Topics in Optimal Liquidation and Contract Theory

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of 166 pages.

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Abstract

The thesis consists of three parts. The first one presents an exhaustive study of three new models arising in the context of the so-called optimal liquidation problem. This is the problem faced by an investor who aims at selling a large number of stock shares within a given time horizon and wants to maximise his expected utility of the cash resulting from the sale. Such an investor has to take into account the impact that his selling strategy has on the underlying stock price. The models studied in the thesis assume that market risk follows a fairly general Lévy process and that the investor has an exponential utility. In each of the three different model formulations, an explicit or semi-explicit expression for the optimal liquidation strategy is derived.

The second part of the thesis presents a study of an optimal liquidation problem embedded in a contractual problem. In particular, a contractual relationship between an investor and a broker is modelled on the basis of a suitable liquidation strategy and the corresponding affected mark-to-market assert price. The analysis of the model determines the broker's compensation and the liquidation strategy that maximise the broker's as well as the investor's expected utilities.

The third part of the thesis studies a continuous time principal-agent problem in which the agent's outside options depend on his past performance. In this new model, even if the agent does not expect any compensation from the principal at all, the agent may still apply work effort with a view to improving his outside options. Formulated as an optimal control and stopping problem for both the agent and the principal, the optimal contract is identified.

Contents

Introduction

This thesis consists of three themes of studies: i) the optimal liquidation problem in different models with stock prices driven by Lévy processes; ii) a liquidation problem involving a contractual agreement between an investor and a broker; and iii) the principal-agent problem taking consideration of the agent's outside options.

I. Optimal liquidation

The optimal liquidation problem considers an investor who aims to sell a large amount of shares within a given time. Rapid sale of shares may depress the stock price, while slicing the big order into many smaller blocks of orders to be executed sequentially over time may take too long to complete the liquidation hence the market volatility risk becomes large. Therefore, the investor needs to find an optimal way to slice the big order over time so that his execution cost get minimised.

A good modelling of the investor's execution price is crucial for studying the optimal liquidation problem. It is common in the literature that an execution price is assumed to have an additive form of an unaffected price and a price impact. The unaffected price is usually interpreted to be the mark-to-market stock price if the investor does not make any trades, while the price impact describes the manner of how the investor's trades can influence the stock price. With this structure of the execution price, Almgren and Chriss (1999, 2000) assume that the price impact is a sum of a so-called permanent price impact and a temporary price impact. The permanent impact hits the stock price fundamentally, and this kind of influence never disappears. The temporary impact changes the stock price only instantaneously, it disappears immediately if the investor stops trading. In particular in Almgren and Chriss (1999, 2000), the permanent impact is described by a linear function of the investor's total size of orders that have been executed, the temporary impact is a linear function of the investor's trading speed, and the unaffected stock price follows a Brownian motion. Formulated as a problem of a mean-variance minimisation of the investor's execution cost over a set of deterministic trading strategies, the optimal liquidation strategy is derived explicitly. Almgren (2003) generalises this model by taking the temporary impact function to be a power function. Then for the power-law temporary impact, Almgren et al. (2005) give out a result of calibration. Following Almgren and Chriss (2000) but instead of the mean-variancce criterion, Schied and Schöneborn (2009) study the optimal liquidation problem with a utility optimisation. In this model, the investor's risk aversion depends on the how many risk-neutral assets he is holding. Schied et al. (2010) prove that in the Almgren-Chriss model, for any investor with a constant absolutely risk aversion (CARA), the optimal liquidation strategy is deterministic.

Instead of the Almgren-Chriss type model, Obizhaeva and Wang (2013) describe the price impact using limit order books. They assume that the price impact is determined by how deep the investor's orders eat into a limit order book as well as how quickly new limit orders refill into the book. But these two factors are essentially determined by the shape of a limit order book together with the speed of it's resilience. With this kind of modelling, the price impact is neither permanent nor temporary, but it is transient meaning that it decays over time due to the new coming limit orders into the book. By considering a flat shaped limit order book with it's resilience following an exponential function of the difference between the current and the unaffected statuses of the limit order book, as well as that the unaffected best bid/ask price following a Brownian motion, Obizhaeva and Wang (2013) solve the problem of minimising the investor's final expect execution cost. Alfonsi et al. (2010) extend this model by concerning a limit order book with a general continuous shape. Predoiu et al. (2011) extend this model further by working on a discrete shaped limit order book and a general resilience function. Based on the setting of Alfonsi et al. (2010), Løkka (2014) solves the optimal liquidation problem for a CARA investor when the unaffected best bid/ask price follows a Brownian motion.

Describing the unaffected stock price using a Brownian motion may allow it to become negative. To deal with this drawback, Gatheral and Schied (2011) study an Almgren-Chriss type model where the unaffected price process is a geometric Brownian motion. By imposing a special optimisation criterion, they compare the geometric Brownian motion model to the linear Brownian motion model, and conclude that the difference between the corresponding optimal liquidation strategies is little. Instead of an additive price impact, to prevent the execution price from being negative, Guo and Zervos (2015) study an optimal liquidation model with multiplicative price impact, in which the unaffected stock price is driven by a geometric Brownian motion.

All of the aforementioned models are about liquidating by submitting scheduled market orders. This kind of strategy may be too aggressive, since in reality submitting limit orders to take some advantage from choosing a preferred execution price is always considered in prior, rather than using market orders directly. The optimal liquidation problem involving using limit orders is studied in Avellaneda and Stoikov (2008), Guéant et al. (2012), Bayraktar and Ludkovski (2014), Cartea and Jaimungal (2015), etc. Moreover, Kratz and Schöneborn (2014, 2015) consider trading involving dark pools for the optimal liquidation problem.

In reality, liquidation can usually finish in a very short time. It is well-known that Lévy processes can provide rather good fits to the distributions of observed stock returns, and this is in particular within short time horizons (see e.g. Madan and Seneta, 1990; Eberlein and Keller, 1995, etc). This therefore motivates us to study the optimal liquidation problem with market risk described by Lévy processes. To reserve the mathematical tractability, we study the liquidation problem that only allows to submit market orders, and the price impact is in an additive form. We establish three models. Two of them are of the Almgren-Chriss type and the rest one involves a limit order book. In all of our three models, a CARA investor is considered.

In the first model, we study the optimal liquidation problem with infinite time horizon in the Almgren-Chriss framework, where the unaffected stock price follows a general Lévy process. The temporary price impact is described by a general function satisfying some conditions which makes the problem to be well-formulated. We suppose the investor wants to maximise the expected utility of the cash received from the sale of his shares, and show that this problem can be reduced to a deterministic optimisation problem which we are able to solve explicitly. In order to compare our results to exponential Lévy models which are supposed to be more natural to describe stock prices, we derive the (linear) Lévy process approximation of such models. In particular we derive expressions for the Lévy process approximation of the exponential variance gamma Lévy process, and study properties of the corresponding optimal liquidation strategy. We find that for the power-law temporary impact function, the optimal strategy is to liquidate so quickly that it may be infeasible in practice. This is because that the power-law price impact doesn't give out big enough penalisations to very large trading speeds. We therefore try to study what kind of temporary price impact is associated with a feasible optimal liquidation strategy in the Lévy model. In particular, we obtain an explicit expression for the connection between the temporary impact function for the Lévy model and the temporary impact function for the Brownian motion model, for which the optimal liquidation strategies from the two models coincide.

In the second model, we consider an Almgren-Chriss type of liquidation model and aim to maximise the expected utility of the investor's cash position at a given finite time. The unaffected stock price follows a general Lévy process. The temporary price impact is described by a general function satisfying some conditions which makes the problem to be well-formulated. We reduce the problem to a deterministic optimisation problem and we derive the optimal liquidation strategy and the corresponding value function in closed forms. It turns out that, if the unaffected asset price has a positive drift, then it might be optimal to wait for a while during selling, or it might be optimal to buy back at the beginning of trading, and price manipulation in the sense of Huberman and Stanzl (2004) is allowed in the case of positive drift. We solve the deterministic optimisation problem using the theory calculus of variations. In particular, we characterise the optimal liquidation strategy using the Beltrami identity which is a first order ordinary differential equation. This characterisation allows us to get a closed-form solution.

In the last model, we consider a general bid limit order book with a general resilience function where the unaffected price process follows a general Lévy process. Our formulation also allows for limit order books with discontinuous shapes which can provide reasonable approximations for limit order books with discrete shapes in reality. It is assumed that the unaffected bid price provides a lower bound for the best ask price and that the bid limit order book is unaffected by the investor's buy orders. These assumptions allow us to exclude any buy orders in the optimal strategy, and they also exclude any price manipulations. The number of available limit orders in the book is assumed to be finite. This limits the investor's strategy in the way that he cannot sell more than currently available bid orders. With an infinite time horizon, we solve the problem of maximising the expected utility of the investor's the finial cash. Due to a certain structure of the market we consider, combining with the CARA utility, we simplify the optimisation problem to be deterministic. Formulated as a two-dimensional singular optimal control problem, we derive an explicit expression for the value function. The optimal intervention boundary completely characterises the optimal liquidation strategy. In particular, this problem provides an example of a solvable two-dimensional singular optimal control problem with an optimal intervention boundary can be discontinuous.

II. Contracted liquidation

The contracted liquidation problem extends the classical optimal liquidation problem by considering additionally a contractual agreement made between an investor (she) and a broker (he). Precisely, instead of concerning an investor is liquidating by herself, she is assumed to be unable to access to the market, a broker is therefore hired to liquidate on behalf of the investor under some conditions stipulated in a contract. The contract specifies a liquidation position, a time to complete the liquidation, how much liquidation proceeds that the broker should deliver to the investor and how much compensation that the investor should pay to the broker. Also, the investor can propose some liquidation strategy that the broker is expected to (or have to) follow. The aim of this problem is to maximise both of the investor's and the broker's expected utilities by finding out the optimal contract offered by the investor as well as the associated optimal liquidation strategy implemented by the broker.

We study this problem in an Almgren-Chriss type of liquidation model (Almgren and Chriss, 2000; Almgren, 2003) embedded in a continuous-time principal-agent model. In terms of an optimal liquidation model, we suppose there is no permanent price impact in the market, the temporary price impact is described by some general function, and the unaffected stock price is driven by a Brownian motion. In addition to the price impact cost, we consider some additional implementation cost depending on the trading speed, which is described by some general function. In terms of a principal-agent model, in our study, the principal is identified by a CARA investor and the agent is identified by a risk-neutral broker. We assume that the liquidation has to finish within a finite time, and that the proceeds from the sale as well as the compensation are paid to each other at the end of liquidation as lump-sums (see e.g. Holmström and Milgrom, 1987; Cvitanić et al., 2006, 2008, 2009, etc, for continuous-time models with lump-sum payments). Depending on whether the investor is able to observe the liquidation strategy that the broker actually implements, we study two different types of contracts which are respectively referred to as the first-best and the second-best (moral hazard) in the literature of the principal-agent problem.

The first-best case assumes that the investor is able to observe which liquidation strategy that the broker implements. The broker therefore has to follow any liquidation strategy recommended by the investor. The optimal contact and the optimal liquidation strategy are explicitly solved out. It shows that under the condition of the optimal contract, the risk-averse investor recommends a trivial liquidation strategy which is usually optimal for a risk-neutral investor in an optimal liquidation model (see e.g. Almgren and Chriss, 2000), and she transfers all of the market volatility risk to the risk-neutral broker via the compensation so that receives a deterministic amount of proceeds.

The second-best case assumes that the investor is unable to observe the liquidation strategy that the broker actually implements. However, she does observe the stock price which is affected by the broker's trades. The investor requires to receive amount of proceeds computed according to her recommended strategy against the observed stock price. Due to the asymmetric information between the investor and the broker, the broker has chance to collect some private benefit throughout the sale, and this is done by taking the difference between the actual proceeds he receives from the liquidation and the amount of money that have to be delivered to the investor. The investor is supposed to seek for an optimal contract which induces the broker to implement the recommended strategy as his optimal choice. Under this condition, the optimal compensation and the optimal recommended liquidation strategy are solved in closed-forms. Similar to the deterministic optimal liquidation strategy for a CARA investor in an optimal liquidation model (see e.g. Schied and Schöneborn, 2009; Schied et al., 2010; Guéant and Royer, 2014; Løkka, 2014) that the optimal recommended strategy also turns out to be deterministic in our contracted liquidation model. An explicit example with a linear temporary price impact function and a quadratic implementation cost function is given. It shows that compare to the Almgren-Chriss optimal liquidation strategy for a CARA investor (see e.g. Schied et al., 2010), the contractual agreement allows the investor to share some market volatility risk with the risk-neutral broker.

The study of the continuous-time principal-agent problem is initiated by Holmström and Milgrom (1987). They introduce a moral hazard model with a finite time horizon, where the compensation is paid at the terminal time as a lump-sum, and both of the principal and the agent have exponential utilities. The agent's optimal effort of working for the principal is derived to be deterministic. Their setting is close to the problem formulation about the contractual agreement in our model, and our optimal implemented liquidation strategy (analogous to the optimal effort) is also deterministic. Within the category of continuous-time moral hazard models, various extensions have been done. In particular, in contrast to the deterministic effort in Holmström and Milgrom (1987), many of researches focus on dynamic incentive contracts, e.g. DeMarzo and Sannikov (2006); Biais et al. (2007); Sannikov (2008); Cvitanić et al. (2008, 2009); Anderson et al. (2017), etc. Cvitanić and Zhang (2012) establish general mathematical frameworks for the principal-agent problem in different contexts, characterising solutions using the forward-backward stochastic differential equations. In contrary to most principal-agent models (e.g. the aforementioned models), the output process in our model, namely the process of liquidation proceeds, has a more meaningful financial structure, rather than just being a general diffusion.

Compare to optimal liquidation models, the investor's recommended strategy in our model is usually referred to as a benchmark strategy in the literature of optimal liquidation. The benchmark liquidation strategy usually determines directly how much the broker have to deliver to the investor when liquidation finishes. The broker can trade against the benchmark to gain some profit (see e.g. Guéant and Royer, 2014; Frei and Westray, 2015, ect). Amoung them, Guéant and Royer (2014) do some indifference pricing for the amount of shares need to be sold with respect to the broker's optimal expected utility maximised over a set of admissible liquidation strategies. This indifference price is quoted by the broker, and it serves as a premium so that the broker agrees to liquidate for the investor. In terms of our model, the aforementioned liquidation models involving benchmarks are equivalent to the broker's problem, where the investor's utility optimisation is completely not considered.

III. Contract with outside options

The principal-agent problem studies how a principal incentivises an agent to manage a project based on some contractual agreement. The principal receives the profit generated from the project, meanwhile compensates the agent for his effort which impacts the profit that the principal receives. Therefore, the aim of this problem is to find an optimal contract and the agent's associated optimal managerial effort.

The continuous-time principal-agent problem is first studied by Holmström and Milgrom (1987) in the context of moral hazard, meaning that the agent can employ some hidden action which is unobservable for the principal. In their finite time horizon model, all payoffs are made as lump-sums at the end of time. Receiving a compensation from the principal, the agent applies effort to maximise his expected utility. Having understood the agent's optimal response, the principal optimises over contracts to maximise her own expected utility. In the context of moral hazard, DeMarzo and Sannikov (2006); Biais et al. (2007) study the optimal structure of financing a company whose manager can employ an unobservable shirking action to reduce the value of the company. They find the optimal contract for which the manager does not reduce the company's value, and implement this optimal contract using realistic financing tools. Sannikov (2008) introduces a continuous-time model for a risk-averse agent and a risk-neutral principal who can terminate the project inefficiently. Cvitanić et al. (2008) extends the Holmström and Milgrom (1987) model by taking the consideration that the principal is allowed to dismiss the agent. When the agent exits, an exogenous payoff is paid to him. Based on DeMarzo and Sannikov (2006), DeMarzo and Sannikov (2017) study a stealing model in which the principal and the agent have different believes about the intrinsic drift of the output process of the project. In addition, when the agent is dismissed by the principal, he gets some outside option whose value depends on both of their believes of the outcome's drift.

DeMarzo and Sannikov (2006); Sannikov (2008) first connect the principal-agent problem to the theory of optimal control. In particular, given any contract, the agent's optimal effort is characterised by his optimal value process which can be taken as a state process of the principal's optimal control problem. The principal then chooses the optimal contract, therefore, the agent's optimal effort associated with this contract is induced at the same time.

In a setting similar to Sannikov (2008), we study a principal-agent problem where the agent receives an outside option when he chooses to stop working for the principal. In contrast to Cvitanić et al. (2008) and DeMarzo and Sannikov (2017), the agent can terminate the contract when the outside option is sufficiently attractive. The value of the outside option depends on agent's past performance. The higher output the agent has produced, the higher value his outside option becomes. Therefore, the agent not only works for the compensation paid by the principal, but also to improve his perspective from the outside option. The value of outside option is assumed to be linear in the present value of the accumulated cash flow generated from the project. This assumption allows the outside option to be incorporated into the agent's running cost, which reduces the dimension of the principal's control problem and effectively enhances the mathematical tractability.

Mathematically, the agent's problem is formulated as a non-Markovian stochastic control and stopping problem. For any given compensation, using a martingale representation technique, the agent's optimal effort and optimal stopping time are characterised by his optimal value process. Then the principal's problem is formulate as a stochastic control problem with mixed classical control and singular control. Similar to Sannikov (2008), the agent's optimal value process is used as the state value for the principal. In contrast to DeMarzo and Sannikov (2006) and Sannikov (2008), where the contract sensitivity is assumed to be bounded, we first show that principal's value function is a unique continuous viscosity solution to the principal's fully nonlinear Hamilton-Jacobi-Bellman variational inequalities (HJBVI) without assuming a bounded control. This result is mathematically interesting in its own right. Imposing an additional assumption on admissible contracts' sensitivity, which makes sure the principal's HJBVI is uniformly elliptic, we upgrade the regularity of the viscosity solution to be twice continuously differentiable. This allows us to derive the optimal contract, and hence the agent's associated optimal effort and optimal time to exist are induced at the same time. It turns out that for the optimal contract, the compensation is paid with a minimum amount such that the agent's optimal value process remains below a certain finite level. The agent's optimal effort is a function of the optimal contract's sensitivity, and the agent is optimal to stop working once his optimal value process drops down to 0.

IV. Structure of the thesis

Part I

Chapter 1: Section 1.1 introduces an Almgren-Chriss type of liquidation model with infinite time horizon and the investor's optimal liquidation problem; Section 1.2 simplifies the problem; solution to the optimal liquidation problem is given in Section 1.3; Section 1.4 gives out an approximation scheme for an liquidation model with exponential Lévy processes, using the model developed in previous sections; some numerical examples are given in Section 1.5; Section 1.6 contains all of the proofs in this chapter.

Chapter 2: Section 2.1 introduces an Almgren-Chriss type of liquidation model with finite time horizons and the investor's optimal liquidation problem; Section 2.2 simplifies the problem; this problem is solved in Section 2.3; Section 2.4 shows that the finite time horizon model converges to the model with infinite time horizon; all of the proofs in this chapter are contained in Section 2.5.

Chapter 3: Section 3.1 introduces a liquidation model in the context of limit order book, and the investor's optimal liquidation problem is introduced; Section 3.2 simplifies the problem; solution to this problem is derived in Section 3.3; all of the proofs in this chapter are given in Section 3.4.

Part II

Chapter 4: Section 4.1 introduces an Almgren-Chriss type of liquidation model with a contractual agreement between an investor and a broker; Section 4.2 studies the first-best contract; the second-best contract is studied in Section 4.3; all of the proofs in this chapter are contained in Section 4.4.

Part III

Chapter 5: Section 5.1 introduces a principal-agent model with agent's outside options; Section 5.2 formulates the agent's and the principal's problems, and the agent's optimal effort as well as the optimal time to exit are solved out for a given contract; the principal's optimal contract is derived in Section 5.3; Section 5.4 contains all of the proofs in this chapter.

Part I

Optimal Liquidation

This part is based on joint works with Dr. Arne Løkka.

Chapter 1

Optimal liquidation trajectories for the Almgren-Chriss model with Lévy processes

1.1 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, which supports a one dimensional, non-trivial, $\mathbb{F}$-adapted Lévy process L. We assume that the Lévy process L possesses the following properties.

Assumption 1.1. L_1 has finite second moment. Moreover, the set $\left\{\delta < 0 \mid \mathbb{E}\left[\mathrm{e}^{\delta L_1}\right] < \infty\right\}$ is non-empty.

For future reference, we observe that this assumption ensures that L_t has finite first and second moments, for all $t \geq 0$. Hence, L admits the decomposition

$$L_t = \mu t + \sigma W_t + \int_{\mathbb{R}} x \left(N(t, dx) - t\nu(dx)\right),$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are two constants, W is a standard Brownian motion, N is a Poisson random measure which is independent of W with compensator $t\nu(dx)$, and ν is the Lévy measure associated with L (see e.g. Kyprianou, 2006). Write

$$\bar{\delta} = \inf\left\{\delta < 0 \mid \mathbb{E}\left[\mathrm{e}^{\delta L_1}\right] < \infty\right\} < 0. \tag{1.1}$$

Assumption 1.1 also ensures that the cumulant generating function of L_1 is finite on the interval $(\bar{\delta}, 0]$. This property will be made extensive use of in the sequel.

We consider an investor who aims to sell a large amount of shares of a single stock in an infinite time horizon. For $t \geq 0$, we denote by Y_t the investor's position in the stock at time t, and let $y \geq 0$ be the investor's initial stock position. We consider the following sets of admissible liquidation strategies.

Definition 1.2. Given an initial share position $y \geq 0$, the set of admissible strategies, denoted by $\mathcal{A}(y)$, consists of all $\mathbb{F}$-adapted, absolutely continuous, non-increasing processes Y satisfying

$$\int_0^\infty \|Y_t\|_{L^\infty(\mathbb{P})}\, dt < \infty \qquad \text{if } \mu \neq 0, \tag{1.2}$$

and

$$\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\, dt < \infty \qquad \text{if } \mu = 0. \tag{1.3}$$

Let $\mathcal{A}_D(y)$ be the set of all deterministic strategies in $\mathcal{A}(y)$.

The reason for operating with different sets of admissibility depending on the drift parameter μ is related to the asymptotic properties of the cumulant generating function of L_1 around 0. If μ is 0 then the cumulant generating function is of order two around zero, while it is of order one if μ is different from zero (the importance of the cumulant generating function of L_1 will be explained later). The integrability conditions in (1.2) and (1.3) make sure that the investor's finial cash position is well-defined (see Proposition 1.5), at the same time, they also rule out some arbitrage in some weak sense (see Remark 1.6).

Let $Y \in \mathcal{A}(y)$. Then there exists an $\mathbb{F}$-adapted, positive-valued process ξ such that Y admits the representation

$$Y_t = y - \int_0^t \xi_s\, ds,$$

i.e. $-\xi_t$ is the time derivative of Y at time t. In the literature of optimal liquidation, the function $t \mapsto Y_t$ is referred to as the liquidation trajectory and the associated process ξ as the liquidation speed (see Almgren and Chriss, 2000; Almgren, 2003, etc). They are identified by each other.

As it is common in the optimal liquidation literature, we refer to the unaffected stock price process the price process observed in the market, if the investor does not trade. Throughout this chapter, we consider that the unaffected stock price process is modelled by the process

$$s + L_t, \quad t \geq 0,$$

where $s > 0$ is some constant which denotes the initial stock price. In reality, liquidation can usually finish in a very short time. It is well-known that Lévy processes can provide rather good fits to the distributions of observed stock returns, and this is in particular within short time horizons. Therefore, the study of liquidation problem with Lévy processes should give out a good result of the optimal liquidation strategy.

Following Almgren and Chriss (1999, 2000) and Almgren (2003), we split a market impact into two components: a permanent impact and a temporary impact. We therefore assume that the stock price at time $t \geq 0$ is given by

$$S_t = s + L_t + \alpha(Y_t - Y_0) - F(\xi_t), \tag{1.4}$$

where $\alpha \geq 0$ is a constant describing the permanent impact and $F : [0, \infty) \to [0, \infty)$ is a function describing the temporary impact. It is common in the literature of optimal liquidation that in a continuous time model, an admissible strategy is assumed to be absolutely continuous (see Schied and Schöneborn, 2009; Schied et al., 2010, etc), and therefore for some $t \geq 0$, the associated liquidation speed might be undefined. As a consequence, the stock price given by (1.4) might be undefined for some $t \geq 0$ as well. However, this is fine in the context of optimal liquidation, because for instance in our study, we only focus on the proceeds from selling shares, which is well-defined with such definition of S (see (1.5)). We assume that F satisfies the following assumptions.

Assumption 1.3. The temporary impact function $F : [0, \infty) \to [0, \infty)$ satisfies that

(i) $F \in C([0, \infty)) \cap C^1((0, \infty))$;

(ii) $F(0) = 0$;

(iii) the function $x \mapsto xF(x)$ is strictly convex on $[0, \infty)$;

(iv) the function $x \mapsto x^2 F'(x)$ is strictly increasing, and it tends to infinity as $x \to \infty$.

In the above assumption, condition (iii) serves for the convexity of the objective function in the optimisation problem we are going to solve (see (1.13)) and hence the uniqueness of solution holds (see Theorem 1.15); condition (iv) ensures that the value function in our optimisation problem is solved in an explicit form (see Proposition 1.14) and the optimal liquidation speed process can be solved in a feedback form (see Theorem 1.15). Assumption 1.3 is satisfied by a large class of functions, for example, $F(x) = \beta x^\gamma$ with $\beta, \gamma > 0$. Under this assumption, we derive the following technical properties of F for future references.

Lemma 1.4. *F is strictly increasing and $\lim_{x\to 0} xF'(x) = 0$. Hence $\lim_{x\to 0} x^2F'(x) = 0$.*

For $t \geq 0$, let C_t^Y denote the cash position of the investor at time t associated with some admissible strategy Y. Denote by $c \in \mathbb{R}$ the investor's initial cash position. Then a direct calculation verifies that his cash position at some finite time T is given by

$$\begin{aligned} C_T^Y &= c - \int_0^T S_t \, dY_t \\ &= c - \big(s - \alpha y\big)\big(Y_T - y\big) + \frac{\alpha}{2}\big(y^2 - Y_T^2\big) - L_T Y_T + \int_0^T Y_{t-} \, dL_t - \int_0^T \xi_t F(\xi_t) \, dt. \end{aligned} \tag{1.5}$$

The next result states that the investor's cash position at the end of time is well-defined.

Proposition 1.5. *For any $Y \in \mathcal{A}(y)$, we have*

(i) $L_T Y_T \to 0$ in $L^2(\mathbb{P})$, as $T \to \infty$;

(ii) $\int_0^\infty Y_{t-} \, dL_t$ is well-defined in $L^1(\mathbb{P})$.

Therefore,

$$C_\infty^Y = c + sy - \frac{1}{2}\alpha y^2 + \int_0^\infty Y_{t-} \, dL_t - \int_0^\infty \xi_t F(\xi_t) \, dt, \qquad a.s., \tag{1.6}$$

for any $Y \in \mathcal{A}(y)$.

From the expression of C_∞^Y, we can make a few observations. The term $c + sy$ can be viewed as the initial mark-to-market wealth of the investor. His total loss due to the permanent impact of trading is given by $\frac{1}{2}\alpha y^2$ which is deterministic and only depends on the initial liquidation size. In particular, it does not depend on the choice of liquidation strategy. The term $\int_0^\infty \xi_t F(\xi_t) \, dt$ represents the total cost due to the temporary impact, and it does depend on the liquidation strategy. The term $\int_0^\infty Y_{t-} \, dL_t$ represents the gain or loss due to the market volatility. A relatively slow liquidation speed reduces the temporary impact, but provides a substantial market volatility risk. The optimal liquidation strategy is therefore a compromise between the loss due to the temporary impact and the market volatility risk. We assume that the investor has a constant absolutely risk aversion (CARA), thus his utility function U satisfies $U(x) = -\exp(-Ax)$, for some constant $A > 0$. Suppose the investor aims to maximise the expected utility of his cash position at the end of time, i.e. he wants to solve

$$\sup_{Y \in \mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty^Y\big)\big]. \tag{1.7}$$

In view of (1.6), this problem takes the form of

$$\inf_{Y \in \mathcal{A}(y)} \mathrm{e}^{-A\widetilde{C}}\, \mathbb{E}\left[\exp\left(-\int_0^\infty AY_{t-}\, dL_t + A\int_0^\infty \xi_t F(\xi_t)\, dt\right)\right], \tag{1.8}$$

where

$$\widetilde{C} = c + sy - \frac{1}{2}\alpha y^2.$$

To solve the above problem, it is sufficient to look at

$$\inf_{Y \in \mathcal{A}(y)} \mathbb{E}\left[\exp\left(-\int_0^\infty AY_{t-}\, dL_t + A\int_0^\infty \xi_t F(\xi_t)\, dt\right)\right]. \tag{1.9}$$

Remark 1.6. Suppose that we do not impose integrability conditions (1.2) and (1.3) on an admissible strategy. Then cash position at time infinity may not be well-defined. In this case, we may consider to solve the problem

$$\sup_{Y \in \mathcal{A}(y)} \mathbb{E}\left[-\exp\left(-A \limsup_{T\to\infty} C_T^Y\right)\right].$$

However, without (1.2) and (1.3), our model admits an arbitrage in some week sense. To see this, we consider the Lévy process L as a standard Brownian motion and consider some stock price $p > s$. Write $\tau_p = \inf\{t \geq 0 \,|\, L_t \geq p\}$ which is finite a.s. (see Rogers and Williams (2000), Lemma 3.6). Suppose Y is an absolutely continuous, non-increasing strategy which consists of a waiting until time τ_p and then decreases to 0 following a deterministic way, i.e. $(Y_{\tau_p+t})_{t\geq 0}$ is a deterministic process starting from y. Such strategy is admissible. Let ξ be the associated speed process. We calculate that

$$
\begin{aligned}
&\sup_{Y\in\mathcal{A}(y)} \mathbb{E}\Big[-\exp\Big(-A\limsup_{T\to\infty} C_T^Y\Big)\Big] \\
\geq\,& \mathbb{E}\Big[-\exp\Big(-A\limsup_{T\to\infty} C_T^Y\Big)\Big] \\
\geq\,& \mathbb{E}\Big[-\exp\Big(-AC_{T+\tau_p}^Y\Big)\Big] \\
=\,& -\exp\bigg(-A\widetilde{C} + A\int_0^T \xi_{t+\tau_p} F\big(\xi_{t+\tau_p}\big)\,dt\bigg)\mathbb{E}\bigg[\exp\bigg(-\int_0^{T+\tau_p} AY_t\,dW_t\bigg)\bigg] \\
=\,& -\exp\bigg(-A\widetilde{C} + A\int_0^T \xi^p_{t+\tau_p} F\big(\xi^p_{t+\tau_p}\big)\,dt\bigg)\mathbb{E}\bigg[\exp\bigg(-AyW_{\tau_p} - \int_{\tau_p}^{T+\tau_p} AY_t^p\,dW_t\bigg)\bigg] \\
=\,& -\exp\bigg(-Ayp - A\widetilde{C} + A\int_0^T \xi^p_{t+\tau_p} F\big(\xi_{t+\tau_p}\big)\,dt\bigg)\mathbb{E}\bigg[\exp\bigg(\int_{\tau_p}^{T+\tau_p} \frac{1}{2}A^2\big(Y_t\big)^2\,dt\bigg)\bigg] \\
=\,& -\exp\bigg(-Ayp - A\widetilde{C} + A\int_0^T \xi_{t+\tau_p} F\big(\xi_{t+\tau_p}\big)\,dt + \int_0^T \frac{1}{2}A^2\big(Y_{t+\tau_p}\big)^2\,dt\bigg),
\end{aligned}
$$

where $\widetilde{C} = c + sy - \frac{1}{2}\alpha y^2$, and notice that the two integrals in the above line are two constants. Taking p to $+\infty$ gives

$$\mathbb{E}\Big[-\exp\Big(-AC_{T+\tau_p}^Y\Big)\Big] = 0.$$

Then Jensen's inequality results in

$$\lim_{p\to\infty} -\exp\Big(-A\mathbb{E}\big[C_{T+\tau_p}^Y\big]\Big) \geq \lim_{p\to\infty} \mathbb{E}\Big[-\exp\Big(-AC_{T+\tau_p}^Y\Big)\Big] = 0,$$

which implies that

$$\lim_{p\to\infty} \mathbb{E}\big[C_{T+\tau_p}^Y\big] = \infty.$$

Therefore, for some large enough p, $C_{T+\tau_p}^Y$ has a strictly positive expectation. This is an arbitrage in the sense that the investor can receive strictly positive proceeds for sure by repeating this strategy. However, Y clearly violates (1.2) and (1.3). $\square$

1.2 Problem simplification

Throughout this section, we reduce problem (1.9) to be a deterministic optimisation problem. Let's first write $\bar{\delta}_A = -\bar{\delta}/A$, where $\bar{\delta}$ is the negative number appearing in (1.1) and A is the risk aversion parameter appearing in the utility function U. We impose further the following assumptions.

Assumption 1.7. The initial stock position y is strictly less than $\bar{\delta}_A$.

Assumption 1.8. The drift μ of the Lévy process L satisfies $\mu \leq 0$.

Assumption 1.7 restricts the investor's maximum initial liquidation position cannot be too large. This assumption helps us to reduce our problem. It ensures the objective function we are going to deal with does not explode, as intuitively market volatility risk associated with a significantly large amount of shares can be severe, which may cause some degeneracy. Assumption 1.8 excludes a degenerate case of our reduced problem (see the discussion after equation (1.14)).

Define function $\kappa_A : [0, \bar{\delta}_A) \to \mathbb{R}$ by $\kappa_A(x) = \kappa(-Ax)$, where κ is the cumulant generating function of L_1. This function will play an important role in the sequel and it has the following properties.

Lemma 1.9. *The function κ_A possesses the following properties*

(i) $\kappa_A(0) = 0$*;*

(ii) κ_A *is strictly convex;*

(iii) if $\mu = 0$*, then* $\lim_{x\to 0} \frac{\kappa_A(x)}{x^2} = K$*, for some constant* $K > 0$*;*

(iv) if $\mu \neq 0$*, then* $\lim_{x\to 0} \frac{\kappa_A(x)}{x} = -A\mu$*.*

Lemma 1.10. *Let Y be a continuous process starting form $y \in [0, \bar{\delta}_A)$. Then*

$$\int_0^\infty \|Y_t\|^i_{L^\infty(\mathbb{P})}\, dt < \infty$$

if and only if

$$\int_0^\infty \kappa_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big)\, du < \infty,$$

where $i = 1$ if $\mu < 0$, and $i = 2$ if $\mu = 0$. Moreover, with $\mu > 0$,

$$\int_0^\infty \|Y_t\|_{L^\infty(\mathbb{P})}\, dt < \infty$$

implies

$$\int_0^\infty \kappa_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big)\, du < \infty.$$

In order to reduce problem (1.9), we also require the following technical result.

Lemma 1.11. *For any $Y \in \mathcal{A}(y)$, the process M^Y given by*

$$M_t^Y = \exp\bigg(\int_0^t -AY_{u-}\, dL_u - \int_0^t \kappa_A(Y_u)\, du\bigg), \qquad t \geq 0, \tag{1.10}$$

is a uniformly integrable martingale.

It follows from Lemma 1.10 and Lemma 1.11 that, for any $Y \in \mathcal{A}(y)$, the process M^Y is a strictly positive martingale closed by M_∞^Y. We can therefore define a new probability measure $\mathbb{Q}^Y$ by

$$\frac{d\mathbb{Q}^Y}{d\mathbb{P}} = M_\infty^Y.$$

Based on the idea in Schied et al. (2010) Theorem 2.8, and with reference to (1.9) and Lemma 1.11, we calculate that

$$\begin{aligned}
&\inf_{Y \in \mathcal{A}(y)} \mathbb{E}\bigg[\exp\bigg(-\int_0^\infty AY_{t-}\, dL_t + A\int_0^\infty \xi_t F(\xi_t)\, dt\bigg)\bigg] \\
&= \inf_{Y \in \mathcal{A}(y)} \mathbb{E}\bigg[\exp\bigg(-\int_0^\infty AY_{t-}\, dL_t - \int_0^\infty \kappa_A(Y_t)dt + \int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)dt\bigg)\bigg] \\
&= \inf_{Y \in \mathcal{A}(y)} \mathbb{E}^{\mathbb{Q}^Y}\bigg[\exp\bigg(\int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\, dt\bigg)\bigg] \\
&\leq \inf_{Y \in \mathcal{A}_D(y)} \exp\bigg[\int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\, dt\bigg].
\end{aligned} \tag{1.11}$$

Now suppose that Y^* is a solution to problem (1.11), then it must also be a solution to problem (1.9), and hence an equality holds in (1.11). This is because that otherwise there must be some $\tilde{Y} \in \mathcal{A}_D(y)$ which coincides with some sample path of some $Y \in \mathcal{A}(y)$ such that

$$\exp\bigg[\int_0^\infty \Big(\kappa_A(\tilde{Y}_t) + A\xi_t F(\tilde{\xi}_t)\Big)\, dt\bigg]$$

$$
\begin{aligned}
&< \mathbb{E}^{\mathbb{Q}^Y}\left[\exp\left(\int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\,dt\right)\right] \\
&< \exp\left[\int_0^\infty \Big(\kappa_A(Y_t^*) + A\xi_t F(\xi_t^*)\Big)\,dt\right].
\end{aligned}
$$

This contradicts with Y^* being a solution to problem (1.11). Therefore, it suffices to solve the problem

$$
V(y) = \inf_{Y\in\mathcal{A}_D(y)} J(Y), \quad y \in [0, \bar{\delta}_A) \tag{1.12}
$$

where V denotes the value function and J is given by

$$
J(Y) = \int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\,dt. \tag{1.13}
$$

If we take $Y \in \mathcal{A}_D(y)$ such that $Y_t = \big(t - \sqrt{y}\big)^2$, for $t \in [0, \sqrt{y}]$, and $Y_t = 0$, for $t > \sqrt{y}$, then it can be checked that

$$
J(Y) = \int_0^{\sqrt{y}} \left(\kappa_A\Big(\big(t-\sqrt{y}\big)^2\Big) + A\big(2\sqrt{y} - 2t\big)F\big(2\sqrt{y} - 2t\big)\right) dt < \infty, \tag{1.14}
$$

which implies that $V < \infty$. Lemma 1.9 implies $\kappa_A \geq 0$, if $\mu \leq 0$. Hence we have $0 \leq V < \infty$, for all $\mu \leq 0$.

Assumption 1.8 excludes some degeneracy. To see this, suppose $\mu > 0$. Then Lemma 1.9 (iv) implies that there exists some constant $z > 0$ such that $-\infty < \kappa_A(z) < 0$. Suppose that the investor's initial stock position is z and let's consider the strategy $Y \in \mathcal{A}_D(z)$ satisfying $Y_t' = -\xi_t = 0$ for $t \in [0, s]$ with some $s > 0$. Then

$$
V(z) \leq \int_0^s \kappa_A(z)\,dt + V(z) = s\kappa_A(z) + V(z).
$$

This can happen only if $V(z) = -\infty$. Let $\bar{Y} \in \mathcal{A}_D(y)$ with $y \geq z$ and set $t_z = \inf\{t \geq 0 \mid \bar{Y}_t = z\} < \infty$. Then

$$
V(y) \leq \int_0^{t_z} \Big(\kappa_A(\bar{Y}_t) + A\bar{\xi}_t F(\bar{\xi}_t)\Big)\,dt + V(z),
$$

which implies that $V(y) = -\infty$. As z can be chosen to be arbitrarily close to zero, it follows that $V(y) = -\infty$, for all $y \in (0, \bar{\delta}_A)$. We therefore conclude that the value function is degenerate when $\mu > 0$. Let $y \in (0, \bar{\delta}_A)$, and suppose (in order to get a contradiction) that there exists an optimal strategy $Y^* \in \mathcal{A}_D(y)$. Define $\tilde{\kappa}_A$ to be the function which is identical to κ_A with $\mu = 0$. Then with reference to the Lévy-Khintchine representation of L (see (1.35)), we can write $\kappa_A(x) = -A\mu x + \tilde{\kappa}_A(x)$. By Assumption 1.3 and Lemma 1.9, we have that $\tilde{\kappa}_A(Y_t^*) + A\xi_t F(\xi_t^*)$ is positive. Thus,

$$V(y) = \int_0^\infty \Big(-A\mu Y_t^* + \tilde{\kappa}_A(Y_t^*) + A\xi_t^* F(\xi_t^*) \Big)\, dt = -\infty, \quad \mu > 0,$$

implies $\int_0^\infty Y_t^*\, dt = \infty$, which contradicts the definition of an admissible strategy. We conclude that if $\mu > 0$, then there is no optimal admissible liquidation strategy.

Before finishing this section, we give out the following two remarks. Remark 1.12 compares the CARA utility to the mean-variance optimisation criterion for our problem, and Remark 1.13 discusses that it is not optimal to buy back during the liquidation in our setting.

Remark 1.12. It is mentioned in Schied et al. (2010) that in the Almgren-Chriss model with Brownian motion describing the unaffected stock price, the problem of optimising the finial cost/reward of a CARA investor over a set of adapted strategies is equivalent to the same problem but with a mean-variance optimisation criterion and over the corresponding set of deterministic strategies. Nevertheless, this is not the case in our model, i.e. this equivalence does not hold if the unaffected stock price is modelled by a general Lévy process. To see this, as we know in our problem, the set of admissible strategies $\mathcal{A}(y)$ can be replaced by $\mathcal{A}_D(y)$. Then in view of (1.8), it suffices to consider

$$\inf_{Y \in \mathcal{A}_D(y)} \mathbb{E}\big[\mathrm{e}^{-AC_\infty^Y}\big],$$

where

$$C_\infty^Y = c + sy - \frac{1}{2}\alpha y^2 + \int_0^\infty Y_{t-}\, dL_t - \int_0^\infty \xi_t F(\xi_t)\, dt.$$

Let's try to express $\mathbb{E}\big[e^{-AC_\infty^Y}\big]$ in terms of $\mathbb{E}[C_\infty^Y]$ and $\mathrm{Var}(C_\infty^Y)$. It can be calculated that

$$\mathbb{E}[C_\infty^Y] = c + sy - \frac{1}{2}\alpha y^2 + \mu \int_0^\infty Y_t\, dt - \int_0^\infty \xi_t F(\xi_t)\, dt$$

and

$$\mathrm{Var}(C_\infty^Y) = \sigma^2 \int_0^\infty Y_t^2\, dt + \int_0^\infty \Big(\int_{\mathbb{R}} Y_t^2 x^2\, \nu(dx) \Big)\, dt.$$

Then,

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(-AC_\infty^Y\big)\big] \\
&= \exp\bigg[-A\mathbb{E}[C_\infty^Y] + \frac{1}{2}A^2\sigma^2\int_0^\infty Y_t^2\,dt + \int_0^\infty\int_{\mathbb{R}}\Big(e^{-AY_tx} - 1 + AY_tx\Big)\,\nu(dx)\,dt\bigg] \\
&= \exp\bigg[-A\mathbb{E}[C_\infty^Y] + \frac{1}{2}A^2\mathrm{Var}(C_\infty^Y) + \int_0^\infty\int_{\mathbb{R}}\Big(e^{-AY_tx} - 1 + AY_tx - \frac{1}{2}A^2Y_t^2x^2\Big)\,\nu(dx)\,dt\bigg].
\end{aligned}
$$

From the above expression, it is clear that the problem is equivalent to

$$
\sup_{Y\in\mathcal{A}_D(y)} \mathbb{E}[C_\infty^Y] - \frac{1}{2}A\mathrm{Var}(C_\infty^Y),
$$

if $\nu(\mathbb{R}) \equiv 0$, i.e. the Lévy process L has no jumps. However, for any general Lévy process, this equivalence does not hold. $\square$

Remark 1.13. Let's suppose that the large investor is allowed to buy shares. In this situation, in order to well-define the finial cash position, in addition to the conditions in Definition 1.2, we assume that any admissible strategy Y satisfies $\lim_{t\to\infty} t\|Y_t\|_{L^\infty(\mathbb{P})} = 0$ (see Lemma 1.22 and proof of Proposition 1.5 for more details). We also suppose Y is positive-valued, $Y_t < \bar{\delta}_A$ for all $t \geq 0$, and it admits $Y_t = y + \int_0^t \xi_u\,du$ with $\xi_t \in \mathbb{R}$. Denote by $\mathcal{A}^\pm(y)$ the set of all admissible strategies, and by $\mathcal{A}_D^\pm(y)$ the collection of all deterministic admissible strategies. Then the liquidation problem can be reduced in a similar way as before to be that

$$
V(y) = \inf_{Y\in\mathcal{A}_D^\pm(y)} \int_0^\infty \Big(\kappa_A(Y_t) + A|\xi_t|F\big(|\xi_t|\big)\Big)\,dt.
$$

Let $Y \in \mathcal{A}_D^\pm(y)$ be a strategy including intermediate buying. Then there exists two time points r and s with $r < s$ such that $Y_r = Y_s$ and $Y_t > Y_r$ for all $t \in (r,s)$. Consider the admissible strategy X such that $X_t = Y_r$ for $t \in (r,s)$ and $X_t = Y_t$ for $t \in [0,r]\cup[s,\infty)$. Then with reference to Lemma 1.9,

$$
\int_r^s \Big(\kappa_A(X_u) + A|\xi_u^X|F\big(|\xi_u^X|\big)\Big)\,du = \kappa_A(X_r)(s-r) < \int_r^s \Big(\kappa_A(Y_u) + A|\xi_u|F\big(|\xi_u|\big)\Big)\,du,
$$

where ξ^X is the speed process associated with X. Therefore, $J(X) < J(Y)$. This shows Y is not optimal. $\square$

1.3 Solution to the problem

With reference to the previous section, recall that the original optimal liquidation problem (1.7) is equivalent to solving

$$V(y) = \inf_{Y \in \mathcal{A}_D(y)} \int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t) \Big)\, dt,$$

with

$$dY_t = -\xi_t\, dt, \qquad Y_0 = y \in [0, \bar{\delta}_A).$$

According to the theory of optimal control, the corresponding Hamilton-Jacobi-Bellman equation is given by

$$\kappa_A(y) + \inf_{x \geq 0} \big\{ AxF(x) - xv'(y) \big\} = 0, \tag{1.15}$$

with associated boundary condition $v(0) = 0$. Define $G : [0, \infty) \to [0, \infty)$ be the inverse function of $x \mapsto x^2 F'(x)$. Assumption 1.3 and Lemma 1.4 together imply that G is a continuous, strictly increasing function satisfying $G(0) = 0$. The following result gives out a classical solution to (1.15).

Proposition 1.14. *Equation (1.15) with boundary condition $v(0) = 0$ has a classical solution given by*

$$v(y) = \int_0^y \left\{ \frac{\kappa_A(u)}{G\big(\frac{\kappa_A(u)}{A}\big)} + AF\left(G\left(\frac{\kappa_A(u)}{A} \right) \right) \right\} du, \qquad 0 \leq y < \bar{\delta}_A. \tag{1.16}$$

The next theorem gives out the optimal liquidation strategy, and it identifies the value function V with the function v in (1.16).

Theorem 1.15. *Let $y \in [0, \bar{\delta}_A)$. Define*

$$\tau = \int_0^y \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\, du. \tag{1.17}$$

Let Y^* satisfy

$$\int_{Y_t^*}^y \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\, du = t, \quad \text{if } t \leq \tau, \qquad \text{and} \qquad Y_t^* = 0, \quad \text{if } t > \tau. \tag{1.18}$$

Then $Y^* \in \mathcal{A}_D(y)$, and its associated speed process ξ^* satisfies

$$\xi_t^* = G\left(\frac{\kappa_A(Y_t^*)}{A}\right), \qquad \textit{for all } t \geq 0. \tag{1.19}$$

Moreover, V in (1.12) is equal to v in (1.16), for all $y \in [0, \bar{\delta}_A)$, and Y^* is the unique optimal liquidation strategy for problem (1.7).

Note that because of the continuity of function G, (1.19) implies that the strategy Y^* in (1.18) is continuously differentiable. Since functions κ_A and G are both strictly increasing, it follows from (1.19) that with a larger stock position at time t, the associated optimal liquidation speed at time t is larger. Moreover, it can be shown by the strict convexity of the cumulant generating function of L_1 that $A \mapsto \kappa_A(x)/A$ is strictly increasing. Hence, the optimal liquidation speed at any time is strictly increasing in the risk aversion parameter A. These two relations coincide with the intuition that with a larger position in stock, the investor potentially encounters bigger risk from the market volatility, as any tiny fluctuation of stock price can be amplified by huge number of shares held, therefore it is optimal to liquidate faster; and that if the investor is more risk averse, then he cares more about the volatility risk, which makes him to employ a liquidation strategy with larger speed of sale. Observe that given an initial stock position $y \in [0, \bar{\delta}_A)$, the quantity τ in (1.17) indicates the first time of the stock position getting 0, if the large investor liquidates following the optimal strategy Y^*. Depending on properties the temporary impact function F, τ may or may not be finite, i.e. it happens in some cases that liquidation can optimally finish in a finite time period, even though there is no restriction on terminal time. The next theorem gives out some sufficient conditions of whether the optimal liquidation strategy Y^* has an endogenous time of termination.

Proposition 1.16. *Under the condition that $y > 0$*

(i) suppose $\mu < 0$ and there exist constants $p < 1$ and $K > 0$ such that $\lim_{x \to 0} x^p F'(x) = K$, then $\tau < \infty$.

(ii) suppose $\mu = 0$ and there exist constants $p < 1$ and $K > 0$ such that $\lim_{x \to 0} x^p F'(x) = K$. If $p \in [0, 1)$, then $\tau = \infty$. If $p < 0$, then $\tau < \infty$.

1.4 Approximation for exponential Lévy model

To model stock prices using Lévy processes, it is more natural to consider exponential Lévy processes (see e.g. Madan and Seneta, 1990; Eberlein and Keller, 1995; Barndorff-Nielsen, 1997, etc). However, due to the mathematical complexity of exponential Lévy processes, the corresponding liquidation model is not tractable. Instead of dealing with an exponential Lévy model directly, we try to approximate such model using the liquidation model established before. To this end, we are going to derive a Lévy process which can be regarded as a linear approximation for a corresponding exponential Lévy process. We show that this Lévy process satisfies all of the assumptions of being a driving process of the unaffected stock price in the liquidation model introduced in previous sections. Therefore, our optimal liquidation strategy derived in the previous section can be regarded as an approximation for the result of the corresponding exponential Lévy model. This linear approximation argument is reasonable since in practice liquidation can usually finish in a very short time period.

Let's first introduce a liquidation model with exponential Lévy processes. Consider a non-trivial, one dimensional, $\mathbb{F}$-adapted Lévy process $\tilde{L}$ which admits the canonical decomposition

$$\tilde{L}_t = \tilde{\mu} t + \tilde{\sigma} \tilde{W}_t + \int_{|z|\geq 1} z \, \tilde{N}(t, dz) + \int_{|z|<1} z \left(\tilde{N}(t, dz) - t\tilde{\nu}(dz) \right), \quad t \geq 0, \tag{1.20}$$

where $\tilde{\mu} \in \mathbb{R}$ and $\tilde{\sigma} \geq 0$ are two constants, $\tilde{W}$ is a standard Brownian motion, $\tilde{N}$ is a Poisson random measure which is independent of $\tilde{W}$ with compensator $t\tilde{\nu}(dz)$, and $\tilde{\nu}$ is the Lévy measure associated with $\tilde{L}$. We assume that $\tilde{L}$ possesses the following properties.

