Excursions of Risk Processes with Inverse Gaussian Processes and their Applications in Insurance

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Declaration

I certify that the thesis I have presented for examination for the Ph.D. degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

Parisian excursion of a Lévy process is defined as the excursion of the process below or above a pre-defined barrier continuously exceeding a certain time length. In this thesis, we study classical and Parisian type of ruin problems, as well as Parisian excursions of collective risk processes generalized on the classical Cramér-Lundberg risk model.

We consider that claim sizes follow mixed exponential distributions and that the main focus is claim arrival process converging to an inverse Gaussian process. By this convergence, there are infinitely many and arbitrarily small claim sizes over any finite time interval. The results are obtained through Gerber-Shiu penalty function employed in an infinitesimal generator and inverting corresponding Laplace transform applied to the generator.

In Chapter 3, the classical collective risk process under the Cramér-Lundberg risk model framework is introduced, and probabilities of ruin with claim sizes following exponential distribution and a combination of exponential distributions are also studied.

In Chapter 4, we focus on a surplus process with the total claim process converging to an inverse Gaussian process. The classical probability of ruin and the joint distribution of ruin time, overshoot and initial capital are given. This joint distribution could provide us with probabilities of ruin given different initial capitals in any finite time horizon.

In Chapter 5, the classical ruin problem is extended to Parisian type of ruin, which requires that the length of excursions of the surplus process continuously below zero reach a predetermined time length. The joint law of the first excursion above zero and the first excursion under zero is studied. Based on the result, the Laplace transform of Parisian ruin time and formulae of probability of Parisian type of ruin with different initial capitals are obtained. Considering the asymptotic properties of claim arrival process, we also propose an

approximation of the probability of Parisian type of ruin when the initial capital converges to infinity.

In Chapter 6, we generalize the surplus process to two cases with total claim process still following an inverse Gaussian process. The first generalization is the case of variable premium income, in which the insurance company invests previous surplus and collects interest. The probability of survival and numerical results are given. The second generalization is the case in which capital inflow is also modelled by a stochastic process, i.e. a compound Poisson process. The explicit formula of the probability of ruin is provided.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation and Literature Review

In recent years, risk models have attracted much attention in insurance businesses. One of the main reasons for this is the increasing importance of risk management and the generated stochastic modelling of financial solvency (for instance, see Frittelli & Scandolo [33], Kaufmann, Gadmer & Klett [49] and Woll [81]). There is one important type of such models that concern ruin theory for insurance companies. The fundamental modern ruin theory goes back to the works of Lundberg and Cramér. In 1903, Lundberg (see Lundberg [57]) first proposed that the Poisson process can be considered as a simple process in solving the first passage time problem and can also be exploited as a model for the claim number process. Lundberg dealt with the modelling of claims that arrive in an insurance company, and also gave advice on how much premium the insurance company needs to charge in order to avoid default. Then in 1930, Cramér [11] extended Lundberg's work to model the ruin for insurance companies as a first passage time problem. Cramér [11] developed collective risk theory using the total claim amount process generated by a Poisson process. The basic risk model is thus called the Cramér-Lundberg risk model or classical risk model.

Over the past century a significant body of literature on ruin theory has been developed. Gerber [37] (see also Gerber [38]) introduced martingale methods in risk theory, which has been a standard technique. There have been a few papers, such as Dassios and Embrechts [20], Dassios and Wu [21] and [23], and Schmidli [69], where the martingale methods have been used to study complicated risk problems. Later, another development of ruin theory appeared, which is Gerber-Shiu expected discounted penalty functions that are used to study the joint distribution of the ruin time, the surplus just before ruin and the overshoot at ruin (see Gerber and Shiu [44], [45] and [46]).

We start from the classical Cramér-Lundberg ruin theory framework (Cramér [11], Lundberg [57]), which is one of the most popular and widely used models in non-life insurance mathematics. This framework considers the risk process in a way in which there is an initial capital for an insurance company, with constant premium rate and total claim amount process following a compound Poisson process. Despite its simplicity, it captures some of the essential features of the total claims amount process and studies how ruin behaves for an insurance company. With relation to possible bankruptcy and reserve capital, the main interest from an insurance company's point of view is the arrival of claim and claim size, which affects the surplus of the insurance company.

Dassios and Embrechts [20] have shown that many important risk processes can be naturally handled within the framework of piecewise deterministic (PD) Markov processes. As also pointed out in Cai et al. [9], the classical Cramér-Lundberg risk process and the compound Poisson process are special cases of PD Markov processes. The class of PD Markov process was first introduced by Davis [24], and it is a general class of time-homogeneous Markov processes that contains deterministic motion and random jumps but no diffusion. When PD Markov processes were introduced, it was soon found that the framework and the developed techniques were important for risk theory. Dassios and Embrechts [20] also showed how to use the framework to solve insurance risk problems. One example they considered is that they allowed the insurance company to borrow money when its surplus is below some barrier level, which it is called the "absolute ruin model". Researchers that followed were Embrechts [30], Davis [26], Embrechts and Schmidli [31], Davis and Vellekoop [25].

In the traditional ruin theory, research on the risk process has been intensively studied, assuming that ruin will immediately occur if the surplus decreases to below zero. In other words, the company is ruined if its surplus becomes negative and falls below a critical threshold level. We refer to Asmussen [1], Bühlmann [8], and Rolski [66] for an intensive study of ruin probability. However, as discussed in Egidio dos Reis [32], the ruin probability is normally very small in practice, and the portfolio that leads to ruin is just one of many existing in the company. The insurance company can have enough funds available to provide support for negative surplus. Therefore, even if ruin occurs the insurance firm can still continue the business with a hope of fast recovery and can survive for some time. That is why new ruin models have been proposed in recent years. Some insurance risk models consider the application of a certain implementation delay when recognizing an insurance company's capital insufficiency, which was inspired by Parisian options (see Chesney et al. [13]). Some comments can be found in Gerber [43], and Egidio dos Reis [32]. There is increased interest in generalized insurance risk models with a redefined event of ruin, which allows the company to stay with a negative surplus with no need to declare ruin and let the company continue their business. Dassios and Embrechts's research [20] is one of the examples of early research, which considered "absolute ruin". They defined that absolute ruin occurs if the drift of process drops below zero. As long as the process drift is above zero, the company could borrow money to continue their business. Gerber and Yang [42], Cai [9], and Cheung [10] also discussed "absolute ruin".

In this thesis, we concentrate on the event of Parisian type of ruin, which was introduced by Dassios and Wu [21], where they consider applying a delay when examining an insurance company's capital inefficiency. In other words, they study Parisian ruin through Parisian excursions of the surplus process. Parisian excursion is defined as the excursion of the surplus process continuously below or under a pre-determined barrier, reaching a pre-defined time length. More precisely, in Dassios and Wu [21], they assume that ruin occurs when the excursion below a pre-defined level continuously exceeds a prescribed length of time $d > 0$. There is another paper by Dassios and Wu [23], where they consider that there is a Parisian delay between a decision to pay a dividend and its implementation. They use a classical surplus process with the claim size being exponentially distributed. When the surplus reaches the pre-determined barrier level $d > 0$, the decision to pay dividends is taken. However the payment is implemented only when the surplus continuously stays above the barrier longer than the pre-determined barrier level $d > 0$, and a dividend is paid at the end of this period. They also obtain an optimal barrier that can maximize the expected present value of dividends.

It is noticeable that Parisian ruin is closely linked with Parisian barrier options. A Parisian barrier option is closed to a standard barrier option, which is defined as an option that is knocked in or out if the underlying asset price process continuously stays above or under a pre-defined barrier for longer than a determined time period $d > 0$ within the lifetime of the option. This means that the option owner does not lose the option if the value of the underlying asset reaches the barrier level but only if it stays long enough above or under the barrier level. The first paper on the Parisian barrier option is from Chesney et al. [13], which studied the practical difference between standard barrier options and the Parisian option. Chesney et al. [13] studied Parisian options of European type options given by the Black and Scholes formula (see Black and Scholes [5]), and they derived a formula of the Laplace transform of the option price with respect to maturity time T by using Brownian excursion theory. The option price needs to be numerically inverted from Laplace transforms.

In Dassios and Wu [21], they study the probability of Parisian type of ruin by considering that the surplus process is under the framework of the classical Cramér-Lundberg risk model. They obtain the explicit formulae of Laplace transform of Parisian ruin time with different initial capital level when claim size has an exponential distribution, and they also study the probability of Parisian type of ruin for small claim sizes and provide a diffusion approximation as well. They also show that when claims are distributed with light tails, an asymptotic formula of the Cramér-Lundberg type is also true. More recently, their results were generalized to general spectrally negative Lévy process in insurance risk (see Czarna and Palmowski [18], Landriault et al. [53] and [54], and Loeffen et al. [56]). In Landriault et al. [53] and [54], they considered the Laplace transform of the ruin time of the Parisian type for a Lévy risk model with bounded variation. They assumed that the excursion is replaced with a stochastic time period with a pre-specified distribution. Explicit results were obtained considering the excursion is exponentially distributed. Czarna and Palmowski [18] and Czarna [19] studied ruin when the surplus process continuously stays below zero for a time length of d or drops below a pre-determined level $a > 0$.

This thesis focuses on the excursions of the Parisian type and the probability of Parisian type of ruin. We extend the study of Dassios and Wu [21] in the context that total claims amount process follows an inverse Gaussian process. The inverse Gaussian process generalizes classical compound Poisson process, and it's a limit of compound Poisson process as claim size follows an inverse Gaussian distribution with one parameter converging to zero. Under the setting of inverse Gaussian process, we consider that there could be infinitely many arbitrarily small claims with any time period. Our approach is based on Dassios and Wu's (see [21]) idea which studies the excursions away from zero of the underlying surplus process. Our main contribution is an explicit formula of the Laplace transform of Parisian ruin time and probability of Parisian type of ruin. We also give an approximation of Parisian ruin probability of Cramér-Lundberg type.

The study of the probability of ruin for classical collective risk process with constant premium rate has been the centre of interest in a number of papers focusing on actuarial risk theory. Most of these articles treat the ruin probability of surplus involving constant premiums. The classical model assumes that the surplus does not receive interest over time. Explicit results have been obtained for certain claim size distributions. However, there is a large part of the surplus of an insurance company that comes from investment income. In the meantime, risk theory with interest income should be studied carefully. In recent years, there have been a few papers in the literature considering premiums whose value depends on current surplus. Some papers argue that considering variable premium income as a function of current surplus is more realistic, taking into account an investment income produced by an insurance company's surplus. Additional to the premium income, the insurance company also receives interest on its surplus. Taylor [78] considered the case where the premium rate continuously changes as a function of the current surplus. Michaud [62] approximated the probability of ultimate ruin by simulating the jumps and the inter arrival times for jumps. Petersen [64] also obtained the ultimate probability of ruin by a simple numerical method.

Under an analogue of the Cramér condition, Sundt and Teugels [76] and [77] discussed the probability of ruin when surplus process has a constant premium rate, constant interest rate and exponential claim sizes in a continuous time within infinite time horizon. They considered the equation of probability of ruin and upper and lower bound. Paulsen and Gjessing [65] studied a classical model perturbed by a diffusion. They obtained a Lundberg type inequality by assuming that there is a stochastic investment income. Klüppelberg and Stadtmüller [51] applied sophisticated analytical analysis to derive an asymptotic formula for the ruin probability. They considered a surplus process with claims following a distribution with a regularly varying tail. Asmussen refined their result in [1], applying the reflected random walk theory to obtain asymptotic formulae for the ruin probability when considering claim sizes follow sub-exponential distribution.

There are other generalizations of the classical risk model. One of the generalizations is that the premium income in risk models is also a stochastic process to keep track of premiums, but independent of the total claim amount process. This idea was first advised by Boucherie [6]. Later on, the model was intensively studied by others. Boikov [7] obtained integral equations and exponential bounds, which are similar to the classical Cramér-Lundberg model. The author also discussed the probability of ruin. Temnov [79] gave a representation for the probability of ruin for surplus process with random premium process. Melnikov [58] removed the deterministic premium rate component, and obtained Laplace transform for the difference between random premium income and total claim amount process and an integro-differential equation for the probability of ruin. Karnaukh [52] obtained a formula for the discounted defective joint probability density function of surplus and deficit at ruin, assuming that the premium sizes have exponential distribution with rate depending on a certain threshold level.

1.2 Organization and Outline of the Thesis

This work is organized as follows.

In chapter 2, we introduce the nomenclature for Laplace transform, inverse of Laplace transform, stochastic processes, random times, and miscellaneous items.

In chapter 3, the classical Cramér-Lundberg risk model is introduced, and the problem of ruin corresponding to infinitesimal generator and Gerber-Shiu expected discounted penalty function are provided as well. The Cramér-Lundberg approximation of probability of ruin is also introduced to give a comprehensive understanding of the asymptotic value of ruin probability. We also show that by inverting corresponding Laplace transforms applied to an infinitesimal generator, explicit formulae of probability of ruin with claim sizes being distributed as exponential distribution and two mixed exponential distributions are obtained. The Laplace transform of the ruin time is also provided under the case of mixed exponentials. The purpose of this chapter is to introduce the reader to some of the mathematical tools that will be useful for original results in subsequent chapters.

Chapter 4 concerns ruin problems with the total claims amount being an inverse Gaussian process, in which there could be infinitely many and arbitrarily small claims within any time interval. We begin with the discussion of Laplace transforms of ruin time and overshoot respectively. Then we study the probability of ruin for non-zero initial capital, as well as the joint distribution of the time of ruin and the overshoot at ruin with zero initial capital. The joint distribution of the ruin time, overshoot and non-zero initial capital is also studied. These results are all derived by using the methods introduced in the previous chapter and by inverting corresponding Laplace transforms.

Chapter 5 extends classical ruin problem to Parisian type of ruin problem. It requires the length of the excursion of a surplus process continuously below zero, reaching a time length predefined. We discuss the joint Laplace transform of the first excursion above zero and the first excursion below zero. Based on the joint Laplace transform, we use a two-state semi-Markov process to obtain the Laplace transforms of Parisian ruin time for zero initial capital and non-zero initial capital. The formulae of the probability of Parisian type of ruin with different initial capitals are also provided. Through considering the asymptotic properties of the total claims arrival process, we also propose an approximation for the probability of the Parisian type of ruin with the initial capital converging to infinity.

In chapter 6, we generalize the risk model to two cases. One considers a variable premium income, which the insurance company invests previous surplus and receives interest. The probability of survival for this risk model is discussed. Another generalization studies a surplus process with stochastic premium income. We are still particularly focusing on total claims following an inverse Gaussian process. The explicit formula of ruin probability is given. For both of these two generalizations, the numerical results of ruin probability and the asymptotic property of approximation for ruin probability when initial capital converges to infinity are also discussed.

Chapter 7 concludes this thesis.

Chapter 2

Nomenclature

We fix a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ for any stochastic Lévy process $\{X_t\}_{t\geq 0}$.

For a function $f(x)$, it is assumed that its Laplace transform $\hat{f}(\xi)$ exists, i.e.

$$\hat{f}(\xi) = \mathcal{L}_\xi\{f(x)\} := \int_0^\infty e^{-\xi x} f(x)\ dx. \tag{2.1}$$

Then the inverse Laplace transform of $\hat{f}(\xi)$, denoted by $\mathcal{L}_\xi^{-1}\{\hat{f}(\xi)\}$ is a function of f, i.e.

$$\mathcal{L}_\xi^{-1}\{\hat{f}(\xi)\} = f(x) \quad \Longleftrightarrow \quad \mathcal{L}_\xi\{f(x)\} = \hat{f}(\xi). \tag{2.2}$$

We consider the inverse Laplace transform with respect to ξ, which is evaluated at the point x.

Similarly for a function $f(x,y)$, we can define the double Laplace transform $\hat{f}(\xi,\beta)$

$$\hat{f}(\xi,\beta) = \mathcal{L}_\beta\mathcal{L}_\xi\{f(x,y)\} := \int_{y=0}^\infty \int_{x=0}^\infty e^{-\xi x} e^{-\beta y} f(x,y)\ dx\ dy. \tag{2.3}$$

Therefore, the inverse double Laplace transform is

$$\mathcal{L}_\beta^{-1}\mathcal{L}_\xi^{-1}\{\hat{f}(\xi,\beta)\} = f(x,y), \tag{2.4}$$

which is evaluated at the points (x,y).

2.1 Stochastic Processes and Random Times

X_t	$=$	$x + ct - \sum_{i=1}^{N_t} Y_i$ – classical surplus process of an insurance company up to time t
x		initial capital of the insurance company, $x \geq 0$
c		constant premium rate, $c > 0$
N_t		number of claims up to time t, $N_t \sim Poisson\left(\frac{\lambda}{\varepsilon}\right)$
Y_i		the ith claim, $Y_i > 0$ and $Y_i \sim IG(\varepsilon, \mu)$
$IG(\varepsilon, \mu)$		inverse Gaussian distribution with parameters $\varepsilon > 0$ and $\mu > 0$
$g(y)$	$=$	$\frac{\varepsilon}{\sqrt{2\pi y^3}} e^{-\frac{(\varepsilon - \mu y)^2}{2y}}$ – probability distribution function of $IG(\varepsilon, \mu)$
$G(y)$	$=$	$\int_0^y g(u)du$ – cumulative distribution function of $IG(\varepsilon, \mu)$
X_t	$=$	$x + ct - Z_t$ – surplus process with constant premium rate $c > 0$ and inverse Gaussian process Z_t up to time t
τ	$=$	$\inf\{t \geq 0 \mid X_t < 0\}$ – the time when ruin occurs
$-X_\tau$		deficit at ruin
$X_{\tau-}$		surplus prior to ruin
W_t		standard Brownian motion
$W_t^{(\nu)}$	$=$	$\sigma W_t + \nu t$ – Brownian motion with drift $\nu \geq 0$ and scaling factor $\sigma > 0$
T_α	$=$	$\inf\{t > 0 \mid W_t^{(\nu)} = \alpha\}$ – first passage time when $W_t^{(\nu)}$ reaches the barrier level $\alpha > 0$
τ_1	$=$	$\tau = \inf\{t \geq 0 \mid X_t < 0\}$ – the time when ruin occurs

to be continued

τ_2	$=$	$\inf\{t-\tau_1 \mid t > \tau_1, X_t \geq 0, X_{\tau_1} < 0\}$ – time elapsed when X_t first goes back above zero after τ_1
g_t^X	$=$	$\sup\{s \leq t \mid \text{sign}(X_s) \neq \text{sign}(X_t)\}$ – last crossing time of 0 before time t
d_t^X	$=$	$\inf\{s \geq t \mid \text{sign}(X_s) \neq \text{sign}(X_t)\}$ – first crossing time of 0 before time t
$\tau_d^{X_t}$	$=$	$\inf\{t > 0 \mid (t - g_t^X)1_{\{X_t<0\}} \geq d\}$ – ruin time of Parisian type of ruin
U_t^X	$=$	$t - g_t^X$ – time spent in current state
$T_{i,k}^X$	$=$	$U_{d_t^X}^X = d_t^X - g_t^X$ – time spent in state i when X_t reaches the state i for the kth time, $i = 1, -1$, and $k = 1, 2, ...$
τ_0^*		stopping time at the end of current excursion above 0
$-X_0^*$		overshoot when previous ruin occurs before time τ^*
τ^*	$=$	$\inf t > 0 \mid X_t \geq 0, X_0^* = -z, z > 0$ – elapsed time when X_t goes back to 0 after previous ruin
$\{\tilde{X}_t\}_{t\geq 0}$		surplus process starting from 0
$\psi_d(x)$	$=$	$\mathbb{P}(\tau_d^{X_t} < \infty) \mid X_0 = x$ – probability of Parisian type of ruin
X_t^δ	$=$	$x + C_t - Z_t$ – surplus process with variable premium income C_t and inverse Gaussian process Z_t up to time t
C_t	$=$	$c + \delta X_t^\delta$ – variable premium income up to time t with constant $c \geq 0$ and interest rate $\delta > 0$

2.2 Miscellaneous

$\mathbb{1}_{\{x\in A\}}$	$=$	$\begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$ – indicator function
$\psi(x)$	$=$	$\mathbb{P}\Big(\inf_{t\geq 0} X_t < 0 \mid X_0 = x\Big)$ – probability of ruin
$w(X_{\tau-}, -X_\tau)$		penalty function, which is bounded and continuous
$\Phi(x,q)$	$=$	$\mathbb{E}\left[e^{-q\tau} w(X_{\tau_-}, -X_\tau)\mathbb{1}_{\{\tau<\infty\}}\right]$ – expected discounted penalty function
$\Phi(x)$	$=$	$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ – cumulative distribution function of standard normal distribution
$\mathcal{A}$		infinitesimal generator
$BvN(h,k;\rho)$	$=$	$\frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{k}\int_{-\infty}^{h} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\} dxdy$ – cumulative distribution function of bivariate normal distribution with correlation coefficient $\rho > 0$
$\{S_t^X\}_{t\geq 0}$	$=$	$\begin{cases} 1 & \text{if } X_t > 0, \\ -1 & \text{if } X_t < 0. \end{cases}$ – two-state semi-Markov process
$\mathcal{H}_n$	$=$	$\{S_0^X, t_0; S_1^X, t_1; ...; S_n^X, t_n\}$ – history of process S_t^X up to time t_n, $n = 0, 1, 2, ...$
$p_{i,j}$		transition density of S_t^X
$\mathbb{P}_{i,j}(t)$	$=$	$\mathbb{P}(T_{i,k}^X < t) = \int_0^t p_{i,j}(s)ds$ – ruin probability in state i
$\bar{\mathbb{P}}_{i,j}(t)$	$=$	$\mathbb{P}(T_{i,k}^X \geq t)$ – survival probability in state i
$\hat{\mathbb{P}}_{i,j}(\beta)$	$=$	$\int_0^\infty e^{-\beta t} p_{i,j}(t)dt$ – Laplace transform of $p_{i,j}(t)$
$\tilde{\mathbb{P}}_{i,j}(\beta)$	$=$	$\int_0^d e^{-\beta t} p_{i,j}(t)dt$

to be continued

A_k		the event that $\tau_d^{X_t}$ is achieved in the kth excursion in state -1
$H_m(x)$	$=$	$(-1)^m e^{\frac{x^2}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(e^{-\frac{x^2}{2}} \right)$ – Hermite polynomial

Chapter 3

Model Definition and Some Preliminaries

Ruin occurs when the surplus process of an insurance company drops below zero at some time point. Deriving the probability of ruin has been a central topic in risk theory literature since the last century. Starting from the classical collective compound Poisson risk model introduced by Cramér and Lundberg (see Cramér [11] and Lundberg [57]), there has been a range of research concentrating on ruin probability and relative ruin problems. Gerber et al. [39] first discussed the probability and severity of ruin for the classical risk model under continuous time. Given a certain initial capital, they derived explicit formula when individual claim amount follows a certain distribution. Subsequently, this work was generalized by Gerber and Dufresne [40], in which they obtained an explicit solution of ruin probability of ruin given certain initial capital when individual claim amount follows a combination of exponential distributions. Asmussen and Albrecher [2] obtained infinite time probability of ruin for the compound Poisson risk model with exponential claims, and several special cases with heavy tails distributions.

In this chapter, we introduce the risk process based on the Cramér-Lundberg risk model, which we use for the thesis. Some important mathematical tools such as infinitesimal generator and Gerber-Shiu expected discounted penalty function (Gerber and Shiu [45]) are introduced as well for subsequent studies. We show how the infinitesimal generator and Gerber-Shiu expected discounted penalty function are used in deriving the probability of ruin. Examples of the results of ruin probability with claim size following exponential distribution and a mixture of two exponential distributions are provided respectively.

3.1 Model Introduction

A basic risk surplus process, X_t, $t \geq 0$, of an insurance company is a model for the time evolution of the surplus of the company. X_t is defined as

$$X_t = x + ct - L_t, \tag{3.1}$$

where $x \geq 0$ is the initial capital, $c \geq 0$ is the constant premium rate, L_t is the accumulated sum of claims up to time t.

Our main focus of the thesis is the study of probability of ruin. The probability of ruin $\psi(x)$ with initial capital x is the probability that the surplus ever drops below zero, i.e.

$$\psi(x) = \mathbb{P}\Big(\inf_{t\geq 0} X_t < 0\Big) = \mathbb{P}\Big(\inf_{t\geq 0} X_t < 0 \mid X_0 = x\Big). \tag{3.2}$$

The probability of ruin before time T is

$$\psi(x, T) = \mathbb{P}\Big(\inf_{0\leq t\leq T} X_t < 0\Big). \tag{3.3}$$

We also refer to $\psi(x)$ and $\psi(x, T)$ as ruin probability with infinite time horizon and finite time horizon respectively. We focus on the study of ruin probability with infinite time horizon throughout the thesis. We shall assume the *net profit condition* ($ct > \mathbb{E}[L_t]$, $\forall t > 0$) for every fixed $x > 0$ to ensure that $\{X_t : t \geq 0\}$ has a positive drift, otherwise the insurance company faces ruin almost surely. From this condition, we can know that insurance companies should choose the premium ct in such a way that the condition holds. This is the way to avoid ruin occurring with probability 1.

This thesis particularly starts from considering the classical collective risk model to evaluate the wealth or surplus of an insurance company, which is modelled by the Cramér-Lundberg

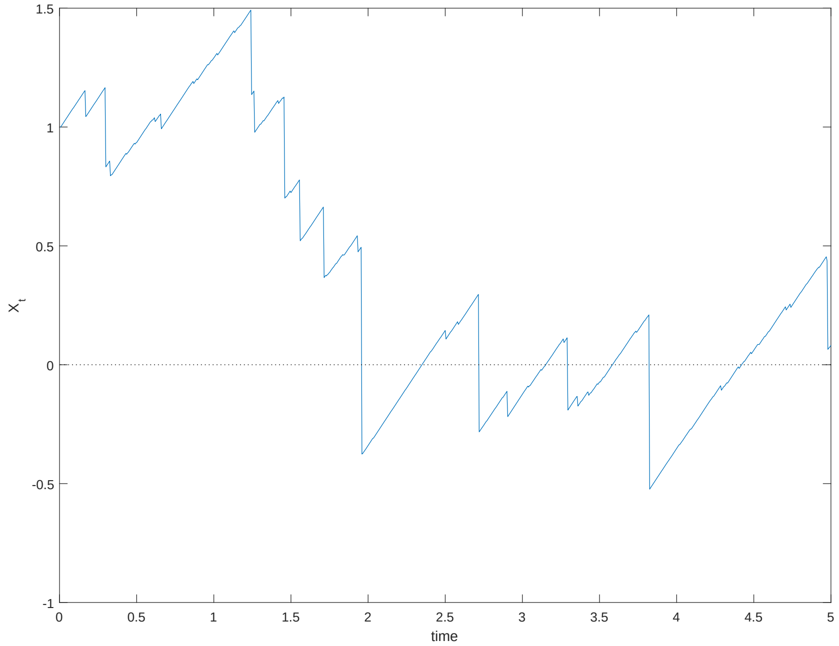


Figure 3.1: A sample path of surplus process X_t.

risk process (see Cramér [11], Lundberg [57], Asmussen [1], and Schmidli [68] for example)

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i. \tag{3.4}$$

In this classical risk process defined in (3.4), $x \geq 0$ is the initial capital, $c \geq 0$ is the constant premium rate, $Z_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process, Y_i, $i = 1, 2, ...$, are strictly positive and independent and identically distributed claim sizes with common distribution function $G(y)$. N_t is the number of claims up to time t, which is a homogeneous Poisson process with intensity λ (i.e. $N_t \sim Poisson(\lambda t)$) and is also independent of the claim Y_i. Claims occur at the arrival times $0 \leq t_1 \leq t_2 \leq ...$ of the homogeneous Poisson process.

Assume that $\mu = \mathbb{E}[Y_1]$. Since N_t and Y_i, $i = 1, 2, ...$, are independent, we can calculate the expectation of $Z_t = \sum_{i=1}^{N_t} Y_i$ as $\mathbb{E}[\sum_{i=1}^{N_t} Y_i] = \lambda t \mu$. Define the safety loading coefficient $\theta > 0$ as (e.g. see Asmussen [1])

$$\theta = \frac{ct}{\mathbb{E}[\sum_{i=1}^{N_t} Y_i]} - 1 = \frac{c - \lambda\mu}{\lambda\mu}, \tag{3.5}$$

which infers that $c = (1+\theta)\lambda\mu$. Positive safety loading coefficient $\theta > 0$ yields that $c > \lambda\mu$. In this case, there is a trend that $X_t \to +\infty$, so we say that there is a net profit condition.

The quantity X_t is actually the insurance company's capital balance at a given time t, and the process $\{X_t : t \geq 0\}$ describes the evolvement of cash-flow in the portfolio over time. The premium income ct describes the inflow of capital into the insurance company up to time t. The total claim amount $\sum_{i=1}^{N_t} Y_i$ is the outflow of capital for the company, which is due to payments for claims occurred in the time period $[0, t]$. If X_t is positive, the insurance company has gained capital. Otherwise, the insurer has lost capital.

In the Cramér-Lundberg model, one of the main goals is to analyze the probability of ruin. The ruin is said to occur if the insurer's surplus ever drops below zero. Therefore, the probability of ruin with initial capital $x \geq 0$ can be defined as

$$\psi(x) = \mathbb{P}\Big(\inf_{t\geq 0} X_t < 0 \mid X_0 = x\Big). \tag{3.6}$$

We also assume (the net profit condition) $c > \lambda\mathbb{E}[Y_1]$ for every fixed $x \geq 0$ to ensure that $\{X_t : t \geq 0\}$ has a positive drift. If $c > \lambda\mathbb{E}[Y_1]$, we may also hope that $\psi(x)$ is different from 1.

Define the stopping time

$$\tau_0 = \inf\{t \geq 0 : X_t < 0\}, \tag{3.7}$$

which denotes the first time when the process falls under the barrier level 0 ($\tau_0 = \infty$ if the set is empty). Thus the probability of ruin in infinite time horizon case is

$$\psi(x) = \mathbb{P}(\tau_0 < \infty \mid X_0 = x), \tag{3.8}$$

whereas in the finite time horizon is (i.e. ruin occurs before time t)

$$\psi(x) = \mathbb{P}(\tau_0 < t \mid X_0 = x). \tag{3.9}$$

For simplification, we denote τ the ruin time if there is no illustration of a non-zero pre-defined barrier level.

In order to calculate the probability of ruin, we introduce the Gerber and Shiu expected discounted penalty function (see Gerber and Shiu [44], [45] and [46]), which has been a standard approach to study ruin problem for the classical risk model. Let $w(X_{\tau_-}, -X_\tau)$ be a bounded continuous function for $0 \leq X_{\tau_-} < \infty$ and $0 \leq -X_\tau < \infty$. The expected discounted penalty function with $q \geq 0$ and initial capital $x \geq 0$ is defined by

$$\begin{aligned}\Phi(x, q) &= \mathbb{E}[e^{-q\tau} w(X_{\tau_-}, -X_\tau)\mathbb{1}_{\{\tau<\infty\}}] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-qt} w(y, z) f(t, y, z \mid x) dt dy dz,\end{aligned} \tag{3.10}$$

where $\mathbb{1}_A$ denotes the indicator function, i.e. $\mathbb{1}_A = 1$ if A is true and $\mathbb{1}_A = 0$ if A is false. q could be interpreted as a force of interest and $w(y, z)$ as some kind of penalty when ruin happens, thus $\Phi(x, q)$ is the expected discounted penalty. $f(t, y, z \mid x)$ is the joint density of the time of ruin τ, the surplus before ruin X_{τ_-}, and the deficit at ruin $-X_\tau$ given initial capital x.

Furthermore, if $w(y, z) = 1$, we can study the distribution of ruin time by inverting the function $\mathbb{E}[e^{-q\tau}\mathbb{1}_{\tau<\infty}]$. If we set $q = 0$ as well, then the expected penalty is just the probability of ruin with starting capital $x > 0$ in infinite time horizon, i.e.

$$\mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}] = \mathbb{P}(\tau < \infty \mid X_0 = x) = \psi(x). \tag{3.11}$$

Dassios and Embrechts [20] have shown that many important risk processes can be naturally handled within the framework of piecewise deterministic (PD) Markov processes. The class of PD Markov process was first introduced by Davis [24], which is a general class of time-homogeneous Markov processes considering deterministic motion and random jumps but no diffusion.

In Dassios and Embrechts [20], a PD process follows a deterministic path and it is determined by a first order differential operator $\mathcal{X}$, until there is a jump, based on an intensity function $\lambda(x)$ or when the process reaches the boundary $\partial\Gamma$ of Γ, and a jump measure $Q(x, B)$, $x \in \Gamma$, $B \in \mathcal{B}(\Gamma)$. The operator $\mathcal{X}$ can be seen as the generator between jumps. $\lambda(x)dt$ is the probability that there is a jump in the time interval $(t, t + dt]$ when $X_t = x$. $Q(x, B)$ is the probability that a point in B is led by a jump. For all functions f defined in the domain of

$\mathcal{A}$, the generator $\mathcal{A}$ is given by (see Dassios and Embrechts [20], pp. 185)

$$\mathcal{A}f(x) = \mathcal{X}f(x) + \lambda(x)\int_{\Gamma}[f(y) - f(x)]Q(x, dy), \tag{3.12}$$

where $\mathcal{X} = \sum_{k=1}^{n} c_k(x)\frac{\partial}{\partial x_k}$. If jumps occurs due to the hitting of the boundary $\partial\Gamma$, f must satisfy the condition

$$f(x) = \int_{\Gamma}[f(y) - f(x)]Q(x, dy), \quad \forall\, x \in \partial\Gamma, \tag{3.13}$$

in order to belong to the domain of $\mathcal{A}$. Therefore, by Dynkin's theorem, if f belongs to the domain of $\mathcal{A}$ and $\mathcal{A}f = 0$, $f(x)$ is a martingale. For more details on generators of piecewise deterministic Markov processes, we refer to Davis [24], Dassios and Embrechts [20], Davis [26] and Rolski et al. [66].

In the thesis, we assume that there is time t as an explicit component of the PD process, in which case that $\mathcal{A}$ can be decomposed as $\frac{\partial}{\partial t} + \mathcal{A}_t$, and $\mathcal{A}_t$ is given in (3.12). We start from the Cramér-Lundberg risk process $\{X_t : t \geq 0\}$ defined in (3.4) as our PD process. Consider the function $f(x, t)$

$$f(x, t) = \mathbb{E}[e^{-\beta\tau}e^{-\kappa(X_{\tau-})-\nu(-X_\tau)}\mathbb{1}_{\{\tau<\infty\}} \mid X_t = x], \tag{3.14}$$

where $\beta \geq 0$, $\kappa \geq 0$, and $\nu \geq 0$. Therefore the infinitesimal generator $\mathcal{A}$ acting on $f(x, t)$ for the surplus process $\{X_t : t \geq 0\}$ becomes

$$\begin{aligned}\mathcal{A}f(x, t) = {} & \frac{\partial f(x, t)}{\partial t} + c\frac{\partial f(x, t)}{\partial x} \\ & + \lambda\left(\int_0^x f(x - y, t)dG(y) + e^{-\beta t}\int_x^{\infty} e^{-\kappa x-\nu(y-x)}dG(y) - f(x, t)\right).\end{aligned} \tag{3.15}$$

The continuous bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, which is defined on the state space of a piecewise-deterministic process. The infinitesimal generator $\mathcal{A}$ is the operator that makes the function f belong to its domain, thus

$$f(X_t, t) - \int_0^t \mathcal{A}f(X_s, s)ds, \tag{3.16}$$

is a martingale. Therefore, solving $\mathcal{A}f = 0$ gives us the martingale $f(X_t, t)$.

Consider the function $f(x)$

$$f(x) = \mathbb{E}[e^{-\beta\tau} e^{-\kappa(X_{\tau-})-\nu(-X_\tau)} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x]. \tag{3.17}$$

It is clear to see that when $\beta = 0$, $\kappa = 0$ and $\nu = 0$, $f(x)$ becomes the probability of ruin given initial capital $x \geq 0$, i.e.

$$f(x) = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}} | X_0 = x] = \mathbb{P}(\tau < \infty | X_0 = x). \tag{3.18}$$

To find the formula of $f(x)$, we can apply Laplace transform of a function $f(x)$ on the positive real line defined by

$$\mathcal{L}\{f(x)\} = \hat{f}(\xi) = \int_0^\infty e^{-\xi x} f(x) dx, \tag{3.19}$$

to $\mathcal{A}f(x,t) \equiv 0$ in order to obtain the Laplace transform $\hat{f}(\xi)$. Notice that $G(y) = \int_0^\infty g(y)dy$ and by applying change of variable $x - y = r$, the Laplace transform of $\int_0^x f(x-y,t)dG(y)$ is

$$\begin{aligned}
&\int_0^\infty e^{-\xi x} \int_0^x f(x-y,t)\ dG(y)dx \\
&= \int_0^\infty \int_0^x e^{-\xi x} f(x-y,t) g(y)\ dydx \\
&= \int_0^\infty g(y) \left(\int_0^\infty f(r) e^{-\xi(y+r)}\ dr \right) dy \\
&= \int_0^\infty e^{-\xi r} f(r) dr \int_0^\infty e^{-\xi y} g(y) dy \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned} \tag{3.20}$$

Meanwhile, the Laplace transform of $\int_x^\infty e^{-\kappa x-\nu(y-x)}dG(y)$ is

$$
\begin{aligned}
&\int_0^\infty e^{-\xi x}\int_x^\infty e^{-\kappa x-\nu(y-x)}\ dG(y)dx\\
=\ &\int_0^\infty e^{-(\xi+\kappa-\nu)x}\int_x^\infty e^{-\nu y}g(y)dy\\
=\ &-\frac{1}{\xi+\kappa-\nu}e^{-(\xi+\kappa-\nu)x}\int_x^\infty e^{-\nu y}g(y)dy\Big|_0^\infty+\frac{1}{\xi+\kappa-\nu}\int_0^\infty e^{-(\xi+\kappa-\nu)x}d\left(\int_x^\infty e^{-\nu y}g(y)dy\right)\\
=\ &\frac{1}{\xi+\kappa-\nu}\int_0^\infty e^{-\nu y}g(y)dy-\frac{1}{\xi+\kappa-\nu}\int_0^\infty e^{-(\xi+\kappa-\nu)x}e^{-\nu x}g(x)dx\\
=\ &\frac{\hat{g}(\nu)-\hat{g}(\xi+\kappa)}{\xi+\kappa-\nu}.
\end{aligned}
\tag{3.21}
$$

Therefore, given the generator in (3.15) and $f(x,t)=e^{-\beta t}f(x)$ with $\beta\geq 0$, applying Laplace transform to $\mathcal{A}f(x,t)\equiv 0$ gives us

$$
-\beta\hat{f}(\xi)+c\xi\hat{f}(\xi)-cf(0)+\lambda\hat{f}(\xi)\hat{g}(\xi)+\lambda\frac{\hat{g}(\nu)-\hat{g}(\xi+\kappa)}{\xi+\kappa-\nu}-\lambda\hat{f}(\xi)=0,\tag{3.22}
$$

which infers that

$$
\hat{f}(\xi)=\frac{cf(0)-\lambda\frac{\hat{g}(\nu)-\hat{g}(\xi+\kappa)}{\xi+\kappa-\nu}}{c\xi-\beta-\lambda(1-\hat{g}(\xi))}.\tag{3.23}
$$

Thus $f(x)$ can be solved by inverting $\hat{f}(\xi)$ with respect to ξ. Explicit formula of $f(x)$ can be derived for the cases of claim sizes following certain distributions.

