Department of Mathematics in

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REFLECTED DIFFUSIONS
AND
PIECEWISE DIFFUSION APPROXIMATIONS OF LEVY PROCESSES

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

This thesis consists of 71 pages.

Abstract

In the first part of the thesis, the solvability of stochastic differential equations with reflecting boundary conditions is studied. Such equations arise in singular stochastic control problems as a way for determining the optimal strategies. The stochastic differential equations represent homogeneous one-dimensional diffusions while the boundaries are given by càdlàg functions. Pathwise solutions are constructed under mild assumptions on the coefficients of the equations. In particular, the solutions are derived as the diffusions' scale functions composed with appropriately time-changed reflected Brownian motions. Several probabilistic properties are addressed and analysed.

In the second part of the thesis, piecewise diffusion approximations of Lévy processes are studied. Such approximating processes have been called Itô semi-diffusions. While keeping the statistical fit to Lévy processes, this class of processes has the analytical tractability of Itô diffusions. At a given time grid, their distribution is the same as the one of the underlying Lévy processes. At times outside the grid, they evolve like homogeneous diffusions. The analysis identifies conditions under which Itô semi-diffusions can be used as an alternative to Lévy processes for modelling financial asset prices. In particular, for a sequence of Itô semi-diffusions determined by a given Lévy process, conditions for the convergence of their finite-dimensional distributions to the ones of the Lévy process are established. Furthermore, for a single Itô semi-diffusion, conditions for the existence of pricing measures are established.

Contents

Part I§

Weak solutions to SDE's reflected in a càdlàg function

1. Introduction

Optimal solutions to stochastic control problems exist under assumptions like continuity of the involved coefficients. These solutions are attainable via dynamic programming when the coefficients are Lipschitz continuous. A comprehensive study on this subject was given by HAUSSMANN & SUO [19, 20]. However, the assumption of (Lipschitz) continuous coefficients was made to include a variety of different kinds of state processes. As mentioned by JACKA [21, Section 4], a tailored study for one-dimensional diffusions with general coefficients would be interesting. In fact, many results concerning diffusions require little assumptions, see for example KARATZAS & RUF [26], MIJATOVIĆ & URUSOV [31] or URUSOV & ZERVOS [46]. Overall, little is known for stochastic control problems beyond continuous coefficients whereas the theory of diffusions handles general measurable coefficients.

In the context of singular stochastic control, optimal solutions are typically described by a certain boundary, see for example KARATZAS [25]. Here, the state space is separated into an acting region and a waiting region. The optimal strategy only acts when the state process tends to fall below the boundary. At the boundary, the optimal strategy uses as little force as possible to keep the process inside of the waiting region. To verify equivalence between the control problem and the boundary problem, a set of stronger assumptions is usually imposed. As those assumptions vastly depend on the specific problem, we rather look at existence in the framework of the boundary problem. Figure 1.1 pictures the situation in a singular stochastic control setting, where Figure 1.2 visualises the situation we have in mind.

§This part is based on joint work with Professor Mihail Zervos.

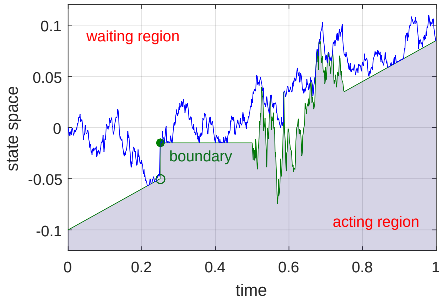

Figure 1.1: singular optimal control

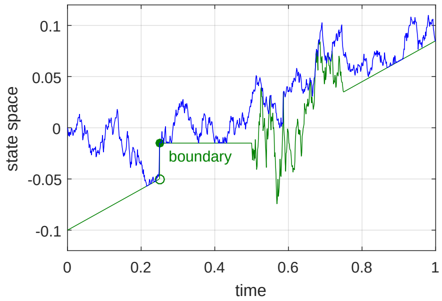

Figure 1.2: a reflected SDE

In our case, the state process follows the dynamical description of a one-dimensional diffusion. Without any boundary, weak solutions exist under mild integrability assumptions, the Engelbert-Schmidt conditions. A detailed proof can be found in KARATZAS & SHREVE [27, Chapter 5.5]. Incorporating the boundary, our idea is to redo the proof and obtain weak solutions under the same mild assumptions.

Fixing ideas, we consider an interval $\mathcal{I}$, a càdlàg function L with values in $\mathcal{I}$, and Borel measurable functions μ, σ fulfilling the Engelbert-Schmidt conditions in $\mathcal{I}$. More precisely, $\sigma^2 > 0$ and $1/\sigma^2, \mu/\sigma^2$ are locally integrable in the interior of $\mathcal{I}$. Our objective is to find a weak solution to the following reflecting stochastic differential equation,

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \mathrm{d}Z + \sigma(X_t)\mathrm{d}W \quad \text{such that}$$
$$X_t \geq L(t) \quad \text{and} \quad Z_t = \int_0^t \mathbb{1}_{X_s = L_s} |\mathrm{d}Z_s| \quad \text{for } t \in \mathbb{R}_+.$$

Varying in the underlying assumptions, there is a rich literature about this kind of equations. In the Lipschitz case with a stochastic L, SŁOMIŃSKI & WOJCIECHOWSKI [44] found strong solutions. For different growth conditions on μ, σ and constant L, LAUKAJTYS & SŁOMIŃSKI [29] and ROZKOSZ & SŁOMIŃSKI [40] considered converging in law sequences of solutions. In contrast to their approaches, our methodology aims to give an explicit representation for the solution.

At first, to simplify the problem, we remove the drift by applying a state transformation. In the terminology of the corresponding literature, the resulting local martingale is called diffusion in natural scale. Interestingly, using similar transformations to normalise the volatility to one, Zhang [47] found strong solutions for linear bounded μ with Lipschitz σ and constant L. His technique is well-known in exact simulation as presented in Beskos et al [3] and Jenkins [23].

In a second step, we construct a solution to the driftless equation by looking at an appropriate time-changed Brownian motion. With this approach, we are able to construct a solution until the diffusion leaves $\mathcal{I}$. In particular, for a solution on the whole timeline, the endpoints of $\mathcal{I}$ need to be inaccessible or absorbing for the diffusion. However, the reflection in the SDE also allows for a reflecting lower endpoint. For a detailed description of the state/time transformations and endpoints, see Revuz & Yor [36, Chapter V.3 and VII.3] and Rogers & Williams [37, Chapter V.6].

The time-change is the main cause of technical issues in this part. Incorporating the boundary, the deterministic function L will be evaluated at the time-change. Figure 1.3 shows our original setting, where Figure 1.4 visualises the setting before a time-change.

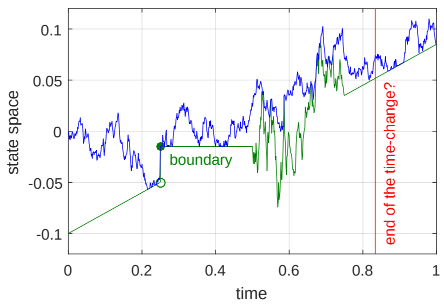

Figure 1.3: time-changed process

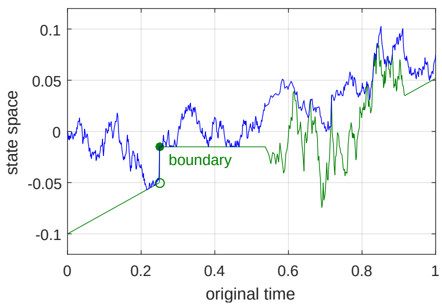

Figure 1.4: reflected Brownian motion

This will lead to a coupled system of equations between the time-change and the reflection. Moreover, known results for the time-change from the unreflected case become unclear with reflection. More precisely, we are interested in the following two questions,

1st question: can we solve the coupled system of time-change and reflection?

2nd question: does the time-change cover the whole timeline?

1st question. The first equation of the coupled system describes a Skorokhod problem. In dimension one, the unique solution is explicitly known, see for example KARATZAS & SHREVE [27, Chapter 3.6]. Since then, research has been focused on multidimensional extensions to this problem, like BURDZY, KANG & RAMANAN [5] or BURDZY & KANG [24]. However, the second equation of the coupled system leads to a functional differential equation involving a running supremum. For an overview on recent developments in this field see CORDUNEANU, LI & MAHDAVI [10]. Using the explicit solution for the first equation, we iteratively construct solutions for piecewise constant càdlàg L. Then, using an approximation argument, we prove existence of a solution to our Skorokhod problem. More precisely, let $y, L\colon \mathbb{R}_+ \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}_+\backslash\{0\}$ such that y, g are continuous, L càdlàg, and $y(0) \geq L(0)$. Given such data, we show that there exist a constant $T_\infty \in \,]0,\infty]$ and a function $\tau\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ such that

$$(i)\ \ \tau(t) = \int_0^t g\Big(y(s) + 0 \vee \sup_{r\leq s} \big(L(\tau(r)) - y(r)\big)\Big)\,\mathrm{d}s \quad \text{for } t < T_\infty,$$

(ii) if $T_\infty < \infty$, then $\lim_{t\to T_\infty} \tau(t) = \infty$.

Interestingly, due to the mild assumptions, we are unable to show meaningful estimates or uniform convergence for the approximation. Instead, we rest on the fact that we work with increasing function. For instance, using a diagonal argument, a sequence of increasing functions always contains a pointwisely converging subsequence. Unfortunately, the choice of the subsequence is based on a list of rational numbers that is going back and forth in time. Therefore, for a given filtration, it becomes an issue to ensure that the limit is adapted. In this context, Bolzano-Weierstrass-like theorems have been studied by DELBAEN & SCHACHERMAYER [11, Chapter 6.3] and BILLINGSLEY [4, Section 13 and 16]. They considered measurably selecting subsequences or tightness respectively. In need to preserve monotonicity of the limit and our lack of estimates, we fall short here. Instead, we use the methodology from FLEMING & SONER [16, Chapter VII.1] to avoid this problem. Assuming uniqueness of the Skorokhod problem, we show that a specific sequence of functions converges already. Hence, choosing a subsequence becomes unnecessary and adaptedness is no issue any more.

2nd question. Having the solution of an optimisation problem in mind, we want to exclude behaviour which is virtually impossible to control. More precisely, we want to exclude the following hypothetical scenario. Assume that the time-change transforms the timeline into a finite interval. Starting from a Brownian motion, we start with a process that fluctuates between $\pm\infty$. In particular, our time-changed Brownian motion will oscillate with infinite frequency between $\pm\infty$ at the end of the finite interval.

It is well-known that for a homogeneous diffusion without reflection the time change covers the whole time line. This is also true in the presence of a reflecting boundary, because locally it evolves like an unaffected homogeneous diffusion. Heuristically, we may assume that the hypothetical scenario occurs, then we are able to see the described oscillation in law. In particular, the law of the reflected homogeneous diffusion changes closer to the end of the finite interval. Due to the fluctuations, the process will be often away from the reflecting boundary. Here, the process evolves as an unaffected homogeneous diffusion. These processes are known to be unique in law, see for example Karatzas & Shreve [27, Theorem 5.7]. Hence, the law depends on the distance to the end of the finite interval and coincides with a law that is independent of it. From this contradiction, we exclude the hypothetical scenario and see that the time-change covers the whole timeline.

Overall, assuming the Engelbert-Schmidt conditions, assuming inaccessible or absorbing endpoints, and assuming that the Skorokhod problem has a unique solution, we are able to prove weak existence of the above reflecting SDE.

A plan for future research includes the task to prove the uniqueness of the Skorokhod problem instead of assuming it. This is equivalent to prove that the functional differential equation $(i), (ii)$ has a unique solution. Heuristically, this can be seen from the following arguments. We formally split the timeline into the parts where the supremum stays constant and where the supremum increases. When the supremum stays constant, there is no feedback and the solution trivial. When the supremum increases, we face an autonomous ODE with positive driver, which is known to have a unique solution. In particular, a solution may be unique. Moreover, it seems intuitive that two different solutions for one Brownian motion y implies the existence of two different solutions for a constant y. As $g > 0$, the differential equation with constant y has a unique solution. Overall, as $g > 0$, we expect to have a unique solution.

The rest of this part is organised as follows. In section 2, we introduce and solve our Skorokhod problem. In section 3, we construct a weak solution to the above reflecting SDE and characterise the endpoints of its state space.

2. The Generalised Skorokhod Equation

In this section, we study the following variation of the classical Skorokhod problem.

2.1 Problem (Generalised Skorokhod Equation). *Let* $y, L\colon \mathbb{R}_+ \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$ *such that* y, g *are continuous,* L *càdlàg, and* $y(0) \geq L(0)$.

Given such data, determine a constant $T_\infty \in \,]0,\infty]$ *and a function* $z\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *such that*

(*i*) z *is increasing and right-continuous,* $z(0) = 0$ *and* $z(t) \in \mathbb{R}_+$ *for all* $t \in [0, T_\infty[$,

(*ii*) *given* $\tau(t) = \int_0^t g(y(s) + z(s))\,\mathrm{d}s$ *and* $\Gamma(t) = \{t \geq s \geq 0 \,|\, y(s) + z(s) = L(\tau(s))\}$,

$$y(t)+z(t) \geq L(\tau(t)) \quad \text{and} \quad z(t) = \int_0^\infty \mathbb{1}_{\Gamma(t)}(s)\,\mathrm{d}z(s) \quad \text{for all } t \in [0, T_\infty[,$$

(*iii*) *if* $T_\infty < \infty$, *then* $\lim_{t\to T_\infty} z(t) = \infty$.

As shown in the following lemma, our generalised Skorokhod problem reduces to the original Skorokhod problem for càdlàg step functions. This lemma would also be true, if we assume that y is only càdlàg and g only locally bounded Borel-measurable.

2.2 Lemma. *Consider a piecewise constant càdlàg function* $L\colon \mathbb{R}_+ \to \mathbb{R}$, *i.e. there are sequences* $(\ell_i), (a_i)$ *from* $\mathbb{R}$ *such that* $\ell_0 \leq y(0)$, $a_0 = 0$, (a_i) *is strictly increasing to* ∞, *and*

$$L = \sum_{i=0}^\infty \ell_i \mathbb{1}_{[a_i, a_{i+1}[}.$$

In this case, the Generalised Skorokhod Equation 2.1 *has a unique solution* (z, T_∞), *which is such that*

$$z(t) = 0 \vee \sup_{s \leq t} \big(L(\tau(s)) - y(s)\big) \quad \text{for all } t < T_\infty.$$

Proof. We set

$$t_0 = 0 \quad \text{and} \quad z_0(t) = 0 \text{ for all } t \geq 0,$$

and define inductively over $i \in \mathbb{N}$ the sequences of times (t_i) and functions (z_i) by

$$z_i(t) = \begin{cases} z_{i-1}(t) & \text{if } t < t_{i-1} \\ z_{i-1}(t_{i-1}) \vee \sup_{t_{i-1} \le s \le t} \big(\ell_i - y(s)\big) & \text{if } t \ge t_{i-1} \end{cases} \quad \text{and} \tag{1}$$

$$t_i = \inf \Big\{ t \ge t_{i-1} \,\Big|\, \int_{t_{i-1}}^{t} g\big(y(s) + z_i(s)\big)\, \mathrm{d}s \ge a_i - a_{i-1} \Big\}. \tag{2}$$

Let Γ_i as in (ii) of 2.1 with z_i in place of z. Then, we have

$$\{t \ge 0 \,|\, \Delta z_i(t) > 0\} \subseteq \Gamma_i(\infty) \cap \{a_j \,|\, i-1 \ge j \ge 1\} \quad \text{and}$$
$$\Delta z_i(a_j) = \Big(\ell_j - \big(y(a_j-) + z(a_j-)\big)\Big)^+ \quad \text{for all } i-1 \ge j \ge 1,$$

Together with Karatzas & Shreve [27, p.210, Lemma 3.6.14] and a simple induction argument, we can see that, the restriction of the function z_i in $[0, t_i]$ is the unique function that satisfies $(i), (ii)$ of 2.1 in $[0, t_i]$. Now, we define

$$T_\infty = \lim_{i \to \infty} t_i \quad \text{and} \quad z(t) = z_i(t) \ \text{ for } t \in [t_{i-1}, t_i[\ \text{ and } i \in \mathbb{N}. \tag{3}$$

As L, g and z_i are locally bounded, we have $t_i > t_{i-1} \ge 0$ for all $i \in \mathbb{N}$, in particular $T_\infty > 0$. Moreover, in case of $T_\infty < \infty$, we have

$$\int_0^{T_\infty} g\big(y(s) + z(s)\big)\, \mathrm{d}s = \sum_{i=1}^{\infty} (a_i - a_{i-1}) = \lim_{i \to \infty} a_i - a_0 = \infty. \tag{4}$$

If z would be bounded in $[0, T_\infty[$, then (4) would be finite. Hence, z is unbounded in $[0, T_\infty[$. Moreover, as z is increasing, $\lim_{t \to T_\infty} z(t) = \infty$. Hence, (iii) of 2.1 is true.

Overall, it follows that the definitions (3) provide the unique solution to the Generalised Skorokhod Equation. □

In preparation for our main theorem in this section, we prove two well-known results about increasing functions. The first one is based on a diagonal argument, and the second one is a variant of the Glivenko-Cantelli theorem. The second one can be found in Parzen [33, p.438, Exercise 10.5.2].

2.3 Lemma. *Every sequence of increasing functions $z_n \colon \mathbb{R}_+ \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, contains a subsequence that converges pointwise to an increasing function $z \colon \mathbb{R}_+ \to \overline{\mathbb{R}}$.*

Proof. To treat finite and infinite limits at the same time, we apply the arcus tangent function to z_n for $n \in \mathbb{N}$. In particular, we assume without loss of generality that (z_n) is a bounded sequence. Let (q_ℓ) be a list of the positive rationals $\mathbb{Q}_+$. We define inductively the following sequences:

$$\ell = 1: \ (a_n^1) \text{ is a sequence such that } \lim_{n\to\infty} z_{a_n^1}(q_1) = \liminf_{n\to\infty} z_n(q_1),$$
$$\ell > 1: \ (a_n^\ell) \text{ is a subsequence of } (a_n^{\ell-1}) \text{ with } \lim_{n\to\infty} z_{a_n^\ell}(q_\ell) = \liminf_{n\to\infty} z_{a_n^{\ell-1}}(q_\ell).$$

If we define $b_n = a_n^n$, for $n \in \mathbb{N}$, then we can see that

$$\tilde{z}(q_\ell) = \lim_{n\to\infty} z_{b_n}(q_\ell) \text{ exists for all } \ell \in \mathbb{N}.$$

As weak inequalities are preserved under limits, $\tilde{z}$ is increasing. Therefore, we can extend $\tilde{z}\colon \mathbb{Q}_+ \to \mathbb{R}$ to an increasing function $\tilde{z}\colon \mathbb{R}_+ \to \mathbb{R}$ by defining

$$\tilde{z}(t) = \inf\left\{\tilde{z}(q_\ell) \,\middle|\, \ell \geq 1 \text{ and } q_\ell \geq t\right\}, \quad \text{for } t \geq 0.$$

In what follows, we denote by $D \subseteq \mathbb{R}_+$ the countable set at which the discontinuities of the increasing function $\tilde{z}$ occur. We show that

$$\tilde{z}(t) = \lim_{n\to\infty} z_{b_n}(t) \quad \text{for all } t \in \mathbb{R}_+ \backslash D. \tag{5}$$

Let $t \in \mathbb{R}_+\backslash D$. Then, for given $\varepsilon > 0$, there exist rationals $s \leq t \leq u$ such that

$$|\tilde{z}(s) - \tilde{z}(t)| \vee |\tilde{z}(u) - \tilde{z}(t)| \leq \varepsilon.$$

The definition of $\tilde{z}$ implies that there exists $N = N(\varepsilon)$ such that

$$|z_{b_n}(s) - \tilde{z}(s)| \vee |z_{b_n}(u) - \tilde{z}(u)| \leq \varepsilon \quad \text{for all } n > N.$$

Moreover, from the monotonicity of the function z_{b_n}, we see that

$$z_{b_n}(s) - \tilde{z}(t) \leq z_{b_n}(t) - \tilde{z}(t) \leq z_{b_n}(u) - \tilde{z}(t).$$

Combining these last observations, we see that

$$|z_{b_n}(t) - \tilde{z}(t)| \leq |z_{b_n}(s) - \tilde{z}(t)| \vee |z_{b_n}(u) - \tilde{z}(t)| < 2\varepsilon \quad \text{for all } n > N.$$

Hence, (5) holds true, because $\varepsilon > 0$ has been arbitrary.

As the set D is countable, a second diagonal argument ensures that there is a subsequence (c_n) of (b_n) such that

$$z(t) = \lim_{n\to\infty} z_{c_n}(t) \;\text{ exists } \quad\text{and}\quad z(t) \in \big[\tilde{z}(t-), \tilde{z}(t)\big] \quad \text{for all } \; t \in D.$$

It follows that the function $z\colon \mathbb{R}_+ \to \mathbb{R}$ defined by this limit for $t \in D$ and by $z(t) = \tilde{z}(t)$ for $t \in \mathbb{R}_+ \backslash D$ is the pointwise limit of (z_{c_n}). ☐

2.4 Lemma. *Pointwise convergence of a sequence of increasing functions* $z_n\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $n \in \mathbb{N}$, *to an increasing continuous function* $z\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *implies that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \big|z_n(t) - z(t)\big| = 0 \quad \textit{for all } \; T > 0 \; \textit{ with } \; z(T) < \infty.$$

Proof. We fix $T > 0$ such that $z(T) < \infty$ and we denote by $\overleftarrow{z}, \overrightarrow{z}\colon [z(0), z(T)] \to \mathbb{R}_+$ the right- and left-continuous inverses of the restriction of z in $[0, T]$. More precisely,

$$\begin{aligned} \overleftarrow{z}(w) &= \inf\big\{x \geq 0 \,\big|\, z(x) > w\big\} \wedge T \quad \text{and} \\ \overrightarrow{z}(w) &= \inf\big\{x \geq 0 \,\big|\, z(x) \geq w\big\} \quad \text{for } \; w \in [z(0), z(T)]. \end{aligned}$$

The continuity of z implies that $\overleftarrow{z}, \overrightarrow{z}$ are strictly increasing. Therefore, $z \circ \overleftarrow{z}$ and $z \circ \overrightarrow{z}$ are the identity function. In view of this observation and the assumption that the sequence (z_n) converges pointwisely to z, we see that

$$\lim_{n\to\infty} z_n(\overleftarrow{z}(w)) = \lim_{n\to\infty} z_n(\overrightarrow{z}(w)) = w \quad \text{for all } \; w \in [z(0), z(T)]. \tag{6}$$

Furthermore, for $t \in [0, T]$, consider $u = \overleftarrow{z}(z(t))$ and $s = \overrightarrow{z}(z(t))$. In particular, $u \geq t \geq s$ and $z(u) = z(t) = z(s)$. This together with the monotonicity of z_n for $n \geq 1$ implies

$$\begin{aligned} |z_n(t) - z(t)| &\leq |z_n(u) - z(u)| \vee |z_n(s) - z(s)| \\ &\leq \sup_{w\in[z(0),z(T)]} |z_n(\overleftarrow{z}(w)) - w| \vee |z_n(\overrightarrow{z}(w)) - w|. \end{aligned} \tag{7}$$

In view of (6) and (7), it is enough to look at the case when $z(w) = w$ for all $w \in [0, T]$.

