CLEARING MODELS FOR SYSTEMIC RISK ASSESSMENT IN INTERBANK NETWORKS

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is my own work save that parts of Chapter 1 and Chapter 2 have been submitted to SSRN as a joint paper with Dr. Luitgard A. M. Veraart (Kusnetsov, Michael and Veraart, Luitgard A. M., “Interbank Clearing in Financial Networks with Multiple Maturities” (April 12, 2018). Available at SSRN: https://ssrn.com/abstract=3161571).

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Abstract

In this thesis I consider the problem of clearing models used for systemic risk assessment in interbank networks. I investigate two extensions of the classical Eisenberg & Noe (2001) model.

The first extension permits the analysis of networks with interbank liabilities of several maturities. I describe a clearing mechanism that relies on a fixed-point formulation of the vector of each bank's liquid assets at each maturity date for a given set of defaulted banks. This formulation is consistent with the main stylised principles of insolvency law, permits the construction of simple dynamic models and furthermore demonstrates that systemic risk can be underestimated by single maturity models.

In the context of multiple maturities, specifying a set of defaulted banks is challenging. Two approaches to overcome this challenge are proposed. The algorithmic approach leads to a well-defined liquid asset vector for all financial networks with multiple maturities. The simpler functional approach leads to the definition of the liquid asset vector that need not exist but under a regularity condition does exist and coincides with the algorithmic approach.

The second extension concerns the non-uniqueness of clearing solutions. When more than one solution exists, the standard approach is to select the greatest solution. I argue that there are circumstances when finding the least solution is desirable. An algorithm for constructing the least solution is proposed. Moreover, the solution is obtainable under an arbitrary lower bound constraint.

In models incorporating default costs, clearing functions can be discontinuous, which renders the problem of constructing the least clearing solution non-trivial. I describe the properties of the construction algorithm by means of transfinite sequences and show that it always terminates. Unlike the construction of the greatest solution, the number of steps taken by the algorithm need not be bounded by the size of the network.

Impact Statement

Since the financial crisis of 2007-8, the central banks and financial regulators in major financial centres throughout the world have been attempting to understand the failure of financial institutions in interconnected systems. It is now well-established that stability of financial systems is a complex dynamic phenomenon that is susceptible to disruption through multiple channels of systemic risk contagion. The precise nature of these sources of systemic risk, their transmission and interactions remain a live area of research. The overarching impact of any significant contribution in this domain is the enabling of regulators to make informed policy decisions to increase financial stability.

One of the principal tools currently used by the central banks in monitoring systemic risk are the periodic stress tests. However, their existing models for system-wide dynamics, including feedback, amplification and spillover mechanisms remain, in some respects, rudimentary. The current models do not use the full wealth of data available to the regulators such as the full maturity breakdown of financial instruments in the system. A key impact stemming from this thesis is the ability to stress test financial systems taking account of the maturity profile of the banks' portfolios. One of the findings in this thesis is that ignoring this information can lead a regulator to underestimate the extent of the systemic risk.

Another weakness of the current stress testing regime is that the exercise is fundamentally static in nature. It does not account for the financial system's ability to respond to the spread of instability. This ignores the impact of the participants' actions that can mitigate or exacerbate the risk. Tackling this weakness requires the development of dynamic models of systemic risk. The ability to account for multiple maturities, introduced in this thesis, provides a basis for such a model.

A recent development in the area has put more focus on the notion of distress, as distinct from default, of financial institutions. The ability to respond to a developing financial crisis when institutions become distressed but before they fail is of clear practical impact on the real economy. Distress models naturally lead themselves to multiple equilibrium states, reflecting the subjective perceptions of the market participants. This differentiates these models from the classical models of default that focus on obtaining unique solutions. The findings in this thesis on least clearing solutions are therefore of benefit to this developing strand of research and thus can have practical impact.

Impact of this thesis has already been obtained by disseminating significant parts of the research contained therein. In particular, some material has been submitted for publication to a major peer-reviewed journal, presented at major international conferences as well as presented at an internal seminar at the Bank of England.

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Thesis Overview

1.1 Motivation

Since the financial crisis of 2007-8 there has been a rapid expansion of literature which aims to explain bank failure in interconnected financial systems (see e.g. Glasserman & Young (2016) for a recent overview). One main modelling goal is to find a suitable contagion mechanism that describes how losses can spread through a financial network. The ultimate objective of such an analysis is to assess the degree of *systemic* risk in a financial network and use this to make informed policy decisions to increase financial stability.

One approach to assess systemic risk in financial networks is to derive *clearing* cash flows between financial institutions and to study which market participants default during the clearing process. Such clearing payments represent the actual payments made by the market participants and are constructed such that they obey certain stylised principles of contract and insolvency law.

A major line of literature adopting this approach follows from and extends the seminal model of Eisenberg & Noe (2001). In Section 1.3 we will provide a brief overview of the major milestones but it is worth emphasising at this stage Rogers & Veraart (2013). This thesis is firmly positioned in this family of models and, in particular, focuses on two extensions of Eisenberg & Noe (2001) and Rogers & Veraart (2013). The next section discusses these extensions in general terms and summarises the key contributions. Section 1.4 then provides an overview of both explicit and implicit assumptions made in Eisenberg & Noe (2001) and Rogers & Veraart (2013) which are relaxed in this thesis.

1.2 Contributions

In Chapter 2 we propose an extension of the Eisenberg & Noe (2001) model of financial networks with only one maturity date to networks with multiple maturity dates. In practice, financial networks do consist of liabilities with different maturity dates. When the clearing process is triggered at the first maturity date long-term debt must not be ignored. We develop clearing mechanisms that account for long-term debt in a way that is consistent with the main principles of insolvency law. This approach is also extended to a multi-period model that can be used as a basis for a full dynamic model of systemic risk. The clearing problem in Chapter 2 is formulated in terms of liquid assets.

In Chapter 3 we propose an extension of the Rogers & Veraart (2013) model, itself a generalisation of the Eisenberg & Noe (2001) model. In contrast to Chapter 2, the clearing problem in Chapter 3 is formulated in terms of payments for ease of comparison with these models. In Rogers & Veraart (2013) it was shown that, in the presence of default costs, clearing payments are not necessarily unique. Similar observations have been made in other related extensions and in such cases it has been conventional to select the greatest clearing payment vector as the preferred solution of the clearing problem. The existence of the least clearing vector is well known in these models but constructing it has been generally assumed to be essentially similar to the greatest clearing vector. In Chapter 3 we show that the problem is not trivial and provide a solution. Moreover, the model that we consider is a generalisation of the Rogers & Veraart (2013) model as we allow for a lower bound constraint which is not in general equal to zero as has been the case in the literature to date.

These two extensions contain seven main contributions to the research into interbank clearing models. Chapter 2 pertains to the first four and Chapter 3 pertains to the remaining three. Chapter 4 concludes by discussing the outlook for further research.

First, in Section 2.2 we introduce the notion of an equilibrium, in terms of liquid assets, achieved by clearing the financial markets at the first maturity date and accounting for long-term liabilities which are due beyond the first maturity date. We also show that that in contrast to the single maturity setting, developing a notion of default in a multiple maturity setting is challenging. A key insight that emerges out of this observation is that characterising the set of banks in default is an integral part of the solution to the clearing problem. This is in contrast to much of the literature where default sets are treated as secondary quantities derived from the clearing cash flows. In particular, we show in Lemma 26 and Remark 30 that under a mild assumption financial systems have at most a finite number of clearing solutions each uniquely determined by a corresponding default set.

Second, in Section 2.3, we introduce two possible approaches to clearing at the first maturity date. We show that these two approaches — algorithmic (Definition 9) and functional (Definition 11) — solve the general equilibrium problem for liquid assets in Propositions 10 and 13. In Section 2.3.3 we describe how the algorithmic approach extends the functional approach, which in turn extends the Eisenberg & Noe (2001) model. Construction of clearing solutions under both approaches is addressed in Section 2.3.4.

Third, we show that the functional approach, used in much of the literature in a single maturity setting, is problematic in a multiple maturity setting. In particular, we elucidate the importance of monotonicity in clearing problems. In general, under the functional approach, the clearing function is not monotonic and may not have a fixed point solution. Nevertheless, we show in Section 2.3.2 that a simple condition, the Monotonicity Condition 14, is sufficient to ensure the existence of a solution.

The fourth contribution is to highlight some applications of the algorithmic approach. In Section 2.3.5

we apply the algorithmic approach to demonstrate how single maturity models can underestimate systemic risk. In Section 2.2.4, we discuss the evolution of the financial system after clearing at the first maturity. In particular, in Section 2.2.4.3, we describe a simple multi-period extensions of our model. Such an extension then captures both the multi-maturity and multi-period aspects and therefore is a basis for a full dynamic model of financial systems.

The fifth contribution is to demonstrate in Section 3.3 that obtaining the least clearing vector of payments in a finite number of steps is not a simple converse of obtaining the greatest clearing vector. In the presence of default costs, as in Rogers & Veraart (2013), the clearing function is not continuous from below. Construction of the least clearing vector of payments then involves the use of transfinite sequences. This is an obstacle that needs to be overcome if we are to achieve this construction in a finite number of steps. We also investigate the effect of imposing an arbitrary lower bound on the clearing vectors, which imposes further complexity when constructing the least clearing vector of payments.

The sixth contribution is to address these difficulties by providing Algorithm 3 in Section 3.4 for computing the least clearing vector in an extension of the Rogers & Veraart (2013) model with arbitrary lower bounds on the clearing vectors.

Finally, we highlight the interpretation-neutral formulation of this extension of the Rogers & Veraart (2013) model. This paves the way to applying interbank clearing models to networks of funding supply as proposed in Hurd (2016). A simple case study using this approach is considered in Section 3.5.2.

1.3 Literature review

The role of complexity and contagion in financial networks has been studied by numerous authors, e.g. Allen & Gale (2000), Gai et al. (2011), Battiston et al. (2012) and David & Lehar (2017). There has been an increasing recognition that there are in fact multiple channels through which network complexity can give rise to systemic risk. Bisias et al. (2012), for example, provide a wide-ranging overview.

In most studies it is assumed that the financial network itself is observable. We will also make this assumption here. Under incomplete information network reconstruction methods could be applied first, see e.g. the Bayesian approach proposed by Gandy & Veraart (2018, 2017) and the references therein.

We focus on one specific channel of contagion, namely the domino effect which arises when complex networks of debt obligations are cleared. This places our work at the intersection of two strands of literature. The first focuses on contagion and domino effects, e.g. Eisenberg & Noe (2001), Cifuentes et al. (2005), Upper (2011), Liu et al. (2012), Elsinger et al. (2013), Cont et al. (2013), Georg (2013) and Elliott et al. (2014). The second investigates clearing, typically in the context of central counterparty clearing in OTC markets. Some contributions from this latter strand include Cont & Kokholm (2014), Duffie et al. (2015), Capponi et al. (2015) and Amini et al. (2015).

This thesis described several extensions of the model proposed by Eisenberg & Noe (2001). While the model in Eisenberg & Noe (2001) was concerned primarily with payment systems, the key ideas have been adapted by numerous authors to model systemic risk in a financial system. The key findings include the existence and construction of clearing solutions and the conditions for their uniqueness. These results rely on a number of simplifying assumptions on clearing, which subsequent authors have attempted to relax. Thus Hurd (2016) clarifies the effect that the external liabilities play, Rogers & Veraart (2013) investigate the effect of liquidation costs, while Elsinger (2011) incorporates cross-holdings and different seniorities of debt. The combined effect of cross-holdings and bankruptcy costs is investigated in Weber & Weske (2017). All these extensions are single period models and hence assume a single maturity for the liabilities.

Recent papers (e.g. Capponi & Chen (2015), Ferrara et al. (2016), Banerjee et al. (2018)) have developed multi-period models. The model in Capponi & Chen (2015) has a "central bank" node and random interbank liabilities. In particular, it highlights the distinction between illiquid and insolvent banks which arises whenever liabilities can become due at different times. This model focuses on the role of liquidity injection policies by the central bank and only tangentially analyses the differences in the default behaviour that arises from this generalisation. Meanwhile, Ferrara et al. (2016) describe how a multi-period system can be cleared simultaneously for every period. Similarly, Banerjee et al. (2018) consider both a discrete and a continuous-time dynamic extension of the Eisenberg & Noe (2001) model. While these models generalise the single period aspect of Eisenberg & Noe (2001), they remain fundamentally single maturity models. Future liabilities are only revealed one period at a time and are not considered as long-term debt at the short-term maturity date, but are rather considered as new short-term debt that started at a later point in time. The clearing mechanism they consider therefore corresponds effectively to a repeated application of a single maturity clearing algorithm.

Another recent direction emphasises the multiple equilibria found in clearing problems. This contrasts the earlier focus on uniqueness results, as surveyed in El Bitar et al. (2017), for example. This focus has been shifting since Rogers & Veraart (2013) which provided an explicit example of a financial system with multiple clearing solutions. Roukny et al. (2018) is a recent attempt to tackle this question systematically. The problem of multiple equilibria is interesting in the context of an alternative interpretation of clearing provided in Glasserman & Young (2014), which views it as dynamic re-valuation of bank assets by the market. This interpretation is particularly interesting in the systemic risk context as different equilibria can be given meaningful interpretation in terms of alternative valuations. Veraart (2017) follows this approach and investigates the effect of pre-default contagion, i.e. contagion that can be triggered prior to the actual default event due to distress and mark-to-market losses. The notions of distress and time-dependent valuation are also developed in Barucca et al. (2016).

1.4 General features of clearing models

1.4.1 Overview of clearing models

There is as yet no unified framework for different extensions of the original Eisenberg & Noe (2001) model. However, heuristically we can talk of interbank clearing models as consisting of five core elements.

First, we need a description of the *financial system* (or, more generally, a *clearing system*) as a weighted directed graph of nodes that represent banks or other financial institutions. The interpretation of the weighted directed edges can vary but most commonly they refer to nominal interbank exposures. This graph is often augmented by other parameters that encode various features of the financial system, such as the seniority structure, default costs etc. For example, the simplest clearing system is given in Eisenberg & Noe (2001) as the pair (a, L) where a is a vector of weights for each node in the graph and L is a weighted adjacency matrix describing the edges and their weights. Note that the same clearing system can have multiple representations. For instance, in Eisenberg & Noe (2001) the clearing system (a, L) can also be given by the triplet $(a, \Pi, \bar{L})$ where Π is a row-substochastic matrix and $\bar{L}$ is a vector of row-sums of the matrix L. An example of additional parameters augmenting the weighted directed graph is given by the default cost constants $\alpha, \beta \in (0, 1]$ as in Rogers & Veraart (2013) so that the full clearing system in that model is given by $(a, \Pi, \bar{L}, \alpha, \beta)$.

The second element is the set of modelling assumptions, typically reflecting stylised principles of contract and insolvency law, and the identification of the *clearing vector* (or, more generally, a *clearing solution*) - the principal financial quantities that the clearing problem is designed to compute. Most clearing models use the stylised assumptions, introduced in Eisenberg & Noe (2001), of limited liability, absolute priority and proportionality. The choice of clearing vector is less standard and, indeed, often clearing problems using different choices of clearing vectors can be shown to be equivalent. For example, Eisenberg & Noe (2001) and Rogers & Veraart (2013) focus on clearing vectors of payments, whereas Elsinger et al. (2006) works with the clearing vectors of equities. The modelling assumptions and the identification of clearing solutions are used to formulate a *clearing function* Φ for the given financial system such that a clearing solution solves some equilibrium problem of the clearing function. Thus Eisenberg & Noe (2001) uses the clearing function $\Phi : [\mathbf{0}, \bar{L}] \to [\mathbf{0}, \bar{L}] : x \mapsto \bar{L} \wedge (a + \Pi^{\top} x)$. In Rogers & Veraart (2013) default costs are a further element which results in a different clearing function.

The third element is the application of the Tarski-Knaster Theorem to demonstrate the existence and structure of the fixed points of Φ. If there are multiple such fixed points then a particular fixed point is identified as the clearing solution of interest. This clearing solution is often chosen on the basis of a financial interpretation. For example, if the edges of a directed graph represent nominal exposures and the solution of interest represents the most conservatives payments made by banks consistently with the insolvency rule (as in e.g. Rogers & Veraart (2013)) then the clearing solution is the *greatest* fixed point of a suitable clearing function.

Fourth, a *fictitious default algorithm* (or, more generally, a *clearing algorithm*) is given to compute the clearing solution in a finite number of steps. For example, both in Eisenberg & Noe (2001) and Rogers & Veraart (2013) it can be shown that the clearing solution for a financial system of N banks can be obtained in at most N steps of the relevant algorithm.

Lastly, the notion of *default* is defined by characterising a special set of banks. Such a default set is often (but, as we show in this thesis, not always) determined completely by the clearing solution. Furthermore, the clearing algorithm can equip this set with more structure by characterising a contagious cascade of bank defaults in a financial system. Both the clearing solution and the cascade of defaults can then be used to assess the extent and the impact of instability in the given financial system.

1.4.2 Common assumptions on clearing functions

The mathematical properties of the clearing function Φ are crucial and dictate much of the behaviour of a given interbank clearing model. In Eisenberg & Noe (2001) Φ is monotonic, continuous and, under the technical condition of *regularity*, contracting. Furthermore, its domain and co-domain $[\mathbf{0}, \bar{L}]$ is a complete lattice. These properties together with the Tarski-Knaster Theorem allow us to obtain several results. Namely, Eisenberg & Noe (2001) show that Φ has the unique fixed point p^* which can be obtained as a monotonically decreasing limit of the sequence $\left(\Phi^n(\bar{L})\right)_{n\geqslant 0}$ where $\bar{L}$ is easily seen as an upper bound for the clearing solution. The fact that this sequence is non-increasing also permits us to characterise the set of banks in default precisely as those banks i for which $p_i^* < \bar{L}_i$.

The existence of the fixed point relies on the Tarski-Knaster Theorem which requires the clearing function to be monotonic from a complete lattice to itself. Meanwhile, continuity of Φ is important for ensuring the uniqueness of the fixed point. For example, the model in Rogers & Veraart (2013) maintains all of the properties of clearing functions mentioned above with the exception that in general the clearing function is not continuous. Hence we are able to obtain examples of clearing systems with multiple fixed points. The Tarski-Knaster Theorem also implies that the set of its fixed points contains the unique greatest element. As mentioned above, it is then conventional in such cases to select this greatest fixed point as the clearing solution. The sequence $\left(\Phi^n(\bar{L})\right)_{n\geqslant 0}$ then converges precisely to the clearing solution. The key feature that allows this to happen is that, although Φ is not continuous, the sequence $\left(\Phi^n(\bar{L})\right)_{n\geqslant 0}$ converges to a fixed point monotonically from above.

The above summary allows us to highlight a number of assumptions in the classical interbank clearing models that this thesis seeks to relax.

Thus in Chapter 2 the introduction of multiple maturities means that the clearing function will not in general be monotonic. This has two far-reaching consequences. Firstly, we are not able to apply the Tarski-Knaster Theorem directly and thus, in general, the classical formulation of the clearing problem may not even have a solution. We provide an alternative formulation of the clearing problem to side-step this issue and obtain an *algorithmic* definition of a clearing solution. Secondly, the sequence $\left(\Phi^n(\bar{L})\right)_{n\geqslant 0}$ is also not monotonic, in general. This means that even though we are able to obtain a generalised clearing vector we cannot use it to fully characterise the default set. Instead, the default set is then obtained as part of the clearing solution.

In Chapter 3 we consider how to obtain the least fixed point in an extension of the Rogers & Veraart (2013) model where the clearing function is monotonic from $[\bar{B}, \bar{L}]$ to itself with $\bar{B}$ being the lower bound on the clearing solution. It has been observed in the literature that in the classical case where $\bar{B} = \mathbf{0}$ the least fixed point can be constructed as a limit of the non-decreasing sequence $\left(\Phi^n(\bar{B})\right)_{n\geqslant 0}$. So far this problem has gotten only marginal attention and we show that it is not trivial for two reasons.

Firstly, if Φ is not continuous then the sequence $\left(\Phi^n(\bar{B})\right)_{n\geqslant 0}$ does not, in general, converge to a fixed point from below. In order to construct the least fixed point it becomes necessary to shift to transfinite sequences. This in turn imposes obstacles to obtaining the clearing solution in a finite number of steps. We are, in fact, able to do so but at a cost. Although the construction in Chapter 3 takes a finite number of steps, this finite number is unbounded and so we cannot, in general, anticipate how many steps would be required. The second problem arises when we consider an extension where the lower bound $\bar{B}$ is no longer assumed to be zero. Although the sequence of vectors $\left(\Phi^n(\bar{B})\right)_{n\geqslant 0}$ remains monotonic, the individual sequences of vector components are not, in general, all non-decreasing. A monotonic sequence of vectors may have some components which are monotonically increasing while other are monotonically decreasing. We explain how to overcome this difficulty.

1.5 Notation, terminology and standard results

In this thesis we adopt the following notational conventions. For $n \in \mathbb{N}$ we denote by $\mathbf{Z}$ and $\mathbf{I}$ the $n \times n$ zero and identity matrices, respectively, while $\mathbf{0}$ and $\mathbf{1}$ denote the n-dimensional vectors of zeros and ones, respectively.

The partial order on vectors is assumed to be component-wise. In other words, for $x, y \in \mathbb{R}^n$, $x \leqslant y$ if and only $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant n$. Similarly, $x < y$ if and only $x_i < y_i$ for all $1 \leqslant i \leqslant n$. The minimum and the maximum of x_i and x_j are denoted by $x_i \wedge x_j$ and $x_i \vee x_j$, respectively, for some $1 \leqslant i, j \leqslant n$. $x \wedge y$ and $x \vee y$ denote the component-wise minimum and maximum of $x, y \in \mathbb{R}^n$. Vectors and matrices will be said to be positive or non-negative if all of their components are positive or non-negative, respectively. In particular, the phrase "positive matrix" will not, in general, be used to refer to positive definite matrices in this thesis.

We use the following notation to define sub-vectors and sub-matrices. For a vector $v \in \mathbb{R}_+^{|\mathcal{N}|}$ and some non-empty index set $\mathcal{A} \subseteq \mathcal{N}$, $v_{\mathcal{A}} \in \mathbb{R}_+^{|\mathcal{A}|}$ denotes the vector given component-wise by $(v_{\mathcal{A}})_i = v_i$ for all $i \in \mathcal{A}$. Similarly, for another non-empty index set $\mathcal{B} \subseteq \mathcal{N}$ and a matrix $M \in \mathbb{R}_+^{|\mathcal{N}|\times|\mathcal{N}|}$, $M_{\mathcal{A}\mathcal{B}} \in \mathbb{R}_+^{|\mathcal{A}|\times|\mathcal{B}|}$ denotes the matrix given component-wise by $(M_{\mathcal{A}\mathcal{B}})_{ij} = M_{ij}$ for all $i \in \mathcal{A}$ and $j \in \mathcal{B}$. As usual, $M_{\mathcal{A}\mathcal{B}}v_{\mathcal{B}}$ is a vector in $\mathbb{R}_+^{|\mathcal{A}|}$ provided $\mathcal{B} \neq \varnothing$. If $\mathcal{B} = \varnothing$, we extend this by convention that $M_{\mathcal{A}\mathcal{B}}v_{\mathcal{B}} := \mathbf{0} \in \mathbb{R}^{|\mathcal{A}|}$.

Since Chapters 2 and 3 use different formulations of the clearing problem, we will use both Ψ and Φ to denote clearing functions. Broadly, Φ (or a variation) is used to denote clearing functions of the form found in Eisenberg & Noe (2001) and Rogers & Veraart (2013) and, typically, corresponding to functions of *payments*. Meanwhile Ψ (or a variation) is used to denote functions of *liquid assets* (principally, in Chapter 2).

Let $(A, \leqslant)$ be a partially ordered set. A function $f : A \to A$ is *monotonically increasing* (respectively, *decreasing*) if $f(x) \geqslant f(y)$ (respectively, $f(x) \leqslant f(y)$) whenever $x \geqslant y$. A sequence $(x^n)_{n\geqslant 0} \subseteq A$ is *monotonically increasing* (respectively, *decreasing*) if $x^n \geqslant x^m$ (respectively, $x^n \leqslant x^m$) whenever $n \geqslant m$. A function or a sequence is *monotonic* if it is either monotonically increasing or monotonically decreasing.

Several results that are used in this thesis are taken to be established and are cited without proof. Further details can be found in the core texts. In particular, Horn & Johnson (1991) and Horn & Johnson (2012) are referred to on results in spectral theory, Karlin & Taylor (1981) for recurrence classes in Markov chains, Newman (2010) on networks and adjacency matrices and Jech (2013) on the theory of ordinals and transfinite induction. The following theorem due to Tarski et al. (1955) is a foundational result that is used throughout the thesis.

Theorem 1 (Tarski-Knaster Theorem)**.** *Let $(A, \leqslant)$ be a complete lattice, $f : A \to A$ a monotonically increasing function and $\mathbf{Fix}(f)$ the set of fixed points of f. Then $\mathbf{Fix}(f)$ is not empty and $(\mathbf{Fix}(f); \leqslant)$ is a complete lattice. In particular the lattice $(\mathbf{Fix}(f); \leqslant)$ contains the unique least and greatest elements.*

A complete lattice $(A, \leqslant)$ is a partially ordered set such that for any subset of A both the infimum and the supremum of the subset are in A.

Remark 2. In particular the closed interval $[a, b] \subseteq \mathbb{R}$ is a complete lattice for any $a, b \in \mathbb{R}$ under the usual ordering of the reals. Similarly, the closed interval $[A, B] \subseteq \mathbb{R}_+^N$ is a complete lattice for any $A, B \in \mathbb{R}^n$ under the component-wise ordering on $\mathbb{R}^n$.

2

Interbank Clearing in Financial Networks with Multiple Maturities

2.1 Introduction

In this chapter we tackle the question of interbank clearing in financial systems with interbank exposures of more than one maturity.

Typically bank default models assume, as e.g. proposed by Eisenberg & Noe (2001), three stylised principles of insolvency law which are common to many jurisdictions. These are the principles of *limited liability*, which says that a financial institution never pays more than it has, *absolute priority* of debt claims, implying that all outstanding debt has to be completely paid off first before shareholders can be considered, and *proportionality*. The principle of proportionality states that the total value of assets paid out in this case is distributed between all the creditors in proportion to the size of their nominal claims.

A crucial nuance of the principle of proportionality is that all liabilities, including future liabilities, are required to be treated equally for the purposes of proportional distribution to creditors. For example, the UK Insolvency Service Technical Manual stipulates that: "*A creditor may prove for a debt where payment would have become due at a date later than the insolvency proceedings [...] and it is only because the company [...] has entered into insolvency proceedings that the debt is claimed by the creditor in advance of its due payment date. Where this occurs, the creditor is entitled to the dividend equally with others [...]*," The Insolvency Service (2010, Chapter 36A, Section 48).

Our model explicitly incorporates this important feature. This contrasts with single maturity models where it is assumed that assets of defaulting banks are distributed to creditors proportionally to the short-term liabilities only. The failure to account for future liabilities in calculating the proportional distributions, leads to an incomplete view of systemic risk in financial systems. We show that two financial systems with the same overall interbank liabilities but different maturity profiles can lead to different clearing outcomes. In particular, it follows that uncertainty about maturity profiles of banks' portfolios is a distinct source of systemic risk that is unaccounted for in single maturity models. Our approach can be used in an analysis of systemic risk to evaluate the effect of such maturity profile uncertainty.

2.2 Clearing in financial systems with multiple maturities

2.2.1 The financial market

We consider a financial market consisting of N banks with indices in $\mathcal{N} = \{1, \dots, N\}$. Banks have liabilities to each other and to external entities which are due at two different maturity dates $0 < T_1 < T_2$. We will later show that we can easily generalise our model to more than two maturities. Hence, time $t = 0$ represents the starting point of the analysis and we model what happens at the two maturity dates $t \in \{T_1, T_2\}$. We assume that all liabilities of the same maturity have the same seniority.

Each bank's liabilities for some maturity can be represented by a *liability matrix*. Together with vectors representing bank's cash assets these are sufficient to describe the financial system at $t = T_1$. These and other related concepts are summarised in Definition 3.

Definition 3 (Financial system)**.**

1. A matrix $M \in \mathbb{R}_+^{N \times N}$ is called a *liability matrix* if, for all $i \in \mathcal{N}$, $M_{ii} = 0$.
2. A *financial system* is given by the tuple $(a, L^{(s)}, L^{(l)}; \gamma)$, where $L^{(s)}$, $L^{(l)}$ are liability matrices with maturity dates T_1 and T_2 respectively, and $a \in \mathbb{R}_+^N$, $\gamma \in (0, 1]$.

   We will refer to the following quantities:

   - the *cash assets* a;
   - the *short-term*, *long-term* and *overall liability matrices* $L^{(s)}$, $L^{(l)}$ and $L := L^{(s)} + L^{(l)}$, respectively;
   - the *short-term*, *long-term* and *overall total nominal liability* vectors $\bar{L}^{(s)} := L^{(s)}\mathbf{1}$, $\bar{L}^{(l)} := L^{(l)}\mathbf{1}$ and $\bar{L} := \bar{L}^{(s)} + \bar{L}^{(l)}$, respectively;
   - the *short-term*, *long-term* and *overall interbank asset vectors* $\bar{A}^{(s)} := (L^{(s)})^\intercal\mathbf{1}$, $\bar{A}^{(l)} := (L^{(l)})^\intercal\mathbf{1}$, $\bar{A} := \bar{A}^{(s)} + \bar{A}^{(l)}$, respectively;
   - the *short-term* and *overall relative liability matrices* $\Pi^{(s)}$ and Π, respectively, which are given by $\Pi_{ij}^{(s)} = \frac{L_{ij}^{(s)}}{\bar{L}_i^{(s)}}$ and $\Pi_{ij} = \frac{L_{ij}}{\bar{L}_i}$ for all $i, j \in \mathcal{N}$ if $\bar{L}_i^{(s)} > 0$ (respectively, $\bar{L}_i > 0$) and $\Pi_{ij}^{(s)} = 0$ (respectively, $\Pi_{ij} = 0$) otherwise;
   - the *bankruptcy cost parameter* γ.

Thus, given a matrix M of liabilities of some maturity, a bank i has an outstanding liability of that maturity to bank j if $M_{ij} > 0$ and the nominal value of this liability is given by M_{ij}. If $M_{ij} = 0$ then i does not owe anything to j and in particular M has a zero diagonal since we assume banks do not

Assets	Liabilities
• Cash assets: a_i	• Short-term interbank liabilities: $\bar{L}_i^{(s)} = \sum_{j=1}^N L_{ij}^{(s)}$
• Short-term interbank loans: $\bar{A}_i^{(s)} = \sum_{j=1}^N L_{ji}^{(s)}$	• Long-term interbank liabilities: $\bar{L}_i^{(l)} = \sum_{j=1}^N L_{ij}^{(l)}$
• Long-term interbank loans: $\bar{A}_i^{(l)} = \sum_{j=1}^N L_{ji}^{(l)}$	• Equity: E_i

Table 2.1: Initial stylised balance sheet at $t = 0$ of bank $i \in \mathcal{N}$.

owe anything to themselves. The i^{th} row sum of M then gives the total nominal value of liabilities of each bank of the relevant maturity and the i^{th} column sum gives the total nominal value of assets of that maturity.

Table 2.1 shows the stylised balance sheet at time $t = 0$ of bank $i \in \mathcal{N}$ where the equity is defined as $E_i := a_i + \bar{A}_i^{(s)} + \bar{A}_i^{(l)} - \bar{L}_i^{(s)} - \bar{L}_i^{(l)}$.

Remark 4. The set of banks $\mathcal{N}$ is assumed to contain a 'sink node', e.g. in this chapter $N \in \mathcal{N}$. This node has no cash assets or liabilities. However other banks may well have liabilities to the sink node. These represent banks' liabilities external to the interbank market but for ease of reference we refer to all entries of the liability matrices as 'interbank' liabilities. In Elsinger (2011) it is pointed out that in order to use a sink node in this manner external liabilities need to be treated as having the same seniority as interbank liabilities; this is indeed our assumption in this thesis.

2.2.2 General equilibrium

In this chapter we formulate a characterisation of an equilibrium achieved by clearing the market at the first maturity date that is based on the requirements of the UK insolvency rules as outlined in The Insolvency Service (2010), which can be heuristically summarised as follows:

- Banks are not required to make any payments either in excess of the total value of their liquidated assets nor the total amount they owe across all maturities.
- Conversely, shareholders are not permitted to retain any value of the defaulting banks as long as any part of any creditor's outstanding claims remains.
- Such claims include both short-term and long-term liabilities, which are treated with the same priority within the same seniority class.
- A bank that is liquidated under the insolvency rules ceases to exist and cannot recover even if liquidators recover sufficient assets to fully compensate all creditors.

Suppose we are at the first maturity date $t = T_1$ and suppose some banks with indices in $\mathcal{D} \subseteq \mathcal{N}$ are in default at $t = T_1$. We postpone the discussion on the cause of these defaults to Section 2.2.3. We will now determine a clearing equilibrium at $t = T_1$. To do so we make two assumptions.

First, we consider the case where a bank j does not default, i.e., $j \in \mathcal{N} \setminus \mathcal{D}$. Then it pays its short-term nominal obligations $\bar{L}_j^{(s)}$ in full; in particular, it pays $L_{ji}^{(s)}$ to every bank i.

Second, we consider a bank j that defaults, i.e., $j \in \mathcal{D}$. Bank j is liable to pay its creditors all of its available liquid asset resources, denoted by v_j, subject to two constraints. Since default is costly and lawyers and other service providers need to be paid, only a fraction $\gamma \in (0, 1]$ of its liquid asset resources reaches its creditors. Furthermore, we now need to consider both its short-term and its long-term liabilities. In general, $\bar{L}_j \geqslant \bar{L}_j^{(s)}$ and if j has any long-term liabilities then $\bar{L}_j > \bar{L}_j^{(s)}$. We assume that the creditors are not entitled to more than the overall total liabilities $\bar{L}_j$.

We therefore need to determine the liquid asset resources v that each bank has at time $t = T_1$. We characterise v in terms of a fixed point problem for a given financial system $(a, L^{(s)}, L^{(l)}; \gamma)$.

Definition 5. Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and $\mathcal{D} \subseteq \mathcal{N}$. Define $\Psi(\cdot; \mathcal{D}) : [\mathbf{0}, a + \bar{A}] \to [\mathbf{0}, a + \bar{A}]$ where $[\mathbf{0}, a + \bar{A}] \subseteq \mathbb{R}_+^N$ and, for each $i \in \mathcal{N}$,

$$\Psi_i(v; \mathcal{D}) := a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}} L_{ji}^{(s)} + \sum_{j \in \mathcal{D}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j).$$

We refer to any vector $v \in [\mathbf{0}, a + \bar{A}]$ satisfying $v = \Psi(v; \mathcal{D})$ as a *general liquid asset vector with respect to* $\mathcal{D}$.

Remark 6. Note that, indeed, $0 \leqslant \Psi(v; \mathcal{D})_i \leqslant a_i + \bar{A}_i$ for all v and i. This follows directly from the fact that for each $i, j \in \mathcal{N}$ and $v \in \mathbb{R}_+^N$, $\Pi_{ji}(\bar{L}_j \wedge \gamma v_j) \leqslant \Pi_{ji}\bar{L}_j = L_{ji}$. Therefore, since $L_{ji}^{(s)} \leqslant L_{ji}$ for all $i, j \in \mathcal{N}$, we have that $\Psi(v; \mathcal{D})_i \leqslant a_i + \sum_{j \in \mathcal{N}} L_{ji} = a_i + \bar{A}_i$.

Recall that, by Remark 2 in Chapter 1, the set $[\mathbf{0}, a + \bar{A}]$ forms a complete lattice under the component-wise ordering of $\mathbb{R}_+^N$.

Definition 5 defines the liquid asset vector with respect to a default set $\mathcal{D}$. In the following we discuss properties of the default set $\mathcal{D}$ before we propose two approaches to define it in Subsection 2.3.1.

2.2.3 Identification of default

Most models based on the Eisenberg & Noe (2001) framework define default by checking whether some value is less than the total nominal short-term liabilities $\bar{L}^{(s)}$. This leads to the following general definition.

Definition 7. Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system of bank $\mathcal{N}$ with the total nominal short-term liabilities vector $\bar{L}^{(s)}$. We define the function D by setting, for each vector $x \in \mathbb{R}_+^N$,

$$D(x) := \{i \in \mathcal{N} \mid x_i < \bar{L}_i^{(s)}\}. \tag{2.1}$$

This allows us to define *fundamental defaults*, i.e., defaults that occur even if everyone is assumed to satisfy their payment obligations. The *fundamental default set* is given by

$$\mathcal{F} := D(a + \bar{A}^{(s)}) = \{i \in \mathcal{N} \mid a_i + \sum_{j \in \mathcal{N}} L_{ji}^{(s)} < \bar{L}_i^{(s)}\}.$$

Fundamental defaults can be read off directly from the stylised balance sheet. It is reasonable to assume that any default set $\mathcal{D}$ satisfies $\mathcal{F} \subseteq \mathcal{D}$. Furthermore it is reasonable to assume that $\mathcal{F} = \varnothing$ implies $\mathcal{D} = \varnothing$.

Nevertheless, $\mathcal{F}$ is too small to be a suitable choice for the default set $\mathcal{D}$. Not all defaults are fundamental defaults. A bank may have interbank assets whose book value is sufficient but contingent on its counterparties avoiding default. If some of the counterparties default this would cause the market value of assets to be adjusted down, making the bank illiquid and thus triggering its default. This type of default is known as a *contagious default* and is well-established as one of the key drivers of systemic risk. These contagious defaults cannot be directly determined from the stylised balance sheet.

To capture some of these contagious defaults, we can ask whether some bank i is *illiquid* in the sense that its liquid assets v_i are insufficient for it to meet its own short-term liabilities in full. The set of such illiquid banks is then given by $D(v)$. We would expect that for any default set $\mathcal{D}$ one should have $D(v) \subseteq \mathcal{D}$. As with the fundamental defaults, the converse is not necessarily true. Since default changes the rules of distribution between counterparties, it may be the case that after a bank defaults its liquid assets exceed its short-term liabilities. However, default is an absorbing state and, once defaulted, a bank cannot recover. Thus $D(v)$ may also be too small to be a suitable choice for the default set $\mathcal{D}$.

Combining these considerations leads to the necessary condition on the default set $\mathcal{D}$:

$$1.)\ \mathcal{D} \supseteq \mathcal{F} \cup D(v), \quad 2.)\ (\mathcal{F} = \varnothing \Rightarrow \mathcal{D} = \varnothing). \tag{2.2}$$

Remark 8. Another simple notion, which can be read directly off a stylised balance sheet and is important from the accounting point of view is one of *insolvency*. A bank i is insolvent if its total nominal liabilities $\bar{L}_i$ exceed its total nominal assets $a_i + \bar{A}_i$. Since $\bar{A} = \bar{A}^{(s)} + \bar{A}^{(l)}$ and $\bar{L} = \bar{L}^{(s)} + \bar{L}^{(l)}$, this is equivalent to saying that $a_i + \bar{A}_i^{(s)} + (\bar{A}_i^{(l)} - \bar{L}_i^{(l)}) < \bar{L}_i^{(s)}$ and so we can write the set of insolvent banks as $D(a + \bar{A}^{(s)} + (\bar{A}^{(l)} - \bar{L}^{(l)}))$.

At the time an insolvent bank's short-term liabilities are due it need not be in default provided it can pay its short-term liabilities. Moreover, a solvent bank may default if it does not have sufficient liquidity to meet its short-term liabilities. Thus $D(a + \bar{A}^{(s)} + (\bar{A}^{(l)} - \bar{L}^{(l)}))$ is not a good candidate for the default set $\mathcal{D}$.

2.3 Clearing at the first maturity

2.3.1 Algorithmic and functional approaches to defining default

In the following we introduce two particular approaches to formalise the notion of default and hence to define the default set $\mathcal{D}$, which we refer to as *the algorithmic approach* and *the functional approach*. In

Section 2.3.2 we will discuss the conditions under which these approaches are well-defined and ensure existence of liquid asset vectors. Alternative definitions of a default set are also possible but we will not investigate them further here.

Algorithmic approach

In the algorithmic approach we will *start* by providing an algorithm which outputs a vector and a set, which we *define* as a liquid asset vector and a default set.

It is similar in spirit to the *Fictitious Default Algorithm (FDA)* developed by Eisenberg & Noe (2001), but in contrast to the FDA we use it to *define* default and the liquid asset vector and do not just use it as a convenient computational tool to calculate a predefined quantity of interest.