We can see that the probability of ruin will converges to zero as the future surplus increases to infinity. Also by the final value theorem we have that $\lim_{x\to\infty}f(x)=\lim_{\xi\to 0}\xi\hat{f}(\xi)=0$. Therefore, in order to ensure the later limitation exists, we let both the denominator and the numerator of $\hat{f}(\xi)$ go to zero, which enable us to find the probability of ruin.

There is also a famous result, Cramér-Lundberg approximation, for the estimation for the probability of ruin (see Asmussen [1] and Minkova [63]). It states that

$$
\psi(x)\sim Ce^{-\gamma x},\qquad x\to\infty,\tag{3.24}
$$

where $C=\theta\mu/(M_Y'(\gamma)-\mu(1+\theta))$, μ is the expectation of claim size distribution, θ is the safety loading coefficient. Recall that $c=(1+\theta)\lambda\mu$. $M_Y(\gamma)$ is the moment generating function of claim size distribution $G(y)$, and the constant $\gamma > 0$ is the positive solution of the Lundberg equation

$$M_Y(\gamma) - 1 = \gamma\mu(1+\theta). \tag{3.25}$$

The constant γ is called the Lundberg exponent or adjustment coefficient. Notice that when $a(x) \sim b(x)$ as $x \to \infty$, this means $\lim_{x\to\infty} a(x)/b(x) = 1$. Therefore,

$$\lim_{x\to\infty} e^{\gamma x}\psi(x) = C = \frac{\theta\mu}{M_Y'(\gamma) - \mu(1+\theta)}. \tag{3.26}$$

The asymptotic formula (3.24) for probability of ruin provides us with an exponential asymptotic estimate for the probability of ruin as $x \to \infty$. It is clear that the rate at which the ruin probabilities reduce depends on the Lundberg coefficient γ.

3.2 Ruin Probabilities with Exponentially Distributed Claim

In this subsection, we study the problem of ruin when the claim sizes follow an exponential distribution and a mixture of two exponential distributions through the infinitesimal generator and Gerber-Shiu expected discounted penalty function.

3.2.1 Exponential Claims

Theorem 3.2.1. *Given the classical risk process defined in (3.4), assume that the claims have an exponential distribution $Y_i \sim Exp(\alpha)$ with rate $\alpha > 0$ and finite moments, i.e.*

$$g(y) = \alpha e^{-\alpha y}, \tag{3.27}$$

then the joint Laplace transform of the ruin time τ and the overshoot $-X_\tau$ with initial capital x is given by

$$\mathbb{E}[e^{-\beta\tau - \nu(-X_\tau)}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = \frac{\lambda\alpha}{c(\alpha+\nu)(\alpha + r_\beta^+)} e^{r_\beta^- x}, \tag{3.28}$$

where $r_\beta^+ > 0$, $r_\beta^- < 0$, *and*

$$r_\beta^\pm = \frac{\beta + \lambda - c\alpha \pm \sqrt{(c\alpha - \beta - \lambda)^2 + 4c\alpha\beta}}{2c}. \tag{3.29}$$

Proof. We first set $\kappa = 0$ in $f(x)$ defined in (3.17), i.e. $f(x) = \mathbb{E}[e^{-\beta\tau - \nu(-X_\tau)}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x]$. Then applying Laplace transform to $\mathcal{A}f(x,t) = 0$ in order to obtain the Laplace transform of $f(x)$. Thus, we have

$$\hat{f}(\xi) = \frac{cf(0) - \lambda\frac{\hat{g}(\nu)-\hat{g}(\xi)}{\xi-\nu}}{c\xi - \beta - \lambda(1-\hat{g}(\xi))}. \tag{3.30}$$

Consider the equation in terms of ξ,

$$\lambda\hat{g}(\xi) = -c\xi + \beta + \lambda, \tag{3.31}$$

with $\hat{g}(\xi) = \frac{\alpha}{\alpha+\xi}$. It is obvious to see that the equation has a positive r_β^+ and a negative root r_β^- since the discriminant is positive.

Second, find $f(0)$. To do this, plug the positive root r_β^+ in the numerator, we have

$$f(0) = \frac{\lambda\alpha}{c(\alpha+\nu)(\alpha+r_\beta^+)}, \tag{3.32}$$

which is just the double Laplace transform of the ruin time τ and the overshoot $-X_\tau$ with initial capital $X_0 = 0$.

Third, we substitute $f(0)$ to the Laplace transform $\hat{f}(\xi)$ and rewrite $\hat{f}(\xi)$ as

$$\hat{f}(\xi) = \frac{\lambda\alpha}{c(\alpha+\nu)(\alpha+r_\beta^+)}\frac{1}{\xi - r_\beta^-} = \frac{f(0)}{\xi - r_\beta^-}. \tag{3.33}$$

Then, inverting $\hat{f}(\xi)$ with respect to ξ gives us the joint distribution of the ruin time τ and the overshoot $-X_\tau$ with initial capital x. ∎

Corollary 3.2.2. *The Laplace transform of ruin time τ with initial capital x is given by*

$$\mathbb{E}[e^{-\beta\tau}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = \frac{\lambda}{c(\alpha + r_\beta^+)} e^{r_\beta^- x}. \tag{3.34}$$

Corollary 3.2.3. *If we set $\beta = 0$ and $\nu = 0$ in (3.28), we get the probability of ruin for initial capital $x > 0$*

$$\psi(x) = \psi(0)e^{-Rx}, \tag{3.35}$$

where $\psi(0) = \frac{\lambda}{c\alpha}$ is the ruin probability with zero initial capital, $R = \alpha - \frac{\lambda}{c}$ is the positive solution of the Lundberg equation.

Corollary 3.2.3 gives us the probability of ruin when the claim size follows an exponential distribution as we know from Gerber & Shiu ([45]), which shows that the ruin probability decays exponentially fast as the initial capital $x \to \infty$.

3.2.2 A Mixture of Two Exponentials Case

We study the problem of ruin when the claim sizes follow a mixture of two exponential distributions, and give the formula of probability of ruin. It is shown that the formula of the ruin probability is similar to the previous exponential case.

Theorem 3.2.4. *Consider the risk process defined in (3.4), assume that the claim size follows mixed two exponential distributions with rates $\alpha_1 > 0$ and $\alpha_2 > 0$, and with weights $0 < p < 1$ and $0 < 1 - p < 1$ respectively, i.e.*

$$g(y) = p\alpha_1 e^{-\alpha_1 y} + (1-p)\alpha_2 e^{-\alpha_2 y}. \tag{3.36}$$

The joint Laplace transform of the ruin time τ and the overshoot $-X_\tau$ with initial capital $x > 0$ is given by

$$\mathbb{E}[e^{-\beta\tau - \nu(-X_\tau)}\mathbb{I}_{\tau<\infty} \mid X_0 = x] = Ae^{r_\beta^- x} + Be^{r_\beta^{--} x}, \tag{3.37}$$

where

$$A = \frac{\lambda}{c(r_\beta^- - r_\beta^{--})}\left\{p\frac{\alpha_1(\alpha_2 + r_\beta^-)}{(\alpha_1+\nu)(\alpha_1+r_\beta^+)} + (1-p)\frac{\alpha_2(\alpha_1+r_\beta^-)}{(\alpha_2+\nu)(\alpha_2+r_\beta^+)}\right\}, \tag{3.38}$$

$$B = \frac{\lambda}{c(r_\beta^{--} - r_\beta^{-})}\left\{p\frac{\alpha_1(\alpha_2 + r_\beta^{--})}{(\alpha_1+\nu)(\alpha_1+r_\beta^+)} + (1-p)\frac{\alpha_2(\alpha_1+r_\beta^{--})}{(\alpha_2+\nu)(\alpha_2+r_\beta^+)}\right\}, \tag{3.39}$$

$r_\beta^+ > 0$, $r_\beta^- < 0$, *and* $r_\beta^{--} < 0$ *are the three roots of the following Lundberg's fundamental equation in terms of* ξ

$$\lambda\hat{g}(\xi) = -c\xi + \beta + \lambda, \tag{3.40}$$

and

$$\hat{g}(\xi) = p\frac{\alpha_1}{(\alpha_1+\xi)} + (1-p)\frac{\alpha_2}{(\alpha_2+\xi)}. \tag{3.41}$$

Proof. We follow the steps in Theorem 3.2.1. Note that given $\hat{g}(\xi)$ in (3.41), it leads the Lundberg equation (3.40) to be a cubic equation with three roots $r_\beta^+ > 0$, $r_\beta^- < 0$ and $r_\beta^{--} < 0$ with $-r_\beta^+ < 0 < -r_\beta^- < \alpha_1 < -r_\beta^{--} < \alpha_2$. To see this, we assume $-\alpha_2 < -\alpha_1$ and

$$h(\xi) = c\xi - \beta - \lambda + \lambda p\frac{\alpha_1}{(\alpha_1+\xi)} + \lambda(1-p)\frac{\alpha_2}{(\alpha_2+\xi)}. \tag{3.42}$$

Note that $h(\xi) \to -\infty$ as $\xi \to -\alpha_1^-$, and $h(\xi) \to +\infty$ when $\xi \to -\alpha_2^+$, which yield there exists a negative root r_β^{--} in the interval $(-\alpha_2, -\alpha_1)$. Meanwhile, $h(\xi) \to +\infty$ as $\xi \to -\alpha_1^+$ and $h(0) = -\beta < 0$ infer that a negative root r_β^- exists in the interval $(-\alpha_1, 0)$. Also note that $h(0) = -\beta < 0$ and $h(\xi) \to +\infty$ as $\xi \to +\infty$ lead that there is a positive root $r_\beta^+ > 0$.

In order to find $f(0)$, plug the positive root r_β^+ in the numerator, i.e.

$$f(0) = \frac{\lambda}{c}\left(p\frac{\alpha_1}{(\alpha_1+\nu)(\alpha_1+r_\beta^+)} + (1-p)\frac{\alpha_2}{(\alpha_2+\nu)(\alpha_2+r_\beta^+)}\right). \tag{3.43}$$

Substitute $f(0)$ in the Laplace transform $\hat{f}(\xi)$, we have

$$\hat{f}(\xi) = \frac{A}{\xi - r_\beta^-} - \frac{B}{\xi - r_\beta^{--}}, \tag{3.44}$$

so inverting $\hat{f}(\xi)$ with respect to ξ provides us with joint Laplace transform. ■

We could also obtain the corresponding probability of ruin.

Corollary 3.2.5. *By setting $\beta = 0$ and $\nu = 0$ in (3.37), it provides us with the probability of ruin with initial capital $X_0 = x$, $x > 0$, i.e.*

$$\psi(x) = \frac{\lambda\mu}{c(r^+ - r^-)}\Big\{(1-q)\alpha_1 + q\alpha_2 - r^-\Big\}e^{-r^- x} + \frac{\lambda\mu}{c(r^+ - r^-)}\Big\{r^+ - (1-q)\alpha_1 - q\alpha_2\Big\}e^{-r^+ x}, \tag{3.45}$$

where

$$\mu = \mathbb{E}[Y_1] = \frac{p}{\alpha_1} + \frac{1-p}{\alpha_2}, \tag{3.46}$$

$$r^{\pm} = \frac{\rho + (\alpha_1 + \alpha_2)\theta \pm \sqrt{(\rho + (\alpha_1 + \alpha_2)\theta)^2 - 4\alpha_1\alpha_2\theta(1+\theta)}}{2(1+\theta)}, \tag{3.47}$$

$$q = \frac{p\alpha_2}{p\alpha_2 + (1-p)\alpha_1}, \qquad \rho = \frac{(1-p)\alpha_1^2 + p\alpha_2^2}{p\alpha_2 + (1-p)\alpha_1}, \tag{3.48}$$

and $\theta > 0$ is the safety loading coefficient with $c > \lambda\mu$.

It is clear that the ruin probability also decays exponentially as the initial capital $x \to \infty$.

Notice that we assume the jump size Y_i is exponentially distributed with rate $\alpha > 0$, it is well-known that the overshoots also have exponential distribution with the same parameter α. At the moment, we suppose the claim size has an identical mixed exponentials distribution defined in (3.36) with two positive intensities α_1 and α_2. Assume that the history of the process up to time t is denoted by $\mathcal{F}_t$, ruin occurs at the time τ and the value of capital $X(\tau - 0)$ just before ruin is equal to a. Suppose the ruining claim, of size Y, is given to be larger than a. Therefore, for each $\mathcal{F}$, a, τ and y, we have

$$\begin{aligned}
&\mathbb{P}(-X_\tau > y \mid \mathcal{F}_\tau) \\
&= \mathbb{P}(Y > y + a \mid Y > a) \\
&= \frac{p\alpha_1 e^{-\alpha_1(y+a)} + (1-p)\alpha_2 e^{-\alpha_2(y+a)}}{p\alpha_1 e^{-\alpha_1 a} + (1-p)\alpha_2 e^{-\alpha_2 a}} \\
&= \frac{p}{p\alpha_1 + (1-p)\alpha_2 e^{-(\alpha_2-\alpha_1)a}}\alpha_1 e^{-\alpha_1 y} + \frac{(1-p)}{p\alpha_1 e^{-(\alpha_1-\alpha_2)a} + (1-p)\alpha_2}\alpha_2 e^{-\alpha_2 y}.
\end{aligned} \tag{3.49}$$

It is obvious that the deficit at ruin $-X_\tau$ also has a distribution of a mixture of two exponential distributions with parameters α_1 and α_2 .

When the claim size is exponentially distributed, a solution for the distribution of the ruin time τ has been known for many years. See corollary 3.2.2 and Asmussen [1] as well as Drekic & Willmot [28] for example. We now consider the case of claim size following the mixture of two exponentials.

Corollary 3.2.6. *Assume the claims have an identical mixed exponential distributions defined in (3.36). The Laplace transform of the ruin time τ is*

$$\mathbb{E}[e^{-\beta\tau}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x > 0] = \frac{e^{r_\beta^- x}}{Ee^{-r^- x} + Fe^{-r^+ x}}, \tag{3.50}$$

where r_β^- is the negative root of the following equation as formulated in theorem (3.2.4),

$$\lambda\left(p\frac{\alpha_1}{(\alpha_1+\xi)} + (1-p)\frac{\alpha_2}{(\alpha_2+\xi)}\right) = -c\xi + \beta + \lambda, \tag{3.51}$$

$$r^\pm = \frac{\rho + (\alpha_1+\alpha_2)\theta \pm \sqrt{(\rho + (\alpha_1+\alpha_2)\theta)^2 - 4\alpha_1\alpha_2\theta(1+\theta)}}{2(1+\theta)}, \tag{3.52}$$

$$E = \frac{\lambda}{c(r^+ - r^-)}\left\{p\frac{\alpha_2 + r^-}{\alpha_1 + r_\beta^-} + (1-p)\frac{\alpha_1 + r^-}{\alpha_2 + r_\beta^-}\right\}, \tag{3.53}$$

and

$$F = \frac{\lambda}{c(r^- - r^+)}\left\{p\frac{\alpha_2 + r^+}{\alpha_1 + r_\beta^-} + (1-p)\frac{\alpha_1 + r^+}{\alpha_2 + r_\beta^-}\right\}. \tag{3.54}$$

Proof. It has been shown in Gerber [43] that $e^{-\beta t + r_\beta^- X_t}$ is a martingale. So by using the optional sampling theorem to this martingale stopped at τ, we have

$$\mathbb{E}[e^{-\beta\tau + r_\beta^- X_\tau} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = e^{r_\beta^- x}. \tag{3.55}$$

Since the distribution of the overshoot $-X_\tau$ is still an exponential distribution with same parameters, and it is independent of the ruin time τ, hence

$$\mathbb{E}[e^{-\beta\tau} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = \frac{e^{r_\beta^- x}}{\mathbb{E}[e^{r_\beta^- X_\tau} \mathbb{I}_{\tau<\infty} \mid X_0 = x]}. \tag{3.56}$$

We need to find the denominator, which is just the Laplace transform of $-X_\tau$ w.r.t. r_β^-. To do this, we just need to let $\beta = 0$ and $\nu = r_\beta^-$ in Theorem (3.2.4), i.e.

$$\mathbb{E}[e^{-r_\beta^-(-X_\tau)} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = Ee^{-r^- x} + Fe^{-r^+ x}, \tag{3.57}$$

which finishes the proof. ∎

Remark. When $\beta = 0$, (3.56) becomes the probability of ruin with initial capital $x > 0$, which satisfies

$$\psi(x) = \frac{e^{r_\beta^- x}}{\mathbb{E}[e^{r_\beta^- X_\tau} \mathbb{I}_{\tau<\infty} \mid X_0 = x]}, \tag{3.58}$$

which also illustrates that the probability of ruin decays exponentially.

Chapter 4

Classical Ruin Problem with Inverse Gaussian Process

In the insurance literature, a lot of attention has been concentrated on the classical risk model in which the total claim amount process follows a compound Poisson process. Ruin probabilities and many other ruin problems such as marginal distribution and the joint distributions of the ruin time, the surplus just before ruin and the deficit at ruin have been intensively studied. A standard method to study together these fundamental ruin problems for the classical risk model is to use an expected discounted penalty function, which was introduced in chapter 3 and by Gerber and Shiu [45] as well. We refer to Lin and Willmot [55] and the references therein for a detailed study of these ruin problems.

Most risk models concentrate on the compound Poisson processes to model the total claim amount process. The interpretation is straightforward, i.e. jump times are the arrival of claims and jump sizes are the sizes of claims. However, a general Lévy process considering infinitely many and arbitrarily small jump sizes in any finite time interval is more complicated to study. Several pieces of research have incorporated general Lévy processes into insurance modelling. One example is that the risk process is perturbed by a diffusion, i.e. a diffusion part is added to the classical risk process. This model was first introduced by Gerber [36] and then generalized by many authors in recent years, e.g. Gerber and Landry [41] and Tsai and Willmot [80]. However, little research has studied the total claim amount process as a limit of the compound Poisson process, without adding any diffusion.

From this chapter, we extend the classical surplus process X_t defined in (3.4), i.e.

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i, \tag{4.1}$$

to the case of the aggregate claim process $\sum_{i=1}^{N_t} Y_i$ actually converging to an inverse Gaussian process Z_t. We first concentrate on the claim sizes Y_i following an inverse Gaussian distribution (abbreviated as $IG(\varepsilon, \mu)$) with parameters $\varepsilon > 0$ and $\mu > 0$. $IG(\varepsilon, \mu)$ has the following probability density function $g(y)$,

$$g(y) = \frac{\varepsilon}{\sqrt{2\pi y^3}} e^{-\frac{(\varepsilon - \mu y)^2}{2y}}, \tag{4.2}$$

for $y \in (0, \infty)$, and its corresponding cumulative distribution function is $G(y)$. It is then assumed that $N_t \sim Poisson\left(\frac{\lambda}{\varepsilon}\right)$. The surplus process X_t thus becomes

$$X_t = x + ct - Z_t. \tag{4.3}$$

Dufresne, Gerber and Shiu ([29]) have shown that Z_t is the limit of compound Poisson process $\sum_{i=1}^{N_t} Y_i$. We will review their method of constructing an inverse Gaussian process by the limit of compound Poisson process in the next subsection.

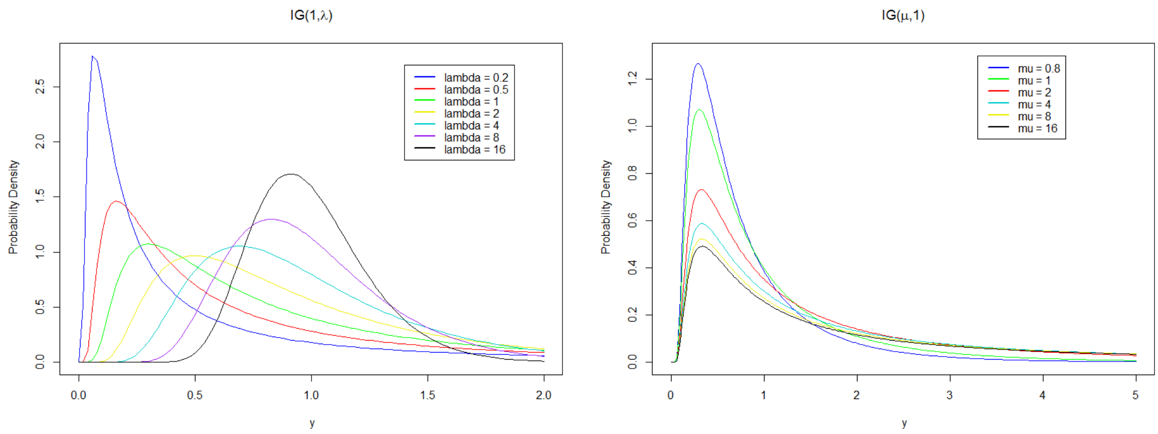
This chapter studies the Laplace transforms of the ruin time and the overshoot respectively, classical probability of ruin, and the joint distribution of the ruin time, the overshoot and initial capital. Next we introduce inverse Gaussian process.

4.1 Inverse Gaussian Process

We shall first introduce IG distribution. Generally, an IG distribution $IG(y; \tilde{\mu}, \lambda)$ is described by two parameters $\tilde{\mu} > 0$ and $\lambda > 0$, and it has support on positive axis with $y > 0$, i.e.

$$g(y) = \frac{\sqrt{\lambda}}{\sqrt{2\pi y^3}} e^{-\frac{\lambda (y - \tilde{\mu})^2}{2\tilde{\mu}^2 y}}. \tag{4.4}$$

It tends to be a Gaussian distribution as $\lambda \to \infty$. Figure 4.1 shows some IG probability densities with different values of parameters.



(a) $IG(1, \lambda)$ (b) $IG(\mu, 1)$

Figure 4.1: Inverse Gaussian densities with different parameter values.

IG distribution $IG(\frac{\alpha}{\nu}, \frac{\alpha^2}{\sigma^2})$ is also the probability density function of

$$T_\alpha = \inf\{t > 0 \mid W_t^{(\nu)} = \nu t + \sigma W_t,\ W_t^{(\nu)} = \alpha\}, \tag{4.5}$$

where W_t is the standard Brownian motion, $\nu > 0$, $\sigma > 0$, and T_a is the first hitting time when $W_t^{(\nu)}$ reaches the barrier $\alpha > 0$.

Inverse Gaussian distribution has been extensively studied by Chhikara & Folks [14], [15] and [16], Chaubey, Garrido & Trudeau [12], Seshadri [72] & [73]. Particular use of IG distribution as a life time model can be found in Gunes, etc. [47] and Singpurwalla [74]. Meanwhile, the hazard rate function of IG is uni-modal, which means it increases from zero to its maximum level and then decreases to a asymptotic constant level. IG distribution can also deal with significantly skewed data following an unclear distribution. Based on these, Chaubey, etc. [12] explains that IG distribution can provide us with a goodness fit of aggregate claims for a wide range of choices of claim distributions. Therefore, IG distribution is believed to be a strong candidate to claim size distribution, which is often used in reliability and survival analysis.

We then introduce inverse Gaussian process. Consider the IG distribution $IG(y; \varepsilon, \mu)$

defined as

$$g(y) = \begin{cases} \frac{\varepsilon}{\sqrt{2\pi y^3}} e^{-\frac{(\varepsilon-\mu y)^2}{2y}} & \text{if } y > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

which is actually the probability density function of (see Applebaum [3] and Kyprianou [50])

$$Z_\varepsilon = \inf\{t > 0 : W_t^{(\mu)} = \mu t + W_t \geq \varepsilon\}. \tag{4.7}$$

That is, Z_ε is a stopping time, and it denotes the first time when a Brownian motion with linear drift $\mu > 0$ hits the barrier level $\varepsilon > 0$. Recall Z_ε is a stopping time with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t$ is generated by $\{W_\varepsilon : \varepsilon \leq t\}$.

We also note that Brownian motion has continuous paths and $B_{Z_\varepsilon} + \mu Z_\varepsilon = \varepsilon$ almost surely. In the meantime, by the strong Markov property of Brownian motion, we also know that $\{W_{Z_\varepsilon+t} + \mu(Z_\varepsilon + t) - \varepsilon : t \geq 0\}$ and W have same law, therefore $\forall\ 0 \leq \varepsilon < t$, we have

$$Z_t = Z_\varepsilon + \tilde{Z}_{t-\varepsilon}, \tag{4.8}$$

where $\tilde{Z}_{t-\varepsilon}$ is an independent copy of $Z_{t-\varepsilon}$. This implies that the process $\{Z_t : t \geq 0\}$ has independent and stationary increments. That is, for each pair of disjoint time intervals (t_1, t_2) and (t_3, t_4) with $t_1 < t_2 < t_3 < t_4$, the random variables $Z_{t_2} - Z_{t_1}$ and $Z_{t_4} - Z_{t_3}$ are independent. Each increment $Z_{t+\varepsilon} - Z_t$ has an inverse Gaussian distribution $IG(\varepsilon, \mu)$. We also note that $Z_0 = 0$ with probability one.

Meanwhile, it is clear to see that Z_t has right continuous paths due to the continuity of paths of $\{W_t + \mu t : t > 0\}$. Also note that almost all paths of Z_t are strictly increasing with jumps, since all the sample paths of $\{W_t + \mu t : t > 0\}$ are continuous and have intervals where paths are decreasing. In other words, $t_1 < t_2 \Rightarrow Z_{t_1} < Z_{t_1}$ almost surely. According to its definition as a sequence of first passage time, Z_t is also the right inverse of the path of process $\{W_t + \mu t : t > 0\}$ from the definition of Z_ε. From this, we call Z_t inverse Gaussian process. Figure 4.2 shows a sample path of inverse Gaussian process.

It is also important to see that the random variable Z_ε is infinitely divisible for each fixed

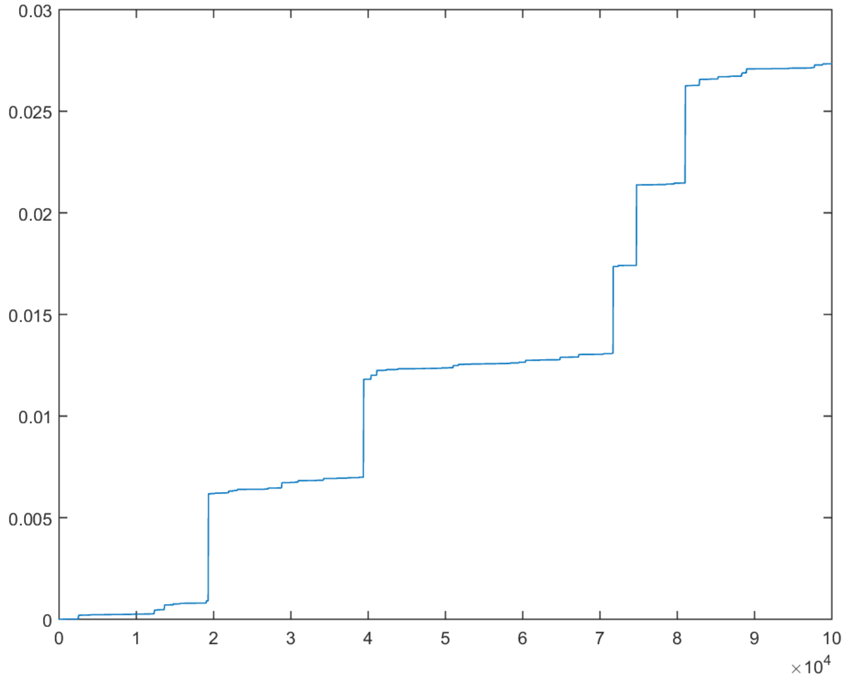


Figure 4.2: A sample path of inverse Gaussian process.

$\varepsilon > 0$ according to the following Lévy-Khintchine formula (see for example Kyprianou [50]). The characteristic exponent of Z_ε is of the form

$$\Psi(\theta) := -\log \mathbb{E}[e^{i\theta Y}] = \varepsilon(\sqrt{\mu^2 - 2i\theta} - \mu), \quad \forall\ \theta \in \mathbb{R}, \tag{4.9}$$

and the corresponding triple (a, σ, Π) is $a = -2\varepsilon\mu^{-1} \int_0^\mu (2\pi)^{-1/2} e^{-y^2/2} dy$, $\sigma = 0$, and the Lévy measure Π corresponding to the IG process Z_t is given by

$$\Pi(dx) = \frac{\varepsilon}{\sqrt{2\pi x^3}} e^{-\frac{\mu^2 x}{2}} dx, \qquad x > 0, \tag{4.10}$$

where $x \in (0, \infty)$. Thus the law of Z_ε can be explicitly calculated as

$$\mu_\varepsilon(dx) = \frac{\varepsilon}{\sqrt{2\pi x^3}} e^{-\frac{(\varepsilon - \mu x)^2}{2x}} dx, \quad \forall x > 0, \tag{4.11}$$

which is just the probability density function of $IG(\varepsilon, \mu)$.

Cont & Tankov [17] also show that

$$\int_0^\infty \Pi(dx) = \infty. \tag{4.12}$$

Equation (4.12) yields that IG process has an infinite number of jumps along any small time interval, which contributes to the infinite activity of IG process. It also infers the strictly increasing property of sample paths of Z_t.

Moreover, Dufresne, Gerber and Shiu ([29]) constructed inverse Gaussian process Z_t as the limit of compound Poisson process $\sum_{i=1}^{N_t} Y_i$. IG process is not a compound Poisson process itself, since the expected number of claims is infinite for each unit time with probability one. However, Z_t is finite in any time interval, because the majority of the claims are very small.

Dufresne, Gerber and Shiu ([29]) started the construction with defining a function Q to construct a general total claims process Z_t with independent, stationary and positive jumps. The function Q is defined as a non-negative and non-increasing function, i.e.

$$Q(x) = \int_x^\infty q(s)ds, \qquad x > 0. \tag{4.13}$$

Furthermore, it is assumed that $q(x) = -Q'(x)$, they specify Q as follows

$$\int_0^\infty xq(x)dx < \infty. \tag{4.14}$$

The process Z_t can be defined from its corresponding Laplace transform

$$\mathcal{L}_\xi\{Z_t\} = \mathbb{E}[e^{-\xi Z_t}] = e^{t\Psi(\xi)}, \tag{4.15}$$

where the exponent Ψ is recognised as the Laplace exponent of a Lévy process whose paths are of finite variation (see Bertoin [4] and Sato [67]). Ψ is given by

$$\Psi(\xi) = \int_0^\infty \left(e^{-\xi x} - 1\right)(-dQ(x)), \qquad \xi > 0. \tag{4.16}$$

$q(dx) = -dQ(x)$ is the Lévy measure of Z_t, then $Q(x)$ is the tail of the process. If $Q(0) < \infty$, Z_t is just a compound Poisson process. Otherwise ($Q(0) = \infty$), Z_t is a process with an infinite number of small claims.

When $Q(0) = \infty$, Z_t can be viewed as the limit of a compound Poisson process. From (4.14) we know that the process Z_t is of finite variation. Also from (4.15), Z_t is a Lévy process with Lévy measure $-dQ(x)$. Dufresne, Gerber and Shiu ([29]) discussed that Z_t can be embedded in a large family of process defined by

$$q(dx) = -dQ(x) = ax^{-\frac{3}{2}}e^{-\frac{x}{2}}dx, \qquad x > 0, \tag{4.17}$$

which leads to an IG process. The IG process is such that its individual distribution has an inverse Gaussian distribution.

In this thesis, we consider asymptotic results of ruin problem when $\varepsilon \to 0$. Under this setting, the intensity of the Poisson process N_t converges to infinity leading to infinite variance of Poisson process, which refers to the infinite activity of the inverse Gaussian process Z_t.

Through the method of generating IG random numbers illustrated by Michael, Schucany & Haas [61], we generate a sample trajectory path of IG process in Figure 4.2, which is approximated by a compound Poisson process with $N_t \sim (\frac{\lambda}{\varepsilon})$ and $\varepsilon \to 0$. Therefore, by the approximation from compound Poisson process, the following convergence holds in probability

$$\mathbb{P}(X_s = x + cs - \sum_{i=1}^{N_s} Y_i > 0) \ \to \ \mathbb{P}(x + cs - Z_s > 0), \qquad \forall\ s \in [0, t], \tag{4.18}$$

as $\varepsilon \to 0$.

Thus, if the claim size $Y_i \sim IG(\varepsilon, \mu)$ where $IG(\varepsilon, \mu)$ defined as in (4.6), the Laplace transform of Y_i w.r.t. ξ is given by

$$\begin{aligned} \hat{g}(\xi) = \mathbb{E}[e^{-\xi Y_i}] &= \int_0^\infty e^{-\xi y} \frac{\varepsilon}{\sqrt{2\pi y^3}} e^{-\frac{(\varepsilon - \mu y)^2}{2y}} dy \\ &= e^{-\varepsilon(\sqrt{\mu^2 + 2\xi} - \mu)}. \end{aligned} \tag{4.19}$$

From the Lévy measure $\Pi(dx)$ of the inverse Gaussian process Z_t, which is calculated in

(4.10), we also have the Laplace transform of Z_t,

$$\mathbb{E}[e^{-\xi Z_t}] = e^{-\frac{\lambda t}{\varepsilon}\int_0^\infty \left(1-e^{-\xi y}\right)\frac{\varepsilon}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2 y}{2}}dy} = e^{-\lambda t(\sqrt{\mu^2+2\xi}-\mu)}. \tag{4.20}$$

4.2 Probability of Ruin

Given the initial capital $X_0 = x$, $x > 0$, this section studies the Laplace transforms of the ruin time τ and the overshoot $-X_\tau$ respectively, and it also provides the explicit formula of the probability of ruin with different values of initial capital.

We first simplify the generator defined in (3.15). The generator actually becomes

$$\begin{aligned}
&\mathcal{A}f(x,t)\\
=&\frac{\partial f(x,t)}{\partial t}+c\frac{\partial f(x,t)}{\partial x}\\
&+\frac{\lambda}{\varepsilon}\left(\int_0^x f(x-y,t)dG(y)+e^{-\beta t}\int_x^\infty e^{-\kappa x-\nu(y-x)}dG(y)-f(x,t)\right)\\
=&\frac{\partial f(x,t)}{\partial t}+c\frac{\partial f(x,t)}{\partial x}+\frac{\lambda}{\varepsilon}\left[\int_0^x\left(f(x,t)-\int_0^y f'(x-v,t)\ dv\right)dG(y)-f(x,t)\right]\\
&+\frac{\lambda}{\varepsilon}e^{-\beta t}\int_x^\infty e^{-\kappa x-\nu(y-x)}dG(y)\\
=&\frac{\partial f(x,t)}{\partial t}+c\frac{\partial f(x,t)}{\partial x}-\frac{\lambda}{\varepsilon}f(x,t)\bar{G}(x)-\frac{\lambda}{\varepsilon}\int_0^x f'(x-v,t)(\bar{G}(v)-\bar{G}(x))\ dv\\
&+\frac{\lambda}{\varepsilon}e^{-\beta t}\int_x^\infty e^{-\kappa x-\nu(y-x)}dG(y)\\
=&\frac{\partial f(x,t)}{\partial t}+c\frac{\partial f(x,t)}{\partial x}-\frac{\lambda}{\varepsilon}f(0,t)\bar{G}(x)-\frac{\lambda}{\varepsilon}\int_0^x f'(x-v,t)\bar{G}(v)\ dv\\
&+\frac{\lambda}{\varepsilon}e^{-\beta t}\int_x^\infty e^{-\kappa x-\nu(y-x)}dG(y).
\end{aligned}\tag{4.21}$$

Substitute the probability density function $g(y)$, thus as $\varepsilon \to 0$, $\mathcal{A}f(x,t)$ becomes

$$\begin{aligned}
&\mathcal{A}f(x,t)\\
=&\frac{\partial f(x,t)}{\partial t}+c\frac{\partial f(x,t)}{\partial x}\\
&-\lambda f(0,t)\int_x^\infty\frac{1}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2}{2}y}\ dy-\lambda\int_0^x f'(x-v,t)\int_v^\infty\frac{1}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2}{2}y}\ dydv\\
&+\lambda e^{-\beta t}\int_x^\infty e^{-\kappa x-\nu(y-x)}\frac{1}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2}{2}y}\ dy.
\end{aligned}\tag{4.22}$$

We will use the generator in (4.22) to find the Laplace transforms of the ruin time and the overshoot respectively, the probability of ruin and the joint law of ruin time, overshoot and initial capital.

4.2.1 Laplace Transform of Ruin Time τ

Proposition 4.2.1. *The Laplace transform of the ruin time τ with $c = 1$, $X_0 = x$, $x > 0$ and $\beta > 0$ is given by*

$$\begin{aligned}
&\mathbb{E}[e^{-\beta\tau} \mid X_0 = x] \\
&= \frac{4\lambda}{\left(\sqrt{(\mu-\lambda)^2+2\beta}+\lambda+\mu\right)} \Bigg\{ \frac{\mu\Phi\left(\mu\sqrt{x}\right)-\mu}{\sqrt{(\mu-\lambda)^2+2\beta}-(\mu+\lambda)} \\
&\quad + \Phi\left(\sqrt{x}\left(\lambda-\sqrt{(\mu-\lambda)^2+2\beta}\right)\right) e^{-x\left(\lambda\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)-\beta\right)} \frac{\sqrt{(\mu-\lambda)^2+2\beta}-\lambda}{\sqrt{(\mu-\lambda)^2+2\beta}-(\mu+\lambda)} \Bigg\}.
\end{aligned} \tag{4.23}$$

Proof. First set $\kappa = 0$, and $\nu = 0$ in $f(x)$ defined in (3.17), i.e. $f(x) = \mathbb{E}[e^{-\beta\tau}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x]$. Next setting $f(x,t) = e^{-\beta t}f(x)$ with $\beta > 0$ and applying Laplace transform to $\mathcal{A}f(x,t) = 0$ provide us with

$$\begin{aligned}
\hat{f}(\xi) &= \frac{cf(0) - \lambda\frac{\sqrt{\mu^2+2\xi}-\mu}{\xi}}{c\xi - \beta - \lambda(\sqrt{\mu^2+2\xi}-\mu)} \\
&= \frac{cf(0) - \lambda\frac{2}{\sqrt{\mu^2+2\xi}+\mu}}{c\xi - \beta - \lambda(\sqrt{\mu^2+2\xi}-\mu)}.
\end{aligned} \tag{4.24}$$

If we use change of variable $\sqrt{\mu^2+2\xi} - \mu = \eta$, $\hat{f}(\xi)$ becomes

$$\hat{f}(\xi) = \frac{cf(0) - \frac{2\lambda}{\eta+2\mu}}{-\beta + c\frac{\eta^2}{2} + (c\mu-\lambda)\eta}. \tag{4.25}$$

Then, we need to find $f(0)$. Consider the equation

$$-\beta + c\frac{\eta^2}{2} + (c\mu - \lambda)\eta = 0, \tag{4.26}$$

which has two roots

$$\eta_\beta^\pm = \frac{\lambda - c\mu \pm \sqrt{(c\mu - \lambda)^2 + 2c\beta}}{c}. \tag{4.27}$$

Plugging the positive root η_β^+ in the numerator in $\hat{f}(\xi)$, we have

$$f(0) = \frac{2\lambda}{\lambda + c\mu + \sqrt{(c\mu - \lambda)^2 + 2c\beta}}. \tag{4.28}$$

Note that by the net profit condition we have $c\mu > \lambda$. This can be deduced from (4.25) by setting $\beta = 0$, resulting in $f(0) = \frac{\lambda}{c\mu}$ and $0 < f(0) < 1$.

