare ω_1^m and $\hat{\omega}$:

$$\hat{\omega} \geq \omega_1^m \Leftrightarrow \frac{3}{50} a\sigma^2 s^2 + \frac{441}{5000} \frac{a^3\sigma^6 s^3}{\bar{e}^2} - \frac{21}{125} \frac{a^2\sigma^4 s^3}{\bar{e}} \geq 0$$

We can rewrite the LHS as $\frac{a\sigma^2 s\left(300\bar{e}^2+441a^2\sigma^4s^2-4200a\sigma^2 s\bar{e}\right)}{5000\bar{e}^2}$, which can be seen as a second-order equation in $\bar{e}$. The discriminant of the numerator is $d = \left(4200^2 - 1200.441\right) a^2\sigma^4 s^2 \equiv \bar{q}^2 a^2\sigma^4 s^2 > 0$. The constant and the second-order term has the same sign, and the first-order term is negative, thus there are two positive roots. The smallest root is equal to $(7-q)\, a\sigma^2 s \approx 0.1 a\sigma^2 s$, with $q = \frac{\bar{q}}{600}$, which is lower than the threshold $\frac{21}{10\left(1+\sqrt{5}\right)} a\sigma^2 s$, and the largest root is $(7+q)\, a\sigma^2 s \approx 14 a\sigma^2 s$. Hence if $\frac{\bar{e}}{a\sigma^2 s} \geq 7+q$, $\Delta_1^* \geq \Delta_1^m$ if $W_{-1} \in [\omega_1^m, \hat{\omega}]$. ■

Proposition 16

Proof. The equilibrium spreads in the constrained competitive and voluntarily constrained cases are given in Corollary 14 and Proposition 13. A time 0:

$$\Delta_0^* \geq \Delta_0^{c_1,u_0} \Leftrightarrow \sqrt{U^m} + \bar{e} \leq \sqrt{Q} + \sqrt{U} \geq 0 \Rightarrow U^m - (Q+U) + \bar{e}^2 \geq \sqrt{QU} - 2\bar{e}\sqrt{U^m}$$

Since $U^m - (Q+U) = 2\bar{e}^2 - 2\bar{e}\sqrt{Q}$, the previous condition rewrites as:

$$2\bar{e}^2 \geq \sqrt{QU} - 2\bar{e}\left[\sqrt{U^m} - \sqrt{Q}\right]$$

I now show that the inequality is not trivially satisfied because the RHS is always positive. To see this, note that $U^m = Q + a^2\sigma^2 s^2 > Q$, hence $\sqrt{U^m} \leq \sqrt{Q} + a\sigma^2 s$ by Cauchy-Schwartz inequality. Hence $0 \leq 2\bar{e}\left[\sqrt{U^m} - \sqrt{Q}\right] \leq 2a\sigma^2 s\bar{e}$. Further, we can write U as $U = \left(\bar{e} - a\sigma^2 s\right)^2 + 4a\sigma^2 x_0 \bar{e}$. Since the financial constraint is binding at time 0 in the competitive case, $W_{-1} = 2x_0\bar{e} - 2a\sigma^2 x_0 (s - x_0)$. Hence $U = Q + 4a^2\sigma^4 x_0 (s-x_0) > Q$, because the second term represents the time-0 trading profit which is positive in equilibrium given that the spread does not close. Since $Q > 0$, $U > Q \Rightarrow UQ > Q^2$. Taking the square root on each side gives: $\sqrt{UQ} \geq Q$. Then using the expression $U = Q + 4a^2\sigma^4 x_0 (s-x_0)$, we have $QU = Q^2 + Q4a^2\sigma^4 x_0 (s-x_0)$. Then, applying Cauchy-Schwartz inequality to $\sqrt{QU}$, we obtain: $Q \leq \sqrt{UQ} \leq Q + 2a\sigma^2\sqrt{Qx_0 (s-x_0)}$. This, combined with the inequalities about U^m, implies that

$$\sqrt{UQ} - \bar{e}\left(\sqrt{U^m} - \sqrt{Q}\right) \geq Q - 2a\sigma^2 s\bar{e} = \bar{e}^2 + a^2\sigma^4 s_2^2 a\sigma^2 W_{-1} \geq 0$$

Next, I consider the time-1 spreads:

$$\Delta_1^* \geq \Delta_1^{c_1,u_0} \;\Leftrightarrow\; U^m \geq U \;\Leftrightarrow\; 2\bar{e}^2 + a^2\sigma^4 s^2 - 2a\sigma^2 s\bar{e} + 2a\sigma^2 W_{-1} \geq 2\bar{e}\sqrt{Q}$$

I first check the sign of the LHS. It is positive if and only if $W_{-1} \geq s\bar{e} - \frac{1}{2}a\sigma^2 s^2 - \frac{\bar{e}^2}{a\sigma^2} = \omega^p - \frac{\bar{e}^2}{a\sigma^2}$. This inequality is always satisfied since by definition $W_{-1} \geq \omega^p$ in the c_1, u_0 equilibrium. I rewrite the LHS as $2a\sigma^2\left(W_{-1} - \left(\omega^p - \frac{\bar{e}^2}{a\sigma^2}\right)\right)$. The above inequality gives:

$$4a^2\sigma^4\left(W_{-1} - \left(\omega^p - \frac{\bar{e}^2}{a\sigma^2}\right)\right)^2 \geq 4\bar{e}^2 Q = 4\bar{e}^2\left(\bar{e} - a\sigma^2 s\right)^2 + 8a\sigma^2\bar{e}^2 W_{-1}$$

Developing each side and skipping some lines of algebra, I find that the previous inequality is equivalent to:

$$W_{-1}^2 - 2\omega^p W_{-1} + s^2\left(\bar{e} - \frac{1}{2}a\sigma^2 s\right)^2 \geq 0$$

Since $s^2\left(\bar{e} - \frac{1}{2}a\sigma^2 s\right)^2 = \left(s\bar{e} - \frac{1}{2}a\sigma^2 s^2\right)^2 = (\omega^p)^2$, the LHS is equal to $(W_{-1} - \omega^p)^2$ which is always positive. This proves the result about time-1 spreads.

Finally, the result about the time-0 spread follows from the facts that under the assumptions of Proposition 16, i) $x_0^{c_1,c_0} \geq x_0^{c_1,u_0}$ and ii) $\Delta_1^* \geq \Delta_1^{c_1,u_0}$.

■

Corollary 19

Proof. I showed in the proof of Proposition 16 that the capital gain is larger in the monopoly case by showing that $U^m \geq U$. Further, recalling that $x_0^{c_1,c0} = \frac{s}{2} - \frac{\bar{e}-\sqrt{Q}}{2a\sigma^2}$ and $x_0^{c_1,u_0} = \frac{s}{2}$, I get:

$$x_0^{c_1,u_0} \leq x_0^{c_1,c_0} \;\Leftrightarrow\; \sqrt{Q} \geq \bar{e} \Rightarrow W_{-1} \geq s\bar{e} - \frac{1}{2}a\sigma^2 s^2 = \omega^p$$

This is always true for the c_1, u_0 equilibrium under consideration. ■

Lemma 11

Proof. Starting from equation (2.11), and using the expression of local investors' demand, $\mathbb{E}_0 (p_1^a) - p_0^A = a\sigma^2 \left(Y_0^A + s_0\right)$, we can rewrite local investors' equilibrium utility as

$$\begin{aligned}
\chi_0^A &= \frac{\left(\mathbb{E}_0\left(p_1^A\right) - p_0^A\right)^2 + (D_1 - p_1)^2}{a\sigma^2} - s\left(\mathbb{E}_0\left(p_1^A\right) - p_0^A + D_1 - p_1\right) - \frac{a\sigma^2}{2}\left[\left(Y_0^A + s\right)^2 + \left(Y_1^A + s_1\right)^2\right] \\
&= \frac{\left(\mathbb{E}_0\left(p_1^A\right) - p_0^A\right)^2 + (D_1 - p_1)^2}{2a\sigma^2} - s\left(\mathbb{E}_0\left(p_1^A\right) - p_0^A + D_1 - p_1\right) \qquad (2.32)
\end{aligned}$$

Given that risk premia are symmetric, we have: $\phi_0^A = D - p_0^A = \frac{\Delta_0}{2}$ and $\mathbb{E}_0\left(-\phi_1^A\right) = \mathbb{E}_0\left(p_1^A - D_1\right) = -\frac{\Delta_1}{2}$. Further, $\mathbb{E}_0(\Delta_1) = \Delta_1$ since the spread, unlike the individual price, does not depend on ϵ_1. Hence local investors' welfare is

$$\chi_0^A = \frac{(\Delta_0 - \Delta_1)^2 + \Delta_1^2}{8a\sigma^2} - \frac{s}{2}\Delta_0 \qquad (2.33)$$

When unconstrained competitive arbitrageurs are present, all spreads are 0, as shown in Proposition 10, hence $\chi_0^{A,*} = 0$. In the autarky situation, local investors are constrained to hold their local asset in equilibrium. Hence using market-clearing and investors' demand functions, we have $Y_1^A = 0$, hence $Y_1^A + s = s$, which implies from investors' demand that $p_1^A = D_1 - a\sigma^2 s$. At time 0, by market-clearing, $Y_0^A = 0$, hence $Y_0^A + s = \frac{\mathbb{E}_0\left(p_1^A\right) - p_0^A}{a\sigma^2} = s$. Hence $p_0^A = \mathbb{E}_0\left(p_1^A\right) - a\sigma^2 s = D - 2a\sigma^2 s$. The prices in market B are opposite, by construction. Plugging this prices into the local investors's welfare function gives $\chi_0^{A,a} = -a\sigma^2 s$.

To rank $\chi_0^{A,m}$ relative to $\chi_0^{A,*}$ and $\chi_0^{A,a}$, note that

$$\begin{cases} 0 \leq \mathbb{E}_0\left(p_1^A\right) - p_0^A \leq 2a\sigma^2 s \\ 0 \leq D_1 - p_1^A \leq 2a\sigma^2 s \end{cases}$$

Using these four inequalities and equation (2.32) yields the result.

Finally, it is clear that in the full insurance and autarky cases, arbitrageurs do not make any profit, while they do in any of the monopolistic cases. ■

Corollaries 20 and 21 Proof. The comparative statics in Corollary 20 obtain by differentiation from equation (2.33). The first part of Corollary 21 follows immediately. The second part about aggregate welfare is proved on an example in the text. ■

Chapter 3

Runs, Asymmetric Price Impact and Predatory Trading

Abstract: Predatory trading is a strategy whereby some traders (predators) amplify or induce adverse price movements of an asset to trigger a margin call on a rival trader's (the prey) position and gain from her subsequent firesales. Given that predation involves a temporary and artificial mis-valuation of the asset, shouldn't smart investors step in and take advantage of predators' price pressure? I show that when they anticipate the prey's firesales, smart investors may actually run on the asset, strengthening, and to some extent substituting to, the predators' price pressure. Further, their reaction leads to a reduction in the prey's price impact, which decreases her already limited ability to support the price and avoid a margin call. This negative feedback loop reduces the cost of predatory trading for predators. The key driver of these results is smart investors' limited risk-bearing capacity. Consequently, I find that predatory trading is likely to occur when smart investors are sufficiently risk-averse or the asset sufficiently risky.

3.1 Introduction

Asset prices can at times exhibit sharp fluctuations. During these episodes, traders marking-to-market or relying on short-term funding (e.g. hedge funds, broker-dealers, investment banks) may become distressed and forced to sell assets at firesale prices. In some cases, it seems that the price movements causing these firesales can be exacerbated by deliberate strategies from traders seeking to profit from a rival's financial difficulties. There is evidence that such predatory trading occurred against LTCM in 1998 (Cai, 2009), and against several hedge funds during the recent financial crisis, in particular in the aftermath of Bear Stearns' and Lehmann Brothers' collapses[1].

Predatory trading strategies consist of two stages. First, some traders (predators) seek to cause or exacerbate price movements decreasing the marked-to-market value of a rival's portfolio. This tightens the prey's financial constraint, eventually leading to firesales. In a second stage, the predators gain by exploiting the firesale prices. Hence, on one hand, predatory trading relies critically on the market being imperfectly liquid: the predators must be able to move asset prices against the prey, and the prey's firesales must also affect prices. On the other hand, market liquidity should also depend on the possibility of predatory trading. Indeed, smart investors should anticipate that liquidity may temporarily dry up if a large trader liquidates her positions.[2] While predators manipulate the price to push their rival into distress, do smart investors absorb the predators' trades, thereby countering the predators' impact, or instead run for the exits, thus magnifying the liquidity dry-up?

Existing theories of predatory trading (e.g. Brunnermeier and Pedersen, 2005, Attari, Mello and Ruckes, 2005) are largely silent on this issue, because they assume that the predators and the prey trade with a competitive fringe of long-term value investors, whose demand is fixed.[3] This implies that these investors are less-than-fully rational in that they disregard future price movements. Without this assumption, it is not clear to which extent the results of these papers would remain or be qualified. In this paper, I show that predatory trading may occur even in the presence of smart investors who understand that the asset can be artificially and temporarily undervalued due to predatory trading-induced price pressure. Since holding the asset is risky, smart investors' willingness to "lean against the wind" and absorb predators' price pressure is limited by their risk-bearing capacity. Hence even if the

[1]For instance, in March 2008, Focus Capital, a New York-based hedge fund specialized in mid-caps, was forced to close in the aftermath of Bear Stearns' collapse. The *Financial Times* wrote: "In a letter to investors, the founders of Focus, Tim OBrien and Philippe Bubb, said it had been hit by 'violent short-selling by other market participants', which accelerated when rumours that it was in trouble circulated." (J. Mackintosh, FT, 4 March 2008). Similarly, in October 2008, the *Financial Times* wrote: "Hedge funds prey on rivals (...) the increasingly cannibalistic activity stems from a wave of redemptions hitting hedge funds " (H. Sender, FT, Oct 2008). See Brunnermeier and Pedersen (2005) for additional anecdotal evidence. Cai's (2009) paper documents dealers' predatory behaviour against LTCM in 1998, using a unique dataset of audit trail transactions.

[2]The financial constraints of a large trader may be known to other market participants. For instance, broker-dealers or lenders have information about the positions and balance sheet of large traders. Regulatory constraints sometimes impose to reveal positions, and although traders' identities may be concealed, market participants can often infer the positions of others from this information. For instance, Amaranth's positions in the natural gas market became known to other traders, who observed from the exchanges data that a single market participant had accumulated very large positions in the futures market (Levin and Coleman, 2007).

[3]That is, long-term value investors in these papers have exogenous downward-sloping demand curves.

mis-valuation is expected to disappear in the future, smart investors are unwilling to take unbounded positions ex-ante because it would expose them to too much risk. In fact, I show that smart investors' reactions to predatory trading may actually reduce the cost of predation for predators.[4]

When the competitive fringe of the market is made of smart, rational investors, the predators' and the prey's price impacts, and more generally market liquidity, not only *affect*, but also *are affected by* the possibility of predatory trading. This two-way relationship can generate self-fulfilling liquidity dry-ups and make predatory trading cheaper. I show that when smart investors expect the prey to fail in the future, current prices adjust to reflect the fact that the prey's firesales will lower the willingness of other market participants to hold the asset. Further, if investors believe that the prey will fail, she loses price impact and her trades move prices less than opposite trades by predators. That is, price impact becomes trader-specific and becomes an increasing function of a large trader's financial strength - or at least the smart investors' perception of it. This reduces the prey's ability to resist predatory trading by supporting prices to avoid reporting a low marked-to-market wealth. Hence the mere anticipation of the prey's firesale can generate a vicious circle in which predatory trading causes smart investors to "rush for the exits", which in turn facilitates predatory trading.

This negative feedback loop materializes in equilibrium when smart investors are sufficiently risk-averse (or equivalently, if the asset is sufficiently risky). A novel prediction of the model is thus that the link between market liquidity and *a trader*'s funding liquidity depends on risk-aversion in the market: in times of high risk-aversion, a trader's price impact becomes an increasing function of her (perceived) funding liquidity, while with low risk-aversion, a trader's price impact is independent of her funding liquidity. This trader-specific prediction complements the results of the limits of arbitrage literature, which predicts a positive link between market liquidity (defined as the spread between two identical assets) and *aggregate* funding liquidity (Gromb and Vayanos, 2002, Brunnermeier and Pedersen, 2009). Another key driver of the equilibrium is the distribution of initial asset ownership. When smart investors start with a small position in the risky asset, an increase in their position increases the probability of predation, and decreases it otherwise.

[4]Note that there are theories of *front-running* with rational market participants (e.g. Pritsker, 2009), whereby strategic traders exploit their advanced knowledge of a rival trader's future liquidation. In this paper, strategic traders engage in *predatory trading*, i.e. they *induce* the need for another trader to liquidate his positions.

The model has three periods, with a risky asset and a risk-free asset. There are three types of market participants: a finite number of predators (e.g. hedge funds, dealers) and one prey (e.g. another hedge fund), the rest of the market being made of a continuum of smart competitive investors. The prey faces a financial constraint: She must liquidate her entire portfolio if her marked-to-market wealth falls below some threshold, e.g. because this triggers margin calls or redemptions. The prey is initially long the asset, so that her financial constraint is likely to bind if the asset price falls below a certain threshold. Finally, I assume that the prey cannot hold more than a certain quantity of the risky asset, i.e. her ability to lever up is limited.[5]

Smart investors are risk-averse and seek to offload a long position in the risky asset in the market, i.e. they demand liquidity. For brevity, I will therefore refer to them as *hedgers*.[6] The predators and the prey are risk-neutral. Hence, in the absence of financial constraints, they would provide hedgers with liquidity by buying the asset. However, being finite in number, they have market power and thus ration liquidity by buying only limited quantities over time. As a result, the asset trades at a discount relative to its fundamental value, i.e. it is imperfectly liquid.

Now consider the effect of the prey's financial constraint. The predators may be tempted to buy less or even short the risky asset in order to ensure that its price is low enough and force the prey to liquidate. Such a strategy involves an opportunity cost: since the asset trades at a discount, the predators would prefer to buy the risky asset by spreading trades over time. However, there is also a benefit from predatory trading. Indeed, eliminating the prey reduces the competition in the provision of liquidity, allowing predators to capture larger rents. Further, the prey's liquidation itself increases the demand for liquidity, which benefits the remaining liquidity providers.

I first study the prey's ability to resist predatory trading by buying the asset in a bid to support its price. This ability may be limited, first, by her leverage constraint, and second - and more interestingly - by the hedgers' anticipations about predatory trading. There are two effects. First, when hedgers expect the prey to liquidate, price impact becomes trader-specific (even though all information is symmetric). If the prey buys the asset to support its price, her trades move the price *less* than opposite orders by predators. Indeed, hedgers

[5] Both the limited borrowing capacity and the marked-to-market wealth constraint may stem from agency frictions arising in the process of delegation of funds by outside investors (Shleifer and Vishny, 1997).

[6] Hedgers may stand for market-makers trying to reduce their inventory, or insurance companies seeking to sell assets following or in anticipation of downgrades or other regulatory constraints.

anticipate that for each share they sell to the prey, with some probability, that share will have to be liquidated in a firesale, reducing future liquidity. Hence selling a share to the prey provides them with only partial, temporary insurance. This reduces the gains from trading with the prey. In this sense, the reactions of rational hedgers to the possibility of predatory trading can be "destabilizing": the mere anticipation of the prey's distress reduces her ability to resist predatory trading.

The second effect is akin to a financial market run: the hedgers are more reluctant to holding the risky asset when they believe that the prey will be distressed. As a result, they are ready to sell their endowment at a lower price. This selling pressure can thus turn into a financial market run, as the hedgers attempt to reduce their asset holdings ahead of the prey's firesale.[7] If the hedgers are sufficiently risk-averse, their run may even be such that predators need not sell the asset: it may be enough for them to reduce the quantity of the asset they buy, i.e. "hoard" liquidity, and let the hedgers' trading push the prey into distress. This implies that short-selling bans may be ineffective to prevent predatory trading, and that there is no direct link between selling an asset and predatory trading.

The hedgers' risk appetite plays a key role in both effects. Their risk appetite depends on the size of their initial position, and the slope of their demand curve (i.e. the product of their risk-aversion and the asset volatility). The change in price impact and the run effect depend primarily on hedgers' risk-aversion. If the hedgers hold no initial positions in the asset, the effects are still present, and is stronger with long positions.

The size of the hedgers' initial position has a non-monotonic effect on the *likelihood* of predatory trading: the likelihood first increases and then decreases with hedgers' initial position. This results from two conflicting effects. On the one hand, the hedgers' behaviour can decrease the cost (to the predators) of predatory trading, because the hedgers' run is stronger. This is especially true if they start with a long position in the risky asset. On the other hand, the hedgers' initial position also affects predators' outside option, which consists in providing rather than withdrawing liquidity: if hedgers generate significant selling pressure (if they have a large enough initial position), liquidity provision is very profitable. As a result, an increase in hedgers' selling pressure (via an increase in their initial position in the risky asset) does not necessarily generate more predatory trading.