Assumption 1.17. We assume that $\tilde{\nu}$ is absolutely continuous with respect to Lebesgue measure, and that

$$\int_{|z|\geq 1} \mathrm{e}^{2z} \, \tilde{\nu}(dz) < \infty. \tag{1.21}$$

Suppose the unaffected stock price is described by the process $\tilde{S}^u$ satisfying

$$\tilde{S}^u_t = \tilde{s} \exp\big(\tilde{L}_t\big), \quad t \geq 0,$$

where $\tilde{s} > 0$ is some constant denoting the initial stock price. Note that (1.21) ensures $\tilde{S}^u_t$ to be square integrable, for all $t \geq 0$ (see e.g. Kyprianou, 2006, Theorem 3.6). Suppose the affected stock price at time $t \geq 0$ is given by

$$\tilde{S}_t = \tilde{s} \exp\bigl(\tilde{L}_t\bigr) + I_t,$$

where $I_t = \alpha(Y_t - Y_0) - F(\xi_t)$ is the price impact at time t appearing in the previous liquidation model with function F satisfying Assumption 1.3 (Gatheral and Schied, 2011, study a liquidation model with the affected stock price in this form with a geometric Brownian motion). By Itô's formula, for all $t \geq 0$, $\tilde{S}_t$ can be rewritten as

$$\tilde{S}_t = \tilde{s} + \int_0^t \tilde{S}^u_{u-} \tilde{m}\, du + \int_0^t \tilde{S}^u_{u-} \tilde{\sigma}\, d\tilde{W}_u + \int_0^t \int_{\mathbb{R}} \tilde{S}^u_{u-} \bigl(\mathrm{e}^z - 1\bigr) \bigl(\tilde{N}(t, dz) - t\tilde{\nu}(dz)\bigr) + I_t,$$

where $\tilde{m} = \tilde{\mu} + \frac{\tilde{\sigma}^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z\mathbb{1}_{\{|z|<1\}})\, \tilde{\nu}(dz)$. In order to approximate the exponential Lévy model, consider the process $\hat{S}$ such that

$$\hat{S}_t = \tilde{s} + \tilde{s}\tilde{m}t + \tilde{s}\tilde{\sigma}\tilde{W}_t + \int_{\mathbb{R}} \tilde{s}\bigl(\mathrm{e}^z - 1\bigr) \bigl(\tilde{N}(t, dz) - t\tilde{\nu}(dz)\bigr) + I_t, \quad t \geq 0,$$

which can be considered as a linear approximation of $\tilde{S}$. Recall that the affected stock price in the preceding model is given by

$$S_t = s + L_t + I_t, \quad t \geq 0,$$

where $L_t = \mu t + \sigma W_t + \int_{\mathbb{R}} x \bigl(N(t, dx) - t\nu(dx)\bigr)$. Comparing this to the expression of $\hat{S}_t$, it can be seen that if we take $s = \tilde{s}$ and choose L to be such that

$$L_t = \tilde{s}\tilde{m}t + \tilde{s}\tilde{\sigma}\tilde{W}_t + \int_{\mathbb{R}} \tilde{s}\bigl(\mathrm{e}^z - 1\bigr) \bigl(\tilde{N}(t, dz) - t\tilde{\nu}(dz)\bigr), \qquad t \geq 0, \tag{1.22}$$

then it follows that

$$\hat{S}_t = \tilde{s} + L_t + I_t, \qquad \text{for all } t \geq 0.$$

We may therefore consider $\hat{S}$ as the affected stock price process in the liquidation model introduced in previous sections. The next proposition verifies that L with the above expression is a Lévy process satisfying Assumption 1.1.

Proposition 1.18. *Let L be given by (1.22). Write $\hat{L} = L/\tilde{s}$. Then $\hat{L}$ is an $\mathbb{F}$-adapted Lévy* *process whose Lévy measure, denoted by $\hat{\nu}$, satisfies*

$$\hat{\nu}(dx) = \frac{1}{x+1}\,\tilde{f}\Big(\ln(x+1)\Big)\,dx, \qquad x > -1,\, x \neq 0.$$

Therefore, L is an $\mathbb{F}$-adapted Lévy process satisfying Assumption 1.1.

Remark 1.19. From equation (1.50) (in the proof of Proposition 1.18) we know that

$$\int_{|x|\geq 1} \mathrm{e}^{ux}\,\hat{\nu}(dx) < \infty, \qquad \text{for all } \; u \leq 0.$$

This implies that $\bar{\delta}$ given by (1.1) is equal to $+\infty$, and therefore, Assumption 1.7 is satisfied for any initial stock position $y > 0$. In other words, if we consider an exponential Lévy model and use the approximation scheme discussed above, we do not need to concern any restriction on the maximum volume of liquidation. □

With L given by (1.22) and $\hat{L}$ defined in Proposition 1.18, in view of (1.12)-(1.13) we consider the optimisation problem

$$V(y) = \inf_{Y\in\mathcal{A}_D(y)} \int_0^{\infty} \Big(\hat{\kappa}_{\tilde{A}}(Y_t) + A\xi_t F(\xi_t)\Big)\,dt, \quad y \geq 0, \tag{1.23}$$

where $A > 0$ denotes the investor's risk aversion, $\tilde{A} = A\tilde{s}$ and $\hat{\kappa}_{\tilde{A}} : [0,\infty) \to [0,\infty)$ is defined by $\hat{\kappa}_{\tilde{A}}(x) = \hat{\kappa}(-\tilde{A}x)$ with $\hat{\kappa}$ being the cumulant generating function of $\hat{L}_1$.

Theorem 1.20. *The unique optimal liquidation speed for problem (1.23) is given by*

$$\xi_t^* = G\bigg(\frac{\hat{\kappa}_{\tilde{A}}(Y_t^*)}{A}\bigg), \quad t \geq 0, \tag{1.24}$$

where $G : [0,\infty) \to [0,\infty)$ is the inverse function of $x \mapsto x^2F'(x)$ and Y^* is the associated unique optimal admissible stock position process satisfying

$$\int_{Y_t^*}^{y} \frac{1}{G\big(\frac{\hat{\kappa}_{\tilde{A}}(u)}{A}\big)}\,du = t, \quad \text{if } t \leq \tau, \qquad \text{and} \qquad Y_t^* = 0, \quad \text{if } t > \tau,$$

with τ defined by

$$\tau = \int_0^y \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\,du.$$

The value function in (1.23) satisfies

$$V(y) = \int_0^y \left\{ \frac{\hat{\kappa}_{\tilde{A}}(u)}{G\big(\frac{\hat{\kappa}_{\tilde{A}}(u)}{A}\big)} + AF\left(G\left(\frac{\hat{\kappa}_{\tilde{A}}(u)}{A} \right) \right) \right\} du, \quad y \geq 0.$$

1.5 Numerical examples

In this section, we give out some numerical examples following the approximation scheme discussed in the previous section. We consider the process $\tilde{L}$ in (1.20) as a variance gamma (VG) Lévy process, which is obtained by subordinating a Brownian motion using a gamma process. Precisely, we consider $\tilde{L}$ to be such that

$$\tilde{L}_t = \theta \tau_t + \rho W_{\tau_t}, \quad t \geq 0,$$

where $\theta \in \mathbb{R}$ and $\rho > 0$ are some constants, W is a standard Brownian motion and τ is a gamma process such that $\tau_t \sim \Gamma\big(\frac{t}{\eta}, \frac{1}{\eta}\big)$ [1], for some constant $\eta > 0$. Then $\tilde{L}$ is a VG Lévy process whose Lévy density is given by

$$\tilde{f}(z) = \frac{1}{\eta |z|} \mathrm{e}^{Cz - D|z|}, \quad z \in \mathbb{R},$$

where

$$C = \frac{\theta}{\rho^2} \qquad \text{and} \qquad D = \frac{\sqrt{\theta^2 + \frac{2\rho^2}{\eta}}}{\rho^2},$$

and its cumulant generating function $\tilde{\kappa}$ admits the expression

$$\tilde{\kappa}(x) = -\frac{1}{\eta} \ln\left(1 - \frac{x^2 \rho^2 \eta}{2} - \theta \eta x \right) \tag{1.25}$$

(see e.g. Cont and Tankov, 2004). It can be shown that Assumption 1.17 is satisfied if $D - C > 2$. We calculate according to Proposition 1.18 that the Lévy measure $\hat{\nu}$ of process

[1] $\Gamma(a, b)$ denotes a gamma distribution with shape parameter $a > 0$ and rate parameter $b > 0$, for which the probability density function is given by $f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \mathrm{e}^{-bx}$, for $x > 0$, where $\Gamma(\cdot)$ is the gamma function. For any $X \sim \Gamma(a, b)$, $\mathbb{E}[X] = \frac{a}{b}$ and $\mathrm{Var}[X] = \frac{a}{b^2}$.

$\hat{L}$ satisfies

$$\hat{\nu}(dx) = \begin{cases} \dfrac{-1}{\eta \ln(x+1)}(x+1)^{C+D-1}\, dx, & x \in (-1,0), \\ \dfrac{1}{\eta \ln(x+1)}(x+1)^{C-D-1}\, dx, & x \in (0,\infty). \end{cases}$$

Therefore, the function $\hat{\kappa}_{\tilde{A}} : [0,\infty) \to [0,\infty)$ in (1.23), denoting it by $\hat{\kappa}^{VG}_{\tilde{A}}$ in the example of VG Lévy process, is given by

$$\hat{\kappa}^{VG}_{\tilde{A}}(u) = -\tilde{A}\tilde{m}u + \int_{-1}^{\infty} \Big(\mathrm{e}^{-\tilde{A}ux} - 1 + \tilde{A}ux \Big)\, \hat{\nu}(dx), \tag{1.26}$$

where the drift parameter $\tilde{m} = \tilde{\kappa}(1)$.

The next result gives out an explicit expression of a lower bound of $\hat{\kappa}^{VG}_{\tilde{A}}$, which will be useful for deciding the limit behaviour of price impact function later.

Proposition 1.21. *For $u \geq 0$, write*

$$\begin{aligned} \underline{\hat{\kappa}}^{VG}_{\tilde{A}}(u) = -\tilde{A}\tilde{m}u + \frac{\mathrm{e}}{\eta}\Bigg[&-\frac{\mathrm{e}^{\tilde{A}u}\tilde{A}u}{C+D+2}\Big(\frac{1}{\tilde{A}u} \wedge 1\Big)^{C+D+2} + \frac{\mathrm{e}^{\tilde{A}u}}{C+D+1}\Big(\frac{1}{\tilde{A}u} \wedge 1\Big)^{C+D+1} \\ &+ \frac{\tilde{A}u}{C+D+2} - \frac{1+\tilde{A}u}{C+D+1}\Bigg], \end{aligned}$$

and in particular, for $u \geq \frac{1}{\tilde{A}}$,

$$\begin{aligned} \underline{\hat{\kappa}}^{VG}_{\tilde{A}}(u) = -\tilde{A}\tilde{m}u + \frac{\mathrm{e}}{\eta}\Bigg[&\Big(\frac{1}{\tilde{A}u}\Big)^{C+D+1} \mathrm{e}^{\tilde{A}u}\Big(\frac{1}{C+D+1} - \frac{1}{C+D+2}\Big) \\ &+ \frac{\tilde{A}u}{C+D+2} - \frac{1+\tilde{A}u}{C+D+1}\Bigg]. \end{aligned}$$

Then we have $\hat{\kappa}^{VG}_{\tilde{A}}(u) \geq \underline{\hat{\kappa}}^{VG}_{\tilde{A}}(u)$, for all $u \geq 0$.

In order to get a comparison between a liquidation model with a VG Lévy process and a liquidation model with a Brownian motion, when $\tilde{L}$ is considered as a Brownian motion, we derive that the function $\hat{\kappa}_{\tilde{A}} : [0,\infty) \to [0,\infty)$ in (1.23), denoting it by $\hat{\kappa}^{BM}_{\tilde{A}}$, has expression

$$\hat{\kappa}^{BM}_{\tilde{A}}(u) = -\tilde{A}\Big(\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}\Big)u + \frac{1}{2}\tilde{A}^2\tilde{\sigma}^2u^2, \tag{1.27}$$

where $\tilde{\mu} \in \mathbb{R}$ and $\tilde{\sigma} > 0$ are some constants which represent drift and volatility of $\tilde{L}$, respectively. In the case of Brownian motion, Assumption 1.17 is always satisfied.

Throughout this section, we use the following reasonable daily data for our VG Lévy process. We take $\theta = -0.002$, $\rho = 0.02$ and $\eta = 0.6$. For more details about empirical studies of parameters of VG stock price model, we refer to Rathgeber et al. (2013). For parameters in the Brownian motion case, in order to get a good comparison, we make the expectation and the second moment of $\mathrm{e}^{\tilde{L}_t}$ when $\tilde{L}$ is considered as a Brownian motion to be the same as when it is considered as a VG Lévy process. Hence, $\tilde{\mu}$ and $\tilde{\sigma}$ in (1.27) are taken to be such that $\tilde{\mu} + \frac{\tilde{\sigma}^2}{2} = \tilde{\kappa}(1)$ and $2\tilde{\mu} + 2\tilde{\sigma}^2 = \tilde{\kappa}(2)$, where $\tilde{\kappa}$ is given by (1.25). Therefore, throughout this section,

$$\tilde{\mu} = 2\tilde{\kappa}(1) - \frac{\tilde{\kappa}(2)}{2} \qquad \text{and} \qquad \tilde{\sigma}^2 = \tilde{\kappa}(2) - 2\tilde{\kappa}(1).$$

Moreover, we choose $\tilde{s} = 100$ for simplicity.

1.5.1 Power-law price impact function

Consider the power-law temporary impact function, i.e. $F : [0, \infty) \to [0, \infty)$ is given by

$$F(x) = \beta x^{\gamma},$$

where $\beta > 0$ and $\gamma > 0$ are constants. This kind of impact function is widely believed and has been well-studied in the literature of price impact (see e.g. Lillo et al., 2003; Almgren et al., 2005, etc). It can be checked that F satisfies Assumption 1.3, and the function G appearing in (1.24) is given by

$$G(x) = \left(\frac{x}{\beta\gamma}\right)^{\frac{1}{\gamma+1}}, \qquad x \geq 0.$$

Applying Proposition 1.16, we see that if $\hat{L}$ is a strict supermartingale, then τ in (1.17) is finite, for all $\gamma > 0$; if $\hat{L}$ is a martingale, then $\tau = \infty$ for $\gamma \in (0, 1]$, and $\tau < \infty$ when $\gamma > 1$. It follows from (1.24) that the optimal liquidation speed takes the expression

$$\xi_t^* = \left(\frac{\hat{\kappa}_{\tilde{A}}(Y_t^*)}{A\beta\gamma}\right)^{\frac{1}{\gamma+1}}, \qquad \text{for all } t \geq 0. \tag{1.28}$$

We adopt the values of β and γ suggested in Almgren et al. (2005) where parameters of the power-law temporary impact are studied empirically. Particularly, we take $\gamma = 0.6$ and choose $\beta = 4.7 \times 10^{-5}$ [2].

Consider a stock with average daily volume 2×10^6. Suppose the investor wants to liquidate a position of 2×10^5 [3] of this stock. Figure 1.1 gives out optimal liquidation trajectories in both VG Lévy process case and Brownian motion case when the risk aversion parameter A takes values of 10^{-6}, 10^{-5} and 10^{-4} [4] . We see that when $A = 10^{-6}$, optimal strategies for two models are almost identical. As A increases, optimal speeds increase in both models, and in particular, speeds increase much faster in VG model for big positions. In each case, liquidation finishes in a short time period, which confirms that the linear approximation scheme of exponential model is reasonable. Now we may make a conclusion that if one believes that the unaffected stock price follows an exponential VG Lévy process and the temporary price impact is described by a 0.6 power-law, then optimal liquidation strategy for the Brownian motion model is suboptimal unless A is very small.

As shown in the first graph of Figure 1.1 that when $A = 10^{-5}$ and $A = 10^{-4}$, at the beginning of liquidation, stock positions drop immediately by a large proportion of its initial value. In order to get more details about these two trajectories, we compute that when $A = 10^{-5}$, time spent on liquidating 40% of 2×10^5 shares is about 0.00018, if the investor follows the optimal strategy for the VG case. Suppose the time parametrisation is the same as clock time, then 0.00018 is just a few seconds. If the investor's risk aversion takes the value 10^{-4}, then according to the optimal strategy for VG model, he spends roughly 1.34×10^{-14}

[2] With our notations, the temporary impact function F in Almgren et al. (2005) is given by $F(x) = \beta x^\gamma = \tilde{S}_{0-}\tilde{\beta}\tilde{\sigma}\left(\frac{x}{\tilde{V}}\right)^\gamma$, where $\tilde{V}$ denotes the daily volume of a given stock, the value of exponent γ is argued to be 0.6 (as the main result in their paper) and $\tilde{\beta}$ is a constant which is suggested to be 0.142. From the values of parameters of the VG Lévy process that we have chosen, it can be calculated that the volatility $\tilde{\sigma}$ in the Brownian motion case is roughly equal to 0.02. Comparing this number to the values of volatilities and daily volumes of stocks provided in examples in Almgren et al. (2005), we may take $\tilde{V} = 2 \times 10^6$ as a reasonable choice. Moreover, we choose $\tilde{s} = 100$ for simplicity. Then β is calculated to be 4.7×10^{-5}.

Note that the empirical study in Almgren et al. (2005) is based on a model parametrised by the volume time which is defined as fractions of a daily volume. Therefore, any results of number regarding time derived from a model with power-law impact function in this section should be interpreted as volume time.

[3] Since the study of parameters of impacts in Almgren et al. (2005) is based on liquidating the amount of shares that weighted as 10% of daily volume, in order to keep consistent with the values of parameters of the temporary impact function that we have chosen, we let the initial stock position to be 2×10^5 which is 10% of the daily volume that we have chosen as explained before.

[4] It seems that these values of A may be too small, however, they are reasonable in a liquidation model, and can be understood as that the investor is not sensitive to any large costs which are insignificant comparing to his total wealth. We refer to Almgren and Chriss (2000) and Almgren (2003) for more details about the risk aversion parameter for the Almgren-Chirss liquidation model.

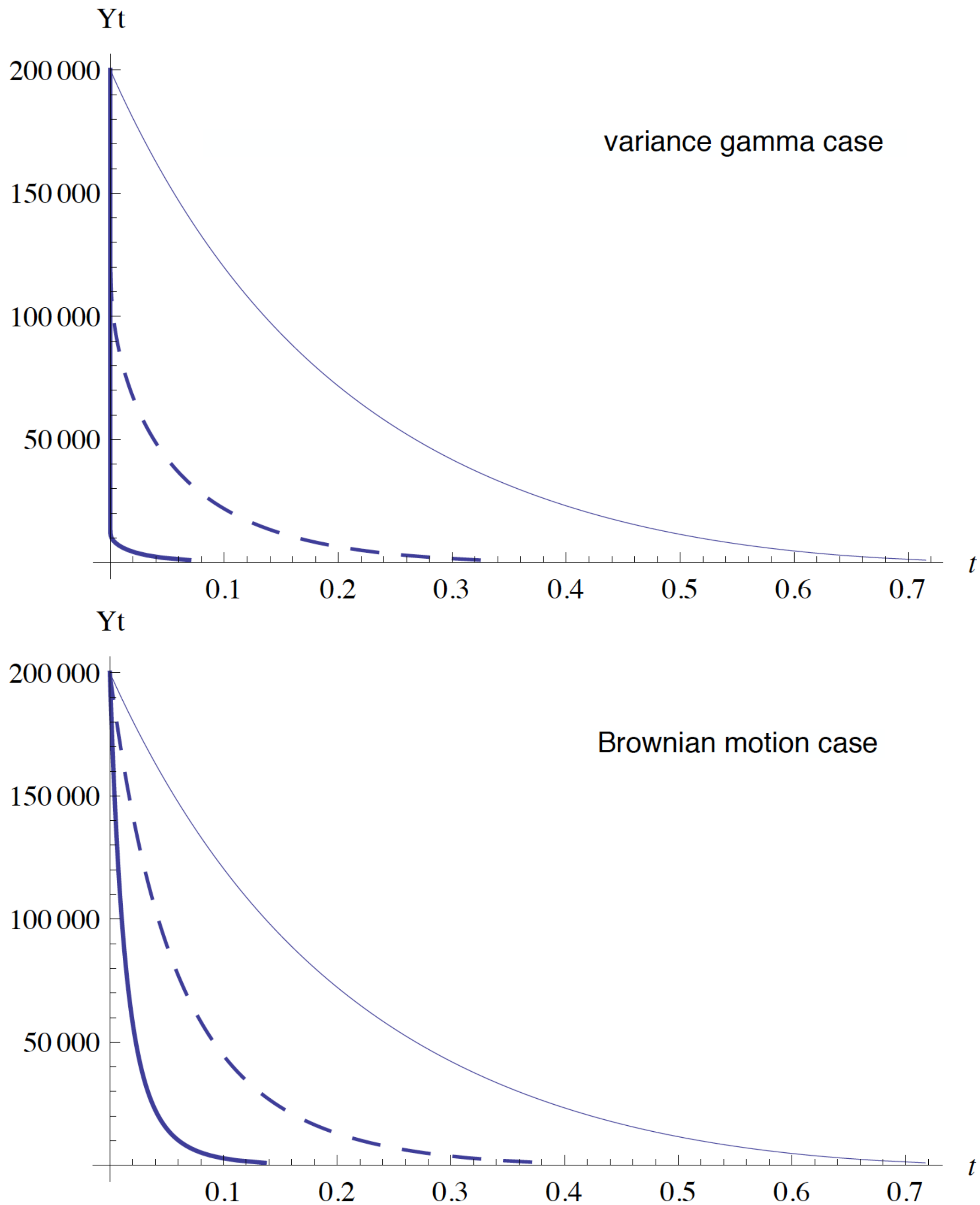

Figure 1.1: Optimal liquidation trajectories for variance gamma Lévy process model and Brownian motion model with 0.6 power-law temporary impact function. Thin curves are for $A = 10^{-6}$, dashed curves are when $A = 10^{-5}$ and thick curves are for $A = 10^{-4}$.

amount of time to liquidate 90% of his initial position.

With a large stock position, due to the nature of jumps of VG Lévy process, as we expect that the investor should liquidate much faster compare to using the optimal strategy from the Brownian motion model. However, as the above examples show that with the 0.6 power-law temporary impact function, in the VG case, optimal liquidation speeds can be too large so that the strategies are infeasible in practice, while speeds in the Brownian motion model stay in a reasonable range. Intuitively, an unreasonably high optimal liquidation speed is due to that price impact for a large trading speed is sub-estimated. In other words, cost resulted from large speeds is too small. This argument can be confirmed by the expression of the optimal liquidation speed in (1.24) that if the temporary impact function F has a small growth rate, then growth rate of function G is large, and therefore optimal speed can be very high, when stock position is large. It is mentioned in Roşu (2009); Gatheral (2010), etc that impact function should be concave for small trading speeds, and for large speeds it is convex. However, to the best of our knowledge, there are no suggestions in price impact literature about what exact kind of function is suitable to describe price impact caused by executing large block orders. Therefore, we next try to explore a mode of growth of the price impact function for which the optimal liquidation speed for the Lévy model is reasonable.

1.5.2 An equivalent relation

We derive a connection between a temporary impact function for the Lévy liquidation model and a temporary impact function for the Brownian motion liquidation model such that the optimal strategy for each model coincide with each other. From this connection, a suitable increasing rate of impact function for the Lévy model is indicated.

Let $F^L : [0,\infty) \to [0,\infty)$ and $F^{BM} : [0,\infty) \to [0,\infty)$ be temporary impact functions, satisfying Assumption 1.3, considered in a Lévy model and a Brownian motion model, respectively. Write $G^L : [0,\infty) \to [0,\infty)$ and $G^{BM} : [0,\infty) \to [0,\infty)$ as the inverse functions of $x \mapsto x^2(F^L)'(x)$ and $x \mapsto x^2(F^{BM})'(x)$, respectively. Then in view of (1.24), the optimal liquidation speed at time t for each model, denoted by ξ_t^L and ξ_t^{BM}, are given by

$$\xi_t^L = G^L\left(\frac{\hat{\kappa}_{\tilde{A}}^L(Y_t^L)}{A}\right) \qquad \text{and} \qquad \xi_t^{BM} = G^{BM}\left(\frac{\hat{\kappa}_{\tilde{A}}^{BM}(Y_t^{BM})}{A}\right),$$

where $\hat{\kappa}_{\tilde{A}}^L$ and $\hat{\kappa}_{\tilde{A}}^{BM}$ are different versions for of $\hat{\kappa}_{\tilde{A}}$, and Y^L and Y^{BM} are corresponding optimal liquidation strategies in each model. Suppose for all $t \geq 0$, $Y_t^* = Y_t^L = Y_t^{BM}$, then

$$G^L\left(\frac{\hat{\kappa}_{\tilde{A}}^L(Y_t^*)}{A}\right) = G^{BM}\left(\frac{\hat{\kappa}_{\tilde{A}}^{BM}(Y_t^*)}{A}\right), \qquad t \geq 0. \tag{1.29}$$

Write $z = G^{BM}\left(\frac{\hat{\kappa}_{\tilde{A}}^{BM}(Y_t^*)}{A}\right)$. So by (1.27) we have

$$Y_t^* = \frac{\tilde{u} + \sqrt{\tilde{u}^2 + 2A\tilde{\sigma}^2 z^2 (F^{BM})'(z)}}{\tilde{A}\tilde{\sigma}^2},$$

where $\tilde{u} = \tilde{\mu} + \frac{\tilde{\sigma}^2}{2}$. Then we obtain from (1.29) that

$$(F^L)'(z) = \frac{1}{Az^2}\,\hat{\kappa}_{\tilde{A}}^L\left(\frac{\tilde{u} + \sqrt{\tilde{u}^2 + 2A\tilde{\sigma}^2 z^2 (F^{BM})'(z)}}{\tilde{A}\tilde{\sigma}^2}\right),$$

which is equivalent to

$$F^L(x) = \int_0^x \frac{1}{Az^2}\,\hat{\kappa}_{\tilde{A}}^L\left(\frac{\tilde{u} + \sqrt{\tilde{u}^2 + 2A\tilde{\sigma}^2 z^2 (F^{BM})'(z)}}{\tilde{A}\tilde{\sigma}^2}\right) dz. \tag{1.30}$$

It can be shown that Assumption 1.3 is satisfied by the above expression. We can therefore conclude that if F^L and F^{BM} satisfy (1.30), then $Y^L = Y^{BM}$, provided that the initial stock positions in both Lévy and Brownian motion models are the same; but if (1.30) does not hold, then the Brownian motion model gives out a suboptimal strategy compare to the solution for the Lévy model.

Suppose F^{BM} in (1.30) follows a power-law such that the optimal speed in Brownian motion case is practically reasonable (this kind of model is indeed used in practice), then the relation in (1.30) tells that for optimal speed in VG case being practically reasonable, the function F^L needs to increase to infinity faster than any power functions. This is because that with VG Lévy process, the lower bound of function $\hat{\kappa}_{\tilde{A}}^{VG}$ given in Proposition 1.21 tends to infinity faster than any power functions. Moreover, (1.30) also indicates that there might be a relationship between the distribution of stock returns and the temporary impact function. We will investigate this in our future study.

1.6 Proofs

Proof of Lemma 1.4. For $\lambda \in (0,1)$ and $x \in (0,\infty)$, Assumption 1.3 (ii) and (iii) imply that $F(\lambda x) < \lambda F(x) < F(x)$, which shows that F is strictly increasing.

The derivative of $x \mapsto xF(x)$, together with the convexity of this function, implies that $\lim_{x\to 0} xF'(x)$ exists. As $F'(x) > 0$, for all $x > 0$, it follows that $\lim_{x\to 0} xF'(x) \geq 0$. Suppose $\lim_{x\to 0} xF'(x) > 0$. Then there exist constants $\bar{x} > 0$ and $c > 0$, such that for all $x \in (0,\bar{x})$,

$$F'(x) > \frac{c}{x}.$$

But then,

$$F(\bar{x}) = \lim_{x\to 0} \int_x^{\bar{x}} F'(u)\,du \geq \lim_{x\to 0} \int_x^{\bar{x}} \frac{c}{u}\,du = \infty,$$

which contradicts the continuity of F. Hence, $\lim_{x\to 0} xF'(x) = 0$, and therefore it follows that $\lim_{x\to 0} x^2F'(x) = 0$. $\square$

The next lemma is used in the proof of Proposition 1.5.

Lemma 1.22. *Let Z be a positive-valued, decreasing process satisfying $\int_0^\infty Z_t dt < \infty$. Then $tZ_t \to 0$, as $t \to \infty$.*

Proof. Suppose $\liminf_{t\to\infty} tZ_t > 0$, then there exists some constant c such that

$$\liminf_{t\to\infty} tZ_t > c > 0.$$

This implies that we can find some $s \geq 0$ such that for all $t \geq s$,

$$Z_t > \frac{c}{t}.$$

It follows therefore

$$\int_s^\infty Z_t\,dt \geq \int_s^\infty \frac{c}{t}\,dt = \infty,$$

which contradicts with $\int_0^\infty Z_t\,dt < \infty$. Thus, we have shown that

$$\liminf_{t\to\infty} tZ_t = 0. \tag{1.31}$$

We know that Z is a decreasing process, which is of finite variation. By Itô's formula we calculate that

$$tZ_t = \int_0^t u\,dZ_u + \int_0^t Z_u\,du.$$

It can be observed that $t \mapsto \int_0^t u dZ_u$ is negative and decreasing while $t \mapsto \int_0^t Z_u du$ is positive and increasing. Then,

$$0 \leq \sup_{t\geq r} tZ_t \leq \sup_{t\geq r} \int_0^t u\,dZ_u + \sup_{t\geq r} \int_0^t Z_u\,du = \int_0^r u\,dZ_u + \int_0^\infty Z_u\,du. \tag{1.32}$$

Also,

$$\inf_{t\geq r} tZ_t \geq \inf_{t\geq r} \int_0^t u\,dZ_u + \inf_{t\geq r} \int_0^t Z_u\,du = \int_0^\infty u\,dZ_u + \int_0^r Z_u\,du. \tag{1.33}$$

Taking r to infinity in (1.33) and (1.32), and by (1.31) we have

$$\begin{aligned} 0 &\leq \limsup_{t\to\infty} tZ_t = \lim_{r\to\infty} \sup_{t\geq r} tZ_t \leq \int_0^\infty u\,dZ_u + \int_0^\infty Z_u\,du, \\ 0 &= \liminf_{t\to\infty} tZ_t = \lim_{r\to\infty} \inf_{t\geq r} tZ_t \geq \int_0^\infty u\,dZ_u + \int_0^\infty Z_u\,du. \end{aligned}$$

Therefore, we conclude that $\lim_{t\to\infty} tZ_t = 0$. $\square$

Proof of Proposition 1.5.

(i) Let f be the characteristic function of L_t, so

$$f(u) = \mathbb{E}[\mathrm{e}^{\mathrm{i}uL_t}] = \mathrm{e}^{t\psi(u)},$$

where $\psi(u)$ is given by the Lévy-Khintchine representation of L. By Assumption 1.1 we know that f, hence ψ, are twice differentiable at 0. Hence, we calculate that $f'(0) = \mathrm{i}\mathbb{E}[L_t] = t\psi'(0)$ and $f''(0) = -\mathbb{E}[L_t^2]$, and therefore,

$$\mathbb{E}[L_t^2] = (\mu t)^2 - \psi''(0)t.$$

Then,

$$\mathbb{E}\big[(L_tY_t)^2\big] \leq \mathbb{E}[L_t^2]\|Y_t\|^2_{L^\infty(\mathbb{P})} = \mu^2\big(t\|Y_t\|_{L^\infty(\mathbb{P})}\big)^2 - \psi''(0)t\|Y_t\|^2_{L^\infty(\mathbb{P})}. \tag{1.34}$$

If $\mu \neq 0$, then for any $Y \in \mathcal{A}(y)$, $\left(\|Y_t\|_{L^\infty(\mathbb{P})}\right)_{t\geq 0}$ and $\left(\|Y_t\|^2_{L^\infty(\mathbb{P})}\right)_{t\geq 0}$ are continuous, positive and decreasing. The integrability condition in (1.2) implies that $\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\, dt < \infty$. Therefore, according to Lemma 1.22 we have

$$\lim_{t\to\infty} t\|Y_t\|_{L^\infty(\mathbb{P})} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} t\|Y_t\|^2_{L^\infty(\mathbb{P})} = 0.$$

Hence, by (1.34) and the finiteness of μ and $\psi''(0)$ we conclude that

$$\lim_{T\to\infty} \mathbb{E}\big[(L_t Y_t)^2\big] = 0.$$

When $\mu = 0$, we get $\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\, dt < \infty$ directly as a condition of admissible strategies. Therefore, the same result follows.

(ii) Using Cauchy-Schwarz inequality and Itô isometry we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T Y_{t-}\, dL_t\right|\right] \\
&\leq |\mu|\mathbb{E}\left[\left|\int_0^T Y_{t-}\, dt\right|\right] + \mathbb{E}\left[\left|\int_0^T Y_{t-} d\Big(\sigma W_t + \int_{\mathbb{R}} x\left(N(t,dx) - t\nu(dx)\right)\Big)\right|\right] \\
&\leq |\mu| \int_0^T \|Y_t\|_{L^\infty(\mathbb{P})}\, dt + \mathbb{E}\left[\left|\int_0^T Y_{t-} d\Big(\sigma W_t + \int_{\mathbb{R}} x\left(N(t,dx) - t\nu(dx)\right)\Big)\right|^2\right]^{\frac{1}{2}} \\
&= |\mu| \int_0^T \|Y_t\|_{L^\infty(\mathbb{P})}\, dt + \left(\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} x^2\nu(dx)\right)^{\frac{1}{2}} \mathbb{E}\left[\int_0^T Y_t^2\, dt\right]^{\frac{1}{2}} \\
&\leq |\mu| \int_0^T \|Y_t\|_{L^\infty(\mathbb{P})}\, dt + \left(\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} x^2\nu(dx)\right)^{\frac{1}{2}} \mathbb{E}\left[\int_0^T \|Y_t\|^2_{L^\infty(\mathbb{P})}\, dt\right]^{\frac{1}{2}}
\end{aligned}$$

From the existence of first and second moments of L_1, we know that μ, σ and $\int_{\mathbb{R}\setminus\{0\}} x^2\nu(dx)$ are all finite. Then, result follows from the integrability conditions in (1.2) and (1.3) of an admissible strategy.

□

Proof of Lemma 1.9.

(i) Let $\psi(u)$ be given by the Lévy-Khintchine representation of L. Then for all $u \in [0, \bar{\delta}_A)$, we have

$$\kappa_A(u) = \psi(\mathrm{i}Au) = -A\mu u + \frac{1}{2}A^2u^2\sigma^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{-Aux} - 1 + Aux\right) \nu(dx). \tag{1.35}$$

Therefore, $\kappa_A(0) = 0$ follows directly.

(ii) Observe that $-A\mu u$, $\frac{1}{2}A^2u^2\sigma^2$ and $\mathrm{e}^{-Aux} - 1 + Aux$ are all convex in u, and in particular that $\frac{1}{2}A^2u^2\sigma^2$ and $\mathrm{e}^{-Aux} - 1 + Aux$ are strictly convex in u. Thus, with reference to (1.35), the strict convexity of κ_A can be concluded from the assumption that L is non-trivial.

(iii) Let $\mu = 0$. In view of (1.35), in order to proof $\lim_{x\to 0} \frac{\kappa_A(x)}{x^2} = K > 0$, it suffices to show that

$$\lim_{u\to 0} \int_{\mathbb{R}} \left(\frac{e^{-Aux} - 1 + Aux}{A^2u^2} \right) \nu(dx) = K',$$

for some constant $K' > 0$. Let $0 < A\bar{u} < \bar{\delta}_A$. It can be checked that for all $u \in (0, \bar{u})$,

$$\left| \frac{e^{-Aux} - 1 + Aux}{A^2u^2} \right| < \frac{x^2}{2}, \quad \text{if } x > 0,$$

and

$$\left| \frac{e^{-Aux} - 1 + Aux}{A^2u^2} \right| < \frac{e^{-A\bar{u}x} - 1 + A\bar{u}x}{A^2\bar{u}^2}, \quad \text{if } x < 0.$$

Because of the finite second moment of L_1 and the fact that $\kappa_A(\bar{u}) < \infty$, both $\frac{x^2}{2}$ and $\frac{e^{-A\bar{u}x}-1+A\bar{u}x}{A^2\bar{u}^2}$ are ν-integrable. Thus, by the dominated convergence theorem, it follows that

$$\lim_{u\to 0} \int_{\mathbb{R}} \left(\frac{e^{-Aux} - 1 + Aux}{A^2u^2} \right) \nu(dx) = \int_{\mathbb{R}} \frac{x^2}{2} \nu(dx) = K',$$

where K' is some strictly positive constant.

(iv) Let $\mu \neq 0$. Then $\lim_{x\to 0} \frac{\kappa_A(x)}{x} = -A\mu$ follows from (1.35) as well as (iii).

□

Proof of Lemma 1.10. Let $\mu = 0$. Then Lemma 1.9 (iii) implies that there exists strictly positive constants $\bar{x}$, C_1 and C_2 such that $C_1x^2 < \kappa_A(x) < C_2x^2$, for all $x \in (0, \bar{x})$. Suppose that $\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\,dt < \infty$. Then Y_t tends to zero as t tends to infinity. Hence, there exists $s > 0$, such that $\|Y_t\|_{L^\infty(\mathbb{P})} \in (0, \bar{x})$, for all $t > s$. Then

$$C_1 \int_s^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\,dt < \int_s^\infty \kappa_A\big(\|Y_t\|_{L^\infty(\mathbb{P})}\big)\,dt < C_2 \int_s^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\,dt, \tag{1.36}$$

from which it follows that $\int_s^\infty \kappa_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big)\,du < \infty$. Since $\|Y_t\|_{L^\infty(\mathbb{P})}$ is bounded for $t \in [0, s]$, we have $\int_0^s \kappa_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big)\,du < \infty$. A similar argument together with the inequality (1.36) also establishes the reverse implication. The proofs regarding the cases of $\mu < 0$ and $\mu > 0$ are similar to above. □

Proof of Lemma 1.11. By Itô's formula and using the expression of κ_A in (1.35) we calculate that

$$\begin{aligned}
M_t^Y =&1 - \int_0^t M^Y_{u-} A Y_{u-}\Big(\big(\mu - \int_{\mathbb{R}} x\,\nu(dx)\big)\,du + \sigma\,dW_u\Big) \\
&- \int_0^t M^Y_{u-}\Big(\kappa_A\big(Y_{u-}\big) - \frac{1}{2}A^2 Y^2_{u-}\sigma^2\Big)\,du \\
&+ \int_0^t \int_{\mathbb{R}} M^Y_{u-}\Big(e^{-AY_{u-}x} - 1\Big)\Big(\big(N(du,dx) - \nu(dx)du\big) + \nu(dx)du\Big) \\
=&1 - \int_0^t M^Y_{u-} A Y_{u-}\sigma\,dW_u + \int_0^t \int_{\mathbb{R}} M_{u-}\Big(e^{-AY_{u-}x} - 1\Big)\big(N(du,dx) - \nu(dx)du\big),
\end{aligned}$$

which shows M is a local martingale. Define

$$X_t = \int_0^t -AY_{u-}\,d\tilde{L}_u \qquad \text{and} \qquad K(\theta)_t = \int_0^t \tilde{\kappa}_A(\theta Y_u)\,du,$$

where $\theta \in [0, 1]$, $Y \in \mathcal{A}(y)$ with $y \in [0, \bar{\delta}_A)$, $\tilde{L}$ is the martingale part of L and $\tilde{\kappa}_A$ is equal to κ_A with $\mu = 0$. It can be checked that the process M^Y in (1.10) can be rewritten as

$$M^Y = \exp\big(X - K(1)\big).$$

With reference to Definition 3.1 and Theorem 3.2 in Kallsen and Shiryaev (2002), in order to show M^Y is a uniformly integrable martingale, it suffices to check that for $\delta \in (0, 1)$,

$$\lim_{\delta \downarrow 0} \sup_{t \in \mathbb{R}_+} \delta \log\left(\mathbb{E}\left[\exp\left(\frac{1}{\delta}\Big((1-\delta)K(1)_t - K(1-\delta)_t\Big)\right)\right]\right) = 0. \tag{1.37}$$

Observe that

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\delta\log\left(\mathbb{E}\left[\exp\left(\frac{1}{\delta}\Big((1-\delta)K(1)_t-K(1-\delta)_t\Big)\right)\right]\right)\\
\leq&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\delta\log\left(\exp\left(\left\|\frac{1}{\delta}\Big((1-\delta)K(1)_t-K(1-\delta)_t\Big)\right\|_{L^\infty(\mathbb{P})}\right)\right)\\
=&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\Big\|(1-\delta)K(1)_t-K(1-\delta)_t\Big\|_{L^\infty(\mathbb{P})}\\
=&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\left\|(1-\delta)\int_0^t\tilde{\kappa}_A(Y_u)\,du-\int_0^t\tilde{\kappa}_A\big((1-\delta)Y_u\big)\,du\right\|_{L^\infty(\mathbb{P})}\\
\leq&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\int_0^t\Big\|(1-\delta)\tilde{\kappa}_A(Y_u)-\tilde{\kappa}_A\big((1-\delta)Y_u\big)\Big\|_{L^\infty(\mathbb{P})}\,du\\
\leq&\lim_{\delta\downarrow 0}\int_0^\infty\Big\|(1-\delta)\tilde{\kappa}_A(Y_u)-\tilde{\kappa}_A\big((1-\delta)Y_u\big)\Big\|_{L^\infty(\mathbb{P})}\,du.
\end{aligned}\tag{1.38}$$

For $\delta\in(0,1)$, we have that

$$\begin{aligned}
&\big\|(1-\delta)\tilde{\kappa}_A(Y_u)-\tilde{\kappa}_A\big((1-\delta)Y_u\big)\big\|_{L^\infty(\mathbb{P})}\\
\leq&\big\|(1-\delta)\tilde{\kappa}_A(Y_u)\big\|_{L^\infty(\mathbb{P})}+\big\|\tilde{\kappa}_A\big((1-\delta)Y_u\big)\big\|_{L^\infty(\mathbb{P})}\\
=&(1-\delta)\tilde{\kappa}_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big)+\tilde{\kappa}_A\big((1-\delta)\|Y_u\|_{L^\infty(\mathbb{P})}\big)\\
\leq&2\tilde{\kappa}_A\big(\|Y_u\|_{L^\infty(\mathbb{P})}\big).
\end{aligned}$$

The last two steps are because $\tilde{\kappa}_A(x)$ is positive and non-decreasing for $x\geq 0$, which follow from Lemma 1.9 (i), (ii) and (iii). According to (1.2) or (1.3) as well as Lemma 1.10, we have

$$\int_0^\infty\tilde{\kappa}_A\big(\|Y_t\|_{L^\infty(\mathbb{P})}\big)dt<\infty.$$

Then, by the dominated convergence theorem, (1.38) gives

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\sup_{t\in\mathbb{R}_+}\delta\log\left(\mathbb{E}\left[\exp\left(\frac{1}{\delta}\Big((1-\delta)K(1)_t-K(1-\delta)_t\Big)\right)\right]\right)\\
\leq&\int_0^\infty\lim_{\delta\downarrow 0}\Big\|(1-\delta)\tilde{\kappa}_A(Y_u)-\tilde{\kappa}_A\big((1-\delta)Y_u\big)\Big\|_{L^\infty(\mathbb{P})}\,du\\
=&0.
\end{aligned}\tag{1.39}$$

On the other hand, the convexity of $\tilde{\kappa}_A(x)$ and $\tilde{\kappa}_A(0) = 0$ imply

$$(1-\delta)\tilde{\kappa}_A(x) \geq \tilde{\kappa}_A\big((1-\delta)x\big), \quad \text{for } \delta \in (0,1),$$

hence,

$$(1-\delta)K(1)_t - K(1-\delta)_t \geq 0.$$

Combining this with (1.39), we get (1.37). $\square$

The next lemma is used in the proofs of Proposition 1.14 and Theorem 1.15.

Lemma 1.23. *Let function F satisfy Assumption 1.3. Then $x \mapsto \frac{x}{G(x)}$ is continuous on $[0,\infty)$, where $G : [0,\infty) \to [0,\infty)$ is the inverse function of $x \mapsto x^2 F'(x)$.*

Proof. Assumption 1.3 and Lemma 1.4 imply that G is continuous and $G(0) = 0$. Therefore, it suffices to check that $\lim_{x\to 0} \frac{x}{G(x)} < \infty$. Let $x = u^2 F'(u)$. Then it follows that $\frac{x}{G(x)} = uF'(u)$. Hence, the result follows from the fact that $u \to 0$, as $x \to 0$, and $\lim_{u\to 0} uF'(u) = 0$ (see Lemma 1.4). $\square$

Proof of Proposition 1.14. We first show that the function v given by (1.16) is continuously differentiable. But it suffices to show that $v'(y) = \frac{\kappa_A(y)}{G\left(\frac{\kappa_A(y)}{A}\right)} + AF\big(G\big(\frac{\kappa_A(y)}{A}\big)\big)$ is continuous on $[0, \bar{\delta}_A)$, and in particular, it suffices to check the continuity of $x \mapsto \frac{x}{G(x)}$, for $x \geq 0$. But this is demonstrated by Lemma 1.23.

Recall that the Hamilton-Jacobi-Bellman equation in our problem is

$$\kappa_A(y) + \inf_{x\geq 0}\big\{AxF(x) - xv'(y)\big\} = 0.$$

In order to prove that v in (1.16) is a solution to this equation, because $AxF(x) - xv'(y)$ is strictly convex in x, it suffices to show that for all $y \in [0, \bar{\delta}_A)$, there exists $x^* \geq 0$ such that

$$Ax^*F'(x^*) + AF(x^*) - v'(y) = 0 \tag{1.40}$$

and

$$\kappa_A(y) + Ax^*F(x^*) - x^*v'(y) = 0, \tag{1.41}$$

where the equality in (1.40) comes from the first-order condition of optimality of the expression

$AxF(x) - xv'(y)$. But with $v'(y) = \frac{\kappa_A(y)}{G\left(\frac{\kappa_A(y)}{A}\right)} + AF\left(G\left(\frac{\kappa_A(y)}{A}\right)\right)$, it can be checked that $x^* = G\left(\frac{\kappa_A(y)}{A}\right)$ satisfies both (1.40) and (1.41). The boundary condition $v(0) = 0$ is a consequence of the expression of $v(y)$ and the continuity of $v(y)$ at $y = 0$. $\square$

Proof of Theorem 1.15. We show the required expression of ξ_t^* in (1.19). We know that when $t \leq \tau$,

$$\int_{Y_t^*}^{y} \frac{1}{G\left(\frac{\kappa_A(u)}{A}\right)}\, du = t,$$

from which it follows that

$$\xi_t^* = -\frac{dY_t^*}{dt} = G\left(\frac{\kappa_A(Y_t^*)}{A}\right), \qquad t \leq \tau.$$

On the other hand, when $t > \tau$, $Y_t^* = 0$. Hence,

$$\xi_t^* = 0 = G\left(\frac{\kappa_A(Y_t^*)}{A}\right), \qquad t > \tau.$$

We next prove that $Y^* \in \mathcal{A}_D(y)$. It is clear that Y^* is deterministic and absolutely continuous. The non-negativity of G implies that Y^* is non-increasing. It remains to show that if $\mu < 0$, then $\int_0^\infty Y_t^*\, dt < \infty$; and if $\mu = 0$, then $\int_0^\infty \left(Y_t^*\right)^2 dt < \infty$. However, with reference to Lemma 1.10, it suffices to check that

$$\int_0^\infty \kappa_A\left(Y_t^*\right) dt = \int_0^\tau \kappa_A\left(Y_t^*\right) dt < \infty.$$

By a change of variable, we have that

$$\int_0^\tau \kappa_A\left(Y_t^*\right) dt = \int_y^0 -\frac{\kappa_A\left(Y_t^*\right)}{G\left(\frac{\kappa_A(Y_t^*)}{A}\right)}\, dY_t^* < \infty,$$

where the finiteness is because of continuity of the integrand on the compact interval $[0, y]$, which is implied by Lemma 1.23.

With reference to (1.40) and (1.41), the function v in (1.16) satisfies

$$\kappa_A(y) + A\xi F(\xi) - \xi v'(y) \geq 0, \qquad \text{for all } \xi \geq 0, \tag{1.42}$$

and an equality holds only when $\xi = G\big(\frac{\kappa_A(y)}{A}\big)$. Let $Y \in \mathcal{A}_D(y)$. Observe that

$$v(Y_T) = v(y) - \int_0^T v'(Y_t)\xi_t \, dt.$$

Taking T to infinity and using the boundary condition $v(0) = 0$, it follows that

$$v(y) = \int_0^\infty v'(Y_t)\xi_t \, dt.$$

Then by (1.42) we have

$$v(y) \leq \int_0^\infty \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big) \, dt. \tag{1.43}$$

Now consider the strategy Y^* in (1.18), which has a speed process ξ^* satisfying $\xi_t^* = G\big(\frac{\kappa_A(Y_t^*)}{A}\big)$, for all $t \geq 0$. Then,

$$\kappa_A\big(Y_t^*\big) + A\xi_t^* F\big(\xi_t^*\big) - \xi_t^* v'\big(Y_t^*\big) = 0, \qquad t \geq 0,$$

hence,

$$v(y) = \int_0^\infty \Big(\kappa_A\big(Y_t^*\big) + A\xi_t F\big(\xi_t^*\big)\Big) \, dt.$$

This together with (1.43) implies that $V(y) = v(y)$, for all $y \in [0, \bar{\delta}_A)$. Therefore, with reference to the analysis after equation (1.11), we get that Y^* is the unique optimal strategy to problem (1.7). $\square$

Proof of Proposition 1.16.

(i) Suppose $\mu < 0$. Let $p < 1$ such that $\lim_{x \to 0} x^p F'(x) = K$ with K being some strictly positive constant. Write $u = x^2 F'(x)$. Then we have

$$\frac{u^{\frac{1}{2-p}}}{G(u)} = \big(x^p F'(x)\big)^{\frac{1}{2-p}}.$$

Taking x to 0, so u tends to 0 as well, and it follows that

$$\lim_{u \to 0} \frac{u^{\frac{1}{2-p}}}{G(u)} = K^{\frac{1}{2-p}}. \tag{1.44}$$

Lemma 1.9 (iv) together with (1.44) gives

$$\lim_{x\to 0} \frac{x^{\frac{1}{2-p}}}{G\big(\frac{\kappa_A(x)}{A}\big)} = K',$$

for some other constant $K' > 0$. Therefore, there exist strictly positive constants K_1, K_2 and $\bar{x}$ such that for all $x \in (0, \bar{x})$,

$$\frac{K_1}{x^{\frac{1}{2-p}}} < \frac{1}{G\big(\frac{\kappa_A(x)}{A}\big)} < \frac{K_2}{x^{\frac{1}{2-p}}}.$$

Integrating and taking limit on each term gives

$$\lim_{x\to 0} \int_x^{\bar{x}} \frac{K_1}{u^{\frac{1}{2-p}}}\, du \le \lim_{x\to 0} \int_x^{\bar{x}} \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\, du \le \lim_{x\to 0} \int_x^{\bar{x}} \frac{K_2}{u^{\frac{1}{2-p}}}\, du.$$

Observe that $p < 1$ implies $\frac{1}{2-p} < 1$, and therefore $\int_0^{\bar{x}} \frac{1}{u^{\frac{1}{2-p}}}\, du < \infty$. Hence,

$$\lim_{x\to 0} \int_x^{\bar{x}} \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\, du < \infty.$$

Then the required result follows from (1.17) and the fact that $\int_{\bar{x}}^{y} \frac{1}{G\big(\frac{\kappa_A(u)}{A}\big)}\, du < \infty$, if the initial stock position $y > \bar{x}$.