Set $c = 1$ for simplicity. Plugging $f(0)$ in $\hat{f}(\xi)$ gives us

$$\begin{aligned}
&\hat{f}(\xi) \\
&= \frac{4\lambda}{(\sqrt{(\mu-\lambda)^2 + 2\beta} + \lambda + \mu)} \frac{1}{\eta + 2\mu} \frac{1}{\sqrt{(\mu-\lambda)^2 + 2\beta} + \eta + \mu - \lambda} \\
&= \frac{4\lambda}{(\sqrt{(\mu-\lambda)^2 + 2\beta} + \lambda + \mu)} \int_0^\infty \int_0^\infty e^{-(\eta+2\mu)v} e^{-(\sqrt{(\mu-\lambda)^2+2\beta}+\eta+\mu-\lambda)w} \, dvdw \\
&= \frac{4\lambda}{(\sqrt{(\mu-\lambda)^2 + 2\beta} + \lambda + \mu)} \\
&\quad \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi x} \frac{v+w}{\sqrt{2\pi x^3}} e^{-\frac{(v+w-\mu x)^2}{2x}} \, dx \; e^{-2\mu v - (\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda)w} \, dvdw,
\end{aligned} \tag{4.29}$$

inverting $\hat{f}(\xi)$ w.r.t ξ gives us

$$\begin{aligned}
&f(x) \\
&= \frac{4\lambda}{(\sqrt{(\mu-\lambda)^2 + 2\beta} + \lambda + \mu)} \\
&\quad \int_0^\infty \int_0^\infty \frac{v+w}{\sqrt{2\pi x^3}} e^{-\frac{(v+w-\mu x)^2}{2x}} \; e^{-2\mu v - (\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda)w} \, dvdw \\
&= \mathbb{E}[e^{-\beta\tau} \mid X_0 = x].
\end{aligned} \tag{4.30}$$

By solving the double integral in (4.30), we have that

$$\int_0^\infty \int_0^\infty \frac{v+w}{\sqrt{2\pi x^3}} e^{-\frac{(v+w-\mu x)^2}{2x}} \; e^{-2\mu v - (\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda)w} \; dvdw$$
$$= \frac{\mu\Phi\left(\mu\sqrt{x}\right)-\mu}{\sqrt{(\mu-\lambda)^2+2\beta}-(\mu+\lambda)}$$
$$+\Phi\left(\sqrt{x}\left(\lambda-\sqrt{(\mu-\lambda)^2+2\beta}\right)\right) e^{-x\left(\lambda\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)-\beta\right)} \frac{\sqrt{(\mu-\lambda)^2+2\beta}-\lambda}{\sqrt{(\mu-\lambda)^2+2\beta}-(\mu+\lambda)}. \tag{4.31}$$

■

4.2.2 Laplace Transform of Overshoot $-X_\tau$

In this subsection, we present an explicit formula of the Laplace transform of the overshoot $-X_\tau$ with initial capital $x > 0$.

Proposition 4.2.2. *The Laplace transform of the overshoot $-X_\tau$ for initial capital $x > 0$ in infinite time horizon is given by*

$$\begin{aligned}\mathbb{E}[e^{-\nu(-X_\tau)}\mathbb{I}_{\tau<\infty} \mid X_0 = x] &= \frac{4\lambda}{c}\frac{1}{\sqrt{\mu^2+2\nu}+\mu}\frac{1}{\sqrt{\mu^2+2\nu}-\mu+\frac{2\lambda}{c}} \\ \cdot\left\{e^{\nu x}\sqrt{\mu^2+2\nu}\bar{\Phi}\left(\sqrt{x(\mu^2+2\nu)}\right)\right. &\left.- e^{\frac{2\lambda}{c}\left(\frac{\lambda}{c}-\mu\right)x}\left(\mu-\frac{2\lambda}{c}\right)\bar{\Phi}\left(\left(\mu-\frac{2\lambda}{c}\right)\sqrt{x}\right)\right\}.\end{aligned} \tag{4.32}$$

Proof. First let $\beta = 0$, $\kappa = 0$, and $\nu > 0$ in $f(x)$ defined in (3.17). Next set $f(x,t) = e^{-\beta t} f(x)$ with $\beta > 0$ and apply Laplace transform to $\mathcal{A}f(x,t) = 0$. Notice that $\hat{g}(\xi) = e^{-\varepsilon(\sqrt{\mu^2+2\xi}-\mu)}$, we then have

$$\hat{f}(\xi) = \frac{cf(0) - \lambda\frac{\sqrt{\mu^2+2\xi}-\sqrt{\mu^2+2\nu}}{\xi-\nu}}{c\xi - \lambda(\sqrt{\mu^2+2\xi}-\mu)}. \tag{4.33}$$

By using final value theorem $\lim_{x\to\infty} f(x) = \lim_{\xi\to 0} \xi\hat{f}(\xi) = 0$, we get

$$f(0) = \frac{2\lambda}{c(\sqrt{\mu^2+2\nu}+\mu)}. \tag{4.34}$$

Plug $f(0)$ in $\hat{f}(\xi)$, we have

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{2\lambda}{\xi} \frac{\frac{1}{\sqrt{\mu^2+2\nu}+\mu} - \frac{1}{\sqrt{\mu^2+2\xi}+\sqrt{\mu^2+2\nu}}}{c - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}} \\
&= \frac{4\lambda}{c} \frac{1}{\sqrt{\mu^2+2\nu}+\mu} \left\{ \frac{1}{\sqrt{\mu^2+2\xi}+\mu-\frac{2\lambda}{c}} \frac{1}{\sqrt{\mu^2+2\xi}+\sqrt{\mu^2+2\nu}} \right\} \\
&= \frac{4\lambda}{c} \frac{1}{\sqrt{\mu^2+2\nu}+\mu} \frac{1}{\sqrt{\mu^2+2\nu}-\mu+\frac{2\lambda}{c}} \\
&\quad \left\{ \frac{1}{\sqrt{\mu^2+2\xi}-\mu+2\mu-\frac{2\lambda}{c}} - \frac{1}{\sqrt{\mu^2+2\xi}-\mu+\sqrt{\mu^2+2\nu}+\mu} \right\} \\
&= \frac{4\lambda}{c} \frac{1}{\sqrt{\mu^2+2\nu}+\mu} \frac{1}{\sqrt{\mu^2+2\nu}-\mu+\frac{2\lambda}{c}} \left\{ \hat{h}_1(\xi) - \hat{h}_2(\xi) \right\}.
\end{aligned} \tag{4.35}
$$

where the last equality is due to the linearity of Laplace transform.

Then invert $\hat{h}_1(\xi)$ and $\hat{h}_2(\xi)$ w.r.t. ξ respectively. Note that

$$
\begin{aligned}
\hat{h}_1(\xi) &= \frac{1}{\sqrt{\mu^2+2\xi}-\mu+2\mu-\frac{2\lambda}{c}} \\
&= \int_0^\infty e^{-(\sqrt{\mu^2+2\xi}-\mu)u-(2\mu-\frac{2\lambda}{c})u} du \\
&= \int_0^\infty \int_0^\infty e^{-\xi x} \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} dx e^{-(2\mu-\frac{2\lambda}{c})u} du,
\end{aligned} \tag{4.36}
$$

thus

$$
\begin{aligned}
h_1(x) &= \int_0^\infty \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} e^{-(2\mu-\frac{2\lambda}{c})u} du \\
&= \frac{1}{\sqrt{2\pi x}} e^{-\frac{\mu^2 x}{2}} - e^{\frac{2\lambda}{c}(\frac{\lambda}{c}-\mu)x} \left(\mu - \frac{2\lambda}{c}\right) \bar{\Phi}\left(\left(\mu - \frac{2\lambda}{c}\right)\sqrt{x}\right).
\end{aligned} \tag{4.37}
$$

Inverting $\hat{h}_2(\xi)$ is similar to the procedure above.

$$
\begin{aligned}
h_2(x) &= \int_0^\infty \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} e^{-(\sqrt{\mu^2+2\nu}+\mu)u} du \\
&= \frac{1}{\sqrt{2\pi x}} e^{-\frac{\mu^2}{2}x} - e^{\nu x}\sqrt{\mu^2+2\nu}\bar{\Phi}\left(\sqrt{x(\mu^2+2\nu)}\right).
\end{aligned} \tag{4.38}
$$

Finally, we have the Laplace transform of the overshoot $-X_\tau$ w.r.t. ν for initial capital

$x > 0$, i.e.

$$\begin{aligned}
&\mathbb{E}[e^{-\nu(-X_\tau)}\mathbb{I}_{\tau<\infty} \mid X_0 = x] \\
&= \frac{4\lambda}{c}\frac{1}{\sqrt{\mu^2+2\nu}+\mu}\frac{1}{\sqrt{\mu^2+2\nu}-\mu+\frac{2\lambda}{c}} \\
&\quad \cdot \left\{e^{\nu x}\sqrt{\mu^2+2\nu}\bar{\Phi}\left(\sqrt{x(\mu^2+2\nu)}\right) - e^{\frac{2\lambda}{c}(\frac{\lambda}{c}-\mu)x}\left(\mu-\frac{2\lambda}{c}\right)\bar{\Phi}\left(\left(\mu-\frac{2\lambda}{c}\right)\sqrt{x}\right)\right\}.
\end{aligned} \tag{4.39}$$

■

Corollary 4.2.3. *The probability density function of $-X_\tau$ with 0 initial capital is given by*

$$f_{-X_\tau|X_0=0}(z) = \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi z}}e^{-\frac{\mu^2}{2}z} - \frac{2\lambda\mu}{c}\bar{\Phi}(\mu\sqrt{z}). \tag{4.40}$$

Proof. The density function of $-X_\tau$ with $X_0 = 0$ can be calculated from inverting its Laplace transform, i.e.

$$f(0) = \mathbb{E}[e^{-\nu(-X_\tau)} \mid X_0 = 0] = \frac{2\lambda}{c(\sqrt{\mu^2+2\nu}+\mu)}. \tag{4.41}$$

Rewrite the Laplace transform as

$$\begin{aligned}
\mathbb{E}[e^{-\nu(-X_\tau)} \mid X_0 = 0] &= \frac{2\lambda}{c}\int_0^\infty e^{-(\sqrt{\mu^2+2\nu}+\mu)u}du \\
&= \frac{2\lambda}{c}\int_0^\infty \int_0^\infty e^{-\nu z}\frac{u}{\sqrt{2\pi z^3}}e^{-\frac{(u-\mu z)^2}{2z}}dz e^{-2\mu u}du,
\end{aligned} \tag{4.42}$$

and then invert it we have

$$\begin{aligned}
f_{-X_\tau|X_0=0}(z) &= \frac{2\lambda}{c}\int_0^\infty \frac{u}{\sqrt{2\pi z^3}}e^{-\frac{(u-\mu z)^2}{2z}}e^{-2\mu u}du \\
&= \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi z}}e^{-\frac{\mu^2}{2}z} - \frac{2\lambda\mu}{c}\bar{\Phi}(\mu\sqrt{z}).
\end{aligned} \tag{4.43}$$

■

4.2.3 Probability of Ruin

Theorem 4.2.4. *Given the total claim amount process following an inverse Gaussian process, the probability of ruin with initial capital $x > 0$ in infinite time horizon is given by*

$$\psi(x) = \bar{\Phi}(\mu\sqrt{x}) - e^{\frac{2\lambda}{c}\left(\frac{\lambda}{c}-\mu\right)x}\left(1 - \frac{2\lambda}{c\mu}\right)\bar{\Phi}\left(\left(\mu - \frac{2\lambda}{c}\right)\sqrt{x}\right), \tag{4.44}$$

where $\Phi(x)$ is the cumulative distribution function for the standard normal distribution, $\bar{\Phi}(x) = 1 - \Phi(x)$.

Proof. Set $\beta = 0$ and apply Laplace transform to $\mathcal{A}f(x,t) \equiv 0$, we have

$$\hat{f}(\xi) = \frac{cf(0) - \lambda \frac{\sqrt{\mu^2+2(\xi+\kappa)}-\sqrt{\mu^2+2\nu}}{\xi+\kappa-\nu}}{c\xi - \lambda\left(\sqrt{\mu^2+2\xi}-\mu\right)}. \tag{4.45}$$

And set $\kappa = 0$ and $\nu = 0$, then we have

$$\hat{f}(\xi) = \frac{cf(0) - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}}{\xi\left(c - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}\right)}. \tag{4.46}$$

Notice that when we set $\beta = 0$, $\kappa = 0$ and $\nu = 0$ in (3.17), i.e.

$$f(x) = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x] = \mathbb{P}(\tau < \infty \mid X_0 = x) = \psi(x), \tag{4.47}$$

which shows that the function $f(x)$ just becomes the probability of ruin $\psi(x)$. We use $f(x)$ for the sake of simplicity.

Equating the denominator to zero, i.e

$$\xi\left(c - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}\right) = 0, \tag{4.48}$$

which has one zero root and one negative root w.r.t. ξ. Then plug the zero root in the numerator of $\hat{f}(\xi)$, we have the probability of ruin with zero capital $f(0) = \frac{\lambda}{c\mu}$. $f(0)$ can be obtained from the final value theorem $\lim_{x\to\infty} f(x) = \lim_{\xi\to 0} \xi\hat{f}(\xi) = 0$ as well. Due to the *net profit condition*, it is easy to check that $0 < f(0) < 1$, which means that if an insurance company starts with zero capital it would not ruin with probability one immediately. Hence, we have $c\mu > \lambda$.

Substituting $f(0) = \frac{\lambda}{c\mu}$ in $\hat{f}(\xi)$, we have

$$\begin{aligned} \hat{f}(\xi) &= \frac{\frac{\lambda}{\mu} - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}}{\xi\left(c - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}\right)} \\ &= \frac{1}{\mu}\left(\frac{1}{\sqrt{\mu^2+2\xi}+\mu-\frac{2\lambda}{c}} - \frac{1}{\sqrt{\mu^2+2\xi}+\mu}\right). \end{aligned} \tag{4.49}$$

Due to the linearity of Laplace transform, $\hat{f}(\xi)$ can be inverted through two parts, i.e.

$$\begin{aligned} \hat{g}(\xi) &= \frac{1}{\sqrt{\mu^2+2\xi}+\mu-\frac{2\lambda}{c}} \\ &= \int_0^\infty \int_0^\infty e^{-\xi x} \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} dx e^{-2(\mu-\frac{\lambda}{c})u} du \\ &= \int_0^\infty e^{-\xi x} \left(\int_0^\infty \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} e^{-2(\mu-\frac{\lambda}{c})u} du\right) dx, \end{aligned} \tag{4.50}$$

then inverting $\hat{g}(\xi)$ yields that

$$\begin{aligned} & g(x) \\ &= \left(\int_0^\infty \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} e^{-2(\mu-\frac{\lambda}{c})u} du\right) \\ &= \exp\left\{\frac{2\lambda^2 x - 2\mu\lambda x c}{c^2}\right\}\left\{\frac{1}{\sqrt{x}}\exp\left(-\frac{(\mu-\frac{2\lambda}{c})^2 x}{2}\right) + \left(\mu - \frac{2\lambda}{c}\right)\left(1 - \Phi\left(\sqrt{x}(\mu - \frac{2\lambda}{c})\right)\right)\right\} \\ &= \frac{1}{\sqrt{2\pi x}} e^{-\frac{\mu^2 x}{2}} - e^{\frac{2\lambda}{c}(\frac{\lambda}{c}-\mu)x}\left(\mu - \frac{2\lambda}{c}\right)\bar{\Phi}\left(\left(\mu - \frac{2\lambda}{c}\right)\sqrt{x}\right). \end{aligned} \tag{4.51}$$

Similarly, we can calculate the second part of the inverse of Laplace transform in terms of ξ,

$$\begin{aligned} h(x) &= L^{-1}\left\{\frac{1}{\sqrt{\mu^2+2\xi}+\mu}\right\} \\ &= \int_0^{\infty} \frac{u}{\sqrt{2\pi x^3}} e^{-\frac{(u-\mu x)^2}{2x}} e^{-2\mu u} du \\ &= \frac{1}{\sqrt{2\pi x}} e^{-\frac{\mu^2 x}{2}} - \mu\left(1-\Phi(\mu\sqrt{x})\right), \end{aligned} \tag{4.52}$$

which finishes the proof. ■

We carry out the asymptotic numerical evaluation of the probability of ruin with varying initial capital $x > 0$ derived in (4.44) in infinite time horizon. The numerical results are shown in Table 4.1. Parameters are set as $\lambda = 1.5$, $\mu = 1$. Different values for the premium rate $c > 0$ are shown.

	$c = 1.65$	$c = 1.95$	$c = 2.25$	$c = 2.4$
x	$\psi(x)$	$\psi(x)$	$\psi(x)$	$\psi(x)$
0.1	0.86045	0.67089	0.54872	0.50271
5	0.35866	0.09349	0.04056	0.02975
10	0.15670	0.01556	0.00413	0.00259
20	0.03000	0.00044	4.67162e-05	2.22824e-05
50	0.00021	1.05144e-08	7.45373e-11	1.66941e-11

Table 4.1: Infinite time ruin probabilities.

As we can see from Table 4.1, the probability of ruin in infinite time horizon decreases when the initial capital grows with same safety loading coefficient, and reduces when the safety loading coefficient increases with same initial capital. Intuitively, the insurance company is less likely to ruin with larger initial capital and higher values of safety loading coefficient.

Remark. The ruin probability $\psi(0)$ can be obtained by plugging 0 in $\psi(x)$, which gives us $\psi(0) = \frac{\lambda}{c\mu}$.

4.3 Joint Distributions of Ruin Time, Overshoot and Initial Capital

In this subsection, we discuss the joint distribution of ruin time τ and overshoot $-X_\tau$ given zero initial capital, and the joint distribution of the ruin time, the overshoot and the non-zero initial capital.

4.3.1 Case $X_0 = 0$

Theorem 4.3.1. *Consider the risk process defined in (4.3), the joint probability density function of the ruin time τ and the overshoot at ruin $-X_\tau$ with zero initial capital is given by*

$$f(t,y) = \frac{\lambda}{c} \frac{1}{\sqrt{2\pi(t+y)^3}} e^{-\frac{\mu^2(t+y)}{2}}. \tag{4.53}$$

Proof. Applying Laplace transform to the generator $\mathcal{A}f(x,t)$ defined in (4.22) and setting $\mathcal{A}f(x,t) \equiv 0$ infer that,

$$\hat{f}(\xi) = \frac{cf(0) - \lambda \frac{\sqrt{\mu^2+2(\xi+\kappa)} - \sqrt{\mu^2+2\nu}}{\xi+\kappa-\nu}}{c\xi - \beta - \lambda\left(\sqrt{\mu^2+2\xi} - \mu\right)}. \tag{4.54}$$

By the final value theorem and setting $\xi \to 0$ yield that

$$f(0) = \frac{2\lambda}{c} \frac{1}{\sqrt{\mu^2+2\xi} + \sqrt{\mu^2+2\nu}}. \tag{4.55}$$

Furthermore, by $\kappa = 0$ and $X_0 = 0$ from (3.17), we have

$$f(0) = \mathbb{E}[e^{-\beta\tau} e^{-\nu(-X_\tau)} \mathbb{I}_{\tau<\infty} | X_0 = 0], \tag{4.56}$$

which is just the double Laplace transform of τ and $-X_\tau$ given zero initial capital. In order to calculate their joint probability density function, we thus invert $f(0)$ in (4.55) w.r.t. ξ and

ν simultaneously, i.e.

$$\begin{aligned}
f(0) &= \frac{2\lambda}{c}\int_0^\infty e^{-2\mu u}e^{-\left(\sqrt{\mu^2+2\xi}-\mu\right)u}e^{-\left(\sqrt{\mu^2+2\nu}-\mu\right)u}du \\
&= \frac{2\lambda}{c}\int_0^\infty e^{-2\mu u}\left(\int_0^\infty e^{-\xi t}\frac{u}{\sqrt{2\pi t^3}}e^{-\frac{(u-\mu t)^2}{2t}}dt\right) \\
&\quad \left(\int_0^\infty e^{-\nu y}\frac{u}{\sqrt{2\pi y^3}}e^{-\frac{(u-\mu y)^2}{2y}}dy\right)du \qquad (4.57)\\
&= \frac{2\lambda}{c}\int_0^\infty\int_0^\infty e^{-\xi t}e^{-\nu y}\left(\int_0^\infty e^{-2\mu u}\frac{u}{\sqrt{2\pi t^3}}e^{-\frac{(u-\mu t)^2}{2t}}\frac{u}{\sqrt{2\pi y^3}}e^{-\frac{(u-\mu y)^2}{2y}}du\right)dtdy \\
&= \mathcal{L}_\xi\mathcal{L}_\nu\{f(t,y)\},
\end{aligned}$$

Note that $t > 0$ and $y > 0$ denote the ruin time τ and the overshoot at ruin $-X_\tau$ respectively, therefore (4.57) is also the double Laplace transform w.r.t. τ and $-X_\tau$. Invert it we obtain their joint density function

$$\begin{aligned}
f(t,y) &= \int_0^\infty \frac{2\lambda}{c}e^{-2\mu u}\frac{u}{\sqrt{2\pi t^3}}e^{-\frac{(u-\mu t)^2}{2t}}\frac{u}{\sqrt{2\pi y^3}}e^{-\frac{(u-\mu y)^2}{2y}}du \\
&= \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi(t+y)}^3}e^{-\frac{\mu^2(t+y)}{2}}\int_0^\infty\frac{1}{\sqrt{2\pi}}z^2e^{-\frac{z^2}{2}}dz \qquad (4.58)\\
&= \frac{\lambda}{c}\frac{1}{\sqrt{2\pi(t+y)}^3}e^{-\frac{\mu^2(t+y)}{2}}.
\end{aligned}$$

■

Corollary 4.3.2. *Due to the symmetry of τ and $-X_\tau$ from their joint density function above, τ and $-X_\tau$ have the identical density formula given by*

$$f_{\tau|X_0=0}(t) = \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi t}}e^{-\frac{\mu^2}{2}t} - \frac{2\lambda\mu}{c}\bar{\Phi}\left(\mu\sqrt{t}\right). \qquad (4.59)$$

4.3.2 Case $X_0 > 0$

Next, we discuss the joint distribution of the ruin time τ, the overshoot at ruin $-X_\tau$ and any non-zero initial capital $x_0 > 0$.

Theorem 4.3.3. *The joint probability density function of the ruin time τ, the overshoot at ruin $-X_\tau$ and the initial capital $X_0 = x_0$, $x_0 > 0$ is given by*

$$\begin{aligned} &f(t,z,x_0) \\ &= 2\lambda t \frac{1}{\sqrt{2\pi t^3}} \frac{1}{\sqrt{2\pi x_0^3}} \frac{1}{\sqrt{2\pi z^3}} e^{-\frac{(\mu-\lambda)^2}{2}t} e^{-\frac{\mu^2}{2}x_0} e^{-\frac{\mu^2}{2}z} \sqrt{\left(\frac{tz}{t+z}\right)^3} \\ &\cdot \left\{ e^{A_2}(H_1(t,z,x_0) + H_2(t,z,x_0)) - e^{A_1}(H_3(t,z,x_0) + H_4(t,z,x_0)) \right\}, \end{aligned} \tag{4.60}$$

where

$$A_1 = exp\left\{ \frac{x_0(t(\lambda-\mu) - 2z\mu)^2}{2(t+z)(x_0+t+z)} + \frac{tz(\mu+\lambda)^2}{2(t+z)} \right\}, \tag{4.61}$$

$$A_2 = exp\left\{ \frac{x_0(t(\lambda-\mu) + 2z\lambda)^2}{2(t+z)(x_0+t+z)} + \frac{tz(\mu+\lambda)^2}{2(t+z)} \right\}. \tag{4.62}$$

Define

$$a = \frac{tz}{t+z}, \qquad c = \frac{x_0(t+z)}{t+x_0+z}, \tag{4.63}$$

$$d_1 = \frac{x_0(t\lambda - t\mu - 2z\mu)}{t+x_0+z}, \qquad d_2 = \frac{x_0(t\lambda - t\mu + 2z\lambda)}{t+x_0+z}, \tag{4.64}$$

then the functions $H_i, i = 1, ..., 4$ can be formulated as

$$\begin{aligned} H_1(t,z,x_0) = &-\frac{1}{t}\sqrt{\frac{ac}{(ac+t^2)^3}} \left(t^3(\mu+\lambda) + acd_2 + 2d_2t^2\right) \\ &\sqrt{2\pi}\bar{\Phi}\left(\frac{(ac+t^2)(\mu+\lambda)}{t\sqrt{c}} - \frac{t^2(\mu+\lambda) + td_2}{\sqrt{c(ac+t^2)}} \right) exp\left\{ -\frac{at^2(\mu+\lambda+\frac{d_2}{t})^2}{2(ac+t^2)} \right\} \\ &+ ce^{-\frac{d_2^2}{2c}}\sqrt{2\pi}\bar{\Phi}\left(\sqrt{a}(\mu+\lambda)\right)\left(\frac{d_2}{t} + \mu + \lambda\right), \end{aligned} \tag{4.65}$$

$$\begin{aligned} H_2(t,z,x_0) = &\left(\sqrt{c}d_2(\mu+\lambda) + \frac{\sqrt{c^3} + \sqrt{c}d_2^2}{t} \right) 2\pi \{ \Phi\left(-\frac{t\sqrt{a}(\mu+\lambda) + \sqrt{a}d_2}{\sqrt{ac+t^2}} \right) \\ &- BvN\left(h_1 = -\frac{t\sqrt{a}(\mu+\lambda) + \sqrt{a}d_2}{\sqrt{ac+t^2}}, k_1 = \sqrt{a}(\mu+\lambda); \rho = -\frac{t}{\sqrt{ac+t^2}} \right) \\ &- \Phi\left(-\frac{d_2}{\sqrt{c}}\right)\bar{\Phi}(\sqrt{a}(\mu+\lambda))\} - \frac{c^2\sqrt{a}}{ac+t^2} exp\left\{ -\frac{(\mu+\lambda)^2 a}{2} - \frac{d_2^2(t^2+1)}{2c(ac+t^2)} \right\}, \end{aligned} \tag{4.66}$$

$$H_3(t,z,x_0) = -\frac{1}{t}\sqrt{\frac{ac^3}{(ac+t^2)^3}}\left(t^3(\mu+\lambda)+acd_1+2d_1t^2\right)$$
$$\sqrt{2\pi}\Phi\left(\frac{(ac+t^2)(\mu+\lambda)}{t\sqrt{c}} - \frac{t^2(\mu+\lambda)-td_1}{\sqrt{c(ac+t^2)}}\right) exp\left\{-\frac{at^2(\mu+\lambda-\frac{d_1}{t})^2}{2(ac+t^2)}\right\}$$
$$+ce^{-\frac{d_1^2}{2c}}\sqrt{2\pi}\bar{\Phi}\left(\sqrt{a}(\mu+\lambda)\right)\left(\frac{d_1}{t}+\mu+\lambda\right), \tag{4.67}$$

$$H_4(t,z,x_0) = \left(\sqrt{c}d_1(\mu+\lambda) - \frac{\sqrt{c^3}-\sqrt{c}d_1^2}{t}\right)2\pi$$
$$\{BvN\left(h_2 = -\frac{t\sqrt{a}(\mu+\lambda)-\sqrt{a}d_1}{\sqrt{ac+t^2}}, k_2=\sqrt{a}(\mu+\lambda); \rho = -\frac{t}{\sqrt{ac+t^2}}\right)$$
$$+\Phi\left(-\frac{d_1}{\sqrt{c}}\right)\bar{\Phi}(\sqrt{a}(\mu+\lambda))\} - \frac{c^2\sqrt{a}}{ac+t^2}exp\left\{-\frac{(\mu+\lambda)^2a}{2}-\frac{d_1^2(t^2+1)}{2c(ac+t^2)}\right\}, \tag{4.68}$$

where BvN stands for Bivariate Normal cumulative distribution function, i.e.

$$BvN(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{k}\int_{-\infty}^{h} exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dxdy, \tag{4.69}$$

with $-\infty < h,k < \infty$*, and correlation coefficient* $-1 < \rho < 1$.

Proof. Step 1: Setting $\kappa = 0$, $\sqrt{\mu^2+2\xi}-\mu = \eta$, $\sqrt{\mu^2+2\nu}-\mu=\gamma$, and applying Laplace transform to the generator in (4.22) infer that

$$\hat{f}(\xi) = \frac{cf(0)-\frac{2\lambda}{\eta+\gamma+2\mu}}{-\beta+c\frac{\eta^2}{2}+(c\mu-\lambda)\eta}; \tag{4.70}$$

Step 2: Find $f(0)$ by letting the denominator to be 0, plug in the positive root η_β^+, where

$$\eta_\beta^+ = \frac{\lambda-\mu+\sqrt{(\mu-\lambda)^2+2c\beta}}{c}; \tag{4.71}$$

so we have

$$f(0) = \frac{2\lambda}{\lambda-c\mu+\sqrt{(c\mu-\lambda)^2+2c\beta}+c\gamma+2c\mu}; \tag{4.72}$$

Note that to ensure $0 < f(0) < 1$, we need $c\mu > \lambda$. This can be deduced from (4.70) that $f(0) = \frac{\lambda}{c\mu}$ by setting $\beta = 0$ and $\nu = 0$.

Step 3: $c = 1$, the Laplace transform of $f(x)$ is

$$\begin{aligned}
&\hat{f}(\xi) \\
&= \frac{4\lambda}{(\sqrt{(\mu-\lambda)^2+2\beta}+\lambda+\gamma+\mu)(\eta+\gamma+2\mu)(\sqrt{(\mu-\lambda)^2+2\beta}+\eta+\mu-\lambda)} \\
&= 4\lambda \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\sqrt{(\mu-\lambda)^2+2\beta}+\lambda+\gamma+\mu)u} e^{-(\eta+\gamma+2\mu)v} e^{-(\sqrt{(\mu-\lambda)^2+2\beta}+\eta+\mu-\lambda)w} \, dudvdw \\
&= 4\lambda \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\sqrt{(\mu-\lambda)^2+2\beta})(u+w)} e^{-\eta(v+w)} e^{-\gamma(u+v)} e^{-\mu(u+2v+w)} e^{-\lambda(u-w)} dudvdw;
\end{aligned} \tag{4.73}$$

Step 4: Rewrite the Laplace transform $\hat{f}(\xi)$ as

$$\begin{aligned}
\hat{f}(\xi) = 4\lambda \int_0^\infty \int_0^\infty \int_0^\infty &\left(\int_0^\infty e^{-(\beta+\frac{(\mu-\lambda)^2}{2})t} \frac{u+w}{\sqrt{2\pi t^3}} e^{-\frac{(u+w)^2}{2t}} dt \right) \\
&\left(\int_0^\infty e^{-(\xi+\frac{\mu^2}{2})x_0} \frac{v+w}{\sqrt{2\pi x_0^3}} e^{-\frac{(v+w)^2}{2x_0}} dx_0 \right) \\
&\left(\int_0^\infty e^{-(\nu+\frac{\mu^2}{2})z} \frac{u+v}{\sqrt{2\pi z^3}} e^{-\frac{(u+v)^2}{2z}} dz \right) e^{-\mu(u+2v+w)} e^{-\lambda(u-w)} dudvdw;
\end{aligned} \tag{4.74}$$

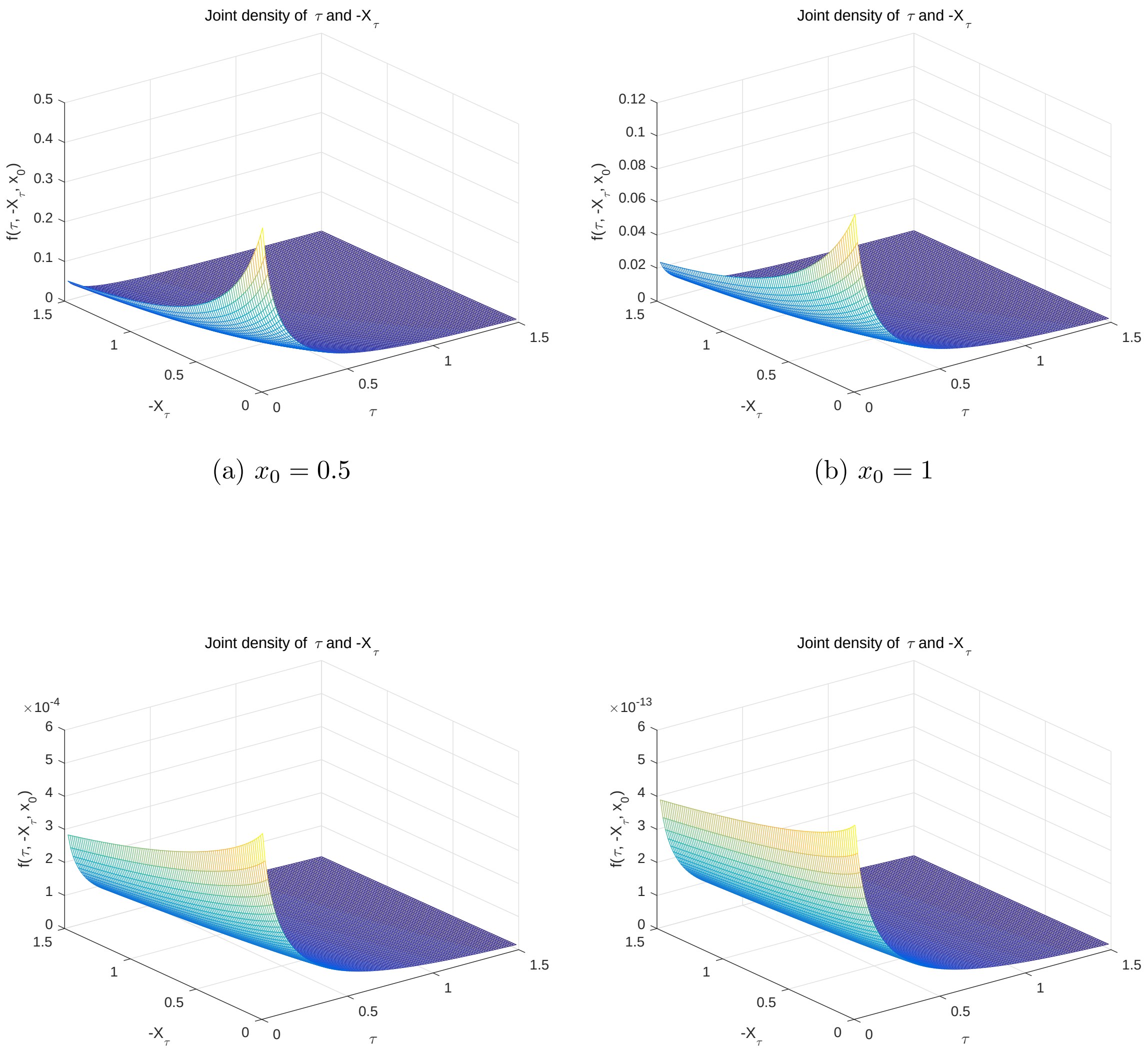
Invert $\hat{f}(\xi)$ w.r.t β, ξ, and γ, we then obtain the joint density $f(t, z, x_0)$,

$$\begin{aligned}
f(t, z, x_0) = 4\lambda &\frac{1}{\sqrt{2\pi t^3}} \frac{1}{\sqrt{2\pi x_0^3}} \frac{1}{\sqrt{2\pi z^3}} e^{-\frac{(\mu-\lambda)^2}{2}t} e^{-\frac{\mu^2}{2}x_0} e^{-\frac{\mu^2}{2}z} \\
&\int_0^\infty \int_0^\infty \int_0^\infty (u+w) e^{-\frac{(u+w)^2}{2t}} (v+w) e^{-\frac{(v+w)^2}{2x_0}} (u+v) e^{-\frac{(u+v)^2}{2z}} \\
&e^{-(\mu+\lambda)u-2\mu v-(\mu-\lambda)w} dudvdw.
\end{aligned} \tag{4.75}$$

■

Remark. This joint probability density function of the triplet can also be viewed as the joint density function of the ruin time τ, the overshoot at ruin $-X_\tau$ given $X_0 = x_0$, i.e. $f_{\tau,-X_\tau|X_0=x_0}(t, z)$.

Given the value of initial capital x_0, we plot the joint probability density function $f(t, z, x_0)$ against the ruin time τ and the overshoot $-X_\tau$ calculated by Matlab. First we set the lower bound q_1 and the upper bound q_2 for τ and $-X_\tau$ respectively. Also denote M the number



(a) $x_0 = 0.5$ (b) $x_0 = 1$

(c) $x_0 = 5$ (d) $x_0 = 30$

Figure 4.3: Joint density function $f(\tau, -X_\tau, x_0)$ with different initial capitals x_0. (Parameters $\mu = 1.2$, $\lambda = 1$, and $c = 1$.)

of time steps and hq_1 the step size with $hq_1 = (q_2 - q_1)/M$. Then, given a value of x_0, we compute $f(t, z, x_0)$ at each step till M steps. It is notable that in the formula of $f(t, z, x_0)$ in Theorem 4.3.3, a Bivariate Normal cumulative distribution function $BvN(h, k; \rho)$ is used. This makes the calculation of $f(t, z, x_0)$ and the probability of ruin easy since there is no need to do numerical integration. To obtain $BvN(h, k; \rho)$, a Matlab function "integral2" may need to use in the calculation. Finally, we record all of the values of $f(t, z, x_0)$ at each step of τ and $-X_\tau$, and plot $f(t, z, x_0)$ by using the Matlab function "mesh". Parameters $\mu = 1.2$, $\lambda = 1$, $c = 1$, initial capital $x_0 = 1$. Other parameters $q_1 = 0.02$, $q_2 = 1.5$, $M = 100$.