Now, we fix any $\varepsilon > 0$ and we consider any points $0 = t_0 \leq t_j < t_{j+1} \leq t_k = T$

such that $t_{j+1} - t_j \le \varepsilon$ for all $k-1 \ge j \ge 0$. The pointwise convergence of (z_n) to the identity function implies that there exists $N = N(\varepsilon)$ such that

$$|z_n(t_j) - t_j| \le \varepsilon \quad \text{for all } k \ge j \ge 0 \text{ and } n \ge N.$$

Given any $n \ge N$ and $t \in [t_j, t_{j+1}]$, we use the above inequality and the assumption that z_n is an increasing function to obtain

$$-2\varepsilon \le z_n(t_j) - t_j - \varepsilon \le z_n(t) - t \le z_n(t_{j+1}) - t_{j+1} + \varepsilon \le 2\varepsilon.$$

It follows that $\sup_{t\in[0,T]} |z_n(t) - t| \le 2\varepsilon$ for all $n \ge N$, and the proof is complete because $\varepsilon > 0$ has been arbitrary. □

Later, we will encounter a sequence of increasing functions for which we are unable to prove uniform convergence. Therefore, we need the following Lemma.

2.5 Lemma. *Let $L_n \colon \mathbb{R}_+ \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of càdlàg functions that converges uniformly from above to $L \colon \mathbb{R}_+ \to \mathbb{R}$. Let $\tau_n \colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$, $n \in \mathbb{N}$, be a sequence of increasing continuous functions that converges pointwisely to a continuous function $\tau \colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$. Moreover, let $\tau(0) = 0$ and let τ be strictly increasing on the set where it is finite. We define*

$$T_\infty = \inf\left\{t \ge 0 \,\middle|\, \tau(t) = \infty\right\}.$$

Then, for any continuous function $y \colon \mathbb{R}_+ \to \mathbb{R}$, there exists a countable set $D \subseteq [0, T_\infty[$ such that for all $t \in [0, T_\infty[\setminus D$ it holds that

$$\lim_{n\to\infty} \sup_{s\le t} \big(L_n(\tau_n(s)) - y(s)\big) = \sup_{s \le t} \big(L(\tau(s)) - y(s)\big). \tag{8}$$

Proof. Let $\vec{\tau} \colon \mathbb{R}_+ \to [0, T_\infty[$ be the strictly increasing continuous inverse of τ. As $y \circ \vec{\tau}$ is continuous, its restriction to a compact is uniformly continuous. Furthermore, according to Lemma 2.4, the sequence (τ_n) converges uniformly to τ in $[0,t]$ for all $t < T_\infty$. In particular, the sequence (τ_n) is uniformly bounded. Combining these two observations, we obtain

$$\begin{aligned}\lim_{n\to\infty} \sup_{s\le t} &\big|(y\circ\vec{\tau})(\tau_n(s)) - y(s)\big| \\ &= \lim_{n\to\infty} \sup_{s\le t} \big|(y\circ\vec{\tau})(\tau_n(s)) - (y\circ\vec{\tau})(\tau(s))\big| = 0 \quad \text{for all } t < T_\infty.\end{aligned} \tag{9}$$

Let $M(t) = L(t) - (y\circ\vec{\tau})(t)$ and $M^*(t) = \sup_{s\le t} M(s)$ for $t \in \mathbb{R}_+$. Let D^*

the countable set of all points at which the discontinuities of M^* occur. As $\vec{\tau}$ is strictly increasing, the set $D = \vec{\tau}(D^*) \subseteq [0, \tau_\infty[$ is also countable. Moreover, for $t \in [0, T_\infty[\setminus D$, the point $\tau(t)$ is a continuity point of M^*. Keeping in mind that (τ_n) and τ are continuous and increasing, as well as (τ_n) converges to τ, we have for all $t \in [0, T_\infty[\setminus D$ that

$$\begin{aligned}\lim_{n\to\infty} \sup_{s\le t} \big(L(\tau_n(s)) - (y \circ \vec{\tau})(\tau_n(s))\big) &= \lim_{n\to\infty} \sup_{s\le t} M(\tau_n(s)) = \lim_{n\to\infty} \sup_{u\le \tau_n(t)} M(u) \\ &= \lim_{n\to\infty} M^*(\tau_n(t)) = M^*(\tau(t)) = \sup_{s\le t} \big(L(\tau(s)) - y(s)\big). \qquad (10)\end{aligned}$$

Now, we can estimate that

$$\begin{aligned}&\lim_{n\to\infty} \big|\sup_{s\le t} \big(L(\tau_n(s)) - y(s)\big) - \sup_{s\le t} \big(L(\tau(s)) - y(s)\big)\big| \\ &\quad\le \lim_{n\to\infty} \big|\sup_{s\le t} \big(L(\tau_n(s)) - y(s)\big) - \sup_{s\le t} \big(L(\tau_n(s)) - (y \circ \vec{\tau})(\tau_n(s))\big)\big| \\ &\qquad + \lim_{n\to\infty} \big|\sup_{s\le t} \big(L(\tau_n(s)) - (y \circ \vec{\tau})(\tau_n(s))\big) - \sup_{s\le t} \big(L(\tau(s)) - y(s)\big)\big| \\ &\quad\overset{(10)}{\le} \lim_{n\to\infty} \sup_{s\le t} \big|(y \circ \vec{\tau})(\tau_n(s)) - y(s)\big| + 0 \\ &\quad\overset{(9)}{=} 0 \quad \text{for all } t \in [0, T_\infty[\setminus D. \qquad (11)\end{aligned}$$

Furthermore, as $L_n \ge L$ for all $n \ge 1$, the following is true for all $t \in [0, T_\infty[$,

$$\begin{aligned}\sup_{s\le t} \big(L(\tau_n(s)) - y(s)\big) &\le \sup_{s\le t} \big(L_n(\tau_n(s)) - y(s)\big) \\ &\le \sup_{s\le t} \big(L_n(\tau_n(s)) - L(\tau_n(s))\big) + \sup_{s\le t} \big(L(\tau_n(s)) - y(s)\big).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\big|\sup_{s\le t} \big(L_n(\tau_n(s)) - y(s)\big) - \sup_{s\le t} \big(L(\tau(s)) - y(s)\big)\big| \\ &\qquad\le ||L - L_n||_\infty + \big|\sup_{s\le t} \big(L(\tau_n(s)) - y(s)\big) - \sup_{s\le t} \big(L(\tau(s)) - y(s)\big)\big|.\end{aligned}$$

According to (11), the term on the far right converges to zero. Hence, as (L^n) converges uniformly to L, the last inequality implies the result. □

Now, we are able to prove our main theorem of this section.

2.6 Theorem. *There exists a solution to the* Generalised Skorokhod Equation 2.1.

Proof. Let (L^n) be a sequence of piecewise constant càdlàg functions that converges uniformly from above to L. For each $n \geq 1$, Lemma 2.2 implies that the Generalised Skorokhod Equation 2.1 has a unique solution $T^n_\infty \in]0,\infty]$ and $z_n \colon \mathbb{R}_+ \to \overline{\mathbb{R}}$ such that

$$z_n(t) = 0 \vee \sup_{s\leq t} \big(L^n(\tau_n(s)) - y(s)\big) \quad \text{and} \tag{12}$$

$$\tau_n(t) = \int_0^t g\big(y(s) + z_n(s)\big)\,\mathrm{d}s \quad \text{for all } t < T^n_\infty. \tag{13}$$

For notational convenience, we extend τ_n to $\mathbb{R}_+$ via $\tau_n(t) = \infty$ for all $t \geq T^n_\infty$. Lemma 2.3 implies that there exists sequences (a_n),(b_n) and increasing functions $\tau, z_\infty \colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ such that (a_n) contains (b_n),

$$\lim_{n\to\infty} \tau_{a_n}(t) = \tau(t), \quad \text{and} \tag{14}$$

$$\lim_{n\to\infty} z_{b_n}(t) = z_\infty(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{15}$$

With the usual convention that $\inf\emptyset = \infty$, we define

$$T_\infty = \inf\big\{t \geq 0 \,\big|\, \tau(t) = \infty\big\} \quad \text{and} \tag{16}$$

$$z(t) = 0 \vee \sup_{s\leq t}\big(L(\tau(s)) - y(s)\big) \quad \text{for all } t < T_\infty. \tag{17}$$

We split the rest of the proof into multiple steps showing that (z, T_∞) is a solution to the Generalised Skorokhod Equation 2.1. The following list gives an overview of the claims we prove in each step,

- *1st step:* z fulfils (i), (ii) of 2.1 in $[0, T_\infty[$,
- *2nd step:* if $\int_0^T g(y(s) + z(s))\,\mathrm{d}s < \infty$, then $\exists\, \varepsilon > 0$ with $\tau(T+\varepsilon) < \infty$,
- *3rd step:* (z, T_∞) fulfils (ii) of 2.1, and $T_\infty > 0$.

1st step. For $t < T_\infty$, the inequality $z_\infty(t) < \infty$ holds true, because in view of (12) it holds that

$$\begin{aligned} z_n(t) &\leq ||L_n - L||_\infty + 0 \vee \sup_{s\leq t} L(\tau_n(s)) + 0 \vee \sup_{s\leq t}(-y(s)) \\ &\leq 1 + 0 \vee \sup_{s\leq \tau(t)+1} L(s) + 0 \vee \sup_{s\leq t}(-y(s)) \quad \text{for big enough } n \geq 1. \end{aligned}$$

Knowing that z_∞ is finite and g, y continuous, we can apply the dominated convergence theorem to (13) using (14) and (15), this yields

$$\tau(t) = \int_0^t g\big(y(s) + z_\infty(s)\big)\,\mathrm{d}s \quad \text{for all } t < T_\infty. \tag{18}$$

Furthermore, as z_∞ is finite and $g > 0$, the function τ is strictly increasing until T_∞. In view of this observation and (12),(14),(15), Lemma 2.5 implies that there exists a countable set $D \subseteq [0, T_\infty[$ such that

$$z_\infty(t) = 0 \vee \sup_{s \le t} \big(L(\tau(s)) - y(s)\big) \quad \text{for all } t \in [0, T_\infty[\,\backslash D.$$

Comparing this with (17) shows

$$z_\infty(t) = z(t) \quad \text{for all } t \in [0, T_\infty[\,\backslash D. \tag{19}$$

Putting (17),(18),(19) together, the following holds true for all $t < T_\infty$,

$$z(t) = 0 \vee \sup_{s \le t} \big(L(\tau(s)) - y(s)\big) \quad \text{and} \quad \tau(t) = \int_0^t g\big(y(s) + z(s)\big)\,\mathrm{d}s. \tag{20}$$

In view of KARATZAS & SHREVE [27, p.210, Lemma 3.6.14], this implies that z fulfils (i) and (ii) of 2.1 in $[0, T_\infty[$.

2nd step. For $T \ge 0$ with $\int_0^T g(y(s) + z(s))\,\mathrm{d}s < \infty$, we introduce

$$\hat{\tau}(t) = \begin{cases} \tau(t) & \text{if } t < T \\ \int_0^T g(y(s) + z(s))\,\mathrm{d}s & \text{if } t = T \end{cases}.$$

In particular, using (20), we are able to define the corresponding $\hat{z}$ in $[0, T]$. Let

$$\tilde{g} = g \wedge \max\big\{g(x) \,\big|\, x \in [-K, K]\big\} \quad \text{with} \quad K = \sup_{t \le T} |y(t)| + 2\hat{z}(T) + 1.$$

Replacing g with $\tilde{g}$, there are $(\tilde{T}_\infty^n), (\tilde{z}_n), (\tilde{\tau}_n), \tilde{T}_\infty, \tilde{z}, \tilde{\tau}$ such that (12),(13),(16),(20) hold. Also, there is a subsequence (c_n) of (b_n) and a countable set $\tilde{D}$ such that (15) hold for $(\tilde{z}_n)$ and $\tilde{z}$ on $[0, \tilde{T}_\infty[\,\backslash \tilde{D}$. With the usual convention that $\inf \emptyset = \infty$, we define

$$S = \inf\big\{t \ge 0 \,\big|\, \tau(t) \ne \tilde{\tau}(t)\big\}. \tag{21}$$

In view of (13), the inequality $\tilde{\tau}_n(t) \le t\,||\tilde{g}^+||_\infty$ holds true for all $t < \tilde{T}_\infty^n$. In particular, if $\tilde{T}^n_\infty < \infty$, equation (12) implies that $\tilde{z}_n$ is bounded in $[0, \tilde{T}^n_\infty[$. As this contradicts (iii) of Problem 2.1, we see that $\tilde{T}^n_\infty = \infty$. Looking back, $\tilde{\tau}_n(t) \leq t\,||\tilde{g}^+||_\infty$ holds for all $t \in \mathbb{R}_+$ and $n \geq 1$. In particular, $\tilde{\tau}$ is finite. Consequently, if $S = \infty$, then τ is finite, hence $\tau(T+\varepsilon) < \infty$ for all $\varepsilon > 0$. Therefore, from here onwards, we only consider the case when $S < \infty$.

As $\tilde{\tau}$ is finite, we see that $\tilde{T}_\infty = \infty$. Hence, $(\tilde{z}_n)$ and $\tilde{z}$ are solutions to their corresponding Generalised Skorokhod Equations 2.1 on the whole real line. As $y, \tilde{z}$ are right-continuous, there is $\delta > 0$ such that

$$\sup_{t\leq\delta} |y(T+t) - y(T)| < 1/4, \quad \text{and} \quad \sup_{t\leq\delta} |\tilde{z}(T+t) - \tilde{z}(T)| < 1/4. \tag{22}$$

Let $\delta \geq \gamma > 0$ such that $\gamma \in \mathbb{R} \setminus \tilde{D}$. Then, in view of (15), we can assume without loss of generality that

$$|\tilde{z}_{c_n}(T+\gamma) - \tilde{z}(T+\gamma)| < 1/4 \quad \text{for all } n \in \mathbb{N}.$$

As $\tilde{z}_n, \tilde{z}$ are increasing, and $\tilde{z}_n(0) = 0$, and $\tilde{z}(0) = 0$, the following estimate holds true,

$$\begin{aligned}\sup_{t\leq T+\gamma} |\tilde{z}_{c_n}(t) - \tilde{z}(t)| &\leq |\tilde{z}_{c_n}(0) - \tilde{z}(T+\gamma)| \vee |\tilde{z}_{c_n}(T+\gamma) - \tilde{z}(0)| \\ &\leq |\tilde{z}(T+\gamma)| + |\tilde{z}_{c_n}(T+\gamma) - \tilde{z}(T+\gamma)| \\ &< |\tilde{z}(T+\gamma)| + 1/4 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{23}$$

In particular, putting (22) and (23) together, we are able to estimate that

$$\begin{aligned}\sup_{t\leq T+\gamma} |y(t) + \tilde{z}_{c_n}(t)| &\leq \sup_{t\leq T+\gamma} |y(t) + \tilde{z}(t)| + \sup_{t\leq T+\gamma} |\tilde{z}_{c_n}(t) - \tilde{z}(t)| \\ &\leq \sup_{t\leq T+\gamma} |y(t)| + 2\tilde{z}(T+\gamma) + |\tilde{z}_{c_n}(T+\gamma) - \tilde{z}(T+\gamma)| \\ &< \sup_{t\leq T} |y(t)| + 2\tilde{z}(T) + 1 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{24}$$

In view of (20) and (21), we can see that $\hat{z} = \tilde{z}$ on $[0, S]$. In particular, if we assume that $S \leq T$, then $\tilde{z}(S) \leq \hat{z}(T)$, thus (24) is bounded by K, and therefore

$$\tilde{g}\big(y(t) + \tilde{z}_{c_n}(t)\big) = g\big(y(t) + \tilde{z}_{c_n}(t)\big) \quad \text{for all } t \leq T+\gamma \text{ and } n \in \mathbb{N}.$$

As $z_n, \tilde{z}_n$ are the unique solutions to their associated Problems 2.1, we see that

$z_{c_n} = \tilde{z}_{c_n}$ on $[0, S+\gamma]$ for all $n \in \mathbb{N}$. In view of (15),(19),(20), we deduce that $\tau = \tilde{\tau}$ on $[0, S+\gamma[$. As this contradicts the definition of S, the assumption is wrong and $S > T$. Therefore, as $\tilde{\tau}$ is finite, we conclude that $\tau(T+\varepsilon) < \infty$ for $S - T > \varepsilon > 0$.

3rd step. Let $T = 0$, then the *2nd step* implies that τ is finite on $[0, \varepsilon[$ for some $\varepsilon > 0$. In view of (16), we see that $T_\infty > 0$.

Let $T_\infty < \infty$. If $\int_0^{T_\infty} g(y(s) + z(s))\,\mathrm{d}s < \infty$, then the *2nd step* implies that $\tau(T_\infty + \varepsilon)$ is finite for some $\varepsilon > 0$. As this contradicts the definition of T_∞, the following holds true,

$$\int_0^{T_\infty} g\big(y(s) + z(s)\big)\,\mathrm{d}s = \infty.$$

Repeating the argument after (4), we see that (z, T_∞) fulfils (iii) of 2.1. □

Even though 2.6 is an existence result, using the same ideas from the proof, we can give the following semi-explicit representation.

2.7 Proposition (Functional Differential Equation). Problem 2.1 *has a (unique) solution if and only if the following functional differential equation has a (unique) solution. For the differential equation one determines a constant* $T_\infty \in\,]0, \infty]$ *and increasing function* $\tau\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *such that*

(i) $\tau(t) = \int_0^t g\Big(y(s) + 0 \vee \sup_{r \le s} \big(L(\tau(r)) - y(r)\big)\Big)\,\mathrm{d}s$ *for* $t < T_\infty$*,*

(ii) *if* $T_\infty < \infty$*, then* $\lim_{t \to T_\infty} \tau(t) = \infty$*.*

In this case a solution to Problem 2.1 is explicitly given by T_∞ *and*

$$z(t) = \begin{cases} 0 \vee \sup_{s \le t} \big(L(\tau(s)) - y(s)\big) & \text{for } t < T_\infty \\ \infty & \text{otherwise} \end{cases}. \tag{25}$$

Proof. First, we consider a solution (z, T_∞) for Problem 2.1, and define τ as in (ii) of 2.1. According to Karatzas & Shreve [27, p.210, Lemma 3.6.14], equation (20) holds. In particular, (i) of Proposition 2.7 is fulfilled. Moreover, if $T_\infty < \infty$, assuming that τ is bounded in $[0, T_\infty[$, equation (20) of Problem 2.1 implies that z is bounded in $[0, T_\infty[$. This contradicts (ii) of 2.1, therefore τ is unbounded in $[0, T_\infty[$. As τ is positive and increasing, this shows that (ii) of Proposition 2.7 is fulfilled.

Second, we consider a solution (τ, T_∞) for the functional differential equation. In this case, equation (20) holds true and the statement thereafter implies that $(i), (ii)$ of Problem 2.1 are fulfilled. Repeating the argument after equation (4), we see that (ii) of Problem 2.1 holds true as well.

Due to (20), uniqueness for Problem 2.1 implies uniqueness for the functional differential equation in the statement. $\square$

2.8 Proposition. *Consider a piecewise locally Lipschitz càdlàg function* $L\colon \mathbb{R}_+ \to \mathbb{R}$ *and a locally Lipschitz function* $g\colon \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$*. Then the* Functional Differential Equation 2.7 *has a unique solution and so has the* Generalised Skorokhod Equation 2.1.

Proof. Let (a_i) be a sequence of points such that $a_0 = 0$, (a_i) is strictly increasing to ∞, and L locally Lipschitz in $[a_i, a_{i+1}[$ for all $i \in \mathbb{N}$. For given $a, b \in \mathbb{R}_+$ and fixed $i \in \mathbb{N}$, we consider the following differential equation

$$\tilde{\tau}(t) = a + \int_q^t g\Big(y(s) + b \vee \sup_{q<r\leq s} \big(L(\tilde{\tau}(r)) - y(r)\big)\Big)\,\mathrm{d}s \quad \text{for } t \in [a_i, a_{i+1}[.$$

For fixed $t \in [a_i, a_{i+1}[$, let K_g, K_L be appropriate Lipschitz constants for g, L in $[a_i, t]$. Given two solutions $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of the above differential equation, the following holds true,

$$\sup_{a_i \leq r \leq t} |\tilde{\tau}_2(r) - \tilde{\tau}_1(r)| \leq K_g K_L \int_{a_i}^t \big(\sup_{a_i \leq r \leq s} |\tilde{\tau}_2(r) - \tilde{\tau}_1(r)| \,\big)\,\mathrm{d}s.$$

Using the Grönwall Lemma, we see that $\sup_{a_i \leq r \leq t} |\tilde{\tau}_2(r) - \tilde{\tau}_1(r)| = 0$ for all $t \in [a_i, a_{i+1}[$. Hence, $\tilde{\tau}_2(t) = \tilde{\tau}_1(t)$ for all $t \in [a_i, a_{i+1}[$. Now, a simple induction argument shows that the following equations have a unique solution,

$$\bar{\tau}(t) = \tau(q) + \int_q^t g\Big(y(s) + \bar{z}(q-) \vee \sup_{q<r\leq s} \big(L(\bar{\tau}(r)) - y(r)\big)\Big)\,\mathrm{d}s \quad \text{and}$$

$$\bar{z}(q-) = 0 \vee \sup_{r<q} \big(L(\bar{\tau}(r)) - y(r)\big) \quad \text{for } t \in \mathbb{R}_+.$$

In view of (25) and (i) of Proposition 2.7, we see that the Functional Differential Equation 2.7 has a unique solution. $\square$

Now, we are interested in adapted solutions to the Generalised Skorokhod Equation 2.1 when we replace the function y with a continuous process Y. More precisely, Problem 2.1 changes in the following way.