(ii) Suppose $\mu = 0$. Observe that (1.44) implies

$$\lim_{x\to 0} \frac{x^{\frac{2}{2-p}}}{G(x^2)} = C,$$

for some constant $C > 0$. Combining this with Lemma 1.9 (iii), we obtain

$$\lim_{x\to 0} \frac{x^{\frac{2}{2-p}}}{G\big(\frac{\kappa_A(x)}{A}\big)} = C',$$

for some other constant $C' > 0$. Then there exist strictly positive constants C_1, C_2 and

$\bar{x}$ such that for all $x \in (0, \bar{x})$,

$$\frac{C_1}{x^{\frac{2}{2-p}}} < \frac{1}{G\left(\frac{\kappa_A(x)}{A}\right)} < \frac{C_2}{x^{\frac{2}{2-p}}}.$$

Therefore,

$$\lim_{x\to 0}\int_x^{\bar{x}} \frac{C_1}{u^{\frac{2}{2-p}}}\, du \le \lim_{x\to 0}\int_x^{\bar{x}} \frac{1}{G\left(\frac{\kappa_A(u)}{A}\right)}\, du \le \lim_{x\to 0}\int_x^{\bar{x}} \frac{C_2}{u^{\frac{2}{2-p}}}\, du.$$

If $p < 0$, then $\frac{2}{2-p} < 1$. Hence $\tau < \infty$ is obtained by the same argument as in (i) of this proof. If $p \in [0, 1)$, then $\frac{2}{2-p} \ge 1$. It follows that $\int_0^{\bar{x}} \frac{1}{G\left(\frac{\kappa_A(u)}{A}\right)}\, du = \infty$, and therefore $\tau = \infty$.

$\square$

Proof of Proposition 1.18. We show that $\hat{L}$ given by

$$\hat{L}_t = \tilde{m}t + \tilde{\sigma}\tilde{W}_t + \int_0^t \int_{\mathbb{R}} \left(\mathrm{e}^z - 1\right)\left(\tilde{N}(dt, dz) - \tilde{\nu}(dz)dt\right), \qquad t \ge 0, \tag{1.45}$$

is a Lévy process. Define a random measure $\hat{N} : \Omega \times \mathcal{B}\big([0, \infty)\big) \otimes \mathcal{B}(\mathbb{R}) \to \mathbb{Z}_+$ and a measure $\hat{\nu} : \mathcal{B}(\mathbb{R}) \to \mathbb{Z}_+$ to be such that if $B \in \mathcal{B}(\mathbb{R})$ and $B \cap (-1, \infty) \ne \emptyset$, then

$$\begin{aligned} \hat{N}(\omega, A, B) &= \tilde{N}\Big(A\,,\, \ln\big(B \cap (-1, \infty) + 1\big)\Big)(\omega), \\ \hat{\nu}(B) &= \tilde{\nu}\Big(\ln\big(B \cap (-1, \infty) + 1\big)\Big); \end{aligned} \tag{1.46}$$

otherwise, they are both equal 0, where $\mathbb{Z}_+$ is the set of all positive integers and $\ln(B \cap (-1, \infty) + 1) = \{\ln(x + 1)\,|\, x \in B \cap (-1, \infty)\}$ (we have for all $A \in \mathcal{B}([0, \infty))$ and $\omega \in \Omega$, $\tilde{N}(A, \{0\})(\omega) = \tilde{\nu}(\{0\}) = 0$). Write $\hat{N}(\cdot, \cdot) = \hat{N}(\omega, \cdot, \cdot)$. Then by writing $x = \mathrm{e}^z - 1$, it follows from (1.45) that

$$\hat{L}_t = \tilde{m}t + \tilde{\sigma}\tilde{W}_t + \int_0^t \int_{\mathbb{R}} x\left(\hat{N}(dt, dx) - \hat{\nu}(dx)dt\right), \qquad t \ge 0. \tag{1.47}$$

With reference to Kallenberg (2001) Corollary 15.7, to prove $\hat{L}$ is a Lévy process, it suffices to show that for any $B \in \mathcal{B}(\mathbb{R})$, $\big(\hat{N}(t,B)\big)_{t\geq 0}$ is a Poisson process with intensity $\hat{\nu}(B)$ satisfying

$$\int_{\mathbb{R}} \big(x^2 \wedge 1\big)\, \hat{\nu}(dx) < \infty. \tag{1.48}$$

But from the definition of $\hat{N}$, it is clear that $\big(\hat{N}(t,B)\big)_{t\geq 0}$ is a Poisson process. Observe that

$$\mathbb{E}[\hat{N}(t,B)] = \mathbb{E}\big[\tilde{N}\big(t, \ln(B\cap(-1,\infty)+1)\big)\big] = t\tilde{\nu}\big(\ln(B\cap(-1,\infty)+1)\big) = t\hat{\nu}(B),$$

which proves that $\hat{\nu}(B)$ is the intensity of $\big(\hat{N}(t,B)\big)_{t\geq 0}$. From Taylor's expansion of $(\mathrm{e}^z-1)^2$, it can be shown that there exist constants $\bar{z}>0$ and $C>0$ such that for all $z\in(-\bar{z},\bar{z})$,

$$\big(\mathrm{e}^z - 1\big)^2 \leq Cz^2.$$

For $\epsilon\in(0,1)$, consider interval $\mathcal{S} = \big(\ln(1-\epsilon)\,,\, \ln(\epsilon+1)\big)$. Then using (1.46) we calculate that for ϵ close enough to 0 so that $\mathcal{S}\subseteq(-\bar{z},\bar{z})$, we have

$$\int_{(-\epsilon,\epsilon)} x^2\, \hat{\nu}(dx) = \int_{\mathcal{S}} \big(\mathrm{e}^z-1\big)^2\, \tilde{\nu}(dz) \leq C\int_{\mathcal{S}} z^2\, \tilde{\nu}(dz) \leq C\int_{(-\bar{z},\bar{z})} z^2\, \tilde{\nu}(dz) < \infty, \tag{1.49}$$

where the finiteness is because of property of Lévy measure $\tilde{\nu}$. Again by (1.46), we obtain

$$\int_{\mathbb{R}\setminus(-\epsilon\,,\,\epsilon)} \hat{\nu}(dx) = \int_{\mathbb{R}\setminus\mathcal{S}} \tilde{\nu}(dz) < \infty,$$

where the finiteness follows from the property of the Lévy measure $\tilde{\nu}$. This implies that $\hat{\nu}\big(\mathbb{R}\setminus(-1,1)\big) < \infty$ and $\hat{\nu}\big((-1,-\epsilon]\cup[\epsilon,1)\big) < \infty$. Since x^2 is bounded on $(-1,-\epsilon]\cup[\epsilon,1)$, together with (1.49), we get

$$\int_{(-1,1)} x^2\, \hat{\nu}(dx) < \infty.$$

Combining this with $\hat{\nu}\big(\mathbb{R}\setminus(-1,1)\big) < \infty$, we get (1.48). We therefore conclude that $\hat{N}$ and $\hat{\nu}$ are Poisson random measure and Lévy measure associated with the Lévy process $\hat{L}$, respectively. Moreover, we calculate from (1.46) that for $x>-1$ and $x\neq 0$,

$$\hat{\nu}(dx) = \tilde{\nu}\big(\, d\big(\ln(x+1)\big)\,\big) = \tilde{f}\big(\ln(x+1)\big)\, d\big(\ln(x+1)\big) = \frac{1}{x+1}\tilde{f}\big(\ln(x+1)\big)dx.$$

The relation $L = \tilde{s}\hat{L}$ shows that L is also a Lévy process. The expression of L in (1.22) shows the adaptedness. Now we check Assumption 1.1 is satisfied by L, but it suffices to check for $\hat{L}$. According to Assumption 1.17, we know $\int_{|z|\geq 1} \mathrm{e}^{2z}\,\tilde{\nu}(dz) < \infty$, and since for any $\epsilon > 0$, $\tilde{\nu}\big(\mathbb{R}\setminus(-\epsilon,\epsilon)\big) < \infty$, it follows that on $[\ln 2, \infty)$, e^{2z} and e^{z} are both $\tilde{\nu}$-integrable and $\tilde{\nu}\big([\ln 2,\infty)\big) < \infty$. Therefore,

$$\int_{|x|\geq 1} x^2\,\hat{\nu}(dx) = \int_{[\ln 2,\infty)} \big(\mathrm{e}^{z}-1\big)^2\,\tilde{\nu}(dz) < \infty,$$

which implies that $\hat{L}_1$ has finite second moment (see e.g. Kyprianou, 2006, Theorem 3.8). Observe that when $u \leq 0$, we have

$$\exp\big(u(\mathrm{e}^{z}-1)\big) \leq 1, \qquad \text{for all } z \geq 0.$$

Hence,

$$\int_{|x|\geq 1} \mathrm{e}^{ux}\,\hat{\nu}(dx) = \int_{[\ln 2,\infty)} \exp\big(u(\mathrm{e}^{z}-1)\big)\,\tilde{\nu}(dz) < \infty, \tag{1.50}$$

from which it follows that $\mathbb{E}[\mathrm{e}^{u\hat{L}_1}] < \infty$, for all $u \leq 0$. $\square$

Proof of Theorem 1.20. This is a direct consequence of Theorem 1.15. $\square$

Proof of Proposition 1.21. For $u \geq 0$, we calculate that

$$\begin{aligned}
&\int_{-1}^{0}\Big(\mathrm{e}^{-\tilde{A}ux} - 1 + \tilde{A}ux\Big)\,\hat{\nu}(dx)\\
=\;&\int_{-1}^{0}\Big(\mathrm{e}^{-\tilde{A}ux} - 1 + \tilde{A}ux\Big)\frac{-1}{\eta\ln(x+1)}(x+1)^{C+D-1}\,dx\\
\geq\;&\frac{\mathrm{e}}{\eta}\int_{-1}^{0}\Big(\mathrm{e}^{-\tilde{A}ux} - 1 + \tilde{A}ux\Big)(x+1)^{C+D}\,dx\\
=\;&\frac{\mathrm{e}}{\eta}\int_{0}^{1}\Big(\mathrm{e}^{-\tilde{A}u(x-1)}x^{C+D}\Big)\,dx + \frac{\mathrm{e}}{\eta}\int_{0}^{1}\Big(\tilde{A}ux^{C+D+1}\Big)\,dx + \frac{\mathrm{e}}{\eta}\int_{0}^{1}\Big(-(1+\tilde{A}u)x^{C+D}\Big)\,dx
\end{aligned} \tag{1.51}$$

where the first inequality is because that $\frac{-1}{(x+1)\ln(x+1)} \geq \mathrm{e}$, for all $-1 < x < 0$, since $(x +$

1) $\ln(x+1)$ is convex with minimum value $-\mathrm{e}^{-1}$. Observe that

$$
\begin{aligned}
&\int_0^1 \left(\mathrm{e}^{-\tilde{A}u(x-1)}x^{C+D}\right) dx \\
&\geq \mathrm{e}^{\tilde{A}u} \int_0^{\frac{1}{\tilde{A}u}\wedge 1} \left((-\tilde{A}ux+1)x^{C+D}\right) dx \\
&= -\frac{\tilde{A}u\mathrm{e}^{\tilde{A}u}}{C+D+2}\left(\frac{1}{\tilde{A}u}\wedge 1\right)^{C+D+2} + \frac{\mathrm{e}^{\tilde{A}u}}{C+D+1}\left(\frac{1}{\tilde{A}u}\wedge 1\right)^{C+D+1}
\end{aligned} \tag{1.52}
$$

and

$$
\int_0^1 \left(\tilde{A}ux^{C+D+1}\right) dx + \int_0^1 \left(-(1+\tilde{A}u)x^{C+D}\right) dx = \frac{\tilde{A}u}{C+D+2} - \frac{1+\tilde{A}u}{C+D+1}, \tag{1.53}
$$

where we have $C+D>0$ and the inequality is because that $\mathrm{e}^{-\tilde{A}ux} \geq -\tilde{A}ux+1$ on interval $\left[0, \frac{1}{\tilde{A}u}\wedge 1\right]$. Therefore, the required result follows from (1.51)-(1.53) and the expression of $\hat{\kappa}_{\tilde{A}}^{VG}$ in (1.26) as well as the fact that $\mathrm{e}^{-\tilde{A}ux} - 1 + \tilde{A}ux$ and $\hat{\nu}$ are positive for all $u \geq 0$ and $x \in \mathbb{R}$. $\square$

Chapter 2

Optimal liquidation in an Almgren-Chriss type model with Lévy processes and finite time horizons

2.1 Problem formulation

We study a finite time horizon version of the liquidation model established in Section 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, which supports a one dimensional, non-trivial, $\mathbb{F}$-adapted Lévy process L. We assume that the Lévy process L satisfies Assumption 1.1 and therefore it admits the decomposition

$$L_t = \mu t + \sigma W_t + \int_{\mathbb{R}} x\, \tilde{N}(t, dx),$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are two constants, W is a standard Brownian motion, $\tilde{N}$ is a compensated Poisson random measure which is independent of W. Let ν denote the Lévy measure associated with L. Recall that the cumulant generating function of L_1, denoted by κ, is finite on the interval $(\bar{\delta}, 0]$, where $\bar{\delta}$ is given in (1.1). Recall further that the function $\kappa_A : [0, \bar{\delta}_A) \to \mathbb{R}$ defined in Section 1.2 is given by $\kappa_A(x) = \kappa(-Ax)$, where $A > 0$ will be referred to as the investor's risk aversion, and $\bar{\delta}_A = -A\bar{\delta}$. The function κ_A satisfies Lemma 1.9. Using the expression in (1.35) of κ_A , one can also check that it is continuously differentiable. Let's denote

$$\underline{\kappa} = \min \kappa_A(x), \quad \text{ and } \quad \underline{y} = \arg\min \kappa_A(x).$$

We see that if $\mu \leq 0$, then $\underline{\kappa} = \underline{y} = 0$; and if $\mu > 0$, then $-\infty < \underline{\kappa} < 0$ and $0 < \underline{y} < \infty$. Figure 2.1, Figure 2.2 and Figure 2.3 (see end of this chapter) give out illustrations of κ_A with different signs of μ.

We consider an investor who aims to sell a large amount of shares of a single stock before a finite time. We denote by Y_t the investor's position in the stock at time t, and by $y \geq 0$ the investor's initial stock position. We define the set of admissible liquidation strategy as follows.

Definition 2.1. Given an initial stock position $y \geq 0$ and a time horizon $T \in (0, \infty)$, the set $\mathcal{A}(T, y)$ of admissible strategies consists of $\mathbb{F}$-adapted, absolutely continuous and positive-valued processes Y with $Y_T = 0$ and $Y_t < \bar{\delta}_A$, for all $t \in [0, T]$. Let $\mathcal{A}_D(T, y)$ denote the set of all deterministic strategies in $\mathcal{A}(T, y)$.

Let $Y \in \mathcal{A}(T, y)$, then there exists an $\mathbb{F}$-adapted process ξ such that Y admits the representation

$$Y_t = y + \int_0^t \xi_s \, ds.$$

The process ξ is called the liquidation speed process associated with strategy Y, and they can be identified by each other. Notice that ξ is $\mathbb{R}$-valued. Same as in Section 1.1, the unaffected stock price is modelled by the process $s + L$, where $s > 0$ is some constant which denotes the stock price at the initial time. Following Almgren and Chriss (2000), Almgren (2003), we assume that the affected stock price at time $t \geq 0$ is given by

$$S_t = s + L_t + \alpha(Y_t - Y_0) + F(\xi_t), \tag{2.1}$$

where $\alpha \geq 0$ is a constant describing the permanent impact and $F : \mathbb{R} \to \mathbb{R}$ is a function describing the temporary impact. We assume that F satisfies the following assumption.

Assumption 2.2. The temporary impact function $F : \mathbb{R} \to \mathbb{R}$ satisfies

(i) F is continuous, and it is twice-continuously differentiable on $\mathbb{R} \setminus \{0\}$;

(ii) $F(0) = 0$;

(iii) the function $x \mapsto xF(x)$ is strictly convex on $\mathbb{R}$;

(iv) there exist constants $K > 0$ and $p < 1$ such that $\lim_{x \to 0} |x|^p F'(x) = K$;

(v) the function $x \mapsto x^2 F'(x)$ is strictly increasing (resp. strictly decreasing) for $x \geq 0$ (resp. $x \leq 0$), and $\lim_{x \to \pm\infty} x^2 F'(x) = \infty$.

In the above assumption, condition (iii) serves for the convexity of the objective function in the optimisation problem we are going to solve and hence the uniqueness of solution holds; condition (iv) is used to well-define the optimal strategy when buying back is involved (see the proof of Lemma 2.6); condition (v) ensures that the value function in our optimisation problem is solved in a closed form and the optimal liquidation speed process can be solved in a feedback form. Similar to Lemma 1.4, it can be derived as consequences of Assumption 2.2 that $F(x)$ is strictly increasing and $\lim_{x\to 0} xF'(x) = 0$. Therefore, $\lim_{x\to 0} x^2F'(x) = 0$. Assumption 2.2 is satisfied by a large class of functions, for example,

$$F(x) = \beta \operatorname{sgn}(x)|x|^{\gamma}, \quad \beta, \gamma > 0,$$

where $\operatorname{sgn}(x)$ denotes the sign of x. We refer to Almgren (2003), Almgren et al. (2005) and Lillo et al. (2003) for both theoretical and empirical studies when the temporary impact function takes the above form. As mentioned in Schied et al. (2010) and Guéant and Royer (2014) that another popular form of the temporary impact function in application would be

$$F(x) = \beta_1 \operatorname{sgn}(x)|x|^{\gamma_1} + \beta_2 \operatorname{sgn}(x)|x|^{\gamma_2},$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2 > 0$.

Let C^Y be the process describing the investor's cash position associated with some $Y \in \mathcal{A}(T, y)$. Let $c \in \mathbb{R}$ be the initial cash position. Then,

$$C_T^Y = c + sy - \frac{1}{2}\alpha y^2 + \int_0^T Y_{t-}\, dL_t - \int_0^T \xi_t F(\xi_t)\, dt. \tag{2.2}$$

We assume the investor has a constant absolute risk aversion (CARA), so his preference between risk and reward/cost is modelled by the utility function U satisfying $U(x) = -\exp(-Ax)$, for some constant $A > 0$. Suppose the investor wants to maximise the expected utility of his cash position at the end of time, i.e. his problem is

$$\sup_{Y\in\mathcal{A}(T,y)} \mathbb{E}\big[U\big(C_T^Y\big)\big]. \tag{2.3}$$

In view of (2.2), this problem takes the form of

$$\inf_{Y\in\mathcal{A}(T,y)} \mathrm{e}^{-A\widetilde{C}}\, \mathbb{E}\bigg[\exp\bigg(-\int_0^T AY_{t-}\, dL_t + A\int_0^T \xi_t F(\xi_t)\, dt\bigg)\bigg], \tag{2.4}$$

where

$$\widetilde{C} = c + sy - \frac{1}{2}\alpha y^2.$$

It then suffices to look at

$$\inf_{Y\in\mathcal{A}(y)} \mathbb{E}\bigg[\exp\bigg(-\int_0^T AY_{t-}\,dL_t + A\int_0^T \xi_t F(\xi_t)\,dt\bigg)\bigg]. \tag{2.5}$$

2.2 Problem simplification

Similar to Section 1.2, we reduce problem (2.5) to be deterministic by a change of measure technique. To this end, for any $Y \in \mathcal{A}(T, y)$, we define the process M^Y to be

$$M_t^Y = \exp\bigg(\int_0^t -AY_{u-}\,dL_u - \int_0^t \kappa_A(Y_u)\,du\bigg), \qquad t \in [0, T].$$

By the same proof of Lemma 1.11, it can be shown that M^Y defines a strictly positive uniformly integrable martingale, and hence a new probability measure $\mathbb{Q}^Y$ can be defined via

$$\frac{d\mathbb{Q}^Y}{d\mathbb{P}} = M_T^Y.$$

With reference to (2.5) and the calculation in (1.11), we obtain that

$$\begin{aligned}
&\inf_{Y\in\mathcal{A}(T,y)} \mathbb{E}\bigg[\exp\bigg(-\int_0^T AY_{t-}\,dL_t + A\int_0^T \xi_t F(\xi_t)\,dt\bigg)\bigg] \\
\le &\inf_{Y\in\mathcal{A}_D(T,y)} \exp\bigg[\int_0^T \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\,dt\bigg].
\end{aligned}$$

Following the same analysis after (1.11), it suffices to solve the problem

$$V(T, y) = \inf_{Y\in\mathcal{A}_D(T,y)} J_T(Y), \tag{2.6}$$

where V denotes the value function with $T > 0$ and $y \in [0, \bar{\delta}_A)$, and J_T is given by

$$J_T(Y) = \int_0^T \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t)\Big)\,dt. \tag{2.7}$$

It can be checked that $\underline{\kappa} T \le V(T, y) < \infty$, for all $(T, y) \in (0, \infty) \times [0, \bar{\delta}_A)$.

The problem in (2.6)-(2.7) might be solved by a dynamic programming approach for which the corresponding Hamilton-Jacobi-Bellman equation is given by

$$-v_t(t,y) + \kappa_A(y) + \inf_{x\in\mathbb{R}}\{AxF(x) + xv_y(t,y)\} = 0$$

with the boundary condition $v(t,0) = 0$ and the singular initial condition

$$\lim_{t\to 0} v(t,y) = \begin{cases} 0, & \text{if } y = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

However, this non-linear partial differential equation is difficult to be solved. (The solutions in different cases to this Hamilton-Jacobi-Bellman equation are given in closed-forms in Section 5.3. From those expressions, we can see that it can be indeed very difficult to solve this partial differential equation directly.) This optimisation problem might also be solved by considering the corresponding Hamiltonian system

$$\begin{aligned} \frac{dY}{dt}(t) &= \mathbf{H}_p\big(Y(t), p(t)\big), \\ \frac{dp}{dt}(t) &= -\mathbf{H}_y\big(Y(t), p(t)\big), \end{aligned}$$

with initial conditions $Y(0) = y$ and $\frac{dY}{dt}(0) = \mathbf{H}_p\big(y, p(0)\big)$, where $\mathbf{H}(y,p) = \sup_x\{xp - \kappa_A(y) - AxF(x)\}$. However, this system of first order ordinary differential equations is also not easy to be solved. In the case of Brownian motion, Theorem 2.14 in Schied et al. (2010) characterises the solution to problem (2.6)-(2.7) using this Hamiltonian system. Instead of these aforementioned approaches, we are going to follow some ideas from the theory of calculus of variations.

2.3 Solution to the problem

Since the objective functional $J_T(\cdot)$ is time-homogeneous, according to the theory of calculus of variations, it suffices to use the Beltrami identity to characterise the optimal strategy (see i.e. Gelfand and Fomin, 2000, Section 4.2). In our problem the corresponding Beltrami identity is

$$K^{T,y} + \kappa_A\big(\hat{Y}_t\big) = A\big(\hat{\xi}_t\big)^2 F'\big(\hat{\xi}_t\big), \quad \text{a.e. } t \in [0,T], \tag{2.8}$$

where $\hat{Y}$ is the candidate of the admissible optimal liquidation strategy, $\hat{\xi}$ is the associated speed process, and $K^{T,y}$ is some constant which is determined by $\hat{Y}_0 = y$ and $\hat{Y}_T = 0$.

Recall that $\underline{y} = \arg\min \kappa_A(x)$, and we have $\underline{y} = 0$ if $\mu \le 0$, and $\underline{y} > 0$ if $\mu > 0$. We then separate the problem into two cases by concerning either $y \ge \underline{y}$ or $y < \underline{y}$. The optimal liquidation strategy in each case will be constructed according to (2.8) (graphs of illustrations of optimal strategies in different cases are given at the end of this chapter), and the value function in each case will be given in a closed form.

2.3.1 Optimal strategy, case 1 ($y \ge \underline{y}$)

Before going into details of the optimal strategy, we make the following primary observation.

Lemma 2.3. *Given an initial stock position $y \ge \underline{y}$, any admissible strategy containing an intermediate buying is not optimal.*

In order to prepare for the construction of an optimal strategy, let's define the continuous function $G^- : [0,\infty) \to (-\infty, 0]$ to be the inverse of $x \mapsto x^2 F'(x)$ restricted on the interval $(-\infty, 0]$. This inverse is well-defined due to Assumption 2.2, and we have that G^- is strictly decreasing and $G^-(0) = 0$. Lemma 2.3 suggests to look for a decreasing strategy. According to (2.8), we seek for a constant $K^{T,y}$ such that

$$\frac{d\hat{Y}_t}{dt} = \hat{\xi}_t = G^-\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t\big)}{A}\right). \tag{2.9}$$

This requires that $K^{T,y} \ge -\min_{t\in[0,T]} \kappa_A\big(\hat{Y}_t\big) = -\underline{\kappa}$, where $\underline{\kappa} = \min \kappa_A(x)$. Equation (2.9) yields (if there is no waiting contained in $\hat{Y}$)

$$\int_0^{\hat{Y}_t} \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T - t, \tag{2.10}$$

and in particular for $y \neq 0$, we look at

$$\int_0^{y} \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T. \tag{2.11}$$

Given $y \in [\underline{y}, \bar{\delta}_A)$, $y \neq 0$ and any constant $K > -\underline{\kappa}$, there is a unique constant $T^K > 0$

satisfying

$$\int_0^y \frac{-1}{G^-\left(\frac{K+\kappa_A(u)}{A}\right)}\,du = T^K.$$

Moreover, since G^- is strictly decreasing and continuous, the mapping $K \mapsto T^K$ is strictly decreasing and continuous on $(-\underline{\kappa}, \infty)$, hence invertible. Write

$$T^y = \int_0^y \frac{-1}{G^-\left(\frac{-\underline{\kappa}+\kappa_A(u)}{A}\right)}\,du, \quad \text{for } y \in [\underline{y}, \bar{\delta}_A),\ y \neq 0, \tag{2.12}$$

which may or may not be finite. Then, when K decreases to $-\underline{\kappa}$, T^K increases to T^y. Therefore, given any $(T, y) \in (0, T^y] \times [\underline{y}, \bar{\delta}_A)$ with $y \neq 0$, we are able to find a unique $K^{T,y} \geq -\underline{\kappa}$ satisfying (2.11) with the corresponding strategy described by (2.10). This strategy is strictly decreasing.

However, if $T > T^y$, (2.11) is impossible to be satisfied by any $K^{T,y} \geq -\underline{\kappa}$. In this situation, it seems that the given time period for liquidation is too long, so we may integrate a period of waiting into the strategy. The expression of $J_T(\cdot)$ indicates that it suffices to consider a waiting when $\hat{Y}_t = \underline{y}$. Taking $\hat{\xi}_t = 0$ and $\hat{Y}_t = \underline{y}$ in (2.9) results in $K^{T,y} = -\underline{\kappa}$. So if $T > T^y$, $\mu \leq 0$ and $y \geq 0$, we have $K^{T,y} = 0$. In this case, we sell until the stock position becomes 0, then keep the stock position to be constantly 0 afterwards. In order to well describe the strategy when $T > T^y$, $\mu > 0$ and $y \geq \underline{y}$, we need to introduce two more quantities of time. For $\mu > 0$, write

$$T^{\underline{y}} = \int_0^{\underline{y}} \frac{-1}{G^-\left(\frac{-\underline{\kappa}+\kappa_A(u)}{A}\right)}\,du \tag{2.13}$$

and

$$T^y_{\underline{y}} = \int_{\underline{y}}^y \frac{-1}{G^-\left(\frac{-\underline{\kappa}+\kappa_A(u)}{A}\right)}\,du, \quad y \geq \underline{y}. \tag{2.14}$$

Thus when $T > T^y$, $\mu > 0$ and $y \geq \underline{y}$, the strategy stays at $\underline{y}$ from time $T^y_{\underline{y}}$ until $T - T^{\underline{y}}$, and it is strictly decreasing satisfying (2.9) at any other time.

Let's formally state the definition of the candidate of the optimal admissible strategy. Figure 2.1 and Figure 2.2 illustrate the strategy in different cases.

Definition 2.4. For any $\mu \in \mathbb{R}$, let $(T, y) \in (0,\infty) \times [\underline{y}, \bar{\delta}_A)$, and T^y, $T^{\underline{y}}$ and $T^y_{\underline{y}}$ be given by (2.12)-(2.14). If $y \in [\underline{y}, \bar{\delta}_A)$ and $y \neq 0$, define the constant $K^{T,y}$ to satisfy

$$\int_0^y \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T \wedge T^y;$$

if $y = 0$, let $K^{T,y} = -\underline{\kappa}$ for all $T \in (0,\infty)$. We call $\hat{Y}$ a candidate of an optimal admissible strategy in Case 1, if it satisfies the following descriptions. Suppose $y = 0$, then there is nothing to liquidate, i.e. we wait until time T. Suppose $y > 0$,

(i) if $T < T^y$, then let $\hat{Y}_t$ be given by (2.10) for all $t \in [0, T]$;

(ii) if $T \geq T^y$ and $\mu \leq 0$, then let $\hat{Y}_t$ be given by

$$\int_0^{\hat{Y}_t} \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T^y - t, \qquad \text{for } t \in [0, T^y],$$

and define $\hat{Y}_t = 0$ for $t \in (T^y, T]$;

(iii) if $T \geq T^y$ and $\mu > 0$, then let $\hat{Y}_t$ satisfy

$$\int_{\underline{y}}^{\hat{Y}_t} \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T^y_{\underline{y}} - t, \qquad \text{for } t \in [0, T^y_{\underline{y}}],$$

define $\hat{Y}_t = \underline{y}$ for $t \in (T^y_{\underline{y}}, T - T^{\underline{y}}]$, and let $\hat{Y}_t$ satisfy (2.10) for $t \in (T - T^{\underline{y}}, T]$.

Note that the integrals appearing in (ii) and (iii) in the above definition can never explode, this is because that they are in the context of $T^y < T < \infty$. The next theorem shows that the candidate strategy given by the above definition is the unique admissible optimal strategy for our liquidation problem.

Theorem 2.5. *For any time horizon $T \in (0,\infty)$ and initial stock position $y \in [\underline{y}, \bar{\delta}_A)$, let the strategy $\hat{Y}$ be given by Definition 2.4. Then $\hat{Y}$ is the unique optimal admissible liquidation strategy for problem (2.3).*

2.3.2 Optimal strategy, case 2 ($y < \underline{y}$)

Note that in this subsection, we only need to consider $\mu > 0$, since for $\mu \leq 0$, we have $\underline{y} = 0$ but $y \geq 0$. In the situation that there is no buying or waiting in the optimial strategy, using the same argument as in Case 1, we are able to find a unique constant $K^{T,y} \geq -\kappa_A(y)$ satisfying (2.11), and we let the corresponding candidate strategy to be given by (2.10). The inequality $K^{T,y} \geq -\kappa_A(y)$ is because $K^{T,y} \geq -\min_{t\in[0,T]} \kappa_A\big(\hat{Y}_t\big)$ is required, and $-\min_{t\in[0,T]} \kappa_A\big(\hat{Y}_t\big) = -\kappa_A(y)$ when $\mu > 0$ with $y < \underline{y}$.

Given any $y \in [0, \underline{y})$, the largest value of time horizon satisfying the situation of no buying or waiting is

$$\tau^y = \int_0^y \frac{-1}{G^-\left(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\right)}\, du. \tag{2.15}$$

Therefore if $T > \tau^y$, we may concern to buy back during liquidation. In order to describe positive trading speeds, analogous to G^-, we introduce the continuous function $G^+ : [0, \infty) \to [0, \infty)$ which is defined as the inverse of $x \mapsto x^2 F'(x)$ when it is restricted on $[0, \infty)$. So G^+ is strictly increasing and $G^+(0) = 0$. We have the following properties of G^+ and G^-.

Lemma 2.6. *For $0 \leq z < y < \underline{y}$,*

$$\int_z^y \frac{1}{G^+\left(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\right)}\, du < \infty \quad \text{and} \quad \int_z^y \frac{-1}{G^-\left(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\right)}\, du < \infty.$$

These two integrals both tend to 0, as $y \to z$.

This lemma implies that $\tau^y < \infty$ and $\tau^0 = 0$. The following lemma will help us to identify the optimal strategy with buying.

Lemma 2.7. *Given an initial position $y < \underline{y}$, it is never optimal to buy back after a period of sale, and it is not optimal to have the stock position being larger than $\underline{y}$ at any time.*

According to the above lemma, if buying back is in the optimal strategy, then it can only happen at the beginning. Then motivated by (2.8), we seek for a constant $K^{T,y}$ such that

$$\frac{d\hat{Y}_t}{dt} = \hat{\xi}_t = G^+\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t\big)}{A}\right)\mathbb{1}_{[0,\theta)}(t) + G^-\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t\big)}{A}\right)\mathbb{1}_{[\theta,T]}(t), \tag{2.16}$$

where $\theta \in [0, T)$ and is defined by $\theta = \inf\{t \geq 0 \,|\, \hat{Y}_t = \kappa_A^{-1}(-K^{T,y})\}$ with κ_A^{-1} being the inverse of κ_A when it is restricted on $[0, \underline{y}]$. Here we must have $\kappa_A^{-1}(-K^{T,y}) \geq y$, which is implied by the requirement that $K^{T,y} \geq -\kappa_A(y)$. Also notice that Lemma 2.7 and $K^{T,y} \geq -\min_{t\in[0,T]} \kappa_A\big(\hat{Y}_t\big)$ implies $\kappa_A^{-1}(-K^{T,y}) \leq \underline{y}$, which is equivalent to $K^{T,y} \leq -\underline{\kappa}$.

Lemma 2.8. *Given $y \in [0, \underline{y})$, the function $T(\cdot; y) : \big(-\kappa_A(y), -\underline{\kappa}\big) \to (0, \infty)$ defined by*

$$T(K; y) = \int_y^{\kappa_A^{-1}(-K)} \frac{1}{G^+\Big(\frac{K+\kappa_A(u)}{A}\Big)}\, du + \int_0^{\kappa_A^{-1}(-K)} \frac{-1}{G^-\Big(\frac{K+\kappa_A(u)}{A}\Big)}\, du$$

is continuous and strictly increasing.

Write

$$\tau_y^{\underline{y}} = \int_y^{\underline{y}} \frac{1}{G^+\Big(\frac{-\underline{\kappa}+\kappa_A(u)}{A}\Big)}\, du. \tag{2.17}$$

Then,

$$\lim_{K\uparrow-\underline{\kappa}} T(K; y) = \tau_y^{\underline{y}} + T^{\underline{y}},$$

and because of Lemma 2.6, it follows that

$$\lim_{K\downarrow-\kappa_A(y)} T(K; y) = \tau^y.$$

Therefore, Lemma 2.8 implies that for any $T \in (\tau^y, \tau_y^{\underline{y}} + T^{\underline{y}}]$, the associated constant $K^{T,y}$ is uniquely determined via $T(K; y) = T$. In this situation we buy until time θ with the strategy satisfying

$$\int_{\hat{Y}_t}^{\kappa_A^{-1}(-K^{T,y})} \frac{1}{G^+\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)}\, du = \theta - t, \tag{2.18}$$

and after time θ, it satisfies (2.10).

Suppose $T > \tau_y^{\underline{y}} + T^{\underline{y}}$, we include a period of waiting into the discussed strategy with buying, and this waiting happens when $\hat{Y}_t = \underline{y}$, which is motivated by the expression of $J_T(\cdot)$.

Let's state the formal definition of the candidate strategy in Case 2, for which Figure 2.3 gives out an illustration.

Definition 2.9. Suppose $(T, y) \in (0, \infty) \times [0, \underline{y})$. Let $T^{\underline{y}}$, τ^y and $\tau_y^{\underline{y}}$ be given by (2.13), (2.15) and (2.17) respectively. For $T \leq \tau^y$, define the constant $K^{T,y}$ to satisfy (2.11) and write $\theta = 0$. When $T > \tau^y$, let $K^{T,y}$ satisfy

$$\int_y^{\kappa_A^{-1}(-K^{T,y})} \frac{1}{G^+\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du + \int_0^{\kappa_A^{-1}(-K^{T,y})} \frac{-1}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du = T \wedge \left(\tau_y^{\underline{y}} + T^{\underline{y}}\right)$$

and write

$$\theta = \int_y^{\kappa_A^{-1}(-K^{T,y})} \frac{1}{G^+\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)}\, du.$$

We call $\hat{Y}$ a candidate of an optimal admissible strategy in Case 2, if it satisfies the following descriptions.

(i) If $T < \tau_y^{\underline{y}} + T^{\underline{y}}$, then let $\hat{Y}_t$ be given by (2.18) when $t \in [0, \theta)$, and let it satisfy (2.10) for $t \in [\theta, T]$.

(ii) If $T \geq \tau_y^{\underline{y}} + T^{\underline{y}}$, then let $\hat{Y}_t$ be given by (2.18) when $t \in [0, \tau_y^{\underline{y}})$, define $\hat{Y}_t = \underline{y}$ for $t \in [\tau_y^{\underline{y}}, T - T^{\underline{y}})$, and let it satisfy (2.10) for $t \in [T - T^{\underline{y}}, T]$.

The following theorem verifies that the candidate strategy given by Definition 2.9 is the unique admissible optimal strategy.

Theorem 2.10. *For any time horizon $T \in (0, \infty)$ and initial stock position $y \in [0, \underline{y})$, let strategy $\hat{Y}$ be given by Definition 2.9. Then $\hat{Y}$ is the unique optimal admissible liquidation strategy for problem (1.7).*

Remark 2.11. Because $\tau^0 = 0$, we see that even when the initial stock position $y = 0$, the strategy defined by Definition 2.9 contains a intermediate buying back, given a strictly positive time horizon. We know that Definition 2.9 gives out the unique optimal strategy, therefore in particular when $y = 0$ and $T > 0$, it is strictly optimal to follow a round-trip strategy. [5] This indicates that when $\mu > 0$, our model allows for the price manipulation in the sense of Huberman and Stanzl (2004), meaning that there exits a round-trip strategy which gives out strictly positive proceeds in average. To see this, let's consider $\mu, T > 0$, and suppose the initial cash position c and the initial stock position y are both equal to 0. Denote

[5] By round-trip strategy we mean any strategy starting and ending at the same position.

by Y^0 the liquidation strategy of doing nothing, and by $\hat{Y}$ the corresponding optimal strategy. Then,

$$-\exp(-Ac) = \mathbb{E}\Big[-\exp\big(-AC_T^{Y^0}\big)\Big] < \mathbb{E}\Big[-\exp\big(-AC_T^{\hat{Y}}\big)\Big] \leq -\exp\Big(-A\mathbb{E}\big[C_T^{\hat{Y}}\big]\Big).$$

Therefore, $0 = c < \mathbb{E}\big[C_T^{\hat{Y}}\big]$. □

2.3.3 Value functions

With expressions of optimal liquidation strategies, we are able to write down corresponding value functions in different cases. To see this, we simply take an optimal strategy into the performance function (2.7). For $\mu \leq 0$, we obtain

$$\begin{aligned} J_T\big(\hat{Y}\big) &= \int_0^T \Big(\kappa_A\big(\hat{Y}_t\big) + A\hat{\xi}_t F\big(\hat{\xi}_t\big)\Big)\, dt \\ &= \int_y^0 \left\{ \frac{\kappa_A\big(\hat{Y}_t\big)}{G^-\Big(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\Big)} + AF\left(G^-\left(\frac{K^{T,y}+\kappa_A\big(\hat{Y}_t\big)}{A}\right)\right)\right\} d\hat{Y}_t. \end{aligned}$$

Therefore, when $\mu \leq 0$, the value function $V : (0,\infty) \times [0,\bar{\delta}_A) \to \mathbb{R}$ admits the expression

$$V(T,y) = \int_0^y \left\{ \frac{-\kappa_A(u)}{G^-\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)} - AF\left(G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)\right)\right\} du, \tag{2.19}$$

where $K^{T,y}$ is given by Definition 2.4. If $\mu > 0$, then the value function takes different forms in different situations. Let's first suppose $(T,y) \in (0,\infty) \times [\underline{y},\bar{\delta}_A)$ or $(T,y) \in (0,\tau^y] \times [0,\underline{y})$. Then there is a waiting period if $(T,y) \in (T^y,\infty) \times [\underline{y},\bar{\delta}_A)$, and it happens between time $T_{\underline{y}}^{y}$ and $T - T^{\underline{y}}$. Taking the optimal strategy into the performance function gives

$$\begin{aligned} J_T\big(\hat{Y}\big) = &\int_y^0 \left\{ \frac{\kappa_A\big(\hat{Y}_t\big)}{G^-\Big(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\Big)} + AF\left(G^-\left(\frac{K^{T,y}+\kappa_A\big(\hat{Y}_t\big)}{A}\right)\right)\right\} d\hat{Y}_t \\ &+ \big(T - T^{\underline{y}} - T_{\underline{y}}^{y}\big)\kappa_A\big(\underline{y}\big)\ \mathbb{1}_{(T^y,\infty)\times[\underline{y},\bar{\delta}_A)}(T,y). \end{aligned}$$

When $(T,y) \in (\tau^y, \infty) \times [0, \underline{y})$, buying back exists, and the strategy possibly has a waiting part between time $\tau_y^{\underline{y}}$ and $T - T^{\underline{y}}$. Thus, it follows that

$$J_T\big(\hat{Y}\big) = \int_y^{\kappa_A^{-1}(-K^{T,y})} \left\{ \frac{\kappa_A\big(\hat{Y}_t\big)}{G^+\Big(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\Big)} + AF\bigg(G^+\bigg(\frac{K^{T,y}+\kappa_A\big(\hat{Y}_t\big)}{A}\bigg)\bigg)\right\} d\hat{Y}_t$$
$$+ \int_{\kappa_A^{-1}(-K^{T,y})}^{0} \left\{ \frac{\kappa_A\big(\hat{Y}_t\big)}{G^-\Big(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\Big)} + AF\bigg(G^-\bigg(\frac{K^{T,y}+\kappa_A\big(\hat{Y}_t\big)}{A}\bigg)\bigg)\right\} d\hat{Y}_t$$
$$+ \big(T - T^{\underline{y}} - \tau_y^{\underline{y}}\big)\kappa_A(\underline{y})\ \mathbb{1}_{(\tau_y^{\underline{y}}+T^{\underline{y}},\,\infty)}(T).$$

Therefore, when $\mu > 0$, the value function $V : (0,\infty) \times [0, \bar{\delta}_A) \to \mathbb{R}$ satisfies that if $(T,y) \in (0,\infty) \times [\underline{y}, \bar{\delta}_A)$ or $(T,y) \in (0, \tau^y] \times [0, \underline{y})$,

$$V(T,y) = \int_0^y \left\{ \frac{-\kappa_A(u)}{G^-\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)} - AF\bigg(G^-\bigg(\frac{K^{T,y}+\kappa_A(u)}{A}\bigg)\bigg)\right\} du$$
$$+ \big(T - T^{\underline{y}} - T_{\underline{y}}^{y}\big)\kappa_A(\underline{y})\ \mathbb{1}_{(T^y,\infty)\times[\underline{y},\bar{\delta}_A)}(T,y);$$

and if $(T,y) \in (\tau^y, \infty) \times [0, \underline{y})$, then

$$V(T,y) = \int_{\kappa_A^{-1}(-K^{T,y})}^{y} \left\{ \frac{-\kappa_A(u)}{G^+\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)} - AF\bigg(G^+\bigg(\frac{K^{T,y}+\kappa_A(u)}{A}\bigg)\bigg)\right\} du$$
$$+ \int_0^{\kappa_A^{-1}(-K^{T,y})} \left\{ \frac{-\kappa_A(u)}{G^-\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)} - AF\bigg(G^-\bigg(\frac{K^{T,y}+\kappa_A(u)}{A}\bigg)\bigg)\right\} du$$
$$+ \big(T - T^{\underline{y}} - \tau_y^{\underline{y}}\big)\kappa_A(\underline{y})\ \mathbb{1}_{(\tau_y^{\underline{y}}+T^{\underline{y}},\,\infty)}(T).$$

where $K^{T,y}$ is given by either Definition 2.4 or Definition 2.9.

2.4 Connection to the infinite time horizon problem

With out loss of generality, define the optimal liquidation strategy to be 0 for $t > T$. Let's examine the limiting behaviour of the liquidation model as time horizon T tends to infinite. With reference to Chapter 1, for the infinite time horizon problem to be well-defined, we have to employ some more conditions on admissible strategies, e.g.

$$\int_0^\infty \|Y_t\|_{L^\infty(\mathbb{P})}\, dt < \infty \qquad \text{if } \mu \neq 0, \tag{2.20}$$

$$\int_0^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\, dt < \infty \qquad \text{if } \mu = 0, \tag{2.21}$$

and

$$\lim_{t\to\infty} t\|Y_t\|_{L^\infty(\mathbb{P})} = 0.$$

If $\mu > 0$, then the optimal strategy in the limit as T tends to infinity never reaches position 0. This kind of strategy gives out a degenerate value function in the limit, which coincides with the situation of $\mu > 0$ discussed in Chapter 1. Now suppose $\mu \le 0$. Denote by $\hat{Y}^T$ the optimal liquidation strategy for time horizon $T < \infty$, and by $\hat{\xi}^T$ the associated speed process. From Definition 2.4, we get $\lim_{T\to\infty} K^{T,y} = 0$. Therefore, the optimal liquidation strategy in the limit satisfies

$$\int_{\lim_{T\to\infty}\hat{Y}_t^T}^{y} \frac{-1}{G^-\left(\frac{\kappa_A(u)}{A}\right)}\, du = t, \qquad \text{if } t \le T^y,$$

and

$$\lim_{T\to\infty} \hat{Y}_t^T = 0, \qquad \text{if } t > T^y.$$

Moreover, from (2.19), the dominated convergence theorem gives that

$$\lim_{T\to\infty} V(T,y) = \int_0^y \left\{ \frac{-\kappa_A(u)}{G^-\left(\frac{\kappa_A(u)}{A}\right)} - AF\left(G^-\left(\frac{\kappa_A(u)}{A}\right)\right)\right\} du, \quad y \in [0, \bar{\delta}_A).$$

Write $\hat{Y}^\infty$, $\hat{\xi}^\infty$ and $V(\infty, y)$ to be the optimal strategy, the optimal speed process and the value function for the infinite time horizon version of our liquidation problem when $\mu \le 0$. Then with reference to Chapter 1 and above results, we have $\lim_{T\to\infty}\hat{Y}_t^T = \hat{Y}_t^\infty$, $\lim_{T\to\infty}\hat{\xi}_t^T = \hat{\xi}_t^\infty$ as well as $\lim_{T\to\infty} V(T,y) = V(\infty,y)$. Write $C^{\hat{Y}^\infty}$ and $C^{\hat{Y}^T}$ to be the corresponding processes of cash positions. It follows that $C_\infty^{\hat{Y}^T}$ converges to $C_\infty^{\hat{Y}^\infty}$ in $L^2(\mathbb{P})$, as T increases to infinity.

To see this, we calculate that

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(C_\infty^{\hat{Y}^T}-C_\infty^{\hat{Y}^\infty}\Big)^2\Big]\\
\le\,&2\mu^2\Big[\int_0^\infty\Big(\hat{Y}_t^T-\hat{Y}_t^\infty\Big)\,dt\Big]^2+2\Big[\int_0^\infty\Big(\hat{\xi}_tF\big(\hat{\xi}_t\big)-\hat{\xi}_t^\infty F\big(\hat{\xi}_t^\infty\big)\Big)\,dt\Big]^2\\
&\quad+\Big(\sigma^2+\int_{\mathbb{R}\setminus\{0\}}x^2\nu(dx)\Big)\int_0^\infty\Big(\hat{Y}_t^T-\hat{Y}_t^\infty\Big)^2\,dt\\
=\,&2\mu^2\Big[\int_0^\infty\Big(\hat{Y}_t^T-\hat{Y}_t^\infty\Big)\,dt\Big]^2+2\Big[\int_y^0\Big\{F\Big(G^-\Big(\frac{K^{T,y}+\kappa_A(u)}{A}\Big)\Big)-F\Big(G^-\Big(\frac{\kappa_A(u)}{A}\Big)\Big)\Big\}\,du\Big]^2\\
&\quad+\Big(\sigma^2+\int_{\mathbb{R}\setminus\{0\}}x^2\nu(dx)\Big)\int_0^\infty\Big(\hat{Y}_t^T-\hat{Y}_t^\infty\Big)^2\,dt,
\end{aligned}
\tag{2.22}
$$

where the first inequality is due to that $(a+b)^2\le 2(a^2+b^2)$ for all $a,b\in\mathbb{R}$, and Itô isometry is used. Since when T increases to infinity, $K^{T,y}$ decreases to 0, and hence $\hat{Y}_t^T$ increases to $\hat{Y}_t^\infty$, combining (2.20), (2.21) and (2.22), the dominated convergence theorem gives the result.

We can therefore regard the infinite time horizon problem as the limit of the finite time horizon problem when T tends to infinity. The Almgren-Chriss liquidation model has a certain robustness.

2.5 Proofs

Proof of Lemma 2.3. Suppose $y\ge\underline{y}$ and $Y\in\mathcal{A}_D(T,y)$ is any strategy including some intermediate buying. Then we can either find two time points r and s with $0\le r<s\le T$ such that $Y_r=Y_s\ge\underline{y}$ and $Y_t>Y_r$ for all $t\in(r,s)$, or there exist p and q with $0<p<q\le T$ such that $Y_p=Y_q<\underline{y}$ and $Y_t<Y_p$ for all $t\in(p,q)$. In the first case, consider the admissible strategy X which consists of a waiting from time r to s, and it is equal to Y at any other time. Then,

$$\kappa_A(X_r)(s-r)<\int_r^s\Big(\kappa_A(Y_u)+A\xi_uF\big(\xi_u\big)\Big)\,du,$$

and therefore, $J_{T,y}(X)<J_{T,y}(Y)$. This shows that Y is not an optimal strategy. For the other case, similarly that the admissible strategy which consists of a waiting from time p to q and being equal to Y at any other time will lead to a desired conclusion. $\square$

Proof of Theorem 2.5. The admissibility conditions given by Definition 2.1 are trivially satisfied by the strategy described by Definition 2.4. Define the function ϕ by $\phi(y,\xi) = \kappa_A(y) + A\xi F(\xi)$. We claim that the Euler-Lagrange equation

$$\frac{d}{dt}\phi_\xi(\hat{Y}_t,\hat{\xi}_t) = \phi_y(\hat{Y}_t,\hat{\xi}_t) \tag{2.23}$$

holds for a.e. $t \in [0,T]$, which requires to show

$$\frac{d}{dt}\phi_\xi(\hat{Y}_t,\hat{\xi}_t) = \kappa_A'(\hat{Y}_t), \qquad \text{for a.e. } t \in [0,T]. \tag{2.24}$$

For $\hat{\xi}_t \neq 0$, we calculate that

$$\begin{aligned}
\phi_\xi(\hat{Y}_t,\hat{\xi}_t) &= AF\big(\hat{\xi}_t\big) + A\hat{\xi}_t F'\big(\hat{\xi}_t\big) \\
&= AF\left(G^-\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t\big)}{A}\right)\right) + \frac{K^{T,y} + \kappa_A\big(\hat{Y}_t\big)}{G^-\left(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\right)}.
\end{aligned}$$

Therefore, a direct differentiation yields (2.24). Consider some time interval where waiting occurs. Then on this interval $\hat{\xi}_t = 0$ and hence $\phi_\xi(\hat{Y}_t,\hat{\xi}_t)$ is a constant. But according to Definition 2.4, on this interval $\hat{Y}_t = \underline{y}$, which is a minimum value of κ_A. Hence on this interval, $\kappa_A'(\hat{Y}_t) = 0 = \frac{d}{dt}\phi_\xi(\hat{Y}_t,\hat{\xi}_t)$.