[7] The difference between the traditional models of market run (e.g. Bernardo and Welch (2004)) and this one is that the probability of the liquidity shock is endogenous. The liquidity shock (the prey's firesale) depends on the first-period price, which is determined in equilibrium.

The analysis has implications for regulation and risk-management. The model predicts that a destabilizing feedback loop can occur when hedgers are sufficiently risk-averse. This prediction is in line with anecdotal evidence that predatory trading occurs during flights-to-liquidity episodes (e.g. LTCM in 1998, predatory activity among hedge funds in 2008). The analysis shows, more precisely, that flight-to-liquidity and predatory trading phenomena feed each others when hedgers are sufficiently risk-averse. If hedgers' risk-aversion in utility proxies for risk-aversion stemming from various constraints limiting the market's risk-bearing capacity, the model suggests that to avoid predatory trading, one should attempt to relax these constraints or provide additional risk-bearing capacity. Since hedgers' risk-aversion translates into high permanent price impact, and assuming that it is possible to classify assets by their coefficient of permanent price impact, another interpretation of the results is that financially-constrained strategic traders are more exposed to predatory trading risk when they hold assets with high permanent price impact.

Finally, the model has also implications for the relation between turnover, liquidity and welfare. The analysis shows that proxying for liquidity by turnover or price impact can be misleading. In a special case of the model where it is socially optimal not to trade because initial endowments are Pareto-efficient, I show that the mere presence of the prey's financial constraint can induce (predatory) trading and thus a positive turnover. Further, although liquidity worsens - the asset trades at a larger discount -, the prey's price impact decreases. Hence in the presence of large investors, traditional measures of market depth can be misleading to assess liquidity and welfare.

This paper departs from the recent literature on predatory trading (Brunnemeier and Pedersen (2005), Attari, Mello and Ruckes (2005), Parida and Venter (2009), Laó (2010)) and front-running (Pritsker (2009), Carlin et al. (2007)) by combining the assumption that all market participants are rational and that the prey's liquidation depends on her marked-to-market wealth.[8] My analysis shows that rational hedgers' optimal behaviour can make predatory trading more likely. My model is close to Pritsker's, who also considers rational market participants, but in a setting with *exogenous* distress, i.e. in which the prey is forced to liquidate at a given time, independently of her marked-to-market wealth. Considering endogenous distress allows me to link the hedgers' optimal behaviour to the probability of predatory trading. It also generates the novel state-dependent link between market liquidity and *a trader*'s funding liquidity.

[8]Note that some of these papers include front-running under the umbrella of predatory trading.

Endogenous distress is also the main difference between this paper and Carlin et al. (2007) and explains why our findings differ. I find that predatory trading is likely to occur when the slope of hedgers' demand curve is steep, while Carlin et al.'s model predicts the opposite. In my setting, high price impact allows predators to move prices to induce the prey's distress. In Carlin et al. (2007), a high price impact allows the prey to retaliate against predators in a repeated interaction.

Modeling all market participants as rational also allows me to connect the literature on predatory trading to that on runs in financial markets and more generally destablizing speculation. The economic force triggering what I call run here is not a sequentiality issue as in Bernardo and Welch (2004), but the prospect of the prey's firesale (i.e. a supply shock) and of the predators' increased market power. A feature common to our models is the market's limited risk-bearing capacity. While Bernardo and Welch assume that hedgers are myopic, in my setting all market participants are rational and forward-looking. DeLong et al. (1990)'s model relies on the presence of positive feedback traders. In my model, the positive feedback stems from the fact that low marked-to-market wealth is followed by the prey's liquidation.

The chapter proceeds as follows. Section 3.2 presents the model. Section 3.3 studies the special case where the hedgers have no endowment in the risky asset. Section 3.4 studies the case with positive endowments. Section 3.5 concludes. The appendix contains the proofs.

3.2 Model

The model has three periods: $t = 0, 1, 2$, and a risky asset, in finite supply $S \geq 0$. It pays off a dividend $\tilde{D}_2$ at $t = 2$, with $\tilde{D}_2 = D + \tilde{\epsilon_1} + \tilde{\epsilon_2}$, $D > 0$. The innovations ϵ_1 and ϵ_2, revealed at $t = 1$ and $t = 2$ respectively, are independent and identically distributed normal variables with mean 0 and variance σ^2. I denote p_t the price of the risky asset. There is a risk-free asset in perfectly elastic supply with return r_f normalised to 0.

There are $n+1$ market participants, divided in two classes: *hedgers* and *strategic traders*. The hedgers are treated as a representative competitive trader (with subscript 0) with exponential utility over final consumption. Their coefficient of absolute risk-aversion is α. The hedgers start with an endowment $X^0_{-1} \geq 0$ in the risky asset.[9]. Since they have CARA

[9]Hedgers may stand for a competitive market-making sector. Their endowment, in this case, represent

preferences, their initial wealth is irrelevant for the problem, hence I assume without loss of generality that they start with cash $B_{-1}^0 = 0$.

The hedgers trade with $n \geq 2$ risk-neutral strategic traders, who start with endowments X_{-1}^i, $i = 1, ..., n$, in the risky asset and B_{-1}^i in cash. For trader $i = 0, 1, ..., n$, x_t^i denotes the time t risky asset trade and X_t^i the end-of-time t position. Strategic traders and the hedgers face the same dynamic budget constraint:

$$\forall i = 0, 1, ..., n, \; W_2^i = C_2^i = B_{-1}^i - x_0^i p_0 - x_1^i p_1 + X_1^i D_2 \tag{3.1}$$

Strategic traders account for the impact of their trades on the price. At time 0 and 1, the hedgers set their demand for the risky asset as a function of its price, and strategic traders compete in quantities (à la Cournot) for the risky asset, taking this demand as given. Strategic traders can be seen as sophisticated investors such as prop trading desks, dealers or hedge funds, who have a superior understanding of the trading environment and the "order-flow", and therefore internalize the impact of their own trades on the price.[10] For simplicity, strategic traders' identities are observable, i.e. trading is not anonymous.[11]

The group of the strategic traders consists of one *prey* (trader 1, "she") and $n - 1$ *predators*. The prey faces financial constraints, while predators do not. In particular, the prey is distressed and must liquidate her position in the risky asset when her marked-to-market wealth is lower than a threshold $\underline{V}$:

Assumption 1 *If $B_0^1 + X_0^1 p_0 \leq \underline{V}$, then $X_1^1 = 0$.*

The prey's liquidation consecutive to a low wealth may follow from large capital outflows as a response to a poor performance. A number of financial constraints are based on prices, e.g. VaR constraints, stop-loss thresholds or high-water marks. The relation between past

market-makers' aggregate inventory, which can result from a temporary order imbalance, in the spirt of Grossman and Miller (1988). Hedgers may also stand for the demand of two groups of local traders subject to endowment shocks in segmented markets, as in Gromb and Vayanos (2002).

[10]For instance, investment banks often have a good understanding of the order-flow. Similarly, Perold reports that LTCM "believed that most of its trading opportunities arose as a result of dislocations in the financial markets caused by institutional demands". The hedge fund "would build models to find mispricings created by such demands, but would also identify the reason for the mispricing before initiating a trade" (Perod (1999)).

[11]See Foucault et al. (2003) and references therein for a description of non-anonymous trading environments . I discuss further the role of this assumption in the model in Section 3.3.3.

performance and fund flows has been documented for both equity and debt financing.[12] Agency concerns resulting from the delegation of funds from investors to strategic traders can rationalize this behaviour: Bolton and Scharfstein (1990) show that a termination threat can arise as a disciplining device in an optimal contract, even if it exposes the agent to predation risk.

In addition to the marked-to-market wealth constraint, the prey faces a leverage constraint. She cannot take a position X_0^1 larger than $\bar{X}$.[13]

Assumption 2 $X_0^1 \leq \bar{X}$

For simplicity, I assume that predators are cash-rich or able to secure better funding conditions and do not face any financial constraints.[14]

Given that all market participants are informed about the prey's constraints, they take into account the possibility of her being distressed in their maximization problems. The hedgers choose trades x_0^0 and x_1^0 to maximize their utility subject to their dynamic budget constraint, while taking prices and the prey's constraints as given. Their problem is given by:

$$\begin{aligned} \max_{x_0^0, x_1^0} -E_0 \exp\left[-\alpha C_2^0\right] \\ s.t.\ C_2^0 = B_{-1}^0 - x_0^0 p_0 - x_1^0 p_1 + X_1^0 D_2 \\ B_0^1 + X_0^1 p_0 \leq \underline{V} \Rightarrow X_1^1 = 0 \\ X_0^1 \leq \bar{X} \end{aligned}$$

Strategic traders maximize their expected wealth by choosing trades x_t^i ($t = 0, 1$ and $i = 1, ..., n$), subject to their dynamic budget constraint, the price schedule which results from the hedgers' demand and market-clearing, and the prey's financial constraints. The

[12]For instance, open-end mutual funds experiencing large outflows after a string of poor returns exert significant price pressure in equity markets (Shleifer and Vishny (1997), Coval and Stafford (2007)). The repo market is also prone to runs (see, e.g. Gorton and Metrick (2010)).

[13]$\bar{X}$ may depend on the prey's initial cash, the first period price, and be correlated with the severity of the wealth constraint.

[14]Strategic traders such as hedge funds may have some leeway in chosing their capital structure. For instance, some hedge funds are able to impose better lock-up periods or gates to their investors than their rivalsand is optimal differentiation in strategic traders' capital structure can arise in equilibrium in an optimal contract setting (Hombert and Thesmar (2009)).

optimization problem of a strategic trader is given by

$$\begin{aligned} \forall i = 1, ..., n, \quad & \max_{x_0^i, x_1^i} E_0 \left[W_2^i \right] \\ s.t.\ C_2^i = B_{-1}^i - x_0^i p_0 - x_1^i p_1 + X_1^i D_2 & \\ hedgers\ demand\ at\ t = 0, 1 & \\ market - clearing\ at\ t = 0, 1 & \\ B_0^1 + X_0^1 p_0 \leq \underline{V} \Rightarrow X_1^1 = 0 & \\ X_0^1 \leq \bar{X} & \end{aligned}$$

Since each strategic trader has price impact and is informed about the prey's financial constraints, these constraints enter not only the prey's optimisation problem, but also that of her rival strategic traders.

Strategic traders have a higher apetite for risk than hedgers. Hence, absent financial constraints, trading is motivated by the hedgers being (strictly) long the risky asset. In that case, the hedgers would offload some of the risk of this position onto the risk-neutral strategic traders. To isolate the effect of the financial constraint in the model, and show how it leads to predatory trading, I start with a special case, in which the hedgers do not initially hold the risky asset.

3.3 Predatory trading vs no trading

In this section, I solve the model in the case where the hedgers have no initial position in the risky asset (i.e. $X_{-1}^0 = 0$), which implies that the strategic traders initially hold all the asset supply. With no risks to hedge, there should be no trading. However, the presence of the financial constraint may generate predatory trading, in particular if the hedgers have a low risk-bearing capacity. In the predatory trading equilibrium, the traders' financial strength (or at least the hedgers' perception of it) affects their price impact. In particular, I show that the prey's price impact decreases, while the predators' increases, which reduces the probability of survival of the prey.

3.3.1 Liquidity rationing during firesales

Since she is initially long the asset, the prey becomes distressed when the price of the asset at time 0 is low. In particular, by rearranging the terms in the marked-to-market wealth constraint, one can see that the prey is in distress when the price falls below $\bar{p}_0$, where $\bar{p}_0$, the prey's distress threshold, is given by

$$\bar{p}_0 \equiv \frac{\underline{V} - B^1_{-1}}{X^1_{-1}} \tag{3.2}$$

Note that the higher the distress threshold is, the more exposed the prey is to a forced liquidation. The threshold is increasing in $\underline{V}$, which measures the severity of the constraint, and decreasing in the amount of cash the prey initially holds, B^1_{-1}. I assume that parameters are such that $0 < \bar{p}_0 < D$, i.e.

Assumption 3 $0 < X^1_{-1} D < \underline{V} - B^1_{-1}$

This assumption implies that the prey remains solvent if the asset trades at its expected value. At time 1, all market participants are aware of whether the prey is in distress or not. The following lemma summarizes the equilibrium at time 1, depending on whether the prey is distressed or not.

Lemma 28 *When the prey is solvent, there is a unique symmetric equilibrium at time* 1*, given by:*

$$\forall i = 1, ..., n, \;\; x^i_1 = \frac{-\sum_{j=1}^n x^j_0}{n+1} \tag{3.3}$$

When the prey is distressed, the unique equilibrium at time 1 *is given by:*

$$\begin{aligned} x^1_1 &= -X^1_0 \\ \forall i = 2, ..., n, \;\; x^i_1 &= \frac{\left(X^1_{-1} + x^1_0\right) - \sum_{j=1}^n x^j_0}{n} \end{aligned} \tag{3.4}$$

Equation (3.3) shows that when the prey is solvent, strategic traders trade in the opposite direction to the time 0 aggregate order flow, $\sum_{j=1}^n x^j_0$. Note that because of imperfect competition, the total order nx_1 does not completely offset the time-0 aggregate order-flow: $|\sum_{j=1}^n x^j_1| \leq |-\sum_{j=1}^n x^j_0|$. If the prey is distressed, she no longer behaves strategically and liquidates her position by submitting an order $x^1_1 = -X^1_0$ at the prevailing market price. In other words, the prey behaves as a liquidity trader. Equation (3.4) shows that the predators take the opposite side of her trade and of the previous aggregate order flow, $\sum_{j=1}^n x^j_0$. They do so, however, only to a certain extent. Indeed, the predators gain market power and can thus limit further the quantity they trade. This can be seen by comparing equations (3.3) and (3.4): for a given supply, strategic traders' aggregate order at time 1 is a fraction $\frac{n}{n+1}$ of the supply in the no-distress case and $\frac{n-1}{n}$ of the supply in the distress case, with for all $n \geq 2$, $\frac{n}{n+1} > \frac{n-1}{n}$. I denote this effect the rationing of liquidity provision. It implies that, during a firesale, the predators do not completely offset the selling/ buying pressure of the distressed prey. Hence, in equilibrium, the hedgers will have to absorb some of the prey's asset firesale. Because there is still uncertainty at time 1 about the fundamental value of the asset, the hedgers are unwilling to hold large quantities. Therefore, at time 0, the hedgers take into account the possibility of the prey's distress when setting their demand.

3.3.2 Run and asymmetric (trader-specific) price impact

Since hedgers understand that predators will ration liquidity further during firesales, their demand changes depending on whether they expect a firesale or not at time 1. This affects the properties of the price schedule (i.e. the inverted demand schedule combined with market-clearing) faced by the predators and the prey at time 0.

Lemma 29 *Let p_0^{nd} and p_0^d denote the price schedule when hedgers expect no-distress and distress, respectively. The price schedule depends on the hedgers' beliefs about future distress as follows:*

$$p_0^{nd} = D + \beta \frac{n+2}{n+1} \sum_{i=1}^{n} x_0^i \tag{3.5}$$

$$p_0^d = D + \beta \sum_{j=1}^{n} x_0^j + \beta \frac{1}{n} \left(\sum_{j=1}^{n} x_0^j - X_0^1 \right) \tag{3.6}$$

Strategic traders' identities are public information, hence, using the dynamics of asset holdings, $X_0^1 = X_{-1}^1 + x_0^1$, equation (3.6) can be rewritten as:

$$p_0^d = D - \beta \frac{1}{n} X_{-1}^1 + \beta \frac{n+1}{n} \sum_{j=2}^{n} x_0^i + \beta x_0^1 \tag{3.7}$$

Comparing equations (3.5) and (3.7) shows that when the hedgers believe that the prey will be distressed, price impact becomes trader-specific. In particular, the prey's trades now move the price less than predators', while all traders have the same price impact when the hedgers expect no distress.

The intuition for this result is that the price impact coefficients reflect the differential marginal gains from trading across different types of strategic traders. If the hedgers think that the prey will have to liquidate and anticipate that they will have to hold some of the prey's position in equilibrium (equation (3.4)), they believe that they will gain marginally less from, say, selling to the prey than to predators at time 0. Indeed, selling to predators has some advantage in terms of hedging: predators will keep this asset until time 2, i.e. until the asset pays off and returns to perfect liquidity. This is not the case when selling to the prey: if the hedgers are right, the asset sold at time 0 to the prey will return to the market at time 1, while the predators will ration liquidity.

Further, equation (3.6) shows that hedgers are now ready to sell the risky asset at a lower price than when they believe that the prey will stay in the market. For instance, consider the following thought experiment: suppose that all strategic traders buy $\hat{x} \geq 0$ in both cases. The overal impact of the trades is $\beta \frac{n^2+n-1}{n}\hat{x}$ in the (anticipated) distress case, and $\beta \frac{n^2+2n}{n+1}\hat{x}$ in the no-distress case. Since $\frac{n^2+2n}{n+1} > \frac{n^2+n-1}{n}$, and given that the constant is lower in the bad scenario, the same purchase translates into a lower price in the distress case than in the no-distress case. The intuition is simply that, in anticipation of the firesale, hedgers are unwilling to hold a long position in the risky asset.

Another way to gain intuition in this effect is to assume that predators and the prey do not trade at time 0, $\forall i = 1, .., n,\ x_0^i = 0$. Then if the hedgers believe that the prey will be solvent, the price is $p_0^{nd} = D$. Since all the asset supply is initially the hands of the predators and the prey, who are risk-neutral, the price must coincide with the expected value of the asset. If the hedgers anticipate the prey to be distressed, the price is $p_0^d = D - \beta \frac{X_{-1}}{n}$. That is, the hedgers, anticipate that the prey will have to liquidate her position, $X_{-1}^1 > 0$, and that because of the predators' liquidity rationing, they will have to hold some of this additional supply. Hence the price adjusts downwards at time 0 in anticipation of this supply shock. In particular, the more concentrated the market is (i.e. the smaller n), and / or the more risk-averse the hedgers are, the larger the discount the price will exhibit at time 0. More concentration means that a tighter rationing of liquidity in the future, which will force the hedgers to absorb more of the supply. Of course, their valuation for holding the additional supply of risky asset decreases with their risk-aversion.I summarize these results as follows:

Lemma 30 *When the hedgers expect the prey to be in distress at $t = 1$,*

- *they are ready to sell at a lower price than when they expect no distress [run] :*

$$p_0^{nd}(\hat{x}) > p_0^d(\hat{x}), \ x_0^1 = x_0^j = \hat{x} \geq 0, \ j = 2, ..., n$$

- *the prey has less price impact than predators [asymmetric price impact] :*

$$\forall j = 2, ..., n, \ \frac{\partial p_0^{nd} / \partial x_0^j}{\partial p_0^{nd} / \partial x_0^1} < \frac{\partial p_0^d / \partial x_0^j}{\partial p_0^d / \partial x_0^1}$$

Note that the run effect is stronger when the prey has a large initial position in the risky asset, since all else equal, its liquidation will hurt the hedgers more in case of distress. The fact that price schedules depend on the hedgers' expectations about the prey's distress has important consequences for the equilibrium determination: the predators' ability to move the price (and the prey's ability to counter them) vary depending on the hedgers' beliefs about future distress.

3.3.3 Equilibria

Taking hedgers' beliefs as given, I determine conditions under which no trading and predatory trading arise in equilibrium.

No trading

Suppose that hedgers anticipate no trading, and thus no distress.[15] It is never in the interest of the prey, who is risk-neutral, to exit the market. The predators, however, may have an incentive to deviate from the no-trading situation to push the prey into distress. This is costly, because it requires to manipulate the price and tighten the prey's financial constraint. But a deviating predator may benefit from the increase in the asset supply resulting from the prey's firesale, and the decrease in competition among the remaining strategic traders.

