On the Running Maximum of Brownian Motion and Associated Lookback Options

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I would like to dedicate this thesis to my loving parents and wife, Yujing.

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Abstract

The running maximum of Brownian motion appears often in mathematical finance. In derivatives pricing, it is used in modelling derivatives with lookback or barrier hitting features. For path dependent derivatives, valuation and risk management rely on Monte Carlo simulation. However, discretization schemes are often biased in estimating the running maximum and barrier hitting time. For example, it is hard to know if the underlying asset has crossed the barrier between two discrete time points when the simulated asset prices are on one side of the barrier but very close.

We apply several martingale methods, such as optional stopping and change of measure, also known as importance sampling including exponential tilting, on simulating the stopping times, and positions in some case, of the running maximum of Brownian motion. This results in more accurate and computationally cheap Monte Carlo simulations.

In the linear deterministic barrier case, close-form distribution functions are obtained from integral transforms. The stopping time and position can hence be simulated exactly and efficiently by acceptance-rejection method. Examples in derivative pricing are constructed by using the stopping time as a trigger event. A differential equation method is developed in parallel to solve for the Laplace transform and has the potential to be extended to other barriers.

In the compound Poisson barrier case, we can reduce the variance and bias of the crossing probabilities simulated by different importance sampling methods. We have also addressed the problem of heavy skewness when applying importance sampling.

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Chapter 1

Introduction

Brownian motion has been used in derivative pricing since Bachelier [1900] and then Black and Scholes [1973] and Merton [1973]. The running maximum of Brownian motion is used in modelling derivatives with lookback and barrier features such as Goldman et al. [1979] and Garman [1989].

Since being introduced to the pricing of options by Boyle [1977], Monte Carlo simulation has been widely used in the valuation of options with complicated features, which would make them difficult to value through a straightforward Black-Scholes-style or lattice based computation. The technique is thus widely used in valuing path dependent structures like lookback- and barrier options, see Glasserman [2003] and Chan and Wong [2015]. Although discretization schemes for generating the sampling paths are convenient to use, the truncation often creates bias for barrier hitting problems. Another choice is to simulate the barrier hitting time directly. For more complicated distributions, rejection sampling can be applied to simulate from the density function exactly, see Von Neumann [1951], Devroye [1986] and Asmussen and Glynn [2007].

Importance sampling, which is closely related to change of measure, is one of the common variance reduction techniques in derivatives pricing. A classical example is to price deeply out-of-money options in Reider [1993]. The idea behind importance sampling is to concentrate on sample paths that contribute most to estimating the required quantity. For instance, the crossing probability for a far-away barrier will be small since most sample paths have not crossed the barrier, so simulating more sample paths that cross the barrier should reduce the estimation variance. Improving simulation of such rare events by importance sampling can be found in Asmussen et al. [1985].

In this research, we demonstrate acceptance-rejection method and importance sampling on several barrier hitting problems regarding to the running maximum of a Brownian motion.

1.1 Research Objectives

This research had two main parts. The first was to develop exact simulation method for the first hitting time, and position, for the running maximum of a Brownian motion by a linear time-dependent barrier. The second was to reduce the simulation bias, and variance, of long horizon problems by a compound Poisson barrier.

We wanted to improve the simulation accuracy of some first hitting time of the running maximum of a Brownian motion as discretization schemes were biased. One way to achieve improvement was to know the close-form distribution function and simulate from it. In the case of linear time-dependent barrier, we solved the analytical distribution functions. Acceptance-rejection method then provided an exact way to simulate from the density function.

In some other cases when the close-form solutions were not available due to the complexity of the problems, we wanted to reduce the bias and variance. Although it was still not exact like the rejection method, importance sampling improved the simulation in specific problems. We demonstrated through the infinite horizon problems of the compound Poisson barrier since it involved truncation in simple discretization schemes.

1.2 Contributions

This thesis presents a number of original contributions in the field of stochastic processes and mathematical finance. These include:

- A martingale for the running maximum of Brownian motion. Close-form distribution functions, including Laplace transform, probability density function and cumulative distribution function, of the first hitting time of the running maximum of a Brownian motion by a linear time-dependent barrier (Chapter 3).

- Close-form distribution functions including double Laplace transform and joint probability density function of the first hitting time and position of the associated reflected Brownian motion. The infinite-horizon probability of the linear time-dependent barrier hitting problem. An acceptance rejection algorithm to simulate the first hitting time and position of the associated reflected Brownian motion exactly (Chapter 3).

- An alternative differential equation approach for the Laplace transform of the linear time-dependent barrier hitting problem (Chapter 4).

- Importance sampling methods for the infinite horizon problem of the compound Poisson barrier hitting problem (Chapter 5).

We give more details on these contributions in the following subsections.

1.2.1 First hitting time by a linear time-dependent barrier

Comparing the level of the running maximum of a Brownian motion and the time-dependent linear barrier can be seen as comparing the running maximum of the return of the underlying asset and a deterministic interest rate. Therefore we can look at derivatives comparing a stock price modelled by a geometric Brownian motion and the present value of a zero coupon bond. The martingale obtained is a function of the current level and the maximum drawdown level so it allows for further pricing problems. The distribution functions are immediate results that lead to the results in the next chapter.

1.2.2 Joint density function and rejection sampling

To price the derivatives described in the previous section, we also need the stopping position of the underlying Brownian motion. Even in a binary payoff structure, the stopped Brownian motion appears in the change of measure. Hence, we solve for the joint distribution function and develop an acceptance-rejection algorithm.

1.2.3 A differential equation approach

The Laplace transform of the first hitting time in the linear barrier problem can also be solved from an ordinary differential equation obtained from the stopped martingale. We hope this alternative approach provides insights for further research with the martingale obtained in the previous chapter.

1.2.4 Importance sampling for infinite horizon

Since the running maximum of a drifted Brownian motion does not hit a compound Poisson barrier for certain, the naive simulation method of such an infinite horizon problem involves a truncation and is biased. We present two importance sampling methods for this problems. Under the new measures, the barrier crossing happens for sure. Also, we do not need to wait for a long time for the boundary crossing to happen in simulations. The importance sampling methods reduce both the bias and variance.

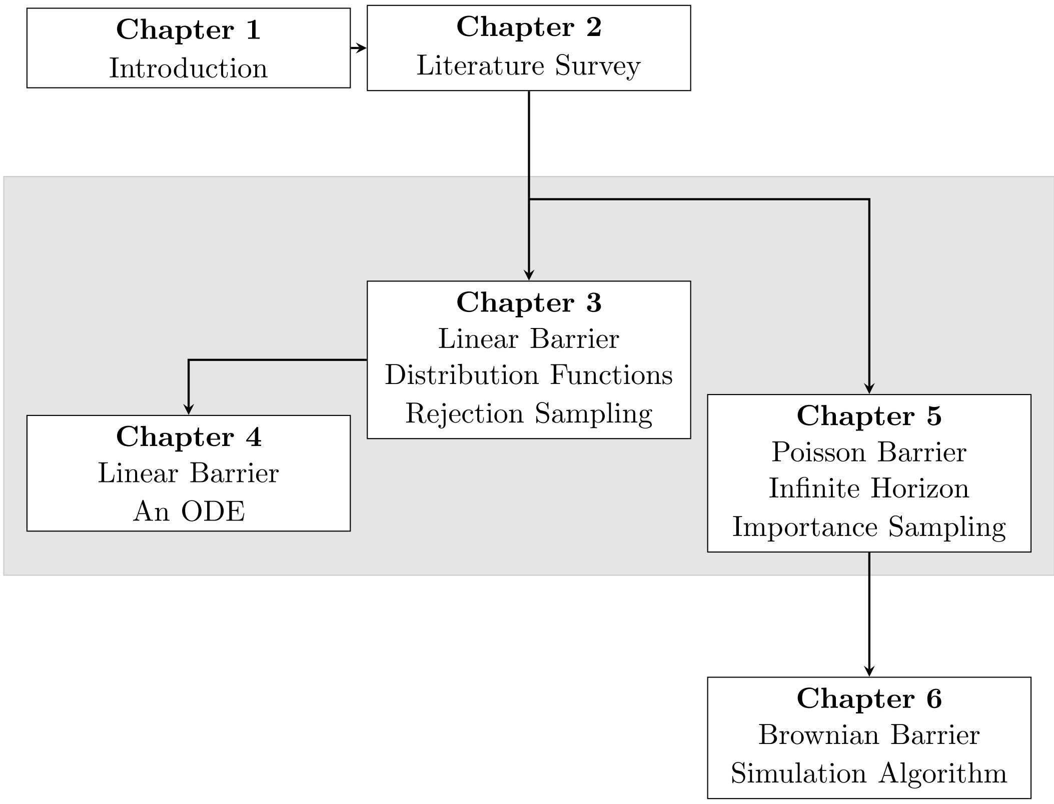

Figure 1.1: Organization of this thesis

1.3 Organization of This Thesis

This document is organized as in Figure 1.1. The body of this thesis consists of the four chapters (in the shaded box in the figure) introduced in the previous section (Section 1.2). The contents of each chapter is as follows:

Chapter 2 Literature Survey. Several simulation methods are introduced for Brownian motion related quantities. The main tools are acceptance-rejection method and importance sampling. First we describe how to simulate some well-known Brownian motion related quantities. Second we demonstrate the application of rejection sampling on more complicated density functions.

Chapter 3 Linear Barrier Stopping time. We start from the infinitesimal generator of a Brownian motion and a reflected Brownian motion. A martingale is obtained from the generator. By applying Optional Stopping Theorem and Fourier sine/cosine transform, the Laplace transform is obtained. By inverting the Laplace transform, we solve the uni-variate probability density function and hence the cumulative distribution function. As the double Laplace transform of the stopping time and the stopped reflected Brownian motion is a function of the uni-variate Laplace transform, plugging the solution from the previous chapter allows us to solve it. Inverting the double Laplace transform gives the joint density function. An iterative discretization method is presented to validate the density function. An acceptance-rejection algorithm is developed to simulate from joint density function exactly. Example applications in derivative pricing are given.

Chapter 4 An Ordinary Differential Equation Approach. We provide an alternative approach for the linear barrier problem. After applying Optional Stopping Theorem to the martingale obtained in chapter 3, we are able rewrite the expectation to an ordinary differential equation. Solving the initial value problem gives the Laplace transform of the stopping time.

Chapter 5 Poisson Barrier Infinite Horizon. Since the infinite horizon problem involves truncation, we provide two importance sampling methods to improve the simulation. First we use exponential tilting. Second we use a more general compound Poisson martingale. We have stressed the problem of extreme skewness when applying importance sampling.

Chapter 6 Brownian Motion Barrier Algorithm. We provide an algorithm for the Brownian motion barrier problem for completeness. Under the discretization scheme, for each time step, the problem is reduced to a first hitting time problem of a Brownian motion. There are three cases for positive, negative and zero drift Brownian barrier.

Chapter 2

A Survey on Simulation Methods

2.1 Introduction

This section first introduces the main simulation techniques used in this thesis. Inverse transform is used for simpler distributions. Acceptance-rejection method is used in the linear time-dependent barrier hitting problem. Importance sampling is used in the compound Poisson barrier hitting problem. We also discuss the potential issues when applying importance sampling. Second we give examples on how to simulate quantities related to Brownian motion. Most examples apply acceptance-rejection method.

2.2 Simulation Methods

In this section, we describe two basic simulation algorithms to generate continuous random variables: inverse transform and acceptance-rejection method. They are well discussed in standard textbooks, e.g. Chan and Wong [2015], Devroye [1986], Glasserman [2003], etc.

2.2.1 Inverse Transform

When the inverse of the cumulative distribution function can be computed, inverse transform can be used to generate from a uniform random variable. The inverse transform algorithm is based on the following classical theorem.

Theorem 1. *Let U be a uniform (0, 1) random variable. For any continuous distribution function F , the random variable X defined by $X = F^{-1}(U)$ has distribution F . Here*

$$F^{-1}(U) = \inf\{x : F(x) \geq U\}$$

.

Proof. Let F denote the distribution of $X = F^{-1}(U)$. Then

$$F(x) = Pr(X \leq x) = Pr(F^{-1}(U) \leq x) = Pr(U \leq F(x)) = F(x).$$

□

Based on the theorem, we can generate the random variable if F^{-1} can be computed.

Algorithm 1 Inverse Transform Algorithm

1: Generate $U \sim U(0, 1)$
2: Set $X = F^{-1}(U)$

2.2.2 Acceptance-Rejection Method

In many cases, the inverse of the cumulative distribution function is not easy to compute. A simple example will be the normal cdf.

Acceptance-rejection methods can be applied to simulate the random variable X from a density $f(x)$ by generating another random variable Y from g and then accept with probability proportional to $f(Y)/g(Y)$. Let c be such that

$$c \geq \frac{f(y)}{g(y)} \quad \forall y$$

.

Theorem 2. *The random variable generated by the acceptance-rejection method has density f. Moreover, the number of iterations this algorithm needs is a geometric random variable with mean c.*

Algorithm 2 Acceptance-rejection Algorithm

1: Generate $Y \sim g$
2: Generate $U \sim U(0, 1)$
3: **if** $U \leq \frac{f(y)}{cg(y)}$ **then** return $X = Y$ **else** go to 1

We want to prove $Pr(Y < y|U \leq h(Y)) = F(Y)$ and $Pr(U \leq h(Y)) = 1/c$.

Proof. We present the proof as in Chan and Wong [2015]. Let $f(x) = cg(x)h(x)$, where $c \geq 1$ is a constant, $g(x)$ is also a pdf and $0 < h(x) \leq 1$. Let Y has a pdf g and $U \sim U(0, 1)$. By Bayes' Theorem,

$$f_Y(x|U \leq h(Y)) = \frac{Pr(U \leq h(Y)|Y = x)g(x)}{Pr(U \leq h(Y))}$$

and

$$Pr(U \leq h(Y)|Y = x) = Pr(U \leq h(x)) = h(x)$$

and also,

$$\begin{aligned} Pr(U \leq h(Y)) =& \int_X Pr(U \leq h(Y)|Y = x)g(x)dx \\ =& \int_X h(x)g(x)dx \\ =& \int_X \frac{f(x)}{c}dx \\ =& 1/c. \end{aligned}$$

Finally, $f_Y(x|U \leq h(Y)) = ch(x)g(x) = f(x)$. □

The art of using acceptance-rejection methods is to find a good envelop density g such that c is minimized so the random variable is easy to sample from g.

2.2.3 Importance Sampling

To improve the simulation efficiency, importance sampling can be used. The method is described in Glasserman [2003], Chan and Wong [2015], etc., as a variance reduction method. It is similar in idea to the acceptance-rejection method in

Section 2.2.2 and attempts to reduce variance by changing the probability measure. The main idea lies in simulating at places where the quantity of interest carries the most information, hence the name importance sampling.

Suppose we are interested in simulating the following expectation

$$\theta = \mathbb{E}[h(\mathbf{X})] = \int h(\mathbf{x})f(\mathbf{x})d\mathbf{x}$$

where $\mathbf{X} = (X_1, ..., X_n)$ is an n-dimensional random vector with a joint density function $f(\mathbf{x}) = f(x_1, ..., x_n)$. A direct simulation is inefficient or not possible. The inefficiency comes from difficulties in simulating $\mathbf{X}$ or the variance of $h(\mathbf{x})$ is too large, or both. One example is the infinite horizon problem described above.

Suppose now we have another density $g(\mathbf{x})$, which is not hard to simulate and match the support (i.e. $f(\mathbf{x}) = 0$ whenever $g(\mathbf{x}) = 0$). Then θ can be estimated by

$$\begin{aligned}\theta =& \mathbb{E}[h(\mathbf{X})] \\ =& \int \frac{h(\mathbf{x})f(\mathbf{x})}{g(\mathbf{x})} g(\mathbf{x})d\mathbf{x} \\ =& \mathbb{E}^g \left[\frac{h(\mathbf{x})f(\mathbf{x})}{g(\mathbf{x})}\right]\end{aligned}$$

where $\mathbb{E}^g$ denotes the expectation of $\mathbf{X} \sim g$. Therefore, the Monte Carlo estimator of θ can be computed by

$$\hat{\theta} = \frac{1}{N}\sum_{i=1}^{N} \frac{h(\mathbf{X}_i)f(\mathbf{X}_i)}{g(\mathbf{X}_i)}$$

The art of importance sampling lies in how to choose g such that $\frac{h(\mathbf{X})f(\mathbf{X})}{g(\mathbf{X})}$ has a smaller variance. It can result in a large increase in variance if the change of measure is not chosen carefully.

Variance for probabilities estimation. We consider estimating by simulation the probability $p = Pr(X \in S)$ with importance sampling. The importance sampling estimator is given by:

$$\hat{p} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{\mathbf{X}_i \in S\}\frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)}$$

where $\mathbf{X}_i \sim g$.

$$\begin{aligned}
Var(\hat{p}) =& \frac{1}{N} Var^g \left(\mathbf{1}\{\mathbf{X}_i \in S\}\frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)} \right) \\
=& \frac{1}{N}\left[\mathbb{E}^g \left(\mathbf{1}\{\mathbf{X}_i \in S\}^2 \frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)}^2 \right) - p^2 \right] \\
=& \frac{1}{N}\left[\mathbb{E} \left(\mathbf{1}\{\mathbf{X}_i \in S\} \frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)} \right) - p^2 \right]
\end{aligned}$$

The last step uses the properties of indicator function and importance sampling. One can choose a density function g such that it minimizes the above variance accordingly. In practice, we often do not know the expectation and p. A naive work-around would be replacing them by simulation estimators and repeating the procedure.

Skewness for probabilities estimation. The long horizon problem which skews the distribution is less straight forward and discussed in Glasserman [2003] and Glynn and L'ecuyer [1995]. Suppose we want to simulate the event $\mathbf{1}\{\mathbf{X} \in S\}$ for $S \subseteq \mathbb{R}^n$ under the original measure ($\mathbf{X} \sim f$). Further if $Pr(\mathbf{X} \in S) = 1$ under the new measure ($\mathbf{X} \sim g$):

$$\mathbb{E}\left(\mathbf{1}\{\mathbf{X} \in S\}\right) = \mathbb{E}^g\left(\frac{f(\mathbf{X})}{g(\mathbf{X})}\right) \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)}$$

Suppose

$$\mathbb{E}^g\left(|\ln(f(\mathbf{X})/g(\mathbf{X}))|\right) < \infty$$

By the law of large numbers, there exists a constant c such that

$$\frac{1}{N}\sum_{i=1}^{N} \ln\frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)} \to \mathbb{E}^g\left(\ln\frac{f(\mathbf{X})}{g(\mathbf{X})}\right) = c$$

By Jensen's inequality, since ln is strictly concave,

$$c \leq \ln\left[\mathbb{E}^g\left(\frac{f(\mathbf{X})}{g(\mathbf{X})}\right)\right] = 0$$

$c = 0$ unless $f = g$. Therefore,

$$\sum_{i=1}^{N} \ln\frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)} \to -\infty$$

exponentiating,

$$\prod_{i=1}^{N} \frac{f(\mathbf{X}_i)}{g(\mathbf{X}_i)} \to 0$$

Therefore, the likelihood ratio goes to 0 but its expectation equals 1 for all N. The likelihood ratio becomes highly skewed, taking increasingly large values with small but non-negligible probability.

In particular, if we want to simulate some stopping time $\mathbf{1}\{\tau < \infty\}$ but

$$\ln\frac{f(t)}{g(t)} \propto -t$$

The likelihood ratio, or Radon-Nikodym derivative, will be skewed if the first hitting time is very large hence the name "Long Horizon Problem".

2.3 Reviews on Simulation Methods related to Brownian Motion

2.3.1 Brownian motion: Extremes and locations of extremes

For a standard Brownian motion W on [0,1], the marginal distributions of the running maximum $M = \sup_{0\leq t\leq 1} W_t$ and its location H for W on $[0,1]$ are well-known (see, e.g., Karatzas and Shreve [1991]). $M \overset{\mathcal{L}}{=} |N|$, where N is a standard normal r.v. and H is arc sine distributed,

$$f_H(x) = \frac{1}{\pi\sqrt{(1-x)x}}, \quad x \in [0,1]$$

The arc-sine, or beta(1/2,1/2) distribution, can be represented equivalently in various forms. Denote C, C' are independent standard Cauchy random variables, N and N' are independent standard normal random variables and G_a and G'_a are independent gamma random variables of shape parameter $a > 0$ and unit scale parameters, the following expressions are equivalent in law (see Devroye [2010]).

$$\frac{G_{1/2}}{G_{1/2}+G'_{1/2}} \overset{\mathcal{L}}{=} \frac{N^2}{N^2+N'^2} \overset{\mathcal{L}}{=} \frac{1}{1+C^2} \overset{\mathcal{L}}{=} \sin^2(2\pi U) \overset{\mathcal{L}}{=} \sin^2(\pi U)$$

$$\overset{\mathcal{L}}{=} \sin^2(\frac{\pi U}{2}) \overset{\mathcal{L}}{=} \frac{1+\cos(2\pi U)}{2} \overset{\mathcal{L}}{=} \frac{1+\cos(\pi U)}{2}$$

In applications, M is rarely needed on its own. It is usually simulated jointly with other processes. For example, M is often generated with W. The distribution function of M conditional on $W = r$ is

$$F(m) = 1 - \exp(\frac{1}{2}(r^2 - (2m-r)^2), \quad m \geq \max(r, 0)$$

Inverting the CDF, with $E \sim \exp(1)$, we have

$$M|W_1 \overset{\mathcal{L}}{=} \frac{1}{2}(r + \sqrt{r^2+2E})$$

This was used by many, such as McLeish [2002], in simulations. Therefore, replacing r by N, the joint law is obtained

$$(M, W_1) \stackrel{\mathcal{L}}{=} (\frac{1}{2}(N + \sqrt{N^2 + 2E}), N)$$

Based on this result, we can simulate the path of the running maximum of Brownian motion from 0 to T with N discrete points.

Algorithm 3 Simulation Algorithm of the running maximum path of a BM

1: Set $j = 0$, $t_0 = 0$, $t_j = jT/N$ and $M_0 = W_0 = 0$
2: Generate $W_{j+1} = W_j + \sqrt{\Delta t}Z$, where $Z \sim N(0, 1)$, set $\Delta W = W_{j+1} - W_j$
3: Generate the local maximum $M_{j+1}^{loc} = \frac{\sqrt{(\Delta W)^2 + 2\Delta tE}}{2} + W_j$, where $E \sim \exp(1)$
4: Set the global maximum $M_{j+1} = \max(M_j, M_{j+1}^{loc})$
5: Repeat until t_N

A path is generated below using 100 discrete time points.