Consider any admissible $Y \neq \hat{Y}$, using the strict convexity of κ_A and $x \mapsto xF(x)$, we compute that

$$\begin{aligned}
\int_0^T \phi(Y_t,\xi_t)\,dt - \int_0^T \phi(\hat{Y}_t,\hat{\xi}_t)\,dt &> \int_0^T \Big(\phi_y(\hat{Y}_t,\hat{\xi}_t)(Y_t - \hat{Y}) + \phi_\xi(\hat{Y}_t,\hat{\xi}_t)(\xi_t - \hat{\xi})\Big)\,dt \\
&= \int_0^T \left(\frac{d}{dt}\Big[\phi_\xi(\hat{Y}_t,\hat{\xi}_t)\Big](Y_t - \hat{Y}) + \phi_\xi(\hat{Y}_t,\hat{\xi}_t)(\xi_t - \hat{\xi})\right)dt \\
&= \int_0^T \frac{d}{dt}\Big[\phi_\xi(\hat{Y}_t,\hat{\xi}_t)(Y_t - \hat{Y})\Big]\,dt \\
&= 0,
\end{aligned}$$

where the last equality is because that Y and $\hat{Y}$ have the same starting and ending values. This combining with the analysis after equation (1.11) shows that $\hat{Y}$ is the unique optimal admissible liquidation strategy to problem (2.3). □

Proof of Lemma 2.6. Observe that by the convexity of κ_A, we have that for all $u \in [z, y]$ and some constant $C > 0$,

$$C(y-u) \leq \frac{-\kappa_A(y) + \kappa_A(u)}{A}.$$

Therefore,

$$\int_z^y \frac{1}{G^+\Big(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\Big)}\, du \leq \int_z^y \frac{1}{G^+\big(C(y-u)\big)}\, du. \tag{2.25}$$

Assumption 2.2 states that there exist constant $p < 1$ and $K > 0$ such that $\lim_{x\to 0} |x|^p F'(x) = K$. Write $u = x^2 F'(x)$. Then we have

$$\frac{u^{\frac{1}{2-p}}}{G^+(u)} = \big(|x|^p F'(x)\big)^{\frac{1}{2-p}}.$$

Taking x to 0, so u tends to 0 as well, and it follows that

$$\lim_{u\to 0} \frac{u^{\frac{1}{2-p}}}{G^+(u)} = K^{\frac{1}{2-p}}.$$

This and (2.25) together with $p < 1$ imply that

$$0 \leq \int_z^y \frac{1}{G^+\Big(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\Big)}\, du \leq \int_z^y \frac{C'}{(y-u)^{\frac{1}{2-p}}}\, du = \frac{C'(2-p)}{1-p}(y-z)^{\frac{1-p}{2-p}} < \infty, \quad C' > 0.$$

Hence, we have $\int_z^y \frac{1}{G^+\left(\frac{-\kappa_A(y)+\kappa_A(u)}{A}\right)}\, du \to 0$, as $y \to z$. Same proof for G^-. □

Proof of Lemma 2.7. For it is not optimal to buy after sale, the proof is exactly the same as the second case in the proof of Lemma 2.3. Consider any $Y \in \mathcal{A}_D(T, y)$ with $y < \underline{y}$, and whose largest value is greater than $\underline{y}$. Then $(Y_t \wedge \underline{y})_{t\in[0,T]}$ is a better admissible strategy. □

Proof of Lemma 2.8. Note that Lemma 2.6 ensures that $T(\cdot; y)$ is well-defined, and in particular, it is real-valued. Let $\delta \in C^\infty(\mathbb{R})$ be a positive-valued function with support $[0, 1]$, satisfying $\int_0^1 \delta(x)\, dx = 1$. For $n \in \mathbb{N}$, write

$$\delta_n(u) = n\delta(nu), \quad \text{for } u \geq 0.$$

Define

$$G_n^+(x) = \int_0^1 G^+(x+u)\delta_n(u)\,du.$$

Thus, $G_n^+ \in C^1([0,\infty))$, and for all $n \in \mathbb{N}$, G_n^+ is a strictly increasing function with $G_n^+(0) > 0$. Since G^+ is continuous and strictly increasing, we have that G_n^+ decreases to G^+ uniformly on compact intervals, as n tends to infinity (see e.g. Folland, 1984). Let's denote

$$T^+(K;y) = \int_y^{\kappa_A^{-1}(-K)} \frac{1}{G^+\big(\frac{K+\kappa_A(u)}{A}\big)}\,du, \quad K \in \big(-\kappa_A(y), -\underline{\kappa}\big).$$

Therefore, by the monotone convergence theorem,

$$\lim_{n\to\infty} \int_y^{\kappa_A^{-1}(-K)} \frac{1}{G_n^+\Big(\frac{K+\kappa_A(u)}{A}\Big)}\,du = T^+(K;y). \tag{2.26}$$

We calculate that

$$\begin{aligned} &\frac{d}{dK}\left[\int_y^{\kappa_A^{-1}(-K)} \frac{1}{G_n^+\Big(\frac{K+\kappa_A(u)}{A}\Big)}\,du\right] \\ &= \frac{-(\kappa_A^{-1})'(-K)}{G_n^+(0)} + \int_y^{\kappa_A^{-1}(-K)} \frac{-(G_n^+)'\Big(\frac{K+\kappa_A(u)}{A}\Big)\kappa_A'(u)}{A(G_n^+)^2\Big(\frac{K+\kappa_A(u)}{A}\Big)} \frac{1}{\kappa_A'(u)}\,du. \end{aligned} \tag{2.27}$$

Observe that for $u \in [y, \kappa_A^{-1}(-K)]$,

$$\frac{1}{\kappa_A'\big(\kappa_A^{-1}(-K)\big)} \le \frac{1}{\kappa_A'(u)} \le \frac{1}{\kappa_A'(y)} < 0.$$

Then we compute from (2.27) that for $K \in \big(-\kappa_A(y), -\underline{\kappa}\big)$,

$$\begin{aligned} &\frac{d}{dK}\left[\int_y^{\kappa_A^{-1}(-K)} \frac{1}{G_n^+\Big(\frac{K+\kappa_A(u)}{A}\Big)}\,du\right] \\ &\ge \frac{-(\kappa_A^{-1})'(-K)}{G_n^+(0)} + \frac{1}{\kappa_A'\big(\kappa_A^{-1}(-K)\big)} \int_y^{\kappa_A^{-1}(-K)} \frac{-(G_n^+)'\Big(\frac{K+\kappa_A(u)}{A}\Big)\kappa_A'(u)}{A(G_n^+)^2\Big(\frac{K+\kappa_A(u)}{A}\Big)}\,du \end{aligned}$$

$$= \frac{-1}{G_n^+\left(\frac{K+\kappa_A(y)}{A}\right)\kappa_A'\left(\kappa_A^{-1}(-K)\right)}$$

$$> 0.$$

But $\frac{-1}{G_n^+\left(\frac{K+\kappa_A(y)}{A}\right)\kappa_A'(\kappa_A^{-1}(-K))}$ is uniformly bounded away from 0 with respect to both n and $K \in (-\kappa_A(y), -\underline{\kappa})$. This together with (2.26) implies that $K \mapsto T^+(K; y)$ is strictly increasing.

Now we prove the continuity of $T^+(\cdot; y)$, and we first show that it is left continuous. For a fixed K and some $\epsilon > 0$, consider the interval $\left(T^+(K; y) - \epsilon, T^+(K; y)\right)$. We claim that there exist some $\bar{\delta} > 0$ such that for all $\delta \in (0, \bar{\delta})$, $T_+^y(K - \delta) \in \left(T^+(K; y) - \epsilon, T^+(; yK)\right)$. To see this, we take $\bar{\delta}$ to be such that

$$\int_y^{\kappa_A^{-1}(-K+\bar{\delta})} \frac{1}{G^+\left(\frac{K+\kappa_A(u)}{A}\right)} du = T^+(K; y) - \epsilon.$$

Then it is clear that

$$T^+(K; y) - \epsilon < T^+(K - \bar{\delta}; y) < T^+(K - \delta; y) < T^+(K; y).$$

This shows the left continuity of $T^+(\cdot; y)$. For the right continuity, observe that for some $\delta > 0$,

$$\begin{aligned}
&\left|T^+(K + \delta; y) - T^+(K; y)\right| \\
\leq &\left|\int_y^{\kappa_A^{-1}(-K-\delta)} \frac{1}{G^+\left(\frac{K+\delta+\kappa_A(u)}{A}\right)} du - \int_y^{\kappa_A^{-1}(-K)} \frac{1}{G^+\left(\frac{K+\delta+\kappa_A(u)}{A}\right)} du\right| \\
&\quad + \left|\int_y^{\kappa_A^{-1}(-K)} \frac{1}{G^+\left(\frac{K+\delta+\kappa_A(u)}{A}\right)} du - \int_y^{\kappa_A^{-1}(-K)} \frac{1}{G^+\left(\frac{K+\kappa_A(u)}{A}\right)} du\right| \\
= &\left|\int_{\kappa_A^{-1}(-K)}^{\kappa_A^{-1}(-K-\delta)} \frac{1}{G^+\left(\frac{K+\delta+\kappa_A(u)}{A}\right)} du\right| + \left|\int_y^{\kappa_A^{-1}(-K)} \left\{\frac{1}{G^+\left(\frac{K+\delta+\kappa_A(u)}{A}\right)} - \frac{1}{G^+\left(\frac{K+\kappa_A(u)}{A}\right)}\right\} du\right|.
\end{aligned}$$

As δ goes to 0, due to Lemma 2.6, the first term in the above line converges to 0, and the second term also tends to 0 by the dominated convergence theorem.

A similar proof verifies the properties of the integral regarding the function G^-, and therefore we make the conclusion that $T(\cdot; y)$ is continuous and strictly increasing. $\square$

Proof of Theorem 2.10. This proof follows exactly the same argument as of the proof of Theorem 2.5 with taking the additional consideration of that for $\hat{\xi}_t \neq 0$,

$$\begin{aligned}
\frac{d}{dt}\phi_\xi(\hat{Y}_t, \hat{\xi}_t) &= \frac{d}{dt}\left[AF\left(\hat{\xi}_t\right) + A\hat{\xi}_t F'\left(\hat{\xi}_t\right)\right] \\
&= \frac{d}{dt}\left[AF\left(G^+\left(\frac{K^{T,y} + \kappa_A\left(\hat{Y}_t\right)}{A}\right)\right) + \frac{K^{T,y} + \kappa_A\left(\hat{Y}_t\right)}{G^+\left(\frac{K^{T,y}+\kappa_A(\hat{Y}_t)}{A}\right)}\right] \\
&= \kappa_A'(\hat{Y}_t).
\end{aligned}$$

$\square$

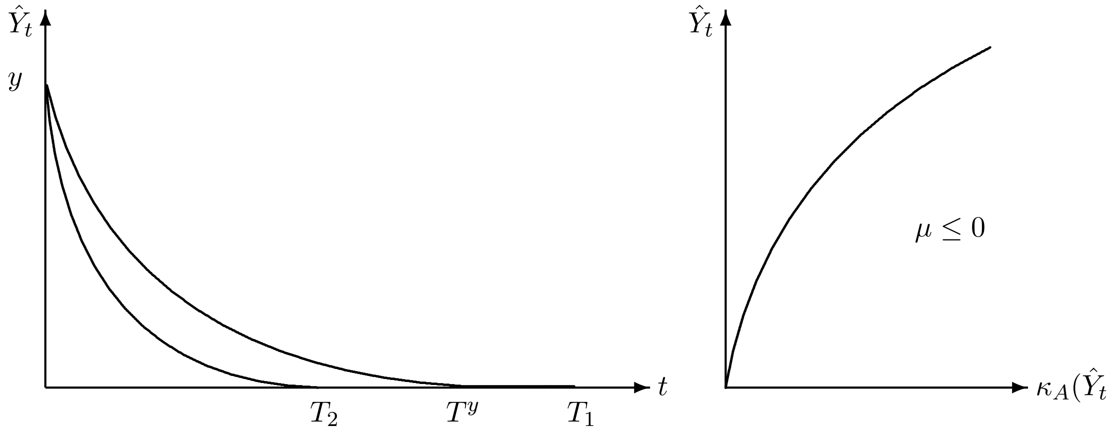

Figure 2.1: An illustration of optimal liquidation trajectories with an initial liquidation position y and different time horizons T_1 and T_2 such that $T_1 > T^y > T_2$.

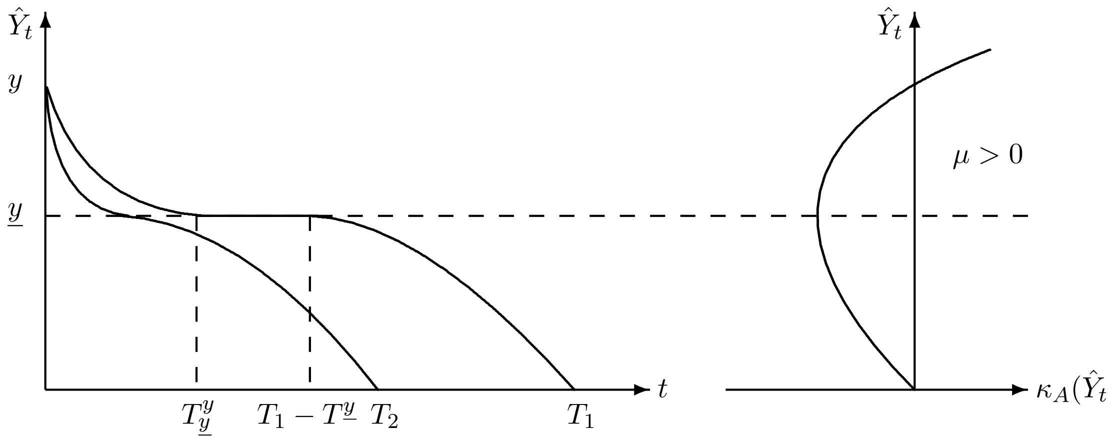

Figure 2.2: An illustration of optimal liquidation trajectories with an initial liquidation position y and different time horizons T_1 and T_2 such that $T_1 > T^y > T_2$.

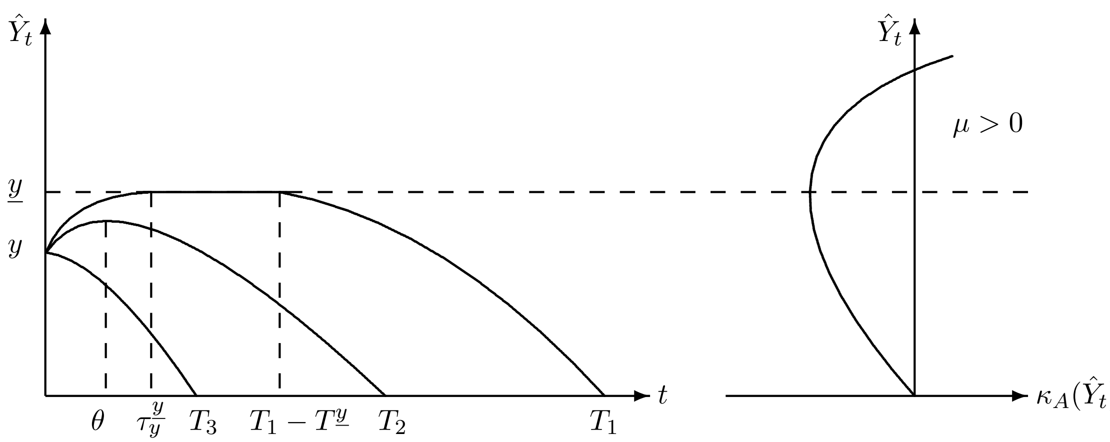

Figure 2.3: An illustration of optimal liquidation trajectories with an initial liquidation position y and different time horizons T_1, T_2 and T_3 such that $T_1 > \tau^y_{\underline{y}} + T^{\underline{y}} > T_2 > \tau^y > T_3$.

Chapter 3

Optimal liquidation in a general one-sided limit order book for a risk averse investor

3.1 Problem formulation

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ be a complete filtered probability space satisfying the usual conditions and supporting a one-dimensional non-trivial Lévy process L.

Assumption 3.1. We assume that L is an $(\mathcal{F}_t)_{t\geq 0}$-martingale, and that there exists some $\delta > 0$ such that $\mathbb{E}\big[\mathrm{e}^{\theta L_1}\big] < \infty$, for $|\theta| < \delta$.

Let κ denote the cumulant generating function of L_1, i.e.

$$\kappa(\theta) = \ln\!\big(\mathbb{E}\big[\mathrm{e}^{\theta L_1}\big]\big), \qquad \theta \in \mathbb{R}.$$

Assumption 3.1 guarantees that the cumulant generating function κ is continuously differentiable on a neighbourhood of 0. With reference to Assumption 3.1 , we notice that the Lévy process L is square integrable, hence admits the representation

$$L_t = \sigma W_t + \int_{\mathbb{R}\setminus\{0\}} z\big(N(t,dz) - t\nu(dz)\big), \qquad t \geq 0,$$

where W is a standard Brownian motion, N is a Poisson random measure which is independent of W with compensator $\pi(t,dz) = t\nu(dz)$, ν denotes the Lévy measure associated with L ((see e.g. Kyprianou, 2006)). The cumulant generating function κ can then be expressed as

$$\kappa(\theta) = \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{\theta z} - 1 - \theta z\Big)\,\nu(dz), \qquad |\theta| < \delta. \tag{3.1}$$

In particular,

$$\kappa'(0) = 0, \qquad \text{and} \qquad \kappa''(0) = \sigma^2 + \int_{\mathbb{R}\setminus\{0\}} z^2 \nu(dz).$$

Moreover, κ is lower semi-continuous (see Ganesh et al., 2004, Lemma 2.3). With reference to (3.1), one can check that κ is strictly convex and continuously differentiable on its effective domain, and it satisfies $\kappa(0) = 0$. Therefore, $\kappa(\theta)$ is strictly decreasing for $\theta < 0$ and strictly increasing for $\theta > 0$. Set

$$\mathbb{R}^+ = [0, \infty) \quad \text{and} \quad \mathbb{R}^- = (-\infty, 0].$$

For any given $A > 0$ define the function $\kappa_A : \mathbb{R}^+ \to [0, \infty]$ by

$$\kappa_A(y) = \kappa(-Ay), \qquad y \geq 0,$$

and set

$$\bar{y}_A = \sup\{y \geq 0 \mid \kappa_A(y) < \infty\}.$$

Then κ_A is strictly increasing, strictly convex, lower semi-continuous and continuously differentiable on $[0, \bar{y}_A]$ with $\kappa_A(0) = 0$. Using (3.1), one can deduce that

$$C_1 y^2 \leq \kappa_A(y) \leq C_2 y^2, \qquad 0 \leq y \leq \epsilon, \tag{3.2}$$

where $\epsilon, C_1, C_2 > 0$. The function κ_A will play a predominant role in the sequel, where the number A will be a parameter describing the large investor's risk aversion.

We consider an investor who aims to sell a large amount of shares of a single stock in an infinite time horizon. Let Y_t denote the number of shares held by the investor at time t. The process Y is called a liquidation strategy, if it converges to 0 a.s., as t goes to infinity. We consider the following set of admissible liquidation strategies.

Definition 3.2. For $y \in \mathbb{R}^+$, let $\mathcal{A}(y)$ denote the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted, predictable, decreasing, càdlàg processes Y, satisfying $Y_{0-} = y$ and

$$\int_0^\infty \kappa_A\big(\|Y_t\|_{L^\infty(\mathbb{P})}\big)\, dt < \infty. \tag{3.3}$$

Moreover, let $\mathcal{A}_D(y)$ denote the set of all deterministic strategies in $\mathcal{A}(y)$.

To describe the investor's execution price, we explicitly model a bid limit order book. We assume that the unaffected bid price process B^0, which is the process describing best bid prices in the market if the investor does not act, is given by

$$B_t^0 = b + L_t, \qquad t \geq 0,$$

where $b > 0$ is the best bid price at the initial time. The Bachelier price model may seem simplistic, but this kind of modelling of unaffected price is widely used in the optimal liquidation literature (see e.g. Almgren and Chriss, 2000; Kissell and Malamut, 2005; Schied and Schöneborn, 2009; Gatheral, 2010, etc). There are studies show that a liquidation model with linear price processes can provide a good approximation for a model with exponential price processes (see e.g. Gatheral and Schied, 2011, and the references there in). In our model, the unaffected bid price is assumed to provide a lower bound for the best ask price and that the best bid price as well as all bid prices are unaffected by the large investor's buy orders (if he is allowed to buy back). These assumptions are satisfied throughout the whole chapter, and they allow us to exclude any buy orders in the optimal trading strategy (see Remark 3.6), and they also exclude price manipulation in our model (see Remark 3.5).

In order to describe the bid limit order book, we consider a measure μ defined on the Borel σ-algebra on $\mathbb{R}^-$, denoted by $\mathcal{B}(\mathbb{R}^-)$. If $\mathcal{S} \in \mathcal{B}(\mathbb{R}^-)$, then $\mu(\mathcal{S})$ represents the number of bid orders with prices in the set $B_t^0 + \mathcal{S} = \{B_t^0 + s \mid s \in \mathcal{S}\}$, provided that the investor didn't act before time t. Notice that the undisturbed bid order book described by μ is relative to unaffected bid prices, it shifts together with the movement of the unaffected market price. Figure 3.1 and Figure 3.2 give out illustrations. We impose the following assumption on μ.

Assumption 3.3. We assume that

(i) there exists some $\bar{x} \in (-\infty, 0)$ such that $\mu((\bar{x}, 0]) = \mu(\mathbb{R}^-) < \infty$;

(ii) μ is absolutely continuous with respect to Lebesgue measure, and is non-zero on any interval properly containing the origin;

(iii) $\mu((x, 0])$ is concave in x.

The first assumption means that there are finitely may bid orders available in the order book and the finite number $\bar{x}$ is equal to the smallest bid price in the book. We know from (ii)

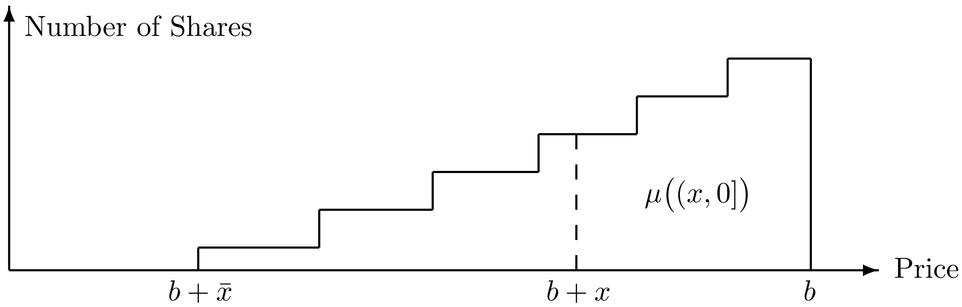

Figure 3.1: An illustration of an undisturbed bid limit order book at time 0.

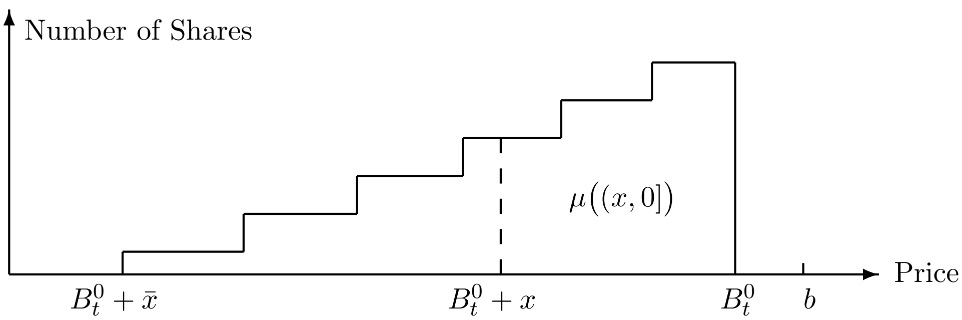

Figure 3.2: An illustration of an undisturbed bid limit order book at time t.

that the right end of the bid order book coincides with the best bid price in the undisturbed bid order book; in other words, one can always sell some amount of shares at the unaffected bid price in an undisturbed bid order book. The concavity of $\mu((x,0])$ tells that if we look at the undisturbed bid order book, there are less bid orders placed at a price which is farther away from the best bid price. Our model captures limit order books with discontinuous shapes which can be used to reasonably approximate discrete shaped limit order books in reality. Compare to models with continuous shaped limit order books (see e.g. Alfonsi et al., 2010; Løkka, 2014, etc), a discontinuous shape is much easier to be calibrated.

Write $\bar{z}=-\mu(\mathbb{R}^-)$. We introduce functions $\phi:[-\infty,0]\to\mathbb{R}^-$ and $\psi:\mathbb{R}^-\to[-\infty,0]$ by

$$\phi(x)=-\mu\big((x,0]\big) \quad \text{and} \quad \psi(z)=\phi^{-1}(z),$$

where $\phi(\psi(z))=z$, for all $z\in[\bar{z},0]$, and $\psi(z)=-\infty$, for all $z<\bar{z}$. As direct consequences of Assumption 3.3, ϕ is convex, ψ is concave, and they are both continuous and strictly

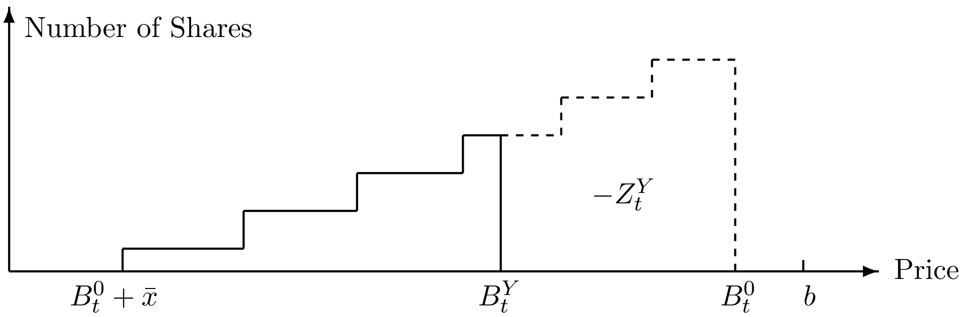

Figure 3.3: An illustration of a disturbed bid limit order book at time t associated with a strategy Y.

increasing when they are finite. They also have the following properties that

$$\phi(0) = \psi(0) = 0; \tag{3.4}$$

$$\int_0^{\bar{z}} \psi(u)\, du < \infty \quad \text{and} \quad \psi(\bar{z}) > -\infty. \tag{3.5}$$

The state of the limit order book changes during trading. The book recovers itself by means that new limit orders allocated at larger bid prices or smaller ask prices. In order to model the dynamic of our bid order book during trading, we need to introduce one more process. For a given strategy Y, let Z^Y be an $\mathbb{R}^-$-valued process such that $-Z_t^Y$ represents the volume spread at time t that is $-Z_t^Y$ is equal to the total number of bid orders which have already been executed subtracted by the total amount of which have refilled in up to time t. We call Z^Y the state process of the bid limit order book associated with a trading strategy Y. Let $Z_{0-}^Y = z$, where $z \geq \bar{z}$ is the initial state of our bid order book. Therefore, we have $\psi(Z_t^Y) = B_t^Y - B_t^0$, where B_t^Y is the best bid price at time t corresponding to Y, and $\psi(Z_t^Y)$ can be understood as the extra price spread at time t, caused by the investor who implements strategy Y (see Figure 3.3). Note that we have defined $\psi(z) = -\infty$, for all $z < \bar{z}$. This implies that the best bid price drops down to $-\infty$, if one sells more than available bids. This in particular will exclude the possibility that the investor making sale while there is no available bid orders. The rate of bid orders refilled into the order book is described by a resilience function $h : \mathbb{R}^- \to \mathbb{R}^-$ which satisfies the following.

Assumption 3.4. We assume the resilience function $h : \mathbb{R}^- \to \mathbb{R}^-$ is increasing and locally Lipschitz continuous. It satisfies $h(0) = 0$ and $h(x) < 0$, for all $x < 0$. We also assume that $1/h$ is a concave function.

Then, we consider the state process Z^Y following the dynamic

$$dZ_t^Y = -h\big(Z_{t-}^Y\big)\,dt + dY_t, \qquad Z_{0-}^Y = z \in \mathbb{R}^-. \tag{3.6}$$

For any admissible strategy Y, we refer to Predoiu et al. (2011) Appendix A, for the existence and uniqueness of a negative, càdlàg and adapted solution to this dynamic. Combining Assumption 3.4 and equation (3.6), we see that the farther the best bid price is away from the unaffected bid price, the larger the resilience speed of the best bid price is. If the investor doesn't trade from time t_1 to t_2, then $\big(Z_t^Y\big)_{t_1<t<t_2}$ satisfies

$$dZ_t^Y = -h(Z_t^Y)\,dt. \tag{3.7}$$

Now define a strictly decreasing function $H : \mathbb{R}^- \to \mathbb{R} \cup \{-\infty\}$ by

$$H(x) = \int_{-1}^{x} \frac{1}{h(u)}\,du. \tag{3.8}$$

Let H^{-1} denote the inverse of H, satisfying $H^{-1}\big(H(x)\big) = x$ for all $x \leq 0$ and $H^{-1}(u) = 0$ for $u \in \big(-\infty\,,\, \lim_{x\to 0-} H(x)\big]$. Then, it can be checked that the process Z given by

$$Z_t = H^{-1}\big(H(Z_0) - t\big) \tag{3.9}$$

has dynamic (3.7). Hence, for any t between time t_1 and t_2, $Z_t^Y = H^{-1}\big(H\big(Z_{t_1}^Y\big) - t + t_1\big)$; and if $Z_{t_2}^Y < 0$, then

$$t_2 - t_1 = H(Z_{t_1}^Y) - H\big(Z_{t_2}^Y\big). \tag{3.10}$$

Suppose the investor's initial cash position is c and that he implements a strategy $Y \in \mathcal{A}(y)$. Then his cash position at time $T > 0$ is

$$C_T(Y) = c - \int_0^T B_{t-}^Y\,dY_t^c - \sum_{0\leq t\leq T} \int_0^{\triangle Y_t} \big\{B_{t-}^0 + \psi\big(Z_{t-}^Y + x\big)\big\}\,dx, \tag{3.11}$$

which corresponds to the best bids offered at all times being executed first so as to match the investor's sell orders, where the first integral is the cost from the continuous component of the liquidation strategy and the sum of integrals gives out total cost due to all block sales. We also suppose the investor has a constant absolute risk aversion (CARA). With initial cash position c, an initial share position y and infinite-time horizon, he wants to maximise the expected utility of his cash position at the final time. Mathematically, the investor's optimal liquidation problem is

$$\sup_{Y \in \mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(Y)\big)\big], \tag{3.12}$$

where the utility function U is given by

$$U(c) = -\mathrm{e}^{-Ac}, \qquad A > 0.$$

This can be seen as a generalisation of the problem considered in Løkka (2014), a risk-averse formulation of the problem considered by Predoiu et al. (2011), and a limit order book equivalent formulation of the optimal liquidation problem studied in Chapter 1.

If $Z_t^Y < \bar{z}$, then $B_t^Y = B_t^0 + \psi(Z_t^Y) = -\infty$. The negative infinity value of best bid price would be unfavoured to the investor. Indeed, (3.11) shows that this brings the investor an infinite cost. Due to this consideration, from now on we restrict ourselves to those admissible strategies Y with $Z_t^Y \geq \bar{z}$, for all $t \geq 0$.

3.2 Problem simplification

In this section, we show that the utility maximisation problem in (3.12) can be reduced to a deterministic optimization problem. This reduction was first explored in Schied et al. (2010), who proved that with a certain market structure and a CARA investor, the optimal liquidation strategy is deterministic. Some results of no price manipulation strategies in our model will also be given in this section.

Let $Y \in \mathcal{A}(y)$. Then it follows from (3.11) that

$$C_T(Y) = c + by - (b + L_T)Y_T + \int_0^T Y_{t-}\, dL_t + \sum_{0 \leq t \leq T} \triangle L_t \triangle Y_t - F_T(Y),$$

where F_T is given by

$$F_T(Y) = \int_0^T \psi\big(Z_{t-}^Y\big)\,dY_t^c + \sum_{0\le t\le T} \int_0^{\triangle Y_t} \psi\big(Z_{t-}^Y + x\big)\,dx. \tag{3.13}$$

It follows from Lemma 1.22 that for any admissible strategy $Y \in \mathcal{A}(y)$, the condition in (3.3) implies

$$\lim_{T\to\infty} T\,\kappa_A\big(\|Y_T\|_{L^\infty(\mathbb{P})}\big) = 0,$$

and with reference to (3.2),

$$\lim_{T\to\infty} \mathbb{E}\big[|L_T Y_T|^2\big] \le \lim_{T\to\infty} \kappa''(0)T\|Y_T\|^2_{L^\infty(\mathbb{P})} \le \lim_{T\to\infty} \kappa''(0)C_1^{-1}T\kappa_A\big(\|Y_T\|_{L^\infty(\mathbb{P})}\big) = 0.$$

We conclude that $B_T^0 Y_T$ tends to 0 in $L^2(\mathbb{P})$ as $T \to \infty$. Set

$$t_\epsilon = \inf\big\{t \ge 0 \mid \|Y_t\|_{L^\infty(\mathbb{P})} \le \epsilon\big\}.$$

From (3.2) and (3.3), it follows that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty Y_{t-}\,dL_t\right)^2\right] &\le \kappa''(0)\left(y^2\|t_\epsilon\|_{L^\infty(\mathbb{P})} + \int_{t_\epsilon}^\infty \|Y_t\|^2_{L^\infty(\mathbb{P})}\,dt\right)\\ &\le \kappa''(0)\left(y^2\|t_\epsilon\|_{L^\infty(\mathbb{P})} + C_1^{-1}\int_{t_\epsilon}^\infty \kappa_A\big(\|Y_t\|_{L^\infty(\mathbb{P})}\big)\,dt\right) < \infty.\end{aligned}$$

Hence, $\int_0^\infty Y_{t-}\,dL_t$ is well-defined in $L^2(\mathbb{P})$. Due to the predictability of Y, we also have that

$$\mathbb{E}\left[\left(\sum_{0\le t} \triangle L_t \triangle Y_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty (\triangle Y_t)^2\,dt\right]\left(\int_{\mathbb{R}\setminus\{0\}} z^2\nu(dz)\right) = 0,$$

which shows the quadratic covariation of jumps of L and Y is almost surely 0, when T goes to infinity. Moreover, note that $F_T(Y) \ge 0$ is an increasing function of T, therefore, F_∞ is a well defined function from the set of càdlàg non-increasing functions into the extended positive real numbers. The final cash position is hence given by

$$C_\infty(Y) = c + by + \int_0^\infty Y_{t-}\,dL_t - F_\infty(Y), \tag{3.14}$$

where $c + by$ gives out the mark-to-market value of the total wealth of the large investor at the beginning of liquidation, $\int_0^\infty Y_{t-}\, dL_t$ represents the cost due to the market volatility risk, and the cost from the price impact resulted from the limited liquidity is described by $F_\infty(Y)$.

Remark 3.5. Suppose we allow for intermediate purchase. Consider any càdlàg adapted strategy which can be decomposed into a pure buy strategy X and a pure sell strategy Y. We assume $X + Y$ satisfies (3.3), $\lim_{t\to\infty} t\kappa_A(||X_t + Y_t||_{L^\infty(\mathbb{P})}) = 0$ and $0 \le X_t + Y_t < \bar{y}_A$, for all $t \ge 0$. Then the cash position associated with strategy $X + Y$ at time infinity, denoted by $C_\infty(X, Y)$, is well-defined in an analogues way of (3.14). We therefore have that

$$
\begin{aligned}
C_\infty(X,Y) &\le \lim_{T\to\infty} C_T(Y) - \int_0^T B^0_{t-}\, dX_t \qquad (3.15)\\
&= c + by + \int_0^\infty \big(X_{t-} + Y_{t-}\big)\, dL_t - F_\infty(Y)\\
&\le c + by + \int_0^\infty \big(X_{t-} + Y_{t-}\big)\, dL_t, \qquad (3.16)
\end{aligned}
$$

where the first inequality is due to the assumption that the unaffected bid price is an lower bound of any best ask price. Taking $y = 0$ shows that the expected cost associated with any round-trip strategy is always positive. Hence, our model doesn't allow for price manipulation in the sense of Huberman and Stanzl (2004).

Note that (3.16) can be derived without concerning any mode of decay of the price impact. The only property of order book which contributes to the absence of price manipulation is that we assume the best ask price gives an upper bounded to the best bid price and the bid limit order book is not affected by the large investor's buy trades. We refer to Alfonsi et al. (2012) and Gatheral et al. (2012) for different model settings which do require conditions on decay of price impact in order to avoid price manipulations. □

Let $Y \in \mathcal{A}(y)$ and define the process M^Y by

$$
M^Y_t = \exp\left(-A \int_0^t Y_{s-}\, dL_s - \int_0^t \kappa_A\big(Y_{s-}\big)\, ds\right), \qquad t \ge 0.
$$

Proposition 1.11 shows that M^Y is a uniformly integrable martingale. We can therefore define a probability measure $\widetilde{\mathbb{P}} = \mathbb{P}^Y$ by

$$
\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = M^Y_\infty.
$$

Based on the idea of Theorem 2.8 in Schied et al. (2010), we calculate that

$$
\begin{aligned}
\sup_{Y\in\mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(Y)\big)\big] &= -\mathrm{e}^{-A(c+by)} \inf_{Y\in\mathcal{A}(y)} \mathbb{E}\left[\exp\left(-A\int_0^\infty Y_{t-}\,dL_t + AF_\infty(Y)\right)\right] \\
&= -\mathrm{e}^{-A(c+by)} \inf_{Y\in\mathcal{A}(y)} \mathbb{E}\left[M_\infty \exp\left(\int_0^\infty \kappa_A\big(Y_{t-}\big)\,dt + AF_\infty(Y)\right)\right] \\
&= -\mathrm{e}^{-A(c+by)} \inf_{Y\in\mathcal{A}(y)} \widetilde{\mathbb{E}}\left[\exp\left(\int_0^\infty \kappa_A\big(Y_{t-}\big)\,dt + AF_\infty(Y)\right)\right] \\
&\le -\mathrm{e}^{-A(c+by)} \inf_{Y\in\mathcal{A}_D(y)} \widetilde{\mathbb{E}}\left[\exp\left(\int_0^\infty \kappa_A\big(Y_{t-}\big)\,dt + AF_\infty(Y)\right)\right] \\
&= -\mathrm{e}^{-A(c+by)} \exp\left(\inf_{Y\in\mathcal{A}_D(y)}\left\{\int_0^\infty \kappa_A\big(Y_{t-}\big)\,dt + AF_\infty(Y)\right\}\right). \qquad (3.17)
\end{aligned}
$$

Then with reference to the analysis after (1.11), it suffices to solve the problem in (3.17).

Remark 3.6. Suppose we allow for intermediate buy trades. Consider a pair of strategies (Y^i, Y^d) which are càdlàg and $(\mathcal{F}_t)$-adapted, and they satisfy $Y^i_{0-} = 0$, $Y^d_{0-} = y$ and $Y^i_\infty = -Y^d_\infty$. Moreover, we assume Y^i is increasing and Y^d is decreasing, and $Y^i + Y^d$ is positive-valued and satisfies (3.3). We also suppose that the cash position at time infinity is well-defined (this requires for instance, $\lim_{t\to\infty} t\kappa_A\big(\|Y^i_t + Y^d_t\|_{L^\infty(\mathbb{P})}\big) = 0$). Consider a non-increasing càdlàg process ξ satisfying $\xi_{0-} = 0$. Then with reference to (3.6), for all $t \ge 0$, we have $\xi_t \le 0$ and

$$
Z^{Y^d+\xi}_t - Z^{Y^d}_t = -\int_0^t \left(h\big(Z^{Y^d+\xi}_{u-}\big) - h\big(Z^{Y^d}_{u-}\big)\right) du + \xi_t.
$$

Suppose there exists $t \ge 0$ such that $Z^{Y^d+\xi}_t > Z^{Y^d}_t$. Let $s = \inf\{t \ge 0 \mid Z^{Y^d+\xi}_t > Z^{Y^d}_t\}$, and let $\delta > 0$ be such that for all $t \in (s, s+\delta]$, $Z^{Y^d+\xi}_t > Z^{Y^d}_t$. This δ exists, since $Z^{Y^d+\xi}$ and Z^{Y^d} are càdlàg . Note that although $Z^{Y^d+\xi}$ and Z^{Y^d} have jumps, we can only have $Z^{Y^d+\xi}_s = Z^{Y^d}_s$. This is because jumps of both $Z^{Y^d+\xi}$ and Z^{Y^d} are negative, and at each time, the jump size of $Z^{Y^d+\xi}$ is always more negative than or equal to that of Z^{Y^d}. Therefore, it follows that

$$
\xi_{s+\delta} \le \xi_s = \int_0^s \left(h\big(Z^{Y^d+\xi}_{u-}\big) - h\big(Z^{Y^d}_{u-}\big)\right) du,
$$

and

$$\xi_{s+\delta} > \int_0^s \Big(h\big(Z_{u-}^{Y^d+\xi}\big) - h\big(Z_{u-}^{Y^d}\big) \Big)\, du + \int_s^{s+\delta} \Big(h\big(Z_{u-}^{Y^d+\xi}\big) - h\big(Z_{u-}^{Y^d}\big) \Big)\, du.$$

However, because h is increasing, the second integral in the above expression is positive. Hence, we get a contradiction, and conclude that for all $t \geq 0$, $Z_t^{Y^d+\xi} \leq Z_t^{Y^d}$. This together with the proof of Lemma 3.1 in Løkka (2014) gives out $F_\infty(Y^d) \leq F_\infty(Y^d + \xi)$. This implies that for any pair $(\widetilde{Y}^i, \widetilde{Y}^d)$ satisfying the same conditions as the aforementioned (Y^i, Y^d), we have $F_\infty(\widetilde{Y}^d \wedge 0) \leq F_\infty(\widetilde{Y}^d)$. Moreover, since κ_A is increasing, it follows that

$$\int_0^\infty \kappa_A\big(\widetilde{Y}_{t-}^i + \widetilde{Y}_{t-}^d\big)\, dt \geq \int_0^\infty \kappa_A\big(\widetilde{Y}_{t-}^d \wedge 0\big)\, dt.$$

If we ignore any impact cost from buying, then it can be shown that our optimal liquidation problem in the case of allowing buying back can be simplified to

$$-\mathrm{e}^{-A(c+by)} \exp\left(\inf_{(Y^i, Y^d)} \left\{ \int_0^\infty \kappa_A\big(Y_{t-}^i + Y_{t-}^d\big)\, dt + AF_\infty(Y^d) \right\} \right),$$

from which it is clear that $(0, \widetilde{Y}^d \wedge 0)$ is a better pair of strategies compared with $(\widetilde{Y}^i, \widetilde{Y}^d)$. According to the above analysis, we can make a conclusion that it is not optimal to buy shares during liquidation in our model. □

Lemma 3.7. *Let F be given by (3.13). Then for every $Y \in \mathcal{A}_D(y)$ and $z \in [\bar{z}, 0]$,*

$$F_\infty(Y) = \int_z^0 \psi(s)\, ds + \int_0^\infty h\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big)\, dt. \tag{3.18}$$

Following the above lemma as well (3.17), we solve the problem

$$V(y,z) = \inf_{Y \in \mathcal{A}_D(y)} \int_0^\infty \Big(\kappa_A\big(Y_{t-}\big) + Ah(Z_{t-}^Y)\psi(Z_{t-}^Y) \Big)\, dt, \tag{3.19}$$

with $y = Y_{0-}$ and $z = Z_{0-}^Y$. Since h and ψ are both negative-valued and $\kappa_A \geq 0$, we have $V \geq 0$. Suppose $y > \bar{y}_A$, which is the smallest upper bound making κ_A to be finite ($\bar{y}_A$ might be $+\infty$). In such case, the investor will make a block sale so that $Y_0 \leq \bar{y}_A$, otherwise Y doesn't satisfy (3.3) and $V(y,z) = \infty$. However, he cannot sell more than $z - \bar{z}$ amount of shares, otherwise $V(y,z)$ will be infinite as well. We shall therefore specify the solvency region to be

$$\mathcal{D} = \left\{ (y,z) \in \mathbb{R}^+ \times [\bar{z}, 0] \;\middle|\; z > y - \bar{y}_A + \bar{z} \right\}.$$

For technical reasons, we don't consider $z = y - \bar{y}_A + \bar{z}$, as the value function may explode along this line.

3.3 Solution to the problem

Our next aim is to derive a solution to the problem in (3.19). The derivation will be based on applying a time-change, and the principle of dynamic programming. With reference to the results in Løkka (2014) and the general theory of optimal control (see e.g. Fleming and Soner, 2006), it is natural to think that there exists a decreasing [6] càglàd function $\beta = \beta^* : \mathbb{R}^+ \to [\bar{z}, 0]$ which separates the (y,z) domain into two different regions; a region where the large investor makes immediate sale and another where he waits. Let β_* denote the càdlàg version of β^*, and set

$$\begin{aligned}
\overline{\mathcal{S}}^{\beta} &= \big\{(y,z) \in \mathcal{D} \mid z \geq \beta_*(y)\big\} \\
\overline{\mathcal{W}}^{\beta} &= \big\{(y,z) \in \mathcal{D} \mid z \leq \beta^*(y)\big\} \cup \big\{(y,z) \mid y = 0\big\} \\
\mathcal{G}^{\beta} &= \overline{\mathcal{S}}^{\beta} \cap \overline{\mathcal{W}}^{\beta}.
\end{aligned}$$

$\overline{\mathcal{S}}^{\beta}$ represents the region for immediate sale, $\overline{\mathcal{W}}^{\beta}$ is the waiting region, and $\mathcal{G}^{\beta}$ is the region of making continuous sale. For $y > 0$, the Hamilton-Jacobi-Bellman equation corresponding to V given by (3.19) takes the form

$$D_y^- v(y,z) + v_z(y,z) = 0, \qquad \text{for } (y,z) \in \overline{\mathcal{S}}^{\beta}, \tag{3.20}$$

$$h(z)v_z(y,z) - \kappa_A(y) - Ah(z)\psi(z) \leq 0, \qquad \text{for } (y,z) \in \overline{\mathcal{S}}^{\beta} \setminus \mathcal{G}^{\beta}, \tag{3.21}$$

and

$$h(z)v_z(y,z) - \kappa_A(y) - Ah(z)\psi(z) = 0, \qquad \text{for } (y,z) \in \overline{\mathcal{W}}^{\beta}, \tag{3.22}$$

[6] Intuitively, when the volume spread is small but the stock position is large, it might be optimal to sell rapidly; on the other hand, if the volume spread is large but the stock position is small, then it might be optimal to wait for a while. This motivates us to make a guess of a decreasing free boundary on the (y,z) domain.

$$D_y^- v(y,z) + v_z(y,z) \le 0, \qquad \text{for } (y,z) \in \overline{\mathcal{W}}^\beta \setminus \mathcal{G}^\beta, \tag{3.23}$$

with associated boundary condition $v(0,z) = A\int_0^z \psi(u)\,du$ for all $z \in [\bar{z}, 0]$, where

$$D_y^- v(y,z) = \lim_{\epsilon \to 0^-} \frac{1}{\epsilon}\Big(v(y+\epsilon, z) - v(y,z)\Big).$$

[7] The equations (3.20)–(3.23) can be motivated as follows. When the large investor is trying to optimise over deterministic strategies, he can either sell a certain number $\triangle > 0$ of shares or wait. Given a state (y,z), it may or may not be optimal to sell $\triangle$ amount of shares, thus

$$v(y,z) \le v\big(y - \triangle, z - \triangle\big),$$

because the share position is decreased from y to $y - \triangle$ due to $\triangle$ number of shares is sold, and at the same time the state of bid order book changes from z to $z - \triangle$. This inequality should hold for all $0 < \triangle \le y$, therefore,

$$\max_{0<\triangle\le y}\big\{v(y,z) - v(y-\triangle, z-\triangle)\big\} \le 0. \tag{3.24}$$

On the other hand, during a period of time $\triangle t > 0$, it may or may not be optimal to wait, hence

$$\begin{aligned} v(y,z) &\le v\big(y, Z_{\triangle t}\big) + \int_0^{\triangle t}\Big(\kappa_A(y) + Ah(Z_{u-})\psi\big(Z_{u-}\big)\Big)\,du \\ &= v(y,z) \\ &\quad + \int_0^{\triangle t}\Big(\kappa_A(y) + Ah(Z_{u-})\big(Z_{u-}\big) - v_z\big(y, Z_{u-}\big)h(Z_{u-})\Big)\,du, \end{aligned}$$

where $dZ_u = -h(Z_u)\,du$, for $0 \le u \le \triangle t$. Multiplying the above inequality by $(\triangle t)^{-1}$ and sending $\triangle t$ to 0, we obtain

$$h(z)v_z(y,z) - \kappa_A(y) - Ah(z)\psi(z) \le 0. \tag{3.25}$$

[7] It will turn out that the value function is continuously differentiable in z, but it is only continuous and admits a one-sided derivative in y (see Proposition 3.12).

Since it is optimal to either sell immediately a certain number of shares or to wait, an equality must hold in either (3.24) or (3.25). We have therefore

$$\max\left\{\max_{0<\triangle\leq y}\left\{v(y,z)-v(y-\triangle,z-\triangle)\right\}\ ,\ h(z)v_z(y,z)-\kappa_A(y)-Ah(z)\psi(z)\right\}=0,$$

and (3.20)–(3.23) follow from this.

We define the liquidation strategy Y^β corresponding to an intervention boundary β as the càdlàg function with the following properties:

(i) If $(y,z)\in\overline{\mathcal{S}}^\beta$, then the investor initially makes a block trade of size $\triangle$ such that $(Y_0^\beta, Z_0^{Y^\beta})=(y-\triangle,z-\triangle)\in\mathcal{G}^\beta$, and set $t_w=0$.

(ii) If $(y,z)\in\overline{\mathcal{W}}^\beta$, then wait until time $t_w=\inf\{t\geq 0\mid Z_t^{Y^\beta}=\beta(y)\}$, where

$$Z_t^{Y^\beta}=z-\int_0^t h\big(Z_u^{Y^\beta}\big)\,du,\qquad 0\leq t\leq t_w.$$

(iii) For $t\geq t_w$, continuously sell shares in such a way that $(Y_t^\beta, Z_t^{Y^\beta})\in\mathcal{G}^\beta$, where

$$Z_t^{Y^\beta}=Z_{t_w}^{Y^\beta}-\int_{t_w}^t h\big(Z_u^{Y^\beta}\big)\,du+Y_t^\beta-Y_{t_w}^\beta,\qquad t\geq t_w.$$

(iv) Stop once $Y_t^\beta=0$.

Figure 3.4 gives out an illustration of such a strategy. We will later characterise an optimal intervention boundary, and prove that the strategy corresponding to such an optimal boundary exists, and is admissible and optimal.