Predators' trade-off. Since all predators have price impact, each of them recognizes he is pivotal for the outcome of the game. Deviating from the no-trading strategy can be profitable, however, only if this leads to the prey's distress, which require to push the price

[15] From equation (3.5), if all strategic traders do not trade (i.e. submit orders $x_0^i = 0$), the asset will trade at the fundamental value - and therefore the prey will not be distressed.

to $\bar{p}_0$. A predator thus faces a trade-off between manipulating the price and gaining from the prey's firesale. The predator's problem is:

$$\begin{array}{rcl} \forall i=2,...,n, \quad \max_{x_0^i} & & E_0\left(B_{-1}^i - x_0^i p_0 + CT_1^i\right) \\ s.t.\ p_0^{nd} & = & D + \beta\dfrac{n+2}{n+1}\sum_{i=1}^{n} x_0^i \\ \forall j \neq i,\ x_0^j & = & 0 \\ p_0 \leq \bar{p}_0 & \Rightarrow & x_1^1 = -X_{-1}^1 \end{array}$$

CT_1^i denotes the continuation payoff of the predator, which is contingent on the prey's distress. From equations (3.3) and (3.4), I get:

$$CT_1^i = \begin{cases} X_0^i D + \frac{\left(-\sum_{j\neq i} x_0^j - x_0^i\right)^2}{(n+1)^2} & \text{if } p_0 > \bar{p}_0 \\ X_0^i D + \frac{\left(X_{-1}^1 - \sum_{j=2,j\neq i}^{n} x_0^j - x_0^i\right)^2}{n^2} & \text{if } p_0 \leq \bar{p}_0 \end{cases}$$

Using the price schedule and the conjectured strategy for the other strategic traders, the predator's problem can be rewritten as follows:

$$\forall i=2,...,n,\ \max_{x_0^i} E_{-1}^i + \beta\left[\underbrace{-\frac{n+2}{n+1}\left(x_0^i\right)^2}_{t=0\ cost} + \underbrace{\frac{\left(-x_0^i\right)^2}{(n+1)^2} I_{p_0>\bar{p}_0}}_{profit\ if\ solvent} + \underbrace{\frac{\left(X_{-1}^1 - x_0^i\right)^2}{n^2} I_{p_0\leq\bar{p}_0}}_{profit\ if\ distressed}\right],$$

with $E_{-1}^i = B_{-1}^i + X_{-1}^i D$, the expected value of the predator's endowment, and I_c a dummy variable that equals one when the condition c is satisfied. This maximization problem illustrates the predator's trade-off. If the predator chooses $x_0^i = 0$, the price will be above the prey's distress threshold $\bar{p}_0$, and the predator's profit is thus 0[16]. If the predator chooses to push the price down to $\bar{p}_0$, he can benefit at time 1 from the decreased competition and the prey's firesale - the numerator of the profit in the distressed case is n^2 instead of $(n+1)^2$, and the numerator increases by $X_{-1}^1 > 0$, the prey's initial position in the asset. However, to trigger the prey's distress, he must short the asset, and this involves a quadratic cost at time 0, $\frac{n+2}{n+1}\left(x_0^i\right)^2$.

[16]Note that since $\forall n \geq 2$, $\frac{n+2}{n+1} > \frac{1}{(n+1)^2}$, all other strategies leading to $p_0 > \bar{p}_0$ are dominated by $x_0^i = 0$.

Ruling out "self-fulfilling" distress. By inspecting the maximization problem, one can also see that the prey's distress can be "self-fulfilling". Namely, ex-ante, it is optimal to take a short position in the asset if one expects a negative supply shock in the future (i.e. the prey's firesale).[17] Since the predators' trades affect prices, the anticipation by a predator that the prey will be distressed at time 1 may indeed lead to a price below $\bar{p}_0$ and trigger the prey's distress. The self-fulfilling distress can be defined more formally as follows:

Definition 2 *Suppose that strategic traders $-i$ choose $x_0^{-i} = 0$. The prey's distress is self-fulfilling if $p_0(\hat{x}_0^i) \leq \bar{p}_0$, where*

$$\hat{x}_0^i = \arg\max_{x_0^i} E_{-1}^i + \beta \left[-\frac{n+2}{n+1} \left(x_0^i\right)^2 + \frac{\left(X_{-1}^1 - x_0^i\right)^2}{n^2} \right]$$

To focus on predatory trading as a strategy aiming at eliminating a rival trader, I rule out self-fulfilling distress by imposing the following condition throughout:

Lemma 31 *There is no self-fulfilling distress if and only if $\beta < \bar{\beta}_{nd}$, where*

$$\bar{\beta}_{nd} = \frac{D - \bar{p}_0}{h_n X_{-1}^1}, \text{ with } h_n = \frac{n+2}{n^3 - 2n^2 - n + 1}$$

On this parameter interval, inducing distress requires a predator to trade

$$x_0^i = \frac{n+1}{n+2} \frac{\bar{p}_0 - D}{\beta} < 0 \tag{3.8}$$

To rule out self-fulfilling distress, one must focus on situations in which the hedgers' demand curve has a flat enough slope, i.e. if $\beta < \bar{\beta}_{nd}$. Intuitively, in this case, the price is not responsive enough to trades, such that a short position taken by a trader anticipating distress does not automatically lead to the prey's firesale. The predator's order, given by equation (3.8) is just enough to push the price to $\bar{p}_0$.

[17] More specifically, if the predator "anticipates" the prey's distress, he expects an increase in the asset supply and less competition in the future. Therefore the marginal cost of buying one more unit at time 1 decreases. Hence it is optimal for the predator to buy less at time 0 (i.e. here, short the asset) and exploit the negative price pressure exerted by the prey's firesale at time 1.

Proposition 18 *There exists a no-trading equilibrium in which the prey remains solvent if and only if* $\beta < \underline{\beta}_{nd}$*, with* $0 < \underline{\beta}_{nd} < \bar{\beta}_{nd}$*. Equilibrium prices are:*

$$p_0 = D \tag{3.9}$$

$$p_1 = D + \epsilon_1 \tag{3.10}$$

This result shows that the no-trading equilibrium holds in the presence of financial constraints only if the slope of the hedgers' demand curve is flat enough. Intuitively, if the slope is steep, a predator can easily move the price against the prey - see equation (3.5) - and this reduces the cost of predation - equation (3.8). Further, a steep slope means that hedgers are reluctant to bear risk (or equivalently that the asset is very risky), implying that the firesale exerts a strong negative pressure on the price at time 1.

Predatory trading

I now assume that the hedgers believe at time 0 that the prey will be distressed in the future. As shown above, the price schedule in this case is:

$$p_0^d = D + \beta \sum_{j=1}^{n} x_0^j + \beta \frac{1}{n} \left(\sum_{j=1}^{n} x_0^j - X_0^1 \right) \tag{3.11}$$

I conjecture that there exists an equilibrium with predatory trading in which the prey's and the predators' strategies are given by:

$$x_0^1 = \bar{X} - X_{-1}^1 \tag{3.12}$$

$$\forall i = 2, ..., n, \; x_0^i = \frac{1}{n-1} \left[X_{-1}^1 + \frac{n}{n+1} \left(\frac{R}{\beta} - \bar{X} \right) \right] \text{ with } R = \bar{p}_0 - D \tag{3.13}$$

These strategies are constructed in a way that, in equilibrium, (i) it is too costly for the prey to stay in the market (i.e. keep the price above $\bar{p}_0$); (ii) in particular, the prey' leverage constraint is binding, and (iii) the predators push the price to the distress threshold $\bar{p}_0$. Further, I continue to assume that the prey's distress is not self-fulfilling. Since the price schedule is different, the condition under which one can rule out self-fulfilling distress are also slightly different:

Lemma 32 *Denote* $a = \frac{\bar{X}}{X_{-1}^1}$ *the prey's leverage capacity (i.e.* $a \geq 1$*). Predatory trading is*

- *never self-fulfilling if* $a > \bar{a}_n$*, where* $\forall n \geq 2$*,* $\bar{a}_n > 1$*.*

- *not self-fulfilling if and only if* $\beta < \bar{\beta}_d$*, if* $a \leq \bar{a}_n$*, where*

$$\bar{\beta}_d = \frac{D - \bar{p}_0}{\rho_{0,n-1} X^1_{-1} - d_n \bar{X}}, \text{ with } \bar{a}_n = \tfrac{(n+1)^2}{n^2-n+2}$$

The lemma shows that the prey's distress can not stem from a self-fulfilling predatory trading strategy if her leverage capacity, a, is large enough. If the prey has enough dry powder, she does not "automatically" fall into distress, because her trades support the price sufficiently. If the prey has little dry powder, i.e. a low, the prey's distress is not self-fulfilling as long as the hedgers' demand curve is not too steep, i.e. if the price is not too responsive to trades.

The prey's problem. The predators' strategy implies that it is too costly for the prey to stay in the market: holding more that $\bar{X}$ in a bid to push the price above $\bar{p}_0$ and avoid distress is infinitely costly for the prey. As a result, the prey's problem is to maximize the proceeds of liquidating her holdings. Taking predators' strategy as given, the prey's problem is:

$$\max_{x_0^1} B^1_{-1} - x^1_0 \left[\bar{p}_0 - \beta\left(\bar{X} - X^1_{-1} - x^1_0\right)\right] + X^1_0 \left[D - \beta \frac{1}{n+1}\left(\bar{X} - \frac{R}{\beta}\right)\right]$$

The prey's liquidation problem involves a simple trade-off between liquidating at time 0 at $\bar{p}_0 - \beta\left(\bar{X} - X^1_{-1} - x^1_0\right)$, or at time 1 at (on average) $D - \beta\frac{1}{n+1}\left(\bar{X} - \frac{R}{\beta}\right)$. Of course, the prey's trade moves the price. If she starts selling from time 0, she will push the price below her distress threshold $\bar{p}_0$. At time 1, however, the average price depends on the prey's position only through predators' strategy, i.e. $\bar{X}$ in this case. This is because trades impact the price permanently.[18] Since the prey exactly offsets her time 0 position, $X^1_0 = X^1_{-1} + x^1_0$, at $t = 1$, her time 0 trade has no effect on the equilibrium price at time 1. It is optimal for the prey to be fully leveraged under the following condition:

Lemma 33 *(prey's optimal liquidation strategy) The prey's best response to predators' conjectured strategy is* $x^1_0 = \bar{X} - X^1_{-1}$ *if* $\beta < \beta_F$*, with* $\beta_F = \frac{D-\bar{p}_0}{\frac{n+2}{n}\bar{X} - \frac{n+1}{n}X^1_{-1}}$*.*

When $\beta \geq \beta_F$, the prey's trade is $\frac{n}{n+1}\frac{D-\bar{p}_0}{\beta} + \frac{n}{2(n+1)}\bar{X} - \frac{1}{2}X^1_{-1}$, i.e. the prey either buys a small amount (if $\frac{n}{n+1}\frac{D-\bar{p}_0}{\beta} + \frac{n}{2(n+1)}\bar{X} \geq \frac{1}{2}X^1_{-1}$) or starts liquidating her position. It is easy to see

[18] This can be seen from equation (3.28) in the appendix.

that this leads to a price below $\bar{p}_0$. This strategy, combined with the predators' conjectured strategy, cannot form an equilibrium: the predators would have an incentive to deviate and sell a bit less while keeping the price below $\bar{p}_0$, because their benefit would be unchanged.[19] Similarly, there cannot be an equilibrium in which the strategies are such that the prey holds less than $\bar{X}$, the predators more than equation (3.13), and the price is less than or equal to $\bar{p}_0$. In this case, the prey would have an incentive, and enough financial slack, to deviate and outbid predators in order to stay in the market. Hence, the only possible predatory equilibrium strategies are those given by equations (3.12)-(3.13). From Lemma 32 and 33, the relevant parameter space for these strategies is $\beta \in \left]0, \bar{\beta}_d \wedge \beta_F\right[$. I show in the appendix that in the special case where $X^0_{-1} = 0$, $\beta_F < \bar{\beta}_d$, so that the relevant interval is $\beta \in]0, \beta_F[$.

Equilibrium. In the conjectured equilibrium strategy, the prey is fully leveraged and has no interest in holding less than $\bar{X}$ (since $\beta < \beta_F$). Hence it is enough to analyze predators' trade-off to determine the equilibrium conditions. Using the same notations as before, and the results of the preliminary analysis, I get the trade-off faced by a predator:

$$\begin{aligned}
\forall i = 2, ..., n, \ \ \max_{x_0^i} E_{-1}^1 + \beta \left[x_0 \frac{n+1}{n} \left(X_{-1}^1 - \sum_{j=2, j\neq i}^{n} x_0^j - x_0^i \right) \right] & \\
+\beta \frac{\left(-\sum_{j=2,j\neq i}^n x_0^j - x_0^i \right)^2}{(n+1)^2} I_{p_0 > \bar{p}_0} & \\
+\beta \frac{\left(X_{-1}^1 - \sum_{j=2,j\neq i}^n x_0^j - x_0^i \right)^2}{n^2} I_{p_0 \leq \bar{p}_0} & \qquad (3.14) \\
s.t. \ \forall j \neq i, \ x_0^j = \frac{1}{n-1} \left[X_{-1}^1 + \frac{n}{n+1} \left(\frac{R}{\beta} - \bar{X} \right) \right] &
\end{aligned}$$

The first line of the maximand shows that, at time 0, the predator faces a quadratic cost, $\beta \frac{n+1}{n} \left(x_0^i\right)^2$. The second line represents the benefit from deviating from the predatory attack. Since other predators' trades exert negative pressure on the price, $\sum_{j=2,j\neq i}^n x_0^j$ is different from zero. If the predator joins the attack, he will, however, benefit from the firesale and the reduced competition in liquidity provision at time 1. Thus a predator "trades-off" the negative price pressure exerted by other predators at time 0, $\sum_{j=2,j\neq i}^n x_0^j$, against the future price pressure exerted by the prey in the following period. If the predator decides to buy while other predators attack the prey, he will rescue the prey, and therefore loses the benefit

[19]Hence, a more "continuous" constraint, in which the amount of selling would depend on the severity of the price drop, may lead to some early liquidation for the prey.

of the firesale.[20] The equilibrium is as follows:

Proposition 19 *There exists a predatory trading equilibrium characterized by equations* (3.12)-(3.13) *iff* $\beta \in \left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$, *with* $\underline{\beta}_d > 0$.

The intuition for this result is simple. If the hedgers' demand curve is steep enough, inducing the prey's distress is not too costly, hence predators engage in predatory trading against the prey. Further, in this case, the prey's firesale is likely to exert strongly negative price pressure, since the hedgers have a limited risk-bearing capacity.

The following comparative static obtains:

Corollary 23 *The equilibrium threshold* $\underline{\beta}_d$ *is lower when the prey is more exposed to the risk of forced liquidation (high* $\underline{V}$*) or has less cash (low* B^1_{-1}*).*

If the prey is more constrained, the cost of the predatory trading strategy is lower, hence the condition on β is less strict.

Since the interval $\left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$ is potentially empty, there can be a concern about the existence of this equilibrium. More generally, given that equilibria depend on the hedgers' beliefs, both types of equilibria may coexist, reducing the predictive ability of the model. To illustrate the results and address these concerns, I study a numerical example.

Coexistence of no-trading and predatory trading equilibria

From Proposition 18 and 19, I get:

Proposition 20 *When* $X^0_{-1} = 0$,

- *The no-trading equilibrium is the only equilibrium for* $\beta \in \left]0, \min\left(\beta_F, \underline{\beta}_d, \underline{\beta}_{nd}\right)\right]$.
- *It coexists with the predatory trading equilibrium on* $\left]\min\left(\beta_F, \underline{\beta}_d, \underline{\beta}_{nd}\right), \min\left(\beta_F, \underline{\beta}_d, \underline{\beta}_{nd}\right)\right[$.
- *Predatory trading is the only equilibrium on* $\left[\min\left(\beta_F, \underline{\beta}_d, \underline{\beta}_{nd}\right), \beta_F\right[$.

[20] I show this point formally in the proof of Proposition 19. Observe also, that since each predator is pivotal, there is no possibility of free-riding on the attack of other predators, especially because the conjectured predatory trading strategies are such that the first-period price reaches exactly $\bar{p}_0$. Predatory trading requires full coordination of the predators in the model.

To understand further in which circumstances equilibria may coexist and when predatory trading is the only equilibrium, I consider:

$$\bar{\beta}_d - \beta_F = \frac{D - \bar{p}_0}{\bar{X}} f(n, a)$$

where the function f is given by equation (3.86) in the appendix. The predatory trading equilibrium is the only equilibrium on a non-empty interval if $f(n, a) > 0$. Since f is monotonically increasing in a, the function implicitly defines a cutoff $a^*(n)$ such that:

$$f(n, a^*(n)) = 0$$

Hence the predatory trading equilibrium exists if $a \leq a^*(n)$. Panel (a) of Figure 1 plots the cutoff a^* (red dotted line), and shows that the predatory trading equilibrium exists when both the number of predators and the prey's leverage capacity are small. Intuitively, if there are many predators, fierce competition during the prey's firesale will quickly erode the benefit of predatory trading - and more quickly than it decreases the cost per predator. Hence coordination on the predatory trading equilibria is more difficult to obtain. When the prey has a high leverage capacity, the cost of inducing distress is high, hence predatory trading is less likely.

The panel (a) of Figure 1 also features a second cutoff $\hat{a}^*(n)$ defined as

$$g(n, \hat{a}^*(n)) = 0, \text{where } \underline{\beta}_{nd} - \underline{\beta}_d = \tfrac{D - \bar{p}_0}{\bar{X}} g(n, a)$$

Since g is monotonically decreasing in a, the no-trading and predatory trading equilibria coexist (that is, $\underline{\beta}_{nd} > \bar{\beta}_d$) when $a \geq \hat{a}^*(n)$, i.e. in the region above the full dark blue line. Hence, it is only when the prey is very constrained in terms of leverage, and the group of predators very concentrated that predatory trading is the only equilibrium. The model therefore delivers a clear prediction in this case, in spite of the self-fulfilling nature of the equilibria.

In the region defined by $a \leq \hat{a}^*(n)$, the model produces the "net" probability of predatory trading (i.e. excluding the region where both equilibria coexist). The following comparative obtains:

Corollary 24 *Suppose that $a \leq \hat{a}^*(n)$ and denote $q(n, a) = 1 - \frac{\underline{\beta}_{nd}}{\beta_F}$. q decreases linearly in a, the prey's leverage capacity.*

It is costly to engage in predatory trading against the prey if she has a lot of dry powder. Hence the probability of predatory trading q decreases in a. To understand the effect of the number of predators, I plot q in Panel (b) of Figure 1. The graph shows that the probability decreases with n, the number of predators, and decreases faster when n is small, a non-linear effect. This is because the benefit of predatory trading decreases as $\frac{1}{n^2}$.

3.3.4 Changing liquidity and the cost of predatory trading

The cost of predatory trading is to push the asset price to the prey's liquidation threshold $\bar{p}_0$, while there are no other motives to trade, if only to short the asset, which has a positive expected payoff. Hence we can define the cost of predation as the distance between the predators' aggregate trade $Q = \sum_{i=2}^{n} x_0^i$ and zero. To understand how the change in price schedule affects the cost of predatory trading, it is interesting to compare the cost that prevails when the hedgers (correctly) anticipate distress, and the cost that predators would have to bear if the hedgers mistakenly believed that the prey will not liquidate. To make this comparison, I fix the prey's strategy and assume that she is fully leveraged, as it is a feature of any predatory equilibrium.[21]

Lemma 34 *Suppose $X_0^1 = \bar{X}$, and let Q^d denote the cost of trading when hedgers anticipate distress and Q^{nd} when they do not. For all parameter values, predators must short less when the hedgers anticipate distress, $Q^d \geq Q^{nd}$, with $Q^{nd} < 0$.*

This result shows that it becomes cheaper for predators to push the prey into distress when hedgers anticipate that the prey will eventually be forced to liquidate her positions. Each unit bought by the prey pushes up the price less than an opposite order by a predator. The asymmetric price impact reflects the hedgers' perceptions of the different traders' financial strength. It depends on the prey's financial condition being known by other traders. Although this effect has not been tested yet, there is some incidental evidence in Cai (2009), who finds that LTCM' price impact was on average lower in the months before receiving margin calls in September 1998 than during the crisis itself.

Another interesting implication of the change in liquidity is that the size of the prey's initial position has an ambiguous effect on predators' time 0 trade, i.e. on the cost of

[21] The condition for this strategy to be optimal given that predators engage in predation would be different. In particular the interval on which this strategy is optimal would decrease. Denoting $\tilde{\beta}_F$ the threshold under the incorrect beliefs, I show in the proof of Lemma 34 that $\tilde{\beta}_F < \beta_F$. I also show that $\bar{\beta}^d > \tilde{\bar{\beta}}^d$, i.e. there is a larger interval under which distress is not self-fulfilling. Because equilibrium conditions change, my result is about the cost of predatory trading, and not the probability of predatory trading.

predatory trading:

Corollary 25 *Denote* $\bar{X} = aX^1_{-1}$*, with* $a \geq 1$*. Then from equation* (3.13)*, the effect of a change in the prey's initial size on predators' aggregate order* Q^d *is:*

$$\frac{\partial Q^d}{\partial X^1_{-1}} = \underbrace{1}_{run\ effect\ >0} + \underbrace{\frac{n}{n+1}}_{\text{diff. price impact "multiplier"} < 1} \underbrace{\left[-a + \frac{1}{\beta}\frac{\partial R}{\partial X^1_{-1}}\right]}_{collateral\ effect\ <0}$$

where $R = \bar{p}_0 - D$*.*

Corollary 25 describes the impact of a small change in the prey's position on the amount predators must trade to push her into distress. The corollary shows that holding a large position in the risky asset may either decrease or increase the cost of predatory trading. Holding a large position strengthens the run effect, because the hedgers anticipate a larger firesale in the following period, and the price has to adjust further downwards ex-ante. This makes it easier for predators to trigger financial distress. At the same time, a larger position means that the prey is richer and that her distress threshold is lower - see equation (3.2), which makes predatory trading more costly. Interestingly, the run effect is 1, while the collateral effect is multiplied by $\frac{n}{n+1} < 1$. This is a consequence of the decrease in price impact the prey experiences in this regime. Hence the decrease in price impact reduces the benefit of holding a large position.