2.9 Problem (Adapted Generalised Skorokhod Equation). *On a filtered space* $(\Omega, \mathcal{F}, (\mathcal{F}_t))$*, consider an adapted continuous process* Y*. Moreover, let* $L\colon \mathbb{R}_+ \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$ *such that* L *is càdlàg,* g *continuous, and* $y(0) \geq L(0)$*.*

Given such data, determine a stopping time $T_\infty > 0$ *and an adapted process* $Z : \Omega \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *such that* $(i), (ii), (iii)$ *of* Problem 2.1 *hold true for all fixed* $\omega \in \Omega$*.*

2.10 Remark. *An adapted solution to* Problem 2.9 *is automatically predictable, i.e.* T_∞ *is a predictable stopping time and* Z *is a predictable process. More precisely, in view of* $(i), (iii)$ *of 2.1, the stopping time* T_∞ *is announced by the stopping times*

$$T_n = \inf \left\{ t > 0 \,\middle|\, Z_t \geq n \right\} \quad \text{for } n \in \mathbb{N}. \tag{26}$$

In view of Proposition 2.7*, the process* Z *is predictable, because it is the maximum of an adapted left-continuous process and a predictable one, namely*

$$0 \vee \bigvee \sup_{s<t} \big(L(\tau(s)) - Y(s) \big) \quad \text{and} \quad L(\tau(t)) - Y(t) \quad \text{for } t < T_\infty.$$

To find an adapted solution to Problem 2.9, we follow the proof of Theorem 2.6. We are unable to apply Lemma 2.3 to the sequence (τ_n) given by (13), as the resulting process may fail to be adapted. Instead, we give a general sufficient condition ensuring that the sequence (τ_n) converges.

2.11 Theorem. *Assume that the* Problem 2.1 *has at most one solution for all continuous functions* y*. Then, the* Problem 2.9 *has a unique adapted solution.*

Proof. In view of Theorem 2.6, there is a unique solution (Z, T_∞) to Problem 2.1 with Y instead of y. Moreover, let τ be defined as in (ii) of 2.1 with the convention that $\tau_t = \infty$ for all $t \geq T_\infty$.

Let (L_n) be a sequence of piecewise constant càdlàg functions that uniformly approximate L from above. In view of (1),(2),(3), there are adapted processes (Z^n) and stopping times (T^n_∞) which fulfil the Generalised Skorokhod Equation 2.1 for (L_n). Trivially, for all events $\omega \in \Omega$ and times $t \in \mathbb{R}_+$, the following inequality holds true,

$$\liminf_{n\to\infty} \int_0^t g\big(Y_s(\omega) + Z_s^n(\omega)\big)\,\mathrm{d}s \le \limsup_{n\to\infty} \int_0^t g\big(Y_s(\omega) + Z_s^n(\omega)\big)\,\mathrm{d}s.$$

Assume that there are ω and t such that the above inequality is strict. In this case, we can choose subsequences such that the different limits are achieved. Following the proof of Theorem 2.6, there are two processes $Z^{\inf}(\omega)$ and $Z^{\sup}(\omega)$ solving Problem 2.1 for $Y(\omega)$ such that

$$\int_0^t g\big(Y_s(\omega) + Z_s^{\inf}(\omega)\big)\,\mathrm{d}s < \int_0^t g\big(Y_s(\omega) + Z_s^{\sup}(\omega)\big)\,\mathrm{d}s.$$

This contradicts the assumption that Problem 2.1 has at most one solution. Hence, the following is true,

$$\tau(t) = \lim_{n\to\infty} \int_0^t g(Y_s + Z_s^n)\,\mathrm{d}s \text{ exists for all } t \in \mathbb{R}_+.$$

In particular, τ is an increasing adapted process. Therefore, in view of (16) and as $\{\infty\}$ is a closed set, T_∞ is a stopping time. Moreover, using (17), we can see that Z is an adapted process. □

3. Construction of a weak solution

In this section, we are interested in one-dimensional diffusions with state space $\mathcal{I}$ that are reflected in a càdlàg function $L\colon \mathbb{R}_+ \to \mathcal{I}$.

More precisely, there are $-\infty < \alpha < \beta \le \infty$ such that $\mathcal{I} = [\alpha, \beta[$ or $\mathcal{I} = [\alpha, \beta]$. The point β belongs to the state space only if it is accessible for the diffusion. However, in both cases, we assume that $L\colon \mathbb{R}_+ \to [\alpha, \beta[$. Processes that are restricted to $\mathcal{I}$ may exit $\mathcal{I}$ in finite time. In particular, there is a stopping time S connected to such an explosion. Now, we are interested in solutions to the reflecting SDE that is determined by the initial condition $L(0) \le X_0 < \beta$ and the requirements

$$(a)\;\; X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + Z_t + \int_0^t \sigma(X_s)\,\mathrm{d}W_s \text{ for } t < S,$$

$$(b)\;\; \beta > X_t \ge L(t) \text{ and } Z_t = \int_{[0,t]} \mathbb{1}_{\{X_u = L(u)\}} |\mathrm{d}Z_u| \text{ for all } t < S,$$

$$(c)\;\; \lim_{t\to S} X_t = \beta \text{ and } X_t = \beta \text{ for all } t \ge S \text{ on } \{S < \infty\}.$$

3.1 Definition. *A weak solution to* (*a*)-(*c*) *is a collection* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, Z, X, S)$. *Here,* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *is a filtered probability space satisfying the usual conditions that supports a standard one-dimensional* $(\mathcal{F}_t)$*-Brownian motion* W. *Furthermore,* Z *and* X *are adapted càdlàg processes and* S *a stopping time such that* (*a*)-(*c*) *hold true.*

We make the following assumption.

3.2 Assumption (Engelbert-Schmidt conditions). *The Borel-measurable functions* $b, \sigma\colon [\alpha, \beta[\to \mathbb{R}$ *are such that*

$$\sigma^2(x) > 0 \quad \textit{and} \quad \int_\alpha^x \frac{1 + |b(u)|}{\sigma^2(u)}\, \mathrm{d}u < \infty \quad \textit{for all } x \in [\alpha, \beta[.$$

Let p be the scale function of the unaffected diffusion, which is given by

$$p(x) = \int_\alpha^x \exp\Big(-\int_\alpha^z \frac{2b(y)}{\sigma^2(y)}\, \mathrm{d}y\Big)\, \mathrm{d}z \quad \text{for } x \in [\alpha, \beta]. \tag{27}$$

Among other things, Assumption 3.2 ensures that p is well-defined, real-valued and strictly increasing in $[\alpha, \beta[$. Its left-hand limit at β coincides with its value $p(\beta)$, which may be ∞. We denote by q the inverse function of p. Both functions are C^1 and have absolutely continuous derivatives. In particular, the derivative q' and its generalised second derivative q'' are such that

$$q'(p(x)) = \frac{1}{p'(x)} \quad \text{and} \quad q''(p(x)) = \frac{2b(x)}{\sigma^2(x)(p'(x))^2} \quad \text{for } x \in [\alpha, \beta[. \tag{28}$$

Furthermore, we consider the following functions for $t \in \mathbb{R}_+$ and $\hat{x} \in [0, p(\beta)[$,

$$\hat{L}(t) = p(L(t)), \quad \text{and} \quad \hat{\sigma}(\hat{x}) = \sigma(q(\hat{x}))\, p'(q(\hat{x})). \tag{29}$$

In view of Assumption 3.2, the function $\hat{\sigma}$ has similar properties as σ, more precisely

$$\hat{\sigma}^2(\hat{x}) > 0 \quad \text{and} \quad \int_0^{\hat{x}} \frac{1}{\hat{\sigma}^2(u)}\, \mathrm{d}u < \infty \quad \text{for all } \hat{x} \in [0, p(\beta)[. \tag{30}$$

In view of (28) and (29), the following holds true for $x \in [\alpha, \beta[$,

$$\hat{\sigma}(p(x))\, q'(p(x)) = \sigma(x) \quad \text{and} \quad \frac{1}{2}\, \hat{\sigma}^2(p(x))\, q''(p(x)) = b(x). \tag{31}$$

We will derive a weak solution to the reflecting SDE (a)-(c) using the solution to the following stochastic Skorokhod problem.

3.3 Problem. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *be a filtered probability space that supports a real-valued continuous adapted process* Y *such that* $Y_0 = p(X_0)$*. Also, let* $\hat{L}\colon \mathbb{R}_+ \to [0, p(\beta)[$ *and* $\hat{\sigma}\colon [0, p(\beta)[\to \mathbb{R}_+$ *be defined as in* (29).

Given such data, determine a stopping time $T > 0$ *and an adapted process* $\bar{Z}$ *such that*

(*i*) $\bar{Z}$ *is increasing and right-continuous with* $\bar{Z}_0 = 0$,

(*ii*) *for all* $t \in [0, T[$ *the process* $A_t = \int_0^t \hat{\sigma}^{-2}(Y_s + \bar{Z}_s)\,\mathrm{d}s < \infty$ *and*

$$p(\beta) > Y_t + \bar{Z}_t \geq \hat{L}(A_t) \quad \textit{and} \quad \bar{Z}_t = \int_{[0,t]} \mathbb{1}_{\{Y_s + \bar{Z}_s = \hat{L}(A_s)\}}\,\mathrm{d}\bar{Z}_s,$$

(*iii*) $\lim_{t \uparrow T} (Y_t + \bar{Z}_t) = Y_T + \bar{Z}_T = p(\beta)$ *on the event* $\{T < \infty\}$.

3.4 Remark. *The process* A *is a continuous and strictly increasing in* $[0, T[$. *In particular,* A *can be continuously extended to a strictly increasing process in* $[0, T]$*. Be aware that we use here continuity in* $\overline{\mathbb{R}}$*, hence* A_T *may be* ∞.

3.5 Lemma. *Suppose that* Assumption 3.2 *holds true and* $\hat{L}$ *is constant. Then, for any* $Y = Y_0 + \bar{W}$ *with* $(\bar{\mathcal{F}}_t)$*-Brownian motion* $\bar{W}$*, where* $(\bar{\mathcal{F}}_t)$ *fulfils the usual conditions, there exists a solution to* Problem 3.3.

Proof. In view of Karatzas & Shreve [27, p.210, Lemma 3.6.14], for the constant $\hat{L}$, there is an adapted continuous increasing process $\bar{Z}$ with $\bar{Z}_0 = 0$ such that

$$Y_t + \bar{Z}_t \geq \hat{L} \quad \text{and} \quad \bar{Z}_t = \int_{[0,t]} \mathbb{1}_{\{Y_s + \bar{Z}_s = \hat{L}\}}\,\mathrm{d}\bar{Z}_s \quad \text{for all } t \in \mathbb{R}_+. \tag{32}$$

Let

$$T = \inf\big\{t > 0 \,\big|\, Y_t + \bar{Z}_t \geq p(\beta)\big\}.$$

As $Y + \bar{Z}$ is continuous, T is a stopping time and $\lim_{t \to T}(Y_t + \bar{Z}_t) = Y_T + \bar{Z}_T = p(\beta)$. In particular, the only claim that remains to show is $A < \infty$ on $[0, T[$.

Let L be the local time of the continuous semimartingale $Y + \bar{Z}$. Thanks to (32), we see that $L_t^x = 0$ for all $(t, x) \in \mathbb{R}_+ \times (\mathbb{R} \setminus [\hat{L}, \infty[)$. Moreover, in view of Revuz & Yor [36, p.225, Theorem VI.1.7], for given t, the process L_t^x is càdlàg and therefore locally bounded with respect to x. In particular, we are able to define the following finite processes,

$$\gamma_t = \sup_{s \le t} \big(Y_s + \bar{Z}_s\big) \quad \text{and} \quad K_t = \sup \big\{ L_t^x \,\big|\, \hat{L} \le x \le \gamma_t \big\} \quad \text{for all } \ t \in \mathbb{R}_+.$$

Using the occupation time formula together with (30), the following holds true,

$$A_t = \int_{\mathbb{R}} \hat{\sigma}^{-2}(x) L_t^x \,\mathrm{d}x \le K_t \int_{\hat{L}}^{\gamma_t} \hat{\sigma}^{-2}(x) \,\mathrm{d}x < \infty \quad \text{for all } \ t < T.$$

Overall, if we identify the constant $\hat{L}$ with its corresponding constant function $\hat{L} \colon \mathbb{R}_+ \to [0, p(\beta)[$, then $(T, \bar{Z})$ is a solution to Problem 3.3. $\square$

Now, we prove our main result.

3.6 Theorem. *Suppose that* Assumption 3.2 *holds true. Moreover, assume that* Problem 3.3 *has a solution for some* $Y = Y_0 + \bar{W}$ *and* $(\bar{\mathcal{F}}_t)$*-Brownian motion* $\bar{W}$ *on a filtered probability space* $(\Omega, \mathcal{F}, (\bar{\mathcal{F}}_t), \mathbb{P})$ *satisfying the usual conditions. Then, the reflecting SDE* (a)*-*(c) *has a weak solution in the sense of* Definition 3.1.

Proof. Let $(T, \bar{Z})$ be a solution to Problem 3.3. Using $\hat{\sigma}^{-2}(x) = \infty$ when $x \ge p(\beta)$, we extend the process A given in (ii) of 3.3 from $[0, T[$ to $\mathbb{R}_+$. Then, we are able to define

$$C_t = \inf \big\{ u \ge 0 \,\big|\, A_u \ge t \big\} \quad \text{for } \ t \in \mathbb{R}_+. \tag{33}$$

As A is an increasing $(\bar{\mathcal{F}}_t)$-adapted process and $[t, \infty]$ is a closed set, the random variable C_u is an $(\bar{\mathcal{F}}_t)$-stopping time for all $u \in \mathbb{R}_+$. Moreover, in view of Remark 3.4, the $\overline{\mathbb{R}}$-valued process A is continuous in $[0, T]$, thus C is strictly increasing in $[0, A_T]$. As A and C are strictly increasing, they are injective in $[0, T]$ and $[0, A_T]$ respectively. Hence,

$$A_{C_t} = t \quad \text{for all } t \le A_T, \quad \text{and} \quad C_{A_t} = t \quad \text{for all } t \le T. \tag{34}$$

We split the rest of the proof into multiple steps. The following list gives an overview of the claims we prove in each step,

- *1st step:* $C < \infty$, C is continuous, and $C_t = C_{A_T}$ for all $t > A_T$ on $\{A_T < \infty\}$,

- *2nd step:* there is a solution for $(Y_0, 0, p(\beta), 0, \hat{\sigma}, \hat{L})$ instead of $(X_0, \alpha, \beta, b, \sigma, L)$,
- *3rd step:* there is a solution for $(X_0, \alpha, \beta, b, \sigma, L)$.

The *2nd step* and the *3rd step* are adapted proofs from Karatzas & Shreve [27, pp.329 – 341, Chapter 5.5].

1st step. In view of (33), for C finite, it is enough to show that A_∞ is infinite. If $p(\beta) < \infty$, using the Lebesgue measure λ we see that $\lambda\{t > 0 \,|\, Y_0 + \bar{W}_t > p(\beta)\} = \infty$. As $Y + \bar{Z} \geq Y_0 + \bar{W}$, we see that $\lambda\{t > 0 \,|\, \hat{\sigma}^{-2}(Y_t + \bar{Z}_t) = \infty\} = \infty$. In particular, $A_\infty = \infty$. Hence, without loss of generality $p(\beta) = \infty$.

If we assume that $\mathbb{P}[A_\infty < \infty] > 0$, then there is some $d \in \mathbb{R}_+$ such that $\mathbb{P}[A_\infty < d] > 0$. In view of Lemma 3.5, there exists a solution to Problem 3.3 when we replace $\hat{L}$ with a constant. Moreover, as C_d is a stopping time, we can paste different solutions at C_d progressively together. In particular, there exists a solution $(\tilde{S}, \tilde{Z})$ to Problem 3.3 when we replace $\hat{L}$ with $\tilde{L}(t) = \hat{L}(t \wedge d)$ for $t \in \mathbb{R}_+$. By construction, $\bar{Z}_t = \tilde{Z}_t$ for all $t < C_d$. Replacing $\bar{Z}$ with $\tilde{Z}$ in (ii) of 3.3, the corresponding process $\tilde{A}$ fulfils $A_t = \tilde{A}_t$ for all $t < C_d$. As $C_d = \infty$ on $\{A_\infty < d\}$, we have $A_\infty = \tilde{A}_\infty$ on $\{A_\infty < d\}$, hence $\mathbb{P}[\tilde{A}_\infty < d] > 0$. Overall, to decide whether $\mathbb{P}[A_\infty < \infty] > 0$ or zero, we can assume without loss of generality that $\hat{L}$ is bounded above.

In view of Karatzas & Shreve [27, p.210, Lemma 3.6.14], as $\hat{L}$ is bounded above, we can see that $Y_t + \bar{Z}_t < \infty$ for all $t \in \mathbb{R}_+$. In view of (iii) of 3.3, this finiteness together with $p(\beta) = \infty$ shows that $T = \infty$.

Moreover, as $\hat{L}$ is bounded above, there is $k > ||\hat{L}^+||_\infty$. We define inductively the following sequence of stopping times. Given $K_0 = 0$, $J_0 = 0$, and $n > 0$, let

$$\begin{aligned} K_n &= \inf\big\{t > J_{n-1} \,\big|\, Y_t + \bar{Z}_t \geq k\big\} \quad \text{and} \\ J_n &= \inf\big\{t > K_n \,\big|\, Y_t + \bar{Z}_t \leq ||\hat{L}^+||_\infty\big\}. \end{aligned} \tag{35}$$

By induction, we show that $K_n, J_n < \infty$. This is obvious for $n = 0$. Now, let $n > 0$. The following holds true,

$$\limsup_{t\to\infty} \bar{W}_t = \infty \quad \text{and} \quad \liminf_{t\to\infty} \bar{W}_t = -\infty.$$

Therefore, there exists for almost all events $\omega \in \Omega$ some time $t > J_{n-1}(\omega)$

such that $\bar{W}_t(\omega) > k - Y_0(\omega)$, in particular $(Y_t + \bar{Z}_t)(\omega) > k$. Hence, $K_n < \infty$. On the other hand, there exists for almost all events $\omega \in \Omega$ some time $t > K_n(\omega)$ such that $\bar{W}_t(\omega) < ||\hat{L}^+||_\infty - (Y_0 + \bar{Z}_{K_n})(\omega)$. If $\bar{Z}_{K_n} = \bar{Z}_t$, then $(Y_t + \bar{Z}_t)(\omega) < ||\hat{L}^+||_\infty$. If $\bar{Z}_{K_n} \neq \bar{Z}_t$, then in view of *(ii)* of 3.3 there exists $s \in]K_n(\omega), t]$ such that $(Y_s + \bar{Z}_s)(\omega) = \hat{L}(s) \leq ||\hat{L}^+||_\infty$. Hence, $J_n < \infty$, which completes the induction proof.

Let $n \in \mathbb{N}$. In view of (35), we have $]K_n, J_n[\subseteq \{Y + \bar{Z} > \hat{L}\}$. In particular, in view of *(ii)* of 3.3, the process $\bar{Z}$ is constant in $[K_n, J_n[$. As jumps of $Y + \bar{Z}$ land on $\hat{L}$ only and $k > ||\hat{L}^+||_\infty$, we see that $Y_{K_n} + \bar{Z}_{K_n} = k$. Moreover, as $\Delta Z \geq 0$, we can see that $\Delta(Y + \bar{Z})_{J_n} = 0$. Hence,

$$Y_t + \bar{Z}_t = k + \bar{W}_t \quad \text{for all } t \in [K_n, J_n].$$

As $J_n < K_{n+1}$ for all $n \in \mathbb{N}$, the sequence of processes $(Y_{(t+K_n)\wedge J_n} + \bar{Z}_{(t+K_n)\wedge J_n})$ is a sequence of independent Brownian motions that are stopped at $||\hat{L}^+||_\infty$. In particular, this sequence is independent and identical distributed. In view of (30) and $\hat{\sigma}^{-2}(x) = \infty$ for $x \geq p(\beta)$, we see that $\hat{\sigma}^{-2} > 0$. Now, an application of the strong law of large numbers reveals that

$$A_\infty \geq \sum_{n\in\mathbb{N}} \int_{K_n}^{J_n} \hat{\sigma}^{-2}(Y_s + \bar{Z}_s)\,\mathrm{d}s = \infty.$$

Let $p(\beta)$ and $\hat{L}$ be general again. As A is strictly increasing on the set where it is finite, C is continuous on the set where it is finite. As $C < \infty$, the process C is continuous.

As $A_\infty = \infty$, the inclusion $\{A_T < \infty\} \subseteq \{T < \infty\}$ holds true. In the following, we work on $\{T < \infty\}$. In view of *(iii)* of 3.3, we see that $Y_T + \bar{Z}_T = p(\beta)$. As T is a stopping time and $\bar{W}$ a Brownian motion, we can see that $\lambda\{\varepsilon > t > 0 \,|\, \bar{W}_{T+t} - \bar{W}_T \geq 0\} > 0$, thus $\lambda\{\varepsilon > t > 0 \,|\, \hat{\sigma}^{-2}(Y_{T+t} + \bar{Z}_{T+t}) = \infty\} > 0$ for all $\varepsilon > 0$. Hence, $A_{T+} = \infty$. In view of (33), we see that $C_t = C_{A_T+}$ for all $t > A_T$. As C is continuous, $C_t = C_{A_T}$ for all $t > A_T$.

2nd step. As C_u is a finite $(\bar{\mathcal{F}}_t)$-stopping time for all $u \in \mathbb{R}_+$, we are able to define

$$\hat{X}_t = Y_{C_t} + \bar{Z}_{C_t}, \quad \hat{Z}_t = \bar{Z}_{C_t} \quad M_t = \bar{W}_{C_t} \quad \text{and} \quad \mathcal{G}_t = \bar{\mathcal{F}}_{C_t} \quad \text{for } t \in \mathbb{R}_+. \tag{36}$$

As C is an increasing continuous process of stopping times, C is a time-change.

Therefore, in view of Revuz & Yor [36, 180f., Propositions V.1.4 and V.1.5], $\hat{X}$ and $\hat{Z}$ are càdlàg $(\mathcal{G}_t)$-adapted processes and M is a continuous $(\mathcal{G}_t)$-local martingale. Now, let

$$S = A_T.$$

In view of (34), we see that $C_S = T$. In particular, as C is strictly increasing in $[0, S]$, we have that $\{u < S\} = \{C_u < C_S\} = \{C_u < T\}$ for all $u \in \mathbb{R}_+$. Keeping in mind that C_u and T are $(\bar{\mathcal{F}}_t)$-stopping times, $\{C_u < T\} \in \mathcal{G}_u$ for all $u \in \mathbb{R}_+$. Putting these two observations together, S is a $(\bar{\mathcal{G}}_t)$-stopping time.