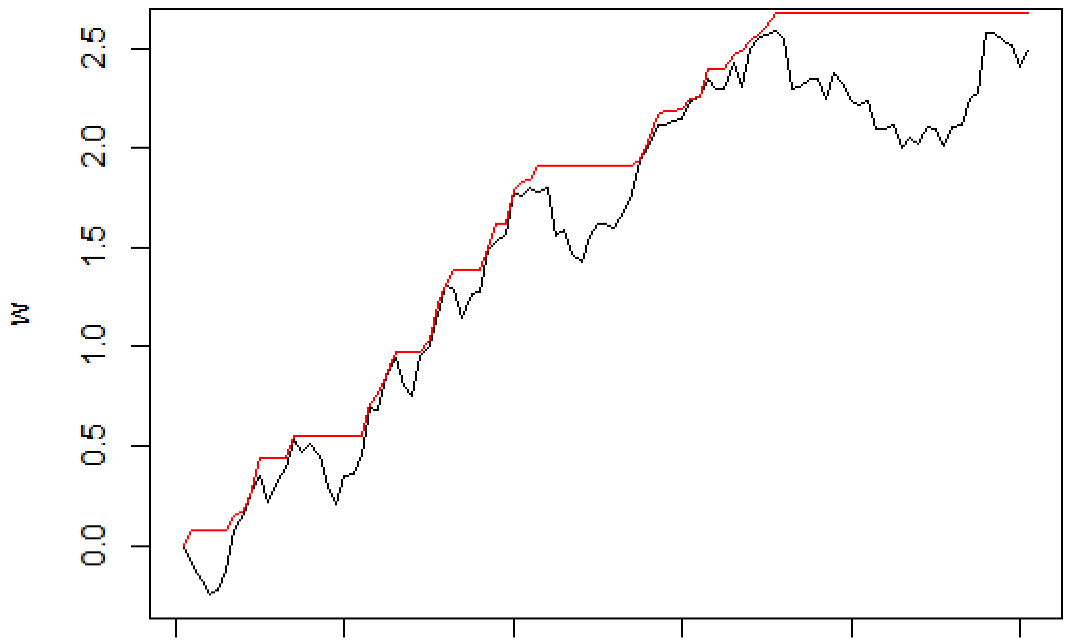

Figure 2.1: A sample path of a Brownian motion and its running maximum

A small difference of the running maximum (the red line) and the running maximum of the discretized Brownian motion (the black line) is observed from the graph. In fact, the running maximum from the discretization of the Brownian motion path is always below the exact running maximum and hence biased.

When $r = 0$, we have the classical result $M \stackrel{\mathcal{L}}{=} \sqrt{E/2}$ from Lévy [1940]. Also,

$$M \stackrel{\mathcal{L}}{=} |N| \stackrel{\mathcal{L}}{=} M - W_1$$

The rightmost result is simply due to Lévy, the process $M_t - W_t$ is a reflected Brownian motion. For $x > 0$, we define the first passage time

$$T_x = \min\{t : W_t = x\}$$

Simulating hitting times and maxima are in fact equivalent computational questions. For $t > 0$,

$$T_x \stackrel{\mathcal{L}}{=} (\frac{x}{M})^2$$

Now consider the joint density of the triple (H, M, W_1), Karatzas and Shreve [1991] showed that the joint density is

$$f(x, m, y) = \frac{m(m-y)}{\pi x^{3/2}(1-x)^{3/2}} \exp(-\frac{m^2}{2x} - \frac{(m-y)^2}{2(1-x)}), \quad m \geq y \in \mathbb{R}, \quad x \in (0,1)$$

Therefore, we can simulate them jointly

$$(H, M, W_1) \stackrel{\mathcal{L}}{=} (H = \frac{1 + cos(2\pi U)}{2}, \sqrt{2HE}, \sqrt{2HE} - \sqrt{2(1-H)E'})$$

An illustration graph from Devroye [2010], X is the location of the maximum M_r from 0 to $B(1) = r$:

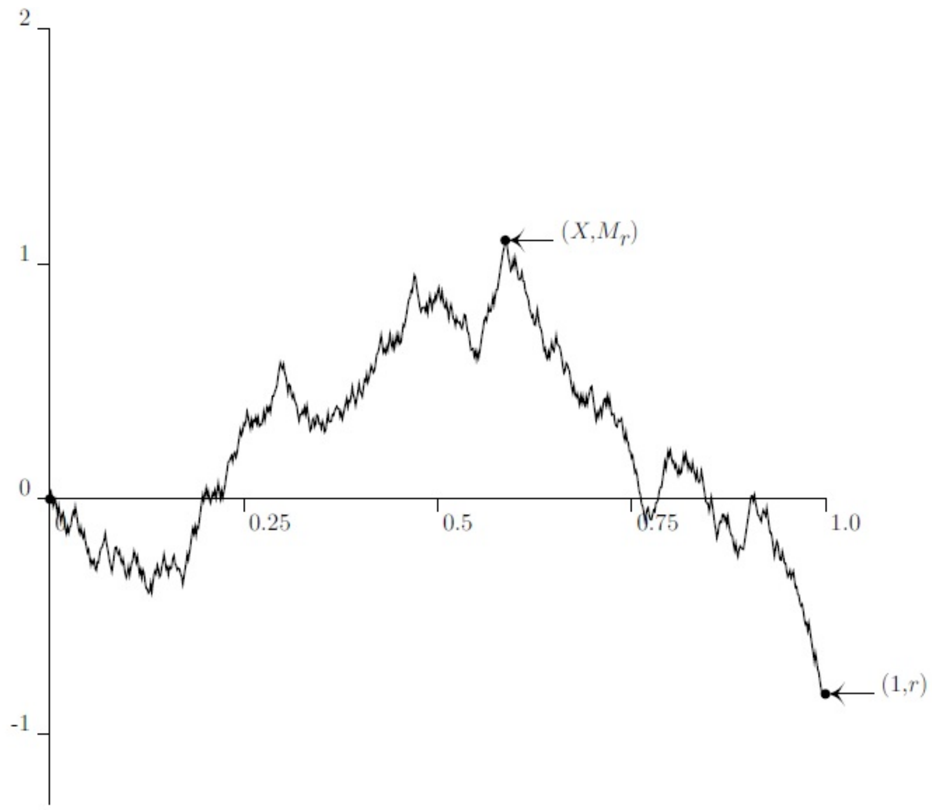

Figure 2.2: A simulation of a Brownian bridge.

2.3.2 Brownian motion: First Exit Time

We define the first exit time of a standard Brownian motion W_t

$$\tau_\delta = \inf\{t > 0; |W_t| > \delta\}$$

For simplicity, we consider $\delta = 1$ here and $\tau := \tau_1$. (For other values, the problem can be reduced to $\delta = 1$.)
From Borodin and Salminen [2002], the Laplace transform of τ

$$\mathrm{E}(\mathrm{e}^{-\lambda\tau}) = \frac{1}{\cos\sqrt{2\lambda}}$$

Inverting the Laplace transform gives the probability density

$$f(t) = \sum_{k=-\infty}^{\infty} (-1)^k g_{(1+2k)}(t), t \geq 0,$$

where

$$g_y(t) = \frac{y}{\sqrt{2\pi t^3}} \exp\{-y^2/2t\}, t \geq 0.$$

A description of the shape of $f(t)$ is given in Burq and Jones [2008].

- The distribution f of τ is unimodal.
- Right tail asymptotics. For any $\epsilon > 0$ we have, for $\gamma = \pi^2/8$,

$$f(t) = o(\mathrm{e}^{-(\gamma-\epsilon)t}), \quad \text{as} \quad t \to \infty.$$

- Left tail asymptotics. For all $n \geq 1$

$$f(t) = o(t^n), \quad \text{as} \quad t \to 0.$$

Define $f_n(t) = \sum_{k=-n}^{n} (-1)^k g_{(1+2k)}(t)$ as a truncation of $f(t)$. The sequence of remainders $\epsilon_n(t) = f(t) - f_n(t), n = 1, 2, ...$ oscillates about 0 for $n \geq N(t) := t(\log 3)/4 \vee 3$.

An acceptance-rejection algorithm is also developed in Burq and Jones [2008]. To complete the algorithm we need the sampling distribution g and constant $a \geq \sup_{t\geq 0} f(t)/g(t)$. The Gamma distribution is used.

$$g(t;\alpha,\lambda) = \frac{\lambda^\alpha t^{\alpha-1}\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}, t \geq 0$$

For any $\epsilon > 0$ we have

$$g(t) = o(\mathrm{e}^{-(\lambda-\epsilon)t}), \quad \text{as} \quad t \to \infty.$$

And for the left tail we have

$$g(t) = o(t^{\alpha-1}), \quad \text{as} \quad t \to 0.$$

As f and g are bounded on $[0,\infty)$, for any $\alpha \geq 0$ and $0 < \lambda < \gamma$, there exists an a such that $f \geq ag$ on $[0,\infty)$.

Numerical minimisation of $\sup_{t\geq 0} f(t)/g(t;\alpha,\lambda)$ subject to $\alpha \geq 0$ and $0 < \lambda < \gamma$ gives, to 6 decimal places, occurred at $t = 0.52495$,

$$a = 1.243707 \quad \text{for} \quad \alpha = 1.088870 \quad \text{and} \quad \lambda = 1.233701.$$

Using the above set of parameters, a graph of $f(t)$ and $ag(t)$ is plotted below.

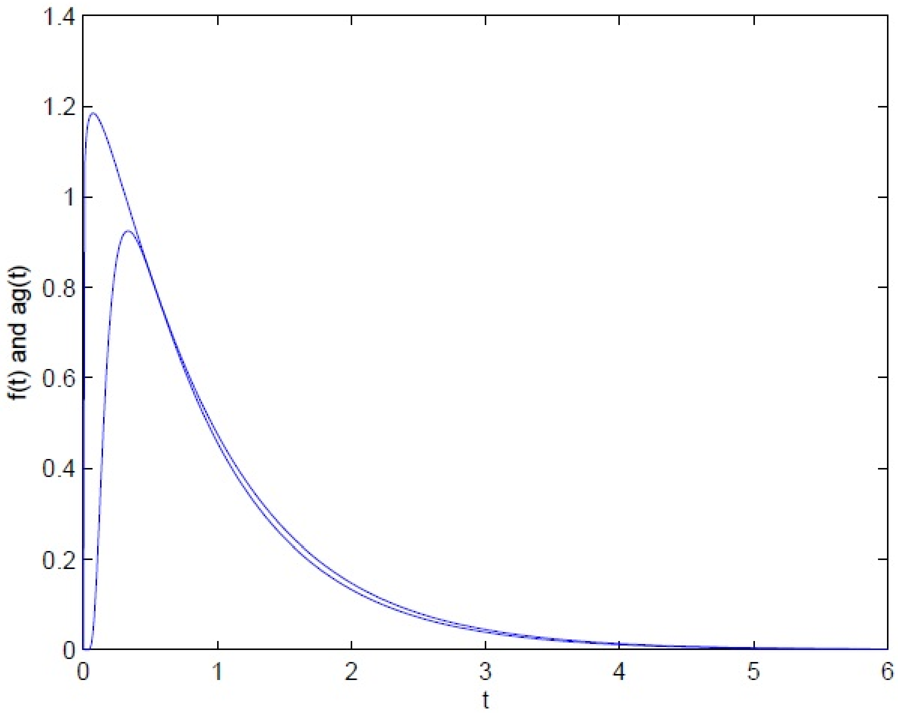

Figure 2.3: The density $f(t)$ of the first exit time τ and dominating curve $ag(t)$ where g is a Gamma(1.088870, 1.233701) density and a= 1.243707.

2.4 Further Examples on Supremum of Brownian Motion

2.4.1 Supremum of reflected standard Brownian motion

From Borodin and Salminen [2002], for $y \geq 0$, $t \geq 0$,

$$\Pr(\sup_{0\leq s\leq t}|W_s| \leq y) = \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\int_{-y}^{y}(\mathrm{e}^{-(z+4ky)^2/2t} - \mathrm{e}^{-(z+2y+4ky)^2/2t})\mathrm{d}z$$

Differentiate w.r.t. y, we obtained the density:

$$\begin{aligned}
f(y) = &\frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\Big[-(4k-1)\mathrm{e}^{-(4k-1)^2y^2/2t} + 2(4k+1)\mathrm{e}^{-(4k+1)^2y^2/2t} \\
&- (4k+3)\mathrm{e}^{-(4k+3)^2y^2/2t}\Big] \\
= &\frac{4\mathrm{e}^{-y^2/2t}}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\Big[(4k+1)\mathrm{e}^{-4k(2k+1)y^2/t}\Big]
\end{aligned}$$

We want to generate $\sup_{0\leq s\leq t}|W_s|$ by an acceptance-rejection method since the inversion of cdf is complicated. By recognizing the expression outside the summation sign, the density function of $|W_t|$ is considered as the envelope function,

$$g(y) = \frac{2\mathrm{e}^{-y^2/2t}}{\sqrt{2\pi t}}, \quad y > 0$$

Next we want to find a constant $c \geq \sup_{y\geq 0} f(y)/g(y)$.

$$\begin{aligned}
&P_x(\sup_{0\leq s\leq t}|W_s| \in dy) \\
\leq &P_x(\sup_{0\leq s\leq t}W_s \in dy, \inf_{0\leq s\leq t}W_s > -y) + P_x(\inf_{0\leq s\leq t}W_s \in dy, \sup_{0\leq s\leq t}W_s < y) \\
\leq &P_x(\sup_{0\leq s\leq t}W_s \in dy) + P_x(\inf_{0\leq s\leq t}W_s \in dy)
\end{aligned}$$

Therefore,

$$\begin{aligned} P_x(\sup_{0\leq s\leq t}|W_s| \in dy) \leq & \frac{2}{\sqrt{2\pi t}}\left[\mathrm{e}^{-\frac{(y-x)^2}{2t}} + \mathrm{e}^{-\frac{(y+x)^2}{2t}}\right] dy \\ =& 2P_x(|W_t| \in dy) \end{aligned}$$

Below is the likelihood ratio for $x = 0$ which is consistent with the result above (≤ 2).

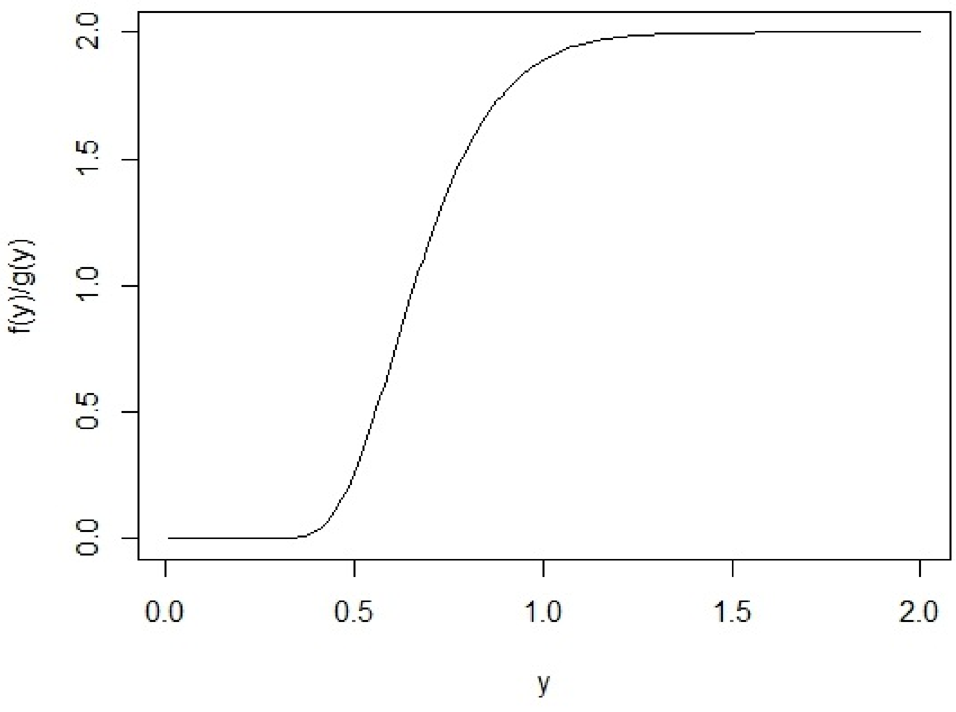

Figure 2.4: The likelihood ratio of pdf of supremum of reflected Brownian motion and reflected Brownian motion

5000 samples of the supremum of reflected Brownian motion are generated by the acceptance-rejection algorithm by R.

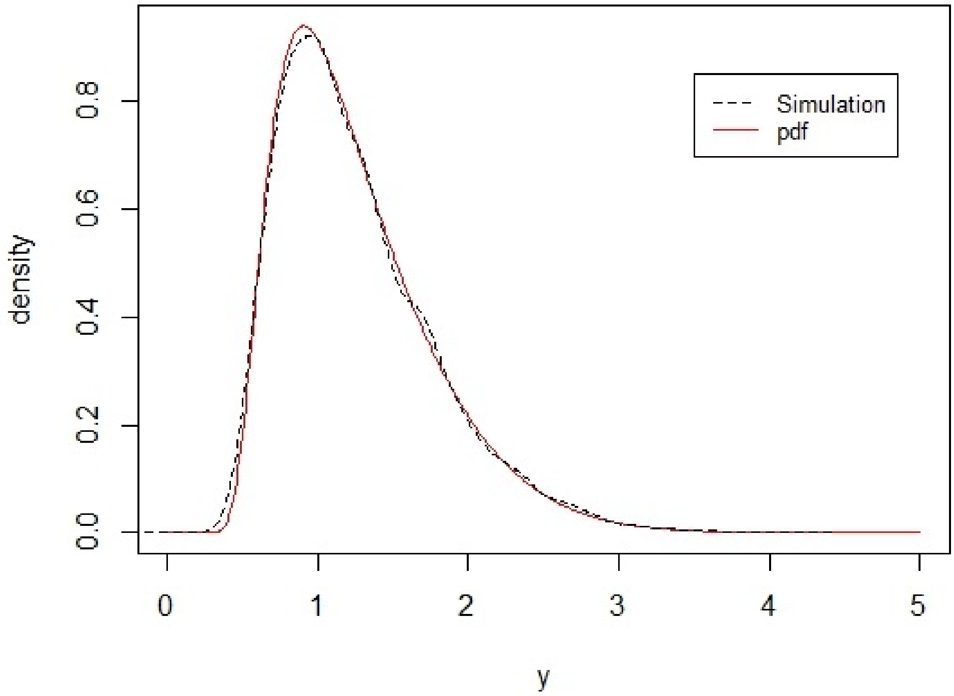

Figure 2.5: 5000 samples of supremum of reflected Brownian motion

2.4.2 Supremum of reflected Brownian motion and its location

The joint distributions of the running maximum M_t^* for $|W_t|$ and its location H^* on $[0,t]$ is (see Borodin and Salminen [2002])

$$P_x(M_t^* \in dy, H^* \in dv, |W_t| \in dz) = 2cc_v(x,y)cc_{t-v}(z,y)dvdydz$$

where

$$cc_v(x,y) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\sqrt{2\pi} v^{3/2}} \left[x + (2k+1)y\right] \mathrm{e}^{-(x+(2k+1)y)^2/2v}$$

Integrating z from 0 to y,

$$\int_0^y cc_{t-v}(z,y)dz = \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2\pi(t-v)}} \left[\mathrm{e}^{-\frac{((2k+2)y)^2}{2(t-v)}} - \mathrm{e}^{-\frac{((2k+1)y)^2}{2(t-v)}}\right]$$

The density of M_t^* and H_t^* on $[0,t]$ becomes

$$f(v,y) = 2cc_v(x,y)\int_0^y cc_{t-v}(z,y)dz, \quad v < t$$

Denote $\overline{M_t} = \sup_{0\le s\le t} W_s$, $\underline{M_t} = \inf_{0\le s\le t} W_s$ and $\overline{H}$ and $\underline{H}$ as the corresponding locations in $[0,t]$. Consider

$$\begin{aligned}
&P_x(M_t^* \in dy, H_t^* \in dv) \\
=&P_x(\overline{M_t} \in dy, \overline{H} \in dv, -\underline{M_t} < y) + P_x(-\underline{M_t} \in dy, \underline{H} \in dv, \overline{M_t} < y) \\
<&P_x(\overline{M_t} \in dy, \overline{H} \in dv) + P_x(-\underline{M_t} \in dy, \underline{H} \in dv) \\
=&\frac{dvdy}{\pi v^{3/2}(t-v)^{1/2}}\left[(y-x)\mathrm{e}^{-\frac{(y-x)^2}{2v}} - (y+x)\mathrm{e}^{-\frac{(y+x)^2}{2v}}\right]
\end{aligned}$$

To simplify the problem, we take $x=0$, then

$$\begin{aligned}
P(M_t^* \in dy, H_t^* \in dv) <&\frac{2y}{\pi v^{3/2}(t-v)^{1/2}}\mathrm{e}^{-\frac{y^2}{2v}}dvdy \\
=&2P(\overline{M_t} \in dy, \overline{H_t} \in dv)
\end{aligned}$$

Therefore, (M_t^*, H_t^*) can be generated with acceptance-rejection method by using the joint density function of $(\overline{H_t}, \overline{M_t})$ as the envelope function.

Below is the simulation by R and compared with envelope random variables.

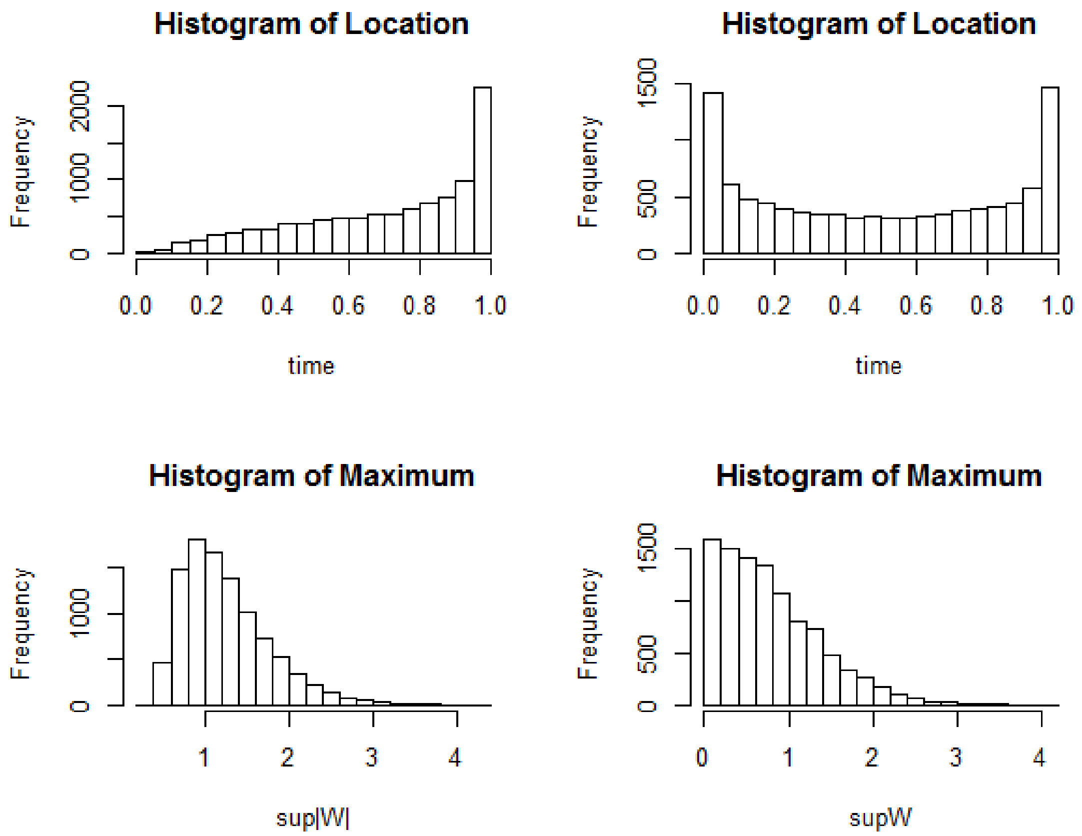

Figure 2.6: 10000 samples of supremum of reflected Brownian motion and locations.

2.4.3 Supremum of Brownian motion with drift

We consider the running maximum M_t^μ of a Brownian motion with drift $W_t + \mu t$ on $[0,1]$. From Borodin and Salminen [2002]

$$P(M_t^\mu < y) = \Phi\left(\frac{y-\mu t}{\sqrt{t}}\right) - \mathrm{e}^{2\mu y}\Phi\left(-\frac{y+\mu t}{\sqrt{t}}\right)$$

By putting $t = 1$ and differentiating w.r.t. y,

$$f(y) = P(M_1^\mu \in dy)/dy = 2\phi(y-\mu) - 2\mu\mathrm{e}^{2\mu y}\Phi(-y-\mu), \quad y \geq 0$$

where ϕ and Φ are the standard normal pdf and cdf respectively.