Figure 4.3 shows the plot of $f(t, z, x_0)$ against τ and $-X_\tau$. The x axis and y axis denote the values of τ and $-X_\tau$ respectively. The z axis denotes the value of $f(t, z, x_0)$. From Figure 4.3, it is clear to see that $f(t, z, x_0)$ decreases as the value of initial capital grows. Intuitively, an insurance company is less likely to get ruined when it possesses initial capital at a higher level. Moreover, when the value of ruin time or the overshoot reduces, $f(t, z, x_0)$ increases as well when the value of x_0 is fixed.

x	$\psi(x;1)$	$\psi(x;2)$	$\psi(x;5)$	$\psi(x;10)$
0.1	0.14011	0.09639	0.04703	0.01745
0.5	0.04355	0.02568	0.01363	0.00773
1	0.01417	0.00998	0.00541	0.00322
5	0.00011	9.0153e-05	6.41341e-05	4.49629e-05
10	1.16715e-06	1.0208e-06	7.98487e-07	5.02671e-07

Table 4.2: Finite time ruin probabilities.

Table 4.2 shows the numerical results of the finite time ruin probabilities when $\lambda = 1$, $\mu = 1.2$, and $c = 1$. We use the notation $\psi(x;t)$ to denote the probability of ruin before time t given initial surplus x when the joint density function is given by (4.60). These results are evaluated by integrating the joint density function over z with different values of initial capital x_0 and ruin times t. We can see from this table that for the fixed ruin time t, the ruin probability is decreasing as the value of initial capital x grows. When x is fixed, ruin is less likely to occur within longer time horizon.

4.4 Appendix to Chapter 4

4.4.1 Detailed Derivation of Theorem 4.3.3

To calculate the triple integral in (4.75), we use change of variables, Jacobian matrix and the corresponding determinant. First set

$$\begin{aligned}\mathcal{Q} = &\int_0^\infty \int_0^\infty \int_0^\infty (u+w)e^{-\frac{(u+w)^2}{2t}}(v+w)e^{-\frac{(v+w)^2}{2x_0}}(u+v)e^{-\frac{(u+v)^2}{2z}} \\ &e^{-(\mu+\lambda)u-2\mu v-(\mu-\lambda)w}dudvdw.\end{aligned} \tag{4.76}$$

Then let

$$\begin{cases} u+w=r \\ v+w=q \\ u+v=m, \end{cases} \tag{4.77}$$

which provides us with the corresponding Jacobian matrix

$$\mathbf{J} = \left(\frac{\partial(u,v,w)}{\partial(r,q,m)}\right) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \tag{4.78}$$

and thus the determinant is

$$|\mathbf{J}| = -\frac{1}{2}. \tag{4.79}$$

Because of $|q-m| \leq r \leq q+m$, we have

$$
\begin{aligned}
\mathcal{Q} &= \frac{1}{2}\int_0^\infty \int_0^\infty \int_{|q-m|}^{q+m} (re^{-\frac{r^2}{2t}})(me^{-\frac{m^2}{2z}-(\mu+\lambda)m})(qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q})drdmdq \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty (-t)(e^{-\frac{(q+m)^2}{2t}} - e^{-\frac{(q-m)^2}{2t}})(me^{-\frac{m^2}{2z}-(\mu+\lambda)m})(qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q})dmdq \\
&= \frac{t}{2}\int_0^\infty \left(\int_0^\infty e^{-\frac{(q-m)^2}{2t}} me^{-\frac{m^2}{2z}-(\mu+\lambda)m}dm\right) qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q}dq \\
&\quad - \frac{t}{2}\int_0^\infty \left(\int_0^\infty e^{-\frac{(q+m)^2}{2t}} me^{-\frac{m^2}{2z}-(\mu+\lambda)m}dm\right) qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q}dq \\
&= \frac{t}{2}(\mathcal{Q}_1 - \mathcal{Q}_2).
\end{aligned}
\tag{4.80}
$$

Then solve $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively.

(i) Solve $\mathcal{Q}_1$ first.

By letting $A = \int_0^\infty e^{-\frac{(q-m)^2}{2t}} me^{-\frac{m^2}{2z}-(\mu+\lambda)m}dm$, thus $\mathcal{Q}_1 = A \cdot qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q}dq$. Then the change of variables $a = \frac{tz}{t+z}$, $b = \frac{tz(\mu+\lambda)-zq}{t+z}$, and $\frac{m+b}{\sqrt{a}} = x$ gives us

$$
\begin{aligned}
A &= \int_0^\infty (\sqrt{a}x - b)e^{-\frac{x^2}{2}}\sqrt{a}dxe^{\frac{b^2}{2a}}e^{\frac{q^2}{2t}} \\
&= \left[a - b\sqrt{2\pi a}\bar{\Phi}\left(\frac{b}{\sqrt{a}}\right)e^{\frac{b^2}{2a}}\right]e^{-\frac{q^2}{2t}},
\end{aligned}
\tag{4.81}
$$

thus

$$
\begin{aligned}
\mathcal{Q}_1 =& \int_0^\infty \left[a - b\sqrt{2\pi a}\bar{\Phi}\left(\frac{b}{\sqrt{a}}\right)e^{\frac{b^2}{2a}}\right] qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}}dq \\
=& a\int_0^\infty qe^{-\frac{(t+x_0)q^2-2x_0(\lambda-\mu)q}{2x_0 t}}dq \\
&- \sqrt{a}\int_0^\infty \left(\frac{tz(\mu+\lambda)}{t+z}q - \frac{z}{t+z}q^2\right)e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}}\left(\int_{\frac{b}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dx\right)e^{\frac{b^2}{2a}}dq \\
=& C_1 - D_1.
\end{aligned}
\tag{4.82}
$$

Next, let $D_1 = E_1 - F_1$, so we have

$$
E_1 = \sqrt{a}\frac{tz(\mu+\lambda)}{t+z}\int_0^\infty qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}}e^{\frac{b^2}{2a}}\left(\int_{\frac{b}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dx\right)dq, \tag{4.83}
$$

and

$$F_1 = \sqrt{a}\frac{z}{t+z}\int_0^\infty q^2 e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}} e^{\frac{b^2}{2a}} \left(\int_{\frac{b}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}\,dx\right) dq. \tag{4.84}$$

Substituting change of variables $c = \frac{x_0(t+z)}{t+z+x_0}$ and $d = \frac{x_0(t\lambda - t\mu - 2z\mu)}{t+z+x_0}$ into E_1,

$$E_1 = e^{A_1}\sqrt{a}\frac{tz(\mu+\lambda)}{t+z}\int_0^\infty q e^{-\frac{(q-d)^2}{2c}}\int_{\frac{b}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}\,dxdq, \tag{4.85}$$

where

$$A_1 = \frac{d^2}{2c} + \frac{(\mu+\lambda)^2}{2}a, \tag{4.86}$$

gives us

$$\begin{aligned} E_1 = e^{A_1}\sqrt{a^3}(\mu+\lambda)\Bigg[&\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_0^\infty q e^{-\frac{(q-d)^2}{2c}}\,dqdx \\ &+ \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-\sqrt{a}(\mu+\lambda)}{\frac{\sqrt{a}}{t}}}^\infty q e^{-\frac{(q-d)^2}{2c}}\,dqdx\Bigg]. \end{aligned} \tag{4.87}$$

In the meantime,

$$\begin{aligned} E_2 &= \int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_0^\infty q e^{-\frac{(q-d)^2}{2c}}\,dqdx \\ &= \int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\,dx\int_0^\infty q e^{-\frac{(q-d)^2}{2c}}\,dq \\ &= \sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda))\left(ce^{-\frac{d^2}{2c}} + \sqrt{c}d\sqrt{2\pi}\bar{\Phi}\left(-\frac{d}{\sqrt{c}}\right)\right). \end{aligned} \tag{4.88}$$

Also

$$\begin{aligned} E_3 =& \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-\sqrt{a}(\mu+\lambda)}{\frac{\sqrt{a}}{t}}}^\infty q e^{-\frac{(q-d)^2}{2c}}\,dqdx \\ =& \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty (y\sqrt{c}+d)e^{-\frac{y^2}{2}}\sqrt{c}\,dydx \\ =& \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty y\sqrt{c}e^{-\frac{y^2}{2}}\sqrt{c}\,dydx + \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty de^{-\frac{y^2}{2}}\sqrt{c}\,dydx \\ =& G_1 + H_1. \end{aligned} \tag{4.89}$$

where

$$L_1 = \sqrt{a}(\mu+\lambda) - \frac{d\sqrt{a}}{t}, \quad M_1 = \frac{\sqrt{ac}}{t}. \tag{4.90}$$

We have

$$\begin{aligned} G_1 &= c\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}} \int_{-\frac{x-L_1}{M_1}}^{\infty} ye^{-\frac{y^2}{2}}dydx \\ &= ce^{-\frac{L_1^2}{2(1+M_1^2)}} \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} \exp\left\{-\frac{(x-\frac{L_1}{a+M_1^2})^2}{2\frac{M_1^2}{1+M_1^2}}\right\}dx \\ &= c\sqrt{2\pi}\frac{M_1}{\sqrt{1+M_1^2}}\Phi\left(\frac{\sqrt{a}(\mu+\lambda)-\frac{L_1}{1+M_1^2}}{\frac{M_1}{\sqrt{1+M_1^2}}}\right)e^{-\frac{L_1^2}{2(1+M_1^2)}}, \end{aligned} \tag{4.91}$$

and

$$\begin{aligned} H_1 &= \sqrt{c}d\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}} \int_{-\frac{x-L_1}{M_1}}^{\infty} e^{-\frac{y^2}{2}}dydx \\ &= \sqrt{c}d\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}} \int_{-\infty}^{\frac{x-L_1}{M_1}} e^{-\frac{y^2}{2}}dydx \\ &= 2\pi\sqrt{c}d\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} \Phi^{'}(x)\Phi\left(\frac{x}{M_1}-\frac{L_1}{M_1}\right)dx \\ &= 2\pi\sqrt{c}d\cdot \mathrm{BvN}\left(-\frac{L_1}{\sqrt{1+M_1^2}}, \sqrt{a}(\mu+\lambda); \rho = -\frac{1}{\sqrt{1+M_1^2}}\right), \end{aligned} \tag{4.92}$$

where $\mathrm{BvN}(h,k;\rho)$ is the joint probability distribution function of random variable's X and Y with correlation ρ.

$$\mathrm{BvN}(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{k}\int_{-\infty}^{h} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dxdy. \tag{4.93}$$

Therefore, we have

$$\begin{aligned} E_1 = e^{A_1}\sqrt{a^3}(\mu+\lambda)\Bigg[&\sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda))\left(ce^{-\frac{d^2}{2c}}+\sqrt{c}d\sqrt{2\pi}\bar{\Phi}\left(-\frac{d}{\sqrt{c}}\right)\right) \\ &+\sqrt{2\pi}c\frac{M_1}{\sqrt{1+M_1^2}}\Phi\left(\frac{\sqrt{1+M_1^2}\sqrt{a}(\mu+\lambda)}{M_1}-\frac{L_1}{M_1\sqrt{1+M_1^2}}\right)e^{-\frac{L_1^2}{2(1+M_1^2)}} \\ &+2\pi\sqrt{c}d\cdot \mathrm{BvN}\left(-\frac{L_1}{\sqrt{1+M_1^2}}, \sqrt{a}(\mu+\lambda); \rho=-\frac{1}{\sqrt{1+M_1^2}}\right)\Bigg]. \end{aligned} \tag{4.94}$$

Now we solve F_1,

$$
\begin{aligned}
F_1 =& e^{A_1}\frac{\sqrt{a^3}}{t}\int_0^\infty q^2 e^{-\frac{(q-d)^2}{2c}}\int_{\frac{b}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dxdq \\
=& e^{A_1}\frac{\sqrt{a^3}}{t}\left[\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_0^\infty q^2 e^{-\frac{(q-d)^2}{2c}}dqdx + \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-\sqrt{a}(\mu+\lambda)}{\frac{\sqrt{a}}{t}}}^\infty q^2 e^{-\frac{(q-d)^2}{2c}}dqdx\right] \\
=& e^{A_1}\frac{\sqrt{a^3}}{t}\left(F_2+F_3\right).
\end{aligned}
\tag{4.95}
$$

By using change of variable $\frac{q-d}{\sqrt{c}}=y$, we have

$$
\begin{aligned}
F_2 &= \int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_0^\infty q^2 e^{-\frac{(q-d)^2}{2c}}dqdx \\
&= \sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda))\int_{-\frac{d}{\sqrt{c}}}^\infty (cy^2+2d\sqrt{c}y+d^2)\sqrt{c}e^{-\frac{y^2}{2}}dy \\
&= \sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda))\left[\sqrt{2\pi}\bar{\Phi}\left(-\frac{d}{\sqrt{c}}\right)(\sqrt{c^3}+\sqrt{c}d^2)+cde^{-\frac{d^2}{2c}}\right],
\end{aligned}
\tag{4.96}
$$

$$
\begin{aligned}
F_3 =& \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty (cy^2+2d\sqrt{c}y+d^2)\sqrt{c}e^{-\frac{y^2}{2}}dydx \\
=& -\sqrt{c^3}\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\left(\frac{x-L_1}{M_1}\right)e^{-\frac{(x-L_1)^2}{2M_1^2}}dx \\
&+(\sqrt{c^3}+\sqrt{c}d^2)\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty e^{-\frac{y^2}{2}}dydx + 2cd\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\int_{-\frac{x-L_1}{M_1}}^\infty ye^{-\frac{y^2}{2}}dydx.
\end{aligned}
\tag{4.97}
$$

Since we have

$$
\begin{aligned}
&-\sqrt{c^3}\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}}\left(\frac{x-L_1}{M_1}\right)e^{-\frac{(x-L_1)^2}{2M_1^2}}dx \\
=& -\frac{\sqrt{c^3}}{M_1}\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} xe^{-\frac{x^2}{2}-\frac{(x-L_1)^2}{2M_1^2}}dx + \sqrt{c^3}\frac{L_1}{M_1}\int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}-\frac{(x-L_1)^2}{2M_1^2}}dx \\
=& \frac{\sqrt{c^3}M_1}{1+M_1^2}\exp\left\{-\frac{[(1+M_1^2)\sqrt{a}(\mu+\lambda)-L_1]^2}{2M_1^2(1+M_1^2)}-\frac{L_1^2}{2(1+M_1^2)}\right\}+\sqrt{c^3}\frac{L_1}{M_1}\frac{G_1}{c},
\end{aligned}
\tag{4.98}
$$

which gives us

$$
\begin{aligned}
F_3 = & \frac{\sqrt{c^3}M_1}{1+M_1^2}\exp\left\{-\frac{[(1+M_1^2)\sqrt{a}(\mu+\lambda)-L_1]^2}{2M_1^2(1+M_1^2)}-\frac{L_1^2}{2(1+M_1^2)}\right\} \\
& -\sqrt{2\pi}\sqrt{c^3}\frac{L_1}{\sqrt{(1+M_1^2)^3}}e^{-\frac{L_1^2}{2(1+M_1^2)}}\Phi\left(\frac{\sqrt{1+M_1^2}\sqrt{a}(\mu+\lambda)}{M_1}-\frac{L_1}{M_1\sqrt{1+M_1^2}}\right) \\
& +\left(\frac{L_1\sqrt{c}}{M_1}+2d\right)\cdot G_1+\left(\frac{c}{d}+d\right)\cdot H_1.
\end{aligned}
\tag{4.99}
$$

Therefore,

$$
F_1 = e^{A_1}\frac{\sqrt{a^3}}{t}\left(F_2+F_3\right), \tag{4.100}
$$

and

$$
\begin{aligned}
D_1 = & E_1 - F_1 \\
= & e^{A_1}\sqrt{a^3}\cdot\left\{\sqrt{2\pi}\frac{cM_1}{\sqrt{1+M_1^2}}\Phi\left(\frac{\sqrt{1+M_1^2}\sqrt{a}(\mu+\lambda)}{M_1}-\frac{L_1}{M_1\sqrt{1+M_1^2}}\right)e^{-\frac{L_1^2}{2(1+M_1^2)}}\right. \\
& \cdot\left(-\frac{\sqrt{c}L_1M_1}{t(1+M_1^2)}+\mu+\lambda-2\frac{d}{t}\right)+\sqrt{2\pi}ce^{-\frac{d^2}{2c}}\bar{\Phi}(\sqrt{a}(\mu+\lambda))\left(\mu+\lambda-\frac{d}{t}\right) \\
& +\left(\sqrt{c}d(\mu+\lambda)-\frac{\sqrt{c^3}}{t}-\frac{\sqrt{c}d^2}{t}\right) \\
& \cdot\left[2\pi\sqrt{c}d\cdot\mathrm{BvN}\left(-\frac{L_1}{\sqrt{1+M_1^2}},\sqrt{a}(\mu+\lambda);\rho=-\frac{1}{\sqrt{1+M_1^2}}\right)+2\pi\bar{\Phi}\left(-\frac{d}{\sqrt{c}}\right)\bar{\Phi}\left(\sqrt{a}(\mu+\lambda)\right)\right] \\
& \left.-\frac{\sqrt{c^3}M_1}{t(1+M_1^2)}\exp\left\{-\frac{[(1+M_1^2)\sqrt{a}(\mu+\lambda)-L_1]^2}{2M_1^2(1+M_1^2)}-\frac{L_1^2}{2(1+M_1^2)}\right\}\right\}.
\end{aligned}
\tag{4.101}
$$

(ii) Then solve $\mathcal{Q}_2$. Let

$$
\begin{aligned}
\mathcal{Q}_2 = & \int_0^\infty\left(\int_0^\infty e^{-\frac{(q+m)^2}{2t}}me^{-\frac{m^2}{2z}-(\mu+\lambda)m}dm\right)qe^{-\frac{q^2}{2x_0}+(\lambda-\mu)q}dq \\
= & a\int_0^\infty qe^{-\frac{(t+x_0)q^2-2x_0(\lambda-\mu)q}{2x_0t}}dq \\
& -\sqrt{a}\int_0^\infty\left(a(\mu+\lambda)q-\frac{a}{t}q^2\right)e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}}e^{\frac{\tilde{b}^2}{2a}}\left(\int_{\frac{\tilde{b}}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dx\right)dq \\
= & C_1 - D_2,
\end{aligned}
\tag{4.102}
$$

where $a = \frac{tz}{t+z}$, $\tilde{b} = \frac{tz(\mu+\lambda)+zq}{t+z}$. In the meantime, let $D_2 = E_2 + F_2$, where

$$E_2 = \sqrt{a^3}(\mu+\lambda)\int_0^\infty q e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}} e^{\frac{\tilde{b}^2}{2a}}\left(\int_{\frac{\tilde{b}}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dx\right)dq, \tag{4.103}$$

and

$$F_2 = \frac{\sqrt{a^3}}{t}\int_0^\infty q^2 e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}} e^{\frac{\tilde{b}^2}{2a}}\left(\int_{\frac{\tilde{b}}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dx\right)dq. \tag{4.104}$$

Substituting the change of variables $c = \frac{x_0(t+z)}{t+z+x_0}$ and $\tilde{d} = \frac{x_0(t\lambda - t\mu + 2z\lambda)}{t+z+x_0}$ into E_2,

$$E_2 = e^{A_2}\sqrt{a^3}(\mu+\lambda)\int_0^\infty q e^{-\frac{(q-\tilde{d})^2}{2c}}\int_{\frac{\tilde{b}}{\sqrt{a}}}^\infty e^{-\frac{x^2}{2}}dxdq, \tag{4.105}$$

where $A_2 = \frac{\tilde{d}^2}{2c} + \frac{(\mu+\lambda)^2}{2}a$. This gives us

$$\begin{aligned}
E_2 &= e^{A_2}\sqrt{a^3}(\mu+\lambda)\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_0^{\frac{tx}{\sqrt{a}}-t(\mu+\lambda)} q e^{-\frac{(q-\tilde{d})^2}{2c}}dqdx \\
&= e^{A_2}\sqrt{a^3}(\mu+\lambda)\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} yce^{-\frac{y^2}{2}}dydx \\
&+ e^{A_2}\sqrt{a^3}(\mu+\lambda)\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\sqrt{c}\tilde{d}dydx \\
&= e^{A_2}\sqrt{a^3}(\mu+\lambda)\left(G_2 + H_2\right),
\end{aligned} \tag{4.106}$$

where

$$L_2 = \sqrt{a}(\mu+\lambda) + \frac{\tilde{d}\sqrt{a}}{t}, \quad M_2 = M_1 = \frac{\sqrt{ac}}{t}. \tag{4.107}$$

So we have

$$\begin{aligned}
G_2 &= \int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}\int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} yce^{-\frac{y^2}{2}}dydx \\
&= -c\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}e^{-\frac{(x-L_2)^2}{2M_2^2}}dx + c\int_{\sqrt{a}(\mu+\lambda)}^\infty e^{-\frac{x^2}{2}}e^{-\frac{\tilde{d}^2}{2c}}dx \\
&= \sqrt{2\pi}ce^{-\frac{\tilde{d}^2}{2c}}\bar{\Phi}(\sqrt{a}(\mu+\lambda)) - \sqrt{2\pi}c\frac{M_2}{\sqrt{1+M_2^2}}\bar{\Phi}\left(\frac{\sqrt{a}(\mu+\lambda) - \frac{L_2}{1+M_2^2}}{\frac{M_2}{\sqrt{1+M_2^2}}}\right)e^{-\frac{L_2^2}{2(1+M_2^2)}},
\end{aligned} \tag{4.108}$$

and

$$
\begin{aligned}
H_2 &= \tilde{d}\sqrt{c}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\,dydx \\
&= \tilde{d}\sqrt{c}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \left(\int_{-\infty}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\,dy - \int_{-\infty}^{-\frac{\tilde{d}}{\sqrt{c}}} e^{-\frac{y^2}{2}}\,dy\right) dx \\
&= \tilde{d}\sqrt{c}\Bigg[\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \int_{-\infty}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\,dydx - \int_{-\infty}^{\sqrt{a}(\mu+\lambda)} e^{-\frac{x^2}{2}} \int_{-\infty}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\,dydx \\
&\quad - \int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\infty}^{-\frac{\tilde{d}}{\sqrt{c}}} e^{-\frac{y^2}{2}}\,dydx\Bigg] \\
&= \tilde{d}\sqrt{c}\Bigg[2\pi\Phi\left(-\frac{L_2}{\sqrt{1+M_2^2}}\right) - 2\pi\mathrm{BvN}\left(-\frac{L_2}{\sqrt{1+M_2^2}}, \sqrt{a}(\mu+\lambda); \rho = -\frac{1}{\sqrt{1+M_2^2}}\right) \\
&\quad - 2\pi\Phi\left(-\frac{\tilde{d}}{\sqrt{c}}\right)\bar{\Phi}(\sqrt{a}(\mu+\lambda))\Bigg].
\end{aligned} \tag{4.109}
$$

Now we solve F_2.

$$
\begin{aligned}
F_2 =& \frac{\sqrt{a^3}}{t}\int_0^{\infty} q^2 e^{-\frac{q^2}{2x_0}+(\lambda-\mu)q-\frac{q^2}{2t}} e^{\frac{\tilde{b}^2}{2a}} \left(\int_{\frac{\tilde{b}}{\sqrt{a}}}^{\infty} e^{-\frac{x^2}{2}}\,dx\right) dq \\
=& e^{A_2}\frac{\sqrt{a^3}}{t}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} (cy^2 + 2\sqrt{c}\tilde{d}y + \tilde{d}^2)\sqrt{c}e^{-\frac{y^2}{2}}\,dydx \\
=& e^{A_2}\frac{\sqrt{a^3}}{t}\Bigg[\sqrt{c^3}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} y^2 e^{-\frac{y^2}{2}}\,dydx \\
& + 2c\tilde{d}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} ye^{-\frac{y^2}{2}}\,dydx + \sqrt{c}\tilde{d}^2\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} \int_{-\frac{\tilde{d}}{\sqrt{c}}}^{\frac{x-L_2}{M_2}} e^{-\frac{y^2}{2}}\,dydx\Bigg] \\
=& e^{A_2}\frac{\sqrt{a^3}}{t}\Bigg[-\sqrt{c^3}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}}\left(\frac{x-L_2}{M_2}\right) e^{-\frac{(x-L_2)^2}{2M_2^2}}\,dx - c\tilde{d}e^{-\frac{\tilde{d}^2}{2c}}\sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda)) \\
& + 2c\tilde{d}\frac{G_2}{c} + \sqrt{c}\tilde{d}^2\frac{H_2}{\sqrt{c}\tilde{d}}\Bigg] \\
=& e^{A_2}\frac{\sqrt{a^3}}{t}\Bigg[F_3 - c\tilde{d}e^{-\frac{\tilde{d}^2}{2c}}\sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda)) + 2\tilde{d}G_2 + \tilde{d}H_2\Bigg],
\end{aligned} \tag{4.110}
$$

and

$$
\begin{aligned}
F_3 =& -\frac{\sqrt{c^3}}{M_2}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} x e^{-\frac{x^2}{2}} e^{-\frac{(x-L_2)^2}{2M_2^2}}\,dx + \sqrt{c^3}\frac{L_2}{M_2}\int_{\sqrt{a}(\mu+\lambda)}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{(x-L_2)^2}{2M_2^2}}\,dx \\
=& -\frac{\sqrt{c^3}}{M_2}\Bigg[1+\frac{M_2^2}{1+M_1^2}\exp\left\{-\frac{[(1+M_2^2)\sqrt{a}(\mu+\lambda)-L_2]^2}{2M_2^2(1+M_2^2)} - \frac{L_2^2}{2(1+M_2^2)}\right\} \\
& - \sqrt{2\pi}\frac{L_2M_2}{\sqrt{(1+M_2^2)^3}} e^{-\frac{L_2^2}{2(1+M_2^2)}}\Phi\left(\frac{\sqrt{1+M_2^2}\sqrt{a}(\mu+\lambda)}{M_2} - \frac{L_2}{M_2\sqrt{1+M_2^2}}\right)\Bigg] \\
& + \sqrt{c^3}\frac{L_2}{\sqrt{1+M_2^2}}\sqrt{2\pi}e^{-\frac{L_2^2}{2(1+M_2^2)}}\bar{\Phi}\left(\frac{\sqrt{1+M_2^2}\sqrt{a}(\mu+\lambda)}{M_2} - \frac{L_2}{M_2\sqrt{1+M_2^2}}\right) \\
=& -\sqrt{c^3}\frac{M_2}{1+M_1^2}\exp\left\{-\frac{[(1+M_2^2)\sqrt{a}(\mu+\lambda)-L_2]^2}{2M_2^2(1+M_2^2)} - \frac{L_2^2}{2(1+M_2^2)}\right\} \\
& + \sqrt{c^3}\sqrt{2\pi}\frac{L_2M_2^2}{\sqrt{(1+M_2^2)^3}} e^{-\frac{L_2^2}{2(1+M_2^2)}}\bar{\Phi}\left(\frac{\sqrt{1+M_2^2}\sqrt{a}(\mu+\lambda)}{M_2} - \frac{L_2}{M_2\sqrt{1+M_2^2}}\right),
\end{aligned}
\tag{4.111}
$$

provide us with

$$
\begin{aligned}
F_2 = e^{A_2}\frac{\sqrt{a^3}}{t}\Bigg[& -\sqrt{c^3}\frac{M_2}{1+M_1^2}\exp\left\{-\frac{[(1+M_2^2)\sqrt{a}(\mu+\lambda)-L_2]^2}{2M_2^2(1+M_2^2)} - \frac{L_2^2}{2(1+M_2^2)}\right\} \\
& + \sqrt{c^3}\sqrt{2\pi}\frac{L_2M_2^2}{\sqrt{(1+M_2^2)^3}} e^{-\frac{L_2^2}{2(1+M_2^2)}}\bar{\Phi}\left(\frac{\sqrt{1+M_2^2}\sqrt{a}(\mu+\lambda)}{M_2} - \frac{L_2}{M_2\sqrt{1+M_2^2}}\right) \\
& - c\tilde{d}e^{-\frac{\tilde{d}^2}{2c}}\sqrt{2\pi}\bar{\Phi}(\sqrt{a}(\mu+\lambda)) + \left(\frac{c}{\tilde{d}} + \sqrt{c}\tilde{d}^2\right)H_2 + 2\tilde{d}G_2\Bigg].
\end{aligned}
\tag{4.112}
$$

Chapter 5

Parisian Excursions with Inverse Gaussian Processes

Traditional risk theory defines that ruin occurs when the surplus process of an insurance company ever drops below zero, so the company is declared as ruined. This definition has been recently generalized to the case of Parisian type of ruin. According to Egidio dos Reis [32] the probability of ruin is usually very small in practice and the insurance company could continue its business if there are enough funds available to support a negative surplus. Therefore, Parisian type of ruin occurs when the surplus stays above or under a pre-defined barrier long enough in a row. From our point of view, the probability of Parisian type of ruin could be a more appropriate measure of risk in practice, providing the possibility for an insurance company to get solvency. Dassios and Wu [21] obtained the solution of Parisian type ruin probability of a classical risk model with exponential claims and for Brownian motion with drift. Dassios and Wu [22] also discussed the Cramér-type asymptotics of Parisian ruin probabilities for the classical risk process.

In this chapter, we consider the surplus process with the total claim amount being an inverse Gaussian process. We begin with the study of the first excursion above zero and the first excursion under zero respectively. By using a two-state semi-Markov process, the Laplace transforms of Parisian ruin time for zero initial capital and non-zero initial capital are derived. Explicit formulae of the probability of Parisian type of ruin with different initial capitals are also provided. By considering the asymptotic properties for the total claims arrival process, we also propose an approximation for the probability of the Parisian type of ruin.

5.1 Joint Laplace transform of τ_1 and τ_2

We study the surplus process defined as in chapter 4, i.e.

$$X_t = x + ct - Z_t, \tag{5.1}$$

where Z_t is an IG process. Define τ_1 as same as the ruin time

$$\tau_1 = \inf\{t \geq 0 \mid X_t < 0\}. \tag{5.2}$$

We also define τ_2 as the first time length after which the surplus process goes back to zero after τ_1, i.e.

$$\tau_2 = \inf\{t - \tau_1 \mid t > \tau_1, X_t \geq 0, X_{\tau_1} < 0\}. \tag{5.3}$$

Recall that the generator simplified in (4.22) is given by

$$\begin{aligned}
&\mathcal{A}f(x,t) \\
= &\frac{\partial f(x,t)}{\partial t} + c\frac{\partial f(x,t)}{\partial x} \\
&- \lambda f(0,t)\int_x^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dy - \lambda \int_0^x f'(x-v,t)\int_v^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dy \\
&+ \lambda e^{-\beta t}\int_x^\infty e^{-\kappa x - \nu(y-x)} \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dy.
\end{aligned} \tag{5.4}$$

Let $f(x,t) = e^{-\beta t} f(x)$ and apply Laplace transform to $\mathcal{A}f(x,t) = 0$, we have $\hat{f}(\xi)$

$$\hat{f}(\xi) = \frac{cf(0) - \lambda\frac{\sqrt{\mu^2+2(\xi+\kappa)} - \sqrt{\mu^2+2\nu}}{\xi+\kappa-\nu}}{c\xi - \beta - \lambda(\sqrt{\mu^2+2\xi} - \mu)}. \tag{5.5}$$

We consider the equation

$$c\xi - \beta - \lambda(\sqrt{\mu^2+2\xi} - \mu) = 0, \tag{5.6}$$

which has two roots

$$r_\beta^\pm = \frac{-c\lambda\mu + c\beta + \lambda^2 \pm \lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta}}{c^2}. \tag{5.7}$$

We first study the joint Laplace transform of τ_1 and τ_2 with different values of initial capital.

5.1.1 For X_t with $X_0 = 0$

Theorem 5.1.1. *The joint Laplace transform of τ_1 and τ_2 with initial capital $X_0 = 0$ is given by*

$$\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2} \mid X_0 = 0] = \frac{2\lambda}{c}\frac{1}{\sqrt{\mu^2+2\beta_1}+\sqrt{\mu^2+2r_{\beta_2}^+}}, \tag{5.8}$$

where $\beta_1 > 0$, $\beta_2 > 0$ and

$$r_{\beta_2}^+ = \frac{-c\lambda\mu + c\beta_2 + \lambda^2 + \lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta_2}}{c^2}. \tag{5.9}$$

Proof. According to Gerber [43], $e^{-\beta t + r_\beta^- X_t}$ is a martingale. Applying the optional stopping theorem to this martingale stopped at τ_1, we have

$$\mathbb{E}[e^{-\beta_1\tau_1 + r_{\beta_1}^- X_{\tau_1}}\mathbb{I}_{\tau_1<\infty} \mid X_0 = 0] = 1. \tag{5.10}$$

Thus if the surplus process recovers from τ_1 and stops at the time $\tau_1+\tau_2$, by using the Markov property of (t, X_t),

$$\mathbb{E}[e^{-\beta_2\tau_2} \mid \mathcal{F}_{\tau_1}] = \mathbb{E}[e^{-\beta_2\tau_2} \mid X_{\tau_1}] = e^{r_{\beta_2}^+ X_{\tau_1}}. \tag{5.11}$$

Then, from the Tower property of expectations,

$$\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2}] = \mathbb{E}[\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2} \mid X_{\tau_1}]] = \mathbb{E}[\mathbb{E}[e^{-\beta_2\tau_2} \mid X_{\tau_1}]e^{-\beta_1\tau_1}] = \mathbb{E}[e^{r_{\beta_2}^+ X_{\tau_1}}e^{-\beta_1\tau_1}]. \tag{5.12}$$

The last expectation in (5.12) is actually the joint Laplace transform of τ_1 and $-X_{\tau_1}$ with $X_0 = 0$. According to Theorem (4.3.1), we have the joint probability density function of the ruin time τ and the overshoot at ruin $-X_{\tau}$,

$$f(t,y) = \frac{\lambda}{c}\frac{1}{\sqrt{2\pi(t+y)^3}}e^{-\frac{\mu^2(t+y)}{2}}. \tag{5.13}$$

Thus the joint Laplace transform of τ_1 and τ_2 from (5.12) is calculated as

$$\begin{aligned}\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2}] &= \mathbb{E}[e^{-r^+_{\beta_2}(-X_{\tau_1})}e^{-\beta_1\tau_1}] \\ &= \int_0^\infty\int_0^\infty e^{-r^+_{\beta_2}y}e^{-\beta_1 t}\frac{\lambda}{c}\frac{1}{\sqrt{2\pi(t+y)^3}}e^{-\frac{\mu^2(t+y)}{2}}dtdy.\end{aligned} \tag{5.14}$$

For simplicity, we let $\beta_1 = a$ and $r^+_{\beta_2} = b$, thus equation (5.14) can be calculated as follows:

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{-r^+_{\beta_2}y}e^{-\beta_1 t}\frac{\lambda}{c}\frac{1}{\sqrt{2\pi(t+y)^3}}e^{-\frac{\mu^2(t+y)}{2}}dtdy \\ =&\frac{\lambda}{c}\int_0^\infty e^{-(\frac{\mu^2}{2}+b)y}\int_0^\infty\frac{1}{\sqrt{2\pi(t+y)^3}}e^{-(\frac{\mu^2}{2}+a)t}dtdy.\end{aligned} \tag{5.15}$$

Let $z = t+y$ and we have

$$\begin{aligned}A &= \int_0^\infty\frac{1}{\sqrt{2\pi(t+y)^3}}e^{-(\frac{\mu^2}{2}+a)t}dt \\ &= e^{(\frac{\mu^2}{2}+a)y}\int_y^\infty\frac{1}{\sqrt{2\pi z^3}}e^{-(\frac{\mu^2}{2}+z)t}dz,\end{aligned} \tag{5.16}$$

and let $B = \frac{\mu^2}{2}+a$, so (5.15) becomes

$$\begin{aligned}&\frac{\lambda}{c}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-by+ay}\left[\frac{2}{\sqrt{y}}e^{-By} - 4\sqrt{B\pi}\bar{\Phi}(\sqrt{2By})\right]dy \\ =&\frac{\lambda}{c}\frac{1}{\sqrt{2\pi}}\left[\int_0^\infty\frac{2}{\sqrt{y}}e^{-(\frac{\mu^2}{2}+b)y} - 4\sqrt{B\pi}\int_0^\infty\bar{\Phi}(\sqrt{2By})e^{(a-b)y}dy\right] \\ =&\frac{2\lambda}{c}\frac{1}{\sqrt{\mu^2+2a}+\sqrt{\mu^2+2b}}.\end{aligned} \tag{5.17}$$

Therefore the joint Laplace transform of τ_1 and τ_2 is,

$$\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2}\mid X_0=0] = \frac{2\lambda}{c}\frac{1}{\sqrt{\mu^2+2\beta_1}+\sqrt{\mu^2+2r^+_{\beta_2}}}. \tag{5.18}$$

■

Corollary 5.1.2. *The probability density function of* τ_2 *conditional on* $X_0 = 0$ *is given by*

$$\begin{aligned} f_{\tau_2|X_0=0}(t) = & \frac{2\lambda(c\mu - 3\lambda)}{c} \exp\left\{-\frac{(c\mu-\lambda)^2 + (c\mu-3\lambda)^2}{2c} t\right\} \Phi\left(\frac{c\mu - 3\lambda}{\sqrt{c}}\sqrt{t}\right) \\ & + \frac{2\lambda}{\sqrt{2\pi c}} \frac{1}{\sqrt{t}} e^{-\frac{(c\mu-\lambda)^2}{2c} t}. \end{aligned} \tag{5.19}$$

Proof. From Theorem 5.1.1, by setting $\beta_1 = 0$, we obtain the Laplace transform of τ_2 conditional on $X_0 = 0$, i.e.

$$\mathbb{E}[e^{-\beta_2 \tau_2} \mid X_0 = 0] = \frac{2\lambda}{c} \frac{1}{\sqrt{\mu^2 + 2r^+_{\beta_2}} + \mu}. \tag{5.20}$$

Inverting this Laplace transform can give us the probability density function of τ_2 conditional on $X_0 = 0$. ■

Corollary 5.1.3. *The joint probability density function of* τ_1 *and* τ_2 *with* $X_0 = 0$ *is given by*

$$\begin{aligned} & f_{\tau_1,\tau_2|X_0=0}(t_1, t_2) \\ = & \frac{\lambda}{\pi}\sqrt{\frac{c^3}{t_1^3 t_2^3}} \exp\left\{-\frac{(c\mu-\lambda)^2 t_2 + c\mu^2 t_1}{2c}\right\} \frac{1}{a^{5/2}} \left[2\sqrt{2\pi}(a+\lambda^2) e^{\frac{\lambda^2}{2a}} \Phi\left(\frac{-\lambda}{\sqrt{a}}\right) - \lambda\sqrt{a}\right], \end{aligned} \tag{5.21}$$

where

$$a = \frac{ct_1 + c^2 t_2}{t_1 t_2}. \tag{5.22}$$

Proof. Given

$$r^+_{\beta_2} = \frac{-c\lambda\mu + c\beta_2 + \lambda^2 + \lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta_2}}{c^2}, \tag{5.23}$$

rewrite the joint Laplace transform of τ_1 and τ_2 as,

$$\begin{aligned}
&\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2} \mid X_0=0] \\
&= 2\lambda\frac{1}{\sqrt{(c\mu-\lambda)^2+2c\beta_2}+\lambda+c\sqrt{\mu^2+2\beta_1}} \\
&= 2\lambda\int_0^\infty \exp\left\{-\sqrt{c}u\sqrt{(\sqrt{c}\mu-\frac{\lambda}{\sqrt{c}})^2+2\beta_2}\right\} e^{-cu(\sqrt{\mu^2+2\beta_1}-\mu)}e^{-(c\mu+\lambda)u}du \\
&= 2\lambda\int_0^\infty\int_0^\infty e^{-\beta_2 t_2}\frac{\sqrt{c}u}{\sqrt{2\pi t_2^3}}\exp\left\{-\frac{\left(\sqrt{c}u-\frac{c\mu-\lambda}{\sqrt{c}}t_2\right)^2}{2t_2}\right\}dt_2 \\
&\quad \int_0^\infty e^{-\beta_1 t_1}\frac{cu}{\sqrt{2\pi t_1^3}}\exp\left\{-\frac{(cu-\mu t_1)^2}{2t_1}\right\}dt_1 e^{-2c\mu u}du.
\end{aligned} \tag{5.24}$$