Using (ii) of 3.3 and (34) when changing variables of integration, the following holds true for $t \in \mathbb{R}_+$,

$$\langle M \rangle_{t\wedge S} = C_{t\wedge S} = \int_0^{C_{t\wedge S}} \mathrm{d}u = \int_0^{C_{t\wedge S}} \hat{\sigma}^2(Y_u + \bar{Z}_u)\,\mathrm{d}A_u = \int_0^{t\wedge S} \hat{\sigma}^2(\hat{X}_u)\,\mathrm{d}u. \quad (37)$$

Using (ii) of 3.3 and $C_S = T$ a second time, the following holds true,

$$\int_0^{t\wedge S} \hat{\sigma}^{-2}(\hat{X}_u)\,\mathrm{d}\langle M \rangle_u = t \wedge S \quad \text{for all } t \in \mathbb{R}_+. \quad (38)$$

Let B be a standard Brownian motion independent of $\mathcal{G}_\infty$. Then, we define the following local martingale and its corresponding filtration for $t \in \mathbb{R}_+$,

$$W_t = \int_0^{t\wedge S} \hat{\sigma}^{-1}(\hat{X}_u)\,\mathrm{d}M_u + B_t - B_{t\wedge S} \quad \text{and} \quad \mathcal{F}_t = \mathcal{G}_t \vee \sigma(B_u,\, u \le t). \quad (39)$$

In view of (38), thanks to Lévy's characterisation theorem, W is an $(\mathcal{F}_t)$-Brownian motion. Without loss of generality, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfies the usual conditions, because a continuous $(\mathcal{F}_t)$-martingale will be a continuous $(\bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon})$-martingale. Furthermore, as $\mathcal{F}_t \supseteq \mathcal{G}_t$ for all $t \in \mathbb{R}_+$, we see that $\hat{X}, \hat{Z}$ are $(\mathcal{F}_t)$-adapted processes and S is an $(\mathcal{F}_t)$-stopping time. Moreover, in view of (36) and (39), the following holds true,

$$\hat{X}_t = Y_0 + \hat{Z}_t + M_t = Y_0 + \hat{Z}_t + \int_0^t \hat{\sigma}(\hat{X}_u)\,\mathrm{d}W_u \quad \text{for all } t < S. \quad (40)$$

Furthermore, using (34) when changing variables of integration in (ii) of 3.3, we obtain

$$p(\beta) > \hat{X}_t \ge \hat{L}(t) \quad \text{and} \quad \hat{Z}_t = \int_{[0,t]} \mathbb{1}_{\{\hat{X}_u = \hat{L}(u)\}}\,\mathrm{d}\hat{Z}_s \quad \text{for all } t \in [0, S[. \quad (41)$$

In view of C continuous and $C_S = T$, we see that $\lim_{t\to S} C_t = T$ on $\{S < \infty\}$. Moreover, in view of the *1st step*, C is constant from S onwards on $\{S < \infty\}$. In particular, in view of (iii) of 3.3, the following holds true,

$$\lim_{t\to S} \hat{X}_t = \beta \quad \text{and} \quad \hat{X}_t = \beta \quad \text{for all } t \geq S \quad \text{on } \{S < \infty\}. \tag{42}$$

3rd step. As defined before, we let $q\colon [0, p(\beta)[\to [\alpha, \beta[$ be the inverse function of p, which is given in (27). As p has a left-hand limit at β that coincides with its value $p(\beta)$, the function q has a left-hand limit at $p(\beta)$ that coincides with β. Now, we define

$$X_t = q(\hat{X}_t) \quad \text{for all } t \in \mathbb{R}_+. \tag{43}$$

Let $\hat{Z}^{\mathrm{c}}$ be the continuous part of $\hat{Z}$. In view of (41) and $\Delta\hat{Z} = \Delta\hat{X}$, the following holds true,

$$\hat{Z}_t = \int_0^t \mathbb{1}_{\{X_u = L(u)\}} \,\mathrm{d}\hat{Z}^{\mathrm{c}}_u + \sum_{0\leq u\leq t} \mathbb{1}_{\{X_u = L(u)\}} \,\Delta\hat{X}_u \quad \text{for all } t \in [0, S[. \tag{44}$$

Using Itô's formula together with (31), we can see that

$$\begin{aligned} X_t &= X_0 + \frac{1}{2}\int_0^t \hat{\sigma}^2(\hat{X}_u) q''(\hat{X}_u)\,\mathrm{d}u + \int_0^t q'(\hat{X}_u)\,\mathrm{d}\hat{Z}^{\mathrm{c}}_u + \sum_{0\leq u\leq t} \big[q(\hat{X}_u) - q(\hat{X}_{u-})\big] \\ &\quad + \int_0^t \hat{\sigma}(\hat{X}_u) q'(\hat{X}_u)\,\mathrm{d}W_u \\ &= X_0 + \int_0^t b(X_u)\,\mathrm{d}u + \int_0^t q'(\hat{X}_u)\,\mathrm{d}\hat{Z}^{\mathrm{c}}_u + \sum_{0\leq u\leq t} \big[q(\hat{X}_u) - q(\hat{X}_{u-})\big] \\ &\quad + \int_0^t \sigma(X_u)\,\mathrm{d}W_u \quad \text{for all } t \in [0, S[. \end{aligned} \tag{45}$$

Now, we define

$$Z_t = \int_0^t q'(\hat{X}_u)\,\mathrm{d}\hat{Z}^{\mathrm{c}}_u + \sum_{0\leq u\leq t} \big[q(\hat{X}_u) - q(\hat{X}_{u-})\big] \quad \text{for } t \in [0, S[. \tag{46}$$

In view of (45) and (46), we see that (a) holds true. Moreover, in view of (46) and $\Delta\hat{X} \geq 0$ and $\hat{Z}, q$ increasing, the process Z is increasing as well. Hence, $\mathrm{d}Z = |\mathrm{d}Z|$. Combining this with (44) and (46), we can see that

$$Z_t = \int_0^t \mathbb{1}_{\{X_u = L(u)\}} q'(\hat{X}_u)\,\mathrm{d}\hat{Z}^{\mathrm{c}}_u + \sum_{0\leq u\leq t} \mathbb{1}_{\{X_u = L(u)\}} \big[q(\hat{X}_u) - q(\hat{X}_{u-})\big]$$

$$= \int_{[0,t]} \mathbb{1}_{\{X_u = L(u)\}} \, \mathrm{d}Z_u = \int_{[0,t]} \mathbb{1}_{\{X_u = L(u)\}} |\mathrm{d}Z_u| \quad \text{for all } t \in [0, S[.$$

As q is strictly increasing in $[0, p(\beta)]$, the inequality in (41) shows that $\beta > X_t \geq L(t)$ for all $t \in [0, S[$. Hence, (b) holds true. Moreover, in view of (42), we see that (c) holds true. As $\hat{X}$ and $\hat{Z}$ are $(\mathcal{F}_t)$-adapted processes, X and Z are $(\mathcal{F}_t)$-adapted processes. As shown before, S is an $(\mathcal{F}_t)$-adapted stopping and we can assume without loss of generality that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfies the usual conditions. Overall, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, Z, X, S)$ is a weak solution to (a)-(c). □

In the Introduction, we mentioned a hypothetical scenario that we wanted to exclude. We imagined a process oscillating with infinite frequency between $\pm\infty$ at the end of a finite interval. Having a look at our construction (36), this scenario corresponds to the case when $C_t = \infty$ for some $t < \infty$. Fortunately, we were able to prove in the *1st step* that $A_\infty = \infty$ and therefore $C_t < \infty$ for all $t < \infty$.

A second look at the above statement shows that the key point is to find a solution to Problem 3.3. The following result describes settings in which we are able to do so.

3.7 Corollary. *The reflecting SDE* (a)-(c) *has a weak solution in the sense of* Definition 3.1 *if* Assumption 3.2 *and one of the following assumptions hold true,*

- (i) *L is constant*
- (ii) *L is a piecewise locally Lipschitz càdlàg function, σ^2 locally Lipschitz and b locally bounded,*
- (iii) *σ^2 is continuous and Problem* 2.1 *has at most one solution.*

Proof. In case of (i), in view of Lemma 3.5, there exists a solution to Problem 3.3. In the following, we show that this is also the case when (ii) or (iii) holds true.

Let (γ_n) be a sequence in $[0, p(\beta)[$ such that $\gamma_n \uparrow p(\beta)$. For given $n \in \mathbb{N}$, let $\hat{\sigma}_n(x) = \hat{\sigma}(0 \vee x \wedge \gamma_n)$ for $x \in \mathbb{R}$. In view of (30), we see that $\hat{\sigma}_n^{-2} \colon \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$. In case of (ii), in view of (28) and (29), the functions $\hat{L}$ and $\hat{\sigma}_n^{-2}$ fulfil the assumptions of Proposition 2.8. In case of (iii), the functions $\hat{L}$ and $\hat{\sigma}_n^{-2}$ fulfil the assumptions of Theorem 2.11.

On a filtered probability space $(\Omega, \mathcal{F}, (\bar{\mathcal{F}}_t), \mathbb{P})$ satisfying the usual conditions, let $Y = Y_0 + \bar{W}$ be an adpated process such that $p(\beta) > Y_0 \geq \hat{L}(0)$ and $\bar{W}$ an $(\bar{\mathcal{F}}_t)$-Brownian motion. According to Proposition 2.8 and Theorem 2.11, the Adapted Generalised Skorokhod Equation 2.9 has a unique adapted solution $(\bar{Z}^n, T^n_\infty)$ for $(Y, \hat{L}, \hat{\sigma}_n^{-2})$. Let

$$T_n = \inf\{t > 0 \,|\, Y_t + \bar{Z}^n_t > \gamma_n\}. \tag{47}$$

From the uniqueness, we can see that $\bar{Z}^n_t = \bar{Z}^m_t$ for all $t < T_n$ and $m > n$. In particular, (T_n) is a sequence of increasing stopping times. Furthermore, we are able to define the following adapted process and the following stopping time,

$$\bar{Z}_t = \bar{Z}^n_t \quad \text{for } t < T_n, \quad \text{and} \quad T = \lim_{n\to\infty} T_n.$$

As $p(\beta) > Y_0$ and $\gamma \uparrow p(\beta)$, we see that $T > 0$. By construction, the pair $(\bar{Z}, T)$ fulfils (i) and (ii) of Problem 3.3 in $[0, T[$.

In the following, we work on the set $\{T < \infty\}$. In particular, as Z is increasing, $\lim_{t\to T}(Y_t + \bar{Z}_t)$ exists. Moreover, as $T_n \leq T$, we see that $T_n < \infty$. Therefore, in view of (47), the following holds true,

$$\gamma_n \geq Y_{T_n-} + \bar{Z}^n_{T_n-} \quad \text{and} \quad Y_{T_n} + \bar{Z}^n_{T_n} \geq \gamma_n.$$

Taking limits on every side shows that $\lim_{t\uparrow T}(Y_t + \bar{Z}_t) = p(\beta)$. We extend $\bar{Z}$ progressively from T onwards by setting

$$\bar{Z}_t = p(\beta) - Y_T \quad \text{for all } t \geq T.$$

Then, the extended $(\bar{Z}, T)$ fulfils (i) and (iii) of Problem 3.3.

Overall, in all cases, there is a solution to Problem 3.3. In particular, Theorem 3.6 implies that the reflecting SDE (a)-(c) has a weak solution. □

The other way around, using the same techniques as in the proof of Theorem 3.6, we get the following remark.

3.8 Remark. *If the reflecting SDE* (a)-(c) *has a weak solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, Z, X, S)$, *then there is a filtered probability space* $(\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \mathbb{P})$ *that supports a Brownian motion* $\bar{W}$ *such that* Problem 3.3 *has a solution* $(T, \bar{Z})$ *for*

$Y = p(X_0) + \bar{W}$ *that fulfils (33),(34),(36),(37),(40),(43).*

Now, we give a characterisation when the boundary point β is inaccessible for the reflecting SDE (a)-(c). To this end, we define

$$v(x) = \int_\alpha^x \frac{2(p(x) - p(u))}{\sigma^2(u)\, p'(u)}\, \mathrm{d}u \quad \text{for } x \in [0, \beta].$$

Recall that β is inaccessible for the unaffected diffusion if and only if $v(x) \uparrow \infty$ for $x \uparrow \beta$, see for example Mijatović & Urusov [31, p.5, equation (20)].

3.9 Theorem. *In the context of* Theorem 3.6*, the following statements hold true:*

(i) *if* $\lim_{x\to\beta} v(x) = \infty$ *and* $\sup_{0\le x\le y} L(x) < \beta$ *for all* $y < \infty$*, then* $S = \infty$*.*

(ii) *if* $\lim_{x\to\beta} v(x) < \infty$ *or* $\sup_{0\le x\le y} L(x) \ge \beta$ *for some* $y < \infty$*, then* $S < \infty$*.*

Proof. (i). If we assume that $\mathbb{P}[S < \infty] > 0$, then there is some $d \in \mathbb{R}_+$ such that $\mathbb{P}[S < d] > 0$. By assumption, we have

$$\ell = \sup_{0\le x\le d} L(x) < \beta.$$

In view of Karatzas & Shreve [27, p.347, Lemma 5.5.26], there is a unique solution ψ to the ODE that satisfies $\psi(\ell) = 1$, $\psi'(\ell) = 0$ and

$$\frac{1}{2}\sigma^2(x)\, \psi''(x) + b(x)\, \psi'(x) - \psi(x) = 0 \quad \text{for all } x \in]\alpha, \beta[. \tag{48}$$

Moreover, this solution is decreasing in $]\alpha, \ell]$ and fulfils

$$1 + v(x) \le \psi(x) \le e^{v(x)} \quad \text{for all } x \in]\alpha, \beta[. \tag{49}$$

Furthermore, let (β_n) be a sequence in $[\alpha, \beta[$ such that $\beta_n \uparrow \beta$. Now, we consider

$$S_n = \inf\left\{t \ge 0 \,\middle|\, X_t \ge \beta_n\right\} \wedge \inf\left\{t \ge 0 \,\middle|\, \int_0^t \sigma^2(X_u)\, \mathrm{d}u \ge n\right\}$$

and define

$$N_t^n = e^{-t\wedge d\wedge S_n}\, \psi(X_{t\wedge d\wedge S_n}) \quad \text{for } n \in \mathbb{N}.$$

For fixed $n \in \mathbb{N}$, using Itô's formula together with (48) and (b), we can see that

$$N_t^n = \psi(X_0) + \int_0^{t\wedge d\wedge S_n} e^{-u}\sigma(X_u)\psi'(X_u)\,\mathrm{d}W_u \\ + \int_0^{t\wedge d\wedge S_n} \mathbb{1}_{\{X_u=L(u)\}}\, e^{-u}\psi'(X_u)\,\mathrm{d}Z_u^{\mathrm{c}} \\ + \sum_{u\le t\wedge d\wedge S_n} \mathbb{1}_{\{X_u=L(u)\}}\, e^{-u}\big[\psi(X_u)-\psi(X_{u-})\big] \quad \text{for } t\in\mathbb{R}_+.$$

Thanks to the definition of S_n and as ψ' is bounded in the interval $[\ell,\beta_n]$, the stochastic integral is a square integrable martingale. As ψ is decreasing in $]\alpha,\ell]$, and as $L(u)\le\ell$ for $u\le d$, and as $\Delta X\ge 0$, the bounded variation part is decreasing. In particular, N^n is a positive supermartingale. Using the supermartingale property together with Fatou's lemma and $S_n\uparrow S$, we can see that

$$\psi(X_0) \ge \liminf_{n\to\infty}\mathbb{E}\big[N_t^n\,\big|\,\mathcal{F}_0\big] \ge \liminf_{n\to\infty}\mathbb{E}\big[N^n_{d\wedge S_n}\,\big|\,\mathcal{F}_0\big] \ge \mathbb{E}\big[e^{-(d\wedge S)}\psi(X_{d\wedge S})\,\big|\,\mathcal{F}_0\big] \\ \ge e^{-d}\,\psi(\beta)\,\mathbb{P}\big[S<d\,\big|\mathcal{F}_0\big].$$

In view of (49) and $\lim_{x\to\beta} v(x)=\infty$, we can see that $\mathbb{P}[\psi(X_0)=\infty]>0$. But, as $\alpha<X_0<\beta$ and ψ is finite in $]\alpha,\beta[$, the term $\psi(X_0)$ is finite. Hence, our original assumption is wrong, and therefore $\mathbb{P}[S<\infty]=0$.

(ii). First, we consider the case when $\sup_{0\le x\le y} L(x)\ge\beta$ for some $y<\infty$. In view of (b), the inequality $S\le y$ holds true, and therefore $\mathbb{P}[S<\infty]=1$.

Second, we consider the case when $\lim_{x\to\beta} v(x)<\infty$. In view of Remark (3.8), there is a Brownian motion $\bar{W}$ such that Problem 3.3 has a solution $(T,\bar{Z})$ for $Y=X_0+\bar{W}$ which fulfils $A_T=S$. Furthermore, let

$$\tilde{T} = \inf\big\{t>0\,\big|\,Y_t\ge p(\beta)\big\}.$$

Using $b(x)=0$ and $\sigma(x)=1$ when $x<\alpha$, we extend p and $\hat{\sigma}$ to $]-\infty,\beta]$. In view of Karatzas & Shreve [27, p.350, Proposition 5.5.32], as $\lim_{x\to\beta} v(x)<\infty$, the unaffected diffusion leaves its state space in finite time at β. In particular, according to Karatzas & Shreve [27, pp.329-341, Chapter 5.5], the following holds true,

$$\int_0^{\tilde{T}} \hat{\sigma}^{-2}(Y_s)\,\mathrm{d}s<\infty. \tag{50}$$

In view of Karatzas & Shreve [27, p.348, Problem 5.5.27], as $\lim_{x\to\beta} v(x)<$

∞, we see that $p(\beta) < \infty$. As $[0, p(\beta)]$ is compact and $\tilde{T} > 0$, applying the occupation time formula for Brownian motion to (50) yields

$$\int_0^{p(\beta)} \hat{\sigma}^{-2}(x)\,\mathrm{d}x < \infty. \tag{51}$$

As $\bar{Z}$ is increasing, the semimartingale $Y + \bar{Z}$ fulfils Protter [35, p.221, Hypothesis IV.7.A]. In particular, $Y + \bar{Z}$ admits a local time L. Thanks to (ii) and (iii) of 3.3, we see that $L_t^x = 0$ for all $(t, x) \in \mathbb{R}_+ \times (\mathbb{R} \setminus [0, \infty[)$. Moreover, according to Protter [35, p.224, Theorem IV.7.75], for given time t, with respect to x, the process L_t^x is càdlàg and therefore locally bounded. Hence,

$$K = \sup\left\{ L_T^x \,\middle|\, \alpha \le x \le \beta \right\} < \infty.$$

Using the occupation time formula together with (51), we can see that

$$S = A_T = \int_{\mathbb{R}} \hat{\sigma}^{-2}(x) L_T^x \,\mathrm{d}x \le K \int_0^{p(\beta)} \hat{\sigma}^{-2}(x)\,\mathrm{d}x < \infty.$$

□

We finish this section with a couple of remarks.

3.10 Remark. *Assume that* Problem 3.3 *has the following uniqueness property. Given two solutions* $(T, \bar{Z})$ *and* $(\tilde{T}, \tilde{Z})$*, then* $T = \tilde{T}$ *and* $\bar{Z}_t = \tilde{Z}_t$ *for all* $t < T$*. For example, this holds, when* Problem 2.1 *has at most one solution for all continuous functions* y*. Then, given initial conditions, a similar proof as in* Karatzas & Shreve [27, p.336, Theorem 5.5.7] *shows that a weak solution to* (a)*-*(c) *is unique in law.*

3.11 Remark (Recipe for a solution). *Let* $L\colon \mathbb{R}_+ \to [\alpha, \beta[$ *be a càdlàg function. Let* $X_0 \ge L(0)$*. Let* $b, \sigma\colon [\alpha, \beta[\to \mathbb{R}$ *be two Borel-measurable functions. And let* W *be a Brownian motion. Then,*

(i) *compute* $(p, q, \hat{\sigma}, \hat{L})$ *according to (27) and (29),*

(ii) *compute a solution* (T, τ) *to the* the Functional Differential Equation 2.7 *for* $Y = p(X_0) + W$ *and* $g = \hat{\sigma}^{-2}$*, but replace* (ii) *of* 2.7 *with*

$$\lim_{t \uparrow T} (Y_t + \bar{Z}_t) = p(\beta) \quad \textit{where} \quad \bar{Z}_t = \sup_{s \le t} \big(L(\tau_s) - Y_s \big) \quad \textit{for } t < T,$$

(*iii*) *compute the inverse function of* τ, *denoted by* C, *and compute* $S = \tau_T$, $\hat{Z}_t = \bar{Z}_{C_t}$ *and* $\hat{X}_t = Y_{C_t} + \hat{Z}_t$ *for* $t \in \mathbb{R}_+$,

(*iv*) *compute* $X_t = q(\hat{X}_t)$ *and* $Z_t = \int_0^t q'(\hat{X}_u)\,\mathrm{d}\hat{Z}_u^{\mathrm{c}} + \sum_{0\le u\le t} \Delta X_u$ *for* $t \in [0, S]$.

Under suitable assumptions, see Corollary 3.7, *the quadruple* (X, Z, S, W) *fulfils* (*a*)-(*c*).

Part II[§]

Piecewise diffusion approximations of Lévy processes

4. Introduction

In reality, asset prices are observed on a discrete time grid rather than observed continuously. Taking into account the randomness, calibrating a model to the asset prices will only fix the distribution at this grid. Therefore, there is no empiric difference between two models which have the same distribution at this grid. In particular, we have the following pictures in mind. Figure 4.1 visualises information we obtain on a grid over time. Figure 4.2 visualises two different models which may have generated the same information.

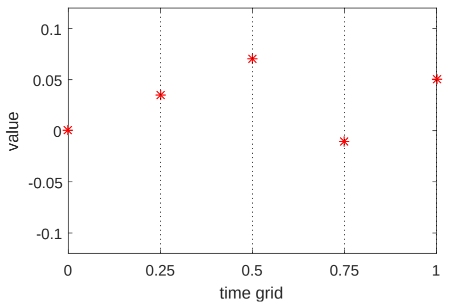

Figure 4.1: information on a grid

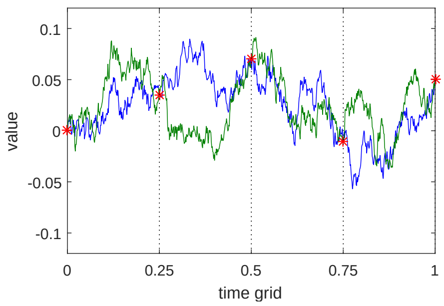

Figure 4.2: two different models

Stochastic modelling incorporating this feature leads to processes with prescribed distributions at grid points and chosen behaviour in between. This approach was first introduced and different ways of constructing these processes were discussed by Precup [34]. It has similarities to the work of Černý, Denkl & Kallsen [6] and Denkl et al. [12]. Here, the authors estimated the error of assuming one model when dealing with a second one.