We consider a fixed financial system $(a, L^{(s)}, L^{(l)}; \gamma)$ and make the crucial modelling assumption that *default is an absorbing state*. In particular, we assume that once a bank enters the default set it will stay there. Furthermore, a bank enters the default set if and only if it has less liquid assets than total short-term liabilities. Algorithm 1 formalises this idea. Thus, for a given financial system $(a, L^{(s)}, L^{(l)}; \gamma)$

Algorithm 1: Algorithmic definition of the default set

---

1 Set $\mathcal{D}^{(0)} = \varnothing$, $v^{(0)} = a + \bar{A}^{(s)}$, $n = 1$.

2 Set

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)} \cup D(v^{(n-1)}).$$

3 If $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$ stop and return $\mathcal{D}^* = \mathcal{D}^{(n-1)}$ and $v^* = v^{(n-1)}$.

4 Else determine the greatest fixed point $v^{(n)}$ satisfying

$$v^{(n)} = \Psi(v^{(n)}; \mathcal{D}^{(n)}), \tag{2.3}$$

where Ψ is defined in Definition 5.

5 Set n=n+1 and go to 2.

Algorithm 1 computes a vector v^* and a set $\mathcal{D}^*$ which will correspond to a liquid asset vector with respect to the default set $\mathcal{D}^*$.

Definition 9. Let $\mathcal{D}^*$ and v^* be the outputs of Algorithm 1. We refer to

- $\mathcal{D}^*$ as the *algorithmic default set*; and
- v^* as the *algorithmic liquid asset vector with respect to* $\mathcal{D}^*$.

Proposition 10. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and let $\mathcal{D}^*$ and v^* be the outputs of Algorithm 1. Then, the algorithmic liquid asset vector v^* is a general liquid asset vector with respect to $\mathcal{D}^*$. Moreover, $\mathcal{D}^*$ satisfies the criteria specified in (2.2).*

Proof. By construction, $v^* = v^{(n)}$ and $\mathcal{D}^* = \mathcal{D}^{(n)}$ for some n and hence $v^* = \Psi(v^*; \mathcal{D}^*)$. Hence, by Definition 5, v^* is a general liquid asset vector with respect to $\mathcal{D}^*$.

To see that the algorithmic default set $\mathcal{D}^*$ satisfies the criteria in (2.2), note that the set of fundamental defaults is given by $\mathcal{F} = D(a + \bar{A}^{(s)}) = D(v^{(0)}) = \mathcal{D}^{(1)} \subseteq \mathcal{D}^*$. Furthermore, if $\mathcal{F} = \varnothing = \mathcal{D}^{(0)}$ then Algorithm 1 terminates with $\mathcal{D}^* = \varnothing$ and $v^* = a + \bar{A}^{(s)} = \Psi(v^*; \varnothing)$. □

Functional approach

We will argue in the following sections that the algorithmic approach is a more general approach that works for any financial system with multiple maturities. However, it is instructive to consider why the more conventional route along the lines of Eisenberg & Noe (2001) is problematic in the multiple maturity setting. To this end we consider an alternative approach where the default set is characterised as a closed-form function $D(v)$ of the liquid asset vector.

Definition 11. Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system. Define $\widetilde{\Psi} : [\mathbf{0}, a + \bar{A}] \to [\mathbf{0}, a + \bar{A}]$ where

$$\widetilde{\Psi}_i(v) := a_i + \sum_{j \in \mathcal{N} \setminus D(v)} L_{ji}^{(s)} + \gamma \sum_{j \in D(v)} \Pi_{ji} v_j. \tag{2.4}$$

We refer to any vector $v \in [\mathbf{0}, a + \bar{A}]$ satisfying $v = \widetilde{\Psi}(v)$ as a *functional liquid asset vector* and the set $D(v)$ as a *functional default set*.

Proposition 12. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system. Then $\widetilde{\Psi}(v) = \Psi(v; D(v))$ for all $v \in [\mathbf{0}, a + \bar{A}]$.*

Proof. Let $v \in \mathbb{R}_+^N$, then for all $j \in D(v) = \{i \in \mathcal{N} \mid v_i < \bar{L}_i^{(s)}\}$ it holds that $\gamma v_j < \bar{L}_j^{(s)} \leqslant \bar{L}_j$ and hence $\bar{L}_j \wedge \gamma v_j = \gamma v_j$. Hence for all $i \in \mathcal{N}$

$$\begin{aligned} \Psi(v, D(v))_i &= a_i + \sum_{j \in \mathcal{N} \setminus D(v)} L_{ji}^{(s)} + \sum_{j \in D(v)} \Pi_{ji} (\bar{L}_j \wedge \gamma v_j) \\ &= a_i + \sum_{j \in \mathcal{N} \setminus D(v)} L_{ji}^{(s)} + \sum_{j \in D(v)} \Pi_{ji} \gamma v_j \\ &= \widetilde{\Psi}_i(v). \end{aligned}$$

□

The following proposition provides the link between functional and general liquid asset vectors.

Proposition 13. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and let v be a functional liquid asset vector. Then v is a general liquid asset vector with respect to $D(v)$.*

Proof. Suppose v be a functional liquid asset vector. Then by Proposition 12, $v = \Psi(v; D(v))$ and hence the result follows by Definition 5. □

We will show that in contrast to the algorithmic liquid asset vector, which exists for all financial systems, a functional liquid asset vector need not exist in a multiple maturity setting.

2.3.2 Existence of liquid asset vectors

Monotonicity of clearing functions is crucial to establishing the existence of clearing solutions. In the single maturity setting monotonicity of clearing functions arises naturally. However, that is no longer the case once the financial system has more than one maturity. Indeed, the functional liquid asset vector does not exist for all financial systems and in those cases the clearing function is not monotonic. Nevertheless, there is a sufficient (but not necessary) monotonicity condition that guarantees the existence of a functional liquid asset vector:

Definition 14 (Monotonicity Condition)**.** Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, with short term and overall relative liability matrix $\Pi^{(s)}$ and Π, respectively. We refer to a financial system as satisfying the *Monotonicity Condition 14* if and only if

$$\Pi^{(s)}_{ij} \geqslant \gamma \Pi_{ij} \ \forall i, j \in \mathcal{N}.$$

From a financial point of view Monotonicity Condition 14 just asserts that for any bank i in the system it is guaranteed that if it defaults it does not pay a larger proportion of its liquid assets to any bank j in the system than its original proportion of short-term liabilities to this particular bank j. From a mathematical point of view, Monotonicity Condition 14 ensures the monotonicity of the function $\widetilde{\Psi}$. Furthermore, it highlights the fact that the distinction between $\Pi^{(s)}$ and Π in our model is a crucial element that is missing in single maturity models.

Remark 15. Note, that networks in which $L^{(s)}_{ij} = 0$ and $L^{(l)}_{ij} > 0$ for some i, j will never satisfy Monotonicity Condition 14. Furthermore, if $\gamma = 1$, Monotonicity Condition 14 implies $\Pi^{(s)} = \Pi$.

Remark 16. Suppose $L^{(l)} = \mathbf{Z}$ where $\mathbf{Z}$ is a zero matrix. Then the short-term and overall nominal liabilities vectors $\bar{L}^{(s)}$ and $\bar{L}$ are equal and hence so are the short-term and overall relative liability matrices $\Pi^{(s)}$ and Π. Thus Monotonicity Condition 14 is always satisfied if $L^{(l)} = \mathbf{Z}$.

The following lemma provides the link between the clearing functions under the two approaches introduced above as well as sufficient conditions for their monotonicity.

Lemma 17. *Let $S = (a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system and $d : \mathbb{R}^N_+ \to \mathcal{P}(\mathcal{N})$ some function, where $\mathcal{P}$ denotes the power set. Let $\Psi^d : [\mathbf{0}, a + \bar{A}] \to [\mathbf{0}, a + \bar{A}]$ be the function given by $x \mapsto \Psi(x; d(x))$ for all $x \in [\mathbf{0}, a + \bar{A}]$.*

1. *Suppose $d \equiv \mathcal{D}$, i.e. $d(x) = \mathcal{D}$ for all $x \in [\mathbf{0}, a + \bar{A}]$ and some fixed $\mathcal{D} \subseteq \mathcal{N}$. Then $\Psi^d = \Psi(\cdot; \mathcal{D})$ and Ψ^d is monotonically increasing.*

2. *Suppose $d = D$, i.e. $d(x) = D(x) = \{i \in \mathcal{N} \mid x_i < \bar{L}^{(s)}_i\}$ for all $x \in [\mathbf{0}, a + \bar{A}]$ and suppose that S satisfies Monotonicity Condition 14. Then $\Psi^d = \widetilde{\Psi}$ and Ψ^d is monotonically increasing.*

Proof. 1. Suppose $d \equiv \mathcal{D}$ for some fixed $\mathcal{D} \subseteq \mathcal{N}$. Then, for each $x \in [\mathbf{0}, a + \bar{A}]$, $\Psi^d(x) = \Psi(x; d(x)) = \Psi(x; \mathcal{D})$ and hence $\Psi^d = \Psi(\cdot; \mathcal{D})$.

Let $x', x \in \mathbb{R}_+$ with $x' \leqslant x$. Define $E(x') := \{i \in \mathcal{D} \mid \gamma x'_i < \bar{L}_i\}$ and, similarly, $E(x)$. Since $\gamma x'_i \leqslant \gamma x_i$ for all $i \in \mathcal{N}$, we see that $E(x) \subseteq E(x') \subseteq \mathcal{N}$. Then, for each $i \in \mathcal{N}$, we have

$$
\begin{aligned}
\Psi_i^d(x') = \Psi_i(x';\mathcal{D}) &= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}} \Pi_{ji}(\bar{L}_j \wedge \gamma x'_j) \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x')} \Pi_{ji}\bar{L}_j + \gamma \sum_{j\in E(x')} \Pi_{ji}x'_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x')} L_{ji} + \gamma \sum_{j\in E(x')} \Pi_{ji}x'_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x'_j + \sum_{j\in E(x')\setminus E(x)} (\Pi_{ji}\gamma x'_j - L_{ji}) \\
&\leqslant a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x'_j \\
&\leqslant a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}\setminus E(x)} L_{ji} + \gamma \sum_{j\in E(x)} \Pi_{ji}x_j \\
&= \Psi_i(x;\mathcal{D}) = \Psi_i^d(x).
\end{aligned}
$$

The first inequality (on the fifth line) follows since $\gamma x'_j < \bar{L}_j$ for $j \in E(x')$ and hence $\Pi_{ji}\gamma x'_j - L_{ji} \leqslant \Pi_{ji}\bar{L}_j - L_{ji} = 0$. The second inequality (on the sixth line) follows since $x' \leqslant x$ by assumption. Therefore Ψ^d is monotonic.

2. Suppose $d = D$. Then, for each $x \in [\mathbf{0}, a + \bar{A}]$, $\Psi^d(x) = \Psi(x; d(x)) = \Psi(x; D(x))$ and hence, by Proposition 12, $\Psi^d = \widetilde{\Psi}$.

   Again, let $x', x \in \mathbb{R}_+$ with $x' \leqslant x$. Note that $D(x) \subseteq D(x') \subseteq \mathcal{N}$. Then, for each $i \in \mathcal{N}$, we have

$$
\begin{aligned}
\Psi_i^d(x') = \widetilde{\Psi}_i(x') &= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \gamma \sum_{j\in D(x')} \Pi_{ji}x'_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \gamma \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}x'_j + \gamma \sum_{j\in D(x)} \Pi_{ji}x'_j \\
&\leqslant a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}^{(s)}x'_j + \gamma \sum_{j\in D(x)} \Pi_{ji}x'_j \\
&\leqslant a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} \Pi_{ji}^{(s)}\bar{L}_j^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x')} L_{ji}^{(s)} + \sum_{j\in D(x')\setminus D(x)} L_{ji}^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus D(x)} L_{ji}^{(s)} + \gamma \sum_{j\in D(x)} \Pi_{ji}x_j \\
&= \widetilde{\Psi}_i(x) = \Psi_i^d(x).
\end{aligned}
$$

   The first inequality (on the third line) follows due to the Monotonicity Condition 14 and the fact that $\gamma \leqslant 1$. The second inequality (on the fourth line) follows because $x' \leqslant x$ by assumption and $x'_j < \bar{L}_j^{(s)}$ for all $j \in D(x')$. Therefore Ψ^d is monotonic.

□

We use Lemma 17 to establish the criteria for existence of algorithmic clearing solutions in Theorem 18 and of functional clearing solutions in Theorem 19. We will discuss the *construction* of the liquid asset vectors under both approaches in Section 2.3.4.

Theorem 18. *Let* $(a, L^{(s)}, L^{(l)}; \gamma)$ *be a financial system. Then, the greatest solution to the fixed-point problem (2.3) exists and lies in* $[\mathbf{0}, a + \bar{A}]$. *Furthermore, Algorithm 1 terminates after a finite number of steps.*

Proof. For each n, $\mathcal{D}^{(n)}$ depends on $v^{(n-1)}$ but not on $v^{(n)}$. Therefore by Lemma 17.1, $\Psi(\cdot; \mathcal{D}^{(n)})$ is monotonic and by Remark 6 is a mapping from a complete lattice to itself. Hence, by the Tarski-Knaster Theorem (Theorem 1, chapter 1), $\Psi(\cdot; \mathcal{D}^{(n)})$ has the greatest fixed point, which lies within the image of $\Psi(\cdot; \mathcal{D}^{(n)})$, i.e. in $[\mathbf{0}, a + \bar{A}]$. In Algorithm 1 this fixed point is denoted $v^{(n)}$. Hence whenever $\mathcal{D}^{(n)}$ is well-defined, $\mathcal{D}^{(n+1)}$ is also well-defined until the algorithm terminates.

In particular, $(\mathcal{D}^{(n)})_{n \geqslant 0}$ is a well-defined and, by construction, non-decreasing sequence of subsets of the finite set $\mathcal{N}$. Hence there exists the least n such that $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$ and so Algorithm 1 terminates after n iterations. □

Theorem 19 (Sufficient condition for the existence of a functional liquid asset vector)**.** *Let* $(a, L^{(s)}, L^{(l)}; \gamma)$ *be a financial system and assume that the Monotonicity Condition 14 is satisfied. Then there exist functional liquid asset vectors* v^- *(the* least functional liquid asset vector*) and* v^+ *(the* greatest functional liquid asset vector*) such that for any functional liquid asset vector* v *we have that* $v^- \leqslant v \leqslant v^+$.

Proof. The result follows directly by the application of the Tarski-Knaster Theorem (Theorem 1, chapter 1) since $\widetilde{\Psi}$ is monotonic by Lemma 17 and is a mapping from a complete lattice to itself by Remark 6. □

The following proposition demonstrates that the Monotonicity Condition 14 is not a necessary condition but nor is it a redundant condition.

Proposition 20.

1. *There exists a financial system that does not satisfy the Monotonicity Condition 14 for which a functional liquid asset vector exists.*

2. *There also exists a financial system that does not satisfy the Monotonicity Condition 14 for which no functional liquid asset vector exists.*

Proof. 1. We first provide one example of a financial system in which the functional liquid asset vector exists even though the Monotonicity Condition 14 is not satisfied.

Let $(a, L^{(s)}, L^{(l)}; 1)$ be a financial system of three banks where

$$a = \begin{pmatrix} 1 \\ 98 \\ 10 \end{pmatrix}, \quad L^{(s)} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^{(l)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 99 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$\bar{L}^{(s)} = \begin{pmatrix} 4 \\ 100 \\ 0 \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} 4 \\ 102 \\ 0 \end{pmatrix}, \quad \Pi^{(s)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{50} & 0 & \frac{49}{50} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{34} & 0 & \frac{33}{34} \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, we see that Monotonicity Condition 14 is not satisfied because $\Pi^{(s)}_{21} = \frac{1}{50} < \frac{3}{102} = \Pi_{21}$. Nevertheless, it can be verified that $(v_1, v_2, v_3)^\top = (3\frac{63}{67}, 99\frac{130}{134}, 109\frac{1}{2})^\top \approx (3.94, 99.97, 109.5)^\top$ is a functional liquid asset vector.

2. Next, we provide an example of a financial system in which the Monotonicity Condition 14 is not satisfied and a functional liquid asset vector does not exist.

   We construct an example with three banks in which only bank 1 is in fundamental default. We set up the network such that this leads to a contagious default of bank 2 which is asset rich. We introduce long-term liabilities in such a way that once bank 2 defaults it repays a much larger proportion of its debt to bank 1 than if it were not in default. This leads to bank 1 being able to pay more than $\bar{L}^{(s)}_1$.

   Let $(a, L^{(s)}, L^{(l)}; 1)$ be a financial system of three banks where

$$a = \begin{pmatrix} 1 \\ 98 \\ 10 \end{pmatrix}, \quad L^{(s)} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^{(l)} = \begin{pmatrix} 0 & 2 & 2 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 4 & 4 \\ 102 & 0 & 98 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Then,

$$\bar{L}^{(s)} = \begin{pmatrix} 4 \\ 100 \\ 0 \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} 8 \\ 200 \\ 0 \end{pmatrix}, \quad \Pi^{(s)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{50} & 0 & \frac{49}{50} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{51}{100} & 0 & \frac{49}{100} \\ 0 & 0 & 0 \end{pmatrix}.$$

   Note that Monotonicity Condition 14 is not satisfied, since for example $\Pi^{(s)}_{21} = \frac{1}{50} < \frac{51}{100} = \Pi_{21}$. Hence, if bank 2 defaults it repays a larger proportion to bank 1 than if it survives. We show in the following that no functional liquid asset vector exists.

   According to Definition 11, bank 3 can never default since it does not have any short-term (or indeed any) liabilities. In particular, since $\widetilde{\Psi}$ is non-negative, we have that $\{i \in \mathcal{N} \mid \widetilde{\Psi}(v)_i < 0\} = \varnothing$ for any v. Hence we need to consider four cases:

   All banks survive. Suppose there exists a functional liquid asset vector v, such that $D(v) = \varnothing$. Hence, $v_i \geqslant \bar{L}^{(s)}_i$ for all i. Then, for all $i \in \mathcal{N}$,

$$v_i = a_i + \sum_{j \in \mathcal{N}} L^{(s)}_{ji}.$$

   Consider $i = 1$. Then $v_1 = 1 + 2 = 3 < 4 = \bar{L}^{(s)}_1$, implying that $1 \in D(v)$ and therefore contradicting the assumption that $D(v) = \varnothing$.

   Only bank 1 defaults. Suppose there exists a functional liquid asset vector v, such that $D(v) =$

$\{i : v_i < \bar{L}_i^{(s)}\} = \{1\}$. Then,

$$v_1 = a_1 + \sum_{j \in \{2,3\}} L_{ji}^{(s)} = 1 + 2 + 0 = 3 < 4 = \bar{L}_1^{(s)},$$
$$v_2 = a_2 + L_{32}^{(s)} + \Pi_{12} v_1 = 98 + 0 + \frac{1}{2} 3 = 99\frac{1}{2} < 100 = \bar{L}_2^{(s)}.$$

Hence $2 \in D(v)$ contradicting the assumption that $D(v) = \{i \in \mathcal{N} : v_i < \bar{L}_i^{(s)}\} = \{1\}$.

Only bank 2 defaults. Suppose there exists a functional liquid asset vector v, such that $D(v) = \{i \in \mathcal{N} : v_i < \bar{L}_i^{(s)}\} = \{2\}$. Then,

$$v_2 = a_2 + \sum_{j \in \{1,3\}} L_{ji}^{(s)} = 98 + 2 + 0 = 100 = \bar{L}_2^{(s)}.$$

Hence $2 \notin D(v)$, contradicting our assumption.

Both bank 1 and bank 2 default. Suppose there exists a functional liquid asset vector v, such that $D(v) = \{i \in \mathcal{N} : v_i < \bar{L}_i^{(s)}\} = \{1, 2\}$. Then,

$$v_1 = a_1 + L_{31}^{(s)} + \Pi_{21} v_2 = 1 + 0 + \frac{51}{100} \cdot v_2,$$
$$v_2 = a_2 + L_{32}^{(s)} + \Pi_{12} v_1 = 98 + \frac{1}{2} \cdot v_1.$$

We then obtain that $(1 - \frac{1 \cdot 51}{2 \cdot 100}) v_1 = 1 + 98 \frac{51}{100}$ and hence $v_1 \approx 68.43 > 4 = \bar{L}_1^{(s)}$. Therefore $1 \notin D(v)$, contradicting our assumption.

Hence, in all cases we get a contradiction and therefore no functional liquid asset vector exists.

□

2.3.3 Relationship between clearing models

In this section we look at the relationship between several clearing models. In particular, we show that the algorithmic approach is indeed a proper generalisation of the functional approach, which in turn generalises the models of Eisenberg & Noe (2001) and Rogers & Veraart (2013).

We introduce a new Algorithm 2 which can be used to construct a functional liquid asset vector under the Monotonicity Condition 14. We then show that under the Monotonicity Condition 14 Algorithm 1 is reduced to Algorithm 2. Therefore the algorithmic liquid asset vector and the algorithmic default set coincide with the functional liquid asset vector and the functional default set under the Monotonicity Condition 14.

The only difference between Algorithm 1 and Algorithm 2 is in step 2 when the new default set is defined. Algorithm 2 only considers banks in default which in the current round have fewer liquid assets than nominal short term liabilities. Algorithm 1 makes the absorbing property of default explicit in the definition, by additionally always keeping those banks in the default set that have defaulted in one of the previous rounds of the algorithm.

The following proposition confirms that Algorithm 2 indeed produces the claimed outputs.

Proposition 21. *Let $\tilde{\mathcal{D}}^*$ and $\tilde{v}^*$ be the output of Algorithm 2. Then, $\tilde{\mathcal{D}}^* = D(\tilde{v}^*)$ and hence $\tilde{\mathcal{D}}^*$ is a functional default set and $\tilde{v}^*$ is a functional liquid asset vector.*

Proof. By construction in Algorithm 2, $\tilde{v}^* = v^{(n)}$ and $\tilde{\mathcal{D}}^* = \mathcal{D}^{(n)}$ for some n. Moreover, $\mathcal{D}^{(n)} = D(v^{(n)}) = D(\tilde{v}^*)$. Hence $v^* = \Psi(v^*; \tilde{\mathcal{D}}^*)$ and so the result follows by Proposition 12. □

Algorithm 2: Functional approach algorithm under Monotonicity Condition

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1 Set $\mathcal{D}^{(0)} = \varnothing$, $v^{(0)} = a + \bar{A}^{(s)}$, $n = 1$.

2 Set

$$\mathcal{D}^{(n)} = D(v^{(n-1)}) = \{i \in \mathcal{N} \mid v_i^{(n-1)} < \bar{L}_i^{(s)}\}.$$

3 If $\mathcal{D}^{(n)} = \mathcal{D}^{(n-1)}$ stop and return $\tilde{\mathcal{D}}^* = \mathcal{D}^{(n-1)}$ and $\tilde{v}^* = v^{(n-1)}$.

4 Else determine the greatest fixed point $v^{(n)}$ satisfying

$$v^{(n)} = \Psi(v^{(n)}; \mathcal{D}^{(n)}), \tag{2.5}$$

where Ψ is defined in Definition 5.

5 Set n=n+1 and go to 2.

By inspection, Algorithm 1 and 2 look similar. Theorem 22 establishes that under the Monotonicity Condition 14 they are in fact the same algorithm.

Theorem 22. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system satisfying Monotonicity Condition 14. Then*

1. *Algorithm 2 produces a monotonically decreasing sequence of vectors $(v^{(n)})_{n\geqslant 0}$ such that $v^{(n)} \leqslant v^{(n-1)} \leqslant a + \bar{A}^{(s)}$ $\forall n \geqslant 1$ and a monotonically increasing sequence of sets $(\mathcal{D}^{(n)})_{n\geqslant 0}$ such that $\mathcal{D}^{(n-1)} \subseteq \mathcal{D}^{(n)}$ $\forall n \geqslant 1$. In particular $\mathcal{D}^{(n)} = D(v^{(n-1)})$ $\forall n \geqslant 1$.*

2. *Algorithms 1 and 2 coincide.*

Proof. The proof uses similar arguments as in Rogers & Veraart (2013, Proof of Theorem 3.7).

1. We prove that $v^{(n)} \leqslant v^{(n-1)} \leqslant a + \bar{A}^{(s)}$ $\forall n \geqslant 1$ and $\mathcal{D}^{(n)} = D(v^{(n-1)})$ $\forall n \geqslant 1$ by induction.

   Note that for all n and $j \in \mathcal{N}$ we have $\bar{L}_j \wedge \gamma v_j^{(n)} \leqslant \gamma v_j^{(n)} \leqslant v_j^{(n)}$. Furthermore, for all n and $j \in D(v^{(n)})$ we also have $v_j^{(n)} < \bar{L}_j^{(s)}$. Therefore, by the Monotonicity Condition 14, for all n, $i \in \mathcal{N}$ and $j \in D(v^{(n)})$ we have that

   $$\Pi_{ji}(\bar{L}_j \wedge \gamma v^{(n)})_j \leqslant \Pi_{ji}^{(s)} \bar{L}_j^{(s)} = L_{ji}^{(s)}. \tag{2.6}$$

   Now let $n = 1$. Then by the definition of the algorithm $\mathcal{D}^{(1)} = \mathcal{D}^{(0)} \cup D(v^{(0)}) = \varnothing \cup D(v^{(0)}) = D(v^{(0)})$. Next we show that $v^{(1)} \leqslant v^{(0)} = a + \bar{A}^{(s)}$.

By (2.6), for all $i \in \mathcal{N}$, we have

$$\begin{aligned}\Psi_i(v^{(0)};\mathcal{D}^{(1)}) &= a_i + \sum_{j\in\mathcal{N}\smallsetminus\mathcal{D}^{(1)}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}^{(1)}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(0)}) \\ &\leqslant a_i + \sum_{j\in\mathcal{N}\smallsetminus\mathcal{D}^{(1)}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}^{(1)}} L_{ji}^{(s)} \\ &= a_i + \sum_{j\in\mathcal{N}} L_{ji}^{(s)} = a_i + \bar{A}_i^{(s)} = v_i^{(0)}.\end{aligned}$$

By Lemma 17.1, $\Psi(\cdot;\mathcal{D}^{(1)})$ is monotonic and so

$$0 \leqslant \Psi^{k+1}(v^{(0)};\mathcal{D}^{(1)}) \leqslant \Psi^{k}(v^{(0)};\mathcal{D}^{(1)}) \leqslant v^{(0)} = a + \bar{A}^{(s)}$$

for all k where Ψ^k is a k-fold composition of Ψ. Since this sequence is bounded from below by zero, the limit $v^{(1)} := \lim_{k\to\infty} \Psi^k(v^{(0)};\mathcal{D}^{(1)})$ exists and solves $v^{(1)} = \Psi(v^{(1)};\mathcal{D}^{(1)})$.

Induction hypothesis: Suppose for an $n \in \mathbb{N}$ it holds that

$$\begin{aligned}&\mathcal{D}^{(n)} = D(v^{(n-1)}),\\ &v^{(n)} \leqslant v^{(n-1)} \leqslant v^{(0)} = a + \bar{A}^{(s)}.\end{aligned}$$

We show that

$$\begin{aligned}&\mathcal{D}^{(n+1)} = D(v^{(n)}),\\ &v^{(n+1)} \leqslant v^{(n)} \leqslant v^{(0)} = a + \bar{A}^{(s)}.\end{aligned}$$

We start with the default sets:

$$\mathcal{D}^{(n+1)} = \mathcal{D}^{(n)} \cup D(v^{(n)}) \overset{\text{ind. hyp. part 1}}{=} D(v^{(n-1)}) \cup D(v^{(n)}) \overset{\text{ind. hyp. part 2}}{=} D(v^{(n)}).$$

Next we consider the vector

$$v^{(n+1)} = \Psi(v^{(n+1)};\mathcal{D}^{(n+1)}) = \Psi(v^{(n+1)};D(v^{(n)})).$$

Then by (2.6), for all $i \in \mathcal{N}$, we have

$$\begin{aligned}&\Psi_i(v^{(n)};\mathcal{D}^{(n+1)}) = \\ &= a_i + \sum_{j\in\mathcal{N}\smallsetminus\underbrace{\mathcal{D}^{(n+1)}}_{D(v^{(n)})}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}^{(n+1)}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) \\ &= a_i + \sum_{j\in\mathcal{N}\smallsetminus D(v^{(n)})} L_{ji}^{(s)} + \sum_{j\in D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) + \sum_{j\in D(v^{(n)})\smallsetminus D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) \\ &\leqslant a_i + \sum_{j\in\mathcal{N}\smallsetminus D(v^{(n)})} L_{ji}^{(s)} + \sum_{j\in D(v^{(n-1)})} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) + \sum_{j\in D(v^{(n)})\smallsetminus D(v^{(n-1)})} L_{ji}^{(s)} \\ &= a_i + \sum_{j\in\mathcal{N}\smallsetminus\underbrace{D(v^{(n-1)})}_{\mathcal{D}^{(n)}}} L_{ji}^{(s)} + \sum_{\underbrace{j\in D(v^{(n-1)})}_{\mathcal{D}^{(n)}}} \Pi_{ji}(\bar{L}_j \wedge \gamma v_j^{(n)}) \\ &= \Psi_i(v^{(n)};\mathcal{D}^{(n)}) = v_i^{(n)} \leqslant a_i + \bar{A}_i^{(s)}.\end{aligned}$$

Again, as before one can show by Lemma 17.1 that

$$0 \leqslant \Psi^{k+1}(v^{(n)}; \mathcal{D}^{(n+1)}) \leqslant \Psi^{k}(v^{(n)}; \mathcal{D}^{(n+1)}) \leqslant v^{(n)} \overset{\text{ind. hyp. part 2}}{\leqslant} a + \bar{A}^{(s)}$$

for all k and hence the limit $v^{(n+1)} := \lim_{k\to\infty} \Psi^{k}(v^{(n)}; \mathcal{D}^{(n+1)})$ exists, solves $v^{(n+1)} = \Psi(v^{(n+1)}; \mathcal{D}^{(n)})$ and in particular satisfies $v^{(n+1)} \leqslant v^{(n)} \leqslant v^{(0)}$.

2. Since the only difference between the two algorithms is the definition of the default sets in Step 2 and we have just proved in (i) that the default sets coincide, both algorithms are indeed identical under the Monotonicity Condition 14.

□

The assumption of Monotonicity Condition 14 is crucial. Without it Algorithm 2 can fail to terminate.

Proposition 23. *There exists a financial system not satisfying the Monotonicity Condition 14 such that the sequence of vectors $\left(v^{(n)}\right)_{n\geqslant 0}$ constructed in Algorithm 2 is not monotonic and Algorithm 2 does not terminate.*

Proof. Let $(a, L^{(s)}, L^{(l)}; 1)$ be as in the proof of Proposition 20.2 where, as mentioned above, Monotonicity Condition 14 fails. Algorithm 2 would fail to terminate since the sequences $v^{(n)}$ and $D(v^{(n)})$ would evolve as follows

$$\begin{aligned}
v^{(0)} &= (3, 100, 110) & D(v^{(0)}) &= \{1\} \\
v^{(1)} &= (3, 99.5, 109.5) & D(v^{(1)}) &= \{1, 2\} \\
v^{(2)} &\approx (68.43, 132.21, 93.43) & D(v^{(2)}) &= \varnothing \\
v^{(3)} &= (3, 100, 110) & D(v^{(3)}) &= \{1\} \\
&\dots
\end{aligned}$$

and it is clear that this sequence would not terminate. □

By Remark 16, a functional liquid asset vector exists for any financial system $(a, L^{(s)}, \mathbf{Z}; \gamma)$ where $\mathbf{Z}$ is a zero matrix. In fact, the system then reduces to a special case of the model by Rogers & Veraart (2013) where the parameters modelling the default costs in Rogers & Veraart (2013) denoted by α, β are all the same and equal to γ, i.e $\gamma = \alpha = \beta$. Proposition 24 formalises this relationship.

Proposition 24. *Let $(a, L^{(s)}, \mathbf{Z}; \gamma)$ be a financial system where $\mathbf{Z}$ is a zero matrix.*

1. *Let v be a functional liquid asset vector. Let q be a vector defined by,*

$$q_i = \begin{cases} \bar{L}_i^{(s)}, & \text{if } i \in \mathcal{N} \setminus D(v), \\ \gamma v_i, & \text{if } i \in D(v), \end{cases}$$

for each $i \in \mathcal{N}$. Then q is a clearing vector in the sense of Rogers & Veraart (2013), i.e., q solves the fixed-point problem:

$$q_i = \begin{cases} \bar{L}_i, & \text{if } a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j \geqslant \bar{L}_j, \\ \gamma a_i + \gamma \sum_{j\in\mathcal{N}} \Pi_{ji} q_j, & \text{if } a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j < \bar{L}_j. \end{cases} \tag{2.7}$$

2. *Let q be a clearing vector in the sense of Rogers & Veraart (2013), i.e., a solution of (2.7). Then $v = a + \Pi^\top q$ is a functional liquid asset vector.*

Proof. Since $L^{(l)} = \mathbf{Z}$, we have that $\bar{L} = \bar{L}^{(s)}$ and $\Pi = \Pi^{(s)}$.

1. For $i \in \mathcal{N}$ we have

$$\begin{aligned} v_i &= a_i + \sum_{j\in\mathcal{N}\setminus D(v)} \underbrace{L_{ji}^{(s)}}_{=\Pi_{ji}\underbrace{\bar{L}_j^{(s)}}_{=q_j}} + \sum_{j\in D(v)} \Pi_{ji} \underbrace{\gamma v_j}_{=q_j} \\ &= a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j. \end{aligned}$$

Hence, $D(v) = \{i \in \mathcal{N} \mid a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j < \bar{L}_i\}$. Hence, for all $i \in D(v)$

$$q_i = \gamma v_i = \gamma a_i + \gamma \sum_{j\in\mathcal{N}} \Pi_{ji} q_j,$$

and for all $i \in \mathcal{N} \setminus D(v)$ we have that $q_i = \bar{L}_i^{(s)} = L_i$. Hence, q satisfies the fixed point equation (2.7).

2. Let q be a solution to (2.7). We show that $v = a + \Pi^\top q$ is a functional liquid asset vector, i.e., $\widetilde{\Psi}(v) = v$. Note that $D(v) = \{i \in \mathcal{N} \mid a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j < \bar{L}_i\}$. Therefore, for all $i \in \mathcal{N}$

$$\begin{aligned} \widetilde{\Psi}_i(v) &= a_i + \sum_{j\in\mathcal{N}\setminus D(v)} \underbrace{L_{ji}^{(s)}}_{=\Pi_{ji}\underbrace{\bar{L}_j^{(s)}}_{=q_j}} + \sum_{j\in D(v)} \Pi_{ji} \underbrace{\gamma v_j}_{=q_j} \\ &= a_i + \sum_{j\in\mathcal{N}} \Pi_{ji} q_j = v_i. \end{aligned}$$

□

If $\gamma = 1$, then $(a, L^{(s)}, \mathbf{Z}; 1)$ is effectively a (single maturity) financial system as defined in Eisenberg & Noe (2001) as the following proposition demonstrates.

Proposition 25. *Let $(a, L^{(s)}, \mathbf{Z}; 1)$ be a financial system where $\mathbf{Z}$ is a zero matrix.*

1. *Let v a functional liquid asset vector. Let $p := \bar{L}^{(s)} \wedge v$. Then p is a clearing vector in the sense of Eisenberg & Noe (2001), i.e., p solves the fixed-point problem*

$$p = \bar{L}^{(s)} \wedge (a + \Pi^\top p). \tag{2.8}$$

2. *Let p be a clearing vector in the sense of Eisenberg & Noe (2001), i.e., a solution of (2.8). Then $v = a + \Pi^\top p$ is a functional liquid asset vector.*

Proof. Since $L^{(l)} = \mathbf{Z}$, we have that $L = L^{(s)}$, $\bar{L} = \bar{L}^{(s)}$ and $\Pi = \Pi^{(s)}$. The result follows directly from Proposition 24 with $\gamma = 1$.

1. Let v be a functional liquid asset vector and $D(v) = \{i \in \mathcal{N} \mid v_i < \bar{L}_i^{(s)}\}$. Hence, with $\gamma = 1$ in Proposition 24, $q = \bar{L}^{(s)} \wedge v$. Furthermore, the fixed point equation (2.7) simplifies to $q = \bar{L}^{(s)} \wedge a + \Pi^\top q$ which is exactly (2.8) and hence the result follows.

2. Similarly, since the fixed point equations (2.7) and (2.8) coincide for $\gamma = 1$ the result follows directly from Proposition 24.

□

2.3.4 Construction of liquid asset vectors

One of the questions we postponed answering was how to construct the liquid asset vectors (and hence default sets) using Algorithms 1 and 2 given that it requires us to compute a solution to the fixed-point problems (2.3) and (2.5), respectively.

In both fixed-point problems, for each n, the relevant set $\mathcal{D}^{(n)}$ is fixed. This leads to the following general lemma, which we will use to construct the solutions to these fixed-point problems.

Lemma 26. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system, $\mathcal{D} \subseteq \mathcal{N}$ some fixed set of $m := |\mathcal{D}|$ banks and $b \in \mathbb{R}_+^m$ some vector. Suppose that*

1. *$\gamma < 1$; or*

2. *$b_i > 0$ for all $i \in \mathcal{D}$.*

Then the system of m linear equations $x_i = b_i + \gamma \sum_{j \in \mathcal{D}} \Pi_{ji} x_j \quad \forall i \in \mathcal{D}$ has a unique non-negative solution.

Proof. The system of m linear equations has a unique solution, $x \in \mathbb{R}_+^m$, if it can be expressed as

$$x = \left(\mathrm{I} - \gamma \left(\Pi_{\mathcal{D}\mathcal{D}}\right)^\top\right)^{-1} b,$$

where $\left(\mathrm{I} - \gamma \left(\Pi_{\mathcal{D}\mathcal{D}}\right)^\top\right)$ is invertible.

We note that $\Pi_{\mathcal{D}\mathcal{D}}$ is a row-substochastic matrix. By Lemma 27.1 below, we only need to consider the case where $\gamma = 1$ and the spectral radius of $\Pi_{\mathcal{D}\mathcal{D}}$ is exactly 1. In this case, by Lemma 27.2 below, there is a set $\mathcal{C} \subseteq \mathcal{D}$ such that $\sum_{j \in \mathcal{C}} \Pi_{ij} = 1$ for each $i \in \mathcal{C}$. By assumption, if $\gamma = 1$ then $b_i > 0$ and so

$$\begin{aligned} x_i &= b_i + \sum_{j \in \mathcal{D}} \Pi_{ji} x_j \\ &\geqslant b_i + \sum_{j \in \mathcal{C}} \Pi_{ji} x_j > \sum_{j \in \mathcal{C}} \Pi_{ji} x_j. \end{aligned}$$

By summing x_i for all $i \in \mathcal{C}$, we arrive at the contradiction

$$\sum_{i \in \mathcal{C}} x_i > \sum_{j \in \mathcal{C}} x_j \left(\sum_{j \in \mathcal{C}} \Pi_{ji} \right) = \sum_{j \in \mathcal{C}} x_j .$$

Thus $\gamma < 1$ and so $\left(\mathrm{I} - \gamma \left(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}}\right)^\top\right)$ is invertible and x is the well-defined and unique solution to the system of linear equations.

Non-negativity of x also follows by Lemma 27.1 below. □

The technical Lemma 27, used in the proof of Lemma 26, is as follows.

Lemma 27. *Suppose* $\Pi \in \mathbb{R}_+^{N \times N}$ *is a row-substochastic matrix,* $0 \leqslant \varrho \leqslant 1$ *its spectral radius and* $0 \leqslant \gamma \leqslant 1$ *a constant.*

1. *If* $\gamma < 1$ *or* $\varrho < 1$ *then the matrix* $(\mathrm{I} - \gamma\Pi^\top)$ *is invertible and* $(\mathrm{I} - \gamma\Pi^\top)^{-1}$ *is non-negative.*
2. *If* $\gamma = 1$ *and* $\varrho = 1$ *then there exists a set* $\mathcal{C} \subseteq \mathcal{N}$ *such that for all* $i \in \mathcal{C}$ *we have that* $\sum_{j \in \mathcal{C}} \Pi_{ij} = 1$.

Proof. 1. If $\gamma = 0$ then $(\mathrm{I} - \gamma\Pi^\top) = \mathrm{I}$, which is clearly invertible with a non-negative inverse. So we assume that $0 < \gamma$.