For different values of μ, the density behaves differently. This can be explained by intuition. If $\mu >> 0$, the running maximum will grow similarly with the underlying Brownian motion. On the other hand, if $\mu << 0$, the running maximum will not grow much.

For $\mu \geq 0$, $f(y) \leq 2\phi(y-\mu)$, therefore, the truncated normal is used as the envelope (see Robert [1995] for truncated normal simulation).

$$g_1(y) = \frac{\phi(y-\mu)}{1-\Phi(-\mu)}, \quad \text{and} \quad \frac{f(y)}{g_1(y)} \leq 2(1-\Phi(-\mu))$$

For $\mu < 0$, $f(y) = 2\phi(y-\mu) + \left[-2\mu\mathrm{e}^{-(-2\mu)y}\right]\Phi(-y-\mu)$ so consider an exponential distribution with $\lambda = -2\mu > 0$,

$$\frac{f(y)}{g_2(y)} = \Phi(-y-\mu) - \frac{\phi(y-\mu)}{\mu\mathrm{e}^{2\mu y}} \leq 1 - \frac{1}{\mu\sqrt{2\pi}}$$

However, the likelihood ratio becomes very large when $\mu \to 0^-$. We need to reconsider another envelope. Reconsider the truncated normal as the envelope.

$$\frac{f(y)}{(1-\Phi(-\mu))g_1(y)} = 2 - 2\mu\frac{\Phi(-\mu-y)}{\phi(y+\mu)}$$

Differentiating w.r.t. y, denote $\lambda(\mu) = 1 - \Phi(-\mu) > 0$

$$\frac{\partial}{\partial y}\left(\frac{f(y)}{\lambda(\mu)g_1(y)}\right) = -2\mu\left[\frac{(y+\mu)\Phi(-\mu-y)}{\phi(y+\mu)} - 1\right] < 0$$

as $x\Phi(x) + \phi(x) = \int_{-\infty}^{x}\Phi(z)dz > 0$. The likelihood ratio f/g_1 is decreasing in y. Therefore,

$$\frac{f(y)}{g_1(y)} \leq \frac{f(0)}{g_1(0)} = 2\left(1 - \frac{\Phi(-\mu)\mu}{\phi(\mu)}\right)(1 - \Phi(-\mu))$$

However, when $\mu << 0$, the likelihood ratio can become very large. Therefore, a combination of g_1 and g_2 is used in the algorithm.

If $\mu > 0$, g_1 is used. If $\mu < 0$, the upper bounds of likelihood ratios f/g_1 and f/g_2 are calculated and the smaller one is used. For example, if $\mu = -.5$, $f/g_1 \leq 1.223048$ and $f/g_2 \leq 1.797885$, g_1 is used for the smaller bound. Some value of μ (slightly less than -0.5) can be solved numerically when the bounds are equal.

Note that all bounds above are smaller than 2. If the running maximum is generated with the terminal value of the Brownian motion, 2 random variables are generated. However, if only the running maximum is needed, the above acceptance-rejection method is more efficient.

The histograms of simulation results are plotted below. Red lines are the pdf functions.

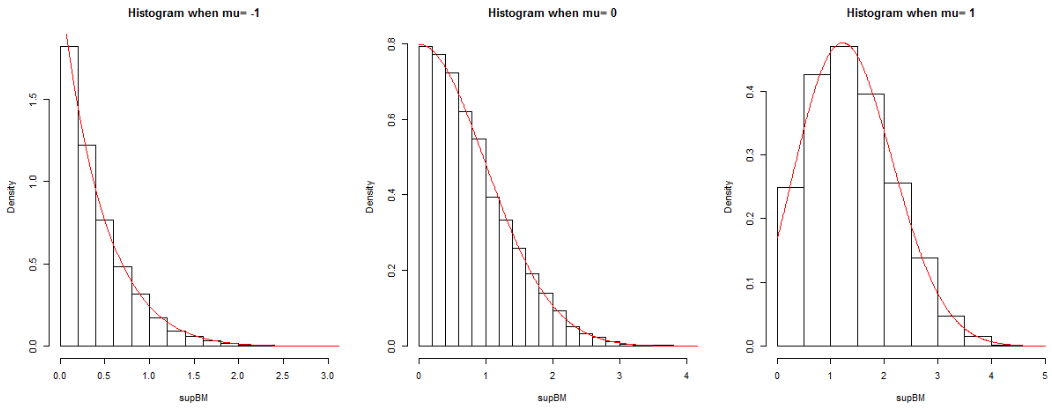

Figure 2.7: 10000 samples of max of Brownian motion with different drifts -1, 0 and 1 respectively.

Chapter 3

The First Hitting Time for the Running Maximum of Brownian Motion by a Linear Barrier

In this chapter, we set up the first hitting time problem for the running maximum of a Brownian motion by a linear time-dependent barrier. We obtain distribution functions and develop an exact simulation algorithm. The problem is motivated by comparing the running maximum of a stock price modelled by a geometric Brownian motion with the present value of a zero coupon bond.

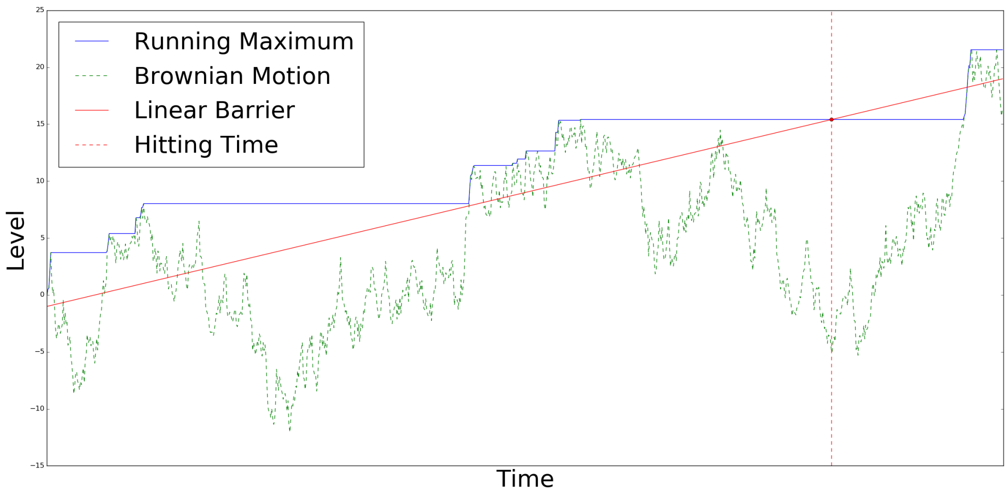

Figure 3.1: A Sample Path of the Linear Barrier Hitting Problem

3.1 Introduction

In the context of crisis management, contingent convertible bonds have been particularly acknowledged for their potential to prevent systematic collapse of important financial institutions Albul et al. [2010]. The concept of contingent convertibles was first introduced in the Harvard Law Review in 1991 10. [1991], following the junk bond crisis of the late 1980s. Following the financial crisis of 2007-08, the idea is considered as a solution for the banking industry. For example, the Basel Committee supports the use of contingent capital as additional Tier 1 Capital Committee et al. [2010].

Contingent convertible bonds can take a variety of different forms such as an option enhanced reverse convertible Pennacchi et al. [2011]. A typical reverse convertible is a short-term note linked to an underlying asset. Before maturity, the security pays the owner the stated coupon rate. At maturity, the owner will receive either the par value or, if the asset value falls, a predetermined number of shares of the underlying. It can be linked to a single asset, an equity index or a basket of indices. In such case, the capital repayment is cash settled, either 100% of principal, or less if the underlying asset falls. For example, at maturity, if the stock closes at or above the initial stock price at issuance, the owner will receive 100% of the original investment amount. If the stock closes below the strike, the owner will get the predetermined number of shares. This means the owner will end up with shares that are worth less than the original investment.

Since contingent convertibles are mandatory converted into equity when banks are in need of a recapitalization, a bankruptcy can be entirely prevented due to quick injection of capital which would be impossible to be obtained elsewhere Pazarbasioglu et al. [2011]. From a bank's standpoint, contingent convertibles are attractive because the coupon payments are tax-deductible and the cost of capital is lower than it is for a share capital increase. In case of conversion, additional equity from debt drives down companys leverage Flannery [2002], Raviv et al. [2004].

One important concern is that the conversion contingent convertibles could produce negative signaling effects leading to potential financial contagion and price manipulation, causing an equity price "death spiral" Pazarbasioglu et al. [2011]. To ease the problem, the "lookback" feature can be added to contingent convertibles by comparing the running maximum of the asset's price to the present value of a zero coupon bond before maturity. In this design, the owner will receive either the principal or, if the maximum of asset price falls below the present value of the zero coupon bond at any time during the life of the note, a predetermined amount of the underlying asset. The conversion is only triggered when the maximum underlying asset's price does not increase for long enough time. The chance of a manipulation is lessened.

Lookback options are path-dependent options whose payoffs depend on not only the final value of the underlying asset but also on the optimal (maximum or minimum) underlying asset's price occurring over the life of the option. The option allows the owner to "look back" over time to determine the payoff.Various formulas have been produced for lookback options Goldman et al. [1979], Garman [1989].

In order to price this kind of 'lookback reverse convertible' notes, we need the distribution of the first hitting time for the running maximum of the underlying process by the present value of the zero coupon bond. The distribution of the running maximum of a standard Brownian has been well studied Karatzas and Shreve [1991]. The price of a zero coupon bond with constant interest rate can be considered as a time dependent barrier. The problem of pricing time-dependent barrier options can be reduced to finding the boundary crossing probabilities for a standard Brownian motion with deterministic function Frishling et al. [1997], Novikov et al. [2003] and Roberts and Shortland [1997]. The linear or (approximated) piece-wise linear boundary crossing problems also have been attacked in a great amount Abundo [2002], Lerche [2013], Scheike [1992] and Wang and Pötzelberger [1997]. Especially in Scheike [1992], Scheike has developed a set of results for a single linear time-dependent boundary similar to our setup but the underlying process is a standard Brownian motion rather than the running maximum of it.

To solve the pricing problem, apart from the trigger time, we need to know also the trigger price of the underlying asset. For contingent convertibles with a lookback feature, the problem can be reduced to find the stopping time of the running maximum (trigger time) and the corresponding stopping position of the Brownian motion or the associated reflected Brownian motion (trigger level).

In the valuation of options with complicated payoff structures, close-form solutions of the option price are often not available. In this case, Monte Carlo Methods are particularly useful. In terms of theory, Monte Carlo valuation relies on risk neutral pricing Glasserman [2003]. The technique applied then, is to generate a large number of price paths for the underlying asset(s), and to then calculate the associated payoff of the option for each path. These payoffs are then averaged and discounted to today. This result is the price of the option. There are more examples in Chan and Wong [2015].

Unfortunately, Monte Carlo simulations, which usually provide a flexible and easy approach, do not perform well in the context of barrier options as the standard methods rely on discretization of the Brownian motion path Geman and Yor [1996], Baldi et al. [1999]. If the option contractually monitors of the underlying price discretely Broadie et al. [1997], Monte Carlo methods provide an unbiased estimator for the price. However, if a contingent convertible is continuously monitored, standard Monte Carlo simulations always give an overestimate of the hitting time and therefore misprice the derivative. In fact, the barrier might have been hit without being detected.

The performance of Monte Carlo simulation can be improved with the help of Brownian bridge concepts Baldi et al. [1999]. Indeed, the simulation algorithm generates the terminal values of the standard Brownian motion then the local running maximum at each interval. However, the scheme has two drawbacks: it is still a discretization scheme with uneven time length; it may take multiple steps when the process is very close to the barrier.

To overcome the limitations, we introduce an acceptance-rejection algorithm based on the joint distribution function of the stopping time and position. The algorithm was first proposed by John von Neumann Von Neumann [1951]. Various applications and alternations of the method have been developed Devroye [1986]. One example is the ziggurat algorithm Marsaglia et al. [2000].

The difficulty of the method is to find an envelop density function such that the likelihood ratio of the joint density function and the envelop density function is bounded above by some constant which should be chosen as small as possible. In our case, a truncated reciprocal inverse Gaussian density function and a truncated Weibull density function are used as the envelop density function. A truncated distribution is a conditional distribution that results from restricting the support of the targeted distribution. An example of truncated distribution is the truncated normal Robert [1995].

The rest of this chapter is organized as follows. The linear time-dependent barrier hitting problem setting is introduced in Section 3.2. We write down the generator in 3.3 and solve for a martingale in 3.4. The distribution functions of the first hitting time and the reflected Brownian motion are presented in Section 3.5. We start from the infinitesimal generator of a standard Brownian motion and a reflected Brownian motion. A martingale of these two quantities is solved from the PDE. Applying Optional Stopping Theorem and Fourier sine/cosine transform, we present the Laplace transform of the first hitting time. Inverting the Laplace transform gives us the probability density function which is integrated to obtain the cumulative distribution function. By plugging in the single Laplace transform to the stopped martingale, we are able to solve the double Laplace transform of the first hitting time and the stopping position of the associated reflected Brownian motion. Inverting the double Laplace gives us the joint density function.

In Section 3.6, two simulation methods are presented. An iterative discretization simulation scheme is developed independently to benchmark and verify the previous results. An acceptance rejection algorithm is developed to simulate the stopping time and position exactly. We are then able to solve the infinite horizon problem for the drifted Brownian motion through the double Laplace transform with a change of measure in Section 3.7. Finally, we provide numerical examples in derivatives pricing in Section 3.8.

3.2 Definitions

In this chapter, we are going to define the running maximum of a standard Brownian motion

$$M_t = \sup_{0<s<t} W_s$$

where W_t is a standard Brownian motion, $t \geq 0$ and $c > 0$. The corresponding first hitting time of a linear barrier is defined as

$$\tau_b = \inf \left\{ t > 0 | M_t - ct = -b \right\} \tag{3.1}$$

with $b > 0$. Therefore, the hitting level is below zero.

If the hitting level is above zero, for some $T > 0$, we consider

$$\{\tau_{-b} > T\} = \{M_t - ct < b, \forall t < T\} = \{W_t - ct < b, \forall t < T\}$$

The problem is reduced to the first hitting time of a Brownian motion with a negative drift (see Borodin and Salminen [2002]). Therefore we focus the case when the hitting level is below zero.

3.3 Infinitesimal Generator of a Reflected Brownian Motion

We consider the Markov process (Y_t, W_t), where the drawdown process $Y_t = M_t - W_t$ is a reflected Brownian motion. The generator $\mathcal{A}$ for some function

$f(t, Y_t, W_t)$ is defined as an operator such that

$$f(t, Y_t, W_t) - \int_0^t \mathcal{A}f(s, Y_s, W_s)ds$$

is a martingale (see Karatzas and Shreve [1991]). Therefore solving

$$\mathcal{A}f = 0$$

subject to certain conditions will provide us with martingales of the form $f(t, Y_t, W_t)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in.

If $\left.\frac{\partial f(t,y,x)}{\partial y}\right|_{y=0} > 0$, and T_0 is some hitting time of Y_t of 0, then $f(Y_{T_0+\delta t}) - f(Y_{T_0}) \geq 0$, hence the expression is a submartingale. Therefore, we need to solve

$$\mathcal{A}f(t, y, x) = \frac{\partial f(t, y, x)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, y, x)}{\partial x^2} + \frac{1}{2}\frac{\partial^2 f(t, y, x)}{\partial y^2} - \frac{\partial^2 f(t, y, x)}{\partial y \partial x} = 0 \quad (3.2)$$

with the boundary condition $\left.\frac{\partial f(t,y,x)}{\partial y}\right|_{y=0} = 0$.

3.4 A Martingale of the Running Maximum of Brownian Motion

Theorem 3. *Let $f(t, Y_t, M_t)$ be the process defined by*

$$f(t, Y_t, M_t) = \mathrm{e}^{\frac{\omega^2}{2}t - \gamma M_t}\left(\cos(\omega Y_t) + \frac{\gamma}{\omega}\sin(\omega Y_t)\right).$$

Then $f(t, Y_t, M_t)$ is a martingale.

Proof. Assume the solution f of (3.2) has the form

$$f(t, y, x) = \mathrm{e}^{\eta t - \gamma(x+y)}h(y)$$

for some function h and constant η and γ. Substitute into (3.2),

$$\mathcal{A}f = \eta f + \frac{1}{2}\mathrm{e}^{\eta t-\gamma(x+y)}h'(y)$$

Solving $\mathcal{A}f = 0$, we obtain an ODE

$$h''(y) + 2\eta h(y) = 0$$

Take $\eta > 0$, $\Delta = -8\eta < 0$, the solution of the ODE

$$h(y) = C_1 \sin\sqrt{2\eta}y + C_2 \cos\sqrt{2\eta}y$$

for some constant C_1, C_2. With the boundary condition

$$\left.\frac{\partial f(t,y,x)}{\partial y}\right|_{y=0} = \mathrm{e}^{\eta t-\gamma x}h'(0) - \gamma \mathrm{e}^{\eta t-\gamma x}h(0) = 0,$$

we have $h'(0) = \gamma h(0)$ and $C_2 = h(0)$. Also

$$h'(0) = \left.\left[\sqrt{2\eta}C_1 \cos\sqrt{2\eta}y - \sqrt{2\eta}C_2 \sin\sqrt{2\eta}y\right]\right|_{y=0} = \sqrt{2\eta}C_1$$

Then we have $C_1 = \frac{\gamma C_2}{\sqrt{2\eta}}$. Set $C_2 = 1$ and let $\omega = \sqrt{2\eta}$ to obtain the solution

$$f(t,y,x) = \mathrm{e}^{\frac{\omega^2}{2}t-\gamma(x+y)}\left(\frac{\gamma}{\omega}\sin\omega y + \cos\omega y\right).$$

□

3.5 Distribution Functions

3.5.1 Laplace Transform of the First Hitting Time

Theorem 4. *For the first hitting time τ_b defined as in (3.1) with $\tau_b^* = \tau_b - b/c$ and $\xi_+ = \sqrt{c^2+2\beta} - c > 0$, when the process starts at zero (i.e. $M_0 = 0$ and*

$Y_0 = 0$, *we have the following Laplace transform:*

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right) = \frac{4c}{\sqrt{c^2+2\beta}+c}\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right), \tag{3.3}$$

where Φ *is the standard normal cdf.*

Proof. Let $\frac{\omega^2}{2} - \gamma c = -\beta$ and $z = x + y - ct$, we obtain the martingale,

$$f(t,y,z) = \exp\left(-\beta t - \frac{\beta + \frac{\omega^2}{2}}{c}z\right)\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c})\sin\omega y + \cos\omega y\right]$$

We can apply the optional stopping theorem to $f(t,y,z)$ with the stopping time τ_b, where τ_b is defined by (3.1) and let $\tau_b^* = \tau_b - \frac{b}{c}$,

$$\begin{aligned}&\mathbb{E}\left\{\exp(-\beta\tau_b^*)\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c})\sin\omega Y_{\tau_b} + \cos\omega Y_{\tau_b}\right]\right\}\\ &= \exp(-\frac{\omega^2}{2}\frac{b}{c})\exp\left(-\frac{\beta+\frac{\omega^2}{2}}{c}m_0\right)\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c})\sin\omega y_0 + \cos\omega y_0\right]\end{aligned}$$

For simplicity, we only consider the case when the process starts at zero, that is $m_0 = 0$ and $y_0 = 0$,

$$\mathbb{E}\left\{\mathrm{e}^{-\beta\tau_b^*}\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c})\sin\omega Y_{\tau_b} + \cos\omega Y_{\tau_b}\right]\right\} = \mathrm{e}^{-\frac{b}{2c}\omega^2} \tag{3.4}$$

Consider the following integrals, for $\xi > 0$,

$$\int_0^\infty \frac{\cos\omega y}{\xi^2+\omega^2}d\omega = \frac{\pi}{2\xi}\mathrm{e}^{-\xi y} \tag{3.5}$$

$$\int_0^\infty \frac{\omega\sin\omega y}{\xi^2+\omega^2}d\omega = \frac{\pi}{2}\mathrm{e}^{-\xi y} \tag{3.6}$$

$$\int_0^\infty \frac{\sin\omega y}{\omega(\xi^2+\omega^2)}d\omega = (1-\mathrm{e}^{-\xi y})\frac{\pi}{2\xi^2} \tag{3.7}$$

On the other hand,

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-\frac{b}{2c}\omega^2}}{\xi^2+\omega^2}d\omega &= \int_0^\infty \int_0^\infty \mathrm{e}^{-\frac{b}{2c}\omega^2-(\xi^2+\omega^2)u}dud\omega \\ &= \int_0^\infty \mathrm{e}^{-\xi^2 u}\int_0^\infty \mathrm{e}^{-(\frac{b}{2c}+u)\omega^2}d\omega du \\ &= \int_0^\infty \frac{1}{2}\sqrt{\frac{\pi}{b/2c+u}}\mathrm{e}^{-\xi^2 u}du\end{aligned}$$

Let $t=\sqrt{u+\frac{b}{2c}}$ and $s^2/2=\xi^2t^2$,

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-\frac{b}{2c}\omega^2}}{\xi^2+\omega^2}d\omega &= \int_{\sqrt{\frac{b}{2c}}}^\infty \mathrm{e}^{-\xi^2t^2}dt\times\sqrt{\pi}\mathrm{e}^{\frac{\xi^2 b}{2c}} \\ &= \pi\mathrm{e}^{\frac{\xi^2 b}{2c}}\int_{\xi\sqrt{\frac{b}{c}}}^\infty \frac{\mathrm{e}^{-s^2/2}}{\xi\sqrt{2\pi}}ds \\ &= \frac{\pi}{\xi}\mathrm{e}^{\frac{\xi^2 b}{2c}}\Phi\left(-\xi\sqrt{\frac{b}{c}}\right)\end{aligned} \tag{3.8}$$

We integrate both sides with the fraction $\frac{1}{\xi^2+\omega^2}$ in (3.4),

$$\int_0^\infty \mathbb{E}\left\{\mathrm{e}^{-\beta\tau_b^*}\left[(\frac{\beta}{c\omega}+\frac{\omega}{2c})\sin\omega Y_{\tau_b}+\cos\omega Y_{\tau_b}\right]\right\}\frac{d\omega}{\omega^2+\xi^2}=\int_0^\infty \frac{\mathrm{e}^{-\frac{b}{2c}\omega^2}}{\omega^2+\xi^2}d\omega$$

As the integral in (3.8) exists, with Fubini's theorem,

$$\mathbb{E}\left\{\mathrm{e}^{-\beta\tau_b^*}\int_0^\infty\left[(\frac{\beta}{c\omega}+\frac{\omega}{2c})\sin\omega Y_{\tau_b}+\cos\omega Y_{\tau_b}\right]\frac{d\omega}{\omega^2+\xi^2}\right\}=\int_0^\infty \frac{\mathrm{e}^{-\frac{b}{2c}\omega^2}}{\omega^2+\xi^2}d\omega$$

Substituting the integrals (3.5), (3.6), (3.7) and (3.8) above,

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\left\{\frac{\beta\pi}{2\xi^2c}+\left[\frac{\pi}{4c}+\frac{\pi}{2\xi}-\frac{\beta\pi}{2\xi^2c}\right]\mathrm{e}^{-\xi Y_{\tau_b}}\right\}\right)=\frac{\pi}{\xi}\mathrm{e}^{\frac{\xi^2 b}{2c}}\Phi\left(-\xi\sqrt{\frac{b}{c}}\right)$$

Rearranging terms, we therefore obtain the double Laplace transform,

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*-\xi Y_{\tau_b}}\right)=\frac{\frac{\mathrm{e}^{\frac{\xi^2 b}{2c}}}{\xi}\Phi\left(-\xi\sqrt{\frac{b}{c}}\right)-\frac{\beta}{2\xi^2 c}\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right)}{\frac{1}{4c}+\frac{1}{2\xi}-\frac{\beta}{2\xi^2 c}} \tag{3.9}$$

If the double Laplace transform exists and is not equal to zero, when the numerator in (3.9) goes to zero, the denominator must go to zero. We observe that $\exists\xi>0$ s.t. $\frac{1}{4c}+\frac{1}{2\xi}-\frac{\beta}{2\xi^2 c}=0$ and the double Laplace transform is still well-defined.