As input variables, we have a time grid and a sequence of distribution functions. The grid is given as an arbitrary sequence of increasing time points (p_i). For tractability, we think of independent processes pasted continuously together. Therefore, instead of modelling the grid points, we use given distri-

[§]This part is based on joint work with Professor Mihail Zervos.

bution functions (F_i) to model the increments in between. In particular, the distributions at grid points are given as convolutions of (F_i). Due to independence of increments, the distribution function of an increment $V_{p_i - p_{i-1}}$ of a Lévy process is a natural choice for F_i. In particular, we have an underlying Lévy model in mind.

For the output process, we have to decide on a behaviour in between grid points. As one of the best known objects in literature, we choose diffusions. Moreover, we have the following possible application in mind. As Lévy processes fit log-return properties and produce the volatility smile, Lévy models are used in finance for pricing options. Cont & Tankov [8], Cont & Voltchkova [9] and Tankov [45] studied Lévy models in great depth. They describe FFT algorithms as the main method for pricing options in these models. Also, they suggest a link to IPDE's, when FFT is impractical. For example, such a situation appears when the characteristic function of the Lévy process has a complicated form. For instance, the class of generalized hyperbolic distributions fails to be convolution closed. Thus, a Lévy process which is generalized hyperbolic at one time may not be generalized hyperbolic at a later time. Therefore, calibration with FFT at one time gives no obvious forecast for a price at a later time. Having IPDE's in mind, for an easier calculus, we would prefer to see PDE's. For diffusions, option pricing is linked to PDE's. Therefore, we introduce the class of so called Itô semi-diffusions.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, an Itô semi-diffusion of (p_i) and $(V_{p_i - p_{i-1}})$ is a continuous process X where

(*i*) $Z_t = X_t - X_{p_i}$ for $p_i \leq t < p_{i+1}$ is a homogeneous Itô-diffusion independent of $\mathcal{F}_{p_i}$,

(*ii*) $Z_{p_{i+1}-} \overset{\mathcal{L}}{=} V_{p_{i+1} - p_i}$ for all $i \in \mathbb{N}_0$.

In this part of the thesis, we study the following construction of an Itô semi-diffusion. We start from a Brownian motion W. Via inverse transform sampling, we generate the increments between grid points using $(W_{p_{i+1}} - W_{p_i})$. To have a process, we replace $W_{p_{i+1}}$ with W_t for $p_i \leq t < p_{i+1}$. As continuous pasting requires the increment processes to start from zero, we introduce drift terms. Let $V^{-1}_{p_{i+1} - p_i}$ be the left inverse of $V_{p_{i+1} - p_i}$ and let Φ be the distribution function of a standard normal random variable. Then, our recipe for

$p_i \le t < p_{i+1}$ is given by

$$Z_t = \left(V^{-1}_{p_{i+1}-p_i} \circ \Phi\right)\left(\Phi^{-1}(V_{p_{i+1}-p_i}(0))\, \frac{p_{i+1}-t}{p_{i+1}-p_i} + \frac{W_t - W_{p_i}}{\sqrt{p_{i+1}-p_i}}\right).$$

The actual Lévy-Itô semi-diffusion X is the continuous pasting of increments of Z between grid points. Figure 4.3 shows a typical sample path of Z. Figure 4.4 shows the corresponding path of X.

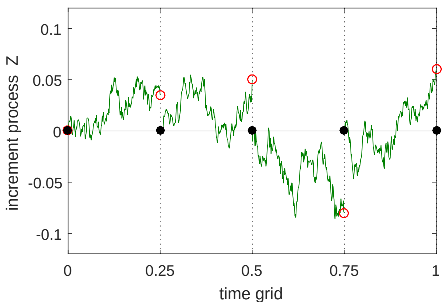

Figure 4.3: sampling of increments

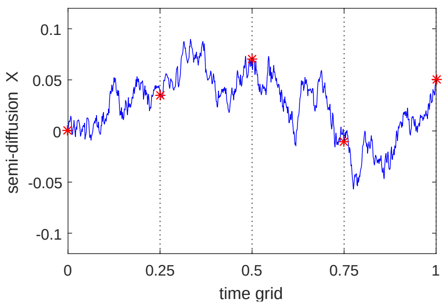

Figure 4.4: Itô semi-diffusion

There are many different ways and generalisations of this construction. For example, we could use an Ornstein-Uhlenbeck process instead of a Brownian motion. For a multidimensional version of Itô semi-diffusions, we could introduce copulas. As we transport the Dirac measure to a prescribed measure, we could impose restrictions related to measure transportation problems. And, for an easier calculus, we could put the drift term outside the function $V^{-1}_{p_{i+1}-p_i} \circ \Phi$. Instead, in order to apply the theory of one-dimensional homogeneous diffusions, we keep it in the above form. In fact, if $V_{p_{i+1}-p_i}$ has enough regularity to apply Itô's formula, then Z becomes a homogeneous diffusion between grid points.

As Lévy models capture real world phenomena and are used for pricing, we are concerned with the following research questions,

1st question: are semi-diffusions close to Lévy processes?

2nd question: can we price with semi-diffusions at all?

1st question. As we work with a grid in continuous time and our preferred behaviour between grid points, the construction of a semi-diffusion appears artificial. For instance, we expect to see a discrepancy when comparing the semi-diffusion with the Lévy model on a finer grid than (p_i). Therefore, we think of semi-diffusions as approximations of realistic models. In particular, we consider a sequence of semi-diffusions where mesh sizes tend to zero. Under suitable assumptions, we will prove convergence of finite-dimensional distributions to the Lévy process. But, in view of Billingsley [4], as we match continuous processes with jump processes, stronger results such as tightness are out of reach.

Heuristically, the assumptions for convergence of finite-dimensional distributions can be deduced in the following way. As we model increments with SDE's driven by Brownian motion, we may assume a scaled Brownian motion in short time. Especially, we expect to see the oscillations of Brownian motion around its starting value zero. Secondly, as the occurrence of finite activity jumps decays exponentially, we expect no impact of such jumps in short time. In fact, after removing any finite activity jumps, V_t needs to be continuous and strictly increasing around zero for all $t > 0$. In terms of V_t's generating triplet (d, σ^2, v) and truncation function $\mathbb{1}_{[-1,1]}$, we prove that convergence of finite-dimensional distributions holds if and only if at least one of the following conditions hold,

$$(i)\ \ \sigma^2 > 0 \ \text{ or } \int_{[-1,1]} |x| \, \mathrm{d}v(x) = \infty, \quad \text{or} \quad (ii)\ \ v(\,]-\infty, 0[\,) = v(\,]0, \infty[\,) = \infty,$$

$$(iii)\ \ \begin{array}{l} \Big[v(\,]0,\infty[\,) = \infty \ \text{ and } \ d - \int_{[-1,1]} x \, \mathrm{d}v(x) < 0 \Big] \quad \text{or} \\ \Big[v(\,]-\infty, 0[\,) = \infty \ \text{ and } \ d - \int_{[-1,1]} x \, \mathrm{d}v(x) > 0 \Big]. \end{array}$$

As indicated by the heuristic argumentation, we establish convergence of finite-dimensional distributions by analysing $V_t(0)$ for $t \downarrow 0$. This will lead to fluctuation theory for Lévy processes as presented by Doney [13] and Rogozin [38]. Also, it will lead to related topics like small time expansions of $V_t(K)$ for $K \neq 0$, i.e. limit of $V_t(K)/t$ for $t \downarrow 0$. Characterising this limit is an exercise in the book of Bertoin [2]. A proof of the exercise can be found in a paper by Rüschendorf & Woerner [42] and correcting remarks in a paper by Figueroa-López & Houdré [15].

2nd question. To ensure theoretical sound pricing rules, semi-diffusion models need to admit local martingale measures. According to Protter [35], Itô diffusions in its natural filtration have unique associated price deflators. Understanding when the price deflators become martingales determines when semi-diffusion models admit local martingale measures.

Determining when the price deflators of diffusions are martingales caused a lot of confusion in the literature, see Mijatović & Urusov [30] and Ruf [41]. Fortunately, the situation has been solved for homogeneous diffusions by Cherny & Urusov [7] and Mijatović & Urusov [31]. In our case, we need to look at associated homogeneous diffusions and determine if these ones exit at $\pm\infty$. Intuitively, as the original price deflators are supermartingales, an exit at ∞ will not appear. By making the right assumptions, we can also ensure that an exit at $-\infty$ will not appear as well. More precisely, our semi-diffusion models admit local martingale measures if and only if

$$\int_{-\infty}^{c} ((\ln V_\tau)'(\ln\ln 1/V_\tau)')(z)\,\mathrm{d}z = -\infty \quad \text{for } \tau = p_{i+1} - p_i \text{ and some } c \in \mathbb{R}.$$

In the end, to verify this condition, we assume that the tail of V_τ admits analytical expressible asymptotics. For Lévy processes, the asymptotics are often known, see for example Gaunt [17], Grigelionis [18] and Rosiński & Sinclair [39]. A tractable description of our final assumption could be, V_τ having a regular enough tail and some negative exponential moment. More precisely, our semi-diffusion models admit local martingale measures if

$$\lim_{z\to-\infty} \frac{V_\tau'(z)}{V_\tau(z)} > 0 \quad \text{for all } \tau = p_{i+1} - p_i.$$

Finding asymptotic results for Lévy processes is a well-known problem in the literature. For example, under abstract conditions on the moments of Lévy measures, Knopova & Kulik [28] gave general asymptotic results for their corresponding Lévy processes. Or, in the context of subexponential distributions, Albin & Sundén [1] and Pakes [32] established links between the asymptotics of Lévy processes and the asymptotics of their corresponding Lévy measures.

A future research plan includes the following three tasks,

1st task: analyse the case of a generalized hyperbolic distribution,

2nd task: simplify the assumptions for existence of local martingale measures,

3rd task: develop a model for the dynamics of volatility smiles.

As explained before, FFT might be impractical for a Lévy model generated from a generalised hyperbolic distribution. We would like to see a comparison between standard methods of pricing these models and our semi-diffusion approach. In particular, we would like to see, if the prices given by the semi-diffusions fit into the prices given by the Lévy model.

Even though, for the existence of local martingale measures, we look for a condition on an integral, we make assumptions on the integrand. Moreover, this assumption holds in a weaker integrated version for even all distributions, more precisely

$$\liminf_{z\to-\infty}\frac{V_\tau(z+h)}{V_\tau(z)}\leq \mathrm{e}^{\alpha h}\leq \limsup_{z\to-\infty}\frac{V_\tau(z+h)}{V_\tau(z)}\quad \text{for all }\ \tau,h>0$$
$$\text{and}\quad \alpha=\sup\big\{a>0\,\big|\int_{\mathbb{R}}\mathrm{e}^{-az}\mathrm{d}V_\tau(z)<\infty\big\}.$$

These observations indicate that the regularity assumptions for the existence of local martingale measures could be weakened.

Lévy models recapture the phenomena of volatility smiles accurately, but are challenging to describe dynamically. Whereas, diffusion models hardly capture the phenomena of volatility smiles, but are described in a dynamical way. Having features from both worlds, we would like to use semi-diffusions to develop models for the dynamics of volatility smiles.

The rest of this part is organised as follows. In section 5, we collect some standalone results for the upcoming analysis. In section 6, we introduce Itô semi-diffusions for Lévy processes. In section 7, we analyse when sequences of semi-diffusions have converging finite-dimensional distribution to the ones of the Lévy process. In section 8, we analyse when sequences of semi-diffusions are tight. In section 9, we analyse when semi-diffusion models admit local martingale measures.

5. Preliminary results

In this section, we collect some lemmas for the upcoming analysis.

Recall that Φ denotes the distribution function of the standard normal distribution. To deal with the non-linear impact of Φ^{-1} in our construction of semi-diffusions, we establish the following asymptotic results.

5.1 Lemma.

(i) $\dfrac{\Phi'(\Phi^{-1}(x))}{(\Phi^{-1}(x))^2}>\beta(x):=\Phi'(\Phi^{-1}(x))-x\,|\Phi^{-1}(x)|>0\quad$ *for all* $x\in\,]0,1/2[\,,$

(ii) $\lim_{x\downarrow 0} \Phi^{-1}(x)\big(\Phi^{-1}(cx) - \Phi^{-1}(x)\big) = -\ln c$ *for all* $c > 0$,

(iii) $\Phi^{-1}(x) \sim \Psi(x) \coloneqq -\sqrt{-2\ln(x)}$ *for* $x \downarrow 0$.

Proof. (i) is a consequence of the following estimate for the normal distribution taken from Feller [14, p.175, Lemma 2],

$$\Phi'(y)\big(\frac{1}{|y|} - \frac{1}{|y|^3}\big) < \Phi(y) < \Phi'(y)\frac{1}{|y|} \quad \text{for } y < 0. \tag{52}$$

Replacing y with $\Phi^{-1}(x)$ for $x \in]0, 1/2[$ and rearranging the terms yields the result.

(ii). For the following argument, let $c > 1$. First, the following holds true,

$$\begin{aligned} 0 < \Phi^{-1}(cx) - \Phi^{-1}(x) &= \int_x^{cx} \frac{1}{\Phi'(\Phi^{-1}(t))}\,\mathrm{d}t \\ &< \int_x^{cx} \frac{1}{\Phi'(\Phi^{-1}(x))}\,\mathrm{d}t = \frac{x\,(c-1)}{x\,|\Phi^{-1}(x)| + \beta(x)} \\ &< \frac{x\,(c-1)}{x\,|\Phi^{-1}(x)|} = \frac{c-1}{|\Phi^{-1}(x)|} \xrightarrow[x\downarrow 0]{} 0, \quad \text{because } \beta > 0. \end{aligned} \tag{53}$$

Furthermore, for $x \downarrow 0$, the following holds true,

$$\begin{aligned} \partial_x\big[(\Phi^{-1}(x))^2\big] &= \frac{2\,\Phi^{-1}(x)}{\Phi'(\Phi^{-1}(x))} = \frac{2\,\Phi^{-1}(x)}{x\,|\Phi^{-1}(x)|}\,\frac{\Phi'(\Phi^{-1}(x)) - \beta(x)}{\Phi'(\Phi^{-1}(x))} \\ &= -\frac{2}{x}\big(1 - \frac{\beta(x)}{\Phi'(\Phi^{-1}(x))}\big) \sim -\frac{2}{x}, \\ \text{because} \quad & 0 < \frac{\beta(x)}{\Phi'(\Phi^{-1}(x))} < \frac{1}{(\Phi^{-1}(x))^2} \xrightarrow[x\downarrow 0]{} 0. \end{aligned} \tag{54}$$

We integrate (54) on both sides. Let $\varepsilon > 0$ and without loss of generality consider x to be so small that $\varepsilon \geq \beta(x)/\Phi'(\Phi^{-1}(x)) > 0$. Then, the following holds true,

$$\begin{aligned} \big|(\Phi^{-1}(cx))^2 - (\Phi^{-1}(x))^2 + 2\ln c\big| &= \Big|\int_x^{cx} -\frac{2}{t}\,\frac{\beta(t)}{\Phi'(\Phi^{-1}(t))}\,\mathrm{d}t\Big| \\ &\leq \varepsilon \int_x^{cx} \frac{2}{t}\,\mathrm{d}t = 2\varepsilon\ln c \xrightarrow[\varepsilon\downarrow 0]{} 0, \\ \text{hence,} \quad \lim_{x\downarrow 0} \big[(\Phi^{-1}(cx))^2 &- (\Phi^{-1}(x))^2\big] = -2\ln c. \end{aligned} \tag{55}$$

Statements (53) and (55) are also true if $0 < c \leq 1$. The case $c = 1$ is trivial, and the case $0 < c < 1$ can be deduced from $c > 1$ by interchanging the roles of x and cx. Overall, in view of (53) and (55), the following holds true,

$$\begin{aligned}
&\Phi^{-1}(x)\big(\Phi^{-1}(cx) - \Phi^{-1}(x)\big) \\
&\qquad = \frac{1}{2}\big[\big((\Phi^{-1}(cx))^2 - (\Phi^{-1}(x))^2\big) - \big(\Phi^{-1}(x) - \Phi^{-1}(cx)\big)^2\big] \\
&\qquad \xrightarrow[x\downarrow 0]{} \frac{1}{2}\big(-2\ln c - 0^2\big) = -\ln c \quad \text{for all } \; c > 0.
\end{aligned}$$

(iii). We apply L'Hôpital's rule to the ratio of $(\Phi^{-1})^2$ and Ψ^2. In view of (54), the following holds true for $x \downarrow 0$,

$$\frac{\partial_x[(\Phi^{-1}(x))^2]}{\partial_x[(\Psi(x))^2]} \sim \frac{-2/x}{-2/x} = 1.$$

This proves the result as Φ^{-1} and Ψ have the same sign for $x \downarrow 0$. ☐

The analysis of convergence of finite-dimensional distributions will lead to fluctuation theory. Our contribution to this field is the following Lemma.

5.2 Lemma. *Let L be a Lévy process with generating triplet (d, σ^2, v) and truncation function $\mathbb{1}_{[-1,1]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

$$(i) \quad \textit{If} \quad \liminf_{t\downarrow 0} \frac{\mathbb{P}[L_t < 0]}{t} < \infty,$$

then $\mathbb{P}[L_t < 0]/t \to v(\,]-\infty, 0[\,)$ and L is a "subordinator + negative compound Poisson process", i.e. $\sigma^2 = 0$ and $v(\,]-\infty,0[\,) < \infty$ and $d - \int_{[-1,1]} x \,\mathrm{d}v(x) \geq 0$.

$$(ii) \quad \textit{If} \quad \liminf_{t\downarrow 0} \frac{\mathbb{P}[L_t \leq 0]}{t} < \infty,$$

then $\mathbb{P}[L_t \leq 0]/t \to v]-\infty, 0[$ and L is a "strictly increasing subordinator + negative compound Poisson process", i.e.

$$\sigma^2 = 0, \quad v(\,]-\infty,0[\,) < \infty, \quad d - \int_{[-1,1]} x \,\mathrm{d}v(x) \geq 0, \quad \textit{and}$$
$$\Big[d - \int_{[-1,1]} x \,\mathrm{d}v(x) > 0 \quad \textit{or} \quad v(\,]0,\infty[\,) = \infty\Big].$$

Proof. To prove *(ii)*, we give a case-by-case study of all possible generating triplets. We divide the triplets into the following categories,

- *1st case:* $\sigma^2 > 0$,

- *2nd case:* $\sigma^2 = 0$, and $v(\,]-\infty,0[\,) = \infty$,

- *3rd case:* $\sigma^2 = 0$, $v(\,]-\infty, 0[\,) < \infty$, and $d - \int_{[-1,1]} x \, \mathrm{d}v(x) < 0$,
- *4th case:* $\sigma^2 = 0$, $v(\,]-\infty, 0[\,) < \infty$, and $d - \int_{[-1,1]} x \, \mathrm{d}v(x) \geq 0$.

1st case. If there is a Brownian component, ROGOZIN [38, p.483, Lemma 1], showed that the distribution of $L_t/\sqrt{t}$ converges for $t \downarrow 0$ to a centred normal distribution. The normal distribution is continuous at zero, i.e.

$$\lim_{t \downarrow 0} \mathbb{P}[L_t \leq 0] = \frac{1}{2}; \quad \text{implying} \quad \lim_{t \downarrow 0} \frac{\mathbb{P}[L_t \leq 0]}{t} = \infty.$$

2nd case. According to BERTOIN [2, p.39, Exercise 1], if $K < 0$ is a point at which v is continuous, then $\mathbb{P}[L_t \leq K]/t \to v(\,]-\infty, K])$ for $t \downarrow 0$. We choose a sequence $K_n \uparrow 0$ of continuous points of v, then

$$\lim_{t \downarrow 0} \frac{\mathbb{P}[L_t \leq 0]}{t} \geq \lim_{n \to \infty} \lim_{t \downarrow 0} \frac{\mathbb{P}[L_t \leq K_n]}{t} = v(\,]-\infty, 0[\,) = \infty.$$

3rd case. As $v(\,]-\infty, 0[\,) < \infty$, the downward jumps of L are summable. Therefore, we can define the following new Lévy process,

$$L^{+a} = L - \sum \Delta L \, \mathbb{1}_{\Delta L \notin [0,a]} \quad \text{for } a > 0. \tag{56}$$

Let $a \in \,]0, 1]$. As L^{+a} has bounded jumps, L^{+a} is integrable. We compute its expectation with its characteristic function and the Lévy-Khintchine formula,

$$\mathbb{E}[L_t^{+a}] = \frac{1}{\mathrm{i}} \, \partial_w \Psi_{L_t^{+a}}(w)\big|_{w=0} = t \left[d - \int_{[-1,0]} x \, \mathrm{d}v(x) - \int_{]a,1]} x \, \mathrm{d}v(x) \right] \quad \text{for } t > 0.$$

By assumption, there is $a \in \,]0, 1]$ such that $\mathbb{E}[L_t^{+a}] < 0$. Thus, we can assume that L^{+a} has only positive jumps and strictly negative expectation. SATO [43, p.345 Chapter 46 and p.350 Propsition 46.8] proved that in this case

$$\mathbb{P}[L_t^{+a} < 0] \geq \frac{1}{16} \quad \text{for } t > 0; \quad \text{hence} \quad \lim_{t \downarrow 0} \frac{\mathbb{P}[L_t^{+a} \leq 0]}{t} = \infty. \tag{57}$$

Using the independence of jumps, the following estimate between $L^{+\infty}$ and L^{+a} holds true,

$$\begin{aligned} \mathbb{P}[L_t^{+\infty} \leq 0] &= \mathbb{P}[L_t^{+a} \leq 0] - \mathbb{P}[L_t^{+a} \leq 0, \, L_t^{+\infty} > 0] \\ &\geq \mathbb{P}[L_t^{+a} \leq 0] - \mathbb{P}[L_t^{+a} \leq 0, \text{ at least one jump bigger } a \text{ in } [0,t]] \\ &= \mathbb{P}[L_t^{+a} \leq 0] - \mathbb{P}[L_t^{+a} \leq 0] \, \mathbb{P}[\text{at least one jump bigger } a \text{ in } [0,t]] \end{aligned}$$

$$= \mathbb{P}[L_t^{+a} \le 0] - \mathbb{P}[L_t^{+a} \le 0]\,(1 - \mathrm{e}^{-t\,v(]a,\infty[)})$$
$$= \mathbb{P}[L_t^{+a} \le 0]\,\mathrm{e}^{-t\,v(]a,\infty[)} \quad \text{for } t > 0. \tag{58}$$

As L is smaller than $L^{+\infty}$, a combination of (57) and (58) gives

$$\lim_{t\downarrow 0} \frac{\mathbb{P}[L_t \le 0]}{t} \ge \lim_{t\downarrow 0} \frac{\mathbb{P}[L_t^{+\infty} \le 0]}{t} \ge \lim_{t\downarrow 0} \frac{\mathbb{P}[L_t^{+a} \le 0]}{t} = \infty.$$

