The London School of Economics and Political Science

Essays on Asset Pricing

Cheng Zhang

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Abstract

This thesis contains three essays on asset pricing. The first chapter examines how introducing an options market affects the liquidity and expected returns of underlying assets when the economy features asymmetric information. I show that introducing derivatives can have opposite effects on underlying asset prices: doing so increases (resp., reduces) prices when the market has relatively more liquidity suppliers (resp., liquidity demanders). Thus the non-monotonic effects of derivatives on underlying assets could reconcile the mixed empirical evidence on options listing effects. Introducing derivatives reduces the price impact of liquidity demanders' trades on the underlying risky asset but has no effect on its price reversal dynamics. In the second chapter, I solve for the equilibrium of a pure-exchange Lucas economy under jump diffusion and populated by one unconstrained agent and one VaR agent in closed form. First, I show that the VaR constraint can generate excess market volatility and the inclusion of the jump component amplifies this effect, which provides a new mechanism to explain the prevalent smirk pattern of Black-Scholes implied volatility in options markets. Second, the VaR constraint pushes up the jump risk premium. Finally, the VaR constraint can generate a decline in the zero coupon bond yields at the VaR horizon, which is consistent with a flight to safety phenomenon taking place during a crisis. The third chapter, co-authored with Chunbo Liu and Zhiping Zhou, documents a positive relationship between funding liquidity and market liquidity in the options market. Further analysis reveals that the positive relationship is mainly driven by short-term and deep out-of-the-money options. Furthermore, liquidity of puts is more sensitive to changes in funding liquidity. In addition, this paper finds a positive relationship between the options market liquidity and VIX, which is in contrast to the negative relationship documented in the equity market.

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Chapter 1

The Effect of Options on Liquidity and Asset Returns

1.1 Introduction

Recent years have seen rapid development of the derivatives market.[1] The effect of these new assets on the whole financial market has led to heated discussion, and the topic has been of extreme importance to practitioners and regulators since the recent financial crisis.[2] Motivated by policy concerns, a large body of empirical work examines how the introduction of derivative assets, such as options, affects the underlying asset.

Most of this research has focused on how derivative securities affect the price level and volatility of underlying assets, and little is known about how they affect market liquidity.[3] Although a few theoretical studies have investigated the effect of derivatives on their underlying assets through the price discovery channel (Cao, 1999; Huang, 2015), the theoretical framework for these empirical findings is incomplete—especially with regard to the effect of derivatives on liquidity of the underlying securities.

[1]In the United States, the trading volume of individual stock options has grown exponentially from 5 million contracts in 1974 to more than 3,845 million contracts in 2014 (http://www.optionsclearing.com/webapps/historical-volume-query).

[2]Some regulators have argued that the complicated new derivatives products, which are designed to provide risk management and liquidity benefits to the financial system, produced exactly the opposite effect during the financial crisis of 2007 to 2010. In response to the 2007 credit crunch and the ensuing liquidity crisis, the Dodd-Frank Wall Street Reform and Consumer Protection Act became federal law, bringing significant changes to financial regulation of the derivative markets.

[3]See, for example, Damodaran and Lim (1991), Fedenia and Grammatikos (1992), and Kumar, Sabin, and Shastri (1998).

This paper examines how introducing financial derivatives affects the underlying asset's price level and liquidity. I find that the advent of options trading improves that liquidity and also the welfare of market participants, but it has a surprisingly non-monotonic effect on the asset's price. Further analysis reveals that the reported effects of derivatives on the asset's liquidity and price are dependent on the measures of illiquidity used and the factors driving asset-specific characteristics. In this regard, the paper offers new and comprehensive guidelines for empirical studies designed to analyze—via the liquidity channel—how securities are affected by the introduction of derivatives.

This study of how derivatives affect liquidity employs a rational expectations equilibrium (REE) model.[4] I start by considering an economy with three dates and two assets as the benchmark (Vayanos and Wang, 2012a,b).[5] Specifically, at dates 0 and 1, agents can trade a risk-free asset and a risky asset that pay off at date 2. At date 0, agents are identical and therefore no trade occurs. At date 1, agents can be one of two types: a *liquidity demander*, who at date 2 will receive a random endowment whose payoff is correlated with the risky asset's payoff; or a *liquidity supplier*, who will receive no such endowment. Only those agents who receive the random endowment observe the covariance between that endowment and the risky asset's payoff. Liquidity demanders can hedge the liquidity shock, which is modeled here as a random endowment, by trading with liquidity suppliers. Therefore, the existence of two agent types results in trade at date 1.[6] In addition, liquidity demanders receive private information about the risky asset's payoff before trade begins at date 1 whereas liquidity suppliers cannot distinguish the private signal from the demanders' liquidity shock.[7]

Next I introduce an options market into the economy. As in Cao and Ou-Yang (2009), the options market consists of a complete set of European call and put options written on the risky asset. At date 0, all agents know that options will be introduced into trade at date 1. Under this setup, I compare the ex ante price and measures of illiquidity of the risky asset before and after options are introduced—a comparison that quantifies the effects of this options market. To

[4]Because options have *no* effect on the underlying assets if the market is complete, competitive, and frictionless (Black and Scholes, 1973), the focus here is on the case of asymmetric information between liquidity demanders and suppliers.

[5]This setting is similar to that described by Grossman and Stiglitz (1980). The only difference is that Grossman and Stiglitz introduce exogenous noise traders whereas Vayanos and Wang (2012a,b) replace the noise traders with rational hedgers who are influenced by a liquidity shock.

[6]Endowment shocks have been modeled as a non-informational trading motive in different forms. See, for example, Diamond and Verrecchia (1981), Wang (1994), O'Hara (2003), and Vayanos and Wang (2012a,b).

[7]Qiu and Wang (2010) employ a more general framework to analyze the asset pricing implications of asymmetric information and endowment shocks.

examine market liquidity, I follow Vayanos and Wang (2012a,b) and consider two widely used empirical measures of illiquidity: price impact λ and price reversal γ.

I find that liquidity demanders who observe the private signal take short positions in the introduced options, whereas liquidity suppliers who learn information through asset prices take long positions in those options. The payoff structure is such that options may yield hedging benefits for the second moment of the risky asset's payoff. Since liquidity demanders have more precise information, they always have less incentive than suppliers to hedge against such volatility. Given that options are in zero net supply, liquidity demanders (resp. suppliers) take short (resp. long) positions in options. Moreover, the trading volume of options increases with the information dispersion across agents because widely dispersed information is followed by high demand for options.

In this paper I establish that options provide hedging benefits and increase risk sharing between liquidity demanders and liquidity suppliers. In my model, liquidity demanders trade to hedge and/or exploit their information. Before options are introduced, uninformed liquidity suppliers are unable to distinguish between these two motives, which reduces the incentive to trade with liquidity demanders. Yet when financial derivatives are available, liquidity suppliers can hedge against uncertainty in the risky asset's payoff and so are more willing to accommodate the trades of informed demanders. Because of this options-enabled increase in risk sharing, their introduction increases all agents' utilities. In other words, each market participant's welfare has improved at date 0.[8]

However, the effects of options on liquidity demanders and suppliers are not symmetric. Intuitively, the relative benefit from more risk-sharing opportunities is determined by the competition within each group. For example, if the market consists mostly of liquidity suppliers then competition within that group is intense; hence the welfare improvement is greater for liquidity demanders than for liquidity suppliers. These results are reversed when the market is dominated by liquidity demanders. Although both agent types benefit when options are introduced, the magnitude of that benefit differs by type and this difference affects the trading incentives of agents at date 0.

More importantly, I show that introducing derivatives has (surprisingly) non-monotonic ef-

[8] The so-called noisy REE models usually introduce exogenous noise trading into the economy. That approach complicates welfare analysis because one cannot compute the welfare of noisy traders. My model replaces noisy traders with rational hedgers whose utility can be calculated, which enables a welfare analysis of the effects of introducing derivatives.

fects on the underlying asset prices—a finding that could reconcile, at least in part, conflicting evidence on options listing and its effect on the underlying asset.[9] When the market has many more liquidity suppliers than demanders, the benefits stemming from options are greater for the latter than the former. At date 0, the identical investor who can anticipate the effects of options is thus less worried about liquidity shock. Hence this investor is more willing to hold the risky asset at date 0, which increases that asset's ex ante price. The opposite effect is observed when the agent population consists mostly of liquidity demanders. The mechanism is robust to derivatives with general payoff structures.

Further analysis shows that the effects of derivatives on the price of their underlying asset are sensitive to the illiquidity measures used and to the particular factors that drive the asset-specific characteristics. For example, if illiquidity is measured by price impact λ and if the cross-sectional variation in λ is driven by the private signal's precision, then introducing options will lower (resp. raise) the prices of stocks that are relatively more (resp. less) liquid. Yet if illiquidity is measured by price reversal γ then the opposite dynamic is observed. When the cross-sectional variation in different illiquidity measures is due to the liquidity shock's precision, one observes different effects of options on the underlying asset's price. These novel implications, which concern how options affect their underlying assets, can be tested empirically.

I also find that introducing an options market reduces the price impact of liquidity demanders' trades on the underlying risky asset—but that it has no effect on price reversal. Options expand the scope of risk sharing between liquidity demanders and suppliers and reduce the price effect per trade captured by λ. Because the introduction of options increases the trade size, the effect of options on overall trade, which is measured by γ, does not change. Furthermore, the liquidity improvement of more liquid (low-λ) stocks is less than that of less liquid (high-λ) stocks. When information becomes more asymmetric, the consequent adverse selection is more severe and so the price impact λ is greater. In other words, high-λ stocks are more subject to information asymmetry. The hedging benefit provided by options is greater for stocks that are more likely to be subject to asymmetric information. That is, the improvement in liquidity is greater for stocks with high λ than for stocks with low λ. The converse effect is observed when illiquidity is instead measured by γ because asymmetric information can *reduce* γ.[10]

[9] Several empirical studies document that an options listing reduces the underlying asset's price (e.g., Conrad, 1989; Detemple and Jorion, 1990; Skinner, 1989), although a few find just the opposite (e.g., Sorescu, 2000; Mayhew and Mihov, 2000).

[10] In the model, price reversal γ captures the importance to price of liquidity shocks. When the private signal

In order to endogenize further the participation decisions of agents, I examine a scenario in which they must pay a participation fee to enter the market at date 1; this fee can be interpreted as learning costs or opportunity costs, much as in Huang and Wang (2009, 2010). A participation fee reduces the participation of liquidity suppliers, which changes the proportion of liquidity suppliers in the population as a whole.[11] I find that the introduction of an options market always reduces the illiquidity discount in the ex ante price and also reduces the expected return of the underlying asset—that is, irrespective of the proportion of liquidity demanders. When options are introduced, more liquidity suppliers are willing to enter the market and to accommodate the trades of liquidity demanders; the result is a decline in the illiquidity discount that a demander would normally require. Moreover, both illiquidity measures decrease after derivatives are introduced, which contrasts to the case *without* costly participation. The reason is that an increased proportion of liquidity suppliers facilitates risk sharing and thus lowers both price impact λ and price reversal γ. Finally, the model predicts that options trading volume is not a monotonic function of the participation cost; rather, it should exhibit an inverse U shape with respect to that cost.

Related Literature. My study is related to the literature that addresses market frictions and liquidity.[12] Most studies on REE with asymmetric information focus on price informativeness. In contrast, market liquidity is the focus of most literature on strategic trading and sequential trading. For example, Biais and Hillion (1994) investigate how derivatives affect market liquidity in a strategic trading model. Vayanos and Wang (2012a,b) take a first step in analyzing how imperfections such as asymmetric information affect ex ante prices and market liquidity in REE models.[13] Inspired by Vayanos and Wang (2012a,b), this paper is the first study of how—in a competitive market—the introduction of an options market affects both the returns and market liquidity of the underlying assets.

This paper contributes also to the theoretical literature on the effects of derivative securities on underlying assets (Grossman, 1988; Back, 1993; Biais and Hillion, 1994; Brenna and Cao,

is extremely precise (i.e., when information asymmetry is severe), the effect of the liquidity shock captured by γ is rather limited.

[11]Because it is the liquidity demanders who face the risk of liquidity shock, they stand to benefit more from participation than do suppliers. Recall also that it is the *relative* measure of participating suppliers and demanders that matters in this model. Thus I limit the analysis to an equilibrium under which liquidity demanders fully participate and liquidity suppliers partially participate (cf. Vayanos and Wang, 2012b).

[12]The literature is vast; Vayanos and Wang (2012b) provide a comprehensive review.

[13]Several recent papers examine the effect of asymmetric information on expected returns (see e.g. O'Hara, 2003; Easley and O'Hara, 2004; Garleanu and Pedersen, 2004; Qiu and Wang, 2010).

1996; Huang and Wang, 1997; Cao, 1999; Chabakauri, Yuan, and Zachariadis, 2014; Huang, 2015; Malamud, 2015). The most closely related to mine is the work of Biais and Hillion (1994), who show that options can help prevent breakdowns caused by the adverse selection problem in a noncompetitive market and thereby make the market more liquid. In Biais and Hillion (1994), market makers are risk neutral and the expected returns on assets are always equal to the risk-free rate. However, the analysis here is conducted in a competitive market and yields empirical implications for how the expected returns of underlying assets are affected by options listing. Brenna and Cao (1996) incorporate a quadratic option into a noisy rational expectations model and find that the derivative allows agents to achieve a Pareto-efficient allocation. However, they find that the underlying asset's price is *not* affected by the option. There are, in addition, several other studies that investigate the effects of derivatives on the underlying asset when the acquisition of information is endogenous. Following Grossman and Stiglitz (1980) and Hellwig (1980), Cao (1999) and Huang (2015) focus on the information acquisition channel and on how introducing options affects the price of risky assets through price informativeness. In contrast, I focus on market liquidity and show that the ex ante price of the risky assets is indeed affected by options—even in the absence of an information acquisition channel (i.e., even if the price informativeness remains unchanged). Furthermore, this paper is one of only a few studies that introduce a set of explicit options into an economy with asymmetric information. Those works include Chabakauri, Yuan, and Zachariadis (2014), who generalize the distribution of asset payoffs, Huang (2015), who focuses on endogenous information acquisition, and Malamud (2015), who examines price discovery under general preferences. My paper is also associated with the strand of literature on financial innovation (Allen and Gale, 1994; Duffie and Rohi, 1995; Dow, 1998; Brock, Hommes, and Wagener, 2009; Dieckmann, 2011; Simsek, 2013a,b; Chabakauri, Yuan, and Zachariadis, 2014).

The rest of the paper is organized as follows. Section 1.2 sets up the model and presents the benchmark case with asymmetric information but without options, and Section 1.3 and 1.4 examine what happens when derivative assets are introduced into the economy. Section 1.5 extends the model further by considering costly participation, after which Section 1.6 discusses some more general derivatives. Section 1.7 concludes. All proofs are given in the appendices.

1.2 Model

1.2.1 Economy

Given the unified framework of liquidity with one risk-free asset and one risky asset as proposed by Vayanos and Wang (2012a,b), I introduce an options market into the economy. To examine the subsequent effects, I compare the price and two measures of the risky asset's illiquidity before and after options are introduced. Before solving in Section 1.4 the equilibrium for the economy with options, I solve the equilibrium in the absence of an options market.

Timeline and Assets. As the benchmark, I consider an economy with three dates, $t = 0, 1, 2$, and two assets. The risk-free asset is in a supply of b shares and pays off one unit of a consumption good at date 2. The supply of the risky asset is $\bar{X} > 0$ shares, each of which pays off D units of the consumption good, where $D \sim N(\bar{D}, 1/h)$. The price of the risky asset is denoted by P_t at date t. With the risk-free asset as numéraire, the price of the risky asset at date 2 is equal to D; that is, $P_2 = D$.

There is a measure one of investors whose utility function over consumption follows a negative exponential utility function with absolute risk aversion coefficient α:

$$-\exp(-\alpha C_2), \tag{1.2.1}$$

where C_2 is the consumption at date 2. All investors are identical at date 0 and are endowed with the per capita supply of both the risk-free asset and the risky asset. Then they become heterogeneous, and that heterogeneity generates trade at date 1. At date 2, all asset payoffs are realized and all investors consume their total wealth.

Investors and Asymmetric Information. At date $\frac{1}{2}$, investors' types are realized. At this interim date, investors learn whether or not they will receive an extra endowment at date 2. A proportion π of these agents encounter the liquidity shock of receiving an additional endowment $z(D - \bar{D})$ of the consumption good at date 2; the remaining proportion $1 - \pi$ of agents receive no extra endowment. Only those who receive the endowment observe the liquidity shock z. Because the endowment received by the proportion π of agents is correlated with D, the agents have an incentive to hedge against the risk exposure induced by that liquidity shock.[14] These

[14] Like Vayanos and Wang (2012a), for simplicity I assume that the endowment is perfectly correlated with the payoff D. If the correlation is not perfect then the results remain qualitatively similar, since the crux of this study is a nonzero correlation.

agents are referred to as liquidity demanders because trades are initiated by their hedging against the liquidity shock. When, for example, a positive endowment shock occurs (i.e., z is positive), liquidity demanders are willing to sell the risky asset to hedge against the risk that the payoff D will be low. The remaining agents accommodate this hedging demand and are referred to as liquidity suppliers. The liquidity shock z is normally distributed with mean zero and precision n—that is, $z \sim N(0, 1/n)$—and is independent of D.[15] In addition to the liquidity shock, liquidity demanders receive a private signal s about the risky asset payoff D before trading at date 1.[16] The signal is

$$s = D + \epsilon, \tag{1.2.2}$$

where ϵ is normally distributed with mean zero and precision m (i.e., $\epsilon \sim N(0, 1/m)$) and is independent of (D, z). To simplify the analysis, I assume that all liquidity suppliers are uninformed about the private signal s; the consequent information asymmetry serves as a benchmark for the following studies.

At date 1, liquidity demanders and liquidity suppliers trade with each other on the basis of their information and the liquidity shock. In this context, liquidity demanders have two trading motives: speculating (informational) and hedging (non-informational). Specifically, demanders can extract profit from the private signal they observe and also wish to hedge against the random endowment they will encounter at date 2. The inability of liquidity suppliers to distinguish between these two motives of demanders reduces the former's incentive to trade with the latter. As shown by Vayanos and Wang (2012a,b), an illiquidity discount in the price of the risky asset at date 0 (i.e., P_0) arises in response to a liquidity shock and is magnified when the economy features asymmetric information.

1.2.2 Asymmetric Information Benchmark without an Options Market

Here I recall the results of Vayanos and Wang (2012a,b) obtained under asymmetric information. It is the benchmark to be used for comparisons with the setting in which an options market has been introduced.

I follow Vayanos and Wang (2012a,b) in first solving the equilibrium at date 1 and then

[15] The endowment $z(D - \bar{D})$ can take large negative values under the normal distribution, which could yield an infinitely negative expected utility. Similarly to Vayanos and Wang (2012a), I assume that the precisions of D and z satisfy $\alpha^2 < nh$ to ensure that utility is finite.

[16] Without loss of generality, I assume that all liquidity demanders are informed. Even if they do not receive the private signal, they can perfectly infer it from the price because they observe the liquidity shock.

working backward to obtain the equilibrium at date 0. Following the model setup, liquidity demanders observe the signal s and know their liquidity status z. As a result, their information comprises the private signal s, the liquidity shock z, and the prices P_0 and P_1; formally, $\mathcal{F}_d = \{s, z, P_0, P_1\}$. At the same time, liquidity suppliers cannot observe the private signal and so their only information is the prices themselves. Hence the information set of these suppliers is $\mathcal{F}_s = \{P_0, P_1\}$. All investors formulate their demand functions conditional on their respective information sets, and the equilibrium price clears the market. I shall denote by X_d and X_s the demand of liquidity demanders and liquidity suppliers (respectively) for the risky underlying asset.

In this study, I assume a linear price function and conjecture that the risky asset's price is a linear function of the signal s and the liquidity shock z:

$$P_1 = A + B(s - \bar{D} - Cz), \tag{1.2.3}$$

where A, B, C are constants. The expectations of liquidity demanders are such that the conditional mean and variance of the risky asset payoff D are[17]

$$\mathrm{E}[D|\mathcal{F}_d] = \bar{D} + \beta_s(s - \bar{D}) \quad \text{and} \quad \mathrm{Var}[D|\mathcal{F}_d] = \frac{1}{h+m}, \tag{1.2.4}$$

where $\beta_s = \frac{m}{h+m}$. At date 1, liquidity demanders maximize their expected utility over the wealth at date 2, $W_{d2} = W_1 + X_d(D - P_1) + z(D - \bar{D})$,[18] and then submit a demand schedule for the risky asset as follows:

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \,\mathrm{Var}[D|\mathcal{F}_d]} - z. \tag{1.2.5}$$

As this equation shows, the demand schedule of liquidity demanders is affected by the liquidity shock z. For instance, if a positive liquidity shock occurs then demanders will attempt to sell the risky asset because they are overly exposed to the risk that the payoff D will be low.

In contrast to liquidity demanders, liquidity suppliers are unable to observe the private signal; for this reason, they can learn about D only by observing the price of the risky asset.

[17] Given that the price P_1 is a coarser indicator than the signal s, no additional information is conveyed by the former than the latter. Furthermore, the liquidity shock z is independent of D and so is not used to compute the conditional expectation and variance of D. It follows that, in the setting without options, we have $\mathrm{E}[D|\mathcal{F}_d] = \mathrm{E}[D|s]$, $\mathrm{Var}[D|\mathcal{F}_d] = \mathrm{Var}[D|s]$, $\mathrm{E}[D|\mathcal{F}_s] = \mathrm{E}[D|P_1]$, and $\mathrm{Var}[D|\mathcal{F}_s] = \mathrm{Var}[D|P_1]$.

[18] Since investors are identical at date 0, it follows that the wealth of a liquidity demander and of a liquidity supplier are the same at date 1: $W_1 = W_{d1} = W_{s1} = W_0 + X_0(P_1 - P_0)$. The wealth of a liquidity supplier at date 2 is $W_{s2} = W_1 + X_s(D - P_1)$, which can be derived from W_{d2} by setting $z = 0$.

Conditional on their information set $\mathcal{F}_s$, the payoff D is normal with mean and variance

$$\mathrm{E}[D|\mathcal{F}_s] = \bar{D} + \beta_P(s - \bar{D} - Cz) \quad \text{and} \quad \mathrm{Var}[D|\mathcal{F}_s] = \frac{1}{h+q}, \tag{1.2.6}$$

respectively; here $\beta_P = \frac{q}{h+q}$ and $q = \left(\frac{1}{m} + C^2\frac{1}{n}\right)^{-1}$. Similarly, liquidity suppliers maximize their expected utility and submit a demand schedule for the risky asset as follows:

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha\,\mathrm{Var}[D|\mathcal{F}_s]}. \tag{1.2.7}$$

After both groups of investors submit their demand schedules, the equilibrium price clears the market and thereby equates investors' aggregate demands and the asset supply $\bar{X}$:

$$\pi X_d + (1-\pi)X_s = \bar{X}. \tag{1.2.8}$$

The equilibrium price of the risky asset is affine in both the private signal and the liquidity shock, per (1.2.3), and the coefficients A, B, C can be written as

$$A = \bar{D} - \frac{\alpha}{h + \pi m + (1-\pi)q}\bar{X}, \quad B = \frac{\pi m + (1-\pi)q}{h + \pi m + (1-\pi)q}, \quad C = \frac{\alpha}{m}. \tag{1.2.9}$$

Substituting the agents' demands into the exponential utility function yields their expected utilities at date 1. The expected utility of a liquidity supplier can be calculated as

$$-\exp\left\{-\alpha\left[W_0 + X_0(P_1 - P_0) + \frac{[\bar{D} + \beta_P(s - \bar{D} - Cz) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_s]}\right]\right\}. \tag{1.2.10}$$

Similarly, one can calculate the expected utility of a liquidity demander as

$$-\exp\left\{-\alpha\left[W_0 + X_0(P_1 - P_0) + z(P_1 - \bar{D}) + \frac{[\bar{D} + \beta_s(s - \bar{D}) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_d]}\right]\right\}. \tag{1.2.11}$$

The expected utility at date 1 is affected not only by the private signal s but also by the liquidity shock z—given that the price P_1 depends on both s and z. I denote by U_s and U_d the expected utilities of (respectively) liquidity suppliers and demanders at the interim date $\frac{1}{2}$, which are the expectations of (1.2.10) and (1.2.11) over (s, z). These interim utilities can be used to derive the identical investor's expected utility at date 0 as the weighted average of suppliers' and demanders' utilities:

$$U \equiv \pi U_d + (1-\pi)U_s. \tag{1.2.12}$$

At date 0, the identical investor chooses X_0 to maximize utility. Then the ex ante price P_0 clears the market and we have $X_0 = \bar{X}$ in equilibrium. As shown in Vayanos and Wang (2012a,b), the following linear equilibrium price exists at date 0:

$$P_0 = \bar{D} - \alpha\frac{1}{h}\bar{X} - \frac{\pi M}{1-\pi+\pi M}\Delta_1\bar{X}, \tag{1.2.13}$$

where

$$\Delta_0 = \frac{(B-\beta_P)^2\left(\frac{1}{h}+\frac{1}{q}\right)}{\pi^2\,\mathrm{Var}[D|\mathcal{F}_s]},$$

$$\Delta_1 = \frac{\alpha^3 B\frac{1}{nh}\left(\frac{1}{h}+\frac{1}{m}\right)}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}},$$

$$\Delta_2 = \frac{\alpha^3\frac{1}{nh^2}\left[1+\frac{(B-\beta_s)^2\left(\frac{1}{h}+\frac{1}{m}\right)}{\mathrm{Var}[D|\mathcal{F}_d]}\right]}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}},$$

and

$$M = \exp\left(\frac{1}{2}\alpha\Delta_2\bar{X}^2\right)\sqrt{\frac{1+\pi^2\Delta_0}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}}}. \tag{1.2.14}$$

The ex ante price P_0 consists of three terms: the expected payoff $\bar{D}$; the risk premium, which compensates investors for the risk they bear; and the illiquidity discount. That discount is itself the product of two other terms. The first of these, $\frac{\pi M}{1-\pi+\pi M}$, can be interpreted as the risk-neutral probability of being a liquidity demander; π is the true probability and M is the ratio of marginal utilities of demanders to those of suppliers. The illiquidity discount's second term, $\Delta_1\bar{X}$, is the discount required if one is certain to be a liquidity demander.

To examine market liquidity, I consider two widely used empirical measures of illiquidity. The first one is based on the price impact of trades by liquidity demanders. More specifically, it is the coefficient derived by regressing the price change (between date 0 and date 1) on the signed volume of liquidity demanders at date 1:

$$\lambda \equiv \frac{\mathrm{Cov}[P_1-P_0, \pi(X_d-\bar{X})]}{\mathrm{Var}[\pi(X_d-\bar{X})]}. \tag{1.2.15}$$

A large λ indicates that trades have a strong price impact, which implies that the market is illiquid. The second measure is based on the autocovariance of price changes between two periods:

$$\gamma \equiv -\text{Cov}[P_2 - P_1, P_1 - P_0], \tag{1.2.16}$$

which is referred to as price reversal. The price reversal measure γ captures the price deviation from fundamental value that a liquidity supplier requires to absorb a liquidity shock. When γ is high, trades generate substantial price deviation and the market is illiquid. In the case of asymmetric information *without* options, these two illiquidity measures are calculated as follows:

$$\lambda = \frac{\alpha \, \text{Var}[D|\mathcal{F}_s]}{(1-\pi)(1-\frac{\beta_P}{B})}; \tag{1.2.17}$$

$$\gamma = B(B-\beta_P)\left(\frac{1}{h}+\frac{1}{q}\right). \tag{1.2.18}$$

The foregoing results on the risky asset's price, the price impact λ, and the price reversal γ follow Vayanos and Wang (2012a,b). These results will be the benchmark for subsequent analysis for the effects of derivatives. In the next section I explore how the introduction of derivative assets affects the equilibrium. Then, in Section 1.5, I account for costly participation and investigate how derivatives affect the underlying asset in an economy with both asymmetric information and participation costs.

1.3 Introduction of A Squared Contract

Before solving in Section 1.4 for the economy with an options market, I first introduce a squared contract whose payoff at date 2 is D^2, where D is the risky asset's payoff, into the economy. In this section, I investigate the impact of the squared contract on asset prices, market illiquidity and the welfare of market participants.

1.3.1 Equilibrium

At date 0, all agents know that the squared contract will be introduced to trade at date 1 and its price at date 1 is denoted by P_{SC}. The net supply of the squared contract is zero. The demands of liquidity demanders and suppliers for this derivative asset are denoted by $X_{d,SC}$ and $X_{s,SC}$. Liquidity demanders and suppliers submit their demand schedules conditional on their new information sets after this derivative is introduced. Specifically, the information set of liquidity demanders at date 1 is $\mathcal{F}_d = \{s, z, P_0, P_1, P_{SC}\}$ whereas that of liquidity suppliers is $\mathcal{F}_s = \{P_0, P_1, P_{SC}\}$. Equipped with the squared contract, the wealth of liquidity demanders at date 2 is given by

$$W_{d2} = W_1 + X_d(D - P_1) + z(D - \bar{D}) + X_{d,SC}(D^2 - P_{SC}), \tag{1.3.1}$$

and the wealth of liquidity suppliers is given by

$$W_{s2} = W_1 + X_s(D - P_1) + X_{s,SC}(D^2 - P_{SC}). \tag{1.3.2}$$

If the derivative asset is redundant and is not traded by investors, then the wealth calculated by (1.3.1) and (1.3.2) is the same as that in the absence of derivatives. Once a squared contract is introduced, there is a partially revealing rational expectations equilibrium.[19] The equilibrium at date 1 with the squared contract is closely related to Brenna and Cao (1996) who introduce a quadratic option that pays off $(D - P_1)^2$.

Proposition 1.3.1. *At date 1, there exists one equilibrium. The underlying risky asset's price is given by*

$$P_1 = \bar{D} - \frac{\alpha}{G}\bar{X} + \frac{\pi m + (1-\pi)q}{G}\left(s - \bar{D} - \frac{\alpha}{m}z\right), \tag{1.3.3}$$

and the price of the squared contract is given by

$$P_{SC} = P_1^2 + \frac{1}{G}. \tag{1.3.4}$$

The liquidity demander's demands for the risky asset and the derivative are

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_d]} - \frac{G - G_d}{\alpha}P_1 - z, \tag{1.3.5}$$

$$X_{d,SC} = \frac{G - G_d}{2\alpha}; \tag{1.3.6}$$

the liquidity suppler's demands for the risky asset and the derivative are

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_s]} - \frac{G - G_s}{\alpha}P_1, \tag{1.3.7}$$

$$X_{s,SC} = \frac{G - G_s}{2\alpha}. \tag{1.3.8}$$

[19]The uniqueness of the equilibrium is left for future research.

In (1.3.3)–(1.3.8), $G_d \equiv \frac{1}{\mathrm{Var}[D|\mathcal{F}_d]} = h + m$, $G_s \equiv \frac{1}{\mathrm{Var}[D|\mathcal{F}_s]} = h + q$, *and* $G \equiv \pi G_d + (1-\pi)G_s = h + \pi m + (1-\pi)q$. *The terms* G_d *and* G_s *represent the conditional precision of* D *for liquidity demanders and suppliers, and* G *denotes the average precision for all investors.*

There are several interesting features of the equilibrium at date 1. First, the squared contract is not redundant and is traded by investors. In equilibrium, liquidity demanders take short positions in derivatives whereas liquidity suppliers take long positions. Given its payoff structure, the squared contract can provide hedging/speculation benefit for the volatility of the risky asset's payoff. Since liquidity demanders have more precise information, they always have less incentive than suppliers to hedge against the volatility. Because the squared contract is in zero net supply, liquidity demanders sell the squared contract to earn profit while liquidity suppliers take long positions to hedge against the volatility. The factor that makes the squared contract nonredundant is the difference in opinions of uncertainty caused by heterogeneous information, which is clearly shown by (1.3.6) and (1.3.8). Second, introducing derivatives into an economy with asymmetric information has no direct effect on the underlying asset's equilibrium price at date 1, which is in line with the result reported in Brenna and Cao (1996). Third, the price of the squared contract is a function of the risky asset's price but carries no additional information; hence its introduction is considered to be "informationally redundant" (Chabakauri, Yuan, and Zachariadis, 2014).

Before the squared contract is available to trade, the wealth of liquidity demanders at date 2 is given by

$$W_{d2} = W_1 + \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \mathrm{Var}[D|\mathcal{F}_d]}(D - P_1) + z(P_1 - \bar{D}), \tag{1.3.9}$$

and the wealth of liquidity suppliers is given by

$$W_{s2} = W_1 + \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \mathrm{Var}[D|\mathcal{F}_s]}(D - P_1). \tag{1.3.10}$$

The marginal rate of substitution of investor i, $i \in \{d, s\}$, between wealth contingent on $D = D_h$

and $D = D_l$ is given by

$$M^i_{hl} = \frac{\exp\big\{-(D_h - \mathrm{E}[D|\mathcal{F}_i])^2 G_i/2\big\}\exp\{-\alpha W_{2i}(D_h)\}}{\exp\big\{-(D_l - \mathrm{E}[D|\mathcal{F}_i])^2 G_i/2\big\}\exp\{-\alpha W_{2i}(D_l)\}}$$
$$= \exp\bigg\{-\frac{1}{2}G_i(D_h - D_l)(D_h + D_l - 2\mathrm{E}[D|\mathcal{F}_i]) - \frac{\mathrm{E}[D|\mathcal{F}_i] - P_1}{\mathrm{Var}[D|\mathcal{F}_i]}(D_h - D_l)\bigg\}$$
$$= \exp\bigg\{-\frac{1}{2}G_i(D_h - D_l)(D_h + D_l - 2P_1)\bigg\}. \tag{1.3.11}$$

As shown by (1.3.11), the marginal rate of substitution is investor specific through G_i, the conditional precision of the investor's posterior beliefs. Therefore, the equilibrium in the economy without the squared contract is not Pareto efficient. Proposition 1.3.2 establishes that the introduction of the squared contract allows agents to achieve a Pareto efficient allocation.[20]

Proposition 1.3.2. *The Pareto efficient allocation is achieved with the introduction of a squared contract.*

Intuitively, the introduced squared contract provides more instruments for hedging volatility which cannot be achieved by merely trading the risky stock, and hence improves the overall allocational efficiency. When the squared contract is available to trade, the market is complete and the Pareto efficient allocation is achieved.

Lemma 1.3.1. *At interim date $t = \frac{1}{2}$, if a squared contract is available then the utilities of liquidity demanders are given by*

$$U_d = \frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}}\mathrm{E}\bigg\{-\exp\bigg[-\alpha\bigg(W_0 + X_0(P_1 - P_0) + z(P_1 - \bar{D}) + \frac{[\bar{D} + \beta_s(s - \bar{D}) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_d]}\bigg)\bigg]\bigg\} \tag{1.3.12}$$

and the utilities of liquidity suppliers are given by

$$U_s = \frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}}\mathrm{E}\bigg\{-\exp\bigg[-\alpha\bigg(W_0 + X_0(P_1 - P_0) + \frac{[\bar{D} + \beta_P(s - \bar{D} + \frac{\alpha}{m}z) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_s]}\bigg)\bigg]\bigg\}. \tag{1.3.13}$$

For $\pi \in (0,1)$ we have that both $\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}} < 1$ and $\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}} < 1$, where E is the expectation

[20] The notion of Pareto efficiency here is ex post. That is, Pareto efficiency is conditional on P_1 and realized random endowments z, where expected utility is calculated using investors' posterior beliefs.

over (s, z).

In the presence of a squared contract, the interim utilities of liquidity demanders and suppliers are the product of two terms: one reflecting the impact of introducing derivative asset and one capturing the interim utilities of agents *before* the derivative is introduced (see Appendix 1.9.1). Introducing a squared contract improves the utilities of both groups of agents thanks to the improved risk sharing that it allows. More importantly, the effects of the financial derivative on the interim utilities of the two groups of agents are not the same: the value of the extra term—which appears when the squared contract is incorporated into the model—depends on the conditional precision of the payoff D, which differs between liquidity demanders and liquidity suppliers. Lemma 1.3.2 formally characterizes this distinction.

Lemma 1.3.2. *Let $\pi^* \in (0, 1)$ be as defined in the Appendix. Then the following statements hold:*

(1) *if $0 < \pi < \pi^*$, then the welfare gain induced by a squared contract is greater for liquidity demanders than for liquidity suppliers;*

(2) *if $\pi^* < \pi < 1$, then the welfare gain induced by a squared contract is less for liquidity demanders than for liquidity suppliers;*

(3) *if $\pi = \pi^*$, then the welfare gain induced by a squared contract for liquidity demanders is the same as that for liquidity suppliers.*

According to this lemma, the welfare gain due to a squared contract is much greater for liquidity demanders than for liquidity suppliers when the market consists predominantly of the latter. It is intuitive that if the population of liquidity suppliers is large then the long side of derivatives is higher than the short side; this, in turn, drives up derivative price and increases the profits of informed liquidity demanders who sell the derivative. Hence the welfare gain for liquidity demanders is greater than that for liquidity suppliers. When the market is dominated by liquidity demanders, these results are reversed. Derivatives benefit both types of agents—but not to the same extent, which affects agents' trading incentives at date 0.

Proposition 1.3.3. *At date 0, the equilibrium price of the risky asset is given by*

$$P_0 = \bar{D} - \alpha \frac{1}{h} \bar{X} - \frac{\pi M^O}{1 - \pi + \pi M^O} \Delta_1 \bar{X}, \tag{1.3.14}$$

where

$$\Delta_0 = \frac{(B-\beta_P)^2\left(\frac{1}{h}+\frac{1}{q}\right)}{\pi^2 \operatorname{Var}[D|\mathcal{F}_s]}, \tag{1.3.15}$$

$$\Delta_1 = \frac{\alpha^3 B \frac{1}{nh}\left(\frac{1}{h}+\frac{1}{m}\right)}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}}, \tag{1.3.16}$$

$$\Delta_2 = \frac{\alpha^3 \frac{1}{nh^2}\left[1+\frac{(B-\beta_s)^2\left(\frac{1}{h}+\frac{1}{m}\right)}{\operatorname{Var}[D|\mathcal{F}_d]}\right]}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}}, \tag{1.3.17}$$

and

$$M^O = \exp\left(\frac{G_s-G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}\exp\left(\frac{1}{2}\alpha\Delta_2\bar{X}^2\right)\sqrt{\frac{1+\pi^2\Delta_0}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}}}. \tag{1.3.18}$$

As shown in the above proposition, introducing derivatives affects only the *ratio* of marginal utilities (liquidity demanders to liquidity suppliers)—that is, M^O in the third term of P_0. The squared contract can improve the welfare of all agents, but this does not ameliorate the aversion to holding the underlying asset at date 0 that results from the two agent groups enjoying different levels of welfare gains. By Lemma 1.3.2, the population of liquidity demanders/suppliers determines the relative benefits for the two groups of agents and is a key component of the illiquidity discount component of P_0. Hence the population π plays an important role in the ex ante price P_0. Proposition 1.3.4 summarizes the results.

Proposition 1.3.4. *Let π^* be as defined in the Appendix. Then the following statements hold:*

(1) *if $0 < \pi < \pi^*$, then P_0 is higher in the presence of a squared contract than in the absence of a squared contract;*

(2) *if $\pi^* < \pi < 1$, then P_0 is lower in the presence of a squared contract than in the absence of a squared contract;*

(3) *if $\pi = \pi^*$, then P_0 is the same in the presence as in the absence of a squared contract.*

This proposition shows that introducing a squared contract has non-monotonic effects—on the ex ante price P_0—that vary with the likelihood of being a liquidity demander. Specifically: when the market consists mainly of liquidity suppliers, the benefits of derivatives are greater for liquidity demanders than for liquidity suppliers. At date 0, the identical investor can anticipate the effects of derivatives and so is less worried about liquidity shock. As a result, they are now more willing to hold the risky asset at date 0; that change leads to a higher ex ante price P_0 and a lower expected return. In the case of a population dominated by liquidity suppliers, a squared contract benefits liquidity demanders more than it does suppliers. In this case, agents are induced to take larger positions in the risky asset, which raises the price at date 0. The same line of reasoning establishes that, when the population is dominated by liquidity demanders, introducing derivatives results in a lower price P_0. More implications of the non-monotonic effects induced by introducing derivatives will be discussed in Section 1.4 which focuses on introducing an options market.