Since ϱ is the spectral radius of Π, it is also the spectral radius of $\Pi^\top$. Since Π is a row substochastic matrix, we have that $\varrho \leqslant 1$. As $\Pi^\top$ is non-negative, standard results for M-matrices (see, for example, Theorem 2.5.3.2 and 2.5.3.17 in Horn & Johnson (1991)) imply that $(\alpha\mathrm{I} - \Pi^\top)$ is invertible with a non-negative inverse if and only if $\alpha > \varrho$. Set $\alpha = \gamma^{-1} > 0$. If $\gamma < 1$ then $\alpha > 1 \geqslant \varrho$ and if $\gamma = 1$ but $\varrho < 1$ then $\alpha = 1 > \varrho$. Hence $(\mathrm{I} - \gamma\Pi^\top) = \alpha^{-1}(\alpha\mathrm{I} - \Pi^\top)$ is invertible with a non-negative inverse.

2. As a standard result in the theory of finite-state Markov chains (see, for example, Theorem 2.1 in Karlin & Taylor (1981)), the number of sets $\mathcal{C} \subseteq \mathcal{N}$ satisfying the property that for all $i \in \mathcal{C}$ $\sum_{j \in \mathcal{C}} \Pi_{ij} = 1$ is equal to the multiplicity of the eigenvalue 1 of Π. Since $\varrho = 1$ by assumption, the multiplicity must be at least 1 and hence at least one such set $\mathcal{C}$ must exist.

□

Equipped with Lemma 26, we now consider the construction of the clearing solution under functional approach.

Proposition 28. *Let* $(a, L^{(s)}, L^{(l)}; \gamma)$ *be a financial system satisfying the Monotonicity Condition 14 such that* $a_i > 0$ *for all* $i \in \mathcal{N}$*. Then, for each* n*, the fixed-point problem (2.5) in Algorithm 2 has a unique non-negative solution given by*

$$v_i^{(n)} = \begin{cases} x_i, & \text{if } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j \in \mathcal{N} \setminus \mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j \in \mathcal{D}^{(n)}} \Pi_{ji} x_j, & \text{if } i \in \mathcal{N} \setminus \mathcal{D}^{(n)}, \end{cases}$$

where $x = \left(\mathrm{I} - \gamma \left(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}}\right)^\top\right)^{-1} \left(a_{\mathcal{D}^{(n)}} + \left(L^{(s)}_{\mathcal{L}^{(n)}\mathcal{D}^{(n)}}\right)^\top \mathbf{1}_{\mathcal{L}^{(n)}}\right)$ *and* $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$.

Proof. By Theorem 22, $\mathcal{D}^{(n)} \subseteq \mathcal{D}^{(n+1)} = D(v^{(n)})$ and, under the Monotonicity Condition 14, $v^{(n)}$ is a fixed point of $\Psi(\cdot;\mathcal{D}^{(n)})$. Then for all $j \in \mathcal{D}^{(n)}$ we have that $\bar{L}_j \wedge \gamma v_j^{(n)} = \gamma v_j^{(n)}$. Therefore the fixed-point problem (2.5) in Algorithm 2 is in a fact a system of linear equations:

$$v_i^{(n)} = \Psi_i(v^{(n)};\mathcal{D}^{(n)}) = a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j\in\mathcal{D}^{(n)}} \Pi_{ji} v_j^{(n)}, \tag{2.9}$$

for $i \in \mathcal{N}$. Moreover, it is sufficient to consider (2.9) only for $i \in \mathcal{D}^{(n)}$. Indeed, if $x \in \mathbb{R}_+^m$, where $m := |\mathcal{D}^{(n)}|$, is some such solution then we can simply set $v_i^{(n)} := x_i$ for $i \in \mathcal{D}^{(n)}$ and $v_i^{(n)} := a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L_{ji}^{(s)} + \gamma \sum_{j\in\mathcal{D}^{(n)}} \Pi_{ji} x_j$ for $i \in \mathcal{N} \setminus \mathcal{D}^{(n)}$.

Setting $b_i := a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L_{ji}^{(s)}$ for each $i \in \mathcal{D}^{(n)}$, we note that $b_i \geqslant a_i > 0$ for all $i \in \mathcal{D}^{(n)}$. Therefore, by Lemma 26, x is a unique solution to the system of linear equations (2.9) for $i \in \mathcal{D}^{(n)}$. In particular, letting $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$, we can write

$$x = \left(\mathrm{I} - \gamma\,(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}})^\top\right)^{-1} b,$$

where $\left(\mathrm{I} - \gamma\,(\Pi_{\mathcal{D}^{(n)}\mathcal{D}^{(n)}})^\top\right)$ is invertible and $b = a_{\mathcal{D}^{(n)}} + \left(L^{(s)}_{\mathcal{L}^{(n)}\mathcal{D}^{(n)}}\right)^\top \mathbf{1}_{\mathcal{L}^{(n)}}$.

Non-negativity of $v^{(n)}$ then follows by Lemma 26 and monotonicity of Ψ (Lemma 17). □

Proposition 28 allows us to explicitly construct functional liquid asset vectors for financial systems satisfying the Monotonicity Condition 14. This in turn lets us construct algorithmic liquid asset vectors for arbitrary financial systems, as shown in the next proposition. The key observation is that for each n in Algorithm 1, the banks in the set $\mathcal{D}^{(n)}$ can be treated as a financial system in its own right.

Proposition 29. *Let $(a, L^{(s)}, L^{(l)}; \gamma)$ be a financial system such that $a_i > 0$ for all $i \in \mathcal{N}$. For each n in Algorithm 1 with $\mathcal{D}^{(n)} \neq \varnothing$ we can construct a financial system S_n of $|\mathcal{D}^{(n)}| + 1$ banks such that S_n satisfies the Monotonicity Condition 14 and $v^{(n)}$, the solution to the fixed-point problem (2.3), is given by*

$$v_i^{(n)} = \begin{cases} x_i, & \text{if } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L_{ji}^{(s)} + \sum_{j\in\mathcal{D}^{(n)}} \Pi_{ji}(\bar{L}_j \wedge \gamma x_j), & \text{if } i \in \mathcal{N} \setminus \mathcal{D}^{(n)}, \end{cases}$$

where x is the greatest functional liquid asset vector of S_n.

The precise form of the system S_n is given in the proof below.

Proof. To simplify the notation we set $m := |\mathcal{D}^{(n)}|$ and in this proof assume that whenever, for some i, we let $1 \leqslant i \leqslant m$ that means that $i \in \mathcal{D}^{(n)}$. In this context, if $i = m+1$ then $i \notin \mathcal{N}$. Moreover, we set $\mathcal{L}^{(n)} := \mathcal{N} \setminus \mathcal{D}^{(n)}$ and let $b \in \mathbb{R}_+^{m+1}$, $\Lambda \in \mathbb{R}_+^{(m+1)\times(m+1)}$ be given by

$$b = \left(\begin{array}{c} a_1 + \sum_{j\in\mathcal{L}^{(n)}} L_{j1}^{(s)} \\ \vdots \\ a_m + \sum_{j\in\mathcal{L}^{(n)}} L_{jm}^{(s)} \\ \hline 0 \end{array}\right), \quad \Lambda = \left(\begin{array}{ccc|c} L_{11} & \cdots & L_{1m} & \frac{\bar{L}_1}{\gamma} - \sum_{k=1}^m L_{1k} \\ \vdots & & & \vdots \\ L_{m1} & \cdots & L_{mm} & \frac{\bar{L}_m}{\gamma} - \sum_{k=1}^m L_{mk} \\ \hline 0 & \cdots & 0 & 0 \end{array}\right),$$

It is clear that $\mathbf{Z}$, the $(m+1)\times(m+1)$ zero matrix, is a liability matrices. To see that Λ is a liability matrix, we need to check that the last column is nonnegative and all other properties follow immediately from the definition. For all $i \in \{1,\dots,m\}$ we have $\gamma\sum_{k=1}^m L_{ik} \leqslant \sum_{k=1}^m L_{ik} \leqslant \sum_{k=1}^N L_{ik} = \bar{L}_i$. Since $\bar{L}_i \geqslant \gamma\sum_{k=1}^m L_{ik} \Leftrightarrow \frac{\bar{L}_i}{\gamma} - \sum_{k=1}^m L_{ik} \geqslant 0$ the last column is indeed nonnegative.

So we define a financial system $S_n := (b,\Lambda,\mathbf{Z};1)$ on the set of $m+1$ banks containing $\mathcal{D}^{(n)}$.

Since S_n has no long-term liabilities, we denote both the short-term and overall total nominal liabilities vector of S_n by $\bar{\Lambda}$ and we immediately see that $\bar{\Lambda}_i = \frac{1}{\gamma}\bar{L}_i$ for $1 \leqslant i \leqslant m$ and $\bar{\Lambda}_{m+1} = 0$. Moreover, the short-term and overall relative liability matrices of S_n are also the same. Denoting them by $\Theta^{(s)}$ and Θ, respectively, we have that $\Theta^{(s)} = \Theta \geqslant 1\cdot\Theta$ and so the Monotonicity Condition 14 is satisfied. Note that for $1 \leqslant i,j \leqslant m$ we have

$$\Theta_{ij} = \frac{\Lambda_{ij}}{\bar{\Lambda}_i} = \frac{\gamma L_{ij}}{\bar{L}_i} = \gamma\Pi_{ij}.$$

Suppose that $x \in \mathbb{R}_+^{m+1}$ is some functional liquid asset vector of S_n with respect to

$$D(x) = \{i \in \{1,\dots,m,m+1\} \mid x_i < \bar{\Lambda}_i\} = \{i \in \mathcal{D}^{(n)} \mid \gamma x_i < \bar{L}_i\},$$

where we used the convention that the m elements of $\mathcal{D}^{(n)}$ are labelled by $1,\dots,m$, and the last equality holds because $\bar{\Lambda}_{m+1} = 0$ and hence the index $m+1$ will never be in the default set.

Since x is a functional liquid asset vector and since $\Lambda^{(s)}_{m+1i} = 0$ for all $i \in \mathcal{D}^{(n)}$, we have for each $i \in \mathcal{D}^{(n)}$

$$\begin{aligned}
x_i = \widetilde{\Psi}_i(x) &= b_i + \sum_{j\in\mathcal{D}^{(n)}\setminus D(x)} \Lambda_{ji} + 1\cdot\sum_{j\in D(x)}\Theta_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j\in\mathcal{D}^{(n)}\setminus D(x)} L_{ji} + \sum_{j\in D(x)}\gamma\Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j\in\mathcal{D}^{(n)}\setminus D(x)} \Pi_{ji}\bar{L}_j + \gamma\sum_{j\in D(x)}\Pi_{ji}x_j \\
&= a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j\in\mathcal{D}^{(n)}}\Pi_{ji}(\bar{L}_j \wedge \gamma x_j) \\
&= \Psi_i(x;\mathcal{D}^{(n)}).
\end{aligned} \tag{2.10}$$

Then, we set

$$v_i^{(n)} = \begin{cases} x_i & \text{for } i \in \mathcal{D}^{(n)}, \\ a_i + \sum_{j\in\mathcal{N}\setminus\mathcal{D}^{(n)}} L^{(s)}_{ji} + \sum_{j\in\mathcal{D}^{(n)}}\Pi_{ji}(\bar{L}_j\wedge\gamma x_j) & \text{for } i \in \mathcal{N}\setminus\mathcal{D}^{(n)}. \end{cases} \tag{2.11}$$

Note that $\Psi(v^{(n)};\mathcal{D}^{(n)})$ does not depend on $v_i^{(n)}$ for $i \in \mathcal{N}\setminus\mathcal{D}^{(n)}$. Hence, from (2.10) we immediately see that $v_i^{(n)} = \Psi_i(v^{(n)};\mathcal{D}^{(n)})$ for all $i \in \mathcal{D}^{(n)}$.
Furthermore, for all $i \in \mathcal{N}\setminus\mathcal{D}^{(n)}$ we have by (2.11) that $v_i^{(n)} = \Psi_i(v^{(n)};\mathcal{D}^{(n)})$.
Hence we have shown that $\Psi((v^{(n)});\mathcal{D}^{(n)}) = v^{(n)}$. □

Remark 30. We showed in Proposition 24 that clearing in the Rogers & Veraart (2013) model can be formulated in terms of the functional liquid asset vector. In that paper it was observed that, unlike in

Eisenberg & Noe (2001), even when $a > 0$ the clearing vectors are not necessarily unique and therefore the same observation must hold of functional liquid asset vectors.

One interesting consequence of Lemma 26 is that it implies that there are at most a finite number of functional liquid asset vectors for any given financial system with $a > 0$. This follows from the fact that there are only a finite number of possible default sets and for each such possible default set there is at most one v satisfying Definition 11.

2.3.5 Uncertainty of the maturity profile

The ability to construct algorithmic liquid asset vectors and default sets for any financial system allows us to demonstrate that the maturity profile of a financial system has a substantial impact on which banks can default.

Proposition 31. *There exists a financial system $S_1 = (a, L^{(s)}, L^{(l)}; \gamma)$ with the algorithmic default set $\mathcal{D}_1^*$ such that the financial system $S_2 := (a, L^{(s)} + L^{(l)}, \mathbf{Z}; \gamma)$, where $\mathbf{Z}$ is a zero matrix, has the algorithmic default set $\mathcal{D}_2^*$ that satisfies $\mathcal{D}_2^* \subsetneq \mathcal{D}_1^*$*

Proof. Let $S_1 = (a, L^{(s)}, L^{(l)}; 1)$ denote the financial system introduced in the proof of Proposition 20.2 and also used in the proof of Proposition 23 above. In Algorithm 1, using the construction in Proposition 29, the sequences $v^{(n)}$ and $D(v^{(n)})$ would evolve as follows

$$
\begin{aligned}
v^{(0)} &= (3, 100, 110) & D(v^{(0)}) &= \{1\} \\
v^{(1)} &= (3, 99\frac{1}{2}, 109\frac{1}{2}) & D(v^{(1)}) &= \{1, 2\} \\
v^{(2)} &= (53\frac{1}{50}, 102, 65\frac{1}{25}) & D(v^{(2)}) &= \{2\}
\end{aligned}
$$

Thus we conclude that $v^* = v^{(2)}$ and $\mathcal{D}_1^* = \{1, 2\}$.

Now let $S_2 = (a, L^{(s)} + L^{(l)}, \mathbf{Z}; 1)$. Then we can verify that the vector v^*, obtained above, is also the unique functional liquid asset vector of S_2 with the functional default set $D(v^*) = \{2\}$. By Remark 16, S_2 satisfies the Monotonicity Condition 14 and hence by Theorem 22 $\mathcal{D}_2^* := \{2\}$ is the algorithmic default set of S_2. □

In Proposition 31 the system S_2 has the same overall interbank liabilities as S_1 but all the interbank liabilities are now short-term liabilities. The proposition shows that if we treat all maturities to be the same then we could end up with the financial system S_2 in which fewer banks default than if we account for the different maturity dates as in S_1. Therefore, this shows that approximating multiple maturity systems by single maturity systems can underestimate the severity of the risk of default. More generally, any uncertainty about the maturity profile in a financial system is itself a potential source of systemic risk.

This observation is particularly pertinent because in practice regulators do not have precise information about the banks' maturity profiles. Typically regulatory reports group liabilities into broad categories without recording the exact maturity dates. According to Langfield et al. (2014), in the UK, "banks report exposures with breakdown by the maturity of the instrument" and "Categories of maturities are: open; less than 3 months; between 3 months and 1 year; between 1 year and 5 years; and more than 5 years. Derivatives are not reported with a maturity breakdown." It is therefore an open question whether these five categories are a sufficient representation of the maturity profile in the UK financial system for the purposes of assessing systemic risk.

2.4 Financial system after the first clearing

2.4.1 Stylised balance sheet after clearing at the first maturity date

Let us denote the financial system $(a, L^{(s)}, L^{(l)}; \gamma)$ that we have been considering so far by $S(0)$ to indicate that it represents the system at time $t = 0$, prior to clearing at $t = T_1$. Following clearing at $t = T_1$ using the algorithmic approach described above, we obtain the algorithmic liquid asset vector and the algorithmic default set, which we now denote by $v^*(T_1)$ and $\mathcal{D}^*(T_1)$. This allows us to formulate a new financial system $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$ of banks in some set $\mathcal{N}(T_1) \subseteq \mathcal{N}$ after clearing at $t = T_1$. The banks that defaulted as part of the clearing at $t = T_1$ are no longer a part of the financial system and so

$$\mathcal{N}(T_1) = \mathcal{N} \setminus \mathcal{D}^*(T_1). \tag{2.12}$$

Note that the sink node $N \in \mathcal{N}$ does not default as it has no liabilities and hence $N \in \mathcal{N}(T_1)$. We assume that the only changes between $t = 0$ and $t = T_1$ are attributable exclusively to the clearing process itself. Thus the new cash assets $a(T_1)$ are just the liquid assets of banks in $\mathcal{N}(T_1)$ less their payments at T_1. Since the banks that do not make their full payments at T_1 default and are not in $\mathcal{N}(T_1)$ it follows that for all $i \in \mathcal{N}(T_1)$,

$$a(T_1)_i = v^*(T_1)_i - \bar{L}_i^{(s)}. \tag{2.13}$$

The new short-term liabilities at T_1 are just the remaining liabilities of the banks in $\mathcal{N}(T_1)$ that were not due at T_1. Thus for all $i, j \in \mathcal{N}(T_1)$ such that $i, j \neq N$

$$L^{(s)}(T_1)_{ij} = L_{ij}^{(l)}. \tag{2.14}$$

With the sink node N the situation is somewhat different. The banks in $\mathcal{N}(T_1)$ that had outstanding long-term liabilities to the banks in $\mathcal{D}^*(T_1)$ that defaulted at T_1 do not escape those liabilities by virtue of the latter defaults. In reality, those liabilities comprise assets of the banks in $\mathcal{D}^*(T_1)$ and these assets typically would be re-distributed at an auction. However, modelling such auctions is outside the scope of this thesis and in order to keep the model clear we simply assume that they are 'acquired' by the sink node N. Moreover, as before, we continue with the assumption that the sink node has no liabilities.

Hence for all $i, j \in \mathcal{N}(T_1)$

$$L^{(s)}(T_1)_{iN} = L^{(l)}_{iN} + \sum_{k \in \mathcal{D}^*(T_1)} L^{(l)}_{ik}, \tag{2.15}$$

$$L^{(s)}(T_1)_{Nj} = L^{(l)}(T_1)_{Nj} = 0. \tag{2.16}$$

In particular, it follows that $\bar{L}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}(T_1) \cup \mathcal{D}^*(T_1)} L^{(l)}_{ij} = \bar{L}^{(l)}_i$ for all $i \in \mathcal{N}(T_1)$.

Furthermore, since these are the only liabilities of banks $\mathcal{N}(T_1)$ at $t = T_1$, we also have that for all $i, j \in \mathcal{N}(T_1)$ there are no new long-term liabilities:

$$L^{(l)}(T_1)_{ij} = 0. \tag{2.17}$$

The following proposition confirms that we have indeed constructed a new financial system.

Proposition 32. *Let $\mathcal{N}(T_1)$ be a set given in (2.12). The tuple $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$ satisfying (2.13) — (2.17) is a financial system.*

Proof. We need to show that $a(T_1)$ is non-negative and $L^{(s)}(T_1)$ and $L^{(l)}(T_1)$ are liability matrices.

By construction of Algorithm 1 $v^*(T_1) = \Psi(v^*(T_1); \mathcal{D}^*(T_1))$ such that $D(v^*(T_1)) \subseteq \mathcal{D}^*(T_1)$. Suppose there is some $i \in \mathcal{N}(T_1)$ such that $v^*(T_1)_i < \bar{L}^{(s)}_i$. Then $i \in D(v^*(T_1))$ and so $i \notin \mathcal{N}(T_1)$. Hence, for all $i \in \mathcal{N}(T_1)$, $a(T_1)_i = v^*(T_1)_i - \bar{L}^{(s)}_i \geqslant 0$.

The fact that $L^{(s)}(T_1)$ and $L^{(l)}(T_1)$ are liability matrices follows from the definitions since it is immediately clear they are non-negative matrices with zero diagonals. □

The stylised balance sheet of each bank except the sink node in this new financial system is given by Table 2.2. The sink node in the new financial system has no cash assets or short-term interbank liabilities and hence $E(T_1)_N = \bar{A}^{(s)}(T_1)_N$. Its short-term interbank loans are given by $\bar{A}^{(s)}(T_1)_N = \sum_{j \in \mathcal{N}(T_1)} L^{(l)}_{jN} + \sum_{j \in \mathcal{N}(T_1)} \sum_{k \in \mathcal{D}^*(T_1)} L^{(l)}_{jk}$.

Assets	Liabilities
• Cash assets: $a(T_1)_i = v^*(T_1)_i - \bar{L}^{(s)}_i$	• Short-term interbank liabilities: $\bar{L}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}} L^{(l)}_{ij}$
• Short-term interbank loans: $\bar{A}^{(s)}(T_1)_i = \sum_{j \in \mathcal{N}(T_1)} L^{(l)}_{ji}$	• Equity: $E(T_1)_i = a(T_1)_i + \bar{A}^{(s)}(T_1)_i - \bar{L}^{(s)}(T_1)_i$

Table 2.2: Stylised balance sheet at $t = T_1$ of bank $i \in \mathcal{N}(T_1) \setminus \{N\}$ after clearing.

2.4.2 Clearing at the second maturity date

The financial system $S(T_1)$, described in Section 2.4.1, can be cleared again by the application of Algorithm 1. In fact, by Remark 16, $S(T_1)$ satisfies the Monotonicity Condition 14 and so can be cleared by the application of the simpler Algorithm 2. Moreover, by Propositions 25 and 24, we can see that at the last maturity the financial system is reducible to the familiar models of Eisenberg & Noe (2001) or Rogers & Veraart (2013).

Let $\tilde{v}^*(T_2)$ and $\tilde{D}^*(T_2)$ be the output of Algorithm 2 applied to the financial system $S(T_1)$. Then, after clearing at $t = T_2$, we obtain a new financial system $S(T_2)$ consisting of banks in the set $\mathcal{N}(T_2) := \mathcal{N}(T_1) \setminus \tilde{D}^*(T_2)$. Since the banks in $\mathcal{N}(T_2)$ have only cash assets and no liabilities, this system is given by $S(T_2) := (a(T_2), \mathbf{Z}, \mathbf{Z}; \gamma)$. Thus S_2 is characterised by the cash assets given by

$$a(T_2)_i = \tilde{v}^*(T_2)_i - \bar{L}^{(s)}(T_1) \qquad \forall i \in \mathcal{N}(T_2).$$

We also have that $\bar{A}^{(s)}(T_2) = \bar{L}^{(s)}(T_2) = \mathbf{0}$ and hence $a(T_2)_i = E(T_2)_i$ for all $i \in \mathcal{N}(T_2)$. Moreover no further clearing of $S(T_2)$ is necessary.

2.4.3 Extension to more than two maturity dates

So far we have focused on financial systems with at most two maturities. However, provided we track the precise maturity profile of all the liabilities amalgamated in the long-term liability matrix $L^{(l)}$, we can readily extend our modelling framework to $n > 2$ maturity dates $0 < T_1 < T_2 < \ldots < T_n$.

We write $L^{(T_i)} \in \mathbb{R}_+^{N \times N}$ for the matrix containing all interbank liabilities maturing at T_i, $i \in \{1, \ldots, n\}$. We then consider an n-maturity financial system as a tuple $S = (a, L^{(T_1)}, L^{(T_2)}, \ldots, L^{(T_n)}; \gamma)$. At $t = 0$ we can define a 2-maturity financial system $S(0) := (a, L^{(s)}, L^{(l)}; \gamma)$ given by $L^{(s)} := L^{(T_1)}$ and $L_{ij}^{(l)} := \sum_{\tau=2}^{n} L_{ij}^{(T_\tau)}$ for all $i, j \in \mathcal{N}$. Then clearing the n-maturity financial system S at time $t = T_1$ reduces to clearing the 2-maturity financial system $S(0)$ at time $t = T_1$ using Algorithm 1 and, using the methodology similar to the one described in Section 2.4.1, produces a new 2-maturity financial system $S(T_1) := (a(T_1), L^{(s)}(T_1), L^{(l)}(T_1); \gamma)$.

The new liquid assets vector $a(T_1)$ is as in (2.13) and only the definition of the new short-term and new long-term interbank liability matrices change so that the liabilities maturing at $t = T_2$ become the new short-term liabilities and all liabilities maturing at $t \geqslant T_3$ are aggregated into the new long-term liabilities. Thus we obtain, for all $i, j \in \mathcal{N}(T_1)$ with $i, j \neq N$, that

$$\begin{aligned}
L^{(s)}(T_1)_{ij} &= L_{ij}^{(T_2)}, \\
L^{(s)}(T_1)_{iN} &= L_{iN}^{(T_2)} + \sum_{k \in \mathcal{D}^*(T_1)} L_{ik}^{(T_2)}, \\
L^{(l)}(T_1)_{ij} &= \sum_{\tau=3}^{n} L_{ij}^{(T_\tau)}, \\
L^{(l)}(T_1)_{iN} &= \sum_{\tau=3}^{n} L_{iN}^{(T_\tau)} + \sum_{k \in \mathcal{D}^*(T_1)} \sum_{\tau=3}^{n} L_{ik}^{(T_\tau)}, \\
L^{(s)}(T_1)_{Nj} &= L^{(l)}(T_1)_{Nj} = 0
\end{aligned}$$

Similarly, we can clear $S(T_2)$ using our methodology for two maturities and then repeat this approach until we reach the point $t = T_{n-1}$ where, for all $i, j \in \mathcal{N}(T_1)$ with $i, j \neq N$,

$$\begin{aligned} L^{(s)}(T_{n-1})_{iN} &= L_{ij}^{(T_n)}, \\ L^{(s)}(T_{n-1})_{Nj} &= 0, \\ L^{(s)}(T_{n-1})_{iN} &= L_{ij}^{(T_n)} + \sum_{k \in \mathcal{D}^*(T_{n-1})} L_{ik}^{(T_n)}, \end{aligned}$$

and $L^{(l)}(T_{n-1}) = \mathbf{Z}$.

This system can now be cleared using Algorithm 2, analogously to what we did in Section 2.4.2. In the end we obtain the last financial system $S(T_n) := (a(T_n), \mathbf{Z}, \mathbf{Z}; \gamma)$ such that $a(T_n) = E(T_n)$ and no further clearing is necessary.

2.5 Conclusion

This chapter has developed a rigorous clearing framework for interbank networks with multiple maturities. We have shown that a vector of clearing cash flows (a vector of liquid assets, in our case) on its own is not sufficient to fully describe the clearing framework. A suitable definition of the set of banks in default is needed. This does not arise naturally from the description of the stylised balance sheets and must be specified as part of the model. We discussed the necessary conditions on such a default set. These conditions are not sufficient and we considered the algorithmic approach and the functional approach as two possible approaches to specifying default.

The functional default set corresponds to the definitions that have been used in prior literature and has a simple functional representation. It does not have an absorbing property and, as a consequence, a liquid asset vector using the functional default set may not exist for every financial system. On the other hand, the algorithmic default set has a more complex algorithmic definition that guarantees that default is an absorbing state. Therefore the algorithmic liquid asset vector can be found for any financial system. We proposed Algorithm 1, which produces a sequence of vectors that converges to the algorithmic liquid asset vector. This sequence of vectors is not in general monotonic but the absorption property of the default sets ensures the algorithm converges in a finite number of steps.

The functional approach has a number of uses despite restrictions on the existence of functional liquid asset vectors. We have shown that for certain types of financial systems the algorithmic approach reduces to the functional approach. Furthermore, we have shown that the functional approach reduces to the models by Eisenberg & Noe (2001) and Rogers & Veraart (2013) if only one maturity is considered. In addition, we have shown that functional liquid asset vectors can be used in the construction of clearing solutions under the algorithmic approach. For these reasons the properties of functional liquid asset vectors are important. We have shown that under a regularity condition functional liquid asset vectors can be characterised as fixed points and a greatest and a least functional liquid asset vectors exist. We have also shown that functional liquid asset vectors are in general not unique but under a mild condition we could show that there can be at most one such vector corresponding to any given default set.

We have illustrated two key applications of Algorithm 1. We demonstrated that the default risk of a bank depends in a non-trivial manner on the precise maturity profile of its liabilities. Relying on the assumption that all interbank liabilities have the same maturity can lead to an inaccurate assessment of risks. Our clearing approach provides a rigorous tool to incorporate different maturities in the clearing process. We also showed how to extend the model to a multi-period one by describing a settlement mechanism, which characterises the stylised balance sheets of the surviving banks after clearing.

3

Least Fixed Point in Clearing Problems

3.1 Introduction

This chapter steps back from the multiple maturity setting of Chapter 2 and focuses on the extension of the Rogers & Veraart (2013) model. In Eisenberg & Noe (2001) it was shown that for regular financial systems the vector of clearing payments obtained by means of the Fictitious Default Algorithm is in fact the unique fixed point of the clearing function. In many extensions the uniqueness of the fixed points is lost. In such cases it is conventional to formulate the clearing problem as a search for the *greatest* fixed point of a suitable clearing function. However, as shown in Rogers & Veraart (2013), this may not be the only fixed point. In this chapter we therefore look at the converse problem of finding the *least* fixed point.

While we focus on the theoretical side of the problem, there is scope for applying the model presented in this chapter. In the classical setting, such as described in Eisenberg & Noe (2001), Rogers & Veraart (2013) and Chapter 2, the interbank network represents nominal exposures. The greatest fixed point of a suitable clearing function gives us the most conservative solution to the clearing problem of finding the vector of effective payments. The conventional rationale is that the market participants will value their counterparties' exposures at the maximal nominal value and then re-value downwards until the clearing values are obtained. However, for the purposes of stress testing carried out by a regulator it may be valuable to compute the worst case solution to the problem. This can be accomplished by finding the least solution to a clearing problem.

An alternative application stems from insights in Hurd (2016) that interbank networks can be interpreted not as networks of exposures but as networks of funding supply. To this end we also consider a simplified model of roll-over credit. We emphasise that this is intended to be a stylised case study of applications of the least fixed point and no claims are made about the suitability of such a model as a model of any specific financial instrument. Nevertheless, a number of financial instruments used in practice, including repurchase agreements and lines of credit, have features that we model. This suggests that there is scope and merit in investigating applications of the least fixed point of clearing models.

3.2 General setting and the least fixed point

We study a class of clearing functions Φ described in Definition 36 below and, in particular, we are interested in the existence and construction of the least fixed point of Φ.

Recall that for a matrix M, $||M||_1$ is the norm given by greatest column sum of the absolute values of the entries of M. For a non-negative M, $||M||_1$ is just the greatest column sum of M.

Definition 33 (Clearing system)**.** Let $\mathcal{N} = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, $a, \bar{B} \in \mathbb{R}_+^N$, $\bar{L} > \mathbf{0} \in \mathbb{R}_+^N$ with $\bar{B} < \bar{L}$ and $\alpha, \beta \in (0, 1]$.

Furthermore, let $\Omega \in \mathbb{R}_+^{N \times N}$ be a matrix with a zero diagonal such that $||\Omega||_1 \leqslant 1$ and for all $j \in \mathcal{N}$ there exists $i \in \mathcal{N}$ with $\Omega_{ji} > 0$. We refer to the tuple $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ as a *clearing system* on $\mathcal{N}$.

Definition 33 extends the definition of a financial system in Rogers & Veraart (2013) (and consequently the classical definition in Eisenberg & Noe (2001)). The main difference is the inclusion of the lower bound $\bar{B}$. In Rogers & Veraart (2013) the clearing process is carried out on $[\mathbf{0}, \bar{L}]$, i.e. the components of clearing vectors are permitted to take any non-negative values as long as they do not exceed the corresponding components of $\bar{L}$. We consider a more general setting where we look for clearing vectors on $[\bar{B}, \bar{L}]$. The second difference is that we do not assume that the matrix Ω is necessarily a relative liability matrix. This allows us to apply our theory in a wider range of scenarios.

Remark 34. Suppose that $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ is a clearing system with $\bar{B} = \mathbf{0}$ and Ω such that $||\Omega||_1 = 1$. Let L be a matrix given by $L_{ij} = \Omega_{ji} \bar{L}_i$ for all i and j. Then the triplet (L, a, α, β) is a "financial system" within the meaning of Rogers & Veraart (2013). Conversely, if (L, a, α, β) is such a financial system and Ω is given by $\Omega_{ij} = L_{ji} / \bar{L}_i$ for $\bar{L}_i > 0$ and $\Omega_{ij} = 0$ otherwise then $(a, \Omega, \mathbf{0}, \bar{L}, \alpha, \beta)$ is a clearing system and $||\Omega||_1 = 1$.

As stated in Chapter 2 Proposition 24, if $\alpha = \beta$ then the financial system (L, a, α, β) in the sense of Rogers & Veraart (2013) is a single maturity financial system in the sense of Chapter 2. Hence a clearing system $(a, \Omega, \mathbf{0}, \bar{L}, \alpha, \beta)$ for $\alpha = \beta$ likewise can be seen as a single maturity financial system.

Definition 35. A clearing system is said to be *regular* if, whenever $||\Omega||_1 = 1$, $(\Omega^\top, \bar{L}, a)$ is a regular financial system within the meaning of Definition 5 in Eisenberg & Noe (2001).

Note that, in particular, a financial system is regular if $||\Omega||_1 < 1$ or $a > 0$.

Definition 36. Let $\mathcal{N} = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$ and $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ a clearing system on $\mathcal{N}$. We then define functions $V_{\alpha,\beta}, V : \mathbb{R}_+^N \to \mathbb{R}_+^N$ as follows:

$$V_{\alpha,\beta}(x) := \alpha a + \beta \Omega x$$
$$V(x) := V_{1,1}(x) = a + \Omega x.$$

The *clearing function* $\Phi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is given by

$$\begin{aligned}\Phi(x)_i &= \begin{cases} \bar{L}_i & \text{if } V(x)_i \geqslant \bar{L}_i \\ \bar{B}_i \vee V_{\alpha,\beta}(x)_i & \text{if } V(x)_i < \bar{L}_i \end{cases} \\ &= \begin{cases} \bar{L}_i & \text{if } (a + \Omega x)_i \geqslant \bar{L}_i \\ \bar{B}_i \vee (\alpha a + \beta \Omega x)_i & \text{if } (a + \Omega x)_i < \bar{L}_i \end{cases}\end{aligned}$$

for each $i \in \mathcal{N}$.

3.2.1 Basic properties

The following propositions give some basic properties of $V_{\alpha,\beta}$, V and Φ which are used throughout this chapter, often without explicit reference.

Proposition 37. *Let $c > 0$ be some constant and Φ and Φ_c be the clearing functions of the clearing systems $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ and $(ca, \Omega, c\bar{B}, c\bar{L}, \alpha, \beta)$, respectively. If x is a fixed point of Φ then cx is a fixed point of Φ_c. Furthermore, if x is a least (respectively, greatest) fixed point of Φ then cx is a least (respectively, greatest) fixed point of Φ_c.*

Proof. Let $i \in \mathcal{N}$ and suppose x is a fixed point of Φ. In particular, $\Phi(x)_i = x_i$. Then

$$\begin{aligned}\Phi_c(cx)_i &= \begin{cases} c\bar{L}_i & \text{if } (ca + \Omega cx)_i \geqslant c\bar{L}_i \\ c\bar{B}_i \vee (\alpha ca + \beta \Omega cx)_i & \text{if } (ca + \Omega cx)_i < c\bar{L}_i \end{cases} \\ &= c \begin{cases} \bar{L}_i & \text{if } (a + \Omega x)_i \geqslant \bar{L}_i \\ \bar{B}_i \vee (\alpha a + \beta \Omega x)_i & \text{if } (a + \Omega x)_i < \bar{L}_i \end{cases} \\ &= c\Phi(x)_i = cx_i.\end{aligned}$$

Hence cx is a fixed point of Φ_c.

Furthermore, since scaling by a positive constant is a strictly monotonic transformation, the ordering of fixed points is preserved. □

Proposition 38.

1. *For any $\alpha, \beta \in (0, 1]$, $V_{\alpha,\beta}$ is continuous and monotonically increasing; i.e. $V_{\alpha,\beta}(x) \geqslant V_{\alpha,\beta}(y)$ whenever $x \geqslant y$.*
2. *For any $\alpha, \beta \in (0, 1]$ and $x \in \mathbb{R}^N$, $V_{\alpha,\beta}(x) \leqslant V(x)$.*
3. *$V(x)_i < \bar{L}_i$ if and only if $\Phi(x)_i < \bar{L}_i$.*
4. *Φ takes values in $[\bar{B}, \bar{L}]$.*
5. *Φ is monotonically increasing.*

Proof. 1. Continuity of $V_{\alpha,\beta}$ follows by composition of continuous functions. For monotonicity, suppose $x \geqslant y$ for some $x, y \in \mathbb{R}^N$. Then $V_{\alpha,\beta}(x) - V_{\alpha,\beta}(y) = \beta\Omega(x-y) \geqslant 0$ since Ω and β are both non-negative and $x-y \geqslant 0$ by assumption. Therefore $V_{\alpha,\beta}(x) \geqslant V_{\alpha,\beta}(y)$ for any $\alpha, \beta \in (0,1]$.

2. Since $\alpha, \beta \in (0,1]$, it is clear that $V(x) = V_{1,1}(x) = a + \Omega x \geqslant \alpha a + \beta\Omega x = V_{\alpha,\beta}(x)$.

3. Let $i \in \mathcal{N}$. If $V(x)_i < \bar{L}_i$ then $\Phi(x)_i = \bar{B}_i \vee V_{\alpha,\beta}(x)_i \leqslant \bar{B}_i \vee V(x)_i$. Since $\bar{B} < \bar{L}$ by assumption, it follows that $\Phi(x)_i < \bar{L}_i$. Conversely, if $\Phi(x)_i < \bar{L}_i$ then $V(x)_i < \bar{L}_i$ as otherwise we would have that $\Phi(x)_i = \bar{L}_i$, by definition.

4. Let $i \in \mathcal{N}$. If $V(x)_i \geqslant \bar{L}_i$ then $\bar{B}_i < \Phi(x)_i = \bar{L}_i$. If $V(x)_i < \bar{L}_i$ then

$$\begin{aligned}\bar{B}_i \leqslant \bar{B}_i \vee V_{\alpha,\beta}(x)_i &= \Phi(x)_i \\ &\leqslant \bar{B}_i \vee V(x)_i < \bar{B}_i \vee \bar{L}_i = \bar{L}_i.\end{aligned}$$

Therefore $\Phi(x)_i \in [\bar{B}_i, \bar{L}_i]$ for all $i \in \mathcal{N}$.

5. Suppose $x \geqslant y$. Letting $\alpha = \beta = 1$ in Proposition 38.1 above, we have that $V(x) \geqslant V(y)$. Let $i \in \mathcal{N}$ and suppose $V(y)_i \geqslant \bar{L}_i$. Then $V(x)_i \geqslant V(y)_i \geqslant \bar{L}_i$. Therefore $\Phi(x) = \bar{L}_i = \Phi(y)$.

Suppose instead that $V(x)_i < \bar{L}_i$. Then $V(y)_i < \bar{L}_i$ and so

$$\begin{aligned}\Phi(x)_i &= \bar{B}_i \vee V_{\alpha,\beta}(x)_i \\ &\geqslant \bar{B}_i \vee V_{\alpha,\beta}(y)_i = \Phi(y)_i.\end{aligned}$$

Finally, suppose that $V(x)_i \geqslant \bar{L}_i > V(y)_i$. Note that $\bar{L} > \bar{B}$ by assumption. Then

$$\begin{aligned}\Phi(x)_i = \bar{L}_i &> \bar{B}_i \vee V(y)_i \\ &\geqslant \bar{B}_i \vee V_{\alpha,\beta}(y)_i = \Phi(y)_i.\end{aligned}$$

This covers all cases and hence $\Phi(x) \geqslant \Phi(y)$ whenever $x \geqslant y$.

□

Proposition 38 tells us that $\bar{L}$ is an upper bound for the values both of Φ and V. However, since Φ is expressed in terms of $V_{\alpha,\beta}$ and $V_{\alpha,\beta}(x) \leqslant V(x)$ for all x, it should be possible to obtain a tighter upper bound for Φ and $V_{\alpha,\beta}$ than $\bar{L}$. The following Lemma 39 characterises such an upper bound, which we denote by $\bar{T}$. $\bar{T}$ is an important vector that we use extensively and, in particular, Lemma 39 also establishes the link between V and $V_{\alpha,\beta}$ through $\bar{T}$.

Lemma 39. *Let* $\bar{T} := \beta\bar{L} + (\alpha - \beta)a$.