Solving the quadratic equation, we denote the roots $\xi_\pm=\pm\sqrt{c^2+2\beta}-c$ and substitute $\xi_+=\sqrt{c^2+2\beta}-c>0$. We have in the numerator

$$\xi_+\mathrm{e}^{-\frac{b}{2c}}\Phi(-\xi_+\sqrt{\frac{b}{c}})=\frac{\beta}{2c}\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right)$$

Rearranging terms to obtain the Laplace transform of τ_b^*

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right)=&\frac{1}{\beta}\left[2\xi_+ c\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right)\right]\\=&\frac{4c}{\sqrt{c^2+2\beta}+c}\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right)\end{aligned}$$

□

3.5.2 Density Function of the First Hitting Time

Then we invert the Laplace transform (3.3) to obtain the density function of τ_b^*.

Theorem 5. *The probability density function of τ_b^* is:*

$$\begin{aligned} f_{\tau_b^*}(s) =& \frac{4c}{\sqrt{s+\frac{b}{c}}}\phi\left(\frac{cs}{\sqrt{s+\frac{b}{c}}}\right)\Phi\left(\sqrt{\frac{bc}{1+\frac{b}{cs}}}\right) \\ &- 4c^2 e^{2bc} Pr\left(X_2 > -2\sqrt{bc}, X_1 > c\sqrt{s} + \frac{2b}{\sqrt{s}}\right), \end{aligned} \tag{3.10}$$

where $X_2 \sim N(0,1)$, $X_1 \sim N(0, 1+\frac{b}{cs})$, $cov(X_1, X_2) = -\sqrt{\frac{b}{cs}}$, ϕ is the standard normal pdf and Φ is the standard normal cdf.

Proof. From (3.3), we have the Laplace transform,

$$\begin{aligned} &\frac{4c}{\sqrt{c^2+2\beta}+c} e^{\frac{\xi_+^2 b}{2c}} \Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right) \\ =& 4c\int_0^\infty e^{-v(\sqrt{c^2+2\beta}+c)}dv \times \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}-\xi_+ t\sqrt{\frac{b}{c}}}dt \\ =& 2c\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty e^{-v(\sqrt{c^2+2\beta}+c)-\frac{t^2}{2}-\xi_+ t\sqrt{\frac{b}{c}}}dvdt \\ =& 2c\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \exp\left(-(v+t\sqrt{\frac{b}{c}})(\sqrt{c^2+2\beta}-c) - 2vc - \frac{t^2}{2}\right)dvdt \end{aligned}$$

By the Laplace transform of inverse gaussian density Zwillinger [2014],

$$\begin{aligned} &\mathcal{L}_\beta^{-1}\left\{\exp\left(-(v+t\sqrt{\frac{b}{c}})(\sqrt{c^2+2\beta}-c)\right)\right\}(s) \\ &= \frac{v+t\sqrt{\frac{b}{c}}}{\sqrt{2\pi s^3}}\exp\left[-\frac{c^2}{2s}\left(s - \frac{v+t\sqrt{\frac{b}{c}}}{c}\right)^2\right] \end{aligned}$$

where $\mathcal{L}_s(f(s))(\beta)$ denotes the Laplace Transform of the function f w.r.t. s and $\mathcal{L}_\beta^{-1}(F(\beta))(s)$ denotes the inverse Laplace Transform of the function F w.r.t. β.

We omit the constant β in notation for convenience and interchange the integrals. Therefore,

$$
\begin{aligned}
&\frac{4c}{\sqrt{c^2+2\beta}+c} \mathrm{e}^{\frac{\xi_+^2 b}{2c}} \Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right) \\
=&2c\sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^\infty \mathcal{L}_s\left\{\frac{v+t\sqrt{\frac{b}{c}}}{\sqrt{2\pi s^3}}\exp\left[-\frac{c^2}{2s}\left(s-\frac{v+t\sqrt{\frac{b}{c}}}{c}\right)^2\right]\right\}\times \mathrm{e}^{-\frac{t^2}{2}-2vc}dvdt \\
=&\mathcal{L}_s\left\{2c\int_0^\infty\int_0^\infty \frac{v+t\sqrt{\frac{b}{c}}}{\pi\sqrt{s^3}}\exp\left(-\frac{v^2}{2s}-(c+\frac{t}{s}\sqrt{\frac{b}{c}})v\right)dv\right. \\
&\left.\times\exp\left(\frac{-t^2}{2}(1+\frac{b}{cs})+\sqrt{bc}t-\frac{c^2 s}{2}\right)dt\right\}
\end{aligned}
\tag{3.11}
$$

By considering the inner integral first, we have

$$
\begin{aligned}
&\int_0^\infty (v+t\sqrt{\frac{b}{c}})\exp\left(-\frac{v^2}{2s}-(c+\frac{t}{s}\sqrt{\frac{b}{c}})v\right)dv \\
&= s+\sqrt{\frac{\pi s}{2}}\left(t\sqrt{\frac{b}{c}}-s(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)\exp\left((c+\frac{t}{s}\sqrt{\frac{b}{c}})^2\frac{s}{2}\right)\operatorname{erfc}\left(\sqrt{\frac{s}{2}}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right) \\
&= s+c\sqrt{\frac{\pi s^3}{2}}\exp\left((c+\frac{s}{2}\frac{t}{s}\sqrt{\frac{b}{c}})^2\right)\operatorname{erfc}\left(\sqrt{\frac{s}{2}}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right),
\end{aligned}
\tag{3.12}
$$

where erfc is the complementary error function.

Substituting (3.12) into the Laplace transform in (3.11),

$$
\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right) &= \frac{4c}{\sqrt{c^2+2\beta}+c}\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right) \\
=&\mathcal{L}_s\left\{\int_0^\infty \frac{2c}{\pi\sqrt{s}}\exp\left(\frac{-t^2}{2}(1+\frac{b}{cs})+\sqrt{bc}t-\frac{c^2s}{2}\right)dt\right. \\
&\left.-\frac{2c^2}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}\operatorname{erfc}\left(\sqrt{\frac{s}{2}}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)dt\right\} \qquad (3.13)
\end{aligned}
$$

Integrating the first part of the above expression,

$$
\begin{aligned}
&\int_0^\infty \frac{1}{\pi\sqrt{s}}\exp\left(\frac{-t^2}{2}(1+\frac{b}{cs})+\sqrt{bc}t-\frac{c^2s}{2}\right)dt \qquad (3.14) \\
=&\frac{1}{\sqrt{2\pi s(1+\frac{b}{cs})}}\exp\left(-\frac{c^2s}{2}+\frac{bc}{2(1+\frac{b}{cs})}\right)\operatorname{erfc}\left(-\sqrt{\frac{bc}{2(1+\frac{b}{cs})}}\right) \\
=&\frac{2}{\sqrt{2\pi s(1+\frac{b}{cs})}}\exp\left(-\frac{c^2s}{2}+\frac{bc}{2(1+\frac{b}{cs})}\right)\Phi\left(\sqrt{\frac{bc}{1+\frac{b}{cs}}}\right)
\end{aligned}
$$

Rewriting the second part as a bivariate normal cdf,

$$
\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}\operatorname{erfc}\left(\sqrt{\frac{s}{2}}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)dt \\
=&2\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}\Phi\left(-\sqrt{s}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)dt \qquad (3.15) \\
=&2\mathrm{e}^{2bc}\int_0^\infty \mathrm{e}^{-\frac{(t-2\sqrt{bc})^2}{2}}Pr(Z_1>\sqrt{s}(c+\frac{t}{s}\sqrt{\frac{b}{c}}))dt
\end{aligned}
$$

Let $Z_2 = t - 2\sqrt{bc}$,

$$
\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}\,\mathrm{erfc}\left(\sqrt{\frac{s}{2}}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)dt \\
=&2\sqrt{2\pi}\mathrm{e}^{2bc}\int_{-2\sqrt{bc}}^{\infty}\phi(z_2)Pr(Z_1-\sqrt{\frac{b}{cs}}Z_2 > c\sqrt{s}+\frac{2b}{\sqrt{s}}|Z_2)dz_2 \\
=&2\sqrt{2\pi}\mathrm{e}^{2bc}Pr\left(Z_2 > -2\sqrt{bc}, Z_1-\sqrt{\frac{b}{cs}}Z_2 > c\sqrt{s}+\frac{2b}{\sqrt{s}}\right) \\
=&2\sqrt{2\pi}\mathrm{e}^{2bc}Pr\left(X_2 > -2\sqrt{bc}, X_1 > c\sqrt{s}+\frac{2b}{\sqrt{s}}\right),
\end{aligned}
\tag{3.16}
$$

where $X_2 = Z_2 \sim N(0,1)$, $X_1 = Z_1 - \sqrt{\frac{b}{cs}}Z_2 \sim N(0, 1+\frac{b}{cs})$, $cov(X_1, X_2) = -\sqrt{\frac{b}{cs}}$. Therefore, combining (3.14) and (3.16), we obtain

$$
\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right) = \mathcal{L}_s\Bigg\{&\frac{4c}{\sqrt{s+\frac{b}{c}}}\phi\left(\frac{cs}{\sqrt{s+\frac{b}{c}}}\right)\Phi\left(\sqrt{\frac{bc}{1+\frac{b}{cs}}}\right) \\
&-4c^2\mathrm{e}^{2bc}Pr\left(X_2 > -2\sqrt{bc}, X_1 > c\sqrt{s}+\frac{2b}{\sqrt{s}}\right)\Bigg\}
\end{aligned}
$$

As the inversion of the Laplace transform is the density function of τ_b^*, the result follows. □

3.5.3 Cumulative Distribution Function of the First Hitting Time

Then we can integrate the density function in (3.10) w.r.t. s to obtain the cumulative distribution of τ_b^*.

Theorem 6. *The cumulative distribution function of* τ_b^* *is*

$$\begin{aligned}
&Pr\left(\tau_b^* < s\right) \\
=&2\int_0^\infty \phi(t)\Phi\left(c\sqrt{s} - t\sqrt{\frac{b}{cs}}\right) dt \\
&- (2 + 4sc^2 + 8bc)\mathrm{e}^{2bc}\int_0^\infty \phi(t - 2\sqrt{bc})\Phi\left(-c\sqrt{s} - t\sqrt{\frac{b}{cs}}\right) dt \\
&+ 2c\mathrm{e}^{2bc}\frac{2sc + 2b}{\sqrt{c(b + cs)}}\phi\left(\frac{cs + 2b}{\sqrt{s + b/c}}\right)\Phi\left(c\sqrt{\frac{bs}{b + cs}}\right) \\
&- 4\sqrt{bc}\mathrm{e}^{2bc}\phi(2\sqrt{bc})\Phi(-c\sqrt{s})
\end{aligned} \tag{3.17}$$

where we can rewrite the integrals of this form

$$\int_0^\infty \phi(at + b)\Phi(ct + d)dt = \frac{1}{a}Pr\left(X_2 > b, X_1 > \frac{bc}{a} - d\right)$$

and $X_2 \sim N(0,1)$, $X_1 \sim N(0, 1 + \frac{c^2}{a^2})$, $cov(X_1, X_2) = \frac{c}{a}$, *for some constants* a, b, c, d.

Proof. From (3.15), we interchange the integrals,

$$\begin{aligned}
&2\int_0^s \int_0^\infty \mathrm{e}^{-\frac{t^2}{2} + 2\sqrt{bc}t}\Phi\left(-\sqrt{u}(c + \frac{t}{u}\sqrt{\frac{b}{c}})\right) dtdu \\
=&2\int_0^\infty \mathrm{e}^{-\frac{t^2}{2} + 2\sqrt{bc}t}\left[\int_0^s \Phi\left(-\sqrt{u}(c + \frac{t}{u}\sqrt{\frac{b}{c}})\right) du\right] dt
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\int_0^s \Phi\left(-\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du \qquad (3.18)\\ =&\Phi\left(-\sqrt{s}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)s+\frac{1}{2}\int_0^s \sqrt{u}(c-\frac{t}{u}\sqrt{\frac{b}{c}})\phi\left(-\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du\end{aligned}$$

First we consider the following integral and let $z=\sqrt{u}$,

$$\begin{aligned}\int_0^s \frac{1}{\sqrt{u}}\phi\left(\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du =&2\int_0^{\sqrt{s}}\phi\left(cz+\frac{t}{z}\sqrt{\frac{b}{c}}\right)dz\\ =&\frac{2\mathrm{e}^{-\sqrt{bct}}}{\sqrt{2\pi}}\int_0^{\sqrt{s}}\exp(-\frac{c^2z^2}{2}-\frac{t^2\frac{b}{c}}{2z^2})dz\end{aligned}$$

Here we apply the formula from Zwillinger [2014], for some constants a and b,

$$\int \mathrm{e}^{-a^2x^2-b^2/x^2}dx=\frac{\sqrt{\pi}}{4a}\left[\mathrm{e}^{2ab}\,\mathrm{erf}(ax+b/x)+\mathrm{e}^{-2ab}\,\mathrm{erf}(ax-b/x)\right] \qquad (3.19)$$

Hence,

$$\begin{aligned}&\int_0^s \frac{1}{\sqrt{u}}\phi\left(\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du\\ =&\frac{2\mathrm{e}^{-\sqrt{bct}}}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{4(\frac{c}{\sqrt{2}})}\left[\mathrm{e}^{\sqrt{bct}}\,\mathrm{erf}\left(\frac{c}{\sqrt{2}}z+\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{z}\right)+\mathrm{e}^{-\sqrt{bct}}\,\mathrm{erf}\left(\frac{c}{\sqrt{2}}z-\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{z}\right)\right]_0^{\sqrt{s}}\\ =&\frac{1}{2c}\left[\mathrm{erf}\left(\frac{c}{\sqrt{2}}\sqrt{s}+\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{\sqrt{s}}\right)-1+\mathrm{e}^{-2\sqrt{bct}}\,\mathrm{erf}\left(\frac{c}{\sqrt{2}}\sqrt{s}-\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{\sqrt{s}}\right)+\mathrm{e}^{-2\sqrt{bct}}\right]\\ =&\frac{1}{c}\left[\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\mathrm{e}^{-2\sqrt{bct}}-\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right] \qquad (3.20)\end{aligned}$$

Then we consider the following integral and let $z = \sqrt{u}$,

$$\int_0^s \sqrt{u}\phi\left(\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du = \frac{2\mathrm{e}^{-\sqrt{bct}}}{\sqrt{2\pi}}\int_0^{\sqrt{s}} z^2\exp(-\frac{c^2z^2}{2}-\frac{t^2\frac{b}{c}}{2z^2})dz$$

From (3.19), we differentiate both sides w.r.t. the constant a to obtain the following formula,

$$\begin{aligned}\int x^2\mathrm{e}^{-a^2x^2-b^2/x^2}dx = &-\frac{x^2}{2a^2}\mathrm{e}^{-a^2x^2-b^2/x^2}-\frac{\sqrt{\pi}}{8a^2}(2b-\frac{1}{a})\mathrm{e}^{2ab}\,\mathrm{erf}(ax+b/x)\\ &+\frac{\sqrt{\pi}}{8a^2}(2b+\frac{1}{a})\mathrm{e}^{-2ab}\,\mathrm{erf}(ax-b/x)\end{aligned} \tag{3.21}$$

Hence,

$$\begin{aligned}&\int_0^s \sqrt{u}\phi\left(\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du\\ =&\left[\frac{-2z}{c^2\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(cz+\frac{t}{z}\sqrt{\frac{b}{c}})^2}-\frac{1}{2c^2}(t\sqrt{\frac{b}{c}}-\frac{1}{c})\,\mathrm{erf}\left(\frac{c}{\sqrt{2}}z+\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{z}\right)\right.\\ &\left.+\frac{1}{2c^2}(t\sqrt{\frac{b}{c}}+\frac{1}{c})\,\mathrm{erf}\left(\frac{c}{\sqrt{2}}z-\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}}\frac{1}{z}\right)\right]_0^{\sqrt{s}}\\ =&\frac{-2\sqrt{s}}{c^2}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)+\frac{1}{c^2}(t\sqrt{\frac{b}{c}}-\frac{1}{c})\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\\ &+\frac{1}{c^2}(t\sqrt{\frac{b}{c}}+\frac{1}{c})\mathrm{e}^{-2\sqrt{bct}}\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\end{aligned} \tag{3.22}$$

Substituting (3.20) and (3.22) into (3.18),

$$
\begin{aligned}
&\int_0^s \Phi\left(-\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)du\\
=&\Phi\left(-\sqrt{s}(c+\frac{t}{s}\sqrt{\frac{b}{c}})\right)s\\
&+\frac{c}{2}\left[\frac{-2\sqrt{s}}{c^2}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)+\frac{1}{c^2}(t\sqrt{\frac{b}{c}}-\frac{1}{c})\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right.\\
&\left.+\frac{1}{c^2}(t\sqrt{\frac{b}{c}}+\frac{1}{c})\mathrm{e}^{-2\sqrt{bc}t}\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right]\\
&-\frac{1}{2c}t\sqrt{\frac{b}{c}}\left[\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\mathrm{e}^{-2\sqrt{bc}t}-\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right]\\
=&\frac{-\sqrt{s}}{c}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)+\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\left(s+\frac{1}{2c}\left(2t\sqrt{\frac{b}{c}}-\frac{1}{c}\right)\right)\\
&+\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\frac{\mathrm{e}^{-2\sqrt{bc}t}}{2c^2}
\end{aligned}
\tag{3.23}
$$

Substituting (3.23) into (3.15),

$$
\begin{aligned}
&2\int_0^s\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}\Phi\left(-\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)dtdu\\
=&2\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}\left[\frac{-\sqrt{s}}{c}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right.\\
&+\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\left(s+\frac{1}{2c}\left(2t\sqrt{\frac{b}{c}}-\frac{1}{c}\right)\right)\\
&\left.+\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\frac{\mathrm{e}^{-2\sqrt{bct}}}{2c^2}\right]dt\\
=&\frac{-2\sqrt{s}}{c}\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)dt &(3.24)\\
&+(2s-\frac{1}{c^2})\int_0^\infty\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}dt &(3.25)\\
&+2\sqrt{\frac{b}{c^3}}\int_0^\infty\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)t\mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}dt &(3.26)\\
&+\frac{1}{c^2}\int_0^\infty\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\mathrm{e}^{-\frac{t^2}{2}}dt &(3.27)
\end{aligned}
$$

Then we need to solve the integrals (3.24) and (3.26) above.

For (3.24), we let $z = t - 2\sqrt{bc}$.

$$
\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)dt\\
=&\mathrm{e}^{2bc}\int_0^\infty \mathrm{e}^{-\frac{(t-2\sqrt{bc})^2}{2}}\phi\left(c\sqrt{s}+\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)dt\\
=&\mathrm{e}^{2bc}\int_{-2\sqrt{bc}}^\infty \mathrm{e}^{-\frac{z^2}{2}}\phi\left(c\sqrt{s}+\frac{2b}{\sqrt{s}}+\sqrt{\frac{b}{cs}}z\right)dz\\
=&\mathrm{e}^{2bc}\sqrt{2\pi}\left[\frac{1}{\sqrt{1+\frac{b}{cs}}}\phi\left(\frac{c\sqrt{s}+\frac{2b}{\sqrt{s}}}{\sqrt{1+\frac{b}{cs}}}\right)\Phi\left(\sqrt{1+\frac{b}{cs}}z+\frac{\sqrt{\frac{b}{cs}}(c\sqrt{s}+\frac{2b}{\sqrt{s}})}{\sqrt{1+\frac{b}{cs}}}\right)\right]_{-2\sqrt{bc}}^\infty\\
=&\ \mathrm{e}^{2bc}\sqrt{2\pi}\sqrt{\frac{cs}{b+cs}}\phi\left(\frac{cs+2b}{\sqrt{s+\frac{b}{c}}}\right)\Phi\left(c\sqrt{\frac{bs}{b+cs}}\right) \qquad (3.28)
\end{aligned}
$$

For (3.26), similarly, with $z = t - 2\sqrt{bc}$,

$$
\begin{aligned}
&\int_0^\infty \Phi\left(-c\sqrt{s} - \frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right) t\mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bc}t}dt \\
=&\mathrm{e}^{2bc}\int_{-2\sqrt{bc}}^\infty (z+2\sqrt{bc})\mathrm{e}^{-\frac{z^2}{2}}\Phi\left(-c\sqrt{s} - \frac{1}{\sqrt{s}}\sqrt{\frac{b}{c}}(z+2\sqrt{bc})\right) dz \\
=&\sqrt{2\pi}\mathrm{e}^{2bc}\left[\frac{\frac{-1}{\sqrt{s}}\sqrt{\frac{b}{c}}}{\sqrt{1+\frac{b}{cs}}}\phi\left(\frac{-c\sqrt{s}-\frac{2b}{\sqrt{s}}}{\sqrt{1+\frac{b}{cs}}}\right)\Phi\left(\sqrt{1+\frac{b}{cs}}z + \frac{\sqrt{\frac{b}{cs}}(c\sqrt{s}+\frac{2b}{\sqrt{s}})}{\sqrt{1+\frac{b}{cs}}}\right)\right. \\
&\left.-\phi(z)\Phi\left(-c\sqrt{s} - \frac{1}{\sqrt{s}}\sqrt{\frac{b}{c}}(z+2\sqrt{bc})\right)\right]_{-2\sqrt{bc}}^\infty \\
&+2\sqrt{bc}\mathrm{e}^{2bc}\int_0^\infty \Phi\left(-c\sqrt{s} - \frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\mathrm{e}^{-\frac{(t-\sqrt{2bc})^2}{2}}dt \\
=&\sqrt{2\pi}\mathrm{e}^{2bc}\left[\phi(2\sqrt{bc})\Phi(-c\sqrt{s}) - \sqrt{\frac{b}{b+cs}}\phi\left(\frac{cs+2b}{\sqrt{s+\frac{b}{c}}}\right)\Phi\left(c\sqrt{\frac{bs}{b+cs}}\right)\right. \\
&\left.+2\sqrt{bc}\int_0^\infty \Phi\left(-c\sqrt{s} - \frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\phi(t-\sqrt{2bc})dt\right] \qquad (3.29)
\end{aligned}
$$

Combining (3.28), (3.29), (3.27) and (3.25), we obtain the integral

$$
\begin{aligned}
&2\int_0^s\int_0^\infty \mathrm{e}^{-\frac{t^2}{2}+2\sqrt{bct}}\Phi\left(-\sqrt{u}(c+\frac{t}{u}\sqrt{\frac{b}{c}})\right)dtdu\\
=&2\sqrt{\frac{b}{c^3}}\sqrt{2\pi}\mathrm{e}^{2bc}\phi(2\sqrt{bc})\Phi(-c\sqrt{s})\\
&-\sqrt{2\pi}\mathrm{e}^{2bc}\left(\frac{2\sqrt{s}}{c}\sqrt{\frac{cs}{b+cs}}+2\sqrt{\frac{b}{c^3}}\sqrt{\frac{b}{b+cs}}\right)\phi\left(\frac{cs+2b}{\sqrt{s+\frac{b}{c}}}\right)\Phi\left(c\sqrt{\frac{bs}{b+cs}}\right)\\
&+(2s-\frac{1}{c^2}+\frac{4b}{c})\sqrt{2\pi}\mathrm{e}^{2bc}\int_0^\infty\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\phi(t-\sqrt{2bc})dt\\
&+\frac{\sqrt{2\pi}}{c^2}\int_0^\infty\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\phi(t)dt
\end{aligned}
\tag{3.30}
$$

For (3.14), we consider the following integral, and again let $z=\sqrt{u}$ and use (3.19),

$$
\begin{aligned}
&\int_0^s\frac{1}{\sqrt{u}}\exp\left(-\frac{t^2b}{2cu}-\frac{c^2u}{2}\right)du=\int_0^{\sqrt{s}}2\exp\left(-\frac{t^2b}{2cz^2}-\frac{c^2z^2}{2}\right)dz\\
=&\frac{1}{c}\sqrt{\frac{\pi}{2}}\left[\mathrm{e}^{\sqrt{bct}}\operatorname{erf}\left(\frac{cz}{\sqrt{2}}+\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}z}\right)+\mathrm{e}^{-\sqrt{bct}}\operatorname{erf}\left(\frac{cz}{\sqrt{2}}-\frac{t\sqrt{\frac{b}{c}}}{\sqrt{2}z}\right)\right]_0^{\sqrt{s}}\\
=&\frac{\sqrt{2\pi}}{c}\left(\mathrm{e}^{-\sqrt{bct}}\Phi\left(c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)-\mathrm{e}^{\sqrt{bct}}\Phi\left(-c\sqrt{s}-\frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right)\right)
\end{aligned}
\tag{3.31}
$$

Substituting (3.31) into (3.14),

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{-t^2/2+\sqrt{bc}t} \int_0^{s} \frac{1}{\sqrt{u}} \exp\left(-\frac{t^2 b}{2cu} - \frac{c^2 u}{2}\right) dudt \\
=&\frac{1}{c}\sqrt{\frac{2}{\pi}}\left(\int_0^{\infty} \mathrm{e}^{-\frac{t^2}{2}} \Phi\left(c\sqrt{s} - \frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right) dt\right. \\
&\left. - \int_0^{\infty} \mathrm{e}^{-\frac{t^2}{2}-2\sqrt{bc}t} \Phi\left(-c\sqrt{s} - \frac{t}{\sqrt{s}}\sqrt{\frac{b}{c}}\right) dt\right)
\end{aligned}
\tag{3.32}
$$

Finally, combining (3.30) and (3.32), we can therefore obtain the result in (3.17). Also, similar to (3.16), the expression can be rewritten a bivariate normal cdf. □

The cumulative distribution functions with $b = 1$ and different values of c are plotted in figure (3.2). The CDFs increase with the slope c as expected.