3.3.5 Implications for liquidity measures

Our analysis has interesting implications for liquidity measures and liquidity proxies. First, from the example above, it is clear that turnover cannot be used as a proxy for liquidity. In the absence of the prey's financial constraints, it is optimal not to trade since the more risk-tolerant investors (the prey and the predators) initially hold the entire asset supply. In that sense, the mere presence of the financial constraint generates "excessive" trading volume. There is a large literature on trading volume and excess trading volume. Heterogeneous information (e.g. Karpoff, 1986) or career concerns (Dasgupta and Prat, 2006) can increase trading volume, among other mechanisms. Here it is the financial constraint and the possibility of default that leads to an increase in trading volume. Interestingly, it is precisely when risk-aversion is high, that is when hedgers are the most unwilling to hold the asset that they end up with some in their hands.

As shown in Lemma 29, predators' price impact increases and the prey's decreases in the predatory trading equilibrium relative to the no-trading equilibrium. Further, the aggregate price impact decreases in the sense that if all traders submit the same order, it pushes up the price less when the hedgers expect a firesale than when they do not. In spite of this, one cannot conclude that the market is more liquid. In our context, trading volume and market depth can thus be misleading indicators of market liquidity. The only consistent measure is the deviation of the transaction price from the risk-neutral value of the asset $\mathbb{E}(D_2)$.

3.4 Predatory trading vs liquidity provision

I now move on to the case where the hedgers start with a long position in the risky asset, i.e. $X_{-1}^0 > 0$. Strategic traders hold the remainder of the supply, and the prey has a long initial position $X_{-1}^1 > 0$. The main effect of strictly positive endowments for the hedgers is to introduce a trading motive between strategic traders and hedgers based on risk-sharing. Thus the no-trading equilibrium is replaced by an equilibrium with imperfect liquidity provision but no distress. In addition, I show that (i) the run effect increases with the hedgers' endowment, decreasing the cost of pushing the prey into distress for predators. At the same time, an increase in the hedgers' endowment increases the benefit of providing liquidity to the hedgers. Because of these conflicting effects, an increase in the hedgers' endowment has an ambiguous impact on the probability of predatory trading. (ii) Run and predatory trading can be so mutually-reinforcing that predators may not have to sell in order to induce the prey's distress: it may be enough for them to hoard liquidity and let the hedgers' run decrease the price.

3.4.1 Equilibria

Liquidity provision

I conjecture that there exists an equilibrium in which all strategic traders buy the asset from the hedgers, thereby providing them with liquidity (that is allowing them to swap the risky, illiquid asset for the safe, liquid asset).

Proposition 21 *Suppose* $0 < \beta < \bar{\beta}_{nd}$. *On this interval, there exists a unique (symmetric) no-distress equilibrium given by*

$$\forall i = 1, ..., n,\ x_0^i = c_{0,n} X_{-1}^0 \tag{3.15}$$

$$x_1^i = c_{1,n} X_{-1}^0 \tag{3.16}$$

iff $\beta < \underline{\beta}_{nd} \wedge \bar{\beta}_{nd}$ *and* $c_{0,n}X_{-1}^0 \leq \bar{X} - X_{-1}^1$.

Equilibrium prices are:

$$p_0 = D - \beta\rho_{0,n}X_{-1}^0 > \bar{p}_0 \tag{3.17}$$

$$p_1 = D + \epsilon_1 - \beta\rho_{1,n}X_{-1}^0 \tag{3.18}$$

with, $\forall n \geq 1$, $c_{0,n} > c_{1,n}$, $\rho_{0,n} > \rho_{1,n}$, $n\left(c_{0,n} + c_{1,n}\right) < 1$.

The coefficients $c_{0,n}$, $c_{1,n}$, $\rho_{0,n}$ *and* $\rho_{1,n}$ *are given by equations* (3.42)-(1.51), *and the thresholds* $\underline{\beta}_{nd}$ *and* $\bar{\beta}_{nd}$ *by equations* (3.56) *and* (3.47) *in the appendix.*

The equilibrium conditions on β given in Proposition 21 are similar to those of Proposition 18, except that the thresholds $\underline{\beta}_{nd}$ and $\bar{\beta}_{nd}$ are now evaluated for $X_{-1}^0 > 0$.[22] The condition $c_{0,n}X_{-1}^0 \leq \bar{X} - X_{-1}^1$ ensures that the equilibrium strategy is feasible for the prey, in spite of her leverage constraint.

The equilibrium has two main features. First, strategic traders ration liquidity in the market. In total, they buy an amount $n\left(c_{0,n} + c_{1,n}\right)X_{-1}^0$, which is lower than the hedgers endowment ($n\left(c_{0,n} + c_{1,n}\right) < 1$, $\forall n \geq 2$). This follows from the oligopolistic nature of the liquidity supply side of the market. Nevertheless, the liquidity rationing is not such that the prey is distressed: the equilibrium price is above $\bar{p}_0$. Second, strategic traders buy the asset slowly, i.e. they spread their trades over both periods. Since trades move prices in a permanent manner, a strategic trader lowers his average purchase price by splitting up trades. However, even with limited competition, there is some pressure to buy ahead of other strategic traders while the price is low. As a consequence, the first period trade is higher than the second period trade: $c_{0,n} > c_{1,n}$ for all $n \geq 2$, and even more so as n increase, as shown by Figure 1.4.

While strategic traders do not engage in predatory trading for $\beta \leq \bar{\beta}_{nd}$, the market is not perfectly liquid on this parameter interval. The risky asset trades at a discount because of imperfect competition and the ensuing rationing of liquidity provision. This discount decreases over time because of the gradual purchases of the strategic traders, and varies as follows:

Corollary 26 *The illiquidity discount in period* t *is* $\Gamma_t = E_t\left(D_2\right) - p_t = \beta\rho_{t,n}X_{-1}^0 > 0$ $(t = 0, 1)$.

[22] I should have written $\bar{\beta}_{ND}^0$ in the zero-endowment case of the previous section. I use the same notations in this section, by a slight abuse of notation.

- *At each period, the discount is larger for a higher risk-aversion coefficient α, a higher riskiness of the asset σ^2, a larger hedging need X_{-1}^0, a smaller number n of strategic traders.*

- *The discount decreases faster when n is small.*

The effect of the number of strategic traders on the speed at which the discount decreases is illustrated by Figure 1.4. The slow adjustment of the price is typical of a "gradual arbitrage", as in Oehmke (2010), except that the illiquidity of the market is endogenous in the present setting. The main driver of this phenomenon is imperfect competition. The perfect competition case, which obtains in the limit case $n \to \infty$, offers an interesting benchmark:

Corollary 27 *When $n \to \infty$, the strategic traders' total first period purchase converges to X_{-1}^0, the hedgers endowment. Their second period total purchases converges to 0. As a consequence, the illiquidity discount goes to 0, strategic traders' trading profits go to 0 and the hedgers certainty equivalent converges to the expected value of his endowment.*

Hence when perfect competition among strategic traders obtains, the market becomes perfectly liquid.

Predatory trading

Price schedule. Suppose that the hedgers believe that the prey will be in distress at time 1, then the price schedule is:

$$p_0^d = D - \beta \frac{n+1}{n} X_{-1}^0 - \beta \frac{1}{n} X_{-1}^1 + \beta \frac{n+1}{n} \sum_{i=2}^{n} x_0^i + \beta x_0^1 \tag{3.19}$$

Equation (3.19) shows that the constant of the price schedule decreases when hedgers have positive endowment. Hence I obtain the following comparative static:

Corollary 28 *The run effect is stronger when hedgers have a positive endowment in the risky asset.*

The intuition is that the hedgers now have a lower marginal valuation for the asset and are thus more eager to offload their risk ahead of the prey's firesale.

Equilibrium. The conjectured predatory trading equilibrium strategy is:

$$\begin{aligned} x_0^1 &= \bar{X} - X_{-1}^1 && (3.20)\\ \forall i = 2, ..., n,\ x_0^i &= \frac{1}{n-1}\left[X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)\right] && (3.21) \end{aligned}$$

The only difference with the no-endowment case is for predators' trade. It needs not be as low, as can see by comparing equations (3.21) and (3.13). This is because the hedgers' run is stronger, pushing the price down further.

I now study the trade-off faced by predators. The predator's maximization problem is the same as 3.14 except that the hedgers' endowment affect the cost, as well as the relative benefit of predatory trading.

$$\begin{aligned} \forall i = 2, ..., n,\ \max_{x_0^i} E_{-1}^i + \beta &\left[x_0^i \frac{n+1}{n}\left(X_{-1}^0 + X_{-1}^1 - \sum_{j=2, j\neq i}^{n} x_0^j - x_0^i\right)\right] \\ &+\beta \frac{\left(X_{-1}^0 - \sum_{j=2,j\neq i}^n x_0^j - x_0^i\right)^2}{(n+1)^2} I_{p_0 > \bar{p}_0} \\ &+\beta \frac{\left(X_{-1}^0 + X_{-1}^1 - \sum_{j=2,j\neq i}^n x_0^j - x_0^i\right)^2}{n^2} I_{p_0 \leq \bar{p}_0} \qquad (3.22)\\ s.t.\ \forall j \neq i,\ x_0^j &= \frac{1}{n-1}\left[X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)\right] \end{aligned}$$

By comparing the maximization problems 3.22 and 3.14, one can see that the cost of predatory trading will be lower (first line). This is caused by the fact that the hedgers run more strongly when they have positive endowments. The benefit from resucing the prey (second line) is higher, and the benefit from predatory trading too. I show in the appendix that the trade-off faced by predators has a simple quadratic form. A predator joins the predatory trading attack if and only if

$$a_d \beta^2 + b_d \beta + c_d \geq 0 \qquad (3.23)$$

where the coefficients are given by equations (3.71)-(3.73) in the appendix. I obtain the following result.

Proposition 22 *Denote* $\theta = \frac{X_{-1}^0}{X_{-1}^1}$ *and* $a = \frac{\bar{X}}{X_{-1}^1}$*, the prey's leverage capacity. There exists an equilibrium with distress given by equations* (3.21)-(3.20) *iff* $\beta \in I_P$*, where* I_P *is as follows:*

- *If* $a \geq \max\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$*, then* $I_P = \left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$
- *If* $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$*, then* $I_P = \left[\underline{\beta}_d, \bar{\beta}_D\right[$
- *If* $\min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right) < a < \max\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$*, then*
  - *If* $\theta > \theta^*$*, then* $I_P = \left[\underline{\beta}_d \wedge \beta_F, \underline{\beta}_{d,2} \wedge \beta_F\right[$,
  - *If* $\theta \leq \theta^*$*, then* $I_P = \left[\underline{\beta}_d \wedge \bar{\beta}_d, \bar{\beta}_d\right[$.

with $\underline{\beta}_d$ *and* $\underline{\beta}_{d,2}$ *the positive roots of equation* (3.23).

The equilibrium price is:

$$p_0 = \bar{p}_0 \tag{3.24}$$
$$p_1 = D + \epsilon_1 - \beta\frac{\bar{X}}{n+1} - \frac{|R|}{n+1} \tag{3.25}$$

Proposition 22 shows that the equilibrium is driven by three factors: the prey's leverage capacity, a, the ratio $\theta = \frac{X^0_{-1}}{X^1_{-1}}$, and the number of predators (since the coefficients m_1, m_2, κ_2 are functions of n)[23] . Intuitively, θ measures the selling pressure caused by the hedgers willingness to share risk relative to that caused by the prey's firesale. The result suggests that predatory trading can occur in equilibrium whether θ is large relative to a or not, i.e. θ plays an ambiguous role. The following comparative statics confirm this observation.

3.4.2 Implications

Hedgers' endowment and probability of predatory trading

Using the results of Proposition 22, I can calculate the probability of predation. The "gross" probability is unadjusted for the fact that the liquidity provision equilibrium can coexist with the predatory trading equilibrium. The "net" probability does take into account the possible coexistence of equilibria. I obtain the following comparative statics with respect to θ.

[23]Note that $\forall n \geq 2$, $\frac{1}{\kappa_2} \leq 1$, and $\max\left(m_2, \frac{1}{\kappa_2}\right) = m_2$. Hence, given that $a \geq 1$, in the special case $X^0_{-1} = 0$, i.e. $\theta = 0$, the equilibrium condition is $\beta \in \left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$, as in Proposition 18.

Corollary 29 *The gross and net probabilities of predation vary as follows.*

- *If* $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$*, denote* $\kappa = \frac{\theta+1}{a}$ *and define the gross probability of predatory trading* $\hat{q}$ *as*

$$\hat{q}(\kappa, n) = \frac{\bar{\beta}_d - \underline{\beta}_d}{\bar{\beta}_d}$$

  $\hat{q}$ *decreases in* κ*, i.e.* $\hat{q}$ *decreases with* θ *on this interval.*

- *If* θ *is small, such that* $a \geq \max\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$*, the equilibrium thresholds are ordered as follows:* $\underline{\beta}_{nd} < \beta_F < \bar{\beta}_d \wedge \bar{\beta}_d$*. Hence the net probability of predation* q *is given by*

$$q(\theta, n, a) = 1 - \frac{\underline{\beta}_{nd}}{\beta_F}$$

  Then for θ *small,* q *increases with* θ*.*

The effect of θ on the probability of predation is non-monotonic[24]. If the hedgers' initial positions relative to the prey's are sufficiently large, then increasing θ decreases the likelihood of predatory trading. However, if θ is initial small, then increasing it may *increase* the probability of predatory trading. There are two conflicting effects at work here. First, the hedgers' initial position determines the equilibrium illiquidity discount. A high discount makes it easier to push the prey into distress. Second, a large endowment raises the opportunity cost of pushing the prey into distress. This is because predatory trading aims at decreasing the price at which strategic traders can buy the asset. However, if the price is already low because the hedgers have large positions to offload, there is a low incentive to engage in predatory trading.

Runs, predatory trading, and short-selling

In Corollary 28, I showed that when X_1^0 is large, hedgers run more. Interestingly, the run can be so strong that the predators may not have to short the asset to trigger the prey's distress.

[24] I checked numerically the "net" probability of predation, i.e. taking into account equilibrium overlap, has typically the same properties as the "gross" probability.

Corollary 30 *(liquidity hoarding) The predators' aggregate order at* $t = 0$ *is* $Q^d = \sum_{i=2}^{n} x_0^i = X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)$.

- *If* $\theta = 0$, *then* $\forall \beta < \beta_F$, $Q^d < 0$, *i.e. predators must short the asset to push the prey into distress.*

- *If* $\theta > 0$, *then if the prey has a small enough leverage capacity (a small enough), there exists* $\beta^h > 0$ *such that for* $\beta \geq \beta^h$, $Q^d \geq 0$, *i.e., it is enough for predators to hoard liquidity to push the prey into distress.*

The second part of the corollary does not state whether β^h satisfies the conditions required on β for predatory trading to occur in equilibrium. However, it is easy to calculate the various thresholds numerically. For instance, when $\theta = 0.3$ and $a = 1.05$, parameters are such that predatory trading is the only equilibrium for $\beta \in \left[\underline{\beta}_{nd}, \bar{\beta}_d \wedge \beta_F\right[$ as long as the number of predators is between 2 and 8. Further, for these parameters, $\beta^h < \bar{\beta}_d \wedge \beta_F$, implying that it is sufficient for predators to restrict liquidity provision, and that they do not have to short the asset. Therefore, when the hedgers are sufficiently risk-averse, they behave as predators' (involuntary) accomplices. More precisely, the possibility of predatory trading induces the hedgers to run, which in turn facilitates predatory trading. Therefore the model provides a natural link between predatory trading and financial market runs. Contrary to models of financial market runs (e.g. Bernardo and Welch (2004), the liquidity shock triggering the run, i.e. the prey's firesale, is endogenous in the model.

An interesting empirical implication of the model is that it may be misleading too look at the trade direction (i.e. buy or sell) in order to identify predators. This implication is in contrast to Brunnermeier and Pedersen's model, in which predators always sell during the predatory phase (time 0 here). Another interesting implication is that short-selling bans may not always be effective in curbing predatory trading. In particular, when the hedgers are sufficiently risk-averse ($\beta \geq \beta_h$), what pushes the prey into distress is that they quickly offload their endowment and predators restrict the quantity they buy.

Price effects

Predatory trading involves a price manipulation in the first period in order to push the prey into distress. Therefore the illiquidity discount is larger than in the no-distress case at time 0 when predators engage in predatory trading. The price effects of predatory trading at time 1 are as follows:

Corollary 31 *In the equilibrium with distress,*

- *The illiquidity discount at $t = 1$ is larger when the prey has a larger capacity, $\frac{\partial \Gamma_1}{\partial \bar{X}} < 0$, and when the prey has more cash or a less severe constraint $\underline{V}$, $\frac{\partial \Gamma_1}{\partial |R|} < 0$.*
- *The price rebounds on average at $t = 1$ and the average rebound is stronger when the prey is less exposed to forced liquidations (e.g. has more cash, or a looser constraint $\underline{V}$), $\frac{E_0(p_1 - p_0)}{\partial |R|} > 0$, and stronger if the prey has a smaller capacity, $\frac{E_0(p_1 - p_0)}{\partial \bar{X}} < 0$.*

If the prey has a large capacity constraint, there is a large firesale at time 1, hence a large discount and a low price rebound, on average. When the prey is not very exposed to forced liquidation, inducing distress requires to push the time 0 price to a very low level. Since price impact is permanent, the time 1 price is also lower in this case. Nevertheless, the average rebound is larger. This is because decreasing the price involves to take low or short positions at time 0, therefore predators must buy more aggressively at time 1, leading to a higher rebound on average.

3.5 Conclusion

I study predatory trading in a model where smart competitive investors (hedgers) understand that capital-rich strategic traders may prey upon a financially constrained competitor. I show that the hedgers' reactions to the possibility of predatory trading can make predation cheaper. This reaction manifests itself through a change in market liquidity, which allows predators to move prices more easily than the prey and increases downward pressure on the price. An important determinant of predatory trading is the hedgers' risk-bearing capacity, because it determines their ability to take the other side of predatory trades and eventually to absorb firesales without causing large market disruptions, and this determines the profitability of predatory trading.

An interesting research avenue is to study the systemic risk created by predatory trading between traders with different levels of capital. Given the mechanisms at work with one prey, one can imagine that the mere prospect of a cascade of failures could trigger a liquidity dry-up which in turn would facilitate predatory trading on multiple preys. At the same time, the possibility of becoming a prey as a result of future market disruptions may limit the willingness of traders with intermediate capital to engage in predation. Hence introducing spillovers from one prey to the other in the analysis should lead to interesting coordination problems. This is left for future research.

3.6 Proofs

The following proofs are given in the case where the hedgers' endowment is $X^0_{-1} \geq 0$. Section 3.6.4 of this appendix contains additional derivations related to the special case where the hedgers have no endowment ($X^0_{-1} = 0$). In my derivations of the equilibrium, I use Lemma 2 in Fardeau (2011). (chapter 1 of this dissertation)

3.6.1 Time-1 subgame equilibrium and price schedules

Lemma 28

Proof. I solve the model backwards. Given the CARA-normal framework of the model, it is convenient to work with certainty equivalents to solve the hedgers' problem.

Date 1. From the viewpoint of date 1, the first innovation ϵ_1 is known, hence $E_1\left(\tilde{D}_2\right) = D + \epsilon_1$ and the hedgers' maximisation problem is

$$CE_1 = \max_{x_1^C} B_0^C - x_1^C p_1 + X_1^C (D + \epsilon_1) - \frac{1}{2}\beta \left(X_1^C\right)^2, \text{with } \beta = \alpha\sigma^2 \tag{3.26}$$

The hedgers' demand at $t = 1$ is thus $X_1^C = \frac{D+\epsilon_1 - p_1}{\beta}$. Inverting the demand curve and imposing market-clearing,

$$\forall t = 0, 1,\ S = X_t^C + \sum_{j=1}^{n} X_t^j \tag{3.27}$$

yields the price schedule faced by strategic traders:

$$p_1 = D + \epsilon_1 - \beta\left(S - \sum_{j=1}^{n} X_1^j\right)$$

Using $X_t^j = X_{t-1}^j + x_t^j$ gives:

$$p_1 = D + \epsilon_1 - \beta\left(S - \sum_{j=1}^{n} X_0^j\right) + \beta\sum_{j=1}^{n} x_1^j \tag{3.28}$$

There are two states of the world at $t = 1$, with and without distress. If there is distress, the prey must liquidate her entire portfolio, i.e. $X_1^1 = 0$, which implies $x_1^1 = -X_0^1$. Otherwise, the prey is free to choose her position.