1. *If there exists some* $x \in \mathbb{R}^N_+$ *such that* $V(x)_i < \bar{L}_i$ *for some* $i \in \mathcal{N}$ *then* $\bar{T}_i \leqslant \bar{L}_i$.

2. *For all* $x \in \mathbb{R}^N_+$ *and* $i \in \mathcal{N}$, $V(x)_i < \bar{L}_i$ *if and only if* $V_{\alpha,\beta}(x)_i < \bar{T}_i$.

3. *For all* $x \in \mathbb{R}^N_+$ *and* $i \in \mathcal{N}$, $\bar{B}_i < \Phi(x)_i < \bar{L}_i$ *if and only if* $\bar{B}_i < \Phi(x)_i = V_{\alpha,\beta}(x)_i < \bar{T}_i$.

Proof. Let $i \in \mathcal{N}$.

1. If $V(x)_i < \bar{L}_i$ for some $i \in \mathcal{N}$ then $a_i \leqslant V(x)_i < \bar{L}_i$. Therefore if $\beta \leqslant \alpha$ then $\bar{T}_i = \beta\bar{L}_i + (\alpha-\beta)a_i \leqslant \beta\bar{L}_i + (\alpha-\beta)\bar{L}_i = \alpha\bar{L}_i \leqslant \bar{L}_i$. If $\beta > \alpha$ then $\bar{T}_i = \beta\bar{L}_i + (\alpha-\beta)a_i \leqslant \beta\bar{L}_i \leqslant \bar{L}_i$.

2. We observe that, since β is positive, $V(x)_i < \bar{L}_i$ if and only if $\beta V(x)_i < \beta\bar{L}_i$. Furthermore, $\beta V(x) = V_{\alpha,\beta}(x) + (\beta-\alpha)a$. So, for all $i \in \mathcal{N}$, $V(x)_i < \bar{L}_i$ if and only if $V_{\alpha,\beta}(x)_i < \beta\bar{L}_i - (\beta-\alpha)a_i = \bar{T}_i$.

3. Let $i \in \mathcal{N}$. By Lemma 39.2, we can now see that $\bar{B}_i < V_{\alpha,\beta}(x)_i < \bar{T}_i$ if and only if $V(x)_i < \bar{L}_i$ and $\bar{B}_i < \bar{B}_i \vee V_{\alpha,\beta}(x)_i$.

   Suppose that $V(x)_i < \bar{L}_i$ and $\bar{B}_i < \bar{B}_i \vee V_{\alpha,\beta}(x)_i$. This implies that $V_{\alpha,\beta}(x)_i = \bar{B}_i \vee V_{\alpha,\beta}(x)_i = \Phi(x)_i$. Therefore $\bar{B}_i < \Phi(x)_i$ and $\Phi(x)_i = V_{\alpha,\beta}(x)_i \leqslant V(x)_i < \bar{L}_i$. Conversely, if we suppose that $\bar{B}_i < \Phi(x)_i < \bar{L}_i$ then by definition we must have that $V(x)_i < \bar{L}_i$ and $\bar{B}_i < \Phi(x)_i = \bar{B}_i \vee V_{\alpha,\beta}(x)_i$.

□

Remark 40. Since Φ takes values in $[\bar{B},\bar{L}]$, we will slightly abuse the notation and use Φ to refer to the function $x \mapsto \Phi(x)$ with $[\bar{B},\bar{L}]$ as both domain and co-domain.

3.2.2 Transfinite sequences

Note that, by Remark 2 in Chapter 1, the interval $[\bar{B},\bar{L}] \subseteq \mathbb{R}^N_+$ is a complete lattice under the component-wise ordering. This allows us to apply the same approach as Eisenberg & Noe (2001) and Rogers & Veraart (2013) to finding the least fixed point of Φ. Namely, we will define a recursive sequence $\left(\Phi^k\right)_{k\geqslant 0}$ of iterates of Φ and apply the Tarski-Knaster Theorem (Theorem 1, Chapter 1) to show that that the limit of this sequence is the fixed point that we seek. In Rogers & Veraart (2013) the sequence starts at $\Phi^0 := \bar{L}$, the upper bound of Φ, and monotonically converges downwards to the greatest fixed point of Φ. Since Φ is continuous from above, it is shown that an ordinary sequence of iterates is sufficient to obtain the required fixed point in the limit.

Rogers & Veraart (2013) observe that in order to obtain the least fixed point of Φ we would need to start the recursive sequence at the lower bound, **0** in their special case. However, considering the lower bound $\bar{B}$ more generally, the usual limit $\lim_{k\to\infty} \Phi^k(\bar{B})$ may fail to be a fixed point of Φ since Φ is not in general continuous from below. Rogers & Veraart (2013) provide an explicit example of this phenomenon and describe how this difficulty can be avoided by ‘restarting’ the sequence from the limit value until a fixed point is obtained. However, since finding the least fixed point is not an objective in Rogers & Veraart (2013), full technical details were not provided. This chapter provides a rigorous description of this approach for the general case of the lower bound $\bar{B}$. It turns out that convenient mathematical tools for treating this problem are transfinite sequences and transfinite induction, i.e. sequences and induction along ordinal numbers.

By way of brief summary[1], ordinal numbers are an extension of natural numbers. Ordinals are well-

[1] Further details can be found in most core texts on set theory such as Jech (2013)

ordered, which means that any set of ordinals is totally ordered and contains the least element. Finite ordinals correspond exactly to the natural numbers and the set $\mathbb{N}$ of natural numbers corresponds to ω, the smallest infinite ordinal. In particular, we will often write $n < \omega$ for some n to mean that n is finite. Infinite ordinals greater than ω exist. Addition and multiplication involving infinite ordinals are not commutative. Ordinals of the form $\mu + 1$ where μ is some ordinal are called *successor ordinals* and include all finite ordinals. Ordinals that are not successor ordinals are called *limit ordinals*. A mapping of ordinals is called a *transfinite sequence*, just as a mapping of natural numbers is an ordinary sequence. Monotonicity of transfinite sequences is defined analogously to that of ordinary sequences (see Section 1.5). In the same way that induction can be used to prove statements about sequences, transfinite induction can be used to prove statements about transfinite sequences. To establish that some proposition P holds for a transfinite sequence $(\phi^{(\mu)})_{\mu \geqslant 0}$, we would need to show that

1. P holds for $\phi^{(0)}$;
2. if P holds for $\phi^{(\mu)}$ then it holds for $\phi^{(\mu+1)}$; and
3. if P holds for $\phi^{(\mu)}$ for all $\mu < \nu$ and some ordinal ν then it holds for $\phi^{(\nu)}$.

Definition 41. Let Φ be a clearing function for some clearing system $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ and Φ^m the m-fold composition of Φ. We define a transfinite sequence $\left(\phi^{(\nu)}\right)_{\nu \geqslant 0}$ by recursion as follows. $\phi^{(0)} := \bar{B}$ and, for μ an ordinal and m an integer, $\phi^{(\mu+m)} := \Phi^m(\phi^{(\mu)})$. Finally, for any limit ordinal μ, $\phi^{(\mu)} := \sup_{\nu<\mu} \phi^{(\nu)}$. In particular, for any ordinal μ for which $\phi^{(\mu)}$ is well-defined, $\phi^{(\mu+\omega)} := \lim_{m\to\infty} \Phi^m(\phi^{(\mu)})$.

The following proposition gives the basic properties of the transfinite sequence $(\phi^{(\mu)})_{\mu \geqslant 0}$ that we use in later results.

Proposition 42.

1. *The transfinite sequence $\left(\phi^{(\mu)}\right)_{\mu \geqslant 0}$ is non-decreasing in μ such that $\bar{B} \leqslant \phi^{(\mu)} \leqslant \bar{L}$ for all μ. In particular, the limit $\lim_{k\to\infty} \phi^{(\mu+k)} = \phi^{(\mu+\omega)}$ is always well-defined for any μ.*
2. *Suppose $\phi^{(\mu)}$ is a fixed point of Φ for some ordinal μ. Then $\phi^{(\mu+\omega)} = \phi^{(\mu+k)} = \phi^{(\mu)}$ for any integer k.*
3. *Suppose $\phi^{(\mu)} = \bar{L}$ for some ordinal μ. Then $\phi^{(\mu)}$ is a fixed point of Φ.*

Proof. 1. We show by transfinite induction that for any ordinal ν the sequence $\left(\phi^{(\mu)}\right)_{\mu \leqslant \nu}$ is non-decreasing. The statement is trivial for $\nu = 0$ as the sequence then contains only a single term. So suppose that $\left(\phi^{(\mu)}\right)_{\mu<\nu}$ is non-decreasing for some ordinal ν.

Let ν be a successor ordinal so that $\nu = \tilde{\mu} + 2$ for some ordinal $\tilde{\mu}$. Then by the inductive hypothesis $\phi^{(\tilde{\mu}+1)} \geqslant \phi^{(\tilde{\mu})}$. Monotonicity of Φ then implies that $\phi^{(\nu)} = \Phi(\phi^{(\tilde{\mu}+1)}) \geqslant \Phi(\phi^{(\tilde{\mu})}) = \phi^{(\tilde{\mu}+1)}$. Hence $\left(\phi^{(\mu)}\right)_{\mu \leqslant \nu}$ is non-decreasing.

Let ν be a successor ordinal so that $\nu = \tilde{\mu} + 1$ for some limit ordinal $\tilde{\mu}$. Then $\phi^{(\tilde{\mu})} = \sup_{\mu<\tilde{\mu}} \phi^{(\mu)}$ and, in particular, $\phi^{(\tilde{\mu})} \geqslant \phi^{(\mu)}$ for any $\mu < \tilde{\mu}$. Then by monotonicity of Φ we ge that

$$\phi^{(\nu)} = \Phi(\phi^{(\tilde{\mu})}) \geqslant \Phi(\phi^{(\mu)}) = \phi^{(\mu+1)}.$$

Since $\mu + 1 \leqslant \tilde{\mu}$ is a successor ordinal and $\tilde{\mu}$ is not a successor ordinal, we obtain that $\mu + 1 < \tilde{\mu}$. Therefore by the inductive hypothesis $\phi^{(\mu+1)} \geqslant \phi^{(\mu)}$. In particular, we get that $\phi^{(\nu)} \geqslant \phi^{(\mu)}$, i.e. $\phi^{(\nu)}$ is an upper bound for $\left(\phi^{(\mu)}\right)_{\mu<\tilde{\mu}}$. But since, by definition, $\phi^{(\tilde{\mu})}$ is the supremum of $\left(\phi^{(\mu)}\right)_{\mu<\tilde{\mu}}$ we get that $\phi^{(\nu)} \geqslant \phi^{(\tilde{\mu})}$. Hence $\left(\phi^{(\mu)}\right)_{\mu\leqslant\nu}$ is non-decreasing.

Now, let ν be a limit ordinal. Then by definition $\phi^{(\nu)} = \sup_{\mu<\nu} \phi^{(\mu)}$ and so $\phi^{(\nu)} \geqslant \phi^{(\mu)}$ for any ordinal μ such that $\mu \leqslant \nu$. Therefore we have shown that for all ordinals ν the sequence $\left(\phi^{(\mu)}\right)_{\mu\leqslant\nu}$ is non-decreasing.

Finally, the fact that the image of Φ is in $[\bar{B}, \bar{L}]$ implies, for any successor ordinal $\mu + 1$, that $\bar{B} \leqslant \phi^{(\mu+1)} \leqslant \bar{L}$. Therefore this is also true for suprema of sets of successor ordinals. Hence $\bar{B} \leqslant \phi^{(\mu)} \leqslant \bar{L}$ for any ordinal μ.

In particular for any μ, $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ is a non-decreasing bounded sequence over integers k. Hence its limit exists and, by definition, equals to $\phi^{(\mu+\omega)}$.

2. For any integer k, we have that $\phi^{(\mu+k)} = \Phi^k(\phi^{(\mu)}) = \phi^{(\mu)}$. Therefore $\phi^{(\mu+\omega)} = \lim_{k\to\infty} \phi^{(\mu+k)} = \lim_{k\to\infty} \phi^{(\mu)} = \phi^{(\mu)}$.

3. By Proposition 42.1 $\phi^{(\mu)} \leqslant \phi^{(\mu+1)}$. Since $\phi^{(\mu+1)} = \Phi(\phi^{(\mu)})$ and Φ takes values in $[\bar{B}, \bar{L}]$, $\phi^{(\mu+1)} \leqslant \bar{L}$. This implies that $\phi^{(\mu)} = \bar{L} = \phi^{(\mu+1)} = \Phi(\phi^{(\mu)})$.

□

Remark 43. The completeness of reals implies that the suprema of subsets of $[\bar{B}, \bar{L}]$ can be approximated by increasing sequences. Crucially, sequences are maps of countable sets. Therefore in considering the transfinite sequence $(\phi^{(\mu)})_{\mu\geqslant 0}$ up to the the least fixed point, we only need to consider countable ordinals. In particular, when applying transfinite induction we will only need to consider the case of ordinals of the form $\mu + 1$ and $\mu + \omega$.

The following theorem is the main result of this section. We show the existence and structure of the least fixed point of Φ.

Theorem 44. *There exists the least fixed point ϕ_* of Φ and it is given by $\phi_* = \phi^{(\nu_*)}$ for some countable ordinal ν_*.*

Proof. 1. *Existence of the least fixed point:* Recall that, by Proposition 38.5, Φ is monotonic and by Remark 2 in Chapter 1 $[\bar{B}, \bar{L}]$ is a complete lattice. The Knaster-Tarski Theorem (Theorem 1, Chapter 1) implies that the set $\mathbf{Fix}(\Phi)$ of fixed points of Φ is a complete lattice and, in particular, contains the unique least element $\phi_* = \inf\{\phi \in [\bar{B}, \bar{L}] \mid \phi \geqslant \Phi(\phi)\}$.

2. *Structure of the least fixed point:* By Proposition 42.1, $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ is non-decreasing and bounded above by $\bar{L}$. Thus there is some countable ordinal ν_* such that $\phi^{(\nu_*)}$ is a fixed point of Φ. Since ordinals are well-ordered, we can assume without loss of generality, that ν_* is the least such ordinal.

   In particular, $\phi^{(\nu_*)} \geqslant \Phi(\phi^{(\nu_*)})$ and so by the property of the infima $\phi^{(0)} = \bar{B} \leqslant \phi_* \leqslant \phi^{(\nu_*)}$. Furthermore, by the well-ordering of ordinals, there is an ordinal η such that $\eta = \sup\{\mu \mid \phi^{(\mu)} \leqslant \phi_*\}$. Then by Definition 41 and Proposition 42.1, $\phi^{(\eta)} = \sup\{\phi^{(\mu)} \mid \phi^{(\mu)} \leqslant \phi_*\}$ so that $\phi^{(\eta)} \leqslant \phi_* \leqslant \phi^{(\nu_*)}$. By monotonicity of Φ we then obtain that $\phi^{(\eta+1)} = \Phi(\phi^{(\eta)}) \leqslant \Phi(\phi_*) = \phi_*$. By definition of $\phi^{(\eta)}$ as a supremum, we have that $\phi^{(\eta)} \geqslant \phi^{(\eta+1)}$ and it then follows that $\phi^{(\eta)} = \phi^{(\eta+1)} = \Phi(\phi^{(\eta)})$, i.e. $\phi^{(\eta)}$ is a fixed point of Φ. But, since we assume that $\phi^{(\nu_*)}$ is the least such element of the transfinite sequence and $\phi^{(\eta)} \leqslant \phi^{(\nu_*)}$, it follows that $\phi^{(\eta)} = \phi^{(\nu_*)}$. Moreover ϕ_* is the least fixed point of Φ. Since $\phi^{(\eta)} \leqslant \phi_*$, it follows that $\phi_* = \phi^{(\eta)}$. Hence we conclude that $\phi_* = \phi^{(\nu_*)}$.

□

Theorem 44 is a non-constructive result. In the remainder of this chapter we will show how to obtain an algorithm for explicitly computing the least fixed point of Φ in a finite number of steps.

3.3 Transfinite sequence decomposition

In this section we describe how to partition the components of the vectors $\phi^{(\mu)}$ in a way that allows us to traverse the sequence $\left(\phi^{(\mu)}\right)_{\mu\leqslant\nu_*}$ in a finite number of steps without skipping the least fixed point. We also introduce the index $\kappa(\mu)$ which allows us to control these jumps along the sequence $\left(\phi^{(\mu)}\right)_{\mu\leqslant\nu_*}$. These techniques are used in Section 3.4 to define a constructive algorithm for finding the least fixed point of Φ.

Definition 45. Let $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ be the transfinite sequence given in Definition 41. We define the following terms of transfinite sequences of sets

$$\begin{aligned}
\hat{\mathcal{L}}^{(\mu)} &:= \{i \in \mathcal{N} \mid \phi_i^{(\mu)} \geqslant \bar{L}_i\},\\
\hat{\mathcal{D}}^{(\mu)} &:= \{i \in \mathcal{N} \mid \phi_i^{(\mu)} < \bar{L}_i\},\\
\hat{\mathcal{A}}^{(\mu)} &:= \{i \in \hat{\mathcal{D}}^{(\mu)} \mid \phi_i^{(\mu)} > \bar{B}_i\},\\
\hat{\mathcal{B}}^{(\mu)} &:= \{i \in \hat{\mathcal{D}}^{(\mu)} \mid \phi_i^{(\mu)} \leqslant \bar{B}_i\}.
\end{aligned}$$

Corollary 46. *Let $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ be the transfinite sequence given in Definition 41. Then*

1. *$(\hat{\mathcal{D}}^{(\mu)})_{\mu\geqslant 0}$ and $(\hat{\mathcal{B}}^{(\mu)})_{\mu\geqslant 0}$ are non-increasing transfinite sequences of sets and $\left(\hat{\mathcal{L}}^{(\mu)}\right)_{\mu\geqslant 0}$ is a non-decreasing sequence of sets; and*

2. *For any two ordinals ν and μ with $\nu < \mu$,*

$$\hat{\mathcal{D}}^{(\mu)} = \{i \in \hat{\mathcal{A}}^{(\nu)} \mid \phi_i^{(\mu)} < \bar{L}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\nu)} \mid \phi_i^{(\mu)} < \bar{L}_i\},$$
$$\hat{\mathcal{B}}^{(\mu)} = \{i \in \hat{\mathcal{D}}^{(\mu)} \cap \hat{\mathcal{B}}^{(\nu)} \mid \phi_i^{(\mu)} \leqslant \bar{B}_i\}.$$

Proof. 1. By Proposition 42.1 $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ is non-decreasing transfinite sequence. It immediately follows that $\left(\hat{\mathcal{D}}^{(\mu)}\right)_{\mu\geqslant 0}$ is a non-increasing sequence of sets and $\left(\hat{\mathcal{L}}^{(\mu)}\right)_{\mu\geqslant 0}$ is a non-decreasing sequence of sets. But since $\left(\hat{\mathcal{D}}^{(\mu)}\right)_{\mu\geqslant 0}$ is non-increasing and $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ is non-decreasing, it also follows that $\left(\hat{\mathcal{B}}^{(\mu)}\right)_{\mu\geqslant 0}$ is a non-increasing sequence of sets.

2. By above we have that $\hat{\mathcal{D}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\nu)}$ and $\hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{B}}^{(\nu)}$. Furthermore, by definition of $\hat{\mathcal{B}}^{(\mu)}$, we have that $\hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\mu)}$ and hence $\hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\mu)} \cap \hat{\mathcal{B}}^{(\nu)}$. Finally by Definition 45, we can see that $\hat{\mathcal{D}}^{(\nu)}$ is a disjoint union of $\hat{\mathcal{A}}^{(\nu)}$ and $\hat{\mathcal{B}}^{(\nu)}$.

□

Remark 47. In contrast to Corollary 46, $\left(\hat{\mathcal{A}}^{(\mu)}\right)_{\mu\geqslant 0}$ is not necessarily a monotonic sequence of sets. For example, if $\mathcal{N} = \{1, 2, 3\}$ it is possible to construct a clearing system such that $\hat{\mathcal{D}}^{(0)} = \hat{\mathcal{D}}^{(1)} = \mathcal{N}$, $\hat{\mathcal{D}}^{(2)} = \varnothing$, $\hat{\mathcal{B}}^{(0)} = \mathcal{N}$ and $\hat{\mathcal{B}}^{(1)} = \hat{\mathcal{B}}^{(2)} = \varnothing$. However, since $\hat{\mathcal{A}}^{(\mu)} = \hat{\mathcal{D}}^{(\mu)} \setminus \hat{\mathcal{B}}^{(\mu)}$ for all μ, we obtain that $\hat{\mathcal{A}}^{(0)} = \varnothing$, $\hat{\mathcal{A}}^{(1)} = \mathcal{N}$ and $\hat{\mathcal{A}}^{(2)} = \varnothing$.

The next proposition is fundamental. It establishes bounds on $\phi^{(\mu)}$.

Proposition 48. *For a clearing system $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ and each ordinal μ, let $\phi^{(\mu)}$, $\hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu)}$ be as in Definition 45 and $\bar{T} = \beta\bar{L} + (\alpha - \beta)a$. Then for each μ the set $\mathcal{N}$ can be partitioned into the sets $\hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu)}$ so that*

$$\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}},$$
$$\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}},$$
$$\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}},$$
$$\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}.$$

Proof. By Definition 45, it is clear that $\mathcal{N} = \hat{\mathcal{D}}^{(\mu)} \cup \hat{\mathcal{L}}^{(\mu)}$ and $\hat{\mathcal{D}}^{(\mu)} = \hat{\mathcal{A}}^{(\mu)} \cup \hat{\mathcal{B}}^{(\mu)}$ with both unions disjoint. Moreover, $\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} \geqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$.

Suppose $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$. Since $(\phi^{(\nu)})_{\nu\geqslant 0}$ is a non-decreasing transfinite sequence, $\phi^{(\nu)}_{\hat{\mathcal{B}}^{(\mu)}} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ for any ordinal ν with $\nu \leqslant \mu$. By transfinite induction we can show that $\phi^{(\nu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. Indeed, for $\nu = 0$, $\phi^{(\nu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. If ν is a successor ordinal then $\nu = \tilde{\mu} + 1$ for some ordinal $\tilde{\mu}$ and $\phi^{(\nu)} = \Phi(\phi^{(\tilde{\mu})})$. Since Φ takes its values in $[\bar{B}, \bar{L}]$, it follows that $\phi^{(\nu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. If ν is a limit ordinal and $\phi^{(\tilde{\mu})}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ for all ordinals $\tilde{\mu}$ satisfying $\tilde{\mu} < \nu$ then $\phi^{(\nu)}_{\hat{\mathcal{B}}^{(\mu)}} = \sup_{\tilde{\mu}} \phi^{(\tilde{\mu})}_{\hat{\mathcal{B}}^{(\mu)}} = \sup_{\tilde{\mu}} \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. By transfinite induction, it follows that $\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ for each ordinal μ.

Similarly for $\hat{\mathcal{L}}^{(\mu)} \neq \varnothing$, we show by transfinite induction that $\phi^{(\nu)}_{\hat{\mathcal{L}}^{(\mu)}} \leqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ for $\nu \leqslant \mu$. For $\nu = 0$, $\phi^{(\nu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{L}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. If ν is a successor ordinal then $\nu = \tilde{\mu} + 1$ for some ordinal $\tilde{\mu}$ and $\phi^{(\nu)} = \Phi(\phi^{(\tilde{\mu})})$.

Since Φ takes its values in $[\bar{B}, \bar{L}]$, it follows that $\phi^{(\nu)}_{\hat{\mathcal{L}}^{(\mu)}} \leqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. If ν is a limit ordinal and $\phi^{(\tilde{\mu})}_{\hat{\mathcal{L}}^{(\mu)}} \leqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ for all ordinals $\tilde{\mu}$ satisfying $\tilde{\mu} < \nu$ then $\phi^{(\nu)}_{\hat{\mathcal{L}}^{(\mu)}} = \sup \phi^{(\tilde{\mu})}_{\hat{\mathcal{L}}^{(\mu)}} \leqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. Hence, $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} \leqslant \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ and so by transfinite induction it follows that $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ for each ordinal μ.

Suppose $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$. The fact that $\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}}$ for each μ follows directly from Definition 45. It remains to show that we also obtain $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ for each ordinal μ. We again proceed by transfinite induction. However, since $\hat{\mathcal{A}}^{(0)} = \varnothing$, we start the induction from the least ordinal $\tilde{\nu}$ such that $\hat{\mathcal{A}}^{(\tilde{\nu})} \neq \varnothing$. Such an ordinal exists by well-ordering.

Let μ be a successor ordinal so that there is some ordinal $\tilde{\mu}$ satisfying $\tilde{\nu} \leqslant \tilde{\mu}$ and $\mu = \tilde{\mu} + 1$. Then $\phi^{(\mu)} = \Phi(\phi^{(\tilde{\mu})})$ and so $\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < \Phi(\phi^{(\tilde{\mu})})_{\hat{\mathcal{A}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}}$. Therefore $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\tilde{\mu})})_{\hat{\mathcal{A}}^{(\mu)}}$ and by Lemma 39.3 we get that $V_{\alpha,\beta}(\phi^{(\tilde{\mu})})_{\hat{\mathcal{A}}^{(\mu)}} < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$. Hence if μ be a successor ordinal then $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$. Now let μ be a limit ordinal. Then $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = \sup_{\nu<\mu} \phi^{(\nu)}_{\hat{\mathcal{A}}^{(\mu)}}$. Suppose that $\phi^{(\nu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ for all ordinals ν satisfying $\tilde{\nu} \leqslant \nu < \mu$. Then $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$. This completes the induction. □

Suppose the partition described in Proposition 48 is constant for the sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$, i.e. $\phi^{(\mu+k)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$, $\phi^{(\mu+k)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ and $\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}}$ for every $k \geqslant 0$. This would allow us to treat $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ as constant for all components i in either $\hat{\mathcal{B}}^{(\mu)}$ or $\hat{\mathcal{L}}^{(\mu)}$. Similarly, for components $i \in \hat{\mathcal{A}}^{(\mu)}$ we would be able to assume that $\phi^{(\mu+k)}_i = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i$, allowing us to use the basic properties of linear maps and the results in Proposition 38.1 and Lemma 39. Overall, Definition 45 could allow us to decompose the sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ into parts with simple and useful properties. Of course in general the partition will not hold for the whole sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$. We introduce the following definition in order to allow us to control the spans of the sequence where it *does* hold.

Definition 49. For a clearing system $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ and each ordinal μ, let $\phi^{(\mu)}$, $\hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu)}$ be as in Definition 45 and $\bar{T} = \beta\bar{L} + (\alpha - \beta)a$.

For each ordinal μ, let

$$\mathcal{V}_T(\mu) := \{k \in \mathbb{N} \cup \{0\} \mid \exists i \in \hat{\mathcal{D}}^{(\mu)}, V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i < \bar{T}_i\}$$
$$\mathcal{V}_B(\mu) := \{k \in \mathbb{N} \cup \{0\} \mid \exists i \in \hat{\mathcal{B}}^{(\mu)}, V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \leqslant \bar{B}_i\}.$$

Thus for each label $l \in \{T, B\}$ we have a set $\mathcal{V}_l(\mu)$ and then let

$$\kappa_l(\mu) := \begin{cases} 0 & \text{if } \mathcal{V}_l(\mu) = \varnothing \\ \max \mathcal{V}_l(\mu) + 1 & \text{if } \max \mathcal{V}_l(\mu) \text{ is finite} \\ \omega & \text{otherwise.} \end{cases}$$

Finally, let

$$\kappa(\mu) = \kappa_B(\mu) \wedge \kappa_T(\mu).$$

Corollary 50. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a clearing system, μ an ordinal and $k \geqslant 0$ an integer.*

1. $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \geqslant \bar{T}_i$ *for all* $i \in \hat{\mathcal{D}}^{(\mu)}$ *if and only if* $\kappa_T(\mu) = 0$.

2. *$V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i > \bar{B}_i$ for all $i \in \hat{\mathcal{D}}^{(\mu)}$ if and only if $\kappa_B(\mu) = 0$.*

3. *Suppose $\hat{\mathcal{D}}^{(\mu)} \neq \varnothing$. Then there is some $i \in \hat{\mathcal{D}}^{(\mu)}$ such that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i < \bar{T}_i$ if and only if $k < \kappa_T(\mu)$.*

4. *Suppose $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$. Then there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \leqslant \bar{B}_i$ if and only if $k < \kappa_B(\mu)$.*

Proof. 1. By definition $\kappa_T(\mu) = 0$ if and only if $\mathcal{V}_T(\mu) = \varnothing$, i.e. if and only if $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \geqslant \bar{T}_i$ for all $i \in \hat{\mathcal{D}}^{(\mu)}$ and $k \geqslant 0$. By Proposition 38.1, the latter condition need only hold for $k = 0$.

2. Similarly, $\kappa_b(\mu) = 0$ if and only if $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i > \bar{B}_i$ for all $i \in \hat{\mathcal{B}}^{(\mu)}$ and $k \geqslant 0$. Again, by Proposition 38.1, the latter condition need only hold for $k = 0$.

3. Let $k < \kappa_T(\mu) < \omega$. Then $\kappa_T(\mu) > 0$ and $\kappa_T(\mu) - 1 \geqslant k$. Suppose that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \geqslant \bar{T}_i$ for all $i \in \hat{\mathcal{D}}^{(\mu)}$. We can write this alternatively as $V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}} \geqslant \bar{T}_{\hat{\mathcal{D}}^{(\mu)}}$. By Proposition 42.1 $\phi^{(\mu+k)} \leqslant \phi^{(\mu+\kappa_T(\mu)-1)}$ and so by Proposition 38.1 $\bar{T}_{\hat{\mathcal{D}}^{(\mu)}} \leqslant V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}} \leqslant V_{\alpha,\beta}(\phi^{(\mu+\kappa_T(\mu)-1)})_{\hat{\mathcal{D}}^{(\mu)}}$. Therefore for all $i \in \hat{\mathcal{D}}^{(\mu)}$ we have that that $V_{\alpha,\beta}\left(\phi^{(\mu+\kappa_T(\mu)-1)}\right)_i \geqslant \bar{T}_i$ and therefore $\kappa_T(\mu) - 1 \notin \mathcal{V}_T(\mu)$, contradicting the definition of $\kappa_T(\mu)$. Since $\hat{\mathcal{D}}^{(\mu)} \neq \varnothing$, it follows that there is some $i \in \hat{\mathcal{D}}^{(\mu)}$ such that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i < \bar{T}_i$.

   Now, let $k < \kappa_T(\mu) = \omega$. Then $k \in \mathcal{V}_T(\mu)$ since otherwise $k \geqslant \max \mathcal{V}_T(\mu)$ by Proposition 38.1, implying that $\kappa_T(\mu) < \omega$. So $\mathcal{V}_T(\mu) \neq \varnothing$ and there must be some $i \in \hat{\mathcal{D}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+k)})_i < \bar{T}_i$.

   The converse statement follows directly by the definition of $\kappa_T(\mu)$. Suppose there is some $i \in \hat{\mathcal{D}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+k)})_i < \bar{T}_i$. Then $k \in \mathcal{V}_T(\mu) \neq \varnothing$. If $\kappa_T(\mu) = \omega$ then $k < \kappa_T(\mu)$, trivially. So suppose that $\kappa_T(\mu) < \omega$. Then $k \leqslant \max \mathcal{V}_T(\mu) = \kappa_T(\mu) - 1$ and hence $k < \kappa_T(\mu)$.

4. Let $k < \kappa_B(\mu) < \omega$. Then $\kappa_B(\mu) > 0$ and $\kappa_B(\mu) - 1 \geqslant k$. Suppose that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i > \bar{B}_i$ for all $i \in \hat{\mathcal{B}}^{(\mu)}$. We can write this alternatively as $V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}} > \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. By Proposition 42.1 $\phi^{(\mu+k)} \leqslant \phi^{(\mu+\kappa_B(\mu)-1)}$ and so by Proposition 38.1 $\bar{B}_{\hat{\mathcal{B}}^{(\mu)}} < V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}} \leqslant V_{\alpha,\beta}(\phi^{(\mu+\kappa_B(\mu)-1)})_{\hat{\mathcal{B}}^{(\mu)}}$. Therefore for all $i \in \hat{\mathcal{B}}^{(\mu)}$ we have that that $V_{\alpha,\beta}\left(\phi^{(\mu+\kappa_B(\mu)-1)}\right)_i < \bar{B}_i$ and therefore $\kappa_B(\mu) - 1 \notin \mathcal{V}_B(\mu)$, contradicting the definition of $\kappa_B(\mu)$. Since $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$, it follows that there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}\left(\phi^{(\mu+k)}\right)_i \leqslant \bar{B}_i$.

   Now, let $k < \kappa_B(\mu) = \omega$. Then $k \in \mathcal{V}_B(\mu)$ since otherwise $k \geqslant \max \mathcal{V}_B(\mu)$ by Proposition 38.1, implying that $\kappa_B(\mu) < \omega$. So $\mathcal{V}_B(\mu) \neq \varnothing$ and there must be some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+k)})_i \leqslant \bar{B}_i$.

   The converse statement follows directly by the definition of $\kappa_B(\mu)$. Suppose there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+k)})_i \leqslant \bar{B}_i$. Then $k \in \mathcal{V}_B(\mu) \neq \varnothing$. If $\kappa_B(\mu) = \omega$ then $k < \kappa_B(\mu)$, trivially. So suppose that $\kappa_B(\mu) < \omega$. Then $k \leqslant \max \mathcal{V}_B(\mu) = \kappa_B(\mu) - 1$ and hence $k < \kappa_B(\mu)$.

□

In particular, if $0 < \kappa(\mu) < \omega$ then $\kappa(\mu)$ identifies the last $k \geqslant 1$ in the sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ where the partition described above holds; in particular, $\phi^{(\mu+\kappa(\mu)+1)}$ is then the first term of the sequence where the partition no longer applies. If $\kappa(\mu)$ is not finite then in fact the partition applies for the whole sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$. Note however, that for $\kappa(\mu) = 0$ the partition may or may not hold for $\phi^{(\mu)}$ since $\phi^{(\mu)}$ is not expressed in terms of $V_{\alpha,\beta}$ if μ is a limit ordinal.

Remark 51. Note that if $\phi^{(\mu)}$ is a fixed point of Φ then $\phi^{(\mu)} = \phi^{(\mu+k)}$ for all finite k and so $\kappa(\mu)$ cannot be finite.

Propositions 55 and 56 below formalise several key properties of $\kappa(\mu)$. We first introduce the following definition which will be used in the proofs of these propositions.

Definition 52. Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a clearing system, μ some ordinal, $N_A := |\hat{\mathcal{A}}^{(\mu)}|$ and $N_B := |\hat{\mathcal{B}}^{(\mu)}|$.

Define $\hat{b}^{(\mu)} \in \mathbb{R}_+^N$:

$$\hat{b}^{(\mu)} := \alpha a + \beta\Omega_{\mathcal{N}\hat{\mathcal{B}}^{(\mu)}}\bar{B}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{L}}^{(\mu)}}\bar{L}_{\hat{\mathcal{L}}^{(\mu)}},$$

using the convention that $\Omega_{\mathcal{N}\hat{\mathcal{B}}^{(\mu)}}\bar{B}_{\hat{\mathcal{B}}^{(\mu)}} := \mathbf{0}$ and $\Omega_{\mathcal{N}\hat{\mathcal{L}}^{(\mu)}}\bar{L}_{\hat{\mathcal{L}}^{(\mu)}} := \mathbf{0}$ if $\hat{\mathcal{B}}^{(\mu)}$ or $\hat{\mathcal{L}}^{(\mu)}$, respectively, are empty.

If $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ then define $\hat{M}^{(\mu)} \in \mathbb{R}_+^{N_A \times N_A}$:

$$\hat{M}^{(\mu)} := \beta\Omega_{\hat{\mathcal{A}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}.$$

If $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ and $(\mathrm{I} - \hat{M}^{(\mu)}) \in \mathbb{R}_+^{N_A \times N_A}$ is invertible then define $\tilde{X}^{(\mu)} \in \mathbb{R}_+^{N_A}$:

$$\tilde{X}^{(\mu)} := (\mathrm{I} - \hat{M}^{(\mu)})^{-1}\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}.$$

If $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$, $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible and moreover $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ then also define $\tilde{Y}^{(\mu)} \in \mathbb{R}_+^{N_B}$:

$$\tilde{Y}^{(\mu)} := \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}(\mathrm{I} - \hat{M}^{(\mu)})^{-1}\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}.$$

Corollary 53 and Lemma 54 give some properties of $\hat{b}^{(\mu)}$ which we use later.

Corollary 53. *Let μ and $\hat{b}^{(\mu)}$ be as in Definition 52 and suppose that $\hat{\mathcal{D}}^{(\mu)} \neq \varnothing$. Then*

$$V_{\alpha,\beta}(\phi^{(\mu)}) = \hat{b}^{(\mu)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}.$$

In particular, if $\hat{\mathcal{A}}^{(\mu)} = \varnothing$ then $V_{\alpha,\beta}(\phi^{(\mu)}) = \hat{b}^{(\mu)}$.

Proof. By Proposition 48, $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$ and $\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. Therefore

$$\begin{aligned}
V_{\alpha,\beta}(\phi^{(\mu)}) &= \alpha a + \beta\Omega\phi^{(\mu)} \\
&= \alpha a + \beta\Omega_{\mathcal{N}\hat{\mathcal{B}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{L}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \\
&= \alpha a + \beta\Omega_{\mathcal{N}\hat{\mathcal{B}}^{(\mu)}}\bar{B}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{L}}^{(\mu)}}\bar{L}_{\hat{\mathcal{L}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \\
&= \hat{b}^{(\mu)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}.
\end{aligned}$$

Furthermore, if $\hat{\mathcal{A}}^{(\mu)} = \varnothing$ then by convention the term $\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}$ is taken to be zero. Therefore $V_{\alpha,\beta}(\phi^{(\mu)}) = \hat{b}^{(\mu)}$. □

Lemma 54. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system such that $\|\Omega\|_1 = 1$. Then, for every ordinal μ, $\sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{b}^{(\mu)}_i > 0$.*

Proof. Let $S \coloneqq \sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{b}^{(\mu)}_i$. By definition of regularity in Eisenberg & Noe (2001), for every $i \in \hat{\mathcal{A}}^{(\mu)}$ either $a_i > 0$ or there is some $j \in \mathcal{N}$ such that $a_j > 0$ and there is a sequence $(i_k)_{0\leqslant k\leqslant n} \subseteq \mathcal{N}$ for some $n \leqslant N$ with $i_0 = i$, $i_n = j$ and $\Omega_{i_k i_{k+1}}\bar{L}_{i_{k+1}} > 0$ for every $0 \leqslant k < n$. In particular, setting $k = 0$, if $a_i = 0$ for some $i \in \hat{\mathcal{A}}^{(\mu)}$ then there is some $i_1 \in \mathcal{N}$ such that $\Omega_{i_0 i_1}\bar{L}_{i_1} > 0$.

Let $i \in \hat{\mathcal{A}}^{(\mu)}$. If $a_i > 0$ then $\alpha a_i > 0$ and so $S \geqslant \hat{b}^{(\mu)}_i \geqslant \alpha a_i > 0$. So suppose that $a_i = 0$. Then let $(i_k)_{0\leqslant k\leqslant n} \subseteq \mathcal{N}$ be a sequence as described above with $i_n = j$. If $j \in \hat{\mathcal{A}}^{(\mu)}$ then $S \geqslant \hat{b}^{(\mu)}_j \geqslant \alpha a_j > 0$. If $j \in \hat{\mathcal{L}}^{(\mu)}$ then there is some $i_1 \in \mathcal{N}$ such that $\Omega_{i_0 i_1}\bar{L}_{i_1} > 0$ and $S \geqslant \hat{b}^{(\mu)}_i \geqslant \Omega_{i_0 i_1}\bar{L}_{i_1} > 0$. Finally, suppose that $j \in \hat{\mathcal{B}}^{(\mu)}$. Again there is some $i_1 \in \mathcal{N}$ such that $\Omega_{i_0 i_1}\bar{L}_{i_1} > 0$ and hence $\Omega_{i_0 i_1} > 0$. Therefore if $\bar{B}_{i_1} > 0$ then $S \geqslant \hat{b}^{(\mu)}_i \geqslant \Omega_{i_0 i_1}\bar{B}_{i_1} > 0$. If $\bar{B}_{i_1} = 0$ then we show that in fact $i_1 \notin \hat{\mathcal{B}}^{(\mu)}$ and therefore without loss of generality we can assume that $j \in \hat{\mathcal{A}}^{(\mu)}$ or $j \in \hat{\mathcal{L}}^{(\mu)}$, which we have shown above implies that $S > 0$.