Proposition 1.3.5. *If a squared contract is available, then the price impact measure is*

$$\lambda = \frac{\alpha \operatorname{Var}[D|\mathcal{F}_s]}{(1-\pi)\big[\pi(m-q)\operatorname{Var}[D|\mathcal{F}_s] + 1 - \frac{\beta_P}{B}\big]} \tag{1.3.19}$$

and the price reversal measure is

$$\gamma = B(B - \beta_P)\left(\frac{1}{h} + \frac{1}{q}\right). \tag{1.3.20}$$

Introducing a squared contract reduces λ but has no effect on γ.

Compared with the symmetric information case, asymmetric information strengthens the price impact λ because the information is dispersed across agents (Vayanos and Wang, 2012a,b). In particular, uninformed liquidity suppliers learn the signal from the price; the consequent learning effect, which corresponds to the term β_p/B, reduces the magnitude of (1.3.19)'s denominator and so increases the price impact λ. Yet if a squared contract is available, then liquidity suppliers can hedge against the uncertainty of the risky asset's payoff despite not observing the signal. In other words, derivatives act as a substitute for information in reducing risks. This effect is clearly reflected by an increase in the magnitude of (1.3.19)'s denominator and a decrease in λ. By Proposition 1.3.1, the squared contract carries no additional information beyond—and has no effect on—the price P_1 of the risky asset. The implication is that introducing a squared contract is unrelated to price reversals of the risky asset. Derivatives reduce the price impact per trade (as captured by λ), yet because their availability increases trade size, the price impact of the *entire* trade (as measured by γ) is unaffected. Therefore, the introduction of derivatives reduces the price impact measure λ but has no effect on the price reversal measure γ.

More importantly, the squared contract can be synthesized using a collection of options

$$D^2 = 2\int_0^{+\infty} (D-K)^+ dK + 2\int_{-\infty}^{0} (K-D)^+ dK. \tag{1.3.21}$$

Then the analysis conducted in this section can be extended to an economy with an options market.

1.4 Introduction of an Options Market

In this section, an options market is introduced into the economy. I examine the effects of the options market on the prices and liquidity of the underlying asset in the presence of asymmetric information. Of particular interest are the illiquidity discount (as reflected in the ex ante price of the risky asset) and two distinct measures of illiquidity.

1.4.1 Equilibrium

To model an options market in a static setting, I follow Cao and Ou-Yang (2009) and define this market as a collection of call and put options. At date 0, all agents know that options will be introduced to trade at date 1 and will expire at date 2. Let K denote the strike price. A call option with strike price K pays off $(D-K)^+$, and its price at date 1 is denoted by P_{CK}. Similarly, the payoff structure of a put option is $(K-D)^+$ and its price at date 1 is denoted by P_{PK}. The net supply of each option is zero. In light of put-call parity, the analysis can be simplified (without loss of generality) by considering only call options with positive strike prices and put options with negative strike prices.[21] I assume that the demand of a liquidity demander for call options with strike prices ranging from K to $K+dK$ is $X_{d,CK}$ and that her demand for put options with strike prices ranging from K to $K+dK$ is $X_{d,PK}$. Likewise, the demand of a liquidity supplier for call options with strike prices ranging from K to $K+dK$ is $X_{s,CK}$ and his demand for put options with strike prices ranging from K to $K+dK$ is $X_{s,PK}$. This paper differs from Cao and Ou-Yang (2009) in that I model asymmetric information and they focus on heterogeneous beliefs. The equilibrium at date 1 with options is closely related to Huang (2015).

Liquidity demanders and suppliers submit their demand schedules conditional on their new

[21] The results are robust to introducing a complete set of calls and puts because call options with negative strike prices can be replicated by a combination of stocks and put options with negative strike prices.

information sets after options are introduced. Specifically, the information set of liquidity demanders at date 1 is $\mathcal{F}_d = \{s, z, P_0, P_1, P_{CK}, P_{PK}\}$ whereas that of liquidity suppliers is $\mathcal{F}_s = \{P_0, P_1, P_{CK}, P_{PK}\}$. If there is an options market, then the wealth of liquidity demanders at date 2 is given by

$$W_{d2} = W_1 + X_d(D - P_1) + z(D - \bar{D}) + \int_0^{+\infty} X_{d,CK}[(D-K)^+ - P_{CK}]\,dK$$
$$+ \int_{-\infty}^{0} X_{d,PK}[(K-D)^+ - P_{PK}]\,dK \tag{1.4.1}$$

and the wealth of liquidity suppliers is given by

$$W_{s2} = W_1 + X_s(D - P_1) + \int_0^{+\infty} X_{s,CK}[(D-K)^+ - P_{CK}]\,dK$$
$$+ \int_{-\infty}^{0} X_{s,PK}[(K-D)^+ - P_{PK}]\,dK. \tag{1.4.2}$$

If options are redundant and are not traded by investors, then the wealth calculated by (1.4.1) and (1.4.2) is the same as that in the absence of options. Once an options market is introduced, there is a partially revealing rational expectations equilibrium—in the prices P_1, P_{CK}, and P_{PK} of different assets and the corresponding demands of each agent type at date 1—as follows.[22]

Proposition 1.4.1. *At date 1, there exists one equilibrium. The price P_1 of the risky asset is given by*

$$P_1 = A + B(s - \bar{D} - Cz), \tag{1.4.3}$$

where

$$A = \bar{D} - \frac{\alpha}{h + \pi m + (1-\pi)q}\bar{X},$$
$$B = \frac{\pi m + (1-\pi)q}{h + \pi m + (1-\pi)q},$$
$$C = \frac{\alpha}{m}.$$

[22] The uniqueness of the equilibrium is left for future research.

The prices P_{CK} and P_{PK} of (respectively) call and put options are given by

$$P_{CK} = (P_1 - K)\mathcal{N}\big(\sqrt{G}(P_1 - K)\big) + \frac{1}{\sqrt{2\pi G}}\exp\left(-\frac{G(P_1 - K)^2}{2}\right) \quad \text{for } K \geq 0, \tag{1.4.4}$$

$$P_{PK} = (K - P_1)\mathcal{N}\big(\sqrt{G}(K - P_1)\big) + \frac{1}{\sqrt{2\pi G}}\exp\left(-\frac{G(K - P_1)^2}{2}\right) \quad \text{for } K < 0, \tag{1.4.5}$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function. The liquidity demander's demands for the risky asset and the corresponding options are

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_d]} - \frac{G - G_d}{\alpha} P_1 - z, \tag{1.4.6}$$

$$X_{d,CK} = \frac{G - G_d}{\alpha}, \tag{1.4.7}$$

$$X_{d,PK} = \frac{G - G_d}{\alpha}. \tag{1.4.8}$$

The liquidity supplier's demands for the risky asset and the corresponding options are

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_s]} - \frac{G - G_s}{\alpha} P_1, \tag{1.4.9}$$

$$X_{s,CK} = \frac{G - G_s}{\alpha}, \tag{1.4.10}$$

$$X_{s,PK} = \frac{G - G_s}{\alpha}. \tag{1.4.11}$$

In (1.4.4)–(1.4.11), $G = h + \pi m + (1-\pi)q$, $G_d = h + m$, $G_s = h + q$, and $\frac{1}{q} = \frac{1}{m} + C^2\frac{1}{n}$. The terms G_d and G_s represent the conditional precision of D for liquidity demanders and suppliers, and G denotes the average precision for all investors.

Similar to the introduction of a squared contract, the introduced options are not redundant securities and are traded by investors. In equilibrium, liquidity demanders take short positions in the options whereas liquidity suppliers take long positions. Given these payoff structures, options are able to provide hedging benefits for the second moment of the risky asset's payoff D. In this sense, options can act as a substitute for information in reducing risk. Since liquidity demanders have more precise information, they always have less incentive than suppliers to hedge against the second moment. Because options are in zero net supply, liquidity demanders sell the options to earn profit while liquidity suppliers take long positions to hedge against the second moment. Second, introducing options into an economy with asymmetric information has no direct effect on the underlying asset's equilibrium price at date 1. Third, option prices are a function of the risky asset's price but carry no additional information; hence the option prices are considered to be "informationally redundant".

Lemma 1.4.1. *At interim date $t = \frac{1}{2}$, if options are available then the utilities of liquidity demanders are given by*

$$U_d = \frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}} \mathrm{E}\Big\{ -\exp\Big[-\alpha\Big(W_0 + X_0(P_1 - P_0) + z(P_1 - \bar{D}) + \frac{[\bar{D} + \beta_s(s - \bar{D}) - P_1]^2}{2\alpha \operatorname{Var}[D|\mathcal{F}_d]} \Big)\Big]\Big\} \tag{1.4.12}$$

and the utilities of liquidity suppliers are given by

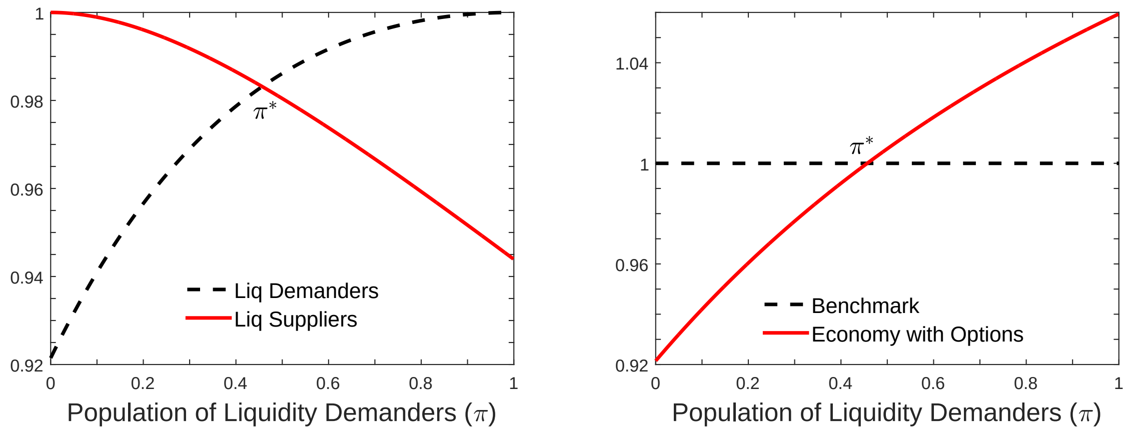
$$U_s = \frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}} \mathrm{E}\Big\{ -\exp\Big[-\alpha\Big(W_0 + X_0(P_1 - P_0) + \frac{[\bar{D} + \beta_P(s - \bar{D} + \frac{\alpha}{m}z) - P_1]^2}{2\alpha \operatorname{Var}[D|\mathcal{F}_s]} \Big)\Big]\Big\}. \tag{1.4.13}$$

For $\pi \in (0,1)$ we have that both $\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}} < 1$ and $\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}} < 1$, where E is the expectation over (s,z).

After an options market is introduced, the interim utilities of liquidity demanders and suppliers are the product of two terms: one reflecting the impact of introducing options and one capturing the interim utilities of agents *before* options are introduced (see Appendix 1.9.1). The left panel of Figure 1.1 illustrates that, for all agents, the additional term induced by options is less than 1. Owing to the negative exponential utility, a less negative number indicates greater utility.[23] Hence introducing options improves the utilities of both groups of agents thanks to the improved risk sharing that options allow. When equipped with options, uninformed liquidity suppliers are more willing to accommodate the trades of informed demanders. More importantly, the effects of these financial derivatives on the interim utilities of the two groups of agents are not the same: the value of the extra term—which appears when options are incorporated into the model—depends on the conditional precision of the payoff D, which differs between liquidity demanders and liquidity suppliers. Lemma 1.4.2 formally characterizes this distinction.

Lemma 1.4.2. *Let $\pi^* \in (0,1)$ be as defined in the Appendix. Then the following statements hold:*

[23] The parameters for all the figures are $h = 1$, $m = 1$, $n = 1$, and $\alpha = 0.7$.



(a) Welfare gains (b) Relative welfare gains

Figure 1.1: Welfare gains from options. The left panel plots the welfare gains for liquidity demanders (the first term in (1.4.12)) and liquidity suppliers (the first term in (1.4.13)); the right panel plots the ratio—demanders/suppliers—of welfare gains. The solid (resp. dashed) line marks the ratio of welfare gains with (resp. without) options.

(1) *if* $0 < \pi < \pi^*$, *then the welfare gain induced by options is greater for liquidity demanders than for liquidity suppliers;*

(2) *if* $\pi^* < \pi < 1$, *then the welfare gain induced by options is less for liquidity demanders than for liquidity suppliers;*

(3) *if* $\pi = \pi^*$, *then the welfare gain induced by options for liquidity demanders is the same as that for liquidity suppliers.*

According to this lemma, the welfare gain due to options is much greater for liquidity demanders than for liquidity suppliers when the market consists predominantly of the latter. It is intuitive that if the population of liquidity suppliers is large then the long side of options is higher than the short side; this, in turn, drives up option price and increases the profits of informed liquidity demanders who sell options. Hence the welfare gain for liquidity demanders is greater than that for liquidity suppliers. When the market is dominated by liquidity demanders, these results are reversed. That is, the relative benefit from more risk-sharing opportunities is determined by the competition within each group. The right panel of Figure 1.1 shows this pattern clearly. Options benefit both types of agents—but not to the same extent, which affects agents' trading incentives at date 0.

Proposition 1.4.2. *At date 0, the equilibrium price of the risky asset is given by*

$$P_0 = \bar{D} - \alpha \frac{1}{h} \bar{X} - \frac{\pi M^O}{1 - \pi + \pi M^O} \Delta_1 \bar{X}, \tag{1.4.14}$$

where

$$\Delta_0 = \frac{(B - \beta_P)^2 \left(\frac{1}{h} + \frac{1}{q}\right)}{\pi^2 \operatorname{Var}[D|\mathcal{F}_s]}, \tag{1.4.15}$$

$$\Delta_1 = \frac{\alpha^3 B \frac{1}{nh} \left(\frac{1}{h} + \frac{1}{m}\right)}{1 + \Delta_0 (1 - \pi)^2 - \alpha^2 \frac{1}{nh}}, \tag{1.4.16}$$

$$\Delta_2 = \frac{\alpha^3 \frac{1}{nh^2} \left[1 + \frac{(B - \beta_s)^2 \left(\frac{1}{h} + \frac{1}{m}\right)}{\operatorname{Var}[D|\mathcal{F}_d]}\right]}{1 + \Delta_0 (1 - \pi)^2 - \alpha^2 \frac{1}{nh}}, \tag{1.4.17}$$

and

$$M^O = \exp\left(\frac{G_s - G_d}{2G}\right) \sqrt{\frac{G_d}{G_s}} \exp\left(\frac{1}{2} \alpha \Delta_2 \bar{X}^2\right) \sqrt{\frac{1 + \pi^2 \Delta_0}{1 + \Delta_0 (1 - \pi)^2 - \alpha^2 \frac{1}{nh}}}. \tag{1.4.18}$$

This proposition states that there are also three terms in the ex ante price P_0 when options are available. Introducing options affects only the *ratio* of marginal utilities (liquidity demanders to liquidity suppliers)—that is, M^O in the third term of P_0. Although introducing derivatives improves agents' welfare and the extent of risk sharing between them, it need not increase their willingness to hold the risky asset at date 0 and thereby raise the price P_0. Intuitively, agents are unsure at date 0 about whether they will become suppliers or demanders. As a result, agents are naturally reluctant to buy the risky asset at date 0 because with probability π they will receive a liquidity shock that could increase their risk exposure. The uncertainty about this liquidity shock is costly to risk-averse agents and hence reduces their willingness to hold the asset at date 0, which explains the illiquidity discount reflected in P_0. Asymmetric information hampers risk sharing and makes liquidity demanders less able to hedge their liquidity risk, which increases the illiquidity discount component of P_0.

Options can improve the welfare of all agents, but this does not ameliorate the aversion to holding the underlying asset at date 0 that results from the two agent groups enjoying different levels of welfare gains (see Figure 1.1). Next I shall explore in detail how introducing an options market affects the illiquidity discount in P_0 as well as the illiquidity measures defined by (1.2.15) and (1.2.16).

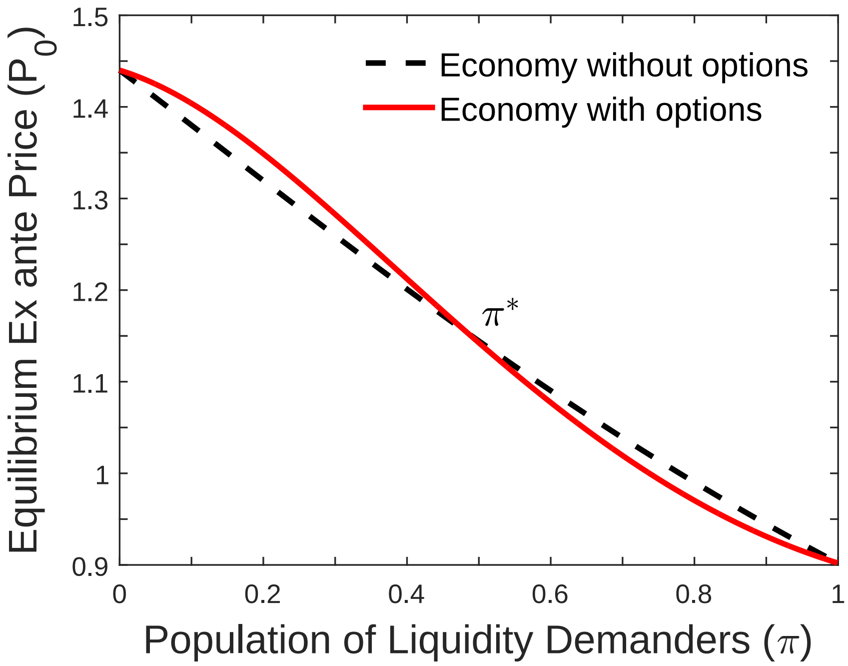


Figure 1.2: Effects of options on the ex ante price of the risky asset. The solid (resp. dashed) line plots the equilibrium ex ante price with (resp. without) options.

1.4.2 Options and Illiquidity

We have seen that introducing options affects liquidity demanders and suppliers in different ways. By Lemma 1.4.2, the population of liquidity demanders/suppliers determines the relative benefits for the two groups of agents and is a key component of the illiquidity discount component of P_0. Hence the population π plays an important role in the ex ante price P_0. Proposition 1.4.3 summarizes the results.

Proposition 1.4.3. *Let π^* be as defined in the Appendix. Then the following statements hold:*

(1) *if $0 < \pi < \pi^*$, then P_0 is higher in the presence of options than in the absence of options;*

(2) *if $\pi^* < \pi < 1$, then P_0 is lower in the presence of options than in the absence of options;*

(3) *if $\pi = \pi^*$, then P_0 is the same in the presence as in the absence of options.*

This proposition shows that introducing an options market has non-monotonic effects on the ex ante price P_0. Specifically: when the market consists mainly of liquidity suppliers (i.e., when liquidity provision is sufficient), the benefits of options are greater for liquidity demanders than for liquidity suppliers. At date 0, the identical investor can anticipate the effects of options and so is less worried about liquidity shock. As a result, they are now more willing to hold the risky asset at date 0; that change leads to a higher ex ante price P_0 and a lower expected return. This mechanism is clearly evident in Figure 1.2. In the case of a population dominated by liquidity suppliers, an options market benefits liquidity demanders more than it does suppliers. In this case, agents are induced to take larger positions in the risky asset, which raises the price at date 0. The same line of reasoning establishes that, when the population is dominated by liquidity demanders, introducing options results in a lower price P_0. Furthermore, P_0 is a decreasing function of π because a lower probability of receiving the liquidity shock translates, at date 0, into both a lower discount and a higher price.

Before options are introduced to trade, two illiquidity measures are affected by the population π of informed liquidity demanders.[24] The price reversal measure γ defined in (1.2.16) captures the price deviation from a fundamental value that liquidity suppliers require to absorb the liquidity shock. A large population of liquidity demanders leads to a large price deviation, which implies that γ is an increasing function of π. As illustrated in Proposition 1.4.3, the effect of an options market depends on π; hence I can use π to establish a link between illiquidity and asset price as a function of whether options are available. I find that the price of relatively more liquid (low-γ) stocks rises in response to the introduction of options; in contrast, the price of relatively less liquid (high-γ) stocks declines. This implication is clearly illustrated in Figure 1.3. If stocks are sorted in terms of price reversal γ then I find, after the corresponding stock options are listed, that more liquid stocks generally have a higher price and a lower expected return; conversely, less liquid stocks tend to have a lower price and a higher expected return.

The results could also be interpreted in other ways. For example, if liquidity provision in the market is scarce (which it usually is during "bad" states) then introducing new assets, such as derivatives, increases the underlying asset's expected return. In contrast, introducing derivatives lowers that asset's expected return when there is sufficient liquidity in the market. As argued by Vayanos and Wang (2012a,b), the liquidity shock could be a consequence of institutional frictions. One can therefore view liquidity demanders as institutional investors who usually have more precise information yet are subject to some frictions (as might result from a change in fund flows). At the same time, liquidity suppliers can be interpreted as designated market makers who supply liquidity.[25] Introducing an options market can increase expected returns of

[24] The price impact measure λ is not a monotonic function of π because it is affected by the extent of risk sharing between the groups of agents. For that reason, in this section the focus is on the price reversal measure γ.

[25] It is worth remarking that the option positions taken by liquidity demanders and suppliers are in line with

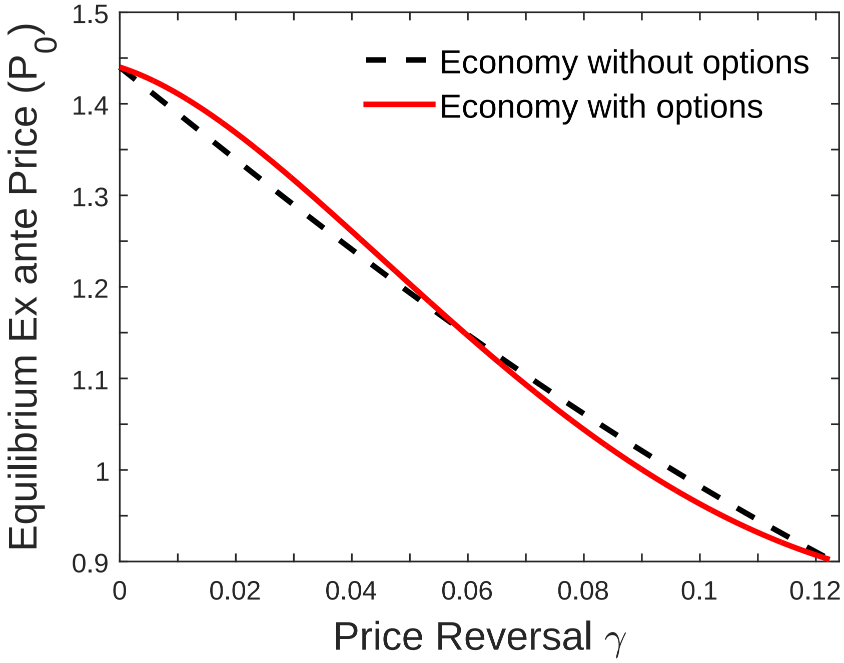


Figure 1.3: Price reversal γ versus equilibrium ex ante price as a function of whether options are (solid line) or are not (dashed line) available; here the population π drives the cross-sectional variation in γ, and P_0.

the underlying stocks when agents include a high proportion of institutional investors (e.g., large firms and firms with extensive coverage by analysts). Conversely, introducing an options market can reduce expected returns of the underlying stocks when agents include a low proportion of institutional investors (e.g., small firms and firms with low analyst coverage). These dynamics yield new implications—which can be tested cross-sectionally—regarding how options affect the underlying assets.

These results illustrate the relationship between illiquidity measures and asset prices when cross-sectional variation in the illiquidity measures (λ and γ) and the illiquidity discount in P_0 is driven by the population π. One can likewise examine this relationship when the cross-sectional variation is driven instead by the precision of the private signal and/or of the liquidity shock (i.e., the parameters m and n). A higher m, the precision of ϵ's distribution in (1.2.2), increases both the illiquidity discount and λ but decreases γ. The intuitive explanation is that, when the private signal is more precise, the consequent adverse selection becomes more severe and thus increases the illiquidity discount and the price impact λ. In contrast, the effects of the

results reported in Garleanu, Pedersen, and Poteshman (2009): market makers take long positions in individual stock options while end users take short positions.

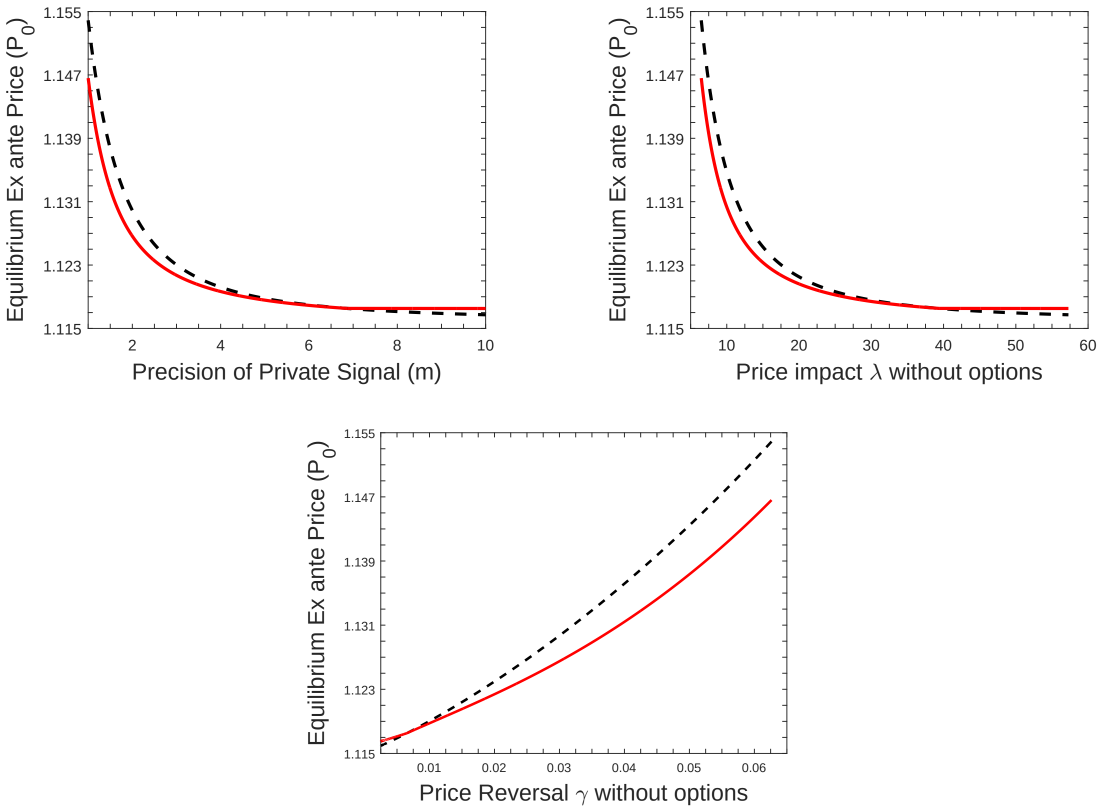


Figure 1.4: Parameters m, λ, and γ (the latter two are illiquidity measures) versus equilibrium ex ante price as a function of whether options are (solid line) or are not (dashed line) available; here the precision parameter m (for the private signal) drives the cross-sectional variation in λ, γ, and P_0 ($\pi = 0.49$).

liquidity shock captured by γ are not that important when the private signal is precise. Hence an increase in m reduces the price reversal γ. Combining the effects of m on P_0, on λ, and on γ yields that the illiquidity measure λ is negatively correlated with the price P_0 even as the illiquidity measure γ is positively correlated with P_0 (this is clearly shown by the dashed lines in Figure 1.4).

After an options market is introduced, Proposition 1.4.3 establishes the existence of a threshold π^* that determines the sign of the effects generated by options. When $\pi = \pi^*$, the welfare gains for liquidity demanders and suppliers are the same. The threshold π^* increases with the parameter m in that liquidity demanders who observe a more accurate private signal can extract more profits thanks to this information advantage, which compensates for the cost of more intense competition due to more demanders. As the parameter m increases, a certain popu-

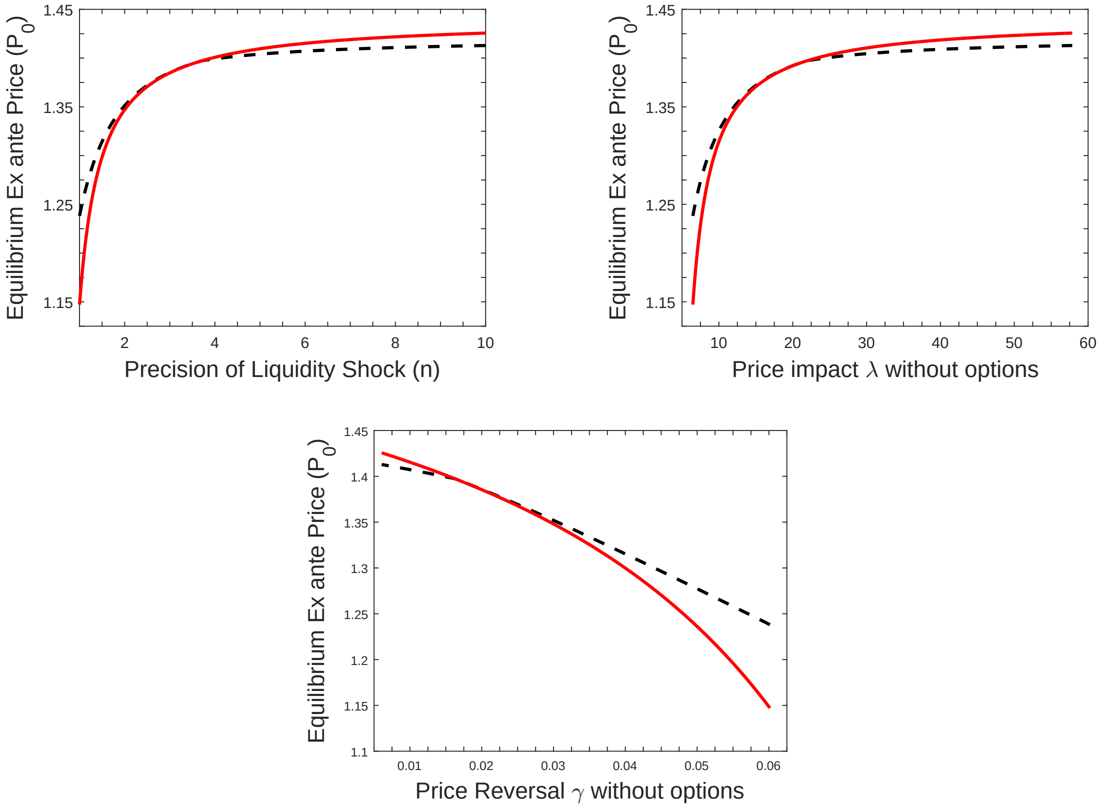


Figure 1.5: Parameters n, λ, and γ (the latter two are illiquidity measures) versus equilibrium ex ante price as a function of whether options are (solid line) or are not (dashed line) available; here the precision parameter n (for the liquidity shock) drives the cross-sectional variation in λ, γ, and P_0 ($\pi = 0.49$).

lation of liquidity demanders close to π^* (say, $\pi = 0.49$) is first larger and then smaller than this threshold π^*. Consequently, one can observe a negative (resp. positive) effect of options on the price of the underlying asset when the private signal is less (resp. more) accurate. This pattern can be seen clearly in the upper left panel of Figure 1.4. Because λ is an increasing function of m, we observe a similar pattern in the upper right panel of that figure. The results are reversed in the lower panel because γ decreases with m. The upper right panel of Figure 1.4 shows that, if illiquidity is measured by λ and if the cross-sectional variation in λ is driven by m, then introducing options lowers the prices of more liquid stocks but raises the price of less liquid stocks. Yet if illiquidity of stocks is measured by γ then the opposite result obtains, as seen in the lower panel of Figure 1.4.

I also examine how the underlying asset price is affected by derivatives if the cross-sectional variations in asset-specific characteristics all stem from the liquidity shock's precision n, which is simply the *reciprocal* of the liquidity shock's magnitude.[26] In contrast to the effects of m, an increase in n reduces both the illiquidity discount and γ because the shock's effect is attenuated as its magnitude becomes smaller. Consider the extreme case where n is infinite. At date 0, agents know that the liquidity shock's size is zero and so they do not expect the price P_0 to reflect an illiquidity discount. The price reversal γ, which captures the effect of a liquidity shock, is likewise attenuated when that shock is of relatively low magnitude. The price impact λ increases with n because a smaller liquidity shock renders price more informative and amplifies the learning effect embedded in λ. After combining the effects of n on P_0, on λ, and on γ, one obtains that the illiquidity measure λ (resp. γ) is positively (resp. negatively) correlated with the price P_0. This relationship between illiquidity and asset price, which is plotted in Figure 1.5, is totally opposite to the one in Figure 1.4.

Just as for the precision m, an increase in the precision parameter n increases π^* as well. One can therefore observe that options have a negative (resp. positive) effect on P_0 when the liquidity shock is large (resp. small); these effects are displayed in the upper left panel of Figure 1.5. The upper right and lower panels of this figure show that, after an options market is introduced, a liquid stock has a lower price if illiquidity is measured by λ but a higher one if illiquidity is measured by γ; these results are similar to those plotted in the corresponding two panels of Figure 1.4.

There is a large body of empirical work that investigates the link between illiquidity and

[26]Because its mean is zero, the liquidity shock's variance (i.e., $1/n$) measures its magnitude.

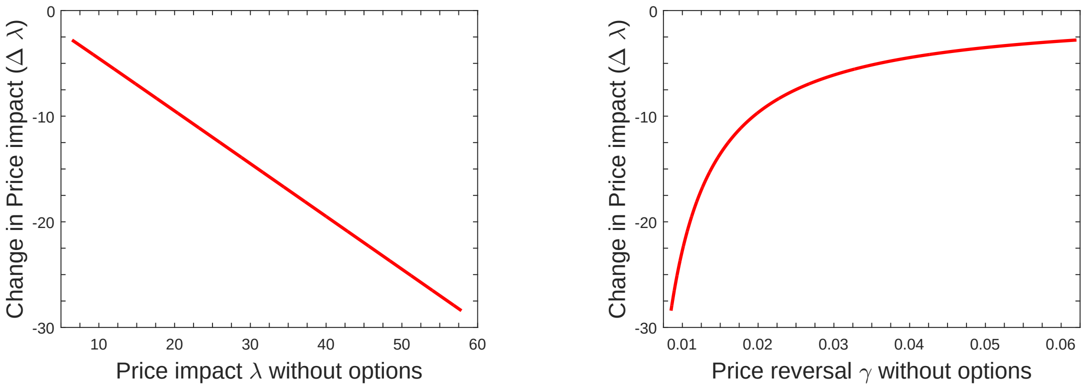


Figure 1.6: Relationship between illiquidity measures and liquidity improvement.

expected asset returns. Among these studies, a basic premise is that a less liquid asset can offer a higher expected return; in other words, there is a positive relationship between illiquidity and expected returns. Figures 1.4 and 1.5 demonstrate that the empirical relationship between illiquidity and asset returns depends on the source of cross-sectional variation and also on how illiquidity is measured. If cross-sectional variation is driven by m, for instance, then the relationship between illiquidity and expected returns is positive when illiquidity is measured by λ but negative when measured by γ. Hence it is critical to identify the factors that drive cross-sectional variation.

Proposition 1.4.4. *If options are available, then the price impact measure is*

$$\lambda = \frac{\alpha \operatorname{Var}[D|\mathcal{F}_s]}{(1-\pi)\left[\pi(m-q)\operatorname{Var}[D|\mathcal{F}_s] + 1 - \frac{\beta_P}{B}\right]} \tag{1.4.19}$$

and the price reversal measure is

$$\gamma = B(B-\beta_P)\left(\frac{1}{h} + \frac{1}{q}\right). \tag{1.4.20}$$

Introducing options reduces λ but has no effect on γ.

This proposition shows that introducing an options market reduces the price impact measure λ but has no effect on the price reversal measure γ. I have shown that the effect of options on the underlying asset's price depends both on precision parameters and on the illiquidity measure chosen. A similar study could be conducted to examine how the beneficial effect of options on liquidity is distributed across stocks of various liquidity levels. As discussed previously, an increase in either m or n raises λ and lowers γ. When information becomes more asymmetric, the consequent adverse selection is more severe and so the price impact λ is greater. In other words, high-λ stocks are more subject to information asymmetry. In comparison to the setting without options, the additional term (viz., $\pi(m-q)\,\mathrm{Var}[D|\mathcal{F}_s]$) in the denominator of (1.4.19) implies that the more asymmetric is information, the greater is the hedging benefit provided by options. Taken together, the liquidity improvement of more liquid (low-λ) stocks is less than that of less liquid (high-λ) stocks. However, if illiquidity is measured by γ then the liquidity improvement of more liquid (low-γ) stocks is *greater* than that of less liquid (high-γ) stocks. This converse effect is observed because asymmetric information can reduce γ. Figure 1.6 plots these results.[27] The findings based on illiquidity as measured by λ are consistent with the empirical evidence reported by Kumar, Sabin, and Shastri (1998). In this sense, the measure λ fits the empirical evidence well and seems to reflect frictions more accurately (see also Vayanos and Wang, 2012a,b).

1.4.3 Welfare

Proposition 1.4.5. *As defined in* (1.2.12)*, an agent's ex ante utility at date 0 is higher in the presence than in the absence of options.*

The identical investor's expected utility at date 0 can be calculated as the weighted average of the liquidity suppliers' and demanders' interim utilities. Introducing options will, of course, increase expected utilities at date 0 because the options improve risk sharing among agents.

1.5 Analysis with Participation Costs

The results described so far all relate to an economy without participation costs. Recall that, in the absence of such costs, introducing derivatives increases the utility of all investors; yet the welfare gains derived by the two groups of agents are of different magnitudes, leading to distinct effects on the ex ante price at date 0. In this section, I endogenize the participation decisions of agents. More specifically, I assume that all investors must pay a fixed cost f in order to participate in the market. My analysis of participation decisions and equilibrium at date 1 is

[27] Because the precision parameters m and n have similar effects on these two illiquidity measures, this figure plots only the case where cross-sectional variation is driven by m.

closely related to the work of Huang and Wang (2009, 2010) and Vayanos and Wang (2012b), who examine a setting in which investors have hedging (non-informational) motives for trading. In contrast, I study the equilibrium when both non-informational and informational motives are present.

1.5.1 Equilibrium without an Options Market

Like Vayanos and Wang (2012b), I seek to establish the existence of an equilibrium in which, at date 1, all liquidity demanders participate in the market but only a (positive) proportion μ of liquidity suppliers participate. Such a *partial participation* equilibrium reflects that only liquidity demanders face the risk of liquidity shock and so can benefit more (than liquidity suppliers) from participation; moreover, of most importance for my model is the *relative* measure of participating suppliers and demanders. I assume that the decision to participate is made ex ante and therefore that investors decide whether or not to pay the cost at date $\frac{1}{2}$—in other words, after learning whether or not they will receive the random endowment but before observing the price at date 1. Once the date-1 price is observed, investors can make decisions contingent on that price. This setup implies that the cost f is a fixed transaction cost, not a participation cost.[28]

I denote by $U_{d,P}$ and $U_{d,NP}$ the interim utilities of participating and non-participating liquidity demanders at date $\frac{1}{2}$ and likewise use $U_{s,P}$ and $U_{s,NP}$ with respect to liquidity suppliers. Much as in the previous section without participation costs, I conjecture a linear price function of the risky asset in the following form:

$$P_1 = A + B(s - \bar{D} - Cz). \tag{1.5.1}$$

Conditional on the information set $\mathcal{F}_d$, liquidity demanders maximize their utilities over the wealth at date 2, $W_{d2} = W_1 + X_d(D - P_1) + z(D - \bar{D}) - f$, and submit a demand schedule for the risky asset:[29]

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_d]} - z. \tag{1.5.2}$$

Similarly, the wealth of liquidity suppliers is $W_{s2} = W_1 + X_s(D - P_1) - f$ and their demand

[28] If the participation cost is introduced at date 0 then it can be viewed as an entry cost. An entry cost similarly reduces the participation rate of agents and thereby reduces the underlying asset's ex ante price. See Huang and Wang (2009) for more details.