Thus, the joint probability density function of τ_1 and τ_2 is

$$\begin{aligned}
&f_{\tau_1,\tau_2}(t_1,t_2) \\
&=\int_0^\infty \frac{\sqrt{c}u}{\sqrt{2\pi t_2^3}}\exp\left\{-\frac{\left(\sqrt{c}u-\frac{c\mu-\lambda}{\sqrt{c}}t_2\right)^2}{2t_2}\right\}\frac{cu}{\sqrt{2\pi t_1^3}}\exp\left\{-\frac{(cu-\mu t_1)^2}{2t_1}\right\}e^{-2c\mu u}du \\
&=\frac{\lambda}{\pi}\sqrt{\frac{c^3}{t_1^3 t_2^3}}\exp\left\{-\frac{(c\mu-\lambda)^2 t_2+c\mu^2 t_1}{2c}\right\}\int_0^\infty u^2\exp\left\{-\frac{(c^2t_1+c^3t_2)u^2-2u(-c\lambda t_1 t_2)}{2ct_1t_2}\right\}du.
\end{aligned} \tag{5.25}$$

Let $a=\frac{ct_1+c^2t_2}{t_1t_2}$, and $b=-\lambda$, so we have

$$\begin{aligned}
&\int_0^\infty u^2\exp\left\{-\frac{(c^2t_1+c^3t_2)u^2-2u(-c\lambda t_1t_2)}{2ct_1t_2}\right\}du \\
&=\int_0^\infty x^2\exp\left\{-\frac{ax^2-2bx}{2}\right\}dx \\
&=\frac{\mathrm{e}^{-\frac{ax^2}{2}}\left(\sqrt{2}\sqrt{\pi}\sqrt{a}\left(b^2+a\right)\mathrm{e}^{\frac{a^2x^2+b^2}{2a}}\operatorname{erf}\left(\frac{ax-b}{\sqrt{2}\sqrt{a}}\right)-2a\left(ax+b\right)\mathrm{e}^{bx}\right)}{2a^3} \\
&=\frac{1}{a^{5/2}}\left[2\sqrt{2\pi}(a+\lambda^2)e^{\frac{\lambda^2}{2a}}\Phi\left(\frac{-\lambda}{\sqrt{a}}\right)-\lambda\sqrt{a}\right].
\end{aligned} \tag{5.26}$$

■

Corollary 5.1.4. *The probability density function of τ_2 conditional on $-X_{\tau_1} = z$ with $z > 0$ and $X_0 = 0$ is given by*

$$f_{\tau_2|-X_{\tau_1}=z}(t_2) = \frac{\lambda z}{\sqrt{2\pi(t_2 c - z)^3}} e^{-\frac{[t_2\lambda - \mu(t_2 c - z)]^2}{2(t_2 c - z)}}. \tag{5.27}$$

Proof. The distribution of τ_2 only depends on the value of X_{τ_1}. Moreover, it has been shown in Gerber [43] that, the Laplace transform of τ_2 conditional on $-X_{\tau_1} = z$ is

$$\mathbb{E}[e^{-\beta_2\tau_2} \mid -X_{\tau_1} = z] = e^{-r^+_{\beta_2} z}. \tag{5.28}$$

So the probability density function of τ_2 conditional on $-X_{\tau_1} = z$ with $z > 0$ can be calculated from inverting the Laplace transform of τ_2 in (5.28), i.e.

$$\begin{aligned}
&\mathbb{E}[e^{-\beta_2\tau_2} \mid -X_{\tau_1} = z] \\
=\ &\exp\left\{\frac{-c\lambda\mu + c\beta_2 + \lambda^2 + \lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta_2}}{c^2}(-z)\right\} \\
=\ &\exp\left\{-\frac{\lambda z}{\sqrt{c^3}}\left[\sqrt{\left(\frac{c\mu-\lambda}{\sqrt{c}}\right)^2 + 2\beta_2} - \frac{c\mu-\lambda}{\sqrt{c}}\right]\right\} e^{-\frac{z}{c}\beta_2} \\
=\ &e^{-\tilde{\varepsilon}(\sqrt{\tilde{\mu}^2+2\beta_2}-\tilde{\mu})} e^{-\frac{z}{c}\beta_2} \\
=\ &e^{-\frac{z}{c}\beta_2} \hat{f}_{\tau_2|-X_{\tau_1}}(\beta_2),
\end{aligned} \tag{5.29}$$

where

$$\tilde{\varepsilon} = \frac{\lambda z}{\sqrt{c^3}}, \qquad \tilde{\mu} = \frac{c\mu - \lambda}{\sqrt{c}}. \tag{5.30}$$

Above Laplace transform indicates that $f_{\tau_2,-X_{\tau_1}}(t_2)$ has a scaled distribution, following an Inverse Gaussian distribution with parameters $\tilde{\varepsilon}$ and $\tilde{\mu}$. Thus, inverting above Laplace transform w.r.t. β_2, we have the probability density function of τ_2 conditional on $-X_{\tau_1} = z$

given by

$$
\begin{aligned}
f_{\tau_2|-X_{\tau_1}=z}(t_2) &= f\left(t_2 - \frac{z}{c}\right) \\
&= \frac{\frac{\lambda z}{\sqrt{c^3}}}{\sqrt{2\pi (t_2 - \frac{z}{c})^3}} \ \exp\left\{ -\frac{\left(\frac{\lambda z}{\sqrt{c^3}} - \frac{(c\mu-\lambda)(t_2 - \frac{z}{c})}{\sqrt{c}}\right)^2}{2(t_2 - \frac{z}{c})} \right\} \\
&= \frac{\lambda z}{\sqrt{2\pi (t_2 c - z)^3}} \ e^{-\frac{[t_2\lambda - \mu(t_2 c - z)]^2}{2(t_2 c - z)}}.
\end{aligned}
\tag{5.31}
$$

■

5.1.2 For X_t with $X_0 > 0$

Theorem 5.1.5. *Let $c = 1$ for the sake of simplicity, the joint Laplace transform of τ_1 and τ_2 with initial capital $X_0 = x$, $x > 0$, is given by*

$$
\begin{aligned}
&\mathbb{E}[e^{-\beta_1\tau_1 - \beta_2\tau_2} \mid X_0 = x] \\
&= \frac{b-\mu}{A-2\mu} e^{-\frac{b^2}{2}x - A\mu x + Abx} \left[e^{BD} \left(\Phi(D) - B\right) + e^{\frac{B^2}{2}} \bar{\Phi}(D - B) \right] + e^{-\frac{2A\mu x - A^2 x}{2}} \Phi(\sqrt{x}(\mu - A)),
\end{aligned}
\tag{5.32}
$$

where

$$
b = \sqrt{\mu^2 + 2r^+_{\beta_2}} + \mu, \qquad A = \sqrt{(\mu-\lambda)^2 + 2\beta_1} + \mu - \lambda, \tag{5.33}
$$

and

$$
D = \sqrt{x}\sqrt{\mu^2 + 2r^+_{\beta_2}}, \qquad B = -\sqrt{x}(A - 2\mu). \tag{5.34}
$$

Proof. We follow the proof method from Theorem 5.1.1, so we have

$$
\mathbb{E}[e^{-\beta_1\tau_1 - \beta_2\tau_2} \mid X_0 = x] = \mathbb{E}[\mathbb{E}[e^{-\beta_1\tau_1 - \beta_2\tau_2} \mid X_{\tau_1}]] = \mathbb{E}[e^{-r^+_{\beta_2}(-X_{\tau_1})} e^{-\beta_1\tau_1} \mid X_0 = x], \tag{5.35}
$$

where $\mathbb{E}[e^{-r^+_{\beta_2}(-X_{\tau_1})} e^{-\beta_1\tau_1} \mid X_0 = x]$ is just the joint Laplace transform of τ_1 and $-X_{\tau_1}$ w.r.t. β_1 and $-r^+_{\beta_2}$, which can be solved from $\mathbb{E}[e^{-\beta\tau - \nu(-X_\tau)} \mid X_0 = x]$ with $\beta = \beta_1$ and $\nu = r^+_{\beta_2}$.

Set $\kappa = 0$ in (3.17), so we have $f(x) = \mathbb{E}[e^{-\beta\tau} e^{-\nu(-X_\tau)} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x]$. We can obtain

$f(x)$ by inverting $\hat{f}(\xi)$ w.r.t. ξ from applying Laplace transform to $\mathcal{A}f(x) = 0$. From Theorem 4.3.3 and the change of variables $\sqrt{\mu^2 + 2\xi} - \mu = \eta$, $\sqrt{\mu^2 + 2\nu} - \mu = \gamma$, we have

$$
\begin{aligned}
&\hat{f}(\xi) \\
&= \frac{4\lambda}{\sqrt{(\mu-\lambda)^2 + 2\beta} + \sqrt{\mu^2 + 2\nu} + \lambda} \left[\frac{1}{\sqrt{\mu^2 + 2\xi} + \sqrt{\mu^2 + 2\nu}} \cdot \frac{1}{\sqrt{(\mu-\lambda)^2 + 2\beta} - \lambda + \sqrt{\mu^2 + 2\xi}} \right] \\
&= A \int_0^\infty e^{-\left(\sqrt{\mu^2+2\xi}+\sqrt{\mu^2+2\nu}\right)u} du \int_0^\infty e^{-\left(\sqrt{(\mu-\lambda)^2+2\beta}-\lambda+\sqrt{\mu^2+2\xi}\right)v} dv \\
&= A \int_0^\infty \int_0^\infty e^{-(u+v)\left(\sqrt{\mu^2+2\xi}-\mu\right)} e^{-u\left(\sqrt{\mu^2+2\nu}+\mu\right)} e^{-v\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)} dudv \\
&= A \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi x} \frac{u+v}{\sqrt{2\pi x^3}} e^{-\frac{(u+v-\mu x)^2}{2x}} dx\ e^{-u\left(\sqrt{\mu^2+2\nu}+\mu\right)} e^{-v\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)} dudv,
\end{aligned}
\tag{5.36}
$$

where $A = \frac{4\lambda}{\sqrt{(\mu-\lambda)^2+2\beta}+\sqrt{\mu^2+2\nu}+\lambda}$. Therefore, we have $f(x)$ as follows

$$
f(x) = A \int_0^\infty \int_0^\infty \frac{u+v}{\sqrt{2\pi x^3}} e^{-\frac{(u+v-\mu x)^2}{2x}}\ e^{-u\left(\sqrt{\mu^2+2\nu}+\mu\right)} e^{-v\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)} dudv. \tag{5.37}
$$

Solving this double integral gives us $f(x) = \mathbb{E}[e^{-\beta\tau} e^{-\nu(-X_\tau)} \mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x]$. ∎

We can derive the joint probability density function of τ_1 and τ_2 given $X_0 = x$ by simultaneously inverting $\mathbb{E}[e^{-\beta_1\tau_1 - \beta_2\tau_2} \mid X_0 = x]$ w.r.t. β_1 and β_2. The calculation could be complicated because of the non-symmetric property of τ_1 and τ_2, so the following method is provided.

Proposition 5.1.6. *The joint probability density function of τ_1 and τ_2 with initial capital $X_0 = x_0 > 0$ is given by*

$$
f_{\tau_1,\tau_2|X_0=x_0}(t_1, t_2) = \int_0^\infty f(t,z) \frac{\lambda z}{\sqrt{2\pi(t_2 c - z)^3}} e^{-\frac{[t_2\lambda - \mu(t_2 c - z)]^2}{2(t_2 c - z)}} dz, \tag{5.38}
$$

where $f(t,z)$ is just the joint probability density function of the first ruin time τ_1, overshoot

$-X_{\tau_1}$ *and initial capital* x_0 *in Theorem 4.3.3, i.e.*

$$\begin{aligned} f(t,z) &= f(t,z,x_0) \\ &= 2\lambda t \frac{1}{\sqrt{2\pi t^3}} \frac{1}{\sqrt{2\pi x_0^3}} \frac{1}{\sqrt{2\pi z^3}} e^{-\frac{(\mu-\lambda)^2}{2}t} e^{-\frac{\mu^2}{2}x_0} e^{-\frac{\mu^2}{2}z} \sqrt{\left(\frac{tz}{t+z}\right)^3} \\ &\quad \cdot \left\{ e^{A_2}(H_1(t,z,x_0)+H_2(t,z,x_0)) - e^{A_1}(H_3(t,z,x_0)+H_4(t,z,x_0)) \right\}, \end{aligned} \tag{5.39}$$

where

$$A_1 = exp\left\{ \frac{x_0(t(\lambda-\mu)-2z\mu)^2}{2(t+z)(x_0+t+z)} + \frac{tz(\mu+\lambda)^2}{2(t+z)} \right\}, \tag{5.40}$$

$$A_2 = exp\left\{ \frac{x_0(t(\lambda-\mu)+2z\lambda)^2}{2(t+z)(x_0+t+z)} + \frac{tz(\mu+\lambda)^2}{2(t+z)} \right\}. \tag{5.41}$$

Define

$$a = \frac{tz}{t+z}, \qquad c = \frac{x_0(t+z)}{t+x_0+z}, \tag{5.42}$$

$$d_1 = \frac{x_0(t\lambda - t\mu - 2z\mu)}{t+x_0+z}, \qquad d_2 = \frac{x_0(t\lambda - t\mu + 2z\lambda)}{t+x_0+z}, \tag{5.43}$$

then the functions $H_i, i = 1, ..., 4$ *can be formulated as*

$$\begin{aligned} H_1(t,z,x_0) = &-\frac{1}{t}\sqrt{\frac{ac}{(ac+t^2)^3}}\left(t^3(\mu+\lambda)+acd_2+2d_2t^2\right) \\ &\sqrt{2\pi}\bar{\Phi}\left(\frac{(ac+t^2)(\mu+\lambda)}{t\sqrt{c}} - \frac{t^2(\mu+\lambda)+td_2}{\sqrt{c(ac+t^2)}}\right) exp\left\{-\frac{at^2(\mu+\lambda+\frac{d_2}{t})^2}{2(ac+t^2)}\right\} \\ &+ ce^{-\frac{d_2^2}{2c}}\sqrt{2\pi}\bar{\Phi}\left(\sqrt{a}(\mu+\lambda)\right)\left(\frac{d_2}{t}+\mu+\lambda\right), \end{aligned} \tag{5.44}$$

$$\begin{aligned} H_2(t,z,x_0) = &\left(\sqrt{c}d_2(\mu+\lambda) + \frac{\sqrt{c^3}+\sqrt{c}d_2^2}{t}\right) 2\pi\{\Phi\left(-\frac{t\sqrt{a}(\mu+\lambda)+\sqrt{a}d_2}{\sqrt{ac+t^2}}\right) \\ &- BvN\left(h_1 = -\frac{t\sqrt{a}(\mu+\lambda)+\sqrt{a}d_2}{\sqrt{ac+t^2}}, k_1 = \sqrt{a}(\mu+\lambda); \rho = -\frac{t}{\sqrt{ac+t^2}}\right) \\ &- \Phi\left(-\frac{d_2}{\sqrt{c}}\right)\bar{\Phi}(\sqrt{a}(\mu+\lambda))\} - \frac{c^2\sqrt{a}}{ac+t^2} exp\left\{-\frac{(\mu+\lambda)^2 a}{2} - \frac{d_2^2(t^2+1)}{2c(ac+t^2)}\right\}, \end{aligned} \tag{5.45}$$

$$
\begin{aligned}
H_3(t,z,x_0) = &-\frac{1}{t}\sqrt{\frac{ac^3}{(ac+t^2)^3}}\left(t^3(\mu+\lambda)+acd_1+2d_1t^2\right) \\
&\sqrt{2\pi}\Phi\left(\frac{(ac+t^2)(\mu+\lambda)}{t\sqrt{c}}-\frac{t^2(\mu+\lambda)-td_1}{\sqrt{c(ac+t^2)}}\right)exp\left\{-\frac{at^2(\mu+\lambda-\frac{d_1}{t})^2}{2(ac+t^2)}\right\} \\
&+ce^{-\frac{d_1^2}{2c}}\sqrt{2\pi}\bar{\Phi}\left(\sqrt{a}(\mu+\lambda)\right)\left(\frac{d_1}{t}+\mu+\lambda\right),
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
H_4(t,z,x_0) = &\left(\sqrt{c}d_1(\mu+\lambda)-\frac{\sqrt{c^3}-\sqrt{c}d_1^2}{t}\right)2\pi \\
&\{BvN\left(h_2=-\frac{t\sqrt{a}(\mu+\lambda)-\sqrt{a}d_1}{\sqrt{ac+t^2}},k_2=\sqrt{a}(\mu+\lambda);\rho=-\frac{t}{\sqrt{ac+t^2}}\right) \\
&+\Phi\left(-\frac{d_1}{\sqrt{c}}\right)\bar{\Phi}(\sqrt{a}(\mu+\lambda))\}-\frac{c^2\sqrt{a}}{ac+t^2}exp\left\{-\frac{(\mu+\lambda)^2a}{2}-\frac{d_1^2(t^2+1)}{2c(ac+t^2)}\right\},
\end{aligned}
\tag{5.47}
$$

where BvN stands for Bivariate Normal cumulative distribution function, i.e.

$$
BvN(h,k;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{k}\int_{-\infty}^{h}exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dxdy, \tag{5.48}
$$

with $-\infty<h,k<\infty$*, and correlation coefficient* $-1<\rho<1$.

Proof. By the law of total probability, the joint probability density function of τ_1 and τ_2 can be written as

$$
\begin{aligned}
f_{\tau_1,\tau_2}(t_1,t_2\mid X_0=x_0) &= \mathbb{P}(\tau_1\in dt_1,\tau_2\in dt_2) \\
&= \int_0^\infty \mathbb{P}(\tau_1\in dt_1,\tau_2\in dt_2,-X_{\tau_1}\in dz) \\
&= \int_0^\infty \mathbb{P}(\tau_1\in dt_1,-X_{\tau_1}\in dz)\mathbb{P}(\tau_2\in dt_2\mid -X_{\tau_1}=z) \\
&= \int_0^\infty f(t,z\mid X_0=x_0)\mathbb{P}(\tau_2\in dt_2\mid -X_{\tau_1}=z) \\
&= \int_0^\infty f(t,z,x_0)\mathbb{P}(\tau_2\in dt_2\mid -X_{\tau_1}=z),
\end{aligned}
\tag{5.49}
$$

where $f(t,z,x_0)$ is the joint probability density function of first ruin time τ_1, overshoot $-X_{\tau_1}$ and initial capital x_0 from Theorem 4.3.3. Meanwhile, $\mathbb{P}(\tau_2\in dt_2\mid -X_{\tau_1}=z)$ is just the probability density function of τ_2 conditional on $-X_{\tau_1}=z>0$ which has been derived from Corollary 5.1.4.

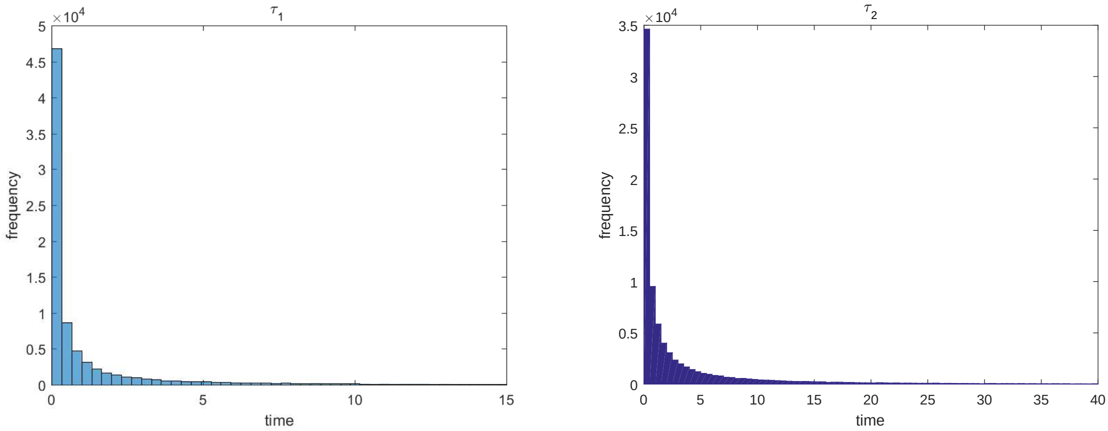
■

Remark. Note that in previous chapters we specify the net profit condition, $c > \frac{\lambda}{\mu}$ (i.e. positive safety loading $\theta > 0$), to ensure that the probability of ruin is less than 1. That is $\psi(x) < 1$ for all $x \geq 0$ and $\psi(x) \to 0$ as $x \to \infty$. This net profit condition results that the surplus process X_t might not go back to 0 after the first ruin time τ_1. However τ_2 exists based on the condition that $\psi(x) = 1$ for all $x \geq 0$ and that X_t is able to go back above 0 after τ_1, which is true only if $c < \frac{\lambda}{\mu}$ holds, i.e. negative safety loading $\theta < 0$. When $\theta < 0$, the premium rate c is reduced in order to make the insurance company continue its business.

Figures 5.1 to 5.9 show the simulation results of distribution of the τ_1 and τ_2 respectively with different settings of initial capital X_0 and inverse Gaussian parameter μ in infinite time horizon. We assume that surplus process X_t is defined as in (4.3), and that the claim size Y_i, $i = 1, 2, ...$, follows $IG(\varepsilon, \mu)$ defined in (4.2). The total claim amount process is an inverse Gaussian process Z_t with $\varepsilon \to 0$. This is to consider infinitely many and arbitrarily small claims over any finite time interval, which is also by the infinite divisibility property of IG process. We choose $\varepsilon = 0.001$. Other parameters are set at $c = 1$, and $\lambda = 3$ under the condition $c < \frac{\lambda}{\mu}$. The number of simulation $N = 100,000$. For the simulation, we simultaneously record the time length when the surplus process X_t first goes cross zero both from above to below and from below to above, which is τ_1 and τ_2 respectively. Then, repeat above procedure for N times to obtain their histograms.

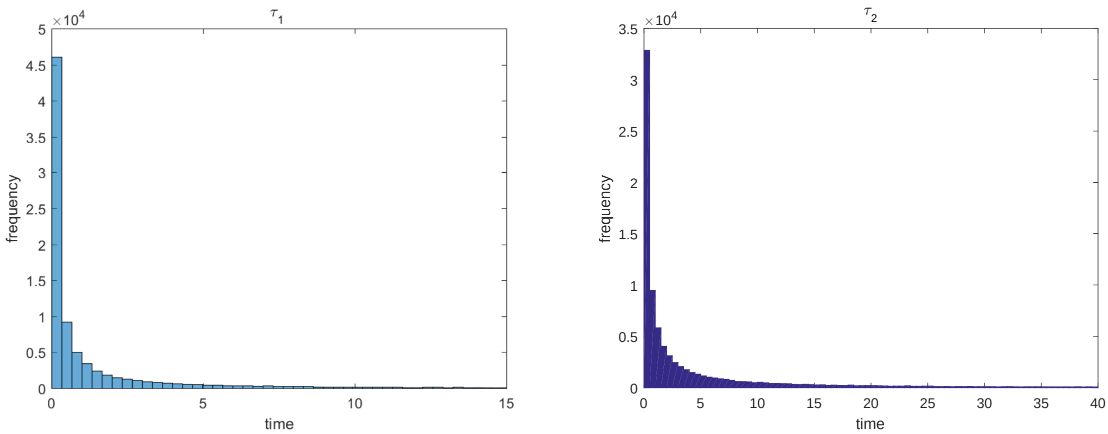
It is noticeable that the claim size depends on the value of μ, so the choices of μ describe the average size of claim, resulting in different behaviours of τ_1 and τ_2. We can easily see from these figures that both τ_1 and τ_2 are skewed to the right with extremely long tails. A distinguishing characteristic is that due to the long tail, extremely large outcomes could occur even when almost all outcome are very small. When $X_0 = 0$, the most values of τ_1 and τ_2 concentrate on the values that are closed to zero. This is because ruin is more likely to occurs within shorter time period. When X_0 increases, ruin is less likely to occur so τ_1 and τ_2 increase accordingly. On the other hand, when considering X_0 is fixed, τ_1 and τ_2 increases as μ increases. This can be easily seen from figure 5.7 and figure 5.8. τ_1 and τ_2 have fatter tails when μ increases. Since the mean of claim size $\mathbb{E}[Y_i] = 1/\mu$, claim size decreases averagely when μ increases. This leads to that ruin is less likely to occur, so it takes longer time to ruin, and τ_1 and τ_2 increase at the same time.

Figures 5.10 to 5.13 show the simulated results of the joint distribution of τ_1 and τ_2 when $c = 1$, $\lambda = 3$, and different settings of initial capital X_0 and μ in infinite time horizon. The number of simulation $N = 100,000$. As we can see from the figures that τ_1 and τ_2 are correlated. As X_0 increases, the correlation between τ_1 and τ_2 strengthens. With the value of μ growing, the correlation between τ_1 and τ_2 also strengthens.



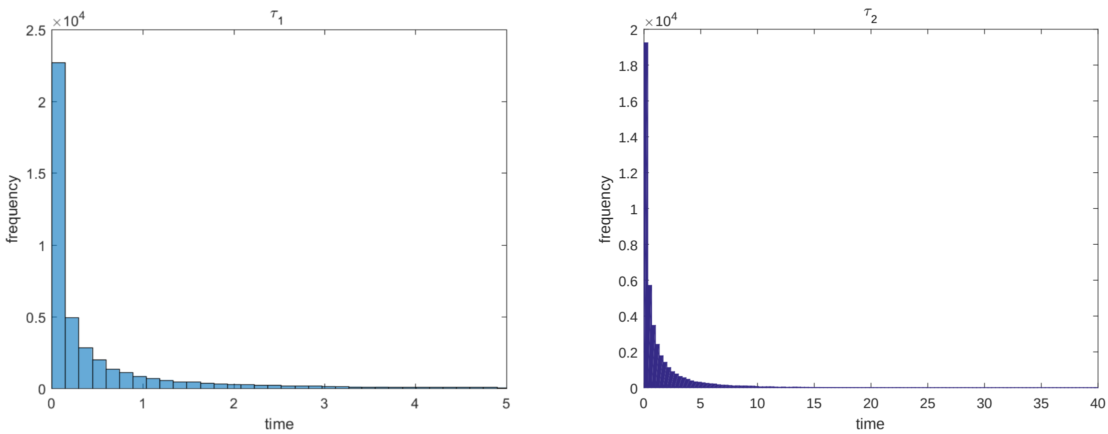
(a) τ_1 (b) τ_2

Figure 5.1: Histograms of τ_1 and τ_2, $X_0 = 0, c = 1, \lambda = 3, \mu = 0.8$



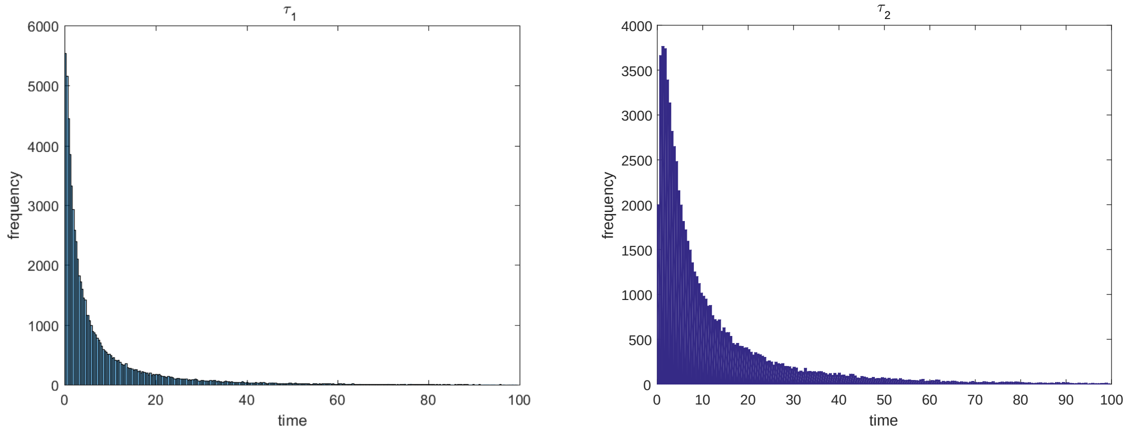
(a) τ_1 (b) τ_2

Figure 5.2: Histograms of τ_1 and τ_2, $X_0 = 0, c = 1, \lambda = 3, \mu = 1.0$



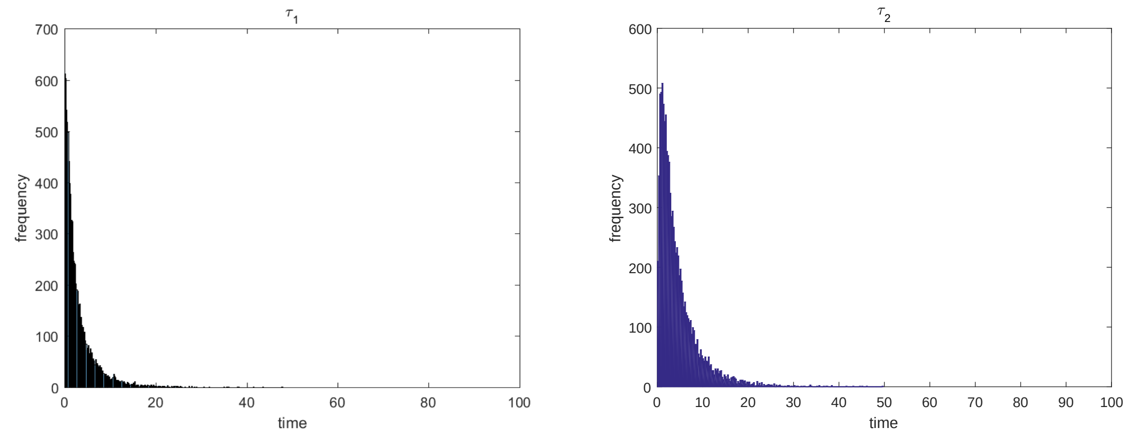
(a) τ_1 (b) τ_2

Figure 5.3: Histograms of τ_1 and τ_2, $X_0 = 0, c = 1, \lambda = 3, \mu = 1.5$



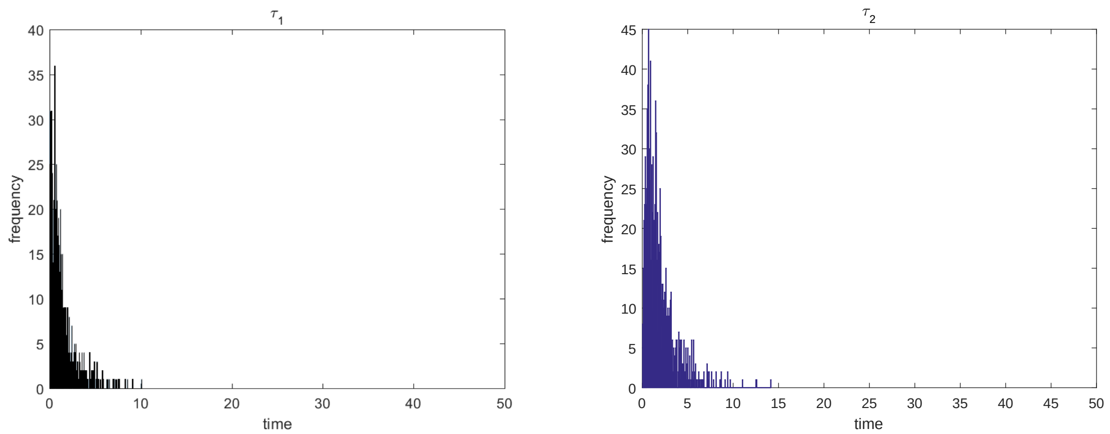
(a) τ_1 (b) τ_2

Figure 5.4: Histograms of τ_1 and τ_2, $X_0 = 1, c = 1, \lambda = 3, \mu = 0.8$



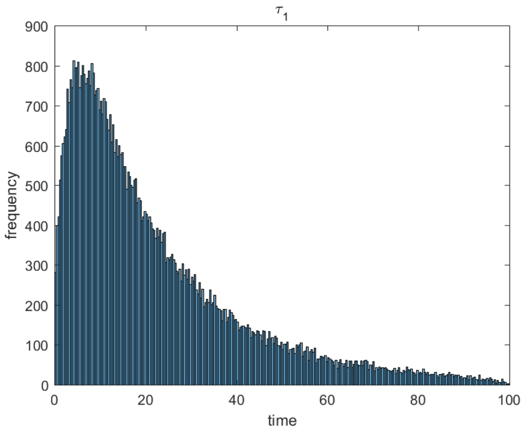
(a) τ_1 (b) τ_2

Figure 5.5: Histograms of τ_1 and τ_2, $X_0 = 1, c = 1, \lambda = 3, \mu = 1.5$

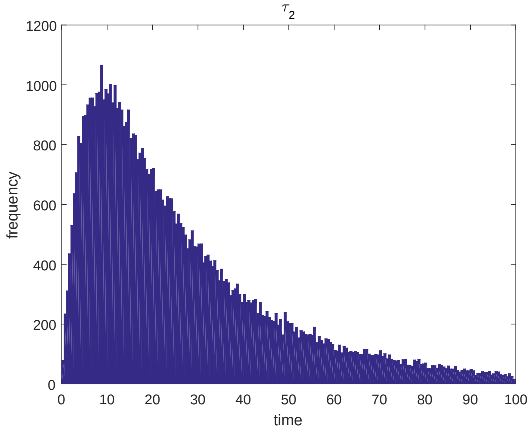


(a) τ_1 (b) τ_2

Figure 5.6: Histograms of τ_1 and τ_2, $X_0 = 1, c = 1, \lambda = 3, \mu = 2.0$

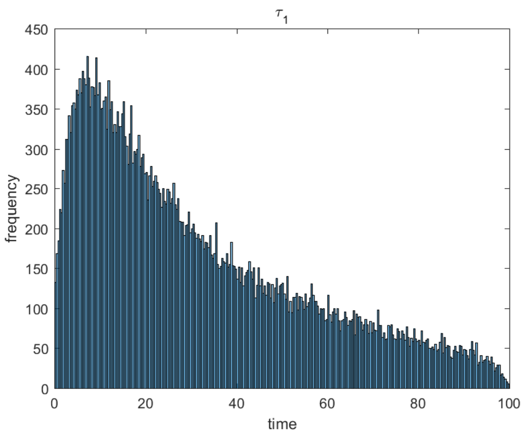


(a) τ_1

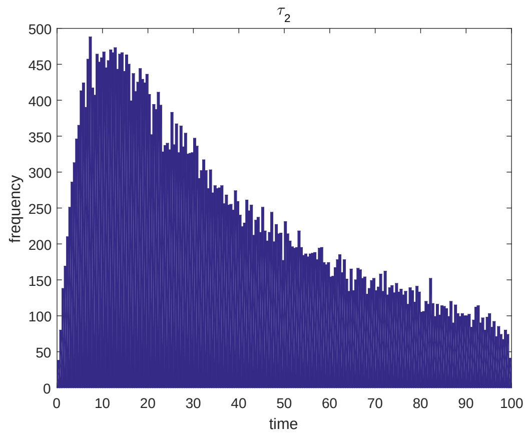


(b) τ_2

Figure 5.7: Histograms of τ_1 and τ_2, $X_0 = 5, c = 1, \lambda = 3, \mu = 0.8$

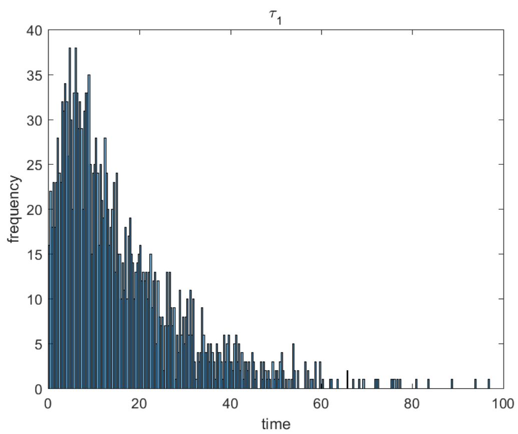


(a) τ_1

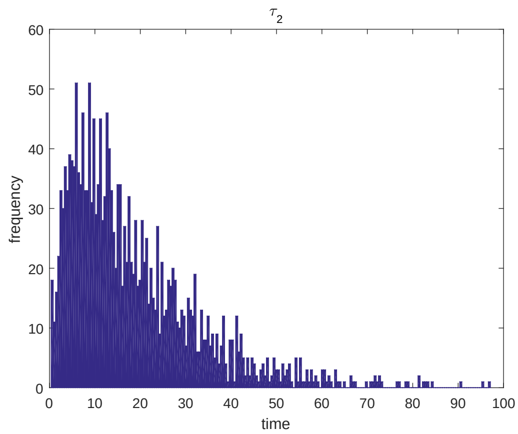


(b) τ_2

Figure 5.8: Histograms of τ_1 and τ_2, $X_0 = 5, c = 1, \lambda = 3, \mu = 1.0$

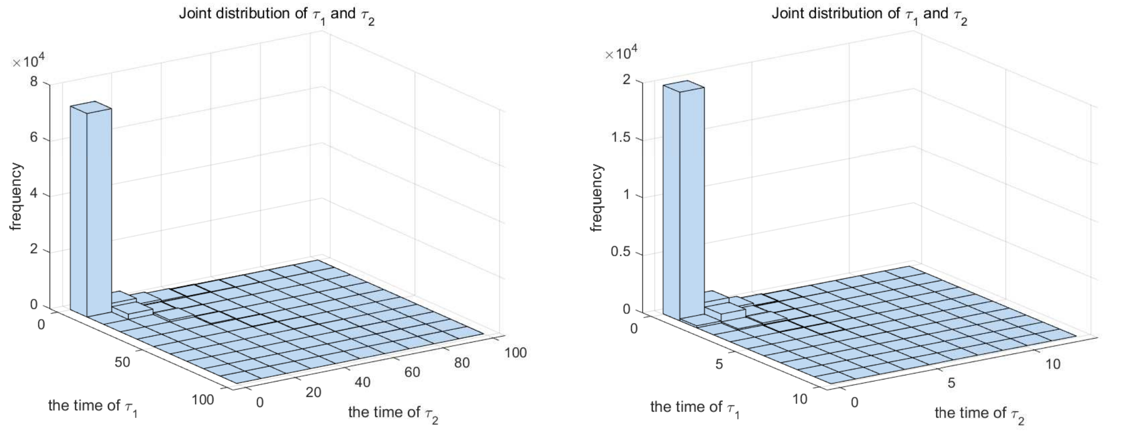


(a) τ_1



(b) τ_2

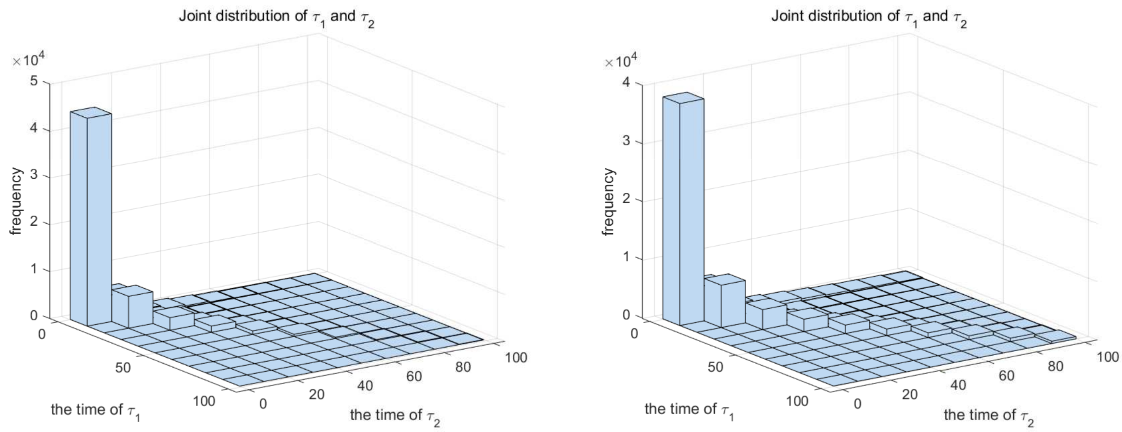
Figure 5.9: Histograms of τ_1 and τ_2, $X_0 = 5, c = 1, \lambda = 3, \mu = 1.3$