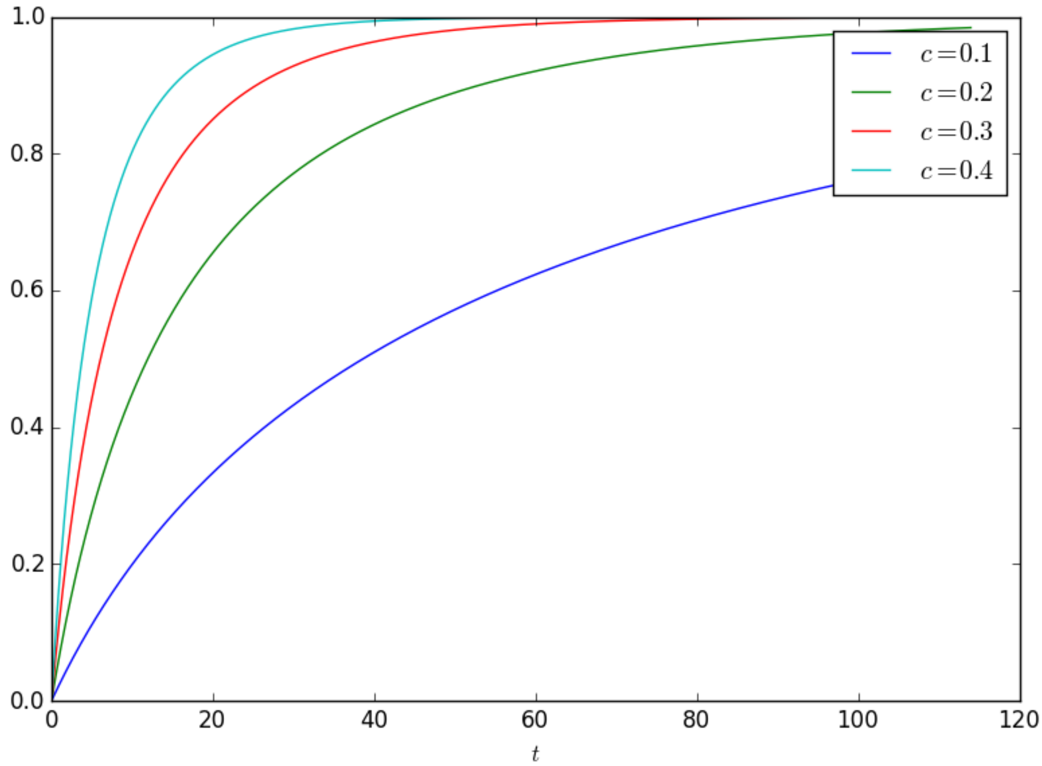

Figure 3.2: Cumulative distribution function of τ_b^* with $b = 1$

3.5.4 The Joint Density Function

Once we have solved the single Laplace transform of the first hitting time τ_b^*, we can substitute it to the double Laplace transform.

Theorem 7. *The double Laplace transform is given by*

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*-\xi Y_{\tau_b}}\right) = \frac{4c}{\xi-\xi_-}\int_0^\infty \phi(t)\left[\frac{\xi \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \xi_+\mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}\right]dt \tag{3.33}$$

where $\phi(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}}$, $\xi_\pm = \pm\sqrt{c^2+2\beta}-c$.

Proof. For the first hitting time τ_b defined as in (3.1) with $\tau_b^* = \tau_b - b/c$ and $\xi_+ = \sqrt{c^2+2\beta}-c > 0$, we have the following Laplace transform (3.3):

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right) = \frac{4c}{\sqrt{c^2+2\beta}+c}\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right),$$

where Φ is the standard normal cdf. The explicit expression of Laplace transform of τ_b^* (3.3) is substituted into the double Laplace transform obtained,

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*-\xi Y_{\tau_b}}\right) &= \frac{\frac{\mathrm{e}^{\frac{\xi^2 b}{2c}}}{\xi}\Phi(-\xi\sqrt{\frac{b}{c}}) - \frac{\beta}{2\xi^2 c}\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*}\right)}{\frac{1}{4c}+\frac{1}{2\xi}-\frac{\beta}{2\xi^2 c}} \\
=&\frac{1}{\frac{1}{4c}+\frac{1}{2\xi}-\frac{\beta}{2\xi^2 c}}\left(\frac{1}{\xi}\int_0^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}-\xi t\sqrt{\frac{b}{c}}}dt - \frac{\beta}{2\xi^2 c}\times\frac{1}{\beta}\left[2\xi_+ c\mathrm{e}^{\frac{\xi_+^2 b}{2c}}\Phi\left(-\xi_+\sqrt{\frac{b}{c}}\right)\right]\right) \\
=&\frac{4c}{(\xi-\xi_+)(\xi-\xi_-)}\left(\int_0^\infty \frac{\xi}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}-\xi t\sqrt{\frac{b}{c}}}dt - \int_0^\infty \frac{\xi_+}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}-\xi_+ t\sqrt{\frac{b}{c}}}dt\right) \\
=&\frac{4c}{\xi-\xi_-}\int_0^\infty \phi(t)\left[\frac{\xi\mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \xi_+\mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}\right]dt
\end{aligned}$$

We leave the integral sign for convenience. □

We can further invert the double Laplace transform to get the joint density function.

Theorem 8. *The joint density function of* τ_b^* *and* Y_{τ_b} *is*

$$f_{\tau_b^*,Y_{\tau_b}}(s,y) = \frac{2c\phi\left(\frac{y+sc}{\sqrt{s}}\right)}{(s+\frac{b}{c})^{3/2}}\left[(y+sc)\exp\left(\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(\frac{\sqrt{\frac{b}{c}}(y+sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right.$$
$$\left. +(y-sc)\exp\left(\frac{\frac{b}{c}(y-sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(-\frac{\sqrt{\frac{b}{c}}(y-sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right] \tag{3.34}$$

Proof. We consider the following integral in (3.33), using integration by parts,

$$\begin{aligned}
&\int_0^\infty \phi(t)\left[\frac{\xi \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \xi_+\mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}\right]dt\\
=&\int_0^\infty \sqrt{\frac{c}{b}}\phi(t)d\left[\frac{\mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}\right]\\
=&\sqrt{\frac{c}{b}}\left(\left[\phi(t)\frac{\mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}\right]_0^\infty + \int_0^\infty \frac{\mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}} - \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}}}{\xi-\xi_+}t\phi(t)dt\right)\\
=&\sqrt{\frac{c}{b}}\int_0^\infty \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}}\left(\frac{1-\mathrm{e}^{-(\xi_+-\xi)t\sqrt{\frac{b}{c}}}}{\xi_+-\xi}\right)t\phi(t)dt\\
=&\sqrt{\frac{c}{b}}\int_0^\infty \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}}\left(\int_0^{t\sqrt{\frac{b}{c}}}\mathrm{e}^{-(\xi_+-\xi)u}du\right)t\phi(t)dt
\end{aligned}$$

Substituting back to the double Laplace transform in (3.33),

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*-\xi Y_{\tau_b}}\right)\\
=&4c\sqrt{\frac{c}{b}}\int_0^\infty\int_0^\infty\int_0^{t\sqrt{\frac{b}{c}}} t\phi(t)\mathrm{e}^{-\xi(v+t\sqrt{\frac{b}{c}}-u)}\mathrm{e}^{-\xi_+u+\xi_-v}dudtdv\\
=&4c\sqrt{\frac{c}{b}}\int_0^\infty\int_0^\infty\int_0^{t\sqrt{\frac{b}{c}}} t\phi(t)\mathrm{e}^{-\xi(v+t\sqrt{\frac{b}{c}}-u)}\mathrm{e}^{-\sqrt{c^2+2\beta}(u+v)-c(v-u)}dudtdv
\end{aligned}$$

Let $y = v + t\sqrt{\frac{b}{c}} - u$ and hence $0 < v < y < v + t\sqrt{\frac{b}{c}} < \infty$,

$$\int_0^\infty \int_0^\infty \int_0^{t\sqrt{\frac{b}{c}}} t\phi(t)\mathrm{e}^{-\xi(v+t\sqrt{\frac{b}{c}}-u)}\mathrm{e}^{-\sqrt{c^2+2\beta}(u+v)-c(v-u)}dudtdv$$
$$= \int_0^\infty \int_0^\infty \int_v^{v+t\sqrt{\frac{b}{c}}} t\phi(t)\mathrm{e}^{-\xi y}\mathrm{e}^{-\sqrt{c^2+2\beta}(2v+t\sqrt{\frac{b}{c}}-y)-c(y-t\sqrt{\frac{b}{c}})}dydtdv$$
$$= \int_0^\infty \mathrm{e}^{-\xi y}\int_0^y \int_{(y-v)\sqrt{\frac{c}{b}}}^\infty t\phi(t)\mathrm{e}^{-\sqrt{c^2+2\beta}(2v+t\sqrt{\frac{b}{c}}-y)-c(y-t\sqrt{\frac{b}{c}})}dtdvdy$$
$$=\mathcal{L}_y\left\{\int_0^y \int_{(y-v)\sqrt{\frac{c}{b}}}^\infty t\phi(t)\mathrm{e}^{-c(y-t\sqrt{\frac{b}{c}})}\mathcal{L}_s\left\{\frac{2v+t\sqrt{\frac{b}{c}}-y}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(2v+t\sqrt{\frac{b}{c}}-y)^2}{2s}-\frac{c^2 s}{2}}\right\}dtdv\right\}$$
$$=\mathcal{L}_y\left\{\mathcal{L}_s\left\{\int_0^\infty \int_{y-t\sqrt{\frac{b}{c}}}^y \frac{2v+t\sqrt{\frac{b}{c}}-y}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(2v+t\sqrt{\frac{b}{c}}-y)^2}{2s}-\frac{c^2 s}{2}-c(y-t\sqrt{\frac{b}{c}})}t\phi(t)\right\}dvdt\right\}$$

We can then integrate the double integrals inside the double Laplace transform to obtain the density function. Consider the following integrals,

$$\int_{y-t\sqrt{\frac{b}{c}}}^y \frac{2v+t\sqrt{\frac{b}{c}}-y}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(2v+t\sqrt{\frac{b}{c}}-y)^2}{2s}}dv = \frac{-1}{2\sqrt{s}}\phi\left(\frac{y+t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right)+\frac{1}{2\sqrt{s}}\phi\left(\frac{y-t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right)$$

and substitute into the double integral,

$$\int_0^\infty \int_{y-t\sqrt{\frac{b}{c}}}^{y} \frac{2v+t\sqrt{\frac{b}{c}}-y}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{(2v+t\sqrt{\frac{b}{c}}-y)^2}{2}+\sqrt{bc}t} t\phi(t)dvdt$$

$$= \int_0^\infty \frac{1}{2\sqrt{s}} \left(\phi\left(\frac{y-t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right) - \phi\left(\frac{y+t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right)\right) t\phi(t)\mathrm{e}^{\sqrt{bct}}dt$$

$$= \frac{1}{2\sqrt{s}}\phi(\frac{y}{\sqrt{s}}) \int_0^\infty \frac{t}{\sqrt{2\pi}} \left[\exp\left(\frac{-1}{2s}\left((s+\frac{b}{c})t^2 - 2t\sqrt{\frac{b}{c}}(y+sc)\right)\right)\right.$$

$$\left. - \exp\left(\frac{-1}{2s}\left((s+\frac{b}{c})t^2 + 2t\sqrt{\frac{b}{c}}(y-sc)\right)\right)\right] dt \qquad (3.35)$$

Consider the first integral,

$$\int_0^\infty t \exp\left(\frac{-1}{2s}\left((s+\frac{b}{c})t^2 - 2t\sqrt{\frac{b}{c}}(y+sc)\right)\right) dt$$

$$= \frac{-\sqrt{s}}{4(s+\frac{b}{c})^{3/2}} \left[\sqrt{2\pi}\left(-2\sqrt{\frac{b}{c}}(y+sc)\right) \mathrm{e}^{\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}} \operatorname{erf}\left(\frac{2(s+\frac{b}{c})t - 2\sqrt{\frac{b}{c}}(y+sc)}{2\sqrt{2}\sqrt{s+\frac{b}{c}}\sqrt{s}}\right)\right.$$

$$\left. +4\sqrt{s+\frac{b}{c}}\sqrt{s} \exp\left(\frac{-1}{2s}((s+\frac{b}{c})t^2 - 2t\sqrt{\frac{b}{c}}(y+sc))\right)\right]_0^\infty$$

$$= \frac{s}{s+\frac{b}{c}} + \frac{\sqrt{s}\sqrt{2\pi}}{(s+\frac{b}{c})^{3/2}}\sqrt{\frac{b}{c}}(y+sc)\exp\left(\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(\frac{\sqrt{\frac{b}{c}}(y+sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right) \qquad (3.36)$$

Similarly, for the second integral,

$$\int_0^\infty t\exp\left(\frac{-1}{2s}\left((s+\frac{b}{c})t^2+2t\sqrt{\frac{b}{c}}(y-sc)\right)\right)dt$$

$$=\frac{s}{s+\frac{b}{c}}-\frac{\sqrt{s}\sqrt{2\pi}}{(s+\frac{b}{c})^{3/2}}\sqrt{\frac{b}{c}}(y-sc)\exp\left(\frac{\frac{b}{c}(y-sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(-\frac{\sqrt{\frac{b}{c}}(y-sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right) \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.35),

$$\int_0^\infty \frac{1}{2\sqrt{s}}\left(\phi\left(\frac{y-t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right)-\phi\left(\frac{y+t\sqrt{\frac{b}{c}}}{\sqrt{s}}\right)\right)t\phi(t)\mathrm{e}^{\sqrt{bc}t}dt$$

$$=\frac{1}{2}\phi\left(\frac{y}{\sqrt{s}}\right)\left[\frac{\sqrt{\frac{b}{c}}(y+sc)}{(s+\frac{b}{c})^{3/2}}\exp\left(\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(\frac{\sqrt{\frac{b}{c}}(y+sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right.$$

$$\left.+\frac{\sqrt{\frac{b}{c}}(y-sc)}{(s+\frac{b}{c})^{3/2}}\exp\left(\frac{\frac{b}{c}(y-sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(-\frac{\sqrt{\frac{b}{c}}(y-sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right]$$

Finally, we combine the integral with other terms and simplify the expression,

$$
\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{-\beta\tau_b^*-\xi Y_{\tau_b}}\right)\\
=&\mathcal{L}_y\left\{\mathcal{L}_s\left\{4c\sqrt{\frac{c}{b}}\times\frac{1}{2}\phi\left(\frac{y}{\sqrt{s}}\right)\mathrm{e}^{-\frac{c^2s}{2}-cy}\left[\frac{\sqrt{\frac{b}{c}}(y+sc)}{(s+\frac{b}{c})^{3/2}}\exp\left(\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(\frac{\sqrt{\frac{b}{c}}(y+sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right.\right.\right.\\
&\left.\left.\left.+\frac{\sqrt{\frac{b}{c}}(y-sc)}{(s+\frac{b}{c})^{3/2}}\exp\left(\frac{\frac{b}{c}(y-sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(-\frac{\sqrt{\frac{b}{c}}(y-sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right]\right\}\right\}\\
=&\mathcal{L}_y\left\{\mathcal{L}_s\left\{\frac{2c\phi\left(\frac{y+sc}{\sqrt{s}}\right)}{(s+\frac{b}{c})^{3/2}}\left[(y+sc)\exp\left(\frac{\frac{b}{c}(y+sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(\frac{\sqrt{\frac{b}{c}}(y+sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right.\right.\right.\\
&\left.\left.\left.+(y-sc)\exp\left(\frac{\frac{b}{c}(y-sc)^2}{2s(s+\frac{b}{c})}\right)\Phi\left(-\frac{\sqrt{\frac{b}{c}}(y-sc)}{\sqrt{s}\sqrt{s+\frac{b}{c}}}\right)\right]\right\}\right\}
\end{aligned}
$$

As the inversion of the double Laplace transform is the joint density function of τ_b^* and Y_{τ_b}, the result follows. □

The joint density function can be simplified to make it easier to program or recognize the structure. Let

$$
\begin{aligned}
m =& sc \\
t =& s + b/c
\end{aligned}
$$

where m and t resemble the stopping position of the running maximum and the original stopping time respectively. Furthermore,

$$
\frac{1}{s} - \frac{b/c}{s(s+b/c)} = \frac{1}{s+b/c} = \frac{1}{t}
$$

which enables further simplification of notations. Let

$$
\begin{aligned}
\frac{1}{\sigma} &= \sqrt{\frac{1}{s} - \frac{1}{t}} = \frac{\sqrt{b/c}}{\sqrt{s}\sqrt{s+b/c}} > 0 \\
d_1 &= \frac{m+y}{\sigma} = \sqrt{\frac{1}{s} - \frac{1}{t}}(m+y) = \frac{\sqrt{b/c}(sc+y)}{\sqrt{s}\sqrt{s+b/c}} > 0 \\
d_2 &= \frac{m-y}{\sigma} = \sqrt{\frac{1}{s} - \frac{1}{t}}(m-y) = \frac{\sqrt{b/c}(sc-y)}{\sqrt{s}\sqrt{s+b/c}} \in \mathbb{R}
\end{aligned}
$$

The density function becomes

$$
\begin{aligned}
f_{\tau_b^*, y_{\tau_b}}(s,y) =& \frac{2c}{t^{3/2}} \phi\left(\frac{\sigma d_1}{\sqrt{s}}\right) \left[\sigma d_1 \mathrm{e}^{\frac{d_1^2}{2}} \Phi(d_1) - \sigma d_2 \mathrm{e}^{\frac{d_2^2}{2}} \Phi(d_2)\right] \\
=& \frac{2c\sigma d_1}{t^{3/2}} \phi\left(\frac{\sigma d_1}{\sqrt{t}}\right) \left[\Phi(d_1) - \frac{d_2}{d_1} \mathrm{e}^{\frac{-2by}{t}} \Phi(d_2)\right]
\end{aligned}
\tag{3.38}
$$

The joint density function with parameters $b = 2$ and $c = 3$ is plotted in figure (3.3) with Python.

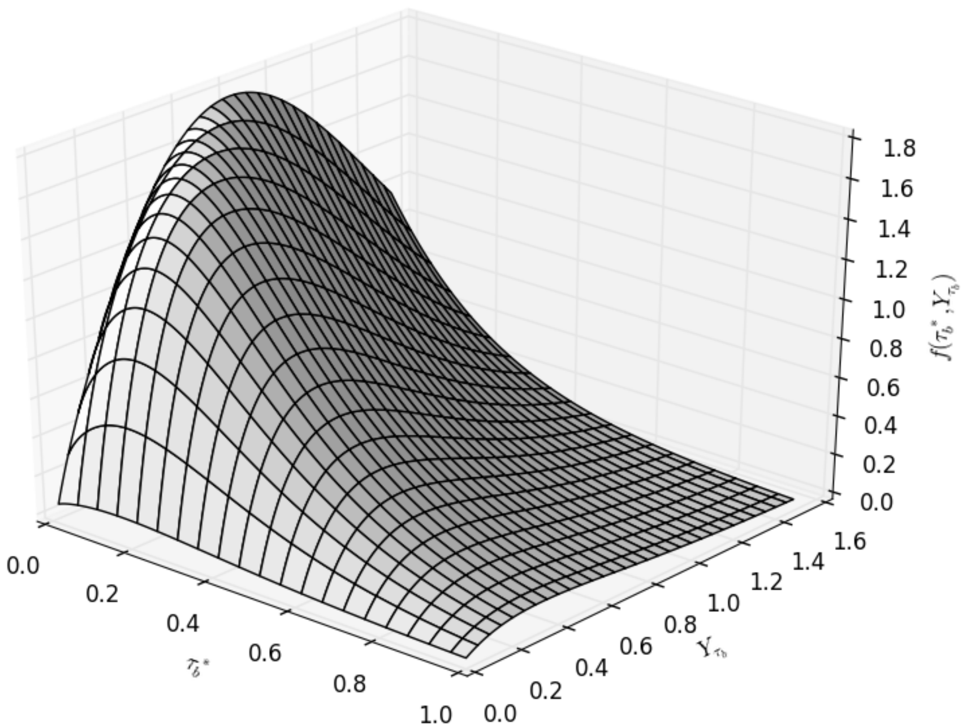

Figure 3.3: Joint density function of (τ_b^*, Y_{τ_b}) with $(b, c) = (2, 3)$

3.6 Simulation Methods

3.6.1 Simulation by Iterative Discretization

The stopping time τ_b can be simulated by an iterative discretization of the paths of Brownian motion and the corresponding running maximum.