[29] The information structure does not differ in the case of participation costs, so agents have the same information sets as before.

schedule is

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \,\mathrm{Var}[D|\mathcal{F}_s]}. \tag{1.5.3}$$

After both groups of investors submit their demand schedules, the equilibrium price clears the market and thus equates investors' total demands and the asset supply $\bar{X}$:

$$\pi X_d + (1-\pi)\mu X_s = [\pi + (1-\pi)\mu]\bar{X}. \tag{1.5.4}$$

The equilibrium price of the risky asset is obtained as follows.

Proposition 1.5.1. *At date 1, the price P_1 of the risky asset in the presence of participation costs is given by*

$$P_1 = A + B(s - \bar{D} - Cz), \tag{1.5.5}$$

here

$$A = \bar{D} - \frac{\alpha}{h + \tilde{\pi}m + (1-\tilde{\pi})q}\bar{X}, \quad B = \frac{\tilde{\pi}m + (1-\tilde{\pi})q}{h + \tilde{\pi}m + (1-\tilde{\pi})q}, \quad \text{and} \quad C = \frac{\alpha}{m} \tag{1.5.6}$$

for $\tilde{\pi} = \frac{\pi}{\pi + (1-\pi)\mu}$.

Comparing (1.2.9) and (1.5.6) reveals that the coefficients in the price function with participation costs take the same form as in the benchmark case without participation costs. Note that $\mathrm{E}[P_1]$, the expected equilibrium price at date 1, depends on the measure μ of participating liquidity suppliers. This finding is in contrast to results for the no-information case studied by Vayanos and Wang (2012b). Substituting the demands of liquidity demanders and liquidity suppliers into their utility functions yields their expected utilities. The agents' interim utilities can be computed as in Section 1.2.2.

To find the equilibrium where all liquidity demanders and a positive fraction of liquidity suppliers participate in the market, I first determine the participation decision by comparing $U_{d,P}$ to $U_{d,NP}$ and $U_{s,P}$ to $U_{s,NP}$. In particular, I discuss several scenarios defined by the participation fraction μ of liquidity suppliers. It is intuitive that all liquidity suppliers choose to enter the market when the participation cost f is low and that none of them will participate when the cost f is high. However, liquidity demanders in this study receive a private signal s in addition to the liquidity shock z. New findings will be compared with results from the no-information case examined in Vayanos and Wang (2012b). I derive the following results concerning the participation decision of liquidity suppliers.

Proposition 1.5.2. *Suppose that all liquidity demanders participate in the market. For c as defined in the Appendix, there are three cases of the liquidity suppliers' participation decisions as follows.*

Case 1: All liquidity suppliers participate in the market if $f \leq f^1 \equiv \frac{\log\left(1+\frac{\pi^2}{[h+q+\pi(m-q)]^2}c\right)}{2\alpha}$.

Case 2: A positive fraction μ *of liquidity suppliers participate in the market if* $f^1 < f < f^2$, *where*

$$\mu = \frac{\pi}{1-\pi}\left\{\frac{m-q}{h+q}\left[\sqrt{\frac{h}{q(e^{2\alpha f}-1)}}-1\right]-1\right\} \quad \textit{and} \quad f^2 \equiv \frac{\log\left(1+\frac{1}{(h+m)^2}c\right)}{2\alpha}.$$

Case 3: No liquidity suppliers participate in the market if $f \geq f^2$.

Given the participation decision of liquidity suppliers, our next step is to find a sufficient condition for all liquidity demanders to enter the market. To ensure their participation it is enough that $U_{d,P}$ be larger than $U_{d,NP}$. The sufficient condition for full participation of liquidity demanders is formalized as follows.

Proposition 1.5.3. (1) *Suppose that a positive fraction of liquidity suppliers participate in the market. Then a sufficient condition for all liquidity demanders to participate is* $\pi \leq (1-\pi)\mu$.

(2) *An equilibrium where all liquidity demanders and a positive fraction of liquidity suppliers participate in the market exists under the sufficient conditions* $\pi \leq \frac{1}{2}$ *and* $f \leq \hat{f}$, *where*

$$\hat{f} \equiv \frac{\log\left[1+\left(4[h+q+\frac{1}{2}(m-q)]^2\right)^{-1}c\right]}{2\alpha}$$

and c is as defined in the Appendix.

This sufficient condition for the existence of a partial participation equilibrium is similar to that given in Vayanos and Wang (2012b). My results differ from theirs in the condition for the fixed participation cost f owing to the presence of asymmetric information in the economy.[30] Proposition 1.5.3 leads to the following statement as regards the ex ante price.

Proposition 1.5.4. *At date 0, the equilibrium price of the risky asset is given by*

$$P_0 = \bar{D} - \alpha\frac{1}{h}\bar{X} - \frac{\pi M^P}{1-\pi+\pi M^P}\Delta_1\bar{X}, \tag{1.5.7}$$

[30] As argued in Vayanos and Wang (2012a,b), there are two equilibria: the one described in the proposition and the one where no investors participate. The latter can be excluded (for details, see Vayanos and Wang, 2012a,b).

where

$$\Delta_0 = \frac{(B-\beta_P)^2\left(\frac{1}{h}+\frac{1}{q}\right)}{(\tilde{\pi})^2\,\mathrm{Var}[D|\mathcal{F}_s]}, \tag{1.5.8}$$

$$\Delta_1 = \frac{\alpha^3 B\frac{1}{nh}\left(\frac{1}{h}+\frac{1}{m}\right)}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}, \tag{1.5.9}$$

$$\Delta_2 = \frac{\alpha^3\frac{1}{nh^2}\left[1+\frac{(B-\beta_s)^2\left(\frac{1}{h}+\frac{1}{m}\right)}{\mathrm{Var}[D|\mathcal{F}_d]}\right]}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}, \tag{1.5.10}$$

and

$$M^P = \exp\left(\frac{1}{2}\alpha\Delta_2\bar{X}^2\right)\sqrt{\frac{1+(\tilde{\pi})^2\Delta_0}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}} \tag{1.5.11}$$

for $\tilde{\pi} = \frac{\pi}{\pi+(1-\pi)\mu}$.

In contrast to the benchmark case studied in Section 1.2.2, the new price P_0 is obtained by replacing π with $\tilde{\pi}$ when evaluating $(\Delta_0, \Delta_1, \Delta_2, M)$. After the equilibrium analysis, I turn to investigate how the illiquidity measures and the illiquidity discount are affected by participation costs when the economy is characterized by information asymmetry. The price impact measure and price reversal measure are calculated as follows.

Proposition 1.5.5. *In the presence of participation costs, the price impact measure is*

$$\lambda = \frac{\alpha[\pi m+(1-\pi)\mu q]}{h(1-\pi)\pi\mu(m-q)} \tag{1.5.12}$$

and the price reversal measure is

$$\gamma = B(B-\beta_P)\left(\frac{1}{h}+\frac{1}{q}\right), \tag{1.5.13}$$

where $B = \frac{\tilde{\pi}m+(1-\tilde{\pi})q}{h+\tilde{\pi}m+(1-\tilde{\pi})q}$.

The new illiquidity measures are akin to those in the benchmark case except that now they are affected by the participation fraction μ. It is therefore natural to posit that introducing options affects illiquidity measures through the participation fraction because options enhance agents' utilities and so induce them to participate. Thus it is crucial to examine the effects of participation costs f and of the participation rate μ on illiquidity measures and on the illiquidity discount; these effects can then be compared to those for the case (discussed in Section 1.5.2) where options are available. The following proposition illustrates these effects.[31]

Proposition 1.5.6. *A decrease in the fixed cost f leads to an increase in the participation rate μ; hence a reduced f lowers both the price impact λ and the price reversal γ while raising the ex ante price P_0.*

1.5.2 Equilibrium with an Options Market

In this section I study how the introduction of an options market, in the presence of participation costs, affects the illiquidity discount and the two illiquidity measures. As shown in Section 1.4, introducing derivatives reduces the price impact measure λ but has no effect on the price reversal measure γ. In the presence of participation costs, however, the introduction of derivatives can affect both liquidity measures through the participation rate μ. The reason is that introducing options attracts more uninformed liquidity suppliers who would otherwise refuse to participate because of the cost f. I formally present the effects of derivatives in the following results.

As in Section 1.4, I introduce a set of call and put options into the economy. I follow similar procedures to solve for the equilibrium at date 0 and date 1, except that only a proportion $\mu \in (0, 1)$ of liquidity suppliers participate. Proposition 1.5.7 characterizes the solution.

Proposition 1.5.7. *At date 1, there exists one equilibrium. The price of the risky asset is given by*

$$P_1 = A + B(s - \bar{D} - Cz), \tag{1.5.14}$$

where

$$A = \bar{D} - \frac{\alpha}{h + \tilde{\pi}m + (1 - \tilde{\pi})q}\bar{X},$$
$$B = \frac{\tilde{\pi}m + (1 - \tilde{\pi})q}{h + \tilde{\pi}m + (1 - \tilde{\pi})q},$$
$$C = \frac{\alpha}{m}.$$

[31] Without loss of generality, I consider only the region $f > f^1$—that is, the region in which only some liquidity suppliers participate. If $f \leq f^1$ then all liquidity demanders and suppliers participate, which reduces to the benchmark case examined in Section 1.2.2.

The prices P_{CK} and P_{PK} of call and put options are given by

$$P_{CK} = (P_1 - K)\mathcal{N}\big(\sqrt{G^P}(P_1 - K)\big) + \frac{1}{\sqrt{2\pi G^P}}\exp\left(-\frac{G^P(P_1-K)^2}{2}\right) \quad \textit{for } K \geq 0,$$
$$P_{PK} = (K - P_1)\mathcal{N}\big(\sqrt{G^P}(K - P_1)\big) + \frac{1}{\sqrt{2\pi G^P}}\exp\left(-\frac{G^P(K-P_1)^2}{2}\right) \quad \textit{for } K < 0,$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function. The liquidity demander's demands for the risky asset and the corresponding options are

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_d]} - \frac{G^P - G_d}{\alpha} P_1 - z, \tag{1.5.15}$$
$$X_{d,CK} = \frac{G^P - G_d}{\alpha}, \tag{1.5.16}$$
$$X_{d,PK} = \frac{G^P - G_d}{\alpha}; \tag{1.5.17}$$

the liquidity supplier's demands for the risky asset and the corresponding options are

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_s]} - \frac{G^P - G_s}{\alpha} P_1, \tag{1.5.18}$$
$$X_{s,CK} = \frac{G^P - G_s}{\alpha}, \tag{1.5.19}$$
$$X_{s,PK} = \frac{G^P - G_s}{\alpha}. \tag{1.5.20}$$

Here $G^P = h + \tilde{\pi}m + (1-\tilde{\pi})q$, $G_d = h + m$, and $G_s = h + q$. The term G^P represents the average precision of D for all investors in the presence of participation costs.

The equilibrium prices of the underlying asset and the derivatives take a form similar to their presentation in Section 1.4 except that the proportion π is replaced by $\tilde{\pi}$. Once again, I limit the scope of analysis to the equilibrium where liquidity demanders fully participate and liquidity suppliers partially participate. The new average precision of D for all investors (G^P) is obtained by replacing the exogenous fraction π with the endogenous fraction $\tilde{\pi}$, where the latter is determined by the fixed participation cost f. Therefore, it is first necessary to identify a sufficient condition for the existence of this equilibrium.

Lemma 1.5.1. *At interim date $t = \frac{1}{2}$, the utilities of liquidity demanders in the presence of* *options are given by*

$$U_d = \frac{\exp\left(\frac{G^P - G_d}{2G^P}\right)}{\sqrt{G^P/G_d}} \mathrm{E}\Big\{ \exp\Big[-\alpha\Big(W_0 + X_0(P_1 - P_0) + z(P_1 - \bar{D}) + \frac{[\bar{D} + \beta_s(s-\bar{D}) - P_1]^2}{2\alpha \operatorname{Var}[D|\mathcal{F}_d]} - f \Big)\Big]\Big\} \tag{1.5.21}$$

and the utilities of liquidity suppliers are given by

$$U_s = \frac{\exp\left(\frac{G^P - G_s}{2G^P}\right)}{\sqrt{G^P/G_s}} \mathrm{E}\Big\{ -\exp\Big[-\alpha\Big(W_0 + X_0(P_1 - P_0) + \frac{[\bar{D} + \beta_P(s-\bar{D}+Cz) - P_1]^2}{2\alpha \operatorname{Var}[D|\mathcal{F}_s]} - f \Big)\Big]\Big\}. \tag{1.5.22}$$

For $\tilde{\pi} \in (0,1)$ *we have that both* $\frac{\exp\left(\frac{G^P - G_d}{2G^P}\right)}{\sqrt{G^P/G_d}} < 1$ *and* $\frac{\exp\left(\frac{G^P - G_s}{2G^P}\right)}{\sqrt{G^P/G_s}} < 1$.

Lemma 1.5.1 states that the interim utility of any agent is the product of two terms after options are introduced, which is just as in the case without participation costs. The first term reflects the effect due to the presence of options, which compensates for the participation cost f and so induces more agents to participate. Hence the previous sufficient condition—for the equilibrium in which options are not available and liquidity suppliers participate partially—still holds.

Proposition 1.5.8. *In the presence of an options market, an equilibrium where all liquidity demanders and a positive fraction of liquidity suppliers participate in the market exists under the sufficient conditions* $\pi \leq \frac{1}{2}$ *and* $f \leq \hat{f}^o$, *where*

$$\hat{f}^o \equiv \frac{\log\left[\left(1 + \frac{1}{4[h+q+\frac{1}{2}(m-q)]^2} c\right)^{\frac{h+q+\frac{1}{2}(m-q)}{h+q}}\right] + \frac{h+q}{h+q+\frac{1}{2}(m-q)} - 1}{2\alpha}$$

and c *is as defined in the Appendix.*

Given this sufficient condition for the equilibrium of partial participation, one can obtain the equilibrium ex ante price at date 0 as follows.

Proposition 1.5.9. *At date 0, the equilibrium price of the risky asset is given by*

$$P_0 = \bar{D} - \alpha \frac{1}{h} \bar{X} - \frac{\pi M^{PO}}{1 - \pi + \pi M^{PO}} \Delta_1 \bar{X}, \tag{1.5.23}$$

where

$$\Delta_0 = \frac{(B-\beta_P)^2\left(\frac{1}{h}+\frac{1}{q}\right)}{(\tilde{\pi})^2\text{Var}[D|\mathcal{F}_s]}, \tag{1.5.24}$$

$$\Delta_1 = \frac{\alpha^3 B\frac{1}{nh}\left(\frac{1}{h}+\frac{1}{m}\right)}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}, \tag{1.5.25}$$

$$\Delta_2 = \frac{\alpha^3\frac{1}{nh^2}\left[1+\frac{(B-\beta_s)^2\left(\frac{1}{h}+\frac{1}{m}\right)}{\text{Var}[D|\mathcal{F}_d]}\right]}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}, \tag{1.5.26}$$

and

$$M^{PO} = \exp\left(\frac{G_s-G_d}{2G^P}\right)\sqrt{\frac{G_d}{G_s}}\exp\left(\frac{1}{2}\alpha\Delta_2\bar{X}^2\right)\sqrt{\frac{1+(\tilde{\pi})^2\Delta_0}{1+\Delta_0(1-\tilde{\pi})^2-\alpha^2\frac{1}{nh}}}. \tag{1.5.27}$$

Proposition 1.5.10. *The introduction of options raises the fraction μ of participating liquidity suppliers.*

When an options market is introduced into the economy, liquidity suppliers are more inclined to provide liquidity. Thus their participation fraction μ increases as a consequence of the welfare gains created by options. Yet when liquidity suppliers participate only partially, their interim utilities are the same as if they do not participate at all. For a liquidity demander, in contrast, expected utility is enhanced by options owing to her full participation (i.e., even before the options were introduced). As a result of these two effects, the identical investors become more willing to hold the risky asset at date 0, which in turn lowers the illiquidity discount and raises the equilibrium ex ante price P_0.

Proposition 1.5.11. *The introduction of options increases the ex ante price P_0, but it reduces both the price impact λ and the price reversal γ.*

This proposition indicates that, if liquidity suppliers participate partially (rather than fully) in the market, then introducing derivatives always raises the ex ante price regardless of the liquidity provision $1-\pi$. This result is in line with the finding of Cao (1999), who studies how stock prices can be affected—through the information acquisition channel—by options listing. At this point, a comparative statics analysis can be used to generate predictions about the effect of participation costs on trading volume in the derivatives market. The trading volume of options is calculated as $V^O = \frac{2\pi(1-\pi)\mu}{(\pi+(1-\pi)\mu)^2}(m-q)$ and depends on the participation cost f.

Proposition 1.5.12. *The trading volume of options exhibits an inverse U shape as a function of the participation cost; thus an increase in the participation cost f raises (resp. lowers) V^O when f is less (resp. greater) than f^o, where f^o is as defined in the Appendix.*

When the participation cost f is high, the participation rate μ of liquidity suppliers is low. In this scenario, reducing the participation cost f leads more suppliers to participate. Then these uninformed liquidity suppliers can buy options (which can be thought of as insurance) in order to hedge the uncertainty of the risky asset's payoff D. Hence the trading volume of options increases, and that volume is a decreasing function of the cost f. If instead the participation cost f is low, then the participation rate μ of liquidity suppliers is high; this scenario implies that the competition among suppliers is intense. So even though a decrease in f attracts more suppliers, the total trading volume of options becomes much lower because each supplier wants to hold a reduced volume of options. Because the long side of options comes from uninformed suppliers, the intense competition under a high participation rate μ makes options expensive. Accordingly, the option trading volume for each liquidity supplier—and thus the total option trading volume—declines. If participation is costly, then this result means that reducing the participation cost could accelerate the growth of an options market (Cao and Ou-Yang, 2009). That implication is consistent with the recent empirical literature on options. In addition, my model offers new predictions about the relation between market participation cost and options trading volume.

The options volume is also an increasing function of the information dispersion, which corresponds to the term $m - q$—that is, the difference in information precision between a liquidity demander and a liquidity supplier. Since the uninformed liquidity suppliers cannot perfectly identify the trading motive of liquidity demanders, they face the risk of trading against the private information of informed demanders. As the adverse selection problem arising from this information asymmetry worsens (higher $m - q$), the uninformed liquidity suppliers are more likely to buy the options as insurance to hedge against the uncertainty; hence the trading volume of options increases. A small peak is evident in the option trading volumes during the financial crisis, a deviation that can be explained by my model.

1.6 Other General Derivative Securities

In previous sections I examined the effect of introducing a set of vanilla options. This section turns to the case of more general derivatives. Specifically, I first analyze what effect the introduction of a generalized straddle has on the ex ante price of the risky asset and on our two liquidity measures; then I check the robustness of this mechanism by examining some concrete examples.

1.6.1 General Derivatives

I consider some general derivatives as modeled by Cao (1999). Specifically: I assume that a general derivative, whose price is denoted by P_G, pays off $g(|D - P_1|)$; here $g(\cdot)$ is a monotonic function.[32] Following the pattern of previous notation, the demand of liquidity demanders and suppliers for this derivative are denoted by (respectively) $X_{d,G}$ and $X_{s,G}$. At date 2, the wealth of liquidity demanders who are equipped with a *generalized straddle* is given by

$$W_{d2} = W_1 + X_d(D - P_1) + X_{d,G}(g(|D - P_1|) - P_G) + z(D - \bar{D}), \tag{1.6.1}$$

and the wealth of liquidity suppliers is given by

$$W_{s2} = W_1 + X_s(D - P_1) + X_{s,G}(g(|D - P_1|) - P_G). \tag{1.6.2}$$

I obtain a partially revealing rational expectations equilibrium that characterizes the effect of introducing the generalized straddle at date 1 (cf. Cao, 1999). This result is stated formally as follows.

Proposition 1.6.1. *At date 1, there exists one equilibrium. The price of the risky asset is given by*

$$P_1 = \bar{D} - \frac{\alpha}{h + \pi m + (1-\pi)q}\bar{X} + \frac{\pi m + (1-\pi)q}{h + \pi m + (1-\pi)q}\left(s - \bar{D} - \frac{\alpha}{m}z\right). \tag{1.6.3}$$

[32] Given that $g(\cdot)$ resides in the positive domain, the quadratic derivative is a special case of this general derivative. A more detailed analysis is given in Section 1.6.2.

Demand for the risky asset is given by

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_d]} - z \tag{1.6.4}$$

for the liquidity demander and by

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha \operatorname{Var}[D|\mathcal{F}_s]} \tag{1.6.5}$$

for the liquidity supplier. The liquidity demander's demand for the general derivative satisfies the equality

$$\int_0^{+\infty} (g(u) - P_G) \exp\left[-\frac{1}{2} G_d u^2 - \alpha X_{d,G}\, g(u) \right] du = 0, \tag{1.6.6}$$

and that of the liquidity supplier satisfies

$$\int_0^{+\infty} (g(u) - P_G) \exp\left[-\frac{1}{2} G_s u^2 - \alpha X_{s,G}\, g(u) \right] du = 0. \tag{1.6.7}$$

As for the general derivative, the market-clearing condition is

$$\pi X_{d,G} + (1 - \pi) X_{s,G} = 0, \tag{1.6.8}$$

where $G_d = h + m$ and $G_s = h + q$.

Proposition 1.6.1 establishes that the introduction of a generalized straddle has no direct influence on the underlying asset's price at date 1, which is the same result as when calls and puts were introduced in Section 1.4. Unlike those vanilla options, a straddle does not affect the demand of agents for the underlying asset. Hence the price impact measure λ and the price reversal measure γ remain unchanged. As before, the introduction of derivatives changes investors' utilities; the next lemma summarizes the results that concern their interim utilities.

Lemma 1.6.1. *At interim date $t = \frac{1}{2}$, if a general derivative is available then the utilities of liquidity demanders are given by*

$$U_d^G = U_d \sqrt{\frac{2G_d}{\pi}} \int_0^{+\infty} \exp\left(-\frac{1}{2} G_d u^2 - \alpha X_{d,G} (g(|u|) - P_G) \right) du \tag{1.6.9}$$

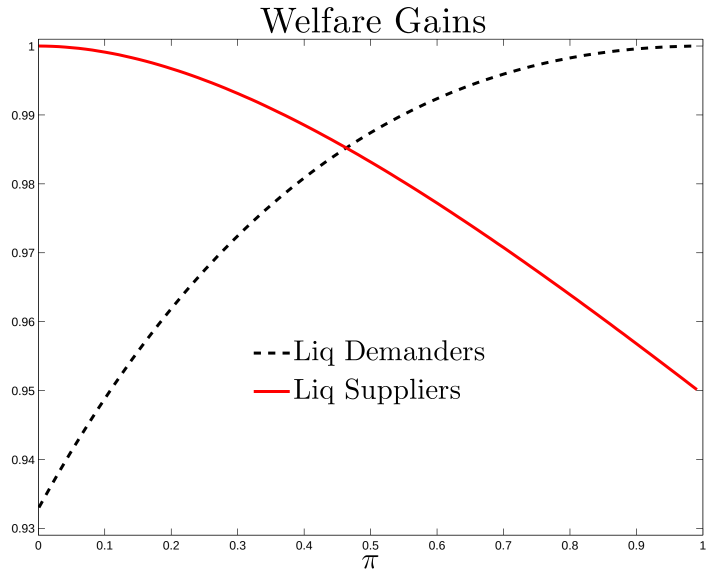


Figure 1.7: Welfare gains from derivatives; $g(y) = y$.

and those of liquidity suppliers are given by

$$U_s^G = U_s \sqrt{\frac{2G_s}{\pi}} \int_0^{+\infty} \exp\left(-\frac{1}{2} G_s u^2 - \alpha X_{s,G}(g(|u|) - P_G) \right) du; \tag{1.6.10}$$

here U_i^G and U_i $(i \in \{s, d\})$ are the interim utilities of agents with and without the general derivatives, respectively.

As suggested by Proposition 1.6.1 and Lemma 1.6.1, it is difficult to obtain analytical solutions for the asset price, demand for the derivative, and interim utilities. I then turn to numerical studies. Figure 1.7 illustrates the welfare gains to the two groups of agents when $g(y) = y$. The welfare gains from derivatives are clearly much greater for liquidity demanders than for liquidity suppliers when the latter dominate the market population (i.e., when π is small).[33] This asymmetric effect of the generalized straddle (as seen graphically in the figure) allows me to confirm the ex ante price results derived previously for the case when vanilla options are introduced to trade. The only difference here is that the two liquidity measures, λ and γ, are not changed by introducing the generalized straddle thanks to the unchanged demand for the underlying asset. To develop a clearer picture of the generalized straddle's effects, I shall next investigate a concrete example of generalized straddles that yields closed-form solutions.

[33] Because of the negative exponential utility, a smaller number indicates a larger welfare gain.

1.6.2 A Quadratic Derivative

Brenna and Cao (1996) analyze the effect of a new quadratic derivative, which can be seen as a special case of the aforementioned generalized straddle. In this section I study the effects of this particular derivative in the presence of a liquidity shock. In particular, I focus on the effects of the quadratic derivative on the illiquidity discount and our two liquidity measures.

In contrast to the case of vanilla options, here I introduce a derivative whose payoff is a quadratic function of the risky asset's payoff D. More specifically, this derivative pays off $(D - P_1)^2$ and its price, prior to expiration, is denoted by P_{G1}. Just as for the vanilla options, this derivative is in zero net supply and expires at date 2. I use $X_{d,G1}$ and $X_{s,G1}$ to denote the demand (of, respectively, liquidity demanders and liquidity suppliers) for this new derivative. If that quadratic derivative is available, then the wealth of liquidity demanders at date 2 is given by

$$W_{d2} = W_1 + X_d(D - P_1) + X_{d,G1}((D - P_1)^2 - P_{G1}) + z(D - \bar{D}) \tag{1.6.11}$$

and that of liquidity suppliers is given by

$$W_{s2} = W_1 + X_s(D - P_1) + X_{s,G1}((D - P_1)^2 - P_{G1}). \tag{1.6.12}$$

I have the following partially revealing rational expectations equilibrium regarding the prices P_1 and P_{G1} of different assets and the corresponding demands of the two groups of agents at date 1.

Proposition 1.6.2. *At date 1, there exists one equilibrium. The underlying risky asset's price is given by*

$$P_1 = \bar{D} - \frac{\alpha}{h + \pi m + (1-\pi)q}\bar{X} + \frac{\pi m + (1-\pi)q}{h + \pi m + (1-\pi)q}\left(s - \bar{D} - \frac{\alpha}{m}z\right), \tag{1.6.13}$$

and the price of the quadratic derivative is given by

$$P_{G1} = \frac{1}{h + \pi m + (1-\pi)q}. \tag{1.6.14}$$

The liquidity demander's demands for the risky asset and the corresponding derivative are

$$X_d = \frac{\mathrm{E}[D|\mathcal{F}_d] - P_1}{\alpha\,\mathrm{Var}[D|\mathcal{F}_d]} - z, \tag{1.6.15}$$

$$X_{d,G1} = \frac{1}{2\alpha}\left(\frac{1}{P_{G1}} - \frac{1}{\mathrm{Var}[D|\mathcal{F}_d]}\right); \tag{1.6.16}$$

the liquidity suppler's demands for the risky asset and the corresponding derivative are

$$X_s = \frac{\mathrm{E}[D|\mathcal{F}_s] - P_1}{\alpha\,\mathrm{Var}[D|\mathcal{F}_s]}, \tag{1.6.17}$$

$$X_{s,G1} = \frac{1}{2\alpha}\left(\frac{1}{P_{G1}} - \frac{1}{\mathrm{Var}[D|\mathcal{F}_s]}\right). \tag{1.6.18}$$

As in Proposition 1.4.1, liquidity demanders take short positions in derivatives while liquidity suppliers take long positions. Proposition 1.6.2 indicates that this result is robust to a derivative with quadratic payoff. However, such a derivative has no effect on demand for the risky asset. In contrast to the nonlinear price function of call and put options illustrated in Proposition 1.4.1, the quadratic derivative's price is the *reciprocal* of the average precision (over all investors) of the risky asset's payoff D (i.e., $1/G$). This difference simply reflects the different payoff structure of these derivatives and does not require any alterations in my mechanism. With regard to interim utilities, the welfare gains stemming from quadratic derivative are of a different magnitude for the two groups of investors; this finding, too, is in line with the results for vanilla options. The following statement formalizes that claim.

Lemma 1.6.2. *At interim date $t = \frac{1}{2}$, if a quadratic derivative is available then the interim utilities of liquidity demanders are given by*

$$U_d = \frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}}\mathrm{E}\Bigg\{-\exp\Bigg[-\alpha\Bigg(W_0 + X_0(P_1 - P_0) + z(P_1 - \bar{D}) + \frac{[\bar{D} + \beta_s(s - \bar{D}) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_d]}\Bigg)\Bigg]\Bigg\} \tag{1.6.19}$$

and the utilities of liquidity suppliers are given by

$$U_s = \frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}}\mathrm{E}\Bigg\{-\exp\Bigg[-\alpha\Bigg(W_0 + X_0(P_1 - P_0) + \frac{[\bar{D} + \beta_P(s - \bar{D} + \frac{\alpha}{m}z) - P_1]^2}{2\alpha\,\mathrm{Var}[D|\mathcal{F}_s]}\Bigg)\Bigg]\Bigg\}. \tag{1.6.20}$$

For $\pi \in (0,1)$ *we have that both* $\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}} < 1$ *and* $\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{G/G_s}} < 1$, *where* E *is the expectation over* (s,z).

For the two groups of agents, the welfare improvement resulting from a quadratic derivative is the same as that from a set of call and put options (the case analyzed previously). Hence the new quadratic derivative will benefit liquidity demanders and suppliers in the same way that options do and will also yield similar equilibrium prices at date 0.

Proposition 1.6.3. *At date 0, the equilibrium price of the risky asset is given by*

$$P_0 = \bar{D} - \alpha \frac{1}{h} \bar{X} - \frac{\pi M^G}{1 - \pi + \pi M^G} \Delta_1 \bar{X}, \tag{1.6.21}$$

where

$$\Delta_0 = \frac{(B - \beta_P)^2 \left(\frac{1}{h} + \frac{1}{q}\right)}{\pi^2 \operatorname{Var}[D|\mathcal{F}_s]}, \tag{1.6.22}$$

$$\Delta_1 = \frac{\alpha^3 B \frac{1}{nh} \left(\frac{1}{h} + \frac{1}{m}\right)}{1 + \Delta_0 (1-\pi)^2 - \alpha^2 \frac{1}{nh}}, \tag{1.6.23}$$

$$\Delta_2 = \frac{\alpha^3 \frac{1}{nh^2} \left[1 + \frac{(B-\beta_P)^2 \left(\frac{1}{h} + \frac{1}{m}\right)}{\operatorname{Var}[D|\mathcal{F}_d]}\right]}{1 + \Delta_0 (1-\pi)^2 - \alpha^2 \frac{1}{nh}}, \tag{1.6.24}$$

and

$$M^G = \exp\left(\frac{G_s - G_d}{2G}\right) \sqrt{\frac{G_d}{G_s}} \exp\left(\frac{1}{2} \alpha \Delta_2 \bar{X}^2\right) \sqrt{\frac{1 + \pi^2 \Delta_0}{1 + \Delta_0 (1-\pi)^2 - \alpha^2 \frac{1}{nh}}}. \tag{1.6.25}$$

Proposition 1.6.4. (1) *For* π^* *as defined in the Appendix: if* $0 < \pi < \pi^*$, *then* P_0 *is higher in the presence than in the absence of a quadratic derivative; if* $\pi^* < \pi < 1$, *then* P_0 *is lower in the presence than in the absence of a quadratic derivative; and if* $\pi = \pi^*$, *then* P_0 *is the same in the presence as in the absence of a quadratic derivative.*

(2) *The introduction of a quadratic derivative alters neither the price impact measure* λ *nor the price reversal measure* γ.

1.7 Conclusion

This paper uses a rational expectations model to examine the effect—on asset returns and liquidity—of introducing an options market. The main results are as follows. First, liquidity demanders (who observe the private signal) take short positions in the introduced options whereas liquidity suppliers (who are informed only by asset prices) take long positions in those options. Second, options provide hedging benefits and increase risk sharing between the demanders and suppliers of liquidity; thus the welfare of market participants is improved. Yet this improvement in welfare is asymmetric across agent types, which affects their trading incentives at date 0. More importantly, I find that introducing derivatives has surprisingly non-monotonic effects on the underlying asset price and so could reconcile the mixed empirical evidence on the effects of options listing. I also find that introducing an options market reduces the price impact λ but leaves the price reversal γ unchanged. Finally, I show that the effects of derivatives on the price and liquidity of the underlying asset are sensitive to the measures of liquidity used and the factors driving asset-specific characteristics. These results constitute new and empirically testable implications concerning how the introduction of financial derivatives affects the underlying asset.

In addition, I endogenize the participation decisions of agents and examine the case where agents must pay a participation fee to enter the market. Such a participation cost naturally reduces the participation of liquidity suppliers and likewise their proportion in the overall agent population. I find that introducing an options market always reduces the illiquidity discount component of the ex ante price P_0 as well as the expected return of the underlying assets—that is, irrespective of the supply of liquidity. Moreover, both illiquidity measures decline after derivatives are introduced. I also provide a new prediction that the trading volume of options exhibits an inverse U shape with respect to the participation cost. Finally, the mechanism proposed here is robust to derivatives with a general payoff structure.

There are several avenues for future research. First, extending the static setting to a dynamic model would be a worthwhile pursuit (see e.g. Cao, 1999). Moreover, because of the CARA-normal framework (adopted to make the model tractable), the options considered in this paper carry no additional information and so are informationally redundant. Relaxing these assumptions and examining the effects of derivatives under more general utility functions and asset payoff distributions may yield findings of considerable interest (see e.g. Chabakauri, Yuan, and Zachariadis, 2014; Malamud, 2015).

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1.9 Appendix

1.9.1 Some useful results from Vayanos and Wang (2012a)

Following Vayanos and Wang (2012a), the interim utilities of liquidity suppliers and demanders at $t = \frac{1}{2}$, denoted by U_s and U_d, can be calculated as follows:

$$\begin{aligned} U_s &= \mathrm{E}\left\{-\exp\left\{-\alpha\left[W_0 + X_0(P_1 - P_0) + \frac{\left[\overline{D} + \beta_P(s - \overline{D} - \frac{\alpha}{m}z) - P_1\right]^2}{2\alpha \mathrm{Var}(D|P_1)}\right]\right\}\right\} \\ &= -\exp(-\alpha F_s)\frac{1}{\sqrt{1 + \frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}\left(\frac{1}{h} + \frac{1}{m} + \frac{\alpha^2}{m^2}\frac{1}{n}\right)}}, \end{aligned} \tag{1.9.1}$$

where

$$F_s = W_0 + X_0(\overline{D} - P_0) - \frac{\alpha}{2h}X_0^2 + \frac{\alpha\frac{(1-B)^2\frac{1}{h^2}}{\mathrm{Var}(D|\mathcal{F}_s)}(X_0 - \overline{X})^2}{2\left[1 + \frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h} + \frac{1}{m} + \frac{\alpha^2}{m^2}\frac{1}{n})\right]}, \tag{1.9.2}$$

and

$$\begin{aligned} U_d &= \mathrm{E}\left\{-\exp\left\{-\alpha\left[W_0 + X_0(P_1 - P_0) + z(P_1 - \overline{D}) + \frac{[\overline{D} + \beta_s(s - \overline{D}) - P_1]^2}{2\alpha \mathrm{Var}(D|\mathcal{F}_d)}\right]\right\}\right\} \\ &= -\exp(-\alpha F_d)\frac{1}{\sqrt{1 + \frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h} + \frac{1}{m})(1 + \alpha^2\frac{1}{m}\frac{1}{n}) - \alpha^2\frac{1}{h}\frac{1}{n}}}, \end{aligned} \tag{1.9.3}$$

where

$$\begin{aligned} F_d =& W_0 + X_0(\overline{D} - P_0) - \frac{\alpha}{h}X_0\overline{X} + \frac{1}{2}\frac{\alpha}{h}\overline{X}^2 \\ &- \frac{\alpha\left\{B^2(\frac{1}{h} + \frac{1}{m})(1 + \alpha^2\frac{1}{m}\frac{1}{n})(X_0 - \overline{X})^2 + (\frac{1}{h} + \frac{1}{m})^2\frac{\alpha^2}{n}\left[\frac{2B\frac{1}{h}X_0\overline{X}}{\frac{1}{h} + \frac{1}{m}} + [\frac{(B-\beta_s)^2}{h\mathrm{Var}(D|s)} - B^2]\overline{X}^2\right]\right\}}{2\left[1 + \frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h} + \frac{1}{m})(1 + \alpha^2\frac{1}{m}\frac{1}{n}) - \alpha^2\frac{1}{h}\frac{1}{n}\right]}. \end{aligned} \tag{1.9.4}$$

Furthermore, it is easy to show that

$$\frac{(B - \beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h} + \frac{1}{m})(1 + \alpha^2\frac{1}{m}\frac{1}{n}) = \Delta_0(1 - \pi)^2, \tag{1.9.5}$$

$$\frac{(B - \beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h} + \frac{1}{m} + \frac{\alpha^2}{m^2}\frac{1}{n}) = \Delta_0\pi^2. \tag{1.9.6}$$

When $X_0 = \overline{X}$, we have four useful equations:

$$\frac{dF_s}{dX_0} = \overline{D} - P_0 - \frac{\alpha}{h}\overline{X}, \tag{1.9.7}$$

$$F_s = W_0 + \overline{X}(\overline{D} - P_0) - \frac{\alpha}{2h}\overline{X}^2, \tag{1.9.8}$$

$$\frac{dF_d}{dX_0} = \frac{dF_s}{dX_0} - \Delta_1\overline{X}, \tag{1.9.9}$$

$$F_d = F_s - \frac{1}{2}\Delta_2\overline{X}^2. \tag{1.9.10}$$

An agent chooses X_0 to maximize his/her utility $U \equiv \pi U_d + (1-\pi)U_s$ at $t = 0$. The first-order condition is given by

$$\pi \exp(-\alpha F_d)\frac{dF_d}{dX_0}\frac{1}{\sqrt{1 + \Delta_0(1-\pi)^2 - \alpha^2\frac{1}{nh}}} + (1-\pi)\exp(-\alpha F_s)\frac{dF_s}{dX_0}\frac{1}{\sqrt{1+\Delta_0\pi^2}} = 0 \tag{1.9.11}$$

$$\Leftrightarrow \pi(\overline{D} - P_0 - \frac{\alpha}{h}\overline{X} - \Delta_1\overline{X})M + (1-\pi)(\overline{D} - P_0 - \frac{\alpha}{h}\overline{X}) = 0. \tag{1.9.12}$$

Then we obtain the equilibrium price of the risky stock as shown in (1.2.13).