(a) $X_0 = 0, c = 1, \lambda = 3, \mu = 0.8$

(b) $X_0 = 0, c = 1, \lambda = 3, \mu = 1.5$

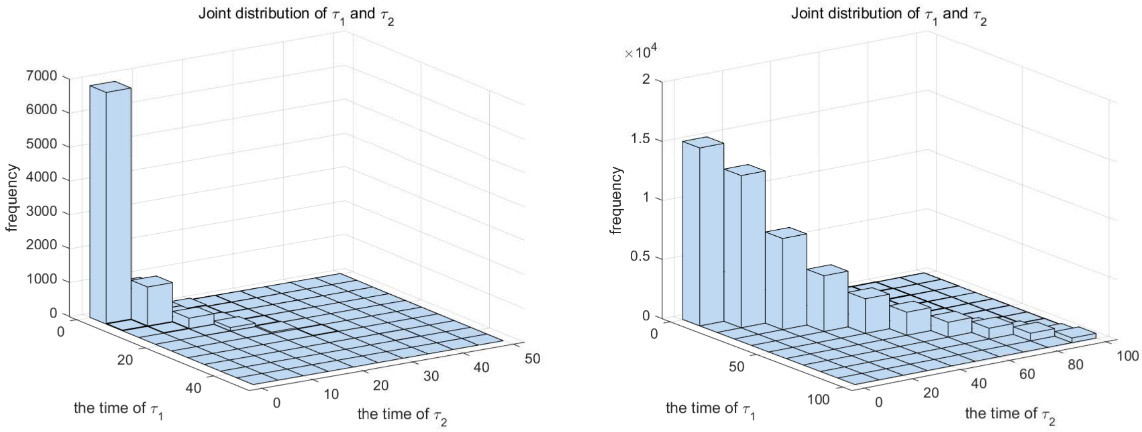
Figure 5.10: Joint distributions of τ_1 and τ_2 with $\mu = 0.8$ and $\mu = 1.5$ respectively



(a) $X_0 = 1, c = 1, \lambda = 3, \mu = 0.8$

(b) $X_0 = 1, c = 1, \lambda = 3, \mu = 1.0$

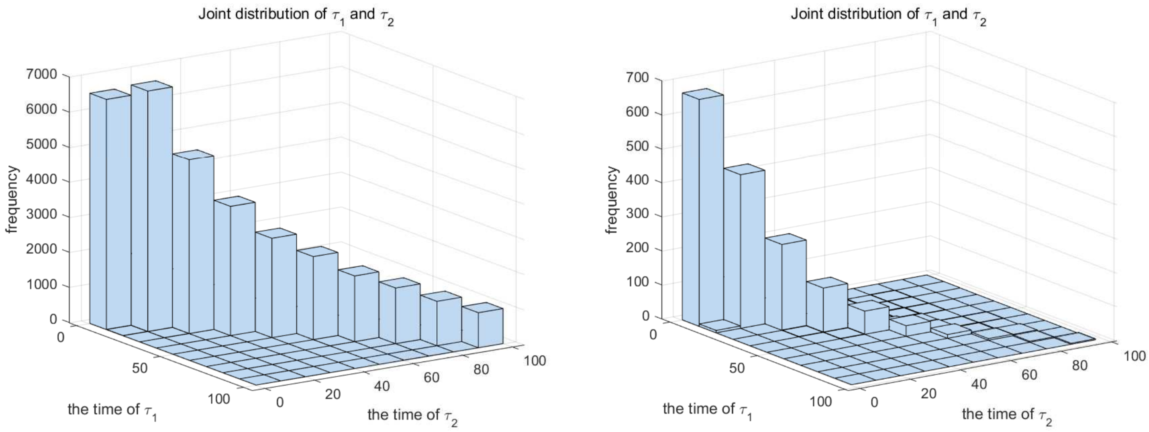
Figure 5.11: Joint distributions of τ_1 and τ_2 with $\mu = 0.8$ and $\mu = 1.0$ respectively



(a) $X_0 = 1, c = 1, \lambda = 3, \mu = 1.5$

(b) $X_0 = 5, c = 1, \lambda = 3, \mu = 0.8$

Figure 5.12: Joint distribution of τ_1 and τ_2



(a) $X_0 = 5, c = 1, \lambda = 3, \mu = 1.0$

(b) $X_0 = 5, c = 1, \lambda = 3, \mu = 1.3$

Figure 5.13: Joint distribution of τ_1 and τ_2

5.2 Laplace Transform of Parisian Excursions

5.2.1 Introduction

Recall the risk process defined as

$$X_t = x + ct - Z_t, \tag{5.50}$$

where Z_t is an Inverse Gaussian process. Also recall that the classical probability of ruin in the infinite time horizon is defined as

$$\psi(x) = \mathbb{P}(\tau < \infty), \tag{5.51}$$

where τ is the ruin time defined as

$$\tau = \inf\{t \geq 0 \mid X_t \leq 0\}. \tag{5.52}$$

In this section, we consider Parisian ruin time, and we aim to find the Laplace transform of Parisian ruin time and the probability of Parisian type of ruin. Parisian type of ruin occurs if the surplus process drops under zero and continuously stays under zero for a pre-defined length $d > 0$.

We use the notations by Dassios & Wu (See [21] for example) to define excursions. For fixed $t > 0$, denote g_t^X the last crossing time of 0 before time t and d_t^X the first crossing time of 0 after time t respectively, i.e.

$$g_t^X = \sup\{s \leq t \mid \text{sign}(X_s) \neq \text{sign}(X_t)\}, \tag{5.53}$$

and

$$d_t^X = \inf\{s \geq t \mid \text{sign}(X_s) \neq \text{sign}(X_t)\}, \tag{5.54}$$

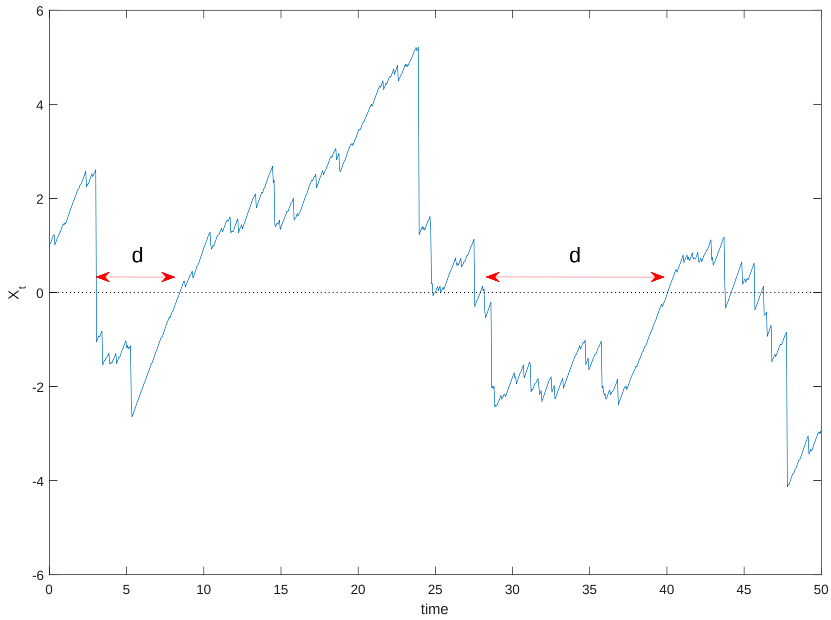


Figure 5.14: Illustration of Parisian ruin time.

where

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0. \end{cases} \tag{5.55}$$

We also assume that $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$.

Therefore, the path between g_t^X and d_t^X is the excursion of the surplus process X_t below or above 0. The time interval (g_t^X, d_t^X) is the excursion time interval straddling time t. We assume $d > 0$, now we define the first time when the duration time of excursion of X_t under 0 reaches the length d, i.e.

$$\tau_d^{X_t} = \inf\{t > 0 \mid (t - g_t^X) 1_{\{X_t < 0\}} \geq d\}. \tag{5.56}$$

Next, we define the probability of the Parisian type of ruin in the infinite time horizon as

$$\mathbb{P}(\tau_d^{X_t} < \infty), \tag{5.57}$$

which is the probability that we are interested in.

5.2.2 Definitions

In this subsection, we introduce a two-state semi-Markov model for this risk process with Inverse Gaussian process, which consists of two states and was proposed by Dassios and Wu [21]. Define the following process S_t^X based on X_t by

$$S_t^X = \begin{cases} 1, & \text{if } X_t > 0 \\ \\ -1, & \text{if } X_t < 0. \end{cases} \tag{5.58}$$

Clearly the definitions (5.53), (5.54) and (5.56) are true similarly for the process S_t^X. We can then define

$$U_t^X = t - g_t^X, \tag{5.59}$$

as the time straddled in the current state. (S_t^X, U_t^X) becomes a Markov process. As a result, S_t^X is a two state semi-Markov process consisting of the state space $\{1, -1\}$, with 1 denoting the state when X_t is above zero and -1 denoting the state when X_t is below zero.

Furthermore, set $T_{i,k}^X$, $i = 1, -1$, and $k = 1, 2, ...$ to be the inter-arrival time that elapses in state i when the process X_t reaches the state i for the k-th time. Hence we have

$$T_{i,k}^X = U_{d_t^X}^X = d_t^X - g_t^X. \tag{5.60}$$

So, $T_{1,k}^X$ denotes the length of excursion above 0 and $T_{-1,k}^X$ denotes the length of excursion below 0.

Due to the strong Markov property of X_t, we note that $T_{-1,k}^X$ depends on the value of the previous overshoot, and the value of the overshoot depends on the initial capital X_0 and the previous ruin time. In addition, $T_{1,k}^X$, $k = 1, 2, ...$, are independent and identically distributed, and $T_{-1,k}^X$, $k = 1, 2, ...$, are also independent and identically distributed. It turns out from the joint Laplace transform of τ_1 and τ_2 in Theorem 5.1.1 that each pair of $(T_{1,k}^X, T_{-1,k}^X)$ are independent and identically distributed for $k = 1, 2, ...$ as well.

Then we define the history of process S_t^X up to time t_n, $n = 0, 1, 2, ...$

$$\mathcal{H}_n = \{S_0^X, t_0; S_1^X, t_1; ...; S_n^X, t_n\}, \tag{5.61}$$

and due to the Markov property of S_t^X, i.e.

$$\mathbb{P}(S_n^X = j, t < T_{i,n}^X < t + \Delta t \mid \mathcal{H}_{n-1}) = \mathbb{P}(S_n^X = j, t < T_{i,n}^X < t + \Delta t \mid S_{n-1}^X), \tag{5.62}$$

the transition densities $p_{i,j}(t)$ for S_t^X can be defined as

$$p_{i,j}(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(S_n^X = i, t < T_{i,n}^X < t + \Delta t \mid S_{n-1}^X = j)}{\Delta t}, \tag{5.63}$$

and

$$\mathbb{P}_{i,j}(t) = \mathbb{P}(T_{i,k}^X < t) = \int_0^t p_{i,j}(s)ds, \tag{5.64}$$

$$\bar{\mathbb{P}}_{i,j}(t) = \mathbb{P}(T_{i,k}^X \geq t). \tag{5.65}$$

We also define the following $\hat{\mathbb{P}}_{i,j}(\beta)$ and $\tilde{\mathbb{P}}_{i,j}(\beta)$ to simplify our notations,

$$\hat{\mathbb{P}}_{i,j}(\beta) = \int_0^\infty e^{-\beta t} p_{i,j}(t)dt, \tag{5.66}$$

$$\tilde{\mathbb{P}}_{i,j}(\beta) = \int_0^d e^{-\beta t} p_{i,j}(t)dt. \tag{5.67}$$

$\mathbb{P}_{i,j}(t)$ gives the probability that the time length for which surplus process X_t stays in state i is not longer than t. It is also noticeable that we set $T_{i,k}^X = \infty$ if the surplus process stays in state i for infinite time period. The net profit condition $c > \frac{\lambda}{\mu}$ yields that $\mathbb{P}(T_{1,k}^X = \infty) > 0$ for $k = 1, 2, ...$

Next, we aim to find transition probabilities $p_{i,j}(t)$ by considering excursions above 0 and excursions under 0 respectively.

(i) For the excursions above 0. We use τ_0^* to define the stopping time at the end of current excursion above 0,

$$\tau_0^* = \inf\{t \geq 0 \mid X_t \leq 0, X_0^* = 0\}, \tag{5.68}$$

where X_0^* is the value at the beginning of current excursion above 0. Thus we have

$$\begin{aligned}
\hat{\mathbb{P}}_{1,-1}(\beta) &= \mathbb{E}\left[e^{-\beta T_{1,k}^X}\right] = \mathbb{E}\left[e^{-\beta\tau_0^*} \mid X_0^* = 0\right] \\
&= \int_0^\infty e^{-\beta x} f_{\tau|X_0=0}(x)\ dx \\
&= \int_0^\infty e^{-\beta x} f_{-X_\tau|X_0=0}(x)\ dx \\
&= \mathbb{E}[e^{-\beta(-X_\tau)} \mid X_0 = 0] \\
&= \frac{2\lambda}{c\left(\sqrt{\mu^2+2\beta}+\mu\right)}.
\end{aligned} \tag{5.69}$$

The third equality is due to the symmetry from the joint probability density of classical ruin time τ and overshoot $-X_\tau$ in Theorem 4.3.1. Inverting $\hat{\mathbb{P}}_{1,-1}(\beta)$ w.r.t. β gives us

$$p_{1,-1}(t) = \frac{2\lambda}{c}\left[\frac{1}{\sqrt{2\pi t}}e^{-\frac{\mu^2 t}{2}} - \mu\ \bar{\Phi}\left(\mu\sqrt{t}\right)\right]. \tag{5.70}$$

(ii) For the excursions below 0. We define τ^* as the elapsed time when the process X_t goes back to zero after previous ruin, i.e.

$$\tau^* = \inf\{t > 0 \mid X_t \geq 0, X_0^* = -z, z > 0\}, \tag{5.71}$$

where $-X_0^*$ is the overshoot when the previous ruin occurs before τ^*. In the meantime, according to Gerber [43], it has shown that

$$\mathbb{E}[e^{-\beta\tau^*} \mid -X_0^* = z] = e^{-r_\beta^+ z}. \tag{5.72}$$

We also note that every excursion below 0 can be seen as starting from an overshoot below 0 with the length $|z|$, and τ^* only depends on the value of $-X_0^*$. Meanwhile, from Corollary 4.2.3, the probability density function of the overshoot $-X_\tau$ with 0 initial capital is given by

$$f_{-X_\tau|X_0=0}(z) = \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi z}}e^{-\frac{\mu^2}{2}z} - \frac{2\lambda\mu}{c}\bar{\Phi}(\mu\sqrt{z}). \tag{5.73}$$

Therefore, we have

$$
\begin{aligned}
\hat{\mathbb{P}}_{-1,1}(\beta) &= \mathbb{E}\left[e^{-\beta T^X_{-1,k}}\right] \\
&= \int_0^\infty \mathbb{E}\left[e^{-\beta\tau^*} \mid -X_0^* = z\right] f_{-X_0^*|X_0=0}(z)\ dz \\
&= \int_0^\infty e^{-r_\beta^+ z} f_{-X_0^*|X_0=0}(z)\ dz \\
&= \mathbb{E}\left[e^{-r_\beta^+(-X_0^*)} \mid X_0 = 0\right] \\
&= \frac{2\lambda}{\sqrt{(c\mu-\lambda)^2 + 2c\beta} + \lambda + c\mu}.
\end{aligned} \tag{5.74}
$$

Inverting $\hat{\mathbb{P}}_{-1,1}(\beta)$ w.r.t. β provides us with

$$
\begin{aligned}
p_{-1,1}(t) = {} & \frac{2\lambda(c\mu-3\lambda)}{c} \exp\left\{-\frac{(c\mu-\lambda)^2 + (c\mu-3\lambda)^2}{2c} t\right\} \Phi\left(\frac{c\mu-3\lambda}{\sqrt{c}}\sqrt{t}\right) \\
& + \frac{2\lambda}{\sqrt{2\pi c}} \frac{1}{\sqrt{t}}\ e^{-\frac{(c\mu-\lambda)^2}{2c}t}.
\end{aligned} \tag{5.75}
$$

5.2.3 Laplace Transform of $\tau_d^{X_t}$

This section gives the Laplace transform of $\tau_d^{X_t}$ for the case with $X_0 = 0$ and the one with $X_0 = x$, $x > 0$.

Theorem 5.2.1. *The Laplace transform of $\tau_d^{X_t}$ with $X_0 = 0$ is given by*

$$
\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] = e^{-\beta d}\ \frac{c}{c-2\lambda} \left(\sqrt{\mu^2 + 2\beta} + \sqrt{\mu^2 + 2r_\beta^+}\ \right)\ \hat{\mathbb{P}}_{1,-1}(\beta)\ \bar{\mathbb{P}}_{-1,1}(d), \tag{5.76}
$$

where $\bar{\mathbb{P}}_{-1,1}(d) = 1 - \mathbb{P}_{-1,1}(d)$,

$$
\begin{aligned}
\mathbb{P}_{-1,1}(d) = {} & \frac{2\lambda}{c\mu-3\lambda}\frac{1}{q}\left\{\frac{1}{2\sqrt{2q+1}}[2\ \Phi(p\sqrt{2q+1}) - 1] - \Phi(p)\ e^{-qp^2} + \frac{1}{2}\right\} \\
& + \frac{2\lambda}{c\mu-\lambda}\left[2\ \Phi\left(\frac{(c\mu-\lambda)^2}{c}d\right) - 1\right],
\end{aligned} \tag{5.77}
$$

$$
p = \frac{cd^2}{(c\mu-3\lambda)^2}, \qquad q = \frac{1}{2} + \frac{(c\mu-\lambda)^2}{2(c\mu-3\lambda)^2}, \tag{5.78}
$$

and

$$\hat{\mathbb{P}}_{1,-1}(\beta) = \frac{2\lambda}{c\left(\sqrt{\mu^2+2\beta}+\mu\right)}. \tag{5.79}$$

Proof. To find the Laplace transform, we first use A_k to denote the event that $\tau_d^{X_t}$ is achieved during the kth excursion under state $S_t^X = -1$. Thus, by the Law of total expectation we have

$$\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[e^{-\beta\tau_d^{X_t}} \mid A_k\right]\mathbb{P}(A_k). \tag{5.80}$$

We also note that conditional on A_k, the Parisian ruin time $\tau_d^{X_t}$ consists of k excursions above 0, $k-1$ excursions under 0, and the last excursion under 0. All of the excursions, including k excursions above 0 and $k-1$ excursions below 0, have time length less than d. The last excursion has length of d. Conditional on A_k, we have

$$\tau_d^{X_t} \mid A_k = \sum_{j=1}^{k-1}\left(T_{1,j}^X + T_{-1,j}^X\right) \ + \ T_{1,k}^X \ + \ d \mid T_{-1,1}^X < d, \dots, T_{-1,k-1}^X < d, T_{-1,k}^X > d. \tag{5.81}$$

By the definition of $P_{i,j}$, $T_{1,j}^X$'s are distributed as $\mathbb{P}_{1,-1}$, $T_{-1,j}^X$'s are distributed as $\mathbb{P}_{-1,1}$. It is also important to note that the pairs of $(T_{1,j}^X, T_{-1,j}^X)$, $j = 1, 2, ..., k$, are independent and distributed as in Theorem 5.1.1 given $X_0 = 0$. $\mathbb{P}_{i,j}$ has probability density function $p_{i,j}$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\beta\tau_d^{X_t}} \mid A_k\right] \\
&= \mathbb{E}\left[e^{-\beta\left(\sum_{j=1}^{k-1}(T_{1,j}^X+T_{-1,j}^X)+T_{1,k}^X+d\right)} \mid T_{-1,1}^X < d, \dots, T_{-1,k-1}^X < d, T_{-1,k}^X > d\right] \\
&= \mathbb{E}\left[e^{-\beta\left(T_{1,1}^X+T_{-1,1}^X\right)} \dots e^{-\beta\left(T_{1,k-1}^X+T_{-1,k-1}^X\right)} e^{-\beta\left(T_{1,k}^X+d\right)} \mid T_{-1,1}^X < d, \dots, T_{-1,k-1}^X < d, T_{-1,k}^X > d\right] \\
&= e^{-\beta d}\left\{\mathbb{E}\left[e^{-\beta\left(T_{1,j}^X+T_{-1,j}^X\right)}\right]\right\}^{k-1} \ \frac{\mathbb{E}\left[e^{-\beta T_{1,j}^X} \mid T_{-1,1}^X < d, \dots, T_{-1,k-1}^X < d, T_{-1,k}^X > d\right]}{[\mathbb{P}_{-1,1}(d)]^{k-1}} \\
&= e^{-\beta d}\left(\frac{2\lambda}{c}\frac{1}{\sqrt{\mu^2+2\beta}+\sqrt{\mu^2+2r_\beta^+}}\right)^{k-1} \frac{\int_0^\infty e^{-\beta t}p_{1,-1}(t)\ dt}{[\mathbb{P}_{-1,1}(d)]^{k-1}},
\end{aligned} \tag{5.82}$$

where

$$r_\beta^+ = \frac{-c\lambda\mu + c\beta + \lambda^2 + \lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta}}{c^2}. \tag{5.83}$$

The last equality comes from the joint Laplace transform in (5.8).

We have also noted that based on the independence of $T_{-1,j}^X$'s and they follow distribution of $\mathbb{P}_{-1,1}$,

$$\mathbb{P}(A_k) = [\mathbb{P}_{-1,1}(d)]^{k-1}(1 - \mathbb{P}_{-1,1}(d)). \tag{5.84}$$

Combining all above, the Laplace transform $\tau_d^{X_t}$ of can be calculated as

$$\begin{aligned}
&\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left[e^{-\beta\tau_d^{X_t}} \mid A_k\right] \mathbb{P}(A_k) \\
&= e^{-\beta d} \sum_{k=1}^{\infty} \left(\frac{2\lambda}{c} \frac{1}{\sqrt{\mu^2+2\beta} + \sqrt{\mu^2 + 2r_\beta^+}}\right)^{k-1} \frac{\int_0^\infty e^{-\beta t} p_{1,-1}(t)\, dt}{[\mathbb{P}_{-1,1}(d)]^{k-1}} [\mathbb{P}_{-1,1}(d)]^{k-1}(1-\mathbb{P}_{-1,1}(d)) \\
&= e^{-\beta d} \sum_{k=1}^{\infty} \left(\frac{2\lambda}{c} \frac{1}{\sqrt{\mu^2+2\beta} + \sqrt{\mu^2 + 2r_\beta^+}}\right)^{k-1} \hat{\mathbb{P}}_{1,-1}(\beta)\, \bar{\mathbb{P}}_{-1,1}(d) \\
&= e^{-\beta d} \frac{c}{c-2\lambda}\left(\sqrt{\mu^2+2\beta} + \sqrt{\mu^2+2r_\beta^+}\right) \hat{\mathbb{P}}_{1,-1}(\beta)\, \bar{\mathbb{P}}_{-1,1}(d).
\end{aligned} \tag{5.85}$$

Furthermore, according to the formula of $p_{-1,1}(t)$ in (5.75), we have that

$$\begin{aligned}
\mathbb{P}_{-1,1}(d) \;&=\; \mathbb{P}(T_{-1,k}^X < d) \;=\; \int_0^d p_{-1,1}(t)\, dt \\
&= \int_0^d \frac{2\lambda(c\mu-3\lambda)}{c} \exp\left\{-\frac{(c\mu-\lambda)^2 + (c\mu-3\lambda)^2}{2c} t\right\} \Phi\left(\frac{c\mu-3\lambda}{\sqrt{c}}\sqrt{t}\right) dt \\
&\quad + \int_0^d \frac{2\lambda}{\sqrt{2\pi c}} \frac{1}{\sqrt{t}}\, e^{-\frac{(c\mu-\lambda)^2}{2c} t}\, dt \\
&= \frac{2\lambda}{c\mu - 3\lambda} \frac{1}{q}\left\{\frac{1}{2\sqrt{2q+1}}[2\Phi(p\sqrt{2q+1}) - 1] - \Phi(p)e^{-qp^2} + \frac{1}{2}\right\} \\
&\quad + \frac{2\lambda}{c\mu-\lambda}\left[2\Phi\left(\frac{(c\mu-\lambda)^2}{c} d\right) - 1\right],
\end{aligned} \tag{5.86}$$

where

$$p = \frac{cd^2}{(c\mu - 3\lambda)^2}, \qquad q = \frac{1}{2} + \frac{(c\mu - \lambda)^2}{2(c\mu - 3\lambda)^2}. \tag{5.87}$$

■

Next we consider the case when $X_0 = x$, $x > 0$. Set $c = 1$ for simplification.

Theorem 5.2.2. *The Laplace transform of $\tau_d^{X_t}$ with $X_0 = x$, $x > 0$ and $c = 1$ is given by*

$$\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] = e^{-\beta d}\ \mathbb{E}\left[e^{-\beta T_{1,1}^X}\right]\bar{\mathbb{P}}_{-1,1}(d) + \mathbb{E}\left[e^{-\beta\left(T_{1,1}^X + T_{-1,1}^X\right)}\right]\mathbb{P}_{-1,1}(d)\ \mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right], \tag{5.88}$$

where $\mathbb{P}_{-1,1}(d)$ is the same as in Theorem 5.2.1, $\mathbb{E}[e^{-\beta T_{1,1}^X}] = \mathbb{E}[e^{-\beta\tau}]$ which is derived from Proposition 4.2.1, i.e.

$$\begin{aligned}
&\mathbb{E}[e^{-\beta T_{1,1}^X}] \\
&= \frac{4\lambda}{\left(\sqrt{(\mu-\lambda)^2 + 2\beta} + \lambda + \mu\right)}\left\{\frac{\mu\Phi\left(\mu\sqrt{x}\right) - \mu}{\sqrt{(\mu-\lambda)^2 + 2\beta} - (\mu+\lambda)} + \right. \\
&\left.\Phi\left(\sqrt{x}\left(\lambda - \sqrt{(\mu-\lambda)^2 + 2\beta}\right)\right) e^{-x\left(\lambda\left(\sqrt{(\mu-\lambda)^2+2\beta}+\mu-\lambda\right)-\beta\right)} \frac{\sqrt{(\mu-\lambda)^2 + 2\beta} - \lambda}{\sqrt{(\mu-\lambda)^2 + 2\beta} - (\mu+\lambda)}\right\},
\end{aligned} \tag{5.89}$$

$\mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right] = \mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right]$ obtained in Theorem 5.2.1, and we have that $\mathbb{E}\left[e^{-\beta\left(T_{1,1}^X + T_{-1,1}^X\right)}\right] = \mathbb{E}\left[e^{-\beta\tau_1 - \beta\tau_2} \mid X_0 = x\right]$ derived from Theorem 5.1.5, i.e.

$$\begin{aligned}
&\mathbb{E}\left[e^{-\beta\left(T_{1,1}^X + T_{-1,1}^X\right)}\right] \\
&= \frac{b-\mu}{A-2\mu} e^{-\frac{b^2}{2}x - A\mu x + Abx}\left[e^{BD}\left(\Phi(D) - B\right) + e^{\frac{B^2}{2}}\bar{\Phi}(D-B)\right] + e^{-\frac{2A\mu x - A^2 x}{2}}\Phi(\sqrt{x}(\mu - A)),
\end{aligned} \tag{5.90}$$

with $b = \sqrt{\mu^2 + 2r_\beta^+} + \mu$, $A = \sqrt{(\mu-\lambda)^2 + 2\beta} + \mu - \lambda$, $D = \sqrt{x}\sqrt{\mu^2 + 2r_\beta^+}$, and $B = -\sqrt{x}(A - 2\mu)$.

Proof. It is important to note that given $X_0 = x$ and $x > 0$, the distribution of the first length of excursion above 0, $T_{1,1}^X$, differs from the distribution of $T_{1,k}^X$, $k = 2, 3, ...$ since these

$T_{1,k}^X$'s start from 0.

We have obtained the Laplace transforms of $T_{1,1}^X$ (i.e. τ defined as classical ruin time) from Proposition 4.2.1 and $-X_{T_{1,1}^X}$ (i.e. the overshoot $-X_\tau$) given $X_0 = x$, $x > 0$ from Proposition 4.2.2 respectively. Thus we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] \\
&= \mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\mathbb{1}_{\{T_{-1,1}^X\}\geq d}\right] + \mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\mathbb{1}_{\{T_{-1,1}^X\}<d}\right] \\
&= e^{-\beta d}\ \mathbb{E}\left[e^{-\beta T_{1,1}^X}\mathbb{1}_{\{T_{-1,1}^X\}\geq d}\right] + \mathbb{E}\left[e^{-\beta\left(T_{1,1}^X+T_{-1,1}^X\right)}\mathbb{1}_{\{T_{-1,1}^X\}<d}\right]\ \mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right],
\end{aligned} \tag{5.91}$$

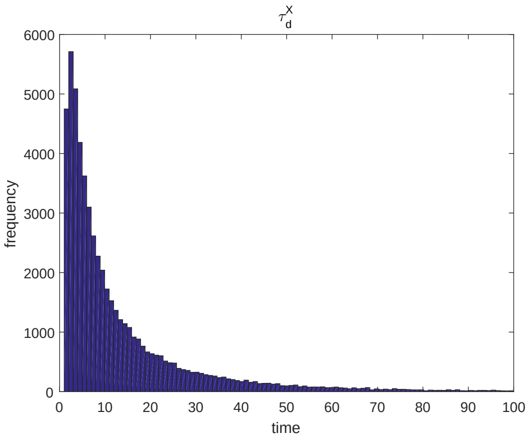
where $\tilde{X}_t$ starts from 0 and it's just the subsequent path of X_t after time $T_{1,1}^X + T_{-1,1}^X$, so $\mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right] = \mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right]$, the later one is obtained from Theorem 5.2.1. Notice that $T_{1,1}^X$ and $T_{-1,1}^X$ are not independent, and we refer their joint Laplace transform with $X_0 = x$, $x > 0$, in Theorem 5.1.5. As a result, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\beta\tau_d^{X_t}}\right] \\
&= e^{-\beta d}\ \mathbb{E}\left[e^{-\beta T_{1,1}^X}\right]\int_d^\infty p_{-1,1}(t)dt + \mathbb{E}\left[e^{-\beta\left(T_{1,1}^X+T_{-1,1}^X\right)}\right]\int_0^d p_{-1,1}(t)dt\ \mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right] \\
&= e^{-\beta d}\ \mathbb{E}\left[e^{-\beta T_{1,1}^X}\right]\bar{\mathbb{P}}_{-1,1}(d) + \mathbb{E}\left[e^{-\beta\left(T_{1,1}^X+T_{-1,1}^X\right)}\right]\mathbb{P}_{-1,1}(d)\mathbb{E}\left[e^{-\beta\tau_d^{\tilde{X}_t}}\right].
\end{aligned} \tag{5.92}$$

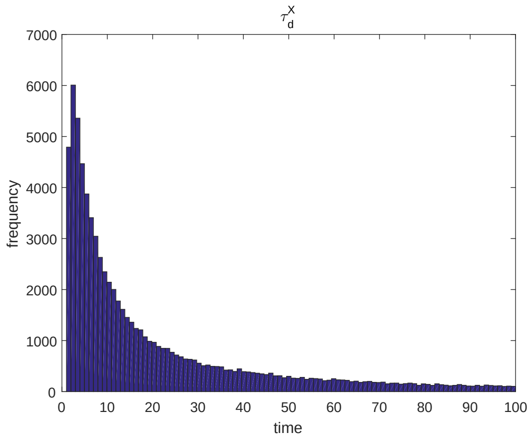
Meanwhile, given $X_0 = x$, $x > 0$, $\mathbb{E}\left[e^{-\beta T_{1,1}^X}\right] = \mathbb{E}\left[e^{-\beta\tau} \mid X_0 = x\right]$, which is calculated in Proposition 4.2.1. ■

Figures 5.15 to 5.18 show the simulation results of the distribution of Parisian ruin time $\tau_d^{X_t}$ when $d = 1$. Parameters are set at $c = 1$, $\lambda = 3$, and different sets of initial capital X_0 and inverse Gaussian parameter μ. The number of simulations $N = 100,000$. We still assume that the surplus process X_t is defined as in (4.3), and claim size Y_i, $i = 1, 2, ...$, follows $IG(\varepsilon, \mu)$ defined in (4.2). The total claim amount process is an inverse Gaussian process Z_t with $\varepsilon \to 0$. We choose $\varepsilon = 0.001$. For each simulation, we record each time when the surplus process X_t ever goes cross zero. Assume that the subscript 1 denotes X_t crosses 0 from above to below, the subscript 2 denotes X_t crosses 0 from below to above, and i denotes the i-th cross. By recording all the time intervals $\tau_{1,i}$ and $\tau_{2,i}$, find the first i that makes $\tau_{2,i} > d$, then $\tau_d^{X_t}$ is the sum of all of the $\tau_{1,i}$ and $\tau_{2,i}$ till $\tau_{2,i} > d$ occurs.

From these figures, we can see that the distribution of Parisian ruin time $\tau_d^{X_t}$ shows similar characteristic to τ_1 and τ_2. $\tau_d^{X_t}$ is skewed to the right with long tail. When μ is fixed, as the initial capital X_0 increases, $\tau_d^{X_t}$ are increasing since ruin is less likely to occur. When X_0 is fixed, $\tau_d^{X_t}$ increases and the distribution of $\tau_d^{X_t}$ has fatter tail as μ increases. Since the mean of claim size $\mathbb{E}[Y_i] = 1/\mu$, claim size decreases averagely when μ increases. This leads to that ruin is less likely to occur, so $\tau_d^{X_t}$ increases.

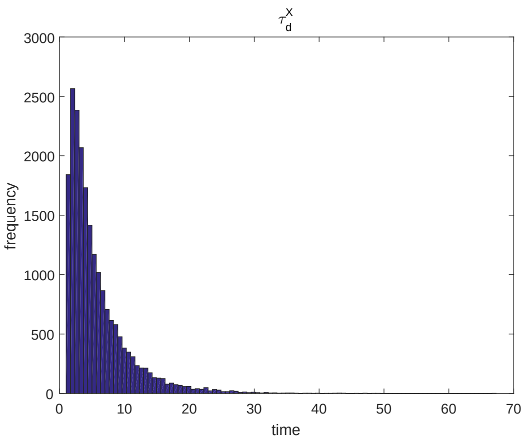


(a) $X_0 = 0, c = 1, \lambda = 3, \mu = 0.8$

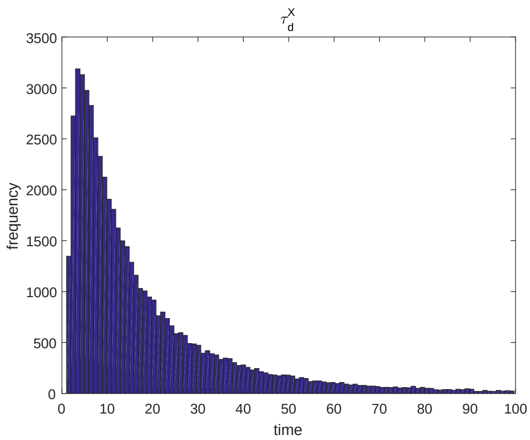


(b) $X_0 = 0, c = 1, \lambda = 3, \mu = 1.0$

Figure 5.15: Simulation result of $\tau_d^{X_t}$

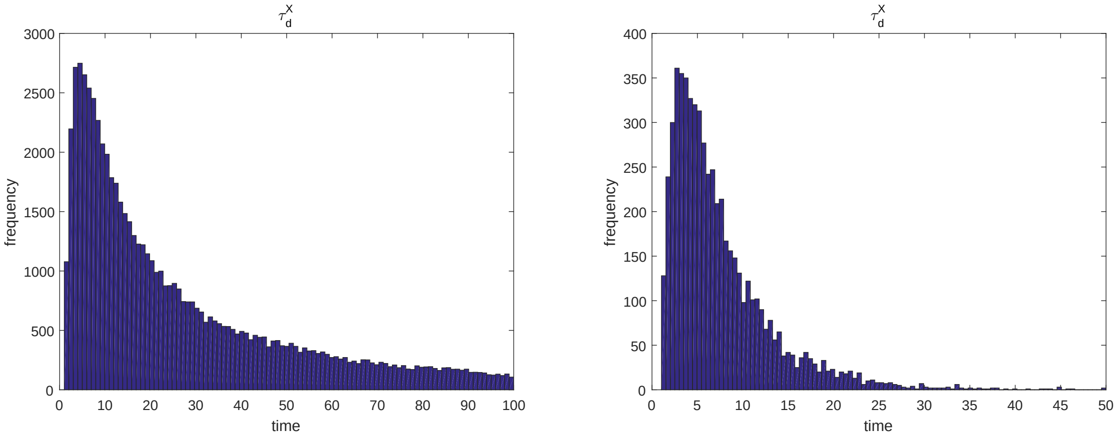


(a) $X_0 = 0, c = 1, \lambda = 3, \mu = 1.5$



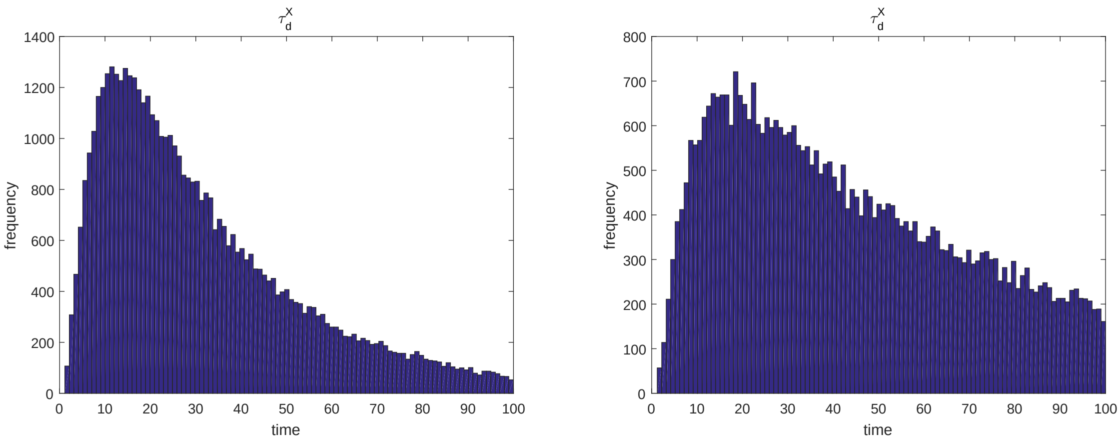
(b) $X_0 = 1, c = 1, \lambda = 3, \mu = 0.8$

Figure 5.16: Simulation result of $\tau_d^{X_t}$



(a) $X_0 = 1, c = 1, \lambda = 3, \mu = 1$ (b) $X_0 = 1, c = 1, \lambda = 3, \mu = 1.5$

Figure 5.17: Simulation result of $\tau_d^{X_t}$



(a) $X_0 = 5, c = 1, \lambda = 3, \mu = 0.8$ (b) $X_0 = 5, c = 1, \lambda = 3, \mu = 1$

Figure 5.18: Simulation result of $\tau_d^{X_t}$

5.3 Parisian Type Ruin Probabilities

In this section, we consider the probabilities of Parisian type of ruin with two cases of initial capital, i.e. $X_0 = 0$ and $X_0 = x$ with $x > 0$.