Let us examine in more details about the strategy corresponding to any given intervention boundary function β. We first need to specify what kind of boundary we are concerning about. We consider any intervention boundary $\beta:\mathbb{R}^+\to[\bar{z},0]$ which is decreasing, càglàd and satisfies $\beta(y)<0$, for all $y>0$. We also require that $\lim_{y\to\infty}\beta(y)=\bar{z}$ and $\beta(0)=0$. It will be shown later that there exists such an optimal intervention boundary which completely characterises the solution to the investor's optimisation problem, and the properties that the optimal boundary may be discontinuous (there might be countably many discontinuities) and not invertible will complicate our analysis quite a lot. Given any intervention boundary β, one may ask whether the corresponding liquidation strategy Y^β exists and is unique. In order to

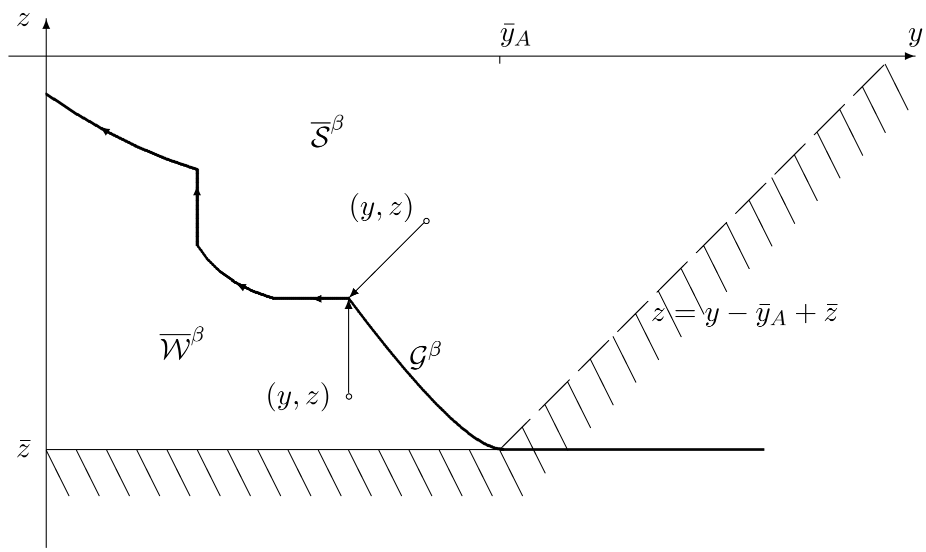

Figure 3.4: An illustration of the strategy Y^β corresponding to a boundary β.

answer this, we need to introduce the following functions relate to β, which will bring benefits to our analysis:

$$\gamma_\beta(y) = \beta(y) - y, \qquad \text{for } y \in \mathbb{R}^+; \tag{3.26}$$
$$\rho_\beta(z) = z - \beta^{-1}(z), \qquad \text{for } z \in [\bar{z}, 0]; \tag{3.27}$$

$$\beta^{-1}(z) = \inf\{y \in \mathbb{R}^+ \,|\, \beta(y) \le z\}, \qquad \text{for } z \in [\bar{z}, 0]; \tag{3.28}$$
$$\gamma_\beta^{-1}(x) = \inf\{y \in \mathbb{R}^+ \,|\, \gamma_\beta(y) \le x\}, \qquad \text{for } x \in \mathbb{R}^-; \tag{3.29}$$
$$\rho_\beta^{-1}(x) = \inf\{z \in [\bar{z}, 0] \,|\, \rho_\beta(z) \ge x\}, \qquad \text{for } x \in \mathbb{R}^-. \tag{3.30}$$

[8] Note that β and γ_β are càglàd , β^{-1} and ρ_β are càdlàg , and γ_β^{-1} as well as ρ_β^{-1} are both continuous. Moreover, β, β^{-1} and γ_β^{-1} are decreasing, γ_β is strictly decreasing, ρ_β is strictly increasing, and ρ_β^{-1} is increasing. Furthermore, it follows directly from the definitions of β^{-1},

[8]It can be checked that for any $x \in \mathbb{R}^-$, $\gamma_\beta^{-1}(x)$ and $\rho_\beta^{-1}(x)$ give out the y-coordinate and the z-coordinate of the intersection of the line $z = y + x$ and $\mathcal{G}^\beta$, respectively.

γ_β, γ_β^{-1}, ρ_β and ρ_β^{-1} that the following three identities hold:

$$\rho_\beta^{-1}(x) = x + \gamma_\beta^{-1}(x), \qquad \text{for all } x \in \mathbb{R}^-; \tag{3.31}$$

$$\gamma_\beta^{-1}\big(\rho_\beta(z)\big) = \beta^{-1}(z), \qquad \text{for all } z \in [\bar{z}, 0]; \tag{3.32}$$

$$\rho_\beta^{-1}\big(\gamma_\beta(y)\big) = \beta(y), \qquad \text{for all } y \in \mathbb{R}^+. \tag{3.33}$$

Also, by the definitions of $\mathcal{G}^\beta$, β and β^{-1}, we see that the set $\mathcal{G}^\beta$ is the union of the graphs of functions β and β^{-1}, restricted in $\mathcal{D}$.

Observe that if $z > \beta(y)$, then the strategy Y^β corresponding to the intervention boundary described by β consists of an initial sale of $\triangle$ number of shares so that $(y - \triangle, z - \triangle)$ is in $\mathcal{G}^\beta$ (see Figure 3.4). Let $Y_{0-}^\beta = y$ and $Y_0^\beta = y - \triangle$. Suppose $(y - \triangle, z - \triangle)$ is on the graph of β. Then $(y - \triangle, z - \triangle) = (y - \triangle, \beta(y - \triangle))$ and this equality is equivalent to

$$\gamma_\beta\big(Y_0^\beta\big) = \beta(Y_0^\beta) - Y_0^\beta = z - y,$$

from which it follows that $Y_0^\beta = \gamma_\beta^{-1}(z-y)$ and $\triangle = y - \gamma_\beta^{-1}(z-y)$. Now suppose $(y-\triangle, z-\triangle)$ is on the graph of β^{-1}, and let $Z_{0-}^{Y^\beta} = z$ and $Z_0^{Y^\beta} = z - \triangle$. Then $(y - \triangle, z - \triangle) = (\beta^{-1}(z - \triangle), z - \triangle)$, which is equivalent to

$$\rho_\beta\big(Z_0^{Y^\beta}\big) = Z_0^{Y^\beta} - \beta^{-1}(Z_0^{Y^\beta}) = z - y,$$

and it follows that $Z_0^{Y^\beta} = \rho_\beta^{-1}(z - y)$ and $\triangle = z - \rho_\beta^{-1}(z - y)$. According (3.31), the number $\triangle$ of shares in both of the aforementioned two cases can be expressed by

$$\triangle = y - \gamma_\beta^{-1}(z - y) = z - \rho_\beta^{-1}(z - y).$$

On the other hand, if $z \leq \beta(y)$, then the strategy Y^β consists of an initial waiting until $\big(Y_t^\beta, Z_t^{Y^\beta}\big)$ being on the graph of β (see Figure 3.4). When there is no action taken, we have $Y_t^\beta = y$, and with reference to (3.9) and (3.10), we obtain $Z_t^{Y^\beta} = H^{-1}\big(H(z) - t\big)$. The first time t_w that the state process is on the graph of β is given by

$$t_w = H(z) - H\big(\beta(y)\big). \tag{3.34}$$

Once the state process (Y^β, Z^{Y^β}) is in the set $\mathcal{G}^\beta$, the strategy Y^β consists of taking minimal actions such that the state process remains in $\mathcal{G}^\beta$ (see Figure 3.4). Therefore, $\big(Y_t^\beta, Z_t^{Y^\beta}\big) =$

$\big(Y_t^\beta, \beta(Y_t^\beta)\big)$ whenever $\beta\big(Y_t^\beta+\big) = \beta\big(Y_t^\beta\big)$. With reference to (3.6), this implies that Y_t^β should solve

$$d\beta\big(Y_t^\beta\big) = -h\big(\beta\big(Y_t^\beta\big)\big)\,dt + dY_t^\beta,$$

which is equivalent to

$$d\gamma_\beta\big(Y_t^\beta\big) = -h\big(\beta\big(Y_t^\beta\big)\big)\,dt.$$

If $\beta^{-1}\big(Z_t^{Y^\beta}\big) = \beta^{-1}\big(Z_t^{Y^\beta}-\big)$, then $\big(Y_t^\beta, Z_t^{Y^\beta}\big) = \big(\beta^{-1}(Z_t^{Y^\beta}), Z_t^{Y^\beta}\big)$. According to (3.6) and the definition of β^{-1}, Z^{Y^β} should solve

$$dZ_t^{Y^\beta} = -h\big(Z_{t-}^{Y^\beta}\big)\,dt.$$

Set

$$t_w = \begin{cases} 0, & \text{if } z > \beta(y), \\ H(z) - H\big(\beta(y)\big), & \text{if } z \le \beta(y), \end{cases} \tag{3.35}$$

and

$$\bar{t} = \inf\{t \ge 0 \mid Y_t^\beta = 0\}. \tag{3.36}$$

Denote by $\{y_n\}_{n\in\mathbb{I}}$ the set of discontinuous points of β. Then $\mathbb{I}$ is countable since β is càglàd. Define $\{t_n\}_{n\in\mathbb{I}}$ by

$$t_n = \inf\big\{t \ge t_w \mid Y_t^\beta = y_n\big\}, \tag{3.37}$$

and $\{s_n\}_{n\in\mathbb{I}}$ by

$$s_n = \inf\big\{t \ge t_w \mid Y_t^\beta < y_n\big\}. \tag{3.38}$$

If $\{t \ge t_w \mid Y_t^\beta = y_n\big\} = \emptyset$, write $t_n = \infty$; and write $s_n = \infty$, if $\{t \ge t_w \mid Y_t^\beta < y_n\big\} = \emptyset$. The following result establishes the existence and uniqueness of such a strategy Y^β corresponding to a given intervention boundary β.

Lemma 3.8. *Let $(y, z) \in \mathcal{D}$ and β be a function of intervention boundary. Suppose h is a* *resilience function satisfying Assumption 3.4, and* H*,* β^{-1}*,* γ_β*,* γ_β^{-1}*,* t_w*,* $\bar{t}$*,* y_n*,* t_n *and* s_n *are given by (3.8), (3.28), (3.26), (3.29) and (3.35)–(3.38) respectively. Let* $\big(Y_t^\beta\big)_{t\geq 0} = \big(Y_{t\wedge\bar{t}}^\beta\big)_{t\geq 0}$*, with* $Y_{0-}^\beta = y$*, which denotes the decreasing càdlàg liquidation strategy corresponding to* β*, and let* $\big(Z_t^{Y^\beta}\big)_{t\geq 0}$*, with* $Z_{0-}^{Y^\beta} = z$*, be the state process of the bid order book associated with* Y^β*. Suppose* Y^β *satisfies the following description:*

(i) If $y = 0$*, then liquidation is completed immediately; otherwise,*

(ii) If $z > \beta(y)$*,*

(a) when $y \in \cup_{n\in\mathbb{I}}\big(z - \beta(y_n) + y_n\,,\, z - \beta(y_n+) + y_n\big]$*, immediately sell* $y - \gamma_\beta^{-1}(z-y)$ *number of shares. This block trade ensures* $Y_0^\beta = \beta^{-1}\big(Z_0^{Y^\beta}\big)$*.*

(b) when $y \in \big(z,\infty\big) \setminus \cup_{n\in\mathbb{I}}\big(z - \beta(y_n) + y_n\,,\, z - \beta(y_n+) + y_n\big]$*, immediately sell* $y - \gamma_\beta^{-1}(z-y)$ *number of shares. This block trade ensures* $Z_0^{Y^\beta} = \beta\big(Y_0^\beta\big)$*.*

Then continuously sell shares so that $\big(Y_t^\beta, Z_t^{Y^\beta}\big) \in \mathcal{G}^\beta$ *for all* $t \in [\,t_w, \bar{t}\,]$*.*

(iii) If $z \leq \beta(y)$*, then wait until time* t_w*. The time* t_w *has the property that* $Z_{t_w}^{Y^\beta} = \beta(y)$*. Continuously sell shares so that* $\big(Y_t^\beta, Z_t^{Y^\beta}\big) \in \mathcal{G}^\beta$ *for all* $t \in [\,t_w, \bar{t}\,]$*.*

Such strategy Y^β *exists and is unique, and it is continuous for all* $t > 0$*. In particular,*

$$Y_t^\beta = y_n \quad \text{for } t \in [\,t_w, \bar{t}\,] \cap \cup_{n\in\mathbb{I}}[t_n, s_n), \tag{3.39}$$

with corresponding $Z_t^{Y^\beta}$ *being the unique solution to*

$$dZ_t^{Y^\beta} = -h\big(Z_t^{Y^\beta}\big)\,dt, \tag{3.40}$$

where

$$Z_{t_w}^{Y^\beta} = \rho_\beta^{-1}(z-y) \;\text{ if } z > \beta(y), \qquad \text{and} \qquad Z_{t_n}^{Y^\beta} = \beta\big(Y_{t_n-}^\beta\big) \;\text{ for } t_n > t_w. \tag{3.41}$$

Moreover,

$$Z_t^{Y^\beta} = \beta\big(Y_t^\beta\big), \qquad \text{for } t \in [\,t_w, \bar{t}\,] \setminus \cup_{n\in\mathbb{I}}[t_n, s_n), \tag{3.42}$$

where Y^β *is the unique solution to*

$$d\gamma_\beta\big(Y_t^\beta\big) = -h\big(\beta\big(Y_t^\beta\big)\big)\,dt, \tag{3.43}$$

with

$$Y^{\beta}_{t_w} = y \text{ if } z \leq \beta(y), \quad Y^{\beta}_{t_w} = \gamma_{\beta}^{-1}(z-y) \text{ if } z > \beta(y), \quad \text{and} \quad Y^{\beta}_{s_n} = y_n \text{ for } s_n > t_w. \tag{3.44}$$

If $t_w > 0$, *then* $Y^{\beta}_t = y$ *and* $Z^{Y^\beta}_t = H^{-1}\big(H(z) - t\big)$, *for* $0 \leq t \leq t_w$.

We can also describe $Z^{Y^\beta}_t$ for $t \in [\bar{t}, \infty)$ that it satisfies (3.40) with initial condition

$$Z^{Y^\beta}_{\bar{t}} = \begin{cases} Z^{Y^\beta}_{t_w}, & \text{if } \bar{t} = t_w, \\ z, & \text{if } \bar{t} < t_w, \\ \beta(0+), & \text{if } \bar{t} > t_w. \end{cases} \tag{3.45}$$

The value of $\beta(0+)$ could determine the finiteness of liquidation period. Precisely, we have that $\beta(0+) < 0$ implies $\bar{t} < \infty$. To see this, it is enough to consider

$$\gamma_\beta\big(Y^\beta_{\bar{t}}\big) - \gamma_\beta\big(Y^\beta_t\big) = \int_t^{\bar{t}} -h\big(\beta\big(Y^\beta_u\big)\big)\, du$$

which follows from (3.43), where there is no waiting period between time t and $\bar{t}$. To get a contradiction, let's suppose $\bar{t} = \infty$. Then it is clear that $\int_t^{\bar{t}} -h\big(\beta\big(Y^\beta_u\big)\big)\, du = \infty$, as $\beta\big(Y^\beta_u\big)$ is bounded away from 0 on the interval $(t, \bar{t})$. However, $\gamma_\beta\big(Y^\beta_{\bar{t}}\big) - \gamma_\beta\big(Y^\beta_t\big)$ is finite. The dynamic of $Z^{Y^\beta}_t$ gives out that Z^{Y^β} is càdlàg and increasing to 0. Moreover, we notice from the continuity of Y^β_t for $t > 0$ that Z^{Y^β} is also continuous for all $t > 0$.

We now progress by deriving an explicit expression for the performance function associated with the strategy Y^β described by Lemma 3.8 for an arbitrary intervention boundary β. As a consequence, an explicit expression for the value function of our problem will be deduced then. For the strategy Y^β with associated state process Z^{Y^β}, given an initial state (y, z), and with reference to (3.19), we define the performance function J_β by

$$J_\beta(y, z) = \int_0^\infty \Big(\kappa_A\big(Y^\beta_t\big) + Ah(Z^{Y^\beta}_t)\psi\big(Z^{Y^\beta}_t\big)\Big)\, dt, \tag{3.46}$$

where $Y^\beta_{0-} = y$, $Z^{Y^\beta}_{0-} = z$ and $(y, z) \in \mathcal{D}$. Since $\kappa_A(0) = 0$, it follows that

$$\int_{\bar{t}}^\infty \Big(\kappa_A\big(Y^\beta_t\big) + Ah(Z^{Y^\beta}_t)\psi\big(Z^{Y^\beta}_t\big)\Big)\, dt = A\int_0^{Z^{Y^\beta}_{\bar{t}}} \psi(u)\, du. \tag{3.47}$$

Therefore, in cases (i) of Lemma 3.8,

$$J_\beta(y,z) = A\int_0^z \psi(u)\,du. \tag{3.48}$$

Lemma 3.9. *Let β, Y^β, Z^{Y^β}, t_w and $\bar{t}$ be defined as the same as in Lemma 3.8. If $t_w < \bar{t}$, then*

$$\begin{aligned}&\int_{t_w}^{\infty}\Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big)\Big)\,dt\\ &= \int_{\beta(0+)}^{Z_{t_w}^{Y^\beta}-Y_{t_w}^\beta}\left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right)du + A\int_0^{\beta(0+)}\psi(u)\,du,\end{aligned}$$

where γ_β^{-1} and ρ_β^{-1} are defined by (3.29) and (3.30), respectively.

In case (ii) (a) of Lemma 3.8, the strategy Y^β consists of an initial sale of $y-\gamma_\beta^{-1}(z-y) = z-\rho_\beta^{-1}(z-y)$ number of shares. The state after the block sale is $\big(Y_0^\beta, Z_0^{Y^\beta}\big) = \big(\beta^{-1}\big(\rho_\beta^{-1}(z-y)\big), \rho_\beta^{-1}(z-y)\big)$, Hence, according to (3.46) and Lemma 3.9,

$$\begin{aligned}J_\beta(y,z) &= J_\beta\Big(\beta^{-1}\big(\rho_\beta^{-1}(z-y)\big)\,,\,\rho_\beta^{-1}(z-y)\Big)\\ &= \int_{\beta(0+)}^{\rho_\beta(\rho_\beta^{-1}(z-y))}\left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right)du + A\int_0^{\beta(0+)}\psi(u)\,du\\ &= \int_{\beta(0+)}^{z-y}\left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right)du + A\int_0^{\beta(0+)}\psi(u)\,du.\end{aligned}$$

In case (ii) (b), we immediately sell $y-\gamma_\beta^{-1}(z-y)$ number of shares at the beginning. The state after the block sale is $\big(Y_0^\beta, Z_0^{Y^\beta}\big) = \big(\gamma_\beta^{-1}(z-y), \beta\big(\gamma_\beta^{-1}(z-y)\big)\big)$. Hence, similar to the above calculation, we have

$$\begin{aligned}J_\beta(y,z) &= J_\beta\Big(\gamma_\beta^{-1}(z-y), \beta\big(\gamma_\beta^{-1}(z-y)\big)\Big)\\ &= \int_{\beta(0+)}^{z-y}\left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right)du + A\int_0^{\beta(0+)}\psi(u)\,du.\end{aligned}$$

Therefore, we conclude that in case (ii) of Lemma 3.8,

$$J_\beta(y,z) = \int_{\beta(0+)}^{z-y} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du + A \int_0^{\beta(0+)} \psi(u)\, du. \tag{3.49}$$

Moreover, in case (iii), $z \le \beta(y)$. So we need to wait until time $t_w > 0$ at which $Z_{t_w}^{Y^\beta} = \beta(y)$. With reference to (3.8) and (3.34), we have

$$t_w = H(z) - H\big(\beta(y)\big) = \int_{\beta(y)}^{z} \frac{1}{h(u)}\, du.$$

Also, observe that

$$\int_0^{t_w} h\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big)\, dt = -\int_0^{t_w} \psi\big(Z_t^{Y^\beta}\big)\, dZ_t^{Y^\beta} = -\int_z^{\beta(y)} \psi(u)\, du.$$

Hence in case (iii), the performance function is given by

$$\begin{aligned} J_\beta(y,z) &= \int_0^{t_w} \left(\kappa_A(y) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \right) dt + J_\beta\big(y, \beta(y)\big) \\ &= \kappa_A(y) \int_{\beta(y)}^{z} \frac{1}{h(u)}\, du - A \int_z^{\beta(y)} \psi(u)\, du \\ &\quad + \int_{\beta(0+)}^{\gamma_\beta(y)} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du + A \int_0^{\beta(0+)} \psi(u)\, du. \end{aligned} \tag{3.50}$$

Although this provides an explicit expression for $J_\beta(y,z)$, it is not obvious to see the information about continuity and differentiability of $J_\beta(y,z)$ in y, since β is only a càglàd function. However, we can calculate further that

$$\begin{aligned} \int_{\beta(0+)}^{\gamma_\beta(y)} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du = \int_0^y & \left(\frac{\kappa_A(u)}{h\big(\beta(u)\big)} + A\psi\big(\beta(u)\big) \right) d\gamma_\beta^c(u) \\ &+ \sum_{0<u<y} \kappa_A(u) \int_{\beta(u)}^{\beta(u+)} \frac{1}{h(x)}\, dx \\ &+ A \sum_{0<u<y} \int_{\beta(u)}^{\beta(u+)} \psi(s)\, ds, \end{aligned}$$

From this expression, as well as

$$\int_{\beta(0+)}^{\beta(y)} \psi(u)\, du = \int_0^y \psi\big(\beta(u)\big)\, d\beta^c(u) + \sum_{0<u<y} \int_{\beta(u)}^{\beta(u+)} \psi(s)\, ds,$$

and

$$\kappa_A(y)H\big(\beta(y)\big) = \kappa_A(0)H\big(\beta(0+)\big) + \int_0^y \kappa_A'(u)H\big(\beta(u)\big)\, du + \int_0^y \frac{\kappa_A(u)}{h\big(\beta(u)\big)}\, d\beta^c(u)$$
$$+ \sum_{0<u<y} \kappa_A(u) \int_{\beta(u)}^{\beta(u+)} \frac{1}{h(x)}\, dx,$$

it follows from (3.50) that the performance function$J_\beta(y,z)$ in case (iii) of Lemma 3.8 admits the expression

$$J_\beta(y,z) = \kappa_A(y)H(z) + A\int_0^z \psi(u)\, du - \int_0^y \left(\frac{\kappa_A(u)}{h\big(\beta(u)\big)} + A\psi\big(\beta(u)\big) + \kappa_A'(u)H\big(\beta(u)\big)\right) du. \tag{3.51}$$

In above calculations, we have assumed the existence and finiteness of $\lim_{u\to 0+} \frac{\kappa_A(u)}{h(\beta(u))}$ and $\lim_{u\to 0+} \kappa_A'(u)H\big(\beta(u)\big)$. We have also used that $\lim_{u\to y-} \kappa_A(u) < \infty$ as well as $\lim_{u\to y-} \kappa_A'(u) < \infty$. The finiteness of $\lim_{u\to 0+} \kappa_A'(u)H\big(\beta(u)\big)$ together with (3.2) implies that $\kappa_A(0)H\big(\beta(0+)\big) = 0$. For an optimal intervention boundary β, all of these properties will be demonstrated below by Lemma 3.11.

Suppose β is an intervention boundary such that Y^β is optimal. Then according to the Hamilton-Jacobi-Bellman equation as well as (3.51), we have

$$D_y^- v(y,z) + v_z(y,z) = \Gamma(z;y) - \Gamma\big(\beta(y);y\big) \le 0, \qquad \text{for all } (y,z) \in \mathcal{D},$$

where

$$\Gamma(x;y) = A\psi(x) + \frac{\kappa_A(y)}{h(x)} + \kappa_A'(y)H(x).$$

Therefore, for any given y, $\beta(y)$ is sufficiently a maximiser of $\Gamma(x;y)$. The next lemma helps us to characterise an intervention boundary β whose value maximises $\Gamma(x;y)$ for a given y, and it will be shown latter that such β is an optimal intervention boundary in our problem.

Lemma 3.10. *For $y \in (0, \bar{y}_A)$, define the function $\Gamma(\cdot\,; y) : [\bar{z}, 0] \to \overline{\mathbb{R}}$ by*

$$\Gamma(x; y) = A\psi(x) + \frac{\kappa_A(y)}{h(x)} + \kappa'_A(y)H(x), \qquad \text{for } x \in (\bar{z}, 0), \tag{3.52}$$

and

$$\Gamma(0; y) = \lim_{x \to 0+} \Gamma(x; y), \qquad \Gamma(\bar{z}; y) = \lim_{x \to \bar{z}} \Gamma(x; y).$$

Let $\beta^* = \beta^*(y)$ and $\beta_* = \beta_*(y)$ denote the functions defined as the largest and smallest $\beta \in [\bar{z}, 0]$ satisfying

$$\max_{x \in [\bar{z}, 0]} \Gamma(x; y) = \Gamma\big(\beta; y\big), \tag{3.53}$$

respectively. Then for all $y \in (0, \bar{y}_A)$, we have $\bar{z} \le \beta_*(y) \le \beta^*(y) < 0$. Furthermore, if $\bar{y}_A < \infty$, write $\beta^*(y) = \beta_*(y) = \bar{z}$, for all $y > \bar{y}_A$. Set

$$\beta^*(0) = 0, \qquad\qquad \beta_*(0) = \lim_{y \to 0^+} \beta_*(y),$$

and

$$\beta^*(\bar{y}_A) = \lim_{y \to \bar{y}_A -} \beta^*(y), \qquad \beta_*(\bar{y}_A) = \lim_{y \to \bar{y}_A +} \beta_*(y).$$

This defines two unique decreasing functions $\beta^*, \beta_* : \mathbb{R}^+ \to [\bar{z}, 0]$ which are càglàd and càdlàg, respectively, and they are left and right-continuous versions of each other.

Lemma 3.11. *Let β^* be given by Lemma 3.10, it follows that if $\lim_{x \to y-} \kappa_A(x) = \infty$ or $\lim_{x \to y-} \kappa'_A(x) = \infty$, then $\lim_{x \to y-} \beta^*(x) = \bar{z}$. Furthermore, we have*

$$\lim_{y \to 0+} \frac{\kappa_A(y)}{h\big(\beta^*(y)\big)} = 0 \qquad \text{and} \qquad \lim_{y \to 0+} \kappa'_A(y)H\big(\beta^*(y)\big) = 0. \tag{3.54}$$

Clearly, the function β^* given in Lemma 3.10 satisfies all of the properties of an intervention boundary that we were concerning. With this intervention boundary, the proposition below provides an explicit expression for the value function which solves (3.20)–(3.23) with associated boundary condition $v(0, z) = A\int_0^z \psi(u)\, du$, for all $z \in [\bar{z}, 0]$, Then as a consequence, we will show that this intervention boundary characterises the optimal liquidation strategy. Before proceeding, we make a few comments on the optimal intervention boundary and the associated optimal liquidation strategy. First of all, the non-increasing property of the boundary essentially means that when the investor makes continuous sale, it is never optimal to implement a trading speed which makes the current best bid price to be decreased. In other words, the sell speed should be at most as large as the current speed of resilience. Therefore, the possible constant parts in the intervention boundary represent the situation that the current volatility risk [9] is too large so that it is optimal to sell as quick as possible in order to reduce the stock position and hence the volatility risk. Moreover, jump parts in the intervention boundary correspond to waiting in the optimal strategy. This can be interpreted as that the current illiquidity cost [10] is relatively large, comparing to the volatility, thus it is optimal to wait so that the best bid price increases to a level which is more preferred by the investor.

Proposition 3.12. *Let $\beta = \beta^*$ denote the largest solution to (3.53), and let γ_β^{-1} and ρ_β^{-1} be the corresponding functions defined by (3.29) and (3.30). Then the function $v : \mathcal{D} \to \mathbb{R}$ given by that for $z > \beta(y)$,*

$$v(y,z) = \int_{\beta(0+)}^{z-y} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du + A \int_0^{\beta(0+)} \psi(u)\, du, \tag{3.55}$$

and for $z \leq \beta(y)$,

$$v(y,z) = \kappa_A(y)H(z) + A\int_0^z \psi(u)\, du - \int_0^y \left(\frac{\kappa_A(u)}{h\big(\beta(u)\big)} + A\psi\big(\beta(u)\big) + \kappa_A'(u)H\big(\beta(u)\big) \right) du, \tag{3.56}$$

is a $C^{0,1}(\mathcal{D})$ solution to (3.20)–(3.23) with the boundary condition $v(0,z) = A\int_0^z \psi(u)\, du$, for all $z \in [\bar{z}, 0]$. Moreover, $D_y^- v(y,z)$ is càglàd in y and continuous in z.

Note that (3.55)-(3.56) agree with (3.48) when $y = 0$. The following theorem verifies that the function v given by (3.55)-(3.56) is equal to the value function V given by (3.19), and that the strategy Y^β corresponding to β characterised by (3.53) is an optimal liquidation strategy. Hence, such a Y^β provides a solution to the utility maximization problem in (3.12).

[9] With reference to (3.14), $\int_0^\infty Y_{t-}\, dL_t$ represents the risk due to market volatility, and this integral corresponds to the term $\int_0^\infty \kappa_A(Y_t)\, dt$ in the simplified optimisation problem (3.19).

[10] The illiquidity cost, or the price impact cost is described by $F_\infty(Y)$ in (3.14), and it corresponds to the term $\int_0^\infty h(Z_t^Y)\psi(Z_t^Y)\, dt$ in the simplified problem (3.19).

Theorem 3.13. *Denote the investor's risk aversion by A, the initial unaffected price by b, and by c the initial cash position. We take β as the largest solution to (3.53) and v to be given by (3.55) and (3.56). Moreover, let V be given by (3.19). Then $v = V$ on $\mathcal{D}$ and*

$$\sup_{Y\in\mathcal{A}(y)} \mathbb{E}\big[U\big(C_\infty(Y)\big)\big] = -\exp\bigg(-A(c+by) + A\int_z^{z-y} \psi(s)\,ds\bigg)\exp\big(v(y,z)\big),$$

where $z = Z^Y_{0-}$ is the initial state of the bid order book and y is the initial share position. The optimal strategy Y^* is equal to $Y^\beta \in \mathcal{A}_D(y)$, where Y^β is the strategy described in Lemma 3.8 corresponding to β with $Y^\beta_{0-} = y$.

3.4 Proofs

Proof of Lemma 3.7. With reference to the dynamic of Z^Y, we calculate that for $z \geq \bar{z}$,

$$\begin{aligned}
\int_0^{Z^Y_T} \psi(u)\,du &= \int_0^z \psi(u)\,du + \int_0^T \psi(Z^Y_{t-})\,dY^c_t \\
&\qquad - \int_0^T h(Z^Y_{t-})\psi(Z^Y_{t-})\,dt + \sum_{0\leq t\leq T}\int_{Z^Y_{t-}}^{Z^Y_{t-}+\triangle Y_t} \psi(u)\,du \\
&= \int_0^z \psi(u)\,du + \int_0^T \psi(Z^Y_{t-})\,dY^c_t \\
&\qquad - \int_0^T h(Z^Y_{t-})\psi(Z^Y_{t-})\,dt + \sum_{0\leq t\leq T}\int_0^{\triangle Y_t} \psi\big(Z^Y_{t-}+u\big)\,du.
\end{aligned}$$

Then,

$$\begin{aligned}
F_T(Y) &= \int_0^T \psi\big(Z^Y_{t-}\big)\,dY^c_t + \sum_{0\leq t\leq T}\int_0^{\triangle Y_t} \psi\big(Z^Y_{t-}+x\big)\,dx \\
&= \int_0^{Z^Y_T} \psi(u)\,du - \int_0^z \psi(u)\,du + \int_0^T h(Z^Y_{t-})\psi(Z^Y_{t-})\,dt \\
&= \int_z^{Z^Y_T} \psi(u)\,du + \int_0^T h(Z^Y_{t-})\psi(Z^Y_{t-})\,dt.
\end{aligned}$$

Notice that for any admissible liquidation strategy Y, we have either Y and Z^Y get 0 at the same time, or Y becomes 0 at some time s while $Z^Y_s < 0$. In the second case, for all $t > s$,

Z^Y satisfies

$$dZ_t^Y = -h(Z_t^Y)\,dt.$$

According to (3.9), we know that the solution to the above dynamic tends to 0, as $t \to \infty$. Therefore, $Z_t^Y \to 0$, as $t \to \infty$ in any case. Then it follows from the above expression of $F_T(Y)$ that

$$F_\infty(Y) = \int_z^0 \psi(u)\,du + \int_0^\infty h(Z_{t-}^Y)\psi(Z_{t-}^Y)\,dt.$$

$\square$

Proof of Lemma 3.8. We first prove that on any time interval I contained in $[\,t_w, \bar{t}\,] \setminus \cup_{n\in\mathbb{I}}[t_n, s_n)$, there exists a unique solution to the dynamic (3.43). But on such I, the process Y^β does not cross any jump of β. Thus, in terms of the function β, we shall only focus on those parts without jumps. Also, it is sufficient to consider Y starting from time 0 (rather than starting at any time in $[\,t_w, \bar{t}\,] \setminus \cup_{n\in\mathbb{I}}[t_n, s_n)$). Write $Y_t^0 = Y_0 > 0$ and

$$Y_t^{k+1} = \gamma_\beta^{-1}\biggl(\Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^k)\bigr)\,du\Bigr\} \wedge \beta(0+)\biggr). \tag{3.57}$$

Let $T \in [0, \infty)$. Then

$$\begin{aligned}
&\sup_{0\le t\le T}\bigl|\,\beta(Y_t^{k+1}) - \beta(Y_t^k)\,\bigr| \\
&= \sup_{0\le t\le T}\biggl|\Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^k)\bigr)\,du\Bigr\} \wedge \beta(0+) - \Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^{k-1})\bigr)\,du\Bigr\} \wedge \beta(0+) \\
&\qquad\qquad + \gamma_\beta^{-1}\biggl(\Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^k)\bigr)\,du\Bigr\} \wedge \beta(0+)\biggr) \\
&\qquad\qquad - \gamma_\beta^{-1}\biggl(\Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^{k-1})\bigr)\,du\Bigr\} \wedge \beta(0+)\biggr)\biggr| \\
&\le 2 \sup_{0\le t\le T}\biggl|\,\Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^k)\bigr)\,du\Bigr\} \wedge \beta(0+) - \Bigl\{\gamma_\beta(Y_0) - \int_0^t h\bigl(\beta(Y_u^{k-1})\bigr)\,du\Bigr\} \wedge \beta(0+)\,\biggr| \\
&\le 2 \sup_{0\le t\le T}\biggl|\int_0^t h\bigl(\beta(Y_u^k)\bigr) - h\bigl(\beta(Y_u^{k-1})\bigr)\,du\biggr| \\
&\le 2L \int_0^T \Bigl|\,\beta(Y_u^k) - \beta(Y_u^{k-1})\,\Bigr|\,du
\end{aligned}$$

$$\leq 2L \int_0^T \sup_{0\leq t\leq u}\Big|\,\beta(Y_t^k) - \beta(Y_t^{k-1})\,\Big|\,du, \tag{3.58}$$

where the first equality is because that when β has no jumps we have $\beta\big(\gamma_\beta^{-1}(x)\big) = x + \gamma_\beta^{-1}(x)$, the first inequality is due the triangle inequality and $|\gamma_\beta^{-1}(x) - \gamma_\beta^{-1}(y)| \leq |x-y|$, and the third inequality is followed from the boundedness of processes $\beta(Y^k)$ and $\beta(Y^{k-1})$ and the locally Lipschitz continuity of h with a Lipschitz constant L. By induction and with reference to (3.58), it can be shown that

$$\sup_{0\leq t\leq T}\big|\,\beta(Y_t^{k+1}) - \beta(Y_t^k)\,\big| \leq \frac{(2LT)^k}{k!}2\big|\beta(Y_0)\big|.$$

Taking k to infinity, we have that $\beta(Y_t^k)$ converges uniformly on $[0,T]$. Define $\beta_t = \lim_{k\to\infty}\beta(Y_t^k)$, for $t\in[0,T]$. Since $T\in[0,\infty)$ is arbitrary, it follows that $\beta_t = \lim_{k\to\infty}\beta(Y_t^k)$ for all $t\in[0,\infty)$. With reference to (3.57), the dominated convergence theorem gives out that for every $t\in[0,\infty)$, $\big(Y_t^k\big)_{k=0}^\infty$ is convergent. We define $Y_t^\beta = \lim_{k\to\infty}Y_t^k$. It can be checked that Y^β decreases to 0. Then since β is continuous, we obtain $\beta_t = \beta(Y_t^\beta)$, for all $t\in[0,\infty)$. Therefore, by sending k to infinity in (3.57), since we only consider Y_t^β before time $\bar{t}$, we have that

$$Y_t^\beta = \gamma_\beta^{-1}\left(\gamma_\beta(Y_0^\beta) - \int_0^t h\big(\beta(Y_u^\beta)\big)\,du\right), \qquad \text{for } t\leq\bar{t}.$$

This shows the existence of solution to the dynamic (3.43) on any time interval contained in $[\,t_w,\bar{t}\,]\setminus\cup_{n\in\mathbb{I}}[t_n,s_n)$. For uniqueness, let's assume that $Y^{(1)}$ and $Y^{(2)}$ satisfy (3.43), where $Y_t^{(1)} = Y_t^{(2)}$ for $0\leq t\leq t_1$, and $Y_t^{(1)} < Y_t^{(2)}$ for $t_1 < t < t_2$. Then for $t_1<t<t_2$,

$$\begin{aligned}
Y_t^{(1)} &= \gamma_\beta^{-1}\left(\gamma_\beta(Y_0^{(1)}) - \int_0^t h\big(\beta(Y_u^{(1)})\big)\,du\right)\\
&\geq \gamma_\beta^{-1}\left(\gamma_\beta(Y_0^{(2)}) - \int_0^t h\big(\beta(Y_u^{(2)})\big)\,du\right)\\
&= Y_t^{(2)},
\end{aligned}$$

which contradicts the assumption that $Y_t^{(1)} < Y_t^{(2)}$ for $t_1<t<t_2$. So the uniqueness holds. The existence and uniqueness of solution to the dynamic in (3.40) on any time interval contained in $[\,t_w,\bar{t}\,]\cap\cup_{n\in\mathbb{I}}[t_n,s_n)$ follow from the locally Lipschitz continuity of function h.

Now let Y^β and Z^{Y^β} be processes satisfying (3.39)–(3.45) with $\big(Y_{0-}^\beta, Z_{0-}^{Y^\beta}\big) = (y,z)\in\mathcal{D}$.

Note that $\big(Y_t^\beta, Z_t^{Y^\beta}\big) \in \mathcal{G}^\beta$ for all $t \in [\,t_w, \bar{t}\,]$. We need to show (3.6) is satisfied. We first focus on the case when $t \leq t_w$. Suppose $z > \beta(y)$, i.e. $t_w = 0$. Then in case (ii) (a),

$$\begin{aligned}
Y_0^\beta - Y_{0-}^\beta &= \beta^{-1}\big(\rho_\beta^{-1}(z-y)\big) - y \\
&= \gamma_\beta^{-1}(z-y) - y \\
&= \big(z - y + \gamma_\beta^{-1}(z-y)\big) - z \\
&= Z_0^{Y^\beta} - Z_{0-}^{Y^\beta},
\end{aligned}$$

where we have used the identity $\beta^{-1}\big(\rho_\beta^{-1}(z-y)\big) = \gamma_\beta^{-1}(z-y)$ which follows from (3.32) and is valid under the condition of (ii) (a). In case (ii) (b), we obtain

$$\begin{aligned}
Z_0^{Y^\beta} - Z_{0-}^{Y^\beta} &= \beta\big(\gamma_\beta^{-1}(z-y)\big) - z \\
&= z - y + \gamma_\beta^{-1}(z-y) - z \\
&= \gamma_\beta^{-1}(z-y) - y \\
&= Y_0^\beta - Y_{0-}^\beta,
\end{aligned}$$

where $\beta\big(\gamma_\beta^{-1}(z-y)\big) = \rho_\beta^{-1}(z-y)$ was used. Suppose $z \leq \beta(y)$, i.e. $t_w > 0$. It can be checked that $Z_t^{Y^\beta} = H^{-1}\big(H(z)-t\big)$ has dynamic (3.40). Because Y_t^β is now constant, (3.6) is satisfied. In the case when $t > t_w$, Y_t^β and $Z_t^{Y^\beta}$ follow (3.39)–(3.45), which satisfy (3.6).

We next prove Y^β is càdlàg and decreasing. Note that by the definitions of t_n, s_n, t_w and $\bar{t}$ and (3.40), (3.43) and the first part of the proof, we have Y_t^β and $Z_t^{Y^\beta}$ are continuous when $(Y_t^\beta, Z_t^{Y^\beta})$ is in each continuous part of the graph of β or β^{-1}, for $t > 0$. Also, each initial condition associated with dynamics (3.40) and (3.43) is chosen to make Y_t^β and $Z_t^{Y^\beta}$ to be continuous at t_n, s_n and t_w when $t_w > 0$. It can also be seen that Y^β and Z^{Y^β} are right continuous at $t = 0$. These together with the well-defined Y_{0-}^β and $Z_{0-}^{Y^\beta}$ imply that Y^β and Z^{Y^β} are continuous for $t > 0$ and they are right-continuous with left-limit at $t = 0$. Y^β decreases to 0 follows from (3.39), (3.40), (3.43), and the first part of this proof.

Finally, $Z_t^{Y^\beta} = H^{-1}\big(H(z) - t\big)$, for $0 \leq t \leq t_w$ follows from (3.9). $\square$

Proof of Lemma 3.9. Let $\{y_n\}_{n\in\mathbb{I}}$ be the set of all points at which the intervention boundary β is discontinuous. Consider a time interval $[t, s] \subseteq [t_n, s_n)$ for some $n \in \mathbb{I}$, where t_n and

s_n are given by (3.37) and (3.38). With reference to (3.6), we note that formally,

$$dt = -\frac{d\rho_\beta\big(Z_t^{Y^\beta}\big)}{h\big(Z_t^{Y^\beta}\big)} \qquad \forall\, t \in [t_n, s_n),$$

and hence,

$$\begin{aligned}
&\int_t^s \Big(\kappa_A\big(Y_r^\beta\big) + Ah\big(Z_r^{Y^\beta}\big)\psi\big(Z_r^{Y^\beta}\big)\Big)\, dr \\
&= \int_s^t \left(\frac{\kappa_A\big(\beta^{-1}(Z_r^{Y^\beta})\big)}{h(Z_r^{Y^\beta})} + A\psi\big(Z_r^{Y^\beta}\big)\right) d\rho_\beta\big(Z_r^{Y^\beta}\big) \\
&= \int_{\rho_\beta(Z_s^{Y^\beta})}^{\rho_\beta(Z_t^{Y^\beta})} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right) du \\
&= \int_{Z_s^{Y^\beta} - Y_{s_n}^\beta}^{Z_t^{Y^\beta} - Y_{t_n}^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right) du, \qquad (3.59)
\end{aligned}$$

where we have used the identity in (3.32). Similarly, since

$$dt = -\frac{d\gamma_\beta\big(Y_t^\beta\big)}{h\big(\beta(Y_t^\beta)\big)} \qquad \forall\, t \in [\,t_w, \bar{t}\,] \setminus \cup_{n\in\mathbb{I}}[t_n, s_n),$$

applying (3.33), it can be calculated that on some time interval $[s,t] \subset [\,t_w, \bar{t}\,] \setminus \cup_{n\in\mathbb{I}}[t_n, s_n)$, for some $n \in \mathbb{I}$,

$$\begin{aligned}
&\int_s^t \Big(\kappa_A\big(Y_r^\beta\big) + Ah\big(Z_r^{Y^\beta}\big)\psi\big(Z_r^{Y^\beta}\big)\Big)\, dr \\
&= \int_{Z_t^{Y^\beta} - Y_t^\beta}^{Z_s^{Y^\beta} - Y_s^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big)\right) du, \qquad (3.60)
\end{aligned}$$

Let $t_w < \bar{t}$. Suppose the number of t_n and s_n in the interval $[\,t_w, \bar{t}\,]$ is equal to $m < \infty$ (possibly $m = 0$). Consider $r_0 \le r_1 < ... < r_m < r_{m+1}$, where $r_0 = t_w$, $r_{m+1} = \bar{t}$ and for $k = 1, ..., m$, r_k are equal to those $t_n, s_n \in [\,t_w, \bar{t}\,]$. We assume $r_1, ..., r_m$ are in an ascending order. Then it follows from (3.59), (3.60) and the continuity of Y_t^β and $Z_t^{Y^\beta}$ when $t > 0$ that

$$\int_{t_w}^{\bar{t}} \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big)\Big)\, dt$$

$$
\begin{aligned}
&= \sum_{k=0}^{m} \int_{r_k}^{r_{k+1}} \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt \\
&= \int_{Z_{\bar{t}}^{Y^\beta}}^{Z_{t_w}^{Y^\beta} - Y_{t_w}^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du.
\end{aligned}
$$

Suppose there are infinitely many t_n and s_n in the interval $[t_w, \bar{t}]$. Let $r \in [t_w, \bar{t}]$ be an accumulative point of the sequence $\{t_n\}_{n\in\mathbb{I}}$. Then with out loss of generality, consider a subsequence $\{t_{n_k}\}_{k=1}^{\infty} \subset [t_w, \bar{t}]$ increases to r. Consider some time interval $[t,s]$ in which r is the only accumulative point of $\{t_n\}_{n\in\mathbb{I}}$. Then, it follows that

$$
\begin{aligned}
&\int_t^s \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt \\
&= \lim_{n\to\infty} \int_t^{t_n} \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt + \int_r^s \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt \\
&= \lim_{n\to\infty} \int_{Z_{t_n}^{Y^\beta} - Y_{t_n}^\beta}^{Z_t^{Y^\beta} - Y_t^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du \\
&\quad + \int_{Z_s^{Y^\beta} - Y_s^\beta}^{Z_r^{Y^\beta} - Y_r^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du, \\
&= \int_{Z_s^{Y^\beta} - Y_s^\beta}^{Z_t^{Y^\beta} - Y_t^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_{t_w}^{\bar{t}} \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt \\
&= \int_{Z_{\bar{t}}^{Y^\beta}}^{Z_{t_w}^{Y^\beta} - Y_{t_w}^\beta} \left(\frac{\kappa_A\big(\gamma_\beta^{-1}(u)\big)}{h\big(\rho_\beta^{-1}(u)\big)} + A\psi\big(\rho_\beta^{-1}(u)\big) \right) du.
\end{aligned}
$$

Therefore the result follows from the above equality as well as (3.45) and (3.47). □

Proof of Lemma 3.10. First notice that for any $y \in (0, \bar{y}_A)$, $\Gamma(x,y)$ is concave in x and this concavity may not be strict. Observe that for $y \in (0, \bar{y}_A)$,

$$
\lim_{x\to 0^-} \Gamma(x; y) = -\infty.
$$

Also, $\Gamma(x;y) \in \mathbb{R}$, for $x \in [\bar{z}, 0)$. These imply that $\bar{z} \leq \beta_*(y) \leq \beta^*(y) < 0$, for all $0 < y < \bar{y}_A$. The largest and smallest solution to (3.53) uniquely define the functions β^* and β_*. For $0 < y < y + \triangle < \bar{y}_A$ and $x \in [\bar{z}, 0)$, we calculate that

$$\frac{d}{dx}\Big[\Gamma(x; y+\triangle) - \Gamma(x;y)\Big] = -\frac{\big(\kappa_A(y+\triangle) - \kappa_A(y)\big)h'(x)}{h^2(x)} + \frac{\kappa_A'(y+\triangle) - \kappa_A'(y)}{h(x)} < 0, \tag{3.61}$$

since κ_A is convex and $\kappa_A'(u) > 0$, for $u > 0$. We want to show that β^* and β_* are decreasing functions. In order to get a contradiction, suppose that there exists $y \in (0, \bar{y}_A)$ and $\triangle > 0$ such that $\beta^*(y+\triangle) > \beta_*(y)$. With reference to (3.61), we obtain

$$\Gamma\big(\beta^*(y+\triangle); y+\triangle\big) - \Gamma\big(\beta^*(y+\triangle); y\big) < \Gamma\big(\beta_*(y); y+\triangle\big) - \Gamma\big(\beta_*(y); y\big).$$

However, this contradicts the definitions of β^* and β_* which imply that

$$\Gamma\big(\beta^*(y+\triangle); y+\triangle\big) \geq \Gamma\big(\beta_*(y); y+\triangle\big) \quad \text{and} \quad \Gamma\big(\beta_*(y); y\big) \geq \Gamma\big(\beta^*(y+\triangle); y\big).$$

Therefore, for all $0 < y < \bar{y}_A$,

$$\beta_*(y+\triangle) \leq \beta^*(y+\triangle) \leq \beta_*(y) \leq \beta^*(y), \tag{3.62}$$

and from which it follows that β^* and β_* are decreasing. By (3.52), we know that for $\bar{z} \leq x < 0$, $\Gamma(x;y)$ is continuous in y. Then for $y \in (0, \bar{y}_A)$, we have

$$\begin{aligned}\Gamma\big(\beta_*(y+); y+\big) = \Gamma\big(\beta_*(y+); y\big) \leq \Gamma\big(\beta_*(y); y\big) = \Gamma\big(\beta_*(y); y+\big)\\ \Gamma\big(\beta^*(y-); y\big) = \Gamma\big(\beta^*(y-); y-\big) \geq \Gamma\big(\beta^*(y); y-\big) = \Gamma\big(\beta^*(y); y\big).\end{aligned}$$

Since β^* and β_* are defined as the largest and smallest maximiser to (3.53) respectively, also because β^* and β_* are decreasing, it follows that $\beta_*(y+) = \beta_*(y)$ and $\beta^*(y-) = \beta^*(y)$. By monotonicity, right limit of β^* and left limit of β_* exist. Hence, we have proved that β^* is càglàd and β_* is càdlàg . The claim that β^* is the càglàd version of β_* and that β_* is the càdlàg version of β^* follows from (3.62). □

Proof of Lemma 3.11. If $y > \bar{y}_A$, then by the definition of β^*, it is certainly true that if $\lim_{x\to y-} \kappa_A(x) = \infty$ or $\lim_{x\to y-} \kappa_A'(x) = \infty$, then $\lim_{x\to y-} \beta^*(x) = \bar{z}$. The rest case is for

$y = \bar{y}_A$. We proof by contradiction and suppose $\beta^*(\bar{y}_A) > \bar{z}$. For any $x \in \big(\bar{z}, \beta^*(\bar{y}_A)\big)$ and $y \in (0, \bar{y}_A)$ such that $\beta^*(y) \geq \beta^*(\bar{y}_A)$, we have

$$
\begin{aligned}
A\psi(x) &\leq A\bigg(\psi(x) - \psi\big(\beta^*(y)\big)\bigg) \\
&\leq \kappa_A(y)\bigg(\frac{1}{h\big(\beta^*(y)\big)} - \frac{1}{h(x)}\bigg) + \kappa_A'(y)\bigg(H\big(\beta^*(y)\big) - H(x)\bigg) \\
&\leq \kappa_A(y)\bigg(\frac{1}{h\big(\beta^*(\bar{y}_A)\big)} - \frac{1}{h(x)}\bigg) + \kappa_A'(y)\bigg(H\big(\beta^*(\bar{y}_A)\big) - H(x)\bigg).
\end{aligned}
$$

Taking y to be arbitrarily close to $\bar{y}_A$ implies $\psi(x) = -\infty$. This means $x < \bar{z}$ which contradicts with $x > \bar{z}$. Hence, we conclude that $\beta^*(\bar{y}_A) = -\infty$.

Now we prove for (3.54). Note that if $\beta^*(0+) < 0$, then (3.54) is obviously true. However, if $\beta^*(0+) = 0$, then

$$
\frac{\kappa_A(y)}{h\big(\beta^*(y)\big)} \;\geq\; \Gamma(x;y) - A\psi\big(\beta^*(y)\big) - \kappa_A'(y)H\big(\beta^*(y)\big) \;\geq\; \Gamma(x;y) - \kappa_A'(y)H\big(\beta^*(y)\big),
$$

from which it follows that for any $x \in (\bar{z}, 0)$,

$$
0 \geq \liminf_{y\to 0+} \frac{\kappa_A(y)}{h\big(\beta^*(y)\big)} \;\geq\; A\psi(x) - \limsup_{y\to 0+} \kappa_A'(y)H\big(\beta^*(y)\big), \tag{3.63}
$$

$$
0 \geq \limsup_{y\to 0+} \frac{\kappa_A(y)}{h\big(\beta^*(y)\big)} \;\geq\; A\psi(x) - \liminf_{y\to 0+} \kappa_A'(y)H\big(\beta^*(y)\big). \tag{3.64}
$$

Therefore,

$$
\begin{aligned}
0 &\geq \limsup_{y\to 0+} \kappa_A'(y)H\big(\beta^*(y)\big) \;\geq\; A\psi(x), \\
0 &\geq \liminf_{y\to 0+} \kappa_A'(y)H\big(\beta^*(y)\big) \;\geq\; A\psi(x).
\end{aligned}
$$

Taking x to 0 and by (3.4), we get $\lim_{y\to 0+} \kappa_A'(y)H\big(\beta^*(y)\big) = 0$. Also, by sending x to 0 in (3.63) and (3.64), $\lim_{y\to 0+} \frac{\kappa_A(y)}{h(\beta^*(y))} = 0$ therefore follows. $\square$

Proof of Proposition 3.12. To show v is continuous, we first prove it is finite. But with reference to (3.47)-(3.51), it suffices to show that the function J_β given by (3.46) is finite with β defined by Lemma 3.10. By the continuity of Y^β and Z^{Y^β} after time 0 and condition (3.5), we have that there exists some $s > 0$ such that

$$\int_0^s \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt < \infty \tag{3.65}$$

and $Y_s^\beta < \bar{y}_A$. According to the condition given by Lemma 3.11 that

$$\lim_{y \to 0^+} \frac{\kappa_A(y)}{h\big(\beta(y)\big)} = 0,$$

it follows that there exists $C_1 > 0$ and $0 < \epsilon < \bar{y}_A$ such that

$$\kappa_A(y) \leq -C_1 h\big(\beta(y)\big), \qquad \text{for all } y \in [0, \epsilon].$$

Since $\psi\big(Z_t^{Y^\beta}\big)$ is bounded for all $t \geq s$ (it increases to 0), this together with the above inequality implies that

$$\begin{aligned} \int_s^\infty \Big(\kappa_A\big(Y_t^\beta\big) + Ah\big(Z_t^{Y^\beta}\big)\psi\big(Z_t^{Y^\beta}\big) \Big)\, dt &\leq \int_s^\infty \Big(- C_1 h\big(\beta(Y_t^\beta)\big) - C_2 h\big(Z_t^{Y^\beta}\big) \Big)\, dt \\ &\leq \int_s^\infty \Big(- C_1 h\big(Z_t^{Y^\beta}\big) - C_2 h\big(Z_t^{Y^\beta}\big) \Big)\, dt \\ &\leq (C_1 + C_2)\big(Y_s^\beta - Z_s^{Y^\beta}\big) < \infty, \end{aligned} \tag{3.66}$$

where $C_2 > 0$ is some constant. Therefore, (3.65) and (3.66) together show that v is finite.