So suppose that $a_i = 0$ for every $i \in \hat{\mathcal{A}}^{(\mu)}$ and in addition for every $j \in \mathcal{N}$ such that $\Omega_{ij}\bar{L}_j > 0$ we also have that $j \in \hat{\mathcal{B}}^{(\mu)}$ with $\bar{B}_j = 0$. By Proposition 48, $j \in \hat{\mathcal{B}}^{(\mu)}$ with $\bar{B}_j = 0$ implies that $\phi^{(\mu)}_j = 0$. Since $\phi^{(\mu)}_j \leqslant \bar{L}_j$ for all $j \in \mathcal{N}$, it follows that for every $j \in \mathcal{N}$ such that $\Omega_{ij} > 0$ we have that $\phi^{(\mu)}_j = 0$. But then $V(\phi^{(\mu)})_i = a_i + \sum_{j\in\mathcal{N}} \Omega_{ij}\phi^{(\mu)}_j = 0$ for each $i \in \hat{\mathcal{A}}^{(\mu)}$. Moreover, for $i \in \hat{\mathcal{A}}^{(\mu)}$, $\Phi(\phi^{(\mu)})_i = \bar{B}_i \vee V_{\alpha,\beta}(\phi^{(\mu)})_i$ and so, since $\bar{L}_i > 0$, $V(\phi^{(\mu)})_i < 0$ and therefore

$$\begin{aligned}\phi^{(\mu+1)}_i = \Phi(\phi^{(\mu)})_i &= \bar{B}_i \vee V_{\alpha,\beta}(\phi^{(\mu)})_i \\ &\leqslant \bar{B}_i \vee V(\phi^{(\mu)})_i \leqslant \bar{B}_i \vee 0 = \bar{B}_i.\end{aligned}$$

But by Proposition 42.1, $\bar{B}_i \leqslant \phi^{(\mu)}_i \leqslant \phi^{(\mu+1)}_i \leqslant \bar{B}_i$ and so $\phi^{(\mu)}_i = \bar{B}_i$. This however contradicts Proposition 48 and the fact that $i \in \hat{\mathcal{A}}^{(\mu)}$ and therefore concludes the proof that $S > 0$. □

Proposition 55 formalises which properties remain constant in the sequence $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ for $k \geqslant 0$ "up to" $\kappa(\mu)$. If $\kappa(\mu) = \omega$ then the properties in Proposition 55 hold for all integers k and Proposition 56 then describes what happens in the limit.

Proposition 55. *Assume the notation of Definitions 45 and 49.*

1. *For any integer k with $0 \leqslant k \leqslant \kappa(\mu)$, $\hat{\mathcal{L}}^{(\mu+k)} = \hat{\mathcal{L}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu+k)} = \hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{A}}^{(\mu+k)} = \hat{\mathcal{A}}^{(\mu)}$.*

2. *For any integer k with $0 \leqslant k < \kappa(\mu)$, if $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ then $\phi^{(\mu+k+1)}_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}}$.*
   If $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ then $\phi^{(\mu+k+1)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} \geqslant V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}}$.
   If $\hat{\mathcal{L}}^{(\mu)} \neq \varnothing$ then $\phi^{(\mu+k+1)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$.

Proof. 1. The statement is trivial for $k = 0$. Suppose then that $i \in \hat{\mathcal{D}}^{(\mu)}$ and there is some integer k with $0 \leqslant k-1 < \kappa(\mu)$. Since $\kappa(\mu) \leqslant \kappa_T(\mu)$, by Corollary 50.3 we have that $V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i < \bar{T}_i$ and Lemma 39.3 implies that $V(\phi^{(\mu+k-1)})_i < \bar{L}_i$. Therefore $i \in \hat{\mathcal{D}}^{(\mu+k)}$.

Suppose further that $i \in \hat{\mathcal{A}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\mu)}$ then, since $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ is non-decreasing, $\phi_i^{(\mu+k)} \geqslant \phi_i^{(\mu)} > \bar{B}_i$. Since $i \in \hat{\mathcal{D}}^{(\mu+k)}$, it follows that $i \in \hat{\mathcal{A}}^{(\mu+k)}$. Hence $\hat{\mathcal{A}}^{(\mu)} \subseteq \hat{\mathcal{A}}^{(\mu+k)}$.

If instead we allow that $i \in \hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\mu)}$ then, by Corollary 50.4, we have that $V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i \leqslant \bar{B}_i$. Since $i \in \hat{\mathcal{D}}^{(\mu+k)}$, it follows that $i \in \hat{\mathcal{B}}^{(\mu+k)}$. Hence $\hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{B}}^{(\mu+k)}$.

Overall, we obtain that $\hat{\mathcal{D}}^{(\mu)} = \hat{\mathcal{A}}^{(\mu)} \cup \hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{A}}^{(\mu+k)} \cup \hat{\mathcal{B}}^{(\mu+k)} = \hat{\mathcal{D}}^{(\mu+k)}$. Since $\hat{\mathcal{D}}^{(\mu)}$ is a non-increasing sequence of sets by Corollary 46.1, $\hat{\mathcal{D}}^{(\mu)} = \hat{\mathcal{D}}^{(\mu+k)}$ and therefore $\hat{\mathcal{L}}^{(\mu)} = \hat{\mathcal{L}}^{(\mu+k)}$. Finally, $\hat{\mathcal{B}}^{(\mu)} \subseteq \hat{\mathcal{B}}^{(\mu+k)}$ then also implies that $\hat{\mathcal{A}}^{(\mu+k)} \subseteq \hat{\mathcal{A}}^{(\mu)}$. Hence $\hat{\mathcal{A}}^{(\mu)} = \hat{\mathcal{A}}^{(\mu+k)}$ and so $\hat{\mathcal{B}}^{(\mu)} = \hat{\mathcal{B}}^{(\mu+k)}$.

2. By Propositions 48 and 55.1 we see that $\phi^{(\mu+k+1)}_{\hat{\mathcal{B}}^{(\mu)}} = \phi^{(\mu+k+1)}_{\hat{\mathcal{B}}^{(\mu+k+1)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. By Corollary 50.4, $V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and hence that $\phi^{(\mu+k+1)}_{\hat{\mathcal{B}}^{(\mu)}} \geqslant V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}}$. Similarly, by Propositions 48 and 55.1 we see that $\phi^{(\mu+k+1)}_{\hat{\mathcal{L}}^{(\mu)}} = \phi^{(\mu+k+1)}_{\hat{\mathcal{L}}^{(\mu+k+1)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$.

We show that $\phi^{(\mu+k+1)}_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}}$ for $0 \leqslant k < \kappa(\mu)$. Suppose $i \in \hat{\mathcal{A}}^{(\mu)} \subseteq \hat{\mathcal{D}}^{(\mu)}$. We already saw in the proof of Proposition 55.1 that this implies that $i \in \hat{\mathcal{D}}^{(\mu+k+1)}$ and $\phi_i^{(\mu+k+1)} > \bar{B}_i$. Therefore for $i \in \hat{\mathcal{A}}^{(\mu)}$ we find that $\phi_i^{(\mu+k+1)} = \Phi(\phi^{(\mu+k)})_i = \bar{B}_i \vee V_{\alpha,\beta}(\phi^{(\mu+k)})_i = V_{\alpha,\beta}(\phi^{(\mu+k)})_i$.

□

Proposition 56. *Suppose that* $\kappa(\mu) = \omega$.

1. *The limit* $\lim_{k\to\infty} V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}}$ *exists and is equal to* $V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{D}}^{(\mu)}}$.
2. *Whenever the corresponding sets* $\hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu)}$ *or* $\hat{\mathcal{L}}^{(\mu)}$ *are non-empty*
   - $\phi^{(\mu+\omega)}_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$;
   - $\phi^{(\mu+\omega)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} \geqslant V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{B}}^{(\mu)}}$ *and* $\bar{T}_{\hat{\mathcal{B}}^{(\mu)}} \geqslant V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{B}}^{(\mu)}}$;
   - $\phi^{(\mu+\omega)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$.

Proof. 1. Let $k \geqslant 0$ be an integer and, in particular, $k < \omega = \kappa(\mu)$. By Proposition 55.2 and non-negativity of $V_{\alpha,\beta}$, $\phi^{(\mu+k+1)}_{\hat{\mathcal{D}}^{(\mu)}} \geqslant V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}} \geqslant \mathbf{0}$. Since $\phi^{(\mu+k+1)}$ increases up to the limit $\phi^{(\mu+\omega)}$ by Proposition 42.1, $\left(V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}}\right)_{k\geqslant 0}$ is a bounded monotonic sequence and hence the limit $\lim_{k\to\infty} V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}}$ exists. Moreover, for $k < \omega = \kappa(\mu)$, we have by Proposition 55.1 that $\hat{\mathcal{A}}^{(\mu+k)} = \hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu+k)} = \hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu+k)} = \hat{\mathcal{L}}^{(\mu)}$. Therefore $\hat{b}^{(\mu+k)} = \hat{b}^{(\mu)}$ and then by continuity of matrices as linear bounded operators we have the following:

$$\begin{aligned} V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{D}}^{(\mu)}} &= \hat{b}^{(\mu)}_{\hat{\mathcal{D}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{D}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu+\omega)}_{\hat{\mathcal{D}}^{(\mu)}} \\ &= \hat{b}^{(\mu)}_{\hat{\mathcal{D}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{D}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}} \lim_{k\to\infty}\phi^{(\mu+k)}_{\hat{\mathcal{D}}^{(\mu)}} \\ &= \lim_{k\to\infty}\left(\hat{b}^{(\mu)}_{\hat{\mathcal{D}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{D}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu+k)}_{\hat{\mathcal{D}}^{(\mu)}}\right) \\ &= \lim_{k\to\infty}\left(\hat{b}^{(\mu+k)}_{\hat{\mathcal{D}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{D}}^{(\mu)}\hat{\mathcal{A}}^{(\mu+k)}}\phi^{(\mu+k)}_{\hat{\mathcal{D}}^{(\mu)}}\right) \\ &= \lim_{k\to\infty}\left(V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{D}}^{(\mu)}}\right). \end{aligned}$$

2. The result follows by the definition of $\phi^{(\mu+\omega)}$. Since $k < \omega = \kappa(\mu)$, the sequence $\left(\phi_i^{(\mu+k)}\right)_{k\geqslant 0}$ is constant for $i \in \hat{\mathcal{B}}^{(\mu)}$ or $i \in \hat{\mathcal{L}}^{(\mu)}$, by Proposition 55.2. Hence the terms of the sequence for $i \in \hat{\mathcal{B}}^{(\mu)}$ or $i \in \hat{\mathcal{L}}^{(\mu)}$ are equal to the limit. Furthermore for $i \in \hat{\mathcal{B}}^{(\mu)}$, by Corollary 50, $V_{\alpha,\beta}(\phi^{(\mu+k)})_i \leqslant \bar{B}_i$ and $V_{\alpha,\beta}(\phi^{(\mu+k)})_i < \bar{T}_i$ for all $k \geqslant 0$. Therefore $V_{\alpha,\beta}(\phi^{(\mu+\omega)})_i \leqslant \bar{B}_i = \phi_i^{(\mu+\omega)}$ and $V_{\alpha,\beta}(\phi^{(\mu+\omega)})_i \leqslant \bar{T}_i$.

   For $i \in \hat{\mathcal{A}}^{(\mu)}$ we have by Proposition 55.2 that

$$\begin{aligned} \phi^{(\mu+\omega)}_{\hat{\mathcal{A}}^{(\mu)}} &= \lim_{k\to\infty}\phi^{(\mu+k+1)}_{\hat{\mathcal{A}}^{(\mu)}} \\ &= \lim_{k\to\infty} V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{A}}^{(\mu)}}. \end{aligned}$$

   Corollary 50 then implies that $\phi^{(\mu+\omega)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$.

□

From Proposition 55, it is clear that on $\hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu)}$ the sequence $\left(\phi^{(\mu+k)}\right)_{0\leqslant k\leqslant\kappa(\mu)}$ is constant. On $\hat{\mathcal{A}}^{(\mu)}$, however, it is not constant as the terms are equal to $V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{A}}^{(\mu)}}$. The difficulty here is that $\phi^{(\mu+k-1)}$ is a vector with components that are not in $\hat{\mathcal{A}}^{(\mu)}$ so we are effectively dealing with some mapping $\mathbb{R}^N_+ \to \mathbb{R}^{N_A}_+$ where $N_A = |\hat{\mathcal{A}}^{(\mu)}|$. This can cause some inconvenience since we cannot easily express $\phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}}$ in terms of $\phi^{(\mu+k-1)}_{\hat{\mathcal{A}}^{(\mu)}}$. We avoid this problem by introducing a related sequence $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ that can support such recursive relationships. In fact, it will prove useful to also introduce a sequence $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ with terms in $\mathbb{R}^{N_B}_+$ where $N_B = |\hat{\mathcal{B}}^{(\mu)}|$.

Lemma 57. *Suppose $b, X(0) \in \mathbb{R}^N_+$ and $A \in \mathbb{R}^{N\times N}_+$ for some N. Let $c := b + (A - \mathrm{I})X(0)$ and let $X : \mathbb{N}\cup\{0\} \to \mathbb{R}^N$ be given by*

$$X(k) := \left[\sum_{j=0}^{k-1} A^j\right] b + A^k X(0).$$

Then for all $k > 0$,

1. $X(k) = X(k-1) + A^{k-1}c$;

2. $X(k) = b + AX(k-1)$; *and*

Proof. 1. We proceed by (ordinary) induction. For $k = 1$ we have

$$\begin{aligned} X(1) &= A^0 b + A^1 X(0) = b + AX(0) \\ &= X(0) + b + (A - \mathrm{I})\, X(0) = X(0) + A^0 c. \end{aligned}$$

Now suppose the claim holds for some $k > 0$. Then

$$\begin{aligned} X(k+1) &= \left[\sum_{j=0}^{k} A^j\right] b + A^{k+1} X(0) = \left[\sum_{j=0}^{k-1} A^j\right] b + A^k b + A^{k+1} X(0) \\ &= \left[\sum_{j=0}^{k-1} A^j\right] b + A^k X(0) + A^k b + A^k (A - \mathrm{I})\, X(0) = X(k) + A^k c. \end{aligned}$$

This completes the induction.

2. We again proceed by induction. For $k = 1$, we have already shown above that $X(1) = b + AX(0)$. Now suppose the claim holds for some $k \geqslant 2$. Then the induction is complete by observing that

$$X(k) = \left[\sum_{j=0}^{k-1} A^j\right] b + A^k X(0) = b + A\left[\sum_{j=0}^{k-2} A^j\right] b + AA^{k-1} X(0) = b + AX(k-1).$$

□

Definition 58. Assume that $\mathcal{A}$ and $\mathcal{B}$ are disjoint subsets of $\mathcal{N}$ with $|\mathcal{A}| = N_A$ and $|\mathcal{B}| = N_B$ so that $N_A + N_B \leqslant N$. Let $A \in \mathbb{R}_+^{N_A \times N_A}$, $b \in \mathbb{R}_+^N$ and $x \in \mathbb{R}_+^{N_A}$.

The sequence $(X(k))_{k \geqslant 0} \coloneqq (X(k; b, A, x, \mathcal{A}))_{k \geqslant 0} \subseteq \mathbb{R}_+^{N_A}$ is defined so that

$$\begin{aligned} X(0) &\coloneqq x \\ X(k) &\coloneqq \left[\sum_{j=0}^{k-1} A^j\right] b_{\mathcal{A}} + A^k x \text{ for all } k \geqslant 1. \end{aligned}$$

The sequence $(Y(k))_{k \geqslant 1} \coloneqq (Y(b, A, x, \mathcal{A}, \mathcal{B}))_{k \geqslant 1} \subseteq \mathbb{R}_+^{N_B}$ is defined so that

$$Y(k) \coloneqq b_{\mathcal{B}} + AX(k-1) \text{ for all } k \geqslant 1.$$

Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and μ an ordinal with $\hat{b}^{(\mu)}$, $\hat{M}^{(\mu)}$ as in Definition 52.

Define $\left(X^{(\mu)}(k)\right)_{k \geqslant 0} \coloneqq \left(X(k; \hat{b}^{(\mu)}, \hat{M}^{(\mu)}, \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}, \hat{\mathcal{A}}^{(\mu)})\right)_{k \geqslant 0}$.

If $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ and $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ then define $\left(Y^{(\mu)}(k)\right)_{k \geqslant 1} \coloneqq \left(Y(k; \hat{b}^{(\mu)}, \hat{M}^{(\mu)}, \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}, \hat{\mathcal{A}}^{(\mu)}, \hat{\mathcal{B}}^{(\mu)})\right)_{k \geqslant 1}$.

If $\hat{\mathcal{A}}^{(\mu)} = \varnothing$ and $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ then define $\left(Y^{(\mu)}(k)\right)_{k \geqslant 1} \coloneqq \left(\hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}}\right)_{k \geqslant 1}$.

The following lemmas (as well as Lemma 57 above) establish the key properties of $\left(X^{(\mu)}(k)\right)_{k \geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k \geqslant 1}$.

Lemma 59. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system, μ an ordinal and $\tilde{X}^{(\mu)}$, $\tilde{Y}^{(\mu)}$ as in Definition 52.*

The limit $\lim_{k \to \infty} \left(X^{(\mu)}(k)\right)_{k \geqslant 0}$ exists if and only if $(\mathrm{I} - M^{(\mu)})$ is invertible. Furthermore, whenever the limit exists $\lim_{k \to \infty} X^{(\mu)}(k) = \tilde{X}^{(\mu)}$ and if $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ the limit $\lim_{k \to \infty} Y^{(\mu)}(k) = \tilde{Y}^{(\mu)}$ also exists.

Proof.

Suppose the limit $\lim_{k\to\infty}\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ exists and denote it by $\hat{X}$. Then

$$\begin{aligned}\hat{X} &= \lim_{k\to\infty} X^{(\mu)}(k)\\ &= \lim_{k\to\infty}\left(\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \hat{M}^{(\mu)}X^{(\mu)}(k-1)\right)\\ &= \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \hat{M}^{(\mu)}\left(\lim_{k\to\infty} X^{(\mu)}(k-1)\right)\\ &= \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \hat{M}^{(\mu)}\left(\lim_{k\to\infty} X^{(\mu)}(k)\right)\\ &= \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \hat{M}^{(\mu)}\hat{X}.\end{aligned}$$

The second line follows by Lemma 57.2 and the third line follows since $\hat{M}^{(\mu)}$ is a bounded linear operator.

Let ϱ be the spectral radius of $\hat{M}^{(\mu)}$. Note that the matrix norm of $\hat{M}^{(\mu)}$ satisfies $\varrho \leqslant \|\hat{M}^{(\mu)}\|_1 \leqslant \beta\|\Omega\|_1$. Since $\|\Omega\|_1 \leqslant 1$ we see that $\varrho \leqslant 1$ too. If $\varrho = 1$ then $\|\hat{M}^{(\mu)}\|_1 = 1$ and, in addition, there is a non-empty set $\mathcal{C} \subseteq \hat{\mathcal{A}}^{(\mu)}$ such that $\sum_{i\in\mathcal{C}} \hat{M}^{(\mu)}_{ij} = 1$ for all $j \in \mathcal{C}$ (e.g. see Karlin & Taylor (1981))[2]. Hence for all $i \in \hat{M}^{(\mu)} \setminus \mathcal{C}$ and $j \in \mathcal{C}$ we have that $\hat{M}^{(\mu)}_{ij} = 0$. Therefore

$$\begin{aligned}\sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{X}_i &= \sum_{i\in\hat{\mathcal{A}}^{(\mu)}}\left(\hat{b}^{(\mu)}_i + \sum_{j\in\hat{\mathcal{A}}^{(\mu)}} \hat{M}^{(\mu)}_{ij}\hat{X}_j\right)\\ &= \sum_{i\in\hat{\mathcal{A}}^{(\mu)}}\left(\hat{b}^{(\mu)}_i + \sum_{j\in\mathcal{C}} \hat{M}^{(\mu)}_{ij}\hat{X}_j\right)\\ &= \sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{b}^{(\mu)}_i + \sum_{j\in\hat{\mathcal{A}}^{(\mu)}} \hat{X}_j \sum_{i\in\mathcal{C}} \hat{M}^{(\mu)}_{ij}\\ &= \sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{b}^{(\mu)}_i + \sum_{j\in\hat{\mathcal{A}}^{(\mu)}} \hat{X}_j.\end{aligned}$$

Therefore $\sum_{i\in\hat{\mathcal{A}}^{(\mu)}} \hat{b}^{(\mu)}_i = 0$. Since $\|\hat{M}^{(\mu)}\|_1 = 1$ this contradicts Lemma 54. Hence $\varrho < 1$. Standard results in spectral theory (e.g. see Horn & Johnson (2012)) imply that $\lim_{k\to\infty}\left(\hat{M}^{(\mu)})\right)^k = \mathbf{Z}$, a zero matrix, and hence that $\|\hat{M}^{(\mu)}\|_1 < 1$ and $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible.

By a simple rearrangement we then get that $(\mathrm{I} - \hat{M}^{(\mu)})\hat{X} = \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}$ and since $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible we obtain that $\hat{X} = (\mathrm{I} - \hat{M}^{(\mu)})^{-1}\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} =: \tilde{X}^{(\mu)}$.

Conversely, if $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible then $\varrho < 1$. Therefore both the series $\sum_{k=0}^{\infty}(\hat{M}^{(\mu)})^k$ converges to $(\mathrm{I} - \hat{M}^{(\mu)})^{-1}$ and the limit $\lim_{k\to\infty}\left(\hat{M}^{(\mu)}\right)^k$ vanishes. Hence

$$\lim_{k\to\infty} X^{(\mu)}(k) = \left[\sum_{k=0}^{\infty}\left(\hat{M}^{(\mu)}\right)^k\right]\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \lim_{k\to\infty}\left(\hat{M}^{(\mu)}\right)^k X^{(\mu)}(0) = (\mathrm{I} - \hat{M}^{(\mu)})^{-1}\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = \tilde{X}^{(\mu)}.$$

Finally, by definition $Y^{(\mu)}(k+1) = \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k)$. By continuity of $\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}$ as a linear

[2] A similar argument was used in the proof of Lemma 27.2. in Chapter 2

operator,

$$\begin{aligned}\lim_{k\to\infty} Y^{(\mu)}(k) &= \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}} \lim_{k\to\infty} X^{(\mu)}(k)\\ &= \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}(\mathrm{I}-\hat{M}^{(\mu)})^{-1}\hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} =: \tilde{Y}^{(\mu)}.\end{aligned}$$

□

Lemma 60. *Let $(a,\Omega,\bar{B},\bar{L},\alpha,\beta)$ be a regular clearing system and μ an ordinal. If $X^{(\mu)}(1) \geqslant X^{(\mu)}(0)$ then the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are non-decreasing.*

Proof. By Lemma 57.1, $X^{(\mu)}(k+1) - X^{(\mu)}(k) = \left(\hat{M}^{(\mu)}\right)^k c$ where $c = \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + (\hat{M}^{(\mu)} - \mathrm{I})X^{(\mu)}(0)$. In particular, $c = \left(\hat{M}^{(\mu)}\right)^0 c = X^{(\mu)}(1) - X^{(\mu)}(0) \geqslant 0$. Since $\left(\hat{M}^{(\mu)}\right)^k$ is also non-negative, this implies that $X^{(\mu)}(k+1) - X^{(\mu)}(k) \geqslant 0$ for all k. Hence $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ is non-decreasing. We then note that for $k > 0$

$$\begin{aligned}&Y^{(\mu)}(k+1) - Y^{(\mu)}(k) =\\ =&\left(\hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k)\right) - \left(\hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k-1)\right)\\ =&\beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}\left(X^{(\mu)}(k) - X^{(\mu)}(k-1)\right).\end{aligned}$$

Since $\beta > 0$, $\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}$ is non-negative and $\left(X^{(\mu)}(k) - X^{(\mu)}(k-1)\right) \geqslant 0$ for all k, it follows that $Y^{(\mu)}(k+1) - Y^{(\mu)}(k) \geqslant 0$ for all k, i.e. $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ is also non-decreasing. □

Lemma 61. *Suppose the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are non-decreasing. Then*

1. *if $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible then, for all k, $X^{(\mu)}(k) \leqslant \tilde{X}^{(\mu)}$ and $Y^{(\mu)}(k) \leqslant \tilde{Y}^{(\mu)}$;*
2. *if there is some $i \in \hat{\mathcal{A}}^{(\mu)}$ such that $X^{(\mu)}(k+1)_i - X^{(\mu)}(k)_i > 0$ for some integer k with $k \geqslant N_A$ then $X^{(\mu)}(l+1)_i - X^{(\mu)}(l)_i > 0$ for arbitrarily many integers l with $l > k$; and*
3. *if there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $Y^{(\mu)}(k+1)_i - Y^{(\mu)}(k)_i > 0$ for some integer k with $k > N_A$ then $Y^{(\mu)}(l+1)_i - Y^{(\mu)}(l)_i > 0$ for arbitrarily many integers l with $l > k$.*

Proof.

1. Lemma 59 implies that the limits $\lim_{k\to\infty} X^{(\mu)}(k) = \tilde{X}^{(\mu)}$ and $\lim_{k\to\infty} Y^{(\mu)}(k) = \tilde{Y}^{(\mu)}$ exist. Since $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are non-decreasing, it follows that every term $X^{(\mu)}(k)$ of the sequence must be bounded above by the limit $\tilde{X}^{(\mu)}$ and every term $Y^{(\mu)}(k)$ of the sequence must be bounded above by the limit $\tilde{Y}^{(\mu)}$.
2. Since $(\hat{M}^{(\mu)})^k$ is non-negative, $X^{(\mu)}(k+1)_i - X^{(\mu)}(k)_i = \sum_{j\in\hat{\mathcal{A}}^{(\mu)}}\left((\hat{M}^{(\mu)})^k\right)_{ij} c_j > 0$ whenever there is some $j \in \hat{\mathcal{A}}^{(\mu)}$ such that $\left((\hat{M}^{(\mu)})^k\right)_{ij} > 0$ and $c_j > 0$. Therefore, it is sufficient to show that $\left((\hat{M}^{(\mu)})^k\right)_{ij} > 0$ for arbitrarily many k satisfying $k \geqslant N_A$.

   Consider the adjacency matrix $A \in \{0,1\}^{N_A\times N_A}$ such that $A_{ij} = 1$ if and only if $\hat{M}^{(\mu)}_{ij} > 0$ and $A_{ij} = 0$ if and only if $\hat{M}^{(\mu)}_{ij} = 0$. Note that $\left(A^k\right)_{ij} > 0$ if and only if $\left(\left(\hat{M}^{(\mu)}\right)^k\right)_{ij} > 0$. A standard observation about adjacency matrices in $\mathbb{R}^{N_A\times N_A}$ (e.g. see 6.2 of Newman (2010)) is that they are in one-to-one correspondence with directed graphs of N_A nodes. Let $G(A)$ be the directed graph corresponding to A.

Another standard observation about powers of adjacency matrices (e.g. see 6.10 of Newman (2010)) is that $\left(A^k\right)_{ij} > 0$ if and only if there is a path[3] in $G(A)$ consisting of exactly $k+1$ (possibly repeating) nodes from node i to node j. Since there are at most N_A distinct nodes in any path in $G(A)$, it follows that for $k \geqslant N_A$ any path of $k+1$ nodes from i to j must contain at least one repeating node. Suppose the length of the path from this node to itself is l. So by repeatedly extending the path from i to j by the l nodes of such a cycle, we can make this path arbitrarily long.

So suppose that $\left(\left(\hat{M}^{(\mu)}\right)^k\right)_{ij} > 0$ for some k satisfying $k \geqslant N_A$. Then $\left(A^k\right)_{ij} > 0$ and there is a path in $G(A)$ of $k+1$ nodes from i to j. Therefore there is a path of $nl+k+1$ nodes from i to j for any integer n. Hence $\left(A^{nl+k}\right)_{ij} > 0$ and so $\left(\left(\hat{M}^{(\mu)}\right)^{nl+k}\right)_{ij} > 0$. Without loss of generality, we can then say that $\left(\left(\hat{M}^{(\mu)}\right)^k\right)_{ij} > 0$ for arbitrarily many k satisfying $k \geqslant N_A$. Therefore we have shown that for such arbitrary k, $X^{(\mu)}(k+1)_i - X^{(\mu)}(k)_i = \left(\left(\hat{M}^{(\mu)}\right)^k\right)_{ij} c_j > 0$.

3. As in the proof of Lemma 60,

$$Y^{(\mu)}(k+1) - Y^{(\mu)}(k) = \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}\left(X^{(\mu)}(k) - X^{(\mu)}(k-1)\right).$$

If there is some k satisfying $k > N_A$ and some $i \in \hat{\mathcal{B}}^{(\mu)}$ with $Y^{(\mu)}(k+1)_i - Y^{(\mu)}(k)_i > 0$ then $\beta\sum_{j\in\hat{\mathcal{B}}^{(\mu)}}\Omega_{ij}\left(X^{(\mu)}(k)_j - X^{(\mu)}(k-1)_j\right) > 0$. Since $\beta > 0$, Ω is non-negative and $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ is non-decreasing, we must have that there is some $j \in \hat{\mathcal{A}}^{(\mu)}$ such that $\Omega_{ij} > 0$ and $X^{(\mu)}(k)_j - X^{(\mu)}(k-1)_j > 0$.

Note that if $k > N_A$ then $k-1 \geqslant N_A$ and so by the previous result, $X^{(\mu)}(k)_j - X^{(\mu)}(k-1)_j > 0$ for arbitrarily many k satisfying $k > N_A$. This implies that $Y^{(\mu)}(k+1)_i - Y^{(\mu)}(k)_i > 0$ for arbitrarily many such k.

□

The remaining results in this section comprise the main tools that will allow us to construct an algorithm for computing the least fixed point of Φ. They characterise the terms of the sequence $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ in terms of $X^{(\mu)}$ and $Y^{(\mu)}$, which can be computed easily. Furthermore, they allow us to identify the spans of the sequence $\left(\phi^{(\mu)}\right)_{\mu\geqslant 0}$ where an infinite number of terms might otherwise have caused us difficulties. However, we can show that precisely on those spans we can compute the terms of our sequence simply by solving systems of linear equations.

Proposition 63 establishes the basic relationship between $X^{(\mu)}$ and $Y^{(\mu)}$ on the one hand and $\phi^{(\mu)}$ and $V_{\alpha,\beta}(\phi^{(\mu)})$ on the other. Lemma 64 looks more closely at the term $X^{(\mu)}(0)$, from which many other

[3] The definition of the term "path" varies with literature. In Newman (2010) and here it refers to an alternating sequence of nodes and edges. A path may visit the same node or edge multiple times. Here we deal with *directed* paths and thus the same node can only be visited by following the direction of the edges.

properties of the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are inherited. Theorem 65 distinguishes the behaviour of the various sequences based on whether $\kappa(\mu)$ is finite or not. Finally, Theorem 66 provides two sufficiency conditions for $\kappa(\mu)$ to be infinite.

Remark 62. Note that $X^{(\mu)}(k)$ is only defined if $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ and $Y^{(\mu)}(k)$ is only defined if $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$. To lighten the notation we will adopt the convention that $X^{(\mu)}(k)$, $Y^{(\mu)}(k)$ and other vectors restricted to an empty set of components are well-defined and that equalities and inequalities between such zero-dimensional vectors always hold vacuously.

Proposition 63. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and μ an ordinal.*

1. *Let k be an integer with $0 \leqslant k \leqslant \kappa(\mu)$. Then $X^{(\mu)}(k) = \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}}$.*
2. *Let k be an integer with $0 < k \leqslant \kappa(\mu)+1$. Then $X^{(\mu)}(k) = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{A}}^{(\mu)}}$ and $Y^{(\mu)}(k) = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{B}}^{(\mu)}}$.*

Proof. 1. We assume that $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$ as otherwise the claim is vaccuous. Since, by definition, $X^{(\mu)}(0) = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}$ the result follows by induction for all integers k with $0 \leqslant k \leqslant \kappa(\mu)$. Suppose that $0 \leqslant k < \kappa(\mu)$. We show that if $X^{(\mu)}(k) = \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}}$ then

$$\begin{aligned} X^{(\mu)}(k+1) &= V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}} \\ &= \phi^{(\mu+k+1)}_{\hat{\mathcal{A}}^{(\mu)}}. \end{aligned}$$

So assume that $X^{(\mu)}(k) = \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}}$. By Proposition 55.1, $\hat{\mathcal{A}}^{(\mu+k)} = \hat{\mathcal{A}}^{(\mu)}$, $\hat{\mathcal{B}}^{(\mu+k)} = \hat{\mathcal{B}}^{(\mu)}$ and $\hat{\mathcal{L}}^{(\mu+k)} = \hat{\mathcal{L}}^{(\mu)}$. Applying Definition 52, we obtain that $\hat{b}^{(\mu+k)} = \hat{b}^{(\mu)}$. By Corollary 53,

$$\begin{aligned} V_{\alpha,\beta}(\phi^{(\mu+k)}) &= \hat{b}^{(\mu+k)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu+k)}}\phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu+k)}} \\ &= \hat{b}^{(\mu)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}} \\ &= \hat{b}^{(\mu)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k). \end{aligned}$$

Since $\hat{\mathcal{A}}^{(\mu)}, \hat{\mathcal{B}}^{(\mu)} \subseteq \mathcal{N}$, it follows by Lemma 57.2 that

$$\begin{aligned} X^{(\mu)}(k+1) &= \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{A}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k) = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}}, \\ Y^{(\mu)}(k+1) &= \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}}X^{(\mu)}(k) = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}}. \end{aligned} \tag{3.1}$$

By Proposition 55.2, it then follows that if $k < \kappa(\mu)$ then $X^{(\mu)}(k+1) = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{A}}^{(\mu)}} = \phi^{(\mu+k+1)}_{\hat{\mathcal{A}}^{(\mu)}}$. Thus, for $0 \leqslant k \leqslant \kappa(\mu)$, we have shown that $X^{(\mu)}(k) = \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)}}$. This completes the induction.

2. Note that in the proof of Proposition 63.1 above we do not use Proposition 55.2 until after we introduced equations (3.1). Until that point we only used Proposition 55.1 which applies for k satisfying $0 \leqslant k \leqslant \kappa(\mu)$ unlike Proposition 63.1 which only applied for k satisfying $0 \leqslant k < \kappa(\mu)$. Moreover we only applied Proposition 55.1 to the first of the two equations (3.1). Therefore,

provided $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$, the claim then follows directly from both equations (3.1) for $0 \leqslant k \leqslant \kappa(\mu)$ by the induction in the proof of Proposition 63.1.

Finally, if $\hat{\mathcal{A}}^{(\mu)} = \varnothing$ then by Corollary 53 $Y^{(\mu)}(k+1) = \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+k)})_{\hat{\mathcal{B}}^{(\mu)}}$ for all $k \geqslant 1$. Meanwhile, $X^{(\mu)}(k) = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{A}}^{(\mu)}}$ is vacuous.

□

Lemma 64. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and μ an ordinal. $X^{(\mu)}(1) \geqslant X^{(\mu)}(0)$ and, in particular, the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are non-decreasing.*

Proof. By definition we have $X^{(\mu)}(0) = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}}$. Suppose first that $\kappa(\mu) \geqslant 1$. By Proposition 63.1, $X^{(\mu)}(1) = \phi^{(\mu+1)}_{\hat{\mathcal{A}}^{(\mu)}}$. Monotonicity of $\left(\phi^{(\mu+k)}\right)_{k\geqslant 0}$ implies that $X^{(\mu)}(0) = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} \leqslant \phi^{(\mu+1)}_{\hat{\mathcal{A}}^{(\mu)}} = X^{(\mu)}(1)$.

Now let $\kappa(\mu) = 0$. By Proposition 63.2, $X^{(\mu)}(1) = V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{A}}^{(\mu)}}$. In addition, by Proposition 48 we also have that $\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < X^{(\mu)}(0) < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}}$ and $X^{(\mu)}(0) \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$.

Let $i \in \hat{\mathcal{A}}^{(\mu)}$ and suppose $V(\phi^{(\mu)})_i \geqslant \bar{L}_i$. By Lemma 39.2 this implies that $X^{(\mu)}(1)_i = V_{\alpha,\beta}(\phi^{(\mu)})_i \geqslant \bar{T}_i$. Hence $X^{(\mu)}(0)_i = \phi^{(\mu)}_i \leqslant \bar{T}_i \leqslant X^{(\mu)}(1)_i$.

Let $i \in \hat{\mathcal{A}}^{(\mu)}$ and suppose $V(\phi^{(\mu)})_i < \bar{L}_i$ and $V_{\alpha,\beta}(\phi^{(\mu)})_i > \bar{B}_i$. By Lemma 39.3 this implies $\phi^{(\mu+1)}_i = V_{\alpha,\beta}(\phi^{(\mu)})_i$ and therefore $\phi^{(\mu+1)}_i = X^{(\mu)}(1)_i$. Hence by Proposition 42.1, $X^{(\mu)}(0)_i = \phi^{(\mu)}_i \leqslant \phi^{(\mu+1)}_i = X^{(\mu)}(1)_i$.

Finally, suppose there is some $i \in \hat{\mathcal{A}}^{(\mu)}$ with $V(\phi^{(\mu)})_i < \bar{L}_i$ and $V_{\alpha,\beta}(\phi^{(\mu)})_i \leqslant \bar{B}_i$. Then by Proposition 38.4, $\Phi(\phi^{(\mu)})_i = \bar{B}_i$ and hence $\phi^{(\mu+1)}_i = \bar{B}_i < \phi^{(\mu)}_i$. This contradicts Proposition 42.1.

Therefore we have shown that for all $i \in \hat{\mathcal{A}}^{(\mu)}$, $X^{(\mu)}(0)_i \leqslant X^{(\mu)}(1)_i$. The rest of the result follows by Lemma 60. □

Theorem 65. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and μ an ordinal.*

1. *Suppose $\kappa(\mu) < \omega$ and let $k = \kappa(\mu) + 1$. Then*
   (a) $\hat{\mathcal{D}}^{(\mu+k)} = \{i \in \hat{\mathcal{A}}^{(\mu)} \mid X^{(\mu)}(k)_i < \bar{T}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\mu)} \mid Y^{(\mu)}(k)_i < \bar{T}_i\}$ *and*
   $\hat{\mathcal{B}}^{(\mu+k)} = \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid Y^{(\mu)}(k)_i \leqslant \bar{B}_i\}$; *and*
   (b) $X^{(\mu)}(k)_{\hat{\mathcal{A}}^{(\mu)} \cap \hat{\mathcal{A}}^{(\mu+k)}} = \phi^{(\mu+k)}_{\hat{\mathcal{A}}^{(\mu)} \cap \hat{\mathcal{A}}^{(\mu+k)}}$ *and*
   $Y^{(\mu)}(k)_{\hat{\mathcal{B}}^{(\mu)} \cap \hat{\mathcal{A}}^{(\mu+k)}} = \phi^{(\mu+k)}_{\hat{\mathcal{B}}^{(\mu)} \cap \hat{\mathcal{A}}^{(\mu+k)}}$.

2. *Suppose $\kappa(\mu) = \omega$. Then*
   (a) $\tilde{X}^{(\mu)} = \phi^{(\mu+\omega)}_{\hat{\mathcal{A}}^{(\mu)}}$ *and*
   $\tilde{Y}^{(\mu)} = V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{B}}^{(\mu)}}$.
   (b) $\left(\mathrm{I} - \hat{M}^{(\mu)}\right)$ *is invertible.*
   (c) $\tilde{X}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$, $\tilde{Y}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ *and* $\tilde{Y}^{(\mu)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$.
   (d) $\hat{\mathcal{D}}^{(\mu+\omega)} = \{i \in \hat{\mathcal{A}}^{(\mu)} \mid \tilde{X}^{(\mu)}_i < \bar{L}_i\} \cup \hat{\mathcal{B}}^{(\mu)}$ *and*
   $\hat{\mathcal{B}}^{(\mu+\omega)} = \hat{\mathcal{B}}^{(\mu)}$.

Proof. 1. Let $0 < k := \kappa(\mu) + 1$ for $\kappa(\mu) < \omega$.