1.9.2 Asymmetric Information with a Squared Contract

Proof of Proposition 1.3.1. To prove that the proposed prices and demands are obtained in equilibrium, we should verify that the market clears and the Euler condition holds for the risky stock and the squared contract. We first prove that the proposed investors' demands do clear the market at the equilibrium prices. Specifically, for the risky stock, it is easy to check that

$$\pi X_d + (1-\pi)X_s = \bar{X}. \tag{1.9.13}$$

Given the demands for the risky stock, checking the market clearing condition for the squared contract is equivalent to verifying that the following relation holds

$$\frac{\pi}{2\alpha}(G - G_d) + \frac{1-\pi}{2\alpha}(G - G_s) = 0. \tag{1.9.14}$$

Then we need to show that the investors' demands are optimal. The liquidity suppliers' wealth at $t = 2$ is given by

$$W_{s2} = W_1 + X_s(D - P_1) + X_{s,SC}(D^2 - P_{SC}). \tag{1.9.15}$$

Using the conjectured equilibrium price of the squared contract, the optimization problem of a liquidity supplier can be written as

$$
\begin{aligned}
&-\mathrm{E}\left[\exp\left(-\alpha W_{s2}\right)|\mathcal{F}_s\right] \\
=&-\frac{1}{\sqrt{1+2\alpha X_{s,SC}\mathrm{Var}[D|\mathcal{F}_s]}}\exp\left[\begin{array}{c}\frac{\mathrm{Var}[D|\mathcal{F}_s]\left(\alpha X_s+2\alpha X_{s,SC}\mathrm{E}[D|\mathcal{F}_s]\right)^2}{2(1+2\alpha X_{s,SC}\mathrm{Var}[D|\mathcal{F}_s])} \\ -\alpha X_{s,SC}\left(\mathrm{E}[D|\mathcal{F}_s]^2-P_1^2-\frac{1}{G}\right)-\alpha(W_1+X_s(\mathrm{E}[D|\mathcal{F}_s]-P_1))\end{array}\right]
\end{aligned} \tag{1.9.16}
$$

Thus, given the equilibrium securities prices, the first-order condition of liquidity suppliers utility maximization problem leads to Equations (1.3.7)-(1.3.8). Along the same line of reasoning, we obtain the liquidity demander's demands for assets as shown in Equations (1.3.5)-(1.3.6). □

Proof of Proposition 1.3.2. When a squared contract is available to trade, the wealth of liquidity demanders at date 2 is given by

$$
W_{d2}=W_1+\left(\frac{\mathrm{E}[D|\mathcal{F}_d]-P_1}{\alpha\mathrm{Var}[D|\mathcal{F}_d]}-\frac{G-G_d}{\alpha}P_1\right)(D-P_1)+z(P_1-\bar{D})+\frac{G-G_d}{2\alpha}\left(D^2-P_1^2-\frac{1}{G}\right), \tag{1.9.17}
$$

and the wealth of liquidity suppliers is given by

$$
W_{s2}=W_1+\left(\frac{\mathrm{E}[D|\mathcal{F}_s]-P_1}{\alpha\mathrm{Var}[D|\mathcal{F}_s]}-\frac{G-G_s}{\alpha}P_1\right)(D-P_1)+\frac{G-G_s}{2\alpha}\left(D^2-P_1^2-\frac{1}{G}\right). \tag{1.9.18}
$$

The marginal rate of substitution of investor i, $i\in\{d,s\}$, between wealth contingent on $D=D_h$ and $D=D_l$ is given by

$$
\begin{aligned}
M_{hl}^i=&\frac{\exp\left\{-(D_h-\mathrm{E}[D|\mathcal{F}_i])^2G_i/2\right\}\exp\{-\alpha W_{2i}(D_h)\}}{\exp\left\{-(D_l-\mathrm{E}[D|\mathcal{F}_i])^2G_i/2\right\}\exp\{-\alpha W_{2i}(D_l)\}} \\
=&\exp\left\{-\frac{1}{2}G_i(D_h-D_l)(D_h+D_l-2\mathrm{E}[D|\mathcal{F}_i])\right. \\
&\left.-\left(\frac{\mathrm{E}[D|\mathcal{F}_i]-P_1}{\mathrm{Var}[D|\mathcal{F}_i]}-(G-G_i)P_1\right)(D_h-D_l)-\frac{G-G_i}{2}(D_h^2-D_l^2)\right\} \\
=&\exp\left\{-\frac{1}{2}G(D_h-D_l)(D_h+D_l-2P_1)\right\}.
\end{aligned} \tag{1.9.19}
$$

As shown by (1.9.19), the marginal rate of substitution is the same for all investors. That is, the introduction of the squared contract allows agents to achieve a Pareto efficient allocation. □

Proof of Lemma 1.3.1. Given the equilibrium in Proposition 1.3.1, the expected utility of a liquidity demander at $t=1$ can be written as

$$
\begin{aligned}
&-\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\\
=&-\frac{1}{\sqrt{\frac{G}{G_d}}}\exp\left\{-\alpha W_1+\alpha z(\overline{D}-P_1)+\frac{G-G_d}{2G}-\frac{G_d}{2}\left(E[D|\mathcal{F}_d]-P_1\right)^2\right\}\\
=&-\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{G/G_d}}\left\{\exp\left[-\alpha\left(W_0+X_0(P_1-P_0)+z(P_1-\overline{D})+\frac{[\,\overline{D}+\beta_s(s-\overline{D})\ -P_1]^2}{2\alpha\mathrm{Var}[D|\mathcal{F}_d]}\right)\right]\right\}.
\end{aligned}
\tag{1.9.20}
$$

Taking the expectation of (1.9.20) over (s,z) yields the interim utility U_d of the liquidity demander at $t=\frac{1}{2}$, i.e. (1.3.12). Similarly, we can obtain the interim utilities of liquidity supplies taking the form of (1.3.13).

As $0<\frac{G}{G_d}<1$, $\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}}$ is a decreasing function of $\frac{G_d}{G}$. Therefore, $\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}}<1$. Likewise, $\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}}$ is an increasing function of $\frac{G_s}{G}$ due to the fact that $\frac{G}{G_s}>1$. Then we get $\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}}<1$. □

Proof of Lemma 1.3.2. The ratio of the additional factor in the interim utilities resulting from introducing a squared contract is given by

$$
\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}}\Big/\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}}=\exp\left(\frac{G_s-G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}. \tag{1.9.21}
$$

Define $\pi^*\equiv\frac{1}{\log\left(1+\frac{m-q}{h+q}\right)}-\frac{h+q}{m-q}$, we have

(1) $\exp\left(\frac{G_s-G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}>1$ only if $\frac{m-q}{h+\pi m+(1-\pi)q}+\log\left(\frac{h+q}{h+m}\right)<0$, which is equivalent to $\pi>\pi^*$.

(2) $\exp\left(\frac{G_s-G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}<1$ only if $\frac{m-q}{h+\pi m+(1-\pi)q}+\log\left(\frac{h+q}{h+m}\right)>0$, which is equivalent to $\pi<\pi^*$.

(3) $\exp\left(\frac{G_s-G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}=1$ only if $\frac{m-q}{h+\pi m+(1-\pi)q}+\log\left(\frac{h+q}{h+m}\right)=0$, which is equivalent to $\pi=\pi^*$.

As $\log\left(1+\frac{m-q}{h+q}\right)<\frac{m-q}{h+q}$, we have $\frac{1}{\log\left(1+\frac{m-q}{h+q}\right)}-\frac{h+q}{m-q}>0$.

□

Proof of Proposition 1.3.3. According to the calculation from Vayanos and Wang (2012a), we compute the expected utilities of liquidity demanders and liquidity suppliers at $t = \frac{1}{2}$ as following. For liquidity suppliers, as shown in Lemma 1.3.1

$$U_s = \frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}} \mathrm{E}\left\{-\exp\left[-\alpha\left(W_0 + X_0(P_1 - P_0) + \frac{[\,\overline{D} + \beta_P(s - \overline{D} + Cz)\;\; - P_1]^2}{2\alpha \mathrm{Var}(D|\mathcal{F}_s)}\right)\right]\right\}, \tag{1.9.22}$$

then we have

$$U_s = -\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}} \exp(-\alpha F_s) \frac{1}{\sqrt{1 + \frac{(B-\beta_{P_1})^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h} + \frac{1}{m} + \frac{\alpha^2}{m^2}\frac{1}{n})}}, \tag{1.9.23}$$

where F_s is the same as the case of Vayanos and Wang (2012a).

For liquidity suppliers, the interim utilities are given as

$$U_d = \frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}} \mathrm{E}\left\{-\exp\left[-\alpha\left(W_0 + X_0(P_1 - P_0) + z(P_1 - \overline{D}) + \frac{[\,\overline{D} + \beta_s(s - \overline{D})\;\; - P_1]^2}{2\alpha \mathrm{Var}(D|\mathcal{F}_d)}\right)\right]\right\}, \tag{1.9.24}$$

then we have:

$$U_d = -\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}} \exp(-\alpha F_d) \frac{1}{\sqrt{1 + \frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h} + \frac{1}{m})(1 + \alpha^2\frac{1}{h}\frac{1}{n}) - \alpha^2\frac{1}{m}\frac{1}{n}}}, \tag{1.9.25}$$

where F_d is the same as the case of Vayanos and Wang (2012a). Based on the above expected utilities, the identical investor's expected utility at $t = 0$ is given by

$$U \equiv \pi U_d + (1 - \pi) U_s. \tag{1.9.26}$$

The first-order condition of the optimization problem is given by

$$\pi \exp(-\alpha F_d) \frac{dF_d}{dX_0} \frac{\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}}}{\sqrt{1 + \Delta_0(1-\pi)^2 - \alpha^2\frac{1}{nh}}} + (1-\pi)\exp(-\alpha F_s)\frac{dF_s}{dX_0} \frac{\frac{\exp\left(\frac{G-G_s}{2G}\right)}{\sqrt{\frac{G}{G_s}}}}{\sqrt{1 + \Delta_0 \pi^2}} = 0 \tag{1.9.27}$$

$$\Leftrightarrow \pi(\overline{D} - P_0 - \frac{\alpha}{h}\overline{X} - \Delta_1 \overline{X}) M^O + (1-\pi)(\overline{D} - P_0 - \frac{\alpha}{h}\overline{X}) = 0. \tag{1.9.28}$$

Then we obtain the ex ante price of the risky asset

$$P_0 = \overline{D} - \frac{\alpha}{h}\overline{X} - \frac{\pi M^O}{1-\pi+\pi M^O}\Delta_1\overline{X}, \tag{1.9.29}$$

where

$$M^O = \exp\left(\frac{G_s - G_d}{2G}\right)\sqrt{\frac{G_d}{G_s}}\exp\left(\frac{1}{2}\alpha\Delta_2\overline{X}^2\right)\sqrt{\frac{1+\pi^2\Delta_0}{1+\Delta_0(1-\pi)^2-\alpha^2\frac{1}{nh}}}. \tag{1.9.30}$$

□

Proof of Proposition 1.3.4. See the proofs of Lemma 1.3.2 and Proposition 1.3.3. □

Proof of Proposition 1.3.5. The signed volume of liquidity suppliers is

$$\begin{aligned}(1-\pi)(X_s-\overline{X}) =&(1-\pi)\left(\frac{\mathrm{E}(D|\mathcal{F}_s)-P_1}{\alpha\mathrm{Var}(D|\mathcal{F}_s)} - \frac{G-G_s}{\alpha}P_1 - \overline{X}\right)\\ =&(1-\pi)\left(\frac{\overline{D}+\beta_P\frac{P_1-A}{B}-P_1}{\alpha\mathrm{Var}(D|\mathcal{F}_s)} - \frac{\pi(m-q)}{\alpha}P_1 - \overline{X}\right).\end{aligned} \tag{1.9.31}$$

According to the definition of the price impact shown in (1.2.15), we have

$$\begin{aligned}\lambda =&\frac{\mathrm{Cov}\left(P_1-P_0,\pi(X_d-\overline{X})\right)}{\mathrm{Var}\left[\pi(X_d-\overline{X})\right]}\\ =&\frac{-\mathrm{Cov}\left(P_1-P_0,(1-\pi)\left(X_s-\overline{X}\right)\right)}{\mathrm{Var}\left[(1-\pi)\left(X_s-\overline{X}\right)\right]}\\ =&\frac{\alpha\mathrm{Var}(D|\mathcal{F}_s)}{(1-\pi)\left[\pi(m-q)\mathrm{Var}(D|\mathcal{F}_s)+1-\frac{\beta_P}{B}\right]}.\end{aligned} \tag{1.9.32}$$

As $m > q$, we have $\lambda < \frac{\alpha\mathrm{Var}(D|\mathcal{F}_s)}{(1-\pi)(1-\frac{\beta_P}{B})}$, which completes the proof of the first part. From Proposition 1.3.1, P_1 is unchanged after the introduction of derivatives. As a result, the price reversal γ is not affected according to the definition of γ. □

1.9.3 Asymmetric Information with an Options Market

Proof of Proposition 1.4.1. To prove that the proposed prices and demands are obtained in equilibrium, we should verify that the market clears and the Euler condition holds for the risky stock and options. We first prove that the proposed investors' demands do clear the market at the equilibrium prices. Specifically, for the risky stock, it is easy to check that

$$\pi X_d + (1-\pi) X_s = \overline{X}. \tag{1.9.33}$$

Given the demands for the risky stock, checking the market clearing condition for options is equivalent to verifying that the following relation holds

$$\frac{\pi}{\alpha}(G - G_d) + \frac{1-\pi}{\alpha}(G - G_s) = 0. \tag{1.9.34}$$

For a liquidity demander, his wealth at $t = 2$ is given by:

$$\begin{aligned} W_{d2} =& W_1 + X_d(D - P_1) + z(D - \overline{D}) \\ &+ \int_0^{+\infty} X_{d,CK}\left[(D-K)^+ - P_{CK}\right] dK + \int_{-\infty}^{0} X_{d,PK}\left[(K-D)^+ - P_{PK}\right] dK. \end{aligned} \tag{1.9.35}$$

Given the proposed equilibrium prices, we then show that the equilibrium demands for the risky stock and call and put options satisfy the first-order conditions.

At $t = 1$, a liquidity demander holds X_d shares of the risky stock to maximize the expected utility conditional on signal s,

$$-\operatorname{E}\exp\left(-\alpha W_{d2}\right). \tag{1.9.36}$$

We compute the terms in (1.9.36) separately. Firstly, it can be easily verified

$$\int_0^{+\infty} (D-K)^+ dK + \int_{-\infty}^{0} (K-D)^+ dK = \frac{D^2}{2}.$$

Then we compute the integration of option prices as follows

$$\begin{aligned}
&\int_0^{+\infty} P_{CK} dK + \int_{-\infty}^{0} P_{PK} dK \\
&= \int_0^{+\infty} (P_1 - K)\mathcal{N}\left(\sqrt{G}\,(P_1 - K)\right) + \int_{-\infty}^{0} (K - P_1)\mathcal{N}\left(\sqrt{G}\,(K - P_1)\right) + \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi G}} \exp\left(-\frac{G(P_1 - K)^2}{2}\right) dK \\
&= \int_0^{+\infty} (P_1 - K)\mathcal{N}\left(\sqrt{G}\,(P_1 - K)\right) + \int_{-\infty}^{0} (K - P_1)\mathcal{N}\left(\sqrt{G}\,(K - P_1)\right) + \frac{1}{G} \\
&= \int_0^{+\infty} (P_1 - K) \int_{-\infty}^{\sqrt{G}(P_1 - K)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx dK + \int_{-\infty}^{0} (K - P_1) \int_{-\infty}^{\sqrt{G}(K - P_1)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx dK + \frac{1}{G} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\sqrt{G}(P_1 - K)} \int_0^{P_1 - \frac{x}{\sqrt{G}}} (P_1 - K) dK \exp\left(-\frac{1}{2}x^2\right) dx \int_{-\infty}^{\sqrt{G}(K - P_1)} \int_{P_1 + \frac{x}{\sqrt{G}}}^{0} (K - P_1) dK \exp\left(-\frac{1}{2}x^2\right) dx \right\} + \frac{1}{G} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{G}(P_1 - K)} \left(\frac{1}{2}P_1^2 - \frac{1}{2G}x^2\right) \exp\left(-\frac{1}{2}x^2\right) dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{G}(K - P_1)} \left(\frac{1}{2}P_1^2 - \frac{1}{2G}x^2\right) \exp\left(-\frac{1}{2}x^2\right) dx + \frac{1}{G} \\
&= \frac{1}{2}P_1^2 - \frac{1}{2G} + \frac{1}{G} \\
&= \frac{1}{2}\left(P_1^2 + \frac{1}{G}\right)
\end{aligned}$$

Substituting above terms into liquidity demander's wealth (1.9.35) yields

$$W_{d2} = W_1 + X_d(D - P_1) + z(D - \overline{D}) + \frac{G - G_d}{2\alpha} D^2 - \frac{G - G_d}{2\alpha}\left(P_1^2 + \frac{1}{G}\right), \tag{1.9.37}$$

and the optimization problem of a liquidity demander can be rewritten as[34]

$$
\begin{aligned}
&-\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\\
&=-\mathrm{E}\left[\exp\left(-\alpha W_1-\alpha X_d(D-P_1)-\alpha z(D-\overline{D})-\frac{G-G_d}{2}D^2+\frac{G-G_d}{2}\left(P_1^2+\frac{1}{G}\right)\right)|s\right]\\
&=-\frac{1}{\sqrt{1+\frac{1}{G_d}(G-G_d)}}*\\
&\exp\left\{...+\alpha X_d P_1-\alpha(X_d+z)\mathrm{E}(D|\mathcal{F}_d)-\frac{G-G_d}{2}\mathrm{E}^2(D|\mathcal{F}_d)+\frac{1}{2}\frac{\left(\alpha(X_d+z)+(G-G_d)\mathrm{E}(D|\mathcal{F}_d)\right)^2}{\left(1+\frac{1}{G_d}(G-G_d)\right)G_d}\right\}.
\end{aligned}
$$

The first order condition with respect to X_d is

$$
\alpha P_1-\alpha\mathrm{E}(D|\mathcal{F}_d)+\alpha\frac{\alpha(X_d+z)+(G-G_d)\mathrm{E}(D|\mathcal{F}_d)}{\left(1+\frac{1}{G_d}(G-G_d)\right)G_d}=0, \tag{1.9.38}
$$

which implies that

$$
X_d=\frac{\mathrm{E}(D|\mathcal{F}_d)-P_1}{\alpha\mathrm{Var}(D|\mathcal{F}_d)}-\frac{G-G_d}{\alpha}P_1-z. \tag{1.9.39}
$$

We have shown that the Euler equation holds for the risky stock at the proposed demand. Next we check whether the Euler condition holds for options. In other words, we need to prove

$$
\mathrm{E}[((D-K)^+-P_{CK})\exp(-\alpha W_{d2})|\mathcal{F}_d]=0, \tag{1.9.40}
$$

$$
\mathrm{E}[((K-D)^+-P_{PK})\exp(-\alpha W_{d2})|\mathcal{F}_d]=0. \tag{1.9.41}
$$

Based on the above results, we can write the liquidity demanders' wealth at $t=2$ as

$$
W_{d2}=W_1+\frac{1}{\alpha}(G_d\mathrm{E}(D|\mathcal{F}_d)-GP_1-z)(D-P_1)+z(D-\overline{D})+\frac{G-G_d}{2\alpha}\left(D^2-P_1^2-\frac{1}{G}\right), \tag{1.9.42}
$$

Substituting (1.9.39) into (1.9.36) yields

$$
\begin{aligned}
&-\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\\
&=-\frac{1}{\sqrt{\frac{G}{G_d}}}\exp\left\{-\alpha W_1+\alpha z(\overline{D}-P_1)+\frac{G-G_d}{2G}-\frac{G_d}{2}\left(E(D|\mathcal{F}_d)-P_1\right)^2\right\}.
\end{aligned} \tag{1.9.43}
$$

[34]To make the first-order condition clear, the constant mean term is dropped in our calculation and is replaced by ....The dropped term is $-\alpha W_{d1}+\alpha z\overline{D}+\frac{G-G_d}{2}\left(P_1^2+\frac{1}{G}\right)$.

Let $x = D - P_1$, $\mu = \mathrm{E}(D|\mathcal{F}_d) - P_1$, and $G_d^{-1} = Var(D - P_1|\mathcal{F}_d)$, we have

$$
\begin{aligned}
&\mathrm{E}\left[(D-K)^{+}\exp(-\alpha W_{d2})|\mathcal{F}_d\right] \\
=&\frac{1}{\sqrt{\frac{G}{G_d}}}\exp\left(-\alpha W_1 + \alpha z(\overline{D} - P_1) + \frac{G-G_d}{2G}\right) \\
&\times \int_{K-P_1}^{+\infty} [x-(K-P_1)]\sqrt{\frac{G_d}{2\pi}}\exp\left(-x(\mu G_d - \frac{1}{2}x(G-G_d)) - \frac{G_d(x-\mu)^2}{2}\right)dx \\
=&\sqrt{\frac{G}{G_d}}\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\int_{K-P_1}^{+\infty}[x-(K-P_1)]\sqrt{\frac{G_d}{2\pi}}\exp\left(-\frac{G}{2}x^2\right)dx \\
=&\sqrt{\frac{G}{G_d}}\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\left\{\frac{\sqrt{G_d}}{G}\frac{1}{\sqrt{2\pi}}\exp(-\frac{G(P_1-K)^2}{2}) + (P_1-K)\sqrt{\frac{G_d}{G}}\mathcal{N}\left(\sqrt{G}(P_1-K)\right)\right\} \\
=&\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right]\left\{\frac{1}{\sqrt{2\pi G}}\exp\left(-\frac{G\left(P_1-K\right)^2}{2}\right) + \left(P_1-K\right)\mathcal{N}\left(\sqrt{G}\left(P_1-K\right)\right)\right\}.
\end{aligned}
\tag{1.9.44}
$$

Combining (1.9.40) and (1.9.44) results in the equilibrium price of call option in (1.4.4). This verifies the proposed prices in the proposition. Following similar procedures, it is easy to demonstrate that for liquidity suppliers, demands of risky asset and options do take the forms in the proposition. □

Proof of Lemma 1.4.1. See the proof of Lemma 1.3.1 □

Proof of Lemma 1.4.2. See the proof of Lemma 1.3.2. □

Proof of Proposition 1.4.2. See the proof of Proposition 1.3.3. □

Proof of Proposition 1.4.3. See the proofs of Lemma 1.4.2 and Proposition 1.4.2. □

Proof of Proposition 1.4.4. See the proof of Proposition 1.4.4. □

Proof of Proposition 1.4.5. The interim utilities of liquidity suppliers and demanders in the presence of options are provided in Lemma 1.4.1, whereas the interim utilities in the absence of options are given in (1.9.1) and (1.9.48). Because the ex ante utility is the expectation of the interim utilities defined in (1.2.12), we can calculate the ratio of utilities of the case with options to the case without options as follows:

$$\frac{\exp\left(\frac{G-G_d}{2G}\right)}{\sqrt{\frac{G}{G_d}}}\exp\left(-\alpha\Delta_1\overline{X}\left(\frac{\pi M^O}{1-\pi+\pi M^O}-\frac{\pi M}{1-\pi+\pi M}\right)\right)\frac{\pi M^O+1-\pi}{\pi M^O+(1-\pi)\frac{M^O}{M}}. \tag{1.9.45}$$

The ratio of ex ante utilities increases in π when the precision of the risky asset payoff is small (i.e. $h < \overline{h}$), whereas it first decreases and then increases in π when the precision of the risky asset payoff is large (i.e. $h > \overline{h}$). The maximum ratio is smaller than or equal to one. As we are using the exponential utility which are negative, the ex ante utility in the presence of options is higher than that in the absence of options. □

1.9.4 Participation Costs

Proof of Proposition 1.5.1. Substituting the demand schedules of liquidity demanders (1.5.2) and liquidity suppliers (1.5.3) into the market clear condition (1.5.4) yields the equilibrium price of the risky asset. □

Proof of Proposition 1.5.2. Following Vayanos and Wang (2012a), the interim utilities of participating liquidity suppliers and demanders, $U_{s,P}$ and $U_{d,P}$, can be calculated as follows:

$$U_{s,P} = -\frac{\exp(-\alpha F_s)}{\sqrt{1+\frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}\left(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n}\right)}}, \tag{1.9.46}$$

where

$$F_s = W_0 + X_0(\overline{D}-P_0) - \frac{\alpha}{2h}X_0^2 + \frac{\alpha\frac{(1-B)^2\frac{1}{h^2}}{\mathrm{Var}(D|\mathcal{F}_s)}(X_0-\overline{X})^2}{2\left[1+\frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n})\right]} - f, \tag{1.9.47}$$

and

$$U_{d,P} = -\frac{\exp(-\alpha F_d)}{\sqrt{1+\frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h}+\frac{1}{m})(1+\alpha^2\frac{1}{m}\frac{1}{n})-\alpha^2\frac{1}{h}\frac{1}{n}}}, \tag{1.9.48}$$

where

$$
\begin{aligned}
F_d =& W_0 + X_0(\overline{D} - P_0) - \frac{\alpha}{h} X_0 \overline{X} + \frac{1}{2}\frac{\alpha}{h}\overline{X}^2 \\
&- \frac{\alpha \left\{ B^2(\frac{1}{h}+\frac{1}{m})(\alpha^2 \frac{1}{m}\frac{1}{n})(X_0 - \overline{X})^2 + (\frac{1}{h}+\frac{1}{m})^2 \frac{\alpha^2}{n} \left[\frac{2B\frac{1}{h}X_0\overline{X}}{\frac{1}{h}+\frac{1}{m}} + [\frac{(B-\beta_s)^2}{h\mathrm{Var}(D|s)} - B^2]\overline{X}^2 \right] \right\}}{2\left[1 + \frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h}+\frac{1}{m})(1+\alpha^2\frac{1}{m}\frac{1}{n}) - \alpha^2 \frac{1}{h}\frac{1}{n}\right]} - f.
\end{aligned}
\tag{1.9.49}
$$

In the case of not participating, the interim utilities of liquidity suppliers are:

$$
U_{s,NP} = -\exp\left\{ -\alpha \left[W_0 + X_0(\overline{D} - P_0) - \frac{1}{2}\alpha X_0^2 \frac{1}{h} \right] \right\}. \tag{1.9.50}
$$

Liquidity suppliers are willing to enter the market if $U_{s,P} \geq U_{s,NP}$, that is

$$
-\frac{\exp(-\alpha F_s)}{\sqrt{1 + \frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}\left(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n}\right)}} \geq -\exp\left\{ -\alpha \left[W_0 + X_0(\overline{D} - P_0) - \frac{1}{2}\alpha X_0^2 \frac{1}{h} \right] \right\} \tag{1.9.51}
$$

$$
\Leftrightarrow \exp(2\alpha f) \leq 1 + \frac{(\pi^*)^2}{[h+q+\pi^*(m-q)]^2} \frac{h^2(m-q)^2}{h+q} \left(\frac{1}{h} + \frac{1}{m} + \frac{\alpha^2}{m^2}\frac{1}{n} \right). \tag{1.9.52}
$$

Let $c = \frac{h^2(m-q)^2}{h+q}\left(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n}\right) = \frac{h}{q}(m-q)^2$. Based on the fact that $\frac{(\pi^*)^2}{[h+q+\pi^*(m-q)]^2}$ increases in π^* and thereby decreases in μ, we have the following three scenarios:

Case 1: if $f \leq f^1 \equiv \frac{\log\left(1+\frac{(\pi)^2}{[h+q+\pi(m-q)]^2}c\right)}{2\alpha}$, then (1.9.52) holds for $\mu = 1$, indicating that all liquidity suppliers participate in the market.

Case 2: if $f^1 < f < f^2$, where $f^2 \equiv \frac{\log\left(1+\frac{1}{(h+m)^2}c\right)}{2\alpha}$, then (1.9.52) holds as an equality for $\mu \in (0,1)$ given by $\mu = \frac{\pi}{1-\pi}\left\{\frac{m-q}{h+q}\left[\sqrt{\frac{h}{q(e^{2\alpha f}-1)}} - 1\right] - 1\right\}$, indicating that a positive fraction of liquidity suppliers participate in the market.

Case 3: if $f \geq f^2$, then (1.9.52) does not hold for any $\mu \in (0,1]$, indicating no liquidity suppliers participate. □

Proof of Proposition 1.5.3. In the presence of participation costs, (1.9.5) and (1.9.6) can be rewritten as

$$\frac{(B-\beta_s)^2}{\mathrm{Var}(D|\mathcal{F}_d)}(\frac{1}{h}+\frac{1}{m})(1+\alpha^2\frac{1}{m}\frac{1}{n}) = \Delta_0(1-\pi^*)^2, \tag{1.9.53}$$

$$\frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n}) = \Delta_0(\pi^*)^2. \tag{1.9.54}$$

Similarly, when $X_0 = \overline{X}$, (1.9.7)-(1.9.10) are rewritten as

$$\frac{dF_s}{dX_0} = \overline{D} - P_0 - \frac{\alpha}{h}\overline{X}, \tag{1.9.55}$$

$$F_s = W_0 + \overline{X}(\overline{D} - P_0) - \frac{\alpha}{2h}\overline{X}^2 - f, \tag{1.9.56}$$

$$\frac{dF_d}{dX_0} = \frac{dF_s}{dX_0} - \Delta_1\overline{X}, \tag{1.9.57}$$

$$F_d = F_s - \frac{1}{2}\Delta_2\overline{X}^2. \tag{1.9.58}$$

In the case of not participating in the market, the interim utilities of liquidity demanders are

$$U_{d,NP} = -\frac{\exp\left\{-\alpha\left[W_0 + X_0(\overline{D} - P_0) - \frac{\alpha\frac{1}{h}}{2(1-\alpha^2\frac{1}{h}\frac{1}{n})}\overline{X}^2\right]\right\}}{\sqrt{1-\alpha^2\frac{1}{h}\frac{1}{n}}} \tag{1.9.59}$$

When liquidity demanders fully participate, we have $U_{d,P} > U_{d,NP}$, that is

$$\exp\left\{2\alpha\left[-\frac{\alpha^3\frac{1}{h^2}\frac{1}{n}}{2(1-\alpha^2\frac{1}{h}\frac{1}{n})}\overline{X}^2 + \frac{1}{2}\Delta_2\overline{X}^2 + f\right]\right\} < \frac{1+\Delta_0(1-\pi^*)^2 - \alpha^2\frac{1}{h}\frac{1}{n}}{1-\alpha^2\frac{1}{h}\frac{1}{n}} \tag{1.9.60}$$

The sufficient condition for (1.9.60) to hold is

$$\frac{\exp\left\{2\alpha\left[-\frac{\alpha^3\frac{1}{h^2}\frac{1}{n}}{2(1-\alpha^2\frac{1}{h}\frac{1}{n})}\overline{X}^2 + \frac{1}{2}\Delta_2\overline{X}^2 + f\right]\right\}}{\exp(2\alpha f)} < \frac{\frac{1+\Delta_0(1-\pi^*)^2-\alpha^2\frac{1}{h}\frac{1}{n}}{1-\alpha^2\frac{1}{h}\frac{1}{n}}}{1+\frac{(B-\beta_P)^2}{\mathrm{Var}(D|\mathcal{F}_s)}(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n})} \tag{1.9.61}$$

$$\Leftrightarrow \exp\left\{2\alpha\left[-\frac{\alpha^3\frac{1}{h^2}\frac{1}{n}}{2(1-\alpha^2\frac{1}{h}\frac{1}{n})}\overline{X}^2 + \frac{1}{2}\Delta_2\overline{X}^2\right]\right\} < \frac{\frac{1+\Delta_0(1-\pi^*)^2-\alpha^2\frac{1}{h}\frac{1}{n}}{1-\alpha^2\frac{1}{h}\frac{1}{n}}}{1+\Delta_0(\pi^*)^2} \tag{1.9.62}$$

It is easy to show that $-\frac{\alpha^3\frac{1}{h^2}\frac{1}{n}}{2(1-\alpha^2\frac{1}{h}\frac{1}{n})}\overline{X}^2 + \frac{1}{2}\Delta_2\overline{X}^2 < 0$, then we transform the sufficient condition as

$$\frac{\frac{1+\Delta_0(1-\pi^*)^2-\alpha^2\frac{1}{h}\frac{1}{n}}{1-\alpha^2\frac{1}{h}\frac{1}{n}}}{1+\Delta_0(\pi^*)^2} > 1, \tag{1.9.63}$$

which is equivalent to

$$\Delta_0(1-\pi^*)^2 > (1-\alpha^2\frac{1}{h}\frac{1}{n})\Delta_0(\pi^*)^2 \tag{1.9.64}$$

Therefore, the sufficient condition is $1-\pi^* \geq \pi^* \Leftrightarrow \pi^* \leq \frac{1}{2} \Leftrightarrow \frac{\pi}{\pi+(1-\pi)\mu} \leq \frac{1}{2} \Leftrightarrow \pi \leq (1-\pi)\mu$. Based on (1.9.52) and $\pi^* \leq \frac{1}{2}$, we have the sufficient condition with respect to f as follows

$$f \leq \widehat{f} = \frac{\log\left(1+\frac{1}{4[h+q+\frac{1}{2}(m-q)]^2}\frac{h^2(m-q)^2}{h+q}(\frac{1}{h}+\frac{1}{m}+\frac{\alpha^2}{m^2}\frac{1}{n})\right)}{2\alpha}. \tag{1.9.65}$$

Then we check the sufficient condition for the existence of the equilibrium where all liquidity demanders and a positive fraction of liquidity suppliers participate in the market.

Since $\pi \leq \frac{1}{2}$, $f^1 \leq \widehat{f} < f^2$. When $f^1 < f \leq \widehat{f}$ and all liquidity demanders participate, then μ is in $(0,1)$ and determined by (1.9.52) as an equality. According to (1.9.65), $f \leq \widehat{f}$ results in $\pi^* \leq \frac{1}{2}$, satisfying the sufficient condition for the full participation of liquidity demanders. When $f \leq f^1$ and all liquidity demanders enter the market, then all liquidity suppliers enter the market as well. On the other hand, $\pi \leq \frac{1}{2}$ implies that the sufficient condition for all demanders to participate, $\pi \leq (1-\pi)\mu$, is satisfied when $\mu = 1$, hence all liquidity demanders participate in the market. □

Proof of Proposition 1.5.4. The investors at $t=0$ choose X_0 to maximize their utilities $U = \pi\max\{U_d, U_{d,NP}\} + (1-\pi)\max\{U_s, U_{s,NP}\}$. The above Proposition implies that $U_d \geq U_{d,NP}$ and $U_s \geq U_{s,NP}$. Then the optimization problem can be rewritten $U = \pi U_d + (1-\pi)U_s$, and the consequent first-order condition is given by

$$\begin{aligned}
&\pi\exp(-\alpha F_d)\frac{dF_d}{dX_0}\frac{1}{\sqrt{1+\Delta_0(1-\pi^*)^2-\alpha^2\frac{1}{nh}}} + (1-\pi)\exp(-\alpha F_s)\frac{dF_s}{dX_0}\frac{1}{\sqrt{1+\Delta_0(\pi^*)^2}} = 0 \\
\Leftrightarrow &\pi(\overline{D}-P_0-\frac{\alpha}{h}\overline{X}-\Delta_1\overline{X})M^P + (1-\pi)(\overline{D}-P_0-\frac{\alpha}{h}\overline{X}) = 0.
\end{aligned}$$

Then we obtain the ex ante price of the risky asset in the presence of participation costs

$$P_0 = \overline{D} - \frac{\alpha}{h}\overline{X} - \frac{\pi M^P}{1-\pi+\pi M^P}\Delta_1\overline{X}. \tag{1.9.66}$$

□

Proof of Proposition 1.5.5. The signed volume of liquidity demanders is

$$\pi(X_d - \overline{X}) = -(1-\pi)\mu(X_s - \overline{X}) = -(1-\pi)\mu\left(\frac{\mathrm{E}(D|\mathcal{F}_s) - P_1}{\alpha \mathrm{Var}(D|\mathcal{F}_s)} - \overline{X}\right) \tag{1.9.67}$$

$$= -(1-\pi)\mu\left(\frac{\overline{D} + \frac{\beta_P}{B}(P_1 - A) - P_1}{\alpha \mathrm{Var}(D|\mathcal{F}_s)} - \overline{X}\right). \tag{1.9.68}$$

The price impact measure is calculated as

$$\begin{aligned}\lambda =& \frac{\mathrm{Cov}\left(P_1 - P_0, \pi(X_d - \overline{X})\right)}{\mathrm{Var}\left[\pi(X_d - \overline{X})\right]} \\ =& \frac{-\mathrm{Cov}\left(P_1 - P_0, (1-\pi)\left(X_s - \overline{X}\right)\right)}{\mathrm{Var}\left[(1-\pi)\left(X_s - \overline{X}\right)\right]} \\ =& \frac{\alpha \mathrm{Var}(D|\mathcal{F}_s)}{(1-\pi)\,\mu\left(1 - \frac{\beta_P}{B}\right)} \qquad (1.9.69) \\ =& \frac{\alpha[\pi m + (1-\pi)\mu q]}{h(1-\pi)\pi\mu(m-q)}. \qquad (1.9.70)\end{aligned}$$

Similarly, the price reversal measure is computed as

$$\gamma = -Cov(D - P_1, P_1 - P_0) \tag{1.9.71}$$

$$= B(B - \beta_P)(\frac{1}{h} + \frac{1}{q}), \tag{1.9.72}$$

where $B = \frac{\pi m + (1-\pi)\mu q}{(\pi + (1-\pi)\mu)h + \pi m + (1-\pi)\mu q}$. □

Proof of Proposition 1.5.6. Firstly, Proposition 1.5.2 indicates that a decrease in f leads to an increase in μ. Secondly, it is easy to show that $\frac{\partial \lambda}{\partial \mu} < 0$ and B is a decreasing function of μ. Then, (1.5.13) implies that $\frac{\partial \gamma}{\partial \mu} < 0$. To prove that the ex ante price is an increasing function of μ, we check each terms of P_0 separately. According to its definition, Δ_0 can be rewritten as $\frac{h(m-q)^2}{q[h+q+\pi^*(m-q)]^2}$, indicating that Δ_0 decreases in π^* and thereby increases in μ. Additionally, $\Delta_0(\pi^*)^2$ increases in π^* and $\Delta_0(1-\pi^*)^2$ decreases in π^*. Further, Eqs. (1.5.9) and (1.5.10) imply that Δ_1 and Δ_2 are both decreasing functions of μ. As a result, M decreases in μ, and we conclude that an increase in μ raises the ex ante price P_0. □

Proof of Proposition 1.5.7, Lemma 1.5.1, Proposition 1.5.9. These proofs are very similar to the proofs in Section 1.4 where agents fully participate. The only difference is that we replace π with π^*. □

Proof of Proposition 1.5.8. Based on the results in Lemma 1.5.1, the condition that liquidity suppliers are willing to participate in the market, i.e. (1.9.52), can be rewritten as

$$\exp(2\alpha f) \leq \left(1 + \frac{(\pi^*)^2}{[h+q+\pi^*(m-q)]^2}c\right) \frac{\frac{G^P}{G_s}}{\exp\left(\frac{G^P - G_s}{G^P}\right)}. \tag{1.9.73}$$

Denote by RHS the right hand side of the above inequality, and we can get that $\frac{\partial RHS}{\partial \pi^*} > 0$ and $\frac{\partial RHS}{\partial \mu} < 0$. Then we have three scenarios similar to those in Proposition 1.5.2:

Case 1: if $f \leq \tilde{f}^1 \equiv \frac{\log\left[\left(1+\frac{(\pi)^2}{[h+q+\pi(m-q)]^2}c\right)\frac{h+q+\pi(m-q)}{h+q}\right]+\frac{h+q}{h+q+\pi(m-q)}-1}{2\alpha}$, then (1.9.73) holds for $\mu = 1$, indicating that all liquidity suppliers participate in the market.

Case 2: if $\tilde{f}^1 < f < \tilde{f}^2$, where $\tilde{f}^2 \equiv \frac{\log\left[\left(1+\frac{1}{(h+m)^2}c\right)\frac{h+m}{h+q}\right]+\frac{h+q}{h+m}-1}{2\alpha}$, then (1.9.73) holds as an equality for $\mu \in (0,1)$, indicating that a positive fraction of liquidity suppliers participate in the market.

Case 3: if $f \geq \tilde{f}^2$, then (1.9.73) does not hold for any $\mu \in (0,1]$, indicating no liquidity suppliers participate.

Here, $\tilde{f}^1 \geq f^1$, and $\tilde{f}^2 \geq f^2$.