Note that each expression given by (3.55) or (3.56) is continuous in y and z. It is therefore sufficient to prove that v is continuous across $\mathcal{G}^\beta$. Write $J_u(y,z)$ to be the expression of $v(y,z)$ given by (3.55), and let $J_l(y,z)$ be the expression in (3.56). Suppose (y,z) is a point on the graph of β, i.e, $z = \beta(y)$. Consider a sequence of points $(y_n, z_n)_{n=1}^\infty$ contained in $\overline{\mathcal{S}}^\beta \setminus \mathcal{G}^\beta$, converging to (y,z). With reference to (3.50) and(3.51), we calculate that

$$\lim_{n \to \infty} v(y_n, z_n) = J_u\big(y, \beta(y)\big) = J_l\big(y, \beta(y)\big) = v\big(y, \beta(y)\big). \tag{3.67}$$

If (y,z) lies on the graph of β^{-1}, i.e, $y = \beta^{-1}(z)$, then using the property that $\beta^{-1}(u) = \beta^{-1}(z)$, for $u \in \big(z, \beta(\beta^{-1}(z))\big)$, direct calculation results (3.67). It therefore can be concluded that v is a continuous function. Differentiating v gives out

$$D_y^- v(y,z) = -\frac{\kappa_A\big(\gamma_\beta^{-1}(z-y)\big)}{h\big(\rho_\beta^{-1}(z-y)\big)} - A\psi\big(\rho_\beta^{-1}(z-y)\big), \qquad z > \beta(y); \tag{3.68}$$

$$v_z(y,z) = \frac{\kappa_A\big(\gamma_\beta^{-1}(z-y)\big)}{h\big(\rho_\beta^{-1}(z-y)\big)} + A\psi\big(\rho_\beta^{-1}(z-y)\big), \qquad z > \beta(y); \tag{3.69}$$

$$D_y^- v(y,z) = \kappa_A'(y)H(z) - \frac{\kappa_A(y)}{h\big(\beta(y)\big)} - A\psi\big(\beta(y)\big) - \kappa_A'(y)H\big(\beta(y)\big), \qquad z \le \beta(y); \tag{3.70}$$

$$v_z(y,z) = \frac{\kappa_A(y)}{h(z)} + A\psi(z), \qquad z \le \beta(y). \tag{3.71}$$

These expressions are left-continuous with right limit in y and continuous in z (all of these expressions are continuous at $(0,0)$, this is guaranteed by (3.54)). Also, it can be checked that for any $(y_n, z_n)_{n=1}^\infty \subseteq \overline{\mathcal{S}}^\beta$, $(y,z) \in \mathcal{G}^\beta$ and $\lim_{n\to\infty}(y_n,z_n) = (y,z)$, we have $v_z(y_n,z_n) \to v_z(y,z)$, as $n \to \infty$. Further, $\lim_{z\to\beta(y)+} D_y^- v(y,z) = D_y^- v\big(y,\beta(y)\big)$. Therefore, we conclude that $v_z(y,z)$ is continuous, and $D_y^- v(y,z)$ is càglàd in y but continuous in z.

Standard calculations show that v satisfies (3.20) and (3.22). When $z = 0$, (3.21) is clearly true. For $z \ne 0$, in order to verify (3.21), we compute that when $z > \beta(y)$,

$$\begin{aligned} & h(z)v_z(y,z) - \kappa_A(y) - Ah(z)\psi(z) \\ =& h(z)\left\{\frac{\kappa_A\big(\gamma_\beta^{-1}(s)\big)}{h\big(\rho_\beta^{-1}(s)\big)} - \frac{\kappa_A(z-s)}{h(z)} + A\big\{\psi\big(\rho_\beta^{-1}(s)\big) - \psi(z)\big\}\right\}, \end{aligned} \tag{3.72}$$

where $s = z - y$. Observe that $h\big(\rho_\beta^{-1}(s)\big) = 0$ implies $y = 0$, but (3.20)–(3.23) are under the condition of $y > 0$. So $h\big(\rho_\beta^{-1}(s)\big)$ is non-zero. By the definition of γ_β^{-1}, we must have $\gamma_\beta^{-1}(s) \in (0, \bar{y}_A)$ if $\beta(\bar{y}_A) = \bar{z}$, or $\gamma_\beta^{-1}(s) \in (0, \bar{y}_A]$ if $\beta(\bar{y}_A) > \bar{z}$. Then according to the limiting behaviour of β in Lemma 3.11, $\kappa_A\big(\gamma_\beta^{-1}(s)\big)$ must be finite, so is $\kappa_A'\big(\gamma_\beta^{-1}(s)\big)$. However, $\kappa_A(z-s)$ may be infinite, but then it follows that (3.72) is negative. Otherwise, if $\kappa_A(y) < \infty$, write

$$G(s;z) = \frac{\kappa_A\big(\gamma_\beta^{-1}(s)\big)}{h\big(\rho_\beta^{-1}(s)\big)} - \frac{\kappa_A(z-s)}{h(z)} + A\big\{\psi\big(\rho_\beta^{-1}(s)\big) - \psi(z)\big\}.$$

Then in order to verify (3.21), it suffices to show $G(s;z) \ge 0$, for all $\rho_\beta^{-1}(s) < z < 0$. $G(s;y)$

can be rewritten as

$$G(s;z) = \Big[\Gamma\big(\rho_\beta^{-1}(s);\gamma_\beta^{-1}(s)\big) - \Gamma\big(z;\gamma_\beta^{-1}(s)\big)\Big] \\ - \kappa_A'\big(\gamma_\beta^{-1}(s)\big)\Big[H\big(\rho_\beta^{-1}(s)\big)\big) - H(z)\Big] + \frac{1}{h(z)}\Big[\kappa_A\big(\gamma_\beta^{-1}(s)\big) - \kappa_A(z-s)\Big], \quad (3.73)$$

where Lemma 3.10 verifies

$$\Gamma\big(\rho_\beta^{-1}(s);\gamma_\beta^{-1}(s)\big) - \Gamma\big(z;\gamma_\beta^{-1}(s)\big) \geq 0. \quad (3.74)$$

We calculate that

$$\frac{1}{h(z)}\Big[\kappa_A\big(\gamma_\beta^{-1}(s)\big) - \kappa_A(z-s)\Big] - \kappa_A'\big(\gamma_\beta^{-1}(s)\big)\Big[H\big(\rho_\beta^{-1}(s)\big) - H(z)\Big] \\ = \int_{\rho_\beta^{-1}(s)}^{z}\left(\frac{\big[\kappa_A(u-s) - \kappa_A\big(\rho_\beta^{-1}(s)-s\big)\big]h'(u)}{h^2(u)} + \frac{\kappa_A'\big(\rho_\beta^{-1}(s)-s\big) - \kappa_A'(u-s)}{h(u)}\right)du \\ \geq 0. \quad (3.75)$$

Therefore, combining (3.73)–(3.75), (3.21) is verified. Furthermore, the definition of β yields

$$\begin{aligned} D_y^- v(y,z) + v_z(y,z) &= \kappa_A'(y)H(z) - \frac{\kappa_A(y)}{h\big(\beta(y)\big)} + A\psi(z-y) - A\psi\big(\beta(y)\big) \\ &\quad - \kappa_A'(y)H\big(\beta(y)\big) + \frac{\kappa_A(y)}{h(z)} + A\psi(z) - A\psi(z-y) \\ &= \Gamma(z;y) - \Gamma\big(\beta(y);y\big) \\ &\leq 0. \end{aligned}$$

This verifies that (3.23) is true.

Finally, the boundary condition is satisfied by (3.55), because for any $u \in [\beta(0+), z]$, we have $\gamma_\beta^{-1}(u) = 0$ and $\rho_\beta^{-1}(u) = u$; and it is trivially satisfied by (3.56). □

Proof of Theorem 3.13. Let δ be a positive-valued $C^\infty(\mathbb{R})$ function with support on $[0,1]$ satisfying $\int_0^1 \delta(x)\,dx = 1$, and define a sequence of functions $\{\delta_n\}_{n=1}^\infty$ by

$$\delta_n(s) = n\,\delta(ns), \quad s \geq 0.$$

We mollify v to obtain a sequence of functions $\{v^{(n)}\}_{n=1}^{\infty}$ which are given by

$$v^{(n)}(y,z) = \int_0^1 v(y-s,z)\,\delta_n(s)\,ds.$$

(One may extend the lower bound of the domain of $v(\cdot,z)$ properly so that $v^{(n)}$ is well-defined at $y=0$.) Then $v^{(n)} \in C^{1,1}(\mathcal{D})$, for all $n \in \mathbb{N}$, and

$$\begin{aligned} v(y,z) &= \lim_{n\to\infty} v^{(n)}(y,z), \\ v_z(y,z) &= \lim_{n\to\infty} v_z^{(n)}(y,z), \\ D_y^- v(y,z) &= \lim_{n\to\infty} v_y^{(n)}(y,z), \end{aligned}$$

where the last equality is due to $D_y^- v(y,z)$ being càglàd in y. Moreover, for every $(y_0,z_0) \in \mathcal{D}$ there exists a $K>0$ such that on the set $\left\{\, (y,z) \in \mathcal{D} \,\middle|\, z \geq y + z_0 - y_0 \,\right\}$,

$$\left| v^{(n)}(y,z) \right| \leq K, \quad n \in \mathbb{N}, \tag{3.76}$$

$$\left| v_y^{(n)}(y,z) \right| \leq K, \quad n \in \mathbb{N}, \tag{3.77}$$

$$\left| v_z^{(n)}(y,z) \right| \leq K, \quad n \in \mathbb{N}. \tag{3.78}$$

(If Y is admissible and $(Y_{0-}, Z_{0-}^Y) = (y_0,z_0)$, then $(Y_t, Z_t^Y) \in \left\{\, (y,z) \in \mathcal{D} \,\middle|\, z \geq y + z_0 - y_0 \,\right\}$, for all $t \geq 0$.) By Itô's formula, we compute that

$$\begin{aligned} & v^{(n)}\big(Y_T, Z_T^Y\big) + \int_0^T \Big(\kappa_A\big(Y_{t-}\big) + Ah\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big) \Big)\,dt \\ &= v^{(n)}(y,z) + \int_0^T \Big(v_y^{(n)}(Y_{t-}, Z_{t-}^Y) + v_z^{(n)}(Y_{t-}, Z_{t-}^Y) \Big)\,dY_t^c \\ &\quad + \int_0^T \Big(\kappa_A\big(Y_{t-}\big) + Ah\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big) - v_z^{(n)}\big(Y_{t-}, Z_{t-}^Y\big)h\big(Z_{t-}^Y\big) \Big)\,dt \\ &\quad + \sum_{0\leq t\leq T} \Big\{ v^{(n)}\big(Y_{t-} + \triangle Y_t, Z_{t-}^Y + \triangle Y_t\big) - v^{(n)}\big(Y_{t-}, Z_{t-}^Y\big) \Big\}, \end{aligned} \tag{3.79}$$

for all $Y \in \mathcal{A}_D(y)$. Observe that for $t \geq 0$,

$$0 \leq -\int_0^t h\big(Z_u^Y\big)\,du = Z_t^Y - Y_t - Z_0^Y + Y_0 \leq y - z.$$

Then, with reference to (3.76)–(3.78), we have

$$\int_0^\infty \sup_{n\in\mathbb{N}} \Big| v_z^{(n)}\big(Y_{t-}, Z_{t-}^Y\big) h\big(Z_{t-}^Y\big) \Big| \, dt \le K(y-z).$$

Similarly,

$$\int_0^\infty \sup_{n\in\mathbb{N}} \Big| v_y^{(n)}\big(Y_{t-}, Z_{t-}^Y\big) + v_z^{(n)}\big(Y_{t-}, Z_{t-}^Y\big) \Big| \, d(-Y_t^c) \le 2Ky$$

and

$$\sum_{0\le t} \sup_{n\in\mathbb{N}} \Big| v^{(n)}\big(Y_{t-} + \triangle Y_t, Z_{t-}^Y + \triangle Y_t\big) - v^{(n)}\big(Y_{t-}, Z_{t-}^Y\big) \Big| \le 2Ky.$$

Hence, by (3.79) and the boundary condition $v(0,z) = A\int_0^z \psi(u)\,du$, the dominated convergence theorem gives out that for any $Y \in \mathcal{A}_D(y)$,

$$\begin{aligned}
&\int_0^\infty \Big(\kappa_A\big(Y_{t-}\big) + Ah\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big) \Big)\, dt \\
&= v(y,z) + \int_0^\infty \Big(D_y^- v\big(Y_{t-}, Z_{t-}^Y\big) + v_z\big(Y_{t-}, Z_{t-}^Y\big) \Big)\, dY_t^c \\
&\quad + \int_0^\infty \Big(\kappa_A\big(Y_{t-}\big) + Ah\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big) - v_z\big(Y_{t-}, Z_{t-}^Y\big)h\big(Z_{t-}^Y\big) \Big)\, dt \\
&\quad + \sum_{t\ge 0} \Big\{ v\big(Y_{t-} + \triangle Y_t, Z_{t-}^Y + \triangle Y_t\big) - v\big(Y_{t-}, Z_{t-}^Y\big) \Big\},
\end{aligned} \tag{3.80}$$

as $n \to \infty$ and $T \to \infty$. According to Proposition 3.12, v satisfies (3.20)–(3.23), and therefore,

$$\int_0^\infty \Big(\kappa_A\big(Y_{t-}\big) + Ah\big(Z_{t-}^Y\big)\psi\big(Z_{t-}^Y\big) \Big)\, dt \ge v(y,z). \tag{3.81}$$

Hence, $V \ge v$.

From from (3.65)-(3.66), we know that with β being the largest solution to (3.53) and Y^β being the strategy described in Lemma 3.8 corresponding to β, Y^β is admissible, in particular (3.3) is satisfied. Therefore, with reference to (3.81), in order to complete the proof, we need to show that (3.81) holds with equality for Y^β. Observe that $\triangle Y^\beta < 0$ only if $t = 0$ and $z > \beta(y)$. But by (3.20) and Proposition 3.12, we have that $D_y^- v(y,z) + v_z(y,z) = 0$, for

$z > \beta(y)$. Therefore,

$$\sum_{t\geq 0}\Big\{v\big(Y_{t-}^{\beta} + \triangle Y_t^{\beta}, Z_{t-}^{Y^{\beta}} + \triangle Y_t^{\beta}\big) - v\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big)\Big\} = 0.$$

For any $z \leq 0$, if $0 \leq t \leq t_w$, where t_w is defined by (3.35), then $d\big(Y_t^{\beta}\big)^c = 0$, hence

$$\int_0^{t_w}\Big(D_y^- v\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big) + v_z\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big)\Big)\, d\big(Y_t^{\beta}\big)^c = 0;$$

if $t > t_w$, then $\big(Y_t^{\beta}, Z_t^{Y^{\beta}}\big) \in \mathcal{G}^{\beta}$, which implies

$$\int_{t_w}^{\infty}\Big(D_y^- v\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big) + v_z\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big)\Big)\, d(Y_t^{\beta})^c = 0.$$

Finally we have

$$\int_0^{\infty}\Big(\kappa_A\big(Y_{t-}^{\beta}\big) + Ah\big(Z_{t-}^{Y^{\beta}}\big)\psi\big(Z_{t-}^{Y^{\beta}}\big) - v_z\big(Y_{t-}^{\beta}, Z_{t-}^{Y^{\beta}}\big)h\big(Z_{t-}^{Y^{\beta}}\big)\Big)\, dt = 0,$$

since the integrand is equal to 0, for all $\big(Y_t^{\beta}, Z_t^{Y^{\beta}}\big) \in \overline{\mathcal{W}}^{\beta}$, and the Lebesgue measure of time taken when $\big(Y_t^{\beta}, Z_t^{Y^{\beta}}\big) \in \overline{\mathcal{S}}_s^{\beta} \setminus \mathcal{G}^{\beta}$ is 0. With reference to (3.80), we therefore conclude that $v = V$ and that $Y^* = Y^{\beta} \in \mathcal{A}_D(y)$ is an admissible optimal liquidation strategy for the optimization problem (3.19), and the result follows from (3.17) as well the analysis after (1.11). $\square$

Part II

Contracted Liquidation

This part is based on a joint work with Prof. Mihail Zervos.

Chapter 4

Optimal liquidation with a contractual agreement

4.1 Contracted liquidation model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which supports a one dimensional, standard Brownian motion B. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by this Brownian motion, and we assume that $\mathbb{F}$ satisfies the usual conditions.

We study a market with a single stock, and this market terminates at some finite time $\overline{T}$. We consider an investor who aims to sell a large amount of shares of this stock within a given finite time. Denote by Y the investor's stock position process, i.e. Y_t is the number of shares held by this investor at time t, and by ξ the associated trading speed process which will be referred to as the liquidation strategy in the sequel. Define the set of admissible liquidation strategies as follows.

Definition 4.1. Given an initial stock position $y > 0$ and a terminal time of liquidation $T \in (0, \overline{T}]$, let ξ be an $\mathbb{F}$-adapted, càdlàg and positive-valued liquidation speed process with finite variation such that the associated stock position process Y satisfies

$$Y_t = y - \int_0^t \xi_t \, dt \geq 0, \quad t \in [0, T],$$

and

$$Y_T = 0.$$

Write $\mathcal{A}(T, y)$ as the set of all admissible liquidation strategies ξ satisfying the above conditions. Denote the set of all deterministic admissible liquidation strategies by $\mathcal{A}_D(T, y)$.

It follows from the above definition that for any $\xi \in \mathcal{A}(T, y)$, the associated stock position process Y is $\mathbb{F}$-adapted, positive and decreasing, and we have the constraint that

$$\int_0^T \xi_s \, ds = y.$$

Note that it might be the case that there exists some $s \in (0, T)$ such that $Y_t = 0$, for all $t \in [s, T]$.

If the investor doesn't trade, we suppose the stock price is described by

$$s + \sigma B_t, \quad t \geq 0,$$

where $s, \sigma > 0$ are the initial stock price and the volatility parameter, respectively. Since the trading volume from this investor is large, due to a lack of enough liquidity in the market, the stock price drops down during the investor's sale. Throughout this chapter, we adopt an Almgren-Chriss type of liquidation model (Almgren and Chriss, 1999, 2000) with absence of the permanent price impact and a general temporary impact being concerned. Precisely, we assume that the affected stock price process S is given by

$$S_t = s + \sigma B_t - F(\xi_t), \quad t \geq 0,$$

where the function $F : [0, \infty) \to [0, \infty)$ describes the price impact in response to the investor's trading speed (i.e. the temporary price impact). Conditions on F will be specified at the end of this section.

In our model interest rate is negligible. This a usual convention for a liquidation model, since in practice liquidation can usually complete in a very short time period. Therefore, if the investor is liquidating shares following some $\xi \in \mathcal{A}(T, y)$, the proceeds she receives up to time t is given by

$$\int_0^t S_u \xi_u \, du = s(y - Y_t) - \sigma B_t Y_t + \sigma \int_0^t Y_u \, dB_u - \int_0^t \xi_u F(\xi_u) \, du.$$

In particular, at the final time of liquidation she receives totally

$$\int_0^T S_u \xi_u \, du = sy + \sigma \int_0^T Y_t \, dB_t - \int_0^T \xi_t F(\xi_t) \, dt. \tag{4.1}$$

The constant sy is the marked-to-market total wealth of the investor at the beginning of liquidation. The stochastic integral $\int_0^T Y_t \, dB_t$ represents the volatility risk encountered by the investor. The total price impact cost is given by the term $\int_0^T \xi_t F(\xi_t) \, dt$.

Now let's suppose the investor is not able to access to the market, and therefore a broker is hired to sell shares. In order to study the interaction between the investor and the broker, we consider a principal-agent problem based on the liquidation problem. From now on, we shall call this investor as the principal (she) and this broker as the agent (he). The agent liquidates shares on behalf of the principal under some contractual agreement. The contract specifies the initial liquidation position, the required terminal time of liquidation and the liquidation strategy that the agent is expected to follow; moreover, manners of paying liquidation proceeds and compensation to each other are also written in the contract. The principal may not be able to observe the actual strategy that the agent implements, and hence the agent may have opportunity to generate some private benefit. We try to find the optimal contract offered by the principal as well as the associated liquidation strategy implemented by the agent. Mathematically, we consider the following agent's problem

$$\sup_{\xi} \mathbb{E}\big[\, U^A\big(\, compensation \;+\; private\ benefit \;-\; cost \,\big)\,\big] \geq \bar{A},$$

where U^A is the agent's utility function, ξ is the implemented liquidation strategy, the term *cost* indicates any implementation cost which is in addition to the price impact cost, and $\bar{A}$ is some constant denoting the agent's participation constraint. The implementation cost is described by some function $H : [0, \infty) \to [0, \infty)$. This function takes liquidation speeds as inputs. Assumptions on H will be specified at the end of this section. We assume that $\bar{A} \geq U^A(0)$ so that the agent only takes any work which can bring him more benefits than doing nothing. The principal's problem is

$$\sup_{\tilde{\xi},\, compensation} \mathbb{E}\big[\, U^P\big(\, proceeds \;-\; compensation \,\big)\,\big],$$

where U^P is the her utility function and $\tilde{\xi}$ is the liquidation strategy recommended by her.

Throughout this chapter, the principal is assumed to have a constant absolutely risk aversion (CARA), and the agent is risk-neutral, i.e. we take

$$U^P(x) = -\exp(-\gamma x) \quad \text{and} \quad U^A(x) = x,$$

where $\gamma > 0$ denotes the principal's risk aversion.

We list some conditions on the price impact function $F : [0, \infty) \to [0, \infty)$ and the implementation cost function $H : [0, \infty) \to [0, \infty)$ as follows.

Condition

(I) The function $x \mapsto xF(x)$ is strictly convex with $F(0) = 0$; if H is non-zero, then it is strictly convex with $H(0) = 0$.

(II) F' is decreasing.

(III) The function $(H' + F)/F'$ has linear growth, i.e. there exists some constant $K > 0$ such that $(H'(x) + F(x))/F'(x) \leq K(1 + x)$, for all $x \geq 0$.

(IV) There exists some constant $K > 0$ such that $(H'(x) + F(x))/F'(x) = Kx$, for all $x \geq 0$.

For future references, notice that Condition (I) implies that F is strictly increasing and $\lim_{x\to 0} xF'(x) = \lim_{x\to 0} x^2F'(x) = 0$; Condition (I) and (IV) together imply that $x \mapsto x^2F'(x)$ is strictly increasing and $\lim_{x\to\infty} x^2F'(x) = \infty$. We group these conditions into different assumptions which will be used in the sequel.

Assumption

(I) $F, H \in C^1((0, \infty))$, and they satisfy Condition (I).

(II) F and H satisfy Condition (I), (II) and (III).

(II) $F, H \in C^2((0, \infty))$, and they satisfy Condition (I), (II) and (IV).

These assumptions are satisfied by a large class of functions. For example,

$$F(x) = \beta_1 x^{\beta_2} \qquad \text{and} \qquad H(x) = \beta_3 x^{\beta_4},$$

with $\beta_1 > 0$ and $\beta_3 \geq 0$, and Assumption (I) is satisfied if $\beta_2 > 0$ and $\beta_4 > 1$; Assumption (II) is satisfied if $\beta_4 - \beta_2 \leq 1$; and Assumption (III) is satisfied if $\beta_4 - \beta_2 = 1$. We refer to

Almgren (2003) for a theoretical study when F takes the above form, and refer to Almgren et al. (2005) and Lillo et al. (2003) for some empirical studies about the price impact function with the above form.

4.2 First-best contract

In this section, we impose Assumption (I) and assume that the principal and the agent share the same information. The agent has to follow any liquidation strategy recommended by the principal. At the terminal time of liquidation, the agent delivers the total proceeds received from the sale to the principal, and he receives a compensation paid by the principal [11]. For a general approach to contimuous-time first-best contracts with lump-sum payments, see Cvitanić et al. (2006). Let C_T be an $\mathcal{F}_T$-measurable random variable which denotes the compensation paid at some terminal time of liquidation $T \leq \overline{T}$. We formulate the principal's problem as

$$\sup_{\xi \in \mathcal{A}(T,y),\, C_T} \mathbb{E}\Big[-\exp\Big\{ -\gamma\Big(\int_0^T S_t \xi_t \, dt - C_T \Big) \Big\} \Big], \tag{4.2}$$

subject to

$$\mathbb{E}\Big[C_T - \int_0^T H(\xi_t)\, dt \Big] \geq \bar{A}, \tag{4.3}$$

where $\bar{A} \geq 0$.

Proposition 4.2. *The optimal liquidation strategy for problem (4.2)-(4.3) is given by*

$$\xi_t^* = \frac{y}{T}, \quad t \in [0, T],$$

and the optimal compensation is

$$C_T^* = \bar{A} + TH(y/T) + \sigma y \int_0^T (1 - t/T)\, dB_t.$$

We see that although the principal is risk averse, the optimal liquidation strategy is a trivial strategy which takes no care about the market volatility risk. This kind of strategy is

[11] A stream form of compensation may also be considered, but in the absence of interest rate, a stream of compensation is equivalent to a compensation in a lump-sum form.

usually optimal for a risk-neutral investor (see e.g. Almgren and Chriss, 2000). The principal optimally choices a trivial strategy which is because that she can transfer all market volatility risk to the risk-neutral agent via the term $\sigma \int_0^T y(1-t/T)\,dB_t$ in the optimal compensation. The principal also pays the agent a fixed amount of money $\bar{A}$ which ensures the agent's minimal requirement is satisfied. The agent's implementation cost $TH(y/T)$ is completely covered by the principal. As a result, the principal is guaranteed to receive

$$sy - \bar{A} - F(y/T)y - TH(y/T),$$

which corresponds to the initial mark-to-market wealth minus the agent's commission charge, the price impact cost as well as the agent's implementation cost; and the agent receives

$$\bar{A} + \sigma y \int_0^T (1-t/T)\,dB_t,$$

which corresponds to the commission fee and the market volatility cost.

4.3 Second-best contract

In practice, the principal may only observe the affected stock price but not the actual strategy implemented by the agent. The agent may also have his private time horizon for completing the liquidation. Throughout this section, we suppose that the principal observes the affected stock price until the terminal time of liquidation required by her, and we denote this time by $T \leq \overline{T}$. We also suppose that the principal asks for a liquidation proceeds calculated according to some strategy recommended by her against the affected stock price she observes. She requires to receive this proceeds at time T, and at the same time she pays the agent amount of compensation denoted by C_T. We refer to Cvitanić et al. (2009) for a general approach to continuous-time second-best contracts with lump-sum payments. Due to the unobservable implemented strategy for the principal and in order to generate some private benefit, the agent can actually follow any strategy which is different from the recommended one and cannot be detected by the principal. However, we assume that even without any supervision, the principal expects the agent to follow her recommendation. Therefore, she has to select an optimal contract which induces the agent to implement the recommended strategy as his optimal choice.

Definition 4.3. Any contract is *incentive compatible*, if with this contract the agent's optimal implemented strategy, if it exists, is identical to the principal's recommendation.

To design an incentive compatible contract, write $\tilde{\xi}$ to be the principal's recommended liquidation strategy, ξ to be the agent's implemented liquidation strategy and $\tilde{Y}$ and Y to be the corresponding stock position processes. The stock price is affected by ξ, and we recall that it satisfies

$$S_t = s + \sigma B_t - F(\xi_t), \quad t \geq 0.$$

Note that if ξ_0 is different from 0, then the affected price process has a jump at the beginning of liquidation. Also, S has jumps whenever ξ has. Therefore, for any given $\tilde{\xi}$, in order to avoid the principal's perceiving of cheating, the agent can only take ξ to be such that $\xi_0 = \tilde{\xi}_0$ and ξ has the same jumps as $\tilde{\xi}$. Denote by $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\geq 0}$ the filtration generated by S, and we assume it satisfies the usual conditions. This filtration captures all of the information accessed by the principal, while the Brownian filtration $\mathbb{F} = (\mathcal{F})_{t\geq 0}$ contains all of the information available to the agent. We have that for any $t \geq 0$, $\mathcal{F}_t^S \subseteq \mathcal{F}_t$, which indicates that the agent is more knowledgeable than the principal, because of his private liquidation strategy ξ. The next definition defines formally the recommended and implemented strategies.

Definition 4.4.

(i) The set of all principal's recommended liquidation strategies, denoted by $\mathcal{A}^R(T, y)$, consists of all $\tilde{\xi} \in \mathcal{A}(T, y)$ such that $\tilde{\xi}$ is $\mathbb{F}^S$-adapted.

(ii) Given any $\tilde{\xi} \in \mathcal{A}^R(T, y)$, the set of all agent's implemented liquidation strategies, denoted by $\mathcal{A}^I(\overline{T}, y; \tilde{\xi})$, consists of all $\xi \in \mathcal{A}(\overline{T}, y)$ such that $\xi_0 = \tilde{\xi}_0$ and ξ has the same jumps as $\tilde{\xi}$.

Remark 4.5. Denote by $T^A \leq \overline{T}$ the agent's private time horizon for completing the liquidation. Then his actual set of admissible strategy should be $\mathcal{A}^I(T^A, y; \tilde{\xi})$ which is a subset of $\mathcal{A}^I(\overline{T}, y; \tilde{\xi})$. For any incentive compatible contract with a recommended strategy $\tilde{\xi}$ satisfying $\tilde{\xi}_t > 0$, for some $t > T^A$, the agent is not able to implement such $\tilde{\xi}$ and hence will not sign such contract. The principal therefore does not need to concern about this case. Otherwise, $\tilde{\xi}$ should be the agent's optimal choice among the set $\mathcal{A}^I(T^A, y; \tilde{\xi})$. But since $\mathcal{A}^I(T^A, y; \tilde{\xi}) \subseteq \mathcal{A}^I(\overline{T}, y; \tilde{\xi})$, it is sufficient for the principal to concern that the agent chooses

$\tilde{\xi}$ as his optimal implemented strategy over the set $\mathcal{A}^I(\overline{T}, y; \tilde{\xi})$. Consequently, the principal does not need to concern about T^A at all. $\square$

For any $\tilde{\xi} \in \mathcal{A}^R(T, y)$ and $\xi \in \mathcal{A}^I(\overline{T}, y; \tilde{\xi})$, the agent's expected private benefit at time T can be expressed as

$$\int_0^T S_t \xi_t \, dt - \int_0^T S_t \tilde{\xi}_t \, dt + \mathbb{E}\left[\int_T^{\overline{T}} S_t \xi_t \, dt - \int_T^{\overline{T}} H(\xi_t) \, dt \,\middle|\, \mathcal{F}_T \right],$$

which corresponds to the difference between the proceeds got from the sale and the money delivered to the principal at time T, plus the expected money received (taking away the implemented cost) by liquidating the amount of shares left at time T (if $Y_T > 0$). The above expression can be rewritten as

$$\sigma \int_0^T \big(Y_t - \tilde{Y}_t\big) \, dB_t + \int_0^T F(\xi_t)(\tilde{\xi}_t - \xi_t) \, dt - \Gamma(Y_T), \tag{4.4}$$

where

$$\Gamma(Y_T) = \mathbb{E}\left[\int_T^{\overline{T}} \Big(\xi_t F(\xi_t) + H(\xi_t) \Big) \, dt \,\middle|\, \mathcal{F}_T \right],$$

which is $\mathcal{F}_T$-measurable and it satisfies $\Gamma(Y_T) \geq 0$ and $\Gamma(0) = 0$.

4.3.1 Agent's problem

In this subsection, we impose Assumtion (II). Base on the assumption that the principal looks for an optimal incentive compatible contract, we first derive heuristically an admissible form of compensation. According to this, we state a formal definition of admissible contracts. Then we study sufficient condition for an admissible contract to be incentive compatible. Finally the set of admissible incentive compatible contracts will be defined.

Given any $\tilde{\xi} \in \mathcal{A}^R(T, y)$ and some $\mathcal{F}_T^S$-measurable compensation C_T, with reference to Remark 4.5 and the expression of private benefit in (4.4), the agent's problem is

$$\sup_{\xi \in \mathcal{A}^I(\overline{T}, y; \tilde{\xi})} \mathbb{E}\left[C_T + \left(\sigma \int_0^T \big(Y_t - \tilde{Y}_t\big) dB_t + \int_0^T F(\xi_t)(\tilde{\xi}_t - \xi_t) dt - \Gamma(Y_T) \right) - \int_0^T H(\xi_t) dt \right]$$

$$= \sup_{\xi \in \mathcal{A}^I(\overline{T}, y; \tilde{\xi})} \mathbb{E}\bigg[C_T + \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt - \Gamma(Y_T)\bigg] \geq \bar{A}, \tag{4.5}$$

where $\bar{A} \geq 0$. Suppose

$$\mathbb{E}\bigg[\bigg|C_T + \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt - \Gamma(Y_T)\bigg|\bigg] < \infty,$$

and write

$$U_t = \mathbb{E}\bigg[C_T + \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt - \Gamma(Y_T)\,\bigg|\,\mathcal{F}_t\bigg].$$

By a martingale representation theorem, we have that

$$\begin{aligned} C_T &= U_0 - \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt + \int_0^T \sigma \hat{Z}_t\,dB_t + \Gamma(Y_T) \\ &= U_0 - \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt + \int_0^T \hat{Z}_t\,d\Big(S_t + F(\xi_t)\Big) + \Gamma(Y_T), \end{aligned}$$

where $\hat{Z}$ is an $\mathbb{F}$-progressively measurable process satisfying $\int_0^T \hat{Z}_t^2\,dt < \infty$ a.s. Since the principal expects the agent to implement $\tilde{\xi} \in \mathcal{A}^R(T, y)$, motivated by the above expression, we consider any compensation with the form

$$C_T = c + \int_0^T H(\tilde{\xi}_t)\,dt + \int_0^T Z_t\,d\Big(S_t + F(\tilde{\xi}_t)\Big), \tag{4.6}$$

where $c \in \mathbb{R}$, $\tilde{\xi} \in \mathcal{A}^R(T, y)$, and Z satisfies $\int_0^T Z_t^2\,dt < \infty$ a.s. and is $\mathbb{F}$-adapted. Note that C_T in (4.6) is $\mathcal{F}_T^S$-measurable, and for any given c, $\tilde{\xi}$ and Z, C_T can be considered as a random function of ξ. Now we are in the position of defining admissible contracts.

Definition 4.6. Any *admissible contract* is the multiplet $\big(T, y, c, \tilde{\xi}, Z\big)$, where

(i) $T \in (0, \overline{T}]$ is the principal's time horizon;

(ii) $y > 0$ is the initial liquidation position;

(iii) $c \in \mathbb{R}$ is some reservation compensation;

(iv) $\tilde{\xi} \in \mathcal{A}^R(T, y)$ is the recommended liquidation strategy;

(v) Z is the sensitivity of the compensation with respect to the affected stock price's randomness, it is $\mathbb{F}$-adapted and satisfies $\int_0^T Z_t^2\,dt < \infty$ a.s.

Therefore, the associated compensation C_T is given by (4.6).

The following result gives out a motivation for the definition of admissible incentive compatible contracts.

Proposition 4.7. *Let* $\big(T, y, c, \tilde{\xi}, Z\big)$ *be an admissible contract such that*

$$c \geq \bar{A} \quad \text{and} \quad \tilde{\xi}_T = 0,$$

and the process Z *admits the dynamic*

$$dZ_t = \frac{1}{F'(\tilde{\xi}_t)}\Big[F(\tilde{\xi}_t) + H'(\tilde{\xi}_t)\Big]\,dt, \quad t \in (0, T], \tag{4.7}$$

with

$$Z_0 = z \in \mathbb{R} \quad \text{and} \quad Z_T \geq 0.$$

Such Z *is increasing and bounded. It follows that the agent's unique optimal implemented liquidation strategy for problem (4.5) is* $\tilde{\xi}$*, and his optimal expected payoff is* c.

Definition 4.8. Any *admissible incentive compatible contract* is an admissible contract $\big(T, y, \bar{A}, \tilde{\xi}, Z\big)$ satisfying the condition of Proposition 4.7.

One can check that with the above definition and in view of Remark 4.5, $\big(T, y, \bar{A}, \tilde{\xi}, Z\big)$ is indeed incentive compatible. In particular, if $\tilde{\xi}_t > 0$ for some $t > T^A$, then the agent is not able to implement the unique optimal strategy $\tilde{\xi}$. Therefore his optimal expected payoff with respect to the set $\mathcal{A}^I(T^A, y; \tilde{\xi})$ is strictly less than $\bar{A}$, and hence he will not sign this contract. Otherwise the incentive compatibility is guaranteed by Proposition 4.7 and $\tilde{\xi}$ is indeed in $\mathcal{A}^I(T^A, y; \tilde{\xi})$.

4.3.2 Principal's problem

We formulate the principal's problem under Assumption (II) which guarantees Definition 4.8 to be well-defined. Having the set of admissible incentive compatible contracts, the principal tries to solve

$$\sup_{\tilde{\xi}, C_T} \mathbb{E}\left[-\exp\left\{-\gamma\left(\int_0^T \tilde{S}_t\tilde{\xi}_t\,dt - C_T\right)\right\}\right], \tag{4.8}$$

where $\gamma > 0$ is the parameter of risk aversion and $\tilde{S}$ is the stock price affected by $\tilde{\xi}$, which admits the expression

$$\tilde{S}_t = s + \sigma B_t - F(\tilde{\xi}_t), \quad t \geq 0. \tag{4.9}$$

Note that it suffices to solve problem (4.8) without the constraint $\tilde{\xi}_T = 0$ for any admissible incentive compatible contract. This is because that this constraint is only at time T which is of Lebesgue measure 0. Then using the expressions in (4.1), (4.4), (4.6) and (4.9), we consider

$$\begin{aligned}
&\inf_{z\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\left[\exp\left(-\gamma\int_0^T \tilde{S}_t\tilde{\xi}_t\,dt + \gamma\bar{A} + \gamma\int_0^T H(\tilde{\xi}_t)\,dt + \gamma\sigma\int_0^T Z_t\,dB_t\right)\right] \\
= &\inf_{z\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\left[\exp\left\{\gamma\big(\bar{A} - sy\big) + \gamma\sigma\int_0^T \big(Z_t - \tilde{Y}_t\big)\,dB_t + \gamma\int_0^T \Big(H(\tilde{\xi}_t) + \tilde{\xi}_t F(\tilde{\xi}_t)\Big)\,dt\right\}\right].
\end{aligned}$$

Then, it is sufficient to look at

$$\inf_{z\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\left[\exp\left\{\gamma\sigma\int_0^T \big(Z_t - \tilde{Y}_t\big)\,dB_t + \gamma\int_0^T \Big(H(\tilde{\xi}_t) + \tilde{\xi}_t F(\tilde{\xi}_t)\Big)\,dt\right\}\right],$$

with dynamics

$$\begin{aligned}
dZ_t &= \frac{1}{F'(\tilde{\xi}_t)}\Big[F(\tilde{\xi}_t) + H'(\tilde{\xi}_t)\Big]\,dt, && Z_0 = z,\ Z_T \geq 0, \\
d\tilde{Y}_t &= -\tilde{\xi}_t\,dt, && \tilde{Y}_0 = y.
\end{aligned}$$

Write $X = Z - \tilde{Y}$. Then this problem is equivalent to

$$\inf_{x\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\left[\exp\left\{\gamma\sigma\int_0^T X_t\,dB_t + \gamma\int_0^T \Big(H(\tilde{\xi}_t) + \tilde{\xi}_t F(\tilde{\xi}_t)\Big)\,dt\right\}\right] \tag{4.10}$$

with

$$dX_t = \left(\frac{1}{F'(\tilde{\xi}_t)}\Big[F(\tilde{\xi}_t) + H'(\tilde{\xi}_t)\Big] + \tilde{\xi}_t\right)dt, \quad X_0 = x,\ X_T \geq 0,$$

where $x = z - y$.

We next show that problem (4.10) can be reduced to be a deterministic optimisation problem. A similar reduction is in Schied et al. (2010). To this end, for $t \in [0, T]$, write

$$M_t^X = \gamma\sigma \int_0^t X_s \, dB_s$$

and

$$\mathcal{E}(M^X)_t = \exp\big(M_t^X - \frac{1}{2}[M^X, M^X]_t\big),$$

where $[M^X, M^X]$ is the quadratic variation process of M^X. Since X is bounded, Novikov's condition is satisfied by M^X. As a result, we can define a probability measure $\mathbb{P}^X$, which is equivalent to $\mathbb{P}$, via

$$\frac{d\mathbb{P}^X}{d\mathbb{P}} = \mathcal{E}(M^X)_T. \tag{4.11}$$

Then following (4.10), we compute that

$$\begin{aligned}
&\inf_{x\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\bigg[\exp\bigg\{\gamma\sigma \int_0^T X_t \, dB_t + \gamma \int_0^T \Big(H(\tilde{\xi}_t) + \tilde{\xi}_t F(\tilde{\xi}_t)\Big)\, dt \bigg\}\bigg] \\
= &\inf_{x\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}\bigg[\exp\bigg\{\gamma\sigma \int_0^T X_t \, dB_t - \frac{1}{2}\int_0^T \gamma^2\sigma^2 X_t^2 \, dt\bigg\} \times \\
&\qquad \times \exp\bigg\{\frac{1}{2}\int_0^T \gamma^2\sigma^2 X_t^2 \, dt + \gamma \int_0^T \Big(H(\tilde{\xi}_t) + \tilde{\xi}_t F(\tilde{\xi}_t)\Big)\, dt \bigg\}\bigg] \\
= &\inf_{x\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}(T;y)} \mathbb{E}^X\bigg[\exp\bigg\{\int_0^T \Big(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \gamma H(\tilde{\xi}_t) + \gamma\tilde{\xi}_t F(\tilde{\xi}_t)\Big)\, dt \bigg\}\bigg] \\
\leq &\inf_{x\in\mathbb{R},\, \tilde{\xi}\in\mathcal{A}_D(T;y)} \exp\bigg\{\int_0^T \Big(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \gamma H(\tilde{\xi}_t) + \gamma\tilde{\xi}_t F(\tilde{\xi}_t)\Big)\, dt \bigg\},
\end{aligned} \tag{4.12}$$

where the last expectation is with respect to the probability measure defined by (4.11). Then with reference to the analysis after (1.11), it suffices to solve

$$\inf_{x\in\mathbb{R}} V(T, x; y) \tag{4.13}$$

with

$$V(T, x; y) = \inf_{\tilde{\xi}\in\mathcal{A}_D(T,y)} \int_0^T \Big(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \gamma H(\tilde{\xi}_t) + \gamma\tilde{\xi}_t F(\tilde{\xi}_t)\Big)\, dt, \tag{4.14}$$

where $T \in (0, \overline{T}]$, $y > 0$ and X satisfies

$$dX_t = \left(\frac{1}{F'(\tilde{\xi}_t)} \Big[F(\tilde{\xi}_t) + H'(\tilde{\xi}_t) \Big] + \tilde{\xi}_t \right) dt, \quad X_0 = x,\ X_T \geq 0. \tag{4.15}$$

4.3.3 Solution to principal's problem

In order to solve the principal's problem, we impose Assumption (III) under which we can write

$$\frac{1}{F'(\xi)} \Big[F(\xi) + H'(\xi) \Big] + \xi = \eta \xi,$$

for some constant $\eta > 1$. By writing $\theta_t = \eta \tilde{\xi}_t$, problem (4.14)-(4.15) is equivalent to

$$V(T, x; y) = \inf_{\frac{\theta}{\eta} \in \mathcal{A}_D(T,y)} \int_0^T \left(\frac{1}{2} \gamma^2 \sigma^2 X_t^2 + \gamma H(\theta_t/\eta) + \gamma F(\theta_t/\eta) \theta_t/\eta \right) dt \tag{4.16}$$

with

$$dX_t = \theta_t \, dt, \quad X_0 = x, \tag{4.17}$$

where the constraints $y = \int_0^T \tilde{\xi}_t \, dt$ and $X_T \geq 0$ are equivalent to

$$X_T = x + \eta y \geq 0.$$

Proposition 4.9. *$V(T, x; y)$ is convex in x, and it attains a global minimum at $x = -\eta y/2$ for all $T \in (0, \overline{T}]$ and $y > 0$.*

According to Proposition 4.9, it suffices to take

$$x = -\eta y/2,$$

therefore,

$$X_T = \eta y/2 > 0.$$

Note that these values are independent of the time horizon for liquidation.

Observe that (4.16)-(4.17) is a standard problem of calculus of variations, where the objective functional is time-homogeneous. Therefore, according to the theory of calculus of variations, one natural way to solve this problem is by considering the Beltrami identity which is a first order ordinary differential equation characterising the optimiser under certain conditions (see, e.g, Gelfand and Fomin, 2000). Let θ^* and X^* be respectively the candidate of the optimiser and its associated state process. Then, the Beltrami identity associated with problem (4.16)-(4.17) is

$$\frac{1}{2}\gamma^2\sigma^2(X_t^*)^2 + \gamma H(\theta_t^*/\eta) - \gamma H'(\theta_t^*/\eta)\theta_t^*/\eta - \gamma F'(\theta_t^*/\eta)(\theta_t^*/\eta)^2 \equiv -K^{T,x}, \quad t \in [0,T],$$

where $K^{T,x}$ is some constant determined by T and x. After rearranging terms, we get

$$\frac{1}{\gamma}\Big(\frac{1}{2}\gamma^2\sigma^2(X_t^*)^2 + K^{T,x}\Big) = \Phi(\theta_t^*/\eta),$$

where $\Phi : [0,\infty) \to [0,\infty)$ is given by

$$\Phi(u) = H'(u)u - H(u) + F'(u)u^2.$$

With reference to Condition (I) and (IV), it can be checked that Φ is a strictly increasing function with $\Phi(0) = 0$. Define $\Psi : [0,\infty) \to [0,\infty)$ to be the inverse of Φ, then Ψ is a strictly increasing function with $\Psi(0) = 0$. It follows therefore

$$\theta_t^* = \eta\Psi\bigg(\frac{\gamma^2\sigma^2(X_t^*)^2 + 2K^{T,x}}{2\gamma}\bigg), \quad t \in [0,T]. \tag{4.18}$$

Note that it must hold that

$$K^{T,x} \geq -\min_{t\in[0,T]} \frac{1}{2}\gamma^2\sigma^2(X_t^*)^2,$$

and with $x = -\eta y/2$ and $X_T^* = \eta y/2$, we must have $K^{T,x} \geq 0$. Then integrating (4.18) yields

$$\int_{-\eta y/2}^{X_t^*} \frac{1}{\eta\Psi\Big(\frac{\gamma^2\sigma^2u^2+2K^{T,x}}{2\gamma}\Big)}\,du = t. \tag{4.19}$$

Sending t to T, we get

$$\int_{-\eta y/2}^{\eta y/2} \frac{1}{\eta\Psi\Big(\frac{\gamma^2\sigma^2u^2+2K^{T,x}}{2\gamma}\Big)}\,du = T. \tag{4.20}$$

For any fixed y, consider the function $T^y(\cdot)$ defined by

$$T^y(K) = \int_{-\eta y/2}^{\eta y/2} \frac{1}{\eta\Psi\left(\frac{\gamma^2\sigma^2u^2+2K}{2\gamma}\right)}\,du, \quad K \geq 0. \tag{4.21}$$

Since Ψ is a continuous strictly increasing function, by the monotone convergence theorem, $T^y(\cdot)$ is a strictly decreasing function, hence invertible. As a result, for any finite T such that $T \leq \sup_K T^y(K)$, there is a unique $K^{T,y}$ satisfying (4.20). However if $\sup_K T^y(K) < T < \infty$, then one may let $\theta^*_t = 0$ over a period of time, and in this case, $K^{T,x} = \arg\max_K T^y(K) = 0$. The next definition defines formally the candidate of the optimal state process for (4.16)-(4.17).

Definition 4.10. For any $y > 0$ and $T \in (0, \overline{T}]$, take $x = -\alpha y/2$, and let $T^y(\cdot)$ be defined by (4.21). If $T^y(0) < T$, then take $K^{T,x} = 0$; otherwise, let $K^{T,x}$ be the unique constant satisfying (4.20). We define the candidate of the optimal state process, X^*, for problem (4.16)-(4.17) to be that

(i) for $t \in [0, \tau]$, X^*_t satisfies (4.19), where $\tau = \inf\{\, t \geq 0 \,|\, X^*_t = 0 \,\}$;

(ii) for $t \in \big(\tau\,,\, \tau + T - T^y(0)\big]$, $X^*_t = 0$;

(iii) for $t \in \big(\tau + T - T^y(0)\,,\, T\big]$,

$$\int_0^{X^*_t} \frac{1}{\eta\Psi\left(\frac{\gamma^2\sigma^2u^2+2K^{T,x}}{2\gamma}\right)}\,du = t - \tau.$$

Theorem 4.11. *Let X^* be defined by Definition 4.10 with the corresponding time derivative θ^* given by (4.18) for all $t \in [0,T]$. Then θ^* is the unique admissible optimiser for problem (4.16)-(4.17). As a result, the principal's optimal admissible incentive compatible contract is $\big(T, y, \bar{A}, \tilde{\xi}^*, Z^*\big)$, where for all $t \in [0,T]$,*

$$\begin{aligned}
\tilde{\xi}^*_t &= \frac{\theta^*_t}{\eta}\mathbb{1}_{[0,T)}(t), \\
Z^*_t &= (1-\eta/2)y + \int_0^t (\eta-1)\tilde{\xi}^*_u\,du, \\
C^*_T &= \bar{A} + \int_0^T H(\tilde{\xi}^*_u)\,du + \int_0^T Z^*_u\,d\Big(\tilde{S}^*_u + F(\tilde{\xi}^*_u)\Big),
\end{aligned}$$

with $\tilde{S}^*$ being the stock price process affected by $\tilde{\xi}^*$.