(a) By Corollary 46.2 we have that

$$\hat{\mathcal{D}}^{(\mu+k)} = \{i \in \hat{\mathcal{A}}^{(\mu)} \mid \phi_i^{(\mu+k)} < \bar{L}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\mu)} \mid \phi_i^{(\mu+k)} < \bar{L}_i\}$$

and

$$\hat{\mathcal{B}}^{(\mu+k)} = \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid \phi_i^{(\mu+k)} \leqslant \bar{B}_i\}.$$

Since $k > 0$, $\phi^{(\mu+k)} = \Phi(\phi^{(\mu+k-1)})$. By Lemma 39.2, we have that, for any $i \in \mathcal{N}$, $\phi_i^{(\mu+k)} < \bar{L}_i$ is equivalent to $V_{\alpha,\beta}\left(\phi^{(\mu+k-1)}\right)_i < \bar{T}_i$. By Proposition 63.2, $V_{\alpha,\beta}\left(\phi^{(\mu+k-1)}\right)_{\hat{\mathcal{A}}^{(\mu)}} = X^{(\mu)}(k)$ and $V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{B}}^{(\mu)}} = Y^{(\mu)}(k)$ and it follows that

$$\hat{\mathcal{D}}^{(\mu+k)} = \{i \in \hat{\mathcal{A}}^{(\mu)} \mid X^{(\mu)}(k)_i < \bar{T}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\mu)} \mid Y^{(\mu)}(k)_i < \bar{T}_i\}.$$

In particular, since we showed above that, for every $i \in \hat{\mathcal{D}}^{(\mu+k)}$, $\phi_i^{(\mu+k)} = \Phi(\phi^{(\mu+k-1)}) < \bar{L}_i$, we also have that $\phi_i^{(\mu+k)} = \bar{B}_i \vee V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i$. Therefore, it follows that

$$\begin{aligned}
\hat{\mathcal{B}}^{(\mu+k)} &= \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid \phi_i^{(\mu+k)} \leqslant \bar{B}_i\} \\
&= \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid \bar{B}_i \vee V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i \leqslant \bar{B}_i\} \\
&= \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i \leqslant \bar{B}_i\} \\
&= \{i \in \hat{\mathcal{D}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)} \mid Y^{(\mu)}(k)_i \leqslant \bar{B}_i\}.
\end{aligned}$$

(b) Since $k > 0$, $\bar{B}_i < \phi_i^{(\mu+k)} < \bar{L}_i$ for $i \in \hat{\mathcal{A}}^{(\mu+k)}$ and so by Lemma 39.3 $\phi_i^{(\mu+k)} = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_i$. By Proposition 63.2 then $\phi_i^{(\mu+k)} = X^{(\mu)}(k)_i$ for all $i \in \hat{\mathcal{A}}^{(\mu+k)} \cap \hat{\mathcal{A}}^{(\mu)}$ and $\phi_i^{(\mu+k)} = Y^{(\mu)}(k)_i$ for all $i \in \hat{\mathcal{A}}^{(\mu+k)} \cap \hat{\mathcal{B}}^{(\mu)}$.

2. Suppose $\kappa(\mu) = \omega$.

(a) By Proposition 63.1 $\phi_{\hat{\mathcal{A}}^{(\mu)}}^{(\mu+k)} = X^{(\mu)}(k)$ for all integers k. By Proposition 42.1 the limit $\phi_{\hat{\mathcal{A}}^{(\mu)}}^{(\mu+\omega)} = \lim_{k\to\infty} \phi_{\hat{\mathcal{A}}^{(\mu)}}^{(\mu+k)}$ exists and hence so must the limit $\tilde{X}^{(\mu)} = \lim_{k\to\infty} X^{(\mu)}(k)$ with $\phi_{\hat{\mathcal{A}}^{(\mu)}}^{(\mu+\omega)} = \tilde{X}^{(\mu)}$. By Lemma 59 the limit $\tilde{Y}^{(\mu)} = \lim_{k\to\infty} Y^{(\mu)}(k)$ then also exists and, by Propositions 56.1 and 63.2, $\tilde{Y}^{(\mu)} = V_{\alpha,\beta}(\phi^{(\mu+\omega)})_{\hat{\mathcal{B}}^{(\mu)}}$.

(b) We have just shown that $\phi_{\hat{\mathcal{A}}^{(\mu)}}^{(\mu+\omega)} = \lim_{k\to\infty} X^{(\mu)}(k)$ and the left-hand side exists by Proposition 42.1. Hence by Lemma 59 we have that $(\mathrm{I} - \hat{M}^{(\mu)})$ is invertible.

(c) The result follows directly by Proposition 56.2 and Theorem 65.2a.

(d) For $i \in \hat{\mathcal{A}}^{(\mu)}$, $\phi_i^{(\mu+\omega)} = \tilde{X}_i^{(\mu)}$ by Theorem 65.2a. For $i \in \hat{\mathcal{B}}^{(\mu)}$, $\phi_i^{(\mu+\omega)} = \bar{B}_i$ by Proposition 56.2. Therefore by Corollary 46.2 we have that

$$\begin{aligned}
\hat{\mathcal{D}}^{(\mu+\omega)} &= \{i \in \hat{\mathcal{A}}^{(\mu)} \mid \phi_i^{(\mu+\omega)} < \bar{L}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\mu)} \mid \phi_i^{(\mu+\omega)} < \bar{L}_i\} \\
&= \{i \in \hat{\mathcal{A}}^{(\mu)} \mid \tilde{X}_i^{(\mu)} < \bar{L}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\mu)} \mid \bar{B}_i < \bar{L}_i\} \\
&= \{i \in \hat{\mathcal{A}}^{(\mu)} \mid \tilde{X}_i^{(\mu)} < \bar{L}_i\} \cup \hat{\mathcal{B}}^{(\mu)}
\end{aligned}$$

and

$$\begin{aligned}\hat{\mathcal{B}}^{(\mu+\omega)} &= \{i \in \hat{\mathcal{D}}^{(\mu+\omega)} \cap \hat{\mathcal{B}}^{(\mu)} \mid \phi_i^{(\mu+\omega)} \leqslant \bar{B}_i\} \\ &= \{i \in \hat{\mathcal{B}}^{(\mu)} \mid \bar{B}_i \leqslant \bar{B}_i\} = \hat{\mathcal{B}}^{(\mu)}\end{aligned}$$

□

Theorem 66. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and μ an ordinal. Suppose that $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$, $\left(\mathrm{I} - \hat{M}^{(\mu)}\right)$ is invertible, $\tilde{X}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$, $\tilde{Y}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\tilde{Y}^{(\mu)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$.*

1. *If $\tilde{X}^{(\mu)} < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ and $\tilde{Y}^{(\mu)} < \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ then $\kappa(\mu) = \omega$.*
2. *If $\kappa(\mu) \geqslant |\hat{\mathcal{A}}^{(\mu)}|$ then $\kappa(\mu) = \omega$.*

Proof. 1. By Definition 52 and Lemma 59, the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ converge to the limits $\tilde{X}^{(\mu)} < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ and $\tilde{Y}^{(\mu)} < \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$, respectively. By Lemma 64, these sequences are non-decreasing and therefore, by Lemma 61.1 $X^{(\mu)}(k) < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$, $Y^{(\mu)}(k) < \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ and $Y^{(\mu)}(k) \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ for all $k \geqslant 0$.

   Suppose to the contrary that $\kappa(\mu) < \omega$ and let $k := \kappa(\mu) + 1 > 0$. Then by Proposition 63.2, $V_{\alpha,\beta}\left(\phi^{(\mu+k-1)}\right)_{\hat{\mathcal{A}}^{(\mu)}} = X^{(\mu)}(k)$ and $V_{\alpha,\beta}\left(\phi^{(\mu+k-1)}\right)_{\hat{\mathcal{B}}^{(\mu)}} = Y^{(\mu)}(k)$. Therefore $V_{\alpha,\beta}\left(\phi^{(\mu+k-1)}\right)_{\hat{\mathcal{D}}^{(\mu)}} < \bar{T}_{\hat{\mathcal{D}}^{(\mu)}}$ with $\hat{\mathcal{D}}^{(\mu)} \neq \varnothing$. By Corollary 50, this must mean that $\kappa(\mu) = k - 1 < \kappa_T(\mu)$ and hence $k - 1 = \kappa(\mu) = \kappa_B(\mu)$. This implies that there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}\left(\phi^{(\mu+\kappa_B(\mu))}\right)_i > \bar{B}_i$. However, we also have that $V_{\alpha,\beta}\left(\phi^{(\mu+\kappa_B(\mu))}\right)_{\hat{\mathcal{B}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu+k-1)})_{\hat{\mathcal{B}}^{(\mu)}} = Y^{(\mu)}(k) \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and therefore such an i does not exist. Hence we conclude that $\kappa(\mu) = \omega$.

2. By Theorem 66.1, we only need to consider the case where there is some $i \in \hat{\mathcal{A}}^{(\mu)}$ such that $\tilde{X}_i^{(\mu)} = \bar{T}_i$ or there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $\tilde{Y}_i^{(\mu)} = \bar{T}_i$.

   By Lemma 64 the sequences $(X^{(\mu)}(k))_{k\geqslant 0}$ and $(Y^{(\mu)}(k))_{k\geqslant 1}$ are non-decreasing. Therefore, by Lemma 59 and Lemma 61.1, they increase up to their respective limits $\tilde{X}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ and $\tilde{Y}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$.

   Suppose to the contrary that $\kappa(\mu) < \omega$ and let $i \in \hat{\mathcal{A}}^{(\mu)}$. Since $\hat{\mathcal{A}}^{(\mu)} \neq \varnothing$, $\kappa(\mu) > 0$. Then by Proposition 63.2 and Corollary 50, we have that $X(\kappa(\mu))_i = V_{\alpha,\beta}(\phi^{(\mu+\kappa(\mu)-1)})_i < \bar{T}_i$. By assumption that $X(\kappa(\mu)+1)_i = \bar{T}_i$, there is some $i \in \hat{\mathcal{A}}^{(\mu)}$ such that $X(\kappa(\mu)+1)_i - X(\kappa(\mu))_i > 0$. Therefore by Lemma 61.2, there is some integer k with $k > \kappa(\mu) \geqslant |\hat{\mathcal{A}}^{(\mu)}|$ such that $X^{(\mu)}(\kappa(\mu) + 1)_i - X^{(\mu)}(\kappa(\mu))_i > 0$. But since $(X^{(\mu)}(k))_{k\geqslant 0}$ is non-decreasing it must follow that $X^{(\mu)}(k + 1)_i > X^{(\mu)}(\kappa(\mu) + 1)_i \geqslant \bar{T}_i$. However this is a contradiction since, by Lemma 61.1, $X^{(\mu)}(k+1) \leqslant \tilde{X}^{(\mu)} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$.

   Similarly, if $i \in \hat{\mathcal{B}}^{(\mu)}$ then, by Proposition 63.2 and Corollary 50, we must have that $Y(\kappa(\mu))_i = V_{\alpha,\beta}(\phi^{(\mu+\kappa(\mu)-1)})_i < \bar{T}_i$. Therefore by assumption that $Y(\kappa(\mu) + 1)_i = \bar{T}_i$ and Lemma 61.3, there is some integer k with $k > \kappa(\mu) \geqslant |\hat{\mathcal{A}}^{(\mu)}|$ such that $Y^{(\mu)}(k+1)_i - Y^{(\mu)}(k)_i > 0$. This implies that $Y^{(\mu)}(k + 1)_i > Y^{(\mu)}(\kappa(\mu) + 1)_i \geqslant \bar{T}_i$, contradicting the fact that $Y^{(\mu)}(k + 1) \leqslant \tilde{Y}^{(\mu)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$, by Lemma 61.1.

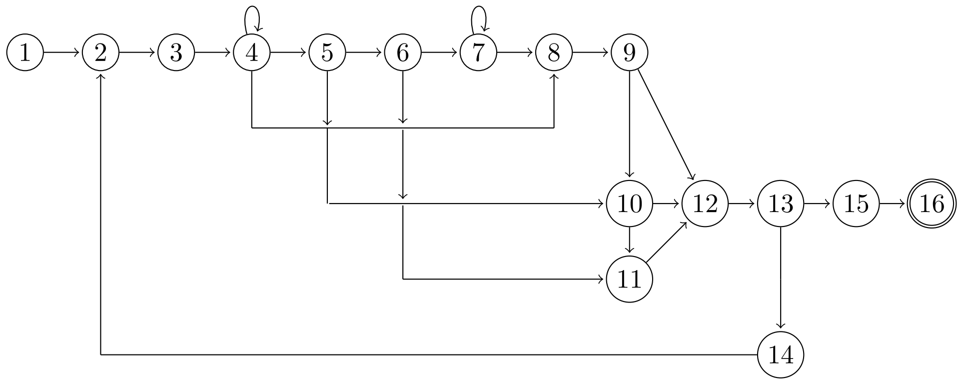

Figure 3.1: Flow of Algorithm 3

Since the assumption that $\kappa(\mu) < \omega$ leads to a contradiction in both cases, we conclude that $\kappa(\mu) = \omega$.

□

3.4 Construction of the least fixed point

As in the previous section, let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system, Φ the corresponding clearing function and $\bar{T} = \beta\bar{L} + (\alpha - \beta)a$. The results established above allow us to formulate Algorithm 3 for computing the least fixed point of Φ. In fact, we will show in Theorem 67 that this algorithm returns the least fixed point in a finite number of steps.

Figure 3.1 depicts the high-level flow of Algorithm 3 for ease of reference. Note that Algorithm 3 adopts a similar convention to Remark 62. That is, we assume that zero-dimensional vectors and matrices are well-defined, the terms containing them are zero and comparisons of such zero-dimensional objects are vacuously true.

Our main claim is that p^* obtained in Algorithm 3 is in fact the least fixed point of Φ and that it can be obtained in a finite number of iterations of the algorithm.

Theorem 67. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system. Then*

1. *Algorithm 3 terminates in a finite number of steps; and*
2. *the vector p^*, obtained as the output of the algorithm, is the least fixed point of Φ.*

We spend the rest of this section proving this theorem. In particular, we will need to understand what happens in the algorithm for different values of n. It should be clear by inspection of Algorithm 3 (and Figure 3.1) that n is initialised to zero and then it is only ever changed at Step 14 where it is incremented by 1 and the algorithm is restarted from Step 2. This process can only be terminated if the algorithm reaches Step 15. Therefore, assuming the algorithm does terminate, while it is running n will satisfy $0 \leqslant n \leqslant n^*$. We call such n an *iteration* of Algorithm 3.

Algorithm 3: Least fixed point clearing algorithm

1. Set $\mathcal{A}^{(-1)} = \mathcal{A}^{(0)} = \mathcal{L}^{(-1)} = \mathcal{L}^{(0)} = \varnothing$, $\mathcal{B}^{(-1)} = \mathcal{B}^{(0)} = \mathcal{D}^{(0)} = \mathcal{N}$, $\tilde{x}^{(-1)} = \tilde{y}^{(-1)} = \mathbf{0}$, $n = 0$
2. Set $b^{(n)} = \alpha a + \beta\Omega_{\mathcal{N}\mathcal{L}^{(n)}}\bar{L}_{\mathcal{L}^{(n)}} + \beta\Omega_{\mathcal{N}\mathcal{B}^{(n)}}\bar{B}_{\mathcal{B}^{(n)}}$ and $M^{(n)} = \beta\Omega_{\mathcal{A}^{(n)}\mathcal{A}^{(n)}}$
3. Set $x^{(0,n)}$ such that $x^{(0,n)}_{\mathcal{A}^{(n)}\cap\mathcal{A}^{(n-1)}} = \tilde{x}^{(n-1)}_{\mathcal{A}^{(n)}\cap\mathcal{A}^{(n-1)}}$ and $x^{(0,n)}_{\mathcal{A}^{(n)}\cap\mathcal{B}^{(n-1)}} = \tilde{y}^{(n-1)}_{\mathcal{A}^{(n)}\cap\mathcal{B}^{(n-1)}}$,
   set $c^{(n)} = b^{(n)}_{\mathcal{A}^{(n)}} + \left(M^{(n)} - \mathrm{I}\right)x^{(0,n)}$ and $y^{(0,n)} = \tilde{y}^{(n-1)}_{\mathcal{B}^{(n)}}$
4. For $k = 0$ to $|\mathcal{A}^{(n)}|$
   (a) Set $x^{(k+1,n)} = x^{(k,n)} + \left(M^{(n)}\right)^k c^{(n)}$ and $y^{(k+1,n)} = b^{(n)}_{\mathcal{B}^{(n)}} + \beta\Omega_{\mathcal{B}^{(n)}\mathcal{A}^{(n)}}x^{(k,n)}$
   (b) If $x^{(k+1,n)} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(k+1,n)} < \bar{T}_{\mathcal{B}^{(n)}}$, $y^{(k+1,n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$ then increment k and continue to Step 4a
   (c) Else increment k and go to Step 8
5. If $\mathcal{A}^{(n)} = \varnothing$ then set $\tilde{y}^{(n)} = b^{(n)}_{\mathcal{B}^{(n)}}$ and go to Step 10
6. If $\det\left(\mathrm{I} - M^{(n)}\right) \neq 0$
   (a) Set $\tilde{x}^{(n)} = \left(\mathrm{I} - M^{(n)}\right)^{-1} b^{(n)}_{\mathcal{A}^{(n)}}$ and set $\tilde{y}^{(n)} = b^{(n)}_{\mathcal{B}^{(n)}} + \beta\Omega_{\mathcal{B}^{(n)}\mathcal{A}^{(n)}}\tilde{x}^{(n)}$
   (b) If $\tilde{x}^{(n)} \leqslant \bar{T}_{\mathcal{A}^{(n)}}$, $\tilde{y}^{(n)} \leqslant \bar{T}_{\mathcal{B}^{(n)}}$ and $\tilde{y}^{(n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$ then go to Step 11
7. While $x^{(k,n)} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(k,n)} < \bar{T}_{\mathcal{B}^{(n)}}$ and $y^{(k,n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$
   (a) Set $x^{(k+1,n)} = x^{(k,n)} + \left(M^{(n)}\right)^k c^{(n)}$ and $y^{(k+1,n)} = b^{(n)}_{\mathcal{B}^{(n)}} + \beta\Omega_{\mathcal{B}^{(n)}\mathcal{A}^{(n)}}x^{(k,n)}$
   (b) Increment k
8. Set $\tilde{x}^{(n)} = x^{(k,n)}$ and $\tilde{y}^{(n)} = y^{(k,n)}$
9. Set $\mathcal{D}^{(n+1)} = \{i \in \mathcal{A}^{(n)} \mid \tilde{x}^{(n)}_i < \bar{T}_i\} \cup \{i \in \mathcal{B}^{(n)} \mid \tilde{y}^{(n)}_i < \bar{T}_i\}$,
   $\mathcal{B}^{(n+1)} = \{i \in \mathcal{D}^{(n+1)} \cap \mathcal{B}^{(n)} \mid \tilde{y}^{(n)}_i \leqslant \bar{B}_i\}$ and go to Step 12
10. Set $\mathcal{D}^{(n+1)} = \{i \in \mathcal{B}^{(n)} \mid \tilde{y}^{(n)}_i < \bar{T}_i\}$, $\mathcal{B}^{(n+1)} = \{i \in \mathcal{D}^{(n+1)} \cap \mathcal{B}^{(n)} \mid \tilde{y}^{(n)}_i \leqslant \bar{B}_i\}$ and go to Step 12
11. Set $\mathcal{D}^{(n+1)} = \{i \in \mathcal{A}^{(n)} \mid \tilde{x}^{(n)}_i < \bar{L}_i\} \cup \mathcal{B}^{(n)}$, $\mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$
12. Set $\mathcal{L}^{(n+1)} = \mathcal{N} \setminus \mathcal{D}^{(n+1)}$ and $\mathcal{A}^{(n+1)} = \mathcal{D}^{(n+1)} \setminus \mathcal{B}^{(n+1)}$
13. If $\mathcal{A}^{(n-1)} = \mathcal{A}^{(n)} = \mathcal{A}^{(n+1)}$, $\mathcal{B}^{(n-1)} = \mathcal{B}^{(n)} = \mathcal{B}^{(n+1)}$, $\mathcal{L}^{(n-1)} = \mathcal{L}^{(n)} = \mathcal{L}^{(n+1)}$ then go to Step 15
14. Else increment n and go to Step 2
15. Set $n^* = n + 1$ and p^* such that $p^*_{\mathcal{B}^{(n^*)}} = \bar{B}_{\mathcal{B}^{(n^*)}}$, $p^*_{\mathcal{L}^{(n^*)}} = \bar{L}_{\mathcal{L}^{(n^*)}}$ and $p^*_{\mathcal{A}^{(n^*)}} = \tilde{x}^{(n^*-1)}$
16. Output p^* and $\mathcal{D}^{(n^*)}$

By inspection, it should be clear that, for each iteration n satisfying $n < n^*$ the sets $\mathcal{A}^{(n)}$, $\mathcal{B}^{(n)}$ and $\mathcal{L}^{(n)}$, the matrix $M^{(n)}$ and the vector $b^{(n)}$ are all defined. In addition the vectors $\tilde{x}^{(n-1)}$ and $\tilde{y}^{(n-1)}$ are defined if $\mathcal{A}^{(n)} \neq \varnothing$ or $\mathcal{B}^{(n)} \neq \varnothing$, respectively.

Remark 68. As mentioned, we can see by inspection of Algorithm 3 that it can only terminate, i.e. reach Step 16, precisely if and only if the algorithm reaches Step 13 and is then pointed to Step 15. Therefore we can use the following statements interchangeably:

- Algorithm 3 terminates;
- n^* is defined in Algorithm 3 at Step 15;
- for the iteration $n = n^* - 1$, $\mathcal{A}^{(n-1)} = \mathcal{A}^{(n)} = \mathcal{A}^{(n+1)}$, $\mathcal{B}^{(n-1)} = \mathcal{B}^{(n)} = \mathcal{B}^{(n+1)}$, $\mathcal{L}^{(n-1)} = \mathcal{L}^{(n)} = \mathcal{L}^{(n+1)}$; and
- for some iteration n, $\mathcal{A}^{(n-1)} = \mathcal{A}^{(n)} = \mathcal{A}^{(n+1)}$, $\mathcal{B}^{(n-1)} = \mathcal{B}^{(n)} = \mathcal{B}^{(n+1)}$, $\mathcal{L}^{(n-1)} = \mathcal{L}^{(n)} = \mathcal{L}^{(n+1)}$.

Lemma 69 states some basic properties of the sets $\mathcal{A}^{(n)}$, $\mathcal{B}^{(n)}$, $\mathcal{L}^{(n)}$ and $\mathcal{D}^{(n)}$, which we will use extensively.

Lemma 69. *Let n be an iteration of Algorithm 3. Then*

1. *$\mathcal{N} = \mathcal{D}^{(n)} \cup \mathcal{L}^{(n)}$ and $\mathcal{D}^{(n)} = \mathcal{A}^{(n)} \cup \mathcal{B}^{(n)}$ where the unions are disjoint; and*
2. *$\mathcal{D}^{(n)} \subseteq \mathcal{D}^{(n-1)}$, $\mathcal{L}^{(n)} \supseteq \mathcal{L}^{(n-1)}$ and $\mathcal{B}^{(n)} \subseteq \mathcal{B}^{(n-1)}$.*

Proof.

1. The claim is true for $n = 0$ (and hence also for $n = -1$) since by definition

$$\mathcal{D}^{(0)} = \mathcal{B}^{(0)} = \varnothing \cup \mathcal{B}^{(0)} = \mathcal{A}^{(0)} \cup \mathcal{B}^{(0)}$$
$$\mathcal{L}^{(0)} = \mathcal{N} \setminus \mathcal{D}^{(0)}.$$

For $n > 0$, at Step 12 $\mathcal{A}^{(n)}$ is defined precisely so that $\mathcal{A}^{(n)} = \mathcal{D}^{(n)} \setminus \mathcal{B}^{(n)}$ and $\mathcal{L}^{(n)} = \mathcal{N} \setminus \mathcal{D}^{(n)}$.

2. The sets $\mathcal{D}^{(n)}$ and $\mathcal{B}^{(n)}$ are defined in Steps 9, 10 or 11, depending on the progression of Algorithm 3 (see Figure 3.1). Note that Step 10 is reached from Step 5, i.e. only if $\mathcal{A}^{(n-1)} = \varnothing$. Therefore in all three steps $\mathcal{D}^{(n)} \subseteq \mathcal{A}^{(n-1)} \cup \mathcal{B}^{(n-1)}$. By Lemma 69.1, $\mathcal{D}^{(n)} \subseteq \mathcal{D}^{(n-1)}$. By taking complements, we then obtain $\mathcal{L}^{(n)} \supseteq \mathcal{L}^{(n-1)}$. Furthermore, in all steps $\mathcal{B}^{(n)}$ is defined as a subset of $\mathcal{B}^{(n-1)}$.

□

The dimensionality of the vectors $\tilde{x}^{(n)}$ and $\tilde{y}^{(n)}$ can change as the iterations increment. The following definition introduces the vector $p^{(n)} \in \mathbb{R}_+^N$ that can be constructed implicitly in each iteration n of Algorithm 3 but has the same dimensionality as p^* and the terms of the sequence $\left(\phi^{(\nu)}\right)_{\nu \geqslant 0}$. This will allow us to establish a relationship between the iterations of Algorithm 3 and the sequence $\left(\phi^{(\nu)}\right)_{\nu \geqslant 0}$. Furthermore, we will be able to characterise p^* as a term of the sequence $\left(p^{(n)}\right)_{n \geqslant 0}$.

Definition 70. Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and let n be an iteration of Algorithm 3. Let vector $p^{(n)} \in \mathbb{R}_+^N$ be given by:

$$\begin{aligned} p^{(n)}_{\mathcal{B}^{(n)}} &:= \bar{B}_{\mathcal{B}^{(n)}}, \\ p^{(n)}_{\mathcal{L}^{(n)}} &:= \bar{L}_{\mathcal{L}^{(n)}}, \\ p^{(n)}_{\mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)}} &:= \tilde{x}^{(n-1)}_{\mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)}}, \\ p^{(n)}_{\mathcal{A}^{(n)} \cap \mathcal{B}^{(n-1)}} &:= \tilde{y}^{(n-1)}_{\mathcal{A}^{(n)} \cap \mathcal{B}^{(n-1)}}. \end{aligned}$$

Remark 71. Note that $p^{(n)}$ in Definition 70 is indeed a vector in $\mathbb{R}_+^N$ in the sense that $p_i^{(n)}$ is well-defined for all $i \in \mathcal{N}$. To see this, observe that, by Lemma 69,

$$\begin{aligned} \left(\mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)}\right) \cup \left(\mathcal{A}^{(n)} \cap \mathcal{B}^{(n-1)}\right) &= \mathcal{A}^{(n)} \cap \left(\mathcal{A}^{(n-1)} \cup \mathcal{B}^{(n-1)}\right) \\ &= \mathcal{A}^{(n)} \cap \mathcal{D}^{(n-1)} \subseteq \mathcal{D}^{(n)}. \end{aligned}$$

Therefore, for each $i \in \mathcal{N}$, exactly one of the equalities in Definition 70 applies.

Also note that if $\mathcal{A}^{(n-1)} = \varnothing$ then $\tilde{x}^{(n-1)}$ is not defined in Algorithm 3 but in that case we also do not need to define $p^{(n)}_{\mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)}}$.

The following theorem establishes the link between Definition 70 and the output of Algorithm 3.

Theorem 72. *Suppose Algorithm 3 terminates so that n^* and p^* are defined. Then $p^* = p^{(n^*)}$.*

Proof. By Definition 70 and construction in Step 15,

$$\begin{aligned} p^{(n^*)}_{\mathcal{B}^{(n^*)}} &= \bar{B}_{\mathcal{B}^{(n^*)}} = p^*_{\mathcal{B}^{(n^*)}} \\ p^{(n^*)}_{\mathcal{L}^{(n^*)}} &= \bar{L}_{\mathcal{L}^{(n^*)}} = p^*_{\mathcal{L}^{(n^*)}}. \end{aligned}$$

By Remark 68 we have that $\mathcal{A}^{(n^*)} = \mathcal{A}^{(n^*-1)}$ and $\mathcal{B}^{(n^*)} = \mathcal{B}^{(n^*-1)}$. Then by Lemma 69.1, $\mathcal{A}^{(n^*)} \cap \mathcal{B}^{(n^*-1)} = \mathcal{A}^{(n^*)} \cap \mathcal{B}^{(n^*)} = \varnothing$. Hence by Definition 70 and construction in Step 15 we have that

$$\begin{aligned} p^{(n^*)}_{\mathcal{A}^{(n^*)}} &= p^{(n^*)}_{\mathcal{A}^{(n^*)} \cap \mathcal{A}^{(n^*-1)}} \\ &= \tilde{x}^{(n^*-1)} = p^*_{\mathcal{A}^{(n^*)}}. \end{aligned}$$

Therefore we can conclude that $p^* = p^{(n^*)}$. □

The following definition will allow us to formalise the relationship between the terms of the ordinary (in fact, finite) sequence $\left(p^{(n)}\right)_{n \geqslant 0}$ and the transfinite sequence $\left(\phi^{(\nu)}\right)_{\nu \geqslant 0}$.

Definition 73. Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and let n be an iteration of Algorithm 3. For some ordinal μ, we will say that *iteration n tracks $\phi^{(\mu)}$* if

1. $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$, $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$, $\mathcal{L}^{(n)} = \hat{\mathcal{L}}^{(\mu)}$;

2. $p^{(n)} = \phi^{(\mu)}$; and

3. if there is some $i \in \mathcal{B}^{(n)}$ such that $\tilde{y}_i^{(n-1)} \geqslant \bar{T}_i > 0$ or $\tilde{y}_i^{(n-1)} > \bar{B}_i$ then $\kappa(\mu) = 0$.

Remark 74. The essence of Definition 73 is to highlight the fact that Algorithm 3 allows us to track certain terms of the sequence $\left(\phi^{(\mu)}\right)_{\nu \geqslant 0}$, with the components of the terms constructed separately for the sets $\mathcal{A}^{(n)}$, $\mathcal{B}^{(n)}$ and $\mathcal{L}^{(n)}$.

The following lemma gives several useful sufficiency conditions that can be applied to Definition 73.

Lemma 75. *Let n be an iteration of Algorithm 3 and μ some ordinal.*

1. *Suppose $\mathcal{D}^{(n)} = \hat{\mathcal{D}}^{(\mu)}$ and $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$. Then the condition in Definition 73.1 is satisfied.*
2. *Suppose the condition in Definition 73.1 is satisfied and $p^{(n)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\mathcal{A}^{(n)}}$. Then the condition in Definition 73.2 is satisfied. In particular, if $\mathcal{A}^{(n)} = \varnothing$ then the condition in Definition 73.2 is satisfied.*
3. *Suppose the condition in Definition 73.1 is satisfied and $\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)}$. Then $\phi^{(\mu)}_{\mathcal{A}^{(n)}} = \tilde{x}^{(n-1)}_{\mathcal{A}^{(n)}}$ if and only if the condition in Definition 73.2 is satisfied.*
4. *Suppose $\tilde{y}^{(n-1)}_{\mathcal{B}^{(n)}} < \bar{T}_{\mathcal{B}^{(n)}}$ and $\tilde{y}^{(n-1)}_{\mathcal{B}^{(n)}} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$. Then the condition in Definition 73.3 is satisfied.*

Proof. 1. From Definition 45 it follows that $\hat{\mathcal{A}}^{(\mu)} = \hat{\mathcal{D}}^{(\mu)} \smallsetminus \hat{\mathcal{B}}^{(\mu)}$. Similarly, by Lemma 69.1, $\mathcal{A}^{(n)} = \mathcal{D}^{(n)} \smallsetminus \mathcal{B}^{(n)}$. Therefore if $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$ and $\mathcal{L}^{(n)} = \hat{\mathcal{L}}^{(\mu)}$ then it follows that $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$.

2. By Proposition 48 and the condition in Definition 73.1 we have

$$\phi^{(\mu)}_{\mathcal{B}^{(n)}} = \phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\mathcal{B}^{(n)}}$$
$$\phi^{(\mu)}_{\mathcal{L}^{(n)}} = \phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\mathcal{L}^{(n)}}.$$

Hence by Definition 73, $\phi^{(\mu)}_{\mathcal{B}^{(n)}} = p^{(n)}_{\mathcal{B}^{(n)}}$ and $\phi^{(\mu)}_{\mathcal{L}^{(n)}} = p^{(n)}_{\mathcal{L}^{(n)}}$. If $p^{(n)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\mathcal{A}^{(n)}}$ then $p^{(n)} = \phi^{(\mu)}$, satisfying Definition 73.2. Note that $p^{(n)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\mathcal{A}^{(n)}}$ is vacuously true if $\mathcal{A}^{(n)} = \varnothing$.

3. If $\mathcal{B}^{(n)} = \mathcal{B}^{(n-1)}$ then, since $\mathcal{D}^{(n)} \subseteq \mathcal{D}^{(n-1)}$, Lemma 69.1 implies that $\mathcal{A}^{(n)} \subseteq \mathcal{A}^{(n-1)}$. Hence $\mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)} = \mathcal{A}^{(n)}$ and $\mathcal{A}^{(n)} \cap \mathcal{B}^{(n-1)} = \mathcal{A}^{(n)} \cap \mathcal{B}^{(n)} = \varnothing$. Therefore $p^{(n)}_{\mathcal{A}^{(n)}} = \tilde{x}^{(n-1)}_{\mathcal{A}^{(n)}}$. Hence $\phi^{(\mu)}_{\mathcal{A}^{(n)}} = \tilde{x}^{(n-1)}_{\mathcal{A}^{(n)}}$ if and only if $p^{(n)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\mathcal{A}^{(n)}}$ and whenever the latter condition holds we can apply Lemma 75.2.

4. The condition in Definition 73.3 is satisfied vacuously since by assumption there is no $i \in \mathcal{B}^{(n)}$ such that $\tilde{y}_i^{(n-1)} \geqslant \bar{T}_i$ or $\tilde{y}_i^{(n-1)} > \bar{B}_i$.

□

The notion of tracking allows us to deploy the full strength of the results in Section 3.3 by equating various quantities introduced in that section with the quantities defined in Algorithm 3. The folowing lemmas are two tools for establishing this correspondence. Lemma 76 gives the properties of the vector $b^{(n)}$ and matrix $M^{(n)}$ which allows us to treat them as a proxies for the vector $\hat{b}^{(\mu)}$ and matrix $\hat{M}^{(\mu)}$, respectively. Lemma 77 gives the properties of $x^{(k,n)}$ and $y^{(k,n)}$ allowing us to use results in Section 3.3 about $X^{(\mu)}(k)$ and $Y^{(\mu)}(k)$.

Lemma 76. *Let n be an iteration of Algorithm 3 and μ an ordinal. Suppose $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$, $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$, $\mathcal{L}^{(n)} = \hat{\mathcal{L}}^{(\mu)}$. Then we obtain the following:*

1. *$b^{(n)} = \hat{b}^{(\mu)}$ and $M^{(n)} = \hat{M}^{(\mu)}$.*
2. *Suppose $\mathcal{A}^{(n)} = \varnothing$. If there is some $i \in \mathcal{B}^{(n)}$ such that $b_i^{(n)} \geqslant \bar{T}_i$ or $b_i^{(n)} > \bar{B}_i$ then $\kappa(\mu) = 0$.*

Proof. 1. Applying the definitions of $b^{(n)}$ and $M^{(n)}$ in Algorithm 3 and of $\hat{b}^{(\mu)}$ and $\hat{M}^{(\mu)}$ in Definition 52, we obtain

$$\begin{aligned} b^{(n)} &= \alpha a + \beta\Omega_{\mathcal{N}\mathcal{B}^{(n)}}\bar{B}_{\mathcal{B}^{(n)}} + \beta\Omega_{\mathcal{N}\mathcal{L}^{(n)}}\bar{L}_{\mathcal{L}^{(n)}} \\ &= \alpha a + \beta\Omega_{\mathcal{N}\hat{\mathcal{B}}^{(\mu)}}\bar{B}_{\hat{\mathcal{B}}^{(\mu)}} + \beta\Omega_{\mathcal{N}\hat{\mathcal{L}}^{(\mu)}}\bar{L}_{\hat{\mathcal{L}}^{(\mu)}} = \hat{b}^{(\mu)} \text{ and} \\ M^{(n)} &= \beta\Omega_{\mathcal{A}^{(n)}\mathcal{A}^{(n)}} \\ &= \beta\Omega_{\hat{\mathcal{A}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}} = \hat{M}^{(\mu)}. \end{aligned}$$

2. By assumption we have $\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n)} = \varnothing$ and by Lemma 76.1 above we have that $b^{(n)} = \hat{b}^{(\mu)}$. Corollary 53 then implies that $V_{\alpha,\beta}(\phi^{(\mu)}) = b^{(n)}$. Hence if there is some $i \in \mathcal{B}^{(n)}$ such that $b_i^{(n)} \geqslant \bar{T}_i$ or $b_i^{(n)} > \bar{B}_i$ then there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu)})_i \geqslant \bar{T}_i$ or $V_{\alpha,\beta}(\phi^{(\mu)})_i > \bar{B}_i$. In the first case we have that $\kappa_T(\mu) = 0$ and in the last case we have that $\kappa_B(\mu) = 0$, by Corollary 50.1 and 50.2. Hence we obtain that $\kappa(\mu) = \kappa_B(\mu) \wedge \kappa_B(\mu) = 0 < \omega$.

□

Lemma 77. *Assume that n^* is well-defined. Let iteration $n < n^*$ track $\phi^{(\mu)}$ for some ordinal μ. Then*

1. *$x^{(0,n)} \leqslant \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(0,n)} \leqslant \bar{T}_{\mathcal{B}^{(n)}}$ and $y^{(0,n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$;*
2. *$x^{(0,n)} = X^{(\mu)}(0)$. Furthermore, if $x^{(K,n)}$ and $y^{(K,n)}$ are defined in Algorithm 3 for some $K \geqslant 1$ then $x^{(k,n)} = X^{(\mu)}(k)$ and $y^{(k,n)} = Y^{(\mu)}(k)$ for all k such that $1 \leqslant k \leqslant K$.*

Proof. 1. The sets $\mathcal{D}^{(n)}$ and $\mathcal{B}^{(n)}$ are defined in one of the Steps 9, 10 or 11. By construction in Step 9, $\tilde{x}_{\mathcal{D}^{(n)}}^{(n-1)} \leqslant \bar{T}_{\mathcal{D}^{(n)}}$, $\tilde{y}_{\mathcal{D}^{(n)}}^{(n-1)} \leqslant \bar{T}_{\mathcal{D}^{(n)}}$ and $\tilde{y}_{\mathcal{D}^{(n)}}^{(n-1)} \leqslant \bar{B}_{\mathcal{D}^{(n)}}$. By construction in Step 10, $\tilde{y}_{\mathcal{D}^{(n)}}^{(n-1)} \leqslant \bar{T}_{\mathcal{B}^{(n)}}$ and $\tilde{y}_{\mathcal{D}^{(n)}}^{(n-1)} \leqslant \bar{B}_{\mathcal{D}^{(n)}}$ and moreover we can observe that Step 10 is only ever reached if $\mathcal{A}^{(n-1)} = \varnothing$. Step 11 can only be reached from Step 6 where again it is clear that $\tilde{x}_{\mathcal{A}^{(n-1)}}^{(n-1)} \leqslant \bar{T}_{\mathcal{A}^{(n-1)}}$, $\tilde{y}_{\mathcal{B}^{(n-1)}}^{(n-1)} \leqslant \bar{T}_{\mathcal{B}^{(n-1)}}$ and $\tilde{y}_{\mathcal{B}^{(n-1)}}^{(n-1)} \leqslant \bar{B}_{\mathcal{B}^{(n-1)}}$ by construction.

   Let $i \in \mathcal{A}^{(n)}$. By construction in Step 3, if $i \in \mathcal{A}^{(n)} \cap \mathcal{A}^{(n-1)}$ then $x_i^{(0,n)} = \tilde{x}_i^{(n-1)} \leqslant \bar{T}_i$ and if $i \in \mathcal{A}^{(n)} \cap \mathcal{B}^{(n-1)}$ then $x_i^{(0,n)} = \tilde{y}_i^{(n-1)} \leqslant \bar{T}_i$ and $x_i^{(0,n)} = \tilde{y}_i^{(n-1)} \leqslant \bar{B}_i$. On the other hand if we let $i \in \mathcal{B}^{(n)}$ then $y_i^{(0,n)} = \tilde{y}_i^{(n-1)} \leqslant \bar{T}_i$ and $y_i^{(0,n)} = \tilde{y}_i^{(n-1)} \leqslant \bar{B}_i$.