Suppose we know how to simulate the running maximum exactly with the Brownian path (see Devroye [2010]). The process $Z_t = M_t - ct$ can not hit the level $-b$ before time $t_1 = b/c$. If the running maximum M_1 at $t_1 = b/c$ remains unchanged, i.e. $M_1 = M_0 = 0$, the $M_1 = -b + ct_1 = 0$ at time t_1. In fact, for further steps, the process Z_{t_j} at step t_j can not hit the level $-b$ before time $t_j = (M_{j-1} + b)/c$. If the running maximum M_j does not increase at $t_j = (M_{j-1}+b)/c$, i.e. $M_j = M_{j-1}$, then $M_j = -b+ct_j = M_{j-1}$ at time t_j. Therefore, we only need to check the process Z_{t_j} at every time point $t_j = (M_{j-1} + b)/c$ for $j = 0, 1, 2....$ An illustration is below.

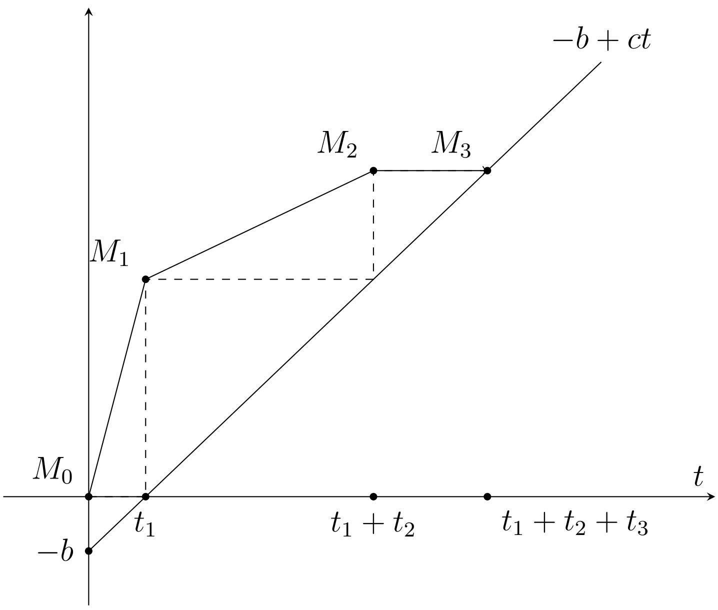

Figure 3.4: An illustration of iterative simulation for linear barrier crossing

Algorithm 4 Simulation Algorithm of τ_b

1: Set $j = 0$, $t_0 = 0$ and $M_0 = W_0 = 0$
2: Calculate the next closest possible hitting time by $t_{j+1} = (M_j + b)/c$
3: Calculate the time interval of simulation by $\Delta t = t_{j+1} - t_j$
4: Generate $W_{j+1} = W_j + \sqrt{\Delta t} Z$, where $Z \sim N(0, 1)$, set $\Delta W = W_{j+1} - W_j$
5: Generate $M^{loc}_{j+1} = \frac{\sqrt{(\Delta W)^2 + 2\Delta t E}}{2} + W_j$, where $E \sim \exp(1)$
6: Set $M_{j+1} = \max(M_j, M^{loc}_{j+1})$
7: **if** $M_{j+1} = M_j$ **then** set $\tau_b = t_{j+1}$ and break **else** set $j = j + 1$ and repeat
8: Set $\tau^*_b = \tau_b - b/c$ if needed

We validate the explicit forms of pdf in (3.10) with simulation results. The distribution functions can be calculated using established bivariate normal cdf algorithms. An algorithm in Python is used here (see Genz and Bretz [2009]).

10000 τ_b^* are generated with different pair of parameters b, c in figure (3.5), (3.6) and (3.7). The histograms are plotted by the simulation results by using the iterative discretization method. The curves are calculated by the corresponding analytical solution in (3.10).

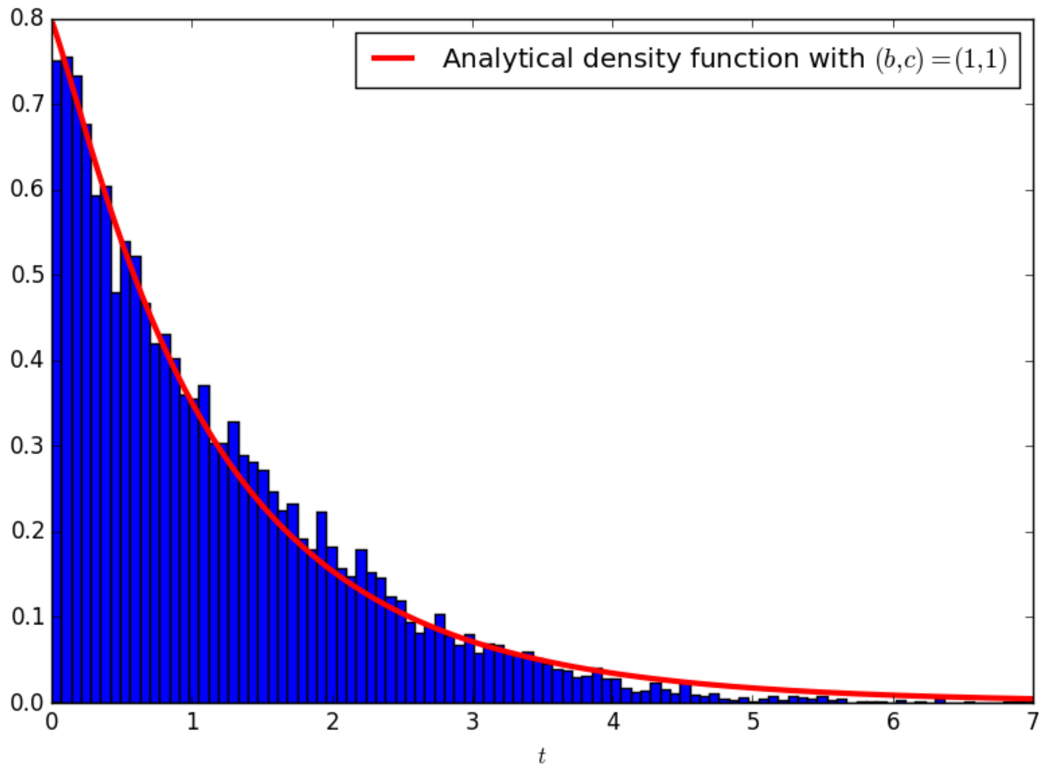

Figure 3.5: Probability density function of τ_b^* with $(b, c) = (1, 1)$

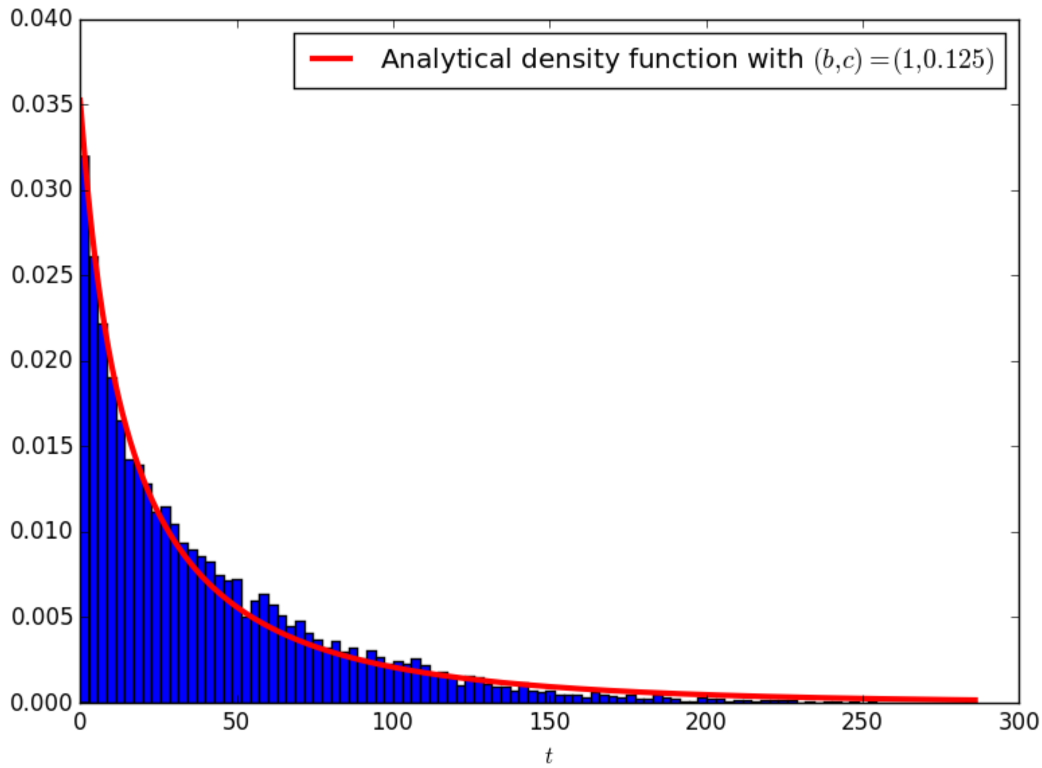

Figure 3.6: Probability density function of τ_b^* with $(b, c) = (1, 0.125)$

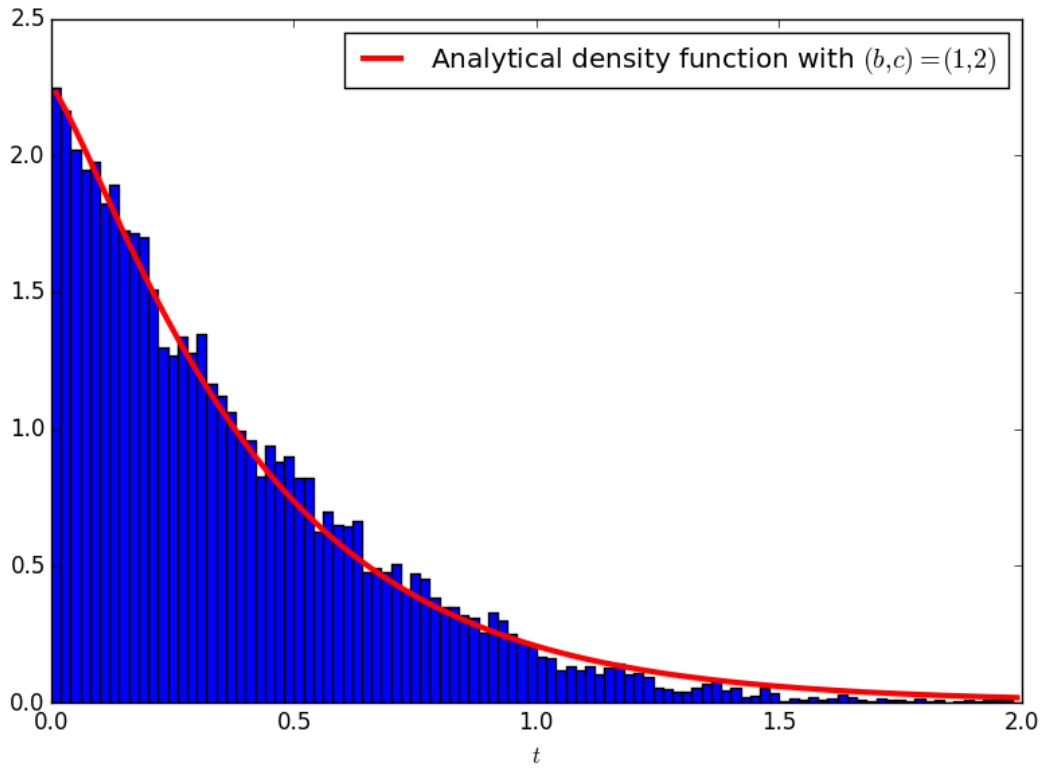

Figure 3.7: Probability density function of τ_b^* with $(b, c) = (1, 2)$

From the graphs, the histograms barely fit the red curves since the simulations are not exact. On the other hand, some consistencies between the histograms and the red curves indicate the results are not too wrong.

3.6.2 An Exact Simulation Algorithm by Rejection Sampling

We want to sample the pair τ_b^* and Y_{τ_b} from the joint density in (3.34) with acceptance-rejection algorithm (see Devroye [1986] for more examples). By doing this, we can avoid the discretization error. We can also improve the speed of simulation as when the path is very close to the barrier, multiple steps may be needed from the previous discretization scheme. Acceptance-rejection algorithm allows us to sample directly from the stopping values.

We need to find an envelope density such that the likelihood ratio is bounded above by some constant.

Theorem 9. *Let f_{RIG} be the reciprocal Inverse Gaussian density function and $f_{Weibull}$ be the Weibull density function, then*

$$\frac{f_{\tau_b^*, y_{\tau_b}}(s,y)}{f_{RIG}\left(s'|s' > \frac{b}{c}\right) f_{Weibull}\left(y'|y' > sc, s\right)} \leq 2Pr(S' > b/c) \tag{3.39}$$

where $s' = s + b/c$, $y' = y + sc$ and $S' \sim RIG(\mu = \frac{c}{b}, \lambda = c^2)$, $Y'|S \sim Weibull(\lambda = \sqrt{2(s+\frac{b}{c})}, k = 2)$. Therefore, the envelope is given by the joint truncated density function of s' and y'.

Proof. Using the notation from (3.38)

$$f_{\tau_b^*, y_{\tau_b}}(s,y) = \frac{2c\sigma d_1}{t^{3/2}} \phi\left(\frac{\sigma d_1}{\sqrt{t}}\right) \left[\Phi(d_1) - \frac{d_2}{d_1} \mathrm{e}^{\frac{-2by}{t}} \Phi(d_2)\right]$$

and noticing that

$$\Phi(d_1) - \frac{d_2}{d_1} \mathrm{e}^{\frac{-2by}{t}} \Phi(d_2) \leq 1 \tag{3.40}$$

The equality holds when either one of the following cases happens,

1. $y \to \infty$, then $d_1 \to \infty$ and $d_2 \to -\infty$, (3.40) $\to \Phi(\infty) - \mathrm{e}^{-\infty}\Phi(-\infty) = 1$
2. $s \to 0$, then $d_1 \to \infty$ and $d_2 \to -\infty$, (3.40) $\to \Phi(\infty) + \mathrm{e}^{-\infty}\Phi(-\infty) = 1$

3. $c \to 0$, then $d_1 \to \frac{y}{\sqrt{s}}$ and $d_2 \to -\frac{y}{\sqrt{s}}$, (3.40) $\to \Phi(\frac{y}{\sqrt{s}}) + \Phi(-\frac{y}{\sqrt{s}}) = 1$

Therefore,

$$f_{\tau_b^*, y_{\tau_b}}(s, y) \leq \frac{2c\sigma d_1}{t^{3/2}} \phi\left(\frac{\sigma d_1}{\sqrt{t}}\right)$$

Using the original notations in (3.34),

$$\begin{aligned} f_{\tau_b^*, y_{\tau_b}}(s, y) \leq & \frac{2c\phi\left(\frac{y+sc}{\sqrt{s+\frac{b}{c}}}\right)}{(s+\frac{b}{c})^{3/2}}(y+sc) \\ = & \frac{2c}{\sqrt{2\pi(s+\frac{b}{c})}}\left[\frac{y+sc}{s+\frac{b}{c}} \exp\left(-\frac{(y+sc)^2}{2(s+\frac{b}{c})}\right)\right] \end{aligned}$$

By letting $y' = y + sc$ and conditioning on s, we recognize the density function of a Weibull distribution with shape parameter k and scale parameter λ,

$$\frac{y+sc}{s+\frac{b}{c}} \exp\left(-\frac{(y+sc)^2}{2(s+\frac{b}{c})}\right) = f_{Weibull}\left(y'; \lambda = \sqrt{2(s+\frac{b}{c})}, k = 2\right)$$

But the support of y' does not match the support of a Weibull random variable since $y' > sc > 0$. We need a truncation with

$$Pr(Y' > sc) = \exp\left(-\frac{c^2 s^2}{2(s+\frac{b}{c})}\right)$$

and

$$f_{Weibull}\left(y'\right) = f_{Weibull}\left(y' | y' > sc\right) Pr(Y' > sc)$$

Hence,

$$\begin{aligned}
f_{\tau_b^*, y_{\tau_b}}(s,y) \leq & \frac{2c}{\sqrt{2\pi(s+\frac{b}{c})}} f_{Weibull}\left(y'; \lambda = \sqrt{2(s+\frac{b}{c})}, k=2\right) \\
= & \frac{2c}{\sqrt{2\pi(s+\frac{b}{c})}} f_{Weibull}\left(y'|y' > sc\right) Pr(Y' > sc) \\
= & \frac{2c}{\sqrt{2\pi(s+\frac{b}{c})}} \exp\left(-\frac{c^2 s^2}{2(s+\frac{b}{c})}\right) f_{Weibull}\left(y'|y' > sc\right)
\end{aligned}$$

By letting $s' = s + b/c$, we recognize the density function of a reciprocal inverse Gaussian distribution with mean $\mu = c/b$ and scale parameter $\lambda = c^2$,

$$\begin{aligned}
\frac{c}{\sqrt{2\pi(s+\frac{b}{c})}} \exp\left(-\frac{c^2 s^2}{2(s+\frac{b}{c})}\right) = & \frac{c}{\sqrt{2\pi s'}} \exp\left(-\frac{c^2\left(1-\frac{c}{b}s'\right)^2}{2s'(\frac{c}{b})^2}\right) \\
= & f_{RIG}\left(s'; \mu = \frac{c}{b}, \lambda = c^2\right)
\end{aligned}$$

Again, a truncation is needed as $s' > b/c > 0$. Therefore,

$$\begin{aligned}
f_{\tau_b^*, y_{\tau_b}}(s,y) \leq & \frac{2c}{\sqrt{2\pi(s+\frac{b}{c})}} \exp\left(-\frac{b^2\left(1-\frac{c}{b}s'\right)^2}{2s'}\right) f_{Weibull}\left(y'|y' > sc\right) \\
= & 2f_{RIG}\left(s'; \mu = \frac{c}{b}, \lambda = c^2\right) f_{Weibull}\left(y'|y' > sc\right) \\
= & 2Pr(S' > b/c) f_{RIG}\left(s'|s' > b/c\right) f_{Weibull}\left(y'|y' > sc\right)
\end{aligned}$$

The result follows after rearranging terms. $\square$

(3.39) provides an acceptance rate and envelope density functions for simulation. The acceptance rate is

$$Pr\left(U \leq \frac{f_{\tau_b^*, y_{\tau_b}}(S,Y)}{f_{RIG}\left(S'|S' > \frac{b}{c}\right) f_{Weibull}\left(Y'|Y' > Sc, S\right)}\right) = \frac{1}{2Pr(S' > b/c)} \geq \frac{1}{2}$$

The ratio looks complicated but can be simplified. Notice

$$\begin{aligned}&Pr(S' > b/c) f_{RIG}(S'|S' > b/c) f_{Weibull}(Y'|Y' > sc)\\ =&\frac{c}{\sqrt{2\pi(s+\frac{b}{c})}}\left[\frac{y+sc}{s+\frac{b}{c}}\exp\left(-\frac{(y+sc)^2}{2(s+\frac{b}{c})}\right)\right]\end{aligned}$$

which is proportional to the density function $f_{\tau_b^*, y_{\tau_b}}(s, y)$.

Now, we can simulate (τ_b^*, Y_{τ_b}) from the basis of $S' \sim$ RIG and $Y'|S' \sim$ Weibull. The truncation here is simply done by throwing away any values outside the wanted support. One can also apply inverse transform or acceptance-rejection techniques to the truncated density.

Python SciPy is used to generate reciprocal Inverse Gaussian and Weibull random variables Jones et al. [2001–]. Here the scaling property of inverse Gaussian random variables $1/S' \sim IG(\mu, \lambda) \Rightarrow 1/\lambda S' \sim IG(\mu/\lambda, 1)$.

Algorithm 5 Acceptance-Rejection Algorithm of (τ_b^*, Y_{τ_b})

1: Generate S' from $f_{RIG}\left(s'; \mu = \frac{c}{b}, \lambda = c^2\right)$. Repeat until $S' > b/c$.
2: Generate $Y'|S'$ from $f_{Weibull}\left(y'; \lambda = \sqrt{2s'}, k = 2\right)$. Repeat until $Y' > sc$.
3: Set $S = S' - b/c$ and $Y = Y' - sc$.
4: Generate a uniform random number $U \sim U(0, 1)$.
5: **if** $U < \frac{f_{\tau_b^*, y_{\tau_b}}(S,Y)}{2Pr(S' > \frac{b}{c}) f_{RIG}\left(S'|S' > \frac{b}{c}\right) f_{Weibull}\left(Y'|Y' > Sc, S\right)}$ **then** set $(\tau_b^*, Y_{\tau_b}) = (S, Y)$
6: **else** go to step 1.

The histograms of 10000 pairs of (τ_b^*, Y_{τ_b}) generated with parameters $(b, c) = (1, 1)$ are plotted in (3.8).

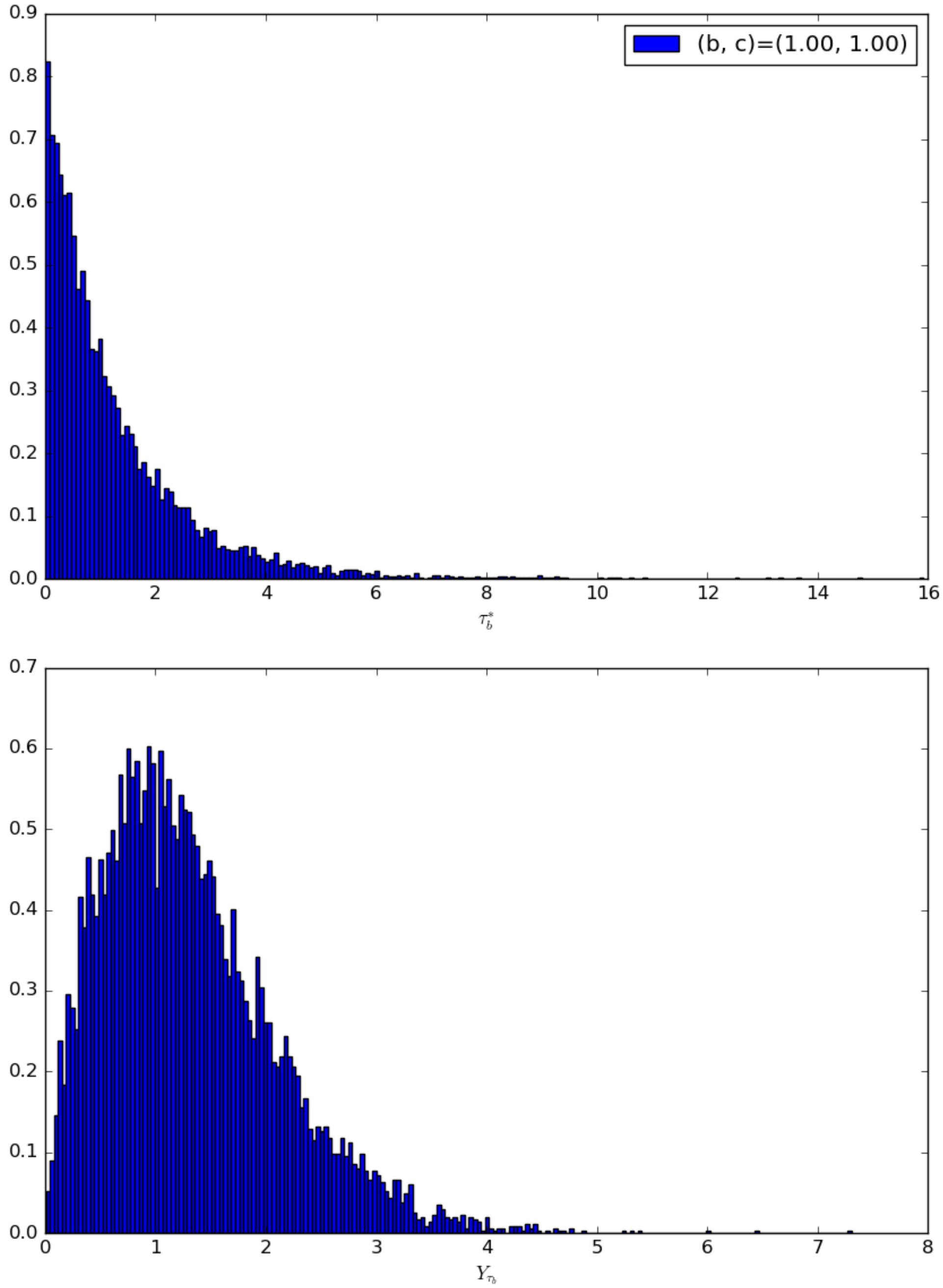

Figure 3.8: Histogram of (τ_b^*, Y_{τ_b}) with $(b, c) = (1, 1)$

10000 τ_b^* are generated by the rejection method with different pairs of parameters b, c. The histograms are plotted in figure (3.9), (3.10) and (3.11). The red curves are calculated by the corresponding analytical solution (3.10) to validate the simulation results.

