The London School of Economics and Political Science

Essays on Market Liquidity

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

In the first chapter of my Thesis I propose a model of front-running in noisy market environment. I demonstrate that even if the front-runner/predator has no initial knowledge about the position of a distressed trader he will be still able to front-run his orders in a linear Bayesian-Nash equilibrium. This is possible because initial orders of the distressed trader tend to reveal his initial position. The contribution of this chapter is also in the analysis of long-term dynamics of predatory trading under Gaussian uncertainty.

Second chapter treats about the dark-pools of liquidity which are highly popular systems that allow participants to enter unpriced orders to buy or sell securities. These orders are crossed at a specified time at a price derived from another market. I present an equilibrium model of coexistence of dark-pools of liquidity and the dealer market. Dealer market provides the immediate execution, whereas the dark-pool of liquidity provides lower cost of trading. Risk-averse agents in equilibrium optimally choose between safe dealer market and cheaper dark-pool of liquidity.

In the third chapter I solve for a partial-equilibrium optimal consumption and investment problem, when one of the investment assets is traded infrequently. Opportunity to trade the "illiquid asset" arises upon the occurrence of a Poisson event. Only when such event occurs a trader is able to change (increase or decrease) her position in the illiquid asset. The investor can consume continuously from the bank account. After deriving HJB equation, I analyze in details the implications of illiquidity on the optimal level of consumption, allocation and welfare. The optimal policy is solved using algorithm from aeronautics.

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All errors are obviously my own.

Contents

Part I

Front-running in an Uncertain Environment

1 Introduction

In many financial market segments it is possible to find large market players that have significantly concentrated risk exposure. In some cases, single market participants are becoming a dominant force, as they employ risk exposure that significantly exceeds that of any other counterpart. Historically, notable examples of such market participants and the implications of the resulting risk concentration have been studied extensively. In 1993, Metallgesellschaft AG built up large gross notional exposure to oil futures contracts. This position, which was equivalent to yearly oil production output in Kuwait, led to losses of over $1.4bn after the value of the contracts went down significantly. This forced Metallgesellschaft to the brink of bankruptcy, which was only avoided after a consortium of banks and debtors agreed to a programme of far-reaching debt restructuring. Another example of a company that built an enormous position in a particular asset class was LTCM, a Greenwich Connecticut-based hedge fund, which amassed enormous spread positions in government bonds and equity derivatives. The gross notional exposure in this case exceeded 1 trillion US dollars[1]. In 1994, the collapse of the hedge fund led to a bail-out financed by major investment banks and coordinated by the Federal Reserve. Other examples of risk concentration include the huge gas/energy exposure of Enron in the late 1990s, the equity derivatives exposure of SocGen in 2008 and J.P. Morgan's coal exposure in 2010.

When a large market participant suffers trading losses, the company may be forced to close-out risk positions after margin calls or investor redemptions. The distress of a large trader, who is forced to liquidate, can be exacerbated by the strategic behaviour of market participants front-running his trades. This is what happened in 1994 when LTCM suffered initial significant losses caused by the Russian debt crisis. Financial institutions aware of LTCM's distress – and armed with knowledge of its trading books – began front-running and significantly amplified the losses of the hedge fund. The collapse of LTCM spurred academic studies of front-running and predation in financial markets, which resulted in a number of excellent academic papers.

The front-running that happened at the time of LTCM's collapse was fairly specific. Front-running investment banks had full knowledge of the trading positions of LTCM, which they gained during bail-out negotiations. This knowledge and the immediate need of LTCM to unravel its position made it much simpler for the front-runners to position themselves appropriately and to benefit from the distress of a collapsing hedge fund. In general, however, front-running happens

[1] For a full account of LTCM's collapse, please refer to the excellent book by Lowenstein (2001).

in an environment where there is no full knowledge about the position of the large trader and the position of the trader does not need to be terminated immediately. An example of such a situation may happen when a hedge fund, highly active in a certain market segment, receives significant redemption requests from its investors. The risky positions of the hedge fund need to be liquidated to satisfy these request, and even though it may be widely known that this certain market participant is active in a particular market segment, the precise position is usually unknown. In particular, potential front-runners may be unaware of whether the large trader has long or short exposure in the asset class at the time he learns he would need to unwind his positions. Also, a large trader may be allowed to unwind his positions over a number of trading days, which complicates front-running even further.

In this paper, I develop a tractable model that allows me to investigate the impact of uncertainty surrounding a large trader's position and the time priority of liquidation on the dynamics of the front-running process. In particular, I show that front-running may occur even though market participants may have no prior information about the large trader's initial position and only know about his need to unwind the position at some future point in time. In equilibrium, the large trader splits liquidation across a number of periods to avoid immediate large price impacts. However, his early trades reveal his initial position and allow other strategic traders to front-run his subsequent orders. I also analyse the value for a large trader of keeping his position secretive.

Another contribution of this paper is a tractable model of multi-period dynamics in front-running. In particular, I show that when a large trader has many periods in which to liquidate his position, other strategic traders can in fact be liquidity providers and decrease the costs of liquidation. However, if the allowed time for liquidation decreases and the large trader needs to decrease his position very quickly, strategic traders become very aggressive in exploiting his vulnerability and front-run his position, eventually leading to elevated trading costs.

2 Related Literature

The model presented in this paper relates directly to the literature on predation. Predatory trading is defined as trading that induces and/or exploits other investors' needs to reduce their positions. In their seminal contribution, Brunnermeier and Pedersen (2004) present the first notable model of predation and front-running. They show that when a trader needs to sell an asset quickly, others, with the knowledge about his position, also sell and subsequently buy back the asset. This leads to overshooting price dynamics and a significant reduction in the level of liquidity available to the distressed trader. The paper also draws conclusions about the systemic impact of predatory trading/front-running. Because of front-running, shocks in the market are amplified and may lead to systemic events. In this respect predatory trading is a source of destabilising speculation, similar in spirit to DeLong, Shleifer, Summers and Waldmann (1990). On the technical side, Brunnermeier and Pedersen (2004) implement their deterministic model in continuous time. They define the exogenous costs of trading that need to be paid by a trader who wants to trade with a higher intensity than the (exogenously-) defined limit. These assumptions are justified by market search frictions in line with Duffie, Gârleanu and Pedersen (2005). More recent academic contributions to the analysis of predatory trading and front-running are those of Attari, Mello and Ruckes (2005), Carlin, Lobo and Viswanathan (2007), Parida and Venter (2010) and Fardeau (2010).

My paper also relates to the literature on limits to arbitrage, for example Shlieifer and Vishney (1997), DeLong et al. (1990), Gromb and Vayanos (2002), Liu and Longstaff (2004), Mukarram, Mello and Ruckes (2006), and an excellent review by Gromb and Vayanos (2010). In this branch of literature, authors investigate how costs and constraints faced by arbitrageurs can prevent them from eliminating mis-pricing and providing liquidity to other investors. These limits lead to deviation of the market price from the fundamentally justified value. In my paper, risk-neutral arbitrageurs fail to bring the price to fundamental equilibrium and even amplify the deviation because of strategic considerations. During periods of distress experienced by a large trader, the rational arbitrageur exploits fragility, removes liquidity from the market and amplifies any deviation of the price from fundamentals. Under certain conditions, an arbitrageur can therefore become a destabilising force in the market. Importantly, the limits to arbitrage in my model are caused not by any exogenous costs or by constraints, but rather result directly from strategic interactions.

My paper also references models of optimal trading in noisy markets, for example Kyle (1985) or Vayanos (2001). In my proposed framework the large distressed trader is attempting to hide his orders in market noise induced by the random orders of noise traders. A potential front-runner needs to filter the order of the large trader from the market noise. The solution to this trading game is in the form of linear Bayesian-Nash equilibrium, which is a standard concept in the literature on trading in noisy markets.

3 Benchmark model

We start by presenting the baseline framework. The model that we consider later in the text includes modifications to this baseline scenario. This approach allows for consistency and helps to investigate how different types of uncertainty affect the optimal trading strategy of the predator or front-runner. The specific feature of the baseline framework is the way we model the residual demand curve, as we assume that a broad market may be represented by one non-strategic, risk-averse trader with a short-term decision horizon. In this respect our model is very similar to those introduced by Brunnermeier and Pedersen (2004), Parida and Venter (2010) and Fardeau (2010). On the one hand, this feature allows for more straightforward comparison with previous models, but on the other makes it possible to focus purely on the effects of predation or front-running. Therefore, we do not consider a residual demand curve that could be formed by trading preferences of the continuum of risk-averse agents with *infinite optimisation horizon* as it was analysed, for example, in Vayanos (2001).

The benchmark model is a full-information dynamic trading game between two risk-neutral strategic agents that face a residual demand curve generated by the trading preferences of short-horizon traders with CARA preferences and who are also influenced by the random orders of liquidity traders. One of the two strategic traders will be referred to as 'prey'. He is characterised by the fact that he is required to close his position in a risky asset, which may be initially long or short, by a certain point in time – a moment that we will call from now on the 'time of liquidation'. Consequently, the trader is not allowed to trade at any time following liquidation. The prey in our model represents a fragile trader, who is required to close his position because of certain specific constraints. The strict requirement of the liquidation restricts the set of possible trading strategies available to him, so the prey may be forced to trade in an adverse market if the time of liquidation is approaching. The distress of the prey will be exacerbated by the actions of a predator – a risk-neutral strategic player, who is not restricted in his trading strategy. The predator, given his knowledge of the position of the prey, will take advantage of his fragility. Below we formalise this baseline model.

3.1 Model

Time is discrete and goes from $t = -n$ to m. The time of liquidation is set at $t = 0$. and the dividend for the risky asset is paid at time $t = m + 1$. The prey trades only in rounds $t \leq 0$. After this point (so for $t > 0$) only the predator is allowed to place orders. In all equations quantities for the predator are marked with the superscript P, whereas the quantities for the prey are denoted with the superscript S. There are two assets in the model. A risky asset pays the dividend after all trading rounds, and the final payout has normal distribution $v \sim N(0, \sigma^2)$. The return on a risk-free asset (interest rate) with perfectly elastic supply is normalised to 0. The predator and prey start trading with initial positions A^P_{-n} and A^S_{-n} in a risky asset. The initial money accounts are assumed (without loss of generality) to be $W^P_{-n} = W^S_{-n} = 0$ as strategic agents are risk-neutral, while the initial positions in the money account do not influence trading dynamics[2]. The strategic objective of these two traders is to maximise expected monetary wealth at the end of all trading rounds ($t = m + 1$). Strategic traders are allowed to submit market orders only, which not only allows us

[2] Of course, this would not be the case if the preferences of traders were of CARA type, for example. In such a case the position in the money account would be an important state variable, which would have an impact on the trading dynamics.

to avoid the complexity of the calculus of variation[3], but also will have important implications when we introduce uncertainty present in the mutations of this model. A liquidity trader submits every period a normally distributed market order that has zero mean and variance σ_L^2, so $u_t \sim N(0, \sigma_L^2)$. Non-strategic short-horizon traders are modelled as CARA-type investors (with the coefficient of absolute risk aversion ρ) with the inventory. Their short-termism is reflected in their expectations concerning the asset's final pay-off. At every point in time t, the non-strategic traders expect that the asset's final pay-off will be realised in time $t+1$. Given this specification, we prove in Appendix A the following proposition:

Proposition 1 *Given our definition of the objective of the short-horizon traders it can be shown that the price process of the risky asset is given by:*

$$p_t = -x_t^M \cdot \beta + p_{t-1} \tag{1}$$

where x_t^M is the total amount of risky asset acquired by the short-term investor in time t and β is a fixed parameter, which depends on the variance of risky asset pay-off and on the risk aversion of the short-term investor.

Proof. In appendix ■

In what follows, we additionally assume that the price process can be generalised to include price convergence – an empirically observed feature of price process in the financial markets[4]. This price convergence may be caused in practice by the trading actions of non-myopic market participants, who consider any deviation of the current price from its fundamental value (which is 0 in this case) as the result of trades made by liquidity traders. This effect emerges in the model through competition between predators (e.g. Fardeau (2010)). In this case, the price process can be described by the following equation

$$p_t = -x_t^M \cdot \beta + \alpha \cdot p_{t-1} \tag{2}$$

where $\alpha \in (0, 1)$ is the constant that measures the strength of the price convergence. The price process in equation (2) will be equivalent to the underlying price process in Parida and Venter

[3]Interested readers are referred to Brunnermeier (2001) page 75 for an approach that does not require the calculus of variation, but works in simpler settings to those in our model, or to Fleming and Soner (2006b) page 33 for an analysis of the Calculus of Variation approach.

[4]Convergence of the price to the mean can be justified by the assumption that there are some (unmodelled) strategic risk-neutral traders who are trading to bring the price of the asset to equilibrium and are unaware of the existence of the strategic trader in distress. The main reason for adding price convergence to the baseline model is to cause all price deviations caused by noise traders to be temporary. The predator, when planning his behaviour, needs to take into account that all opportunities resulting from noise trading are temporary, so he must decide whether his trading is more focused on exploiting these opportunities or on exacerbating the weakness of the prey.

(2010) and Brunnermeier and Pedersen (2004) for $\alpha = 0$ and to the underlying price process in Fardeau (2010) for $\alpha = 1$. Parameter α measures the strength of the long-term price effect. If α is close to 1, then there is persistence in the market impact of the trades. If α is closer to zero, then the price impact of the trades is mostly temporary. Parameter β is the measure of the immediate price impact.

As we shall soon see, the price process is a natural object of consideration in our settings. Although in practice we could solve the model directly in the space of asset holdings[5] and treat the price process implicitly, in some modifications of our model holdings are not observable to agents directly and any inference must be based solely on the price dynamics, which is public knowledge.

3.2 General solution

The baseline model is solved recursively. First, we solve for the value function and dynamics of trading in periods $t > 0$, so for the time when only the predator is allowed to trade. Given this solution, we can solve for the more complicated dynamics in periods $t < 0$. In these derivations we use the generalised price process from equation (2).

After liquidation ($t > 0$)

When the prey is absent, the predator trades only with the broad market and liquidity traders. In this case in given period k, the market clearing condition implies that $x_k^P + u_k + x_k^M = 0$, where x_k^M is the amount of risky asset acquired by short-term investors, u_k an order submitted by liquidity traders and x_k^P the order submitted by the predator. In the last period $t = n$ the programme of the predator is therefore

$$\max_{x_n^P} E\left[(A_n^P + x_n^P) \cdot v - x_n^P \cdot p_n + W_n^P\right] \tag{3}$$

where the dynamics of prices are given by $p_n = (x_n^P + u_n)\beta + \alpha p_{n-1}$ [6]and A_n^P is the position in risky assets of the predator after time $t = n - 1$. By plugging the equation for p_n back to equation (3) and making use of the fact that $E(v) = E(u_n) = 0$, we see that the problem for the predator in the last period is equivalent to

$$\max_{x_n^P} -(x_n^P)^2\beta - x_n^P \cdot \alpha p_{n-1} + W_n^P$$

with the corresponding FOC

$$x_n^P = -\frac{\alpha p_{n-1}}{2\beta}$$

If the price in previous period p_{n-1} is above the fundamental value of the asset – 0 – the predator

[5]We would denote by e_t^M and W_t^M the current position of the broad market in the risky asset and the cash account, respectively.

[6]The equation is equivalent to $(x_n^p + u_n) = \frac{p_n - \alpha p_{n-1}}{\beta}$

is submitting a sell order in the current period. The size of the order depends on market depth $\frac{1}{\beta}$. The higher the depth of the market, the bigger is the order of the predator. Knowing the optimal order of the predator in the last period, we can calculate the value function for the predator at time $t = n$, which is simply

$$V_n^P(p_{n-1}, W_n^P) = \frac{\alpha^2}{4\beta} \cdot (p_{n-1})^2 + W_n^P \tag{4}$$

The value function does not depend on the amount of risky assets accumulated by the predator in earlier periods because of the risk-neutrality of the predator and the fact that $E(v) = 0$. The value function is the quadratic function in the past deviation of the market price from its fundamental value. This particular feature will characterise the value function of the predator in all periods $t > 0$. In the last trading round $t = n$, the variance of the order placed by the liquidity trader does not affect the value function of the predator.

The properties of the value functions and optimal market orders of the predator may be summarised by the following proposition:

Proposition 2 *The value function and optimal market orders for the predator in periods t>0 are given by*

$$V_k^P(p_{k-1}, W_k^P) = G_k \cdot (p_{k-1})^2 + W_k^P + N_k \tag{5}$$

$$x_k^P = \left(\frac{2 \cdot G_k}{\alpha} - \frac{\alpha}{\beta}\right) \cdot p_{k-1}$$

where the law of motions of G and N is given recursively by

$$G_k = \frac{\alpha^2}{4\beta} \cdot \frac{1}{(1 - G_{k+1}\beta)}$$

$$N_k = N_{k+1} + G_{k+1} \cdot \beta^2 \sigma_u^2$$

with the boundary condition

$$G_n = \frac{\alpha^2}{4\beta}$$

$$N_n = 0$$

Proof. In appendix ■

The interpretation of G_k and N_k is straightforward. Variable G_k measures the benefit the trader can enjoy by trading optimally the current price deviation. As the number of trading rounds that are left decreases ($n - k$ decreases), the predator has less time to take advantage of the price deviation, so he needs to trade quicker and the value of G_k is lower. N_k gives us the expected value of future trading opportunities that may be caused by random orders placed by the liquidity traders, who place random orders that may lead to deviation of the price from the fundamentals. The predator will take advantage of these future deviations and their value (in expectation) for his utility is equal to N_k.

We can now calculate the limiting values of G_k that would prevail if $n \to \infty$:

Corollary 1 *For any fixed k*

$$\begin{aligned} G_k &\to \frac{1-\sqrt{1-\alpha^2}}{2\beta} \\ x_k^P &\to \left(\frac{1-\sqrt{1-\alpha^2}-\alpha^2}{\alpha\beta}\right) p_{k-1} \\ N_k &\to +\infty \\ as\ n &\to +\infty \end{aligned} \tag{6}$$

The value of N_k diverges to infinity as the number of periods in which the liquidity trader can effect price divergence becomes infinite. This means that the predator will have (in expectations) an infinite number of opportunities to trade on the convergence of the price to the fundamentals.

Before liquidation $(t \leqslant 0)$

We now consider the dynamics of trading for the periods in which the prey trades along with the predator. In this simple model, we assume a full information environment in that the predator knows in any period the current inventory of the prey (we will relax this assumption in a subsequent model). The model is solved backwards for the sub-game perfect Nash Equilibrium. First, we outline a solution method for general values α and G_1, and then later we consider more tractable version with $\alpha = 1$ and G_1 equal to its limiting value from equation (6). For this more tractable version we solve the model for an arbitrary number of periods.

We consider (for mathematical clarity) a general form of the value function as shown in Proposition 2 in period $t = 1$, so for the first period after the liquidation

$$V_1^P(p_0, W_1^P) = G_1 (p_0)^2 + W_1^P$$

and the problem of the predator at time $t = 0$ is

$$\max_{x_0^P} E_0[V_1^P(p_0, W_1^P)] \tag{7}$$

subject to

$$\begin{aligned} p_0 &= (x_0^P - A_0^S + u_0)\cdot\beta + \alpha\cdot p_{t-1} \\ W_1^P &= W_0^P - p_0 x_0^P \end{aligned}$$

where A_0^S is the position of the prey in a risky asset when he reaches $t = 0$. As the prey needs to liquidate, he has to close this position; consequently, he submits order $-A_0^S$. The first order condition of programme (7) is given by

$$x_0^P = -\frac{(p_{-1}\alpha - A_0^S\beta)}{2\beta}\cdot\left(2 - \frac{1}{1-G_1\beta}\right) \tag{8}$$

and as $\frac{\alpha^2}{4\beta} \leqslant G_1 \leqslant \frac{1}{2\beta}$ and $0 \leqslant \alpha \leqslant 1$, we know that $1 - G_1\beta \geqslant \frac{1}{2}$, so the second part of equation (8) is greater than zero. The predator will submit the buy order only if $(p_{-1}\alpha - A_0^S\beta) < 0$, i.e. only if the price in the previous period was far below fundamentals or if there is a large expected sell order from the prey (the predator provides liquidity in this case). Plugging the optimal order x_0^P back into equation (7), we end up with the value function for period $t = 0$

$$V_0^P(p_{-1}, W_0^P, A_0^S) = \frac{\left(\alpha p_{-1} - \beta A_0^S\right)^2}{4\beta(1-G_1\beta)} + W_0^P + G_1\beta^2\sigma_u^2$$

which has a fairly similar structure to the value function from Proposition 1. In fact, there is no proper game between the predator and prey in period 0, as the only strategy allowed for the prey is to sell his entire holdings of the risky asset. As we are now dealing with full-information settings (the predator knows exactly the amount that will be sold by the prey), his problem in period $t = 0$ is not greatly different from those in all subsequent periods.

The proper game takes place in periods $t < 0$, during which the prey may decide about the size of his trade. Below we solve for SPNE for the game played by the prey and the predator. In period $t = -1$ the problem of the prey is

$$\max_{x_{-1}^S} E_1\left[p_{-1}x_{-1}^S + p_0(-A_{-1}^S - x_{-1}^S)\right] \tag{9}$$

subject to standard price dynamics. Element A_{-1}^S is the prey's holding of the risky asset in period $t = -1$. In equation (9) the prey is just maximising the expected value of the future pay-off from selling the inventory. The only unknowns in this equation at time $t = -1$ are prices p_{-1} and p_0

$$\begin{aligned} p_{-1} &= \alpha p_{-2} + (x_{-1}^S + x_{-1}^P + u_1)\beta \\ p_0 &= \alpha(\alpha p_{-2} + (x_{-1}^S + x_{-1}^P + u_1)\beta) + (x_0^P + (-A_{-1}^S - x_{-1}^S) + u_0)\beta \end{aligned} \tag{10}$$

By substituting (8) for x_0^P into (10), substituting it back into (9) and then taking the first order condition, we end up with

$$x_{-1}^S = -\frac{(p_{-2}\alpha + x_{-1}^P\beta)}{2\beta}\left(1 - \frac{1}{(3-\alpha-2G_1\beta)}\right) - \frac{A_{-1}^S(2-\alpha)}{2(3-\alpha-2G_1\beta)} \tag{11}$$

This equation shows that the optimal order of the prey in period $t = -1$ can be decomposed into two parts. First is the arbitraging component and second is the hedging component[7]. The arbitraging component implies that even though the prey is aware of the necessity to close his position in the risky asset, he is also trying to exploit trading opportunities that are induced by the trading of noise traders. For the same period t=-1 the problem of the predator is

$$\max_{x_{-1}^P} EV_0^P(p_{-1}, W_0^P, A_0^S)$$

subject to

$$\begin{aligned} A_0^S &= A_{-1}^S + x_{-1}^S \\ W_0^P &= W_{-1}^P - p_{-1}x_{-1}^P \\ p_{-1} &= \alpha p_{-2} + (x_{-1}^S + x_{-1}^P + u_1)\beta \end{aligned}$$

The first order condition of the problem is given by

$$x_{-1}^P = -\frac{p_{-2}\alpha}{2\beta}\left(1 - \frac{\alpha^2}{(4-\alpha^2-4G_1\beta)}\right) - \frac{\alpha A_{-1}^S}{(4-\alpha^2-4G_1\beta)} - \frac{x_{-1}^S}{2}\left(1 + \frac{2\alpha-\alpha^2}{(4-\alpha^2-4G_1\beta)}\right) \tag{12}$$

and then by combining equations (12) and (11) we can calculate the equilibrium values of market orders for the prey and predator, which are given by

$$x_{-1}^P = \frac{-p_2\alpha(8-(4-\alpha)\alpha(1+\alpha) - 12G_1\beta + 2G_1\alpha(2+\alpha)\beta + 4G_1^2\beta^2) - A_{-1}^S\beta(\alpha(6+(1-\alpha)\alpha - 6G_1\beta) - 4 + 4G_1\beta)}{\beta(20 - \alpha(8+(3-\alpha)\alpha) - 32G_1\beta + 2G_1\alpha(4+\alpha)\beta + 12G_1^2\beta^2)}$$

$$x_{-1}^S = \frac{-2p_2\alpha(1-G_1\beta)(2-\alpha-2G_1\beta) - A_{-1}^S\beta(8 - 8G_1\beta - \alpha(6+\alpha-\alpha^2-6G_1\beta))}{\beta(20 - \alpha(8+(3-\alpha)\alpha) - 32G_1\beta + 2G_1\alpha(4+\alpha)\beta + 12G_1^2\beta^2)}$$

3.2.1 Interpretation

The baseline model with front-running in periods $t = -1, 0$ is already relatively rich and allows for interesting observations. In particular, it is interesting to see that the prey can make an order in

[7] The second part of equation (11) reflects the need of the prey to close his position in the risky asset before

period $t=-1$, which instead of decreasing his risk position will increase it. This would happen if following condition is satisfied

$$\frac{p_2}{A_{-1}^S} < \frac{-\beta(8-8G_1\beta-\alpha(6+\alpha-\alpha^2-6G_1\beta))}{2\alpha(1-G_1\beta)(2-\alpha-2G_1\beta)}$$

In this case the prey will accumulate risk, as the arbitraging component in equation (11) outweighs the hedging component. In such a situation, the prey will be willing to trade the deviation of the price from the fundamentals, even though this would increase his risky position. In this case the prey considers the deviation of the price from the fundamentals to be an opportunity that could warrant an increase in the risk present in his books, even though it is shortly before the time of liquidation.

It may also happen in this simple model whereby the predator is a provider of liquidity and in fact helps the prey instead of applying pressure. The condition for this behaviour is

$$\frac{p_2}{A_{-1}^S} < \frac{-\beta(\alpha(6+(1-\alpha)\alpha-6G_1\beta)-4+4G_1\beta)}{\alpha(8-(4-\alpha)\alpha(1+\alpha)-12G_1\beta+2G_1\alpha(2+\alpha)\beta+4G_1^2\beta^2)}$$

In this case the predator finds the exploiting price deviation caused by noise trading more profitable than predation. However, in normal market conditions, when deviation of the price for fundamentals is small, the predator may submit an order that drains the market from any liquidity available for the prey and move the price in an unfavourable direction, so the prey subsequently closes his position at the worst price.