- First case: no distress (nd). A strategic trader's value function is defined as

$$\begin{aligned}\forall i = 1,...,n,\ J_1^{i,nd} &= \max_{x_1^i} E_1\left[B_0^i - x_1^i p_1 + X_1^i \tilde{D}_2\right] \\ s.t.\ p_1 &= D + \epsilon_1 - \beta\left(S - \sum_{j=1}^n X_0^j\right) + \beta\sum_{j=1}^n x_1^j\end{aligned}$$

Plugging the constraint in the maximand gives:

$$\forall i = 1,...n,\ J_1^{i,nd} = \max_{x_1^i} B_0^i + X_0^i\left(D + \epsilon_1\right) + x_1^i\left[S - \sum_{j=1}^n X_0^j - \sum_{j\neq i}^n x_1^j - x_1^i\right],$$

where, $\forall j = 1,...,n,\ X_0^j$ has been determined in the previous period. Taking the first-order condition, solving for its zero and rearranging terms, we get:

$$\forall i = 1,...,n,\ x_1^i + \sum_{j=1}^n x_1^j = S - \sum_{j=1}^n X_0^j \tag{3.29}$$

Collecting the n equations and using matrix notation gives

$$\left(I + \mathbf{1}\right).\mathbf{x_1} = \left(S - \sum_{j=1}^n X_0^j\right).1,$$

where $\mathbf{1}$ is a (n,n) matrix of 1's, $\mathbf{x_1} = (x_1^1,...,x_1^n)$ and 1 is a vector of 1's. The lines and columns of the matrix $A = I + \mathbf{1}$ are linearly independent. Thus the matrix is invertible with inverse A^{-1} and multiplying on both sides from the left by A^{-1} gives the unique equilibrium in the subgame:

$$\forall i = 1,...,n,\ x_1^i = \frac{S - \sum_{j=1}^n X_0^j}{n+1} \tag{3.30}$$

Plugging this quantity into the strategic trader's value function $J_1^{i,nd}$ gives

$$J_1^{i,nd} = B_0^i + X_0^i\left(D + \epsilon_1\right) + \beta\frac{\left(S - \sum_{j=1}^n X_0^j\right)^2}{\left(n+1\right)^2} \tag{3.31}$$

The strategic trader's value function is the expected payoff on his date 0 positions in the riskfree and risky assets, plus the continuation payoff $\beta\frac{\left(S-\sum_{j=1}^n X_0^j\right)^2}{(n+1)^2}$. Using equations (3.26) and (3.30), the hedgers' certainty equivalent is:

$$CE_1^{nd} = B_0^C + X_0^C \left(D + \epsilon_1 - \beta \frac{S - \sum_{j=1}^n X_0^j}{n+1} \right) + \beta \frac{\left(S - \sum_{j=1}^n X_0^j \right)^2}{2\left(n+1\right)^2} \tag{3.32}$$

- Second case: prey is in distress (d).

In this case, $X_1^1 = 0$, hence $x_1^1 = -X_0^1$. Given that $X_1^1 = 0$, the problem of a predator is

$$\forall i = 2, ..., n, \; J_1^{i,d} = \max_{x_1^i} E_1 \left(B_0^i - x_1^i p_1 + X_1^i \tilde{D}_2 \right)$$
$$s.t. \; p_1 = D + \epsilon_1 - \beta \left(S - \sum_{i=2}^n X_1^i \right)$$

Repeating the same steps as above, I get the unique equilibrium in the subgame:

$$x_1^1 = -X_0^1 \tag{3.33}$$
$$\forall i = 2, ..., n, \; x_1^i = \frac{S - \sum_{j=2}^n X_0^j}{n} \tag{3.34}$$

Strategic trader's value function and the hedgers' certainty equivalent are given by:

$$\forall i = 2, ..., n, \; J_1^{i,d} = B_0^i + X_0^i \left(D + \epsilon_1\right) + \beta \frac{\left(S - \sum_{j=2}^n X_0^j \right)^2}{n^2} \tag{3.35}$$
$$CE_1^d = B_0^C + X_0^C \left(D + \epsilon_1 - \beta \frac{S - \sum_{j=2}^n X_0^j}{n} \right) + \beta \frac{\left(S - \sum_{j=2}^n X_0^j \right)^2}{2n^2} \tag{3.36}$$

■

Lemma 30

Proof. **Date 0.** I now solve for the hedgers' demand at date 0, depending on the hedgers' beliefs about the state at $t = 1$.

- First case: The hedgers believe that the prey will be solvent at $t = 0$. The hedgers' maximisation problem at $t = 0$, using $B_0^C = B_{-1}^C - x_0^C p_0$ and equation (3.32), is

$$\max_{x_0^C} E_0 - \exp -\alpha \left(-x_0^C p_0 + X_0^C \left(D + \tilde{\epsilon_1} - \beta \frac{S - \sum_{j=1}^n X_0^j}{n+1} \right) + \beta \frac{\left(S - \sum_{j=1}^n X_0^j \right)^2}{2\left(n+1\right)^2} \right),$$

where $\tilde{\epsilon_1}$ is random. Using the projection theorem for normals, the problem simplifies to maximising the hedgers' date-0 certainty equivalent:

$$CE_0 = \max_{x_0^C} -x_0^C p_0 + X_0^C \left(D - \beta \frac{S - \sum_{j=1}^n X_0^j}{n+1} \right) + \beta \frac{\left(S - \sum_{j=1}^n X_0^j \right)^2}{2\,(n+1)^2} - \frac{1}{2}\frac{\beta}{2}\left(X_0^C\right)^2 \quad (3.37)$$

From the first-order condition I get the hedgers' demand function at $t = 0$:

$$X_0^C = \frac{D - \beta \frac{S - \sum_{j=1}^n X_0^j}{n+1} - p_0}{\beta}$$

Inverting the demand, imposing market-clearing (equation (3.27)):

$$p_0^{nd} = D - \beta \frac{n+2}{n+1} \left[S - \sum_{j=1}^n X_0^j \right]$$

Using the accounting identity:

$$S = X_{-1}^0 + \sum_{j=1}^n X_{-1}^j \quad (3.38)$$

gives the date-0 price functional when the hedgers anticipate no distress:

$$p_0^{nd} = D - \beta \frac{n+2}{n+1} X_{-1}^0 + \beta \frac{n+2}{n+1} \sum_{j=1}^n x_0^j \quad (3.39)$$

With $X_{-1}^0 = 0$, equation (3.39) corresponds to equation (3.5) given in the text.

- Second case: Suppose that the hedgers believe the prey will be in distress at $t = 1$. Using equation (3.36), solving for the hedgers' date 0-maximisation problem and using equation (3.38), I get:

$$p_0^d = D - \beta \frac{n+1}{n} X_{-1}^0 + \beta \sum_{j=1}^n x_0^j + \beta \frac{1}{n} \left(\sum_{j=1}^n x_0^j - X_0^1 \right)$$

Strategic traders' identities are public information, hence, using the dynamics of asset holdings, $X_0^1 = X_{-1}^1 + x_0^1$, this equation can be rewritten as:

$$p_0^d = D - \beta\frac{n+1}{n}X_{-1}^0 - \beta\frac{1}{n}X_{-1}^1 + \beta\frac{n+1}{n}\sum_{j=2}^{n} x_0^i + \beta x_0^1 \tag{3.40}$$

Setting $X_{-1}^0 = 0$ gives equation (3.6) in the text. Lemma 30 follows immediately from equations (3.39) and (3.40) and arguments given in the text. ■

3.6.2 Liquidity provision equilibrium

Lemma 31

Proof. Suppose that the hedgers believe that the prey will not be distressed. Since the hegders are rational, their beliefs must be correct in equilibrium. I now determine under which condition strategic traders' actions are consistent with the hedgers' beliefs.

At date 0, a strategic trader's problem is:

$$\begin{aligned}
\forall i = 1,...,n,\ J_0^{i,nd} &= \max_{x_0^i} E_0\left[B_{-1}^i - x_0^i p_0 + X_0^i\left(D + \tilde{\epsilon}_1\right) + \beta\frac{\left(S - \sum_{j=1}^n X_0^j\right)^2}{(n+1)^2}\right] \\
s.t.\ p_0^{nd} &= D - \beta\frac{n+2}{n+1}X_{-1}^0 + \beta\frac{n+2}{n+1}\sum_{j=2}^{n} x_0^j + \beta\frac{n+2}{n+1}x_0^1 \\
B_0^1 + X_0^1 p_0 &\leq \underline{V} \Rightarrow X_1^1 = 0 \\
X_0^1 &\leq \bar{X}
\end{aligned}$$

The second constraint corresponds to Assumption 1 (marked-to-market wealth constraint), the third constraint to Assumption 2 (leverage constraint). I first derive the equilibrium that would prevail in the absence of these two financial constraints, and then derive under which conditions this equilibrium holds in the presence of the constraints.

Ignoring the second and third constraints, plugging the first constraint into the maximand and using equation (3.38) gives

$$J_0^{i,nd} = \max_{x_0^i} E_{-1}^i + \beta\left[\frac{n+2}{n+1}x_0^i\left(X_{-1}^0 - \sum_{j\neq i}^{n} x_0^j - x_0^i\right) + \frac{\left(X_{-1}^0 - \sum_{j\neq i}^n x_0^j - x_0^i\right)^2}{(n+1)^2}\right]$$

with $E_{-1}^i = B_{-1}^i + X_{-1}^i D$. From the first-order condition, I get:

$$\forall i \in \{1, ..., n\},\ x_0^i + \frac{n^2+3n}{(n+1)^2}\sum_{j=1}^{n} x_0^j = \frac{n^2+3n}{(n+1)^2} X_{-1}^0 \tag{3.41}$$

Solving this system of n equations with n unknowns, I get the unique equilibrium in this subgame (in absence of constraints)

$$\forall i = 1, ..., n,\ x_0^i = \frac{n^2+3n}{n^3+4n^2+3n+2} X_{-1}^0 = c_{0,n} X_{-1}^0 \tag{3.42}$$

From equation (3.30), I find the date 1 equilibrium trade:

$$\forall i = 1, ..., n,\ x_1^i = \frac{n+2}{n^3+4n^2+3n+2} X_{-1}^0 = c_{1,n} X_{-1}^0 \tag{3.43}$$

After some simple algebra, I obtain the equilibrium prices:

$$p_0 = D - \beta \frac{(n+2)^2}{n^3+4n^2+3n+2} X_{-1}^0 = D - \beta \rho_{0,n} X_{-1}^0 \tag{3.44}$$

$$p_1 = D + \epsilon_1 - \beta \frac{n+2}{n^3+4n^2+3n+2} X_{-1}^0 = D + \epsilon_1 - \beta \rho_{1,n} X_{-1}^0 \tag{3.45}$$

Further, using (3.42) and (3.43), I compute the payoff (skipping two lines of algebra):

$$\begin{aligned} J_0^{nd} &= E_{-1}^i + \beta \pi_{0,n} \left(X_{-1}^0\right)^2 \\ \text{with } \pi_{0,n} &= \frac{(n^2+3n+1)(n+2)^2}{(n^3+4n^2+3n+2)^2} \end{aligned} \tag{3.46}$$

Let us now consider the problem with the financial constraints. I conjecture that the equilibrium trade is given by equation (3.42). An obvious condition on parameters is that $X_{-1}^1 + c_{0,n} X_{-1}^0 \leq \bar{X}$.

In the presence of the financial constraints, one must check for two types of deviations. First, the prey may opt for a voluntary liquidation. The prey being risk-neutral, it is easy to show that she will never voluntarily liquidate, therefore I skip the proof.

Second, a strategic trader may turn predator and exploit the prey's constraints to trigger a forced liquidation[25]. Doing so affects strategic traders' continuation payoff, which becomes

[25] Note that since the equilibrium is unique in the absence of financial constraints, this is the only deviation

$\frac{\left(S-\sum_{j=2}^{n}X_0^j\right)^2}{n^2}$.

Let's compute the payoff from exploiting the prey's financial constraints for predator i:

$$\begin{aligned}
J_0^{i,nd,dev} &= \max_{x_0^i} E_{-1}^i + \beta\left[\frac{n+2}{n+1}x_0^i\left(S-\sum_{j=1}^{n}X_0^j\right)+\frac{\left(S-\sum_{j=2}^{n}X_0^j\right)^2}{n^2}\right] \\
s.t.\ \forall j\neq i,\ x_0^j &= c_{0,n}X_{-1}^0 \\
p_0 &\leq \bar{p}_0
\end{aligned}$$

where $i\in\{2,...,n\}$. Using (3.38), this problem can be rewritten as

$$\begin{aligned}
\max_{x_0^i} \quad & \beta\frac{n+2}{n+1}x_0^i\left[X_{-1}^0-\sum_{j=1,j\neq i}^{n}x_0^j-x_0^i\right]+\beta\frac{\left[X_{-1}^0+X_{-1}^1-\sum_{j=2,j\neq i}^{n}x_0^j-x_0^i\right]^2}{n^2} \\
s.t.\ \forall j\neq i,\ x_0^j \quad &= \quad c_{0,n}X_{-1}^0 \\
p_0 \quad &\leq \quad \bar{p}_0
\end{aligned}$$

Note that in the second constraint, p_0 depends on the strategy of predator i and on the postulated strategy of other strategic traders, $\forall j\neq i,\ x_0^j=c_{0,n}X_{-1}^0$. I first determine under which condition a predatory deviation is costly, i.e. under which condition the Lagrangian of the second (price) constraint is strictly positive.

Let's first ignore the constraint $p_0\leq\bar{p}_0$ and solve for the zero of the first-order condition. I get:

$$x_0^{i,dev}=\frac{n^5+5n^4+4n^3-10n^2-11n-2}{\left(n^3+2n^2-n-1\right)\left(n^3+4n^2+3n+2\right)}X_{-1}^0-\frac{n+1}{n^3+2n^2-n-1}X_{-1}^1$$

As a consequence,

$$\frac{n+2}{n+1}\left[X_{-1}^0-\sum_{j=1}^{n}x_0^j\right]=H_1X_{-1}^0+H_2X_{-1}^1$$

with $H_1=\frac{n(n+2)\left(n^4+5n^3+8n^2+6n+3\right)}{(n+1)(n^3+2n^2-n-1)(n^3+4n^2+3n+2)}$ and $H_2=\frac{n+2}{n^3+2n^2-n-1}$. This, in turn, implies that

one must check for.

$p_0 \leq \bar{p}_0$ iff

$$\beta \geq \bar{\beta}_{nd} = \frac{|R|}{H_1 X_{-1}^0 + H_2 X_{-1}^1}, \text{with } R = \bar{p}_0 - D \tag{3.47}$$

Therefore, I will now focus on the parameter space $\beta < \bar{\beta}_{nd}$.

On this interval, pushing the prey into distress requires for a predator to set:

$$p_0^{nd} = \bar{p}_0$$

That is, predator i must choose $x_0^{i,dev}$ such that

$$D - \beta \frac{n+2}{n+1} X_{-1}^0 + \beta \frac{n+2}{n+1} \sum_{j=1, j \neq i}^{n} x_0^j + \beta \frac{n+2}{n+1} x_0^{i,dev} = \bar{p}_0$$

where $\forall j \neq i, x_0^j = c_{0,n} X_{-1}^0$. Rearranging the terms, I get:

$$x_0^{i,dev} = \frac{n+1}{n+2} \frac{R}{\beta} + \frac{2\left(n^2 + 3n + 1\right)}{n^3 + 4n^2 + 3n + 2} X_{-1}^0 \tag{3.48}$$

This achieves the proof of Lemma 31 ■

Proposition 21

Proof. Building on Lemma 31, I calculate the new continuation payoff of the strategic traders.

$$X_{-1}^0 + X_{-1}^1 - \sum_{j=2}^{n} x_0^j = X_{-1}^1 + \frac{n^2 + 3n}{n^3 + 4n^2 + 3n + 2} X_{-1}^0 - \frac{n+1}{n+2} \frac{R}{\beta} \tag{3.49}$$

Therefore, using equations (3.48) and (3.49), and developping and rearranging terms, predator i gets the following payoff from pushing the prey into distress:

$$\begin{aligned} J_0^{i,nd,dev} &= E_{-1}^i + \beta \frac{(n+3)^2}{(n^3+4n^2+3n+2)^2} \left(X_{-1}^0\right)^2 \\ +\beta &\left[\frac{1}{n^2}\left(X_{-1}^1\right)^2 + \frac{2(n+3)}{n(n^3+4n^2+3n+2)} X_{-1}^1 X_{-1}^0\right] \\ -R &\left[\frac{2(n^4+5n^3+8n^2+6n+3)}{n(n+2)(n^3+4n^2+3n+2)} X_{-1}^0 + \frac{2(n+1)}{n^2(n+2)} X_{-1}^1\right] \\ &- \frac{(n+1)(n^3+2n^2-n-1)}{n^2(n+2)^2} \frac{R^2}{\beta} \end{aligned} \tag{3.50}$$

Hence, predator i prefers buying over preying iff $J_0^{i,nd} \geq J_0^{i,nd,dev}$. Using equations (3.46) and (3.50), it is equivalent to:

$$a_{nd}\beta^2 + b_{nd}\beta + c_{nd} \geq 0 \tag{3.51}$$

$$\text{where } a_{nd} = \lambda_1 \left(X_{-1}^0\right)^2 - \lambda_2 \left(X_{-1}^1\right)^2 - \lambda_3 X_{-1}^1 X_{-1}^0 \tag{3.52}$$

$$b_{nd} = R\left[\lambda_4 X_{-1}^0 + \lambda_5 X_{-1}^1\right] < 0 \tag{3.53}$$

$$c_{nd} = \lambda_6 R^2 > 0 \tag{3.54}$$

with $\lambda_1 = \frac{n^4+7n^3+16n^2+10n-5}{(n^3+4n^2+3n+2)^2}$, $\lambda_2 = \frac{1}{n^2}$, $\lambda_3 = \frac{2(n+3)}{n(n^3+4n^2+3n+2)}$, $\lambda_4 = \frac{2(n^4+5n^3+8n^2+6n+3)}{n(n+2)(n^3+4n^2+3n+2)}$, $\lambda_5 = \frac{2(n+1)}{n^2(n+2)}$, $\lambda_6 = \frac{(n+1)(n^3+2n^2-n-1)}{n^2(n+2)^2}$. Note that for all $k = 1, ..., 6,$, for all $n \geq 2$, $\lambda_k > 0$.

The discriminant of the LHS of inequality (3.51) is

$$\Delta_{nd} = R^2 \left[A_1 \left(X_{-1}^0\right)^2 + A_2 \left(X_{-1}^1\right)^2 + A_3 X_{-1}^1 X_{-1}^0\right] \tag{3.55}$$

with $A_1 = \lambda_4^2 - 4\lambda_1\lambda_6 = \frac{4(3n^6+39n^5+104n^4+170n^3+125n^2+36n+4)}{n^2(n+2)^2(n^3+4n^2+3n+2)^2} > 0$, $A_2 = \lambda_5 + 4\lambda_6\lambda_2 > 0$, $A_3 = 2\lambda_4\lambda_5 + 4\lambda_6\lambda_3 > 0$. Hence for all $n \geq 2$, $\Delta_{nd} > 0$, which guarantees that there are always two real roots, β_1, β_2. Since the sign of b_{nd} and c_{nd} is known, the sign of equation (3.51) depends on the sign of a_{nd}.

Using $\theta = \frac{X_{-1}^0}{X_{-1}^1}$, I rewrite equation (3.52) as

$$a_{nd} = \left(X_{-1}^1\right)^2 \left[\lambda_1\theta - \lambda_3\theta - \lambda_2\right]$$

The discriminant of the equation in parenthesis is $\Delta_a = \lambda_3^2 + 4\lambda_1\lambda_2 > 0$. Since $\lambda_1 > 0$ and

$-\lambda_2 < 0$, there is a positive and a negative root. The positive root is given by

$$\bar{\theta} = \frac{\lambda_3 + \sqrt{\Delta_a}}{2\lambda_1}$$

and since $\theta \geq 0$, the sign of a_{nd} is striclty negative iff $\theta \in \left[0, \bar{\theta}\right[$ and positive iff $\theta > \bar{\theta}$.

I can now determine the equilibrium:

- If $0 \leq \theta < \bar{\theta}$, the no distress equilibrium exists iff $\beta < \beta_1 \wedge \bar{\beta}_{nd}$, with $\beta_1 = -\frac{b_{nd}+\sqrt{\Delta_{nd}}}{2a_{nd}}$.
- If $\theta > \bar{\theta}$, the no distress equilibrium exists iff $\beta < \beta_1 \wedge \bar{\beta}_{nd}$ or $\beta > \beta_2 \wedge \bar{\beta}_{nd}$, with $\beta_2 = \frac{-b_{nd}+\sqrt{\Delta_{nd}}}{2a_{nd}}$.

Using equations (3.52)-(3.54), equation (3.55), and the change of variable $\theta = \frac{X_{-1}^0}{X_{-1}^1}$, the roots are given by

$$\beta_1 = \frac{|R|}{X_{-1}^1} \frac{(\lambda_4\theta + \lambda_5) - \left[A_1\theta^2 + A_3\theta + A_2\right]^{\frac{1}{2}}}{2\left(\lambda_1\theta^2 - \lambda_3\theta - \lambda_2\right)} \equiv \underline{\beta}_{nd} \tag{3.56}$$

$$\beta_2 = \frac{|R|}{X_{-1}^1} \frac{(\lambda_4\theta + \lambda_5) + \left[A_1\theta^2 + A_3\theta + A_2\right]^{\frac{1}{2}}}{2\left(\lambda_1\theta^2 - \lambda_3\theta - \lambda_2\right)} \tag{3.57}$$

I now show that in the second case ($\theta > \bar{\theta}$), the second root, β_2, does not satisfy the parameter restriction $\beta < \bar{\beta}_{nd}$, where $\bar{\beta}_{nd}$ is given by equation (3.47).