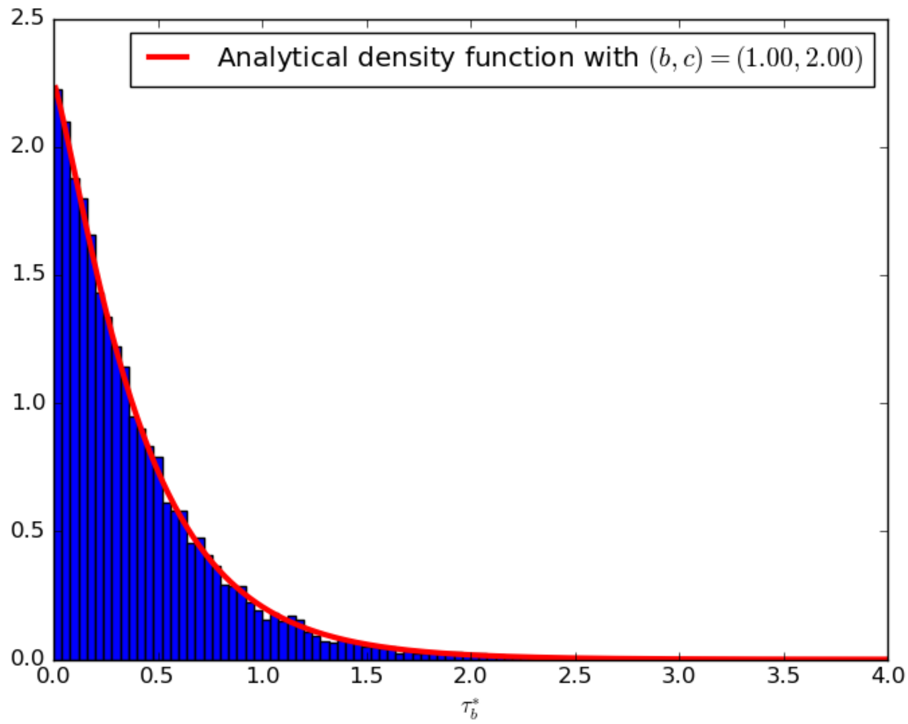

Figure 3.9: Histogram of simulated τ_b^* with $(b, c) = (1, 2)$

- Empirical acceptance rate: 0.7980
- Simulated mean stopping time: 0.4074
- Approximated mean stopping time by Simpson's rule: 0.4067

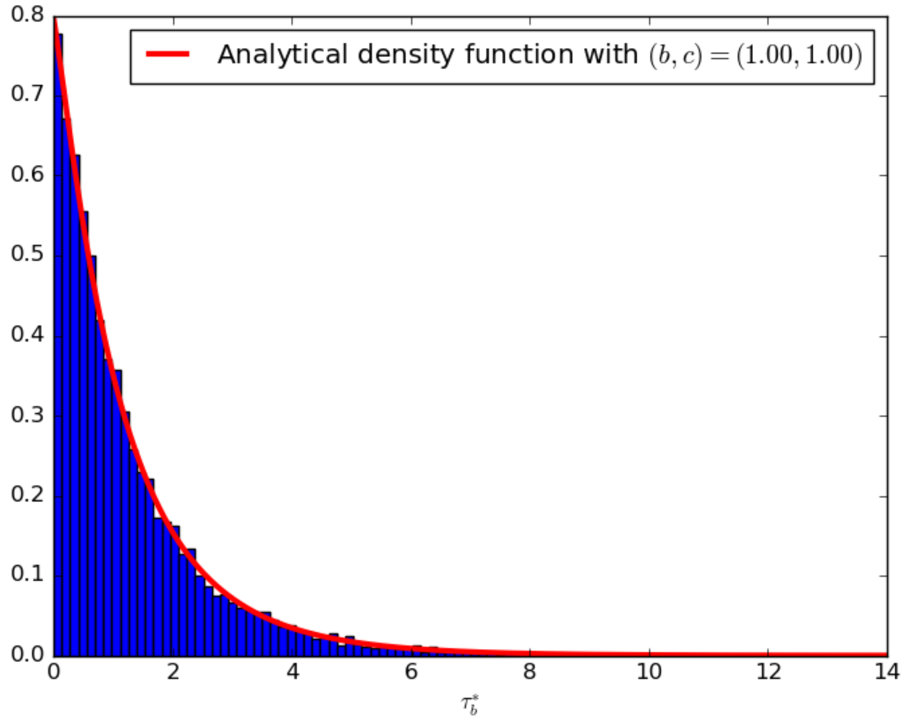

Figure 3.10: Histogram of simulated τ_b^* with $(b, c) = (1, 1)$

- Empirical acceptance rate: 0.7443
- Simulated mean stopping time: 1.2826
- Approximated mean stopping time by Simpson's rule: 1.2950

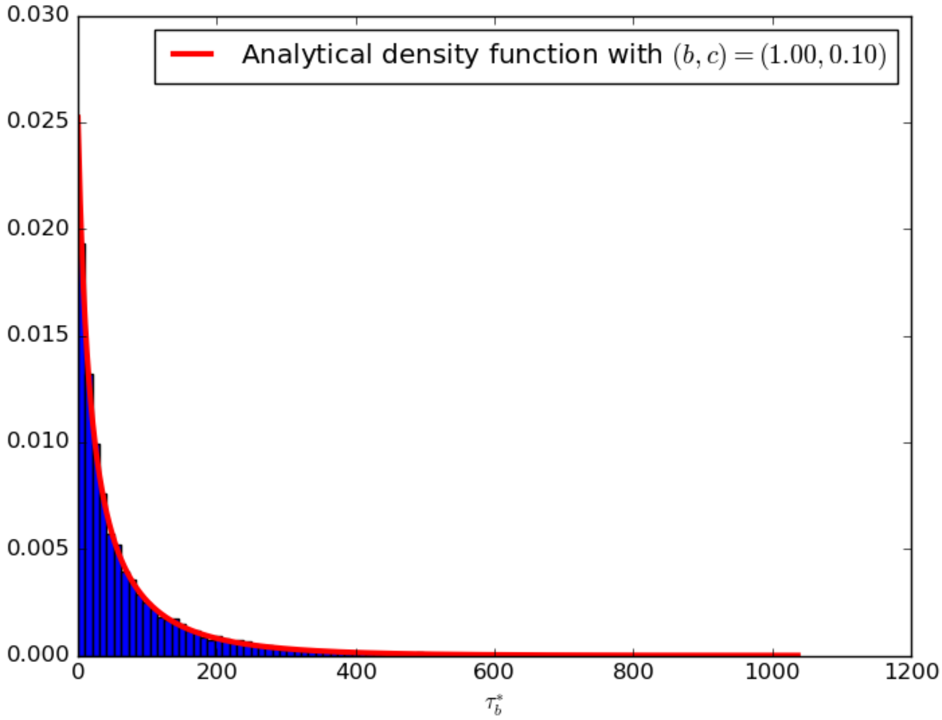

Figure 3.11: Histogram of simulated τ_b^* with $(b, c) = (1, 0.1)$

- Empirical acceptance rate: 0.6055
- Simulated mean stopping time: 74.9207
- Approximated mean stopping time by Simpson's rule: 74.9204

The simulated means are close to the numerical integrated mean and the empirical acceptance rates are quite high (> 0.5).

3.7 Infinite Horizon Problem

We are also interested in if the process will eventually hit the barrier. Also, we want to know the condition when τ_b is almost surely finite.

Theorem 10. *For the first hitting time τ_b defined in (3.1), $Pr(\tau_b < \infty) = 1$*

Proof. Define the super-maximum $Z_t = M_t - ct$, We consider the martingale in (3),

$$f(t, Y_t, Z_t) = \exp\left(-\beta t - \frac{\beta + \frac{\omega^2}{2}}{c} Z_t\right)\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin \omega Y_t + \cos \omega Y_t\right]$$

and apply Optional Stopping Theorem with the finite stopping time $\tau_b \wedge T$ for some fixed $T > 0$,

$$\mathbb{E}(f(\tau_b \wedge T, Y_{\tau_b \wedge T}, Z_{\tau_b \wedge T})) = f(0, Y_0, Z_0) = 1$$

since we have $Z_0 = Y_0 = 0$. On the other hand,

$$\mathbb{E}(f(\tau_b \wedge T, Y_{\tau_b \wedge T}, Z_{\tau_b \wedge T})) = \mathbb{E}\left(f(\tau_b, Y_{\tau_b}, Z_{\tau_b})\mathbf{1}\left\{\tau_b < T\right\}\right) + \mathbb{E}\left(f(T, Y_T, Z_T)\mathbf{1}\left\{\tau_b \geq T\right\}\right)$$

Since $\{Z_T \geq -b, \forall T \leq \tau_b\}$,

$$\mathbb{E}\left(f(T, Y_T, Z_T)\mathbf{1}\left\{\tau_b \geq T\right\}\right) \leq \mathbb{E}\left(\mathrm{e}^{-\beta T + \frac{\beta + \frac{\omega^2}{2}}{c} b}\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin \omega Y_T + \cos \omega Y_T\right]\mathbf{1}\left\{\tau_b \geq T\right\}\right)$$

As $\sin(\cdot)$ and $\cos(\cdot)$ are bounded and for $\beta > 0$, we then let $T \to \infty$ with bounded convergence theorem,

$$\lim_{T\to\infty} \mathbb{E}\left(f(T, Y_T, Z_T)\mathbf{1}\left\{\tau_b \geq T\right\}\right) = \mathbb{E}\left(\lim_{T\to\infty} f(T, Y_T, Z_T)\mathbf{1}\left\{\tau_b \geq T\right\}\right) = 0$$

Therefore,

$$1 = \lim_{T\to\infty} \mathbb{E}\left(f(\tau_b, Y_{\tau_b}, Z_{\tau_b})\mathbf{1}\left\{\tau_b < T\right\}\right) = \mathbb{E}\left(f(\tau_b, Y_{\tau_b}, Z_{\tau_b})\mathbf{1}\left\{\tau_b < \infty\right\}\right).$$

Finally let $\omega \to 0$ and $\beta \to 0$, $\frac{\sin \omega Y_{\tau_b}}{\omega Y_{\tau_b}} \to 1$,

$$\mathbb{E}\left(\mathbf{1}\left\{\tau_b < \infty\right\}\right) = 1$$

The result follows. $\square$

Suppose we have a Brownian motion with drift. Intuitively, the first hitting time of the corresponding super-maximum should depend on the drift. If the drift is too large, the super-maximum may never hit the barrier. The first hitting time is inaccessible.

Theorem 11. *Suppose W_t^μ is a Brownian motion with drift μ under $\mathbb{P}$ and $M_t^\mu = \max_{0<s<t} W_s^\mu$. We define the first hitting time of the super-maximum $Z_t^\mu = M_t^\mu - ct$ as*

$$\tau_b^\mu = \inf\left\{t > 0 | Z_t^\mu = -b\right\}$$

and $\tau_b^{\mu *} = \tau_b^\mu - b/c$,

$$Pr\left(\tau_b^{\mu *} < \infty\right) = \begin{cases} 1, & \text{if } \mu < c \\ \Phi\left(-\mu\sqrt{\frac{b}{c}}\right) - \frac{\mu - 2c}{\mu} \mathrm{e}^{-2b(\mu - c)} \Phi\left(-(\mu - 2c)\sqrt{\frac{b}{c}}\right), & \text{otherwise} \end{cases}$$

Proof. Define $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{e}^{\mu W_t - \mu^2 \frac{t}{2}}$, W_t is a standard Brownian Motion under $\mathbb{Q}$, apply the Girsanov theorem (see Karatzas and Shreve [1991]), for some fixed time T,

$$\begin{aligned}
Pr(\tau_b^{\mu *} < T) =& \mathbb{E}^{\mathbb{P}}\left(\mathbf{1}\left\{\tau_b^{\mu *} < T\right\}\right) = \mathbb{E}^{\mathbb{Q}}\left(\mathbf{1}\left\{\tau_b{}^* < T\right\} \mathrm{e}^{\mu W_T - \frac{\mu^2 T}{2}}\right) \\
=& \mathbb{E}^{\mathbb{Q}}\left(\mathbf{1}\left\{\tau_b{}^* < T\right\} \mathrm{e}^{\mu W_{\tau_b^*} - \frac{\mu^2 \tau_b^*}{2}} \mathrm{e}^{\mu(W_T - W_{\tau_b}) - \frac{\mu^2 (T - \tau_b^*)}{2}}\right) \\
=& \mathbb{E}^{\mathbb{Q}}\left(\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{\mu(W_T - W_{\tau_b}) - \frac{\mu^2 (T - \tau_b^*)}{2}} \middle| \tau_b^*\right) \mathbf{1}\left\{\tau_b{}^* < T\right\} \mathrm{e}^{\mu W_{\tau_b} - \frac{\mu^2 \tau_b^*}{2}}\right) \\
=& \mathbb{E}^{\mathbb{Q}}\left(\mathbf{1}\left\{\tau_b{}^* < T\right\} \mathrm{e}^{\mu W_{\tau_b} - \frac{\mu^2 \tau_b^*}{2}}\right)
\end{aligned}$$

Let $T \to \infty$ and since $Pr(\tau_b < \infty) = 1$ under $\mathbb{Q}$,

$$\begin{aligned} Pr(\tau_b^{\mu *} < \infty) =& \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{\mu W_{\tau_b} - \frac{\mu^2 \tau_b^*}{2}}\right) \\ =& \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-(\frac{1}{2}\mu^2 - c\mu)\tau_b^* - \mu Y_{\tau_b} - \frac{b}{2c}\mu^2}\right) \end{aligned}$$

If we consider $\xi_\pm = \pm\sqrt{c^2 + 2\beta} - c := \xi_\pm(\beta)$ as a function of β,

$$\xi_+\left(\frac{1}{2}\mu^2 - c\mu\right) = \begin{cases} -\mu & \text{if } \mu < c \\ \mu - 2c & \text{if } \mu \geq c \end{cases} \quad \xi_-\left(\frac{1}{2}\mu^2 - c\mu\right) = \begin{cases} \mu - 2c & \text{if } \mu < c \\ -\mu & \text{if } \mu \geq c \end{cases}$$

From the double Laplace transform of τ_b^* and Y_{τ_b} (3.33),

$$\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\beta\tau_b^* - \xi Y_{\tau_b}}\right) = \frac{4c}{\xi - \xi_-}\int_0^\infty \phi(t)\left[\frac{\xi \mathrm{e}^{-\xi t\sqrt{\frac{b}{c}}} - \xi_+ \mathrm{e}^{-\xi_+ t\sqrt{\frac{b}{c}}}}{\xi - \xi_+}\right] dt$$

We substitute $\beta = \frac{1}{2}\mu^2 - c\mu$ and $\xi = \mu$. If $\mu < c$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-(\frac{1}{2}\mu^2 - c\mu)\tau_b^* - \mu Y_{\tau_b}}\right) &= \int_0^\infty \phi(t)\left[\mu \mathrm{e}^{-\mu t\sqrt{\frac{b}{c}}} + \mu \mathrm{e}^{\mu t\sqrt{\frac{b}{c}}}\right] dt \\ =& \left[\frac{\mathrm{e}^{\frac{b}{2c}\mu^2}}{2}\,\mathrm{erf}\left(\frac{\mu\sqrt{\frac{b}{c}} + t}{\sqrt{2}}\right) + \frac{\mathrm{e}^{\frac{b}{2c}\mu^2}}{2}\,\mathrm{erf}\left(\frac{-\mu\sqrt{\frac{b}{c}} + t}{\sqrt{2}}\right)\right]_0^\infty \\ =& \mathrm{e}^{\frac{b}{2c}\mu^2} - \frac{\mathrm{e}^{\frac{b}{2c}\mu^2}}{2}\left[\mathrm{erf}\left(\frac{\mu\sqrt{\frac{b}{c}}}{\sqrt{2}}\right) + \mathrm{erf}\left(-\frac{\mu\sqrt{\frac{b}{c}}}{\sqrt{2}}\right)\right] \\ =& \mathrm{e}^{\frac{b}{2c}\mu^2} \end{aligned}$$

as the error function is an odd function. Therefore, if $\mu < c$, $Pr(\tau_b^{\mu *} < \infty) = 1$.

If $\mu \geq c$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-(\frac{1}{2}\mu^2 - c\mu)\tau_b^* - \mu Y_{\tau_b}}\right) \\
=&\frac{1}{\mu}\int_0^\infty \phi(t)\left[\mu\mathrm{e}^{-\mu t\sqrt{\frac{b}{c}}} - (\mu-2c)\mathrm{e}^{-(\mu-2c)t\sqrt{\frac{b}{c}}}\right]dt \\
=&\frac{1}{\mu}\left[\frac{\mu}{2}\mathrm{e}^{\frac{b}{2c}\mu^2}\operatorname{erf}\left(\frac{\mu\sqrt{\frac{b}{c}}+t}{\sqrt{2}}\right) - \frac{\mu-2c}{2}\mathrm{e}^{\frac{b}{2c}(\mu-2c)^2}\operatorname{erf}\left(\frac{(\mu-2c)\sqrt{\frac{b}{c}}+t}{\sqrt{2}}\right)\right]_0^\infty \\
=&\mathrm{e}^{\frac{b}{2c}\mu^2}\Phi\left(-\mu\sqrt{\frac{b}{c}}\right) - \frac{1}{\mu}(\mu-2c)\mathrm{e}^{\frac{b}{2c}(\mu-2c)^2}\Phi\left(-(\mu-2c)\sqrt{\frac{b}{c}}\right)
\end{aligned}$$

Therefore, if $\mu \geq c$,

$$Pr\left(\tau_b^{\mu *} < \infty\right) = \Phi\left(-\mu\sqrt{\frac{b}{c}}\right) - \frac{1}{\mu}(\mu-2c)\mathrm{e}^{-2b(\mu-c)}\Phi\left(-(\mu-2c)\sqrt{\frac{b}{c}}\right)$$

Note that when $\mu \to c$, $Pr\left(\tau_b^{\mu *}\right) \to 1$ and also when $\mu \to \infty$, $Pr\left(\tau_b^{\mu *}\right) \to 0$. $\square$

3.8 Applications in Derivatives

The result can be used to price derivatives with a lookback trigger event. Some payoff structure with a lookback feature can be found in Musiela and Rutkowski [2006], Hull [2006], etc. Lookbacks are appealing to investors, but very expensive when compared with regular options. As with barrier options, the value of a lookback option is liable to be sensitive to the frequency with which the asset price is observed for the purposes of computing the maximum or minimum. The derivative described in this section assume that the asset price is observed continuously.

Below are the financial notations:
S_t: Underlying asset price at time t
D: Face value of a zero-coupon bond
τ: Trigger time
$\tilde{S}_t$: Maximum price of the asset before time t
r: Risk free rate

We assume S follows a geometric Brownian motion with constant risk free rate r under the risk-neutral measure $\mathbb{Q}$, set

$$X_t = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)t + W_t^{\mathbb{Q}}, \quad M_t = \sup_{0<s<t} X_s, \quad b = \frac{1}{\sigma}\left(\ln\frac{S_0}{D} + rT\right), \quad c = \frac{r}{\sigma}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under the risk-neutral measure. We have

$$S_t = S_0 \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^{\mathbb{Q}}} = S_0 \mathrm{e}^{\sigma X_t}$$

If we consider the following trigger event at time τ for some fixed time T:

$$\{\tau > T\} = \{M_t - ct > -b, \quad \forall t < T\} = \left\{ \sup_{0\le t\le T} S_t > D\mathrm{e}^{-r(T-t)}, \quad \forall t < T \right\}$$

For this trigger event:

- If we want $b > 0$, $S_0 > D\mathrm{e}^{-rT}$. The current stock price must be higher than the present value of the bond.
- Since $\tau \ge b/c$, to make the conversion meaningful i.e. $Pr(\tau \le T) > 0$, we need $b/c < T$, i.e. $S_0 < D$.

The price of a lookback derivative at time zero, V_0, with some payoff function $\Psi(\tau, S_\tau, S_T, \tilde{S}_T)$ can be written as the following:

$$V_0 = \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-rT} \Psi(\tau, S_\tau, S_T, \tilde{S}_T) \mathbf{1}\{\tau \le T\} \right]$$

We denote by $\mathcal{F}_t = \sigma(X_s, s \le t)$ the natural filtration of the Brownian motion $(X_t, t \ge 0)$. Then τ is an $\mathcal{F}_t$-stopping time.