3.3 Multi-period numeric solution

In the previous subsection we showed analytical results for the general price process for one period before liquidation. Unfortunately, in this specification it is very difficult to analyse analytically the dynamics of trading for a greater number of periods before the liquidation. Nevertheless, this can be achieved numerically for given numerical parameter values (α, β, G_1, n). For this purpose we use the following proposition, which is proved in the appendix. The proposition states that if the final period value function is of a particular quadratic form (which is the case in our model), then the value functions for earlier periods are also of the same quadratic form, with parameters that can be calculated by backward induction.

Proposition 3 *If the value functions of the prey and the predator in given period $t<0$ (so strictly before the liquidation) can be presented as*

$$\begin{aligned} V_t^P(p_{t'}, W_t^P, A_0^S) &= \overline{A}_t^P \cdot p_{t'}^2 + \overline{B}_t^P \cdot p_{t'} A_t^S + \overline{C}_t^P \cdot \left(A_t^S\right)^2 + W_t^P + N_t^P \\ V_t^S(p_{t'}, W_t^S, A_0^S) &= \overline{A}_t^S \cdot p_{t'}^2 + \overline{B}_t^S \cdot p_{t'} A_t^S + \overline{C}_t^S \cdot \left(A_t^S\right)^2 + W_t^S + N_t^S \end{aligned}$$

where $\overline{A}_t^P, \overline{B}_t^P, \overline{C}_t^P, N_t^P$ *and* $\overline{A}_t^S, \overline{B}_t^S, \overline{C}_t^S, N_t^S$ *are constants, then the optimal orders of the prey and the predator in period* $t' = t - 1$ *are given by*

$$x_{t'}^P = A_{t'}^S \overline{H}_t^P + p_{t''} \overline{I}_t^P \tag{13}$$

$$x_{t'}^S = A_{t'}^S \overline{H}_t^S + p_{t''} \overline{I}_t^S$$

where $\overline{H}_t^P, \overline{H}_t^S, \overline{I}_t^P, \overline{I}_t^S$ *are the functions of* $\overline{A}_t^P, \overline{B}_t^P, \overline{C}_t^P, N_t^P$ *and* $\overline{A}_t^S, \overline{B}_t^S, \overline{C}_t^S, N_t^S$ *only and the value functions for period* $t' = t - 1$ *are in the form*

$$\begin{aligned} V_{t'}^P(p_{t''}, W_{t'}^P, A_{t'}^S) &= \overline{A}_{t'}^P \cdot p_{t''}^2 + \overline{B}_t^P \cdot p_{t''} A_{t'}^S + \overline{C}_{t'}^P \cdot \left(A_{t'}^S\right)^2 + W_{t'}^P + N_{t'}^P \\ V_t^S(p_{t''}, W_{t'}^S, A_{t'}^S) &= \overline{A}_{t'}^S \cdot p_{t''}^2 + \overline{B}_t^S \cdot p_{t''} A_{t'}^S + \overline{C}_{t'}^S \cdot \left(A_{t'}^S\right)^2 + W_{t'}^S + N_{t'}^S \end{aligned}$$

where $\overline{A}_{t'}^P, \overline{B}_{t'}^P, \overline{C}_{t'}^P, N_{t'}^P$ *and* $\overline{A}_{t'}^S, \overline{B}_{t'}^S, \overline{C}_{t'}^S, N_{t'}^S$ *are the functions of* $\overline{A}_t^P, \overline{B}_t^P, \overline{C}_t^P, N_t^P$ *and* $\overline{A}_t^S, \overline{B}_t^S, \overline{C}_t^S, N_t^S$ *only.*

Proof. In appendix ■

After proving Proposition 3 we outlined a detailed method of mapping the parameters of the value function from period t to earlier period t'. The complexity of these transformations is the prime reason why the elegant closed-form solution for this model for periods before liquidation does not exist[8].

Trading dynamics in the longer term can, however, be analysed relatively easily if we iterate the mapping from Proposition 3 for some numerical boundary values. Setting $\overline{A}_0^P, \overline{B}_0^P, \overline{C}_0^P, \overline{A}_0^S, \overline{B}_0^S, \overline{C}_0^S$ equal to the values calculated in the previous section [9]

$$\overline{A}_0^P = \tfrac{\alpha^2}{4\beta(1-G_1\beta)} \quad \overline{B}_0^P = \tfrac{-\alpha\beta}{2\beta(1-G_1\beta)} \quad \overline{C}_0^P = \tfrac{\beta}{4(1-G_1\beta)}$$

$$\overline{A}_0^S = 0 \qquad \overline{B}_0^S = \tfrac{\alpha}{2-2G_1\beta} \qquad \overline{C}_0^S = \tfrac{\alpha}{2-2G_1\beta}$$

we can analyse how any variations in model parameters affect the dynamics of trading. In particular, we are interested in the effects of changing the value of α on the dynamics of the front-running process.

3.3.1 Variations in α

By changing α we are affecting exogenous price convergence. If α is zero, then the market price instantly adjusts to fundamentals. In such a case neither predator nor prey can benefit from

[8] Even though the elegant solution is not available, all of the results we have here are closed-form. A brave reader can calculate the optimal trading for any possible period in closed form.

[9] These final values are boundary conditions.

trading the price convergence. It is interesting to observe that in such a market front-running is an inefficient strategy. As both predator and trader trades have only an immediate impact on the market, and big orders are absorbed quickly in the market, a predator in the market with a small α provides liquidity to his prey. This can be seen in the table below, which highlights the values of $\overline{H}_t^P$ from equation (13) as a function of α and t for $G1 = 0.25$ and $\beta = 1$

	$\overline{H}_t^P$				
α	0	0.25	0.5	0.75	1
t=0	0.3333	0.3333	0.3333	0.3333	0.3333
t=-1	0.2353	0.1646	0.0676	-0.0710	-0.2857
t=-2	0.1420	0.1194	0.0783	-0.0029	-0.2020
t=-3	0.1023	0.0891	0.0616	-0.0028	-0.2200
t=-4	0.0800	0.0707	0.0505	-0.0008	-0.2253
t=-5	0.0658	0.0585	0.0426	-0.0000	-0.2298
t=-6	0.0559	0.0498	0.0366	0.0002	-0.2337
t=-7	0.0486	0.0433	0.0320	0.0002	-0.2370
t=-8	0.0430	0.0384	0.0285	0.0002	-0.2397

$\overline{H}_t^P$ is an increasing function of α. The higher α, the more persistent are the price effects and the stronger incentive of the predator to front-run the orders of the prey. The persistence of the price also affects the behaviour of the prey. In an environment where α is high, the prey starts to close out his position early in order to avoid costly execution close to the final time of liquidation. However, if the price is less sticky, the prey is not afraid to close his position closer to the final liquidation, where he gets additional help from the predator.

	$\overline{H}_t^S$				
α	0	0.25	0.5	0.75	1
t=0	-1.0000	-1.000	-1.000	-1.000	-1.000
t=-1	-0.4706	-0.4346	-0.3919	-0.3149	-0.2857
t=-2	-0.2841	-0.2962	-0.3090	-0.3274	-0.3793
t=-3	-0.2046	-0.2195	-0.2387	-0.2694	-0.3668
t=-4	-0.1601	-0.1736	-0.1934	-0.2295	-0.3650
t=-5	-0.1316	-0.1434	-0.1619	-0.1990	-0.3630
t=-6	-0.1118	-0.1220	-0.1389	-0.1750	-0.3606
t=-7	-0.0972	-0.1061	-0.1214	-0.1558	-0.3582
t=-8	-0.0859	-0.0939	-0.1077	-0.1401	-0.3558

As shown above, by changing the value of α we can change the dynamics of the trading game. For α close to 1, the dynamics mimic what we could have seen in the model by Brunnermeier and Pedersen (2004) – the prey is closing his position at the constant rate and the predator is draining liquidity from the market. On the other hand, when α is close to 0, we have dynamics similar to the one in the model proffered by Parida and Venter (2010) – in which case our earlier predator/front-runner now becomes a provider of liquidity.

4 Uncertainty about the size of the initial position

In the benchmark case presented in the previous section, we analysed a dynamic trading game between the predator and prey in a full information environment. In reality, however, the predator does not usually possess perfect knowledge about the position of the prey; instead, this knowledge needs to be inferred from observed price dynamics. In this section, we present a model of an equilibrium trading, when the position of the prey is not common knowledge. The model is a modification of the baseline scenario presented in the previous section.

4.1 Model

The model set-up closely matches the benchmark model. The time in the economy goes from $t = -n = -2$ to $t = m$. The dividend on the risky asset is paid at time $t = m + 1$. The prey is allowed to trade in periods $t \leqslant 0$ only. We call time $t = 0$ the time of liquidation – the prey needs to close his position (which may be long or short) up to this point in time. For $t > 0$ the amount of risky assets held in the portfolio of the prey must be equal to zero. The predator is not aware of the initial size of the position of the prey, and he only knows that this position is drawn from the certain distribution $A^S_{-2}\tilde{}N(0, \sigma_{A^S_{-2}})$, which is common knowledge. As in the benchmark model, we have non-strategic traders in the market and their existence implies the following price dynamics

$$p_t = -x^M_t \cdot \beta + p_{t-1} \tag{14}$$

where x^M_t is the amount of assets acquired by a non-strategic trader. The addition of noise traders completes the specification of the model.

In this set-up it is clear why modelling a price process as in equation (14) is a more natural task than setting up the model in the space of asset allocations – prices are directly observable by market participants, whereas asset allocation is not common knowledge.

4.2 Solution

Our aim is to establish a dynamic linear Bayesian-Nash equilibrium for the dynamic trading game specified above. The solution of the dynamics of trading for periods $t > 0$ is the same as in the baseline scenario. For periods $t > 0$ the prey trades only with a non-strategic trader and with a noise trader – there is no uncertainty coming from the prey's trades, so the solution of the trade dynamics is exactly the same as in the baseline scenario. We can therefore directly recall and apply our earlier results. Assuming an infinite horizon and zero price convergence ($\alpha = 1$), we have from equation (6)

$$V^P_1(p_0, W^P_1) = \frac{1}{2\beta} \cdot (p_0 - 0)^2 + W^P_1 \tag{15}$$

where V_1^P denotes the value function of the predator at time $t = 1$ and W_1^P is the cash account of the predator. The value function of the predator at time $t = 1$ depends on the deviation of the price from the fundamentals in earlier periods. As there is no other strategic market participant in periods $t > 0$, the value function and the dynamics of trading are exactly the same as in our baseline scenario.

At $t = 0$ the prey needs to liquidate his position, so the optimal (and only allowed) trade of the prey at time $t = 0$ is

$$x_0^S = -A_0^S$$

where $-A_0^S$ is the position of the prey in risky assets at the beginning of time $t = 0$. This order is deterministic from the point of view of the prey – due to the constraint, he knows exactly how much he will need to liquidate in the last period. This knowledge is, however, not available to the predator, who only has an estimate of the amount of assets the prey will be buying or selling in the last period. Let us assume for the moment (we will later show that this is a case in equilibrium) that this estimate is at time $t = 0$ of the form $A_0\tilde{}N(\mu_{A_0}, \sigma_{A_0}^2)$. In addition to this estimate about the position of the prey in the risky assets, the predator can condition his order on the price of the risky asset observed in the past periods, which is common knowledge. The problem for the predator is therefore

$$\max_{x_0^P} E(V_1^P(p_0, W_1^P)) \tag{16}$$

subject to

$$p_0 = (x_0^S + x_0^P + u_0) \cdot \beta + p_{-1}$$

and

$$x_0^S = -A_0 \sim N(-\mu_{A_0}, \sigma_{A_0}^2) \tag{17}$$

which after the substitution to equation (15) and then into equation (16) yields the problem of the predator in period $t = 0$, namely

$$\max_{x_0^P} E(\frac{1}{2\beta} \cdot ((x_0^S + x_0^P + u_0) \cdot \beta + p_{-1})^2 + W_1^P)$$

The optimal order of the predator is described by the following proposition, which we prove in the appendix.

Proposition 4 *When there is no exogenous price convergence ($\alpha = 1$) the optimal order of the predator at time of liquidation ($t = 0$) is to refrain from trading. The predator thus submits an order of zero size.*

Proof. In appendix ■

The fact that the predator does not submit any order at the time of liquidation depends crucially on the assumed zero price convergence ($\alpha = 1$). Intuitively, the predator refrains from trading, because he knows that the price deviation (caused by the final liquidation of the position by the prey) will be long-lived. The predator will have many future periods to effectively take advantage of the price deviation and therefore does not need to rush to exploit this opportunity. Of course, if $\alpha < 1$, then the order of the predator would be different – he will be a provider of liquidity to the prey, as described in section 3 for the baseline model (this effect is similar to what is observed in Brunnermeier and Pedersen's model).

In proving Proposition 4 on page 37 we derived the value function of the predator at time $t = 0$, which is

$$V_0^P(p_{-1}, W_0^P, \mu_{A_0}, \sigma_{A_0}^2) = W_0^P + \frac{p_{-1}^2 - 2\mu_{A_0} p_{-1}\beta + \beta^2(\mu_{A_0}^2 + \sigma_{A_0}^2 + \sigma_L^2)}{2\beta}$$

where p_{-1} is the price of the asset in $t = -1$, W_0^P is the value of the predator's money account at time $t = 0$ and $\mu_{A_0}, \sigma_{A_0}^2$ are the moments of rational belief about the position of the prey.

From the above calculations we understand that the equilibrium at time $t = 0$ is fairly straightforward, as the prey just liquidates his position in the risky assets, so therefore there is no real optimisation. The predator on the other hand optimally selects (for any beliefs) to refrain from trading. The dynamics of trading becomes more interesting in period $t = -1$, when the prey can choose the amount he will trade.

As we are looking for a linear Bayesian equilibrium, we posit that the optimal order of the prey at time $t = -1$ is a linear function of his asset holdings

$$x_{-1}^S = a_1 + b_1 A_{-1}$$

The predator does not know the position of the prey in risky assets A_{-1} (and therefore x_{-1}^S) with certainty, but only has rational belief (resulting from the Bayesian updating in the prior periods $t = -2$) about the current value of the assets held by the predator, namely $A_{-1} \sim N(\mu_{A_{-1}}, \sigma_{A_{-1}}^2)$.

4.2.1 Bayesian updating

The predator updates his beliefs regarding the position of the prey in risky assets by observing the price that realises in the market. Using information on the previous price p_{-2}, the value of his order x_{-1}^P and the characteristics of the market (summarised in the price process) and the conjectured equilibrium, the predator updates his beliefs about A_{-1}. From the price process we have

$$p_{-1} = (x_{-1}^P + x_{-1}^S + u_{-1})\beta + p_{-2}$$

which implies that the signal of the predator is

$$S_{-1} = x^S_{-1} + u_{-1} = \frac{p_{-1} - p_{-2}}{\beta} - x^P_{-1}$$

which after utilising the conjectured equilibrium gives

$$S_{-1} = a_1 + b_1 A_{-1} + u_{-1}$$

Proposition 5 *Given the conjectured strategy of the prey and the signal S_{-1} observed by the predator, the Bayesian beliefs of the predator in period $t = 0$ are given by*

$$\begin{aligned}(A_0|S_{-1}) \quad \sim \quad & N[a_1 + (1 + b_1) \cdot (\mu_{A_{-1}} + \frac{b_1 \cdot \sigma^2_{A_{-1}}}{b_1^2 \cdot \sigma^2_{A_{-1}} + \sigma^2_u}(S_{-1} - a_1 - b_1\mu_{A_{-1}})), \\ & (1 + b_1)^2 \cdot (\sigma^2_{A_{-1}} - \frac{b_1^2 \cdot \sigma^4_{A_{-1}}}{b_1^2 \cdot \sigma^2_{A_{-1}} + \sigma^2_u})]\end{aligned}$$

Proof. In appendix ■

The important point here is that the updated beliefs of the predator in period $t = 0$ are normal and therefore consistent with what we assumed at the beginning of this subsection.

4.2.2 Problem of the predator at time $t = -1$

The programme of the predator at time $t = -1$ is

$$\max_{x^P_{-1}} E(V^P_0(p_{-1}, W^P_0, \mu_{A_0}, \sigma^2_{A_0})) \Leftrightarrow \max_{x^P_{-1}} E(W^P_0 + \frac{p^2_{-1} - 2\mu_{A_0} p_{-1}\beta + \beta^2(\mu^2_{A_0} + \sigma^2_{A_0} + \sigma^2_L)}{2\beta}) \quad (18)$$

where

$$\begin{aligned}E(W^P_0) \quad &= \quad W^P_{-1} - p_{-1}x^P_{-1} \\ p_{-1} \quad &= \quad p_{-2} + \beta(x^S_{-1} + x^P_{-1} + u_{-1})\end{aligned}$$

The solution to the problem of the predator is summarised by the following proposition.

Proposition 6 *Given the conjectured strategy of the prey, the optimal order of the predator at time $t = -1$ is a function of his expectations about the current position of the prey in risky assets*

$$x^P_{-1} = -a_1 - (1 + b_1)\mu_{A_{-1}}$$

Proof. In appendix ■

Given the conjectured equilibrium, the predator puts the order which is exactly minus the expected position of the prey in risky assets at time $t = 0$.

4.2.3 Problem of the prey at time $t = -1$

As highlighted earlier in Proposition 4, the optimal order of the predator at time $t = 0$ is equal to 0, irrespective of what the predator believes. The programme of the prey is therefore

$$\max_{x_{-1}^S} E(W_{-1}^S - p_{-1}x_{-1}^S - p_0(-(A_{-1} + x_{-1}^S)))$$

where

$$\begin{aligned} p_{-1} &= p_{-2} + \beta(x_{-1}^S + x_{-1}^P + u_{-1}) = p_{-2} + \beta(x_{-1}^S - a_1 - (1+b_1)\mu_{A_{-1}} + u_{-1}) \\ p_0 &= p_{-2} + \beta(x_{-1}^S - a_1 - (1+b_1)\mu_{A_{-1}} + u_{-1}) + \beta(x_0^S + u_0) \end{aligned}$$

which after substitution gives

$$\begin{aligned} &\max_{x_{-1}^S} E(W_{-1}^S - (p_{-2} + \beta(x_{-1}^S - a_1 - (1+b_1)\mu_{A_{-1}} + u_{-1}))x_{-1}^S \\ &-(p_{-2} + \beta(x_{-1}^S - a_1 - (1+b_1)\mu_{A_{-1}} + u_{-1}) + \beta(x_0^S + u_0))(-(A_{-1} + x_{-1}^S))) \end{aligned} \tag{19}$$

Taking the first order conditions yields

$$x_{-1}^S = -\frac{A_{-1}}{2}$$

which implies that the prey liquidates half of his position in period $t = -1$ – irrespective of the beliefs of the predator. Substituting the optimal order back into equation (19) and simplifying yields, the value function of the prey at time $t = -1$

$$V_{-1}^S(A_{-1}, p_{-2}, W_{-1}^S, \mu_{A_{-1}}) = A_{-1}p_{-2} + W_{-1}^S - \frac{1}{4}A_{-1}(3A_{-1} + 2\mu_{A_{-1}})\beta$$

The value function of the prey depends on the current amount of risky assets held by the predator, the most recent market price and the beliefs of the predator at $t = -1$ about the position of the prey in the risky assets. In equilibrium the beliefs of the predator are known to the prey. The reason for this can be explained easily in following steps. First, the a priori distribution of risky assets held by the prey is common knowledge. Second, (as we shall show on page 29) the equilibrium order of the prey in period $t = -2$ is deterministic and depends only on the known parameters of the a priori distribution. The prey, who submits the order in period $t = -2$, knows exactly his order and therefore can recover the value of the noise traders' order. The prey can also solve for how the beliefs of the predator were updated, and in any period is able to disentangle the order of the predator from the order of the noise trader. Conversely, the predator is unable to fully disentangle the order of the prey from that of the noise trader. One important element here is that the beliefs of the predator are not deterministic from the point of view of the prey, as these are affected by the random orders of the noise traders. However, the prey has full knowledge of how much the noise traders have traded and is also aware of how the beliefs of the predator have changed. The ability of the prey to know the beliefs of the predator at each point in time is specific to our pure strategy equilibrium – mixed strategy equilibria will in general not allow for this scenario.

The form of the order of the prey allows him to recover instantly undetermined coefficients, which are

$$\begin{aligned} a_1 &= 0 \\ b_1 &= -\frac{1}{2} \end{aligned}$$

Substituting these coefficients (which determine the Bayesian-Nash equilibrium of a trading game) into the value function of the predator, we have[10]

$$V_{-1}^{P}(p_{-2}, W_{-1}^{P}, \mu_{A_{-1}}, \sigma_{A_{-1}}^{2}) = W_{-1}^{P} - \mu_{A_{-1}} p_{-2} + \frac{p_{-2}^{2}}{2\beta} + U_{-1}(\mu_{A_{-1}}, \sigma_{A_{-1}}^{2})$$

4.2.4 Problem of the predator at time $t = -2$

At time $t = -2$ we are again positing the order of the prey as linear in his asset holdings

$$x_{-2}^{S} = a_2 + b_2 A_{-2}$$

given this aspect, the problem of the predator is

$$\max_{x_{-2}^{P}} E\left[W_{-1}^{P} - \mu_{A_{-1}} p_{-2} + \frac{p_{-2}^{2}}{2\beta} + U_{-1}(\mu_{A_{-1}}, \sigma_{A_{-1}}^{2})\right] \tag{20}$$

The problem of the predator summarised in equation (20) is equivalent to the problem in period $t = -1$ from equation (18), and the solution to this problem closely matches that of the earlier problem, as shown in the following proposition.

Proposition 7 *Given the conjectured strategy of the pre,y the optimal order of the predator at time*

[10] More details are provided in the proof of Proposition 6.

$t=-2$ is a function of his expectations about the current position of the prey in risky assets

$$x^P_{-2} = -a_2 - (1+b_2)\mu_{A_{-2}}$$

Proof. In appendix ■

Again, given the conjectured strategy equilibrium, the predator puts the order, which is exactly minus the expected position of the prey, in risky assets at time $t=-1$.

4.2.5 Problem of the prey at time $t=-2$

The programme of the predator is

$$\max_{x^S_{-2}} E\left[V^S_{-1}(A_{-1}, p_{-2}, W^S_{-1}, \mu_{A_{-1}})\right] = \max_{x^S_{-2}} E\left[A_{-1}p_{-2} + W^S_{-1} - \frac{1}{4}A_{-1}(3A_{-1} + 2\mu_{A_{-1}})\beta\right] \quad (21)$$

where the prey maximises the expected value of the value function at time $t=-1$ subject to the price process and the dynamics of the expectations of the predator. Intuitively the prey wants the price in the next periods to be favourable (if the prey has assets to sell, he wants the price in period $t=-1$ to be as high as possible and vice versa), but also wants to steer the beliefs of the predator, so he is not subject to intensive front-running in period $t=-1$.

Given the suggested strategy of the prey, the beliefs of the predator will be updated according to the following formula[11]

$$\mu_{A_{-1}} = \mu_{A_{-2}} + \frac{b_2\sigma^2_{A_{-2}}}{b_2^2\sigma^2_{A_{-2}} + \sigma^2_u}\left(x^B_{-2} + u_{-2} - a_2 - b_2 \cdot \mu_{A_{-2}}\right)$$

with expected value

$$E(\mu_{A_{-1}}) = \mu_{A_{-2}} + \frac{b_2\sigma^2_{A_{-2}}}{b_2^2\sigma^2_{A_{-2}} + \sigma^2_u}\left(x^B_{-2} + \mathbf{0} - a_2 - b_2 \cdot \mu_{A_{-2}}\right)$$

It is important to note that when solving for the optimal order of the predator, we are allowing for any possible order, not necessarily one consistent with the conjectured linear equilibrium. Later, we solve for coefficients a_2 and b_2, which would make the order of the predator consistent with the equilibrium.

Equation (21) can be rewritten in a more convenient form

$$\max_{x^S_{-2}} E\left[W^S_{-2} - p_{-2}\cdot x^B_{-2} + (A_{-2} + x^B_{-2})p_{-2} - \beta\frac{3}{4}(A_{-2} + x^B_{-2})^2 + \beta\frac{1}{2}\mu_{A_{-1}}(A_{-2} + x^B_{-2})\right]$$

[11] More details about Bayesian updating can be found in the proof of Proposition 5 on page (38).

The solution to this problem is given by the following proposition, proven in the appendix.

Proposition 8 *Given the conjectured strategy, the optimal order of the prey at time $t = -2$ is*

$$x^S_{-2} = \frac{a_2 b_2 \sigma^2_{A_{-2}} - (a_2 + (1+b_2)\mu_{A_{-2}})\sigma^2_u}{b_2(2+5b_2)\sigma^2_{A_{-2}} + 3\sigma^2_u} + A_{-2}\left(-\frac{b_2(1+2b_2)\sigma^2_{A_{-2}} - \sigma^2_u}{b_2(2+5b_2)\sigma^2_{A_{-2}} + 3\sigma^2_u}\right)$$

where $\mu_{A_{-2}}$ is the mean of the a priori distribution of the amount of the risky assets held by the prey, while $\sigma^2_{A_{-2}}$ is the variance of this distribution.

Proof. In appendix ■

In order to calculate the undetermined coefficients we need to solve the system of two following equations:

$$a_2 = \frac{a_2 b_2 \sigma^2_{A_{-2}} - (a_2 + (1+b_2)\mu_{A_{-2}})\sigma^2_u}{b_2(2+5b_2)\sigma^2_{A_{-2}} + 3\sigma^2_u} \tag{22}$$

$$b_2 = -\frac{b_2(1+2b_2)\sigma^2_{A_{-2}} + \sigma^2_u}{b_2(2+5b_2)\sigma^2_{A_{-2}} + 3\sigma^2_u} \tag{23}$$

Although an analytical solution is possible for a wide range of coefficients, we focus now on the case for which a priori distribution of risky assets held by the predator has the following parameters $\mu_{A_{-2}} = 0$ and $\sigma^2_{A_{-2}} = 1$. This implies that the predator has no initial knowledge about whether the prey is more likely to have a long or short position in the risky asset. In this case equation (22) becomes

$$a_2 = \frac{a_2 b_2 - a_2 \sigma^2_u}{b_2(2+5b_2) + 3\sigma^2_u}$$

where one of the solutions is $a_2 = 0$.