Since the denominator of β_2 is strictly positive when $\theta > \bar{\theta}$, $\beta_2 - \bar{\beta}_{nd} < 0$ is, after rearranging terms, equivalent to:

$$(\lambda_4 H_1 - 2\lambda_1)\,\theta^2 + (\lambda_5 H_1 + \lambda_4 H_2 + 2\lambda_3)\,\theta + (\lambda_5 H_2 + 2\lambda_2) + (H_1\theta + H_2)\,U_\theta^{\frac{1}{2}} < 0$$

where $U_\theta = A_1\theta^2 + A_3\theta + A_2$. Since for all $n \geq 2$, $\lambda_4 H_1 - 2\lambda_1 > 0$ and since all other coefficients are also positive, this condition is never satisfied for any $\theta \geq 0$, hence for any $\theta > \bar{\theta}$. Hence $\beta_2 > \bar{\beta}_{nd}$.

As a result, the necessary and sufficient condition for the existence of the no distress equilibrium is $\beta < \underline{\beta}_{nd} \wedge \bar{\beta}_{nd}$. ∎

Corollaries 26 and 27

Proof. The results follow directly from calculations in the proof of Proposition 21. ∎

3.6.3 Predatory trading equilibrium and comparative statics

I conjecture that the predators' equilibrium predatory trade is

$$\forall j = 2, ..., n, \; x_0^j = \frac{1}{n-1}\left[X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)\right] \tag{3.58}$$

Using equation (3.34), this implies that their date-1 trade is

$$\forall j = 2, ..., n, \; x_1^j = \frac{1}{n+1}\left(\bar{X} - \frac{R}{\beta}\right) \tag{3.59}$$

which leads to the following price:

$$p_1 = D + \epsilon_1 - \frac{\beta}{n+1}\left(\bar{X} - \frac{R}{\beta}\right)$$

I assume that the hedgers believe that the prey will be distressed. I first determine conditions under which the prey's conjectured strategy is optimal given the predators' conjectured strategy.

Lemma 33

Proof. The prey's problem. The predators' conjectured strategy implies the following first-period price (as a function of the prey's trade):

$$p_0 = \bar{p}_0 - \beta\left[\bar{X} - X_{-1}^1 - x_0^1\right] \tag{3.60}$$

Since the predators' strategy is constructed so that the prey can not outbid predators, the prey's problem given predators' trade is to maximise the proceeds of liquidation. Hence the prey's maximisation problem is:

$$\begin{aligned}
\max_{x_0^1} \quad & E_0 \quad \left[B_{-1}^1 - x_0^1 p_0 - x_1^1 p_1 + X_1 D_2\right] && (3.61)\\
s.t.\; X_1^1 &= 0 \\
x_0^1 &\leq \bar{X} - X_{-1}^1 \\
p_0 &= \bar{p}_0 = D - \beta\left[\bar{X} - X_{-1}^1 - x_0^1\right] \\
p_1 &= D + \epsilon_1 - \frac{\beta}{n+1}\left(\bar{X} - \frac{R}{\beta}\right)
\end{aligned}$$

Plugging the first and last two constraints into the maximand, this problem can be rewritten as:

$$\begin{array}{rcl} \max_{x_0^1} & & B_{-1}^1 - x_0^1 \left[\bar{p}_0 - \beta\left[\bar{X} - X_{-1}^1 - x_0^1\right]\right] + X_0^1\left[D - \beta\frac{1}{n+1}\left[\bar{X} - \frac{R}{\beta}\right]\right] \\ s.t.\ x_0^1 & \leq & \bar{X} - X_{-1}^1 \end{array}$$

Writing the Lagrangian of the problem and solving for the zero of the first-order condition gives:

$$x_0^1 = \begin{cases} \frac{n}{2(n+1)}\frac{|R|}{\beta} + \frac{1}{2}\left[\frac{n}{n+1}\bar{X} - X_{-1}^1\right] & \text{if } \beta < \beta_F \\ \bar{X} & \text{otherwise,} \end{cases}$$

$$\text{where } \beta_F = \frac{|R|}{\frac{n+2}{n}\bar{X} - \frac{n+1}{n}X_{-1}^1} \tag{3.62}$$

$\Rightarrow$ A necessary condition for the conjectured strategy to be a Nash equilibrium is $\beta < \beta_F$. ■

Lemma 32

Proof. The predators' problem. The predators' conjectured strategy (3.58) is constructed assuming that predation is costly and that predators behave symmetrically. I.e., the conjectured strategy is such that predators choose a quantity leading to $p_0^d = \bar{p}_0$, with $x_0^1 = \bar{X} - X_{-1}^1$.

A necessary condition for this conjectured strategy to be a Nash equilibrium is that the Lagrangian of the first constraint in the following problem is zero.

$$\begin{array}{rcl} \max_{x_0^i} & & \beta x_0^i\left[\frac{n+1}{n}\left(S - \sum_{j=2}^{n} X_0^j\right) - X_0^1\right] + \beta\frac{\left(S - \sum_{j=2}^n X_0^j\right)^2}{n^2} \\ s.t.\ p_0 & \leq & \bar{p}_0 \\ X_0^1 & = & \bar{X} \end{array} \tag{3.63}$$

The problem can be rewritten as

$$\max_{x_0^i} \beta x_0^i\left[\frac{n+1}{n}\left(X_{-1}^0 + X_{-1}^1 - \sum_{j=2}^{n} x_0^j\right) - \bar{X}\right] + \beta\frac{\left(X_{-1}^0 + X_{-1}^1 - \sum_{j=2}^n x_0^j\right)^2}{n^2}$$
$$s.t.\ p_0 \leq \bar{p}_0$$

After writing the Lagrangian of the problem and solving for the equilibrium, I get:

$$x_0^i = \begin{cases} \frac{1}{n-1}\left[X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)\right] & \text{if } a > \frac{\rho_{0,n-1}}{d_n} \text{ or if } \beta < \bar{\beta}_d \text{ when } a \leq \frac{\rho_{0,n-1}}{d_n} \\ \frac{n^2+n-2}{n^3+n^2-2n+2}\left(X_{-1}^0 + X_{-1}^1\right) - \frac{n^2}{n^3+n^2-2n+2}\bar{X} & \text{otherwise,} \end{cases}$$

$$\text{with } \bar{\beta}_d = \frac{|R|}{\rho_{0,n-1}\left(X_{-1}^0 + X_{-1}^1\right) - d_n \bar{X}} \tag{3.64}$$

where $\rho_{0,n-1} = \frac{(n+1)^2}{n^3+n^2-2n+2}$, $d_n = \frac{n^2-n+2}{n^3+n^2-2n+2}$, and $a = \frac{\bar{X}}{X_{-1}^1}$ is the prey's spare leverage capacity. Note that symmetry is imposed when the Lagrangian of the constraint is zero, while it is the unique outcome when the constraint is not binding.

$\Rightarrow$ A necessary condition for the conjectured strategy to be a Nash equilibrium is $\beta < \beta_d$ if $a \leq \frac{\rho_{0,n-1}}{d_n}$. ■

Propositions 19 and 22

Proof. The payoff of the conjectured strategy for predators is, using equations (3.58) and (3.59):

$$J_0^{i,D} = E_{-1}^i + \beta\frac{\bar{X}^2}{(n+1)^2} - R\left[\frac{1}{n-1}\left(X_{-1}^0 + X_{-1}^1\right) - \frac{n^2-n+2}{(n-1)(n+1)^2}\bar{X}\right] - \frac{n^2+1}{(n-1)(n+1)^2}\frac{R^2}{\beta} \tag{3.65}$$

Payoff from deviating: "rescuing" the prey. Predator i may not join the predatory attack and "rescue" the prey. All predators are pivotal, hence this rescue implies a change in the continuation payoff from $\frac{S-\sum_{j=2}^n X_0^j}{n^2}$ to $\frac{S-\sum_{j=1}^n X_0^j}{(n+1)^2}$.

The strategy of a deviating predator solves the following problem:

$$\begin{aligned} J_0^{i,d,dev} &= \max_{x_0^i} \; \beta x_0^i \left[\frac{n+1}{n}\left(S - \sum_{j=2} X_0^j\right) - X_0^1\right] + \beta\frac{\left(S - \sum_{j=2}^n X_0^j\right)^2}{(n+1)^2} \\ s.t.\ \forall j \neq i,\ x_0^j &= \frac{1}{n-1}\left[X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right)\right] \\ X_0^1 &= \bar{X} \\ p_0 &> \bar{p}_0 \end{aligned}$$

Using equation (3.38), and plugging the first and second constraints into the maximand, the maximisation problem boils down to

$$\begin{aligned} J_0^{i,d,dev} &= \max_{x_0^i} \quad \beta x_0^i \left[\frac{n+1}{n(n-1)} \left(X_{-1}^0 + X_{-1}^1 \right) - \frac{n-2}{n-1} \frac{R}{\beta} - \frac{n+1}{n} x_0^i - \frac{1}{n-1} \bar{X} \right] \\ &+ \quad \frac{\beta}{(n+1)^2} \left[\frac{1}{n-1} \left(X_{-1}^0 + X_{-1}^1 \right) - \frac{n(n-2)}{n^2-1} \frac{R}{\beta} - \frac{2n-1}{n^2-1} \bar{X} - x_0^i \right]^2 \\ s.t.\ p_0 &> \quad \bar{p}_0 \end{aligned}$$

Writing the Lagrangian and solving for the first-order condition (ignoring the price constraint for now), I get the strategy of a deviating ("rescuing") predator:

$$\begin{aligned} x_0^{i,dev} &= \frac{n^3+3n^2+n+1}{2(n-1)(n^3+3n^2+2n+1)} \left(X_{-1}^0 + X_{-1}^1 \right) \\ &- \frac{n(n^3+3n^2-n+3)}{2(n^2-1)(n^3+3n^2+2n+1)} \bar{X} - \frac{n(n-2)(n^3+3n^2+n+1)}{2(n^2-1)(n^3+3n^2+2n+1)} \frac{R}{\beta} \end{aligned} \tag{3.66}$$

It is easy albeit algebraically tedious to check that $\beta < \bar{\beta}_d$ implies that $p_0 > \bar{p}_0$, so that the Lagrangian of the price constraint is always zero.

To compute the payoff of the rescue for predator i, it is convenient to calculate the following quantities:

$$\frac{n+1}{n} \left(X_{-1}^0 + X_{-1}^1 - \sum_{j=2}^{n} x_0^j \right) - \bar{X} = z_1 \left(X_{-1}^0 + X_{-1}^1 \right) - z_2 \bar{X} - z_3 \frac{R}{\beta} \tag{3.67}$$

where $z_1 = \frac{(n+1)\left(n^3+3n^2+3n+1\right)}{2n(n-1)(n^3+3n^2+2n+1)}$, $z_2 = \frac{n^3+3n^2+5n-1}{2(n-1)(n^3+3n^2+2n+1)}$, $z_3 = \frac{(n-2)\left(n^3+3n^2+3n+1\right)}{2(n-1)(n^3+3n^2+2n+1)}$. and

$$\frac{X_{-1}^0 - \sum_{j=1} x_0^j}{n+1} = z_1' \left(X_{-1}^0 + X_{-1}^1 \right) - z_2' \bar{X} - z_3' \frac{R}{\beta} \tag{3.68}$$

with $z_1' = \frac{n^3+3n^2+3n+1}{2(n^2-1)(n^3+3n^2+2n+1)}$, $z_2' = \frac{3n^4+7n^3+3n^2-3n-2}{2(n-1)(n+1)^2(n^3+3n^2+2n+1)}$, $z_3' = \frac{n(n-2)\left(n^3+3n^2+3n+1\right)}{2(n-1)(n+1)^2(n^3+3n^2+2n+1)}$.

From equations (3.66)-(3.68), skipping some algebra, the payoff of rescuing the prey is:

$$\begin{aligned} J_0^{i,d,dev} = \beta \left[w_1 \bar{X}^2 + w_2 \left(X_{-1}^0 + X_{-1}^1 \right) - w_3 \left(X_{-1}^0 + X_{-1}^1 \right) \bar{X} \right] \\ - R \left[w_4 \left(X_{-1}^0 + X_{-1}^1 \right) - w_5 \bar{X} \right] + w_6 \frac{R^2}{\beta} \end{aligned} \tag{3.69}$$

with $w_1 = \frac{n^{10}+9n^9+43n^8+114n^7+155n^6+98n^5+41n^4+50n^3+28n^2-15n+4}{4(n-1)^2(n+1)^4(n^3+3n^2+2n+1)^2}$, $w_2 = \frac{(n+1)^4}{4n(n1)^2(n^3+3n^2+2n+1)}$, $w_3 = \frac{n^6+9n^5+23n^4+24n^3+7n^2+n-1}{2(n-1)^2(n^3+3n^2+2n+1)^2}$, $w_4 = \frac{(n-2)(n+1)^3}{2(n-1)^2(n^3+3n^2+2n+1)}$, $w_5 = \frac{n^2(n-2)\left(n^5+9n^4+23n^3+25n^2+10n+4\right)}{2(n+1)(n-1)^2(n^3+3n^2+2n+1)^2}$, $w_6 = \frac{n(n-2)^2(n+1)^2\left(n^4+4n^3+4n^2+3n+1\right)}{4(n-1)^2(n^3+3n^2+2n+1)^2}$.

The conjectured predatory trades form a Nash equilibrium iff $\forall i = 2, ..., n,\ J_0^{i,d} \geq J_0^{i,d,dev}$. From equations (3.65) and (3.69), this is equivalent to

$$
\begin{aligned}
a_d\beta^2 + b_d\beta + c_d &\geq 0 && (3.70)\\
\text{with } a_d &= e_1\bar{X}^2 - e_2\left(X_{-1}^0 + X_{-1}^1\right)^2 + e_3\bar{X}\left(X_{-1}^0 + X_{-1}^1\right) && (3.71)\\
b_d &= -R\left[e_4\left(X_{-1}^0 + X_{-1}^1\right) - e_5\bar{X}\right] && (3.72)\\
c_d &= -e_6R^2 && (3.73)
\end{aligned}
$$

and $e_1 = \frac{1}{(n+1)^2} - w_1$, $e_2 = w_2$, $e_3 = w_3$, $e_4 = \frac{1}{n-1} - w_4$, $e_5 = \frac{n^2-n+2}{(n-1)(n+1)^2} - w_5$, $e_6 = \frac{n^2+1}{(n-1)(n+1)^2} + w_6$

$$
\begin{aligned}
e_4 &= \frac{n^4+3n^3+n^2+3n}{2(n-1)^2(n^3+3n^2+2n+1)} && (3.74)\\
e_5 &= \frac{n^9-4n^7+24n^6+79n^5+56n^4+14n^3-12n^2-10n-4}{2(n-1)^2(n+1)^2(n^3+3n^2+2n+1)^2} && (3.75)
\end{aligned}
$$

It is clear that $c_d < 0$. Let us now study the signs of b_d and a_d.

Sign of b_d

$$
b_d \geq 0 \Leftrightarrow \kappa \geq \frac{e_5}{e_4}, \text{ where } \kappa = \frac{X_{-1}^0 + X_{-1}^1}{\bar{X}} \tag{3.76}
$$

Further, from equations (3.74)-(3.75), $\forall n \geq 2$, $\frac{e_5}{e_4} = \frac{n^9-4n^7+24n^6+79n^5+56n^4+14n^3-12n^2-10n-4}{(n+1)^2(n^3+3n^2+2n+1)(n^4+3n^3+n^2+3n)}$ and $\frac{e_5}{e_4} \leq 1$.

Sign of a_d

Using the variable $\kappa = \frac{X_{-1}^0+X_{-1}^1}{\bar{X}}$, I rewrite equation (3.71) as:

$$
a_d = \bar{X}^2\left[e_1 - e_2\kappa^2 + e_3\kappa\right]
$$

For $n = 2$, $e_1 < 0$, $e_2 > 0$, $e_3 > 0$. When $n > 2$, all coefficients are strictly positive. Thus,

- If $n = 2$, there are two positive roots, $\kappa_1 = \frac{e_3-\sqrt{\delta}}{2e_2}$ and $\kappa_2 = \frac{e_3+\sqrt{\delta}}{2e_2}$, where $\delta = e_3^2+4e_2e_1$.
- If $n > 2$, there is a positive and a negative roots, with $\kappa_1 < 0$ and $\kappa_2 > 0$.

Hence, $a_d > 0 \Leftrightarrow$

- $\kappa \in \left]\kappa_1, \kappa_2\right[$, if $n = 2$
- $\kappa \in \left]0, \kappa_2\right[$, if $n > 2$.

<u>*Discriminant*</u>

The discriminant of equation (3.70) is:

$$\begin{aligned} \Delta_d &= R^2 \left[r_1 \left(X_{-1}^0 + X_{-1}^1\right)^2 + r_2\bar{X}\left(X_{-1}^0 + X_{-1}^1\right) + r_3\bar{X}^2\right] \\ \text{i.e., } \Delta_d &= R^2\bar{X}^2 \left[r_1\kappa^2 + r_2\kappa + r_3\right] \end{aligned} \tag{3.77}$$

with $r_1 = e_4^2 - 4e_6e_2$, $r_2 = 4e_6e_3 - 2e_5e_4$, $r_3 = e_5^2 + 4e_6e_1$. $\forall n \geq 2$, $r_1 > 0$, and $r_2 > 0$. Further, $r_3 < 0$ for $n = 2$ and $r_3 > 0$ for $n > 2$.[26]

Hence if $n = 2$, the equation $r_1\kappa^2 + r_2\kappa + r_3$ has two solutions:

$$\begin{aligned} \kappa_1^d &= \frac{-r_2 + \sqrt{\Delta_d}}{2r_1} \approx 0.1 \\ \kappa_2^d &= \frac{-r_2 - \sqrt{\Delta_d}}{2r_1} < 0, \text{where } \Delta_d = r_2^2 - 4r_1r_3 \end{aligned}$$

If $n > 2$, then all coefficients r_i being strictly positive, $\Delta_D > 0$ for any κ. Hence,

- If $n = 2$, then $\Delta_d < 0$ for $\kappa \in \left[0, \kappa_1^d\right[$. If $\kappa > \kappa_1^d \approx 0.1$, then $\Delta_d > 0$.
- If $n > 2$, then $\Delta_d > 0$.

Equilibrium

The equilibrium is determined by the sign of equation (3.70) and the parameter restrictions β_F and $\bar{\beta}_d$, given by equations (3.62) and (3.64), respectively.

[26]For the sake of brevity, I did not reproduce the analytical expression of the coefficients r_i. I check the signs numerically for $n = 2$ to $n = 150$.

When $\Delta_d > 0$, equation (3.70) has two real roots given by

$$\underline{\beta}_d = \frac{\sqrt{\Delta_d} - b_d}{2a_d} \tag{3.78}$$

$$\underline{\beta}_{d,2} = -\frac{b_d + \sqrt{\Delta_d}}{2a_d} \tag{3.79}$$

It is easy to see that if $a_d > 0$, $\beta_2 < 0$, and if $a_d < 0$, $\beta_2 > \underline{\beta}_d > 0$. Using $\kappa = \frac{X^0_{-1}+X^1_{-1}}{\bar{X}}$, equations (3.78) and (3.79) and (3.71)-(3.73), the roots can be rewritten as:

$$\underline{\beta}_d = \frac{|R|}{\bar{X}} \frac{Z_\kappa^{\frac{1}{2}} - (e_4\kappa - e_5)}{2\,(e_1 - e_2\kappa^2 + e_3\kappa)} \tag{3.80}$$

$$\beta_2 = -\frac{|R|}{\bar{X}} \frac{Z_\kappa^{\frac{1}{2}} + (e_4\kappa - e5)}{2\,(e_1 - e_2\kappa^2 + e_3\kappa)} \tag{3.81}$$

where $Z_\kappa = r_1\kappa^2 + r_2\kappa + r_3$.

I first study the sign of equation (3.70) independently of the parameter restrictions.