4.3.4 Example

For some constants $\beta_1 > 0$ and $\beta_2 \geq 0$, take $F(x) = \beta_1 x$ and $H(x) = \beta_2 x^2$. Then in view of (4.16)-(4.17), we solve

$$V(T, x; y) = \inf_{\frac{\theta}{\eta} \in \mathcal{A}_D(T,y)} \int_0^T \left(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \frac{\gamma\beta_1}{2\eta}\theta_t^2 \right) dt$$

with

$$dX_t = \theta_t \, dt = \eta\tilde{\xi}_t \, dt, \quad X_0 = x = -\eta y/2,$$

where $\eta = 2(\beta_1 + \beta_2)/\beta_1$. Then $\Psi(u) = \sqrt{u}/\sqrt{\beta_1 + \beta_2}$, and (4.19) is equivalent to

$$\sqrt{\frac{\gamma\beta_1}{2\eta}} \int_{-\eta y/2}^{X_t^*} \frac{1}{\sqrt{\frac{\gamma^2\sigma^2 u^2}{2} + K^{T,x}}} \, du = t.$$

Write $Q = \gamma\sigma/\sqrt{2K^{T,x}}$ and $R = \sqrt{\eta\gamma\sigma^2}/\sqrt{\beta_1}$. After integrating and rearranging terms of the above expression we obtain

$$X_t^* = \frac{1}{Q}\sinh\big[Rt + \operatorname{arsinh}(-Q\eta y/2)\big] = \frac{\sqrt{1 + Q^2(\eta y/2)^2}}{Q}\sinh(Rt) - \cosh(Rt)\eta y/2.$$

Therefore, by taking $t = T$, it follows that

$$\frac{\sqrt{1 + Q^2(\eta y/2)^2}}{Q} = \frac{\big(1 + \cosh(RT)\big)\eta y/2}{\sinh(RT)},$$

and then

$$X_t^* = \frac{\eta y}{2\sinh(RT)}\big[\sinh(Rt) - \sinh\big(R(T-t)\big)\big] = -\frac{\eta y \sinh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}.$$

Differentiating with respect to t gives out

$$\theta_t^* = -\frac{R\eta y \cosh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}.$$

Substituting X^* and θ^* back into the expression of $V(T,x;y)$, we get

$$V\big(T,-\eta y/2;y\big)=\frac{\gamma^2\sigma^2\eta^2y^2}{4R}\coth(RT/2). \tag{4.22}$$

Now we verify the constant $K^{T,x}\geq 0$. To this end, we consider (4.18), and it is equivalent to

$$K^{T,x}=\frac{R^2\gamma\beta_1}{4\eta\sinh^2(RT)}\big[\cosh(RT)+1\big]\eta^2y^2>0.$$

Therefore, according to the above solution to the principal's problem, her optimal admissible incentive compatible contract is $\big(T,y,\bar{A},\tilde{\xi}^*,Z^*\big)$, where for all $t\in[0,T]$,

$$\begin{aligned}
\tilde{\xi}^*_t&=\frac{Ry\cosh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}\mathbb{1}_{[0,T)}(t),\\
Z^*_t&=\frac{(1-\eta)y\sinh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}+\frac{y}{2},\\
C^*_T&=\bar{A}+\frac{\beta_2^2R^2y^2}{8\sinh(RT/2)}\left(\frac{\sinh(RT)}{R}+T\right)\\
&\quad+\int_0^T\left(\frac{\sigma(1-\eta)y\sinh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}+\frac{\sigma y}{2}\right)dB,
\end{aligned}$$

and the optimal stock position process satisfies

$$\tilde{Y}^*_t=\frac{y\sinh\big(R(T-2t)/2\big)}{2\sinh(RT/2)}+\frac{y}{2},\quad t\in[0,T].$$

We compare our solution to the solution for the corresponding liquidation model without any contract. Using the notations in our model, with reference to Schied et al. (2010), the optimal liquidation speed process and the associated stock position process in an Almgren-Chriss type of liquidation model with a CARA investor are

$$\bar{\xi}_t=\frac{\bar{R}y\cosh\big(\bar{R}(T-t)/2\big)}{2\sinh(\bar{R}T/2)}\qquad\text{and}\qquad\bar{Y}_t=\frac{y\sinh\big(\bar{R}(T-t)/2\big)}{\sinh(\bar{R}T/2)},$$

where $t\in[0,T]$ and $\bar{R}=\sqrt{2\gamma\sigma^2}/\sqrt{\beta_1+\beta_2}$. Figure 4.1 compares the solution for the second-best contract and the solution for no contract. In the case of second-best contract, the liquidation speed is small at the beginning, which corresponds to a strategy takes less care of the market volatility risk. This is because that the principal transfers some volatility to the agent via the compensation. At the end of liquidation, the speed in the case with contract becomes big, this is due to that the liquidation has to finish by time T.

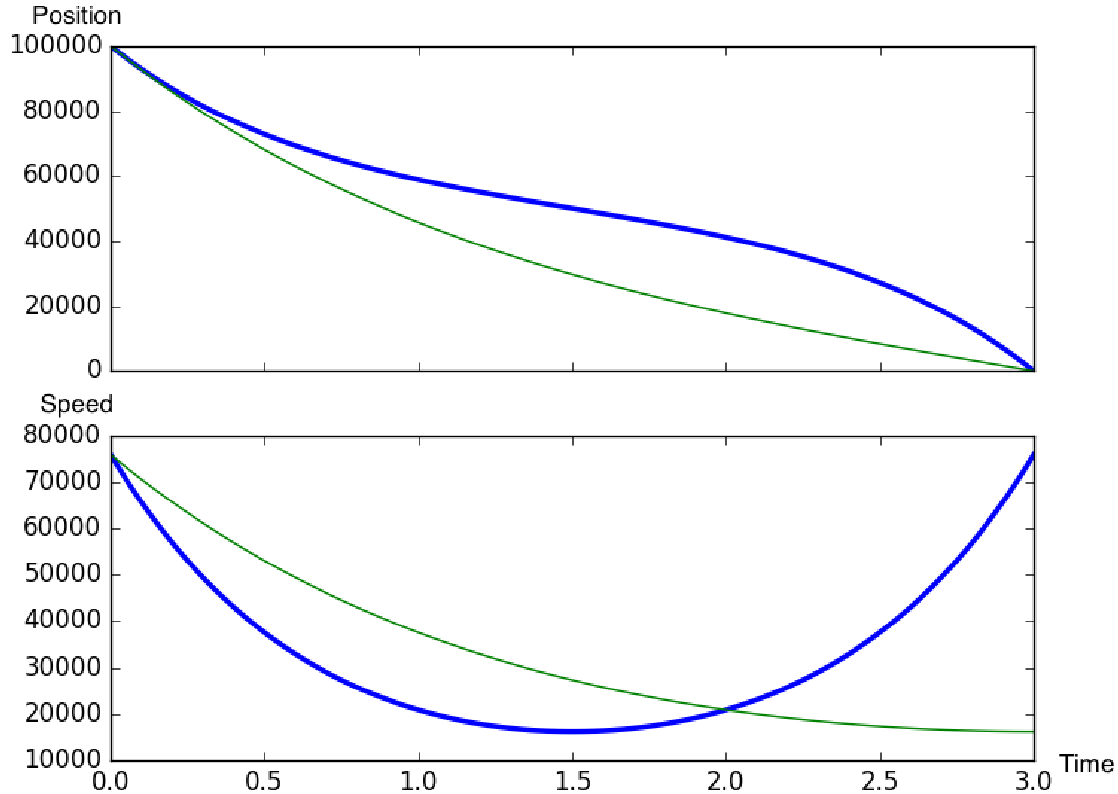

Figure 4.1: These two graphs are for $y = 10^5$, $T = 3$, $\gamma = 10^{-4}$, $\sigma = 0.01$, $\beta_1 = 10^{-8}$, $\beta_2 = 10^{-9}$, $\bar{A} = -1$ and $s = 50$. The upper graph plots the optimal stock positions, and the lower graph plots the optimal liquidation speeds. The thick curves are for the case with the second-best contract, and the thin curves are for the case with out any contract.

4.4 Proofs

Proof of Proposition 4.2. We first prove that

$$\xi_t^* = \frac{y}{T}, \quad t \in [0, T],$$

is a solution to problem

$$\inf_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[\int_0^T \Big(\xi_t F(\xi_t) + H(\xi_t)\Big)\, dt\Big].$$

To this end, with $\lambda = F(y/T) + F'(y/T)y/T + H'(y/T)$, consider

$$\inf_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[\int_0^T \Big(\xi_t F(\xi_t) + H(\xi_t)\Big)\, dt\Big] = \inf_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[\int_0^T \Big(\xi_t F(\xi_t) + H(\xi_t) - \lambda\xi_t\Big)\, dt\Big] + \lambda y,$$

where the equality is because of the constraint $y = \int_0^T \xi_t\, dt$. Using first order condition and convexity of the integrand in the above expression, we obtain that ξ^* is the solution.

To solve problem (4.2)-(4.3), by taking

$$\rho = \gamma \exp\Big\{\gamma\Big(\bar{A} - sy + F(y/T)y + TH(y/T)]\Big)\Big\},$$

we make the observation that

$$\begin{aligned}
&\sup_{\xi\in\mathcal{A}(T,y),\, C_T} \mathbb{E}\Big[-\exp\Big(-\gamma\int_0^T S_t\xi_t\, dt + \gamma C_T\Big)\Big] \\
\le\ &\sup_{\xi\in\mathcal{A}(T,y),\, C_T} \mathbb{E}\Big[-\exp\Big(-\gamma\int_0^T S_t\xi_t\, dt + \gamma C_T\Big) + \rho\Big(C_T - \int_0^T H(\xi_t)\, dt\Big)\Big] - \rho\bar{A} \qquad (4.23)\\
=\ &\sup_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[-\exp\Big(-\gamma\int_0^T S_t\xi_t\, dt + \gamma C_T^*(\xi)\Big) + \rho\Big(C_T^*(\xi) - \int_0^T H(\xi_t)\, dt\Big)\Big] - \rho\bar{A}\\
=\ &\sup_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[-\frac{\rho}{\gamma} + \frac{\rho}{\gamma}\log\Big(\frac{\rho}{\gamma}\Big) + \rho sy + \rho\sigma\int_0^T Y_t\, dB_t - \rho\int_0^T \Big(\xi_t F(\xi_t) + H(\xi_t)\Big)\, dt\Big] - \rho\bar{A}\\
=\ &-\frac{\rho}{\gamma} + \frac{\rho}{\gamma}\log\Big(\frac{\rho}{\gamma}\Big) + \rho sy - \rho\inf_{\xi\in\mathcal{A}(T,y)} \mathbb{E}\Big[\int_0^T \Big(\xi_t F(\xi_t) + H(\xi_t)\Big)\, dt\Big] - \rho\bar{A}\\
=\ &-\frac{\rho}{\gamma} + \frac{\rho}{\gamma}\log\Big(\frac{\rho}{\gamma}\Big) + \rho sy - \rho\int_0^T \Big(\xi_t F(\xi_t^*) + H(\xi_t^*)\Big)\, dt - \rho\bar{A},
\end{aligned}$$

where

$$C_T^*(\xi) = \frac{1}{\gamma}\log\Big(\frac{\rho}{\gamma}\Big) + sy + \sigma\int_0^T Y_t\, dB_t - \int_0^T \xi_t F(\xi_t)\, dt.$$

In the above computation, the inequality is because of (4.3), the first equality is due to the first order condition as well as the convexity of $x \mapsto -\exp\big(-\gamma\int_0^T S_t\xi_t\, dt + \gamma x\big) + \rho\big(x - \int_0^T H(\xi_t)\, dt\big)$, the second equality is a direct computation and the last equality uses the result from the first part of this proof. It can be checked that

$$C_T^*(\xi^*) - \int_0^T H(\xi_t^*)\,dt = \bar{A}.$$

Therefore, with $\xi = \xi^*$ and $C_T = C_T^*(\xi^*)$, (4.23) attains an equality, which yields the result. $\square$

Proof of Proposition 4.7. The process Z is increasing follows the positivity of its time derivative. To show Z is bounded, it suffices to have the observation that for all $t \in [0, T]$,

$$|Z_t| \leq |z| + \int_0^t \left| \frac{1}{F'(\tilde{\xi}_s)} \Big[F(\tilde{\xi}_s) + H'(\tilde{\xi}_s) \Big] \right| ds \leq K + K \int_0^t \tilde{\xi}_s\,ds \leq K + Ky,$$

where $K > 0$ is some constant, and we used the assumptions that $x \mapsto (F(x) + H'(x))/F'(x)$ has linear growth, $\tilde{\xi} \geq 0$ and that $\int_0^T \tilde{\xi}_t\,dt = y$.

For any contract $\big(T, y, c, \tilde{\xi}, Z\big)$ satisfying the condition of this proposition, by writing $\dot{Z}_t = dZ_t/dt$, we maximise

$$\begin{aligned}
&\mathbb{E}\bigg[C_T + \int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) - H(\xi_t)\Big\}\,dt - \Gamma(Y_T)\bigg] \\
&= c + \mathbb{E}\bigg[\int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) + H(\tilde{\xi}_t) - H(\xi_t)\Big\}\,dt + \int_0^T Z_t\,d\big(F(\tilde{\xi}_t) - F(\xi_t)\big) - \Gamma(Y_T)\bigg] \\
&= c + \mathbb{E}\bigg[\int_0^T \Big\{F(\xi_t)(\tilde{\xi}_t - \xi_t) + H(\tilde{\xi}_t) - H(\xi_t) - \big(F(\tilde{\xi}_t) - F(\xi_t)\big)\dot{Z}_t\Big\}\,dt \\
&\qquad - Z_T F(\xi_T) - \Gamma(Y_T)\bigg] \\
&= c + \mathbb{E}\bigg[\int_0^T \Big\{\int_{\tilde{\xi}_t}^{\xi_t} \Big(F'(u)\tilde{\xi}_t - F'(u)u - F(u) - H'(u) + F'(u)\dot{Z}_t\Big)\,du\Big\}\,dt \\
&\qquad - \Big(Z_T F(\xi_T) + \Gamma(Y_T)\Big)\bigg],
\end{aligned} \tag{4.24}$$

where the term $\int_0^T Z_t \sigma\,dB_t$ vanishes in expectation. Since the functions $x \mapsto xF(x)$ and H are strictly convex, F' is decreasing, as well as the fact that $\dot{Z} + \tilde{\xi} \geq 0$, it follows that

$$F'(x)\tilde{\xi}_t - F'(x)x - F(x) - H'(x) + F'(x)\dot{Z}_t$$

is strictly decreasing in x, and it is equal to 0 when $x = \tilde{\xi}_t$. Therefore, the time integral in (4.24) is less than or equal to 0, and it is equal to 0 only if $\xi_t = \tilde{\xi}_t$, for $t \in [0,T]$. Moreover, we have $Z_T F(\xi_T) + \Gamma(Y_T) \geq 0$, and it is equal to 0 if $\xi_t = \tilde{\xi}_t$, for $t \in [0,T]$. As a result, the expression in (4.24) not greater than c, and it attains c uniquely when $\xi_t = \tilde{\xi}_t$, for all $0 \leq t \leq T$. $\square$

Proof of Proposition 4.9. For any $x^{(1)}$, $x^{(2)} \in \mathbb{R}$ and $\frac{\theta^{(1)}}{\eta}, \frac{\theta^{(2)}}{\eta} \in \mathcal{A}_D(T, x^{(1)})$, consider the state processes $X^{(1)}$ and $X^{(2)}$ satisfying

$$\begin{aligned} dX_t^{(1)} &= \theta_t^{(1)}\,dt, \quad X_0^{(1)} = x^{(1)}, \\ dX_t^{(2)} &= \theta_t^{(2)}\,dt, \quad X_0^{(2)} = x^{(2)}. \end{aligned}$$

Let $\lambda \in (0,1)$, and write $x = \lambda x^{(1)} + (1-\lambda)x^{(2)}$, $X = \lambda X^{(1)} + (1-\lambda)X^{(2)}$ and $\theta = \lambda\theta^{(1)} + (1-\lambda)\theta^{(2)}$. Then,

$$dX_t = \theta_t\,dt, \quad X_0 = x,$$

and $\frac{\theta}{\eta} \in \mathcal{A}_D(T, y)$. By the convexity of H and $x \mapsto xF(x)$, we compute that

$$\begin{aligned} V(T,x;y) &\leq \int_0^T \left(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \gamma H(\theta_t/\alpha) + \gamma F(\theta_t/\alpha)\theta_t/\alpha\right) dt \\ &\leq \lambda \int_0^T \left(\frac{1}{2}\gamma^2\sigma^2\big(X_t^{(1)}\big)^2 + \gamma H\big(\theta_t^{(1)}/\eta\big) + \gamma F\big(\theta_t^{(1)}/\eta\big)\theta_t^{(1)}/\eta\right) dt \\ &\quad + (1-\lambda) \int_0^T \left(\frac{1}{2}\gamma^2\sigma^2\big(X_t^{(2)}\big)^2 + \gamma H\big(\theta_t^{(2)}/\eta\big) + \gamma F\big(\theta_t^{(2)}/\eta\big)\theta_t^{(2)}/\eta\right) dt. \end{aligned}$$

Since $\theta^{(1)}$ and $\theta^{(2)}$ are arbitrary, it follows that

$$V(T,x;y) \leq \lambda V\big(T, x^{(1)}; y\big) + (1-\lambda)V\big(T, x^{(2)}; y\big).$$

Then the arbitrariness of $x^{(1)}$ and $x^{(2)}$ implies the convexity of $x \mapsto V(T,x;y)$.

To show the remaining result, suppose $x < -\eta y/2$. Let θ be any admissible process for problem (4.16)-(4.17) with the corresponding state process denoted by X. Let

$$\tau = \inf\{t \geq 0 \,|\, X_t \geq -\eta y/2\}.$$

If $\tau = \infty$, consider the state process $\tilde{X}$ starting from $-\eta y/2$ with time derivative process being equal to θ. Then,

$$
\begin{aligned}
&\int_0^T \left(\frac{1}{2}\gamma^2\sigma^2 X_t^2 + \gamma H(\theta_t/\eta) + \gamma F(\theta_t/\eta)\theta_t/\eta\right) dt \\
&> \int_0^T \left(\frac{1}{2}\gamma^2\sigma^2 \tilde{X}_t^2 + \gamma H(\theta_t/\eta) + \gamma F(\theta_t/\eta)\theta_t/\eta\right) dt \\
&\geq V(T, -\eta y/2; y),
\end{aligned}
\tag{4.25}
$$

since $|X_t| > |\tilde{X}_t|$, for all t. If $\tau < \infty$, consider the process $\tilde{\theta}$ such that $\tilde{\theta}_t = \theta_{\tau+t}$, for $t \in [0, T-\tau)$; $\tilde{\theta}_{T-\tau+t} = \theta_t$, for $t \in [0,\tau)$; and $\tilde{\theta}_t = \theta_t$, for all $t \geq T$. Denote by $\tilde{X}$ the state process associated with $\tilde{\theta}$, starting from $-\alpha y/2$. Clearly $\tilde{\theta}/\eta \in \mathcal{A}_D(T, -\eta y/2)$. Then we have

$$
\begin{aligned}
\int_0^T \left(\gamma H(\theta_t/\eta) + \gamma F(\theta_t/\eta)\theta_t/\eta\right) dt &= \int_0^T \left(\gamma H(\tilde{\theta}_t/\eta) + \gamma F(\tilde{\theta}_t/\eta)\tilde{\theta}_t/\eta\right) dt, \\
\int_\tau^T \frac{1}{2}\gamma^2\sigma^2 X_t^2\, dt &= \int_0^{T-\tau} \frac{1}{2}\gamma^2\sigma^2 \tilde{X}_t^2\, dt, \\
\int_0^\tau \frac{1}{2}\gamma^2\sigma^2 X_t^2\, dt &> \int_{T-\tau}^T \frac{1}{2}\gamma^2\sigma^2 \tilde{X}_t^2\, dt,
\end{aligned}
$$

where the last inequality is due to that $X_t^2 > \alpha y/2$, for $t < \tau$, and $X_t^2 < \alpha y/2$, for $t \in [T-\tau, T)$. These three equations give out the same inequalities as in (4.25). Taking infimum of (4.25) over all admissible processes yields

$$
V(T, x; y) \geq V(T, -\eta y/2; y). \tag{4.26}
$$

A similar argument with taking consideration about $\tau = \inf\{t \geq 0 \,|\, X_t \geq \eta y/2\}$ applies to the case of $x > -\eta y/2$. Thus, the required result follows. $\square$

Proof of Theorem 4.11. The admissibility is a direct consequence of the construction of X^*. Define the function ϕ by

$$
\phi(x,\theta) = \frac{1}{2}\gamma^2\sigma^2 x^2 + \gamma H(\theta/\eta) + \gamma F(\theta/\eta)\theta/\eta.
$$

We claim that the Euler-Lagrange equation

$$\frac{d}{dt}\phi_\theta(X_t^*, \theta_t^*) = \phi_x(X_t^*, \theta_t^*), \qquad \text{a.e. } t \in [0, T]$$

holds, which requires to show that

$$\frac{d}{dt}\phi_\theta(X_t^*, \theta_t^*) = \gamma^2\sigma^2 X_t^*, \qquad \text{a.e. } t \in [0, T]. \tag{4.27}$$

For $\theta_t^* \neq 0$, we calculate that

$$\begin{aligned}
\phi_\theta(X_t^*, \theta_t^*) &= \frac{\gamma}{\eta}\Big(H'(\theta_t^*/\eta) + F(\theta_t^*/\eta) + F'(\theta_t^*/\eta)\theta_t^*/\eta\Big) \\
&= \frac{\gamma}{\theta_t^*}\Big(\Phi(\theta_t^*/\eta) + H(\theta_t^*/\eta) + F(\theta_t^*/\eta)\theta_t^*/\eta\Big) \\
&= \frac{\gamma}{\theta_t^*}\Big(\frac{\gamma^2\sigma^2(X_t^*)^2 + 2K^{T,x}}{2\gamma} + H(\theta_t^*/\eta) + F(\theta_t^*/\eta)\theta_t^*/\eta\Big),
\end{aligned}$$

and a direct differentiation verifies (4.27). Consider some interval on which $\theta_t^* = 0$ a.e. Then on this interval, $\phi_\theta(X_t^*, \theta_t^*)$ is a constant, and according to Definition 4.10, $X_t^* = 0$. As a result, (4.27) holds true on this interval. For any process $\theta \neq \theta^*$ with $\theta/\eta \in \mathcal{A}_D(T, y)$ and X being the associated state process, we compute by the strict convexity of H and $x \mapsto xF(x)$ that

$$\begin{aligned}
&\int_0^T \phi(X_t, \theta_t)\, dt - \int_0^T \phi(X_t^*, \theta_t^*)\, dt \\
&> \int_0^T \Big(\phi_x(X_t^*, \theta_t^*)(X_t - X_t^*) + \phi_\theta(X_t^*, \theta_t^*)(\theta_t - \theta_t^*)\Big)\, dt \\
&= \int_0^T \Big(\frac{d}{dt}\Big[\phi_\theta(X_t^*, \theta_t^*)\Big](X_t - X_t^*) + \phi_\theta(X_t^*, \theta_t^*)(\theta_t - \theta_t^*)\Big)\, dt \\
&= \int_0^T \frac{d}{dt}\Big[\phi_\theta(X_t^*, \theta_t^*)(X_t - X_t^*)\Big]\, dt \\
&= 0,
\end{aligned}$$

where the last equality is because that X and X^* have the same starting values as well as the same ending values. This shows that θ^* is the unique optimiser.

Then the rest results follow directly from the analysis of the simplification of the principal's problem. □

Part III

Contract with Outside Options

This part is based on a joint work with Dr. Hao Xing.

Chapter 5

Optimal contract under reputation concern

5.1 Model setting

Let $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$ be a filtered probability space, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by a one-dimensional standard Brownian motion B, and it satisfies the usual conditions of completeness and right-continuous. For any two stopping times $\tau \leq \sigma$, $\int_\tau^\sigma$ stands for the integral on the interval $[\tau, \sigma]$. All processes are of right-continuous with left-limit. For any process X and stopping time τ, denote $\Delta X_\tau = X_\tau - X_{\tau-}$. Write $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$, $t \geq 0$ and $\mathbb{E}[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_0]$.

We consider a Principal who hires an Agent to manage a project. Both of them are risk neutral. Given Agent's effort α, this project generates a cash flow whose accumulated value X^α follows the dynamic

$$dX_t^\alpha = (\mu + \alpha_t)dt + \sigma dB_t, \quad X_0^\alpha = 0, \tag{5.1}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are constants. We assume that Agent's effort α is a real-valued process taking values in some compact interval

$$A = [\underline{\alpha}, \overline{\alpha}]$$

with $0 \in A$. Agent's effort is costly. The cost of effort α_t at time t is $g(\alpha_t)$, for some cost function g which is continuously differentiable, strictly convex and satisfies $g(0) = 0$.

Any contract offered by Principal can be identified by a non-decreasing process C with $C_{0-} = 0$, which denotes the cumulative compensation received by Agent. While working for Principal, Agent explores some outside employment opportunity whose expected value may depend on his past performance. Given a contract C, Agent applies certain effort and also determines when to stop working for Principal, in order to maximise his expected discounted payoff received from both Principal and his outside option. Agent's optimal value is

$$U_{0-} = \sup_{(\tau,\alpha)} \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\gamma t}\Big(dC_t - g(\alpha_t)\, dt\Big) + \mathrm{e}^{-\gamma\tau} R(\tau, X_\cdot^\alpha)\right],$$

where $\gamma > 0$ is Agent's discount rate, $R(t, X_\cdot^\alpha)$ denotes the value of Agent's outside option (or reputation) at time t. In this chapter, we consider

$$R(t, X_\cdot^\alpha) = \beta \tilde{X}_t^\alpha, \tag{5.2}$$

where $\tilde{X}_t^\alpha = e^{\gamma t}\int_0^t e^{-\gamma s} dX_s^\alpha$ describes the present value of the accumulated cash flow generated from Principal's project, and $\beta > 0$ is a constant multiplication factor. Therefore, Agent's outside option is an increasing function of $\tilde{X}_t^\alpha$. The special form of outside option in (5.2) allows us to incorporate the value of the outside option into the running cost, therefore, his optimal value can be rewritten as

$$U_{0-} = \sup_{(\tau,\alpha)} \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\gamma t}\Big(dC_t + \Big(\beta\mu + \beta\alpha_t - g(\alpha_t)\Big)\, dt\Big)\right], \tag{5.3}$$

where $\beta\mu + \beta\alpha - g(\alpha)$ can be regarded as Agent's net running cost of his effort α. We can see from the above expression that even when $C \equiv 0$, it is optimal for Agent to employ some positive effort, if $g'(0) < \beta$. The outside option motivates Agent to work for Principal even if Principal does not pay Agent, since Agent's effort can improve the value of the project, hence enhances the value of his outside option. Agent works for Principal only when $U_{0-} \geq \bar{A}$, for some constant $\bar{A} > 0$ describing Agent's reservation utility.

We assume that Principal only observes the cash flow X^α. Let $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\geq 0}$ be the filtration generated by X^α and satisfy the usual conditions. Denote by r Principal's discount rate, and suppose that

$$r \in (0,\gamma)$$

which indicates that Agent is more impatient than Principal. Principal chooses some $\mathbb{F}^X$-adapted compensation process C to maximise her expected discounted payoff

$$\begin{aligned}
&\sup_C \mathbb{E}\Big[\int_0^{\tau^*} \mathrm{e}^{-rt}\Big(d\alpha^*_t - dC_t\Big) + \int_0^{\infty} \mathrm{e}^{-rt}\Big(dX^{\alpha^*}_t - d\alpha^*_t\Big)\Big] \\
&= \sup_C \mathbb{E}\Big[\int_0^{\tau^*} \mathrm{e}^{-rt}\Big(d\alpha^*_t - dC_t\Big)\Big] + \frac{\mu}{r},
\end{aligned} \tag{5.4}$$

where (τ^*,α^*) is Agent's optimal strategy associated with a contract C, μ/r represents the expected value of the project without hiring Agent, and the first term in (5.4) represents the additional expected value provided by Agent's contribution.

5.2 Agent and Principal's optimisation problems

Agent's problem (5.3) is an optimal control-stopping problem, and for any given contract C, his optimal strategy can be characterised by his optimal value process. To see this, we first specify the set of Agent's admissible strategies as follows.

Definition 5.1. For any given contract C, we call (τ,α) *admissible* if α is an $\mathbb{F}$-progressively measurable process taking values in A, and τ is an $\mathbb{F}$-stopping time satisfying

$$\mathbb{E}\Big[\int_0^{\tau} \mathrm{e}^{-\gamma t} dC_t\Big] < \infty.$$

The class of admissible strategies is denoted by $\mathcal{A}(C)$.

We define Agent's *optimal value* process as

$$U_t = \text{ess sup}_{(\tau,\alpha)\in\mathcal{A}_t(C)} U^{\tau,\alpha}_t,$$

where $\mathcal{A}_t(C)$ is the class of admissible strategies (τ,α) such that $\tau \geq t$, $\alpha_s = \tilde{\alpha}_s$ for some admissible effort $\tilde{\alpha} \in \mathcal{A}(C)$ and any $s \in [0,T]$, and

$$U^{\tau,\alpha}_t = U^{\tau,\alpha}_{t-} - \Delta C_t$$

$$= \mathbb{E}\Big[\int_t^\tau \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \Big(\beta\mu + \beta\alpha_s - g(\alpha_s)\Big)\,ds\Big)\,\Big|\,\mathcal{F}_t\Big] - \Delta C_t.$$

Since $\mathbb{E}\big[\int_0^\tau e^{-\gamma s} dC_s\big] < \infty$ and α is bounded, we have $U_t < \infty$.

The next result derives the dynamic of Agent's optimal value process. Let us first recall the *martingale principle*: the process $\tilde{U}_t = e^{-\gamma t}U_t + \int_0^t e^{-\gamma s}\big(dC_s + (\beta\mu + \beta\alpha_s - g(\alpha_s))ds\big)$ is a supermartingale on $[0,\tau]$ for arbitrary admissible effort α, and is a martingale for the optimal effort. Meanwhile, we expect from the optimal stopping theory that Agent's optimal stopping time is the first time that U drops below zero. Introduce

$$g^*(p) = \min_{\alpha\in A}\{g(\alpha) - p\alpha\}$$

$$\hat{\alpha}(z) = \arg\min_{\alpha\in A}\{g(\alpha) - (z/\sigma + \beta)\alpha\} = \begin{cases} (g')^{-1}(z/\sigma + \beta), & z/\sigma + \beta \in [g'(\underline{\alpha}), g'(\overline{\alpha})] \\ \underline{\alpha}, & z/\sigma + \beta < g'(\underline{\alpha}) \\ \overline{\alpha}, & z/\sigma + \beta > g'(\overline{\alpha}) \end{cases}.$$

Lemma 5.2. *For a given contract C and any admissible effort α, suppose that there exists an $\mathbb{F}^X$-progressively measurable process Z, satisfying $\int_0^t Z_s^2 ds < \infty$ for any $t > 0$, such that the process $\hat{U}^\alpha$ defined via*

$$d\hat{U}_t^\alpha = [\gamma\hat{U}_t^\alpha + g^*\big(Z_t/\sigma + \beta\big) - \beta\mu - \mu Z_t/\sigma]\,dt + Z_t/\sigma\,dX_t^\alpha - dC_t, \tag{5.5}$$

together with the stopping time $\tau_0^\alpha = \inf\{t \geq 0 : \hat{U}_t^\alpha \leq 0\}$, satisfies that $\hat{U}^\alpha_{\cdot\wedge\tau_0^\alpha}$ is of class D and the transversality condition

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-\gamma t}\hat{U}_t^{\hat{\alpha}(Z)}\mathbb{1}_{\{t\leq\tau_0^{\hat{\alpha}(Z)}\}}\Big] = 0. \tag{5.6}$$

Moreover, $\Delta C_{\tau_0^\alpha} \leq \hat{U}^\alpha_{\tau_0^\alpha-}$ when $\tau_0^\alpha < \infty$. Then

$$U_t = \hat{U}_t^{\hat{\alpha}(Z)}, \qquad \textit{for any } t \leq \tau_0^{\hat{\alpha}(Z)},$$

and $(\tau_0^{\hat{\alpha}(Z)}, \hat{\alpha}(Z))$ is Agent's optimal strategy.

Denote $\tau_0 = \tau_0^{\hat{\alpha}(Z)}$ as Agent's optimal stopping time for a given contract satisfying the above Lemma.

Remark 5.3. The assumption $\Delta C_{\tau_0^\alpha} \leq \hat{U}^\alpha_{\tau_0^\alpha -}$ in Lemma 5.2 implies that $U_{\tau_0} = 0$, whenever $\tau_0 < \infty$. In other words, the compensation Agent receives at τ_0 cannot drop Agent's optimal value process below zero. This restriction of the compensation process does not reduce Principal's value. Indeed, for any C such that $\Delta C_{\tau_0} > U_{\tau_0 -}$ and $\tau_0 < \infty$, Agent stops at τ_0. But the contract which agrees with C up to τ_0- and pays Agent U_{τ_0-} at τ_0 gives Principal a larger value than C does. □

In order to solve Principal's problem, motivated by Lemma 5.2, we consider the following set of admissible contracts.

Definition 5.4. Define Principal's admissible contract (Z, C) to be such that

(i) Z is an $\mathbb{F}^X$-progressively measurable process satisfying $\int_0^t Z_t^2 \, dt < \infty$, for all $t \geq 0$;

(ii) C is an $\mathbb{F}^X$-adapted, non-decreasing, càdlàg process with $C_{0-} = 0$ and satisfies $\mathbb{E}\big[\int_0^{\tau_0^\alpha} \mathrm{e}^{-rt} \, dC_t\big] < \infty$, and $\Delta C_{\tau_0^\alpha} \leq \hat{U}^\alpha_{\tau_0^\alpha -}$ when $\tau_0^\alpha < \infty$;

(iii) $\hat{U}^\alpha$ is given by

$$d\hat{U}^\alpha_t = [\gamma \hat{U}^\alpha_t + g^*\big(Z_t/\sigma + \beta\big) - \beta\mu - \mu Z_t/\sigma] \, dt + Z_t/\sigma \, dX^\alpha_t - dC_t, \quad \hat{U}^\alpha_{0-} = u \in \mathbb{R},$$

and $\hat{U}^\alpha_{\cdot \wedge \tau_0^\alpha}$ is of class D satisfying $\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-rt} \hat{U}^{\hat{\alpha}(Z)}_t \mathbb{1}_{\{t \leq \tau_0^{\hat{\alpha}(Z)}\}}\Big] = 0$,

where $\tau_0^\alpha = \inf\{t \geq 0 \, : \, \hat{U}^\alpha_t \leq 0\}$ and α is any admissible effort of Agent. Denote the set of all admissible contracts by $\mathcal{P}(u; \mathbb{F}^X)$.

It can be checked that the condition of Lemma 5.2 is satisfied by any admissible contract. With $(Z, C) \in \mathcal{P}(u; \mathbb{F}^X)$, Agent is induced to apply the $\mathbb{F}^X$-adapted effort $\hat{\alpha}(Z)$. This makes $\mathbb{F} = \mathbb{F}^X$, and at the same time Agent's value process satisfies the following dynamic

$$dU_t = [\gamma U_t + G(Z_t)] \, dt + Z_t \, dB - dC_t, \quad U_{0-} = u, \tag{5.7}$$

where

$$G(z) = g(\hat{\alpha}(z)) - \beta\hat{\alpha}(z) - \beta\mu.$$

Similar to Sannikov (2008), Agent's optimal value process will be used as the state process for Principal's problem. With reference to Lemma 5.2, Principal's optimisation problem (5.4) is reduced to

$$\sup_{u \geq \bar{A}} V(u) + \frac{\mu}{r}, \tag{5.8}$$

where the value function $V : [0, \infty) \to \mathbb{R}$ is given by

$$V(u) = \sup_{(C,Z) \in \mathcal{P}(u;\mathbb{F})} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} \Big(\hat{\alpha}(Z_s)\, ds - \, dC_s \Big) \right] \tag{5.9}$$

with the state process U satisfying (5.7).

5.3 Main results

In this section we present our main results. Under parameter restrictions, we first show that the value function V in (5.9) is a unique continuous viscosity solution to the associated Hamilton-Jabobi-Bellman variational inequalities (HJBVI). Then with an additional condition on Agent effort set A, we show further that V is twice-continuously differentiable. Finally, we obtain Principal's optimal contract. For future references, we list all further assumptions on functions $\hat{\alpha}(\cdot)$, $G(\cdot)$ and $g(\cdot)$ as follows.

Assumption

(I) $\max_{z \in \mathbb{R}} \{\hat{\alpha}(z) - G(z)\} \geq 0$,

(II) $g'(\underline{\alpha}) > \beta$,

(III) $g \in C^3(\mathbb{R})$, g''' is positive and $z/g''\big((g')^{-1}(z/\sigma + \beta)\big)$ is strictly increasing.

From the theory of optimal control, it is expected that V in (5.9) satisfies the following corresponding HJBVI

$$\min\left\{ rv(u) - \max_{z \in \mathbb{R}} \left\{ \hat{\alpha}(z) + \big(\gamma u + G(z)\big) v'(u) + \frac{1}{2} z^2 v''(u) \right\} \ , \ v'(u) + 1 \right\} = 0, \quad u > 0, \tag{5.10}$$

with associated boundary condition $v(0) = 0$. For future development, let's impose Assumption (I). Then the following proposition provides an upper bound and a lower bound to V.

Proposition 5.5. *Let functions $\overline{\psi}, \underline{\psi} : [0,\infty) \to \mathbb{R}$ defined by*

$$\overline{\psi}(u) = -u + \frac{1}{r}\max_{z\in\mathbb{R}}\{\hat{\alpha}(z) - G(z)\},$$
$$\underline{\psi}(u) = -u.$$

Then we have

$$\underline{\psi}(u) \leq V(u) \leq \overline{\psi}(u), \quad \text{for } u \geq 0. \tag{5.11}$$

When $\max_z\{\alpha(z) - G(z)\} = 0$, then $V(u) = -u$. In this case, Principal is optimal to pay Agent at the beginning with a size of compensation making Agent to stop immediately, which can be seen by substituting $C_0 = u$ into the expression in (5.9).

The following result characterises V as a viscosity solution to (5.10).

Theorem 5.6. *The value function V is a unique continuous viscosity solution to (5.10) satisfying the growth constraint* (5.11)*.*

Remark 5.7. The result that V is a (discontinuous) viscosity solution is standard, and it follows from the dynamic programming principle. The novelty in Theorem 5.6 is the uniqueness. When the control variable Z is unbound, the uniqueness requires some careful treatment, and is proved only when the Hamiltonian satisfies certain specific structural condition (see e.g. Lio and Ley, 2011, Assumption (A)(iii)). To prove the uniqueness here, we first prove that V is a viscosity solution to

$$\min\left\{rv(u) - \max_{z\in I}\{\hat{\alpha}(z) + (\gamma u + G(z))v^{'}(u) + \tfrac{1}{2}z^2 v^{''}(u)\}, -v^{''}(u), v^{'}(u) + 1\right\} = 0, \tag{5.12}$$

where $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. Here the concavity of v reduces the control Z to a compact interval. Then using a similar argument to Lio and Ley (2006), we prove that V is a unique viscosity solution of (5.12) satisfying (5.11). $\square$

In order to describe V further, let us define $\widetilde{\psi} : [0,\infty) \to \mathbb{R}$ via

$$\widetilde{\psi}(x) = -\frac{\gamma}{r}x + \frac{1}{r}\max_{z\in\mathbb{R}}\{\hat{\alpha}(z) - G(z)\},$$

and write

$$u_b = \inf\big\{u \geq 0 \,:\, V(u) \geq \widetilde{\psi}(u)\big\}.$$

To prove some smoothness of V, we need to impose Assumption (II). Under this condition, the interval I in (5.12) avoids 0, therefore (5.12) is uniformly elliptic.

Proposition 5.8. *V is a $C^2([0,\infty))$ solution to*

$$\begin{aligned}
rV(u) - \max_{z\in\mathbb{R}}\Big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2V''(u)\Big\} > 0, &\quad u \in (u_b,\infty),\\
V'(u) + 1 = 0, &\quad u \in [u_b,\infty),\\
rV(u) - \max_{z\in\mathbb{R}}\Big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2V''(u)\Big\} = 0, &\quad u \in (0,u_b],\\
V'(u) + 1 > 0, &\quad u \in (0,u_b),
\end{aligned}$$

with boundary condition $V(0) = 0$.

In general, applying Itô's formula and making the verification argument work only requires V to be smooth enough in the interior of its domain. In the above theorem, the result that the C^2 property of V is extended to the boundary is used to show the property of Lipschitz continuity for the function $Z^*(\cdot)$ defined by the following lemma.

Lemma 5.9. *Write $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. For $x \in [0,u_b]$, take*

$$Z^*(x) = \arg\max_{z\in I}\Big\{\hat{\alpha}(z) + G(z)V'(x) + \frac{1}{2}z^2V''(x)\Big\}.$$

Then under the condition of Assumption (III), for any admissible effort α of Agent, there exists a unique $\mathbb{F}^X$-adapted process $\hat{U}^{\alpha,*}$ with $\hat{U}^{\alpha,*}_{0-} = u \in \mathbb{R}$, and a unique non-decreasing, càdlàg, $\mathbb{F}^X$-adapted process C^* with $C^*_{0-} = 0$ such that

$$\begin{aligned}
d\hat{U}^{\alpha,*}_t &= [\gamma\hat{U}^{\alpha,*}_t + g^*\big(Z^*_t/\sigma + \beta\big) - \beta\mu - \mu Z^*_t/\sigma]\,dt + Z^*_t/\sigma\,dX^\alpha_t - dC^*_t,\\
\int_0^{\tau_0^{\alpha,*}} \mathbb{1}_{\{\hat{U}^{\alpha,*}_t < u_b\}}dC^*_t &= 0,\\
\hat{U}^{\alpha,*}_t &\in [0,u_b] \quad \text{for all } 0 \le t \le \tau_0^{\alpha,*},
\end{aligned}$$

where $\tau_0^{\alpha,*} = \inf\{t \ge 0 : \hat{U}^{\alpha,*}_t \le 0\}$ and $Z^* = Z^*(\hat{U}^{\alpha,*})$. If $u > u_b$, then $C^*_0 = u - u_b$ and $\hat{U}^{\alpha,*}_0 = u_b$; if $u \le u_b$, then C^* is continuous.

With reference to the above lemma, the next theorem identifies Principal's optimal contract.

Theorem 5.10. *Under Assumption (III), let Z^*, C^*, $\hat{U}^{\alpha,*}$ and $\tau_0^{\alpha,*}$ be given by Lemma 5.9. Then (Z^*, C^*) belongs to $\mathcal{P}(u;\mathbb{F}^X)$. This contract is optimal for Principal's problem (5.9), and it induces Agent to apply the effort $\hat{\alpha}(Z^*)$ and to stop at $\tau_0^{\hat{\alpha}(Z^*),*}$. Principal's optimal contract is completely characterised by*

$$\begin{aligned}\hat{U}_t^{\alpha,*} = u^* &+ \int_0^t [\gamma \hat{U}_s^{\alpha,*} + g^*\big(Z_s^*/\sigma + \beta\big) - \beta\mu - \mu Z_s^*/\sigma]\, ds \\ &+ \int_0^t Z_s^*/\sigma \, dX_s^\alpha - \int_0^t dC_t^*, \qquad\qquad t \le \tau_0^{\alpha,*},\end{aligned}$$

where the starting value $u^* \ge \bar{A}$ is solved from the problem (5.8). The existence of u^* is guaranteed, and in particular, $u^* = \bar{A}$, if $\bar{A} \ge u_b$.

5.4 Proofs

Proof of Lemma 5.2. For any admissible effort process α, consider

$$\tilde{U}_t := \int_0^t \mathrm{e}^{-\gamma s}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\, ds\Big) + \mathrm{e}^{-\gamma(t\wedge\tau_0^\alpha)} \hat{U}^\alpha_{t\wedge\tau_0^\alpha}, \quad t \ge 0.$$

Then,

$$\begin{aligned} d\tilde{U}_t &= \mathrm{e}^{-\gamma t}\, dC_t + \mathrm{e}^{-\gamma t}\big[\beta\mu + \beta\alpha_t - g(\alpha_t)\big]\, dt - \gamma \mathrm{e}^{-\gamma t}\hat{U}_t^\alpha \, dt + \mathrm{e}^{-\gamma t}\, d\hat{U}_t^\alpha \\ &= \mathrm{e}^{-\gamma t}\, dC_t + \mathrm{e}^{-\gamma t}\big[\beta\mu + \beta\alpha_t - g(\alpha_t)\big]\, dt - \gamma \mathrm{e}^{-\gamma t}\hat{U}_t^\alpha \, dt \\ &\quad + \mathrm{e}^{-\gamma t}\big[\gamma \hat{U}_t^\alpha + g^*\big(Z_t/\sigma + \beta\big) - \beta\mu - \mu Z_t/\sigma\big]\, dt + \mathrm{e}^{-\gamma t} Z_t(\mu + \alpha_t)/\sigma \, dt \\ &\quad + \mathrm{e}^{-\gamma t} Z_t\, dB_t - \mathrm{e}^{-\gamma t}\, dC_t \\ &= \mathrm{e}^{-\gamma t}\Big\{ g^*\big(Z_t/\sigma + \beta\big) - \big[g(\alpha_t) - \big(Z_t/\sigma + \beta\big)\alpha_t\big]\Big\}\, dt + \mathrm{e}^{-\gamma t} Z_t\, dB_t \\ &\le \mathrm{e}^{-\gamma t} Z_t\, dB_t. \end{aligned} \tag{5.13}$$

For any fixed $t \ge 0$, let α be the effort process given by $\alpha_s = \hat{\alpha}(Z_s)$ for $s \in [0, t)$, and $\alpha_s = \tilde{\alpha}_s$ for $s \ge t$, where $\tilde{\alpha}$ an arbitrary effort. Consider some $(\tau, \alpha) \in \mathcal{A}_t(C)$ and some localising sequence of finite stopping times $\{\tau_n\}_{n=1}^\infty$ such that each process $\int_0^{\tau_n\wedge\cdot} \mathrm{e}^{-\gamma s} Z_s\, dB_s$ is a martingale. Then we have for $t \le \tau_0^\alpha$,

$$
\begin{aligned}
&\hat{U}_t^\alpha \\
&= \mathrm{e}^{\gamma t}\tilde{U}_t - \int_0^t \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big) \\
&\geq \lim_{n\to\infty}\mathbb{E}_t\bigg[\mathrm{e}^{\gamma t}\tilde{U}_{\tau\wedge\tau_n} - \int_t^{\tau\wedge\tau_n}\mathrm{e}^{-\gamma(s-t)}Z_s\,dB_s - \int_0^t \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big)\bigg] \\
&= \lim_{n\to\infty}\mathbb{E}_t\bigg[\int_t^{\tau\wedge\tau_n}\mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big) + \mathrm{e}^{-\gamma(\tau\wedge\tau_n\wedge\tau_0^\alpha - t)}\hat{U}^\alpha_{\tau\wedge\tau_n\wedge\tau_0^\alpha}\bigg] - \Delta C_t \\
&= \mathbb{E}_t\bigg[\int_t^{\tau}\mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big) + \mathrm{e}^{-\gamma(\tau\wedge\tau_0^\alpha - t)}\hat{U}^\alpha_{\tau\wedge\tau_0^\alpha}\bigg] - \Delta C_t, \qquad (5.14)
\end{aligned}
$$

where we applied the dominated convergence theorem in the last equality, this is valid since $\hat{U}^\alpha_{\cdot\wedge\tau_0^\alpha}$ hence $\mathrm{e}^{-\gamma(\cdot\wedge\tau_0^\alpha)}\hat{U}^\alpha_{\cdot\wedge\tau_0^\alpha}$ is of class D, and that

$$
\begin{aligned}
&\mathbb{E}_t\bigg[\bigg|\int_t^{\tau\wedge\tau_n}\mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big)\bigg|\bigg] \\
\leq\,&\mathbb{E}_t\bigg[\int_t^{\tau}\mathrm{e}^{-\gamma(s-t)}\,dC_s\bigg] + \mathbb{E}_t\bigg[\int_t^{\infty}\mathrm{e}^{-\gamma(s-t)}\Big|\beta\mu + \beta\alpha_s - g(\alpha_s)\Big|\,ds\bigg] < \infty.
\end{aligned}
$$

Observe that according to the definition of τ_0^α and the condition $\Delta C_{\tau_0^\alpha} \leq \hat{U}^\alpha_{\tau_0^\alpha -}$, we have $\hat{U}^\alpha_{\tau\wedge\tau_0^\alpha} \geq 0$. Then it follows from (5.14) that

$$
\hat{U}_t^\alpha \geq \mathbb{E}_t\bigg[\int_t^{\tau}\mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\alpha_s - g(\alpha_s)\big]\,ds\Big)\bigg] - \Delta C_t. \qquad (5.15)
$$

On the other hand, equality is uniquely attained in (5.13) if $\alpha = \hat{\alpha}(\hat{Z})$. Write $\tau_0 = \tau_0^{\hat{\alpha}(Z)}$. Then,

$$
\begin{aligned}
\hat{U}_t^{\hat{\alpha}(Z)} &= \mathrm{e}^{\gamma t}\tilde{U}_t - \int_0^t \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\hat{\alpha}(Z_s) - g(\hat{\alpha}(Z_s))\big]\,ds\Big) \\
&= \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_t\bigg[\mathrm{e}^{\gamma t}\tilde{U}_{\tau_0\wedge\tau_n\wedge m} - \int_t^{\tau_0\wedge\tau_n\wedge m}\mathrm{e}^{-\gamma(s-t)}Z_s\,dB_s \\
&\qquad - \int_0^t \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\hat{\alpha}(Z_s) - g(\hat{\alpha}(Z_s))\big]\,ds\Big)\bigg] \\
&= \lim_{m\to\infty}\mathbb{E}_t\bigg[\int_t^{\tau_0\wedge m}\mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\hat{\alpha}(Z_s) - g(\hat{\alpha}(Z_s))\big]\,ds\Big)
\end{aligned}
$$

$$+ \mathrm{e}^{-\gamma(\tau_0\wedge m-t)}\hat{U}^{\hat{\alpha}(Z)}_{\tau_0\wedge m}\Bigg] - \Delta C_t$$

$$= \mathbb{E}_t\left[\int_t^{\tau_0} \mathrm{e}^{-\gamma(s-t)}\Big(dC_s + \big[\beta\mu + \beta\hat{\alpha}(Z_s) - g(\hat{\alpha}(Z_s))\big]\, ds\Big)\right] - \Delta C_t, \tag{5.16}$$

where the last equality is because of the dominated convergence theorem, the definition of τ_0 and the transversality condition (5.6). This equality also shows $\mathbb{E}[\int_0^{\tau_0} \mathrm{e}^{-\gamma t}\, dC_s] < \infty$ which implies $(\tau_0, \hat{\alpha}(Z))$ is admissible. Combining (5.15) and (5.16), we get

$$U_t = \text{ess sup}_{\tau,\alpha} U_t^{\tau,\alpha} = U_t^{\tau_0,\hat{\alpha}(Z)} = \hat{U}_t^{\hat{\alpha}(Z)}.$$

□

Proof of Proposition 5.5. Since $\gamma > r$, a direct calculation verifies that $\overline{\psi}$ is a supersolution to (5.10). This implies that for any $z \in \mathbb{R}$,

$$r\overline{\psi} - \hat{\alpha}(z) - \big(\gamma u + G(z)\big)\overline{\psi}' - \frac{1}{2}z^2\overline{\psi}'' \geq 0.$$

Let (Z, C) be an admissible control with U being the corresponding controlled state process starting from $u \geq 0$. By Itô's formula and the above inequality, it follows that

$$\begin{aligned}
&\mathrm{e}^{-r(\tau_0\wedge T)}\,\overline{\psi}(U_{\tau_0\wedge T}) \\
=&\overline{\psi}(u) + \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\Big(-r\overline{\psi}(U_t) + \big(\gamma U_t + G(Z_t)\big)\overline{\psi}'(U_t) + \frac{1}{2}Z_t^2\overline{\psi}''(U_t)\Big)\, dt \\
&\quad - \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\overline{\psi}'(U_t)\, dC_t^c + \sum_{0\leq t\leq \tau_0\wedge T} \mathrm{e}^{-rt}\Delta\overline{\psi}(U_t) \\
&\quad + \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\overline{\psi}'(U_t) Z_t\, dB_t \\
\leq&\overline{\psi}(u) - \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\hat{\alpha}(Z_t)\, dt + \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\, dC_t - \int_0^{\tau_0\wedge T} \mathrm{e}^{-rt} Z_t\, dB_t,
\end{aligned}$$

and then

$$\mathbb{E}\left[\int_0^{\tau_0\wedge T} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z_t)\, dt - dC_t\Big) + \mathrm{e}^{-r(\tau_0\wedge T)}\,\overline{\psi}(U_{\tau_0\wedge T})\right] \leq \overline{\psi}(u),$$

where the stochastic integral vanishes in expectation by a localisation sequence argument.