Equation (23) is a cubic equation

$$b_2^2(4+5b_2)\sigma^2_{A_{-2}} + b_2(3\sigma^2_u + \sigma^2_{A_{-2}}) + \sigma^2_u = 0 \Leftrightarrow b_2^2(4+5b_2) + b_2(3\sigma^2_u + 1) + \sigma^2_u = 0 \tag{24}$$

which needs to be solved using a standard approach (the depressed cubic method). Cubic equations of this type always have three different roots, some of which can be complex. Fortunately, this equation in our setting for the given value of σ^2_u has only one real solution, with others having an imaginary component, so we have a unique linear equilibrium for any value of σ^2_u. The solution to the problem (24) is given by

$$
\begin{aligned}
b_2^* &= -\frac{4}{15} + \frac{2^{\frac{1}{3}}(45\sigma_u^2 - 1)}{15(-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}}\text{`} \\
&\quad - \frac{(-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}}{15 \cdot 2^{\frac{1}{3}}}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
b_2^{**} &= -\frac{4}{15} + \frac{(1 + i \cdot \sqrt{3})(45\sigma_u^2 - 1)}{15 \cdot 2^{\frac{2}{3}} \cdot (-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}} \\
&\quad - \frac{(1 - i \cdot \sqrt{3})(-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}}{30 \cdot 2^{\frac{1}{3}}}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
b_2^{***} &= -\frac{4}{15} + \frac{(1 - i \cdot \sqrt{3})(45\sigma_u^2 - 1)}{15 \cdot 2^{\frac{2}{3}} \cdot (-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}} \\
&\quad - \frac{(1 + i \cdot \sqrt{3})(-52 + 135\sigma_u^2 + 15\sqrt{3}\sqrt{4 - 20\sigma_u^2 - 9\sigma_u^4 + 540\sigma_u^6})^{\frac{1}{3}}}{30 \cdot 2^{\frac{1}{3}}}
\end{aligned} \tag{27}
$$

The relationship between the volatility of the orders of noise traders and coefficient b_2 in the linear Bayesian equilibrium is presented in Figure 1 below. The lower the volatility of the noise traders' orders, the smaller fraction of his total risk position the prey unravels in period $t = -2$, which can be understood intuitively. If there is very little noise in the market induced by the activity of the noise traders, then the predator learns about the position of the prey relatively quickly from the price process. The prey knowing about this element decreases the fraction of risky position he unwinds in period $t = -2$ to avoid subsequent predation. The prey attempts to hide his order in the market noise, so the predator is effectively learning very little about the position of the prey and is unable to front-run orders in period $t = -1$.

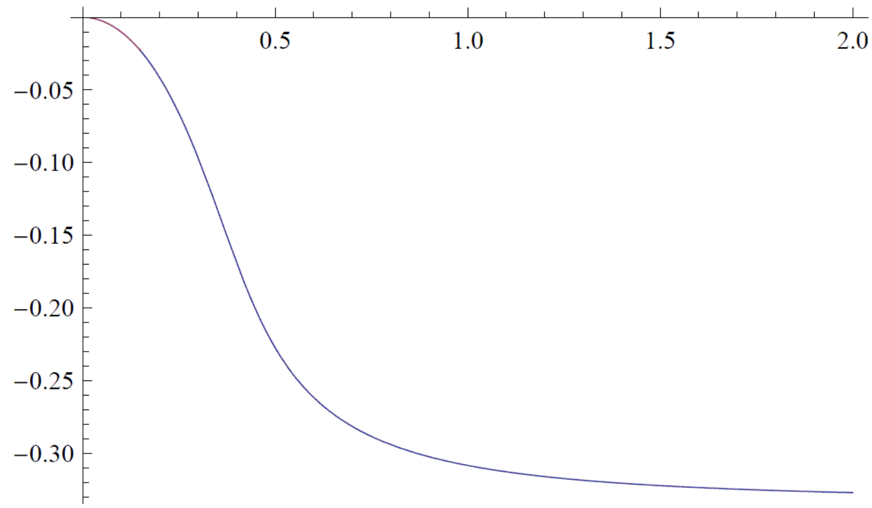


Figure 1: Horizontal axis σ_u^2; Vertical axis b_2;

When the variance of the market noise increases, coefficient b_2 approaches $1/3$, which is the amount of the position the prey will decide to liquidate if there is no predator in the market at all. A high level of variance in the orders of the noise traders σ_u^2 implies that the orders of the prey are very well hidden in the noise and the predator is unable to learn about the prey's position and subsequently front-run his position. A higher level of noise in the market allows risk-neutral prey to close his position at a cost lower than expected in the market, where the amount of noisy trading is low.

5 Conclusion

In this part of my thesis, I study front-running in dynamic noisy markets. In the full-information environment presented in Section 3, the random orders submitted by noise traders present an opportunity that can be exploited by the prey and also by the predator if there is exogenous price convergence in the market. As described in Section 3, the prey can in some circumstances increase his risky position, particularly if the current price provides a significant opportunity that can be exploited. Depending on the strength of an exogenous price convergence[12], the predator can either drain liquidity from the market or (strong exogenous price convergence) provide liquidity to the prey and allow him to decrease the overall expected costs of trading. At the end of Section 3 I show the method used to analyse effectively the dynamics of trading in multi-period settings. I- show the recursive dynamics of the value functions of the prey and predator and their optimal orders in Proposition 3 on page 18. The baseline model, in addition to adding randomness induced by the

[12] As discussed in Section 3, any price convergence can be meant to reflect the market power of some un-modelled strategic agents, who are unaware of the fragility of prey.

noise traders, bridges the gap between the models of Parida and Venter (2010), who assume ultra-quick price convergence[13] and the model of Brunnermeier and Pedersen (2004), who assume long-lived price effects. I am showing the dynamics of trading for different levels of price convergence.

In section 4 I present the model, which allows me to investigate whether front-running is possible when the predator does not have full information about the initial size of the prey's position in risky assets. It transpires that front-running is possible even in this case, and I show the linear Bayesian-Nash equilibrium in which the predator learns about the prey's initial position from his initial orders and then front-runs him in subsequent periods. Moreover, I show that the higher the noise in the market caused by noise traders, the lower the expected costs of closing the position for the prey. If the market is noisy the prey is able to hide in this noise and the predator fails to learn about the position of the prey, so that front-running is not significant. To the best of my knowledge this is the first model in the literature that focuses on this aspect of predation. My results imply that keeping one's own position secret is an important issue for the prey. Furthermore, keeping this position secret during the liquidation of the position allows the prey to close his position in the expectation of higher prices and to avoid front-running.

[13] In the Parida and Venter (2010) model there are no long-lived price effects. Price instantly converges to the fundamental value.

6 Appendix to Part I

6.1 Proof of Proposition 1

At any point in time, a short-term horizon trader starts with the endowment of risky assets from previous period A^M_{-1} and a certain amount of cash on hand A^M_{-1}. Given his expectations concerning the asset's final pay-off, his optimisation problem is given by

$$\max_x E\left[U((x + A^M_{-1}) \cdot v + W^M_{-1} - x \cdot p)\right]$$

which, because of CARA preferences and the normality of A, is equivalent to

$$\max_x \left[(x + A^M_{-1})E(v) + W^M_{-1} - x \cdot p - \frac{\rho}{2}(x + A^M_{-1})^2 \cdot Var(v)\right]$$

The first order condition of this problem is

$$p = -(x + A^M_{-1}) \cdot \rho Var(v)$$

which we rewrite for convenience as

$$p = -(x + A^M_{-1})\beta \tag{28}$$

where β is a positive constant that summarises an impact of the coefficient of absolute risk aversion ρ and the variance of the final asset pay-off.

We now know that condition will hold in any period, in particular in period $t = -1$. We can therefore lag equation (28) and write

$$p_{-1} = -(x_{-1} + A^M_{-2})\beta \tag{29}$$

Then, by applying the fact that $x_{-1} + A^M_{-2} = A^M_{-1}$ and combining equations (28) and (29), we end up with

$$p = -x\beta + p_{-1}$$

which with the observation that this equation applies to all periods completes the proof.

QED

6.2 Proof of Proposition 2

Assuming that the value function of the predator at $t = k + 1$ is given by

$$V^P_{k+1}(p_k, W^P_{k+1}) = G_{k+1} \cdot (p_k)^2 + W^P_{k+1} + N_{k+1} \tag{30}$$

then the programme of the predator at $t = k$ is

$$\max_{x_k} E_k \left[V_{k+1}^P(p_k, W_{k+1}^P) \right] \Leftrightarrow \max_{x_k} E_k \left[G_{k+1} \cdot (p_k)^2 + W_{k+1}^P + N_{k+1} \right] \tag{31}$$

subject to price dynamics

$$p_k = (x_k^P + u_k) \cdot \beta + \alpha \cdot p_{k-1} \tag{32}$$

where $E_k[]$ is the expectation operator with respect to knowledge at time $t = k$.

By substituting the price process from equation (32) into equation (31) we get

$$\begin{aligned} V_k^P(p_{k-1}, W_k^P) &= \max_{x_k} E_k[G_{k+1}((x_k^P + u_k)\beta + \alpha p_{k-1})^2 + \\ &\quad W_k^P - x_k^P((x_k^P + u_k) \cdot \beta + \alpha p_{k-1}) + N_{k+1}] \end{aligned} \tag{33}$$

and then by taking FOC and observing that $E_k[u_k^2] = \sigma_L^2$ we have

$$x_k^P = -\frac{p_{k-1} \cdot \alpha - 2G_{k+1} \cdot p_{k-1} \cdot \alpha\beta}{2\beta(G_{k+1}\beta - 1)} \tag{34}$$

After substituting(34) into (33) for x_k^P, and the application of some straightforward (but tedious) algebra, we have

$$V_k^P(p_{k-1}, W_k^P) = \frac{\alpha^2}{4\beta(1 - G_{k+1}\beta)} \cdot (p_{k-1})^2 + W_k^P + N_{k+1} + G_{k+1}\beta^2\sigma_L^2$$

which we can rewrite as

$$V_k^P(p_{k-1}, W_k^P) = G_k \cdot (p_{k-1})^2 + W_k^P + N_k$$

where

$$\begin{aligned} G_k &= \frac{\alpha^2}{4\beta} \cdot \frac{1}{(1 - G_{k+1}\beta)} \\ N_k &= N_{k+1} + G_{k+1} \cdot \beta^2\sigma_u^2 \end{aligned} \tag{35}$$

Solving equation (35) for G_{k+1} and substituting it into (34) yields

$$x_k^P = \left(\frac{2 \cdot G_k}{\alpha} - \frac{\alpha}{\beta} \right) \cdot p_{k-1}$$

The value function in period $t = n$ (equation (4) on page 13) is of the form assumed in equation (30), so in line with the standard induction argument, Proposition 2 follows.

QED

6.3 Proof of Proposition 3

We suggest that the value function of the predator and prey at certain point in time t is equal to

$$\begin{aligned} V_t^P(p_{t'}, W_t^P, A_0^S) &= \overline{A}_t^P \cdot p_{t'}^2 + \overline{B}_t^P \cdot p_{t'} A_t^S + \overline{C}_t^P \cdot \left(A_t^S\right)^2 + W_t^P + N_t^P \\ V_t^S(p_{t'}, W_t^S, A_0^S) &= \overline{A}_t^S \cdot p_{t'}^2 + \overline{B}_t^S \cdot p_{t'} A_t^S + \overline{C}_t^S \cdot \left(A_t^S\right)^2 + W_t^S + N_t^S \end{aligned} \tag{36}$$

where with t' we denote time $t^{'} = t - 1$. All other annotations are the same as before. At time t' both agents maximise the expected value of their respective value function

$$\begin{aligned} \max_{x_{t'}^P} E_{t'}[V_t^P(p_{t'}, W_t^P, A_t^S)] &= \max_{x_{t'}^P} E_{t'}[\overline{A}_t^P \cdot p_{t'}^2 + \overline{B}_t^P \cdot p_{t'} A_t^S + \overline{C}_t^P \cdot \left(A_t^S\right)^2 + W_t^P + N_t^P] \\ \max_{x_{t'}^S} E_{t'}[V_t^S(p_{t'}, W_t^S, A_t^S)] &= \max_{x_{t'}^S} E_{t'}[\overline{A}_t^S \cdot p_{t'}^2 + \overline{B}_t^S \cdot p_{t'} A_t^S + \overline{C}_t^S \cdot \left(A_t^S\right)^2 + W_t^S + N_t^S] \end{aligned}$$

subject to

$$\begin{aligned} p_{t'} &= \alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'}) \\ A_t^S &= A_{t'}^S + x_{t'}^S \\ W_t^P &= W_{t'}^P - x_{t'}^P p_{t'} \\ W_t^S &= W_{t'}^S - x_{t'}^S p_{t'} \end{aligned}$$

which after substitution into the original equations gives, respectively,

$$\begin{aligned} &\max_{x_{t'}^P} E_{t'}[\overline{A}_t^P \left(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})\right)^2 + \overline{B}_t^P \left(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})\right)(A_{t'}^S + x_{t'}^S) \\ &+\overline{C}_t^P \cdot \left(A_{t'}^S + x_{t'}^S\right)^2 + W_{t'}^P - x_{t'}^P(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})) + N_t^P] \end{aligned} \tag{37}$$

$$\begin{aligned} &\max_{x_{t'}^S} E_{t'}[\overline{A}_t^S \left(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})\right)^2 + \overline{B}_t^S \left(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})\right)(A_{t'}^S + x_{t'}^S) \\ &+\overline{C}_t^S \cdot \left(A_{t'}^S + x_{t'}^S\right)^2 + W_{t'}^S - x_{t'}^S(\alpha p_{t''} + \beta(x_{t'}^S + x_{t'}^P + u_{t'})) + N_t^S] \end{aligned} \tag{38}$$

The first order conditions for the problems of prey and predators yield, respectively,

$$
\begin{aligned}
x^P_{t'} &= -p_{t''} \cdot \frac{\alpha - 2\alpha\beta\overline{A}^P_t}{2\beta\left(1-\overline{A}^P_t\beta\right)} + A^S_{t'} \cdot \frac{\beta\overline{B}^P_t}{2\beta\left(1-\overline{A}^P_t\beta\right)} - x^S_{t'} \cdot \frac{\beta - \overline{B}^P_t\beta - 2\overline{A}^P_t\beta^2}{2\beta\left(1-\overline{A}^P_t\beta\right)} \\
x^S_{t'} &= -p_{t''}\frac{-\alpha + \alpha\overline{B}^S_t + 2\overline{A}^S_t\alpha\beta}{2(\overline{C}^S_t - \beta + \overline{B}^S_t\beta + \overline{A}^S_t\beta^2)} + A^S_{t'}\frac{-2\overline{C}^S_t - \overline{B}^S_t\beta}{2(\overline{C}^S_t - \beta + \overline{B}^S_t\beta + \overline{A}^S_t\beta^2)} - x^P_{t'}\frac{-\beta + \overline{B}^S_t\beta + 2\overline{A}^S_t\beta^2}{2(\overline{C}^S_t - \beta + \overline{B}^S_t\beta + \overline{A}^S_t\beta^2)}
\end{aligned}
$$

which may be conveniently rewritten as

$$
\begin{aligned}
x^P_{t'} &= -p_{t''}\overline{D}^P_t + A^S_{t'}\overline{E}^P_t - x^S_{t'}\overline{F}^P_t \\
x^S_{t'} &= -p_{t''}\overline{D}^S_t + A^S_{t'}\overline{E}^S_t - x^P_{t'}\overline{F}^S_t
\end{aligned}
$$

and under equilibrium (Nash) this implies that

$$
x^P_{t'} = \frac{A^S_{t'}(\overline{E}^P_t - \overline{E}^S_t\overline{F}^P_t) + p_{t'}(\overline{D}^S_t\overline{F}^P_t - \overline{D}^P_t)}{1 - \overline{F}^P_t\overline{F}^S_t}
$$

$$
x^S_{t'} = \frac{A^S_{t'}(\overline{E}^S_t - \overline{E}^P_t\overline{F}^S_t) + p_{t'}(\overline{D}^P_t\overline{F}^S_t - \overline{D}^S_t)}{1 - \overline{F}^P_t\overline{F}^S_t}
$$

which again can be conveniently rewritten as

$$
x^P_{t'} = A^S_{t'}\overline{H}^P_t + p_{t'}\overline{I}^P_t
$$

$$
x^S_{t'} = A^S_{t'}\overline{H}^S_t + p_{t'}\overline{I}^S_t
$$

which after substituting back into equations (37) and (38) will yield

$$
\begin{aligned}
V^P_{t'}(p_{t''}, W^P_{t'}, A^S_{t'}) \quad = \quad & (\mathbf{p}_{t''})^2 \cdot [\overline{C}^P_t (\overline{I}^S_t)^2 + (\alpha + (\overline{I}^P_t + \overline{I}^S_t)\beta)(\overline{B}^P_t \overline{I}^S_t - \overline{I}^P_t + \overline{A}^P_t \alpha + \overline{A}^P_t (\overline{I}^P_t + \overline{I}^S_t)\beta)] \\
& \\
& + \mathbf{A}^S_{t'} \mathbf{p}_{t''} \cdot [2\overline{C}^P_t (1 + \overline{H}^S_t)\overline{I}^S_t - \overline{H}^P_t \alpha + \overline{B}^P_t (1 + \overline{H}^S_t)\alpha + \\
& + \overline{B}^P_t (\overline{I}^P_t + \overline{H}^S_t \overline{I}^P_t + \overline{I}^S_t + \overline{H}^P_t \overline{I}^S_t + 2\overline{H}^S_t \overline{I}^S_t)\beta + \\
& - (2\overline{H}^P_t \overline{I}^P_t + \overline{H}^S_t \overline{I}^P_t + \overline{H}^P_t \overline{I}^S_t - 2\overline{A}^P_t (\overline{H}^P_t + \overline{H}^S_t)\alpha)\beta + 2\overline{A}^P_t (\overline{H}^P_t + \overline{H}^S_t)(\overline{I}^P_t + \overline{I}^S_t)\beta^2)] \\
& \\
& + (\mathbf{A}^S_{t'})^2 \cdot [\overline{C}^P_t (1 + \overline{H}^S_t)^2 + (\overline{H}^P_t + \overline{H}^S_t)\beta(\overline{B}^P_t - \overline{H}^P_t + \overline{B}^P_t \overline{H}^S_t + \overline{A}^P_t (\overline{H}^P_t + \overline{H}^S_t)\beta)] \\
& \\
& + N^P_t + W^P_{t'} + \overline{A}^P_t \beta^2 \sigma^2
\end{aligned}
$$

and

$$
\begin{aligned}
V^S_{t'}(p_{t''}, W^S_{t'}, A^S_{t'}) \quad = \quad & (\mathbf{p}_{t''})^2 \cdot [\overline{C}^S_t \left(\overline{I}^S_t\right)^2 + (\alpha + (\overline{I}^P_t + \overline{I}^S_t)\beta)(\overline{I}^S_t (\overline{B}^S_t + \overline{A}^S_t \beta - 1) + \overline{A}^S_t (\alpha + \overline{I}^P_t \beta))] \\
& \\
& + \mathbf{A}^S_{t'} \mathbf{p}_{t''} \cdot [2\overline{C}^S_t (1 + \overline{H}^S_t) I^S_t - \overline{H}^S_t \alpha + \overline{B}^S_t (1 + \overline{H}^S_t)\alpha + \\
& + \overline{B}^S_t (\overline{I}^P_t + \overline{H}^S_t \overline{I}^P_t + \overline{I}^S_t + \overline{H}^P_t \overline{I}^S_t + 2\overline{H}^S_t \overline{I}^S_t)\beta + \\
& - (\overline{H}^S_t \overline{I}^P_t + \overline{H}^S_t \overline{I}^S_t + 2\overline{H}^S_t \overline{I}^S_t - 2\overline{A}^S_t (\overline{H}^P_t + \overline{H}^S_t)\alpha)\beta + 2\overline{A}^S_t (\overline{H}^P_t + \overline{H}^S_t)(\overline{I}^P_t + \overline{I}^S_t)\beta^2)] \\
& \\
& + (\mathbf{A}^S_{t'})^2 \cdot [\overline{C}^S_t (1 + \overline{H}^S_t)^2 + (\overline{H}^P_t + \overline{H}^S_t)\beta(\overline{B}^S_t - \overline{H}^S_t + \overline{B}^S_t \overline{H}^S_t + \overline{A}^S_t (\overline{H}^P_t + \overline{H}^S_t)\beta)] \\
& \\
& + N^S_t + W^S_{t'} + \overline{A}^S_t \beta^2 \sigma^2
\end{aligned}
$$

so value functions for the period t' are again of the form as in equation (36)

$$
\begin{aligned}
V^P_{t'}(p_{t''}, W^P_{t'}, A^S_{t'}) &= \overline{A}^P_{t'} \cdot p^2_{t''} + \overline{B}^P_t \cdot p_{t''} A^S_{t'} + \overline{C}^P_{t'} \cdot \left(A^S_{t'}\right)^2 + W^P_{t'} + N^P_{t'} \\
V^S_t(p_{t''}, W^S_{t'}, A^S_{t'}) &= \overline{A}^S_{t'} \cdot p^2_{t''} + \overline{B}^S_t \cdot p_{t''} A^S_{t'} + \overline{C}^S_{t'} \cdot \left(A^S_{t'}\right)^2 + W^S_{t'} + N^S_{t'}
\end{aligned}
$$

QED

6.4 Proof of Proposition 4

The programme of the predator at time $t = 0$ is

$$\max_{x_0^P} E(\frac{1}{2\beta} \cdot ((x_0^S + x_0^P + u_0) \cdot \beta + p_{-1})^2 + W_1^P) \tag{39}$$

subject to beliefs about the position of the prey

$$x_0^S = -A_0 \sim N(-\mu_{A_0}, \sigma_{A_0}^2)$$

and the distribution of the orders of the noise traders given by

$$u_0 \sim N(0, \sigma_L^2).$$

The value of the predator's money account at time $t = 1$ is a function of his order at time $t = 0$ and the price of the asset at time zero – p_0. Equation (39) can be therefore written as

$$\max_{x_0^P} E(\frac{1}{2\beta} \cdot (p_0^2) + W_0^P - p_0 \cdot x_0^P) \Longleftrightarrow \max_{x_0^P} \frac{1}{2\beta} E(p_0^2) - x_0^P E(p_0) \tag{40}$$

By substituting the price process and by taking expectations we have

$$E(p_0) = (x_0^S - \mu_{A_0}) \cdot \beta + p_{-1}$$

and

$$\begin{aligned}
E(p_0^2) &= \\
&= E(p_{-1}^2 - 2\beta A_0 p_{-1} + 2p_{-1} x_0^P \beta + A_0^2 \beta^2 - 2A_0 u_0 \beta^2 + u_0^2 \beta^2 - 2A_0 x_0^P \beta^2 + 2u_0 x_0^P \beta^2 + (x_0^P)^2 \beta^2) \\
&= p_{-1}^2 - 2\beta \mu_{A_0} p_{-1} + 2p_{-1} x_0^P \beta + (\mu_{A_0}^2 + \sigma_{A_0}^2)\beta^2 + \sigma_L^2 \beta^2 - 2\mu_{A_0} x_0^P \beta^2 + (x_0^P)^2 \beta^2
\end{aligned}$$

By substituting $E(p_0)$ and $E(p_0^2)$ into equation (40), and then by taking first order conditions, we know that

$$x_0^P = 0$$

which concludes the proof of the proposition.

By substituting the optimal order of the prey to equation (39), and then by taking expectations, we can calculate the value function of the predator at time $t = 0$, which is

$$V_0^P(p_{-1}, W_0^P, \mu_{A_0}, \sigma_{A_0}^2) = W_0^P + \frac{p_{-1}^2 - 2\mu_{A_0} p_{-1} \beta + \beta^2 (\mu_{A_0}^2 + \sigma_{A_0}^2 + \sigma_L^2)}{2\beta} \tag{41}$$

QED

6.5 Proof of Proposition 5

In proving this proposition we use a standard theory on Bayesian updating. Readers are referred to Brunnermeier (2001) page 12 for more details.

We know that the beliefs of the predator about the risky assets held by the prey are given by $A_{-1} \sim N(\mu_{A_{-1}}, \sigma^2_{A_{-1}})$. The signal observable by the predator is $S_{-1} = a_1 + b_1 A_{-1} + u_{-1}$. Therefore, the mean vector and variance covariance matrix can be rewritten as

$$\mu = \begin{bmatrix} \mu_{A_{-1}} \\ \mu_{S_{-1}} \end{bmatrix} = \begin{bmatrix} \mu_{A_{-1}} \\ a_1 + b_1\mu_{A_{-1}} + 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma^2_{A_{-1}} & b_1\sigma^2_{A_{-1}} \\ b_1\sigma^2_{A_{-1}} & b_1^2\sigma^2_{A_{-1}} + \sigma^2_u \end{bmatrix}$$

then by application of the projection theorem we have

$$(A_{-1}|S_{-1}) \sim N[\mu_{A_{-1}} + \frac{b_1\sigma^2_{A_{-1}}}{b_1^2\sigma^2_{A_{-1}} + \sigma^2_u}(S_{-1} - (a_1 + b_1\mu_{A_{-1}})), \sigma^2_{A_{-1}} - \frac{b_1^2\sigma^4_{A_{-1}}}{b_1^2\sigma^2_{A_{-1}} + \sigma^2_u}]$$

As the order of the prey is (conjectured) to be $x^S_{-1} = a_1 + b_1 A_{-1}$, the amount of risky assets in the prey's portfolio at time $t = 0$ is equal to $A_0 = A_{-1}(1 + b_1) + a_1$, and the beliefs of the predator are given by

$$\begin{aligned} (A_0|S_{-1}) \quad \sim \quad & N[a_1 + (1 + b_1) \cdot (\mu_{A_{-1}} + \frac{b_1\sigma^2_{A_{-1}}}{b_1^2\sigma^2_{A_{-1}} + \sigma^2_u}(S_{-1} - (a_1 + b_1\mu_{A_{-1}}))), \\ & (1 + b_1)^2 \cdot (\sigma^2_{A_{-1}} - \frac{b_1^2\sigma^4_{A_{-1}}}{b_1^2\sigma^2_{A_{-1}} + \sigma^2_u})] \end{aligned} \tag{42}$$

QED

6.6 Proof of Proposition 6

The problem of the predator at time $t = -1$ is

$$\max_{x^P_{-1}} E(V^P_0(p_{-1}, W^P_0, \mu_{A_0}, \sigma^2_{A_0})) \Leftrightarrow \max_{x^P_{-1}} E(W^P_0 + \frac{p^2_{-1} - 2\mu_{A_0}p_{-1}\beta + \beta^2(\mu^2_{A_0} + \sigma^2_{A_0} + \sigma^2_L)}{2\beta}) \tag{43}$$

where

$$
\begin{aligned}
E(W_0^P) &= W_{-1}^P - E(p_{-1})x_{-1}^P \\
p_{-1} &= p_{-2} + \beta(x_{-1}^S + x_{-1}^P + u_{-1})
\end{aligned}
$$

and μ_{A_0} and $\sigma^2_{A_0}$ result from the Bayesian updating outlined in Proposition 5.