2. By our convention, if $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)} = \varnothing$ then $x^{(0,n)} = X^{(\mu)}(0)$ vacuously. So suppose that $\mathcal{A}^{(n)} \neq \varnothing$. By construction in Step 3 and Definition 70,

$$\begin{aligned} x_{\mathcal{A}^{(n-1)}}^{(0,n)} &= \tilde{x}_{\mathcal{A}^{(n)}\cap\mathcal{A}^{(n-1)}}^{(n)} = p_{\mathcal{A}^{(n)}\cap\mathcal{A}^{(n-1)}}^{(n)} \\ x_{\mathcal{B}^{(n-1)}}^{(0,n)} &= \tilde{y}_{\mathcal{A}^{(n)}\cap\mathcal{B}^{(n-1)}}^{(n)} = p_{\mathcal{A}^{(n)}\cap\mathcal{B}^{(n-1)}}^{(n)}. \end{aligned}$$

Hence, by Definition 73, $x^{(0,n)} = p^{(n)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\mathcal{A}^{(n)}}$. But then by the definition of $X^{(\mu)}(0)$ we have that $x^{(0,n)} = \phi^{(\mu)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = X^{(\mu)}(0)$.

Note that $X^{(\mu)}(k)$ and $Y^{(\mu)}(k)$ are defined for all $k \geqslant 1$ and $x^{(1,n)}$ and $y^{(1,n)}$ are defined for all iterations n. Therefore, since $x^{(k,n)}$ and $y^{(k,n)}$ are defined recursively, they are well-defined for $1 \leqslant k \leqslant K$ whenever $x^{(K,n)}$ and $y^{(K,n)}$ are defined. Since $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$, $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$, $\mathcal{L}^{(n)} = \hat{\mathcal{L}}^{(\mu)}$, we can observe by comparing Definition 58 and Algorithm 3 that, for $1 \leqslant k \leqslant K$, $x^{(k,n)}$ is just a relabelling of $X^{(\mu)}(k)$ and $y^{(k,n)}$ is a relabelling of $Y^{(\mu)}(k)$.

□

Lemmas 76 and 77 do not tell us much about the interaction of Algorithm 3 with the other key quanitites introduced in Section 3.3 — $\kappa(\mu)$, $\tilde{X}$ and $\tilde{Y}$. The following definition provides us with an analogue of $\kappa(\mu)$. We can use it to control for the properties of the terms of the sequence $(p^{(n)})_{n\geqslant 0}$ without running into problems caused by the fact that $\kappa(\mu)$ can take the non-finite value ω.

Definition 78. Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system and n an iteration of Algorithm 3. Assume that Step 14 is reached. Denote by $K(n)$ the value of $k - 1$ at the beginning of Step 14 in iteration n.

Note that $K(n)$ is finite whenever it is defined but at the cost of not being a well-defined object in all cases. In particular, $K(n)$ will not be well-defined if Step 14 is not reached. The main point of concern where this might happen is the loop in Step 7. If the condition of the loop is always true then Step 14 is not reached. We will later show that, in fact, we can circumvent this difficulty.

The next, and final, definition provides some terminology for talking about the different states of Algorithm 3. This will allows us to formulate the proof of Theorem 67 concisely.

Definition 79. Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system. Assume that n^* is defined in Algorithm 3 and let n be an iteration with $n < n^*$. We say that iteration n satisfies the *bound condition* if

1. $\mathcal{A}^{(n)} \neq \varnothing$ and $\left(\mathrm{I} - M^{(n)}\right)$ is invertible so that $\tilde{x}^{(n)} := \left(\mathrm{I} - M^{(n)}\right)^{-1} b^{(n)}_{\mathcal{A}^{(n)}}$ and $\tilde{y}^{(n)} := b^{(n)}_{\mathcal{B}^{(n)}} + \beta\Omega_{\mathcal{B}^{(n)}\mathcal{A}^{(n)}}\tilde{x}^{(n)}$; and
2. $\tilde{x}^{(n)} \leqslant \bar{T}_{\mathcal{A}^{(n)}}$, $\tilde{y}^{(n)} \leqslant \bar{T}_{\mathcal{B}^{(n)}}$ and $\tilde{y}^{(n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$.

We then say that iteration n is in

- the *degenerate termination state* if $\mathcal{A}^{(n)} = \varnothing$ and either $\mathcal{B}^{(n)} = \varnothing$ or both $b^{(n)}_{\mathcal{B}^{(n)}} < \bar{T}_{\mathcal{B}^{(n)}}$ and $b^{(n)}_{\mathcal{B}^{(n)}} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$;
- the *simple transition state* if it is not in the degenerate termination state and $K(n)$ is well-defined with $K(n) \leqslant |\mathcal{A}^{(n)}|$;
- the *unbounded transition state* if it does not satisfy the bound condition and $K(n)$ is well-defined with $K(n) > |\mathcal{A}^{(n)}|$;

- the *limit state* if it satisfies the bound condition and $K(n)$ is well-defined with $K(n) > |\mathcal{A}^{(n)}|$.

For brevity we will refer to either the simple transition state or the unbounded transition state as a *transition state*.

We can see by inspection of Figure 3.1 that Algorithm 3 can complete each iteration in four different ways that correspond to the four states in Definition 79. The degenerate termination state corresponds to the case where the algorithm reaches Step 5 and is then redirected to Step 10. We will see that this state can only occur within the last two iterations of the algorithm, which motivates the name. The limit state corresponds to the case where the algorithm reaches Step 6 where it is redirected to Step 11. We will see that the quantities constructed in this state will be the limit vectors $\tilde{X}$ and $\tilde{Y}$. Finally, the two transition states correspond to the case where the algorithm is ultimately pointed to Step 9. In the simple transition state the algorithm is redirected from Step 4 to Step 8, bypassing the loop in Step 7. Since the loop in Step 4 can only run at most $|\mathcal{A}^{(n)}|$ times, the number of steps between Step 4 and Step 9 is bounded. In the unbounded transition state the loop in Step 7 is not bypassed and the algorithm can run an unbounded (but still finite) number of steps.

The following proposition demonstrates that these four states are indeed an appropriate way of characterising the state of Algorithm 3.

Proposition 80. *Suppose n^* is defined and let $n < n^*$ be such that iteration n of Algorithm 3 tracks $\phi^{(\tilde{\mu})}$ for some ordinal μ. Then the four states in Definition 79 are pairwise mutually exclusive and jointly exhaustive.*

Proof. By definition, the degenerate termination state and the simple transition state are mutually exclusive. The simple transition state is mutually exclusive with either the unbounded transition state or the limit state due to the constraints on $K(n)$. The unbounded transition state and the limit state are mutually exclusive due to the mutually exclusive requirements for the bound condition. The degenerate termination state and the limit state are mutually exclusive since the bound condition requires that $\mathcal{A}^{(n)} \neq \varnothing$.

It remains to show that the degenerate termination state and the unbounded transition state are mutually exclusive. Suppose that iteration n is in the unbounded transition state with $\mathcal{A}^{(n)} = \varnothing$ as otherwise the contradiction is trivial by definition. In particular, Algorithm 3 must reach the loop in Step 7. By our convention $y^{(k+1,n)} = y^{(k,n)} = b^{(n)}_{\mathcal{B}^{(n)}}$ for all k. By Remark 68, $n < n^*$ implies that Algorithm 3 terminates and, in particular, so must the loop in Step 7. Hence there must be some k such that $y_i^{(k,n)} = b_i^{(n)} \geqslant \bar{T}_i$ or $y_i^{(k,n)} = b_i^{(n)} > \bar{B}_i$. But that implies that iteration n is not in the degenerate termination state. Hence all four states are pairwise mutually exclusive.

To see that the four states are jointly exhaustive, first note that if $\mathcal{A}^{(n)} = \varnothing$ then we showed that $y^{(k,n)} = b^{(n)}_{\mathcal{B}^{(n)}}$ for all k. If iteration n is not in the degenerate termination state then there is some $i \in \mathcal{B}^{(n)}$ such that $y_i^{(0,n)} \geqslant \bar{T}_i$ or $y_i^{(0,n)} > \bar{B}_i$. Then k is incremented so that $k = 1$ and Algorithm 3 proceeds to terminate. Hence $K(n) = k - 1 = 0 \leqslant |\mathcal{A}^{(n)}|$ and hence iteration n is in the simple transition state. Thus the degenerate termination state and the simple transition state exhaust all cases if $\mathcal{A}^{(n)} = \varnothing$. So suppose $\mathcal{A}^{(n)} \neq \varnothing$ and assume that iteration n is not in the limit state. If $K(n)$ is well-defined then, by definition, it follows that iteration n must be in the simple transition state if $K(n) \leqslant |\mathcal{A}^{(n)}|$ or in the unbounded transition state if $K(n) > |\mathcal{A}^{(n)}|$.

It therefore remains to consider what happens if $|\mathcal{A}^{(n)}| \neq \varnothing$ and $K(n)$ is not well-defined. This can happen either if Step 7 is not reached at all or it is reached but the loop in Step 7 does not terminate. Step 7 can fail to be reached only in Steps 4c or 6b. The former would imply that $K(n)$ is well-defined and the latter would imply that iteration n is in the limit state, which we have assumed is not the case.

Therefore we suppose that iteration n is in the unbounded transition state and the loop in Step 7 does not terminate. In particular, for all $k > |\mathcal{A}^{(n)}|$ we have that $x^{(k,n)} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(k,n)} < \bar{T}_{\mathcal{B}^{(n)}}$ and $y^{(k,n)} \leqslant \bar{T}_{\mathcal{B}^{(n)}}$, $y^{(k,n)} < \bar{T}_{\mathcal{B}^{(n)}}$. Then Lemma 77.2 implies that $x^{(k,n)} = X^{(\mu)}(k)$ and $y^{(k,n)} = Y^{(\mu)}(k)$ for all $k > |\mathcal{A}^{(n)}|$. In particular, the sequences $\left(X^{(\mu)}(k)\right)_{k\geqslant 0}$ and $\left(Y^{(\mu)}(k)\right)_{k\geqslant 1}$ are bounded and, by Lemma 64, non-decreasing. Hence their limits exist. Lemma 59 then implies that $\left(\mathrm{I} - \hat{M}^{(\mu)}\right)$ is invertible and the limits are $\tilde{X}$ and $\tilde{Y}$, respectively. By Lemma 76 it then follows that iteration n satisfies the bound condition, contradicting the assumption that it is in the unbounded transition state. Therefore if n is in the unbounded transition state then the loop in Step 7 must terminate. Hence $K(n)$ is well-defined, showing that the four states in Definition 79 exhaust all possibilities. □

Lemmas 81 and 82 use the terminology of Definition 79 to characterise the properties of Algorithm 3 and relate it to the notions introduced in the previous section.

Lemma 81. *Assume that n^* is well-defined. Let iteration $n < n^*$ track $\phi^{(\mu)}$ for some ordinal μ.*

1. *If iteration n is in the limit state then $\tilde{x}^{(n)} = \tilde{X}^{(\mu)}$ and $\tilde{y}^{(n)} = \tilde{Y}^{(\mu)}$.*
2. *If iteration n is in a transition state then $K(n) \geqslant 0$ and $x^{(k,n)}$ and $y^{(k,n)}$ are defined for all integers k satisfying $0 \leqslant k \leqslant K(n) + 1$.*

   If $K(n) > 0$ then for all integers k satisfying $0 < k \leqslant K(n)$, $x^{(k,n)} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(k,n)} < \bar{T}_{\mathcal{B}^{(n)}}$ and $y^{(k,n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$. Moreover, either there is some $i \in \mathcal{A}^{(n)}$ such that $x_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or there is some $i \in \mathcal{B}^{(n)}$ such that $y_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or $y_i^{(K(n)+1,n)} > \bar{B}_i$.
3. *If iteration n is in a transition state then $K(n) \leqslant \kappa(\mu)$. If moreover $\kappa(\mu) < \omega$ then $K(n) = \kappa(\mu)$.*

Proof. 1. By assumption of Definition 73, $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$, $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$, $\mathcal{L}^{(n)} = \hat{\mathcal{L}}^{(\mu)}$. If iteration n is in the limit state then

$$\begin{aligned}
\tilde{x}^{(n)} &= (\mathrm{I} - M^{(n)})^{-1} b^{(n)}_{\mathcal{A}^{(n)}} \\
\tilde{y}^{(n)} &= b^{(n)}_{\mathcal{B}^{(n)}} + \beta \Omega_{\mathcal{B}^{(n)}\mathcal{A}^{(n)}} (\mathrm{I} - M^{(n)})^{-1} b^{(n)}_{\mathcal{A}^{(n)}}.
\end{aligned}$$

Lemma 76.1 then implies that

$$\begin{aligned}
\tilde{x}^{(n)} &= (\mathrm{I} - \hat{M}^{(\mu)})^{-1} \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = \tilde{X}^{(\mu)} \\
\tilde{y}^{(n)} &= \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} + \beta \Omega_{\hat{\mathcal{B}}^{(\mu)}\hat{\mathcal{A}}^{(\mu)}} (\mathrm{I} - \hat{M}^{(\mu)})^{-1} \hat{b}^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = \tilde{Y}^{(\mu)}.
\end{aligned}$$

2. If iteration n is in the a transition state then the loop in Step 4 must be completed at least once and so at the beginning of Step 14, $k \geqslant 1$ and hence $K(n) = k - 1 \geqslant 0$.

   For $k = 0$ $x^{(0,n)}$ and $y^{(0,n)}$ are defined in Step 2. Now suppose that $K(n) > 0$ and k is some integer satisfying $0 < k \leqslant K(n) + 1$. By Definition 78 this implies that $x^{(k,n)}$ and $y^{(k,n)}$ are defined.

   Let $0 < k \leqslant K(n)$. If iteration n is in the simple transition state then the loop at Step 4 terminates precisely when $k + 1 = K(n) + 1$ and the algorithm then proceeds to the next iteration. If iteration n is in the unbounded transition state then the loop at Step 4 does not terminate the iteration but rather proceeds on to the loop in Step 7 which terminates precisely when $k+1 = K(n)+1$. It then follows by the conditions in those loops that $x^{(k,n)} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(k,n)} < \bar{T}_{\mathcal{A}^{(n)}}$ and $y^{(k,n)} \leqslant \bar{B}_{\mathcal{A}^{(n)}}$. Moreover, by the same conditions we also have that either there is some $i \in \mathcal{A}^{(n)}$ such that $x_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or there is some $i \in \mathcal{B}^{(n)}$ such that $y_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or $y_i^{(K(n)+1,n)} > \bar{B}_i$.

3. First we show that $K(n) \leqslant \kappa(\mu)$. For $K(n) = 0$, $K(n) \leqslant \kappa(\mu)$ trivially and so we can let $K(n) > 0$. Suppose then that $K(n) \geqslant \kappa(\mu) + 1$ with $\kappa(\mu)$ finite. Therefore, for any k such that $0 < k \leqslant \kappa(\mu) + 1$, $x^{(k,n)}$ and $y^{(k,n)}$ are defined and moreover by Lemma 77.2 $X^{(\mu)}(k) = x^{(k,n)} < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$, $Y^{(\mu)}(k) = y^{(k,n)} < \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$ and $Y^{(\mu)}(k) = y^{(k,n)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. By Proposition 63.2 this means, in particular, that $V_{\alpha,\beta}(\phi^{(\mu+\kappa(\mu))})_{\hat{\mathcal{D}}^{(\mu)}} < \bar{T}_{\hat{\mathcal{D}}^{(\mu)}}$ and $V_{\alpha,\beta}(\phi^{(\mu+\kappa(\mu))})_{\hat{\mathcal{B}}^{(\mu)}} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. Corollary 50.1 and 50.2 then imply that $\kappa_T(\mu) > \kappa(\mu)$ and $\kappa_B(\mu) > \kappa(\mu)$. Hence we get the contradiction that $\kappa(\mu) = \kappa_T(\mu) \wedge \kappa_B(\mu) > \kappa(\mu)$. This shows that $K(n) < \kappa(\mu) + 1$, i.e. $K(n) \leqslant \kappa(\mu)$.

   Suppose $\kappa(\mu) < \omega$. To show that $K(n) = \kappa(\mu)$, we obtain a contradiction by allowing $K(n) < \kappa(\mu)$. Then $x^{(K(n)+1,n)}$ and $y^{(K(n)+1,n)}$ are defined and $x^{(K(n)+1,n)} = X^{(\mu)}(K(n)+1)$, $y^{(K(n)+1,n)} = Y^{(\mu)}(K(n) + 1)$ by Lemma 77.2. Corollary 50 and Proposition 63.2 give us that $x^{(K(n)+1,n)} = V_{\alpha,\beta}(\phi^{(\mu+K(n))})_{\mathcal{A}^{(n)}} < \bar{T}_{\mathcal{A}^{(n)}}$, $y^{(K(n)+1,n)} = V_{\alpha,\beta}(\phi^{(\mu+K(n))})_{\mathcal{B}^{(n)}} < \bar{T}_{\mathcal{B}^{(n)}}$ and $y^{(K(n)+1,n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$. But if so then the loop at Step 4 must complete fully and Algorithm 3 must then proceed to the loop at Step 7, which does not terminate. Hence $x^{(K(n)+2,n)}$ is defined, contradicting Definition 78 that requires the loop to terminate. Therefore $\kappa(\mu) \leqslant K(n)$ and hence $K(n) = \kappa(\mu) < \omega$.

□

Lemma 82. *Let $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be a regular clearing system. Let iteration n track $\phi^{(\mu)}$ for some $n < n^*$ and some ordinal μ.*

1. *If iteration n is in a transition state then $\kappa(\mu) < \omega$. In particular, if iteration n is in the simple transition state with $\mathcal{A}^{(n)} = \varnothing$ then $\kappa(\mu) = 0$.*

2. *If iteration n is in either the degenerate termination state or the limit state then $\kappa(\mu) = \omega$.*

Proof. 1. Suppose iteration n is in a transition state state. Then $K(n)$ is well-defined and by Lemma 77.2, $X^{(\mu)}(K(n) + 1) = x^{(K(n)+1,n)}$ and $Y^{(\mu)}(K(n) + 1) = y^{(K(n)+1,n)}$. If $K(n) \leqslant |\mathcal{A}^{(n)}|$ then iteration n is in the simple transition state and the loop at Step 4 must terminate before completing. If $K(n) > |\mathcal{A}^{(n)}|$ then iteration n is in the unbounded transition state and the loop at Step 4 must complete fully with Algorithm 3 then proceeding to the loop at Step 7, which must terminate. In either case, there is some $i \in \mathcal{A}^{(n)}$ such that $X^{(\mu)}(K(n)+1)_i \geqslant \bar{T}_i$ or some $i \in \mathcal{B}^{(n)}$ such that $Y^{(\mu)}(K(n)+1)_i \geqslant \bar{T}_i$ or $Y^{(\mu)}(K(n)+1)_i > \bar{B}_i$.

Suppose to the contrary that $\kappa(\mu) = \omega$. Then $K(n)+1 < \kappa(|mu)$ and by Lemma 77.2 and Proposition 63.2 $x^{(K(n)+1,n)} = V_{\alpha,\beta}(\phi^{(\mu+K(n))})_{\hat{\mathcal{A}}^{(\mu)}}$ and $y^{(K(n)+1,n)} = V_{\alpha,\beta}(\phi^{(\mu+K(n))})_{\hat{\mathcal{B}}^{(\mu)}}$. Then by Lemma 81.2 there is some $i \in \mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+K(n))})_i \geqslant \bar{T}_i$ or some $i \in \mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu+K(n))})_i \geqslant \bar{T}_i$ or $V_{\alpha,\beta}(\phi^{(\mu+K(n))})_i > \bar{B}_i$. By Corollary 50.1 and 50.2, in the first two cases we have that $\kappa_T(\mu) \leqslant K(n)$ and in the last case we have that $\kappa_B(\mu) \leqslant K(n)$. Hence we obtain the contradiction that $\kappa(\mu) = \kappa_B(\mu) \wedge \kappa_B(\mu) \leqslant K(n) < \omega$. Therefore, if iteration n is in a transition state then $\kappa(\mu) < \omega$.

In particular, if $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\mu)} = \varnothing$ then there is some $i \in \mathcal{B}^{(n)}$ such that $b_i^{(n)} = y_i^{(0,n)} \geqslant \bar{T}_i$ or $b_i^{(n)} = y_i^{(0,n)} > \bar{B}_i$ as, otherwise, iteration n would be in the degenerate termination state. Lemma 76.2 then implies that $\kappa(\mu) = 0 < \omega$.

2. Suppose iteration n is in the limit state. Then the bound condition is satisfied and by Lemma 81.1 we get that $\tilde{X}^{(\mu)} = \tilde{x}^{(n)}$ and $\tilde{Y}^{(\mu)} = \tilde{y}^{(n)}$. To establish a contradiction, suppose further that $\kappa(\mu) < \omega$. Hence by Lemma 81.3 above $\kappa(\mu) = K(n)$. By definition of the limit state $K(n) > |cA^{(n)}| = |\hat{\mathcal{A}}^{(\mu)}|$ and so $\kappa(\mu) \geqslant |\hat{\mathcal{A}}^{(\mu)}|$. By the bound condition we then obtain that $\tilde{X}^{(\mu)} = \tilde{x}^{(n)} \leqslant \bar{T}_{\hat{\mathcal{A}}^{(\mu)}}$, $\tilde{Y}^{(\mu)} = \tilde{y}^{(n)} \leqslant \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\tilde{Y}^{(\mu)} = \tilde{y}^{(n)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. Theorem 66.2 then implies that $\kappa(\mu) = \omega$.

   Finally, suppose that iteration n is in the degenerate termination state. Then $\mathcal{A}^{(n)} = \varnothing$ and we have that $\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. We then compute $\Phi(\phi^{(\mu)})$. Since $\left(\phi^{(\nu)}\right)_{\nu \geqslant 0}$ is non-decreasing and bounded above by $\bar{L}$, we get that $\Phi(\phi^{(\mu)})_{\hat{\mathcal{L}}^{(\mu)}} = \phi^{(\mu+1)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. If $\hat{\mathcal{B}}^{(\mu)} \neq \varnothing$ then by Corollary 53 and Lemma Lemma 76.1 $V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} = \hat{b}^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = b^{(n)}_{\mathcal{B}^{(n)}}$ and so by the definition of the degenerate termination state, $b^{(n)}_{\mathcal{B}^{(n)}} < \bar{T}_{\mathcal{B}^{(n)}} = \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\hat{b}^{(n)}_{\hat{\mathcal{B}}^{(\mu)}} = b^{(n)}_{\mathcal{B}^{(n)}} \leqslant \bar{B}_{\mathcal{B}^{(n)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$. Hence by Lemma 39 $V(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{B}}^{(\mu)}}$ and so $\Phi(\phi^{(\mu)})_{\hat{\mathcal{D}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} \wedge V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} = \phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}}$. Therefore $\Phi(\phi^{(\mu)}) = \phi^{(\mu)}$ and $\phi^{(\mu)}$ is a fixed point of Φ. Hence by Remark 51, $\kappa(\mu) = \omega$.

□

Lemma 83. *Let $n < n^*$ and suppose that iteration n of Algorithm 3 tracks $\phi^{(\tilde{\mu})}$ for some ordinal $\tilde{\mu}$.*

1. *If $\kappa(\tilde{\mu}) < \omega$ then iteration $n+1$ tracks $\phi^{(\tilde{\mu}+\kappa(\tilde{\mu})+1)}$.*
2. *If $\kappa(\tilde{\mu}) = \omega$ then iteration $n+1$ tracks $\phi^{(\tilde{\mu}+\omega)}$.*

Proof. 1. Let $\kappa(\tilde{\mu}) < \omega$. By Lemma 82.1, iteration n is in a transition state. By inspection of Algorithm 3, the sets $\mathcal{D}^{(n+1)}$ and $\mathcal{B}^{(n+1)}$ are defined at Step 9. $K(n)$ is well-defined and, by Lemma 81.3, $\kappa(\tilde{\mu}) = K(n)$. Let $K := \kappa(\tilde{\mu})+1$ and $\mu := \tilde{\mu}+\kappa(\tilde{\mu})+1 = \tilde{\mu}+K$. Then $K = K(n)+1$ and $x^{(K,n)} = \tilde{x}^{(n)}$, $y^{(K,n)} = \tilde{y}^{(n)}$.

By Lemma 77.2 it follows that $X^{(\tilde{\mu})}(K) = \tilde{x}^{(n)}$ and $Y^{(\tilde{\mu})}(K) = \tilde{y}^{(n)}$. We then compare the sets $\hat{\mathcal{D}}^{(\mu)}$ and $\hat{\mathcal{B}}^{(\mu)}$ with Step 9 and observe by Theorem 65.1a that

$$\begin{aligned}
\hat{\mathcal{D}}^{(\mu)} &= \hat{\mathcal{D}}^{(\tilde{\mu}+K)} \\
&= \{i \in \hat{\mathcal{A}}^{(\tilde{\mu})} \mid X^{(\tilde{\mu})}(K)_i < \bar{T}_i\} \cup \{i \in \hat{\mathcal{B}}^{(\tilde{\mu})} \mid Y^{(\tilde{\mu})}(K)_i < \bar{T}_i\} \\
&= \{i \in \mathcal{A}^{(n)} \mid \tilde{x}_i^{(n)} < \bar{T}_i\} \cup \{i \in \mathcal{B}^{(n)} \mid \tilde{y}_i^{(n)} < \bar{T}_i\} \\
&= \mathcal{D}^{(n+1)}, \text{ and} \\
\hat{\mathcal{B}}^{(\mu)} &= \hat{\mathcal{B}}^{(\tilde{\mu}+K)} \\
&= \{i \in \hat{\mathcal{D}}^{(\mu)} \cap \hat{\mathcal{B}}^{(\tilde{\mu})} \mid Y^{(\tilde{\mu})}(K)_i \leqslant \bar{B}_i\} \\
&= \{i \in \mathcal{D}^{(n+1)} \cap \mathcal{B}^{(n)} \mid \tilde{y}_i^{(n)} \leqslant \bar{B}_i\} \\
&= \mathcal{B}^{(n+1)}.
\end{aligned}$$

Since $\hat{\mathcal{D}}^{(\mu)} = \mathcal{D}^{(n+1)}$ and $\hat{\mathcal{B}}^{(\mu)} = \mathcal{B}^{(n+1)}$, Lemma 75.1 implies that the condition in Definition 73.1 is satisfied and, in particular, $\hat{\mathcal{L}}^{(\mu)} = \mathcal{L}^{(n+1)}$ and $\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n+1)}$.

By Lemma 75.2, it is sufficient to show that $p^{(n+1)}_{\mathcal{A}^{(n+1)}} = \phi^{(\mu)}_{\mathcal{A}^{(n+1)}}$ to establish that Definition 73.2 is satisfied. Note that we must have that at least one of $\hat{\mathcal{A}}^{(\tilde{\mu})}$ and $\hat{\mathcal{B}}^{(\tilde{\mu})}$ is non-empty. Otherwise, $\phi^{(\tilde{\mu})} = \bar{L}$ which must be a fixed point of Φ by Proposition 42.3 and hence $\kappa(\tilde{\mu}) = \omega$ by Remark 51. Suppose that $\mathcal{A}^{(n)} = \hat{\mathcal{A}}^{(\tilde{\mu})} \neq \varnothing$. Then by Theorem 65.1b

$$\begin{aligned}
p^{(n+1)}_{\mathcal{A}^{(n+1)} \cap \mathcal{A}^{(n)}} = \tilde{x}^{(n)}_{\mathcal{A}^{(n+1)} \cap \mathcal{A}^{(n)}} &= X^{(\tilde{\mu})}(K)_{\mathcal{A}^{(n+1)} \cap \mathcal{A}^{(n)}} \\
= \phi^{(\tilde{\mu}+K)}_{\mathcal{A}^{(n+1)} \cap \mathcal{A}^{(n)}} &= \phi^{(\mu)}_{\mathcal{A}^{(n+1)} \cap \mathcal{A}^{(n)}}.
\end{aligned}$$

Suppose instead that $\mathcal{B}^{(n)} = \hat{\mathcal{B}}^{(\tilde{\mu})} \neq \varnothing$. Then Theorem 65.1b implies that

$$\begin{aligned}
p^{(n+1)}_{\mathcal{A}^{(n+1)} \cap \mathcal{B}^{(n)}} = \tilde{y}^{(n)}_{\mathcal{A}^{(n+1)} \cap \mathcal{B}^{(n)}} &= Y^{(\tilde{\mu})}(K)_{\mathcal{A}^{(n+1)} \cap \mathcal{B}^{(n)}} \\
= \phi^{(\tilde{\mu}+K)}_{\mathcal{A}^{(n+1)} \cap \mathcal{B}^{(n)}} &= \phi^{(\mu)}_{\mathcal{A}^{(n+1)} \cap \mathcal{B}^{(n)}}.
\end{aligned}$$

This confirms that Definition 73.2 is satisfied.

By construction above we have that, for all $i \in \mathcal{B}^{(n+1)}$, $\tilde{y}_i^{(n)} < \bar{T}_i$ and $\tilde{y}_i^{(n)} \leqslant \bar{B}_i$. Lemma 75.4 states that the condition in Definition 73.3 is satisfied. Hence we have confirmed all elements of Definition 73 and have shown that iteration $n+1$ tracks $\phi^{(\tilde{\mu}+\kappa(\tilde{\mu})+1)}$.

2. Let $\kappa(\tilde{\mu}) = \omega$ and $\mu := \tilde{\mu} + \omega$. By Proposition 80 and Lemma 82.2, iteration n is in the limit state or the degenerate termination state.

   Suppose iteration n is in the limit state. By Theorem 65.2b and Lemma 81.1 we get that $\tilde{x}^{(n)} = \tilde{X}^{(\tilde{\mu})}$. We can observe by inspection of Algorithm 3 that the sets $\mathcal{D}^{(n+1)}$ and $\mathcal{B}^{(n+1)}$ are defined at Step 11. Theorem 65.2c and 2d then gives us that

$$\begin{aligned}
\hat{\mathcal{D}}^{(\mu)} &= \hat{\mathcal{D}}^{(\tilde{\mu}+\omega)} \\
&= \{i \in \hat{\mathcal{A}}^{(\tilde{\mu})} \mid \tilde{X}_i^{(\tilde{\mu})} < \bar{L}_i\} \cup \hat{\mathcal{B}}^{(\tilde{\mu})} \\
&= \{i \in \mathcal{A}^{(n)} \mid \tilde{x}_i^{(n)} < \bar{L}_i\} \cup \mathcal{B}^{(n)} \\
&= \mathcal{D}^{(n+1)},
\end{aligned}$$

and $\hat{\mathcal{B}}^{(\mu)} = \mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$.

By Lemma 75.1, the condition in Definition 73.1 is satisfied and, in particular, $\hat{\mathcal{L}}^{(\mu)} = \mathcal{L}^{(n+1)}$ and $\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n+1)}$. Theorem 65.2a gives us that $\phi^{(\mu)}_{\mathcal{A}^{(n)}} = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\tilde{\mu})}} = \tilde{X}^{(\tilde{\mu})} = \tilde{x}^{(n)}$. By Lemma 75.3, this is sufficient to establish that Definition 73.2 is satisfied.

It remains to establish that the condition in Definition 73.3 holds. By Theorem 65.2c, $\tilde{Y}^{(\mu)} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\mathcal{B}^{(n+1)}}$ so suppose there is some $i \in \mathcal{B}^{(n+1)}$ such that $\tilde{y}^{(n)}_i \geqslant \bar{T}_i$. Note that by Theorem 65.2a we have $V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\tilde{\mu})}} = \tilde{Y}^{(\tilde{\mu})} = \tilde{y}^{(n)}$. Therefore, since $\hat{\mathcal{B}}^{(\mu)} \supseteq \hat{\mathcal{B}}^{(\tilde{\mu})}$, there is some $i \in \hat{\mathcal{B}}^{(\mu)}$ such that $V_{\alpha,\beta}(\phi^{(\mu)})_i \geqslant \bar{T}_i$. Then, by Corollary 50.2, $\kappa_B(\mu) = 0$ and hence $\kappa(\mu) = 0$. Therefore we have confirmed all elements of Definition 73 if $\hat{\mathcal{A}}^{(\tilde{\mu})} \neq \varnothing$.

Now suppose instead that iteration n is in the degenerate termination state. Then $\mathcal{A}^{(n)} = \varnothing$, $\tilde{y}^{(n)} = b^{(n)}_{\mathcal{B}^{(n)}}$ and $\mathcal{D}^{(n)} = \mathcal{B}^{(n)}$. By definition, either $\mathcal{B}^{(n)} = \varnothing$ or $\tilde{y}^{(n)} < \bar{T}_{\mathcal{B}^{(n)}}$ and $\tilde{y}^{(n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$. If $\mathcal{B}^{(n)} = \varnothing$ then $\mathcal{D}^{(n)} = \varnothing$ and so by Lemma 69.2 $\mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$ and $\mathcal{D}^{(n+1)} = \mathcal{D}^{(n)}$. If $\mathcal{B}^{(n)} \neq \varnothing$ then the sets $\mathcal{D}^{(n+1)}$ and $\mathcal{B}^{(n+1)}$ are defined at Step 10 and we can see that, again, $\mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$ and $\mathcal{D}^{(n+1)} = \mathcal{D}^{(n)}$. In either case, $\mathcal{A}^{(n+1)} = \mathcal{A}^{(n)}$ and $\mathcal{L}^{(n+1)} = \mathcal{L}^{(n)}$. Theorem 65.2d implies that $\hat{\mathcal{D}}^{(\mu)} = \hat{\mathcal{B}}^{(\tilde{\mu})} = \mathcal{B}^{(n)} = \mathcal{B}^{(n+1)} = \mathcal{D}^{(n+1)}$ and $\hat{\mathcal{B}}^{(\mu)} = \hat{\mathcal{B}}^{(\tilde{\mu})} = \mathcal{B}^{(n)} = \mathcal{B}^{(n+1)}$. By Lemma 75.1, the condition in Definition 73.1 is satisfied and, in particular, $\hat{\mathcal{L}}^{(\mu)} = \mathcal{L}^{(n+1)}$ and $\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n+1)} = \varnothing$. By Lemma 75.2, this is sufficient to establish that Definition 73.2 is satisfied.

Since $\tilde{y}^{(n)} < \bar{T}_{\mathcal{B}^{(n)}} = \bar{T}_{\mathcal{B}^{(n+1)}}$ and $\tilde{y}^{(n)} \leqslant \bar{B}_{\mathcal{B}^{(n)}} = \bar{B}_{\mathcal{B}^{(n+1)}}$, Lemma 75.4 implies that the condition in Definition 73.3 is also satisfied. Hence we have confirmed all elements of Definition 73 if $\hat{\mathcal{A}}^{(\tilde{\mu})} = \varnothing$. In particular, we have shown that if $\kappa(\tilde{\mu}) = \omega$ then iteration $n+1$ tracks $\phi^{(\tilde{\mu}+\omega)}$.

□

Corollary 84. *Iteration 0 tracks $\phi^{(0)}$ and therefore, for every $n \leqslant n^*$, iteration n of Algorithm 3 tracks $\phi^{(\mu)}$ for some ordinal μ.*

Proof. Let $n = 0$. Then by definition $\mathcal{A}^{(0)} = \varnothing$ and $y^{-1} = \mathbf{0}$. By Lemma 75.2 and 75.4 all conditions of Definition 73 are satisfied and so iteration 0 tracks $\phi^{(0)}$. The result then follows by induction and Lemma 83. □

We are now in a position to prove the first part of Theorem 67.

Proof of Theorem 67.1. By Corollary 84, every iteration tracks $\phi^{(\mu)}$ for some ordinal μ. By Proposition 80 we know that each iteration is in one of four states. In all states apart from the unbounded transition state, simple observation of Algorithm 3 (or Figure 3.1) shows that it proceeds sequentially in a finite number of steps until the end of the iteration. If iteration n is in the unbounded transition state then $K(n)$ is well-defined and so the loop in Step 7 will not run more than $K(n) - |\mathcal{A}^{(n)}|$ times. Hence in all states each iteration is completed in a finite number of steps.

By Lemma 69.2 $(\mathcal{D}^{(n)})_{n\geqslant 0}$ and $(\mathcal{B}^{(n)})_{n\geqslant 0}$ are non-decreasing sequences of sets. If for any iteration n, $\mathcal{D}^{(n-1)} = \mathcal{D}^{(n)} = \mathcal{D}^{(n+1)} = \mathcal{D}^{(n)}$ and $\mathcal{B}^{(n-1)} = \mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$ then we will also have $\mathcal{A}^{(n-1)} = \mathcal{A}^{(n+1)} =$

$\mathcal{A}^{(n)}$ and $\mathcal{L}^{(n-1)} = \mathcal{L}^{(n+1)} = \mathcal{L}^{(n)}$. Then by Remark 68 Algorithm 3 will terminate. The only alternative to this is that $\mathcal{D}^{(n-1)} \subsetneq \mathcal{D}^{(n)}$, $\mathcal{D}^{(n+1)} \subsetneq \mathcal{D}^{(n)}$, $\mathcal{B}^{(n-1)} \subsetneq \mathcal{B}^{(n)}$ or $\mathcal{B}^{(n+1)} \subsetneq \mathcal{B}^{(n)}$. However, since these are subsets of the finite set $\mathcal{N}$ and the sequences of these sets are monotonic, this case can only occur a finite number of times before the sequences become constant.

Hence Algorithm 3 proceeds in a finite number of iterations, each of which completes in a finite number of steps. □

Now that we have shown that Algorithm 3 always terminates, we can characterise the final iterations of the algorithm using the following lemmas. This will then let us conclude that the algorithm outputs a fixed point of Φ.

Lemma 85. *Suppose iteration n is in a transition state. Then $n < n^* - 2$.*

Proof. Note that according to Remark 68 n^* is well-defined precisely whenever at Step 13 it is established that $\mathcal{A}^{(n^*-2)} = \mathcal{A}^{(n^*-1)} = \mathcal{A}^{(n^*)}$, $\mathcal{B}^{(n^*-2)} = \mathcal{B}^{(n^*-1)} = \mathcal{B}^{(n^*)}$ and $\mathcal{L}^{(n^*-2)} = \mathcal{L}^{(n^*-1)} = \mathcal{L}^{(n^*)}$.

Suppose iteration n is in a transition state so that Lemma 81.2 implies that either there is some $i \in \mathcal{A}^{(n)}$ with $x_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or there is some $i \in \mathcal{B}^{(n)}$ with $y_i^{(K(n)+1,n)} \geqslant \bar{T}_i$ or $x_i^{(K(n)+1,n)} > \bar{B}_i$. Furthermore, for a transition state we have $\tilde{x}^{(n)} = x^{(K(n)+1,n)}$ and $\tilde{y}^{(n)} = y^{(K(n)+1,n)}$ and Algorithm 3 reaches Step 9. In particular,

$$\mathcal{D}^{(n+1)} = \{i \in \mathcal{A}^{(n)} \mid \tilde{x}_i^{(n)} < \bar{T}_i\} \cup \{i \in \mathcal{B}^{(n)} \mid \tilde{y}_i^{(n)} < \bar{T}_i\}$$
$$\mathcal{B}^{(n+1)} = \{i \in \mathcal{D}^{(n+1)} \cap \mathcal{B}^{(n)} \mid \tilde{y}_i^{(n)} \leqslant \bar{B}_i\}.$$

Therefore $\mathcal{A}^{(n+1)} = \{i \in \mathcal{A}^{(n)} \mid \tilde{x}_i^{(n)} < \bar{T}_i\} \cup \{i \in \mathcal{B}^{(n)} \mid \bar{B}_i < \tilde{y}_i^{(n)} < \bar{T}_i\}$. If there is some $i \in \mathcal{A}^{(n)}$ with $\tilde{x}_i^{(n)} \geqslant \bar{T}_i$ then $\mathcal{A}^{(n+1)} \neq \mathcal{A}^{(n)}$. If there is some $i \in \mathcal{B}^{(n)}$ with $\tilde{y}_i^{(n)} \geqslant \bar{T}_i$ or $\tilde{y}_i^{(n)} > \bar{B}_i$ then $\mathcal{B}^{(n+1)} \neq \mathcal{B}^{(n)}$. In either event we have that $n + 1 \neq n^* - 1$ and $n + 1 \neq n^*$. □

Lemma 86. *Suppose iteration n is in the degenerate termination state and tracks $\phi^{(\mu)}$ for some ordinal μ. Then $\phi^{(\mu)}$ is a fixed point of Φ and if $n < n^*$ then iteration $n+1$ is also in the degenerate termination state.*

Proof. By definition of the degenerate termination state we have $\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n)} = \varnothing$ and $\tilde{y}^{(n)} = b^{(n)}_{\mathcal{B}^{(n)}} < \bar{T}_{\mathcal{B}^{(n)}}$ and $b^{(n)}_{\mathcal{B}^{(n)}} \leqslant \bar{B}_{\mathcal{B}^{(n)}}$. Therefore by Definition 73, $\phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. Proposition 38 implies that $\Phi(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} \geqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and $\Phi(\phi^{(\mu)})_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}}$. Suppose that, for some $i \in \hat{\mathcal{B}}^{(\mu)}$, $\Phi(\phi^{(\mu)})_i > \bar{B}_i$. Then $V_{\alpha,\beta}(\phi^{(\mu)})_i > \bar{B}_i$. By Corollary 53, $V_{\alpha,\beta}(\phi^{(\mu)}) = \hat{b}^{(\mu)} = b^{(n)}$ and so $b_i^{(n)} > \bar{B}_i$, contradicting the definition of the degenerate termination state. Therefore $\Phi(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and hence $\Phi(\phi^{(\mu)}) = \phi^{(\mu)}$, i.e. $\phi^{(\mu)}$ is a fixed point of Φ.