4^th case. Let $L^{+\infty}$ be defined as in (56). Let $C = \sum \Delta L\, \mathbb{1}_{\Delta L < 0}$ and $v_- = v\,\mathbb{1}_{]-\infty,0[}$. We denote by $v_-^{\underline{k}}$ the k-th convolution of v_-, and mean by $v_-^{\underline{0}}$ the Dirac delta distribution. Then, using little-o notation for $t \downarrow 0$, the following holds true,

$$\begin{aligned}\frac{\mathbb{P}[L_t \le 0]}{t} &= \frac{\mathbb{P}[L_t^{+\infty} \le -C_t]}{t} = \frac{1}{t} \int_{\mathbb{R}} \mathbb{P}[L_t^{+\infty} \le -c]\,\mathrm{d}\mathbb{P}_{C_t}(c)\\ &= \frac{\mathrm{e}^{-t\,v(]-\infty,0[)}}{t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{\mathbb{R}} \mathbb{P}[L_t^{+\infty} \le -c]\,\mathrm{d}v_-^{\underline{k}}(c)\\ &= \mathrm{e}^{-t\,v(]-\infty,0[)} \Big(\frac{\mathbb{P}[L_t^{+\infty} \le 0]}{t} + \int_{]-\infty,0[} \mathbb{P}[L_t^{+\infty} \le -c]\,\mathrm{d}v(c) + \mathrm{o}(1) \Big).\end{aligned}$$

As $d - \int_{[-1,1]} x\,\mathrm{d}v(x) \ge 0$, the inequality $L^{+\infty} \ge 0$ holds true. Using continuity in probability and dominated convergence, we can see that $\int_{]-\infty,0[} \mathbb{P}[L_t^{+\infty} \le -c]\,\mathrm{d}v(c) \to v(\,]-\infty,0[\,)$ for $t \downarrow 0$. Hence,

$$\frac{\mathbb{P}[L_t \le 0]}{t} = \frac{\mathbb{P}[L_t^{+\infty} = 0]}{t} + v(\,]-\infty,0[\,) + \mathrm{o}(1) \quad \text{for } t \downarrow 0. \tag{59}$$

If $d - \int_{[-1,1]} x\,\mathrm{d}v(x) > 0$, then $L^{+\infty}$ is strictly increasing. If $v(\,]0,\infty[\,) = \infty$, then $L_t^{+\infty}$ is continuously distributed for $t > 0$. In both cases we can see that

$$\mathbb{P}[L_t^{+\infty} = 0] = 0 \quad \text{for } t > 0. \quad \text{In view of (59),} \quad \lim_{t\downarrow 0} \frac{\mathbb{P}[L_t \le 0]}{t} = v(\,]-\infty,0[\,).$$

Otherwise, $d - \int_{[-1,1]} x\,\mathrm{d}v(x) = 0$ and $v(\,]0,\infty[\,) < \infty$. Here, $L^{+\infty}$ is a positive compound Poisson process. Therefore,

$$\mathbb{P}[L_t^{+\infty} = 0] = \mathrm{e}^{-t\,v(\,]-\infty,0[\,)} \quad \text{for } t > 0. \quad \text{In view of (59),} \quad \lim_{t\downarrow 0} \frac{\mathbb{P}[L_t \le 0]}{t} = \infty.$$

We conclude the proof with the remark that (i) follows from a similar argumentation as (ii). Here, the *4^th case* can be studied in a whole, as $L^{+\infty} \ge 0$ implies $\mathbb{P}[L_t^{+\infty} < 0] = 0$ for $t > 0$. □

6. Lévy-Itô semi-diffusions

In this section, we introduce the additive time-homogeneous Itô semi-diffusions from Precup [34, Chapter 4].

We fix a partition (p_i) of the positive half line $[0,\infty[$, i.e. $0 = p_0 < p_i < p_j \to \infty$ for $0 < i < j \to \infty$. Moreover, we fix a sequence of distribution functions (F_i).

6.1 Definition (Itô semi-diffusion). *A continuous process X on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called (homogeneous additive) Itô semi-diffusion of (F_i) given (p_i) if*

(i) $Z_t = X_t - X_{p_i}$ for $p_i \le t < p_{i+1}$ is a homogeneous Itô-diffusion independent of $\mathcal{F}_{p_i}$

(ii) $Z_{p_{i+1}-} \overset{\mathcal{L}}{=} F_i$, for all $i \in \mathbb{N}_0$.

For a tractable construction of a semi-diffusion with explicit formulas, we make the following assumption.

6.2 Assumption. *$F_i(0) \in]0,1[$ and $F_i^{-1}(F_i(0)) = 0$, where we use the left inverse $F_i^{-1}(y) = \inf\{x \in \mathbb{R} \mid F_i(x) \ge y\}$ for $i \in \mathbb{N}_0$.*

Consider a Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and denote by Φ the distribution function of a standard normal distribution. Now, we define Z and the corresponding continuous X using $Z_t = X_t - X_{p_i}$ for $p_i \le t < p_{i+1}$ by

$$Z_t = \Big(F_i^{-1} \circ \Phi\Big)\Big(\Phi^{-1}(F_i(0))\,\frac{p_{i+1} - t}{p_{i+1} - p_i} + \frac{W_t - W_{p_i}}{\sqrt{p_{i+1} - p_i}}\Big). \tag{60}$$

Assumption 6.2 ensures that Z is well-defined, more precisely

- $F_i(0) \in]0,1[$ implies $\Phi^{-1}(F_i(0)) \in \mathbb{R}$, in particular (60) is finite,
- $F_i^{-1}(F_i(0)) = 0$, hence (60) vanishes at p_i, thus $Z_t = X_t - X_{p_i}$ holds at p_i.

Definition (60) gives almost an Itô semi-diffusion, except that Z might fail to be a homogeneous Itô diffusion between grid points, more precisely

- (i) of 6.1 is partially true, as Z in $[p_i, p_{i+1}[$ is independent from $\mathcal{F}_{p_i}$ by construction,

- (ii) of 6.1 holds, as F_i^{-1} has at most countable many discontinuities and as W_t is continuously distributed. More precisely, $Z_{p_{i+1}-} = (F_i^{-1} \circ \Phi)((W_{p_{i+1}} - W_{p_i})/\sqrt{p_{i+1} - p_i})$ $\mathbb{P}$-as. In particular, $Z_{p_{i+1}-}$ is distributed like F_i by inverse transform sampling.

For a dynamical description of Z, we make the following assumption.

6.3 Assumption. *F_i' exists, is absolutely continuous and strictly positive.*

6.4 Remark. *Assumption 6.3 implies Assumption 6.2.*

Under the Assumption 6.3, Itô's formula shows that Z is in fact a homogeneous Itô diffusion between grid points. In particular, under Assumptions 6.3 the definition (60) gives an Itô semi-diffusion. Moreover, in this case, the following holds true,

$$\mathrm{d}Z_t = \mu_i(Z_t)\mathrm{d}t + \sigma_i(Z_t)\mathrm{d}W_t \quad \text{for } p_i \le t < p_{i+1}, \quad \text{with}$$

$$\sigma_i(z) = \frac{\Phi'(\Phi^{-1}(F_i(z)))}{\sqrt{p_{i+1} - p_i} F_i'(z)} \quad \text{and} \tag{61}$$

$$\mu_i(z) = -\frac{\sigma_i(z)^2}{2}\Big(\frac{2\Phi^{-1}(F_i(0)) + \Phi^{-1}(F_i(z))}{\sigma_i(z)\sqrt{p_{i+1} - p_i}} + \frac{F_i''(z)}{F_i'(z)}\Big).$$

In the rest of this part, we will concentrate on the special case of F_i being the distribution function of an increment of length $p_{i+1} - p_i$ of a given Lévy process.

6.5 Definition (Lévy construction). *Consider*

- V_t *distribution function of an increment of length* $t \ge 0$ *of a given Lévy process,*
- $F_i = V_{p_{i+1} - p_i}$ *for all* $i \in \mathbb{N}_0$ *such that* Assumption 6.2 *holds.*

A Lévy construction X of V given (p_i) and Brownian motion W is the continuous pasting of $Z_t = X_t - X_{p_i}$ for $p_i \le t < p_{i+1}$ that is defined by formula (60).

6.6 Definition (Lévy-Itô semi-diffusion). *A Lévy-Itô semi-diffusion X of V given (p_i) and W is a Lévy construction X of V given (p_i) and W such that* Assumption 6.3 *holds for* $F_i = V_{p_{i+1} - p_i}$.

7. Convergence of finite-dimensional distributions

In this section, we study sequences of Lévy constructions for a fixed Lévy process. We look for conditions on the underlying infinite divisible distribution, such that the finite-dimensional distributions of the sequences converge to the ones of the Lévy process.

For the sequence of grids (p_i^n), we assume that

$$\lim_{n\to\infty} \sup_{i\in\mathbb{N}_0} |p_{i+1}^n - p_i^n| = 0. \tag{62}$$

Beyond (62), we have no preferences concerning the structure of the grids. In fact, we use the freedom to choose grids to establish if and only if conditions for our main result.

7.1 Theorem (convergence of finite-dimensional distributions). *The finite-dimensional distributions of sequences of Lévy constructions of V converge for all grid sequences (p_i^n) with (62) to the ones of the Lévy process of V, if and only if*

$$\lim_{\tau\downarrow 0} \frac{V_\tau(0)}{\tau} \wedge \frac{1-V_\tau(0)}{\tau} = \infty. \tag{63}$$

Let (d,σ^2,v) be the generating triplet and $\mathbb{1}_{[-1,1]}$ the corresponding truncation function of V_t for some $t>0$. Then (63) holds if and only if any of the following conditions holds true,

$$(i)\ \ \sigma^2>0 \ \ or \ \int_{[-1,1]} |x|\,\mathrm{d}v(x)=\infty, \quad or \quad (ii)\ \ v(\,]-\infty,0[\,)=v(\,]0,\infty[\,)=\infty,$$

$$(iii)\ \ \begin{matrix}\Big[v(\,]0,\infty[\,)=\infty \ \ and\ \ d-\int_{[-1,1]} x\,\mathrm{d}v(x)<0\Big] \quad or \\ \Big[v(\,]-\infty,0[\,)=\infty \ \ and\ \ d-\int_{[-1,1]} x\,\mathrm{d}v(x)>0\Big].\end{matrix}$$

Proof. Let (X^n) be a sequence of Lévy constructions of V given (p_i^n) and W on $(\Omega,\mathcal{F},\mathbb{P})$. Furthermore, let $Z_t^n = X_t^n - X_{p_i}^n$ for $p_i \le t < p_{i+1}$. For ease of notation, let L be a Lévy process corresponding to V, and introduce for $t\ge 0$ the functions

$$p^n(t) = \max\{(p_i^n)_{i\in\mathbb{N}_0}\,|\,p_i^n \le t\} \quad \text{and} \quad q^n(t) = \min\{(p_i^n)_{i\in\mathbb{N}_0}\,|\,t<p_i^n\}. \tag{64}$$

We will encounter the following technical conditions equivalent to the convergence of finite-dimensional distributions. Let $\lambda_K = \lambda_K(\tau) = \Phi^{-1}(V_\tau(K))$ for

$K \in \mathbb{R}$ and $\tau > 0$. For all $K \neq 0$,

$$\tau_n \downarrow 0 \quad \Rightarrow \quad \lambda_K(\tau_n) - \lambda_0(\tau_n) \neq 0 \quad \text{for } n \text{ big enough,} \tag{65}$$

$$\tau_n \downarrow 0 \text{ with } \big(\frac{\lambda_K - \lambda_0}{\lambda_0}\big)(\tau_n) \in]0,1[\quad \Rightarrow \lim_{n\to\infty} \big(\lambda_0 \left(\lambda_K - \lambda_0\right)\big)(\tau_n) = \infty. \tag{66}$$

We split the proof into multiple steps. The following list gives an overview of the claims we prove in every step,

- *1st step:* if $(Z_t^n) \xrightarrow[n\to\infty]{\mathcal{L}} 0$ for all $t > 0$, then X^n has converging finite-dimensional distributions to the ones of L,
- *2nd step:* (65), (66) imply $(Z_t^n) \xrightarrow[n\to\infty]{\mathcal{L}} 0$,
- *3rd step:* if $(X_t^n) \xrightarrow[n\to\infty]{\mathcal{L}} L_t$ for all $t > 0$ and all (p_i^n) with (62), then (65) and (66) are fulfilled,
- *4th step:* condition (63) implies (65) and (66),
- *5th step:* (65), (66) imply condition (63),
- *6th step:* condition (63) holds if and only if at least one of $(i), (ii), (iii)$ in the statement of the theorem holds.

1st step. Assuming (Z_t^n) in law to zero for all $t > 0$. We prove by induction over J that $(X_{t_j}^n)_{j=1}^J$ in law to $(L_{t_j})_{j=1}^J$ for all increasing sequences $(t_j)_{j=1}^J$.

$J = 1:$ The process L is stochastically continuous, i.e. $X^n_{p^n(t_1)} \stackrel{\mathcal{L}}{=} L_{p^n(t_1)} \xrightarrow{as.} L_{t_1}$. Hence, Slutsky's theorem implies that

$$X_{t_1}^n = X_{p^n(t_1)}^n + Z_{t_1}^n \xrightarrow[n\to\infty]{\mathcal{L}} L_t + 0.$$

$J > 1:$ As L is stochastic continuous, we can see that $X^n_{q^n(t_{J-1})} - X^n_{p^n(t_{J-1})} \stackrel{\mathcal{L}}{=} L_{q^n(t_{J-1})} - L_{p^n(t_{J-1})} \xrightarrow{as.} 0$. For the induction, we assume that

$$\big((X_{t_j}^n)_{j=1}^{J-1}, X^n_{p^n(t_J)} - X^n_{q^n(t_{J-1})}\big) \xrightarrow[n\to\infty]{\mathcal{L}} \big((L_{t_j})_{j=1}^{J-1}, L_{t_J} - L_{t_{J-1}}\big).$$

Furthermore, we observe that

$$X_{t_J}^n = Z_{t_J}^n + X^n_{p^n(t_J)} - X^n_{q^n(t_{J-1})} + X^n_{q^n(t_{J-1})} - X^n_{p^n(t_{J-1})} + X^n_{t_{J-1}} - Z^n_{t_{J-1}}.$$

Using Slutsky's theorem together with the above two statements completes the induction proof, as we can see that

$$(X^n_{t_J}, (X^n_{t_j})_{j=1}^{J-1}) \xrightarrow[n\to\infty]{\mathcal{L}} (L_{t_J}, (L_{t_j})_{j=1}^{J-1}) = (L_{t_j})_{j=1}^{J}.$$

2nd step. For $\tau > 0$, we introduce the following process,

$$Z^\tau_s = \Big(V_\tau^{-1} \circ \Phi\Big)\Big(\lambda_0 \, \frac{\tau - s}{\tau} + \frac{W_s}{\sqrt{\tau}}\Big) \quad \text{for } s \in [0,\tau]. \tag{67}$$

Then, the following holds true,

$$(Z^\tau_s)_{0\le s<\tau} \overset{\mathcal{L}}{=} (Z^n_t)_{p^n_i \le t < p^n_{i+1}} \quad \text{for } \tau = p^n_{i+1} - p^n_i. \tag{68}$$

Let $K < 0$. As we use the left inverse V_τ^{-1} of V_τ, the inequality $V_\tau^{-1}(y) \le x$ holds true if and only if $y \le V_\tau(x)$, for given $y, x \in \mathbb{R}$. In particular, the following holds true,

$$\begin{aligned}
&\mathbb{P}[Z^\tau_s \le K] = \Phi\big((\tau/s)^{1/2}(\lambda_K - \lambda_0(1 - s/\tau))\big) \quad \text{for } s \in \,]0,\tau], &(69)\\
&\mathbb{P}[Z^\tau_0 \le K] = 0 = \lim_{s\downarrow 0} \mathbb{P}[Z^\tau_s \le K], \quad \text{since (65) implies } \lambda_K < \lambda_0.
\end{aligned}$$

Thus, the function $s \mapsto \mathbb{P}[Z^\tau_s \le K]$ is continuous in $[0,\tau]$ and smooth in $]0,\tau[$. In order to maximise this function, we analyse the boundary points and search for stationary points in the interior of $[0,\tau]$. The boundary points are

$$\begin{aligned}
s = 0: &\quad \mathbb{P}[Z^\tau_0 \le K] = \mathbb{P}[\,0 \le K] = 0, &(70)\\
s = \tau: &\quad \mathbb{P}[Z^\tau_\tau \le K] = \Phi(\lambda_K) = V_{p^n_{i+1} - p^n_i}(K) \xrightarrow[n\to\infty]{} 0, \quad \text{since (62).} &(71)
\end{aligned}$$

In view of (69) and Φ strictly increasing, finding the stationary points of $s \mapsto \mathbb{P}[Z^\tau_s \le K]$ in $]0,\tau[$ is equivalent to find the stationary points of

$$f(s) = (\tau/s)^{1/2}(\lambda_K - \lambda_0(1 - s/\tau)) \quad \text{in } s \in \,]0,\tau[.$$

Applying the first and second order condition to this optimisation problem gives

$$\begin{aligned}
f'(s) &= \frac{(\lambda_0 - \lambda_K)\tau + \lambda_0 s}{2\tau^2 (s/\tau)^{3/2}}, \text{ i.e. } \widetilde{s} = \tau \, \frac{\lambda_K - \lambda_0}{\lambda_0} \text{ only stationary point,} &(72)\\
f''(s) &= \frac{3(\lambda_K - \lambda_0)\tau - \lambda_0 s}{4\tau^3 (s/\tau)^{5/2}}, \text{ i.e. } f''(\widetilde{s}) = \frac{2(\lambda_K - \lambda_0)}{4\tau^3(\widetilde{s}/\tau)^{5/2}} < 0, \text{ since (65).} &(73)
\end{aligned}$$

Now, we rather emphasise that τ depends on n for given $t > 0$ and look at

$\tau_n = p^n_{i+1} - p^n_i$. We split the sequence (τ_n) into three subsequences according to $(\lambda_K - \lambda_0)/\lambda_0 \le 0$ or $(\lambda_K - \lambda_0)/\lambda_0 \in \,]0,1[$ or $(\lambda_K - \lambda_0)/\lambda_0 \ge 1$.

If $(\lambda_K - \lambda_0)/\lambda_0 \le 0$, then $\widetilde{s} < 0$. In particular, as f is well-defined in the positive half line only, we cannot use the results from the above optimisation. However, as $\lambda_K < \lambda_0$ since (65), the inequality $(\lambda_K - \lambda_0)/\lambda_0 \le 0$ gives $\lambda_0 \ge 0$. Hence,

$$\begin{aligned}\sup_{s\in[0,\tau_n]} \mathbb{P}[Z^{\tau_n}_s \le K] &\overset{(69),(70)}{=} \sup_{s\in]0,\tau_n]} \Phi\big((\tau_n/s)^{1/2}(\lambda_K - \lambda_0(1 - s/\tau_n))\big) \\ &\le \sup_{s\in]0,\tau_n]} \Phi\big((\tau_n/s)^{1/2}\lambda_K\big) = \Phi(\lambda_K) \xrightarrow[n\to\infty]{(71)} 0. \qquad (74)\end{aligned}$$

If $(\lambda_K - \lambda_0)/\lambda_0 \ge 1$, then $\widetilde{s} \ge \tau_n$. In particular, (72) and (73) show that f is increasing in $[0, \tau_n]$, thus τ_n is again the point of global maximum. Hence, (74) holds also in this case. If $(\lambda_K - \lambda_0)/\lambda_0 \in \,]0,1[\,$, then (72) and (73) give

$$\sup_{s\in[0,\tau_n]} \mathbb{P}[Z^{\tau_n}_s \le K] = \Phi\big(-2\,(\lambda_0\,(\lambda_K - \lambda_0))^{1/2}\big) \xrightarrow[n\to\infty]{(66)} 0.$$

As all subsequences converge in the same way, the following holds true,

$$\lim_{n\to\infty} \sup_{s\in[0,\tau_n]} \mathbb{P}[Z^{\tau_n}_s \le K] = 0.$$

In a similar way, for $K > 0$ we can see that $\sup_{s\in[0,\tau_n]} \mathbb{P}[Z^{\tau_n}_s > K] \to 0$ for $n \uparrow \infty$. Even though, we use here $\theta_K = \Phi^{-1}(1 - V_\tau(K))$ instead of $\lambda_K = \Phi^{-1}(V_\tau(K))$, the conditions (65), (66) keep the same. More precisely, as $\Phi^{-1}(x) = -\Phi^{-1}(1-x)$ for $x \in [0,1]$, the following equalities hold true,

$$\frac{\lambda_K - \lambda_0}{\lambda_0} = \frac{\theta_K - \theta_0}{\theta_0}, \quad \lambda_0\,(\lambda_K - \lambda_0) = \theta_0\,(\theta_K - \theta_0).$$

Combining the statements for $K < 0$ and $K > 0$ into one equation, we see that

$$\lim_{n\to\infty} \sup_{s\in[0,\tau_n]} \mathbb{P}[|Z^{\tau_n}_s| > K] = 0 \quad \text{for } K \ne 0.$$

In view of (68), this yields $\mathbb{P}[|Z^n_t| > K] \le \sup_{s\in[0,\tau_n]} \mathbb{P}[|Z^{\tau_n}_s| > K] \to 0$ for $n \to \infty$ and $K \ne 0$.