We can rewrite equation (43) as

$$
\max_{x_{-1}^P} E(W_0^P + \frac{p_{-1}^2 - 2\mu_{A_0}p_{-1}\beta}{2\beta}) + E(\frac{\beta}{2}(\mu_{A_0}^2 + \sigma_{A_0}^2 + \sigma_L^2)) \tag{44}
$$

and observe (given that equation (42) of Proposition 5 does not depend on the behaviour of the predator) that the second part of this equation will not depend on the submitted order of the predator x_{-1}^P, so the problem of the predator is equivalent to

$$
\max_{x_{-1}^P} W_{-1}^P - E(p_{-1})x_{-1}^P + \frac{1}{2\beta}E(p_{-1}^2) - E(\mu_{A_0}p_{-1})
$$

By using the price process in the equation above we have

$$
E(p_{-1})x_{-1}^P = x_{-1}^P E(p_{-2} + \beta(a_1 + b_1 A_{-1} + x_{-1}^P + u_{-1})) = x_{-1}^P(p_{-2} + \beta(a_1 + b_1\mu_{A_{-1}} + x_{-1}^P))
$$

$$
\begin{aligned}
E(p_{-1}^2) &= E((p_{-2} + \beta(a_1 + b_1 A_{-1} + x_{-1}^P + u_{-1}))^2) = \\
&= (p_{-2})^2 + 2p_{-2}(a_1 + b_1\mu_{A_{-1}} + x_{-1}^P)\beta \\
&\quad +\beta^2(a_1^2 + b_1^2(\mu_{A_{-1}}^2 + \sigma_{A_{-1}}^2) + 2b_1\mu_{A_{-1}}^2 x_{-1}^P + (x_{-1}^P)^2 + 2a_1(b_1\mu_{A_{-1}} + x_{-1}^P) + \sigma_u^2)
\end{aligned}
$$

$$
\begin{aligned}
E(\mu_{A_0}p_{-1}) &= E[(a_1 + (1+b_1)(\mu_{A_{-1}} + \frac{b_1\sigma_{A_{-1}}^2}{b_1^2\sigma_{A_{-1}}^2 + \sigma_u^2}(b_1(A_{-1} - b_1\mu_{A_{-1}}))))p_{-1}] = \\
&= ((1+b_1)\mu_{A_{-1}} + a_1)(p_{-2} + (a_1 + b_1\mu_{A_{-1}} + x_{-1}^P)
\end{aligned}
$$

Putting the above back into the value function, and taking the first order conditions with respect to x_{-1}^P, yields

$$
x_{-1}^P = -a_1 - (1+b)\mu_{A_{-1}}
$$

In addition to this we have (after some tedious algebra)

$$E(\mu_{A_0}^2) = (a_1 + (1+b_1)\mu_{A_{-1}})^2 + \frac{b_1^4(1+b_1)^2\sigma_{A_{-1}}^4}{b_1^2\sigma_{A_{-1}}^2 + \sigma_u^2} - \frac{b_1^4(b_1-1)(1+b_1)^3\sigma_{A_{-1}}^4}{(b_1^2\sigma_{A_{-1}}^2 + \sigma_u^2)^2}$$

and

$$E(\sigma_{A_0}^2) = (1+b_1)^2(\sigma_{A_{-1}}^2 - \frac{b_1^2\sigma_{A_{-1}}^4}{b_1^2\sigma_{A_{-1}}^2 + \sigma_u^2})$$

Then, by substituting the above into equation (44) and inserting an optimal order of the predator, we have the value function of the predator at time $t = -1$

$$V_{-1}^P(p_{-2}, W_{-1}^P, \mu_{A_{-1}}, \sigma_{A_{-1}}^2) = W_{-1}^P - \mu_{A_{-1}}p_{-2} + \frac{p_{-2}^2}{2\beta} + U_{-1}(\mu_{A_{-1}}, \sigma_{A_{-1}}^2)$$

where $U_{-1}(,)$ is the function (introduced to clear the notation) that depends on $\mu_{A_{-1}}$ and $\sigma_{A_{-1}}^2$ only.

QED

6.7 Proof of Proposition 7

The problem of the predator at time $t = -2$ is

$$\max_{x_{-2}^P} E\left[W_{-1}^P - \mu_{A_{-1}}p_{-2} + \frac{p_{-2}^2}{2\beta} + U_{-1}(\mu_{A_{-1}}, \sigma_{A_{-1}}^2)\right] \tag{45}$$

where

$$\begin{aligned} E(W_{-1}^P) &= W_{-2}^P - E(p_{-2})x_{-2}^P \\ p_{-2} &= p_{-3} + \beta(x_{-2}^S + x_{-2}^P + u_{-2}) \end{aligned}$$

and $\mu_{A_{-1}}$ and $\sigma_{A_{-1}}^2$ are the results of the Bayesian updating outlined in Proposition 5.

We can rewrite equation (45) as

$$\max_{x_{-2}^P} E\left(W_{-1}^P - \mu_{A_{-1}}p_{-2} + \frac{p_{-2}^2}{2\beta}\right) + E\left(U_{-1}(\mu_{A_{-1}}, \sigma_{A_{-1}}^2)\right) \tag{46}$$

and observe (given that equation (42) of Proposition 5 does not depend on the behaviour of the predator) that the second part of this equation will not depend on the submitted order of the predator x^P_{-1}, so the problem of the predator is equivalent to

$$\max_{x^P_{-2}} W^P_{-2} - E(p_{-2})x^P_{-2} - E(\mu_{A_{-1}} p_{-2}) + \frac{1}{2\beta} E(p^2_{-2})$$

By using the same arguments as in Proposition 6 we have

$$E(p_{-2})x^P_{-2} = x^P_{-2}(p_{-3} + \beta(a_2 + b_2\mu_{A_{-2}} + x^P_{-2}))$$

$$\begin{aligned} E(p^2_{-2}) &= (p_{-3})^2 + 2p_{-3}(a_2 + b_2\mu_{A_{-2}} + x^P_{-2})\beta \\ &+\beta^2(a_2^2 + b_2^2(\mu^2_{A_{-2}} + \sigma^2_{A_{-2}}) + 2b_2\mu^2_{A_{-2}} x^P_{-2} + (x^P_{-2})^2 + 2a_2(b_2\mu_{A_{-2}} + x^P_{-2}) + \sigma^2_u) \end{aligned}$$

$$E(\mu_{A_{-1}} p_{-2}) = ((1+b_1)\mu_{A_{-2}} + a_2)(p_{-3} + (a_2 + b_2\mu_{A_{-2}} + x^P_{-2})$$

Putting the above back into the value function, and taking the first order conditions with respect to x^P_{-2}, yields (after some tedious algebra)

$$x^P_{-2} = -a_2 - (1+b)\mu_{A_{-2}}$$

In addition to this we have

$$E(\mu^2_{A_{-1}}) = (a_2 + (1+b_2)\mu_{A_{-2}})^2 + \frac{b_2^4(1+b_2)^2\sigma^4_{A_{-2}}}{b_2^2\sigma^2_{A_{-2}} + \sigma^2_u} - \frac{b_2^4(b_2-1)(1+b_2)^3\sigma^4_{A_{-2}}}{(b_2^2\sigma^2_{A_{-2}} + \sigma^2_u)^2}$$

and

$$E(\sigma^2_{A_{-1}}) = (1+b_2)^2(\sigma^2_{A_{-2}} - \frac{b_2^2\sigma^4_{A_{-2}}}{b_2^2\sigma^2_{A_{-2}} + \sigma^2_u})$$

Then, by substituting the above into equation (46) and inserting an optimal order of the predator, we have the value function of the predator at time $t = -2$

$$V^P_{-2}(p_{-3}, W^P_{-2}, \mu_{A_{-2}}, \sigma^2_{A_{-2}}) = W^P_{-2} - \mu_{A_{-2}} p_{-3} + \frac{p^2_{-3}}{2\beta} + U_{-2}(\mu_{A_{-2}}, \sigma^2_{A_{-2}})$$

QED

6.8 Proof of Proposition 8

The programme of the prey at time $t = -2$ is

$$\max_{x_{-2}^S} E\left[W_{-2}^S - p_{-2} \cdot x_{-2}^S + (A_{-2} + x_{-2}^S)p_{-2} - \beta\frac{3}{4}(A_{-2} + x_{-2}^S)^2 + \beta\frac{1}{2}\mu_{A_{-1}}(A_{-2} + x_{-2}^S)\right] \qquad (47)$$

given the dynamics of prices

$$p_{-2} = p_{-3} + \beta(x_{-2}^S + x_{-2}^P + u_{-2})$$

and the dynamics of the beliefs of the predator

$$a_2 + (1 + b_2) \cdot (\mu_{A_{-2}} + \frac{b_2\sigma_{A_{-2}}^2}{b_2^2\sigma_{A_{-2}}^2 + \sigma_u^2}(x_{-2}^S + u_{-2} - (a_2 + b_2\mu_{A_{-2}})))$$

By substituting everything back into equation (47) and by taking expectations we have (after simplification)

$$\max_{x_{-2}^S} A_{-2}p_{-2} + W_{-2}^S - \frac{1}{4}(A_{-2} + x_{-2}^S)\beta\left(3(A_{-2} + x_{-2}^S) + 2(a_2 + \frac{(1 + b_2)(b_2\sigma_{A_{-2}}^2(x_{-2}^S - a_2) + \mu_{A_{-2}}\sigma_u^2}{b_2^2\sigma_{A_{-2}}^2 + \sigma_u^2}\right)$$

Taking first order conditions of the equation above yields (after simplification) we have

$$x_{-2}^S = \frac{b_2(a_2 - A_{-2}(1 + 2b_2))\sigma_{A_{-2}}^2 - (a_2 + A_{-2} + (1 + b_2)\mu_{A_{-2}})\sigma_u^2}{b_2(2 + 5b_2)\sigma_{A_{-2}}^2 + 3\sigma_u^2}$$

which after appropriate reshuffling of the terms yields

$$x_{-2}^S = \frac{a_2b_2\sigma_{A_{-2}}^2 - (a_2 + (1 + b_2)\mu_{A_{-2}})\sigma_u^2}{b_2(2 + 5b_2)\sigma_{A_{-2}}^2 + 3\sigma_u^2} + A_{-2}\left(-\frac{b_2(1 + 2b_2)\sigma_{A_{-2}}^2 + \sigma_u^2}{b_2(2 + 5b_2)\sigma_{A_{-2}}^2 + 3\sigma_u^2}\right)$$

QED

Part II
Dark Pools of Liquidity

1 Introduction

The past decade has been marked by significant changes in the set-up of financial markets. New terms such as dark pools of liquidity, algorithmic trading or flash orders are now helping to define new mechanisms that take an important role in the everyday execution of orders across global exchanges.

This recent revolution was made possible mainly because of constant improvements in IT and communication technologies that became cheap and reliable. Nowadays, almost every investment bank can afford to set up up its own trading platform, which can provide many functions of well-established global exchanges. Thanks to the current state of technology, these venues allow for the reliable processing of submitted orders and effective matching. One of the types of venues that has become particularly popular in recent years is the **dark pool of liquidity**. Also known under the name of 'Crossing Networks' (CN), dark pools of liquidity are defined by SEC as *"systems that allow participants to enter un-priced orders to buy and sell securities, these orders are crossed at a specified time at a price derived from another market"*. Based on research by Ian Domowitz and Yegerman (2008), as of 2008 over 40 independent dark pools operated in the US alone, and there were more than 60 dark pools worldwide in total. The growing number of venues is accompanied by the growing total volume of the trades executed within dark pools of liquidity. According to research by Tabb (2004), the overall volume matched in dark pools in 2003 was between 5-10% of the total volume, which is expected to increase to over 20% of the globally traded volume in 2011 (Tabb (2010)).

The emergence and growth of dark pools of liquidity was fuelled by RegNMS, a fairly new security regulation in the US which requires that every trading venue ensures the best possible execution. The regulation created common rules for various liquidity providers that helped to standardise the trading space. This standardisation and the need of institutional investors to hide their trading activity resulted in the growth of a number of trading venues and in an increase in the volume cleared by them.

The rapid expansion of these alternative trading venues would not be possible without another mechanism that defines today's marketplace – **algorithmic trading**, also known as 'algo trading' or 'automatic trading'. Algorithmic trading is a general name for methods involved in using computer programs to drive trading decisions or any aspects such as a quantity or timing. Although many algos differ significantly – some are designed for automatic execution, arbitrage or pure speculation – their common feature is that they require a limited amount of human intervention and are able to process enormous sets of data. It is estimated (Lati (July 2009)) that as of 2009 almost 73% of the US's equity trading volume was executed through algos. Algos designed for optimal execution are able to trade simultaneously in a number of different trading venues in order to gather all dispersed liquidity and then execute large block orders. These are the algos that made trading across dozens of venues possible, and therefore allowed for the increase in the number of dark pools.

In this paper I propose a tractable equilibrium model of the coexistence and competition for liquidity between dark pools of liquidity and regular (dealer-oriented) exchanges. In regular exchanges dealers present their firm bid and offer prices at which clients can trade. Clients executing their orders on the exchange through a specialist can be sure their order will be executed at a corresponding bid or ask price. However, as a specialist deals with both informational frictions (as in Glosten and Milgrom (1985)) and inventory-carrying costs (as in Garman (1976) or Stoll (1978)), investors need to pay some positive costs for trading with a competitive market-maker. Clients who are willing to avoid these costs can attempt to execute the trade in the dark pool, which would potentially allow them to avoid the need to pay the half-spread as a transaction cost[14]. Unfortunately, dark pools do not guarantee execution. Because of imbalances in the dark pool of liquidity, orders may not be executed, which makes an investor prone to the risk of a price change in subsequent periods. If he is very risk-averse, he may be better off trading through a dealer in the market.

The model I present analyses the balance between a safe execution through a dealer and low-cost, uncertain execution in the dark pool. I show how these two trading venues can coexist in a stable equilibrium. In comparison to currently available models, I endogenise both dealers' spread and the matching probability in the dark pool.

Before progressing with the model it is important to outline the most important practical features of dark pools. For a full analysis and systematic categorisation of dark pools, please refer to Mittal (2008). For the matter of this paper the relevant factors are as follows.

Pricing of the crosses – as written in the definition of the crossing network by SEC, *"orders are crossed at a specified time at a price derived from another market"*. This essentially means that the price at which the orders in the dark pool are crossed was established outside of this venue. Typically, the price at which the orders are cleared is the **mid** price from the regular exchange, which renders the execution/crossing price in the dark pool uninformative about any order imbalance within the pool. In order to avoid the possibility of manipulation, the time of sampling the reference mid-price for the crosses can be selected randomly. For example, it may be that the mid-price selected for the cross is sampled at a random time of +/- 5 minutes before or after the cross. As the crossing price is not an equilibrium price, some orders remain unmatched –

[14] Assuming that all orders are crossed in the dark pool at the mid-price in the market.

which leads to the second important feature of dark pools.

Allocation rule – in the case of an imbalance between the buy and sell orders in the pool, not all orders will be matched. Pools tend to differ in how they handle orders. Most common rules for the selection of orders for matching are: pro-rata (orders from the excess side of the market are matched pro-rata for all participants), time/volume preference (orders entered earlier have a higher probability of being matched) or purely random matching (orders from the excess side are selected randomly for matching).

Access to the dark pool – some pools limit the access to specific group of traders. One rule that is particularly commonly used gives access to the pool only to institutional investors with a certain historical volume of trades, while some other pools are only available to selected clients of the manager of the pool. The important point here is that access to the dark pools can be limited by the manager at his discretion.

Crossing Times – the majority of dark pools allow for a single cross every day. However, some pools provide semi-continuous crossing for incoming orders (at the price derived from the major exchange). In this paper, I focus on pools that provide the functionality of a single crossing during a day.

2 Literature review

In this paper I analyse interactions and competition between dealer markets and crossing networks. Dealer markets have been analysed extensively by Garman (1976), Stoll (1978), Amihud and Mendelson (1980), Ho and Stroll (1981), Amihud and Mendelson (1982), Ho and Stroll (1983), Mildenstein and Schleef (1983), Glosten and Milgrom (1985), Grossman and Miller (1988) and Biais (1993). The set-up of the dealer market I use in the model in this paper is related closely to the set-up presented by Glosten and Milgrom (1985). The presence of traders with superior information leads in my model to a positive bid-ask spread, even though the specialist is risk-neutral and makes zero expected profits – as such, the spread in the market is a consequence of adverse selection. Alternatively, it might be possible to analyse the model in which the bid-ask spread was induced by the inventory-carrying costs of a market-maker, as established in Ho and Stroll (1981).

Although the first papers explicitly analysing crossing networks are dated to the beginning of 2000s (e.g. Hendershott and Mendelson (2000)), some relevant contributions were presented much earlier. In particular, the stream of literature analysing the market impact of large block trades is particularly relevant for the analysis of dark pools of liquidity. Big institutional traders in many cases use crossing networks to execute large block orders and avoid market impacts. Easley and O'Hara (1987) investigate the effects of trade size on security prices and show that dealers' optimal pricing strategies must depend on trade size, with large trades being made at less favourable prices.

This is because adverse selection arises, as informed traders prefer to trade larger sizes if both small and large orders are priced uniformly. Dark pools of liquidity give large traders a chance to avoid this price discrimination. In this respect, crossing networks play a similar role to syndicated trades analysed by Burdett and O'Hara (1987) or the up-stair market analysed by Grossman (1992).

Competition between trading venues for order flow and issues of the market and order flow fragmentation were analysed in a number of papers by notable authors such as Chowdhry and Nanda (1991), Parlour and Seppi (2003) or more recently Rahi and Zigrand (2010). The basic conclusion from the literature is that an increase in competition between the exchanges can lead to a narrowing of bid-ask spreads, but order flow fragmentation can have the opposite effect and lead to increases in the bid-offer spread. The relationship between the strengths of these two opposing forces is crucial for the model presented in my paper. It is important to mention that specific features of dark pools of liquidity introduce new observations that expand on and extend the results of earlier studies. Dark pools of liquidity do not directly affect price discovery, and orders submitted to the pool are expected to have generally lower price impacts than those executed through conventional means [15].

The earliest influential model on competition between the dealer market and the dark pool was presented by Hendershott and Mendelson (2000). They directly model interactions between the crossing networks and the dealer market in a static set-up. A random number (drawn from geometric distribution) of uninformed and informed traders decide to submit orders to the dealer market in which the number of market-makers operate. All dealers are in Bertrand price competition, so any market-maker can capture any fraction of the market by offering the lowest possible spread. A trader's choice to trade depends on both trader-specific characteristics (like his valuation or impatience to trade) and on endogenous characteristics such as the probability of execution in the crossing network dealer's half spread etc. Four possible equilibria may exist in the model: an equilibrium with no trading at all, equilibria with trading only in the crossing network, an equilibrium with an exclusive dealer market trading and a combination of crossing networks (CNs), trading and dealer market (DM). The model I present in this paper is in many respects similar to that created by Hendershott and Mendelson (2000), although I refine and extend some of the results presented in their article. In particular, I add two period dynamics, the risk aversion of traders and I solve explicitly for matching probabilities in the dark pool.

The first influential dynamic model of competition between the regular markets and dark pools of liquidity was presented by Hans Degryse and Wuyts (2009). Authors adapt the model by

[15] The price impact of the orders submitted through the dark pool is, however, still non-zero. There is a second order effect. When a big institutional investor routes his order through the dark pool, and therefore decreases liquidity in the exchange, this action has an indirect impact on the price discovery process of the exchange, i.e. the price in the exchange can become more volatile because of the lower liquidity. Also in this situation the market volume starts to convey important information – lower volume can indicate that more orders are being executed outside of the exchange.

Parlour (1998) to study the dynamics of order flow to crossing networks and the dealer market. In the model, traders are assumed to arrive sequentially and randomly in the marketplace during one of two trading days. Each trader is characterised by his specific (and random) impatience to trade. Upon arrival the trader decides between submitting the order to the dark pool of liquidity or to the regular DM, where the spread is set exogenously to one tick. All trades in the crossing network are matched at the end of each trading day and at the mid-price derived from the DM (CN does not contribute to price discovery). Each trader at the moment of his decision as to whether he trades through the DM or CN observes the order book of the CN and can assess the probability of execution in the dark pool. Execution in the dark pool assumes time priority, so orders that are entered earlier have a higher probability of execution. Importantly, the optimal strategies of the traders are non-stationary and time-dependent – traders trading later have a lower probability of their orders being executed in the dark pool, so they are less keen to submit their orders to this venue. The main findings of the paper are as follows: increases in the DM's relative spread (which in this model is assumed to be exogenous) increase order flow to the dark pool, which should therefore be more successful in the markets, where the spreads are wide. In this set-up, traders with higher willingness to trade prefer to go through the DM, which provides them with instant execution. More patient traders choose to trade through the dark pool, which is cheaper in expectation. The model suggested by Hans Degryse and Wuyts (2009) has some similar features to the one I present in this paper. In particular, their model is dynamic, as it has multiple trading rounds. In addition, the probability of matching in the dark pool is endogenous. My model extends the set-up by providing the risk aversion of traders, and the impatience of traders is endogenous rather then exogenous – as in the model of Hans Degryse and Wuyts (2009). In addition to this, I assume the existence of informed traders, which allows me to endogenise the spread in the DM and allows me to study the informational impact of dark pools of liquidity.

Although there is quite a substantial number of empirical studies on the interaction of the trading systems (for a full overview, please see Biais, Glosten and Spatt (2005)), dark pools are relatively secretive entities, and so the low availability of data means a relative paucity of empirical studies. Jennifer Conrad and Wahal (2003) use proprietary data and examine institutional orders and trades filled by alternative electronic trading systems. Their dataset contains information about institutional trades between the first quarter of 1996 and the first quarter of 1998. These datasets allow one to distinguish between orders filled by external crossing systems, electronic communication networks (ECNs) and traditional brokers. They find that external crossing systems are used largely to execute orders in listed stocks, while ECNs concentrate on Nasdaq stocks. On average, broker-filled orders are also larger and have a longer duration and higher fill rates than orders executed by alternative trading systems. Controlling for variation in order characteristics, difficulty and endogeneity in the choice of trading venue, they find that realised execution costs are, however, substantially lower for external crossing systems and ECNs.

3 The Model

In this section I first describe the set-up of the model economy. I define assets available for trading, market participants and available trading venues. Later, I solve the optimisation problem of market participants and derive equilibrium in the economy.

One security is available for trading. This asset is a claim to a risky dividend that will be paid in the last period of the trading game. The value of the final risky dividend v is either $v_h > 0$ or -v_h, where a priori probability is $P(v = v_h) = 0.5$.

There are three periods. In period $t = 1$, all agents submit their orders either to the risk-neutral market-maker or to the dark pool. Orders that are submitted to the market-maker are cleared with certainty at the prices quoted by the market-maker. Orders submitted to the dark pool are attempted to be matched and (if matched) executed at the mid-price taken from the dealer market (price equal to the average bid-ask shown by the market-maker)[16]. If there is an imbalance between buy and sell orders in the dark pool, some orders are not executed and agents, in order to satisfy their exogenous liquidity needs, are forced to resubmit their orders to the market-maker in period $t = 2$. In the case of an imbalance between buy and sell orders, the agents for whom orders are not executed are chosen randomly[17]. In period $t = 3$ the risky asset pays off.

In the economy there are n uninformed agents and 1 informed agent. The uninformed agents, similar to classical noise traders, tend to have exogenous liquidity needs independent of the liquidity needs of other uninformed agents. An uninformed agent is required to buy a single unit of risky asset with probability $\frac{p_u}{2}$ or to sell it with probability $\frac{p_u}{2}$ and will not be required to trade with probability $1-p_u$. The liquidity need of the uninformed trader is random and exogenous. The only endogenous decision of the uninformed trader concerns the choice of trading venue to which his order is submitted in the first place. In this respect our specification of uninformed traders differs from the classical definition suggested by DeLong et al. (1990), where noise traders do not conduct any optimisation at all and are in fact only automated. We can think of our uninformed agents as traders executing their clients' orders – they are required by clients to buy or sell securities, but can choose the venue they use to execute the order. This is exactly how dealers manage their positions

[16] The price setting mechanism assumed can be thought of as a good approximation of the price-setting mechanisms implemented by dark pools of liquidity. In the majority of dark pools, additional mechanisms are used to avoid market manipulation. For example, the price at which all orders are crossed in the dark pool is equal to the average dealer market mid-price over a certain period before actual crossing. Alternatively, crossing prices can be set at a randomly selected time before or after actual crossing in the dark pool.

[17] For instance, if there is n buy orders and m sell orders, where $n > m$, $n - m$ buy orders will not be executed. The agents whose buy orders will not be executed will be chosen randomly – so every agent who submits a buy order has the same probability of having the order executed.

– they receive orders from clients and execute these orders in the market by using the appropriate trading venues. Uninformed traders are assumed to be risk-averse and therefore have preferences for certain executions over others if both can be conducted at the same price. We assume that each uninformed trader has the CARA utility function with the coefficient of absolute risk aversion γ. By choosing the trading venue in the first period an uninformed trader maximises the expected utility of his terminal wealth

$$\max_{v \in \{MM, DP\}} E(U(W))$$

where MM denotes trading (in the first period) through the market-maker and DP denotes trading through the dark pool of liquidity. In equilibrium, the risk-averse uninformed trader submits his order to the market-maker with probability q^* (q^* is endogenous). The trade is submitted to the dark pool with probability $(1 - q^*)$. In an alternative specification of the model we could suggest that proportion q^* of the market-maker goes to the MM – the results in this case would remain essentially unchanged.