Since $\mathcal{A}^{(n)} = \varnothing$, $\mathcal{D}^{(n)} = \mathcal{B}^{(n)}$. The sets $\mathcal{D}^{(n+1)}$ and $\mathcal{B}^{(n+1)}$ are defined in Step 10 and we have that $\mathcal{D}^{(n+1)} = \{i \in \mathcal{B}^{(n)} \mid \tilde{y}_i^{(n)} < \bar{T}_i\} = \mathcal{B}^{(n)} = \mathcal{D}^{(n)}$ and $\mathcal{B}^{(n+1)} = \mathcal{B}^{(n)}$. Hence $\mathcal{A}^{(n+1)} = \mathcal{A}^{(n)} = \varnothing$ and $\mathcal{L}^{(n+1)} = \mathcal{L}^{(n)}$, which also implies that $b^{(n+1)} = b^{(n)}$. Hence iteration $n + 1$ is again in the degenerate termination state. □

Lemma 87. *Suppose iterations n^*-2 and n^*-1 are both in the limit state and iteration n^*-1 tracks $\phi^{(\mu)}$ for some ordinal μ. Then $\phi^{(\mu)}$ is a fixed point of Φ.*

Proof. By Remark 68 and Definition 73, we have that

$$\hat{\mathcal{B}}^{(\mu)} = \mathcal{B}^{(n^*-1)} = \mathcal{B}^{(n^*-2)};$$
$$\hat{\mathcal{A}}^{(\mu)} = \mathcal{A}^{(n^*-1)} = \mathcal{A}^{(n^*-2)};$$
$$\hat{\mathcal{L}}^{(\mu)} = \mathcal{L}^{(n^*-1)} = \mathcal{L}^{(n^*-2)}; \text{ and}$$
$$\hat{\mathcal{D}}^{(\mu)} = \mathcal{D}^{(n^*-1)} = \mathcal{D}^{(n^*-2)}.$$

In particular, it follows that $b^{(n^*-1)} = b^{(n^*-2)}$ and $M^{(n^*-1)} = M^{(n^*-2)}$. Also note that since iteration n^*-1 is in the limit state, $\mathcal{A}^{(n^*-1)} \neq \varnothing$ and so $K(n^*-1) > |\mathcal{A}^{(n^*-1)}| \geqslant 1$.

Since $\mathcal{A}^{(n^*-2)} \cap \mathcal{B}^{(n^*-2)} = \varnothing$, Definition 73 implies that $\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = p^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}} = \tilde{x}^{(n^*-2)}$. Since iteration n^*-2 is in the limit state we have that

$$\begin{aligned}(\mathrm{I} - M^{(n^*-1)})\tilde{x}^{(n^*-2)} &= (\mathrm{I} - M^{(n^*-2)})\tilde{x}^{(n^*-2)} \\ &= b^{(n^*-2)}_{\mathcal{A}^{(n^*-2)}} = b^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}}\end{aligned}$$

and so

$$\tilde{x}^{(n^*-2)} = b^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}} + M^{(n^*-1)}\tilde{x}^{(n^*-2)};$$
$$\tilde{y}^{(n^*-2)} = b^{(n^*-2)}_{\mathcal{B}^{(n^*-2)}} + \beta\Omega_{\mathcal{B}^{(n^*-2)}\mathcal{A}^{(n^*-2)}}\tilde{x}^{(n^*-2)}.$$

Therefore

$$\phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}} = b^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}} + M^{(n^*-1)}\tilde{x}^{(n^*-2)} = b^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}} + M^{(n^*-1)}\phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}};$$
$$\tilde{y}^{(n^*-2)} = b^{(n^*-1)}_{\mathcal{B}^{(n^*-1)}} + \beta\Omega_{\mathcal{B}^{(n^*-1)}\mathcal{A}^{(n^*-1)}}\tilde{x}^{(n^*-2)}.$$

By Corollary 53 and Lemma 76.1, $V_{\alpha,\beta}(\phi^{(\mu)}) = \hat{b}^{(\mu)} + \beta\Omega_{\mathcal{N}\hat{\mathcal{A}}^{(\mu)}}\phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} = b^{(n^*-1)} + \beta\Omega_{\mathcal{N}\mathcal{A}^{(n^*-1)}}\phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}}$. In particular,

$$V_{\alpha,\beta}(\phi^{(\mu)})_{\mathcal{A}^{(n^*-1)}} = b^{(n^*-1)}_{\mathcal{A}^{(n^*-1)}} + M^{(n^*-1)}\phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}} = \phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}};$$
$$V_{\alpha,\beta}(\phi^{(\mu)})_{\mathcal{B}^{(n^*-1)}} = b^{(n^*-1)}_{\mathcal{B}^{(n^*-1)}} + \beta\Omega_{\mathcal{B}^{(n^*-1)}\mathcal{A}^{(n^*-1)}}\tilde{x}^{(n^*-2)} = \tilde{y}^{(n^*-2)}.$$

By Proposition 48, $\bar{B}_{\hat{\mathcal{A}}^{(\mu)}} < \phi^{(\mu)}_{\mathcal{A}^{(n^*-1)}} = \phi^{(\mu)}_{\hat{\mathcal{A}}^{(\mu)}} < \bar{L}_{\hat{\mathcal{A}}^{(\mu)}}$ and so $\Phi(\phi^{(\mu)})_{\hat{\mathcal{A}}^{(\mu)}} = V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{A}}^{(\mu)}} = \phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}}$. Furthermore, $\tilde{y}^{(n^*-2)} < \bar{T}_{\mathcal{B}^{(n^*-1)}}$ and $\tilde{y}^{(n^*-2)} \leqslant \bar{B}_{\mathcal{B}^{(n^*-1)}}$ since otherwise $\kappa(\mu) = 0$ by the condition in Definition 73.3. But if $\kappa(\mu) = 0$ then by Lemma 81.3 we get the contradiction $1 < K(n^*-1) \leqslant \kappa(\mu) = 0$. Hence $V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} < \bar{T}_{\hat{\mathcal{B}}^{(\mu)}}$ and $V_{\alpha,\beta}(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} \leqslant \bar{B}_{\hat{\mathcal{B}}^{(\mu)}}$ and so by Lemma 39 we get that $\Phi(\phi^{(\mu)})_{\hat{\mathcal{B}}^{(\mu)}} = \bar{B}_{\hat{\mathcal{B}}^{(\mu)}} = \phi^{(\mu)}_{\hat{\mathcal{B}}^{(\mu)}}$. Finally, by Propositions 38 and 48, $\Phi(\phi^{(\mu)})_{\hat{\mathcal{L}}^{(\mu)}} = \bar{L}_{\hat{\mathcal{L}}^{(\mu)}} = \phi^{(\mu)}_{\hat{\mathcal{L}}^{(\mu)}}$. Therefore $\Phi(\phi^{(\mu)}) = \phi^{(\mu)}$ and so $\phi^{(\mu)}$ is a fixed point of Φ. $\square$

We are now in a position to prove that Algorithm 3 outputs a fixed point of Φ.

Proposition 88. *The output of Algorithm 3, p^*, is a fixed point of Φ.*

Proof. By Theorem 67.1 Algorithm 3 terminates and so by Theorem 72 $p^* = p^{(n^*)}$. By Corollary 84, let μ be an ordinal such that iteration $n^* - 1$ tracks $\phi^{(\mu)}$. Then $p^* = \phi^{(\mu)}$.

Note that by Lemma 85 iteration $n^* - 2$ is not in a transition state. If iteration $n^* - 2$ is in the degnerate termination state then by Lemma 86 iteration $n^* - 1$ is also in the degenerate termination state and p^* is a fixed point of Φ. If iteration $n^* - 2$ is in the simple transition state and iteration $n^* - 1$ is in the degenerate termination state then again Lemma 86 tells us that p^* is a fixed point of Φ. Finally, if both iterations $n^* - 2$ and $n^* - 1$ are in the limit state then Lemma 87 implies that p^* is a fixed point of Φ. Hence, in all cases, p^* is a fixed point of Φ. □

To conclude the proof of Theorem 67.2, we need to show that the fixed point p^* is in fact the least fixed point of Φ, whose existence is ensured by Theorem 44. We first need the following lemma.

Lemma 89. *Suppose iteration n tracks $\phi^{(\tilde{\mu})}$ and iteration $n+1$ tracks $\phi^{(\mu)}$ for some ordinals $\tilde{\mu}$ and μ. If ν is an ordinal such that $\tilde{\mu} \leqslant \nu \leqslant \mu$ with $\phi^{(\nu)}$ a fixed point of Φ then $\phi^{(\nu)} = \phi^{(\mu)}$.*

Proof. Suppose iteration n is in a transition state. We can assume that $\nu < \mu$ as otherwise $\phi^{(\nu)} = \phi^{(\mu)}$ trivially. Then by Lemma 82.1, $\kappa(\tilde{\mu}) < \omega$. Furthermore, by Lemma 83.1 we can assume that $\mu = \tilde{\mu} + \kappa(\tilde{\mu}) + 1$. Then, since $\tilde{\mu} \leqslant \nu < \mu$, there is some integer k satisfying $0 \leqslant k \leqslant \kappa(\tilde{\mu})$ such that $\nu = \tilde{\mu} + k$. Hence $V_{\alpha,\beta}(\phi^{(\nu)}) = V_{\alpha,\beta}(\phi^{(\tilde{\mu}+k)})$ and so, by Corollary 50, $\kappa(\nu) = \kappa(\tilde{\mu}) - k \leqslant \kappa(\tilde{\mu}) < \omega$. Therefore $\phi^{(\nu)}$ is not a fixed point of Φ as per Remark 51.

So if $\phi^{(\nu)}$ a fixed point of Φ then iteration n is either in the degenerate transition state or in the limit state. By Lemma 82.2, $\kappa(\tilde{\mu}) = \omega$ and by Lemma 83.2 we can assume that $\mu = \tilde{\mu} + \omega$. Clearly, if $\nu = \mu$ then $\phi^{(\nu)} = \phi^{(\mu)}$ so suppose ν satisfies $\tilde{\mu} \leqslant \nu < \mu = \tilde{\mu} + \omega$. Then there is some integer k with $\nu = \tilde{\mu} + k$. Therefore $\nu + \omega = \tilde{\mu} + k + \omega = \tilde{\mu} + \omega$. Then by Proposition 42.2 $\phi^{(\mu)} = \phi^{(\tilde{\mu}+\omega)} = \phi^{(\nu+\omega)} = \phi^{(\nu)}$. □

We can now conclude the proof of Theorem 67

Proof of Theorem 67.2. By Proposition 88, p^* is a fixed point of Φ. Theorem 44 guarantees the existence of the least fixed point of Φ and confirms that such least fixed point is of the form $\phi^{(\nu_*)}$ for some countable ordinal ν_*.

By Corollary 84, let μ_* be an ordinal such that iteration $n^* - 1$ tracks $\phi^{(\mu_*)}$. Then by Theorem 72 $p_* = \phi^{(\mu_*)}$. Note that by Proposition 42.1 $\nu_* \leqslant \mu_*$, since $\phi^{(\mu_*)} = p^*$ is a fixed point of Φ and $\phi^{(\nu_*)}$ is the least fixed point of Φ. If $n^* - 1 = 0$ then by Corollary 84, $\nu_* = \mu_* = 0$ and so $\phi^{(\mu_*)} = \phi^{(\nu_*)}$. If $n^* - 1 > 0$ then by Corollary 84, $n^* - 2$ tracks $\phi^{(\tilde{\mu})}$ for some ordinal $\tilde{\mu}$. By Lemma 89, $\phi^{(\mu_*)} = \phi^{(\nu_*)}$. In particular, it follows that $p^* = \phi^{(\mu_*)} = \phi^{(\nu_*)}$. □

3.5 Examples of least fixed points

3.5.1 Least fixed points in Rogers & Veraart (2013)

In this section we apply Algorithm 3 to a clearing system $(a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ on $\mathcal{N} = \{1, 2\}$. In particular, Ω is fully characterised by Ω_{12} and Ω_{21}.

Remark 90. Let $a = \mathbf{1}$, $\bar{B} = \mathbf{0}$, $\Omega_{12} = \Omega_{21} = 1$, $\bar{L}_1 = K_1$, $\bar{L}_2 = K_2$ for some K_1, K_2 and $\alpha = \beta = \frac{1}{2}$. Then the clearing system corresponds to the financial system described in Example 3.3 in Rogers & Veraart (2013) and the function Φ in that paper coincides with the function Φ in this chapter.

Example 91. Let the clearing system be as in Remark 90 with $K_1 = K_2 = 2.2$. Then $\bar{L} = 2.2 \cdot \mathbf{1}$ and $\bar{T} = 1.1 \cdot \mathbf{1}$.

In iteration 0, $p^{(0)} = \bar{B} = \mathbf{0}$. Furthermore, $\tilde{y}^{(0)} = b^{(0)} = \frac{1}{2}\mathbf{1}$. Also, $y^{(0,1)} = \frac{1}{2}\mathbf{1} > \bar{B}$. Hence iteration 0 is in the simple transition state.

In iteration 1, $p^{(1)} = \frac{1}{2}\mathbf{1}$. Furthermore, $\mathcal{D}^{(1)} = \mathcal{A}^{(1)} = \mathcal{N}$, $\mathcal{B}^{(1)} = \mathcal{L}^{(1)} = \mathcal{L}^{(0)} = \varnothing$, $b^{(1)} = \frac{1}{2}\mathbf{1}$, $M^{(1)} = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $c^{(1)} = \frac{1}{4}\mathbf{1}$. Then $x^{(0,1)} = \frac{1}{2}\mathbf{1}$, $x^{(1,1)} = \frac{3}{4}\mathbf{1}$, $x^{(2,1)} = \frac{7}{8}\mathbf{1}$ and $x^{(3,1)} = \frac{15}{16}\mathbf{1}$. Therefore $K(1) > 2 = |M^{(1)}|$ and $\tilde{x}^{(1)} = (\mathrm{I} - M^{(1)})^{-1} b^{(1)} = \mathbf{1}$. In particular, iteration 1 is in the limit state.

In iteration 2, $p^{(2)} = \mathbf{1}$. Furthermore, $\mathcal{D}^{(2)} = \mathcal{A}^{(2)} = \mathcal{D}^{(1)} = \mathcal{A}^{(1)} = \mathcal{N}$, $\mathcal{B}^{(2)} = \mathcal{B}^{(1)} = \mathcal{L}^{(2)} = \mathcal{L}^{(1)} = \varnothing$, $b^{(2)} = b^{(1)}$, $M^{(2)} = M^{(1)}$ and $c^{(2)} = \mathbf{0}$. Therefore $x^{(k,2)} = \mathbf{1}$ for all k and so $K(2) > 2 = |M^{(2)}|$ and $\tilde{x}^{(2)} = \mathbf{1}$. In particular, iteration 2 is again in the limit state.

In iteration 3, $p^{(3)} = \mathbf{1}$. Furthermore, $\mathcal{D}^{(3)} = \mathcal{D}^{(2)} = \mathcal{N}$ and $\mathcal{B}^{(3)} = \mathcal{B}^{(2)} = \varnothing$. Hence by the same argument as in iteration 2, iteration 3 is in the limit state. In particular, it follows that $n^* = 4$ and $p^* = \mathbf{1}$.

Thus $\mathbf{1}$ is the least fixed point, agreeing with the finding in Rogers & Veraart (2013) where it was shown that $\mathbf{1}$ and $2.2 \cdot \mathbf{1}$ are the two fixed points of Φ.

The next example demonstrates that while Algorithm 3 terminates in a finite number of steps, this finite number can be arbitrarily large.

Example 92. Let the clearing system be as in Example 91 with the exception that $\beta = 1$ (while $\alpha = \frac{1}{2}$, as before) and $K_1 = K_2 = K + 1$ for some $K > 2$. Then $\bar{L} = (K + 1)\mathbf{1}$ and $\bar{T} = (K + \frac{1}{2})\mathbf{1}$. As in Example 91, iteration 0 is in the simple transition state, $p^{(0)} = \mathbf{0}$ and $p^{(1)} = \frac{1}{2}\mathbf{1}$. However, since $\beta = 1$, $\mathrm{I} - M^{(1)} = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ and so $\det(\mathrm{I} - M^{(1)}) = 0$. It can then be shown that $\mathcal{A}^{(1)} = \mathcal{N}$ and $x^{(k,1)} < \bar{T}$ for all $k < 2K + 1$. Hence $K(1) \geqslant 2K > 2|\mathcal{A}^{(n)}|$ and therefore iteration 1 is in the unbounded transition state.

3.5.2 Roll-over cascades

In this section, we consider an example where the lower bound of the clearing function is strictly greater than zero. To this end we will briefly introduce a simplified model of roll-over credit.

We consider a set of banks $\mathcal{N}$ with starting cash assets given by a vector $\tilde{a}$. We assume that in normal times, at maturity of the loan L_{ij}, the creditor j will roll over its loan to i provided i can make a partial repayment at some rate h_j. The rate h_j, set by the creditor j, is the same for all of its counterparties. We assume that $0 < \bar{h}_j \leqslant h_j \leqslant 1$ where $\bar{h}_j$ is some minimal repayment rate. The upper bound of 1 corresponds to the notion of h being a rate. A bank may require a higher repayment rate than the minimal rate $\bar{h}_j$ in times of stress but it will not require a lower repayment as $\bar{h}$ is set by the interbank market risk tolerance and this threshold is in place to ensure that j remains profitable and does not perpetually extend loans to potentially insolvent banks. In particular, the repayment rate h_j may differ from $\bar{h}_j$ if any counterparty i cannot make their minimal repayment or if j itself suffers a liquidity shortage.

The roll-over mechanism ensures that the banking system maintains a sufficient level of overall funding liquidity. The rate h_j is a proxy measure for how confident j is that its counterparties are liquid and can continue meeting their obligations. If j is not confident then it can increase h_j potentially necessitating its counterparties to raise their funding elsewhere, for example by withdrawing their own lending. Provided $h_j < 1$, j retains some confidence in its counterparties and this confidence allows the debtor counterparties to raise further funding by leveraging their liquid assets. However if $h_j = 1$ then it requires immediate repayment of its loans. In such cases the counterparties are assumed to be undergoing a liquidity shock and will not be able to raise further funding by leveraging their assets sufficiently quickly.

Given the roll-over repayment rates h, the position of each bank i is described as follows:

- Cash assets: $\tilde{a}_i$
- Roll-over repayment from each debtor j: $L_{ji}h_i$
- Rolled over loan to j: $L_{ji}(1 - h_i)$
- Roll-over repayment to each creditor k: $L_{ik}h_k$
- Rolled over loan from k: $L_{ik}(1 - h_k)$

The rolled over loans are transacted "on paper". Only the minimal payments are made out of the actual liquid asset reserves. Thus i has to be able to make total repayments of $\sum_k L_{ik}h_k$. The total liquid assets that it has available to meet these repayments are given by $\tilde{a}_i + \sum_j L_{ji}h_i$. However, as mentioned above we assume that this amount can be leveraged by i. In a reduced form we assume that, provided it is not undergoing a liquidity shock, i can make its repayments from a total supply of value equal to $\lambda\left(\tilde{a}_i + h_i \sum_j L_{ji}\right)$ where $\lambda \geqslant 1$ is a leverage multiplier. On the other hand if i is undergoing a liquidity shock then it needs to make its repayments from a total supply of value equal to $\tilde{a}_i + h_i \sum_j L_{ji}$.

Provided i is not undergoing a liquidity shock, the net value of liquid assets retained in the bank is given by

$$w_i = \lambda\left(\tilde{a}_i + h_i \sum_j L_{ji}\right) - \sum_k L_{ik}h_k.$$

We assume that i seeks to ensure that $w_i \geqslant \underline{\mathrm{w}}_i$ for some minimal value $\underline{\mathrm{w}}_i \geqslant 0$ chosen by i. If $\underline{\mathrm{w}}_i > \lambda \tilde{a}_i$ then i seeks to maintain a buffer in excess of its leveraged cash assets in order to absorb possible loss of value. Thus we obtain the following expression

$$\lambda \left(\tilde{a}_i + h_i \sum_j L_{ji} \right) - \sum_k L_{ik} h_k \geqslant \underline{\mathrm{w}}_i,$$

which we rewrite as

$$h_i \geqslant \frac{1}{\lambda} \frac{\underline{\mathrm{w}}_i - \lambda \tilde{a}_i}{\sum_j L_{ji}} + \frac{1}{\lambda} \sum_k \frac{L_{ik}}{\sum_j L_{ji}} h_k.$$

The right-hand side of the inequality is a lower bound on h_i. We then assume that this lower bound is attained since h should only exceed $\bar{h}$ in times of stress and hence we would like to find the lowest possible consistent value of h.

We set $a_i := \frac{\underline{\mathrm{w}}_i - \lambda \tilde{a}_i}{\sum_j L_{ji}}$ and $\Omega_{ij} := \frac{L_{ij}}{\sum_j L_{ji}}$ for each i, j. Recalling that $\bar{h}_j \leqslant h_j \leqslant 1$ and leveraging is permitted only if $h < 1$ whenever $\lambda = 1$, we obtain the fixed point problem $h = \Phi(h)$ where

$$\Phi(h)_i = \begin{cases} 1 & \text{if } (a + \Omega h)_i \geqslant 1 \\ \bar{h}_i \vee \left(\frac{1}{\lambda} a + \frac{1}{\lambda} \Omega h \right)_i & \text{if } (a + \Omega h)_i < 1 \end{cases}$$

for each i. Note that if $\underline{\mathrm{w}} > \lambda \tilde{a}$ then $a > 0$. What we have is a cleating system $(a, \Omega, \mathbf{1}, \bar{h}\mathbf{1}, \frac{1}{\lambda}, \frac{1}{\lambda})$.

Example 93. Let $S1 = (a, \Omega, \bar{B}, \bar{L}, \alpha, \beta)$ be the clearing system given in Example 91 and consider the clearing system $S2 = (ca, \Omega, c\bar{B}, c\bar{L}, \alpha, \beta)$ where $c = \frac{1}{2.2}$. Then by Proposition 37 the least fixed point of the clearing function of $S2$ is given by $\frac{1}{2.2}\mathbf{1}$. Now let $S3 = (ca, \Omega, \frac{1}{2}\mathbf{1}, c\bar{L}, \alpha, \beta)$. $S3$ is the same clearing system as $S2$ but, instead of the the lower bound of $\mathbf{0}$, the clearing function of $S3$ has the lower bound $\frac{1}{2}\mathbf{1} > \frac{1}{2.2}\mathbf{1}$, i.e. a lower bound which is greater than the least fixed point of $S2$. Applying Algorithm 3, we find that the least fixed point of the clearing function of $S3$ is $\frac{1}{2}\mathbf{1}$.

Now consider the roll-over credit model where the reserve of liquid value that the banks wish to maintain is given by $\underline{\mathrm{w}} := 2\tilde{a} + \frac{1}{2.2}\mathbf{1}$ for some vector of cash assets $\tilde{a}$ and the minimal roll-over rate is given by $\bar{h} = \frac{1}{2.2}\mathbf{1}$. Then we can readily see that the effective roll-over rate h is in fact given by the fixed point of the clearing function associated with $S3$. Namely, $h = \frac{1}{2}\mathbf{1}$.

3.6 Conclusion

In this chapter we developed Algorithm 3 for constructing the least fixed point of a class of clearing functions in a finite number of steps. The class of clearing functions considered is larger that the class described in Rogers & Veraart (2013). In particular, we extended the clearing problem in that paper by allowing for an arbitrary lower bound on the values taken by the clearing function. As in Rogers & Veraart (2013), the clearing problem considered allows for default costs, which renders the corresponding clearing function discontinuous from below.

The discontinuity from below and the choice of the lower bound do not play a role in the more common problem of constructing the greatest fixed point of the clearing function. For this reason these features have not been previously analysed in detail in the literature. We show that they introduce a number of obstacles and the problems of construction of the greatest and the least fixed points are not simple converses of each other. Nevertheless, we are able to describe the structure of the least fixed points by means of transfinite sequences. However, there is a cost to constructing the least fixed points in a finite number of steps. Algorithm 3 is more complicated than corresponding algorithms for the greatest fixed points. Moreover, we are not able to obtain a bound on the number of steps it takes the algorithm to terminate.

The problem discussed in this chapter is primarily of theoretical interest. Nevertheless there is scope for applications in the context of systemic risk assessment. In particular, we described a simple model of roll-over credit where the least fixed point corresponds to the natural clearing solution of interest.

4

Discussion and Outlook

In this thesis we considered several extensions of the Eisenberg & Noe (2001) model of interbank network clearing. These problems, both as mathematical problems and models for systemic risk assessment, are interesting in their own right but moreover, in answering the questions posed, we have also opened up further avenues of research.

4.1 Dynamic models

As discussed in Section 1.3, recent literature has made advances in developing multi-period models of systemic risk. In part, these advances have been driven by the desire to develop a full controlled dynamic model of systemic risk. Financial regulators, in particular, are very interested in dynamic models because such models would allow them to evaluate the effects of various policy proposals and the market's responses to them. The multiple maturity model described in Chapter 2 is another significant milestone towards that objective. We briefly described a simple *uncontrolled* system in Section 2.4.3.

The next steps would involve developing a control theory by deciding on a set of actions that financial institutions can choose from as they move forward in time. Examples of such actions could be new borrowing or lending activities. For such dynamic models one could then also include stochastic dynamics for some of the quantities of interest.

One of the conceptual difficulties that has reoccurred in systemic risk literature is the precise timing of different events and the conflict this can cause with the static nature of stylised balance sheets that are typically used to formulate the models. This difficulty would only be exacerbated once control actions are introduced.

Figure 4.1 is a simple tool that can be used to alleviate this issue in the development of a full dynamic model. It is a snapshot of a timeline, describing times $t \in \{t_i \mid 0 \leqslant i \leqslant 6\}$ with the focus on the two maturities T_1 and T_1. The timeline in Figure 4.1 is an extension of the much simpler timeline considered in Chapter 2 where we assumed that $t_0 = 0$, $t_1 = t_2 = t_3$ and $t_4 = t_5 = t_6$ so that $t \in \{0, T_1, T_2\}$.

At time $t = t_0$ each bank's stylised balance sheet is 'unconstrained' in the sense that the nearest maturity

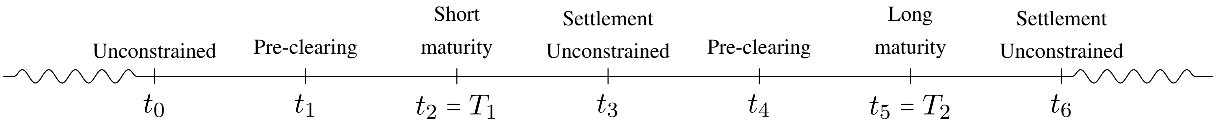

Figure 4.1: Sample timeline

T_1 is sufficiently far in the future that banks are free to carry out day-to-day trading and to respond to the anticipated maturity of short-term liabilities. Between $t = t_0$ and $t = t_1$ the stylised balance sheet of each bank can change due to various factors such as the effect of interest rates, fluctuations in asset value and active adjustments of the balance sheet through changes in the bank's portfolio. The graph topology of the network can change substantially during this period. Asset liquidation can be introduced into the model with the accompanying risk of firesales. Repurchase agreements and other asset purchases can create new links in the networks or introduce cross-holding effects as described in Elsinger (2011), for example. Thus the interval between $t = t_0$ and $t = t_1$ is the principal period that would need to be accounted for in detail in a full dynamic model.

The cumulative effect of actions taken by the banks as well as any changes due to other effects is summarised in their stylised balance sheets as they stand at $t = t_1$. In particular, if no actions take place between $t = t_0$ and $t = t_1$, then such a timeline describes the perspective in Chapter 2. Short-term liabilities mature at $t = t_2 = T_1$ and we assume that no market activity takes place between $t = t_1$ and $t = t_2$. This assumption stems from the fact that, should a bank fail to honour some of its obligations at $t = t_2$, the insolvency proceedings will wind back in time to establish the balance sheet immediately prior to such a default. We take this point to be at $t = t_1$.

At $t = t_2 = T_1$ banks reconcile their mutual nominal obligations by making payments in satisfaction of their short-term liabilities by *clearing* them as described in detail in Chapter 2. No market activity is assumed to take place between $t = t_2$ and $t = t_3$. This time period represents technical adjustments to the financial system where the banks that had defaulted at $t = t_2 = T_1$ leave the financial system and the surviving banks reclassify their remaining liabilities as the new short-term and long-term liabilities. We refer to this transition as *settlement* and it was described in detail in Section 2.4. In Section 2.4 we adopted a reduced approach of reassigning the assets of the defaulted banks to the sink node. As discussed there, in reality some sort of auction process would be involved and the assets are likely to be reassigned to some of the surviving banks. Any model seeking to simmulate this process would be situated in the period between $t = t_2$ and $t = t_3$.

At $t = t_3$ the financial system returns to an 'unconstrained' state (as at $t = t_0$) and the banks can resume their market activity between $t = t_3$ and $t = t_4$. Similarly, $t = t_4$ represents the last opportunity for the financial system to respond prior to the remaining liabilities maturing at $t = t_5 = T_2$ and settlement at $t = t_6$. This approach is clearly periodic and thus can be extended to an arbitrarily long timeline with any number of maturities.

4.2 Notions of default and distress

Another 'holy grail' of systemic risk research is the development of a unified theory that can account for several major channels of contagion and financial instability. Currently the models used to describe default contagion and contagion due to fire sales, for example, use very different frameworks which prevents us from understanding the combined effect of multiple channels of contagion. Such an ideal unified framework is currently some way from being achieved. Indeed, even within the domain of a single channel, such as the default contagion discussed in this thesis, there is often no single coherent framework.

Different strands of research within the interbank clearing literature use similar, but nonetheless different, definitions of clearing functions and the nature of the clearing solutions themselves varies. In Chapter 2, for instance, the clearing solution focused on the vector of liquid assets whereas in Chapter 3 it was more convenient to work with the vector of payments. Often these choices are a matter of convenience and by means of a simple transformation we can obtain an equivalent clearing problem using a different quantity of interest. But again, to our best knowledge, no theory has been put forward which would answer upfront which quantities lend themselves to such equivalent formulations and which do not.

The view that has been developed in this thesis (and in Chapter 2, in particular) is that we should seek to formalise the theory of clearing not around the notion of clearing vectors but rather around the notion of default sets. As soon as a clearing problem uses a non-monotonic clearing function we lose the ability to represent the default set as a function of the clearing solution. Thus already we need to include the default set as *part* of the clearing solution. Moreover, as shown in Lemma 26 in Chapter 2, under some mild conditions a default set can be shown to *determine* the clearing solution. Indeed, this approach has also been adopted in some of the most recent literature such as Roukny et al. (2018). Once we shift the focus from clearing vectors to default sets as a fundamental notion, promising avenues open up.

As was discussed in Chapter 1, one of the core properties of the Eisenberg & Noe (2001) model and its various successors is the uniqueness of clearing solutions. This property is often lost once some of the strong assumptions of the original model are relaxed. It is not even clear that the uniqueness of clearing solutions, in the context of systemic risk assessment, is necessarily a desirable property to have since it suppresses the subjective nature of valuations. A key insights stemming from this thesis is that the idea of non-unique clearing solutions ought to be embraced and new theory developed to understand the structure of such solutions. Chapter 3 was one step in that direction. Now that we have methodologies for constructing both the greatest and least clearing solutions, a natural question arises about how to treat the fixed points that lie between these extrema. Adopting default sets as a fundamental notions offers one possible approach. Unlike clearing vectors, there is only a finite number of possible default sets for a given financial system. Following the analogy of Lemma 26 in Chapter 2, intermediate clearing solutions can be identified with default sets that lie between the least and greatest possible default sets.

Centring the analysis on the notion of the default set also allows us to re-examine the suitability of various assumptions. For example, as discussed in this thesis, an implicit assumption made in many interbank clearing models is that default is an absorbing state. In the context of solvency contagion, this is a reasonable assumption and in Chapter 2 we made it explicit. It is the absorbing nature of default that allowed us to circumvent the difficulties posed by non-monotonicity of the clearing function and obtain Algorithm 1. However, the absorption assumption should not be made in the related field of *distress* contagion as discussed in Barucca et al. (2016) and Veraart (2017), for example. Unlike a bank in default, a distressed bank may well recover. This does not present difficulties as long as the assumptions of monotonicity is maintained. However, analysing models of distress contagion without assuming either the monotonicity of clearing functions or the absorption property is a challenging problem. The Tarski-Knaster Theorem would not apply and the solutions would not necessarily form a complete lattice.

4.3 Policy implications

In addition to opening up further theoretical research, Chapter 2 also has interesting policy implications.

In Section 2.3.5 we demonstrated that maturity profile uncertainly can, in principle, act as a novel source of systemic risk. This observation can be tested, given access to regulatory data. In particular, the question of whether the five maturity categories described in Langfield et al. (2014) are sufficient can be answered.

One of the implications of the non-monotonicity of the clearing function described in Chapter 2 is that it is possible to construct a financial system where a smaller default set is obtained by, initially, reducing the liquid assets vector. In effect, this describes a sacrificial effect where a solvent asset-rich bank is sacrificed freeing up liquid assets to rescue multiple weaker banks. While this, of course, is not a desirable phenomenon, it provides a novel tool for testing the effects of policy proposals that seek to distribute liquidity more evenly through a financial system.

As shown in this thesis, significant advances and promising directions of further inquiry maintain systemic risk as a vibrant area of research.

Bibliography

Allen, F. & Gale, D. (2000). Financial contagion. *Journal of Political Economy* **108**, 1–33.

Amini, H., Cont, R. & Minca, A. (2016). Resilience to contagion in financial networks. *Mathematical Finance* **26**, 329–365.

Amini, H., Filipović, D. & Minca, A. (2015). Systemic risk and central clearing counterparty design. *Swiss Finance Institute Research Paper* **13**.

Banerjee, T., Bernstein, A. & Feinstein, Z. (2018). Dynamic clearing and contagion in financial networks. Preprint, Available at arXiv: arXiv:1801.02091.

Barucca, P., Bardoscia, M., Caccioli, F., D'Errico, M., Visentin, G., Battiston, S. & Caldarelli, G. (2016). Network valuation in financial systems. Preprint, Available at SSRN: http://ssrn.com/abstract=2795583.

Battiston, S., Delli Gatti, D., Gallegati, M., Greenwald, B. & Stiglitz, J. E. (2012). Liaisons dangereuses: Increasing connectivity, risk sharing, and systemic risk. *Journal of Economic Dynamics and Control* **36**, 1121–1141.

Bisias, D., Flood, M., Lo, A. W. & Valavanis, S. (2012). A survey of systemic risk analytics. *Annual Review of Financial Economics* **4**, 255–296.

Capponi, A. & Chen, P.-C. (2015). Systemic risk mitigation in financial networks. *Journal of Economic Dynamics and Control* **58**, 152–166.

Capponi, A., Cheng, W. A. & Rajan, S. (2015). Enterprise value dynamics in centrally cleared markets. Preprint, Available at SSRN: http://ssrn.com/abstract=2542684.

Chen, N., Liu, X. & Yao, D. D. (2016). An optimization view of financial systemic risk modeling: Network effect and market liquidity effect. *Operations Research* **64**, 1089–1108.

Cifuentes, R., Ferrucci, G. & Shin, H. S. (2005). Liquidity risk and contagion. *Journal of the European Economic Association* **3**, 556–566.

Cont, R. & Kokholm, T. (2014). Central clearing of otc derivatives: bilateral vs multilateral netting. *Statistics & Risk Modeling* **31**, 3–22.

Cont, R., Moussa, A. & Santos, E. B. (2013). Network structure and systemic risk in banking systems. In *Handbook of Systemic Risk* (eds. J.-P. Fouque & J. A. Langsam), Cambridge University Press.

David, A. & Lehar, A. (2017). Imperfect renegotiations in interbank financial networks. Forthcoming, Available at SSRN: http://ssrn.com/abstract=1108870.

Duffie, D., Scheicher, M. & Vuillemey, G. (2015). Central clearing and collateral demand. *Journal of Financial Economics* **116**, 237–256.

Eisenberg, L. & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science* **47**, 236–249.

El Bitar, K., Kabanov, Y. M. & Mokbel, R. (2017). On uniqueness of clearing vectors reducing the systemic risk. *[Informatics and its Applications]* **11**, 109–118.

Elliott, M., Golub, B. & Jackson, M. O. (2014). Financial networks and contagion. *American Economic Review* **104**, 3115–3153.

Elsinger, H. (2011). Financial networks, cross holdings, and limited liability. Oesterreichische Nationalbank Working Paper 156.

Elsinger, H., Lehar, A. & Summer, M. (2006). Risk assessment for banking systems. *Management Science* **52**, 1301–1314.

Elsinger, H., Lehar, A. & Summer, M. (2013). Network models and systemic risk assessment. In *Handbook on Systemic Risk* (eds. J.-P. Fouque & J. A. Langsam), Cambridge University Press.

Ferrara, G., Langfield, S., Liu, Z. & Ota, T. (2016). Systemic illiquidity in the interbank network. Tech. Rep. 586, Bank of England.

Gai, P., Haldane, A. & Kapadia, S. (2011). Complexity, concentration and contagion. *Journal of Monetary Economics* **58**, 453–470.

Gandy, A. & Veraart, L. A. M. (2017). A Bayesian methodology for systemic risk assessment in financial networks. *Management Science* **63**, 4428 – 4446.

Gandy, A. & Veraart, L. A. M. (2018). Adjustable network reconstruction with applications to CDS exposures. In Press, Accepted Manuscript.

Georg, C.-P. (2013). The effect of the interbank network structure on contagion and common shocks. *Journal of Banking & Finance* **37**, 2216–2228.

Glasserman, P. & Young, H. P. (2014). How likely is contagion in financial networks? *Journal of Banking & Finance* **50**, 383–399.

Glasserman, P. & Young, H. P. (2016). Contagion in financial networks. *Journal of Economic Literature* **54**, 779–831.

Horn, R. A. & Johnson, C. R. (1991). *Topics in Matrix Analysis*. Cambridge University Press, Cambridge.

Horn, R. A. & Johnson, C. R. (2012). *Matrix analysis*. 2nd ed., Cambridge university press.

Hurd, T. (2016). *Contagion! The Spread of Systemic Risk in Financial Networks*. Springer.

Jech, T. (2013). *Set theory*. Springer Science & Business Media.

Karlin, S. & Taylor, H. E. (1981). *A Second Course in Stochastic Processes*. Elsevier.

Kusnetsov, M. & Veraart, L. A. M. (2018). Interbank clearing in financial networks with multiple maturities. Available at SSRN: https://ssrn.com/abstract=3161571.

Langfield, S., Liu, Z. & Ota, T. (2014). Mapping the UK interbank system. *Journal of Banking & Finance* **45**, 288–303.

Liu, M., Feng, M. & Staum, J. (2012). Systemic risk components in a network model of contagion. *IIE Transactions* **48**, 501–510.

Liu, M. & Staum, J. (2010). Sensitivity analysis of the Eisenberg–Noe model of contagion. *Operations*

Research Letters **38**, 489–491.

Newman, M. E. J. (2010). *Networks : an introduction*. Oxford University Press.

Rogers, L. C. G. & Veraart, L. A. M. (2013). Failure and rescue in an interbank network. *Management Science* **59**, 882–898.

Roukny, T., Battiston, S. & Stiglitz, J. E. (2018). Interconnectedness as a source of uncertainty in systemic risk. *Journal of Financial Stability* 93–106.

Tarski, A. et al. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics* **5**, 285–309.

The Insolvency Service (2010). *The Insolvency Service Technical Manual*. UK Government. Available at: https://www.insolvencydirect.bis.gov.uk/technicalmanual/Ch25-36/Chapter

Upper, C. (2011). Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability* **7**, 111–125.

Veraart, L. A. M. (2017). Distress and default contagion in financial networks. Available at SSRN: https://ssrn.com/abstract=3029702.

Weber, S. & Weske, K. (2017). The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk* **2**, 9.