If $n > 2$, $\Delta_d > 0$, hence the equation has two real roots. From the signs of a_d and b_d, there are two thresholds for κ in this case: κ_2 and $\frac{e_5}{e_4}$. Since for all $n \geq 2$, $\kappa_2 \geq 1$ and $\frac{e_5}{e_4} < 1$, it is clear that $\kappa_2 > \frac{e_5}{e_4}$. Then the sign of equation (3.70) is as follows:

- If $\kappa \in \left[0, \frac{e_5}{e_4}\right[$, $a_d > 0$, $b_d < 0$, $c_d < 0$, hence $\beta_2 < 0$, $\underline{\beta}_d > 0$ and $a_d\beta^2 + b_d\beta + c_d \geq 0 \Leftrightarrow \beta > \underline{\beta}_d$
- If $\left[\frac{e_5}{e_4}, \kappa_2\right[$, $a_d > 0$, $b_d > 0$, $c_d < 0$, then $\beta_2 < 0$, $\underline{\beta}_d > 0$ and $a_d\beta^2 + b_d\beta + c_d \geq 0 \Leftrightarrow \beta > \underline{\beta}_d$.
- If $\kappa > \kappa_2$, then $a_d < 0$, $b_d < 0$, and $c_d < 0$ and $a_d\beta^2 + b_d\beta + c_d \geq 0 \Leftrightarrow \beta \in \left[\underline{\beta}_d, \underline{\beta}_{d,2}\right[$

When $n = 2$, there are four thresholds κ_1^d, κ_1, $\frac{e_5}{e_4}$ and κ_2, in increasing order. For $\kappa \geq \frac{e_5}{e_4}$, the analysis is similar to the case where $n > 2$. For $\kappa < \frac{e_5}{e_4}$, the intervals are as follows:

- If $\kappa \in \left[0, \kappa_1^d\right[$, $a_d < 0$, $b_d < 0$, $c_d < 0$, and $\Delta_d < 0$, hence $a_d\beta^2 + b_d\beta + c_d < 0$ and there is no predatory trading equilibrium.
- If $\kappa \in \left[\kappa_1^d, \kappa_1\right[$, then $\Delta_d > 0$, but since $a_d < 0$, $b_d < 0$, $c_d < 0$, there are two negative roots, and therefore, there is no predatory trading equilibrium. This case can be grouped with the previous one.

- If $\kappa \in \left[\kappa_1, \frac{e_5}{e_4}\right[$, then $a_d > 0$, $b_d < 0$, $c_d < 0$ and $\Delta_d > 0$. Then $\beta_2 < 0$, $\underline{\beta}_d > 0$ and $a_d\beta^2 + b_d\beta + c_d \geq 0 \Leftrightarrow \beta > \underline{\beta}_d$. Thus this case can be grouped with the one in which $\kappa > \frac{e_5}{e_4}$.

$\Rightarrow$ The $n = 2$ case is thus the same as the $n > 2$ case, except for $\kappa < \kappa_1$.

I now determine the intervals of the predatory trading equilibrium, taking into account the parameter restrictions β_F and $\bar{\beta}_d$, given by equations (3.62) and (3.78), respectively.

Position of β_F relative to $\bar{\beta}_d$

From equations (3.62) and (3.78):

$$\bar{\beta}_d > \beta_F \Leftrightarrow a \geq m_1\theta + m_2 \tag{3.82}$$

with $m_1 = \frac{n(n+1)^2}{n^4+4n^3-n^2+4}$ and $m_2 = \frac{n^4+3n^3+n^2+n+2}{n^4+4n^3-n^2+4}$

Note that $m_2 = 1$ when $n = 2$ and $m_2 < 1$ when $n > 2$.
$\Rightarrow$ If $\theta = 0$ (i.e. $X^0_{-1} = 0$), $\bar{\beta}_d > \beta_F \Leftrightarrow a \geq m_2$, which is always true since $a \geq 1$.
$\Rightarrow$ Proposition 19 follows from this remark and the analysis below.

Intervals of the predatory trading equilibrium

The analysis of equation (3.70) gives necessary and sufficient conditions in terms of the variable κ, whereas the parameter restrictions for β_F and $\bar{\beta}_d$ are expressed in terms of θ. Noting that[27]:

$$\kappa = \frac{\theta + 1}{a} \tag{3.83}$$

I rewrite all the conditions in terms of a and θ.

The thresholds in terms of κ are κ_1 (for $n = 2$ only), $\frac{e_5}{e_4}$ and κ_2. Hence using equation (3.83), the corresponding thresholds in terms of a are, in increasing order, $\frac{1}{\kappa_1}\theta + \frac{1}{\kappa_1}$, $\frac{e_4}{e_5}\theta + \frac{e_4}{e_5}$ and $\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}$.

[27]Using the definition of κ (3.76) and the following notations: $\theta = \frac{X^0_{-1}}{X^1_{-1}}$, $a = \frac{\bar{X}}{X^1_{-1}}$

I now compare these thresholds to the condition (3.82). For all $n \geq 2$, $\frac{e_4}{e_5} > m_2 > m_1$, $\frac{1}{\kappa_1} > m_2 > m_1$. Therefore, $\forall n \geq 2$,

$$\begin{cases} \frac{e_4}{e_5}\theta + \frac{e_4}{e_5} > m_1\theta + m_2 \\ \frac{1}{\kappa_1}\theta + \frac{1}{\kappa_1} > m_1\theta + m_2 \end{cases}$$

Further, $\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2} > m_1\theta + m_2$ is equivalent to

$$\theta > \theta^* = \frac{m_2 - \frac{1}{\kappa_2}}{\frac{1}{\kappa_2} - m_1}$$

Since $\forall n \geq 2$, $m_2 > \frac{1}{\kappa_2} > m_1$, $\theta^* > 0$. Hence, combining the equilibrium conditions and the parameter restrictions yields, $\forall n > 2$

- If $a \geq \max\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, then $I_P = \left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$
- If $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, then $I_P = \left[\underline{\beta}_d \wedge \bar{\beta}_d, \underline{\beta}_{d,2} \wedge \bar{\beta}_d\right[$
- If $\min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right) < a < \max\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, then
  - If $\theta > \theta^*$, then $I_P = \left[\underline{\beta}_d \wedge \beta_F, \underline{\beta}_{d,2} \wedge \beta_F\right[$,
  - If $\theta \leq \theta^*$, then $I_P = \left[\underline{\beta}_d \wedge \bar{\beta}_d, \bar{\beta}_d\right[$.

If $n = 2$, there is an additional case: if $a \geq \frac{1}{\kappa_1}\theta + \frac{1}{\kappa_1}$, there is no predatory trading equilibrium.

In the second case, $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, it is possible to refine the boundaries of the interval I_P and show that it is non-empty, thereby proving the existence of the equilibrium in this case.

Existence conditions

I first show that $\underline{\beta}_d < \bar{\beta}_d$. This case is interesting for $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, hence the interval I consider is $\kappa > \kappa_2$. Using (3.80) and (3.64), and rearranging terms, I get

$$\underline{\beta}_d - \bar{\beta}_d = \frac{|R|}{\bar{X}} \frac{g_2(\kappa)}{(\rho_{0,n-1}\kappa - d_n)(e_1 - e_2\kappa^2 + e_3\kappa)} \quad (3.84)$$

$$\text{with } g_2(\kappa) = (\rho_{0,n-1}\kappa - d_n) Z_\kappa^{\frac{1}{2}} + B_1\kappa^2 + B_2\kappa - B_3 \quad (3.85)$$

where $\forall n \geq 2$, $B_1 = 2e_2 - \rho_{0,n-1}e_4 < 0$, $B_2 = e_5\rho_{0,n-1} + d_n e_4 - 2e_3 < 0$, $B_3 = 2e_1 + d_n e_5 > 0$.[28]

The denominator of equation (3.84) is negative when $\kappa > \kappa_2$, thus $\underline{\beta}_d - \bar{\beta}_d < 0$ iff $g_2(\kappa) \geq 0$. To determine the sign of g_2, I first study its first derivative:

$$g_2'(\kappa) = \rho_{0,n-1} Z_\kappa^{\frac{1}{2}} + (\rho_{0,n-1}\kappa - d_n) \frac{Z_\kappa'}{Z_\kappa^{\frac{1}{2}}} + 2B_1\kappa + B_2$$

The first term of the derivative is positive for any $\kappa > 0$. The second term is also positive, because $\forall n \geq 2$, $\frac{d_n}{\rho_{0,n-1}} < \kappa_2$ and $Z_\kappa' = 2r_1\kappa + r_2 > 0$ for any $\kappa > \kappa_2 > 0$ (r_1 and r_2 being positive for any $n \geq 2$). The third term, however, is negative, because B_1 and B_2 are negative. I will show that $\forall \kappa > \kappa_2$, $g_2'(\kappa) > 0$. To show this, it is enough to show that $\rho_{0,n-1} Z_\kappa^{\frac{1}{2}} + 2B_1\kappa + B_2 \geq 0$.

Since $Z_\kappa = r_1\kappa^2 + r_2\kappa + r_3$ (see equation (3.80)), the following holds for any $\kappa > \kappa_2$:

$$Z_\kappa \geq r_1\kappa^2 + r_2\kappa_2 + r_3$$

and therefore $\rho_{0,n-1}\sqrt{Z_\kappa} \geq \rho_{0,n-1}\sqrt{r_1\kappa + r_2\kappa_2 + r_3}$, which implies that

$$\rho_{0,n-1}\sqrt{Z_\kappa} + 2B_1\kappa + B_2 \geq \rho_{0,n-1}\sqrt{r_1\kappa^2 + r_2\kappa_2 + r_3} + 2B_1\kappa + B_2$$

Given that $\forall n \geq 2$, $\rho_{0,n-1}\sqrt{r_1} \geq -2B_1$, the function on the RHS of the inequality is increasing in κ. Hence for $\kappa > \kappa_2$, $\rho_{0,n-1}\sqrt{Z_\kappa} + 2B_1\kappa + B_2 > \rho_{0,n-1}\sqrt{r_1\kappa_2^2 + r_2\kappa_2 + r_3} + 2B_1\kappa_2 + B_2$. The right-hand side of the inequality is positive for all $n \geq 2$, hence $\forall \kappa > \kappa_2$, $\forall n \geq 2$, $g_2'(\kappa) > 0$ and g_2 is increasing on this interval. As a result, one can minor this function by $g_2(\kappa_2)$, with $\forall n \geq 2$, $g_2(\kappa_2) > 0$.

Hence $\forall \kappa > \kappa_2$, $\underline{\beta}_d < \bar{\beta}_d$.

Using a similar reasoning, one can show that $\underline{\beta}_{d,2} > \bar{\beta}_d$ when $\kappa > \kappa_2$. From equations (3.81) and (3.64), $\underline{\beta}_{d,2} < \bar{\beta}_d$ is equivalent to $h_2(\kappa) > 0$, with

$$h_2(\kappa) = -(\rho_{0,n-1} - d_n)\sqrt{Z_\kappa} + B_1\kappa^2 + B_2\kappa - B_3$$

The function $-(\rho_{0,n-1} - d_n)\sqrt{Z_\kappa}$ is always negative, as well as $B_1\kappa^2 + B_2\kappa - B_3$. Thus $\forall \kappa > \kappa_2$, $\underline{\beta}_{d,2} > \bar{\beta}_d$. ■

[28]For the remainder of the proof, I rely again on calculations for the coefficients which are functions of n.

Corollary 29

Proof. Suppose that $a \leq \min\left(\frac{1}{\kappa_2}\theta + \frac{1}{\kappa_2}, m_1\theta + m_2\right)$, and consider $p(\kappa) = 1 - \hat{q}(\kappa) = \frac{\beta_d}{\bar{\beta}_d}$. From equations (3.80) and (3.64), we can write

$$p(\kappa) = \frac{(\rho_{0,n-1}\kappa - d_n)\left(Z_\kappa^{\frac{1}{2}} - (e_4\kappa - e_5)\right)}{2(e_1 - e_2\kappa^2 + e_3\kappa)}$$

Hence the first derivative w.r.t. κ, after regrouping terms, is

$$p^{'}(\kappa) = \frac{(e_1 - e_2\kappa^2 + e_3\kappa)(\rho_0 Z_\kappa + (\rho_0 - d_n)(2r_1\kappa + r_2)) - (e_3 - 2e_2\kappa)(\rho_0\kappa - d_n) 2Z_\kappa}{2Z_\kappa^{\frac{1}{2}}}$$
$$+ (e_5\rho_0 + e_4 d_n - 2e_4\rho_0\kappa)\left(e_1 - e_2\kappa^2 + e_3\kappa\right) + (2e_2\kappa - e_3)\left(-e_4\rho_0\kappa^2 (e_5\rho_0 + e_4 d_n)\kappa - d_n e_5\right)$$

It is enough to show that p is increasing when $\kappa \geq \kappa_2$. I start by developing and rearranging terms of the numerator in the first line. Using that $Z_\kappa = r_1\kappa^2 + r_2\kappa + r_3$, I get after a few calculations that the numerator is equal to $H_1\kappa^4 + H_2\kappa^3 + H_3\kappa^2 + H_4\kappa + H_5$, with

$$\begin{aligned} H_1 &= e_2 r_1 \rho_0;\ H_2 = 2e_2\rho_0 r_2 - 2r_1 d_2 e_2 + r_1 e_3 \rho_0 \\ H_3 &= 3r_3 e_2 \rho_0 - 2r_2 d_n e_2 + 3r_1\rho_0 e_1 + e_3\rho_0 r_2;\ H_4 = 2r_2\rho_0 e_1 - r_3 e_3 \rho_0 - 4r_3 d_n e_2 - e_1 r_1 d_n \\ H_5 &= 2r_1 e_3 d_n - e_1 r_2 d_n \end{aligned}$$

Now consider the second line in $p^{'}$ and rearrange terms. This gives: $H_6\kappa^2 - H_7\kappa + H_8$, with

$$H_6 = e_2(e_5\rho_0 + e_4 d_n) + e_3 e_4 \rho_0;\ H_7 = 2e_4 e_1 \rho_0 + 2e_2 d_n e_5;\ H_8 = e_1(e_5\rho_0 + e_4 d_n) + d_n e_5 e_3$$

Hence the sign of $p^{'}$ is the same as the sign of

$$\phi_\kappa = H_1\kappa^4 + H_2\kappa^3 + H_3\kappa^2 + H_4\kappa + H_5 + 2Z_\kappa^{\frac{1}{2}}\left(H_6\kappa^2 - H_7\kappa + H_8\right)$$

Calculating the coefficients H_i, which are functions of n, we find that H_1, H_2, H_3, H_6 and H_8 are positive for any $n \geq 2$. However, for $n \geq 2$, H_4 is negative, H_7 is positive and H_5 becomes negative for $n \geq 4$. Given the signs of the coefficients, to show that $p^{'}$ is positive for $\kappa \geq \kappa_2$, it is enough to show $H_3\kappa^2 + H_4\kappa + H_5 \geq 0$ and $H_6\kappa^2 - H_7\kappa + H_8$ on this interval.

First, consider $H_3\kappa^2 + H_4\kappa + H_5 \geq 0$. Since $H_3 > 0$, it is increasing for $\kappa \geq -\frac{H_4}{2H_3}$, which calculations show is smaller than κ_2. Further, I find that for any $n \geq 2$, $H_3(\kappa_2)^2 + H_4\kappa_2 + H_5 > 0$. Next, consider $H_6\kappa^2 - H_7\kappa + H_8$ and apply the same steps. H_6 is positive and the function peaks in $\frac{H_7}{2H_6}$, which I find is smaller than κ_2 for $n \geq 2$. Further, I find that

$H_6\kappa^2 - H_7\kappa + H_8 > 0$. As a result, $p^{'}$ is positive for $\kappa \geq \kappa_2$, hence $\hat{q}$ is decreasing on its interval.

■

Corollary 30

Proof. I start with $\theta > 0$:

$$Q^d \geq 0 \quad \Leftrightarrow \quad X_{-1}^0 + X_{-1}^1 + \frac{n}{n+1}\left(\frac{R}{\beta} - \bar{X}\right) \geq 0$$
$$\Leftrightarrow \quad X_{-1}^0 + X_{-1}^1 - \frac{n}{n+1}aX_{-1}^1 \geq \frac{n}{n+1}\frac{|R|}{\beta}, \text{ using } \bar{X} = aX_{-1}^1$$

With a small enough, $X_{-1}^0 + \left(1 - \frac{n}{n+1}a\right)X_{-1}^1 > 0$. Hence,

$$Q^d \geq 0 \;\Leftrightarrow\; \beta \geq \beta^h \equiv \frac{n}{n+1}\frac{|R|}{X_{-1}^0 + \left(1 - \frac{n}{n+1}a\right)X_{-1}^1}$$

If $\theta = 0$, we need to prove that $\beta^h \geq \beta_F$. Using the expression for β_F from Lemma 33, we get:

$$\beta^h \geq \beta_F \;\Leftrightarrow\; \frac{n+1}{n} - a \leq \frac{n+2}{n}a - \frac{n+1}{n} \;\Leftrightarrow\; \frac{2(n+1)}{n} \leq \frac{2(n+1)}{n}a$$

Since $a \geq 1$, this inequality is always satisfied. ■

Corollary 31

Proof. Using Proposition 22, we get:

$$E_0(p_1 - p_0) = D - \bar{p}_0 - \frac{\beta}{n+1}\bar{X} - \frac{|R|}{n+1} = \frac{n|R|}{n+1} - \frac{\beta}{n+1}\bar{X}$$

Thus $E_0(p_1 - p_0) \geq 0 \Leftrightarrow\; n|R| - \beta\bar{X} > 0 \;\Leftrightarrow\; \beta < \frac{n|R|}{\bar{X}}$. Since$\beta < \beta_F = \frac{n|R|}{(n+2)\bar{X} - (n+1)X_{-1}^1}$, and $\beta_F \leq \frac{n|R|}{\bar{X}} \;\Leftrightarrow\; X_{-1}^1 \leq \bar{X}$, we have $E_0(p_1) \geq p_0$. Clearly, $E_0(p_1 - p_0)$ increases with —R— and decreases with X_{-1}^1.

The illiquidity discount at time 1, $\Gamma_1 = -\frac{\beta\bar{X} + |R|}{n+1}$. Hence Γ_1 is decreasing in $\bar{X}$ and $|R|$.

■

3.6.4 Additional derivations for the no trading case

Proposition 18

Proof. From Proposition 22, the driver of the equilibrium is the position of a relative to $\max\left(m_2, \frac{1}{\kappa_2}\right)$ and $\min\left(m_2, \frac{1}{\kappa_2}\right)$.

If $X_{-1}^0 = 0$, then $\theta = 0$, and the equilibrium condition simplifies as follows:

- Since $\forall n \geq 2$, $m_2 > \frac{1}{\kappa_2}$ and since $\frac{1}{\kappa_2} \leq 1 \leq a$, the case $a < \min\left(m_2, \frac{1}{\kappa_2}\right)$ does not exist.
- Further, $\forall n \geq 2, m_2 \geq 1$, hence the case $\min\left(m_2, \frac{1}{\kappa_2}\right) < a < \max\left(m_2, \frac{1}{\kappa_2}\right)$ does not exist either.

The only remaining case is thus $a \geq \max\left(m_2, \frac{1}{\kappa_2}\right) = m_2$. Since $\frac{1}{\kappa_2} < m_2 \leq 1$ for all $n \geq 2$, the condition on a is always satisfied. Hence if $\theta = 0$, the equilibrium condition for the equilibrium with predatory trading is $\beta \in \left[\underline{\beta}_d \wedge \beta_F, \beta_F\right[$. ■

Proposition 20

Proof. The equilibrium with distress occurs on a non-empty interval iff $\underline{\beta}_d < \beta_F$. Using equations (3.80) and (3.62):

$$\begin{aligned} \underline{\beta}_d - \beta_F &= \frac{|R|}{\bar{X}} f(n,a) \\ \text{with } f(n,a) &= \frac{(u_1 - u_{2a})\left(\sqrt{\gamma_{3a}} - \gamma_{5a}\right) - 2\gamma_{6a}}{2\gamma_{6a}(u_1 - u_{2a})} \end{aligned} \tag{3.86}$$

Similarly, using equations (3.80) and (3.56), I get:

$$\begin{aligned} \underline{\beta}_d - \underline{\beta}_{nd} &= \frac{|R|}{\bar{X}} g(n,a) \\ \text{with } g(n,a) &= \frac{\lambda_2\left(\sqrt{\gamma_{3a}} - \gamma_{5a}\right) - a\gamma_{6a}\left(\sqrt{A_2} - \lambda_5\right)}{2\gamma_{6a}\lambda_2} \end{aligned} \tag{3.87}$$

The no-trading and predatory trading equilibria coexist iff $g(n,a) > 0$. ■

Lemma 34

Proof. We can recover Q^d from equation (3.13): $Q^d = \frac{n}{n+1}\frac{R}{\beta} - \frac{n}{n+1}\bar{X} + X_{-1}^1$. Using p_0^{nd} from Lemma 29, $p_0^{nd}\left(Q^{nd}, \bar{X}\right) = \bar{p}_0 \Leftrightarrow Q^{nd} = \frac{n+1}{n+2}\frac{R}{\beta} - \bar{X} + X_{-1}^1$. Thus

$$Q^{nd} \geq Q^d \Leftrightarrow \frac{1}{(n+1)(n+2)}\frac{R}{\beta} \geq \frac{1}{n+1}\bar{X}$$

The left-hand side is strictly negative, while the right-hand side is strictly positive. Hence $Q^{nd} < Q^d$. Further, note that since $\bar{X} > X_{-1}^1$, $Q^{nd} < 0$.