An informed agent has full prior knowledge of the terminal dividend of the risky asset. For tractability purposes I assume that the informed agent is submitting only a single order to the dealer market. He is always buying in the market when he knows that the dividend will be positive and selling in the market and when he knows that the dividend will be negative. The set-up of the informed agent is therefore identical to that introduced in Glosten and Milgrom (1985). In addition, I assume, without the loss of generality, that the informed agent is allowed to trade only in the first period. The model can be easily extended to accommodate variable numbers of informed agents submitting their orders to the dealer market, and also their ability to trade in both periods.

In the model, I analyse the trading game with variable numbers of agents. Number n, which is an integer reflecting the size of the countable population of uninformed traders, is an important model parameter because it allows for an explicit analysis of how the number of trading agents entering the dark pool of liquidity affects the probability of order matching in this dark pool. Intuitively, if the number of uninformed agents (who independently submit buy and sell orders with equal probability) entering the dark pool of liquidity increases, the probability of having the order executed in this venue increases. In theory, if the number of uninformed agents entering the dark pool tended to infinity, the probability of having the order executed in the dark pool would tend to one[18] By increasing or decreasing n we are also able to check how changes in the composition of market participants (the ratio of informed to uninformed agents – 1 to n) affects market equilibrium.

The dealer market is represented by a single risk-neutral, competitive market-maker, who before

[18]It can be shown that if the number of uninformed agents in the dark pool tended to infinity the ratio between the number of traders wishing to sell and buy would tend to one by law of large number and the probabilistic independence.

seeing the orders of the agents commits to the bid and ask prices at which he will execute any order that may come to him. Uninformed agents have therefore (in equilibrium) full certainty about the price at which their orders will be executed in the first period $t = 1$ in the dealer market. In the second period the market-maker updates his beliefs about the final pay-off using the information about orders in the first period, and the spreads reflect all information the market-maker learned in this initial trading round. An important element of this set-up is that the market-maker does not internalise information within a single time period[19]. Alternatively, we could assume that the market-maker sets up bid and ask prices after he sees all the orders that have been submitted in the trading round. However, in this set-up the dealer market would not provide a certain price for the execution, which is empirically the most beneficial feature of the market, i.e. the ability to trade on firm prices quoted by the market-maker. We can rationalise our set-up – in which the market-maker commits to bid and ask prices before seeing the orders – by considering the maker-maker to be in fact a composition of a large number of market-makers, each of whom receives orders from informed or uninformed agents independently. Each of these market-makers can have a maximum one client in a single period – the bid and ask prices quoted by such a single entity would be equal to the prices quoted by the market-maker in our set-up. Each of these competitive market-makers will set a spread that will provide him with zero profits in expectation – the same condition as for the market-maker in our set-up. After the initial trading round, all market-makers process market-wide information about the observed orders and refine their beliefs about the true asset pay-off. These can be thought of as happening through the inter-dealer broker market.

3.1 The Market-Maker's Quoted Prices

In the first period the bid and ask prices quoted by the market-maker are given by

Proposition 9 *Given the conjectured equilibrium-strategy profile* (q^*) *the* BID *and* ASK *prices quoted in the first period are given by:*

$$\begin{aligned} P_{ASK1} &= v_h \cdot (\frac{1}{np_u q^* + 1}) \\ P_{BID1} &= -v_h \cdot (\frac{1}{np_u q^* + 1}) \end{aligned} \tag{48}$$

where q^* *is an equilibrium (endogenous) probability of a market user choosing to trade through the market-maker, while* p_u *is the probability that a given market-user will be required to trade.*

Proof. In appendix ■

[19] The market-maker is not able to update his spreads as new orders are coming in, so we assume that he sets the spread and all orders appear at the same time.

Obviously the spread of the market-maker's quote widens if the order flow becomes dominated by the informed agent (q or p_u decreases). If q is low, the probability of trading through the market-maker by the uninformed agents is low, so the spread quoted by the market-maker is relatively wide. Our market-maker in this case is expecting that most likely he will be trading with the informed agent, and a wide spread reflects information cost. The important point here is that from a single period perspective a decrease in q^* decreases the benefits of trading in the dealer market – spreads that effectively represent the costs of trading through the dealer market increase. There is, however, yet another multi-period implication of variation in q. Changes in value q lead not only to a widening/narrowing spread in the current period, but also have an impact on the ability of the market-maker to learn about the true asset value, which affects quoted prices in the following periods. The higher q, the less likely the market-maker is to learn about the true asset value after he sees all orders in the first period. This effect will be very important for the model, so I will be discussing it in detail.

After the first trading round the market-maker observes the realised orders $\{k_{-1}, k_1\}$, where k_{-1} is the number of sell orders and k_1 the number of buy orders submitted to him in period $t = 1$. Using this information he updates beliefs about the true asset's value (which is either v_h or -v_h). We know that the price quoted by the market-maker in the second period will be equal to[20]:

$$\begin{aligned} P_2 &= E(v|\{k_{-1}, k_1\}) = \\ &= v_h P(v = v_h|\{k_{-1}, k_1\}) - v_h P(v = -v_h|\{k_{-1}, k_1\}) \end{aligned}$$

where

$$\begin{aligned} P(v &= v_h|\{k_{-1}, k_1\}) \\ &= \frac{P(\{k_{-1}, k_1\}|v = v_h)P(v = v_h)}{P(\{k_{-1}, k_1\}, \text{buy}|v = v_h)P(v = v_h) + P(\{k_{-1}, k_1\}, \text{buy}|v = -v_h)P(v = -v_h)} \end{aligned}$$

If k_{-1} and k_1 are both greater than zero we have that

$$P(\{k_{-1}, k_1\}, \text{buy}|v = v_h) = (\frac{p_u}{2}q^*)^{k_1 - 1}(\frac{p_u}{2}q^*)^{k_{-1}}$$

$$P(\{k_{-1}, k_1\}, \text{buy}|v = -v_h) = (\frac{p_u}{2}q^*)^{k_1}(\frac{p_u}{2}q^*)^{k_{-1} - 1}$$

so it can be easily seen that

$$P_2 = E(v|\{k_{-1}, k_1\}) = 0$$

[20] Please bear in mind that in the current set-up of the model only uninformed agents are trading in the second period, so there will be no bid-ask spread.

If $k_{-1} = 0$ then

$$P_2 = v_h$$

If $k_1 = 0$ then

$$P_2 = -v_h$$

We therefore have (given the equilibrium probability q^*) the probability of both $k_{-1}$and k_1 being greater than zero equal to

$$P(k_{-1} > 0, k_1 > 0) = P(v = -v_h)P(k_1 > 0|v = -v_h) + P(v = v_h)P(k_{-1} > 0|v = v_h)$$

where we have

$$P(k_1 > 0|v = -v_h) = P(k_{-1} > 0|v = v_h) = 1 - (1 - \frac{p_u}{2}q^*)^n$$

so

$$P(k_{-1} > 0, k_1 > 0) = 1 - (1 - \frac{p_u}{2}q^*)^n \tag{49}$$

Equation (49) gives the probability of the event that the true value will not be revealed after the first trading round. The true value will be revealed if the market-maker observes that all orders are of the same sign, which would mean that the informed agent must have submitted the order that was observed (I assumed that the informed agent is always sending an order to MM and that this order is consistent with the agent's knowledge) and the true value of the dividend can be then easily deduced.

Calculated probability in equation (49) is a decreasing function of q and an increasing function of n (if $q \cdot p_u > 0$). This result implies that an increase in the probability of the uninformed agent going to the MM decreases the probability of the information about the true dividend being fully revealed in the first period. As I will show later, this actually increases the attractiveness of opportunistic trading in the dark pool. If information about the true dividend is not revealed for sure after the first trading round, an uninformed agent will prefer to trade in the dark pool. If his order is executed in the dark pool, he will save on the spread. If his order is not executed in the dark pool, he simply resubmits the order to the MM in period $t = 2$, where he likely receives a price not much different to the price in the first period. Conversely, if it is very likely that information on the true dividend is revealed in the first period, then trading through the dark pool becomes less attractive. Nonetheless, an uninformed agent can save on the bid-ask spread if he trades through the dark pool. However, if he is unlucky and his order is not executed in the dark pool, he risks trading in period $t = 2$ where the price from $t = 1$ perspective is very volatile (is either v_h or -v_h). As our agent is risk-averse he may prefer to submit his order to the market-maker in order to enjoy a certain execution.

Intuitively, the migration of orders from the dealer market to the dark pool by decreasing the liquidity increases empirical market volatility, which can act against uninformed agents if the dark pool fails to execute its orders. Therefore, by moving his order from the regular exchange to the dark pool of liquidity, an uninformed agent has impact on market liquidity, which increases the risks he may bear in period $t = 2$.

3.2 Matching in the dark pool

All orders submitted to the dark pool are considered for matching. However, as an imbalance between the buy and sell order can occur, some orders will not be executed and will be forwarded into the dealer market in period $t = 2$We assume that every order in the dark pool has a priori the same probability of being successfully executed, which is consistent with the mechanics of a number of dark pools of liquidity[21]. The operator of the dark pool does not favour any of the market participants.

Below, we calculate the probability of a match for an order that was submitted by one uninformed agent into the dark pool of liquidity. This probability is conditional on the conjectured probability q^*. Because of the discrete nature of the problem the resulting equations are the finite sums of probabilities.

First, consider the probability of matching a buy order (exactly the same reasoning applies to the sell order) if we have k agents already in the dark pool. If k_s of these orders are sell orders, then the probability of our buy order being executed in the dark pool is

$$P(k, k_s) = MIN\left[1, \frac{k_s}{k + 1 - k_s}\right]$$

The probability of having k_s sell orders in the pool of k orders is

$$\binom{k}{k_s} \cdot \left(\frac{1}{2}\right)^{k_s} \left(\frac{1}{2}\right)^{k-k_s} = \binom{k}{k_s} \cdot \left(\frac{1}{2}\right)^{k}$$

where the first part is a familiar binomial term k over k_s. Therefore, we find that the probability of having an order executed in the dark pool, when there are k agents in the dark pool (each submitting independent orders to the dark pool – either buy or sell with equal probability) is

$$P(k) = \sum_{i=0}^{k} \binom{k}{i} \cdot \left(\frac{1}{2}\right)^{k} \cdot MIN\left[1, \frac{i}{k + 1 - i}\right]$$

[21] Some dark pools give time priority – orders that are received earlier have a higher chance of being executed than orders that come in late. This set-up is analysed by Hans Degryse and Wuyts (2009).

It is fairly easy to show, by virtue of the argument presented earlier, that $P(k)$ goes to one when k goes to infinity.

Another complication, however, is that k also is a random variable on its own, depending on the value of equilibrium quantity q^*. Higher q^* implies that uninformed agents prefer to trade through the market-maker and k will be expected to be lower. Agents do not have certainty about how many uninformed agents will appear in the dark pool. In equilibrium, each uninformed agent will be trading through the dark pool of liquidity with probability $p_u(1-q^*)$, so if we have $n-1$ agents that could enter the dark pool (abstracting from the current agent), the probability of having s of them in the dark pool is

$$P(s \text{ agents in the dark pool}) = \binom{n-1}{s}(p_u(1-q^*))^s(1-p_u(1-q^*))^{n-1-s}$$

The probability of having an order executed in the dark pool is therefore

$$\Pr = \sum_{s=0}^{n-1}\left[\binom{n-1}{s}(p_u(1-q^*))^s(1-p_u(1-q^*))^{n-1-s}\cdot\left[\sum_{i=0}^{s}\binom{s}{i}\cdot\left(\frac{1}{2}\right)^s\cdot MIN\left[1,\frac{i}{s+1-i}\right]\right]\right] \tag{50}$$

It can be seen relatively easily that the probability of having an order executed in the dark pool is a decreasing function q. Higher q implies that more orders are routed to the market-maker and there are less orders in the dark pool, which essentially suggests that the probability of having a match in the dark pool is less likely. Another important observation is that the higher number of uninformed agents n leads to a higher matching probability in the dark pool. These two observations are illustrated in Figure 1 below.

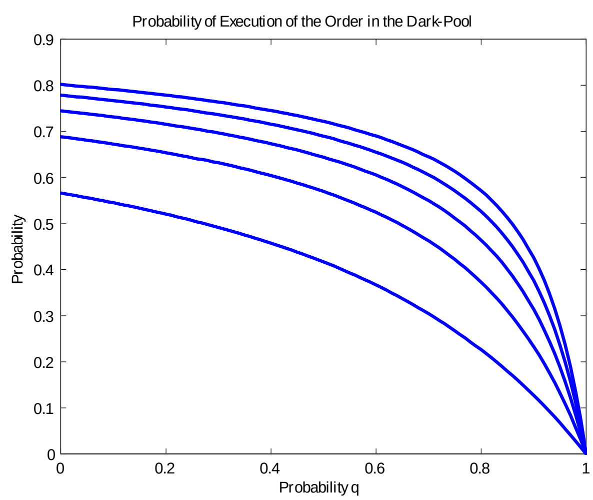


Figure 1: Matching probabilities in the dark pool depending on the number of uninformed agents

($n = 10, 20, 30, 40, 50$). For the same probability q higher n implies higher matching probability. Assumed $p_u = 0.32$.

3.3 The decision of an uninformed agent

An uninformed agent can decide on which trading venue will execute his orders. He can submit his order directly to the market-maker or to the dark pool in the first period. The advantage of executing the order through the market-maker is that there is certainty of execution. In equilibrium, the price at which an investor will clear his order with MM is known with certainty. However, bid and ask prices quoted by the risk-neutral market-maker are not equal – they entail a positive spread, which reflects the information cost borne by the market-maker. This cost can be avoided if the order is matched in the dark pool. If the uninformed trader is lucky, his order will be executed in the dark pool at a mid-price and he will However, if the order of the uninformed trader is not executed in the dark pool, it will be forwarded to the market-maker at time $t = 2$. Unfortunately, the price that will be quoted in period $t = 2$ is random from the perspective of the uninformed agent in $t = 1$. It is possible that trading in period $t = 1$ will be informative regarding the true asset pay-off, and the quoted price in $t = 2$ will be very different from the price the market-maker quoted in the $t = 2$. This shows the quintessence of the decision of the uninformed agent, as he needs to choose between the certain (but costly) execution at time $t = 1$ and the more risky, but potentially cheaper, option through the dark pool. Every single uninformed agent, by changing the probability of entering the dark pool, has an impact on the relative attractiveness of these two trading venues.

3.3.1 Execution of the order through the market-maker

If an uninformed trader submits his buy order (same argument for the sell order) to the market-maker, he knows that in equilibrium (when all agents submit their orders to the market-maker with probability q^*) his order will be executed at the price equal to

$$P_{ASK1} = v_h \cdot (\frac{1}{np_u q^* + 1})$$

so if he buys one unit of the asset from the market-maker his total wealth at the end of period $t = 3$ will be

$$W = -v_h \cdot (\frac{1}{np_u q^* + 1})$$

and the expected utility will be therefore equal to

$$E_{MM}(U(W)) = -\exp\left(-\gamma \cdot (-v_h \cdot (\frac{1}{np_u q^* + 1}))\right) \tag{51}$$

The CARA utility function has been chosen as an example of the parametrised utility function, which allows us to observe how the sets of equilibria are affected by the risk aversion of an individual agent. It is, however, worth mentioning that results from this paper are valid for any utility functions that display the risk aversion of individual agents.

From the above equation we see that the final realised dividend will have no impact on the expected utility of an uninformed agent. When an uninformed agent trades through the market-maker he satisfies his liquidity need before the true dividend is revealed. After these liquidity needs are satisfied through the dealer an uninformed agent is fully hedged and has no exposure to final risky dividend.

Execution through the market-maker allows for certain execution, but an uninformed investor needs to bear the cost of trading, as exemplified in the standard framework of Glosten and Milgrom (1985).

3.3.2 Execution in the dark pool of liquidity

If an uninformed trader decides to submit his order into the dark pool he faces the risk that it will not be executed and he will need to forward it to the market-maker in period $t = 2$. Uncertainty about the asset's true pay-off may be by then revealed, and the uninformed trader may risk trading at very unfavourable terms.

Essentially, three options are possible:

If the order of the uninformed agent is executed in the dark pool, he trades at zero price[22] and his terminal wealth is simply $\mathbf{0}$.

If the order of the uninformed agent is not executed in the dark pool, and asset uncertainty not resolved, he still trades at zero cost in period $t = 2$. Zero spread in this case is a direct result of the assumption that an informed trader places an order only in time $t = 1$. In the second period the spread is equal to zero, as there are no informational costs of trading.

The last and the most risky possibility from the point of view of an uninformed trader is when the order of the uninformed agent has not been executed in the dark pool and at the same time the uncertainty about the true asset pay-off has been fully resolved. In the second period an agent will be forced to trade at either price – v_h or $-v_h$ – which from the perspective of the first period will have an equal probability.

I summarise the expected utility for an uninformed trader that decides to go to the dark pool of liquidity through the following proposition.

Proposition 10 *Given the conjectured equilibrium-strategy profile* (q^*) *the expected utility of trad-*

[22] In the assumed set-up of the dark pool the execution price in DP is calculated as the average of the ask and bid price observed in the dealer market, which is according to equation (48) equal to zero.

ing through the dark pool by the uninformed trader is given by

$$\begin{aligned} E_{DP}(U(W)) &= \Pr\cdot(-\exp(-\gamma\cdot 0)) + \\ &(1-\Pr)\cdot\Big[\Big(1-(1-\frac{p_u}{2}q^*)^{n-1}\Big)\cdot(-\exp(-\gamma\cdot 0)) + \\ &(1-\frac{p_u}{2}q^*)^{n-1}\cdot\frac{1}{2}\cdot(-\exp(-\gamma\cdot -v_h)-\exp(-\gamma\cdot +v_h))\Big] \end{aligned} \tag{52}$$

Proof. In appendix ■

Each line in equation (52) corresponds to a relevant scenario outlined above, and P is the probability of match (depending on q^*) as calculated in equation **(49)** on page 52.

From a partial equilibrium perspective, if P is equal to 1 and the execution in the dark pool is perfectly certain, the option of executing through the dark pool dominates the execution through the market-maker. If the execution in the dark pool becomes uncertain, the uninformed agent may start to favour certain executions in the dealer market depending on his risk aversion.

It is also worth mentioning that an investor with a linear utility function will always (weakly) prefer opportunistic trading through the dark pool. In expectation that the price of the asset in period $t = 2$ is equal to 0 and therefore the expected cost of trading is also zero, trading opportunistically through the dark pool of liquidity is (for risk neutral investor) always a better solution than trading through the dealer market, as it allows for saving on the informational cost charge, which is the spread quoted by the market-maker in the first trading round.

3.4 Equilibrium

In this subsection I recover the symmetric equilibria (pure and mixed) of the trading game defined in previous sections.

3.4.1 Definition of symmetric equilibria

I define the mixed strategy equilibrium of the economy as follows:

Definition 1 *The Symmetric Pure Strategy Bayesian equilibrium is a collection of* $P_{ASK1}(q^*, p_u, v_h, n)$, $P_{BID1}(q^*, p_u, v_h, n)$, $\Pr(q^*, p_u, n)$, $P_2(q^*, p_u, v_h, n, k_{-1}, k_1)$ *such that:*

1. A competitive market-maker sets the bid and ask prices P_{ASK1} *and* P_{BID1}*, so he makes zero profits in expectations while trading with a counterpart.*

2. All uninformed risk-averse agents trade optimally through the market-maker or through the dark pool of liquidity.

3. The informed agent trades through the dealer market in period $t = 1$ *only.*

4. The market price of the asset in period $t = 2$, defined as $P_2(q^*, p_u, v_h, n, k_{-1}, k_1)$, utilises (through Bayesian updating) all information available to the market-maker after trading in period $t = 1$.

Mixed strategy equilibria are defined as follows:

Definition 2 *The Symmetric Mixed Strategy Bayesian equilibrium of the economy is a collection of $q^*, P_{ASK1}(q^*, p_u, v_h, n)$, $P_{BID1}(q^*, p_u, v_h, n)$, $\Pr(q^*, p_u, n)$, $P_2(q^*, p_u, v_h, n, k_{-1}, k_1)$ such that:*

1. All uninformed agents submit their orders to the market-maker with probability q^* and to the dark pool of liquidity with probability $(1 - q^*)$.

2. A competitive market-maker sets the bid and ask prices P_{ASK1} and P_{BID1}, so he makes zero profits in expectations while trading with a counterpart.

3. Probability q^* maximises the utility for each uninformed risk-averse agent given prices quoted by the market-maker and the probability of execution in the dark pool $\Pr(q^*, p_u, n)$. For mixed strategy equilibria we understand that $E_{MM}(U(W(q^*))) = E_{DP}(U(W(q^*)))$

4. The informed agent trades through the dealer market in period $t = 1$ only.

5. The market price of the asset in period $t = 2$, defined as $P_2(q^*, p_u, v_h, n, k_{-1}, k_1)$, utilises (through Bayesian updating) all information available to the market-maker after trading in period $t = 1$..

Conditions 1 to 5 define conditions for the Bayesian Nash-Equilibria for the dynamic trading game. Condition 3 defines the optimality of the decision of each uninformed agent. Condition 4 sets the equilibrium condition for the action of the competitive market-maker. Condition 5 implies rational Bayesian updating.

3.4.2 Pure strategy equilibria

For any choice of initial parameters (n, γ, p_u, v_h) there is always one symmetric equilibrium in pure strategies $q^* = 0$ (I denote this equilibrium q^*_{DP}). In this case, all uninformed traders submit their orders only to the dark pool in the first place. If this happens we know that

$$\begin{aligned} P_{ASK1} &= v_h \cdot (\frac{1}{np_u 0 + 1}) = v_h \\ P_{BID1} &= -v_h \cdot (\frac{1}{np_u 0 + 1}) = -v_h \end{aligned}$$

so the spread quoted by the market-maker is the widest expected and the market-maker expects that he will be trading only with the informed agent. Therefore, no agent has any incentive to deviate from a $q^* = 0$ equilibrium strategy. It is possible that the uninformed agent's order submitted to the dark pool will be executed in the dark pool, and in the worst case (when the order is not matched in dark pool) the uninformed trader will be able to trade at no worse a price than that quoted by the market-maker in the first period.

The second potentially obvious candidate for the symmetric equilibrium is an equilibrium with $q^* = 1$ (I denote this equilibrium q^*_{MM}). This, however, is an equilibrium only for a specific choice of model parameters (n, γ, p_u, v_h). In the model an investor may prefer to submit his order to the dark pool, even though he knows definitely that this order will not be executed within DP (all other players will submit their orders to the market-maker in the first period and the dark pool will be empty). The trader knows, though, that there will be no informed trader in the second period, so the market-maker will be quoting prices without the spread. The possible risk of the single trader's decision to go to the dark pool is that it may happen that uncertainty about the asset value will be resolved in the first period and in the second period he may face a price which is significantly different from that quoted by the market-maker. If the risk aversion of the uninformed agent is sufficiently low, he may prefer to trade opportunistically through the dark pool of liquidity. Therefore, in general, a pure strategy equilibrium with $q^* = 1$ will exist only if investors are sufficiently risk-averse. In fact, we can provide an implicit condition for $q^* = 1$ to be an equilibrium. The condition for $q^* = 1$ to be the equilibrium is[23]

$$\begin{aligned} -\exp\left(-\gamma \cdot (-v_h \cdot (\frac{1}{np_u + 1}))\right) \quad &\geqslant \quad \left(1 - (1 - \frac{p_u}{2})^{n-1}\right) \cdot (-\exp(-\gamma \cdot 0)) \\ &\quad + (1 - \frac{p_u}{2})^{n-1} \cdot \frac{1}{2} \cdot (-\exp(-\gamma \cdot -v_h) - \exp(-\gamma \cdot +v_h)) \end{aligned} \tag{53}$$

If the coefficient of risk aversion is low and the above condition is not satisfied, then $q = 1$ is not an equilibrium. Agents prefer to deviate and go to the dark pool of liquidity instead of staying with the market-maker.

3.4.3 Mixed strategy symmetric equilibria

In addition to at least one pure-strategy symmetric equilibrium we may have equilibria in mixed strategies where $0 < q^* < 1$. Because of the discrete set-up of the model it is possible to find these values only numerically using a simple Newton algorithm – for given model parameters I am looking for q^* that would equalise uninformed traders' benefits of executing the order through the market-maker or submitting the order to the dark pool of liquidity.

The first mixed strategy equilibrium emerges when agents are risk-averse, but the condition from equation (53) is still not satisfied. In this case, as discussed, $q = 1$ cannot be an equilibrium

[23] Here, I just combine equations (51) and (52) and set matching probability Pr equal to zero (as there is no chance the order will be matched in the dark pool).

and agents consequently prefer to deviate and route their order to the dark pool. However, as q decreases and agents symmetrically submit the order to the dark pool of liquidity with higher probability, going to the dark pool becomes more risky. This is because now there are effectively less agents trading with the market-maker and there is a higher chance that the true asset value will be revealed in period $t = 2$, leading to high uncertainty about the market-maker's price in $t = 2$. By further decreasing probability q we will at some point equalise the benefits of going to the market-maker and to the dark pool. This would be a mixed strategy equilibrium with probability q_1^*. In Figure 2, this equilibrium occurs for $q_1^* = 0.63$. For this probability all agents are indifferent to entering the dark pool or dealing with the market-maker; therefore, this equilibrium can be sustained. If we now slightly decrease q (so we decrease the probability of going to the market-maker), we find that investors are beginning to prefer MM. This is because the probability of revealing the true asset value increases and the potential high swing of the price in period $t = 2$ poses a larger threat to the trader that submits his order to the dark pool of liquidity. An opposite effect happens when q probability is increased, as the threat of a highly uncertain price in the second period is lower and agents can benefit from execution in the dark pool of liquidity. The properties of these equilibria therefore make it robust in the sense of Selten (1983)'s trembling hand perfection. Another interesting observation about this equilibrium is that a decrease in the level of risk aversion (γ in the model) leads to a decrease in equilibrium probability q_1^*. If agents are less risk-averse they are less concerned with the risks of submitting orders through the dark pool of liquidity. This result can be observed when we compare Figure 2 and figure 3. A decrease in risk aversion leads to a decrease in the q_1^* probability.