Sending T to infinity, due to the dominated convergence theorem, the transversality condition and the fact that $\overline{\psi}$ is linear and $\overline{\psi}(0) \geq 0$ we have therefore,

$$\mathbb{E}\left[\int_0^{\tau_0} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z_t)\,dt - dC_t\Big)\right] \leq \overline{\psi}(u).$$

Hence it follows that $V \leq \overline{\psi}$, since (Z, C) is arbitrary.

To prove the lower bound, for any given $u \in [0, \infty)$, take C to be such that $C_0 = u$ and $dC_t = 0$, for all $t > 0$. Let Z be any admissible control. Denote by U the corresponding controlled state process starting from u. Then, $\tau_0 = 0$ and

$$V(u) \geq \mathbb{E}\left[\int_0^{\tau_0} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z_t)\,dt - dC_t\Big)\right] = -u = \underline{\psi}(u).$$

$\square$

The local boundedness of V guarantees that the upper semi-continuous (u.s.c.) and the lower semi-continuous (l.s.c.) envelops of the principal's value function V are well-defined. The following two lemmas show that V is both a viscosity supersolution and a viscosity subsolution to (5.10). Hence, it is a viscosity solution to (5.10). The proofs of these two lemmas follow a standard approach (see e.g. Pham, 2009, etc).

Lemma 5.11. *The function $V : [0, \infty) \to \mathbb{R}$ is a (discontinuous) viscosity supersolution to (5.10).*

Proof. Write V_* to be the l.s.c. envelop of V, i.e., $V_*(u) = \liminf_{x \to u} V(x)$, for all $u \geq 0$. Let $\bar{u} > 0$ and $\phi \in C^2((0, \infty))$ be such that

$$0 = \min(V_* - \phi)(u) = (V_* - \phi)(\bar{u}),$$

so we have $V \geq V_* \geq \phi$, and there exists a sequence of numbers $(u_n)_{n=1}^{\infty}$ contained in $[0, \infty)$ such that as $n \to \infty$, $u_n \to \bar{u}$ and $V(u_n) \to V_*(\bar{u})$. Let $\xi_n = (V - \phi)(u_n)$, so $\xi_n \geq 0$ and $\xi_n \to 0$, as $n \to \infty$. Denote

$$\rho_n = \sqrt{\xi_n} + \frac{1}{n}\mathbb{1}_{\{\xi_n = 0\}}.$$

Let (Z, C) be an admissible control with Z being equal to some constant z. Write $U^{u_n, Z, C}$ as the corresponding controlled state process starting from u_n. Define

$$\tau_n = \inf\big\{t \geq 0 \,:\, |U_t^{u_n,Z,C} - u_n| \geq K\big\} \quad \text{and} \quad \theta_n = \rho_n \wedge \tau_n \wedge \tau_0$$

. Then it follows from the dynamic programming principle that

$$\begin{aligned}
\phi(u_n) + \xi_n &= V(u_n) \\
&\geq \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\hat{\alpha}(z)\,dt - dC_t\Big) + \mathrm{e}^{-r\theta_n} V\big(U_{\theta_n}^{u_n,Z,C}\big)\bigg] \\
&\geq \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\hat{\alpha}(z)\,dt - dC_t\Big) + \mathrm{e}^{-r\theta_n} \phi\big(U_{\theta_n}^{u_n,Z,C}\big)\bigg]
\end{aligned}$$

Applying Itô's formula to $\mathrm{e}^{-r\theta_n}\phi\big(U_{\theta_n}^{u_n,Z,C}\big)$, together with the above inequality, we obtain that

$$\begin{aligned}
\frac{\xi_n}{\rho_n} \geq \mathbb{E}\bigg[&\frac{1}{\rho_n}\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(-r\phi\big(U_t^{u_n,Z,C}\big) + \hat{\alpha}(z) \\
&+ \big(\gamma U_t^{u_n,Z,C} + G(z)\big)\phi'\big(U_t^{u_n,Z,C}\big) + \frac{1}{2}z^2\phi''\big(U_t^{u_n,Z,C}\big)\Big)\,dt \\
&- \frac{1}{\rho_n}\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\phi'\big(U_t^{u_n,Z,C}\big) + 1\Big)\,dC_t^c + \frac{1}{\rho_n}\sum_{0\leq t\leq\theta_n} \mathrm{e}^{-rt}\Big(\Delta\phi\big(U_t^{u_n,Z,C}\big) - \Delta C_t\Big)\bigg],
\end{aligned} \tag{5.17}$$

where $\int_0^{\theta_n} \mathrm{e}^{-rt}\phi'\big(U_t^{u_n,Z,C}\big)z\,dB_t$ vanishes in expectation, since $\int_0^{\theta_n\wedge\cdot} \mathrm{e}^{-rs}\phi'\big(U_s^{u_n,Z,C}\big)z\,dB_s$ is a uniformly integrable martingale. Now suppose $\phi'(\bar{u}) + 1 < 0$. Note that if we assume in addition that C has a jump at time 0, then (5.17) still holds, and it follows therefore,

$$\frac{1}{\rho_n}\sum_{0\leq t\leq\theta_n} \mathrm{e}^{-rt}\Big(\Delta\phi\big(U_t^{u_n,Z,C}\big) - \Delta C_t\Big) \xrightarrow{n\to\infty} \infty.$$

As a consequence, (5.17) gives out a contradiction, because $\frac{\xi_n}{\rho_n} \to 0$, as $n \to \infty$. We therefore conclude that $\phi'(\bar{u}) + 1 \geq 0$. On the other hand, if we assume in addition that $C_t = 0$, for $0 \leq t \leq \theta_n$, then (5.17) still holds for such C. By the continuity of paths of $U^{u_n,Z,C}$, for $0 \leq t \leq \theta_n$, the mean value theorem, the dominated convergence theorem and (5.17), we have

$$r\phi(\bar{u}) - \hat{\alpha}(z) - \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) - \frac{1}{2}z^2\phi''(\bar{u}) \geq 0,$$

where z is arbitrary. Hence,

$$r\phi(\bar{u}) - \max_{z\in\mathbb{R}}\Big\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\Big\} \geq 0.$$

□

Lemma 5.12. *The function $V : [0,\infty) \to \mathbb{R}$ is a (discontinuous) viscosity subsolution to (5.10).*

Proof. Write V^* to be the u.s.c. envelop of V, i.e., $V^*(u) = \limsup_{x\to u} V(x)$, for all $u \geq 0$. Let $\bar{u} > 0$ and $\phi \in C^2((0,\infty))$ be such that

$$0 = \max(V^* - \phi)(u) = (V^* - \phi)(\bar{u}).$$

So we have $V \leq V^* \leq \phi$. Let's suppose the contrary that

$$h_1(\bar{u}) := r\phi(\bar{u}) - \max_{z\in\mathbb{R}}\Big\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\Big\} > 0$$

and

$$h_2(\bar{u}) := \phi'(\bar{u}) + 1 > 0.$$

Then we must have $\phi''(\bar{u}) \leq 0$, otherwise $h_1(\bar{u}) = -\infty$. Consequently,

$$h_1(\bar{u}) := r\phi(\bar{u}) - \max_{z\in I}\Big\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\Big\} > 0,$$

where $I := [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$, since both $\hat{\alpha}$ and G are bounded functions. It follows from the maximum theorem (see, e.g. Aliprantis and Border, 2006, Theorem 17.31) that $\bar{u} \mapsto \max_{z\in I}\big\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\big\}$ is a continuous function. Hence, there exists an interval $I(\bar{u};K) = \big\{u \geq 0 : |u - \bar{u}| < K\big\}$ on which $h_1 > \epsilon$ and $h_2 > \epsilon$, where $K, \epsilon > 0$ are some constants. Let $(u_n)_{n=1}^{\infty}$ be a sequence of numbers contained in $I(\bar{u};K)$ such that as $n \to \infty$, $u_n \to \bar{u}$ and $V(u_n) \to V^*(\bar{u})$. Write $\xi_n = \phi(u_n) - V(u_n)$, so $\xi_n \to 0$, as $n \to \infty$. Denote

$$\rho_n = \sqrt{\xi_n} + \frac{1}{n}\mathbb{1}_{\{\xi_n=0\}}.$$

Now for each u_n, consider an $\epsilon\rho_n/2 + m(\rho_n)$-optimal admissible control (Z^n, C^n) such that

$\Delta C^n_t = 0$, for $t < \rho_n$, where $m(\cdot)$ is some positive strictly decreasing function satisfying $m(0) = 0$. [12] Write U^{u_n,Z^n,C^n} as the corresponding controlled state process starting from u_n. Let

$$\tau_n = \inf\{t \geq 0 \,:\, U^{u_n,Z^n,C^n}_t \notin I(\bar{u}; K)\} \quad \text{and} \quad \theta_n = \rho_n \wedge \tau_n \wedge \tau_0.$$

Then we compute that

$$\begin{aligned}
\phi(u_n) - \xi_n &= V(u_n) \\
&\leq \mathbb{E}\bigg[\int_0^{\tau_0} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z^n_t)\,dt - dC^n_t\Big)\bigg] + \frac{\epsilon\rho_n}{2} + m(\rho_n) \\
&\leq \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z^n_t)\,dt - dC^n_t\Big) + \mathrm{e}^{-r\theta_n} V\big(U^{u_n,Z^n,C^n}_{\theta_n}\big)\bigg] + \frac{\epsilon\rho_n}{2} + m(\rho_n) \\
&\leq \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\hat{\alpha}(Z^n_t)\,dt - dC^n_t\Big) + \mathrm{e}^{-r\theta_n} \phi\big(U^{u_n,Z^n,C^n}_{\theta_n}\big)\bigg] + \frac{\epsilon\rho_n}{2} + m(\rho_n) \\
&= \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(-r\phi\big(U^{u_n,Z^n,C^n}_t\big) + \hat{\alpha}(Z^n_t) \\
&\qquad + \big(\gamma U^{u_n,Z^n,C^n}_t + G(Z^n_t)\big)\phi'\big(U^{u_n,Z^n,C^n}_t\big) + \frac{1}{2}z^2\phi''\big(U^{u_n,Z^n,C^n}_t\big)\Big)\,dt \\
&\qquad - \int_0^{\theta_n} \mathrm{e}^{-rt}\Big(\phi'\big(U^{u_n,Z^n,C^n}_t\big) + 1\Big)\,d(C^n)^c_t \\
&\qquad + \sum_{0\leq t\leq\theta_n} \mathrm{e}^{-rt}\Big(\Delta\phi\big(U^{u_n,Z^n,C^n}_t\big) - \Delta C^n_t\Big)\bigg] + \phi(u_n) + \frac{\epsilon\rho_n}{2} + m(\rho_n) \\
&< \phi(u_n) + \frac{\epsilon\rho_n}{2} + m(\rho_n) + \mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\Big(-r\phi\big(U^{u_n,Z^n,C^n}_t\big) + \hat{\alpha}(Z^n_t) \\
&\qquad + \big(\gamma U^{u_n,Z,C}_t + G(Z^n_t)\big)\phi'\big(U^{u_n,Z^n,C^n}_t\big) + \frac{1}{2}z^2\phi''\big(U^{u_n,Z^n,C^n}_t\big)\Big)\,dt\bigg] \\
&< \phi(u_n) + \frac{\epsilon\rho_n}{2} + m(\rho_n) - \epsilon\,\mathbb{E}\bigg[\int_0^{\theta_n} \mathrm{e}^{-rt}\,dt\bigg] \\
&= \phi(u_n) + \frac{\epsilon\rho_n}{2} + m(\rho_n) - \epsilon\,\mathbb{E}\bigg[\frac{1}{r}\big(1 - \mathrm{e}^{-r\theta_n}\big)\bigg],
\end{aligned}$$

where the term $\int_0^{\theta_n} \mathrm{e}^{-rt}\phi'\big(U^{u_n,Z^n,C^n}_t\big)Z^n_t\,dB_t$ vanishes in expectation, the strict inequalities

[12] Let $(Z^n, \tilde{C}^n)$ be an $\epsilon\rho_n/2$-optimal admissible control. Take $C^n_t = 0$ for $t < \rho_n$, $\Delta C^n_{\rho_n} = \mathrm{e}^{\gamma\rho_n}\int_0^{\rho_n}\mathrm{e}^{-\gamma t}\,d\tilde{C}^n_t$ and $dC^n_t = d\tilde{C}^n_t$ for $t > \rho_n$. Then it can be shown that (Z^n, C^n) is an $\epsilon\rho_n/2 + \rho_n(\mathrm{e}^{\gamma\rho_n} - 1)2K$-optimal admissible control, i.e., $m(x) = x(\mathrm{e}^{\gamma x} - 1)2K$.

follow from that $h_1, h_2 > \epsilon$ on $I(\bar{u}; K)$. Therefore,

$$-\frac{\xi_n}{\rho_n} < \frac{\epsilon}{2} - \epsilon\, \mathbb{E}\left[\frac{1}{r\rho_n}\left(1 - \mathrm{e}^{-r\theta_n}\right)\right].$$

Sending $n \to \infty$, by the dominated convergence theorem, we obtain

$$0 \le \frac{\epsilon}{2} - \epsilon \lim_{n\to\infty} \frac{\mathbb{E}[\theta_n]}{\rho_n}. \tag{5.18}$$

Observe that for large n and $\delta = (\bar{u} \wedge K)/2$,

$$\mathbb{P}[\tau_n \wedge \tau_0 \le \rho_n] \le \mathbb{P}\left[\sup_{0\le t<\rho_n} \left|U_t^{u_n,Z^n,C^n} - u_n\right| \ge \delta\right] \le \frac{1}{\delta^2}\mathbb{E}\left[\sup_{0\le t<\rho_n} \left|U_t^{u_n,Z^n,C^n} - u_n\right|^2\right],$$

where the upper bound tends to 0, as $n \to \infty$ (see, e.g. Pham, 2009, Theorem 1.3.16). Also, we have that $\mathbb{E}[\theta_n] \ge \rho_n \mathbb{P}[\tau_n \wedge \tau_0 > \rho_n]$, so

$$\mathbb{P}[\tau_n \wedge \tau_0 > \rho_n] \le \frac{\mathbb{E}[\theta_n]}{\rho_n} \le 1,$$

Therefore, $\lim_{n\to\infty} \frac{\mathbb{E}[\theta_n]}{\rho_n} = 1$, and (5.18) results in a contradiction. $\square$

The following lemma shows that any viscosity solution to (5.10) is also a viscoity solution to (5.19). The main purpose of this result is to reduce values of the control Z to be on the compact interval $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. This reduction serves for the proof of the comparison principle given by Lemma 5.14.

Lemma 5.13. *Any viscosity supersolution (resp. subsolution) to (5.10) is also a viscosity supersolution (resp. subsolution) to*

$$\min\left\{rv(u) - \max_{z\in I}\{\hat{\alpha}(z) + (\gamma u + G(z))v^{'}(u) + \tfrac{1}{2}z^2 v^{''}(u)\}, -v^{''}(u), v^{'}(u) + 1\right\} = 0, \tag{5.19}$$

where $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$.

Proof. Supersolution: Let V be a viscosity supersolution to (5.10) and write V_* as its l.s.c. envelop. Then Lemma 5.11 gives out that for any $\bar{u} > 0$ and $\phi \in C^2((0,\infty))$ such that $0 = \min(V_* - \phi)(u) = (V_* - \phi)(\bar{u})$, we have

$$r\phi(\bar{u}) - \max_{z\in\mathbb{R}}\left\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\right\} \ge 0 \quad \text{and} \quad \phi'(\bar{u}) + 1 \ge 0.$$

It must hold that $\phi''(\bar{u}) \leq 0$, otherwise, $rV(\bar{u}) - \max_{z\in\mathbb{R}} \big\{\hat{\alpha}(z) + \big(\gamma\bar{u} + G(z)\big)\phi'(\bar{u}) + \frac{1}{2}z^2\phi''(\bar{u})\big\} = -\infty$. Therefore,

$$\min\left\{r\phi(\bar{u}) - \max_{z\in\mathbb{R}}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}, -\phi^{''}(\bar{u}), \phi^{'}(\bar{u}) + 1\right\} \geq 0,$$

and then

$$\min\left\{r\phi(\bar{u}) - \max_{z\in I}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}, -\phi^{''}(\bar{u}), \phi^{'}(\bar{u}) + 1\right\} \geq 0.$$

Subsolution: Let V^* be the u.s.c. envelop of V. For any $\bar{u} > 0$, let $\phi \in C^2((0,\infty))$ such that $0 = \max(V^* - \phi)(u) = (V^* - \phi)(\bar{u})$. If $\phi^{''}(\bar{u}) \geq 0$, we already have

$$\min\left\{r\phi(\bar{u}) - \max_{z\in I}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}, -\phi^{''}(\bar{u}), \phi^{'}(\bar{u}) + 1\right\} \leq 0.$$

If $\phi^{''}(\bar{u}) < 0$, we have

$$\max_{z\in\mathbb{R}}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\} = \max_{z\in I}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}.$$

Therefore

$$\min\left\{r\phi(\bar{u}) - \max_{z\in\mathbb{R}}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}, \phi^{'}(\bar{u}) + 1\right\} \leq 0$$

implies

$$\min\left\{r\phi(\bar{u}) - \max_{z\in I}\{\hat{\alpha}(z) + (\gamma\bar{u} + G(z))\phi^{'}(\bar{u}) + \tfrac{1}{2}z^2\phi^{''}(\bar{u})\}, -\phi^{''}(\bar{u}), \phi^{'}(\bar{u}) + 1\right\} \leq 0.$$

$\square$

The following comparison principle is proved using a similar argument to Lio and Ley (2006).

Lemma 5.14. *Let $u : [0,\infty) \to \mathbb{R}$ (resp. $v : [0,\infty) \to \mathbb{R}$) be an u.s.c viscosity subsolution (resp. l.s.c. viscosity supersolution) to (5.10). Suppose $u(x) + x$ and $v(x) + x$ are bounded functions. Suppose further that $(u - v)(0) \leq 0$, then $(u - v)(x) \leq 0$, for all $x \geq 0$.*

Proof. Write $w_1(x) = u(x) + x$ and $w_2(x) = v(x) + x$. With reference to Lemma 5.13, it can be shown that w_1 and w_2 are respectively viscosity subsolution and viscosity supersolution to

$$\min\Big\{rw(x) - \gamma x w'(x) + (\gamma - r)x - \max_{z\in I}\{\hat{\alpha}(z) + G(z)(w^{'}(x) - 1) + \tfrac{1}{2}z^2 w^{''}(x)\},$$
$$-w^{''}(x), w^{'}(x)\Big\} = 0,$$

where $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. Let $\bar{x} > 0$ and $\phi \in C^2((0,\infty))$ be such that $w_1 - w_2 - \phi$ achieves a unique maximum value at $\bar{x}$ over some closed ball

$$\bar{B}(\bar{x}; R) := \{x \geq 0 \,|\, |x - \bar{x}| \leq R,\ R > 0\}.$$

Consider $\Theta_n(x,y) = \phi(x) + n^2|x-y|^2$ and

$$\begin{aligned} M_n &:= \max_{x,y\in\bar{B}(\bar{x};R)} \big\{w_1(x) - w_2(y) - \Theta_n(x,y)\big\} \\ &= w_1(x_n) - w_2(y_n) - \Theta_n(x_n, y_n), \end{aligned}$$

for some x_n, $y_n \in \bar{B}(\bar{x}; R)$. As $n \to \infty$, (x_n, y_n) converges up to a subsequence to some $(\hat{x}, \hat{y})$. Since for all n, M_n is lower bounded by $w_1(\bar{x}) - w_2(\bar{x}) - \phi(\bar{x})$, we must have $\hat{x} = \hat{y}$. Moreover, we have

$$w_1(x_n) - w_2(y_n) - \phi(x_n) - n^2|x_n - y_n|^2 \geq w_1(\hat{x}) - w_2(\hat{x}) - \phi(\hat{x}),$$

and then

$$\lim_{n\to\infty} n^2|x_n - y_n|^2 \leq \lim_{n\to\infty} w_1(x_n) - w_2(y_n) - \phi(x_n) - w_1(\hat{x}) + w_2(\hat{x}) + \phi(\hat{x}) \leq 0,$$

where the last inequality follows from the upper semi-continuity of $w_1 - w_2$. Thus,

$$\lim_{n\to\infty} n^2|x_n - y_n|^2 = 0.$$

It follows therefore,

$$\begin{aligned} w_1(\hat{x}) - w_2(\hat{x}) - \phi(\hat{x}) &\geq \lim_{n\to\infty} w_1(x_n) - w_2(y_n) - \phi(x_n) - n^2|x_n - y_n|^2 \\ &\geq w_1(\bar{x}) - w_2(\bar{x}) - \phi(\bar{x}) \\ &\geq w_1(\hat{x}) - w_2(\hat{x}) - \phi(\hat{x}). \end{aligned}$$

This together with $\bar{x}$ being the unique maximiser implies $\hat{x} = \bar{x}$ and also that $M_n \to w_1(\bar{x}) - w_2(\bar{x}) - \phi(\bar{x})$, as $n \to \infty$.

Observe that $D_x\Theta(x_n, y_n) = 2n^2(x_n - y_n) + \phi'(x_n)$ and $-D_y\Theta(x_n, y_n) = 2n^2(x_n - y_n)$. Using Θ_n as the test function, in view of Ishii's lemma (see, e.g. Pham, 2009), we have for all $\rho > 0$,

$$\begin{aligned} \big(\, 2n^2(x_n - y_n) + \phi'(x_n) \,,\, A_n \,\big) &\in \bar{\mathcal{J}}^2_+ w_1(x_n), \\ \big(\, 2n^2(x_n - y_n) \,,\, B_n \,\big) &\in \bar{\mathcal{J}}^2_- w_2(y_n), \end{aligned}$$

and

$$\begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \leq M + \rho M^2,$$

where

$$M = D^2\Theta(x_n, y_n) = \begin{pmatrix} \phi''(x_n) + 2n^2 & -2n^2 \\ -2n^2 & 2n^2 \end{pmatrix},$$

A_n, $B_n \in \mathbb{R}$, and $\bar{\mathcal{J}}^2_+ w_1(x_n)$ (resp. $\bar{\mathcal{J}}^2_- w_2(x_n)$) is the second order limiting superjet for w_1 (resp. limiting subjet for w_2) at point x_n. Then,

$$A_n - B_n = tr\Big(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix}\Big) \leq \phi''(x_n) + m(\rho n^4), \tag{5.20}$$

where m is some positive strictly decreasing function with $m(0) = 0$.

According to the characterisation of viscosity supersolution (resp. viscosity subsolution) in terms of subjet (resp. superjet) (see, e.g. Pham, 2009), it follows that

$$\begin{aligned} \min\Big\{ & rw_1(x_n) - \gamma x_n\big(2n^2(x_n - y_n) + \phi'(x_n)\big) + (\gamma - r)x_n \\ & \qquad - \max_{z \in I}\big\{\hat{\alpha}(z) + G(z)\big(2n^2(x_n - y_n) + \phi(x_n) - 1\big) + \frac{1}{2}z^2 A_n\big\}, \\ & - A_n \,,\, 2n^2(x_n - y_n) + \phi'(x_n)\Big\} \leq 0, \end{aligned}$$

and

$$\min\Big\{rw_2(x_n) - \gamma y_n 2n^2(x_n - y_n) + (\gamma - r)y_n - \max_{z\in I}\big\{\hat{\alpha}(z) + G(z)\big(2n^2(x_n - y_n) - 1\big) + \frac{1}{2}z^2 B_n\big\}, - B_n\,,\, 2n^2(x_n - y_n)\Big\} \geq 0,$$

Taking the difference between these two inequalities gives out

$$\begin{aligned}
0 \geq \min\Big\{& r\big(w_1(x_n) - w_2(y_n)\big) - 2\gamma n^2(x_n - y_n)^2 - \gamma x_n\phi'(x_n) + (\gamma - r)(x_n - y_n) \\
&- \max_{z\in I}\big\{G(z)\phi'(x_n) + \frac{1}{2}z^2(A_n - B_n)\big\}\,,\, -(A_n - B_n)\,,\, \phi'(x_n)\Big\} \\
\geq \min\Big\{& r\big(w_1(x_n) - w_2(y_n)\big) - 2\gamma n^2(x_n - y_n)^2 - \gamma x_n\phi'(x_n) + (\gamma - r)(x_n - y_n) \\
&- \max_{z\in I}\big\{G(z)\phi'(x_n) + \frac{1}{2}z^2\big(\phi''(x_n) + m(\rho n^4)\big)\big\}\,,\, -\big(\phi''(x_n) + m(\rho n^4)\big)\,,\, \phi'(x_n)\Big\},
\end{aligned}$$

where the second inequality follows from (5.20). First sending ρ to 0 then taking $n \to \infty$ results in

$$0 \geq \min\Big\{r\big(w_1(\bar{x}) - w_2(\bar{x})\big) - \gamma\bar{x}\phi'(\bar{x}) - \max_{z\in I}\big\{G(z)\phi'(\bar{x}) + \frac{1}{2}z^2\phi''(\bar{x})\big\}\,,\, -\phi''(\bar{x})\,,\, \phi'(\bar{x})\Big\}.$$

where we have used the fact that $(x_n, \rho) \to \max_{z\in I}\big\{G(z)\phi'(x_n) + \frac{1}{2}z^2\big(\phi''(x_n) + m(\rho n^4)\big)\big\}$ is continuous according to the maximum theorem (see, e.g. Aliprantis and Border, 2006, Theorem 17.31). Therefore, for $x > 0$, $(w_1 - w_2)(x)$ is a viscosity subsolution to

$$\min\Big\{rw(x) - \gamma xw'(x) - \max_{z\in I}\big\{G(z)w'(x) + \frac{1}{2}z^2w''(x)\big\}\,,\, -w''(x)\,,\, w'(x)\Big\} = 0. \tag{5.21}$$

Now we construct an unbounded viscosity supersolution to (5.21). To this end, we look at the function $\Phi(x) = \eta(x + C)^{r/\gamma}$ with $x \geq 0$, $C > \frac{\|G\|_\infty}{\gamma}$ and $\eta > 0$. Denote $\zeta = \|w_1\|_\infty + \|w_2\|_\infty - (w_1 - w_2 - \Phi)(0)$, so $\Phi\big((\zeta/\eta)^{\gamma/r} - C\big) = \zeta$. It can be checked that for $x \in [0, (\zeta/\eta)^{\gamma/r} - C]$,

$$\Phi'(x) \geq \frac{r}{\gamma}\zeta^{1-\gamma/r}\eta^{\gamma/r},$$

$$-\Phi''(x) \geq \frac{r(\gamma-r)}{\gamma^2}\zeta^{1-2\gamma/r}\eta^{2\gamma/r},$$

$$r\Phi(x) - \gamma x\Phi'(x) - \max_{z\in I}\left\{G(z)\Phi'(x) + \frac{1}{2}z^2\Phi''(x)\right\} \geq r\Big(C - \frac{\|G\|_\infty}{\gamma}\Big)\zeta^{1-\gamma/r}\eta^{\gamma/r}.$$

Therefore, Φ on $x \in [0, (\zeta/\eta)^{\gamma/r} - C]$ is a strict supersolution to (5.21).

For any $x > (\zeta/\eta)^{\gamma/r} - C$,

$$(w_1 - w_2 - \Phi)(x) < \|w_1\|_\infty + \|w_2\|_\infty - \zeta = (w_1 - w_2 - \Phi)(0).$$

Hence, the function $w_1 - w_2 - \Phi$ attains a maximum on $[0, (\zeta/\eta)^{\gamma/r} - C]$. Suppose this maximum value is attained at $\tilde{x}$ and that $\tilde{x} > 0$. Then the viscosity subsolution property of $w_1 - w_2$ to (5.21) implies that

$$\min\left\{r\Phi(\tilde{x}) - \gamma x\Phi'(\tilde{x}) - \max_{z\in I}\left\{G(z)\Phi'(\tilde{x}) + \frac{1}{2}z^2\Phi''(\tilde{x})\right\},\, -\Phi''(\tilde{x}),\, \Phi'(\tilde{x})\right\} \leq 0,$$

which contradicts with it being a strict supersolution. We conclude consequently that $\tilde{x} = 0$. It follows then

$$(w_1 - w_2 - \Phi)(x) \leq (w_1 - w_2 - \Phi)(0) \leq 0, \quad x \geq 0.$$

Sending η to 0 results in $(w_1 - w_2)(x) \leq 0$, for all $x \geq 0$. □

Proof of Theorem 5.6. The value function V being a viscosity solution to (5.10) follow directly from Lemma 5.11 and Lemma 5.12. Suppose v is another viscosity solution to (5.10) satisfying the growth condition (5.11). Denote by v^* (resp. v_*) the u.s.c. (resp. l.s.c.) envelop of v, and write V^* (resp. V_*) to be the u.s.c. (resp. l.s.c.) envelop of V. Then according to Lemma 5.14, we have that $V^* \leq v_* \leq v^* \leq V_* \leq V^*$, which shows that $v = V$ and V is continuous. □

Proof of Proposition 5.8. Lemma 5.13 implies that V is a viscosity supersolution to $-v'' \geq 0$, and hence it is a concave function, by Touzi (2012) Lemma 5.24. It follows therefore there exists some $\hat{u} \in [0, \infty]$ such that $V'_+ > -1$ on $(0, \hat{u})$ and $V'_+ = -1$ on $(\hat{u}, \infty)$, since V is a viscosity solution to (5.10), where V'_+ denotes the right derivative of V. Now suppose $\hat{u} < u_b$, then for $u \in (\hat{u}, u_b)$, $V'(u) = -1$. As a result, for all $u \in (\hat{u}, u_b)$,

$$
\begin{aligned}
& rV(u) - \max_{z\in\mathbb{R}}\Big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2V''(u)\Big\} \\
<\,& r\widetilde{\psi}(u) + \gamma u - \max_{z\in\mathbb{R}}\{\hat{\alpha}(z) - G(z)\} \\
=\,& 0.
\end{aligned}
$$

However, this contradicts with V being a supersolution to (5.10), and we therefore conclude that for all $u \in (0, u_b)$, $V'_+(u) > -1$. This implies that V satisfies $V' + 1 > 0$ on $(0, u_b)$ in viscosity sense, and hence $rV(u) - \max_{z\in\mathbb{R}}\big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2V''(u)\big\} = 0$ on the same interval.

Define $v : \mathbb{R}^+ \to \mathbb{R}$ by

$$
v(u) := \begin{cases} V(u), & \text{for } u \in [0, u_b), \\ -u + u_b + \widetilde{\psi}(u_b), & \text{for } u \in [u_b, \infty). \end{cases}
$$

Let's first try to show $v \leq V$. But it suffices to prove $v(u) \leq V(u)$, for $u \in (u_b, \infty)$. Since V is a viscosity supersolution to the equation $v' + 1 \geq 0$ and by Touzi (2012) Lemma 5.23, $u \mapsto V(u)+u$ is non-decreasing. Therefore, the result follows from that $v(u)+u$ is constant, for $u \geq u_b$. Secondly, we show $v \geq V$. It can be checked that v restricted on (u_b, ∞) is a viscosity supersolution to (5.10) with $v(u_b) = \widetilde{\psi}(u_b)$. Thus, $v \geq V$ on (u_b, ∞) can be verified by the same argument as in the proof of Proposition 5.5 showing $\overline{\psi}$ is an upper bound of V. We therefore have $v = V$ on $[0, \infty)$. Then, $rV(u) - \max_{z\in\mathbb{R}}\big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2V''(u)\big\} > 0$ and $V'(u) + 1 = 0$ on (u_b, ∞) follow by a direct computation.

We proceed to prove the C^2 property of V. But it suffices to prove V is twice-continuously differentiable on $[0, u_b)$ as well as at the single point $\{u_b\}$. According to Theorem 1 and Footnote 21 in Strulovici and Szydlowski (2015), the differential equation

$$
rv(u) - \max_{z\in I}\Big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)v'(u) + \frac{1}{2}z^2v''(u)\Big\} = 0,
$$

with boundary condition $v(0) = 0$ and $v(u_b) = \widetilde{\psi}(u_b)$ has a unique twice-continuously differentiable solution on $[0, u_b)$, where $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. Let's denote the solution by W. Moreover, by Lemma 7 and the proof of Proposition 1 in Strulovici and Szydlowski

(2015), W is continuously differentiable on $[0, u_b)$. Now observe that

$$-\frac{1}{2}W''(u) \geq \frac{-rW(u) + \hat{\alpha}(z) + \big(\gamma u + G(z)\big)W'(u)}{z^2}, \quad \text{for all } z \in I,\ u \in (0, u_b).$$

But given any $u \in (0, u_b)$, there exists $z_u \in I$ such that

$$-\frac{1}{2}W''(u) = \frac{-rW(u) + \hat{\alpha}(z_u) + \big(\gamma u + G(z_u)\big)W'(u)}{z_u^2}.$$

Thus, for $u \in (0, u_b)$,

$$-\frac{1}{2}W''(u) = \max_{z \in I} \frac{-rW(u) + \hat{\alpha}(z) + \big(\gamma u + G(z)\big)W'(u)}{z^2}.$$

Since $\{-rW(u) + \hat{\alpha}(z) + \big(\gamma u + G(z)\big)W'(u)\}/z^2$, as a function of (z, u), is continuous on $I \times [0, u_b)$, by the maximum theorem (see, e.g. Aliprantis and Border, 2006, Theorem 17.31), W'' is continuous on $[0, u_b)$. On the other hand, Lemma 5.13 indicates that V is a viscosity supersolution to

$$rv(u) - \max_{z \in I}\Big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)v'(u) + \frac{1}{2}z^2 v''(u)\Big\} = 0$$

with boundary condition $v(0) = 0$ and $v(u_b) = \widetilde{\psi}(u_b)$. Therefore by a comparison theorem (see, e.g. Touzi, 2012, Theorem 5.18), $V = W$ on $[0, u_b]$.

The concavity of V ensures the existence of left and right derivatives of $V(u_b)$, denoted by $V'_-(u_b)$ and $V'_+(u_b)$ respectively. Moreover, $V'_-(u_b) \geq V'_+(u_b) = -1$. To get the C^1 property of V at u_b, we prove by contradiction and assume the contrary that $V'_-(u_b) > V'_+(u_b)$. Consider the test function $\phi_n \in C^2((0,\infty))$ defined by

$$\phi_n(u) = V(u_b) + \eta(u - u_b) - \frac{n}{2}(u - u_b)^2, \tag{5.22}$$

where $\eta \in \big(V'_+(u_b), V'_-(u_b)\big)$. Then, $\phi'_n(u_b) = \eta > -1$, $\phi''_n(u_b) = -n$, and u_b is a local maximum of $V - \phi_n$ with $(V - \phi_n)(u_b) = 0$. According to Lemma 5.13, the viscosity subsolution of V to (5.19) implies that

$$rV(u_b) - \max_{z \in I}\Big\{\hat{\alpha}(z) + \big(\gamma u_b + G(z)\big)\eta - \frac{1}{2}z^2 n\Big\} \leq 0, \tag{5.23}$$

where $I = [(g'(\underline{\alpha}) - \beta)\sigma, (g'(\overline{\alpha}) - \beta)\sigma]$. Since $g'(\underline{\alpha}) > \beta$, I is a compact interval bounded away from 0. Therefore, taking n to be sufficiently large in the above expression results in a contradiction, and we conclude that V is continuously differentiable at u_b.

It now remains to show V is twice-continuously differentiable at u_b. Observe that for $u \in (0, u_b)$, with $z_u \in I$ being the optimiser of $\max_{z\in\mathbb{R}}\big\{\hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u) + \frac{1}{2}z^2 V''(u)\big\}$, we have

$$0 \geq \frac{1}{2} z_u^2 V''(u) = rV(u) - \hat{\alpha}(z_u) - (\gamma u + G(z_u))V'(u).$$

Sending u to u_b and taking a convergent subsequence of z_u result in

$$\begin{aligned} 0 \geq \frac{1}{2} z_{u_b}^2 V''(u_b-) &= rV(u_b) + \gamma u_b - \hat{\alpha}(z_{u_b}) + G(z_{u_b}) \\ &\geq r\widetilde{\psi}(u_b) + \gamma u_b - \max_{z\in\mathbb{R}}\{\hat{\alpha}(z) - G(z)\} = 0, \end{aligned}$$

where $z_u^2 > 0$. Hence, $V''(u_b-) = V''(u_b+) = 0$. □

The next lemma shows the Lipschitz continuity of the function $Z^*(\cdot)$ defined by Lemma 5.9. The proof follows a similar argument to Proposition 6 in Strulovici and Szydlowski (2015).

Lemma 5.15. *Under the condition of Lemma 5.9, $Z^*(\cdot)$ is Lipschitz continuous on* $[0, u_b]$.

Proof. Let's first prove $Z^*(\cdot)$ is a continuous function. To this end, write

$$h(z, u) = \hat{\alpha}(z) + G(z)V'(u) + \frac{1}{2}z^2 V''(u), \quad z \in I,\ u \in [0, u_b].$$

We calculate that for all $z \in I$,

$$\begin{aligned} \hat{\alpha}'(z) &= \frac{1}{\sigma g''(\hat{\alpha}(z))} > 0, &\qquad \hat{\alpha}''(z) &= -\frac{g'''(\hat{\alpha}(z))\hat{\alpha}'(z)}{\sigma\big(g''(\hat{\alpha}(z))\big)^2} \leq 0, \\ G'(z) &= \frac{z\hat{\alpha}'(z)}{\sigma} > 0, &\qquad G''(z) &= \frac{\hat{\alpha}'(z)}{\sigma} - \frac{G'(z)g'''(\hat{\alpha}(z))}{\sigma\big(g''(\hat{\alpha}(z))\big)^2} > 0, \end{aligned}$$

and

$$h_z(z, u) = \hat{\alpha}'(z) + G'(z)V'(u) + zV''(u),$$

where $G''(z) > 0$ is due to the assumption that $z/g''\big((g')^{-1}(z/\sigma + \beta)\big)$ is strictly increasing.

Then,

$$
\begin{aligned}
h_{zz}(z,u) &= \hat{\alpha}''(z) + G''(z)V'(u) + V''(u) \\
&= \hat{\alpha}''(z) + \frac{\hat{\alpha}'(z)V'(u)}{\sigma} - \frac{G'(z)g'''(\hat{\alpha}(z))V'(u)}{\sigma\big(g''(\hat{\alpha}(z))\big)^2} + V''(u) \\
&= \hat{\alpha}''(z) + \frac{h_z(z,u) - \hat{\alpha}'(z)}{z} - \frac{G'(z)g'''(\hat{\alpha}(z))V'(u)}{\sigma\big(g''(\hat{\alpha}(z))\big)^2} \\
&= \hat{\alpha}''(z) + \frac{h_z(z,u) - \hat{\alpha}'(z)}{z} - \frac{g'''(\hat{\alpha}(z))}{\sigma\big(g''(\hat{\alpha}(z))\big)^2}\Big(h_z(z,u) - \hat{\alpha}'(z) - zV''(u)\Big) \\
&= \hat{\alpha}''(z) + \Big(h_z(z,u) - \hat{\alpha}'(z)\Big)\left(\frac{1}{z} - \frac{g'''(\hat{\alpha}(z))}{\sigma\big(g''(\hat{\alpha}(z))\big)^2}\right) + \frac{g'''(\hat{\alpha}(z))zV''(u)}{\sigma\big(g''(\hat{\alpha}(z))\big)^2} \\
&= \hat{\alpha}''(z) + \Big(h_z(z,u) - \hat{\alpha}'(z)\Big)\frac{G''(z)}{G'(z)} + \frac{g'''(\hat{\alpha}(z))zV''(u)}{\sigma\big(g''(\hat{\alpha}(z))\big)^2}.
\end{aligned}
$$

Therefore, for $h_z(z,u) \le 0$,

$$
h_{zz}(z,u) \le -\hat{\alpha}'(z)\frac{G''(z)}{G'(z)} < 0.
$$

This implies that for any fixed $u \in [0, u_b]$, there is at most one $z \in I$ such that $h_z(z,u) = 0$. Consequently, for any $u \in [0, u_b]$, $Z^*(u)$ is an I-valued number, and hence $Z^*(\cdot)$ is a function. According to Theorem 17.31 and Theorem 17.11 in Aliprantis and Border (2006), the function $Z^*(\cdot)$ has compact range and closed graph. Hence, it is continuous.

Following a similar argument in the proof of Proposition 5.8 gives out that

$$
-\frac{1}{2}V''(u) = \max_{z\in I} \frac{-rV(u) + \hat{\alpha}(z) + \big(\gamma u + G(z)\big)V'(u)}{z^2}, \quad u \in [0, u_b].
$$

Then by Madan and Seneta (1990) Corollary 4, V''' exists on $(0, u_b)$ and

$$
V'''(u) = \frac{2rV'(u) - 2\hat{\alpha}\big(Z^*(u)\big) - 2\gamma\big(V'(u) + uV''(u)\big) - G\big(Z^*(u)\big)V''(u)}{\big(Z^*(u)\big)^2}, \quad u \in (0, u_b),
$$

which is continuous. The above expression of V''' clearly can be continuously extended onto the interval $[0, u_b]$. Therefore, the function $h_{zu}(Z^*(u), u)$ is continuous hence bounded on $[0, u_b]$. For the Lipschitz property of $Z^*(\cdot)$, it is sufficient to show that the restriction of $Z^*(\cdot)$ to the union of non-empty open intervals on which the values of $Z^*(\cdot)$ are in the interior of I

has bounded derivative. According to the previous part of this proof, along this restriction, we have $h_z(Z^*(u), u) = 0$, $h_{zz}(Z^*(u), u)$ and $h_{zu}(Z^*(u), u)$ are both continuous, $h_{zu}(Z^*(u), u)$ is bounded and $h_{zz}(Z^*(u), u)$ is bounded away from 0. Therefore, applying the implicit function theorem, we have that along this restriction,

$$(Z^*)'(u) = -\frac{h_{zu}(Z^*(u), u)}{h_{zz}(Z^*(u), u)},$$

which is continuous and bounded. $\square$

Proof of Lemma 5.9. Note that

$$d\hat{U}_t^{\alpha,*} = [\gamma \hat{U}_t^{\alpha,*} + g^*\big(Z_t^*/\sigma + \beta\big) - \beta\mu - \mu Z_t^*/\sigma]\,dt + Z_t^*/\sigma\,dX_t^\alpha - \,dC_t^*$$

can be rewritten as

$$\begin{aligned} d\hat{U}_t^{\alpha,*} = [&\gamma \hat{U}_t^{\alpha,*} + G\big(Z^*(\hat{U}_t^{\alpha,*})\big) - \hat{\alpha}(Z^*(\hat{U}_t^{\alpha,*}))Z^*(\hat{U}_t^{\alpha,*})/\sigma + \alpha_t Z^*(\hat{U}_t^{\alpha,*})/\sigma]\,dt \\ &+ Z^*(\hat{U}_t^{\alpha,*})\,dB_t - \,dC_t^*. \end{aligned} \tag{5.24}$$

We first show this stochastic differential equation with reflection has a unique $\mathbb{F}$-adapted strong solution. According to Theorem 4.1 in Tanaka (1979), it suffices to show that the coefficients are Lipschitz continuous functions with linear growth. But since $Z^*(\cdot)$, $G(\cdot)$, $G'(\cdot)$, $\hat{\alpha}(\cdot)$, $\hat{\alpha}'(\cdot)$ and α are all bounded, it suffices to prove $Z^*(\cdot)$ is Lipschitz continuous, which is demonstrated by Lemma 5.15.

Now it remains to show $\hat{U}^{\alpha,*}$ and C^* are also $\mathbb{F}^X$-adapted. By exactly the same proof of Theorem 4.1 in Tanaka (1979), one can find two sequences of processes $\{\hat{U}^n\}_{n=0}^\infty$ and $\{C^n\}_{n=1}^\infty$ such that for all $n \geq 1$ and $0 \leq t \leq \hat{\tau}_0^n$,

$$\begin{aligned} \hat{U}_t^0 &= \hat{U}_0^{\alpha,*}, \\ \hat{U}_t^n &= \hat{U}_0^{\alpha,*} + \int_0^t \left[\gamma \hat{U}_s^{n-1} + g^*\big(Z^*(\hat{U}_s^{n-1})/\sigma + \beta\big) - \beta\mu - \mu Z^*(\hat{U}_s^{n-1})/\sigma\right] ds \\ &\quad + \int_0^t Z^*(\hat{U}_s^{n-1})/\sigma\,dX_s^\alpha - \int_0^t dC_s^n, \\ \int_0^{\hat{\tau}_0^n} \mathbb{1}_{\{\hat{U}_t^n < u_b\}} dC_t^n &= 0, \\ \hat{U}_t^n &\in [0, u_b], \end{aligned}$$

where $\hat{\tau}_0^n = \inf\{t \geq 0 : \hat{U}_t^n \leq 0\}$, C^n is non-decreasing càdlàg starting from 0, and $\hat{U}^n$ (resp. C^n) converges uniformly on compact intervals to $\hat{U}^{\alpha,*}$ (resp. C^*) a.s. as n tends to infinity. Therefore, for all $n \geq 1$, $\hat{U}^n$ and C^n both adapt to $\mathbb{F}^X$, this is essentially because $\int_0^\cdot Z^*(\hat{U}_s^{n-1})/\sigma \, dX_s^\alpha$ is $\mathbb{F}^X$-adapted and C^n is given by

$$C_t^n = X_t^n - \max_{0 \leq s \leq t}(X_s^n \wedge u_b), \quad 0 \leq t \leq \hat{\tau}_0^n,$$

where

$$\begin{aligned} X_t^n = \hat{U}_0^{\alpha,*} &+ \int_0^t \left[\gamma \hat{U}_s^{n-1} + g^*\big(Z^*(\hat{U}_s^{n-1})/\sigma + \beta\big) - \beta\mu - \mu Z^*(\hat{U}_s^{n-1})/\sigma\right] ds \\ &+ \int_0^t Z^*(\hat{U}_s^{n-1})/\sigma \, dX_s^\alpha. \end{aligned}$$

Thus, both $\hat{U}^{\alpha,*}$ and C^* are $\mathbb{F}^X$-adapted. □

Proof of Theorem 5.10. In the context of Lemma 5.9, write $\hat{U}^* = \hat{U}^{\hat{\alpha}(Z^*),*}$ and $\tau_0^* = \tau^{\hat{\alpha}(Z^*),*}$. This proof is based on the result of Lemma 5.2 that for $(Z^*, C^*) \in \mathcal{P}(u; \mathbb{F}^X)$, Agent applies effort $\hat{\alpha}(Z^*)$. Therefore, $\mathbb{F} = \mathbb{F}^X$ and $\hat{U}^*$ satisfies

$$d\hat{U}_t^* = [\gamma \hat{U}_t^* + G(Z_t^*)]\, dt + Z_t^* \, dB - dC_t^*.$$

Then the classical argument of verification goes through.

We first show $(Z^*, C^*) \in \mathcal{P}(u; \mathbb{F}^X)$. By the proof of Lemma 5.15 we know $Z^*(\cdot)$ is continuous. This together with the adaptedness of $\hat{U}^{\alpha,*}$ implies that the process Z^* is $\mathbb{F}^X$-progressively measurable. The boundedness of Z^* implies $\int_0^t \mathrm{e}^{-2rt}(Z_s^*)^2 \, ds < \infty$, for all $t \geq 0$. Since the state process $U^{\alpha,*}_{\cdot \wedge \tau_0^{\alpha,*}}$ is bounded, it is of class D and the associated transversality condition is satisfied. Following the dynamic in (5.24),

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^{\tau_0^{\alpha,*}} \mathrm{e}^{-rs} \, dC_s^*\bigg] \\ &= u + \mathbb{E}\bigg[\int_0^{\tau_0^{\alpha,*}} (\gamma - r)\mathrm{e}^{-rs} U_s^{\alpha,*} \, ds + \int_0^{\tau_0^{\alpha,*}} \mathrm{e}^{-rs}\Big(G\big(Z_s^*\big) - \hat{\alpha}(Z_s^*)Z_s^*/\sigma + \alpha_t Z_s^*/\sigma\Big)\, ds\bigg] < \infty, \end{aligned}$$

where the first equality is because $\mathrm{e}^{-r\tau_0^{\alpha,*}} U^{\alpha,*}_{\tau_0^{\alpha,*}} = 0$ and $\int_0^{\cdot \wedge \tau_0^{\alpha,*}} \mathrm{e}^{-rs} Z_s^* \, dB_s$ is a uniformly integrable martingale, and the finiteness follows from the boundedness of integrands. Therefore, combining with the result in Lemma 5.9, we have $(Z^*, C^*) \in \mathcal{P}(u; \mathbb{F}^X)$.

Using the condition that $r < \gamma$, it can be checked that any $(Z, C) \in \mathcal{P}(u; \mathbb{F}^X)$ also satisfies Lemma 5.2. Hence Agent applies effort $\hat{\alpha}(Z)$. Write $\hat{U} = \hat{U}^{\hat{\alpha}(Z)}$, then it satisfies

$$d\hat{U}_t = [\gamma \hat{U}_t + G(Z_t)]\, dt + Z_t\, dB - dC_t.$$

Consider any $(Z, C) \in \mathcal{P}(u; \mathbb{F}^X)$. Let $\{\tau_n\}_{n=1}^{\infty}$ be a localisation sequence of stopping times such that each $\int_0^{\tau_n \wedge \cdot} \mathrm{e}^{-rt} V'(U_t) Z_t\, dB_t$ is a martingale. Then,

$$\begin{aligned}
&\mathrm{e}^{-r(\tau_0 \wedge T \wedge \tau_n)}\, V(\hat{U}_{\tau_0 \wedge T \wedge \tau_n}) \\
&= V(u) + \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} \Big(-rV(\hat{U}_t) + \big(\gamma \hat{U}_t + G(Z_t)\big) V'(\hat{U}_t) + \frac{1}{2} Z_t^2 V''(\hat{U}_t) \Big)\, dt \\
&\quad - \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} V'(\hat{U}_t)\, dC_t^c + \sum_{0 \le t \le \tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} \Delta V(\hat{U}_t) \\
&\quad + \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} V'(\hat{U}_t) Z_t\, dB_t \\
&\le V(u) - \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} \hat{\alpha}(Z_t)\, dt + \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt}\, dC_t - \int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} Z_t\, dB_t.
\end{aligned}$$

It follows that

$$\begin{aligned}
V(u) &\ge \lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\Big[\int_0^{\tau_0 \wedge T \wedge \tau_n} \mathrm{e}^{-rt} \Big(\hat{\alpha}(Z_t)\, dt - dC_t \Big) + \mathrm{e}^{-r(\tau_0 \wedge T \wedge \tau_n)}\, V(\hat{U}_{\tau_0 \wedge T \wedge \tau_n}) \Big] \\
&= \mathbb{E}\Big[\int_0^{\tau_0} \mathrm{e}^{-rt} \Big(\hat{\alpha}(Z_t)\, dt - dC_t \Big) \Big], \qquad (5.25)
\end{aligned}$$

where the equality is due to the dominated convergence theorem, the transversality condition on $\hat{U}$, and the fact that V is bounded by two linear functions going through 0.

On the other hand, consider $(Z^*, C^*) \in \mathcal{P}(u; \mathbb{F}^X)$. It follows from Proposition 5.8 and a similar computation as above that

$$\mathbb{E}\Big[\int_0^{\tau_0^*} \mathrm{e}^{-rt} \Big(\hat{\alpha}(Z_t^*)\, dt - dC_t^* \Big) \Big] = V(u).$$

This together with (5.25) implies that (Z^*, C^*) is optimal.

With the optimal choice of $(Z^*, C^*) \in \mathcal{P}(u; \mathbb{F}^X)$ Principal chooses an optimal starting point u^* by solving the problem (5.8). The existence of u^* is guaranteed by the concavity of V on $[\bar{A}, \infty)$. Since V is strictly decreasing on (u_b, ∞), we have $u^* = \bar{A}$, if $\bar{A} \ge u_b$. □

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