As for the participation decision of liquidity demanders, the additional term in the interim utilities reflecting the introduction of options ensures that (1.9.60) holds. Following the proof in Proposition 1.5.3, we are able to achieve a similar sufficient condition for the equilibrium where all liquidity demanders and a positive fraction μ of liquidity suppliers participate. □

Proof of Proposition 1.5.10. Under the sufficient condition for the equilibrium where liquidity suppliers partially participate, (1.9.73) holds as an equality for $\mu \in (0,1)$. the introduction of derivative enhances the expected utilities of liquidity suppliers and thus increases the participation rate μ. □

Proof of Proposition 1.5.11. This proof is similar to the proof of Proposition 1.4.3 and 1.5.6. □

Proof of Proposition 1.5.12. The trading volume is calculated as follows

$$V^O = \frac{2\pi(1-\pi)\mu}{(\pi+(1-\pi)\mu)^2}(m-q). \tag{1.9.74}$$

When the participation cost f decreases, μ will increase. It is easy to show that V^O exhibits a hump shape as a function of μ. Then we can complete the proof. $\square$

1.9.5 Asymmetric Information with General Derivatives

Generalized Straddles

Proof of Proposition 1.6.1 and Lemma 1.6.1. For a liquidity demander, his wealth at $t = 2$, i.e. (1.6.1), can be rewritten as:

$$W_{d2} = W_1 + z(P_1 - \overline{D}) + (X_d + z)(D - P_1) + X_{d,G}\left(g(|D - P_1|) - P_G\right), \tag{1.9.75}$$

The expected utilities of liquidity demanders at $t = 1$ can be expressed as

$$\begin{aligned}
&- \mathrm{E}(e^{-\alpha W_{d2}}|\mathcal{F}_d) \\
=& - e^{-\alpha\left(W_1 + z(P_1 - \overline{D}) - X_{d,G}P_G\right)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \mathrm{Var}(u|\mathcal{F}_d)}} e^{-\frac{1}{2}\frac{(u - \mathrm{E}(u|\mathcal{F}_d))^2}{\mathrm{Var}(u|\mathcal{F}_d)}} e^{-\alpha\left[(X_d + z)u + X_{d,G}g(|u|)\right]} du \\
=& - e^{-\alpha\left(W_1 + z(P_1 - \overline{D}) - X_{d,G}P_G\right)} \sqrt{\frac{G_d}{2\pi}} \\
&\times \int_{-\infty}^{+\infty} \exp\left(- \left[\alpha(X_d + z) - G_d(\mathrm{E}(D|\mathcal{F}_d) - P_1)\right] u - \frac{1}{2}G_d u^2 - \frac{1}{2}G_d(\mathrm{E}(D|\mathcal{F}_d) - P_1)^2 - \alpha X_{d,G}g(|u|)\right) du,
\end{aligned} \tag{1.9.76}$$

where $u \equiv D - P_1$, $\mathrm{E}(u|\mathcal{F}_d) = \mathrm{E}(D|\mathcal{F}_d) - P_1$ and $\mathrm{Var}(u|\mathcal{F}_d) = \mathrm{Var}(D|\mathcal{F}_d) = G_d^{-1}$. Dropping irrelevant terms, we obtain two Euler equations for liquidity demanders:

$$\int_{-\infty}^{+\infty} u \exp\left\{ -[\alpha(X_d + z) - G_d(\mathrm{E}(D|\mathcal{F}_d) - P_1)]u - \frac{1}{2}G_d u^2 - \alpha X_{d,G}g(|u|) \right\} du = 0,$$

$$\int_{-\infty}^{+\infty} (g(|u|) - P_G) \exp\left\{ -[\alpha(X_d + z) - G_d(\mathrm{E}(D|\mathcal{F}_d) - P_1)]u - \frac{1}{2}G_d u^2 - \alpha X_{d,G}g(|u|) \right\} du = 0.$$

When $X_d = \frac{\mathrm{E}(D|\mathcal{F}_d) - P_1}{\alpha \mathrm{Var}(D|\mathcal{F}_d)} - z$, the linear term in the first Euler equation vanishes, and we obtain an odd function of u as the integrand. Then the integral on the left-hand side of the Euler equation with regard to the underlying asset is equal to zero. Along the same line of reasoning, we employ the Euler equation for liquidity suppliers and the market clearing condition to obtain the demands of liquidity suppliers and the equilibrium price for the risky asset. Substituting the demands for the risky asset into the Euler equations for the general derivative yields (1.6.6) and (1.6.7). Finally, we get the price of the general derivatives through the market clearing condition.

Based on the above results, we can write the liquidity demanders' expected utilities at $t = 2$ as

$$\begin{aligned}
&-\mathrm{E}(e^{-\alpha W_{d2}}|\mathcal{F}_d) \\
=&-e^{-\alpha\left(W_1+z(P_1-\overline{D})\right)}\sqrt{\frac{G_d}{2\pi}}\times 2\int_0^{+\infty}\exp\left(-\frac{1}{2}G_d u^2-\frac{1}{2}G_d(\mathrm{E}(D|\mathcal{F}_d)-P_1)^2-\alpha X_{d,G}\left(g(|u|)-P_G\right)\right)du \\
=&-e^{-\alpha\left(W_0+X_0(P_1-P_0)+z(P_1-\overline{D})+\frac{1}{2}\frac{(\mathrm{E}(D|\mathcal{F}_d)-P_1)^2}{\alpha\mathrm{Var}(D|\mathcal{F}_d)}\right)}\sqrt{\frac{2G_d}{\pi}}\int_0^{+\infty}\exp\left(-\frac{1}{2}G_d u^2-\alpha X_{d,G}\left(g(|u|)-P_G\right)\right)du.
\end{aligned} \tag{1.9.77}$$

Then we get (1.6.9) and (1.6.10). □

Quadratic Derivatives

Proof of Proposition 1.6.2. The liquidity demanders' wealth at $t = 2$ is given by

$$W_{d2} = W_1 + X_d(D-P_1) + X_{d,G1}((D-P_1)^2 - P_G) + z(D-\overline{D}), \tag{1.9.78}$$

The optimization problem of a liquidity demander can be written as

$$-\mathrm{E}\left[\exp\left(-\alpha W_{d2}\right)|\mathcal{F}_d\right] \tag{1.9.79}$$

$$= -\frac{1}{\sqrt{1+2\alpha X_{d,G1}\mathrm{Var}(D|\mathcal{F}_d)}}\exp\left[\begin{array}{c}\frac{\mathrm{Var}(D|\mathcal{F}_d)\left[\alpha(X_d+z)-\frac{\mathrm{E}(D|\mathcal{F}_d)-P_1}{\mathrm{Var}(D|\mathcal{F}_d)}\right]^2}{2(1+2\alpha X_{d,G1}\mathrm{Var}(D|\mathcal{F}_d))}-\frac{1}{2}\frac{(\mathrm{E}(D|\mathcal{F}_d)-P_1)^2}{\mathrm{Var}(D|\mathcal{F}_d)} \\ -\alpha(W_1 - X_{d,G1}P_G + z(P_1-\overline{D}))\end{array}\right] \tag{1.9.80}$$

Then we have the demands for the stock and the derivative in the following forms

$$X_d = \frac{\mathrm{E}(D|\mathcal{F}_d)-P_1}{\alpha\mathrm{Var}(D|\mathcal{F}_d)} - z, \tag{1.9.81}$$

$$X_{d,G1} = \frac{1}{2\alpha}\left(\frac{1}{P_{G1}} - \frac{1}{\mathrm{Var}(D|\mathcal{F}_d)}\right). \tag{1.9.82}$$

Along the same line of reasoning, we have

$$X_s = \frac{\mathrm{E}(D|\mathcal{F}_s) - P_1}{\alpha \mathrm{Var}(D|\mathcal{F}_s)}, \tag{1.9.83}$$

$$X_{s,G1} = \frac{1}{2\alpha}\left(\frac{1}{P_{G1}} - \frac{1}{\mathrm{Var}(D|\mathcal{F}_s)}\right). \tag{1.9.84}$$

Employing the market clearing conditions for both stocks and derivatives markets, we can complete the proof. □

Proof of Lemma 1.6.2. See the proof of Lemma 1.4.1 regarding the vanilla options. □

Proof of Proposition 1.6.3. See the proof of Proposition 1.4.2 regarding the vanilla options. □

Proof of Proposition 1.6.4. See the proof of Proposition 1.4.3 regarding the vanilla options. □

Chapter 2

Dynamic Equilibrium with Rare Events and Value-at-Risk Constraint

2.1 Introduction

The past few years have witnessed a global financial meltdown in which financial institutions have incurred substantial trading losses. These losses have raised a heated discussion on the massive failures of risk measurement and management in the financial industry. At the heart of risk management practices is the use of Value-at-Risk (VaR) to measure and control tail risks. As an easily understandable and interpretable measure of risk, VaR has been extensively used by financial as well as nonfinancial firms, and over the past 15 years, it has become established as the industry and regulatory standard in measuring market risk. Because of this widespread popularity among practitioners and regulators, it is of interest to study the asset pricing implications of the risk management practice governed by the prevalent VaR constraint. The distinction between the VaR constrained agents and unconstrained agents generates an observed heterogeneity in the financial markets, which is contrasted to the assumed heterogeneity in beliefs that is common in the equilibrium pricing literature.[1] Moreover, the recent crisis highlights the importance of incorporating jump components in theoretical frameworks as a proxy for rare events.[2] However, previous works studying the effectiveness of VaR-based risk manage-

[1] Previous studies assume agents have different beliefs on disaster magnitude and intensity, which have served as the building block of numerous theoretical models established in the literature. See, for example, Liu, Pan, and Wang (2004), Bates (2008), Dieckmann (2011), Chen, Joslin, and Tran (2012), Chen, Joslin, and Ni (2014), among others.

[2] Recent research has shown that a model of rare disasters can explain the equity premium and a wide range of other macro and asset pricing puzzles. See, for example, Rietz (1988), Liu, Pan, and Wang (2004), Barro (2006),

ment systems only consider diffusive risk.[3] Therefore, we build a model under jump diffusion and attempt to see how this feature affects the asset dynamics in equilibrium.

In this paper, we investigate the asset pricing implications of a heterogeneity that is induced by the VaR constraint faced by many institutional investors. We employ jump components to model rare events in the financial market, such as the Lehman Brothers bankruptcy in the recent crisis. Specifically, we solve the equilibrium of a pure-exchange Lucas economy that is populated by one unconstrained agent and one VaR constrained agent and explore how the introduction of VaR constraints affects the market volatility, the Black-Scholes implied volatility curve, the jump risk premium, and the term structure of interest rates.

Our main results are as follows. First, we find that the VaR constraint leads to excess stock market volatility, which is consistent with Basak and Shapiro (2001), and more importantly, the introduction of the jump component intensifies this effect.[4] In this sense, we provide a new mechanism to explain the prevalent "smirk" pattern of the Black-Scholes implied volatility in the option markets. Numerous models have been proposed to explain the shape of the implied volatility, such as stochastic volatility models (Hull and White, 1987; Heston, 1993), local volatility models (Dupire, 1994; Derman and Kani, 1994), jump diffusion models (Merton, 1976; Bates, 1991), and general equilibrium models (Liu, Pan, and Wang, 2004; Li, 2013). As an alternative explanation, our model reveals that the premium for the at-the-money (ATM) options can be rationalized when the economy is dominated by the VaR agent. Further, a sharp downward trend is exhibited in the VaR economy as the option type goes from the out-of-the-money (OTM) puts to the out-of-the-money (OTM) calls. More importantly, the downward implied volatility curve steepens as the VaR agent dominates. Comparing volatility curves across various jump scenarios yields an important observation that rarer and more severe negative jumps result in a more pronounced smirk pattern. This is also one of our motivations for the inclusion of the jump component. When the economy becomes more restricted by the VaR constraint, the jump risk premium is pushed up because the VaR agent ignores the tail risk and takes excessive equity exposure.

Second, our model sheds some light on flight to safety episodes. In bad economic conditions,

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Gabaix (2012), Wachter (2013), Martin (2013), among others.

[3]See, for example, Basak and Shapiro (2001) and Alexander and Baptista (2006). Note that, in this sense they rely on normal market distributions and ignore the empirically observed "skewed" and "fat-tailed" features of stock return distributions, which can be generated by jump components.

[4]To make our model comparable with the case without jump, we calibrate the market volatility by taking into account the impacts of the jump component. More details are discussed in the following sections.

investors are inclined to hold less risky and more liquid securities, such as bonds, thereby resulting in low bond yields, which is known as a flight to safety (see e.g., Longstaff, 2004; Vayanos, 2004; Caballero and Krishnamurthy, 2008; Beber, Brandt, and Kavajecz, 2009; Routledge and Zin, 2009). In particular, Caballero and Krishnamurthy (2008) and Routledge and Zin (2009) find that the uncertainty aversion plays an important role in explaining this empirical fact. By contrast, our model shows that the imposition of the VaR constraint results in a drop in the risk-free rate in bad states. Because of the upward jump in the pricing kernel induced by the VaR constraint, the bond price goes up and the yield drops at the VaR horizon. Intuitively, facing bad states of the world, financial intermediations demand safer and more liquid assets to meet the VaR constraints. After the VaR horizon, the zero coupon bond yield gradually returns to its steady level, leading to an upward sloping term structure.

This article is related to the strand of literature on dynamic asset allocation and asset pricing under risk measure constraints in a variety of settings, which includes Basak and Shapiro (2001), Cuoco, He, and Isaenko (2008) and Shi and Werker (2012). The closest to our paper is Basak and Shapiro (2001), who study the institutional investors' optimal portfolio and wealth polices subject to a VaR constraint and find that in the worst state of the world, the VaR agent takes on larger risk than the unconstrained agent and consequently increases the stock market volatility. We extend their paper by adding jump component. This is motivated by the ample evidence of jumps in stock returns (see e.g., Bakshi, Cao, and Chen, 1997; Eraker, Johannes, and Polson, 2003) and the substantial impact of jump risk on portfolio choice and risk management (see e.g., Duffie, Pan, and Singleton, 2000; Liu, Longstaff, and Pan, 2003; Liu and Pan, 2003) documented in the literature. We find that the addition of the jump component further amplifies the stock market volatility, yielding a better fit to the volatility smirk in the options market than the Basak and Shapiro's model.

This paper also builds on the literature on rare events and the corresponding disagreement over inferences regarding disasters. The model of rare disasters is shown to explain a wide range of asset pricing puzzles: Liu, Pan, and Wang (2004) set up an equilibrium model in which a representative agent is averse to the uncertainty of rare events and explore the implications for option smirks. Bates (2008) examines how the presence of heterogeneous beliefs toward crash risk helps explain various option pricing anomalies. Dieckmann (2011) explores the asset pricing implications in an equilibrium model in which log-utility investors have heterogeneous beliefs about the likelihood of rare events. Chen, Joslin, and Tran (2012) consider an equilibrium model with two Constant Relative Risk Aversion (CRRA) agents who disagree about rare event risk. They study the risk sharing induced by heterogeneous beliefs and show that a small proportion of optimistic investors can substantially reduce the impact of disaster risk on stock prices. Piatti (2014) extends Chen, Joslin, and Tran (2012) by specifying a Lucas endowment economy with multiple trees. More recently, the effect of the heterogeneous Epstein-Zin preference with a rare event is examined by Chabakauri (2014). In contrast to previous studies that assume the existence of heterogeneity in beliefs or preferences, we investigate the asset pricing implications of an observed heterogeneity in financial markets, that is, the one generated by the VaR constraints faced by many institutional investors.

The rest of the paper is organized as follows. Section 2.2 sets up the model. Section 2.3 characterizes the optimization problem of two CRRA agents with and without the constraint. Section 2.4 investigates the market equilibrium. Section 2.5 explores the model implications for asset dynamics and Section 2.6 concludes. All proofs are given in the appendices.

2.2 Model

We consider a continuous-time pure exchange economy with one consumption good (Lucas, 1978) and assume that this Lucas economy is populated by two types of agents: one agent subject to the VaR constraint and one unconstrained agent. Both have CRRA utility functions and derive utility from consumption streams over their lifetime $[0, T']$. The VaR constraint at horizon T can be formulated as

$$P\left[W\left(T\right) \leq \underline{W}\right] \leq \alpha, \tag{2.2.1}$$

where the "floor" $\underline{W}$ and the loss probability α are exogenously specified. The VaR constraint requires that the probability that the institutional investor's wealth at the horizon falls below the floor wealth $\underline{W}$ be α or less. Without loss of generality, we assume that the VaR horizon, T, is shorter than the VaR agent's investment horizon, T'. There are two reasons for this assumption. First, as shown by Basak and Shapiro (2001), the VaR agent's wealth at the VaR horizon is discontinuous over the states of world. This implies that a discontinuity in the exogenous terminal consumption provision is needed to clear the goods market, which seems too restrictive. To solve this problem, we make the aforementioned assumption so that the VaR constraint is imposed on the intermediate wealth, which need not be directly governed by an exogenous consumption supply. Second, the assumption is reasonable from a practical point of view, as the VaR horizon does not necessarily coincide with the investment (consumption) horizon. In order to model rare events, we adopt a jump-diffusion model. The total endowment (or dividend) dynamics are given as follows:

$$\frac{\mathrm{d}\delta(t)}{\delta(t-)} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t + \left(e^{Z_t} - 1\right) \mathrm{d}N_t \tag{2.2.2}$$

where B is a standard Brownian motion and N is a Poisson process with constant intensity λ. In the absence of the jump component, the endowment is a standard geometric Brownian motion with a constant growth rate μ and a constant volatility σ. Jump arrivals are captured by the Poisson process N with intensity λ. Given a jump occurring at time t, the jump magnitude is Z_t, which is normally distributed with mean μ_J and standard deviation σ_J. As a result, the mean percentage jump in the aggregate endowment is $k = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$. For the purpose of modeling undesirable events, we focus our studies on negative jumps, that is $k \leq 0$. Jump amplitudes Z_t and Z_s are independent if $t \neq s$, and all three random shocks B, N, and Z are independent of each other. A noteworthy feature of (2.2.2) is that the dividend growth is no longer normally distributed because the presence of a negative jump risk produces skewness. To this end, the inclusion of the jump component allows us to relax the assumption of the normal market conditions and generates a large tail risk, which is consistent with the empirically observed stock return distribution. Following Naik and Lee (1990), this specification of the aggregate endowment provides a parsimonious way to incorporate both normal and rare events.

The market is assumed to be complete. Specifically, there are three securities available to investors: a riskless bond in zero net supply, a risky stock in one unit net supply (which is the claim to the stream of dividends generated by the Lucas tree), and a zero net supply option written on the risky stock. We denote by B, S, O the price processes of the bond, the stock, and the option, respectively. Finally, the state price density ξ is defined as a strictly positive process such that ξB, ξS, and ξO are martingales.

2.3 Optimization under VaR constraint

In this section, we first state the VaR agent's optimization problem. The institutional investor has CRRA utility over intertemporal consumption $c(t)$ and is subject to the VaR constraint.

The optimization problem of the VaR agent can be formulated as follows:

$$\max_{(c(t),W(T-))} E\left[\int_0^{T'} \frac{c(t)^{1-\gamma}}{1-\gamma} dt\right] \tag{2.3.1}$$

$$\text{subject to} \quad E\left[\int_0^{T} \xi(t)\, c(t)\, dt + \xi(T-)\, W(T-)\right] \leqslant \xi(0)\, W(0)\,, \tag{2.3.2}$$

$$E\left[\int_T^{T'} \xi(t)\, c(t)\, dt|\mathcal{F}_T\right] \leqslant \xi(T-)\, W(T-)\,, \tag{2.3.3}$$

$$P\left(W(T-) \leq \underline{W}\right) \leq \alpha. \tag{2.3.4}$$

The VaR constraint is imposed on the left limit of time-T wealth to maintain the standard convention of right continuity of wealth processes. Obviously, the optimization of the VaR agent is similar to Merton's problem except for the additional VaR constraint. The static budget constraint is split into two components, before and after the VaR horizon, as shown in equations (2.3.2) and (2.3.3), to capture the effect of the VaR constraint on the optimization. Following Basak and Shapiro (2001), we adopt the convex-duality approach (see e.g., Karatzas and Shreve, 1998) to incorporate the VaR constraint and solve this problem using the martingale representation method of Cox and Huang (1989). Proposition 2.3.1 characterizes the optimal solutions for the constrained agent and the unconstrained agent.[5]

Proposition 2.3.1. *The optimal consumption policies and time-T optimal wealth of the two agents are*

$$c(t) = \left(y\xi(t)\right)^{-\frac{1}{\gamma}}, t \in [0, T'], \tag{2.3.5}$$

$$c_{VaR}(t) = \begin{cases} \left(y_{VaR1}\xi(t)\right)^{-\frac{1}{\gamma}} & t \in [0, T), \\ \left(y_{VaR2}\xi(t)\right)^{-\frac{1}{\gamma}} & t \in [T, T'], \end{cases} \tag{2.3.6}$$

$$W(T-) = \frac{1}{\xi(T-)} y^{-\frac{1}{\gamma}} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt|\mathcal{F}_T\right] \tag{2.3.7}$$

$$W_{VaR}(T-) = \begin{cases} \frac{1}{\xi(T-)} y_{VaR1}^{-\frac{1}{\gamma}} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt|\mathcal{F}_T\right] & \text{if } \xi(T-) < \underline{\xi} \\ \underline{W} & \text{if } \underline{\xi} \leqslant \xi(T-) < \overline{\xi}, \\ \frac{1}{\xi(T-)} y_{VaR1}^{-\frac{1}{\gamma}} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt|\mathcal{F}_T\right] & \text{if } \overline{\xi} \leqslant \xi(T-) \end{cases} \tag{2.3.8}$$

[5]In what follows, we use the subscript "VaR" to represent the VaR agent. For example, c_{VaR} represents the consumption of the VaR agent.

where the Lagrange multipliers y, y_{VaR1}, *and the* $\mathcal{F}_T-$*measurable random variable* y_{VaR2} *satisfy*

$$\xi(0)W(0) = y^{-\frac{1}{\gamma}} * E\left[\int_0^{T'} \xi(t)^{1-\frac{1}{\gamma}}dt\right], \tag{2.3.9}$$

$$\xi(0)W_{VaR}(0) = E\left[\begin{array}{l}\left[\xi(T-)\underline{W} - y_{VaR1}^{-\frac{1}{\gamma}}E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt|\mathcal{F}_T\right]\right] \\ *1_{\left\{\frac{1}{\underline{W}}y_{VaR1}^{-\frac{1}{\gamma}}E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt|\mathcal{F}_T\right]\leqslant\xi(T-)<\bar{\xi}\right\}}\end{array}\right] + y_{VaR1}^{-\frac{1}{\gamma}} * E\left[\int_0^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt\right], \tag{2.3.10}$$

$$y_{VaR2}^{-\frac{1}{\gamma}} * E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt\right] = \left[\xi(T-)\underline{W} - y_{VaR1}^{-\frac{1}{\gamma}}E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt|\mathcal{F}_T\right]\right] *1_{\left\{\frac{1}{\underline{W}}y_{VaR1}^{-\frac{1}{\gamma}}E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt|\mathcal{F}_T\right]\leqslant\xi(T-)<\bar{\xi}\right\}} + y_{VaR1}^{-\frac{1}{\gamma}} * E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt\right], \tag{2.3.11}$$

and $\underline{\xi} \equiv \frac{1}{\underline{W}}y_{VaR1}^{-\frac{1}{\gamma}}E\left[\int_T^{T'}\xi(t)^{1-\frac{1}{\gamma}}dt|\mathcal{F}_T\right]$, *where* $\bar{\xi}$ *is defined by* $P\left(\xi(T-)\geq\bar{\xi}\right)\equiv\alpha$. *The VaR constraint is binding if and only if* $\underline{\xi}<\bar{\xi}$.

Proposition 2.3.1 shows that the optimal consumption strategy is governed by the Lagrange multipliers, which can be thought of as the shadow price of the state. In accordance with the separation of the VaR agent's budget constraint, the VaR agent gives different weightings to pre-VaR and post-VaR periods, which are reflected by the distinction between y_{VaR1} and y_{VaR2}. If the Lagrange multiplier before the VaR horizon is larger than that after the VaR horizon, i.e., $y_{VaR1} > y_{VaR2}$, the VaR agent is insuring himself. By contrast, if the two Lagrange multipliers are equal, the VaR constraint slackens and the optimization problem collapses to an unconstrained one, the solution to which is shown in equation (2.3.5). It is worth noting that for the constrained agent, the post-horizon consumption not only provides him with utility but also contributes to meeting the VaR constraint.

Moreover, if the VaR constraint is binding, the VaR agent's optimal horizon wealth can be classified into three distinct regions: in both the regions of "good states" $\left[\xi(T-)<\underline{\xi}\right]$ and "bad states" $\left[\xi(T-)\geqslant\bar{\xi}\right]$, his terminal wealth is decreasing in $\xi(T-)$, while in the region of "intermediate states" $\left[\underline{\xi}\leqslant\xi(T-)<\bar{\xi}\right]$, his terminal wealth is kept constant at the portfolio insurance level, which is $\underline{W}$. The definition of the upper bound $\bar{\xi}$ implies that the probability under the bad states region stays constant at α. In contrast to the case in partial equilibrium (Zhang, Zhou, and Zhou, 2016), the upper bound for the pricing kernel is no longer exogenous as the only exogenous process is the dividend process generated by the Lucas tree.

Figure 2.1 illustrates the optimal terminal wealth of the VaR agent and of the unconstrained agent.[6] Consistent with Proposition 2.3.1, in both regions of good states and bad states, the VaR agent behaves like the unconstrained agent. In contrast, in the intermediate states the VaR agent adopts a portfolio insurance strategy as the portfolio insurance (PI) agent does. A striking feature of the VaR agent's horizon wealth strategy is that he leaves the bad states fully uninsured as they are very costly to insure against; his wealth is even lower than the unconstrained agent's wealth in the worst state for any given $\xi(T-)$. In other words, the VaR agent ignores losses in the upper tail of the $\xi(T-)$ distribution. This worse performance of the VaR agent unveils a shortcoming of VaR that it creates incentive to take on tail risk. As one of the motivations of this study, it is interesting to investigate how this feature drives the dynamics of asset prices when risk management becomes more relevant and rare events are more likely to occur.

2.4 Market Equilibrium

In this section, we use the solutions of agents' optimization shown in Proposition 2.3.1 to investigate market equilibrium. Applying the goods market clearing conditions yields the equilibrium state price density. Proposition 2.4.1 solves the equilibrium state price density and its dynamics.

Proposition 2.4.1. *The equilibrium state price density is given by*

$$\xi\left(t\right)=\begin{cases}\left(\frac{\delta(t)}{y^{-\frac{1}{\gamma}}+y_{VaR1}^{-\frac{1}{\gamma}}}\right)^{-\gamma}, t\in[0,T),\\ \left(\frac{\delta(t)}{y^{-\frac{1}{\gamma}}+y_{VaR2}^{-\frac{1}{\gamma}}}\right)^{-\gamma}, t\in[T,T'],\end{cases} \tag{2.4.1}$$

where y, y_{VaR1}, y_{VaR2} satisfy the budget constraints specified in Proposition 1. The jump size of the equilibrium state price density at the VaR horizon is

$$\eta=\ln\left(\xi\left(T-\right)/\xi\left(T\right)\right)=-\gamma\ln\left(\left(y^{-\frac{1}{\gamma}}+y_{VaR2}^{-\frac{1}{\gamma}}\right)/\left(y^{-\frac{1}{\gamma}}+y_{VaR1}^{-\frac{1}{\gamma}}\right)\right)\leq 0. \tag{2.4.2}$$

[6]If not explicitly stated otherwise, all numerical illustrations are based on the assumption that both agents are initially endowed with half of the total wealth in what follows.

Moreover, for $s \in [0,T)$ and $t \in [s,T')$, the equilibrium state price density is given by

$$\xi(t) = \xi(s)\exp\left\{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(t-s) - \gamma\sigma(B_t - B_s) + \sum_{i=N_s+1}^{N_t}(-\gamma Z_i) - \eta 1_{\{t\geq T\}}\right\}. \quad (2.4.3)$$

Proposition 2.4.1 presents the equilibrium state price density in the economy populated by one institutional agent subject to the VaR constraints and one unconstrained agent. It is worth emphasizing the importance of the upward jump in the pricing kernel, namely $e^{-\eta}$. The distinction between the VaR economy and the benchmark economy is that the equilibrium state price density in the former economy possesses a potential jump at the VaR horizon. As shown in equation (2.4.1), a jump in the pricing kernel at the VaR horizon happens if the VaR constraint is binding, which means that the VaR agent is in the insured states with wealth $\underline{W}$. This horizon wealth is the claim against the post-horizon consumption, and the VaR agent intends to increase the post-horizon consumption since he has a much higher floor wealth as a result of postponing consumption to meet the VaR constraint. Therefore, the pricing kernel needs an upward jump to counteract this upward demand in consumption in order to clear the goods market; otherwise, the VaR agent will consume too much in the post-VaR period with the insured wealth $\underline{W}$, and the market clearing conditions are unlikely to be satisfied. It is important to note that the upward jump in the pricing kernel at the VaR horizon is conditional on whether the VaR constraint is binding, but not on the Poisson jump in the endowment.

At the VaR horizon, the VaR agent's optimal wealth can be classified into three regions (see Figure 2.1). Accordingly, the Lagrange multipliers also fall into three distinct regions as illustrated in Figure 2.2. The left panel is for the Lagrange multiplier in the unconstrained case, and the right panel is for the Lagrange multipliers in the constrained case. In the intermediate region, as the state of the world gets worse, the constraint becomes tighter and the shadow price of such a constraint then increases. For the VaR agent, the Lagrange multiplier after the VaR horizon, y_{VaR2}, decreases when the state deteriorates since the insured wealth $\underline{W}$ is fixed and is more valuable in bad economic conditions. Comparing our results with those in Basak and Shapiro (2001) reveals how the introduction of rare events amplifies the jump in the equilibrium pricing kernel.[7][8]

Let W_{em} be the price of the equity market portfolio, which is defined as the aggregate optimally invested wealth in risky securities. Its market dynamics can be represented by

$$\frac{\mathrm{d}W_{em}(t)+\delta(t)\,\mathrm{d}t}{W_{em}(t)}=\mu_{em}(t)\,\mathrm{d}t+\sigma_{em}(t)\,\mathrm{d}B_t+\left(e^{Z_t}-1\right)\mathrm{d}N_t+\eta\mathrm{d}M_t, \tag{2.4.4}$$

where M_t is a (right-continuous) step function defined by $M_t \equiv 1_{\{t\geq T\}}$, so that dM_t is a measure assigning unit mass to time T, and $k=e^{\mu_J+\frac{1}{2}\sigma_J^2}-1$. Proposition 2.4.2 presents the equity market drift and volatility in equilibrium and contrasts them with the benchmark (B) economy without constrained agents.

Proposition 2.4.2. *The equilibrium market price, volatility, and risk premium in a benchmark economy with only unconstrained agents, $\forall t\in[0,T']$, are given by*

$$W_{em}^{B}(t)=\delta(t)\frac{\left\{e^{A(T'-t)}-1\right\}}{A},\ \ \sigma_{em}^{B}(t)=\sigma,\ \ \mu_{em}^{B}(t)=\mu-A,$$

where $A\equiv-(\gamma-1)\left(\mu-\frac{\gamma}{2}\sigma^2\right)+\lambda\left(e^{-(\gamma-1)\mu_J+\frac{1}{2}(\gamma-1)^2\sigma_J^2}-1\right)$. *Before the VaR horizon, the equilibrium market price in the economy with one CRRA-utility long-lived VaR agent and one CRRA-utility unconstrained agent is*

$$\begin{aligned}
W_{em}^{VaR}(t)=&\ \delta(t)\left[\frac{e^{A(T-t)}-1}{A}+e^{-\eta\left(1-\frac{1}{\gamma}\right)}\frac{e^{A(T'-t)}-e^{A(T-t)}}{A}\right]\\
&+\underline{W}\sum_{n=0}^{\infty}p(n)\left[e^{\Pi(t)-\gamma\Psi(n)}\left\{\begin{array}{c}\mathcal{N}\left(-d_2\left(\bar{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\\-\mathcal{N}\left(-d_2\left(\underline{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}\right]\\
&-\frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}}+y_{VaR1}^{-\frac{1}{\gamma}}}e^{-\eta\left(1-\frac{1}{\gamma}\right)}\delta(t)\frac{\left\{e^{A(T'-T)}-1\right\}}{A}\\
&*\sum_{n=0}^{\infty}p(n)\left[e^{\Gamma(t)+(1-\gamma)\Psi(n)}\left\{\begin{array}{c}\mathcal{N}\left(-d_1\left(\bar{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\\-\mathcal{N}\left(-d_1\left(\underline{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}\right],
\end{aligned} \tag{2.4.5}$$

[7] To make our model comparable with the case without a jump, we calibrate the market volatility by taking into account the impacts of the jump component. Let σ denote the volatility associated with the diffusive component, λ the jump arrival intensity, and μ_J and σ_J the mean and standard deviation of the random jump magnitude. Then, the total market volatility is calculated as $\sqrt{\sigma^2+\lambda\left(\mu_J^2+\sigma_J^2\right)}$. All of the following results stemming from Basak and Shapiro (2001) are obtained using the calibrated market volatility instead of the diffusive volatility σ.

[8] Unreported results show that even if the jump parameter values are mild, such as $\mu_J=-0.02$ and $\lambda=0.5$, the jump in the equilibrium pricing kernel can be twice as large as the case without a jump.

where $\mathcal{N}(.)$ is the standard normal cumulative distribution function and

$$\begin{aligned}
\Psi(n) &\equiv n\mu_J, \\
\Gamma(t) &\equiv \left((1-\gamma)^2\frac{\sigma_n^2}{2} + (1-\gamma)\left(\mu - \frac{1}{2}\sigma^2\right)\right)(T-t), \\
\Pi(t) &\equiv \left(\gamma^2\frac{\sigma_n^2}{2} - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)\right)(T-t), \\
p(n, T-t) &= \frac{\exp(-\lambda(T-t))\,[\lambda(T-t)]^n}{n!}, \\
d_1(x) &\equiv d_2(x) + \sigma_n\sqrt{T-t}, \\
d_2(x) &\equiv \frac{\ln\frac{\delta(t)}{x} + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma_n\sqrt{T-t}}, \\
\sigma_n^2 &\equiv \sigma^2 + \sigma_J^2\frac{n}{(T-t)}.
\end{aligned}$$

The market volatility ratio between the VaR economy and the benchmark economy without VaR constraints is given by

$$\sigma_{em}^{VaR}(t) = q_{em}^{VaR}(t)\,\sigma_{em}^{B}(t),$$

where

$$\begin{aligned}
q_{em}^{VaR}(t) = {} & 1 - \frac{W}{W_{em}^{VaR}(t)}\sum_{n=0}^{\infty}p(n)\left[e^{\Pi(t)-\gamma\Psi(n)}\left\{\begin{array}{l}\mathcal{N}\left(-d_2(\bar{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\\ -\mathcal{N}\left(-d_2(\underline{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}\right] \\
& - \frac{W}{W_{em}^{VaR}(t)}\sum_{n=0}^{\infty}p(n)\left[\frac{e^{\Pi(t)-\gamma\Psi(n)}}{\sigma_n\sqrt{T-t}}\left\{\begin{array}{l}\phi\left(-d_2(\bar{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\\ -\phi\left(-d_2(\underline{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}\right] \\
& + \frac{\frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}}e^{-\eta\left(1-\frac{1}{\gamma}\right)}\frac{\left\{e^{A(T'-T)}-1\right\}}{A}\delta(t)}{W_{em}^{VaR}(t)} \\
& * \sum_{n=0}^{\infty}p(n, T-t)\left[\frac{e^{\Gamma(t)+(1-\gamma)\Psi(n)}}{\sigma_n\sqrt{T-t}}\left\{\begin{array}{l}\phi\left(-d_1(\bar{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\\ -\phi\left(-d_1(\underline{\delta}) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}\right] \quad (2.4.6)
\end{aligned}$$

After the VaR horizon, market prices, volatility, and risk premia in both economies are identical. Consequently, before the VaR horizon,

(i) $W_{em}^{VaR}(t) > W_{em}^{B}(t)$,

(ii) $\sigma_{em}^{VaR}(t) > \sigma_{em}^{B}(t)$ *and* $\mu_{em}^{VaR}(t) > \mu_{em}^{B}(t)$ *under some certain regions.*

As shown in Proposition 2.4.2, the equilibrium market volatility in the benchmark economy coincides with the endowment volatility, while the equilibrium risk premium is the difference between the endowment growth rate and a component A, which is negative for $\gamma > 1$ and $\lambda > 0$. In the case of log utility and no jumps, A becomes zero and the equilibrium risk premium reduces to that in Basak and Shapiro's case. The market portfolio price in the VaR economy is higher than that in the benchmark economy. This follows from the fact that the VaR agent values post-horizon dividends more than pre-horizon consumption as these dividends help him meet the VaR constraint. The pre-horizon value of the equity market is then increased since equities are claims against the post-horizon dividends, which is clearly illustrated in Figure 2.3.

Consistent with Basak and Shapiro (2001), equation (2.4.5) shows that the optimal time-t wealth consists of three components: a myopic component that maximizes the Sharpe ratio and represents the optimal wealth of the benchmark economy (in case of $\eta = 0$) and two option components that correspond to a long position in an option whose payoff is the floor wealth and a short position in an option whose payoff is related to the endowments. In contrast to Basak and Shapiro (2001), the option prices in the presence of jump risk do not immediately follow from the Black-Scholes option pricing formula but rather are computed as the expectation of the Black-Scholes option prices conditional on jumps realized with respect to the jump risk factor. The distinction between the two cumulative distribution functions in equation (2.4.5) captures the probability that one adopts the insurance strategy. Here n denotes the number of jumps and $p(n)$ is the corresponding probability. The first line of equation (2.4.5) collapses to the benchmark case when $\eta = 0$. Note that even for the case of log preference, the equity price is affected by the heterogeneity induced by the VaR constraint, which is different from the case of heterogeneity in beliefs (see e.g., Basak, 2005). Given the equilibrium market price, we can easily derive the market volatility in the VaR economy and the volatility ratio between two economies, which is denoted by q_{em}^{VaR}. In what follows, some numerical illustrations are presented to facilitate the understanding of the market equilibrium.

Figure 2.4 shows the optimal wealth of the two agents in the VaR economy at time t prior to the VaR horizon. Apparently, the optimal pre-horizon wealth of the unconstrained agent is a linear function of the dividend. In both the good and bad states, the optimal pre-horizon wealth of the VaR agent behaves similarly to that of the unconstrained agent. In contrast, in the intermediate region, the VaR agent's wealth is much higher than the unconstrained agent's wealth because he just begins to insure against the intermediate state. This option-like payoff is also documented in Weinbaum (2009) who shows that the heterogeneity in risk aversion makes agents demand non-linear payoffs.

Figure 2.5 illustrates the time-t market volatility in the VaR economy relative to the benchmark economy. The volatility in the benchmark economy stays constant, which is reflected by the straight dashed line against the dividend level. In the two extreme states, the market volatility in the VaR economy acts similarly to the benchmark economy. In between, it fluctuates considerably; when the dividend is not too low, the VaR agent elevates the demand for risky assets to achieve the portfolio insurance level, but when the dividend is already very low, it is unlikely to satisfy the VaR constraint and he simply behaves like the unconstrained agent. Thus, the fluctuation of the volatility in the VaR economy is due to insuring against the intermediate states. A comparison between our model and Basak and Shapiro's model reveals that the introduction of the jump component magnifies the oscillation of market volatility. Figure 2.6 shows a sensitivity analysis of q_{em}^{VaR} to the VaR constraint parameters α and $\underline{W}$. Tighter VaR constraint (lower α and higher $\underline{W}$) makes the market more volatile.

2.5 Equilibrium Implications

In this section, we discuss several implications of our model. Specifically, we investigate the shape of the implied volatility curve in the options market, the amplification effect on the jump risk premium, and the upward sloping term structure after the VaR horizon.