The second mixed strategy equilibrium emerges for low levels of q. In the previous paragraph I claimed that for a certain level of q a decrease in q leads to an increase of attractiveness for the MM relative to DP. However, at some point if we decrease the value of q further, the MM starts to lose its allure because the costs of trading with the market-maker become punitively high (as very few uninformed agents trade with the market-maker, the spreads widen a lot), while the probability of executing an order through the dark pool increases. At some point the cost of trading through the market-maker is so high that a risk-averse agent becomes indifferent to DP and the MM. In Figure 1 such an equilibrium exits for $q^* = 0.32$.

For some values of the model parameters, mixed strategies may not exist. In particular, if agents are not greatly concerned with risk, they might prefer the dark pool for all values of q (figure 3), so both the first and the second equilibria described above would not exist (please remember that for risk-neutral investors there is only one equilibrium $q^* = 0$). For high levels of risk aversion, when condition (53) is satisfied, investors prefer the MM for most q values and only decide to join the dark pool when everyone else enters it with very high probability (figure 4). In this case the only mixed strategy equilibrium is of the q_2^* type.

3.4.4 Social welfare implications

In the model I present equilibria for which $0 < q^* < 1$ (equilibria in which uninformed agents trade also through the dark pool) are characterised by a worse level of expected utility for risk-averse agents than if all these agents traded through the market-maker only. This claim can be easily proven using a simple argument. For all mixed strategy equilibria we have that

$$E_{MM}(U(W(q^*))) = E_{DP}(U(W(q^*)))$$

what combined with the fact that

$$\frac{\partial}{\partial q} E_{MM}(U(W(q^*))) > 0$$

that can be deducted from equation (51) on page 55 implies that the agent's utility in equilibrium increases with q. It is therefore possible to rank the equilibria that I identified in subsection 3.4.3.

The gradation of utility can be seen in the upper left graph in Figure 2 – equilibria with the lower level of q^* provide lower level of utility to uninformed agents. As the utility of an informed agent also decreases with decreasing q (the spread quoted by the dealer increases), the introduction of the dark pool has a negative impact on overall social welfare. There are two intuitive explanations for this effect. First, the introduction of the dark pool of liquidity leads to a fragmentation of the marketplace and therefore decreases risk sharing in the dealer market. The competitive, risk-neutral market-maker prices the informational risk competitively and distributes its costs across all clients. If q^* is lower, the market-maker expects a lower order flow and therefore distributes the informational costs across a smaller number of informed agents, which leads to lower utility for each of these agents. The second important element that needs to be taken into consideration when the dark pool is introduced into the market is the fact that the it does not provide certainty of execution – some orders submitted to the dark pool may be not executed, which forces the uninformed agent to submit the order in period $t = 2$, when the price is volatile. By introducing the dark pool of liquidity, uninformed risk-averse investors need to bear the risk of uncertain execution and price volatility in period $t = 2$. In the market with a market-maker only, this risk is taken over by the risk-neutral market-maker.

As described in section 3.4.2, $q = 1$ is not necessarily a symmetric (pure strategy) equilibrium. In this case the introduction of the dark pool of liquidity leads necessarily to an overall loss of welfare, as investors in any equilibrium start to use it – $(q^* < 1)$.

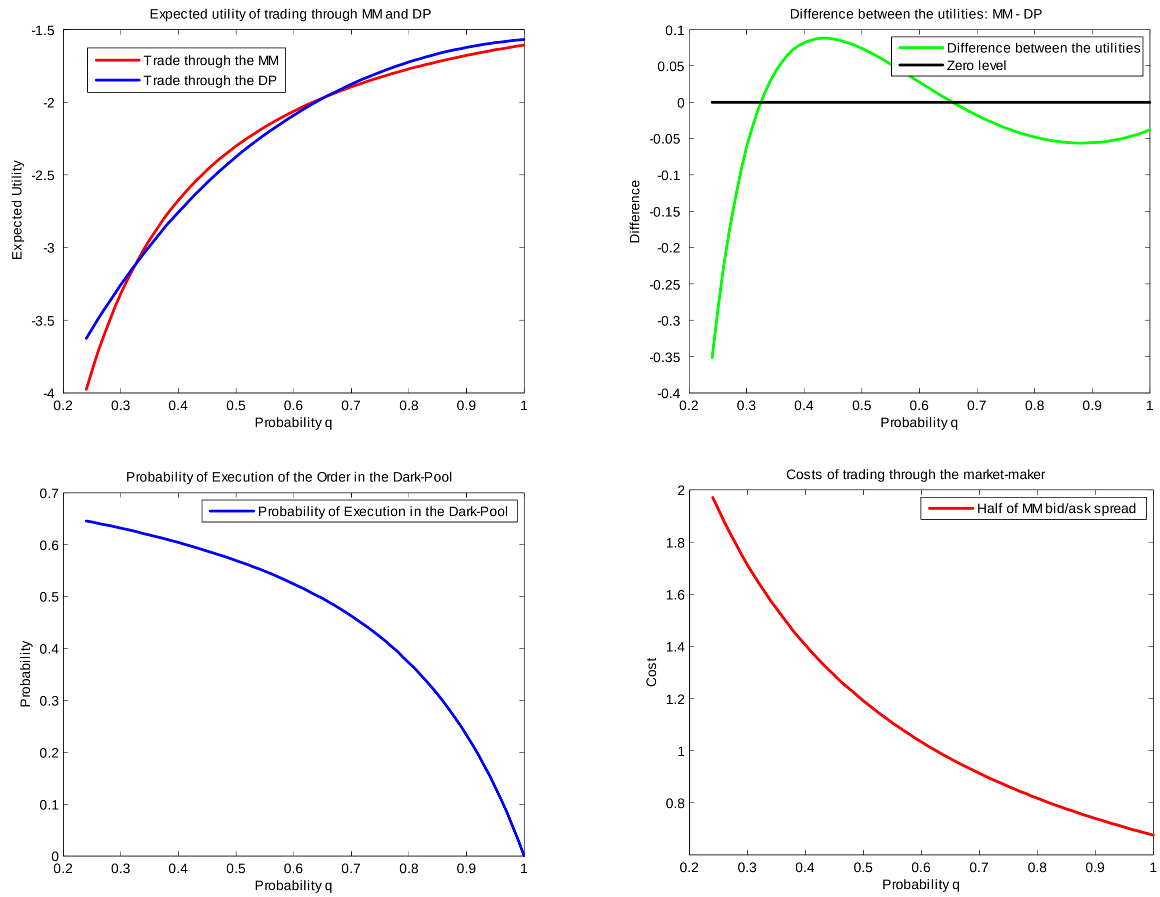


Figure 2: $p_u = 0.32, \boldsymbol{\gamma} = \mathbf{0.70}, n = 20, v_h = 5$

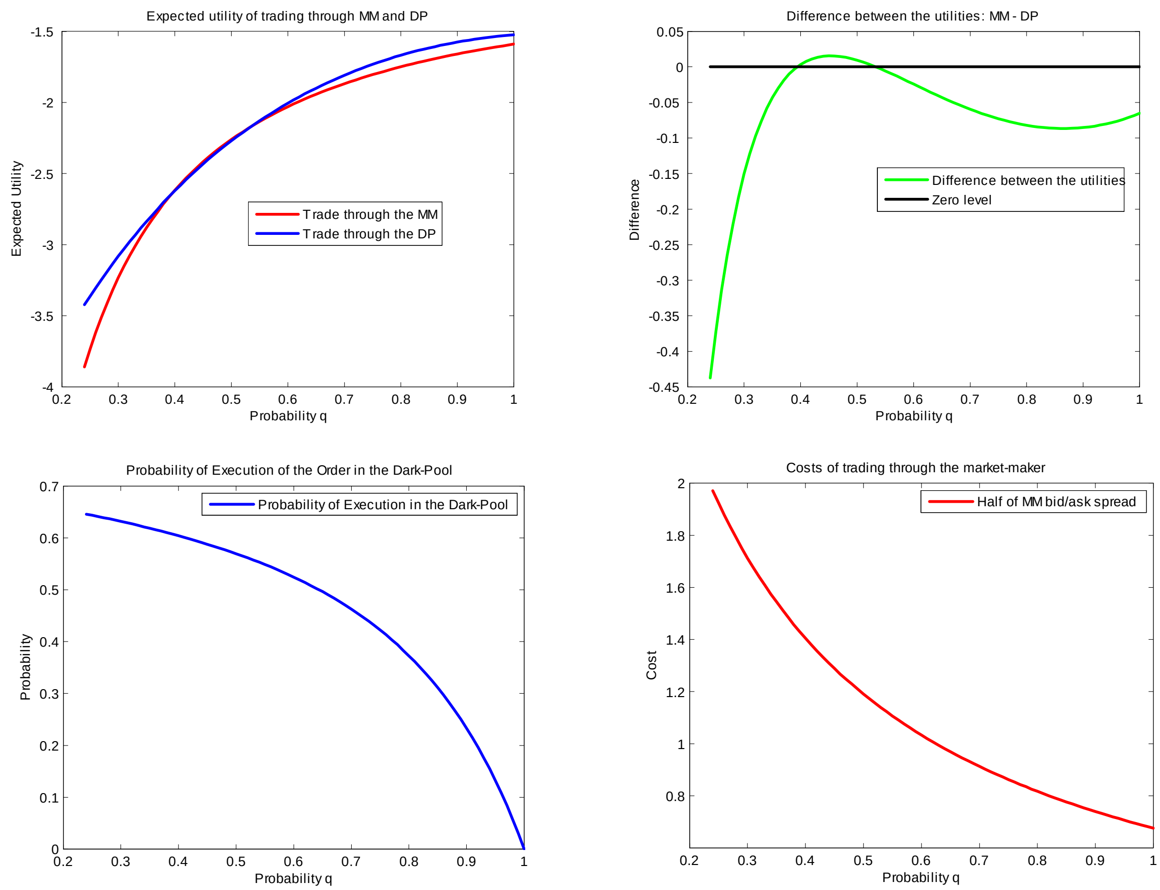
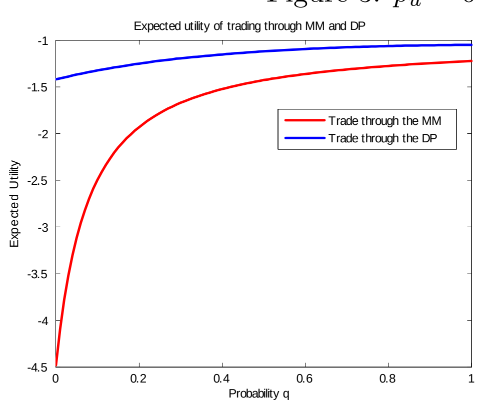


Figure 3: $p_u = 0.32, \boldsymbol{\gamma} = \mathbf{0.68}, n = 20, v_h = 5$



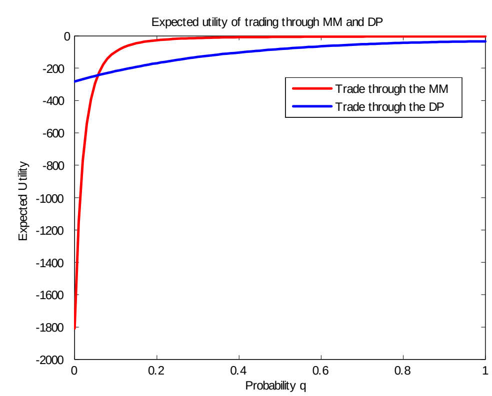


Figure 4 (left): $p_u = 0.32, \boldsymbol{\gamma} = \mathbf{0.32}, n = 20, v_h = 5$

Figure 5 (right): $p_u = 0.32, \boldsymbol{\gamma} = \mathbf{1.5}, n = 20, v_h = 5$

3.4.5 Existence of symmetric equilibria

Depending on certain model parameters, some equilibria mentioned in the previous subsections do not actually exist. In particular, the set of available equilibria depends on the level of agents' risk aversion $\boldsymbol{\gamma}$ and the probability of having an exogenous liquidity need by the uninformed trader p_u. A high level of risk aversion favours equilibria in which agents trade mainly through the market-maker. Lower risk aversion, on the other hand, supports equilibria in which uninformed agents trade through the dark pool of liquidity. Probability p_u has a similar impact on the existence of equilibria in the model. Furthermore, high p_u favours dark pool equilibria, while low p_u supports market-maker equilibria.

The impact of $\boldsymbol{\gamma}$ and p_u on the existence of equilibria is illustrated in Figure 6 below. The red region denotes these pairs of $\boldsymbol{\gamma}$ and p_u for which equilibria $q^*_{DP}, q^*_{MM}, q^*_2$ can be sustained in the model economy. In the blue region there is only one equilibrium: q^*_{DP}. In the region denoted by the green colour we have the following equilibria: q^*_{DP}, q^*_1, q^*_2.

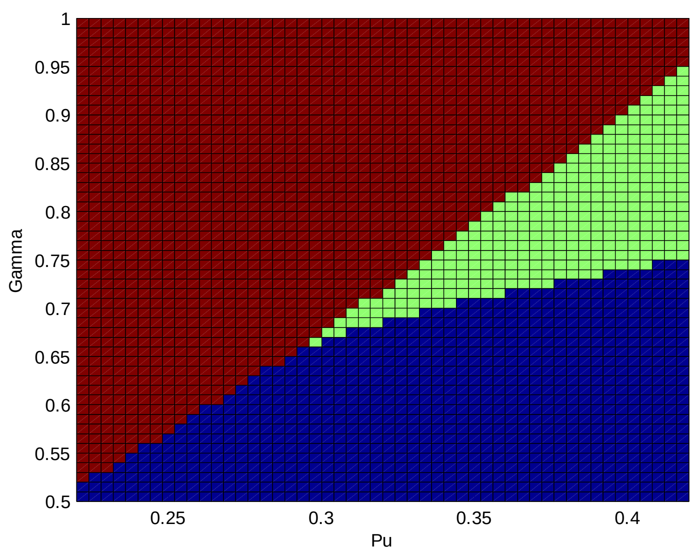


Figure 6. Equilibria for various γ and p_u

4 Concluding Remarks

This paper examines the implications of introducing an alternative trading venue into the market-place. In a presented model I show how dark pools of liquidity can coexist in equilibrium with the standard dealer market. During regular exchanges dealers present their firm bid and offer prices. A client executing an order on the exchange through a specialist can be sure his order will be executed at a corresponding bid or ask price. Investors willing to avoid paying a fee (spread) can decide to move the order to a dark pool, which is a cheaper – but riskier – way of executing orders. In my model the risk aversion of the investors is a force that drives the final market equilibrium. In a market where investors are very risk-averse, agents have a tendency to execute their orders through the market-maker. Conversely, low risk aversion favours services of the dark pool of liquidity.

My model can be compared to the model of Hendershott and Mendelson (2000). Similar to their model, I also have multiple equilibria. The set-up of my model is, however, significantly different in that I have two period dynamics, the agents in my model are risk-averse and the spread quoted by the market-maker is endogenous. Moreover, I solve explicitly for the matching probability in the dark pool – something that is assumed in the majority of papers to be exogenous.

An important observation is the fact that the decision of a market participant whether to go to a dark pool or execute the order through a market-maker has inter-temporal implications. If an investor decides to route his order to the dark pool, he increases the probability that the true value of the asset will be revealed in the current period and therefore increases expected volatility of the price in a subsequent period, which can have negative implications for him if his order is not executed in the dark pool of liquidity.

I find that equilibria in which investors execute their orders through the dark pool of liquidity lead in general to lower social welfare than equilibria in which all investors trade through the market-maker. The main reason for this is market segmentation and inefficient risk sharing.

5 Appendix to Part II

5.1 Proof of proposition 9

We find that

$$\begin{aligned} P_{ASK1} &= v_h \cdot P(v = v_h|\text{buy}) - v_h \cdot P(v = -v_h|\text{buy}) = \\ &= v_h \cdot (2P(v = v_h|\text{buy}) - 1) \end{aligned}$$

where

$$P(v = v_h|\text{buy}) = \frac{P(\text{buy}|v = v_h)P(v = v_h)}{P(\text{buy}|v = v_h)P(v = v_h) + P(\text{buy}|v = -v_h)P(v = -v_h)}$$

$$P(v = v_h|\text{buy}) = \frac{\frac{n}{n+1}\frac{p_u q^*}{2} + \frac{1}{n+1}}{\frac{n}{n+1}\frac{p_u q^*}{2} + \frac{1}{n+1} + \frac{n}{n+1}\frac{p_u q^*}{2}} = \frac{np_u q^* + 2}{2np_u q^* + 2}$$

so we have after substitution

$$P_{ASK1} = v_h \cdot (\frac{1}{np_u q^* + 1})$$

Calculations for the ask price are equivalent.

QED

5.2 Proof of proposition 10

For the uninformed trader submitting the order to the dark pool of liquidity there are three possibilities:

1. His order is executed in the dark pool. In this case his terminal wealth will be equal to zero and the utility will be

$$U(0) = -\exp(-\gamma \cdot 0)$$

which will happen in an equilibrium with probability $P(q^*)$..

2. With probability $1 - P(q^*)$ an order submitted to the dark pool will not be executed. In this case there are two further possible options. The true asset nature can be fully revealed after trading in the dealer market in $t = 1$, which will happen with probability[24]

$$(1 - \frac{p_u}{2}q^*)^{n-1}$$

[24] Please refer to the derivation of equation (49) on page **52**.

In this case the price at which the uninformed trader will execute his order will be either v_h or $-v_h$ with equal probability. The expected utility of such a scenario is

$$\frac{1}{2} \cdot (-\exp(-\gamma \cdot -v_h) - \exp(-\gamma \cdot +v_h))$$

If the nature of the asset is not fully revealed after trading in period $t = 1$, then the uninformed trader will trade at a price equal to zero.

Summing up all elements, we find that

$$\begin{aligned} E_{DP}(U(W)) &= P \cdot (-\exp(-\gamma \cdot 0)) + \\ &\quad (1-P) \cdot [\left(1 - (1 - \frac{p_u}{2} q^*)^{n-1}\right) \cdot (-\exp(-\gamma \cdot 0)) + \\ &\quad (1 - \frac{p_u}{2} q^*)^{n-1} \cdot \frac{1}{2} \cdot (-\exp(-\gamma \cdot -v_h) - \exp(-\gamma \cdot +v_h))] \end{aligned} \tag{54}$$

QED

Part III

Portfolio Problem with Infrequent Trading

1 Introduction

Traditional continuous time asset-pricing and asset allocation theory is based on the assumption that all assets are perfectly liquid and can be traded by economic agents continuously. In particular, it is assumed that market participants are allowed to rebalance their portfolios at any time. In reality, however, completing an asset transaction may require a significant amount of time. Economic agents may need time to coordinate (search for each other), agree on the price and then finalise the transaction, which for some asset classes can require a substantial amount of time. All of these delaying factors become even more severe in volatile markets, when the frequency of trading decreases and it may take longer to find a counter-party for a trade. Rational investors, when making their investment/consumption decisions, need to take into account any future effects of illiquidity.

Typical examples of illiquid assets are real estate, private equity and hedge fund investments. In the case of real estate and private equity investment, significant waiting time for the completion of a transaction is linked to the unique nature of underlying. For example, when an investor sells a house, he needs to find a buyer who is interested and will accept specific characteristics of the property such as its location, quality, etc. The process of finding a counter-party for such transactions can be long -winded and characterised by uncertainty. In reality it may take many months before the transaction in this asset class is completed. In addition, investors in private equity need to take into account the variable time it takes them to enter/exit the investments. Unwinding an investment in an unlisted company requires either finding a specialist buyer (who will manage the newly acquired company) or a long process of IPO. In addition, investors investing in hedge funds have limited ability to move their capital from the fund in which they have invested, as the majority of hedge funds require long-term capital commitment. Many money managers contractually allow themselves to restrict the outflow of funds at their discretion[25]. Rational investors into private equity or hedge funds, who foresee limited liquidity in their investment, require an additional premium that will compensate for this fact.

Issues surrounding optimal trading in illiquid markets have motivated an increasing amount of research in the past few years. This paper contributes to this research threefold. First, I analyse in detail the partial equilibrium optimal investment/consumption problem of an agent allowed to

[25] This restriction, known as "gating", was frequently used in the middle of the financial crisis in 2008. Hedge fund managers restricted redemptions from entering their funds, arguing that many assets of the funds were very illiquid and they wanted to avoid unwinding the position at a fire-sale price.

allocate his capital to illiquid assets. I clearly highlight the mechanisms and show explicitly how optimal portfolio choice depends on the level of illiquidity. Second, I propose an effective numerical algorithm to solve for the optimal policy. In order to find an answer to this degenerate elliptic boundary value problem I modify the algorithm proposed by Kushner (1968). Third, I present general equilibrium and welfare implications from earlier partial equilibrium results.

Different definitions of illiquidity can be found in the literature. For this paper, I assume that it is the inability to trade an asset continuously. In my model investors can trade the asset only at random moments, which are governed by a Poisson random process. When an investor has an opportunity to trade, he is doing so at the fundamental value of the underlying asset[26]. Any rebalancing of the position in the illiquid asset can be done only at random moments in time. Nonetheless, the investor is allowed to consume from his cash account continuously, and any shocks to his total wealth will be partially absorbed by a change in the rate of consumption from the cash account. Another definition of illiquidity that is sometimes considered in the literature assumes that it is the inability to trade at a fair price rather than the inability to trade at all. An investor facing this type of illiquidity is allowed to trade at any point in time, but he pays high transaction costs when making the transaction. In this set-up the illiquidity of the asset is simply reflected in the additional transaction cost an investor needs to pay to trade the asset. Probably the true nature of this phenomenon is somewhere in between whereby investors willing to close their position are required to pay significant transaction costs (are trading at fire-sale price) if they want to close the position instantly, or need to wait a random time for the opportunity to trade at the fair fundamental price.

2 Related Literature

This paper relates to several branches in the literature. First it is directly connected to classical literature on optimal investment/consumption in continuous time. Models presented in seminal papers by Merton (1969) and Merton (1971) most likely provided the first successful applications of mathematical optimal control of Markov processes in economics. This initial research was extended by Richard (1975), Kim and Omberg (1996), Fleming and Zariphopoulou (1991), Karatzas, Lehoczky and Shreve (1987) and others. The continuous time approach with stochastic volatility is based on the approach of Heston (1993). Models presented in this classical branch of literature allowed in general for the closed form solution of the underlying stochastic optimisation problem, which enabled their applications to appear in general equilibrium settings, as in the model by Merton (1973).

Investment/consumption problems with transaction costs were studied in detail by, amongst

[26] To be defined in detail later

others, Davis and A.R.Norman (1990), Dumas and Luciano (1989), Constantinides (1986), Vayanos (1998) (in general equilibrium settings) and Liu and Loewenstein (2002). Transaction fees decrease investors' allocations to assets, which are costly to trade. Models show that economic agents facing transaction costs prefer a more conservative allocation, where a bigger portion of their total wealth is devoted to the liquid asset, as it would otherwise be implied by a standard model such as that posited by Merton (1971). Importantly, however, the transaction cost needs to be considerable to observe large deviations from the baseline model. A framework with short-sell constraints was studied extensively by Cvitanic and Karatzas (1992) and Chabakauri (2009).

My paper also relates to the literature on search in financial markets by Darrell Duffie (2005), Vayanos and Wang (2007), Vayanos and Weill (2008) and Garleanu (2009). In these models agents are required to wait for a random time before their orders are executed, which leads to a deviation of equilibrium prices from the perfectly liquid case. An important point about this branch of literature is that it focuses on the general equilibrium implications of illiquidity and not on the portfolio selection of the individual agent. The majority of the results are obtained assuming that the law of large numbers holds, which allows one to disregard the second-order considerations of a single economic agent.

The most closely related papers to this paper are by Rogers and Zane (1998), Schwartz and Tebaldi (2006), Pham and Tankov (2008) and Longstaff (2009). Rogers and Zane (1998) analysed a very similar problem to the one I present in this paper. The solution presented relies on an linear approximation, which is approximately valid only for small levels of illiquidity, so any deviation from the standard result of Merton (1971) is relatively minor. Schwartz and Tebaldi (2006) showed a series expansion solution to a related and simplified[27] HJB problem. This paper is also closely linked to the literature on numerical methods for solving Stochastic Control Problems. In particular, I present an efficient numerical algorithm for solving the underlying HJB equation, which is a modification of Kushner (1968)'s numerical algorithm. This method was later refined by Fitzpatrick and Fleming (1991) and Chellathurai and Draviam (2007).

A general overview of the issues concerning illiquidity can be found in the work by Vayanos and Wang (2010), who review the current state of illiquidity theory. A good introduction to HJB equations and stochastic optimal control can be found in a book by Fleming and Soner (2006a).

[27] In their model there is an illiquid asset for which trading is not allowed until known a priori time T, when the asset needs to be consumed. Similar to my model an investor adjusts the rate of consumption depending on the value of the illiquid asset. Illiquidity in the model by Schwartz and Tebaldi (2006) does not have a Poisson nature as in the model I present.

3 Standard infinite horizon Merton problem

I first start by reviewing the original results of Merton (1969), in order to reference them later and show differences in the case of an illiquid asset. I first fix the probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $F = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. All stochastic processes involved in this paper are defined on the stochastic basis $(\Omega, \mathcal{F}, F, P)$.

An investor with a fixed time horizon of T has access to two assets. He can invest money in the bank at the deterministic short rate of interest $r$so he has access to a risk-free cash account growing deterministically as

$$dA_t = A_t r dt$$

The investor can also invest in a risky asset with price process S_t,, which is assumed to be governed by standard geometric Brownian motions

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where B_t is a Wiener process adapted to filtration F..