5.3.1 Probability of Ruin with $X_0 = 0$

If there is no initial reserve, i.e. $X_0 = 0$, the surplus process becomes

$$X_t = ct - Z_t. \tag{5.93}$$

We denote the probability of Parisian type of ruin to be

$$\psi_d(x) = \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = x). \tag{5.94}$$

It is obvious to see that $\psi_d(x) < \psi(x)$ since $\tau_d^{X_t} > \tau_1$.

Theorem 5.3.1. *If $X_0 = 0$, we have that*

$$\psi_d(0) = \frac{\lambda}{c\mu - \lambda K(d)}(1 - K(d)), \tag{5.95}$$

where $K(d) = \mathbb{P}_{-1,1}(d)$ calculated as in (5.86), i.e.

$$\begin{aligned} K(d) = {} & \frac{2\lambda}{c\mu - 3\lambda}\frac{1}{q}\left\{\frac{1}{2\sqrt{2q+1}}[2\Phi(p\sqrt{2q+1}) - 1] - \Phi(p)e^{-qp^2} + \frac{1}{2}\right\} \\ & + \frac{2\lambda}{c\mu - \lambda}\left[2\Phi\left(\frac{(c\mu - \lambda)^2}{c}d\right) - 1\right], \end{aligned} \tag{5.96}$$

with

$$p = \frac{cd^2}{(c\mu - 3\lambda)^2}, \qquad q = \frac{1}{2} + \frac{(c\mu - \lambda)^2}{2(c\mu - 3\lambda)^2}, \tag{5.97}$$

and

$$r_\beta^{\pm} = \frac{-c\lambda\mu + c\beta + \lambda^2 \pm \lambda\sqrt{(c\mu - \lambda)^2 + 2c\beta}}{c^2}. \tag{5.98}$$

Proof. When $X_0 = 0$, we have $\psi(0) = \frac{\lambda}{c\mu}$ by applying the final value theorem $\lim_{x\to\infty} f(x) = \lim_{\xi\to 0} \xi\hat{f}(\xi) = 0$. Also from Corollary 4.2.3, the density of overshoot $-X_\tau$ with 0 initial capital is given by

$$f_{-X_\tau|X_0=0}(z) = \frac{2\lambda}{c}\frac{1}{\sqrt{2\pi z}}e^{-\frac{\mu^2}{2}z} - \frac{2\lambda\mu}{c}\bar{\Phi}(\mu\sqrt{z}). \tag{5.99}$$

Furthermore, we use τ^* defined in (5.71) and by the property in Gerber [43] pp. 116, we have

$$\mathbb{E}[e^{-\beta\tau^*} \mid -X_0^* = z] = e^{-r_\beta^+ z}. \tag{5.100}$$

Denote $k(t)$ the density of the first excursion below 0. Then the Laplace transform of $k(t)$ can be calculated as

$$\begin{aligned}
\hat{k}(\beta) &= \int_0^\infty e^{-\beta t}k(t)dt \\
&= \int_0^\infty \mathbb{E}[e^{-\beta\tau^*} \mid -X_0^* = z] f_{-X_\tau|X_0=0}(z)dz \\
&= \int_0^\infty e^{-r_\beta^+ z} f_{-X_\tau|X_0=0}(z)dz \\
&= \mathbb{E}\left[e^{-r_\beta^+(-X_\tau)} \mid X_0 = 0\right] \\
&= \frac{2\lambda}{c\left(\sqrt{\mu^2+2r_\beta^+}+\mu\right)}.
\end{aligned} \tag{5.101}$$

Then define the cumulative distribution function of τ^* as

$$K(d) = \mathbb{P}(\tau^* < d) = \int_0^d k(t)dt, \tag{5.102}$$

therefore

$$\begin{aligned}
K(d) &= \int_0^d \mathcal{L}_\beta^{-1}\left\{\hat{k}(\beta)\right\}\ dt \\
&= \int_0^d \mathcal{L}_\beta^{-1}\left\{\frac{2\lambda}{\sqrt{(c\mu-\lambda)^2+2c\beta}+\lambda+c\mu}\right\}\ dt \\
&= \int_0^d \frac{2\lambda(c\mu-3\lambda)}{c}\exp\left\{-\frac{(c\mu-\lambda)^2+(c\mu-3\lambda)^2}{2c}t\right\}\Phi\left(\frac{c\mu-3\lambda}{\sqrt{c}}\sqrt{t}\right)\ dt \\
&\quad + \int_0^d \frac{2\lambda}{\sqrt{2\pi c}}\frac{1}{\sqrt{t}}\ e^{-\frac{(c\mu-\lambda)^2}{2c}t}\ dt \\
&= \frac{2\lambda}{c\mu-3\lambda}\frac{1}{q}\left\{\frac{1}{2\sqrt{2q+1}}[2\Phi(p\sqrt{2q+1})-1]-\Phi(p)e^{-qp^2}+\frac{1}{2}\right\} \\
&\quad + \frac{2\lambda}{c\mu-\lambda}\left[2\Phi\left(\frac{(c\mu-\lambda)^2}{c}d\right)-1\right],
\end{aligned} \tag{5.103}$$

where

$$p = \frac{cd^2}{(c\mu - 3\lambda)^2}, \qquad q = \frac{1}{2} + \frac{(c\mu - \lambda)^2}{2(c\mu - 3\lambda)^2}. \tag{5.104}$$

Notice that $K(d)$ is just $\mathbb{P}_{-1,1}(d)$ obtained in (5.86).

We next find the distribution of the number of excursions N below 0, which actually follows a geometric distribution, i.e.

$$\mathbb{P}(N = k) = (1 - \psi(0))\,(\psi(0))^k = \left(1 - \frac{\lambda}{c\mu}\right)\left(\frac{\lambda}{c\mu}\right)^k, \tag{5.105}$$

for $k = 0, 1, 2, ...,$.

Let D be the largest excursion blow 0, we have that

$$\begin{aligned} \mathbb{P}(D \le d) &= \sum_{k=0}^{\infty} (K(d))^k \left(1 - \frac{\lambda}{c\mu}\right)\left(\frac{\lambda}{c\mu}\right)^k \\ &= \frac{1 - \frac{\lambda}{c\mu}}{1 - \frac{\lambda}{c\mu}K(d)}. \end{aligned} \tag{5.106}$$

Therefore, the probability of Parisian type of ruin is

$$\psi_d(0) = 1 - \mathbb{P}(D \le d) = \frac{\frac{\lambda}{\mu}\bar{K}(d)}{c - \frac{\lambda}{\mu}K(d)} = \lambda \frac{1 - K(d)}{c\mu - \lambda K(d)}. \tag{5.107}$$

■

Remark. It is clear to verify that $\psi(0) = \frac{\lambda}{c\mu}$ by taking $d \to 0$ in (5.95). Also we have $\psi(0) > \psi_d(0)$ since $c\mu > \lambda$.

5.3.2 Ruin Probability and Asymptotic Ruin Probability with $X_0 > 0$

In this subsection, we study the probability of Parisian type of ruin and asymptotic result of the Parisian ruin probability when $X_0 = x$, $x > 0$.

Theorem 5.3.2. *For the surplus process X_t with $X_0 = x$ and $x > 0$, the probability of ruin* *of Parisian type is*

$$\psi_d(x) = \psi(x) - \frac{c\mu - \lambda}{c\mu - \lambda K(d)} \int_0^\infty \int_0^d f_{\tau_1,\tau_2|X_0=x}(t_1,t_2)dt_2dt_1, \tag{5.108}$$

where $K(d) = \mathbb{P}_{-1,1}(d)$ *calculated as in (5.86), i.e.*

$$\begin{aligned} K(d) = &\frac{2\lambda}{c\mu - 3\lambda}\frac{1}{q}\left\{\frac{1}{2\sqrt{2q+1}}[2\Phi(p\sqrt{2q+1}) - 1] - \Phi(p)e^{-qp^2} + \frac{1}{2}\right\} \\ &+ \frac{2\lambda}{c\mu - \lambda}\left[2\Phi\left(\frac{(c\mu-\lambda)^2}{c}d\right) - 1\right], \end{aligned} \tag{5.109}$$

and $\psi(x)$ *is the classical probability of ruin we discuss in Theorem 4.2.4, i.e.*

$$\psi(x) = \bar{\Phi}(\mu\sqrt{x}) - e^{\frac{2\lambda}{c}(\frac{\lambda}{c}-\mu)x}\left(1 - \frac{2\lambda}{c\mu}\right)\bar{\Phi}\left(\left(\mu - \frac{2\lambda}{c}\right)\sqrt{x}\right). \tag{5.110}$$

Proof. The Parisian ruin probability with x initial capital can be written as

$$\begin{aligned} &\psi_d(x) \\ &= \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = x) \\ &= \mathbb{P}(\tau_d^{X_t} < \infty, \tau < \infty, \tau^* < d \mid X_0 = x) + \mathbb{P}(\tau_d^{X_t} < \infty, \tau < \infty, \tau^* \geq d \mid X_0 = x) \\ &= \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x)\ \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0) + \mathbb{P}(\tau < \infty, \tau^* \geq d \mid X_0 = x). \end{aligned} \tag{5.111}$$

Note that the last equation results from the strong Markov property of X_t. Meanwhile, $\mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0)$ is just the Parisian ruin probability with 0 initial capital, which has been obtained from Theorem 5.3.1. Furthermore, we have

$$\mathbb{P}(\tau < \infty, \tau^* \geq d \mid X_0 = x) = \psi(x) - \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x), \tag{5.112}$$

which provides us with

$$\begin{aligned} \psi_d(x) &= \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x)\ \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0) \\ &\quad + \psi(x) - \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x) \\ &= \psi(x) - [1 - \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0)]\ \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x). \end{aligned} \tag{5.113}$$

We also have that

$$\mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x) = \int_0^\infty \int_0^d f_{\tau_1,\tau_2|X_0=x}(t_1,t_2)dt_2dt_1, \tag{5.114}$$

where $f_{\tau_1,\tau_2|X_0=x}(t_1,t_2)$ is the joint probability density function of τ_1 and τ_2 with $X_0 = x$. $f_{\tau_1,\tau_2|X_0=x}(t_1,t_2)$ can be obtained from jointly inverting $\mathbb{E}[e^{-\beta_1\tau_1-\beta_2\tau_2} \mid X_0 = x]$ w.r.t. β_1 and β_2 in Theorem 5.1.5. Therefore we have proved (5.108). ■

It is also interesting to consider the asymptotic probability of ruin when the initial capital $x \to \infty$.

Theorem 5.3.3. *Consider the surplus process X_t with $X_0 = x$ and $x > 0$, if $x \to \infty$, we have the following asymptotic Parisian ruin probability*

$$\psi_d(x) \sim C_d\, e^{-\gamma x}, \tag{5.115}$$

where

$$\gamma = \frac{2\lambda(c\mu - \lambda)}{c^2}, \tag{5.116}$$

$$C_d = C\left\{\frac{c\mu - \lambda K(d) - \mu\gamma Q(d)}{c\mu - \lambda K(d)}\right\}, \tag{5.117}$$

$$C = \frac{c\mu - \lambda}{2\lambda\mu - c\mu\sqrt{\mu^2 - 2\gamma}}\,\sqrt{\mu^2 - 2\gamma}, \tag{5.118}$$

and

$$\begin{aligned} &Q(d) \\ &= \frac{c^3}{2\lambda^2}\left(1 - e^{-\frac{2\lambda^2 t}{c}}\right)\left\{\Phi\left(\frac{(c\mu-\lambda)t - 2\lambda}{\sqrt{ct}}\right) + e^{\frac{4\lambda(c\mu-\lambda)}{c}}\Phi\left(\frac{(c\mu-\lambda)t + 2\lambda}{\sqrt{ct}}\right)\right\}. \end{aligned} \tag{5.119}$$

Proof. First, by applying the strong Markov property of X_t, we rewrite the Parisian ruin probability as

$$\begin{aligned}\psi_d(x) = \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x)\ \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0) \\ +\ \mathbb{P}(\tau < \infty, \tau^* \geq d \mid X_0 = x).\end{aligned} \tag{5.120}$$

Define that

$$h(d) = e^{\gamma x}\ \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x). \tag{5.121}$$

γ is the adjustment coefficient which is the unique positive root of the following Lundberg equation

$$c\gamma + \lambda\left(\sqrt{\mu^2 - 2\gamma} - \mu\right) = 0, \tag{5.122}$$

therefore the positive root is

$$\gamma = \frac{2\lambda(c\mu - \lambda)}{c^2}. \tag{5.123}$$

Consider the limit of Laplace transform of $h(d)$ when $x \to \infty$, i.e.

$$\begin{aligned}&\lim_{x\to\infty} \hat{h}(\beta) \\ =& \lim_{x\to\infty} \int_0^\infty e^{-\beta d} e^{\gamma x}\ \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x)\ \mathrm{d}d \\ =& \lim_{x\to\infty} e^{\gamma x}\mathbb{E}\left[\frac{e^{-\beta\tau^*}}{\beta}\mathbb{1}_{\{\tau<\infty\}} \mid X_0 = x\right] \\ =& \lim_{x\to\infty} \int_0^\infty e^{\gamma x}\ \mathbb{E}\left[\frac{e^{-\beta\tau^*}}{\beta} \mid -X_\tau = z\right] \mathbb{P}(\tau < \infty, -X_\tau \in dz \mid X_0 = x) \\ =& \lim_{x\to\infty} \int_0^\infty e^{\gamma x}\ \mathbb{E}\left[\frac{e^{-\beta\tau^*}}{\beta} \mid -X_\tau = z\right] \mathbb{P}(-X_\tau \in dz \mid \tau < \infty, X_0 = x)\ \mathbb{P}(\tau < \infty \mid X_0 = x).\end{aligned} \tag{5.124}$$

From Gerber [43] p. 116, we know that $e^{-\beta t - r_\beta^+ X_t}$ is a martingale,

$$\mathbb{E}[e^{-\beta\tau^*} \mid -X_\tau = z] = e^{-r_\beta^+ z}. \tag{5.125}$$

In the meantime, based on the result of Theorem 2 on p.234 – 235 in Schmidli [70], we can show that

$$\begin{aligned}
&\lim_{x\to\infty} \mathbb{P}(-X_\tau \in dz \mid \tau < \infty, X_0 = x) \\
&= \frac{\lambda\mu}{c\mu-\lambda}\gamma \int_0^\infty e^{\gamma x}\bar{G}(x+z)dx \\
&= \frac{\lambda\mu}{c\mu-\lambda} e^{-\gamma z} \int_z^\infty \bar{G}(y)de^{\gamma y} \\
&= \frac{\lambda\mu}{c\mu-\lambda} e^{-\gamma z}\left(-\bar{G}(z)e^{\gamma z} + \int_z^\infty e^{\gamma y} g(y)dy\right).
\end{aligned} \tag{5.126}$$

We refer to p.4 in Schmidli [71] about the Cramér-Lundberg approximation,

$$\lim_{x\to\infty} e^{\gamma x}\mathbb{P}(\tau < \infty \mid X_0 = x) \ = \ C, \tag{5.127}$$

where

$$C = \frac{\theta \ \mathbb{E}[Y_1]}{M_Y'(\gamma) - (1+\theta) \ \mathbb{E}[Y_1]}, \tag{5.128}$$

with θ being the safety loading coefficient and $c = (1+\theta)\lambda\mathbb{E}[Y_1]$. As $Y_i \sim IG(\varepsilon, \mu)$ and $\varepsilon \to 0$, we have

$$C = \frac{c\mu - \lambda}{2\lambda\mu - c\mu\sqrt{\mu^2 - 2\gamma}} \ \sqrt{\mu^2 - 2\gamma}. \tag{5.129}$$

Then $\lim_{x\to\infty} \hat{h}(\beta)$ becomes

$$\begin{aligned}
&\lim_{x\to\infty} \int_0^\infty \frac{e^{-r_\beta^+ z}}{\beta} \ \mathbb{P}(-X_\tau \in dz \mid \tau < \infty, X_0 = x) \ e^{\gamma x} \ \mathbb{P}(\tau < \infty \mid X_0 = x) \\
&= \int_0^\infty \frac{e^{-r_\beta^+ z}}{\beta} \lim_{x\to\infty} \mathbb{P}(-X_\tau \in dz \mid \tau < \infty, X_0 = x) \lim_{x\to\infty} e^{\gamma x} \ \mathbb{P}(\tau < \infty \mid X_0 = x) \\
&= \frac{C}{\beta}\frac{\lambda\mu}{c\mu-\lambda} \int_0^\infty e^{-r_\beta^+ z} \int_0^\infty \gamma e^{\gamma x}\bar{G}(x+z) \ dxdz \\
&= \frac{C}{\beta}\frac{\lambda\mu}{c\mu-\lambda}\left(\frac{\hat{g}(-\gamma) - \hat{g}(r_\beta^+)}{\gamma + r_\beta^+} - \frac{1-\hat{g}(r_\beta^+)}{r_\beta^+}\right) \\
&= \frac{C\mu\gamma}{c\mu-\lambda} \ \frac{1}{r_\beta^+(r_\beta^+ + \gamma)} \\
&= \frac{C\mu\gamma}{c\mu-\lambda} \ \frac{c^3}{\beta\left(2\lambda^2 + c\beta + 2\lambda\sqrt{(c\mu-\lambda)^2 + 2c\beta}\right)},
\end{aligned} \tag{5.130}$$

which provides us with

$$\begin{aligned}&\lim_{x\to\infty} e^{\gamma x}\ \mathbb{P}(\tau<\infty,\tau^*<d \mid X_0=x)\\&=\frac{C\mu\gamma}{c\mu-\lambda}\ Q(d),\end{aligned} \tag{5.131}$$

where

$$\begin{aligned}Q(d) &= \mathcal{L}_\beta^{-1}\left\{\frac{c^3}{\beta\left(2\lambda^2+c\beta+2\lambda\sqrt{(c\mu-\lambda)^2+2c\beta}\right)}\right\}\\
&= c^3\int_0^t \mathcal{L}_\beta^{-1}\left\{\frac{1}{2\lambda^2+c\beta+2\lambda\sqrt{(c\mu-\lambda)^2+2c\beta}}\right\}\\
&= c^3\int_0^t \mathcal{L}_\beta^{-1}\left\{\frac{1}{c}\int_0^\infty\int_0^\infty e^{-\beta x}e^{-\beta x}\ e^{-\frac{2\lambda^2}{c}x}\frac{\tilde{\varepsilon}}{\sqrt{2\pi y^3}}e^{-\frac{(\tilde{\varepsilon}-\tilde{\mu}y)^2}{2y}}\,dxdy\right\}\\
&= c^2\int_0^t \mathcal{L}_\beta^{-1}\left\{\mathcal{L}_\beta\left\{\mathcal{L}_\beta\{f(x,y)\}\right\}\right\}\\
&= c^2\int_0^t e^{-\frac{2\lambda^2}{c}x}\frac{\tilde{\varepsilon}}{\sqrt{2\pi y^3}}e^{-\frac{(\tilde{\varepsilon}-\tilde{\mu}y)^2}{2y}}\,dxdy\\
&= \frac{c^3}{2\lambda^2}\left(1-e^{-\frac{2\lambda^2 t}{c}}\right)\left\{\Phi\left(\frac{(c\mu-\lambda)t-2\lambda}{\sqrt{ct}}\right)+e^{\frac{4\lambda(c\mu-\lambda)}{c}}\Phi\left(\frac{(c\mu-\lambda)t+2\lambda}{\sqrt{ct}}\right)\right\},\end{aligned} \tag{5.132}$$

with

$$\tilde{\varepsilon}=\frac{2\lambda}{\sqrt{c}},\qquad \tilde{\mu}=\frac{c\mu-\lambda}{\sqrt{c}}. \tag{5.133}$$

Therefore,

$$\mathbb{P}(\tau<\infty,\tau^*<d \mid X_0=x)\ \sim\ e^{-\gamma x}\frac{C\mu\gamma}{c\mu-\lambda}\ Q(d). \tag{5.134}$$

It is also noticeable that

$$\mathbb{P}(\tau<\infty,\tau^*\geq d \mid X_0=x)=\ \psi(x)-\mathbb{P}(\tau<\infty,\tau^*<d \mid X_0=x), \tag{5.135}$$

which yields

$$\mathbb{P}(\tau<\infty,\tau^*\geq d \mid X_0=x)\ \sim\ Ce^{-\gamma x}\left(1-\frac{\mu\gamma}{c\mu-\lambda}\ Q(d)\right). \tag{5.136}$$

Combining with (5.120) and the result of $\mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0)$ from Theorem 5.3.1 we have

$$\begin{aligned}
\psi_d(x) &= \mathbb{P}(\tau < \infty, \tau^* < d \mid X_0 = x)\ \mathbb{P}(\tau_d^{X_t} < \infty \mid X_0 = 0) \\
&\quad + \ \mathbb{P}(\tau < \infty, \tau^* \geq d \mid X_0 = x) \\
&\sim e^{-\gamma x} \frac{C\mu\gamma}{c\mu - \lambda}\ Q(d) \frac{\lambda(1 - K(d))}{c\mu - \lambda K(d)}\ + Ce^{-\gamma x}\left(1 - \frac{\mu\gamma}{c\mu - \lambda}\ Q(d)\right) \\
&= e^{-\gamma x} C \left\{ \frac{c\mu - \lambda K(d) - \mu\gamma Q(d)}{c\mu - \lambda K(d)} \right\}.
\end{aligned} \tag{5.137}$$

■

Remark. It is obvious to check that $C_d < C$ under the net profit condition $c > \frac{\lambda}{\mu}$, which infers that $\psi_d(x) < \psi(x)$.

Chapter 6

Surplus Processes with Variable Premium Income and Stochastic Premium Income

This chapter first studies the probability of survival for an insurance company, of which surplus process consists of variable premium income. In other words, an insurance company reinvests its current surplus and collects interest. The formula of survival probability and the numerical results of probability of ruin are provided.

Then the second part of this chapter considers the case that premium income rate is no longer a linear function of time, but a stochastic process independent of the total claim amount process. The explicit formula and the numerical results of probability of ruin are given as well.

6.1 Surplus Process with Variable Premium Income

In this subsection, we study the problem of ruin with more general type of premium. Let X_t^δ denote the value of the surplus at time t, which is defined as

$$X_t^\delta = x + C_t - Z_t, \tag{6.1}$$

where $x \geq 0$ is the initial capital, $\{C_t, t \geq 0\}$ is the generalized premium up to time t, $\{Z_t, t \geq 0\}$ is the inverse Gaussian process that we discussed in previous chapter. Assume that the premium income depends on the current surplus, earning interest at a constant force $\delta > 0$, i.e.

$$C(X_t^\delta) = c + \delta X_t^\delta, \tag{6.2}$$

where $c \geq 0$ is the constant premium rate defined as in precious chapters. This means that the insurance company receives premium at a constant rate δ, and the premium income at time t is a linear function of the surplus X_t^δ. We assume that $c(0) > 0$, and $c(x) > 0$ for $x > 0$. Another example of variable premium income is that premiums are charged by the level of the current surplus. We refer Michaud [62] for the detailed discussion of this example.

In the case defined in (6.2), the surplus process X_t^δ satisfies the equation (see Sundt and Teugels, [76] and [77])

$$X_t^\delta = xe^{\delta t} + c\int_0^t e^{\delta y}dy - \int_0^t e^{\delta(t-y)}dZ_y, \tag{6.3}$$

with $X_0^\delta = x$ and

$$\int_0^t e^{\delta y}dy = \begin{cases} t, & \text{if } \delta = 0 \\ \frac{1}{\delta}\left(e^{\delta t - 1}\right), & \text{if } \delta > 0. \end{cases} \tag{6.4}$$

When $\delta = 0$, this is the case that considers the classical surplus process, which has been intensively discussed in a vast literature. We focus on the case that $\delta > 0$. Define the infinite time probability of ruin of the insurance company at some time beginning with initial capital x as

$$\psi_\delta(x) = \mathbb{P}\left(\inf_{t \geq 0} X_t^\delta < 0 \mid X_0^\delta = x\right). \tag{6.5}$$

It is obvious to see that if $X_v^\delta \geq 0$ for all $v \leq t$, then $X_v^\delta \geq X_v$ for all $v \leq t$. X_v is the classical surplus process with constant premium rate at time $v \geq 0$. This implies that $\psi_\delta(x) \leq \psi(x)$. On other words, the $\psi(x)$ in the classical case $\delta = 0$ provides with an upper bound for the probability of ruin in the general case $\delta > 0$.

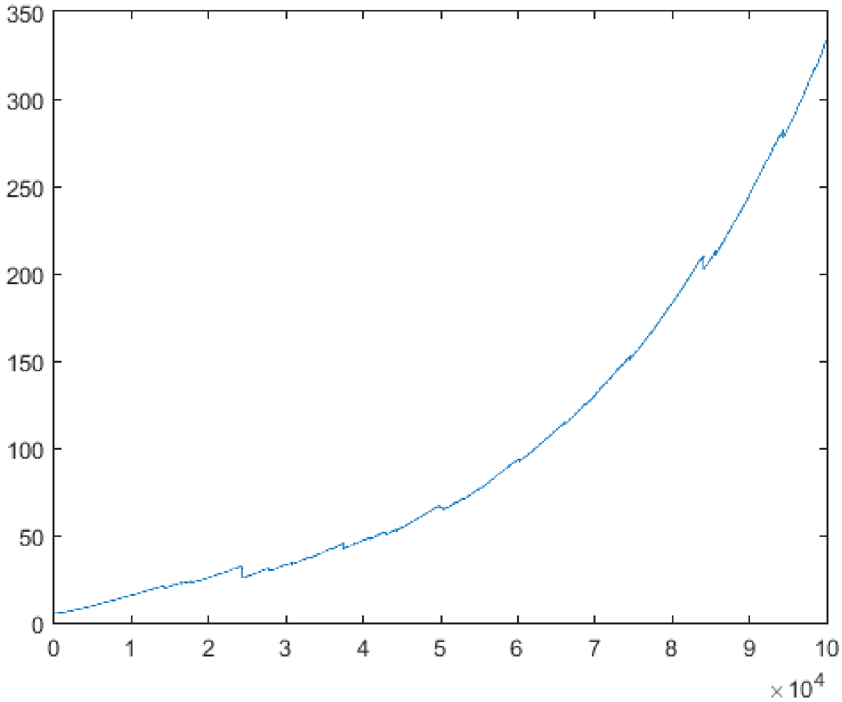


Figure 6.1: A sample path of surplus process with variable premium income.

When $\delta = 0$, we have $X_t^\delta = X_t = x + ct - Z_t$. It has been shown that (See e.g. Sundt [75]) $X_t \to \infty$ as $t \to \infty$ under the net profit condition $c > \frac{\lambda}{\mu}$, while $X_t \to -\infty$ as $t \to \infty$ if $c < \frac{\lambda}{\mu}$. This follows that $\psi(x) = 1$ for all x if $c < \frac{\lambda}{\mu}$. It is noticeable that $\psi(x) = 1$ when $c = \frac{\lambda}{\mu}$. When $\delta > 0$, Sundt and Teugels [76] showed that $\psi_\delta(x)$ is no longer obviously equal to 1 if $c < \frac{\lambda}{\mu}$. However it is clear that $\psi_\delta(x) = 1$ if $c \leq -\delta x$. That is, the interest received from investing previous surplus would not be sufficient to cover the negative premium, thus the surplus would become negative sooner or later.

Theorem 6.1.1. *If $u_t = x_t + \frac{c}{\delta}$, the probability of survival with initial value $u > 0$ for an insurance company is given by*

$$\begin{aligned} f(u) = {} & K(2\mu)^m e^{-2\mu a} + K\left(\frac{1}{\sqrt{u}}\right)^m H_{m-1}\left(\frac{a+\mu u}{\sqrt{u}}\right)\varphi\left(\frac{a+\mu u}{\sqrt{u}}\right) \\ & + (-1)^{m+1}K\left(\frac{1}{\sqrt{u}}\right)^m b^m e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}}) \\ & + (-1)^{m+1}K\left(\frac{1}{\sqrt{u}}\right)^m \varphi(\frac{a+\mu u}{\sqrt{u}})\sum_{k=0}^{m-1}(-1)^k H_k\left(\frac{a+\mu u}{\sqrt{u}}\right)b^{m-1-k}, \end{aligned} \tag{6.6}$$

where $K = (2\mu)^{-\frac{2\lambda\mu}{\delta}} e^{\frac{4\lambda\mu}{\delta}}$, $a = \frac{2\lambda}{\delta}$, $b = -2\mu\sqrt{u}$, $m = [\frac{2\lambda\mu}{\delta}]$ *is the integer part of* $\frac{2\lambda\mu}{\delta}$, $\Phi(x)$ *is the probability distribution function for standard normal distribution, and* $\Phi'(x) = \varphi(x)$.

Proof. The infinitesimal generator $\mathcal{A}$ applied to $f(x)$ becomes

$$\mathcal{A}f(x) = (c+\delta x)\frac{\partial f(x)}{\partial x} + \frac{\lambda}{\varepsilon}\left\{\int_0^{x+\frac{c}{\delta}} f(x-y)dG(y) - f(x)\right\}, \tag{6.7}$$

with $\lim_{x\to\infty} f(x) = 1$. In this case, $f(x)$ is the probability of survival given initial surplus $x \geq 0$. Let $u = x + \frac{c}{\delta}$, so the generator is

$$\mathcal{A}f(u) = \delta u\frac{\partial f(u)}{\partial u} + \frac{\lambda}{\varepsilon}\left\{\int_0^{u} f(u-y)dG(y) - f(u)\right\}. \tag{6.8}$$

Simplifying this generator as

$$\begin{aligned}\mathcal{A}f(u) = \delta u\frac{\partial f(u)}{\partial u} &- \lambda f(0)\int_u^\infty \frac{1}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2}{2}y}\,dy \\ &- \lambda\int_0^u f'(x-v)\int_v^\infty \frac{1}{\sqrt{2\pi y^3}}e^{-\frac{\mu^2}{2}y}\,dydv.\end{aligned} \tag{6.9}$$

Applying Laplace transform to $\mathcal{A}f(u) = 0$, we have the following equation in terms of $\hat{f}(\xi)$,

$$-\delta\frac{\mathrm{d}(\xi\hat{f}(\xi))}{\mathrm{d}\xi} - \lambda\hat{f}(\xi)\left(\sqrt{\mu^2+2\xi} - \mu\right) = 0, \tag{6.10}$$

which can be seen as an ordinary differential equation in terms of $\xi\hat{f}(\xi)$. Solving this equation and using the theorem of final value $\lim_{u\to\infty} f(u) = \lim_{\xi\to 0}\xi\hat{f}(\xi) = 1$ give us

$$\hat{f}(\xi) = (2\mu)^{-\frac{2\lambda\mu}{\delta}} e^{\frac{2\lambda\mu}{\delta}} \frac{(\sqrt{\mu^2+2\xi}+\mu)^{\frac{2\lambda\mu}{\delta}}}{\xi} e^{-\frac{2\lambda}{\delta}\sqrt{\mu^2+2\xi}}. \tag{6.11}$$

Therefore, we have the Laplace transform for the probability of survival $f(u)$, which can be obtained by inverting $\hat{f}(\xi)$.

Notice that when the power $\frac{2\lambda\mu}{\delta}$ could be relatively large due to possible small values of interest rate $\delta > 0$, the decimal part does not affect $\hat{f}(\xi)$ too much, for which we only need to focus on corresponding integer part $m = [\frac{2\lambda\mu}{\delta}]$. Thus $\hat{f}(\xi)$ can be rewritten as

$$\hat{f}(\xi) = K\frac{1}{\xi}(\sqrt{\mu^2 + 2\xi} + \mu)^m e^{-a(\sqrt{\mu^2+2\xi}+\mu)}, \tag{6.12}$$

where $K = (2\mu)^{-\frac{2\lambda\mu}{\delta}} e^{2\frac{2\lambda\mu}{\delta}}$, $a = \frac{2\lambda}{\delta}$. Therefore,

$$\begin{aligned}
\hat{f}(\xi) &= K\frac{1}{\xi}(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(e^{-a(\sqrt{\mu^2+2\xi}+\mu)}\right) \\
&= K\frac{1}{\xi}(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\int_0^\infty e^{-\xi t}\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}dt\right) \\
&= K\frac{1}{\xi}(-1)^m \int_0^\infty e^{-\xi t}\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}\right)dt \\
&= \frac{1}{\xi}\int_0^\infty e^{-\xi t}\left[K(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}\right)\right]dt \\
&= \frac{1}{\xi}\hat{h}(\xi),
\end{aligned} \tag{6.13}$$

where $\hat{h}(\xi)$ is the Laplace transform of $h(t)$ and

$$h(t) = K(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}\right). \tag{6.14}$$

Note that for a given probability density function $g(y)$ with $g(y) = G'(y)$

$$\frac{1}{\xi}\hat{g}(\xi) = \int_0^\infty e^{-\xi y}G(y)dy = \int_0^\infty e^{-\xi y}\left(\int_0^y g(t)dt\right)dy = \hat{G}(\xi). \tag{6.15}$$

Thus, for $\hat{f}(\xi)$, we have

$$\hat{f}(\xi) = \frac{1}{\xi}\hat{h}(\xi) = \int_0^\infty e^{-\xi u}H(u)du = \int_0^\infty e^{-\xi u}\left(\int_0^u h(t)dt\right)du, \tag{6.16}$$

which gives us the probability of survival

$$\begin{aligned}
f(u) &= \int_0^u h(t)dt \\
&= \int_0^u K(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}\right)dt \\
&= K(-1)^m\int_0^u \frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}\right)dt \\
&= K(-1)^m\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\int_0^u \frac{a}{\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2t}}dt\right).
\end{aligned} \tag{6.17}$$

The integrand in equation (6.17) is the just the probability density function of

$$\tau_a = \inf\{t > 0 \mid W_t^{(-\mu)} = W_t - \mu t = a\}, \tag{6.18}$$

where W_t is Brownian motion, τ_a is the first hitting time when the drifted Brownian motion $W_t^{(-\mu)}$ touches the barrier $a > 0$. As a result, the probability of survival $f(u)$ in equation (6.17) becomes

$$\begin{aligned} f(u) &= K(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}a^m} \left(\mathbb{P}(\tau_a < u)\right) \\ &= K(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}a^m} \left(\mathbb{P}(\max_{0\le s\le u} (W_t - \mu t) \ge a)\right) \\ &= K(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}a^m} \left(1 - \Phi(\frac{a+\mu u}{\sqrt{u}}) + e^{-2\mu a} - e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}})\right) \\ &= K(-1)^m(-2\mu)^m e^{-2\mu a} + \\ &\quad K(-1)^{m+1}\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\Phi(\frac{a+\mu u}{\sqrt{u}})\right) + K(-1)^{m+1}\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}})\right), \end{aligned} \tag{6.19}$$

where $\Phi(x)$ is the probability distribution function for standard normal distribution, and $\Phi'(x) = \varphi(x)$. Also we have

$$\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(\Phi(\frac{a+\mu u}{\sqrt{u}})\right) = (-1)^{m-1}\left(\frac{1}{\sqrt{u}}\right)^m H_{m-1}\left(\frac{a+\mu u}{\sqrt{u}}\right)\varphi\left(\frac{a+\mu u}{\sqrt{u}}\right), \tag{6.20}$$

and

$$\begin{aligned} &\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}})\right) \\ &= \left(\frac{1}{\sqrt{u}}\right)^m \Bigg\{ b^m e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}}) + \varphi(\frac{a+\mu u}{\sqrt{u}})\Big[b^{m-1} + (-1)^1 H_1\left(\frac{a+\mu u}{\sqrt{u}}\right) b^{m-2} \\ &\quad + (-1)^2 H_2\left(\frac{a+\mu u}{\sqrt{u}}\right) b^{m-3} +, \ldots, +(-1)^{m-1} H_{m-1}\left(\frac{a+\mu u}{\sqrt{u}}\right)\Big]\Bigg\}, \end{aligned} \tag{6.21}$$

where $b = -2\mu\sqrt{u}$, and $H_m(x)$ is the Hermite polynomial defined as

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(e^{-\frac{x^2}{2}}\right). \tag{6.22}$$

	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.1$
x	$\psi_\delta(x)$	$\psi_\delta(x)$	$\psi_\delta(x)$
0	0.95646	0.89221	0.78798
1	0.94342	0.85876	0.71555
3	0.90991	0.77864	0.56421
5	0.86611	0.68581	0.42395
7	0.81248	0.58741	0.30750
9	0.75048	0.49029	0.21757
10	0.71701	0.44398	0.18178
20	0.36796	0.12861	0.02717
30	0.13714	0.02858	0.00405
40	0.04119	0.00572	0.00065
50	0.01083	0.00111	0.00011

Table 6.1: Infinite time ruin probabilities. $\lambda = 1$, $c = 1.5$ and $\mu = 0.5$.