2.5.1 Implied Volatility Curve in the VaR Economy

As documented in the existing literature,[9] at-the-money (ATM) options are priced with a premium and out-of-the-money (OTM) options with an even higher premium, leading to a smirk pattern of the implied volatility curve. Several previous studies employ model uncertainty or heterogeneous beliefs to explain this puzzling shape of the volatility curve implied by the option prices (see e.g., Liu, Pan, and Wang, 2004; Bates, 2008; Li, 2013). In this section, we show that the presence of the VaR constraint provides a new mechanism through which such a smirk pattern is generated.

Equipped with the equilibrium state price density, we can price any derivative securities in this economy. Let C_t be the time-t price of a European-style call option written on risky stock,

[9] See e.g., Jackwerth and Rubinstein (1996) and Rubinstein (1994).

S_τ, with strike price K maturing at time τ.[10] By no arbitrage condition, we have

$$C_t = E_t \left[\frac{\xi_\tau}{\xi_t} \max\left(S_\tau - K, 0\right) \right]. \tag{2.5.1}$$

Alternatively, we can use risk-neutral valuation to price derivatives. Specifically, the ex-dividend stock price of the VaR economy shown in Proposition 2.4.2 can be rewritten as

$$\frac{\mathrm{d}S_t}{S_{t-}} = (r-q)\mathrm{d}t + q_{em}^{VaR}\sigma \mathrm{d}B_t^Q + q_{em}^{VaR}\left(e^{Z_t} - 1\right)\mathrm{d}N_t - e^{-\eta}\lambda^Q k^Q \mathrm{d}t, \tag{2.5.2}$$

where $\lambda^Q = \lambda e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2}$, $k^Q = (1+k)e^{-\gamma\sigma_J^2} - 1$ and $r = \gamma\mu - \frac{\gamma(\gamma+1)}{2}\sigma^2 - \lambda\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right)$.[11] Applying the option pricing method proposed in Merton (1976), we obtain the price of European option prices.[12] Specifically, we price one-month European-style options, both calls and puts, with the strike to spot prices ratios ranging from 0.9 to 1.1 and quote the option prices in terms of the Black-Scholes implied volatility.

To quantitatively study the optimal portfolio strategies under the VaR constraint in a jump diffusion model, we carry out some numerical experiments. Here, we consider three different jump cases: $\mu_J = -1\%$ jumps once every three years, $\mu_J = -10\%$ jumps once every 25 years, and $\mu_J = -20\%$ jumps once every 100 years. The price of an option can have different sensitivities to both infinitesimal and large changes in the price of the underlying stock. In other words, it is capable of providing separate exposures to both the diffusive and jump risks and therefore is non-redundant. For example, the deep OTM put options are more sensitive to the tail risk and are effective in distinguishing the two risk factors.

Figure 2.7 depicts the Black-Scholes implied volatility curves with varying initial wealth shares of the VaR agent or equivalently the extent to which the economy is restricted by the VaR constraint in three jump size scenarios.[13] In the first case with jump amplitude $\mu_J = -1\%$,

[10] The security market clearing conditions imply that the aggregate wealth of this economy is equal to the total wealth of the two agents invested in the securities. Both bond and option are in zero net supply. Thus, the equilibrium price of the risky stock is the aggregate wealth of the economy, that is, $S_\tau = W_{em}^{VaR}$.

[11] In the following analysis, the dividend yield is set at $q = 3\%$. Put differently, we are not using the equilibrium dividend yield without much loss of generality. The dividend yield is a bit complicated as it is time varying in our setting due to the finite horizon. However, we can change the model setup to examine the infinite horizon case. Alternatively, we can set the horizon T' that is sufficiently large compared with the maturity of the options.

[12] The option prices in the case of jump diffusion are a weighted average of BS option prices. Please refer to, for example, Merton (1976) and Bates (2008) for more details.

[13] Unlike in the models with heterogeneous beliefs, the Lagrange multiplier cannot be explicitly obtained in our setting because it is complicated to track the wealth distribution between two agents at time t. To address this issue, we instead use initial wealth to characterize the equilibrium jump risk premium without loss of generality (see, e.g., Bates, 2008).

the total market volatility is $\sqrt{\sigma^2 + \lambda\left(\mu_J^2 + \sigma_J^2\right)} = 15.19\%$. In the absence of the VaR constraint, the BS implied volatility of the ATM options is very close to the volatility of the underlying asset, implying that the model cannot capture the premium embedded in ATM options. On the contrary, with an increasing wealth share of the VaR agent, our setting can predict the premium for the ATM options.

Now we turn to examine the option prices across moneyness. As illustrated in Figure 2.7, while in the benchmark economy the implied volatility curve is rather flat, a sharp downward trend is exhibited in the VaR economy as the option type goes from the OTM puts to the OTM calls. More importantly, the downward implied volatility curve steepens as the VaR agent dominates. This is consistent with the excess volatility that is observed in Figure 2.5; the introduction of the VaR constraint amplifies the volatility of the underlying asset, especially in the bad state, and therefore drives up the price of the OTM put. Another crucial ingredient that explains the smirk pattern is the jump component in the equilibrium pricing kernel as illustrated in equation (2.5.2). When the economy becomes more restricted by the VaR constraint, the jump risk premium is pushed up. As mentioned previously, the price of OTM put options can effectively reflect the jump risk premium. In this sense, the BS implied volatility backed out from the deep OTM put increases relative to the ATM option. Consequently, we end up with a steep volatility curve, which is in line with the volatility smirk documented in the literature. Comparing the volatility curves across the four panels of Figure 2.7 yields the important observation that as the negative jump becomes rarer and more severe, the smirk pattern becomes more pronounced. This is consistent with one of our motivations for the inclusion of the jump component. As illustrated in the top left panel, the VaR constraint alone cannot generate a quantitatively reasonable level of implied volatility empirically observed in financial markets. It is worth emphasizing that the maturity of the option does not have to match the investment horizon as the rebalancing of the portfolio is allowed at each future point in time.

2.5.2 Jump Risk Premium

In this section, we will discuss how the presence of the VaR constraint affects the jump risk premium. Using the equilibrium state price density in equation (2.4.3), one can easily derive the jump risk premium in the VaR economy,

$$\text{Jump Risk Premium} = e^{-\eta}\frac{\lambda^Q}{\lambda} = e^{-\eta}e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2}, \tag{2.5.3}$$

where λ and λ^Q are the jump intensity associated with the jump process N under the physical measure P and the risk neutral measure Q, respectively. A larger λ^Q over λ results in a higher market price of the jump risk. Likewise, as the jump amplitude of the pricing kernel induced by the VaR constraint becomes larger, $(e^{-\eta})$, the jump risk premium becomes higher. If $\eta = 0$, then our results reduce to those of Naik and Lee (1990). That is, the jump risk premium is given by λ^Q over λ. As depicted in Figure 2.8, the extra component $e^{-\eta}$ resulting from the VaR constraint pushes up the jump risk premium, while the ratio of λ^Q to λ is determined by the jump parameters regardless of the wealth distribution of the two agents.

Figure 2.8 illustrates how the initial wealth distribution of the two agents affects the equilibrium price of the jump risk. When the wealth is mainly owned by the VaR agent, the amplification effect on the jump risk premium imposed by the VaR constraint is manifest. The jump risk premium in the VaR economy is about four times as large as that in the unconstrained economy, which is set at one. The intuition is that when the economy becomes more restricted by the VaR constraint, the jump risk premium is enlarged due to the fact that the VaR agent ignores the tail risk and takes excessive equity exposure.

2.5.3 Term Structure of Interest Rates

In this section, we explore the equilibrium term structure of interest rates in the VaR economy. In the absence of the VaR constraint, interest rates stay constant across different maturities, which is clearly shown in the left panel of Figure 2.9. In contrast, the constraint-induced jump in the pricing kernel alters the shape of the term structure after the VaR horizon. The price of a zero-coupon bond with a face value of \$1 maturing at date τ is denoted by $B(t,\tau)$. For $\tau \in (t, T)$, the bond prices are

$$\begin{aligned} B(t,\tau) &= \frac{1}{\xi(t)} E_t\left[\xi(\tau)\right] \\ &= \exp\left\{-\gamma\left(\mu - \frac{\gamma+1}{2}\sigma^2\right)(T-t) + \lambda(T-t)\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right)\right\}, \end{aligned} \tag{2.5.4}$$

and the zero coupon bond yields are

$$\begin{aligned} y(t,\tau) &= -\frac{\ln B(t,\tau)}{(\tau - t)} \\ &= \gamma\mu - \frac{\gamma(\gamma+1)}{2}\sigma^2 - \lambda\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right). \end{aligned} \tag{2.5.5}$$

However, when the maturity exceeds the VaR horizon, e.g., $T \leq \tau \leq T'$, the bond prices are given by

$$
\begin{aligned}
B(t,\tau) &= \frac{1}{\xi(t)} E_t\left[\xi(\tau)\right] \\
&= e^{-\eta} \exp\left\{-\gamma\left(\mu - \frac{\gamma+1}{2}\sigma^2\right)(T-t) + \lambda(T-t)\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right)\right\}, \qquad (2.5.6)
\end{aligned}
$$

and the zero coupon bond yields then become

$$
\begin{aligned}
y(t,\tau) &= -\frac{\ln B(t,\tau)}{(\tau - t)} \\
&= \frac{\eta}{\tau - t} + \gamma\mu - \frac{\gamma(\gamma+1)}{2}\sigma^2 - \lambda\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right). \qquad (2.5.7)
\end{aligned}
$$

As shown in equation (2.5.5), before the VaR horizon, the zero coupon bond yields are independent of τ and thus the term structure is completely flat. By contrast, when the bond maturity is longer than the VaR horizon, the bond price increases. This follows from the fact that the pricing kernel has a potential upward jump and the zero coupon bond becomes more valuable for the agents because its payoff remains unchanged irrespective of whether the jump takes place or not. We can see from equation (2.5.6) that the extra component $e^{-\eta}$ is greater than or equal to one as η is a non-positive number. As a consequence, the zero coupon bond yields likely experience a drop at the VaR horizon. This is consistent with flight to safety phenomenon where in times of economic distress, investors rebalance their portfolios toward less risky and more liquid securities, especially in the bond markets (see e.g., Longstaff, 2004; Vayanos, 2004; Caballero and Krishnamurthy, 2008; Beber, Brandt, and Kavajecz, 2009; Routledge and Zin, 2009). In particular, both Caballero and Krishnamurthy (2008) and Routledge and Zin (2009) emphasize uncertainty aversion as a central ingredient in flight to safety episodes. In contrast, our model reveals that flight to safety can also be attributed to the VaR constraints imposed on financial intermediations as they need safe and liquid assets to satisfy the constraints at the VaR review. After the VaR horizon, the interest rates converge to the original level and display an upward trend as the maturity τ increases. As shown in equation (2.5.7), the first component is driven by the VaR constraint and so it does not emerge before the VaR horizon. Figure 2.9 illustrates the pattern of the term structure of the interest rates described above.

2.6 Conclusion

In this paper, we study the implications of the VaR constraint imposed by regulators for asset dynamics in a general equilibrium model with rare events. First, our model amplifies the fluctuation of the equilibrium market volatility generated by the model in Basak and Shapiro (2001) and provides an explanation for excess stock market volatility. Second, we offer a new mechanism that helps explain the prevalent implied volatility smirk in the option market. Further, the VaR constraint can explain the dramatic increase in the jump risk premium in the bad economic conditions. Finally, the VaR constraint results in a potential drop in the zero coupon bond yields at the VaR horizon. After the VaR horizon, bond yields converge to the original level, and thus, the term structure exhibits an upward sloping trend.

There are several avenues for future research. First, it would be interesting to examine what happens in the option market. In the current framework, the consumption share is a function of endowment only. Thus, no other assets are needed besides the aggregate endowment claim, and agents invest all wealth in the stock market. Without the demand for options, the risk sharing between the two types of agents can hardly be studied (see e.g., Chen, Joslin, and Tran, 2012; Chen, Joslin, and Ni, 2014). Second, allowing for time-varying jump intensity will likely produce more realistic asset dynamics (see e.g., Wachter, 2013).

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2.8 Appendix

Proof of Proposition 1

Applying the standard martingale representation approach yields optimal consumption policies and optimal horizon wealth for an unconstrained agent. Regarding the solutions for the VaR agents, one can see the Proof of Proposition 1 in the Appendix of Basak and Shapiro (2001) for reference since we completely follow Basak and Shapiro (2001) in deriving optimal horizon wealth for agents subject to such a constraint.

Proof of Proposition 2

Using the goods market clearing conditions, we get the equilibrium state price density. Applying Itô's lemma to equation (2.4.1) yields the process for the pricing kernel in our VaR economy, namely equation (2.4.3).

Proof of Proposition 3

Given the equilibrium state price density given in equation (2.4.3), for $s \in [t, T)$,

$$\begin{aligned}
E_t[\xi(s)] &= \xi(t)\exp\left\{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(s-t) + \frac{1}{2}\gamma^2\sigma^2(s-t)\right\} \\
&\quad * \sum_{n=0}^{\infty}\left[E\exp\Sigma_n\left(-\gamma\left(\mu_J + \sigma_J\varepsilon_2\right)\right)|N_s - N_t = n\right]\Pr\left(N_s - N_t = n\right) \\
&= \xi(t)\exp\left\{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(s-t) + \frac{1}{2}\gamma^2\sigma^2(s-t)\right\} \\
&\quad * \left\{\sum_{n=0}^{\infty}\frac{\left(\varphi_{-\gamma(\mu_J+\sigma_J\varepsilon_2)}\lambda(s-t)\right)^n}{n!}e^{-\lambda(s-t)}\right\} \\
&= \xi(t)\exp\left\{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(s-t) + \frac{1}{2}\gamma^2\sigma^2(s-t)\right\} \\
&\quad * \exp\left\{\lambda(s-t)\left(\varphi_{-\gamma(\mu_J+\sigma_J\varepsilon_2)} - 1\right)\right\} \\
&= \xi(t)\exp\left\{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(s-t) + \frac{1}{2}\gamma^2\sigma^2(s-t)\right\} \\
&\quad * \exp\left\{\lambda(s-t)\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right)\right\} \\
&= \xi(t)\exp\left\{-\gamma\left(\mu - \frac{\gamma+1}{2}\sigma^2\right)(s-t) + \lambda(s-t)\left(e^{-\gamma\mu_J + \frac{1}{2}\gamma^2\sigma_J^2} - 1\right)\right\},
\end{aligned} \tag{2.8.1}$$

where φ is the moment generating function. Similarly, it is easy to verify

$$E_t\left[\xi(s)^{1-\frac{1}{\gamma}}\right] = \xi(t)^{1-\frac{1}{\gamma}} \exp\left\{\begin{array}{c} -(\gamma-1)\left(\mu - \frac{\gamma}{2}\sigma^2\right)(s-t) \\ +\lambda(s-t)\left(e^{-(\gamma-1)\mu_J + \frac{1}{2}(\gamma-1)^2\sigma_J^2} - 1\right) \end{array}\right\}.$$

Let $A \equiv -(\gamma-1)\left(\mu - \frac{\gamma}{2}\sigma^2\right) + \lambda\left(e^{-(\gamma-1)\mu_J+\frac{1}{2}(\gamma-1)^2\sigma_J^2} - 1\right)$. Then, the above equation implies

$$\begin{aligned} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt|\mathcal{F}_T\right] &= \int_T^{T'} E\left(\xi(t)^{1-\frac{1}{\gamma}} dt|\mathcal{F}_T\right) \\ &= \xi(T)^{1-\frac{1}{\gamma}} \int_T^{T'} e^{A(t-T)} dt \\ &= \xi(T)^{1-\frac{1}{\gamma}} \frac{e^{A(T'-T)} - 1}{A}. \end{aligned} \tag{2.8.2}$$

Therefore, the equilibrium price for the market portfolio in a benchmark economy with only unconstrained CRRA agents can be computed $\forall t \in [0, T']$,

$$\begin{aligned} W_{em}^B(t) &= \frac{1}{\xi(t)} E\left[\int_t^{T'} c(t)\,\xi(s)\,ds|\mathcal{F}_t\right] \\ &= \frac{1}{\xi(t)} y^{-\frac{1}{\gamma}} \left[\int_t^{T'} E\left(\xi(s)^{1-\frac{1}{\gamma}} ds|\mathcal{F}_t\right)\right] \\ &= \frac{1}{\xi(t)} y^{-\frac{1}{\gamma}} \xi(t)^{1-\frac{1}{\gamma}} \frac{e^{A(T'-t)} - 1}{A} \\ &= \delta(t) \frac{e^{A(T'-t)} - 1}{A}. \end{aligned} \tag{2.8.3}$$

Applying Itô's lemma yields

$$\begin{aligned} dW_{em}^B(t) &= \frac{e^{A(T'-t)} - 1}{A} d\delta(t) - e^{A(T'-t)}\delta(t)\,dt \\ &= W_{em}^B(t)\left\{\mu dt + \sigma dB_t + \left(e^{Z_t} - 1\right) dN_t\right\} - \left(AW_{em}^B(t) + \delta(t)\right) dt \\ &= W_{em}^B(t)\left\{(\mu - A)\,dt + \sigma dB_t + \left(e^{Z_t} - 1\right) dN_t\right\} - \delta(t)\,dt. \end{aligned}$$

Thus, the drift and volatility of the benchmark economy are given by $\mu_{em}^B(t) = \mu - A$ and $\sigma_{em}^B(t) = \sigma$. In contrast to the benchmark economy, there is an upward jump in the equilibrium pricing kernel, which is captured by $-\eta$ in the VaR economy. Therefore, the relation between aggregate endowment and state price density is $\delta(T-) = \left(y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}\right)\xi(T-)^{-\frac{1}{\gamma}}$, which is in contrast to $\delta(T-) = (y\xi(T-))^{-\frac{1}{\gamma}}$ in the benchmark economy. The optimal horizon wealth for the VaR agents in both regions of good and bad states can be rewritten as

$$\begin{aligned} & \frac{1}{\xi(T-)} y_{VaR1}^{-\frac{1}{\gamma}} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt | \mathcal{F}_T\right] \\ = & \frac{1}{\xi(T-)} y_{VaR1}^{-\frac{1}{\gamma}} \xi(T)^{1-\frac{1}{\gamma}} \frac{e^{A(T'-T)} - 1}{A} \\ = & \frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}} \delta(T-) e^{-\eta\left(1-\frac{1}{\gamma}\right)} \frac{e^{A(T'-T)} - 1}{A}. \end{aligned}$$

Therefore, the optimal wealth of the VaR agent at the horizon is rewritten as

$$W_{VaR}(T-) = \begin{cases} \frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}} \delta(T-) \frac{e^{A(T'-T)}-1}{A} & \text{if } \delta(T-) > \overline{\delta} \\ \underline{W} & \text{if } \underline{\delta} < \delta(T-) \leqslant \overline{\delta} \\ \frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}} \delta(T-) \frac{e^{A(T'-T)}-1}{A} & \text{if } \delta(T-) \leqslant \underline{\delta} \end{cases} .$$

Similarly, one can get the lower bound for the pricing kernel, which characterizes the regions of horizon wealth

$$\begin{aligned} \xi(T-) &< \frac{1}{\underline{W}} y_{VaR1}^{-\frac{1}{\gamma}} E\left[\int_T^{T'} \xi(t)^{1-\frac{1}{\gamma}} dt | \mathcal{F}_T\right] \\ \xi(T-) &< \left(\underline{W} e^{\eta\left(1-\frac{1}{\gamma}\right)} y_{VaR1}^{\frac{1}{\gamma}} \frac{A}{e^{A(T'-T)} - 1}\right)^{-\gamma}. \end{aligned}$$

Thus, using this budget constraint, the lower bound for equilibrium state price density is

$$\underline{\xi} \equiv \left(y_{VaR1}^{\frac{1}{\gamma}} \underline{W} e^{\eta\left(1-\frac{1}{\gamma}\right)} \frac{A}{e^{A(T'-T)} - 1}\right)^{-\gamma}.$$

This bound can be rewritten in terms of the upper bound for aggregate endowment

$$\overline{\delta} \equiv \frac{y^{-1/\gamma} + y_{VaR1}^{-1/\gamma}}{y_{VaR1}^{-1/\gamma}} \underline{W} e^{\eta\left(1-\frac{1}{\gamma}\right)} \frac{A}{e^{A(T'-T)} - 1}.$$

Given that $\overline{\xi}$ is defined by $P\left(\xi(T-) \geq \overline{\xi}\right) \equiv \alpha$, we get the lower bound of dividend

$$\underline{\delta} \equiv \overline{\xi}^{-1/\gamma}\left(y^{-1/\gamma} + y_{VaR1}^{-1/\gamma}\right).$$

Equipped with the upper and lower bounds for aggregate endowment and pricing kernel, we can fully characterize the distribution of the optimal horizon wealth for the agents subject to the VaR constraint. Consequently, using the market clearing conditions, we can determine the price for the market portfolio before the VaR horizon.

Before calculating the equilibrium market price in an economy with one VaR-constrained agent and one unconstrained agent, we first compute the unconstrained agent's prehorizon wealth. According to the definition that the wealth is a claim against all future consumption, the unconstrained agent's prehorizon wealth is given by

$$
\begin{aligned}
W(t) &= \frac{1}{\xi(t)} E\left[\int_t^{T'} c(t)\,\xi(s)\,ds|\mathcal{F}_t\right] \\
&= \frac{1}{\xi(t)} E\left[\int_t^{T} + \int_T^{T'} c(t)\,\xi(s)\,ds|\mathcal{F}_t\right] \\
&= \frac{1}{\xi(t)} E\left[\int_t^{T} y^{-\frac{1}{\gamma}}\xi(s)^{1-1/\gamma}\,ds|\mathcal{F}_t\right] + \frac{1}{\xi(t)} E_t\left[E_T\left[\int_T^{T'} y^{-\frac{1}{\gamma}}\xi(s)^{1-1/\gamma}\,ds\right]\right] \\
&= (y\xi(t))^{-\frac{1}{\gamma}}\left[\frac{e^{A(T-t)}-1}{A} + e^{-\eta\left(1-\frac{1}{\gamma}\right)}\frac{e^{A(T'-t)}-e^{A(T-t)}}{A}\right] \\
&= \frac{y^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}}\delta(t)\left[\frac{e^{A(T-t)}-1}{A} + e^{-\eta\left(1-\frac{1}{\gamma}\right)}\frac{e^{A(T'-t)}-e^{A(T-t)}}{A}\right].
\end{aligned}
\tag{2.8.4}
$$

Similarly, we can easily compute the VaR agents prehorizon wealth,

$$
\begin{aligned}
W^{VaR}(t) &= \frac{1}{\xi(t)} E\left[\int_t^{T} c_{VaR}(t)\,\xi(s)\,ds + \xi(T-)\,W_{VaR}(T-)\,|\mathcal{F}_t\right] \\
&= \frac{1}{\xi(t)} y_{VaR1}^{-\frac{1}{\gamma}} E_t\left[\int_t^{T'} \xi(s)^{1-\frac{1}{\gamma}}\,ds\right] \\
&\quad + \frac{1}{\xi(t)} E_t\left[\xi(T-)\left[\underline{W} - \frac{1}{\xi(T-)} y_{VaR1}^{-\frac{1}{\gamma}} E_T\left[\int_T^{T'} \xi(s)^{1-\frac{1}{\gamma}}\,ds\right]\right]|\underline{\delta} < \delta(T-) \leqslant \overline{\delta}\right] \\
&= \frac{1}{\xi(t)} y_{VaR1}^{-\frac{1}{\gamma}} E_t\left[\int_t^{T'} \xi(s)^{1-\frac{1}{\gamma}}\,ds\right] \\
&\quad + \frac{1}{\xi(t)} \underline{W} E_t\left[\left(\frac{\delta(T-)}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}}\right)^{-\gamma} |\underline{\delta} < \delta(T-) \leqslant \overline{\delta}\right] \\
&\quad - \frac{1}{\xi(t)} E_t\left[e^{-\eta\left(1-\frac{1}{\gamma}\right)} y_{VaR1}^{-\frac{1}{\gamma}}\left(\frac{\delta(T-)}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}}\right)^{1-\gamma}\frac{e^{A(T'-T)}-1}{A}|\underline{\delta} < \delta(T-) \leqslant \overline{\delta}\right].
\end{aligned}
\tag{2.8.5}
$$

We compute the terms in (2.8.5) separately

$$E_t\left[\delta\left(T-\right)^{1-\gamma}|\underline{\delta}<\delta\left(T-\right)\leqslant\overline{\delta}\right]=E_t\left[E_t\left[\delta\left(T-\right)^{1-\gamma}|\underline{\delta}<\delta\left(T-\right)\leqslant\overline{\delta},\sigma\left(\Sigma_{i=1}^{n}Z_i\right)\right]\right],$$

where the conditioning σ-algebra $\sigma\left(\Sigma_{i=1}^{n}Z_i\right)$ is generated by the random variable $\Sigma_{i=1}^{n}Z_i$.

It is easy to see $ln\xi\left(s\right)$ follows a normal distribution conditional on both $\mathcal{F}_t$ and $\sigma\left(\Sigma_{i=1}^{n}Z_i\right)$, $\forall s\in\left[0,T'\right)$

$$ln\xi\left(s\right)|\mathcal{F}_t,\sigma\left(\Sigma_{i=1}^{n}Z_i\right)\sim\mathcal{N}\left(ln\xi\left(t\right)-\gamma\left(\mu-\frac{1}{2}\sigma^2\right)\left(s-t\right)-\gamma n\mu_J-\eta 1_{\{s\geq T\}},\gamma^2\left(\sigma^2\left(s-t\right)+n\sigma_J^2\right)\right).$$

Analogously, $ln\delta\left(s\right)$ follows another normal distribution conditional on both $\mathcal{F}_t$ and $\sigma\left(\Sigma_{i=1}^{n}Z_i\right)$, $\forall s\in\left[0,T'\right)$

$$ln\delta\left(s\right)|\mathcal{F}_t,\sigma\left(\Sigma_{i=1}^{n}Z_i\right)\sim\mathcal{N}\left(ln\delta\left(t\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\left(s-t\right)+n\mu_J,\left(\sigma^2\left(s-t\right)+n\sigma_J^2\right)\right). \quad (2.8.6)$$

Let $C\equiv ln\delta\left(t\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\left(T-t\right)+n\mu_J=ln\delta\left(t\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\left(T-t\right)+\Psi\left(n\right)$ and $\Psi\left(n\right)\equiv n\mu_J$. Then, (2.8.6) implies

$$\begin{aligned}
&E_t\left[\delta\left(T-\right)^{1-\gamma}|\underline{\delta}<\delta\left(T-\right)\leqslant\overline{\delta},\sigma\left(\Sigma_{i=1}^{n}Z_i\right)\right]\\
&=\int_{\ln\underline{\delta}}^{\ln\overline{\delta}}e^{(1-\gamma)\ln\delta(T-)}\frac{1}{\sqrt{2\pi\left(\sigma^2\left(T-t\right)+n\sigma_J^2\right)}}e^{-\frac{\left[\ln\delta(T-)-C\right]^2}{2\left(\sigma^2(T-t)+n\sigma_J^2\right)}}d\ln\delta\left(T-\right)\\
&=\exp\left(\left(1-\gamma\right)^2\frac{\sigma^2\left(T-t\right)+n\sigma_J^2}{2}+C\left(1-\gamma\right)\right)\int_{\ln\underline{\delta}}^{\ln\overline{\delta}}\frac{e^{-\frac{\left[\ln\delta(T-)-\left(C+\left(\sigma^2(T-t)+n\sigma_J^2\right)(1-\gamma)\right)\right]^2}{2\left(\sigma^2(T-t)+n\sigma_J^2\right)}}}{\sqrt{2\pi\left(\sigma^2\left(T-t\right)+n\sigma_J^2\right)}}d\ln\delta\left(T-\right)\\
&=\delta\left(t\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)^2\frac{\sigma^2\left(T-t\right)+n\sigma_J^2}{2}+\left(1-\gamma\right)\left(\mu-\frac{1}{2}\sigma^2\right)\left(T-t\right)+\left(1-\gamma\right)n\mu_J\right)\\
&*\left\{\begin{array}{c}\mathcal{N}\left(\frac{\ln\overline{\delta}-ln\delta(t)-\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)-n\mu_J-\left(\sigma^2(T-t)+n\sigma_J^2\right)(1-\gamma)}{\sqrt{\sigma^2(T-t)+n\sigma_J^2}}\right)\\-\mathcal{N}\left(\frac{\ln\underline{\delta}-ln\delta(t)-\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)-n\mu_J-\left(\sigma^2(T-t)+n\sigma_J^2\right)(1-\gamma)}{\sqrt{\sigma^2(T-t)+n\sigma_J^2}}\right)\end{array}\right\}\\
&=\delta\left(t\right)^{1-\gamma}e^{\Gamma(t)+(1-\gamma)\Psi(n)}\left\{\begin{array}{c}\mathcal{N}\left(-d_1\left(\overline{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\\-\mathcal{N}\left(-d_1\left(\underline{\delta}\right)-\frac{\Psi(n)}{\sigma_n\sqrt{T-t}}+\gamma\sigma_n\sqrt{T-t}\right)\end{array}\right\}, \quad (2.8.7)
\end{aligned}$$

where $\Gamma\left(t\right)\equiv\left(\left(1-\gamma\right)^2\frac{\sigma_n^2}{2}+\left(1-\gamma\right)\left(\mu-\frac{1}{2}\sigma^2\right)\right)\left(T-t\right)$, $d_1\left(x\right)\equiv d_2\left(x\right)+\sigma_n\sqrt{T-t}$, $d_2\left(x\right)\equiv$

$\frac{\ln\frac{\delta(t)}{x}+\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma_n\sqrt{T-t}}$, and $\sigma_n^2 \equiv \sigma^2 + \sigma_J^2 \frac{n}{(T-t)}$. Likewise, we have

$$E_t\left[\delta(T-)^{-\gamma}\,|\underline{\delta} < \delta(T-) \leqslant \overline{\delta}, \sigma\left(\Sigma_{i=1}^n Z_i\right)\right] \\ = \delta(t)^{-\gamma}\, e^{\Pi(t)-\gamma\Psi(N_t)} \left\{ \begin{array}{c} \mathcal{N}\left(-d_2\left(\overline{\delta}\right) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \\ -\mathcal{N}\left(-d_2\left(\underline{\delta}\right) - \frac{(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \end{array} \right\}, \tag{2.8.8}$$

where $\Pi(t) \equiv \left(\gamma^2\frac{\sigma_n^2}{2} - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)\right)(T-t)$. Substituting (2.8.7) and (2.8.8) into (2.8.5) yields

$$\begin{aligned} W^{VaR}(t) &= \frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}}\delta(t)\left[\frac{e^{A(T-t)}-1}{A} + e^{-\eta\left(1-\frac{1}{\gamma}\right)}\frac{e^{A(T'-t)} - e^{A(T-t)}}{A}\right] \\ &\quad + \underline{W}\sum_{n=0}^{\infty} p(n)\left[e^{\Pi(t)-\gamma\Psi(n)}\left\{ \begin{array}{c} \mathcal{N}\left(-d_2\left(\overline{\delta}\right) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \\ -\mathcal{N}\left(-d_2\left(\underline{\delta}\right) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \end{array} \right\}\right] \\ &\quad - \frac{y_{VaR1}^{-\frac{1}{\gamma}}}{y^{-\frac{1}{\gamma}} + y_{VaR1}^{-\frac{1}{\gamma}}} e^{-\eta\left(1-\frac{1}{\gamma}\right)}\delta(t)\frac{\left\{e^{A(T'-T)}-1\right\}}{A} \\ &\quad * \sum_{n=0}^{\infty} p(n)\left[e^{\Gamma(t)+(1-\gamma)\Psi(n)}\left\{ \begin{array}{c} \mathcal{N}\left(-d_1\left(\overline{\delta}\right) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \\ -\mathcal{N}\left(-d_1\left(\underline{\delta}\right) - \frac{\Psi(n)}{\sigma_n\sqrt{T-t}} + \gamma\sigma_n\sqrt{T-t}\right) \end{array} \right\}\right], \end{aligned}$$

where $p(n, T-t) = \frac{\exp(-\lambda(T-t))[\lambda(T-t)]^n}{n!}$ captures the probability that n jumps arrive from t to T.

The security market clearing conditions imply that the aggregate wealth of this economy is equal to the total wealth of the two agents invested in the securities and that both bond and option are in zero net supply. Thus, the equilibrium price of the market portfolio is the aggregate wealth of the economy. We compute the equilibrium market price in an economy with one constrained agent and one unconstrained agent as the sum of their wealth, which is shown in equation (2.4.5).

Applying Itô's lemma to (2.4.5), we get (2.4.6).

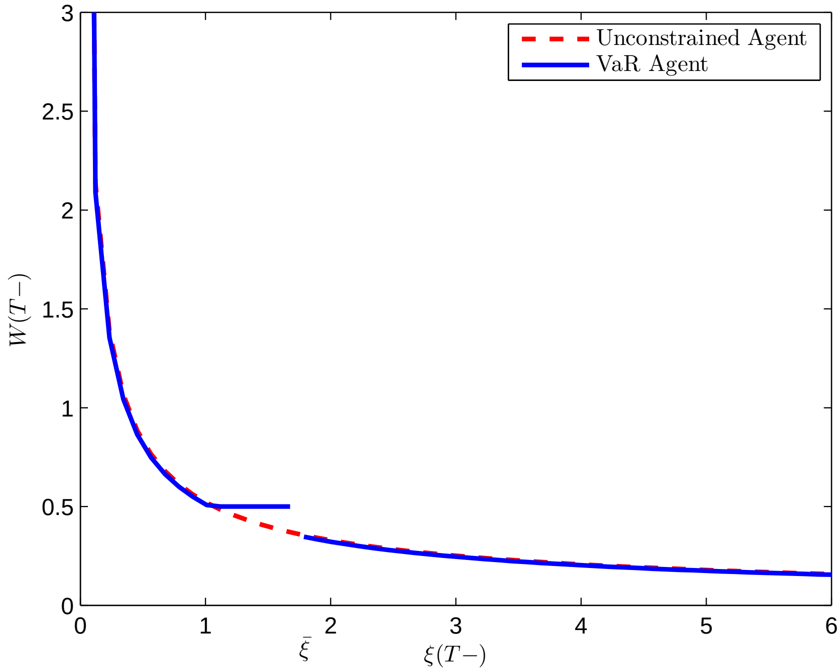


Figure 2.1: Optimal horizon wealth of two agents. The figure plots the optimal horizon wealth of the VaR agent (solid line) and the unconstrained agent (dashed line). $\bar{\xi}$ is the upper bound of the equilibrium state price density separating the intermediate region and the bad state region.

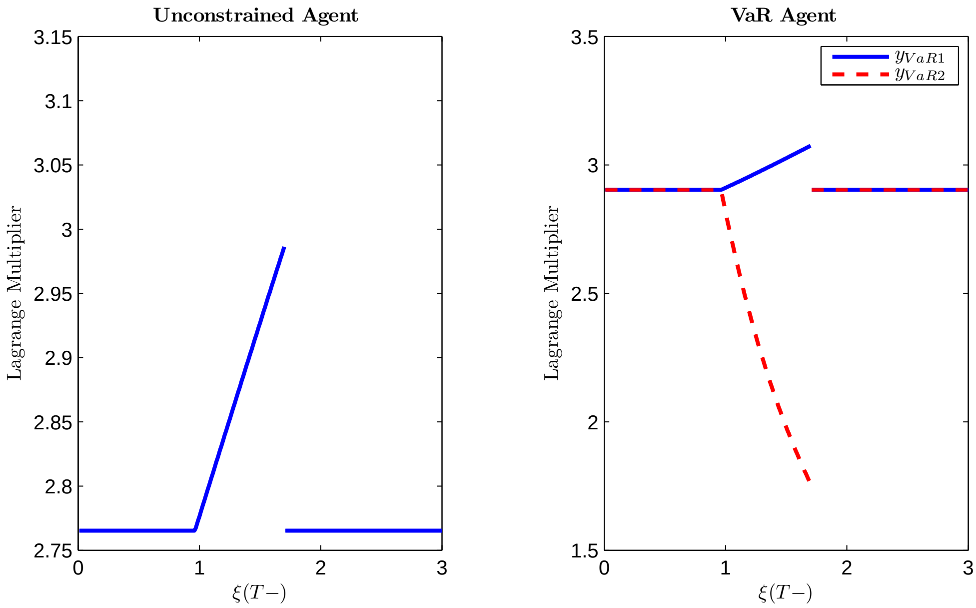


Figure 2.2: Lagrange multipliers of two agents. The left panel plots the lagrange multiplier of the unconstrained agent, and the right panel depicts the lagrange multipliers for the VaR agent. In the right panel, the solid line is for the VaR agent before the VaR horizon (y_{VaR1}) and the dashed line is for the VaR agent after the horizon (y_{VaR2}).

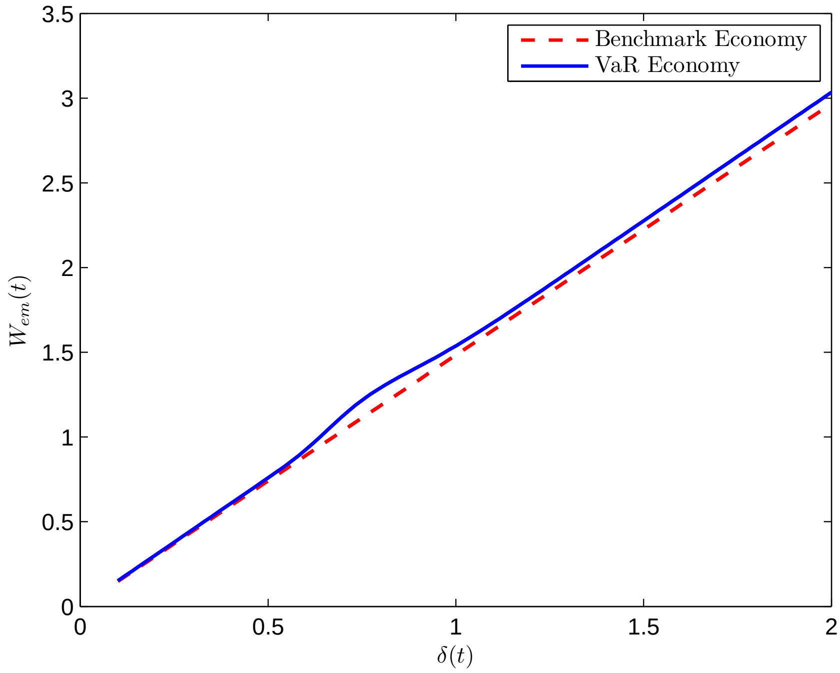


Figure 2.3: Equilibrium time-t market prices of two economies. This figure plots the equilibrium market price of the VaR economy (solid line) and the benchmark economy (dashed line).

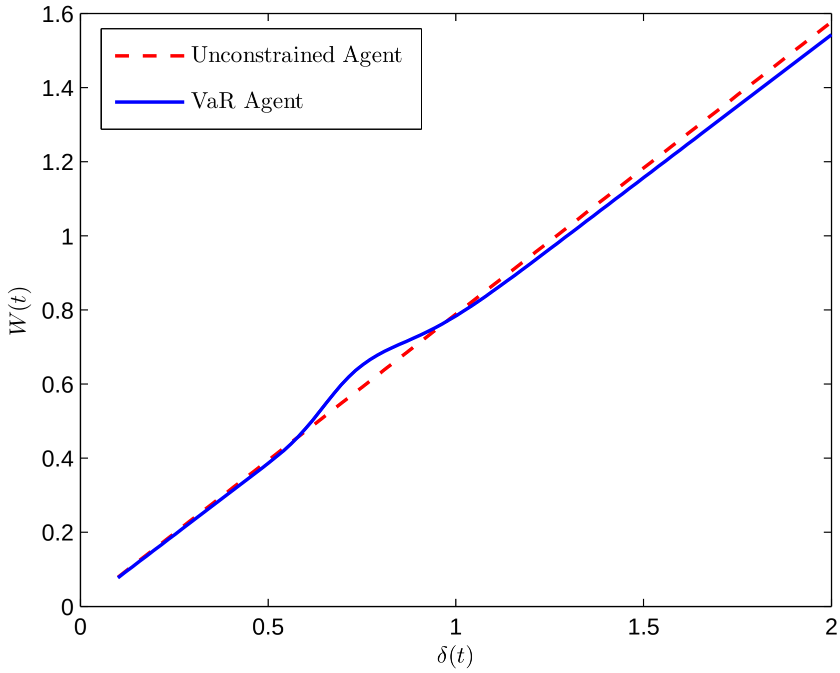


Figure 2.4: Optimal time-t wealth of two agents. This figure plots the optimal time-t wealth of the VaR agent (solid line) and the unconstrained agent (dashed line).