The objective of an investor is to maximise the expected utility of future consumption

$$V(W_0) = E\left[\int_0^{+\infty} u(c_t)dt\right]$$

for admissible control $\{c_t, a_t\}$, where c_t is the instantaneous rate of consumption and a_t the amount of risky asset the investor holds in his portfolio at time t . The Bellman equation for the Merton problem in differential form is given by

$$V(W_t) = \max_{c_t, a_t}\left(u(c_t)dt + E\left(\frac{1}{1+\rho dt}\cdot V(W_{t+dt})\right)\right) \tag{55}$$

which is subject to the law of motion in regard to an agent's wealth

$$dW_t = W_t\left(rdt + a_t(\mu - r)dt + a_t\sigma dB_t\right) - c_t dt$$

and non-negativity of wealth and consumption

$$c_t > 0, W_t > 0 \ a.s.\ P$$

Equation (55) may be rewritten in the following form:

$$0 = \max_{c_t, a_t} u(c_t)dt + E\left(V(W_{t+dt}) - V(W_t)\right) - \rho dt V(W_t)$$

which, after applying Ito lemma results in the corresponding HJB equation

$$0 = \max_{c_t, a_t} u(c_t) + V_W(W_t)W_t(r + a_t(\mu - r)) - V_W(W_t)c_t + \frac{1}{2}a_t^2 W_t^2 \sigma^2 V_{WW}(W_t) - \rho V(W_t)$$

First-order conditions for this problem are

$$u'(c_t) = V_W(W_t)$$

and

$$-\frac{V_W(W_t)(\mu - r)}{W_t \sigma^2 V_{WW}(W_t)} = a_t$$

For $u(c_t) = \frac{c_t^\gamma}{\gamma}$ we end up with the following highly non-linear PDE

$$\begin{aligned} 0 &= \frac{(V_W(W_t))^{\frac{\gamma}{\gamma-1}}}{\gamma} + V_W(W_t)W_t(r - \frac{V_W(W_t)(\mu - r)^2}{W_t \sigma^2 V_{WW}(W_t)}) - (V_W(W_t))^{\frac{\gamma}{\gamma-1}} \\ &+ \frac{1}{2}\left(\frac{V_W(W_t)(\mu - r)}{W_t \sigma^2 V_{WW}(W_t)}\right)^2 W_t^2 \sigma^2 V_{WW}(W_t) - \rho V(W_t) \end{aligned}$$

By conjecturing on the solution to the power utility form

$$V(W_t) = \frac{K W_t^\gamma}{\gamma}$$

and substituting it back into PDE we find

$$\begin{aligned} 0 &= \frac{\left(K W_t^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}}}{\gamma} + K W_t^{\gamma-1} W_t(r - \frac{K W_t^{\gamma-1}(\mu - r)^2}{W_t \sigma^2 (\gamma - 1) K W_t^{\gamma-2}}) - \left(K W_t^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}} \\ &+ \frac{1}{2}\left(\frac{K W_t^{\gamma-1}(\mu - r)}{W_t \sigma^2 (\gamma - 1) K W_t^{\gamma-2}}\right)^2 W_t^2 \sigma^2 (\gamma - 1) K W_t^{\gamma-2} + \rho \frac{K W_t^\gamma}{\gamma} \end{aligned}$$

which can be simplified to

$$0 = \frac{K^{\frac{1}{\gamma-1}}}{\gamma} + \left(r - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2(\gamma - 1)}\right) - K^{\frac{\gamma}{\gamma-1}} + \rho\frac{1}{\gamma}$$

From this point we can recover the value of K , which is equal to

$$K = \left(\frac{\rho - \gamma r}{(1 - \gamma)} - \frac{1}{2}\frac{\gamma(\mu - r)^2}{\sigma^2(1 - \gamma)^2}\right)^{\gamma-1}$$

while the value function can be presented as

$$V(W_t) = \left(\frac{\rho - \gamma r}{(1-\gamma)} - \frac{1}{2}\frac{\gamma(\mu - r)^2}{\sigma^2(1-\gamma)^2}\right)^{\gamma-1} \frac{W_t^\gamma}{\gamma} \tag{56}$$

The transversality condition holds (if $K > 0$), so we have a solution to the Merton problem. From FOCs we find that the allocation to risky asset is

$$a_t = \frac{(\mu - r)}{\sigma^2(1-\gamma)} \tag{57}$$

and the intensity of consumption

$$c_t = K^{\frac{1}{\gamma-1}} W_t \tag{58}$$

The results imply that an investor invests a constant fraction of his wealth into a risky asset. His rate of consumption is also a constant fraction of his total wealth. In this consumption/investment problem the optimal rate of consumption follows an Ito process whereby

$$dc_t = c_t\left(\left(r - K^{\frac{1}{\gamma-1}} + \frac{(\mu - r)^2}{\sigma^2(1-\gamma)}\right)dt + \frac{(\mu - r)}{\sigma(1-\gamma)}dB_t\right)$$

Important observations from this solution are as follows. The rate of consumption is a smooth process without jumps (this will not be the case for the case of illiquid assets). The only state variable is wealth W_t of the investor. As the portfolio composition can be adjusted instantly the only thing that matters for the investor is the total sum of the assets. In this instance, when one of the assets is illiquid there will be two state variables − the allocation to a liquid and an illiquid asset. Another important observation is the fact that the desired portfolio composition is constant through time, as described by equation (57). The desired composition of assets depends on the relative risk and reward offered by the risky asset. In particular, equation (57) can prescribe either a short ($(\mu - r) < 0$) or long position in the underlying risky asset. In the illiquid case the composition between assets is not constant through time. In particular, if the value of illiquid assets increases, an investor is not able to instantaneously adjust the level of his cash account to reflect the higher value of his total wealth. Instead, he needs to wait for the opportunity to rebalance his portfolio.

4 Merton problem with infrequently traded asset

The problem I would like to solve is the optimal consumption and investment that occurs when one of the investment assets is traded infrequently. The opportunity to trade the “illiquid asset" arises upon the occurrence of a Poisson event. Only when such an event occurs is the trader able to change (increase or decrease) his position in the illiquid asset. The investor can, however, consume continuously from the bank account between the times that he is allowed to trade in the illiquid asset. An important element in our set-up is that the investor observes the fair value of the illiquid asset continuously and not only at the times at which he is able to trade it. This is one of the differences between my model and that of Pham and Tankov (2008), who assume that the investor is only observing the value of the illiquid asset at the time he is allowed to trade it. Although the value of the risky asset is internally governed by a Brownian log-normal process, the investor is only observing discreet, randomly spaced valuations of this process. This difference has important implications for the solution. In my model, as an investor is able to observe the fair value of his risky asset at each point in time, the intensity of the consumption between random trading opportunities will be a function of the time from the last trading opportunity (as in Pham and Tankov (2008)), but also the fair value of illiquid underlying. I believe that this set-up is consistent with the nature of asset illiquidity; for example, let us consider an investor with his wealth divided between illiquid housing and an amount of liquid cash. If he observes that the housing market is suffering significant losses and fair values are going down, he will decrease his level of consumption in response to these changes. Consequently, I assume that even though the investor is not able to trade the asset instantly, he is able to estimate its fair value at each point in time.

Holdings of the liquid asset are denoted by L_t, while those of the illiquid asset will be denoted by I_t. An agent is able to trade illiquid assets only at random times $\{\tau_k\}$ governed by the Poisson process with intensity λ adapted to Poisson filtration $\Im$. Obviously, in this case we cannot treat total wealth $W_t = I_t + L_t$ as a state variable, as evident in Merton (1969). Now, there are two state variables – L_t and I_t – and the value function is a function of both their sum and ratio[28]. This can be understood intuitively when we compare two possible situations of an investor. It is possible that an investor has a very small fraction of his wealth held in a liquid asset, and even though he may have considerable total wealth he is not able to consume as much as he would like [29]; otherwise, he may zero-out all the money he has in his bank account and will “starve". The investor with the same level of total wealth, but with a more balanced split, will be better-off because he will be able to enjoy higher level of consumption, thus reflecting his higher level of wealth.

Formally, the optimisation problem of the investor is (we assume without loss of generality that, when the economy starts. the composition of the investor's portfolio is given):

[28] The value function is time homogeneous, as I consider the infinite horizon problem and the Poisson process as memoryless.

[29] I will shortly prove that in this set-up an investor is not allowed to borrow in excess of the wealth of his liquid assets.

$$V(L_t, I_t) = \max_{\{c,a\}} E \int_0^{+\infty} e^{-\rho t} u(c_t) dt \tag{59}$$

subject to a law of motion

$$\text{If } \ t \notin \{\tau_k\} \text{ then} \begin{cases} dL_t = (rL_t - c_t)dt \\ dI_t = \mu I dt + \sigma I dB_t \end{cases} \tag{60}$$

$$\text{If } \ t \in \{\tau_k\} \text{ then } L_t + I_t = L_{t-} + I_{t-} \text{ and } (L_t, I_t) = \underset{L_t+I_t=L_{t-}+I_{t-}}{\arg\max} V(L, I) \tag{61}$$

and non-negativity of the total wealth process[30]:

$$L_t + I_t > 0 \text{ a.s. } P$$

The first observation is that the process involving both L_t and I_t cannot be expected to be necessarily continuous. In fact, both are jump-diffusion processes, where the values of both L_t and I_t jump to optimum values at the times when an investor is given the opportunity to rebalance. At these moments an investor adjusts the ratio between the L_t and I_t to the desired optimum.

From the initial set-up of the problem we can show important observations about both L_t and I_t processes, which are summarised in the propositions below:

Proposition 11 *The agent always sets $L_t > 0$ (the agent never borrows).*

Proof. Non-negativity of the total wealth process implies that an investor can only choose controls $\{c_t, a_t\}$ such that $L_t + I_t > 0$ a.s. P. Let us now assume that the agent at time $t = t^*$ opened a short position (borrowed) in a liquid risk-less asset and $-L_{t^*} = \epsilon_{t^*}$, where $\epsilon_{t^*} > 0$. For any intensity λ let us denote a stopping time for the next Poisson event with τ^*. We find that for any g, $P(\tau^* > t^* + g) > 0$. We also find (from the properties of the geometric Brownian motion) $P(I_{t^*+g} < \epsilon_{t^*}) > 0$ for any g,, so we have $P(L_{t^*+g} + I_{t^*+g} < 0) > 0$,, which contradicts the non-negativity of the wealth condition. ■

The intuition behind Proposition 11 is as follows. An investor is not allowed to finance his investment in illiquid assets by borrowing in liquid assets, as he may be not able to liquidate the position in the illiquid asset quickly enough when the value of the illiquid asset falls. In this case an investor can end up in a position whereby his total wealth is negative, which will violate the non-negativity of the wealth condition and ultimately a transversality condition. In the standard case of Merton (1971) an investor can borrow to finance a risky asset (this will happen if $a_t > 1$

[30] This is a standard requirement used in the literature whereby an investor cannot have negative total wealth, as he may be not able to satisfy his future obligations.

in equation (57) on page 73. Nevertheless, this is only possible because of negative shocks to risky assets (i.e. the value of the risky asset going down), and as such an investor is able to decrease exposure to this asset instantly and therefore avoid the risk of bankruptcy. In an illiquid case this can turn out to be impossible with positive probability.

Proposition 12 *The agent never sets $I_t < 0$ (the agent never goes short in the illiquid asset).*

Proof. The proof of this proposition is similar to that of the previous proposition. Non-negativity of the total wealth process implies that an investor can only choose controls $\{c_t, a_t\}$ such that $L_t + I_t > 0$ a.s. P. Let us now assume that the agent at time $t = t^*$ opened a short position in the illiquid asset and $-I_{t^*} = \epsilon_{t^*}$, where $\epsilon_{t^*} > 0$. Without losing generality we assume that holding the liquid asset at time t^* is L_{t^*}. For any intensity λ let us denote a stopping time for the next Poisson event with τ^*. We find that for any g, $P(\tau^* > t^* + g) > 0$. From the equation relating to the dynamics of liquid wealth (60) we find that[31] $L_{t^*+g} \leqslant L_{t^*}e^{rg}$. We also find (from the properties of geometric Brownian motion) $P(-I_{t^*+g} > L_{t^*}e^{rg}) > 0$ for any g, so we have $P(L_{t^*+g} + I_{t^*+g} < 0) > 0$,, which contradicts the non-negativity of the wealth condition. ∎

The intuition behind this proposition is straightforward. An investor in our set-up will never , as his potential losses on this position are uncapped. If an investor makes a long-only investment into an illiquid asset the maximum he may lose before he is able to re-trade the assets is equivalent to I_t – the current allocation to the illiquid asset. However, in the case of the short position an investor may suffer losses, which are in excess of his total wealth when the value of the asset he is shorting rises significantly.

An important conclusion from Proposition 11 and Proposition 12 is that exposure to an illiquid asset equates to $a_t \in [0, 1)$. In particular, we cannot expect the closed form equation for a_t to be of the same form as the standard problem of Merton (1969), which I presented in equation (57) on page 73.

Proposition 13 *(Homothetic property of the value function) For $u(c) = c^\gamma/\gamma$ the value function*[32] *is homogeneous to degree γ in I_t and L_t, $V(\pi I_t, \pi L_t) = \pi^\gamma V(I_t, L_t)$. For $u(c) = \ln(c)$ we find $V(\rho I_t, \rho L_t) = \frac{1}{\rho}\ln(\pi)V(I_t, L_t)$.*

Proof. This property of the value function can be easily established from the definition. First, let us denote with $\Theta(x, y)$ the class of admissible policies starting at (L_t, I_t). From equations (60) and (61) we see that

$$\Theta(\pi x, \pi y) = \{(\pi c, \pi a) : (c, a) \in \Theta(x, y)\}$$

[31] With equality only for the infeasible case of $c_t = 0$ for $t \in (t^*, t^* + g)$.
[32] The set of felicity functions that I consider in this paper is HARA.

By combining this observation with equation (59) we see that

$$V(\pi x, \pi y) = \max_{\Theta(\pi x, \pi y)} E \int_0^{+\infty} e^{-\rho t} u(c_t) dt = \max_{\Theta(x,y)} E \int_0^{+\infty} e^{-\rho t} u(\pi c_t) dt = V$$

where V is equal to $\pi^\gamma V(x,y)$ for $u(c) = c^\gamma/\gamma$ (as $u(\pi c) = \pi^\gamma u(c)$) and V is equal to $V(x,y) + \ln(\pi)/\rho$ for $u(c) = \ln(c)$ (as $\ln(\pi c) = \ln \pi + \ln c$). ■

The property stated by Proposition 13 is instrumental in helping to show the validity of Proposition 14, which is crucial for an effective numerical solution to the investor's problem.

Proposition 14 *The optimal solution to the problem* $\max\limits_{L+I=w} V(L,I)$ *always gives the same ratio* $\frac{I^*}{L^*}$ *for any* $w > 0$.

Proof. Proof follows directly from the homogeneity of $V(L,I)$. The first order conditions for optimal solution (L^*, I^*) are $V_L(L^*, I^*) = V_I(L^*, I^*)$. However, homogeneity implies that $(L^*/w, I^*/w)$ will satisfy the first order conditions of a related problem $\max\limits_{L+I=1} V(L,I)$. By setting $w \in R^+$ we complete proof of the proposition. ■

An important implication of Proposition 14 is that an investor, whenever he is allowed to re-trade an illiquid asset, brings back the ratio of L_t/I_t to the same long-term optimum. At the occurrence of Poisson events both L_t and I_t jump, so their ratio is adjusted back to optimum. This implies that processes for both L_t and I_t are jump diffusions with perfectly predictable jump sizes. The dynamics of the ratio of L_t to I_t can be compared to those of relevant ratios in the Merton problem with transaction costs (as in the model by Davis and A.R.Norman (1990)). Regarding the problem with transaction costs, the ratio is adjusted whenever it deviates significantly from the optimum ratio. The investor is passive if the ratio is in an "inaction" area, and starts to trade when it is out of it, in order to bring it back to the inaction area.

Although Proposition 14 implies that there is a fixed ratio with which an investor can bring back his allocations to liquid and illiquid assets, it is important to remember that this ratio can be different to a standard Merton solution, which is implied by Proposition 11 and Proposition 12. I provide an efficient algorithm to solve for this ratio numerically.

For the clarity of further analysis I define function Φ as

$$\Phi(L_t + I_t) = \max_{L^* + I^* = L_t + I_t} V(L_*, I_*)$$

where the function is the continuation utility, given the opportunity to re-trade the asset.

4.1 Derivation of the HJB Equation

We start with the derivation of the Hamiltonian Jacobi Bellman equation for problem (59). The problem for the investor presented in differential form is

$$
\begin{aligned}
V(L_t, I_t) \quad = \quad & \max_{c_t}\{u(c_t)dt + E(\frac{1}{1+\rho dt}\cdot(\frac{1}{1+\lambda dt}V(L_{t+dt}, I_{t+dt}) \\
& +(1-\frac{1}{1+\lambda dt})\cdot\Phi(L_{t+dt}+I_{t+dt})))
\end{aligned} \tag{62}
$$

where $\Phi()$ is a function defined in a previous paragraph. By multiplying both sides by $((1+\rho dt)(1+\lambda dt))$ we have

$$
(1{+}\rho dt)(1{+}\lambda dt)V(L_t, I_t) = \max_{c_t}\left((1+\rho dt)(1+\lambda dt)u(c_t)dt + E\left(V(L_{t+dt}, I_{t+dt}) + \lambda dt\Phi(L_{t+dt}+I_{t+dt})\right)\right)
$$

which can be simplified by cancelling terms of order dt^2. This results after simplifications in

$$
0 = \max_{c_t}\left(u(c_t)dt + E\left(V(L_t, I_t) + dV(L_t, I_t) + \lambda dt\Phi(L_{t+dt}+I_{t+dt}) - (1+(\rho+\lambda)dt)V(L_t, I_t)\right)\right)
$$

which after applying the Ito formula leads to

$$
\begin{aligned}
0 \quad = \quad & \max_{c_t}((u(c_t)dt + E[(\left(V_{L_t}(rL_t - c_t) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2\right)dt \\
& +V_{I_t}\sigma dB_t) + \lambda dt\Phi(L_{t+dt}+I_{t+dt}) - (\rho+\lambda)V(L_t, I_t)dt]
\end{aligned}
$$

which by taking expectations simplifies to

$$
\begin{aligned}
0 \quad = \quad & \max_{c_t}(u(c_t)dt + (V_{L_t}(rL_t - c_t) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2)dt \\
& +\lambda dt\Phi(L_{t+dt}+I_{t+dt}) - (\rho+\lambda)V(L_t, I_t)dt)
\end{aligned}
$$

dividing by dt

$$
0 = \max_{c_t}\left(u(c_t) + \left(V_{L_t}(rL_t - c_t) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2\right) + \lambda\Phi(L_{t+dt}+I_{t+dt}) - (\rho+\lambda)V(L_t, I_t)\right)
$$

and taking the limit as $dt \to 0$ results in the HJB Equation

$$
0 = \max_{c_t}\left(u(c_t) + \left(V_{L_t}(rL_t - c_t) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2\right) + \lambda\Phi(L_t+I_t) - (\rho+\lambda)V(L_t, I_t)\right) \tag{63}
$$

The corresponding first-order condition of the optimisation problem is:

$$u'(c_t) = V_{L_t}$$

4.1.1 Case of logarithmic utility function

Below, I calculate the form of HJB equation for the logarithmic felicity function. By substituting the felicity function

$$u(c_t) = \ln(c_t)$$

into equation (63) we have

$$0 = \max_{c_t} \left(\ln(c_t) + \left(V_{L_t}(rL_t - c_t) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2 \right) + \lambda\Phi(L_t + I_t) - (\rho + \lambda)V(L_t, I_t) \right)$$

while the first-order condition is

$$\frac{1}{c_t} = V_{L_t}$$

which implies following the HJB equation that

$$0 = \ln(\frac{1}{V_{L_t}}) + \left(V_{L_t}(rL_t - \frac{1}{V_{L_t}}) + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2 \right) + \lambda\Phi(L_t + I_t) - (\rho + \lambda)V(L_t, I_t)$$

4.1.2 The power utility function

For the case of the power utility function we find

$$u(c_t) = \frac{(c_t)^\gamma}{\gamma}$$

and the corresponding first-order condition

$$(c_t)^{\gamma-1} = V_{L_t}$$

$$c_t = (V_{L_t})^{\frac{1}{\gamma-1}}$$

which after substitution into the HJB equation and basic simplifications give

$$0 = (\frac{1}{\gamma} - 1)\,(V_{L_t})^{\frac{\gamma}{\gamma-1}} + rL_tV_{L_t} + V_{I_t}\mu I_t + \frac{1}{2}V_{I_tI_t}\sigma^2 I^2 + \lambda\Phi(L_t + I_t) - (\rho + \lambda)V(L_t, I_t)$$

4.1.3 Using homogeneity and the ratio $R_t = \frac{I_t}{L_t}$

To simplify the corresponding HJB equation we can use the property stated in Proposition 13 – the homogeneity of the value function. In particular, we can simplify the HJB PDE to a simpler ODE.

Interestingly, we can achieve this by using the ratio of either I_t/L_t or L_t/I_t. These two different choices result in different algebraic forms of the ODE. I start the analysis of the case with $R_t = \frac{I_t}{L_t}$. We find

$$V(I_t, L_t) = L_t^{\gamma} \cdot V(1, \frac{I_t}{L_t}) = L_t^{\gamma} \cdot V(1, R_t) = L_t^{\gamma} \cdot Z(R_t) \tag{64}$$

where R_t is the ratio of illiquid to liquid assets in the investor's portfolio. In this formulation we have

$$\begin{aligned} V_{L_t} &= \gamma L_t^{\gamma-1} V(1, \frac{I_t}{L_t}) + L_t^{\gamma} V_{R_t}(1, \frac{I_t}{L_t}) \cdot \left(-\frac{I_t}{L_t^2}\right) \\ &= \gamma L_t^{\gamma-1} V(1, \frac{I_t}{L_t}) + L_t^{\gamma-1} V_{R_t}(1, \frac{I_t}{L_t}) \cdot \left(-\frac{I_t}{L_t}\right) \\ &= L_t^{\gamma-1} \big(\gamma Z(R_t) - Z'(R_t) \cdot R_t\big) \end{aligned}$$

$$V_{I_t} = L_t^{\gamma} V_{R_t}(1, \frac{I_t}{L_t}) \cdot \frac{1}{L_t} = L_t^{\gamma-1} Z'(R_t)$$

$$V_{I_t I_t} = L_t^{\gamma-2} Z''(R_t)$$

which after substitution back into the HJB gives

$$\begin{aligned} 0 = &\left(\frac{1}{\gamma}\right) (L_t^{\gamma-1} \big(\gamma Z(R_t) - Z'(R_t) \cdot R_t\big))^{\frac{\gamma}{\gamma-1}} \\ &+ (L_t^{\gamma} \big(\gamma Z(R_t) - Z'(R_t) \cdot R_t\big)) r \\ &- (L_t^{\gamma-1} \big(\gamma Z(R_t) - Z'(R_t) \cdot R_t\big))^{\frac{\gamma}{\gamma-1}} \\ &+ L_t^{\gamma} Z'(R_t) \mu R_t \\ &+ \frac{1}{2} L_t^{\gamma} Z''(R_t) \sigma^2 R_t^2 \\ &+ \lambda \Phi(L_t + I_t) \\ &- (\rho + \lambda) L_t^{\gamma} \cdot Z(R_t) \end{aligned}$$

which after simplifications yields

$$\begin{aligned} 0 = &\left(\frac{1}{\gamma} - 1\right) \big(\gamma Z(R_t) - Z'(R_t) \cdot R_t\big)^{\frac{\gamma}{\gamma-1}} \\ &+ (\gamma r - \rho - \lambda) Z(R_t) \\ &+ Z'(R_t)(\mu - r) R_t \\ &+ \frac{1}{2} Z''(R_t) \sigma^2 R_t^2 \\ &+ \frac{1}{L_t^{\gamma}} \lambda \Phi(L_t + I_t) \end{aligned} \tag{65}$$

However, this is still not an ODE because of the $\Phi(L_t + I_t)$ component.

Function Φ used in previous equations is defined (as outlined earlier) as

$$\begin{aligned} \Phi(L_t + I_t) &= \max_{L_*, I_*} V(L_*, I_*) \\ s.t.\ L_* + I_* &= L_t + I_t \end{aligned}$$

Corresponding first-order conditions require that $V_L(L_*, I_*) = V_I(L_*, I_*)$. Given Proposition 14 the solution of the above optimisation problem will give the constant ratio between L_* and I_*. Let us denote this unknown ratio with A^* (A^*, which will be a function of the parameters of the $\gamma, r, \sigma^2, \lambda, \mu$. model). At the optimum point we therefore find

$$\frac{I_*}{L_*} = A^*$$

$$A^* L_* + L_* = L_t + I_t$$

$$L_* = \frac{(L_t + I_t)}{1 + A^*}$$

$$I_* = \frac{(L_t + I_t)A^*}{1 + A^*}$$

so

$$\Phi(L_t + I_t) = V\left(\frac{(L_t + I_t)}{1 + A^*}, \frac{(L_t + I_t)A^*}{1 + A^*}\right)$$

while using the homogeneity of V we have

$$\Phi(L_t + I_t) = \left(\frac{(L_t + I_t)}{1 + A^*}\right)^{\gamma} V\left(1, A^*\right)$$

and

$$\Phi(L_t + I_t) = L_t^{\gamma}\left(\frac{1 + R_t}{1 + A^*}\right)^{\gamma} Z(A^*) \tag{66}$$

where we used the assumed ratio of $R_t = \frac{I_t}{L_t}$. By substituting this equation into (65) we have

$$\begin{aligned} 0 \quad = \quad & (\frac{1}{\gamma} - 1)\left(\gamma Z(R_t) - Z'(R_t) \cdot R_t\right)^{\frac{\gamma}{\gamma - 1}} \\ & +(\gamma r - \rho - \lambda) Z(R_t) \\ & +Z'(R_t)(\mu - r) R_t \\ & +\frac{1}{2} Z''(R_t)\sigma^2 R_t^2 \\ & +\lambda\left(\frac{1 + R_t}{1 + A^*}\right)^{\gamma} Z(A^*) \end{aligned} \tag{67}$$

which is a non-linear Ordinary Differential Equation for the ratio $R_t = \frac{I_t}{L_t}$..