To understand this impact of the change in price schedule on the equilibrium conditions, I redo the analysis of Lemma 33 based on the no-distress price schedule, following identical steps. The prey's problem is

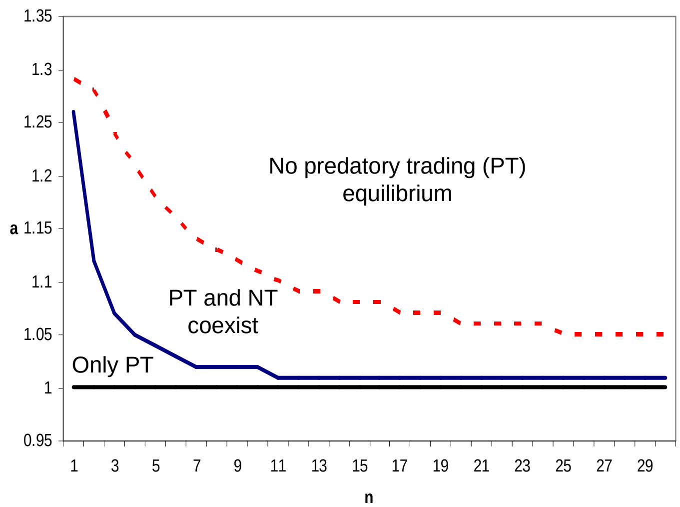
$$\max_{x_0^1, x_0^1 \leq \bar{X} - X_{-1}^1} B_{-1}^1 - x_0^1 \left[\bar{p}_0 - \beta \frac{n+2}{n+1} \left(\bar{X} - X_{-1}^1 - x_0^1\right)\right] + X_0^1 \left[D - \frac{\beta}{n+1} \left(\bar{X} - \frac{R}{\beta}\right)\right]$$

I write the Lagrangian of the problem and solve for the zero of the first-order condition (assuming the Lagrangian multiplier is 0). I get:

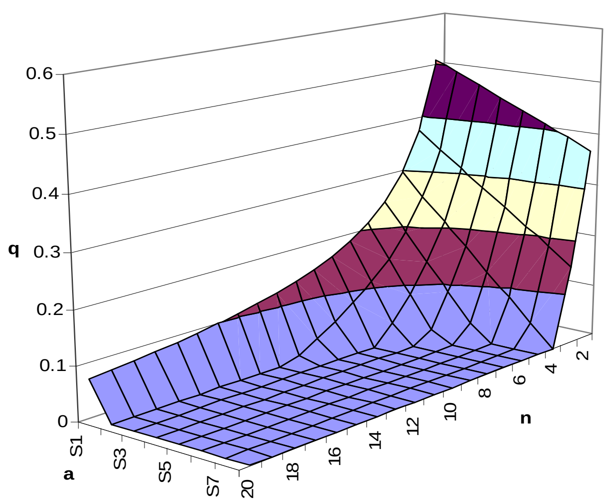
$$x_0^1 = \frac{n}{2\left(n+2\right)} \frac{|R|}{\beta} + \frac{1}{2} \left(\frac{n+1}{n+2} \bar{X} - X_{-1}^1\right)$$

Hence the constraint on the prey's position is not binding if $\frac{n}{2(n+2)} \frac{|R|}{\beta} + \frac{1}{2} \left(\frac{n+1}{n+2} \bar{X} - X_{-1}^1\right) \leq \bar{X} - X_{-1}^1$, which is equivalent to $\beta < \tilde{\beta}_F \equiv \frac{|R|}{\frac{n+3}{n} \bar{X} - \frac{n+2}{n} X_{-1}^1}$. In the proof of Lemma 33, I show that $\beta_F = \frac{|R|}{\frac{n+2}{n} \bar{X} - \frac{n+1}{n} X_{-1}^1}$, hence $\beta_F > \tilde{\beta}_F$.

Similarly, one can predict how the condition for ruling out self-fulfilling distress would change. Since predators have less price impact when the price schedule is p_0^{nd}, it will harder, conditional on distress, to trigger it, thus there should be a larger interval on which predatory trading is not self-fulfilling. In other words, $\tilde{\bar{\beta}}^d > \bar{\beta}^d$. ■

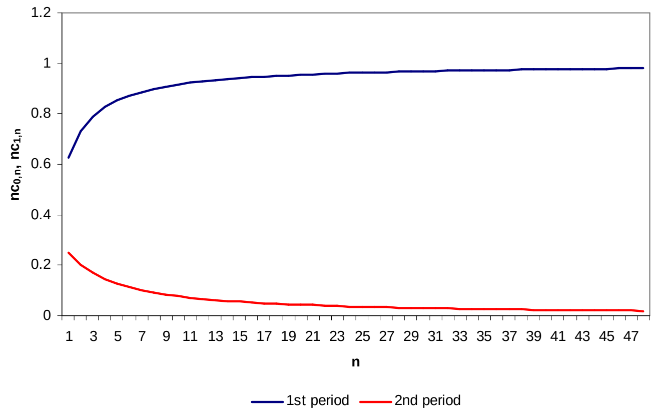


(a) Equilibria with or without distress

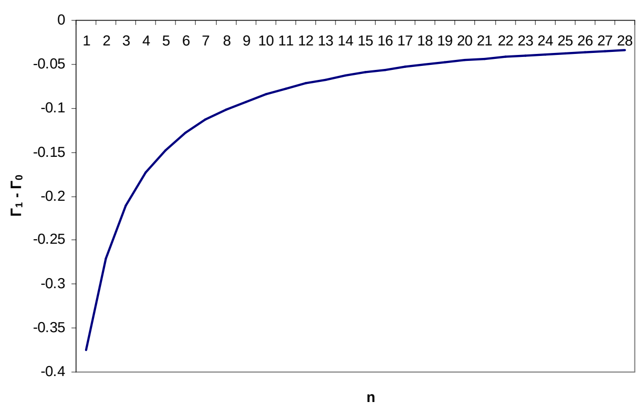


(b) "Net" probability of predatory trading

Figure 3.1: Coexistence of equilibria and "net" probability of predatory trading as a function of the number of predators n, and the prey's leverage capacity, $a = \frac{\bar{X}}{X_{-1}^1}$. In Panel (b), a varies from 1 (S1) to 1.07 (S7). The calculations assume that β is uniformly distributed between 0 and β_F.



(a) Equilibrium trades



(b) Convergence

Figure 3.2: Equilibrium trades and speed of convergence of the price towards the fundamental value of the asset as a function of the number of strategic traders in the no-distress equilibrium.

Bibliography

Acharya, V., and L. H. Pedersen (2005): "Asset Pricing with Liquidity Risk," *Journal of Financial Economics*, 77, 375–410.

Acharya, V., and S. Viswanathan (2011): "Leverage, Moral Hazard and Liquidity," *Journal of Finance*, 66, 99–138.

Allen, F. (2001): "Presidential Address: Do Financial Institutions Matter?," *Journal of Finance*, 56 (4), 1165–1175.

Allen, F., and G. Gorton (1992): "Stock Price Manipulation, Market Microstructure and Asymmetric Information," *European Economic Review*, 36, 624–630.

Allen, F., L. Litov, and J. Mei (2006): "Large Investors, Price Manipulation, and Limits to Arbitrage," *Review of Finance*, 10, 655–693.

Amihud, Y., and H. Mendelson (1986): "Asset Pricing and the Bid-Ask Spread," *Journal of Financial Economics*, 17, 223–249.

——— (2002): "Illiquidity and Stock Returns: Cross-Section and Time Series Effects," *Journal of Financial Markets*, 5, 31–56.

Amihud, Y., H. Mendelson, and L. H. Pedersen (2005): "Liquidity and Asset Prices," *Foundations and Trends in Finance*, 1, 269–364.

Attari, M., and A. Mello (2006): "Financially Constrained Arbitrage in Illiquid Markets," *Journal of Economic Dynamics and Control*, 30, 2793–2822.

Attari, M., A. Mello, and M. Ruckes (2005): "Arbitraging Arbitrageurs," *Journal of Finance*, 60(5), 2471–2511.

Basak, S. (1997): "Consumption Choice and Asset Pricing with Non-Price Taking Agents," *Economic Theory*, 10, 437–462.

BAUMOL, W. (1982): "Contestable Markets: An Uprising in the Theory of Industry Structure," *American Economic Review*, 72(1), 1–15.

BERGSTRESSER, D., R. C., AND S. SHENAI (2011): "Financial Guarantors and the 2007-2009 Credit Crisis," Harvard Business School, Working Paper, 2011.

BERNARDO, A., AND I. WELCH (2004): "Liquidity and Financial Market Runs," *Quarterly Journal of Economics*, pp. 135–158.

BERSTIMAS, D., AND A. LO (1998): "Optimal Control of Execution Costs," *Journal of Financial Markets*, 1, 1–50.

BIAIS, B., D. MORTIMORT, AND J. ROCHET (2000): "Competing Mechanisms in a Common Value Environment," *Econometrica*, 68(4), 799–837.

BOLTON, P., AND D. SCHARFSTEIN (1990): "A Theory of Predation Based on Agency Problems in Financial Contracting," *American Economic Review*, 80(1), 93–106.

BRADY, N., AND ET AL. (1988): *Report of the Presidential Task Force on Market Mechanisms*. Washington: U.S. Government Printing Office.

BRUNNERMEIER, M. (2009): "Deciphering the Liquidity and Credit Crunch 2007-2008," *Journal of Economic Perspectives*, 23, 77–100.

BRUNNERMEIER, M., S. NAGEL, AND L. H. PEDERSEN (2008): "Carry Trades and Currency Crashes," *NBER Macroeconomics Annual*, 23, 313–347.

BRUNNERMEIER, M., AND L. PEDERSEN (2005): "Predatory Trading," *Journal of Finance*, 60(4), 1825–1863.

——— (2009): "Market Liquidity and Funding Liquidity," *Review of Financial Studies*, 22(6), 2201–2238.

BULOW, J. (1982): "Durable Goods Monopolists," *Journal of Political Economy*, 90, 314–332.

CAI, F. (2009): "Trader Exploitation of Order Flow Information during the LTCM Crisis," *Journal of Financial Research*, XXXII(3), 261–284.

CARLIN, B., M. SOUSA-LOBO, AND S. VISWANATHAN (2007): "Episodic Liquidity Crises: Cooperative and Predatory Trading," *Journal of Finance*, 62(5), 2235–2274.

CGFS (1999): *A Review of Financial Market Events in Autumn 1998*. Committee on the Global Financial System, Bank for International Settlements, Basle, Switzerland.

Chen, J., S. Hanson, H. Hong, and J. Stein (2008): "Do Hedge Funds Exploit Mutual-Fund Distress?," Harvard Business School, Working Paper.

Chen, Z., W. Stanzl, and M. Watanabe (2002): "Price Impact Costs and the Limits of Arbitrage," Yale University Working Paper.

Chevalier, J., and G. Ellison (1997): "Risk Taking by Mutual Funds as a Response to Incentives," *Journal of Political Economy*, 105(6), 1167–1200.

Coase, R. (1972): "Durability and Monopoly," *Journal of Law and Economics*, 15, 143–149.

Comerton-Forde, C., T. Hendershott, C. Jones, P. Moulton, and M. Seasholes (2011): "Time Variation in Liquidity: The Role of Market-Maker Inventories and Revenues," *Journal of Finance*, 65(1), 295–331.

Coval, J., and E. Stafford (2007): "Asset Fire Sales (and Purchases) in Equity Markets," *Journal of Financial Economics*, 86(2), 479–512.

Dasgupta, A., and A. Prat (2006): "Financial Equilibrium with Career Concerns," *Theoretical Economics*, 1, 67–93.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990): "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance*, 45(2), 379–395.

DeMarzo, P., and B. Urosevic (2007): "Optimal Trading and Asset Pricing with a Large Shareholder," *Journal of Political Economy*, forthcoming.

Dennert, J. (1993): "Price Competition between Market-Makers," *Review of Economic Studies*, 60, 735–751.

Duffie, D. (2010): "Presidential Address: Asset Price Dynamics with Slow-Moving Capital," *Journal of Finance*, 65(4), 1237–1267.

Duffie, D., and B. Strulovici (2010): "Capital Mobility and Asset Pricing," Stanford GSB working paper.

Edelstein, R., A. Sureda-Gomill, B. Urosevic, and N. Wonder (2010): "Ownership Dynamics with Multiple Insiders: The Case of REITS," *Real Estate Economics*, 38 (1), 57–90.

Fardeau, V. (2011a): "Dynamic Strategic Arbitrage," LSE Working Paper.

Fardeau, V. (2011b): "Predatory Trading with Smart Investors: Run and Asymmetric Price Impact," London School of Economics Working Paper.

Foster, F. D., and S. Viswanathan (2011): "A Theory of Intraday Variations in Volumes, Variances and Trading Costs in Securities Markets," *Review of Financial Studies*, 4, 595–624.

Foucault, T., S. Moinas, and E. Theissen (2003): "Does Anonymity Matter in Electronic Limit Order Markets?," HEC, Working Paper.

Froot, K., and E. Dabora (1999): "How are Stock Prices Affected by the Location of Trade," *Journal of Financial Economics*, 53 (2), 189–216.

Fudenberg, D., and J. Tirole (1987): "Understanding Rent Dissipation: On the Use of Game Theory in Industrial Organization," *American Economic Review*, 77(2), 176–183.

Gabaix, X., A. Krishnamurthy, and O. Vigneron (2007): "Limits of Arbitrage: Theory and Evidence from the Mortgage-Backed Securities Market," *Journal of Finance*, 62(2), 557–595.

Garleanu, N., L. Pedersen, and A. Poteshman (2009): "Demand-Based Option Pricing," *Review of Financial Studies*, 22(10), 4259–4299.

Garleanu, N., and L. H. Pedersen (2011): "Margin-Based Asset Pricing and the Law of One Price," *Review of Financial Studies*, 24, 1980–2022.

Gorton, G., and A. Metrick (2010): "Securitized Banking and the Run on Repo," Yale School of Management, Working Paper.

Gromb, D., and D. Vayanos (2002): "Equilibrium and Welfare in Markets with Financially-Constrained Arbitrageurs," *Journal of Financial Economics*, 66, 361–407.

——— (2009a): "The Dynamics of Financially Constrained Arbitrage," LSE and INSEAD, Working Paper.

——— (2009b): "Financially Constrained Arbitrage and Cross-Market Contagion," LSE and INSEAD, Working Paper.

——— (2010): "Limits of Arbitrage: The State of the Theory," *Annual Review of Financial Economics*, 2, 251–275.

Grossman, S., and M. Miller (1988): "Liquidity and Market Structure," *Journal of Finance*, 38, 617–633.

Grossman, S., and J. Stiglitz (1980): "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70(3), 393–408.

Gul, F., H. Sonnenschein, and R. Wilson (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39.

He, Z., and A. Krishnamurthy (2011): "Intermediary Asset Pricing," Chicago Booth and Northwesthern University Working Paper.

——— (forthcoming): "A Model of Capital and Crises," *Review of Economic Studies.*

Hombert, J., and D. Thesmar (2009): "Limits of Limits of Arbitrage: Theory and Evidence," HEC Working Paper.

Hu X., J. P., and J. Wang (2011): "Noise as Information for Liquidity," Princeton and MIT working paper.

Karpoff, J. (1986): "A Theory of Trading Volume," *Journal of Finance*, 41(5), 1069–1087.

Kihlstrom, R. (2000): "Monopoly Power in Dynamic Securities Markets," Wharton Working Paper.

Kondor, P. (2009): "Risk in Dynamic Arbitrage: Price Effects of Convergence Trading," *Journal of Finance*, 64 (2), 638–658.

Krishnamurthy, A. (2002): "The Old Bond-New Bond Spread," *Journal of Financial Economics.*

Kyle, A. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53, 1315–1336.

——— (1989): "Informed Speculation With Imperfect Competition," *Review of Economic Studies*, 56, 317–356.

KYLE, A., AND S. VISWANATHAN (2008): "How to Define Illegal Price Manipulation," *American Economic Review, Papers and Proceedings*, 98(2), 274–279.

KYLE, A., AND W. XIONG (2001): "Contagion as a Wealth Effect," *Journal of Finance*, 56(4), 1401–1440.

KYRILLOS, B., AND P. TUFANO (1994): *Leland O'Brien Rubinstein Associates Incorporated: Portfolio Insurance*. Harvard Business Case.

LAMONT, O., AND R. THALER (2003): "Anomalies: The Law of One Price in Financial Markets," *Journal of Economic Perspectives*, 17 (4), 191–202.

LA'O, J. (2010): "Predatory Trading and Credit Freeze," Massachusettes Institute of Technology, Working Paper.

LEVIN, C., AND N. COLEMAN (2007): *Excessive Speculation in the Natural Gas Market*. Permanent Subcommittee on Investigations, US Senate.

LIU, J., AND F. LONGSTAFF (2011): "Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities," *Review of Financial Studies*, 17, 611–641.

LOEWENSTEIN, M., AND G. WILLARD (2011): "The Limits of Investor Behavior," *Journal of Finance*, 61, 231–258.

LOU, D. (2011): "A Flow-Based Explanation for Return Predictability," London School of Economics Working Paper.

LOU, D., H. YAN, AND J. ZHANG (2011): "Anticipated and Repeated Shocks in Liquid Markets," London School of Economics Working Paper.

LOWENSTEIN, R. (2000): *When Genius Failed: The Rise and Fall of Long-Term Management*. Random House, New York.

MARTIN, A., D. SKEIE, AND E. VON THADDEN (2011): "Repo Runs," FRB of New York Staff Report n. 44.

MITCHELL, M., L. H. PEDERSEN, AND T. PULVINO (2007): "Slow Moving Capital," *American Economic Review Papers and Proceedings*, forthcoming.

Mitchell, M., and T. Pulvino (forthcoming): "Arbitrage Crashes and the Speed of Capital," *Journal of Financial Economics*.

Mitchell, M. L., T. C. Pulvino, and E. Stafford (2002): "Limited Arbitrage in Equity Markets," *Journal of Finance*, 57, 551–584.

Oehmke, M. (2010a): "Gradual Arbitrage," Columbia University, Working Paper.

——— (2010b): "Liquidating Illiquid Collateral," Columbia Business School Working Paper.

Parida, S., and G. Venter (2009): "Financially Constrained Strategic Trading and Predation in Illiquid Markets," London School of Economics Working Paper.

Pastor, L., and R. Stambaugh (2003): "Liquidity Risk and Expected Stock Returns," *Journal of Political Economy*, 111, 642–685.

Pérold, A. (1999): *Long-Term Capital Management, L.P.* Harvard Business Case.

Perotti, E., and K. Spier (1993): "Capital Structure as a Bargaining Tool: the Role of Leverage in Contract Renegotiation," *American Economic Review*, 83(5), 1131–1141.

Pflueger, C., and L. Viceira (2011): "An Empirical Decomposition of Risk and Liquidity in Nominal and Inflation-Indexed Government Bonds," Harvard Business School Working Paper.

Plantin, G., and H. S. Shin (2011): "Carry Trades, Monetary Policy and Speculative Dynamics," Toulouse School of Economics and Princeton Working Paper.

Pritsker, M. (2009): "Large Investors: Implications for Equilibrium Asset Returns, Shock Absorption, and Liquidity," Finance and Economics Discussion Series, Federal Reserve Board, Washington, D.C.

Rostek, M., and M. Weretka (2011): "Dynamic Thin Markets," University of Wisconsin-Madison working paper.

Ruf, T. (2011): "Limits to Arbitrage and the Skewness Risk Premium in Options Market," UBC Working Paper.

Seasholes, M., and C. Liu (2011): "Trading Imbalances and the Law of One Price," HKUST Working Paper.

Shleifer, A. (1986): "Do Demand Curves for Stock Slope Down?," *Journal of Finance*, 41, 579–590.

Shleifer, A., and R. Vishny (1997): "The Limits of Arbitrage," *Journal of Finance*, 52, 35–55.

Stockey, N. (1981): "Rational Expectations and Durable Goods Pricing," *Bell Journal of Economics*, 12, 112–128.

Tufano, P. (1989): "Financial Innovation and First-Mover Advantages," *Journal of Financial Economics*, 25, 213–240.

Vayanos, D. (1999): "Strategic Trading and Welfare in a Dynamic Market," *Review of Economic Studies*, 66, 219–254.

——— (2001): "Strategic Trading in a Dynamic Noisy Market," *Journal of Finance*, 56, 131–171.

Vayanos, D., and J. Wang (forthcoming): "Liquidity and Asset Prices under Asymmetric Information and Imperfect Competition," *Review of Financial Studies*.

Weill, P. (2007): "Leaning Against the Wind," *Review of Economic Studies*, 70, 1329–1354.

Xiong, W. (2001): "Convergence Trading with Wealth Effects: an Amplification Mechanism in Financial Markets," *Journal of Financial Economics*, 62, 247–292.

Zigrand, J.-P. (2004): "A General Equilibrium Analysis of Stategic Arbitrage," *Journal of Mathematical Economics*, 40, 923–952.

——— (2006): "Endogenous Market Integration, Manipulation and Limits to Arbitrage," *Journal of Mathematical Economics*, 42 (3), 301–314.