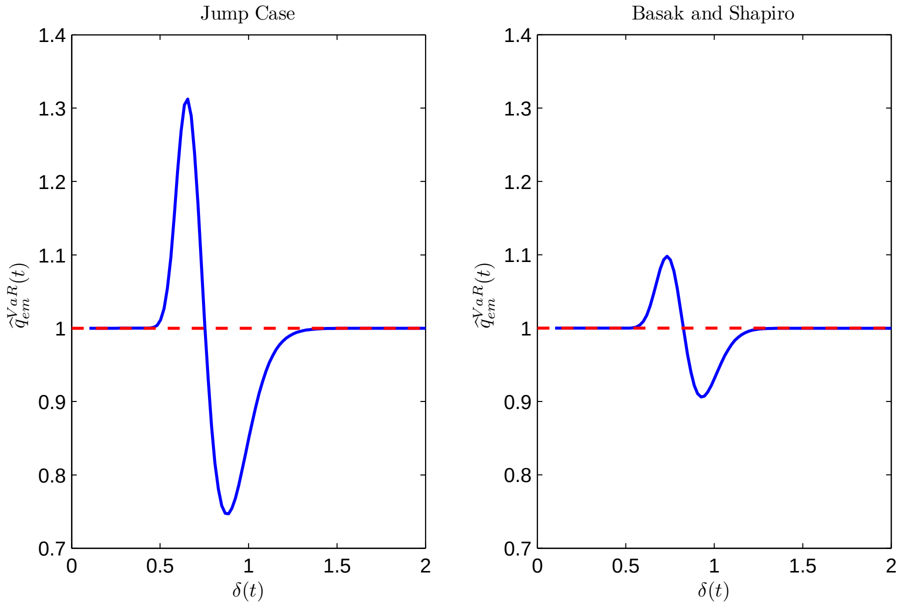


Figure 2.5: Optimal time-t market volatility ratio. This figure plots the market volatility ratios of the VaR economy (solid line) and the benchmark economy (dashed line). The left panel represents the case with a jump component and the right panel the case without (Basak and Shapiro). To compare our model with the case without jump, the market volatility is calibrated by taking into account the impacts of the jump component.

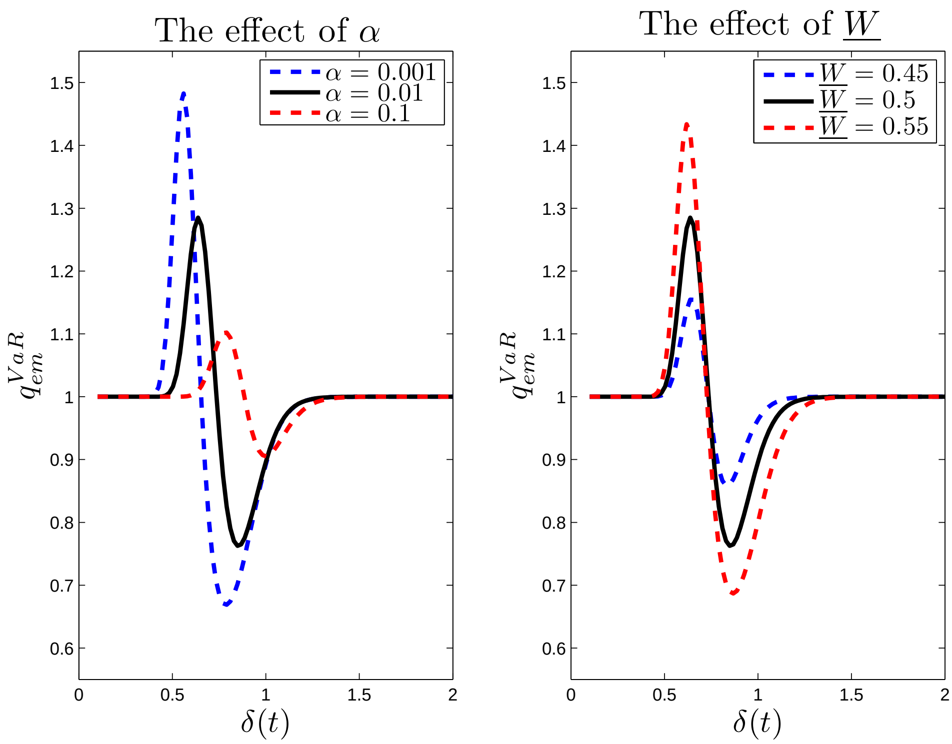
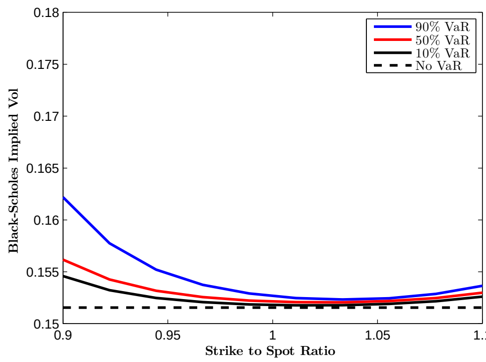
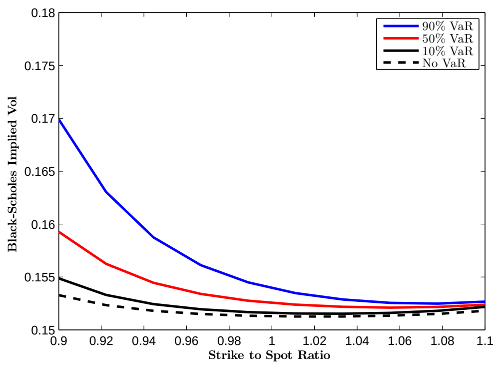


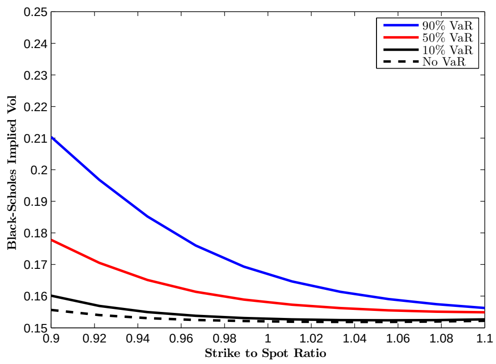
Figure 2.6: Behavior of optimal time-t market volatility ratio with respect to VaR parameters. The left panel plots the optimal time-t market volatility ratio for varying levels of the VaR probability α. The right panel plots the optimal market volatility ratio for varying levels of the floor wealth $\underline{W}$.



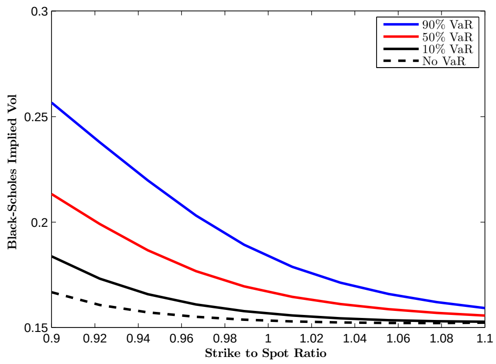
(a) No Jump



(b) $\lambda = 1/3, \mu_J = -1\%$



(c) $\lambda = 1/25, \mu_J = -10\%$



(d) $\lambda = 1/100, \mu_J = -20\%$

Figure 2.7: Equilibrium implied volatility curve. This figure plots the implied volatility curves of the VaR economy with various initial wealth distributions (three solid lines) and the benchmark economy (dashed line). The four panels depict results for different jump cases. To compare our model with the case without jump, the market volatility in the top left panel is calibrated by taking into account the impacts of the jump component.

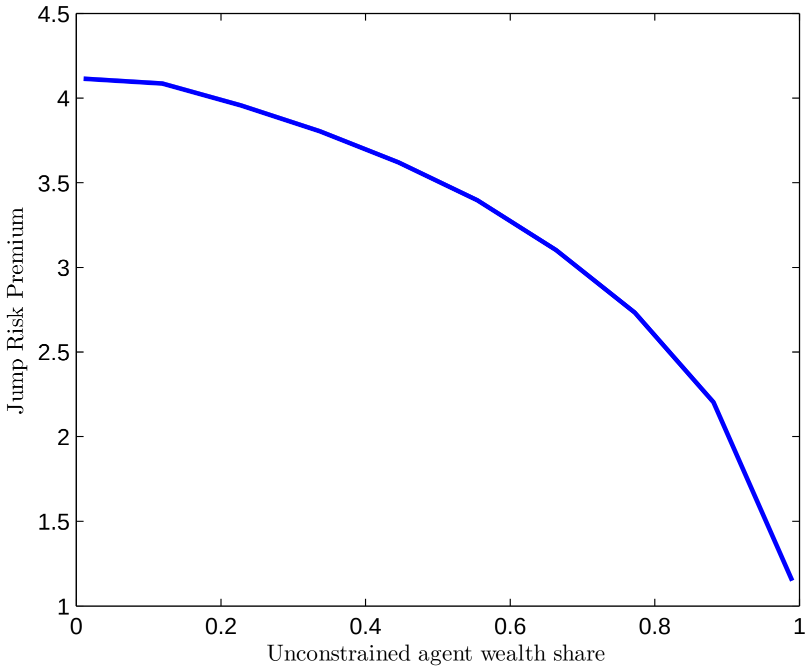


Figure 2.8: Jump risk premium in the VaR economy. This figure plots the jump risk premium in the VaR economy against the wealth share of the unconstrained agent.

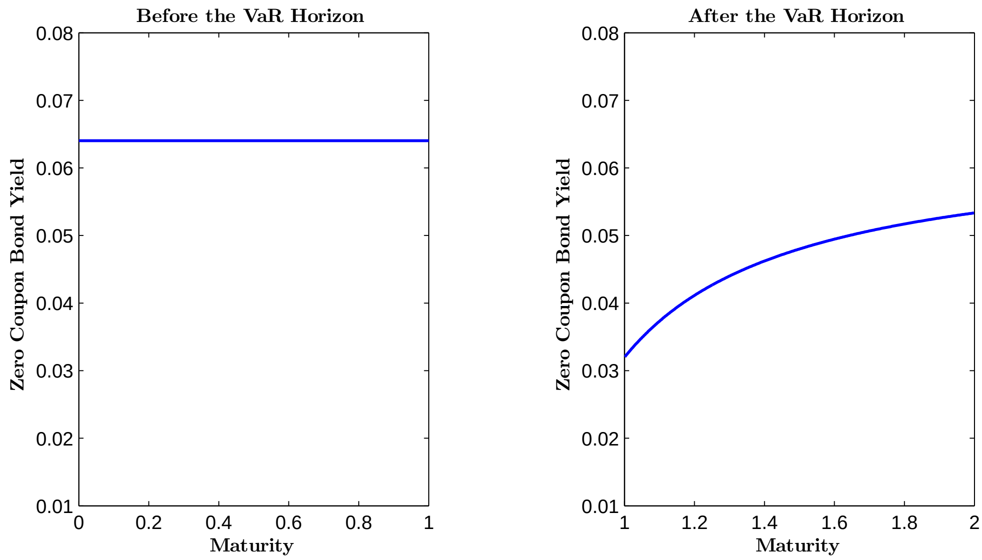


Figure 2.9: Equilibrium term structure of interest rates. This figure depicts the term structure of interest rates in the VaR economy. The left panel and the right panel show the zero coupon bond yields before and after the VaR horizon, respectively.

Chapter 3

From Funding Liquidity to Market Liquidity: Evidence from the Index Options Market

3.1 Introduction

During the 2008 crisis, and especially in the periods when Lehman Brothers and other important financial institutions failed, funding available to banks and non-financial firms was in short supply. A number of institutions failed because they had difficulties in raising funds in illiquid markets. It is thus timely and fitting to examine the dynamic changes in market liquidity in regard to changes in funding liquidity. This paper provides empirical evidence that options market liquidity is strongly influenced by funding liquidity during periods of high market uncertainty. More specifically, we find that liquidity in the S&P 500 index options market is positively correlated with funding liquidity, after controlling for VIX, a broad-based measure of market uncertainty.[1]

A number of theoretical studies examine the link between market declines and asset illiquidity. Based on the idea that market liquidity depends on the capital of financial intermediaries, Gromb and Vayanos (2002) show that when arbitrageurs have enough wealth, they fully absorb other investors' supply shocks and thus provide market liquidity, but this situation is not so if

[1]VIX is calculated by the Chicago Board Options Exchange (CBOE), which measures the implied volatility of options on the S&P 500 index. It is often referred to as the fear index or the fear gauge.

arbitrageurs are less well capitalized. They explicitly point out that arbitrageurs act as intermediaries by providing liquidity to other investors. Brunnermeier and Pedersen (2009) elaborate on the relationship between funding liquidity and market liquidity (FL-ML) and show that the two notions are mutually reinforcing, leading to liquidity spirals. They argue that a huge market-wide decline in prices reduces the ease with which market makers can obtain funding, which feeds back as higher comovement in market liquidity during recessions. Garleanu and Pedersen (2007) argue that tighter risk management reduces liquidity, which further tightens risk management. This feedback effect helps explain the connection between sudden drops in liquidity and increased volatility. In his 2010 AFA presidential address, Duffie (2010) argues that the financial crisis and slow movement of investment capital increased the cost of intermediation and thus led to increases in trading spreads. Moreover, Duffie (2012) points out that the 2008 financial crisis not only affected banks' lending function, but it also had a major impact on market liquidity. He further argues that investors and issuers of securities found it more costly to raise capital and obtain liquidity for their existing positions during the recent financial crisis.

The implications of these recent important theoretical findings have not been fully investigated from an empirical point of view and to date, to the best of our knowledge, there has not been a thorough empirical analysis of the relationship between market liquidity and funding liquidity over a long period of time. The relationship between funding liquidity and market liquidity in the stock market has been tested by Hameed, Kang, and Viswanathan (2010) with precrisis data. After the 2008 crisis, researchers have paid increased attention to the investigation of the relationship between funding liquidity and market liquidity of different financial markets, including the stock market (see e.g., Hu, Jain, and Jain, 2013), the corporate bond market (see e.g., Dick-Nielsen, Gyntelberg, and Lund, 2013), and the foreign exchange market (see e.g., Coffey, Hrung, and Sarkar, 2009; Mancini, Ranaldo, and Wrampelmeyer, 2013). However, none of the previous studies have examined the dynamics of funding liquidity and options market liquidity.

This paper presents one of the first systematic empirical studies of liquidity in the S&P 500 index options market and analyzes the impact of funding liquidity on the index options market liquidity during the recent financial crisis. We measure liquidity in the index option market on a daily basis, relate index options market liquidity to measures of funding liquidity as well as liquidity of equity markets, and provide solid evidence to support the theoretical predictions of Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009). This paper tests and validates the following hypotheses: H1: option market liquidity is positively correlated with funding liquidity, and this effect is more prominent during periods of high market uncertainty; and H2: the market liquidity of call and put options responds differently to funding liquidity.

We compute options liquidity using a comprehensive database. Ranging from January 2003 to January 2012, our sample includes the financial crisis and is thus highly relevant for analyzing liquidity. Following Chordia, Roll, and Subrahmanyam (2000) and Cao and Wei (2010), we use the proportional bid-ask (PBA) spread as our measure of index options liquidity. We compute the PBA spread by dividing the difference between ask and bid quotes by the midquote. We use the TED spread, the difference between the three-month LIBOR and the three-month U.S. Treasury bill rate, as a proxy for the level of funding liquidity.[2]

We retrieve the residual of the TED spread from an OLS regression of the TED spread on VIX in order to isolate the effect of funding liquidity from the influence of market-wide uncertainty. The residual from the aforementioned regression, $Residual_{TED|VIX}$, is then included in an ARMAX model as an exogenous regressor along with VIX to examine the relation between liquidity in the index options market and funding costs. A positive relationship between the PBA spread and the $Residual_{TED|VIX}$ is found for the whole sample period, with the coefficient is statistically significant at the 1% level.[3] In particular, a one standard deviation increase in the $Residual_{TED|VIX}$ translates into an increase in bid-ask spread of 0.49 basis point, which is about 11% of its standard deviation. Our empirical findings lend support to the first hypothesis that market liquidity declines when liquidity providers face high funding costs.

We then examine whether the effect of funding liquidity on options market liquidity depends on market uncertainty. This conjecture is tested by interacting the TED spread with VIX. We find that shocks to funding liquidity positively affect options market liquidity only when VIX is high enough, which implies that our main findings are very likely to be "conditional". This "conditional" effect can be observed for both call and put options and for options with different

[2] It's common to employ the TED spread as a proxy for funding liquidity (see e.g., Brunnermeier, Nagel, and Pedersen, 2008; Brunnermeier and Pedersen, 2009; Hameed, Kang, and Viswanathan, 2010; Boyson, Stahel, and Stulz, 2010). An alternative proxy for funding liquidity in this paper is the LIBOR-OIS spread: the difference between the LIBOR and the overnight index swap rate (OIS). The results based on the LIBOR-OIS spread are similar and are available upon request.

[3] An ARMAX is estimated to fit the time series of the PBA spread, which is serially correlated and can be explained by exogenous variables, such as VIX and the $Residual_{TED|VIX}$. AIC and BIC are employed to determine the optimal number of lags of autoregressive and moving average terms.

characteristics.[4] For instance, only when VIX is higher than 28% can we observe that call options liquidity exhibits a significant deterioration following an increase in funding costs. For put options, the corresponding threshold value of VIX is 14%. Considering the sample median of VIX is 18%, our findings indicate that put options market liquidity reacts to funding liquidity shocks in a less "conditional" way. Since market uncertainty (VIX) stays at relatively high levels even for a long time after the crisis, the "conditional" effect we document cannot be attributed to the financial crisis.

Cao and Wei (2010) show that the effect of market movements on options' liquidity differs between calls and puts. Specifically, the liquidity of calls mostly responds to upward market movements while the liquidity of puts responds mostly to downward movements. One can therefore expect that the liquidity of put options mostly responds to funding liquidity during periods of high market uncertainty. Our results show that the liquidity of puts and calls indeed responds asymmetrically to funding liquidity.

We further split the whole sample according to maturity and moneyness (the ratio between the strike price and the underlying spot price) of each option to study how the effect of funding liquidity on options market liquidity is distributed across options of various maturities and moneyness levels. We maintain the same specification of the ARMAX model, linking the option market liquidity to funding liquidity and VIX. This exercise is related to the growing literature on the information content of option trading (see e.g., Vijh, 1990; Easley, O'hara, and Srinivas, 1998; Jayaraman, Frye, and Sabherwal, 2001; Cao, Chen, and Griffin, 2003). Trading deep out-of-the money options benefits from high leverage although these options are generally less liquid with high proportional bid-ask spreads. In the presence of superior information, however, the leverage effect may dominate the liquidity consideration. Similarly, to avoid a high option premium, one may prefer a short-term option over a long-term one as the former offers high leverage. One can therefore expect that the relationship between funding liquidity and market liquidity of options is mainly driven by short-term and deep out-of-the-money options. First, we document a positive relationship between the PBA spread of short-term options and the $Residual_{TED|VIX}$. A reduction of funding liquidity is followed by a lower liquidity level of short maturity options. Second, our results show that the $Residual_{TED|VIX}$ is positively related to

[4]We are very cautious in generalizing this conclusion to a broader set of assets because of the distinctive feature of options as well as its distinctive relation with market uncertainty, compared with other types of financial instruments.

the PBA spread and significant in explaining the liquidity of options with different moneyness.

Finally, to show that our results are not driven by specific samples, we conduct several robustness tests. First, we split the whole sample into three sub-periods: precrisis period (01/2003-07/2007), crisis (08/2007-06/2009) and postcrisis period (07/2009-01/2012). After controlling for the interaction term between the TED spread and VIX, the interaction term is significantly positive during the crisis period. This suggests that the effect of funding liquidity on options market liquidity is more prominent when market uncertainty is high, consistent with the theoretical predictions of Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009) that market liquidity is affected by funding liquidity. Second, we reexamine the main findings using weekly data. Compared with daily time series, weekly data are characterized by a much lower volatility and therefore allow us to make more reliable inferences. In general, the results are similar to what is found for the daily sample. It is also worth noting that the $Residual_{TED|VIX}$ is always positively related to the PBA spread. Probably due to the lower volatility of weekly data, the coefficient magnitude of the $Residual_{TED|VIX}$ has declined compared with the main results. In short, results using the weekly sample have confirmed the main findings.

Our paper is related to the growing literature that investigates the relationship between funding liquidity and market liquidity. Chordia, Sarkar, and Subrahmanyam (2005) examine liquidity movements in stock and Treasury bond markets with daily data and build a link between macro liquidity, or money flows, and micro or transactions liquidity. Using a dummy variable as a proxy for the period of low funding liquidity, Hameed, Kang, and Viswanathan (2010) test the relationship between funding liquidity and market liquidity in the stock market but only cover precrisis data. Hu, Jain, and Jain (2013) explore the non-linear FL-ML relationship in the stock market and show that the relationship weakens after the enactment of the Volcker Rule. Dick-Nielsen, Gyntelberg, and Lund (2013) investigate how funding liquidity affects the bond market liquidity in Denmark. They find that the ease of obtaining term funding in the money markets determines the liquidity in the bond market, for both long- and short-term bonds. Mancini, Ranaldo, and Wrampelmeyer (2013) use intraday trading and order data to measure liquidity in the foreign exchange (FX) market and show that negative shocks in funding liquidity lead to significantly lower FX market liquidity and systematic FX liquidity comoves with equity liquidity.

Our paper is also related to the literature focusing on liquidity in the options market. While extensive literature studying liquidity in the equity markets exists, liquidity in the options market is much less known, although the options market is by far one of the most important markets.[5] For instance, Jameson and Wilhelm (1992) show that the bid-ask spread of options is determined by the ability of market makers to rebalance options positions as well as uncertainty regarding the return volatility of the underlying stock. Employing a simultaneous equation system, George and Longstaff (1993) examine how trading activities of call (put) options depend on the bid-ask spread of calls (puts) as well as puts (calls). They find that the bid-ask spread negatively affects trading volume, and calls and puts are substitutes in terms of trading activities. A recent paper by Wei and Zheng (2010) studies the relation between trading activities and bid-ask spread on the individual options level. Using data on inventory positions of market makers, Wu, Liu, Lee, and Fok (2014) consider the price risk for market makers and show that price risk is not significantly related to option spreads, which seems to be consistent with the prediction of derivative hedging theory. Using Ivy DB's OptionMetrics data, Cao and Wei (2010) examine the commonality among various liquidity measures such as the bid-ask spread, volumes and price impact. In addition, they establish that the options liquidity responds asymmetrically to upward and downward market movements. Furthermore, several studies investigate the effect of liquidity in derivative prices. In an extended Black-Scholes economy, Cetin, Jarrow, Protter, and Warachka (2006) derive the pricing of options with illiquid underlying assets. Their empirical results support the conjecture that liquidity costs account for a significant portion of the option price. Bongaerts, de Jong, and Driessen (2011) develop a theoretical asset pricing model of liquidity effects in derivative markets and test the pricing of liquidity for the credit default swap market. Using the OTC euro interest rate cap and floor data, Deuskar, Gupta, and Subrahmanyam (2011) find that illiquid options trade at higher prices relative to liquid options.

The rest of the paper is organized as follows. In Section 3.2, we describe the data, define the liquidity measures and report the summary statistics. Section 3.3 presents the main results concerning the dynamics of market liquidity and funding liquidity. Some additional robustness tests are provided in Section 3.4. Section 3.5 concludes.

[5] In the United States, the trading volume of individual stock options has grown exponentially from 5 million contracts in 1974 to more than 3,727 million contracts in 2015 (http://www.optionsclearing.com/webapps/historical-volume-query).

3.2 Data and Variables

In this section, we first discuss the data used to construct liquidity measures. Then we describe how to calculate these measures and report the summary statistics.

3.2.1 Data

Our data consist of daily closing bid and ask quotes, daily volume and open interest on the S&P 500 index options. We cover the period from January 17, 2003, to January 31, 2012, for a total of 2,263 trading days, including the height of the recent crisis in the fall of 2008. We extract the options data from the OptionMetrics IVY DB, which includes daily best bid and ask closing quotes, open interest and volume for each option. We then apply several filters to minimize possible data errors. To eliminate outliers and options with non-standard features, we discard options with missing implied volatilities. Further, we drop observations violating basic no-arbitrage conditions. We also remove all options with zero bid prices. In the S&P 500 index options sample, we end up with 223,447 observations, of which 104,502 are calls and 118,945 are puts. We have an average of about 99 options per day.

In addition to the whole sample analysis, we split the entire sample according to several characteristics of options to obtain a clearer picture. Following Bakshi, Cao, and Chen (1997), we first classify the time-to-maturity, measured in calendar days to expiration ($ADTE_{i,t}$), into three categories: short-term with less than 60 days, medium-term with more than or equal to 60 days and less than or equal to 180 days, and long-term with more than 180 days. Second, we categorize the moneyness ($m_{i,t}$, the ratio between strike price and the underlying spot price) of options into five groups as follows: Deep-out-of-the-money (DOTM) if the contract is a call and $m_{i,t} > 1.06$ or if the contract is a put and $m_{i,t} < 0.94$, Out-of-the-money (OTM) if the contract is a call and $1.01 < m_{i,t} \leq 1.06$ or if the contract is a put and $0.94 \leq m_{i,t} < 0.99$, At-the-money (ATM) if $0.99 \leq m_{i,t} \leq 1.01$ for either puts or calls, In-the-money (ITM) if the contract is a call and $0.94 \leq m_{i,t} < 0.99$ or if the contract is a put and $1.01 < m_{i,t} \leq 1.06$, and Deep-in-the-money (DITM) if the contract is a call and $m_{i,t} < 0.94$ or if the contract is a put and $m_{i,t} > 1.06$ (Goncalves and Guidolin, 2006).

Following Cao and Wei (2010), we compute the proportional bid-ask spread (PBA) by dividing the difference between the ask and bid quotes by the mid-quote. Then we employ a volume-weighted average of the proportional spreads within each day and use this average to implement our analysis.[6]

The funding liquidity measure used in this paper is the TED spread, which is from the Federal Reserve Bank in St. Louis. Our proxy for market-wide uncertainty is the Chicago Board Options Exchange Volatility Index (VIX) which is frequently used as a proxy for investors' fear and uncertainty in financial markets. Figure 3.1-3.3 depicts the evolution of options liquidity (PBA), the TED spread and VIX from January 2003 to January 2012. Both the TED spread and VIX shoot up during the financial crisis. However, the options market bid-ask spread seems to reach its lowest level in the crisis. Before the crisis when both the TED and VIX stay at low levels, the options bid-ask spread is almost twice as high as its level during the crisis. The "cooling down" in the boom and "heating up" in the crisis of options market transactions point directly to the distinctive features of this market.

3.2.2 Summary Statistics

Table 3.1 presents the summary statistics of both the bid-ask spread of various types of options and the key explanatory variables. In addition to summarizing the whole sample, we also divide the sample period into three sub-periods, namely, the precrisis period (01/2003-07/2007), the crisis period (08/2007-06/2009) and the postcrisis period (07/2009-01/2012). During the whole sample period, the mean bid-ask spread for all options is 12 basis points. It is consistent with the calculation in Cao and Wei (2010) who find a 13-bps bid-ask spread during the period from 1996 to 2004. Since our sample period has spanned the financial crisis when the options market was relatively more liquid, the PBA spread should be lower in this study. Several things are worth noting. First, compared with call options, put options have a smaller bid-ask spread in terms of both the mean and the median. The higher liquidity for puts in our paper might be attributed to the high transaction activities during the financial crisis. Second, options in general become more liquid during the financial crisis. For instance, options are traded with a proportional bid-ask spread as high as 13 basis points before the crisis, and 12 bps after the crisis. During the crisis, the bid-ask spread narrows down to 10 bps. Moreover, this pattern

[6] The option liquidity measures are defined in the Appendix. In an unreported analysis, we weigh the bid-ask spread by the corresponding open interest and find that all results in the paper are qualitatively similar.

applies to different types of options, such as options with different maturities and moneyness.[7]

Panel B contains the mean, median and standard deviation for independent regressors used in this paper. It is observed that both the TED spread and VIX increase dramatically as the crisis unfolds. The TED spread is more than 10 times higher after the crisis, indicating that the funding liquidity suddenly drops in this period. We also notice that VIX is almost twice as high as that before the crisis, implying that the market uncertainty perceived by investors increases after the inception of the crisis. We also examine the dynamics of stock market liquidity, using both bid-ask spreads and volume as the measure. The US stock market has an average bid-ask spread of 14 cents when equally weighted and 2 cents when value weighted. The mean daily volume and dollar volume are 6.3 billion trades and 189 billion dollars. In contrast to the options market, the stock market has become much less liquid during the financial crisis.

3.3 Empirical Results

To model the relationship between options market liquidity and funding liquidity as well as market uncertainty, we first have to test for the stationarity of these several time series. The Augmented Dickey Fuller (ADF) test result is shown in Table 3.2, revealing that all of the variables of interest are stationary. The null hypothesis of unit root is rejected at 1% for all of our series. Therefore, we choose ARMAX to model the effect of funding liquidity and market uncertainty on the options market liquidity.[8] Given the high correlation between the TED spread and VIX (0.776 over the whole sample period), we isolate the effect of funding liquidity from the influence of market uncertainty. We adopt a two-step procedure in which only that part of the TED spread which is orthogonal to VIX is used to predict options market liquidity. Specifically, we run OLS regression in the first step where the TED spread is regressed on VIX:

$$TED_t = \alpha_0 + \alpha_1 VIX_t + \mu_t \tag{3.3.1}$$

[7] An exception is deep-in-the-money options, which are traded with higher bid-ask spreads during the crisis compared with those before the crisis.

[8] For each specification in this paper, we also run OLS regressions with Newey and West (1987) standard errors. The number of lags used to calculate Newey-West standard errors is set to be seven, the closest integer to the fourth root of the number of observations in our main sample, as suggested by Greene (2011). All results are qualitatively similar to those generated from ARMAX and thus are not tabulated for conciseness. Results from Newey-West regressions are available upon request.

We then obtain the residual $Residual_{TED_t|VIX_t}$ from Equation (3.3.1) and include it in an ARMAX(p,q) model where the independent variable is the PBA spread of the index options market:

$$PBAspread_t = \beta_0 + \sum_{i=1}^{p} \pi_i PBAspread_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \beta_1 Residual_{TED_{t-1}|VIX_{t-1}} + \beta_2 VIX_{t-1} + \epsilon_t \tag{3.3.2}$$

where p and q are the number of lags for autoregressive and moving average terms. β_1 and β_2 are the coefficient of the residual of TED spread and VIX, respectively. We use AIC and BIC to determine the optimal number of lags.

3.3.1 Main Results

Table 3.3 shows the results of the ARMAX regressions linking the liquidity in the options market to funding costs and VIX over the entire sample period from January 17, 2003, to January 31, 2012. After controlling for the lagged proportional bid-ask (PBA) spread, we first regress the PBA spread on the $Residual_{TED_t|VIX_t}$ and VIX using simple OLS. As shown in column (1), a significantly negative relationship between the PBA spread and VIX is documented. In terms of magnitude, when market uncertainty increases by one standard deviation (9.89%) yesterday, the current option market bid-ask spreads decline by 1.34 basis points, which is about 31% of its standard deviation. This effect is in contrast to what Mancini, Ranaldo, and Wrampelmeyer (2013) find in the currency market when an increase in market uncertainty is followed by a decline in FX market liquidity. It arises from the fact that the convex payoff structure is such that options can yield hedging benefits for volatility, which is more pronounced during periods of high market uncertainty. Note that this effect is the net of controlling for one lag of the proportional bid-ask spread. The magnitude of this effect estimated from OLS regressions without controlling for lags of the PBA spread is actually doubled.

In line with Hypothesis 1, we find a significantly positive relationship between the current PBA spread and the previous day's $Residual_{TED|VIX}$. The coefficient is statistically significant at the 1% level. One-standard deviation increase in $Residual_{TED|VIX}$ (0.32%) can be translated into an increase in options bid-ask spread as high as 0.26, which is about 6% of the standard deviation of the PBA spread.

In column (2), we estimate an ARMAX model with four autoregressive and three moving average terms which generates the baseline result of this paper. The $Residual_{TED|VIX}$ and VIX lagged for one day are included as exogenous variables. The results are similar to those in the OLS estimation. However, the magnitude of the effect of the $Residual_{TED|VIX}$ on the bid-ask spread is about twice as high as in the first column, highlighting the necessity of taking into account the autocorrelation within the PBA spread at higher orders. Here, one standard deviation increase in $Residual_{TED|VIX}$ leads to an increase in the bid-ask spread as large as 0.49 basis point, which is about 11% of the standard deviation of the PBA spread. Therefore, after controlling for VIX, the options market liquidity declines when liquidity providers face higher funding costs, consistent with the theoretical prediction of Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009).

In columns (3) and (4), we further distinguish between call and put options to see whether the pattern of how funding liquidity and market uncertainty affect options liquidity differs between calls and puts. As in previous regressions, we find a significantly negative relationship between the PBA spread and VIX, in both the call and the put samples. This finding suggests that rising market-wide uncertainty contributes to a lower bid-ask spread in both call and put options. The magnitude of VIX's effect on the PBA spread is higher in the put options sample. For call options, one-standard-deviation increase in VIX is followed by a decrease in the PBA spread by 0.14 ($-9.89 \times 0.074/5.21 = -0.140$) standard deviation. For puts, the sensitivity of options liquidity to uncertainty is higher. A one-standard-deviation shock to VIX at time $t-1$ leads to a change in the PBA of puts as large as 0.109 standard deviation ($-9.89 \times 0.051/4.62 = -0.109$) at time t.

Interestingly, we only find a positive relationship between the market liquidity and the funding liquidity for the subsample of puts. A one-standard-deviation increase in the $Residual_{TED|VIX}$ is followed by an increase in the PBA spread as large as 0.65 basis points, which is equivalent to 14% of its standard deviation. This effect is higher than that for options in the whole sample. It lends support to Hypothesis 2. Overall, the results using daily options market liquidity support our hypothesis that market liquidity deteriorates when the supply of capital is tight. We also show that the options market becomes more liquid during periods of high market uncertainty.

We then examine whether the effect of funding liquidity on the index options market liquidity is more prominent during periods of high uncertainty. We test this prediction by interacting the TED spread with VIX. If the effect is magnified when market-wide uncertainty is high, one can expect the interaction term to be significantly positive. As shown in columns (5)-(7), the interaction term is positive and statistically significant, irrespective of the sample we use. However, the TED spread alone becomes insignificantly negative, indicating that the effect of funding liquidity on index options market liquidity is very likely to be conditional in nature. In other words, the index options market becomes illiquid following a shock to funding liquidity only when market-wide uncertainty is high. For instance, as indicated in column (5), when VIX is below 17%, the PBA spread of index options as a whole reacts negatively to shocks to the TED spread.[9] This relationship, however, becomes positive as VIX stays above 17%. According to columns (6) and (7), the market liquidity for call and put options is positively correlated with funding liquidity only when VIX is larger than 28% and 14%, respectively. Obviously, put options market liquidity reacts to funding liquidity shocks in a less "conditional" way, showing that put options react more sensitively to shocks to funding liquidity. These results are consistent with the empirical findings of Cao and Wei (2010), who argued that the liquidity of puts and calls respond asymmetrically to market movements. Specifically, they document that put options' liquidity responds mostly to downward movements.

3.3.2 Subsample Analysis

To study the dynamics of options market liquidity in regard to changes in funding liquidity, we further split the sample in three ways: (1) options with short, medium and long maturity; (2) options with different moneyness; (3) call and put options with different moneyness.

Options Maturity

In Table 3.4, we split the sample according to maturity (Short-term if $ADTE_{i,t} < 60$; Medium-term if $60 \leq ADTE_{i,t} \leq 180$; Long-term if $180 \leq ADTE_{i,t}$). To avoid a high option premium, one may prefer a short-term option over a long-term one, as the former offers high leverage and is generally more liquid. One can therefore expect that the FL-ML relationship is mainly driven by short-term options. Table 3.4 shows the results of ARMAX regressions linking the liquidity of options with various maturities to funding costs and VIX over the entire sample

[9] When VIX stays at 17.5% (1.35/0.077 = 17.53), the marginal effect of the TED spread on the PBA spread is close to zero. The effect thus turns negative when VIX is below 17%.

period. Again, VIX is strongly negatively related to short, medium, and long maturity options. Also, the magnitude of the effect of VIX on the bid-ask spread of short maturity options is much higher than that of medium and long maturity options. As a result, the liquidity of different maturity options responds asymmetrically to market movements.

In column (1), we document a positive relationship between the PBA spread of short maturity options and the $Residual_{TED|VIX}$, which is significant at the 10% level. Thus, a reduction of funding liquidity is followed by a lower liquidity level for short maturity options, but the coefficients are not significant for options with longer maturities. This lends support to our conjecture that the relationship between funding liquidity and market liquidity is mainly driven by short-term options.

As in Section 3.3.1, we include the interaction term between the TED spread and VIX to examine whether the effect of funding liquidity on options' liquidity is more pronounced during periods of high market uncertainty. Columns (4)-(6) show that the market liquidity for short, medium and long maturity options increases with funding liquidity only when VIX is larger than 18.6%, 29.9% and 33.7%, respectively. This further demonstrates that short maturity options are the most sensitive to changes in funding costs, and thus the FL-ML relationship is mainly driven by short maturity options.

Options Moneyness

Next, we split our sample according to the extent of options moneyness. Options are divided into five categories based on the moneyness of each option, namely deep out of the money (DOTM), out of the money (OTM), at the money (ATM), in the money (ITM) and deep in the money (DITM). Trading deep out-of-the money options benefits from high leverage although these options are generally less liquid with high proportional bid-ask spreads. In the presence of superior information, however, the leverage effect may dominate the liquidity consideration. So, we expect that the relationship between FL-ML is mainly driven by deep out-of-the-money options. Our new results, shown in Table 3.5, indicate that both DOTM and DITM options liquidity responds significantly to changes in the $Residual_{TED|VIX}$. Coefficient estimates for the $Residual_{TED|VIX}$ are significant in columns (1) and (5) at the 5% level.

Moreover, we find that although market uncertainty continues to have significant impact on options liquidity, the effect exhibits substantial heterogeneity among options with different moneyness. Specifically, the coefficient estimates of VIX indicate that the DOTM and OTM options become more liquid following an increase in market uncertainty, whereas the response of DITM options is just the opposite. Coefficient estimates in columns (1)-(2) and (6)-(7) indicate that DOTM and OTM options' liquidity is significantly negatively correlated with VIX. The coefficient becomes positive in columns (5) and (10) in which the PBA spread of DITM options is the dependent variable. While there might not be one single explanation for this phenomenon, one candidate could be that investors are less likely to trade the DITM options, which become more costly during periods of high volatility. This arises from the fact that DITM options are unable to hedge uncertainty but are with a high premium compared with ATM or OTM options. As a result, the liquidity of these options decreases with VIX.

Columns (6)-(10) indicate that the conditional effect of funding liquidity seems to exist only in DOTM, OTM and ATM options. The liquidity of in-the-money options appears to be very insensitive to changes in funding costs, as it is not related to the TED spread either in an unconditional or a conditional way. The liquidity of DITM options, however, is only related to the TED spread unconditionally. In other words, the effect of the TED spread on the liquidity of DITM options does not vary with VIX.