3rd step. Here, we assume that (65), (66) fails to hold for some given $K \ne 0$

and $\tau_n \downarrow 0$. We choose time points (ξ_n) such that

$$\text{if (65) fails}: \quad 0 < \xi_n < \tau_n \quad \text{and} \quad (\xi_n/\tau_n)^{1/2}\lambda_0(\tau_n) \xrightarrow[n\to\infty]{} 0, \quad \text{and}$$
$$\text{if (66) fails}: \quad \xi_n = \widetilde{s} \quad \text{the optimal point of } f \text{ in }]0,\tau_n[\,, \quad \text{see (72).}$$

For given $t > 0$, we expand $(t - \xi_n, t - \xi_n + \tau_n)$ to grids with (62) such that the original sequence becomes a sequence of consecutive grid points. In view of (69), the following holds true,

$$\text{if (65) fails}: \ \mathbb{P}[\operatorname{sgn}(K)\, Z_t^n \geq |K|\,] \geq \Phi\big(-\operatorname{sgn}(K)\,(\xi_n/\tau_n)^{1/2}\lambda_0\big) \xrightarrow[n\to\infty]{} \Phi(0),$$
$$\text{if (66) fails}: \ \mathbb{P}[\operatorname{sgn}(K)\, Z_t^n \geq |K|\,] \geq \Phi\big(-2\,(\lambda_0\,(\lambda_K - \lambda_0))^{1/2}\big) \underset{n\to\infty}{\not\longrightarrow} 0.$$

In particular, (Z_t^n) fails to converge in distribution to zero. But, if (X_t^n) converges in distribution to L_t, then (Z_t^n) needs to converge in distribution to zero for $t > 0$. To see this, we look at the corresponding characteristic functions and use the independence between $X^n_{g^n_t}$ and Z^n_t.

$$\text{As} \quad \varphi_{X_t^n} = \varphi_{X^n_{p^n(t)}}\varphi_{Z^n_t}, \quad \varphi_{X_t^n} \xrightarrow[n\to\infty]{} \varphi_{L_t}, \quad \text{if} \quad \varphi_{X^n_{p^n(t)}} \xrightarrow[n\to\infty]{} \varphi_{L_t},$$
$$\text{then} \quad \varphi_{Z_t^n} \xrightarrow[n\to\infty]{} 1.$$

4^{th} step. It is impossible that $\sigma^2 = 0$ and $v(\,]-\infty,0[\,) < \infty$ and $v(\,]0,\infty[\,) < \infty$, as

$$d - \textstyle\int_{[-1,1]} x\,\mathrm{d}v(x) > 0 \quad \text{would contradict condition (63), since } (ii) \text{ of 5.2},$$
$$d - \textstyle\int_{[-1,1]} x\,\mathrm{d}v(x) \leq 0 \quad \text{would contradict condition (63), since } (i) \text{ of 5.2}.$$

Therefore, at least one of the following statements holds true,

$$\sigma^2 > 0 \quad \text{or} \quad v(\,]-\infty,0[\,) = \infty \quad \text{or} \quad v(\,]0,\infty[\,) = \infty. \tag{75}$$

Sato [43, p.152, Theorem 24.10] described the support of infinite divisible distributions. Comparing his results with (75) and Assumption 6.2, the following holds true,

$$V_\tau(K) \neq V_\tau(0) \quad \text{for } K \neq 0 \text{ and } \tau > 0; \quad \text{implying (65).}$$

Let $K < 0$ and $\tau_n \downarrow 0$ with $((\lambda_K - \lambda_0)/\lambda_0)(\tau_n) \in \,]0,1[\,$. In particular, $\lambda_K/\lambda_0 \in \,]1,2[\,$. Furthermore, as $V_{\tau_n}(K) \to V_0(K) = 0$, we see that $\lambda_K \to$

$-\infty$. Putting these two things together,

$$\lim_{n\to\infty} \lambda_0(\tau_n) = -\infty, \quad \text{i.e.} \quad \lim_{n\to\infty} V_{\tau_n}(0) = 0. \tag{76}$$

Moreover, let $K < \bar{K} < 0$ be a point at which v is continuous. Condition (63) ensures that

$$\frac{V_\tau(K)}{V_\tau(0)} \le \frac{V_\tau(\bar{K})}{V_\tau(0)} = \frac{V_\tau(\bar{K})}{\tau}\,\frac{\tau}{V_\tau(0)} \underset{\tau\downarrow 0}{\longrightarrow} v(\,]-\infty,\bar{K}])\times 0, \quad \text{i.e.} \quad \frac{V_\tau(K)}{V_\tau(0)} \underset{\tau\downarrow 0}{\longrightarrow} 0.$$

Thus, for a fixed $\varepsilon > 0$, the inequality $V_{\pi_n}(K) \le \varepsilon V_{\pi_n}(0)$ holds true for all big enough n. Applying (iv) of Lemma 5.1 to the sequence (76) gives

$$\begin{aligned}
&\lambda_0(\lambda_K - \lambda_0) = (-\lambda_0)(\lambda_0 - \lambda_K) \\
&\ge \big(-\Phi^{-1}(V_{\tau_n}(0))\big)\big(\Phi^{-1}(V_{\tau_n}(0)) - \Phi^{-1}(\varepsilon\, V_{\tau_n}(0))\big) \\
&= \Phi^{-1}(V_{\tau_n}(0))\big(\Phi^{-1}(\varepsilon V_{\tau_n}(0)) - \Phi^{-1}(V_{\tau_n}(0))\big) \underset{n\to\infty}{\longrightarrow} -\ln\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} \infty; \qquad (77)
\end{aligned}$$

implying (66) for $K < 0$.

A similar argumentation yields (66) for $K > 0$.

5th step. We prove that (66) fails to hold if (65) holds and $\liminf_{\tau\downarrow 0} \dfrac{V_\tau(0)}{\tau} < \infty$. According to (ii) of 5.2, the underlying Lévy process is the sum of a strictly increasing subordinator and a negative compound Poisson process, and

$$\frac{V_\tau(0)}{\tau} \underset{\tau\downarrow 0}{\longrightarrow} v(\,]-\infty,0[\,) < \infty. \tag{78}$$

If $v(\,]-\infty,0[\,) = 0$, then the process would coincide with the strictly increasing subordinator, thus $V_\tau(0) = 0$ for $\tau > 0$. As this contradicts Assumption 6.2, we see instead that $v(\,]-\infty,0[\,) > 0$. Therefore, there is $K < 0$ at which v is continuous such that

$$\frac{V_\tau(K)}{\tau} \underset{\tau\downarrow 0}{\longrightarrow} v(\,]-\infty,K]) > 0. \tag{79}$$

In view of (78) and (79), there is $\delta > 0$ such that $V_\tau(K) > \delta\, V_\tau(0)$ for all small enough τ. Since (78), $V_\tau(0) \to 0$ for $\tau \downarrow 0$. Therefore, we can apply (ii) of Lemma 5.1 to $V_\tau(0)$,

$$\lambda_0(\lambda_K - \lambda_0) < \Phi^{-1}(V_\tau(0))\,\big(\Phi^{-1}(\delta\, V_\tau(0)) - \Phi^{-1}(V_\tau(0))\big)$$

$$\xrightarrow[\tau\downarrow 0]{} -\ln\delta < \infty. \tag{80}$$

As $V_\tau(0) \to 0$, we see that $\lambda_0 \to -\infty$. In view of (80), $\limsup_{\tau\downarrow 0}\lambda_0(\lambda_K - \lambda_0)/(\lambda_0^2) \le 0$, in particular

$$\Big(\frac{\lambda_K - \lambda_0}{\lambda_0}\Big)(\tau) < 1 \quad \text{for small enough } \tau > 0. \tag{81}$$

Furthermore, as $V_\tau(0) \to 0$, we see that $\lambda_0 < 0$. From (65), we see that $\lambda_K - \lambda_0 < 0$. Combining these inequalities yields

$$\Big(\frac{\lambda_K - \lambda_0}{\lambda_0}\Big)(\tau) > 0 \quad \text{for small enough } \tau > 0. \tag{82}$$

In view of (80),(81),(82), condition (66) fails to hold.

Similarly, (66) fails to hold if (65) holds and $\liminf_{\tau\downarrow 0}(1 - V_\tau(0))/\tau < \infty$. Here, according to (i) of 5.2, the underlying Lévy process is a positive compound Poisson process minus a subordinator. If $v(\,]0,\infty[\,) = 0$, then $V_\tau(0) = 1$ for all $\tau > 0$, which contradicts Assumption 6.2. Therefore, there is $K > 0$ such that $\limsup_{\tau\downarrow 0}\lambda_0(\lambda_K - \lambda_0) < \infty$. However, as $V_\tau(0) \to 1$, hence $\lambda_0 \to \infty$, and we can see that $(\lambda_K - \lambda_0)/\lambda_0 \in\,]0,1[\,$.

6^{th} step. If $\sigma^2 > 0$ or (ii) or (iii) in the statement of the theorem holds, then Lemma 5.2 implies directly that condition (63) holds. Let $\int_{[-1,1]}|x|\,\mathrm{d}v(x) = \infty$, where without loss of generality $\int_{[-1,0]}|x|\,\mathrm{d}v(x) = \infty$, thus $v]{-\infty},0[\, = \infty$. Then, the following holds true,

$$\begin{aligned}&(1 - V_\tau(0))/\tau \to \infty, \quad \text{since } d - \textstyle\int_{[-1,1]} x\,\mathrm{d}v(x) \not\le 0 \text{ and } (ii) \text{ of 5.2},\\ &V_\tau(0)/\tau \to \infty, \quad \text{since } v]{-\infty},0[\, = \infty \text{ and } (ii) \text{ of 5.2, for } \tau\downarrow 0.\end{aligned}$$

Hence, any of $(i),(ii),(iii)$ in the statement of the theorem is sufficient for condition (63).

We divide the triplet cases excluded by $(i),(ii),(iii)$ into three new cases $(iv),(v),(vi)$. Here we assume that $\sigma^2 = 0$ and $\int_{[-1,1]}|x|\,\mathrm{d}v(x) < \infty$.

$$(iv) \quad v(\,]{-\infty},0[\,) < \infty \quad \text{and} \quad v(\,]0,\infty[\,) < \infty.$$

If $d - \int_{[-1,1]} x\,\mathrm{d}v(x) > 0$, then (ii) of Lemma 5.2 implies $V_\tau(0)/\tau \not\to \infty$ for $\tau\downarrow 0$. Otherwise, $d - \int_{[-1,1]} x\,\mathrm{d}v(x) \le 0$ and (i) of Lemma 5.2 implies

$(1 - V_\tau(0))/\tau \nrightarrow \infty$ for $\tau \downarrow 0$.

$$(v) \quad v(\,]-\infty,0[\,) < \infty, \quad v(\,]0,\infty[\,) = \infty \quad \text{and} \quad d - \int_{[-1,1]} x \,\mathrm{d}v(x) \geq 0.$$

Here, $V_\tau(0)/\tau \nrightarrow \infty$, since (ii) of Lemma 5.2.

$$(vi) \quad v(\,]-\infty,0[\,) = \infty, \quad v]0,\infty[< \infty \quad \text{and} \quad d - \int_{[-1,1]} x \,\mathrm{d}v(x) \leq 0.$$

In this case, (i) of Lemma 5.2 implies $(1 - V_\tau(0))/\tau \nrightarrow \infty$ for $\tau \downarrow 0$. Hence, $(iv), (v), (vi)$ exclude condition (63), in particular one of $(i), (ii), (iii)$ in the statement of the theorem is necessary for condition (63). □

Having the proof in mind, we can justify the heuristic argumentation which was given in the introduction. Due to the oscillations of Brownian motion, we argued that V_t needs to be continuous and strictly increasing around zero for all $t > 0$. An example of a Lévy process with distribution functions fulfilling Assumption 6.2 but being discontinuous at zero is a

"compound Poisson process with state space $\mathbb{R}$".

Here, for any given $K > 0$, condition (66) fails to hold. For a fixed $t > 0$, we choose (p_i^n) accordingly to the *3rd step* of the above proof. Then the corresponding sequence (Z_t^n) resembles properties of the normal distribution. For K continuity point of v, the following holds true,

$$\lim_{n\to\infty} \mathbb{P}[Z_t^n > K] = \Phi\big(-2\sqrt{\log(v(\,]0,\infty[\,)/v(\,]K,\infty[\,))}\big) \xrightarrow[K\downarrow 0]{} \Phi(0) = 1/2.$$

Due to exponential decay, we argued that removing finite activity jumps will not change the nature of the convergence result. An example of a Lévy process with V_t continuous and strictly increasing for all $t > 0$ yet misses condition (63) is

"infinite activity subordinator + negative compound Poisson process".

After removing the compound Poisson process, the distribution functions are not strictly increasing around zero anymore. Looking at the *3rd case* of the proof of Lemma 5.2, finiteness and infiniteness of $\mathbb{P}[\,\cdot \leq 0]/t$ for $t \downarrow 0$ is an invariant for processes suitable to our analysis. More precisely, let L be the original process and $L^{+\infty}$ the subordinator, then

$$\mathbb{P}[L_t^{+\infty} \leq 0] \leq \mathbb{P}[L_t \leq 0] \leq (1 - \mathrm{e}^{-t\, v(]-\infty,0[)}) + \mathbb{P}[L_t^{+\infty} \leq 0]\, \mathrm{e}^{-t\, v(]-\infty,0[)}.$$

On the other hand, drifts behave differently. A drift affects all paths at all times, whereas finite activity jumps affect the paths randomly. An example of a Lévy process close to the previous one that fulfils condition (63) is

"infinite activity subordinator + strictly negative drift".

More precisely, the process fulfils (*iii*) of Theorem 7.1.

Turning away from specific examples, we now analyse the suitable generating triplets from a quantitative point of view. For instance, V_t and V_t^{-1} become more regular than Assumption 6.2 suggest. Since Sato [43], p.152, Theorem 24.10, any of $(i), (ii), (iii)$ of Theorem 7.1 imply that for $t > 0$, the following holds true,

- $x \mapsto V_t(x)$ is strictly increasing and continuous in $V_t^{-1}(\,]0,1[\,)$,
- $x \mapsto V_t^{-1}(x)$ is continuous and strictly increasing in $]0,1]$, (83)
- Assumption 6.2 holds for $F_i = V_{p_{i+1}-p_i}$.

Furthermore, the generating triplets in Theorem 7.1 are linked to canonical properties of Lévy processes: (*i*) holds when the Lévy process is of unbounded variation, (*ii*) holds when downward and upward jumps are of infinite activity, and (*iii*) is a technical case due to drift effects.

We conclude this section by listing Lévy processes that are used in practise and fulfil condition (63).

7.2 Example. *The assumptions of Theorem 7.1 are fulfilled by the distribution functions of the following processes,*

- *Lévy jump diffusions,*
- *generalized hyperbolic processes (GH, VGP, NIG),*
- *stable processes of infinite activity (TS, MTS, CGMY),*
- *generalized z-processes (Meixner processes).*

For more details, see Gaunt [17], Grigelionis [18], *and* Rosiński & Sinclair [39].

8. A note on tightness in the Skorokhod space

As in the previous section, we study sequences of Lévy constructions for a fixed Lévy process. To establish weak convergence, we assume converging finite-dimensional distributions, and look for conditions which ensure tightness.

At first, we show that any jump is incompatible with ucp convergence.

8.1 Lemma. *Let* (X^n) *be a sequence of Lévy constructions of* V *given* (p_i^n) *on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* V *and* (p_i^n) *fulfil condition* (63) *and* (62) *respectively. Considering* $Z_t^n = X_t^n - X_{p_i^n}^n$ *for* $p_i^n \leq t < p_{i+1}^n$ *and the underlying Lévy measure* v, *then the following holds true*

$$\liminf_{n\to\infty} \mathbb{P}\big[\sup_{t\leq T} |Z_t^n| > K\big] \geq 1 - \mathrm{e}^{-\frac{T}{2}v([-K,K]^c)} \quad \textit{for all } T, K > 0. \tag{84}$$

Proof. Let $\tau > 0$. For $K \in \mathbb{R}$, let $\lambda_K = \lambda_K(\tau) = \Phi^{-1}(V_\tau(K))$. For Brownian motion W, we define Z_s^τ as in (67).

Let $K < 0$. We are going to estimate $P_\tau(K) = \mathbb{P}[\inf_{s\leq\tau} Z_t^\tau \geq K]$. In view of (83) as well as Φ continuous and strictly increasing, the $\inf_{s\leq\tau}$ commutes with $V_\tau^{-1} \circ \Phi$. Using a formula from Jeanblanc et al. [22, pp.145-147], we can see that

$$\begin{aligned} P_\tau(K) &= \mathbb{P}\big[\inf_{s\leq\tau} \big(V_\tau^{-1}\circ\Phi\big)\big(\lambda_0\,(\tau-s)/\tau + W_s/\sqrt{\tau}\big) \geq K\big] \\ &= \mathbb{P}\big[\inf_{s\leq\tau} \big(W_s - \lambda_0 s/\sqrt{\tau}\big) \geq \sqrt{\tau}(\lambda_K - \lambda_0)\big] \\ &= 1 - V_\tau(K) - \mathrm{e}^{-2\lambda_0(\lambda_K-\lambda_0)}\Phi(\lambda_K - 2\lambda_0). \end{aligned} \tag{85}$$

For the next step, we assume that $V_\tau(K) > 0$. Here, $P_\tau(K) < 1$, in particular we can divide by $V_\tau(K)$ and $1 - P_\tau(K)$. As $V_\tau(K) \to 0$, the argument in (77) implies that $\lambda_0(\lambda_K - \lambda_0) \to \infty$, thus $P_\tau(K) \to 1$ for $\tau \downarrow 0$. This yields

$$\lim_{\tau\downarrow 0} \frac{\ln P_\tau(K) - \ln 1}{P_\tau(K) - 1} = (\ln x)'\big|_{x=1} = 1.$$

Having a second look at (85), we can see that

$$\frac{1 - P_\tau(K)}{V_\tau(K)} = 1 + \mathrm{e}^{-2\lambda_0(\lambda_K-\lambda_0)}\frac{\Phi(\lambda_K - 2\lambda_0)}{V_\tau(K)} \geq 1. \tag{86}$$

Moreover, a statement from BERTOIN [2, p.39, Exercise 1], implies that

$$\liminf_{\tau\downarrow 0} \frac{V_\tau(K)}{\tau} \geq v(\,]-\infty, K[\,). \tag{87}$$

Combining the above three statements, we see that

$$\begin{aligned}\liminf_{\tau\downarrow 0}\Big(-\frac{\ln P_\tau(K)}{\tau}\Big) &= \liminf_{\tau\downarrow 0} \frac{\ln P_\tau(K)-\ln 1}{P_\tau(K)-1}\,\frac{1-P_\tau(K)}{V_\tau(K)}\,\frac{V_\tau(K)}{\tau}\\ &\geq v(\,]-\infty, K[\,).\end{aligned} \tag{88}$$

Before proceeding further, let $V_\tau(K)=0$. Here, $P_\tau(K)=1$ and (87) gives $v(\,]-\infty,K[\,)=0$. Hence, inequality (88) is trivially fulfilled. In particular, (88) holds regardless of the values of $P_\tau(K)$.

Proceeding with our analysis, in view of (88), there is for any $\varepsilon>0$ some $\Gamma(\varepsilon)>0$ such that for $0<\tau<\Gamma(\varepsilon)$, we have

$$P_\tau(K) \leq \exp\big(\tau(\varepsilon - v(\,]-\infty, K[\,))\big).$$

In view of (62), we choose n such that $\sup_{i\in\mathbb{N}_0}|p^n_{i+1}-p^n_i|<\Gamma(\varepsilon)$. Let $(p^n_i)_{i=0}^{I_n}$ be the list of consecutive grid points up to time T and define $\tau^n_i = p^n_i - p^n_{i-1}$. Then

$$\begin{aligned}\mathbb{P}\big[\inf_{t\leq T} Z^n_t \geq K\big] &\leq \mathbb{P}\big[\inf_{t\leq p^n_{I_n}} Z^n_t \geq K\big] = \prod_{i=1}^{I_n}\mathbb{P}\big[\inf_{]p^n_{i-1},p^n_i]} Z^n_t \geq K\big] \overset{(68)}{=} \prod_{i=1}^{I_n} P_{\tau^n_i}(K)\\ &\leq \exp\Big((\varepsilon - v]-\infty,K[)\sum_{i=1}^{I_n}\tau^n_i\Big) = \exp\big(p^n_{I_n}(\varepsilon - v(\,]-\infty,K[\,))\big).\end{aligned}$$

After taking $n\to\infty$ followed by $\varepsilon\downarrow 0$, the following holds true,

$$\limsup_{n\to\infty}\mathbb{P}\big[\inf_{t\leq T} Z^n_t \geq K\big] \leq \exp\big(-T\,v(\,]-\infty,K[\,)\big).$$

Using similar arguments for $K>0$, we can see that

$$\limsup_{n\to\infty}\mathbb{P}\big[\sup_{t\leq T} Z^n_t \leq K\big] \leq \exp\big(-T\,v(\,]K,\infty[\,)\big).$$

We conclude the proof by combining the statements for $K<0$ and $K>0$

into the following estimate,

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}\big[\sup_{t\le T}|Z_t^n|\le K\big] &\le \exp\big(-T\,v(]-\infty,-K[)\big)\wedge\exp\big(-T\,v(]K,\infty[)\big)\\ &\le \exp\big(-\frac{T}{2}\,v([-K,K]^c)\big)\quad\text{for } K>0.\end{aligned}$$

□

Having the proof in mind, if (86) has a unique limit for $\tau\downarrow 0$, then (84) can be replaced with an equality instead of an estimate. For instance, in view of the *1st case* of Lemma 5.2, if the underlying Lévy process has a Brownian component, then

$$\lim_{\tau\downarrow 0}\frac{\log V_\tau(0)}{\log\tau}=0.$$

Using (52), and (*iii*) of Lemma 5.1, and Bertoin [2, p.39, Exercise 1], this yields

$$\lim_{\tau\downarrow 0}\big(\mathrm{e}^{-2\lambda_0(\lambda_K-\lambda_0)}\frac{\Phi(\lambda_K-2\lambda_0)}{V_\tau(K)}\big)=1.$$

If in addition K is a point at which v is continuous, then the following holds true,

$$\lim_{n\to\infty}\mathbb{P}\big[\inf_{t\le T}Z_t^n\le K\big]=1-\mathrm{e}^{-Tv(]-\infty,K[)}.$$

Under the same setting, for $K>0$ at which v is continuous, a similar argumentation yields

$$\lim_{n\to\infty}\mathbb{P}\big[\sup_{t\le T}Z_t^n> K\big]=1-\mathrm{e}^{-Tv(]K,\infty[)}.$$

Nonetheless, the estimate (84) is enough for our following main result of this section.

8.2 Theorem. *Consider a sequence of Lévy constructions* (X^n) *of* V *given* (p_i^n). *Let* V *and* (p_i^n) *fulfil condition* (63) *and* (62) *respectively. If the underlying Lévy process has jumps, then* (X^n) *fails to be tight.*

Proof. Let $T,K>0$ be fixed. Moreover, consider $Z_t^n=X_t^n-X_{p_i}^n$ for $p_i\le t<p_{i+1}$.

In view of the definition of tightness in the Skorokhod space, we are going to estimate the càdlàg modulus for X^n. Assuming that (X^n) is tight, we show that $\mathbb{P}[\sup_{t\le T}|Z_t^n|>K]\to 0$ for $n\to\infty$. This would contradict Lemma 8.1

and establish the result.