Therefore, the probability of survival $f(u)$ is calculated as

$$\begin{aligned} f(u) &= K(-1)^m(-2\mu)^m e^{-2\mu a} + K\left(\frac{1}{\sqrt{u}}\right)^m H_{m-1}\left(\frac{a+\mu u}{\sqrt{u}}\right)\varphi\left(\frac{a+\mu u}{\sqrt{u}}\right) \\ &= K(2\mu)^m e^{-2\mu a} + K\left(\frac{1}{\sqrt{u}}\right)^m H_{m-1}\left(\frac{a+\mu u}{\sqrt{u}}\right)\varphi\left(\frac{a+\mu u}{\sqrt{u}}\right) \\ &+ (-1)^{m+1}K\left(\frac{1}{\sqrt{u}}\right)^m b^m e^{-2\mu a}\Phi(\frac{a-\mu u}{\sqrt{u}}) \\ &+ (-1)^{m+1}K\left(\frac{1}{\sqrt{u}}\right)^m \varphi(\frac{a+\mu u}{\sqrt{u}})\sum_{k=0}^{m-1}(-1)^k H_k\left(\frac{a+\mu u}{\sqrt{u}}\right) b^{m-1-k}, \end{aligned} \tag{6.23}$$

where $K = (2\mu)^{-\frac{2\lambda\mu}{\delta}} e^{\frac{4\lambda\mu}{\delta}}$, $a = \frac{2\lambda}{\delta}$, $b = -2\mu\sqrt{u}$, and $m = [\frac{2\lambda\mu}{\delta}]$. ■

Table 6.1 illustrates the probabilities of ruin with different values of initial capital x and different interest rates δ of surplus. Parameters are set at $\lambda = 1$, $c = 1.5$ and $\mu = 0.5$. The table shows that the ruin probability decreases as either initial capital grows or interest rate grows. It can be seen that if the initial capital is sufficiently large, the ruin probability becomes very small. Intuitively, an insurance company is less likely to default when it possesses the corresponding initial capital or receives interest rate at a higher level.

It is interesting to compare the ruin probability $\psi_\delta(x) = 1 - f(x)$ (which can be derived from (6.23)) with the case of claim size following an exponential distribution with mean $\tilde{\mu}$. Sundt and Teugels [76] discussed the probability of ruin for the exponential case, in which the ruin probability is exponentially bounded. The explicit formula is given by

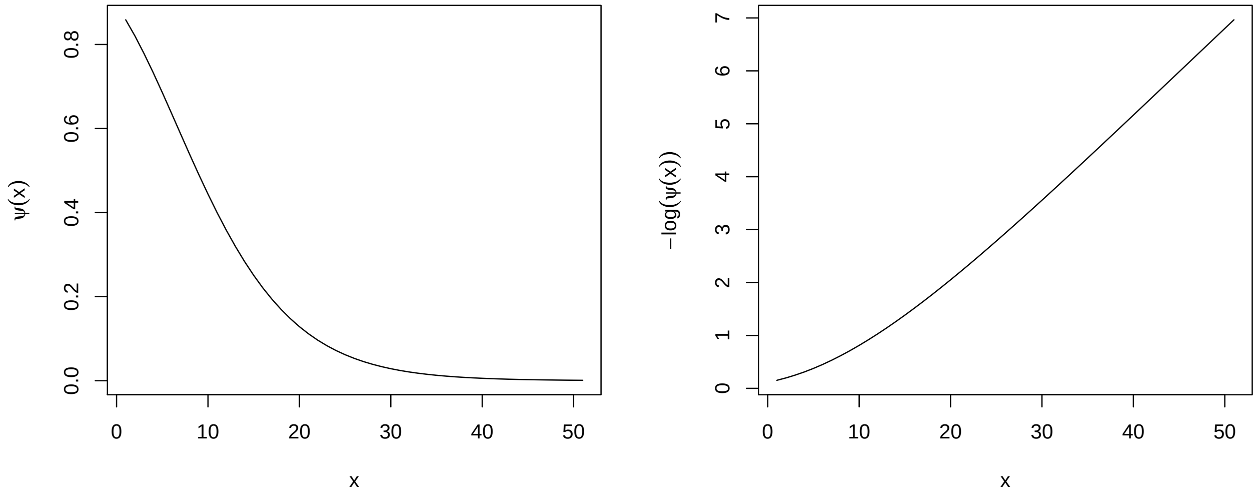
$$\psi_\delta(x) = \frac{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\tilde{\mu}} + \frac{x}{\tilde{\mu}}\right)}{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\tilde{\mu}}\right) + \frac{c}{\lambda}\left(\frac{c}{\delta\tilde{\mu}}\right)^{\frac{\lambda}{\tilde{\mu}}} e^{-\frac{c}{\delta\tilde{\mu}}}}, \tag{6.24}$$

where $\Gamma(a, b)$ is the incomplete gamma function defined as

$$\Gamma(a, b) = \int_b^\infty x^{a-1} e^{-x}\, dx. \tag{6.25}$$

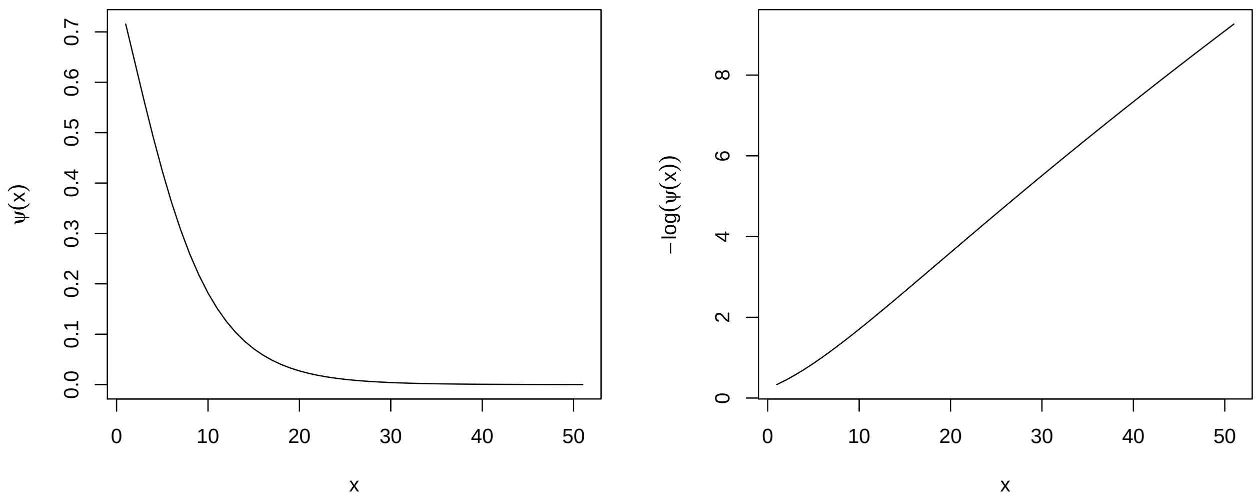
Michaud [62] obtained an estimate result of probability of ruin for the exponential case by considering the duality between the surplus process and the single-server queue. The results show that with initial capital increasing from 0 to 10, the ruin probability decays more quickly than our inverse Gaussian case. The decreasing behaviour results from the exponential decay of the claim size distribution.

Figures 6.2 and 6.3 show the plots of probability of ruin $\psi_\delta(x)$ (i.e. $1-f(x)$, $f(x)$ is derived from Theorem 6.1.1) and $-\log\psi_\delta(x)$ respectively according to the different values of initial capital x. We can see from figures 6.2 (a) and 6.3 (a) that with the initial capital x increasing, the probability of ruin decays exponentially. This can be seen from the formula of survival probability in (6.6). According to the Cramér-Lundberg approximation of ruin probability in (3.24), when $x \to \infty$, $\psi(x) \sim Ce^{-\gamma x}$ where γ is the positive root of a corresponding Lundberg equation. So we plot $-\log\psi_\delta(x)$ to check the linearity of $-\log\psi_\delta(x)$ with different values of x. It is interesting to see that $-\log\psi_\delta(x)$ slightly deviates from the corresponding straight line when the values of x are relatively small (see figures 6.2 (b) and 6.3 (b)). As x increases to large values, $-\log\psi_\delta(x)$ tends to show linearity.



(a) $\psi_\delta(x)$ (b) $-\log(\psi_\delta(x))$

Figure 6.2: Plots of probability of ruin $\psi_\delta(x)$ and $-\log(\psi_\delta(x))$ against x, $\delta = 0.05$.



(a) $\psi_\delta(x)$ (b) $-\log(\psi_\delta(x))$

Figure 6.3: Plots of probability of ruin $\psi_\delta(x)$ and $-\log(\psi_\delta(x))$ against x, $\delta = 0.1$.

6.2 Surplus Process with Stochastic Premium Income

This section studies the probability of ruin of an insurance company when we consider a generalized risk model. The premium income is no longer just a simple linear function of time but a stochastic premium income modelled by a compound Poisson process.

Define the surplus process as

$$X_t = x + \sum_{i=1}^{M_t} H_i - Z_t. \tag{6.26}$$

$x \geq 0$ is the initial capital. $\sum_{i=1}^{M_t} H_i$ is a compound Poisson process representing the fluctuations in the risk premium, H_i, i=1,2,..., are strictly positive and independent and identically distributed with common exponential distribution, i.e. $H_i \sim \exp(\alpha)$ with $\alpha > 0$. M_t is the number of jumps H_i up to time t which is a homogeneous Poisson process with intensity ρ and is also independent of the total claim process Z_t. Z_t is the same inverse Gaussian process that we introduced in previous chapter. Assume that the $\sum_{i=1}^{M_t} H_i$ and Z_t are independent. We should also specify the net profit condition, $\mathbb{E}[\sum_{i=1}^{M_t} H_i] - \mathbb{E}[Z_t] > 0$, i.e. $\frac{\rho}{\alpha} - \frac{\lambda}{\mu} > 0$.

The surplus process defined in (6.26) is a special case of the model described in Huzak et al. [48]. It incorporates the risk models discussed in Furrer [34], Yang and Zhang [82], Morales [59] and Garrido and Morales [35]. It is also a special case of the studies of Morales and Schoutens [60] and Doney and Kyprianou [27].

Given a surplus process X_t defined in (6.26), the simplified generator becomes

$$\begin{aligned}\mathcal{A}f(x,t) = {} & \frac{\partial f(x,t)}{\partial t} + \rho\left(\int_0^\infty f(x+y,t)\alpha e^{-\alpha y}dy - f(x,t)\right) \\ & - \lambda f(0,t)\int_x^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\,dy - \lambda \int_0^x f'_x(x-v,t)\int_v^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\,dydv \\ & + \lambda e^{-\beta t}\int_x^\infty e^{-\kappa x - \nu(y-x)}\frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\,dy.\end{aligned} \tag{6.27}$$

Theorem 6.2.1. *Consider the surplus process defined in (6.26), $H_i \sim exp(\alpha)$, $M_t \sim Poisson(\rho)$,*

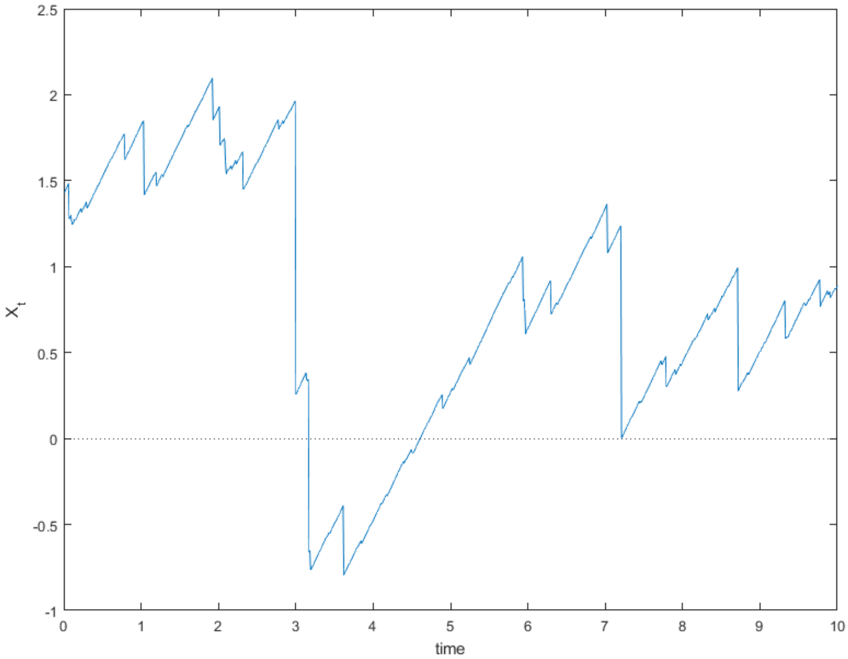


Figure 6.4: A sample path of surplus process with stochastic premium income.

the total claim process Z_t *is the inverse Gaussian process that we have discussed in previous chapters. When* $X_0 = x$ *and* $x > 0$*, the probability of ruin with initial capital* x *is given by*

$$\begin{aligned} f(x) = & \frac{2C_1}{2\mu+\eta^+}\left\{(\mu+\eta^+)\ e^{\frac{(\eta^+)^2+2\mu\eta^+}{2}x}\ \Phi((\mu+\eta^+)\sqrt{x})+\mu\ \bar{\Phi}(\mu\sqrt{x})\right\} \\ & +\frac{2C_2}{2\mu+\eta^-}\left\{(\mu+\eta^-)\ e^{\frac{(\eta^-)^2+2\mu\eta^-}{2}x}\ \Phi((\mu+\eta^-)\sqrt{x})+\mu\ \bar{\Phi}(\mu\sqrt{x})\right\}, \end{aligned} \tag{6.28}$$

where

$$\eta^{\pm} = \frac{-\left(2\mu+\frac{\rho}{\lambda}\right)\pm\sqrt{\left(2\mu-\frac{\rho}{\lambda}\right)^2+8\alpha}}{2}, \tag{6.29}$$

and

$$C_1 = \frac{\eta^+ + 2\mu + \frac{\alpha}{\mu}}{\eta^+ - \eta^-}, \qquad C_2 = \frac{\eta^- + 2\mu + \frac{\alpha}{\mu}}{\eta^- - \eta^+}. \tag{6.30}$$

Proof. It is also remarkable that ruin will immediately occur with $X_0 = 0$, i.e. $f(0) = 1$, due to the arbitrarily small jump from the claim. Suppose at time T_1, the premium process $\sum_{i=1}^{M_t} H_i$ is waiting for a jump H_{T_1}, but the claim process $\sum_{i=1}^{N_t} Y_i$ has jumped to Y_{T_1} without waiting. Therefore, ruin occurs with probability 1 if $X_0 = 0$.

Set $f(x,t) = e^{-\beta x} f(x)$, then let $\beta = 0$, the generator becomes

$$\begin{aligned}\mathcal{A}f(x) = &\rho\left(\int_0^\infty f(x+y)\alpha e^{-\alpha y} dy - f(x)\right)\\ &-\lambda\int_x^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dy - \lambda\int_0^x f'(x-v)\int_v^\infty \frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dydv \\ &+\lambda\int_x^\infty e^{-\kappa x - \nu(y-x)}\frac{1}{\sqrt{2\pi y^3}} e^{-\frac{\mu^2}{2}y}\, dy.\end{aligned} \tag{6.31}$$

In the meantime, setting $\kappa = 0$, $\nu = 0$ and applying Laplace transform to $\mathcal{A}f(x) = 0$ gives us

$$\hat{f}(\xi) = \frac{\rho\frac{\alpha}{\alpha-\xi}\hat{f}(\alpha) - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}}{\frac{\rho\alpha}{\alpha-\xi} - \rho - \lambda(\sqrt{\mu^2+2\xi}-\mu)}. \tag{6.32}$$

When $\xi \to 0$, we have $\rho\hat{f}(\alpha) = \frac{\lambda}{\mu}$. Thus by the change of variable $\sqrt{\mu^2+2\xi} - \mu = \eta$, we have

$$\begin{aligned}\hat{f}(\xi) &= \frac{\frac{\lambda}{\mu}\frac{\alpha}{\alpha-\xi} - \frac{2\lambda}{\sqrt{\mu^2+2\xi}+\mu}}{\rho\frac{\xi}{\alpha-\xi} - \lambda(\sqrt{\mu^2+2\xi}-\mu)}\\ &= \frac{2}{\lambda}\frac{1}{\eta}\frac{\frac{\lambda\alpha}{\mu} - \frac{2\lambda}{\eta+2\mu}\left(\alpha - \frac{1}{2}\eta^2 - \mu\eta\right)}{\eta^2 + \left(2\mu + \frac{\rho}{\lambda}\right)\eta + \frac{2}{\lambda}(\mu\rho - \lambda\alpha)}\\ &= 2\frac{1}{\eta+2\mu}\frac{\eta + (2\mu + \frac{\alpha}{\mu})}{\eta^2 + \left(2\mu + \frac{\rho}{\lambda}\right)\eta + \frac{2}{\lambda}(\mu\rho - \lambda\alpha)},\end{aligned} \tag{6.33}$$

Consider the equation

$$\eta^2 + \left(2\mu + \frac{\rho}{\lambda}\right)\eta + \frac{2}{\lambda}(\mu\rho - \lambda\alpha) = 0, \tag{6.34}$$

which has two roots

$$\eta^{\pm} = \frac{-\left(2\mu + \frac{\rho}{\lambda}\right) \pm \sqrt{\left(2\mu - \frac{\rho}{\lambda}\right)^2 + 8\alpha}}{2}. \tag{6.35}$$

Therefore, $\hat{f}(\xi)$ can be calculated as

$$\begin{aligned}\hat{f}(\xi) &= 2\frac{1}{\eta+2\mu}\left(\frac{C_1}{\eta-\eta^+} + \frac{C_2}{\eta-\eta^-}\right)\\ &= 2\frac{1}{\eta+2\mu}\frac{C_1}{\eta-\eta^+} + 2\frac{1}{\eta+2\mu}\frac{C_2}{\eta-\eta^-},\end{aligned} \tag{6.36}$$

where

$$C_1 = \frac{\eta^+ + 2\mu + \frac{\alpha}{\mu}}{\eta^+ - \eta^-}, \qquad C_2 = \frac{\eta^- + 2\mu + \frac{\alpha}{\mu}}{\eta^- - \eta^+}. \tag{6.37}$$

Since $\kappa = 0$, $\nu = 0$, $\hat{f}(\xi)$ is just the Laplace transform of the probability of ruin. Invert $\hat{f}(\xi)$ w.r.t. ξ, we can obtain the ruin probability for non-zero initial capital. Substitute $\sqrt{\mu^2 + 2\xi} - \mu = \eta$ and rewrite $\hat{f}(\xi)$ (6.36) as

$$\begin{aligned}
&\hat{f}(\xi) \\
&= 2\frac{1}{\sqrt{\mu^2 + 2\xi} + \mu}\frac{C_1}{\sqrt{\mu^2 + 2\xi} - \mu - \eta^+} + 2\frac{1}{\sqrt{\mu^2 + 2\xi} + \mu}\frac{C_2}{\sqrt{\mu^2 + 2\xi} - \mu - \eta^-} \\
&= 2\hat{h}_1(\xi) + 2\hat{h}_2(\xi).
\end{aligned} \tag{6.38}$$

Thus, $f(x)$ can be obtained from inverting $\hat{h}_1(\xi)$ and $\hat{h}_2(\xi)$ respectively, which is due to the linearity of Laplace transform.

We show the steps of inverting $\hat{h}_1(\xi)$. Rewrite $\hat{h}_1(\xi)$ as

$$\begin{aligned}
\hat{h}_1(\xi) &= \frac{1}{\sqrt{\mu^2 + 2\xi} + \mu}\frac{C_1}{\sqrt{\mu^2 + 2\xi} - \mu - \eta^+} \\
&= \frac{C_1}{2\mu + \eta^+}\left(\frac{1}{\sqrt{\mu^2 + 2\xi} - \mu - \eta^+} - \frac{1}{\sqrt{\mu^2 + 2\xi} + \mu}\right).
\end{aligned} \tag{6.39}$$

First invert $\frac{1}{\sqrt{\mu^2+2\xi}-\mu-\eta^+}$ as

$$\begin{aligned}
&\mathcal{L}_\xi^{-1}\left\{\frac{1}{\sqrt{\mu^2 + 2\xi} - \mu - \eta^+}\right\} \\
=&\mathcal{L}_\xi^{-1}\left\{\int_0^\infty e^{-\left(\sqrt{\mu^2+2\xi}-\mu-\eta^+\right)u} du\right\} \\
=&\mathcal{L}_\xi^{-1}\left\{\int_0^\infty e^{-\xi x}\int_0^\infty \frac{u}{\sqrt{2\pi x^3}}\, e^{-\frac{(u-\mu x)^2}{2x}} e^{\eta^+ u} du dx\right\} \\
=&\int_0^\infty \frac{u}{\sqrt{2\pi x^3}}\, e^{-\frac{(u-\mu x)^2}{2x}} e^{\eta^+ u} du \\
=&\frac{1}{\sqrt{2\pi x}} e^{-\frac{\mu^2}{2}x} + (\mu + \eta^+)\; e^{\frac{(\eta^+)^2 + 2\mu\eta^+}{2}x}\; \Phi((\mu + \eta^+)\sqrt{x}).
\end{aligned} \tag{6.40}$$

Then invert $\frac{1}{\sqrt{\mu^2+2\xi}+\mu}$ as

$$\mathcal{L}_{\xi}^{-1}\left\{\frac{1}{\sqrt{\mu^2+2\xi}+\mu}\right\}=\frac{1}{\sqrt{2\pi x}}e^{-\frac{\mu^2}{2}x}-\mu\bar{\Phi}(\mu\sqrt{x}). \tag{6.41}$$

Therefore, $\hat{h}_1(\xi)$ can be inverted as

$$h_1(x)=\frac{C_1}{2\mu+\eta^+}\left((\mu+\eta^+)\; e^{\frac{(\eta^+)^2+2\mu\eta^+}{2}x}\;\Phi((\mu+\eta^+)\sqrt{x})+\mu\bar{\Phi}(\mu\sqrt{x})\right). \tag{6.42}$$

$\hat{h}_2(\xi)$ can be inverted similarly.

■

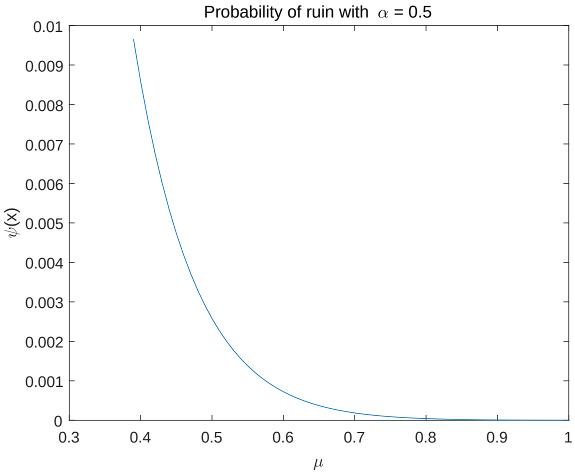
Remark. By the probability of ruin $f(x)$ in theorem 6.2.1, it is easy to check that when $x \to 0$, $f(x) \to 1$.That is, ruin will immediately occur if $X_0 = 0$.

Table 6.2 shows the numerical results of the probability of ruin with different premium sizes and different claim sizes respectively. For premium income, we consider large premium sizes (i.e. $H_i \sim \text{Exp}(\alpha)$, $\alpha = 0.5$, $i = 1, 2, ...$) and small premium sizes (i.e. $\alpha = 3$) respectively. For claim amount, we consider small claim sizes (i.e. $\mathbb{E}[Y_i] = 1/\mu$, $\mu = 1$, $i = 1, 2, ...$) and large premium sizes (i.e. $\mu = 1/3$) respectively. It is first obvious that the insurance company is more likely to ruin with initial capital x increasing. Under the case of large premium sizes (i.e. $\alpha = 0.5$), the probability of ruin grows when the claim sizes increases. The cases of small premium sizes behaviour similarly. In the meantime, if we consider large claim sizes (i.e. $\mu = 1/3$), the probability of ruin grows as well when the premium sizes decreases.

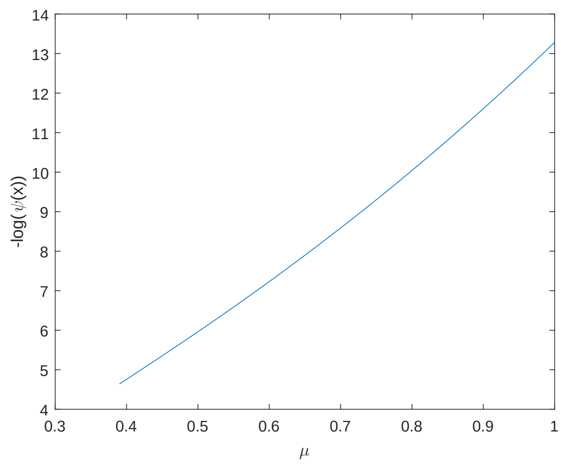
Figures 6.5 and 6.6 show the plots of probability of ruin $\psi(x)$ (i.e. $f(x)$ derived from Theorem 6.2.1) and $-\log\psi(x)$ respectively according to the change of the parameter μ when $x = 15$. Other parameter are set as $\rho = 5$, $\lambda = 0.5$. We consider the evolutions of $\psi(x)$ and $-\log\psi(x)$ when the claim size parameter μ changes under the net profit condition. We can see from figures 6.5 (a) and 6.6 (a) that with μ increasing, the probability of ruin decays exponentially. This can be seen from the formula of ruin probability in (6.42). According to the Cramér-Lundberg approximation of ruin probability in (3.24), when $x \to \infty$, $\psi(x) \sim Ce^{-\gamma x}$

	$\alpha = 0.5$		$\alpha = 3$	
	$\mu = 1$	$\mu = 1/3$	$\mu = 1$	$\mu = 1/3$
x	$\psi(x)$	$\psi(x)$	$\psi(x)$	$\psi(x)$
0	1	1	1	1
0.1	0.2082	0.3215	0.3740	0.9139
0.3	0.1014	0.2274	0.2433	0.8936
0.5	0.0656	0.1913	0.1835	0.8820
0.7	0.0467	0.1695	0.1453	0.8731
0.9	0.0349	0.1541	0.1181	0.8654
1	0.0306	0.1478	0.1071	0.8619
3	0.0043	0.0869	0.0219	0.8099
5	9.2474e-04	0.0616	0.0057	0.7704
7	2.3388e-04	0.0463	0.0016	0.7359
9	6.4158e-05	0.0360	4.9106e-04	0.7046
10	3.4295e-05	0.0320	2.7250e-04	0.6899

Table 6.2: Infinite time ruin probabilities $\psi(x)$.

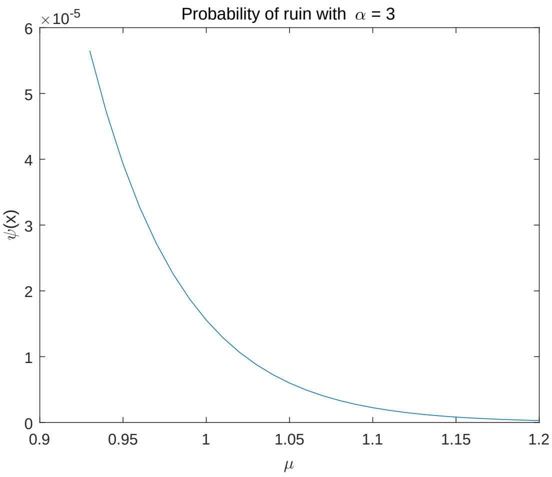


(a) Probability of ruin $\psi(x)$

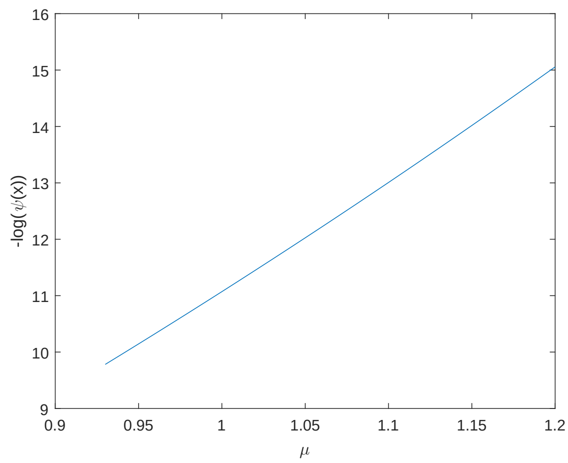


(b) $-\log \psi(x)$

Figure 6.5: Plots of probability of ruin $\psi(x)$ and $-\log \psi(x)$ w.r.t μ, $\alpha = 0.5$



(a) Probability of ruin $\psi(x)$

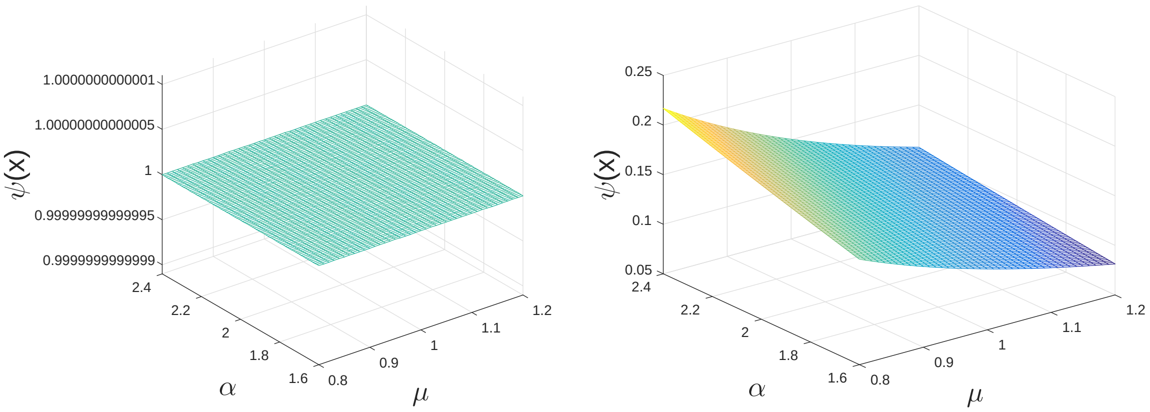


(b) $-\log \psi(x)$

Figure 6.6: Plots of probability of ruin $\psi(x)$ and $-\log \psi(x)$ w.r.t μ, $\alpha = 3$

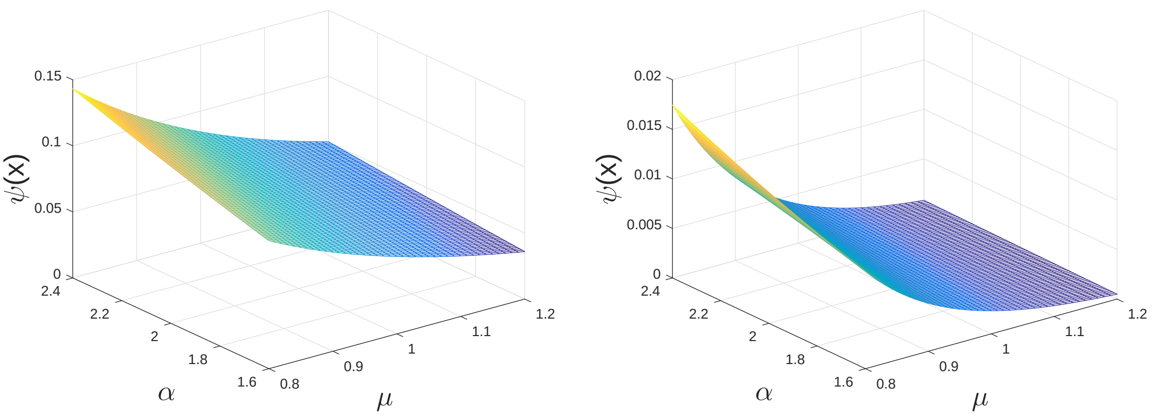
where γ is the positive root of a corresponding Lundberg equation. So we plot $-\log\psi(x)$ to check the linearity of $-\log\psi(x)$ with different values of μ. It is interesting to see that $-\log\psi(x)$ slightly deviates from the corresponding straight line (see figures 6.5 (b) and 6.6 (b)). We can see from the formula of probability of ruin in (6.28), the logarithm of $f(x)$ is nearly linear with respect to μ, and the term $e^{\frac{(\eta^+)^2+2\mu\eta^+}{2}x}$ dominates logarithm of $f(x)$ with μ increases.

Figures 6.7 to 6.9 show the 3-D plots of probability of ruin $\psi(x)$ with respect to μ and α with different values of initial capital x. Other parameter are set as $\rho = 5$ and $\lambda = 0.5$. We study the evolutions of $\psi(x)$ when the claim size parameter μ and premium parameter α change simultaneously. It is clear to see that the insurance company ruins with probability 1 given zero initial capital $x = 0$. With x increasing, the probability of ruin decreases. When μ and α are fixed, the insurance company is less likely to ruin with larger initial capital x. When x is fixed, ruin is less likely to occur with smaller claim size (i.e. larger μ) or larger premium size (i.e. smaller α).



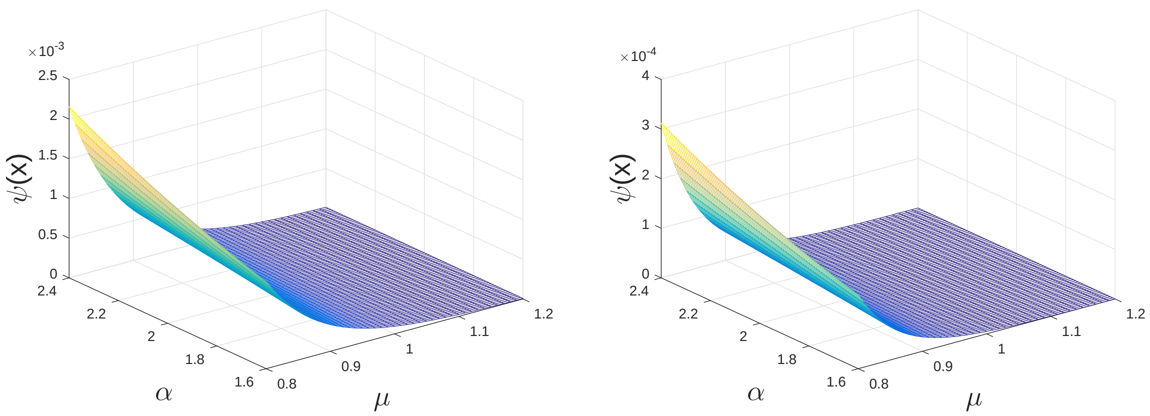
(a) $x = 0$ (b) $x = 0.5$

Figure 6.7: Plots of probability of ruin $\psi(x)$ w.r.t μ and α, $x = 0$ and $x = 0.5$



(a) $x = 1$ (b) $x = 5$

Figure 6.8: Plots of probability of ruin $\psi(x)$ w.r.t μ and α, $x = 1$ and $x = 5$



(a) $x = 10$ (b) $x = 15$

Figure 6.9: Plots of probability of ruin $\psi(x)$ w.r.t μ and α, $x = 10$ and $x = 15$

Chapter 7

Conclusion

This thesis investigates classical ruin and Parisian type of ruin problems in insurance risk management. The focus is on inverse Gaussian process and Parisian excursion theory.

There are several main results in this thesis. First, classical ruin problems are studied. We begin by studying ruin probabilities of a classical collective risk process, in which claim sizes are driven by an exponential distribution and a mixture of two exponential distributions respectively. Then, we extend the total claim amount process to the case that it follows an inverse Gaussian process, which considers that there could be infinitely many and arbitrarily small claims over any finite time interval. The Laplace transforms of the ruin time and the overshoot given different initial capitals are derived respectively. These Laplace transforms provide the essentially primary calculations for the study of Parisian type of ruin.

Furthermore, the joint distribution of the ruin time, the overshoot and non-zero initial capital has also been studied. We present a closed-form of the probability of ruin for non-zero initial capital. Our results mainly rely on a piecewise deterministic Markov model and Gerber-Shiu expected discounted penalty function.

Then we study Parisian type of ruin through Parisian excursions, which are the excursions that continuously exceed a certain length. Parisian type of ruin is a generalization of the classical ruin, with the advantage of being highly adaptable to insurance companies' beliefs in practice. The Laplace transforms of the first excursion below zero and the first excursion above zero are obtained respectively. We also discuss their dependence via their joint Laplace transform and joint probability density function. By using a piecewise deterministic semi-Markov process, we present the Laplace transform of the Parisian ruin time for zero initial capital and for non-zero initial capital respectively. Then the explicit formulae of the probabilities of Parisian type of ruin are derived. We also obtain an asymptotic Parisian ruin probability when initial capital converges to infinity. This asymptotic result is similar to the Cramér-Lundberg approximation, in which ruin probability decays at an exponential rate.

Furthermore, we extend our work to two cases. One is that we consider variable premium income, which studies the probability of survival when the insurance company invests its surplus and collects interest. Another is that we discuss a surplus process with stochastic premium income (i.e. a compound Poisson process) but no linear income. The probability of ruin with non-zero initial capital for the latter case is presented.

The study of Parisian ruin time can be used to pricing Parisian options whose underlying asset stays continuously above or under a certain barrier level and reach a pre-defined length. Further research can be done to find the price of one-sided Parisian barrier options, with the underlying asset price process being written on the surplus. Dassios and Wu [21] had obtained the Laplace transform of Parisian down-and-in call option price w.r.t. the maturity of the option, where they considered the claim size follows an exponential distribution. Their result could be generalized to the case that the total claim amount process is an inverse Gaussian process.

Another direction of further research might be to explore further into stopping times including both the length, the height and the number of excursions, as these could provide the number of claims and corresponding severity for an insurance company. It would also be great to further apply some of these results to mathematical finance such as option pricing.

Regarding the ruin probability of an insurance company with stochastic premium income, one can also include a negative linear premium. Assume that the intensity rate of premium jumps is as same as the constant rate of the linear premium. When the rate converges to infinity, the stochastic premium income subtracted by the linear premium converges to Brownian motion. Therefore, ruin could occur either due to a large jump from claim or due to the negative movement from Brownian motion. In other words, ruin occurs by creeping. That is to say when a spectrally negative Lévy process started from positive enters negative for the first time, this may do so either by a jump or continuously. It would also be interesting to look into exploring the classical probability of ruin as well as Parisian type ruin probability for this case. The corresponding pricing problems of Parisian option could also be interesting to study.

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