Call and Put Moneyness

We further examine whether the effect of funding liquidity on options liquidity is different between call and put options with different moneyness. We hence split each of the call and put subsample into five categories based on the moneyness of each option.[10] Panel A of Table 3.6 shows the results of ARMAX regressions on the liquidity of call options with different extents of moneyness. Different from the main results in Table 3.3 for call options, the liquidity of ATM, ITM and DITM call options is found to react to shocks to funding liquidity in a significant way. As shown in columns (3)-(5), the coefficient of the $Residual_{TED|VIX}$ is positive and statistically significant, especially when DITM call options liquidity is the dependent variable.

Columns (6)-(10) investigate the "conditional" effect of funding liquidity on the liquidity of options with various moneyness. While the coefficient of the interaction term is positive in all

[10]For details regarding the definition of moneyness categories for call and put options, see the variable definition table in the Appendix.

specifications, it is only significant for OTM, ATM and ITM calls. For these three types of call options, funding costs start to have a positive impact on options' liquidity when VIX is above 26%-30%.

We then turn to put options in Panel B. Similar to call options, the estimated coefficients of the $Residual_{TED|VIX}$ are significantly positive except for the DOTM and OTM options. Another finding in common between call and put options is that the response of options' liquidity to shocks on VIX varies substantially across options moneyness. One distinctive finding for put options is the "conditional" effect. Although not all types of put options' liquidity responds to shocks to funding cost in a significant way, the conditional effect prevails. As indicated in columns (6)-(10), the interaction term is always positive and highly significant. The turning point of VIX around which the effect of funding costs on puts liquidity becomes positive lies between 23% and 30%. Surprisingly, the DITM put options exhibit a totally different way of responding to funding liquidity shocks than other types of options. The impact of the TED spread on the liquidity of DITM puts is significantly positive. As indicated by the significantly positive interaction term, the above positive effect is then reinforced by increasing market uncertainty.

3.3.3 Relation to the Liquidity of the US Equity Market

In this section, we control for stock market liquidity to rule out the possibility that the effect observed is due to the relation between funding liquidity and liquidity in the equity market. There are a number of reasons to expect a connection between equity and index options market liquidity. For instance, liquidity exhibits comovement across asset classes and can be driven by common influences of the systemic shocks to the liquidity of the equity market. In particular, the liquidity of the underlying assets is closely related to that of the corresponding derivatives. We use three variables to proxy for aggregate stock market liquidity, namely the bid-ask spread, the trading volume and the dollar volume. The data are from CRSP. The method used to calculate stock liquidity measures can be found in the Appendix.

Table 3.7 shows the results of the ARMAX regressions linking options market liquidity, stock market liquidity, and funding liquidity. We use equally- and value-weighted stock market bid-ask spread as equity liquidity measures. The first two columns report the results of regressions in which the bid-ask spread is used to proxy for stock liquidity. Consistent with the results of

Cao and Wei (2010), the liquidity of the options market is closely linked to that of the equity market.[11] The coefficient of the stock's bid-ask spread is positive, though only the value-weighted measure is statistically significant. Importantly, the coefficients on the $Residual_{TED|VIX}$ are still significant and their signs are positive. This implies that the effect of funding liquidity on options liquidity remains after considering the possible channel through the equity market liquidity. In columns (3) and (4), we use volume and dollar volume as proxies for stock liquidity, and we obtain similar results. After controlling for VIX and the equity market liquidity, there is still a positive relationship between the options market liquidity and funding liquidity. Then we include the interaction term between the TED spread and VIX. As shown in columns (5)-(8), the coefficient of the TED spread becomes insignificantly negative, but the coefficient on the interaction term is significantly positive. These findings are similar to that shown in Table 3.3.

3.4 Robustness

To show that our results are not driven by specific samples, we conduct several robustness tests in this section. First, we test the relationship between funding liquidity and market liquidly using different sample periods. Second, we test our hypotheses using weekly data.

3.4.1 Split Sample Pre-Post Financial Crisis

Table 3.8 examines the FL-ML relationship during different sample periods, using an ARMAX model. The whole sample period is divided into three sub-periods: the precrisis period (01/2003-07/2007), the crisis period (08/2007-06/2009) and the postcrisis period (07/2009-01/2012). In Panel A, we use VIX and $Residual_{TED|VIX}$ as independent variables. Columns (1)-(3) display results of the precrisis period. Columns (4)-(6) and columns (7)-(9) are for the crisis and postcrisis periods, respectively. We find that the PBA spreads of the precrisis and postcrisis periods are negatively related to VIX. Thus, an increase in market uncertainty is followed by higher options market liquidity. This is in line with what we find using the whole sample. Different to specifications in Panel A, Panel B controls for the interaction term between the lagged TED spread and lagged VIX. As shown in Panel B, the interaction term is not statistically

[11] Note that the positive coefficient seems to be inconsistent with the pattern of the bid-ask spread of options and stocks shown in Figure 3.1 and 3.4. However, the positive correlation is net of the effect of other factors, which affect both the stock market liquidity and options liquidity, such as VIX and the TED spread.

significant during the precrisis period and postcrisis period, while it is significantly positive during the crisis period, irrespective of the options sample we use. This suggests that the effect of funding liquidity on options market liquidity is more prominent when market uncertainty is high, consistent with the theoretical predictions of Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2009) that market liquidity is affected by funding liquidity.

3.4.2 Weekly Data Sample

In Table 3.9, we re-examine the main findings using weekly data. Compared with the daily time series, weekly data are characterized by a much lower volatility and therefore allow us to make more reliable inferences. Here the proportional bid-ask spread of options is calculated on the weekly level. Weekly VIX and the $Residual_{TED|VIX}$ are used as exogenous regressors correspondingly. In general, the results are similar to what is found for the daily sample. Consistent with the results in Table 3.3, an increase in market uncertainty in week $t-1$ is followed by a decrease in the PBA spread in week t, irrespective of sample used. It is also worth noting that the $Residual_{TED|VIX}$ is always positively related to the PBA spread. Probably due to the lower volatility of weekly data, the coefficient magnitude of the $Residual_{TED|VIX}$ has declined, compared with the main results. In columns (5)-(7), we investigate the "conditional" effect using weekly data. Again, we observe a significant effect of the TED spread on the liquidity of options, which is especially large for put options. All in all, results using the weekly sample have confirmed the main findings.

3.5 Conclusion

Funding liquidity and its impact on market liquidity have become a major focus of the academic literature. Most studies investigate the relationship between funding liquidity and market liquidity from a theoretical point of view. For instance, Brunnermeier and Pedersen (2009) explain that a large market-wide decline in prices reduces the ease with which market makers can obtain funding, which feeds back as higher comovement in market liquidity during recessions. Recently, some studies have emerged to examine the FL-ML relationship in stocks, corporate bonds, and foreign exchange markets. However, none of the previous works study the relationship of funding liquidity and options market liquidity during the crisis. This paper presents one of the first empirical studies of liquidity in the S&P 500 index options market, and studies the FL-ML relationship.

Using data on the S&P 500 index options traded on the CBOE market covering the period from January 17, 2003 to January 31, 2012, we establish convincing evidence of a positive relationship between funding liquidity and options market liquidity during periods of high market uncertainty. More specifically, we find a positive relationship between the PBA spread and the $Residual_{TED|VIX}$, and the coefficient is statistically significant. These empirical findings lend support to the hypothesis that market liquidity declines when liquidity providers face high funding costs during the periods of high market uncertainty.

This paper serves as a first step toward understanding the relationship between funding liquidity and index options market liquidity during periods of high market uncertainty. It opens up several avenues for future research. One natural extension would be the in-depth examination of the relationship of funding liquidity and individual options market liquidity. Another area of future research would be to investigate the effect of funding constraints on the pricing of index options.

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Appendix: Variable Definitions

Variable	Definition
Dependent variables	
Proportional bid-ask spread (PBA)	$$\frac{\sum_{j=1}^{J} VOL_j * \frac{ask_j - bid_j}{\left(ask_j + bid_j\right)/2}}{\sum_{j=1}^{J} VOL_j}$$ where j is one specific trade
Trading volume (VOL)	$$\sum_{j=1}^{J} VOL_j$$
Dollar trading volume (DVOL)	$$\sum_{j=1}^{J} VOL_j * (ask_j + bid_j)/2$$
Independent variables	
Bid-ask spread of stocks (equally weighted)	$$\frac{\sum_{i=1}^{N} (ask_i - bid_i)}{N}$$ where i is one specific stock
Bid-ask spread of stocks (value weighted)	$$\frac{\sum_{i=1}^{N} w_i (ask_i - bid_i)}{\sum_{i=1}^{N} w_i}$$ where w_i is the capitalisation of stock i
TED Spread	The difference between the three-month LIBOR and the three-month U.S. T-bills
VIX	CBOE S&P 500 volatility index
VXO	CBOE S&P 100 volatility index
VXN	CBOE NASDAQ-100 volatility index
VXD	CBOE Dow Jones Industrial Average (DJIA) volatility index

Table 3.1: Summary statistics

This table shows summary statistics for options liquidity and other variables used in this study. The whole sample period is divided into three sub-periods, the pre-crisis period (01/2003-07/2007), crisis period (08/2007-06/2009) and post-crisis (07/2009-01/2012) period. As shown in Panel A, proportional bid-ask spread is used as the measure of options liquidity and is expressed in basis points. Aside from showing the liquidity of calls and puts, this table also reports the basic statistics of liquidity measures for options with different maturity and moneyness. Panel B presents summary statistics of the independent variables used in this paper.

Sample period	*Whole sample*			*Before crisis*			*During crisis*			*After crisis*		
Statistics	Mean	Med.	Std.	Mean	Med.	Std.	Mean	Med.	Std.	Mean	Med.	Std.
Panel A: Proportional bid-ask spread (basis points)												
All options	11.84	11.4	4.34	12.82	12.37	4.75	9.69	9.18	3.52	11.68	11.54	3.46
Call options	12.67	11.99	5.21	13.14	12.25	5.89	10.81	10.25	4.17	13.19	12.92	4.26
Put options	11.19	10.44	4.62	12.54	11.71	5.13	8.86	8.12	3.49	10.53	10.29	3.44
Short-maturity	13.64	13.09	5.1	14.62	14.06	5.49	11.36	10.79	4.21	13.59	13.12	4.38
Medium-maturity	7.82	7.61	2.87	8.52	8.25	2.94	6.29	5.79	2.51	7.7	7.5	2.54
Long-maturity	4.96	4.76	1.81	5.31	5.08	1.93	3.95	3.72	1.66	5.07	4.75	1.4
Out of the money	12.6	11.88	5.08	13.88	13.16	5.47	9.86	9.35	3.64	12.34	11.78	4.39
At the money	6.92	6.88	2.11	7.1	7.16	1.96	6.2	5.85	2.19	7.12	6.99	2.18
In the money	4.96	4.84	1.56	4.82	4.84	1.32	4.65	4.28	1.87	5.45	5.22	1.59
Deep out of the money	19.94	18.29	8.52	21.45	19.57	9.33	17.26	16.01	8.1	19.24	18.22	6.54
Deep in the money	2.69	2.47	1.08	2.16	2.12	0.57	2.95	2.49	1.52	3.17	3.13	0.86
Panel B: Independent variables												
TED Spread (%)	0.48	0.41	0.5	0.1	0.1	0.18	1.09	0.91	0.56	0.7	0.61	0.17
VIX (%)	21.09	18.53	9.89	15.42	14.32	4.61	31.5	26.22	13.06	23.46	22.22	6.21
Stock bid-ask spread (ew)	0.14	0.12	0.36	0.13	0.13	0.02	0.23	0.17	0.79	0.09	0.08	0.03
Stock bid-ask spread (vw)	0.02	0.02	0.02	0.02	0.02	0.01	0.04	0.03	0.02	0.01	0.01	0
Stock volume	6.3	5.71	2.64	4.29	4.18	0.96	8.66	8.33	2.54	8.1	7.82	1.89
Stock dollar volume	189.21	189.03	84.14	128.74	118.05	49.11	273.21	260.8	70.62	234.06	225.97	57.03

Table 3.2: Stationarity test for key variables

This table shows results of stationarity test for key variables used in this paper, namely proportional bid-ask spread (PBA), dollar volume, the TED spread, VIX and several stock market liquidity measures. Augmented Dickey-Fuller test statistics and the 1% critical value are reported in Columns (3) and (4). The corresponding p-value is shown in the last column. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

Variable	# observations	Dickey-Fuller test statistic	1% value	p-value
Bid-ask spread	2263	-21.678***	-3.430	0.000
Volume	2263	-19.657***	-3.430	0.000
Dollar volume	2263	-12.075***	-3.430	0.000
TED spread	2263	-3.873***	-3.430	0.002
VIX	2263	-4.557***	-3.430	0.000
BAClose_ew	2263	-46.758***	-3.430	0.000
BAHL_ew	2263	-16.498***	-3.430	0.000
Stock_Volume	2263	-11.009***	-3.430	0.000
Stock_Dollar	2263	-11.038***	-3.430	0.000

Table 3.3: Funding liquidity and the options market liquidity

This table shows results of ARMAX regressions linking liquidity in the options market to funding costs and market uncertainty. The proxy for market uncertainty is VIX which is the CBOE S&P 500 volatility index. Our measure for funding costs is the TED spread which is the difference between three-month LIBOR rate and U.S. Treasury bill with the same maturity. We adopt a two-step procedure in which only that part of TED spread which is orthogonal to VIX is used to predict options market liquidity. Specifically, we run an OLS regression in the first step where TED spread is regressed on VIX: $TED_t = \alpha_0 + \alpha_1 VIX_t + \mu_t$. The residual from the aforementioned regression, $Residual_{TED_t|VIX_t}$, is then included in an ARMAX(p,q) model as an exogenous regressor along with VIX. Both VIX and the residual of TED spread are lagged for one period (day). The optimal number of lags of AR and MA terms is selected according to BIC and AIC information criterion. Columns (1)-(4) include lagged VIX and lagged $Residual_{TED|VIX}$ as exogenous regressors, while Columns (5)-(7) add the interaction term between VIX and TED spread into the regression. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of moving averaging terms are not displayed. T-statistics are shown below the coefficient estimates inside parentheses. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Sample	All	All	Call	Put	All	Call	Put
$Residual_{TED_{t-1}\|VIX_{t-1}}$	0.812***	1.527**	0.644	2.036**			
	(3.40)	(1.98)	(0.61)	(2.40)			
TED_{t-1}					-1.350	-1.856	-1.283
					(-1.15)	(-1.28)	(-1.03)
VIX_{t-1}	-0.136***	-0.057**	-0.074*	-0.051*	-0.218***	-0.179***	-0.249***
	(-10.60)	(-2.04)	(-1.96)	(-1.68)	(-5.64)	(-3.55)	(-5.95)
$TED_{t-1} \times VIX_{t-1}$					0.077***	0.067**	0.091***
					(2.92)	(2.05)	(3.52)
AR							
1	0.419***	2.385***	1.524***	2.114***	3.352***	1.466***	2.079***
	(21.97)	(33.52)	(16.86)	(13.78)	(74.70)	(13.75)	(12.70)
2		-2.226***	-0.525***	-1.572***	-4.508***	-1.395***	-1.510***
		(-15.50)	(-5.84)	(-6.26)	(-37.96)	(-11.08)	(-5.63)
3		0.914***		0.457***	2.857***	1.340***	0.430***
		(8.05)		(4.56)	(24.09)	(11.39)	(4.01)
4		-0.074**			-0.701***	-0.413***	
		(-2.24)			(-15.69)	(-4.25)	
Model	OLS	ARMAX(4,3)	ARMAX(2,2)	ARMAX(3,3)	ARMAX(4,4)	ARMAX(4,4)	ARMAX(3,3)
N	2263	2263	2263	2263	2263	2263	2263

Table 3.4: Funding liquidity and the liquidity of options with different maturities

This table shows results of ARMAX regressions on liquidity of options with different maturities. An option is considered to be short-term if $ADTE_{i,t} < 60$, medium-term if $60 \leq ADTE_{i,t} \leq 180$, long-term if $180 \leq ADTE_{i,t}$. Columns (1)-(3) include lagged VIX and lagged $Residual_{TED|VIX}$ as exogenous regressors, while Columns (4)-(6) add the interaction term between VIX and TED spread into the regression. T-statistics are shown below the coefficient estimates inside parentheses. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of moving averaging terms are not displayed. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1) Short	(2) Medium	(3) Long	(4) Short	(5) Medium	(6) Long
$Residual_{TED_{t-1}\|VIX_{t-1}}$	1.572*	-0.149	-0.249			
	(1.72)	(-0.30)	(-0.82)			
TED_{t-1}				-1.752	-1.675**	-1.178***
				(-1.36)	(-2.57)	(-2.84)
VIX_{t-1}	-0.092***	-0.068***	-0.026***	-0.274***	-0.131***	-0.058***
	(-2.71)	(-4.27)	(-2.65)	(-6.08)	(-4.64)	(-3.14)
$TED_{t-1} \times VIX_{t-1}$				0.094***	0.056***	0.035***
				(3.19)	(3.56)	(3.71)
AR						
1	2.366***	-0.375	-0.617***	2.340***	-1.050***	-0.632***
	(32.60)	(-0.58)	(-7.77)	(29.46)	(-72.29)	(-7.08)
2	-2.222***	0.945***	0.490***	-2.170***	0.816***	0.474***
	(-15.78)	(12.74)	(8.19)	(-14.38)	(28.87)	(7.15)
3	0.946***	0.295	0.785***	0.918***	0.909***	0.758***
	(8.69)	(0.50)	(10.36)	(8.00)	(60.25)	(8.90)
4	-0.091***	-0.030	0.076***	-0.089***		0.086***
	(-2.77)	(-0.95)	(2.60)	(-2.60)		(2.87)
ARMAX(p,q)	(4,3)	(4,3)	(4,3)	(4,3)	(3,4)	(4,3)
N	2263	2263	2257	2263	2263	2257

Table 3.5: Funding liquidity and the liquidity of options with different moneyness

This table shows results of ARMAX regressions on the liquidity of options with different extent of moneyness. An option is considered to be deep-out-of-the-money (DOTM) if the contract is a call and $m_{i,t} < 1.06$ or if the contract is a put and $m_{i,t} < 0.94$, out-of-the-money (OTM) if the contract is a call and $1.01 < m_{i,t} \leq 1.06$ or if the contract is a put and $0.94 \leq m_{i,t} < 0.99$, at-the-money (ATM) if $0.99 \leq m_{i,t} \leq 1.01$ for either puts or calls, in-the-money (ITM) if the contract is a call and $0.94 \leq m_{i,t} < 0.99$ or if the contract is a put and $1.01 < m_{i,t} \leq 1.06$, deep-in-the-money (DITM) if the contract is a call and $m_{i,t} < 0.94$ or if the contract is a put and $m_{i,t} > 1.06$. Columns (1)-(5) include lagged VIX and lagged $Residual_{TED|VIX}$ as exogenous regressors, while Columns (6)-(10) add the interaction term between VIX and TED spread into the regression. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of moving averaging terms are not displayed. T-statistics are shown below the coefficient estimates inside parentheses. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM
$Residual_{TED_{t-1}\|VIX_{t-1}}$	3.136**	1.682*	0.494	0.101	0.410***					
	(2.16)	(1.85)	(1.36)	(0.43)	(3.30)					
TED_{t-1}						-1.956	-1.478	-0.837	-0.076	0.553***
						(-1.17)	(-1.09)	(-1.58)	(-0.22)	(2.71)
VIX_{t-1}	-0.152**	-0.057*	0.017	0.020***	0.049***	-0.539***	-0.243***	-0.052***	0.009	0.039***
	(-2.46)	(-1.81)	(1.36)	(2.58)	(11.04)	(-7.25)	(-5.47)	(-2.67)	(0.58)	(4.38)
$TED_{t-1} \times VIX_{t-1}$						0.152***	0.091***	0.040***	0.006	-0.005
						(3.01)	(3.32)	(3.42)	(0.78)	(-1.14)
AR										
1	1.451***	2.484***	1.392***	0.956***	0.979	-0.528***	2.338***	1.235***	0.953***	-0.485***
	(4.21)	(43.11)	(32.20)	(23.41)	(1.03)	(-11.20)	(33.78)	(67.96)	(23.14)	(-22.99)
2	-0.686	-2.426***	-1.137***	-0.952***	-0.035	0.059	-2.086***	-0.108***	-0.951***	-0.094***
	(-1.24)	(-19.55)	(-38.29)	(-29.33)	(-0.04)	(1.17)	(-18.62)	(-3.91)	(-29.37)	(-5.45)
3	0.345	1.030***	1.471***	0.955***		0.739***	0.746***	-0.065**	0.953***	0.358***
	(1.40)	(10.36)	(49.92)	(27.52)		(20.93)	(15.26)	(-1.99)	(26.75)	(19.12)
4	-0.112***	-0.089***	-0.728***	-0.087***				-0.065***	-0.086***	0.923***
	(-2.78)	(-3.04)	(-17.22)	(-3.16)				(-3.04)	(-3.11)	(48.35)
ARMAX(p,q)	(4,3)	(4,3)	(4,4)	(4,3)	(2,2)	(3,3)	(3,4)	(4,1)	(4,3)	(4,4)
N	2263	2263	2261	2261	1638	2263	2263	2261	2261	1638

Table 3.6: Funding liquidity and the liquidity of call and put options with different moneyness

This table shows results of ARMAX regressions on the liquidity of call and put options with different extent of moneyness. Panel A examines the liquidity of call options with different moneyness and Panel B put options. An option is considered to be deep-out-of-the-money (DOTM) if the contract is a call and $m_{i,t} > 1.06$ or if the contract is a put and $m_{i,t} < 0.94$, out-of-the-money (OTM) if the contract is a call and $1.01 < m_{i,t} \leq 1.06$ or if the contract is a put and $0.94 \leq m_{i,t} < 0.99$, at-the-money (ATM) if $0.99 \leq m_{i,t} \leq 1.01$ for either puts or calls, in-the-money (ITM) if the contract is a call and $0.94 \leq m_{i,t} < 0.99$ or if the contract is a put and $1.01 < m_{i,t} \leq 1.06$, deep-in-the-money (DITM) if the contract is a call and $m_{i,t} < 0.94$ or if the contract is a put and $m_{i,t} > 1.06$. Columns (1)-(5) in both panels include lagged VIX and lagged $Residual_{TED|VIX}$ as exogenous regressors, while Columns (6)-(10) add the interaction term between TED spread and VIX into the regression. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of autoregressive and moving averaging terms are not displayed. T-statistics are shown below the coefficient estimates inside parentheses. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Moneyness	DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM
Panel A: Call options										
$Residual_{TED_{t-1}\|VIX_{t-1}}$	3.900	1.297	0.427*	0.292*	0.327***					
	(1.56)	(1.59)	(1.88)	(1.76)	(3.40)					
TED_{t-1}						0.933	-2.369	-1.165***	-0.461	0.087
						(0.23)	(-1.53)	(-2.92)	(-1.52)	(0.49)
VIX_{t-1}	-0.322**	-0.133***	0.011	0.038***	0.041***	-0.510***	-0.281***	-0.046***	0.011	0.027***
	(-2.40)	(-3.22)	(0.99)	(4.98)	(10.27)	(-2.99)	(-5.02)	(-3.04)	(0.92)	(3.97)
$TED_{t-1} \times VIX_{t-1}$						0.074	0.090***	0.039***	0.018***	0.005
						(0.67)	(2.78)	(4.80)	(3.02)	(1.45)
ARMAX(p,q)	(4,1)	(4,4)	(2,2)	(2,3)	(2,2)	(4,1)	(4,4)	(2,2)	(4,4)	(2,3)
N	2182	2207	2205	2178	1286	2182	2207	2205	2178	1286
Panel B: Put options										
$Residual_{TED_{t-1}\|VIX_{t-1}}$	1.251	0.569	0.475**	0.459**	0.385***					
	(1.15)	(1.31)	(2.15)	(2.46)	(3.36)					
TED_{t-1}						-2.656	-1.823**	-0.954**	-0.524	0.367*
						(-1.42)	(-2.35)	(-2.38)	(-1.56)	(1.70)
VIX_{t-1}	-0.209***	-0.071***	0.003	0.017	0.037***	-0.357***	-0.162***	-0.051***	-0.022	0.014
	(-3.71)	(-3.05)	(0.23)	(1.62)	(4.99)	(-4.79)	(-4.92)	(-2.87)	(-1.33)	(1.20)
$TED_{t-1} \times VIX_{t-1}$						0.099***	0.061***	0.035***	0.023***	0.008*
						(2.66)	(4.34)	(4.10)	(3.40)	(1.86)
ARMAX(p,q)	(3,3)	(4,3)	(2,2)	(2,2)	(4,4)	(3,3)	(3,4)	(2,2)	(2,3)	(4,1)
N	2207	2207	2205	2116	766	2207	2207	2205	2116	766

Table 3.7: Funding liquidity and options liquidity: controlling for stock market liquidity

This table shows results of ARMAX regressions linking liquidity in the option market to stock market liquidity. *BAClose_ew* and *BAClose_vw* are equally- and value-weighted closing stock bid-ask spread for the whole market. The other two exgoenous variables are *Stock_Volume* and *Stock_Dollar_Volume* which denote the trading volume and dollar volume of the whole stock market, both scaled down by 1 billion dollars. Columns (1)-(4) include lagged VIX, lagged $Residual_{TED|VIX}$ and stock market liquidity measures as exogenous regressors, while Columns (5)-(8) add the interaction term between lagged VIX and lagged TED spread into the regression. T-statistics are shown below the coefficient estimates inside parentheses. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of moving averaging terms are not displayed. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$Residual_{TED_{t-1}\|VIX_{t-1}}$	1.527*	1.450*	1.391*	1.450*				
	(1.94)	(1.77)	(1.66)	(1.70)				
TED_{t-1}					-0.994	-0.952	-1.365	-1.803
					(-0.87)	(-0.83)	(-1.16)	(-1.49)
VIX_{t-1}	-0.057**	-0.071**	-0.083**	-0.083**	-0.208***	-0.215***	-0.235***	-0.252***
	(-1.97)	(-2.28)	(-2.42)	(-2.48)	(-5.57)	(-5.77)	(-5.77)	(-6.04)
BAClose_ew	0.029				0.036			
	(0.04)				(0.05)			
BAClose_vw		25.011***				24.409***		
		(3.15)				(3.23)		
Stock_Volume			0.280***				0.283***	
			(4.95)				(5.14)	
Stock_Dollar_Volume				0.009***				0.010***
				(5.32)				(5.73)
$TED_{t-1} \times VIX_{t-1}$					0.070***	0.067***	0.076***	0.089***
					(2.87)	(2.70)	(2.90)	(3.22)
AR								
1	2.384***	2.396***	2.373***	2.379***	2.354***	2.379***	2.337***	-0.037
	(33.50)	(36.61)	(33.89)	(34.56)	(30.47)	(34.66)	(30.19)	(-0.17)
2	-2.225***	-2.256***	-2.204***	-2.216***	-2.166***	-2.225***	-2.135***	-0.416***
	(-15.49)	(-16.90)	(-15.69)	(-15.97)	(-14.18)	(-16.06)	(-14.12)	(-3.14)
3	0.913***	0.937***	0.904***	0.910***	0.884***	0.924***	0.869***	0.352***
	(8.04)	(8.76)	(8.12)	(8.25)	(7.38)	(8.35)	(7.36)	(2.94)
4	-0.074**	-0.078**	-0.074**	-0.074**	-0.073**	-0.079**	-0.072**	0.134**
	(-2.23)	(-2.46)	(-2.23)	(-2.27)	(-2.10)	(-2.41)	(-2.08)	(2.23)
ARMAX(p,q)	(4,3)	(4,3)	(4,3)	(4,4)	(4,3)	(4,3)	(4,3)	(4,4)
N	2263	2263	2263	2263	2263	2263	2263	2263

Table 3.8: Funding liquidity and options liquidity: different sample periods

This table examines the relation between funding liquidity and options market liquidity during different sample periods, using an ARMAX model. The whole sample period is divided into three sub-periods, the pre-crisis period (01/2003-07/2007), crisis (08/2007-06/2009) and post-crisis (07/2009-01/2012) period. Columns (1)-(3) display results for all types of options, call and put options during pre-crisis period. Columns (4)-(6) and Columns (7)-(9) are for the crisis and post-crisis period, respectively. Different to specifications in Panel A, Panel B controls for the interaction term between lagged TED spread and lagged VIX. T-statistics are shown below the coefficient estimates inside parentheses. For brevity, the coefficients of autoregressive and moving averaging terms are not displayed. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

Panel A: Funding liquidity and options market liquidity

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
		pre-crisis			crisis			post-crisis	
Sample	All	Call	Put	All	Call	Put	All	Call	Put
$Residual_{TED_{t-1}\|VIX_{t-1}}$	0.349	-0.592	-1.165	-1.330	-1.711	-1.230	-2.716	-0.504	-2.886
	(0.21)	(-0.26)	(-0.80)	(-1.31)	(-1.41)	(-1.25)	(-1.21)	(-0.17)	(-1.36)
VIX_{t-1}	-0.377***	-0.269*	-0.598***	-0.062**	-0.085***	-0.048*	-0.230***	-0.173	-0.209***
	(-3.63)	(-1.83)	(-6.66)	(-2.21)	(-2.72)	(-1.78)	(-2.95)	(-1.63)	(-2.89)
ARMAX(p,q)	(1,4)	(3,4)	(3,4)	(1,1)	(1,1)	(1,2)	(4,4)	(4,4)	(4,4)
N	1139	1139	1139	472	472	472	652	652	652

Panel B: The interaction between TED spread and VIX

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
		pre-crisis			crisis			post-crisis	
Sample	All	Call	Put	All	Call	Put	All	Call	Put
TED_{t-1}	5.767	1.319	5.917	-3.480**	-4.648***	-3.160**	-5.322	-2.648	-13.290
	(1.33)	(0.23)	(1.39)	(-2.33)	(-2.80)	(-2.26)	(-0.69)	(-0.29)	(-1.52)
$TED_{t-1} \times VIX_{t-1}$	-0.247	-0.086	-0.330	0.076*	0.100**	0.070**	0.247	0.093	0.458
	(-1.13)	(-0.30)	(-1.44)	(1.90)	(2.00)	(2.03)	(0.77)	(0.24)	(1.25)
VIX_{t-1}	-0.405***	-0.251**	-0.570***	-0.149*	-0.195*	-0.130*	-0.311	-0.219	-0.425*
	(-5.67)	(-2.50)	(-10.12)	(-1.87)	(-1.89)	(-1.73)	(-1.42)	(-0.82)	(-1.68)
ARMAX(p,q)	(1,4)	(3,4)	(3,4)	(1,1)	(1,1)	(1,2)	(4,4)	(4,4)	(4,4)
N	1139	1139	1139	472	472	472	652	652	652

Table 3.9: Funding liquidity and options market liquidity: Weekly data

This table uses weekly data and shows results of OLS and ARMAX regressions linking the liquidity in the option market to funding costs and VIX. In addition to studying the liquidity of options as a whole, we also conduct analysis on call and put options separately. Columns (1)-(4) include lagged VIX and lagged $Residual_{TED|VIX}$ as exogenous regressors, while Columns (5)-(7) add the interaction term between TED spread and VIX into the regression. T-statistics are shown below the coefficient estimates inside parentheses. Below the exogenous regressors are several autoregressive terms for each ARMAX model. For brevity, the coefficients of moving averaging terms are not displayed. ***, ** and * denote significance level at 1 %, 5 % and 10 %.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Option Type	All	All	Call	Put	All	Call	Put
$Residual_{TED_{t-1}\|VIX_{t-1}}$	0.521**	0.922*	1.070*	1.010*			
	(2.11)	(1.75)	(1.94)	(1.72)			
TED_{t-1}					-0.940	-0.426	-1.317
					(-0.91)	(-0.38)	(-1.11)
VIX_{t-1}	-0.112***	-0.103***	-0.099***	-0.112***	-0.176***	-0.170***	-0.206***
	(-7.38)	(-4.19)	(-3.74)	(-3.40)	(-5.03)	(-4.42)	(-4.60)
$VIX_{t-1} \times TED_{t-1}$					0.043*	0.035	0.055**
					(1.80)	(1.26)	(2.09)
AR							
1	0.477***	-0.660***	-0.802***	1.143***	-0.942***	-0.729***	1.138***
	(12.14)	(-11.58)	(-15.95)	(19.37)	(-73.38)	(-12.04)	(20.16)
2		0.992***	1.035***	-0.159***	0.918***	0.940***	-0.152***
		(17.75)	(19.48)	(-2.92)	(51.40)	(16.41)	(-2.91)
3		0.773***	0.856***		0.979***	0.847***	
		(16.90)	(20.39)		(80.85)	(15.28)	
4		-0.149***	-0.127***			-0.092*	
		(-2.65)	(-2.65)			(-1.84)	
Model	OLS	ARMAX(4,3)	ARMAX(4,3)	ARMAX(2,1)	ARMAX(3,4)	ARMAX(4,3)	ARMAX(2,1)
N	470	470	470	470	470	470	470

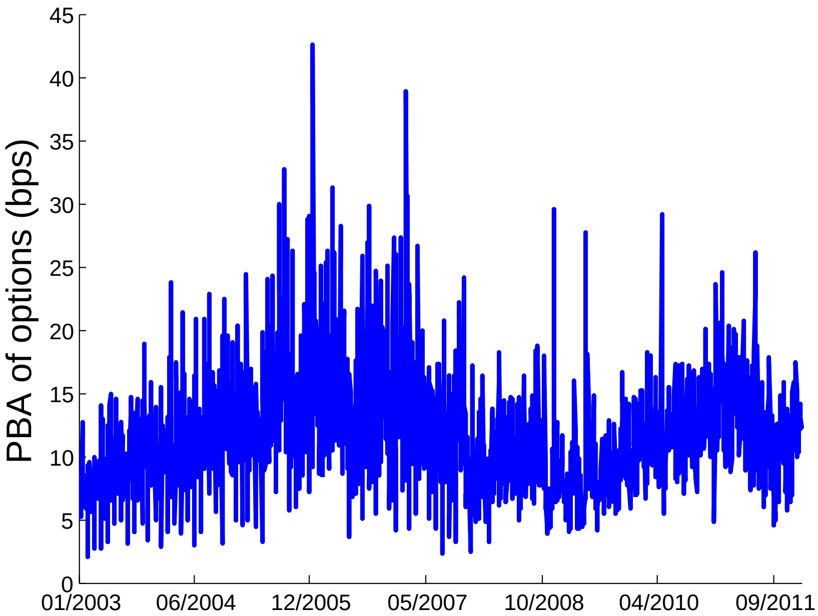

Figure 3.1: The evolution of options market liquidity. This figure illustrates the evolution of the index option market liquidity from January 2003 to January 2012. The proportional bid-ask spread is used as proxy for options market liquidity and the definition can be found in the Appendix.

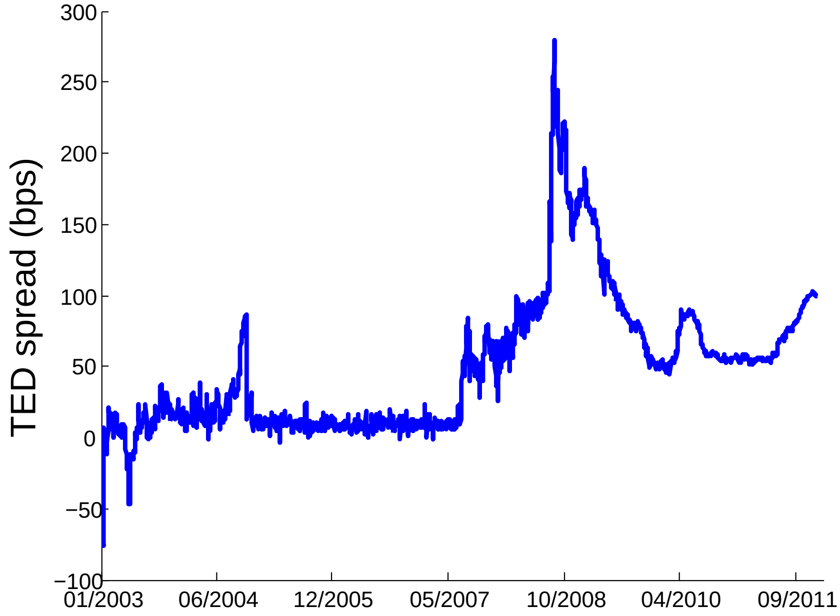

Figure 3.2: The evolution of the TED spread. This figure illustrates the evolution of the TED spread from January 2003 to January 2012.

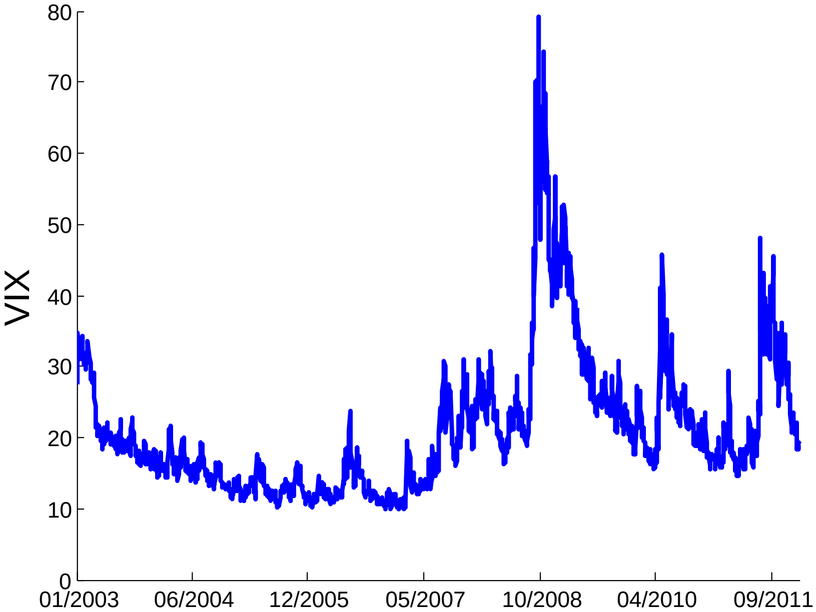

Figure 3.3: The evolution of VIX. This figure illustrates the evolution of VIX (%) from January 2003 to January 2012.

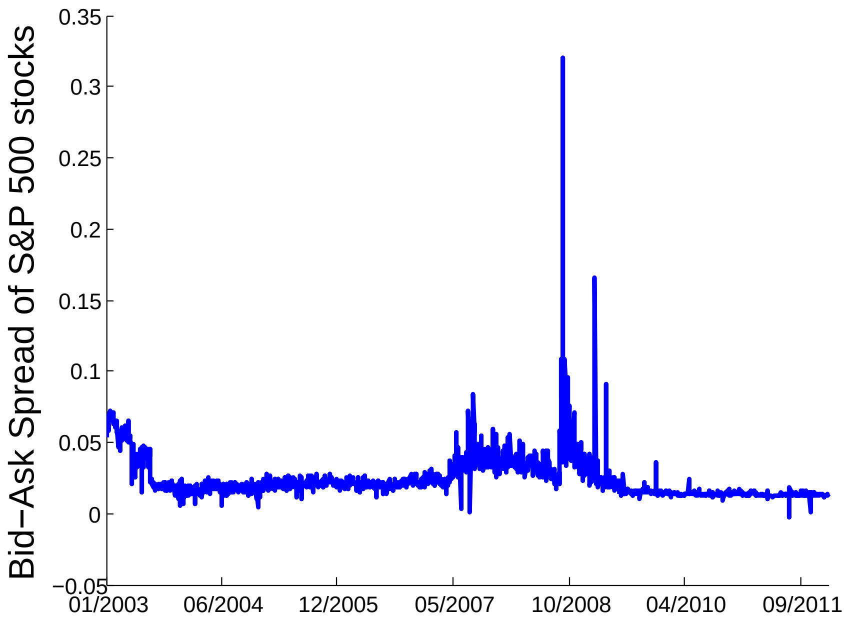

Figure 3.4: The evolution of stock market liquidity. This figure illustrates the evolution of stock market liquidity from January 2003 to January 2012. The proportional bid-ask spread for S&P 500 stocks is used as proxy for stock market liquidity.