4.1.4 Using homogeneity and the ratio $R_t = \frac{L_t}{I_t}$

Alternatively, we can define ratio R_t as L_t/I_t for calculating the ODE. Using homogeneity of the value function we find that

$$V(I_t, L_t) = I_t^{\gamma} \cdot V(1, \frac{L_t}{I_t}) = I_t^{\gamma} \cdot F(R_t)$$

$$V_{L_t} = I_t^{\gamma} \cdot F'(R_t) \cdot \frac{1}{I_t} = I_t^{\gamma-1} \cdot F'(R_t)$$

$$V_{I_t} = \gamma I_t^{\gamma-1} \cdot F(R_t) - I_t^{\gamma} \cdot F'(R_t) \cdot \frac{L_t}{I_t^2} = I_t^{\gamma-1} \left[\gamma F(R_t) - F'(R_t)R_t\right]$$

$$V_{I_tI_t} = I_t^{\gamma-2} \left[(\gamma^2 - \gamma)F(R_t) - (2\gamma - 2)F'(R_t) + F''(R_t)R_t^2\right]$$

which can be substituted back into the HJB to get

$$\begin{aligned} 0 \quad = \quad & (\frac{1}{\gamma} - 1)I_t^{\gamma} \cdot \left(F'(R_t)\right)^{\frac{\gamma}{\gamma-1}} \\ & + rI_t^{\gamma} \cdot F'(R_t)R_t \\ & + I_t^{\gamma} \left[\gamma F(R_t) - F'(R_t)R_t\right] \mu \\ & + \frac{1}{2}I_t^{\gamma} \left[(\gamma^2 - \gamma)F(R_t) - (2\gamma - 2)F'(R_t)R_t + F''(R_t)R_t^2\right] \sigma^2 \\ & + \lambda\Phi(L_t + I_t) \\ & - (\rho + \lambda)I_t^{\gamma} \cdot F(R_t) \end{aligned}$$

which after simplification leads to

$$\begin{aligned} 0 \quad = \quad & (\frac{1}{\gamma} - 1) \cdot \left(F'(R_t)\right)^{\frac{\gamma}{\gamma-1}} \\ & + (r - \mu - (\gamma - 1)\sigma^2) \cdot F'(R_t)R_t \\ & + \frac{1}{2}F''(R_t)R_t^2\sigma^2 \\ & - (\rho + \lambda - \mu\gamma - \frac{1}{2}(\gamma^2 - \gamma)\sigma^2)F(R_t) \\ & + (1/I^{\gamma})\, \lambda\Phi(L_t + I_t) \end{aligned}$$

This can be further simplified after

derived in equation (66). After appropriate substitution we get

$$\begin{aligned}
0 \;=\;& (\frac{1}{\gamma}-1)\cdot\left(F'(R_t)\right)^{\frac{\gamma}{\gamma-1}} \\
&+(r-\mu-(\gamma-1)\sigma^2)\cdot F'(R_t)R_t \\
&+\frac{1}{2}F''(R_t)R_t^2\sigma^2 \\
&-(\rho+\lambda-\mu\gamma-\frac{1}{2}(\gamma^2-\gamma)\sigma^2)F(R_t) \\
&+\lambda\left(\frac{1+R_t}{1+A^*}\right)^{\gamma}F(A^*)
\end{aligned} \tag{68}$$

so for a given fixed A^* the problem boils down to solving the non-linear ODE of the form

$$E = A\cdot(y'(x))^{\alpha} + B\cdot y(x) + C\cdot y'(x)x + D\cdot y''(x)x^2 + E\cdot(1+x)^{\gamma}$$

5 Numerical Solution

Unfortunately, despite a significant amount of work, I could not find closed-form solutions to either of the non-linear ODEs presented in equations (67) and (68). I suspect that such closed-form solutions are unlikely to exist given Proposition 11 and Proposition 12, which suggest singularities at both $a_t = 0$ and $a_t = 1$. In order to provide some insights into optimal policies it is therefore necessary to provide an efficient algorithm to solve for the optimal control problem numerically.

Initially, I started the numerical analysis by applying the standard explicit Runge-Kutta[33] method. Unfortunately, it transpired that initial conditions for the ODEs are very difficult to specify and therefore the solution tends to be highly unstable. As an alternative I applied the Crank-Nicolson[34] method directly to the HJB and PDE. This algorithm unfortunately did not work well either. This method is well suited for linear PDEs like heat equations, but does not converge for problems of highly non-linear PDEs like the one I am solving in this paper.

The final – and successful – approach came after applying the modified Controlled Markov Chain method. This algorithm, developed by Kushner (1968), was initially used to solve degenerated elliptic non-linear PDEs in aeronautics, but later found applications in economics and finance (e.g. Fitzpatrick and Fleming (1991)). The method is based on the discretisation of the Bellman equation (62) and the solution to this discrete problem on the grid. It is possible to show, using the viscosity property of the value function, that a discrete solution converges under mild regularity conditions to the continuous solution[35].

[33] Details regarding the Runge-Kutta method can be found in an excellent survey by Butcher (2003).

[34] Details about the Crank-Nicolson method can be found in a book by Duffie (2001) in Chapter 12.

[35] For more details regarding proof, please review the paper of Kushner (1968) or chapter IX in the book by Fleming

We start with stating the discretised Bellman Equation

$$V_t(I_t, L_t) = \max_{c_t} \left\{ u(c_t)\Delta t + e^{-\rho\Delta t} E((1 - e^{-\lambda\Delta t}) V_{t+\Delta t}(I_{t+\Delta t}, L_{t+\Delta t}) + e^{-\lambda\Delta t} \Phi_{t+\Delta t}(I_{t+\Delta t}, L_{t+\Delta t})) \right\}$$

where Δt is a selected time step for discretisation. By defining a new control variable $e_t = c_t/L_t$ (rate of consumption over total liquid wealth), and by using the property outlined in Proposition 14 and the result summarised in equation (66), we can transform the previous equation into the following discrete equation:

$$\begin{aligned} Z_t(R_t) &= \max_{e_t} \{ u(e_t)\Delta t + \frac{L^\gamma_{t+\Delta t}}{L^\gamma_t} e^{-(\rho+\lambda)\Delta t} E((e^{\lambda\Delta t} - 1) Z_{t+\Delta t}(R_{t+\Delta t}) \\ &\quad + \max_{A_{t+\Delta t}} \left(\frac{1 + R_{t+\Delta t}}{1 + A_{t+\Delta t}} \right)^\gamma Z_{t+\Delta t}(A_{t+\Delta t}) \} \end{aligned} \tag{69}$$

where $Z_t(R_t) = Z_t(I_t/L_t) = V_t(I_t/L_t, 1) = V_t(I_t, L_t)/L^\gamma_t$. From equation (60) on page 60 we observe that in discrete form we have

$$L_{t+\Delta t} - L_t = (rL_t - c_t)\Delta t \Leftrightarrow L_{t+\Delta t} = (1 + r\Delta t)L_t - c_t\Delta t$$

which implies that

$$\frac{L_{t+\Delta t}}{L_t} = (1 + r\Delta t) - e_t\Delta t$$

We can substitute this into the previous equation for $Z_t(R_t)$. We then end up with a discrete Bellman Equation for the single variable R_t

$$\begin{aligned} Z_t(R_t) &= \max_{e_t} \{ u(e_t)\Delta t + ((1 + r\Delta t) - e_t\Delta t)^\gamma \cdot e^{-(\rho+\lambda)\Delta t} E((e^{\lambda\Delta t} - 1) Z_{t+\Delta t}(R_{t+\Delta t}) \\ &\quad + \max_{A_{t+\Delta t}} \left(\frac{1 + R_{t+\Delta t}}{1 + A_{t+\Delta t}} \right)^\gamma Z_{t+\Delta t}(A_{t+\Delta t}) \} \end{aligned} \tag{70}$$

In my continuous time model, in the absence of Poisson event R_t is a standard controlled Ito process. Given the dynamics of L_t and I_t

$$\begin{aligned} dL_t &= (rL_t - c_t)dt \\ dI_t &= \mu I_t dt + \sigma I_t dB_t \end{aligned}$$

and Soner (2006a).

we can calculate the dynamics of R_t using Ito lemma

$$dR_t = d\left(\frac{I_t}{L_t}\right) = R_t \cdot (\mu - r + e_t)dt + R_t \cdot \sigma dB_t \tag{71}$$

Using the insights of Kushner (1968) and Fitzpatrick and Fleming (1991) we can approximate the controlled diffusion above with a controlled Markov Chain. Given selected grid size[36] $h > 0$, we set the state space of our Markov Chain as grid $\{ih : 0 \leqslant i \leqslant N\}$. Control-dependent transition probabilities on the grid are taken to be equal to

$$\begin{aligned} P^{e_t}_{ii+1} &= \left\{\frac{1}{2}\sigma^2(i\cdot h)^2 + h\left[(i\cdot h)\cdot(\mu + e_t)\right]\right\}/Q \\ P^{e_t}_{ii-1} &= \left\{\frac{1}{2}\sigma^2(i\cdot h)^2 + h\left[(i\cdot h)\cdot(r)\right]\right\}/Q \\ P^{e_t}_{ii} &= 1 - P^{e_t}_{ii+1} - P^{e_t}_{ii-1} \end{aligned}$$

for $1 \leqslant i \leqslant N-1$ and $e_t \leqslant Nh$. The boundary probabilities are equal to

$$\begin{aligned} P_{00} &= 1 \\ P_{NN-1} &= \left\{\frac{1}{2}\sigma^2(Nh)^2 + h\left[(Nh)\cdot(r)\right]\right\}/Q \\ P_{NN} &= 1 - P^{e_t}_{NN-1} \end{aligned}$$

where P_{00} implies that 0 is an absorbing state[37]. Q is a normalising constant taken to be

$$Q = (Nh)^2\sigma^2 + Nh^2\left[\mu + e_{\max} + r\right]$$

where $e_{\max}$ is the maximum value of the intensity of consumption from the liquid wealth that I consider in the numerical algorithm. The above transition probabilities and a scaling factor modify the scheme proposed by Kushner (1968), who shows that for appropriate transition probabilities and time step Δt the Markov Chain will have first and second moments closely matching those of the continuous Ito process. The transition probabilities specified above, combined with the time step $\Delta t = h^2/Q$, lead to a Markov Chain with first and second moments closely matching those of

[36] Which is the selected granularity of the state space of R_t.

[37] When R_t is equal to zero an investor does not invest into an illiquid asset. Therefore, no dynamics exist for a stochastic process that could move R_t from zero unless there is an occurrence of the Poisson event.

the continues stochastic process described by equation (71). This can be shown by observing that

$$\begin{aligned} E(\Delta R_t) &= E(R_{t+1} - R_t) \\ &= h \cdot \left\{ \frac{1}{2}\sigma^2 (i \cdot h)^2 + h\left[(i \cdot h) \cdot (\mu + e_t)\right] \right\} /Q - h \cdot \left\{ \frac{1}{2}\sigma^2 (i \cdot h)^2 + h\left[(i \cdot h) \cdot (r)\right] \right\} /Q \\ &= h^2 \left\{ \left[(i \cdot h) \cdot (\mu + e_t - r)\right] \right\} /Q = \Delta t \cdot \left[(R_t) \cdot (\mu + e_t - r)\right] \end{aligned}$$

$$\begin{aligned} Var(\Delta R_t) &= E(\Delta R_t^2) - E(\Delta R_t)^2 = \\ &= \Delta t \cdot \left\{ \sigma^2 (i \cdot h)^2 + h\left[(i \cdot h) \cdot (\mu + e_t + r)\right] \right\} + \Delta t \cdot \left[(R_t) \cdot (\mu + e_t - r)\right] \\ &= \Delta t \cdot \sigma^2 (R_t)^2 + \Delta t \cdot h \cdot (R_t) \cdot (\mu + e_t + r) + \Delta t^2 \cdot \left[(R_t) \cdot (\mu + e_t - r)\right] \end{aligned}$$

where terms $\Delta t \cdot h$ and Δt^2 converge to zero quicker than Δt when the time step of the approximation approaches zero.

Given these transition probabilities we can implement standard techniques to solve for the optimal policy and value function from equation (70). Two basic solution methods utilised are value function iterations and the Howard Policy-Improvement Algorithm. The method I implemented to solve the problem numerically was a value function iteration for the given assumed terminal utility $Z_T(R_T)$. The algorithm for the method[38] iterates the value function by using equation (70) and transition probabilities of the process (71). The important point in the algorithm is that at each iteration $A_{t+\Delta t}$ the ratio of illiquid to liquid assets that maximises

$$\max_{A_{t+\Delta t}} \left(\left(\frac{1 + R_{t+\Delta t}}{1 + A_{t+\Delta t}} \right)^{\gamma} Z_{t+\Delta t}(A_{t+\Delta t}) \right)$$

is calculated. This is the value of the value function that will be attained if an investor is given an opportunity to re-trade the risky asset. After I obtain the solution to the value function $Z_t(R_t)$, the value function $V_t(L_t, I_t)$ can be quickly recovered using equation (64) on page 80. The optimal consumption level c_t can be recovered from relationship $e_t \cdot L_t = c_t$.

In the numerical algorithm I need to assume the terminal value function $V_T(L_T, I_T)$. I consider two cases. In the first I assume that an investor will not be able to gain any benefit from an illiquid asset for $t > T$.This assumption is equivalent to saying that after T there will be no re-trade

[38] The value function iteration code is available for download on my personal website: http://personal.lse.ac.uk/~zurawski

opportunity. In this case the terminal value function is

$$V_T(L_T, I_T) = \left(\frac{\rho - \gamma r}{(1-\gamma)}\right)^{\gamma-1} \frac{L_t^\gamma}{\gamma} \tag{72}$$

Alternatively, I can assume that that for $t > T$ the illiquid asset becomes perfectly liquid and then the terminal value function is equal to that of the standard Merton problem (as in equation (56) on page 73), where the investor gets utility from both his liquid and illiquid wealth

$$V_T(L_T, I_T) = V_{Merton}(L_T + I_T)$$

As I will show shortly, the choice of terminal value function has important consequences for an investor's optimal allocation between liquid and illiquid assets as the time approaches T.

6 Results

Numerical analysis allows me to present a number of interesting observations on optimal consumption and investment when one of the assets is illiquid. To gain insights into the real-life impact of illiquidity I set model parameters to values that are close to those observed empirically:

Interest Rate	$r = 3\%$ per annum
Discount Factor	$\rho = 10\%$ per annum
Drift of Risky Asset	$\mu = 12\%$ per annum
Volatility of Risky Asset	$\sigma^2 = 15\%$ per annum
CRRA Coefficient	$1 - \gamma = 2$

The time step in each numerical solution is set as equal to one day (1/360). For these coefficients, optimal allocation in the Merton problem will be

$$a_t = \frac{(\mu - r)}{\sigma^2(1-\gamma)} = 0.3$$

which implies that an investor will keep 0.3 of his liquid wealth in the risky asset if he is allowed to reallocate freely between his liquid and illiquid assets at each point in time.

6.1 Optimal ratio between a liquid and illiquid asset

The first observation that I make concerns the limiting case of our problem. In particular, we can expect that if the risky asset becomes very liquid, then the solution to our problem – and indeed the optimal allocation – converges to what we would expect in the standard Merton problem. This is, of course, subject to the solution to the Merton problem satisfying conditions imposed by Proposition 13 and Proposition 14 on page 8. For the model parameters outlined at the beginning of the section, the terminal value function from equation (72) and the coefficient $\lambda = 2000$, the allocation between the risky and risk-free asset is very close to the value of 0.3 implied by the fully liquid case. As can be seen in Figure 1, most of the time an investor optimally invests around 30% of his liquid wealth into the illiquid asset (if he is allowed to re-trade). Only when he approaches time T, after which time he definitely will not be able to trade in his illiquid asset, does he rapidly decrease his allocation to the illiquid asset. The interesting observation is that the result very closely matches the liquid case/Merton result. Allocating an agent who is allowed to trade frequently closely matches the allocation in a perfectly liquid case.

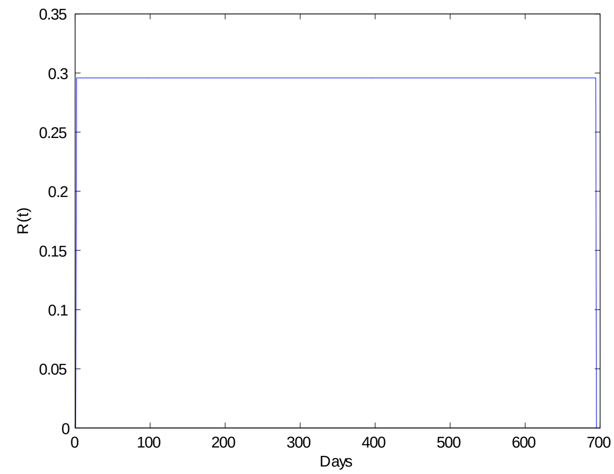


Figure 2: The optimal allocation of wealth to an illiquid asset ($r = 3\%$, $\rho = 10\%$, $\mu = 12\%$, $\sigma^2 = 15\%$, $1 - \gamma = 2$, $\lambda = 20000$)

However, if I start to adjust the liquidity of the asset I can see the impact of illiquidity on the optimal allocation. In Figure 2 I can see the optimal allocation of an investor, who has an asset for which it takes on average two days to re-trade ($\lambda = 365/2$). Although the initial part (far from time T) is not greatly changed and is still close to the perfectly liquid value of 0.3 when an investor approaches time T, he optimally decreases his position in the illiquid asset and in the last two to three weeks before the final time T, after which he will not be able to trade the illiquid asset he decides to totally unwind his position. Although the illiquidity of the asset has important consequences just before T, it seems that the Merton result is relatively robust against minor levels of liquidity.

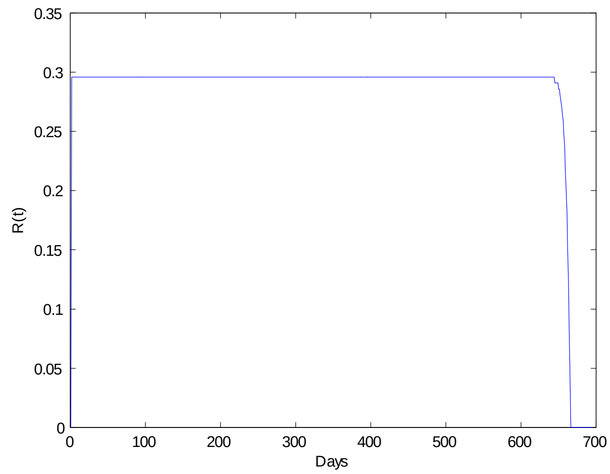


Figure 2: The optimal allocation of wealth to illiquid asset ($r = 3\%$, $\rho = 10\%$, $\mu = 12\%$, $\sigma^2 = 15\%$, $1 - \gamma = 2$, $\lambda = 182$)

However, if we increase the level of liquidity even further and assume that λ is equal to $365/31$, we can see that the optimal allocation in the illiquid case starts to deviate significantly from the standard Merton result. A selected level of liquidity implies an average time between re-trades equal to a month, and can be compared to the time it takes to sell a car or other similar items. For this level of liquidity an investor decreases his long-term allocation to the illiquid asset from 0.3 to just over 0.25 of his total wealth (at the time when he is able to re-trade). Lower optimal allocation to the risky asset in the long-term reflects the investor's concern described in previous sections. The value of the risky asset can start to drop significantly and an investor may have no ability to decrease exposure in the risky asset before a large portion of his total wealth is wiped out.

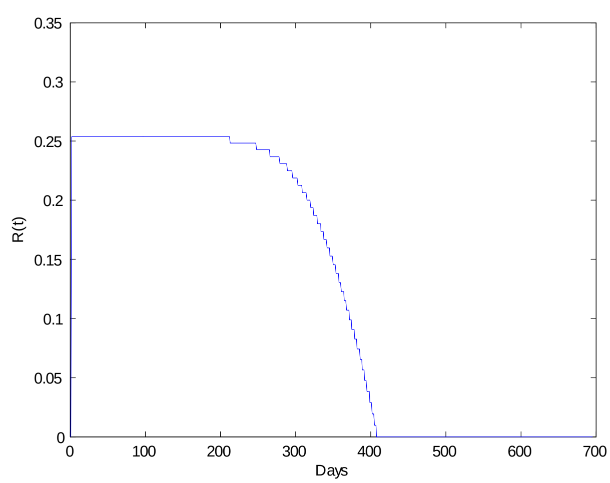


Figure 3: The optimal allocation of wealth to illiquid asset ($r = 3\%$, $\rho = 10\%$, $\mu = 12\%$, $\sigma^2 = 15\%$, $1 - \gamma = 2$, $\lambda = 11$)

Of course, this effect is more pronounced if the liquidity decreases even further. In Figure 4 I present the results for $\lambda = 1$, which can be assumed a good indication of the illiquidity of real estate (one year to re-trade). For this level of illiquidity the optimal allocation is less than half of the optimal allocation of the fully liquid case (0.11 vs 0.3).

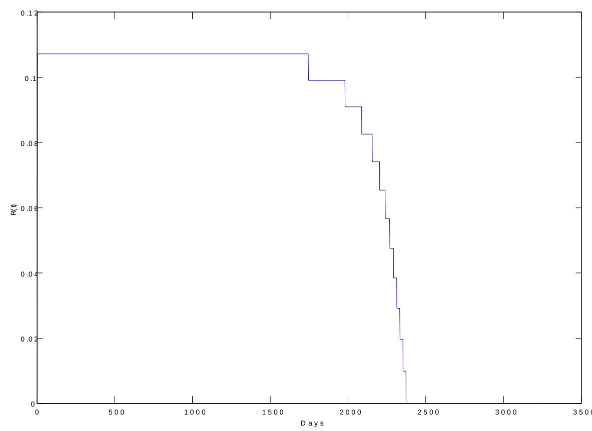


Figure 4: The optimal allocation of wealth to illiquid asset ($r = 3\%$, $\rho = 10\%$, $\mu = 12\%$, $\sigma^2 = 15\%$, $1 - \gamma = 2$, $\lambda = 1$)

6.2 Optimal consumption

Another important observation concerns the shape of the consumption function of an investor. In a perfectly liquid case we find that the ratio of the rate of consumption to total wealth is a constant. In particular, from equation (58) on page 73 we find that the rate of consumption is a constant fraction of total wealth

$$\frac{c_t}{W_t} = \left(\frac{\rho - \gamma r}{(1-\gamma)} - \frac{1}{2} \frac{\gamma(\mu - r)^2}{\sigma^2 (1-\gamma)^2} \right)$$

where all elements on the right-hand side are just the model parameters.

In an environment with illiquid assets the result above no longer holds. The relative rate of consumption not only depends on the total level of wealth, but also on how this wealth is spread between liquid and illiquid assets. A propensity to consume from total wealth actually depends on the ratio of $R_t = I_t / L_t$. To exemplify this point I show that

$$\frac{c_t}{W_t} = \frac{e_t L_t}{I_t + L_t} = e_t \cdot \frac{1}{R_t} = e_t(R_t) \cdot \frac{1}{R_t}$$

which allows me to reuse the values of e_t (which is itself a function of R_t) to present the propensity to consume from total wealth as a function of ratio R_t. For the numerical example presented in Figure 5 I use the values from the model parameters in the previous subsection ($r = 3\%$, $\rho = 10\%$, $\mu = 12\%$, $\sigma^2 = 15\%$, $1 - \gamma = 2$, $\lambda = 1$). For these coefficients the optimal level of consumption from liquid wealth in a perfectly liquid (Merton) case is $c_t/W_t = 0.0770$, which essentially implies that an investor consumes around 8% of his liquid wealth each year. In an illiquid case this value is the function of coefficient R_t, as shown in the figure below.

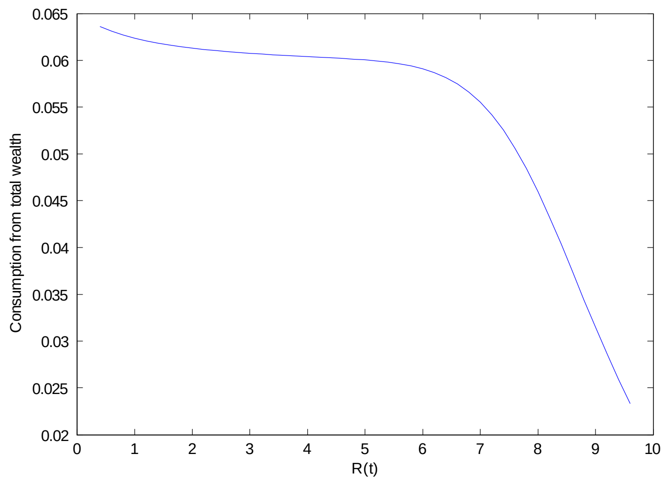


Figure 5: Optimal consumption from total wealth

The rate of consumption is uniformly lower than in the fully liquid case, as the investor consumes less than in the fully liquid case. Interestingly, the level of consumption from total wealth is a decreasing function of R_t. As ratio R_t increases, the more the wealth of the investor is found in the illiquid asset. At some point, when more and more of the investor's wealth is placed into the illiquid asset, he needs to decrease the ratio of the rate of consumption to his total wealth; otherwise, he may run down his liquid assets very quickly and then if the opportunity to re-trade the asset does not come quickly (and borrowing is excluded on the basis of Proposition 11) he will “starve". This high R_t solution corresponds to the situation of all investors who maybe quite rich, but because most of their wealth is frozen in illiquid assets their consumption does not reflect their perceived wealth[39].

[39] I am thinking here for example about entrepreneurs, who may possess ventures of significant value but because it may take a long time to cash them in the level of consumption of such an investor is significantly lower that one would expect from someone with that level of total wealth.

7 Conclusions

In this paper I analyse investment and consumption problems involved with illiquid assets. An investor is allowed to invest into an asset that is not perfectly liquid – he can decrease or increase a position in a risky asset only on the occurrence of a random Poisson event. The limited liquidity of the investment has a number of important implications, which I outline in the paper. In particular, I show that an investor will never open an illiquid asset in a position that will be greater than his total wealth. In addition, I prove that an investor will never short an illiquid asset, as this could potentially lead to his bankruptcy if the value of the assets increases significantly before he can re-trade. In my paper, I derive the HJB equation of the investor and propose an efficient numerical algorithm to solve for the optimal policy function. I show through calibrated examples that a higher level of illiquidity leads to the lower long-term target allocation of wealth to an illiquid asset. In particular, an allocation to the illiquid asset is lower than in a perfectly liquid benchmark Merton case. I show in the model that the consumption process is characterised by jumps of perfectly predictable size (jumps in the consumption rate happen at the same time as those found in Poisson events). Moreover, I show that the investor's rate of consumption depends not only on his total wealth (as in the Merton case), but also on the ratio of wealth he has in illiquid and liquid assets. If an investor places a significant fraction of his wealth into an illiquid asset, the rate of consumption from his total wealth is low.

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