Let $\delta > 0$. In view of (62), we choose n large enough such that $\sup_{i\in\mathbb{N}_0} |p^n_{i+1} - p^n_i| \leq \delta$. Furthermore, let $U > 0$ such that $U - T > \delta$. Now, consider $(t_j)_{j=0}^J$ from $[0, U]$ with

$$0 = t_0 < t_{j_1} < t_{j_2} < t_J = U \quad \text{for } 0 < j_1 < j_2 < J, \quad \text{and} \qquad (89)$$
$$\min_{0\leq j\leq J} |t_{j+1} - t_j| > \delta.$$

Then any $[p^n_i, p^n_{i+1}[$ interval is covered by a union of at most two consecutive $[t_j, t_{j+1}[$ intervals unless it exceeds the point U. For $[p^n_i, p^n_{i+1}[\subseteq [t_j, t_{j+1}[$, we have that

$$\sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| \geq \sup_{t\in[p^n_i,p^n_{i+1}[} |X^n_t - X^n_{p^n_i}| = \sup_{t\in[p^n_i,p^n_{i+1}[} |Z^n_t|. \qquad (90)$$

In the other case when $p^n_i \in [t_j, t_{j+1}[$ and $p^n_{i+1} \in [t_{j+1}, t_{j+2}[$, we have

$$\begin{aligned}\sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| &= \sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| \vee \sup_{s,t\in[t_{j+1},t_{j+2}[} |X^n_t - X^n_s| \\ &\geq \sup_{t\in[p^n_i,t_{j+1}[} |X^n_t - X^n_{p^n_i}| \vee \sup_{t\in[t_{j+1},p^n_{i+1}[} |X^n_t - X^n_{t_j+1}| \\ &= \sup_{t\in[p^n_i,t_{j+1}[} |Z^n_t| \vee \sup_{t\in[t_{j+1},p^n_{i+1}[} |Z^n_t - Z^n_{t_j+1}| \\ &\geq \sup_{t\in[p^n_i,t_{j+1}[} |Z^n_t| \vee \sup_{t\in[t_{j+1},p^n_{i+1}[} |Z^n_t| - |Z^n_{t_j+1}|. \end{aligned} \qquad (91)$$

Let $a, b, c \in \mathbb{R}$ such that $a \leq b$ then $b \vee (c - a) \geq c/2$, in particular $b \vee (c - a) \geq b \vee c/2 \geq (b \vee c)/2$. As Z^n is continuous in $[p^n_i, t_{j+1}[$, we have $|Z^n_{t_j+1}| \leq \sup_{t\in[p^n_i,t_{j+1}[} |Z^n_t|$. In view of (91), these two observations imply

$$\sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| \geq \frac{1}{2} \sup_{t\in[p^n_i,p^n_{i+1}[} |Z^n_t|. \qquad (92)$$

Let $p^n(U)$ be the grid point that was defined in (64). As $U - T > \delta$, we can see that $U \geq p^n(U) > T$. In view of (90) and (92), when taking the maximum over i and j, we get

$$\max_{0\leq j\leq J} \sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| \geq \frac{1}{2} \sup_{t\leq p^n(U)} |Z^n_t| \geq \frac{1}{2} \sup_{t\leq T} |Z^n_t|.$$

Let ω denote the càdlàg modulus as defined in Billingsley [4, pp.121-122].

Moreover, we denote by $\Pi_U(\delta)$ the set of all partitions like (89) for given δ, U but variable J. Then the following holds true,

$$\omega_U(\delta, X^n) = \inf_{\Pi_U(\delta)} \max_{0\le j\le J} \sup_{s,t\in[t_j,t_{j+1}[} |X^n_t - X^n_s| \ge \frac{1}{2}\sup_{t\le T}|Z^n_t|.$$

If (X^n) is tight, then we would have

$$0 = \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}[\,\omega_U(\delta, X^n) > K] \ge \limsup_{n\to\infty} \mathbb{P}\big[\,\frac{1}{2}\sup_{t\le T}|Z^n_t| > K\big].$$

□

9. Existence of local martingale measures

In this section, we study exponentials of Lévy-Itô semi-diffusions. We look for conditions such that the discounted exponentials admit local martingale measures.

For SDE's, this is a well-known problem. The following Lemma recapitulates the canonical link between stochastic exponentials and local martingale measures in our context.

9.1 Lemma. *Let X be a Lévy-Itô construction of V given (p_i) and W on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $r, T \ge 0$. Let μ_i, σ_i be defined as in* (61) *with $F_i = V_{p_{i+1}-p_i}$. Moreover, consider*

$$\hat\sigma_i = \frac{r - \mu_i - \sigma_i^2/2}{\sigma_i}. \tag{93}$$

Then, there is $\mathbb{Q} \approx \mathbb{P}$ such that $t \mapsto \exp(X_t - rt)$ is a $\mathbb{Q}$-local-martingale in $[0,T]$ if and only if, for $Z_t = X_t - X_{p_i}$ with $p_i \le t < p_{i+1}$, the following holds true,

$$\mathcal{E}\Big(\int_{p_i} \hat\sigma_i(Z)\,\mathrm{d}W\Big) \quad \textit{is a } \mathbb{P}\textit{-martingale in } [p_i, p_{i+1}\wedge T]. \tag{94}$$

Proof. Let $(\mathcal{F}^W_t)$ be the completed natural filtration of W.

At first, we consider the case when $\mathbb{Q}$ exists. Here, let $Y_t = \mathbb{E}[\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}|\,\mathcal{F}^W_t]$ for $t \in [0,T]$. As $t \mapsto \exp(X_t - rt)$ is adapted to the Brownian filtration, the tower property of conditional expectation ensures that

$$t \mapsto \mathrm{e}^{X_t - rt} Y_t \quad \text{is a } \mathbb{P}\text{-martingale with respect to } (\mathcal{F}^W_t) \tag{95}$$

According to Revuz & Yor [36, p.200, Theorem V.3.4], there exists a φ such that

$$\mathrm{d}Y_t = \varphi_t \mathrm{d}W_t.$$

Using Itô's formula together with the dynamical description (61) of X, we can see that

$$\begin{aligned} &\mathrm{d}\big(\mathrm{e}^{X_t - rt} Y_t\big) - \mathrm{e}^{X_t - rt}\mathrm{d}Y_t - \mathrm{e}^{X_t - rt} Y_t\, \sigma_i(Z_t)\mathrm{d}W_t \\ &= \mathrm{e}^{X_t - rt}\big(Y_t\big(\mu_i(Z_t) - r + \sigma_i^2(Z_t)/2\big) + \varphi_t\, \sigma_i(Z_t)\big)\mathrm{d}t \quad \text{for } p_i \le t < p_{i+1} \wedge T. \end{aligned}$$

By construction and (95), the left side of the equation describes a continuous local martingale with respect to $(\mathcal{F}_t^W)$. On the other hand, the right side of the equation describes a bounded variation process. Therefore, both processes are constant, in particular

$$Y_t\big(\mu_i(Z_t) - r + \sigma_i^2(Z_t)/2\big) + \varphi_t\, \sigma_i(Z_t) = 0 \quad (\mathrm{d}t \otimes L^2)\text{-as.}$$

In view of Assumption 6.3 and formula (61), we have $\sigma_i > 0$, hence

$$\mathrm{d}Y_t = Y_t\, \hat{\sigma}_i(Z_t)\mathrm{d}W_t. \tag{96}$$

As Y is the density process of an equivalent measure, $Y > 0$ and $Y_0 = 1$. There is at most one strictly positive solution to the SDE (96) with initial condition $Y_0 = 1$. This follows from $\mathrm{d}(Y_t^1/Y_t^2) = 0$ for two such solutions Y^1, Y^2. Moreover, the solution to (96) with $Y_0 = 1$ is given by

$$Y_t = Y_{p_i} \mathcal{E}_t\Big(\int_{p_i} \hat{\sigma}_i(Z)\,\mathrm{d}W\Big) \quad \text{for } p_i \le t < p_{i+1} \wedge T. \tag{97}$$

As Y is continuous and strictly positive, we can replace $t < p_{i+1} \wedge T$ with $t \le p_{i+1} \wedge T$. Using the martingale property of Y together with the independence between grid points, we conclude that

$$\mathbb{E}\Big[\mathcal{E}_t\Big(\int_{p_i} \hat{\sigma}_i(Z)\,\mathrm{d}W\Big)\Big] = 1 \quad \text{for } p_i \le t \le p_{i+1} \wedge T. \tag{98}$$

As $\mathcal{E}(\int_{p_i} \hat{\sigma}_i(Z)\,\mathrm{d}W)$ is the stochastic exponential of a local-martingale with respect to the filtration $(\mathcal{F}_t)$, it is a supermartingale for the same filtration. Thus, (98) implies (94).

Now we consider the case when condition (94) holds. Here, we define Y inductively via (97). By assumption and independence between grid points,

Y is a martingale. Moreover, as a stochastic exponential, Y fulfils (96). In view of the Girsanov theorem, $Y_T = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ defines a probability measure $\mathbb{Q}$ as claimed. $\square$

9.2 Remark. *The probability measure $\mathbb{Q}$ is uniquely defined on the sigma algebra $\mathcal{F}_T^W$ that is generated by the Brownian motion W that is used in the definition of X.*

As we work with homogeneous diffusions, we can apply results by Mijatović & Urusov [31]. In the following, we summarise the assumptions and the statement of their Corollary 2.2 in our context. Assumption 6.3 implies that $\sigma_i^2 > 0$ and $1/\sigma_i^2$, μ_i/σ_i^2 are locally integrable. In particular, Z is the unique in law solution of (61) and does not exit its state space since (60). Moreover, as Z is a càdlàg process and $\hat{\sigma}_i^2$ locally integrable, the stochastic exponential (94) is well-defined and never hits zero. Furthermore, as $\hat{\sigma}_i^2/\sigma_i^2$ is locally integrable, the following SDE permits a unique in law weak solution,

$$\mathrm{d}\bar{Z}_s = (\mu_i + \hat{\sigma}_i\sigma_i)(\bar{Z}_s)\,\mathrm{d}s + \sigma_i(\bar{Z}_s)\,\mathrm{d}W_s \quad \text{with } \bar{Z}_{p_i} = 0 \text{ for } p_i \le t < p_{i+1}. \tag{99}$$

Now, Mijatović & Urusov [31, p.7, Corollary 2.2] showed that (94) is a martingale if and only if $\bar{Z}$ does not exit its state space. This can be expressed in terms of Feller's test for explosion. For $c, x \in \mathbb{R}$ and the convention $\int_c^x = -\int_x^c$, we define

$$h(x) = \exp\big(-2\int_c^x \frac{\mu_i + \hat{\sigma}_i\sigma_i}{\sigma_i^2}(z)\,\mathrm{d}z\big), \quad v(x) = \int_c^x \int_z^x \frac{h(y)}{h(z)\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z. \tag{100}$$

Then, according to Mijatović & Urusov [31, p.5, equation (20)],

$$\text{(94) is a martingale if and only if } v(\pm\infty) = \infty. \tag{101}$$

Having this characterisation in mind, we are able to prove now our main result of this section.

9.3 Theorem. *Let X be a Lévy-Itô semi-diffusion of V given (p_i) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then, there is for all $T \ge 0$ a $\mathbb{Q} \approx \mathbb{P}$ such that $t \mapsto \exp(X_t)$ is a $\mathbb{Q}$-local-martingale in $[0,T]$ if and only if, for some $c \in \mathbb{R}$ and all $p_i < T$, the* *following holds true,*

$$\int_{-\infty}^{c} ((\ln 1/V_\tau)'(\ln\ln 1/V_\tau)')(z)\,\mathrm{d}z = \infty \quad \textit{for all } \tau = p_{i+1} - p_i. \tag{102}$$

In this case, there is for all $r, T \geq 0$ *a* $\mathbb{Q}^r \approx \mathbb{P}$ *such that* $t \mapsto \exp(X_t - rt)$ *is a* $\mathbb{Q}^r$*-local-martingale in* $[0, T]$.

Proof. Let $c \in \mathbb{R}$, then consider the following condition,

$$\int_{-\infty}^{c} \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z = \infty. \tag{103}$$

In view of Lemma 9.1, and (100), (101), we split the proof into the following three steps,

- *1st step:* $v(\infty) = \infty$,
- *2nd step:* $v(-\infty) = \infty$ is equivalent to condition (103),
- *3rd step:* condition (103) is equivalent to condition (102).

1st step. In view of (93) and (99), we see that the drift coefficient of $\bar{Z}$ is bounded above. More precisely, there is a constant $b > 0$ and a function $z \mapsto \eta(z) \geq 0$ such that

$$\mu_i + \hat{\sigma}_i\sigma_i = r - \sigma_i^2/2 = b - \eta. \tag{104}$$

For $x \geq c$, we introduce the following function

$$f(x) = \exp\big(2b\int_c^x \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z\big).$$

As f is increasing, the following holds true,

$$\begin{aligned}
v(x) &= \int_c^x\int_z^x \frac{h(y)}{h(z)\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z \\
&= (2b)^{-1}\int_c^x\int_z^x \frac{f'(z)}{f(y)}\exp\big(2\int_z^y \frac{\eta_i(u)}{\sigma_i^2(u)}\,\mathrm{d}u\big)\,\mathrm{d}y\,\mathrm{d}z \\
&\geq (2b)^{-1}\int_c^x\int_z^x f'(z)/f(y)\,\mathrm{d}y\,\mathrm{d}z, \quad \text{since } f'(z)/f(y) \geq 0 \text{ and } \eta_i \geq 0, \\
&= (2b)^{-1}\int_c^x 1 - f(c)/f(y)\,\mathrm{d}y \\
&\geq (2b)^{-1}\int_{c+\varepsilon}^x 1 - f(c)/f(c+\varepsilon)\,\mathrm{d}y \quad \text{for } x - c > \varepsilon > 0,
\end{aligned}$$

$$= (2b)^{-1}(1 - f(c)/f(c+\varepsilon))(x - (c+\varepsilon)) \xrightarrow[x\to\infty]{} \infty.$$

2nd step. At first, we consider the case when $v(-\infty) = \infty$. Here, we look at $r = 0$ only. We see that

$$\infty \xleftarrow[x\to-\infty]{} v(x) = \int_c^x \int_z^x \frac{h(y)}{h(z)\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z = \int_x^c \int_x^z \frac{\mathrm{e}^{-(z-y)}}{\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_x^c \frac{1-\mathrm{e}^{-(z-x)}}{\sigma_i^2(z)}\,\mathrm{d}z \le \int_x^c \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z.$$

Now, we consider the case when condition (103) holds. Here, we consider any $r \ge 0$. For $x \le c$, the following holds true,

$$v(x) = \int_c^x \int_z^x \frac{h(y)}{h(z)\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z = \int_x^c \int_x^z \frac{\mathrm{e}^{-(z-y)}}{\sigma_i^2(z)}\,\exp\big(2r\int_y^z \frac{1}{\sigma_i^2(z)}\,\mathrm{d}u\big)\,\mathrm{d}y\,\mathrm{d}z$$
$$\ge \int_x^c \int_x^z \frac{\mathrm{e}^{-(z-y)}}{\sigma_i^2(z)}\,\mathrm{d}y\,\mathrm{d}z = \int_x^c \frac{1-\mathrm{e}^{-(z-x)}}{\sigma_i^2(z)}\,\mathrm{d}z \;\ge\; \big(1-\mathrm{e}^{x/2}\big)\int_{x/2}^c \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z$$
$$\xrightarrow[x\to-\infty]{} \int_{-\infty}^c \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z \overset{(103)}{=} \infty.$$

3rd step. We consider β, Ψ from Lemma 5.1. In view of (iii) of Lemma 5.1, using little-o notation, the following holds true,

$$\int_{-\infty}^c \frac{1}{\sigma_i^2(z)}\,\mathrm{d}z = \tau \int_{-\infty}^c \big[\frac{V_\tau'(z)}{\Phi'(\Phi^{-1}(V_\tau(z)))}\big]^2\,\mathrm{d}z$$
$$= \tau \int_{-\infty}^c \big[\frac{V_\tau'(z)}{\Phi'(\Phi^{-1}(V_\tau(z)))}\,\frac{\Psi(V_\tau(z))}{\Psi(V_\tau(z))}\,\frac{\Phi'(\Phi^{-1}(V_\tau(z))) - \beta(V_\tau(z))}{V_\tau(z)|\Phi^{-1}(V_\tau(z))|}\big]^2\,\mathrm{d}z$$
$$= \tau \int_{-\infty}^c \big[\frac{V_\tau'(z)}{\Psi(V_\tau(z))V_\tau(z)}\,\frac{\Psi(V_\tau(z))}{|\Phi^{-1}(V_\tau(z))|}\,\frac{\Phi'(\Phi^{-1}(V_\tau(z)) - \beta(V_\tau(z))}{\Phi'(\Phi^{-1}(V_\tau(z)))}\big]^2\,\mathrm{d}z$$
$$= \tau \int_{-\infty}^c \frac{(V_\tau'(z))^2}{-2\ln V_\tau(z)\,(V_\tau(z))^2}\,\big[\frac{\Psi(V_\tau(z))}{|\Phi^{-1}(V_\tau(z))|}\,\frac{\Phi'(\Phi^{-1}(V_\tau(z)) - \beta(V_\tau(z))}{\Phi'(\Phi^{-1}(V_\tau(z)))}\big]^2\,\mathrm{d}z$$
$$= \frac{\tau}{2}\int_{-\infty}^c (\ln 1/V_\tau(z))'(\ln\ln 1/V_\tau(z))'\,\big[(\mathrm{o}(1)-1)\cdot(1-\mathrm{o}(1))\big]^2\,\mathrm{d}z.$$

□

An inspection of the last proof gives rise to the following simple results.

9.4 Remark. *Consider the homogeneous diffusion* $\mathrm{d}Z_t = \mu(Z_t)\mathrm{d}t + \sigma(Z_t)\mathrm{d}W_t$ *such that* $\sigma^2 > 0$ *and* $1/\sigma^2$, μ/σ^2 *are locally integrable. Then, the following statements are true,*

- *if μ is bounded above, then Z will not exit at ∞,*

- *if μ^2/σ^2, σ^2 are locally integrable and Z does not exit its state space, then $t \mapsto \exp(Z_t)$ admits a local martingale measure on $[0,T]$ if and only if*

$$\int_{-\infty}^{c} \frac{1}{\sigma^2(z)}\,\mathrm{d}z = \infty \quad \textit{for some constant } c \in \mathbb{R}.$$

Looking at condition (102), we observe that $(\ln 1/V_\tau)'$ and $(\ln\ln 1/V_\tau)'$ integrates to ∞. Therefore, condition (102) will hold if one of them is asymptotically bounded away from zero. Turning our attention to tail asymptotics, we recall that V_τ is said to have a lower semi-heavy tail, if there are $a(\tau) \in \mathbb{R}$ and $b(\tau), C(\tau) > 0$ with

$$V_\tau'(z) \sim C(\tau)\,|z|^{a(\tau)} \exp(b(\tau)z) \quad \text{for } z \to -\infty. \tag{105}$$

Moreover, V_τ is said to have a lower fat tail, if there are $a(\tau) < 1$ and $C(\tau) > 0$ with

$$V_\tau'(z) \sim C(\tau)\,|z|^{a(\tau)} \quad \text{for } z \to -\infty. \tag{106}$$

9.5 Proposition (sufficient condition). *Condition* (102) *is satisfied if V_τ has a lower semi-heavy tail for all $\tau > 0$. On the other hand, condition* (102) *fails to hold if V_τ has a lower fat tail for all $\tau > 0$.*

Proof. First, we consider the case when V_τ has a lower semi-heavy tail as given in (105). In particular, for $z \downarrow -\infty$, we can see that the following holds true,

$$\begin{aligned}\frac{V_\tau(z)}{V_\tau'(z)} \sim \int_{-\infty}^{z} \frac{|y|^{a(\tau)}\mathrm{e}^{b(\tau)y}}{|z|^{a(\tau)}\mathrm{e}^{b(\tau)z}}\,\mathrm{d}y &= \int_{-\infty}^{0} (1 + x/z)^{a(\tau)}\mathrm{e}^{-b(\tau)x}\,\mathrm{d}x \\ &\xrightarrow[z\to-\infty]{} \int_{-\infty}^{0} \mathrm{e}^{-b(\tau)x}\,\mathrm{d}x = 1/b(\tau).\end{aligned}$$

Let $0 < \gamma < b(\tau)$. Then, we can assume without loss of generality that $\gamma < V_\tau'(z)/V_\tau(z)$ for all $z \le c$. Hence,

$$\int_{-\infty}^{c} (\ln 1/V_\tau)'(z)\,(\ln\ln 1/V_\tau)'(z)\,\mathrm{d}z \ge \gamma \int_{-\infty}^{c} -(\ln\ln 1/V_\tau)'(z)\,\mathrm{d}z = \infty.$$

Now, we consider the case when V_τ has a lower fat tail as given in (106).

In particular,

$$\frac{V_\tau(z)}{V_\tau'(z)} \sim \int_{-\infty}^{0} (1+x/z)^{a(\tau)}\,\mathrm{d}x = \frac{z}{1+a(\tau)}$$

$$\text{i.e.}\quad V_\tau'(z)/V_\tau(z) \sim (1+a(\tau))/z \quad \text{for } z \to -\infty.$$

Without loss of generality we can assume that $c < 0$ and for some $\varepsilon > 0$ we can assume that $V_\tau'(z)/V_\tau(z) \leq (1+\varepsilon)(1+a(\tau))/z$ for all $z \leq c$. Hence,

$$\int_{-\infty}^{c} (\ln 1/V_\tau)'(z)\,(\ln\ln 1/V_\tau)'(z)\,\mathrm{d}z = \int_{-\infty}^{c} \frac{(V_\tau'(z)/V_\tau(z))^2}{\ln 1/V_\tau(z)}\,\mathrm{d}z$$
$$\leq \int_{-\infty}^{c} (V_\tau'(z)/V_\tau(z))^2\,\mathrm{d}z = (1+\varepsilon)^2 \int_{-\infty}^{c} ((1+a(\tau))/z)^2\,\mathrm{d}(z) < \infty.$$

□

9.6 Corollary (existence of local martingale measures). *Let X be a Lévy-Itô semi-diffusion of V on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. If V_τ has a lower semi-heavy tail for all $\tau > 0$, then*

$$\forall\, r, T \geq 0 \ \ \exists\, \mathbb{Q} \approx \mathbb{P}: \ \ t \mapsto \exp(X_t)/\exp(rt) \ \ \textit{is a } \mathbb{Q}\textit{-local-martingale in } [0,T].$$

Proof. Application of Theorem 9.3 together with Proposition 9.5. □

We conclude this section by listing Lévy processes that are used in practise with semi-heavy tails. In particular, all their Lévy-Itô semi-diffusions admit local martingale measures.

9.7 Example. Assumption 6.3 *and condition* (102) *are fulfilled by the distribution functions of the following processes,*

- *generalized hyperbolic processes (GH, VGP, NIG),*
- *tempered stable processes (TS, MTS, CGMY),*
- *generalized z-processes (Meixner processes).*

For more details, see Gaunt [17], Grigelionis [18], *and* Rosiński & Sinclair [39].

On the other hand, the Cauchy process is a Lévy process with smooth distribution function having lower fat tails. In particular, all its Lévy-Itô constructions fail to have local martingale measures for $r = 0$.

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