Let $m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$. Define $\tilde{\mathbb{Q}}$ by $\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} = \mathrm{e}^{mX_t - m^2\frac{t}{2}}$, X_t is a standard Brownian Motion under $\tilde{\mathbb{Q}}$. Applying Girsanovs Theorem, we have

$$\begin{aligned} V_0 =& \mathbb{E}^{\mathbb{Q}} \left\{ \mathrm{e}^{-r\tau} \mathbb{E} \left[\mathrm{e}^{-r(T-\tau)} \Psi(\tau, S_\tau, S_T, \tilde{S}_T) | \mathcal{F}_\tau \right] \mathbf{1}\{\tau \le T\} \right\} \\ =& \mathrm{e}^{-rT} \mathbb{E}^{\tilde{\mathbb{Q}}} \left\{ \mathbb{E} \left[\mathrm{e}^{mX_T - m^2\frac{T}{2}} \Psi(\tau, S_\tau, S_T, \tilde{S}_T) | \mathcal{F}_\tau \right] \mathbf{1}\{\tau \le T\} \right\} \\ =& \mathrm{e}^{-(r+\frac{m^2}{2})T} \mathbb{E}^{\tilde{\mathbb{Q}}} \left\{ \mathbb{E} \left[\mathrm{e}^{mX_T} \Psi(\tau, S_\tau, S_T, \tilde{S}_T) | \mathcal{F}_\tau \right] \mathbf{1}\{\tau \le T\} \right\} \end{aligned}$$

Denote $\tilde{S}_{\tau,T} = \sup_{\tau\le t\le T} S_t$ and $M_{\tau,T} = \sup_{\tau\le s\le T} X_s$, then we have

$$\tilde{S}_T = \tilde{S}_\tau \vee \tilde{S}_{\tau,T} \quad \text{and} \quad \tilde{S}_{\tau,T} = S_\tau \mathrm{e}^{\sigma M_{\tau,T}}$$

Substituting into Ψ,

$$\begin{aligned}
\mathrm{e}^{(r+\frac{m^2}{2})T}V_0 =& \mathbb{E}^{\tilde{\mathbb{Q}}}\left\{\mathbb{E}\left[\mathrm{e}^{mX_T}\Psi(\tau, S_\tau, S_T, \tilde{S}_\tau \vee \tilde{S}_{\tau,T})|\mathcal{F}_\tau\right]\mathbf{1}\{\tau \leq T\}\right\} \\
=& \mathbb{E}^{\tilde{\mathbb{Q}}}\left\{\mathrm{e}^{mX_\tau}\mathbb{E}\left[\mathrm{e}^{mX_{T-\tau}}\Psi(\tau, S_\tau, S_T, \tilde{S}_\tau \vee \tilde{S}_{\tau,T})|\mathcal{F}_\tau\right]\mathbf{1}\{\tau \leq T\}\right\}
\end{aligned}$$

Rewrite the inner expectation as an integral,

$$\begin{aligned}
&\mathrm{e}^{(r+\frac{m^2}{2})T}V_0 \\
=& \mathbb{E}^{\tilde{\mathbb{Q}}}\left\{\mathbf{1}\{\tau \leq T\}\mathrm{e}^{mX_\tau}\int_0^\infty\int_{-\infty}^u \mathrm{e}^{mz}\Psi(\tau, S_\tau, S_\tau\mathrm{e}^{\sigma z}, \tilde{S}_\tau \vee S_\tau\mathrm{e}^{\sigma u})f_{M,X}(u,z)dzdu\right\}
\end{aligned} \tag{3.41}$$

where

$$f_{M,X}(u,z) = \frac{2(2u-z)}{(T-\tau)\sqrt{2\pi(T-\tau)}}\mathrm{e}^{-\frac{(2u-z)^2}{2(T-\tau)}}, \quad u \geq 0, x \leq u$$

is the joint distribution of the running maximum M and the corresponding standard Brownian motion X starting at zero ($X_0 = 0$) at time $T - \tau$.

In particular, if $\Psi(\tau, S_\tau, S_T, \tilde{S}_T) = \Psi(\tau, S_\tau, S_T)$ does not depend on $\tilde{S}_T$.

$$\begin{aligned}
&\mathrm{e}^{(r+\frac{m^2}{2})T}V_0 \\
=& \mathbb{E}^{\tilde{\mathbb{Q}}}\left\{\mathbf{1}\{\tau \leq T\}\mathrm{e}^{mX_\tau}\int_{-\infty}^\infty \mathrm{e}^{mz}\Psi(\tau, S_\tau, S_\tau\mathrm{e}^{\sigma z})f_X(z)dz\right\}
\end{aligned}$$

where

$$f_X(z) = \frac{1}{\sqrt{2\pi(T-\tau)}}\mathrm{e}^{-\frac{z^2}{2(T-\tau)}}, \quad -\infty < z < \infty.$$

(3.41) motivates a simulation algorithm to price the derivative.

Algorithm 6 Simulation Method for the Lookback Derivative

1: Generate (τ, X_τ) from Algorithm 5
2: **if** $\tau > T$ **then** $V_0^{(i)} = 0$ and repeat from step 1 **else**
3: Calculate $S\tau = S_0 \mathrm{e}^{\sigma X_\tau}$ and $\tilde{S}\tau = S_0 \mathrm{e}^{\sigma(-b+c\tau)}$
4: Generate $X_T = X_\tau + \sqrt{T-\tau}Z$, where $Z \sim N(0,1)$
5: Generate $M_{\tau,T} = \frac{\sqrt{(X_T - X_\tau)^2 + 2(T-\tau)E}}{2} + X_\tau$, where $E \sim \exp(1)$
6: Calculate $S_T = S_\tau \mathrm{e}^{\sigma(X_T - X_\tau)}$, and $\tilde{S}_{\tau,T} = S_\tau \mathrm{e}^{\sigma M_{\tau,T}}$ and hence $\tilde{S}_T = \tilde{S}_\tau \vee \tilde{S}_{\tau,T}$
7: Calculate $V_0^{(i)} = \mathrm{e}^{-(r+\frac{m^2}{2})T} \mathrm{e}^{m X_T} \Psi(S_\tau, \tau, S_T, \tilde{S}_T)$
8: Repeat step 1-7 for N times to obtain $V_0^{(1)}$, $V_0^{(2)}$, ..., $V_0^{(N)}$
9: Return the simulated price $\hat{V}_0 = \frac{1}{N}\sum_{i=1}^N V_0^{(i)}$

In this chapter, we continue to solve the joint density function of the first hitting time and the associated reflected Brownian motion and provide an acceptance-rejection algorithm. The infinite horizon problems are solved as an immediate result. Since the problem is motivate from derivatives pricing, financial claims with look-back features are given, including contingent convertibles.

3.8.1 Trigger Probability

One crucial concern for such a derivative is the probability of the trigger event happening before the maturity. Under the risk-neutral measure $\mathbb{Q}$, the probability is

$$\mathbb{Q}(\tau \leq T) = \mathbb{E}^{\mathbb{Q}}(\mathbf{1}\{\tau \leq T\}) = \mathbb{E}^{\tilde{\mathbb{Q}}}(\mathrm{e}^{mX_\tau - \frac{m^2}{2}\tau}\mathbf{1}\{\tau \leq T\})$$

which can also be estimated by the simulation scheme. Intuitively, the trigger (knock-in) probability increases with the slope c and the intercept $-b$ of the linear barrier. Both $b = \frac{1}{\sigma}\left(\ln \frac{S_0}{D} + rT\right)$ and $c = \frac{r}{\sigma}$ increase with r and decrease with σ. The effects of volatility and interest rate are mixed on the trigger level.

The trigger probability, on the other hand, is related to binary options whose prices are discounted trigger probabilities. Binary options can be of particular interest as they are not regulated by the FCA but the Gambling Commission, at the time of writing this dissertation.

One way to decide the knock-in level is to think D is a function of b and hence $D = S_0\mathrm{e}^{rT-b\sigma}$. In particular, we can set $b \approx 0$ and $b > 0$, so the trigger level solely depends on $c = r/\sigma$. Then

$$D \approx F_T = S_0\mathrm{e}^{rT}$$

which is the forward price of the underlying asset. The financial interpretation is straight forward: you want to bet on the difference between the running maximum and the forward price of the underlying asset.

The following graph plots the trigger probabilities for different volatility σ, for parameters $S_0 = 100$, $T = 5$, $r = 0.05$, with $D = F = S_0 e^{rT}$. These values are obtained by simulating 20000 samples.

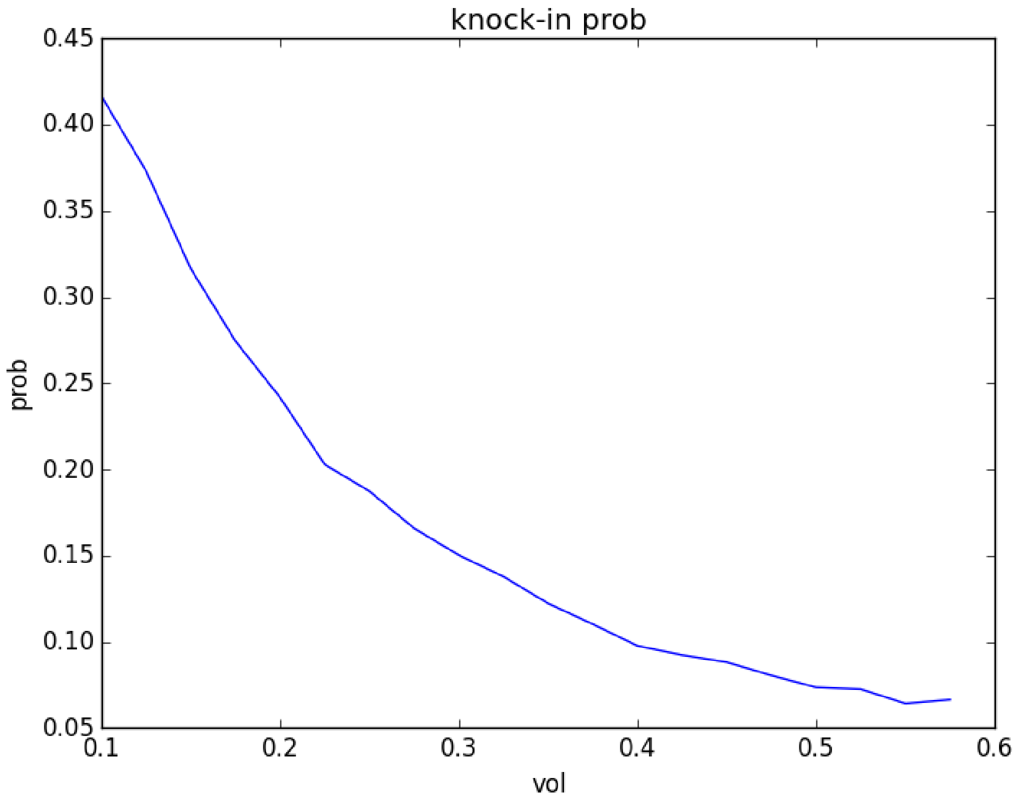

Figure 3.12: Trigger Probabilities of Lookback Derivatives

Volatility may drive the stock price up or down. Since the running maximum increases with volatility, the trigger probability decreases as expected.

3.8.2 Barrier Options

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.

The trigger event described above can be embedded into the option contract as a "down-and-in" feature to reduce the initial premium. A down-and-in option is a knock-in option which comes into existence only when the underlying asset price reaches a barrier. Here we have the running maximum of the asset price instead.

A standard down-and-in call option payoff will have the following form:

$$C_T^1 = (S_T - K)^+ \mathbf{1}\{\tau \leq T\}$$

The price of it will be

$$\begin{aligned} C_0^1 =& \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-rT} (S_T - K)^+ \mathbf{1}\{\tau \leq T\} \right] \\ =& \mathbb{E}^{\mathbb{Q}} \left\{ \mathbf{1}\{\tau \leq T\} \mathrm{e}^{-r\tau} \mathbb{E} \left[\mathrm{e}^{-r(T-\tau)} (S_T - K)^+ | \mathcal{F}_\tau \right] \right\} \end{aligned}$$

By the strong Markov property of Brownian motion (see Karatzas and Shreve [1991]), the inner expectation can be recognized as a vanilla call.

Here we introduce the Black-Scholes formula for option pricing Black and Scholes [1973]. The price of call option for a non-dividend paying asset at time 0 is

$$\begin{aligned} C_0(S_0, K, T) =& \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-rT} (S_T - K)^+ \right] \\ =& S_0 \Phi(d_1) - K \mathrm{e}^{-rT} \Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_0}{K} + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Substituting, we have

$$\begin{aligned}C_0^1 =& \mathbb{E}^{\mathbb{Q}}\left\{\mathbf{1}\{\tau \le T\}\mathrm{e}^{-r\tau}C_\tau(S_\tau, K, T-\tau)\right\} \\ =& \mathbb{E}^{\mathbb{Q}}\left\{\mathbf{1}\{\tau \le T\}\mathrm{e}^{-r\tau}\left[S_\tau\Phi(d_1') - K\mathrm{e}^{-r(T-\tau)}\Phi(d_2')\right]\right\}\end{aligned} \tag{3.42}$$

where

$$d_1' = \frac{\ln\frac{S_\tau}{K} + (r+\sigma^2/2)(T-\tau)}{\sigma\sqrt{T-\tau}}, \quad d_2 = d_1 - \sigma\sqrt{T-\tau}, \quad S_\tau = S_0\mathrm{e}^{\sigma X_\tau}$$

Forward start options are options that will start at some time in the future. Similar to (3.42), the price of a down-and-in forward start option FT start at τ with payoff function

$$FS_T = (S_T - S_\tau)^+\mathbf{1}\{\tau \le T\}$$

is

$$FS_0 = \mathbb{E}^{\mathbb{Q}}\left\{\mathbf{1}\{\tau \le T\}\mathrm{e}^{-r\tau}C_\tau(S_\tau, S_\tau, T-\tau)\right\}$$

where $C_\tau(S_\tau, S_\tau, T-\tau)$ denotes the Black-Schole price of a vanilla call option at time τ with initial asset price S_τ, strike S_τ and maturity $T-\tau$.

3.8.3 Lookback Options

Lookback options are an example of path-dependent options. Option contracts whose payoff at expiry depends not only on the price of the underlying asset at a certain time point (usually maturity), but also on asset price fluctuations during the options' lifetimes.

A standard lookback put option and its pricing formula are described in Musiela and Rutkowski [2006], whose terminal payoff equals

$$LP_T = (\tilde{S}_T - S_T)^+ = (\tilde{S}_T - S_T)$$

Note that a lookback option is not a genuine option contract since the (European) lookback option is always exercised by its holder at its expiry date. This contract can be expensive too.

The trigger event described above can be embedded into the contract as a "down-and-in" feature to reduce the initial premium. A down-and-in option is a knock-in option which comes into existence only when the underlying asset price reaches a barrier. Here we have the running maximum of the asset price instead. An example of barrier lookback put is described below:

$$BLP_{0,\tau} = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}(\tilde{S}_\tau - S_\tau)\mathbf{1}\{\tau \leq T\}\right]$$

The owner will receive the difference between the running maximum of the asset price and asset price at the trigger time τ.

Let $m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$. Define $\tilde{\mathbb{P}}$ by $\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} = \mathrm{e}^{mX_t - m^2\frac{t}{2}}$, X_t is a standard Brownian Motion under $\tilde{\mathbb{Q}}$. Applying Girsanovs Theorem, we have

$$\begin{aligned} BLP_{0,\tau} =& \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathrm{e}^{mX_\tau - m^2\frac{\tau}{2}}\mathrm{e}^{-r\tau}(\tilde{S}_\tau - S_\tau)\mathbf{1}\{\tau \leq T\}\right] \\ =& \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathrm{e}^{mX_\tau}\mathrm{e}^{-(r+\frac{m^2}{2})\tau}(D\mathrm{e}^{-r(T-\tau)} - S_0\mathrm{e}^{\sigma X_\tau})\mathbf{1}\{\tau \leq T\}\right] \end{aligned}$$

Since the trigger time τ is random and hence the cash flow, it increases the reinvestment risk. Another contract can be structured for the owner to receive the receive the difference between the running maximum of the asset price and asset price at maturity:

$$BLP_{0,T} = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-rT}(\tilde{S}_T - S_T)\mathbf{1}\{\tau \leq T\}\right]$$

By substituting $\Psi(S_\tau, \tau, S_T, \tilde{S}_T) = \tilde{S}_T - S_T$ into (3.41), we have

$$\begin{aligned} & BLP_{0,T} \\ =& \mathrm{e}^{-(r+\frac{m^2}{2})T}\mathbb{E}^{\tilde{\mathbb{Q}}}\left\{\mathbf{1}\{\tau \leq T\}\mathrm{e}^{mX_\tau}\int_0^\infty\int_{-\infty}^u \mathrm{e}^{mz}(\tilde{S}_\tau \vee S_\tau\mathrm{e}^{\sigma u} - S_\tau\mathrm{e}^{\sigma z}))f_{M,X}(u,z)dzdu\right\} \end{aligned}$$

3.8.4 An example of Contingent Convertibles

The payoff of Contingent Convertibles (CoCo) would be equal to the value of a bond if conversion never happens before maturity. If a trigger event occurs, the owner would get the shares. Therefore, the price of CoCo at time $t = 0$ has the following form (see Musiela and Rutkowski [2006] for pricing contingent claims):

$$CoCo_0 = \mathbb{E}^{\mathbb{Q}}\left[S_\tau \mathrm{e}^{-r\tau}\mathbf{1}\{\tau \leq T\} + D\mathrm{e}^{-rT}\mathbf{1}\{\tau > T\}\right] \tag{3.43}$$

A lookback feature can be added to the contingent claim. The contingent claim then is analogous to a reverse convertible but with a "lookback" trigger event, which depends on the maximum of the asset price over the lifetime and also the present value of the zero coupon bond. The Lookback Reverse Convertible can be structured as a combination of longing a zero coupon and shorting a barrier lookback put option with the trigger time as the maturity.

$$\begin{aligned}
CoCo_0 =& \mathbb{E}^{\mathbb{Q}}\left[S_\tau \mathrm{e}^{-r\tau}\mathbf{1}\{\tau \leq T\} + D\mathrm{e}^{-rT}\mathbf{1}\{\tau > T\}\right] \\
=& \mathbb{E}^{\mathbb{Q}}\left[S_\tau \mathrm{e}^{-r\tau}\mathbf{1}\{\tau \leq T\} + D\mathrm{e}^{-rT}(1 - \mathbf{1}\{\tau \leq T\})\right] \\
=& D\mathrm{e}^{-rT} + \mathbb{E}^{\mathbb{Q}}\left[(S_\tau \mathrm{e}^{-r\tau} - D\mathrm{e}^{-rT})\mathbf{1}\{\tau \leq T\}\right] \\
=& D\mathrm{e}^{-rT} - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}[D\mathrm{e}^{-r(T-\tau)} - S_\tau]\mathbf{1}\{\tau \leq T\}\right] \\
=& D\mathrm{e}^{-rT} - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}[\tilde{S}_\tau - S_\tau]\mathbf{1}\{\tau \leq T\}\right] \\
=& D\mathrm{e}^{-rT} - BLP_{0,\tau}
\end{aligned}$$

where $\mathbb{Q}$ denotes the risk-neutral measure.

Obviously, we can see $CoCo_0 \leq D\mathrm{e}^{-rT}$. The value is the highest when it is not converted.

Then we can carefully apply Girsanov's Theorem and the result in the previous sections to obtain the price by simulation. Let $m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$. Define $\tilde{\mathbb{Q}}$ by $\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} = \mathrm{e}^{mX_t - m^2\frac{t}{2}}$, X_t is a standard Brownian Motion under $\tilde{\mathbb{Q}}$. Then

$$\begin{aligned} CoCo_0 =& D\mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\sigma Y_\tau}\mathbf{1}\{\tau \le T\} + \mathbf{1}\{\tau > T\}\right] \\ =& D\mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[1 - (1 - \mathrm{e}^{-\sigma Y_\tau})\mathbf{1}\{\tau \le T\}\right] \\ =& D\mathrm{e}^{-rT}\left\{1 - \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathrm{e}^{mX_\tau - m^2\frac{\tau}{2}}(1 - \mathrm{e}^{-\sigma Y_\tau})\mathbf{1}\{\tau \le T\}\right]\right\} \end{aligned}$$

This form is more desirable for simulation as when τ is very large,

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathrm{e}^{mX_\tau - m^2\frac{\tau}{2}}\mathbf{1}\{\tau > T\}\right] \to 1$$

but

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln(\mathrm{e}^{mX_\tau - m^2\frac{\tau}{2}}\mathbf{1}\{\tau > T\})\right] \to -\infty$$

can be very skewed and hence increase the variance of the price.

Table 3.1, 3.2 and 3.3 give the prices of the lookback convertibles for different initial stock prices S_0, with parameters $T = 5$, $r = 0.04$, $\sigma = 0.4$, $D = 100$, unless specified. The present value of a zero coupon bond with these parameters is $De^{-rT} = 81.87$. These values are obtained by simulating 20000 samples.

Table 3.1: Prices of Lookback Convertibles with Different Maturities

T	S_0 =90	S_0 =92	S_0 =94	S_0 =96	S_0 =98
2	-	-	88.910883	90.854598	91.095172
3	82.088587	83.776462	85.467776	87.372622	87.651429
5	77.919373	78.382724	79.251274	79.703791	80.999520
10	64.851386	64.877435	64.894619	66.068075	66.172912

Table 3.2: Prices of Lookback Convertibles with Different Interest Rates

r	S_0 =90	S_0 =92	S_0 =94	S_0 =96	S_0 =98
0.03	81.671804	82.625188	83.735580	84.386375	85.266319
0.04	77.611750	78.633984	79.485007	80.303148	80.995793
0.05	73.789333	74.679791	75.547243	76.310873	77.075869
0.06	70.016678	70.808653	71.851603	72.541492	73.297076

Table 3.3: Prices of Lookback Convertibles with Different Volatilities

σ	S_0 =90	S_0 =92	S_0 =94	S_0 =96	S_0 =98
0.2	74.157754	75.586224	77.059301	78.818261	80.306868
0.3	76.687399	77.821685	78.722629	79.776830	80.867256
0.4	77.463987	78.462941	79.491323	80.173495	80.986845
0.5	78.110482	78.939914	79.658690	80.399851	81.156018

We can explain the sensitivities from the tables.

- The price increases with the initial stock price. A large initial stock price will defer the trigger event.
- The price decreases with the maturity. The instrument can be seen as longing a zero coupon bond and shorting a down-and-in lookback put option. Increasing the maturity decreases the present value of the bond and increases the option time value. It also allows more time for the trigger event to happen.
- The price decreases with the interest rate. Similarly, increasing the interest rate decreases the present value of the bond. On the other hand, a change in interest rates generally has a minor overall impact on the pricing of options. Especially in this set-up, the strike of the lookback option is not fixed but depends on the maximum of the stock price.
- The price increases with the volatility. High volatility pushes up the running maximum of the stock price and has two effects on the option position. First, a large running maximum will delay the trigger event. Second, the payoff if triggered will be higher. It seems that the probability of triggering has a larger effect than the payoff.

Chapter 4

A Differential Equation Approach for the Laplace Transform of the First Hitting Time for the Running Maximum Brownian Motion by a Linear Barrier

4.1 Introduction

In the previous chapters, the Laplace transform of of the first hitting time τ_b^* is solved by sine and cosine transform of the solution of the infinitesimal generator. The trick is to recognize the existence of the Laplace transform.

In this chapter, we present another approach which rewrites the expectation of the stopped martingale as an integral equation. With another integral transform, the equation is reduced to an ODE. Solving the initial value problem of the ODE gives the Laplace transform of the τ_b^*. Here we have to recognize the maximum of Brownian motion only hits the barrier when it is not increasing.

This approach is developed in parallel and hopes to provide some insights for further research.

4.2 An Ordinary Differential Equation of the Laplace Transform

Refer to (3.1), the corresponding first hitting time of a linear barrier is defined as

$$\tau_b = \inf\left\{t > 0 | M_t - ct = -b\right\}$$

Theorem 12. *Let the joint density function of τ_b^* and Y_{τ_b} be $p(t, y)$ and denote*

$$\hat{P}(y) = \int_y^\infty \int_0^\infty \mathrm{e}^{-\beta t} p(t, u) dt du$$

then it satisfies

$$\frac{-1}{2c}\hat{P}''(y) - \hat{P}'(y) + \frac{\beta}{c}\hat{P}(y) = \sqrt{\frac{2}{\pi}\frac{c}{b}}\mathrm{e}^{-\frac{c}{2b}y^2}$$

with the boundary conditions $\hat{P}(\infty) = 0$ and $\hat{P}'(0) = 0$.

Proof. We start with the stopped martingale in (3.4)

$$\mathbb{E}\left\{\mathrm{e}^{-\beta\tau_b^*}\left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin \omega y_{\tau_b} + \cos \omega y_{\tau_b}\right]\right\} = \mathrm{e}^{-\frac{b}{2c}\omega^2}$$

Let the joint density function of τ_b^* and Y_{τ_b} be $p(t, y)$ and denote

$$\hat{p}(\beta, y) = \int_0^\infty \mathrm{e}^{-\beta t} p(t, y) dt$$

Rewrite LHS of (3.4),

$$
\begin{aligned}
&\mathbb{E}\left\{ \mathrm{e}^{-\beta\tau_b^*} \left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin\omega y_{\tau_b} + \cos\omega y_{\tau_b} \right] \right\} \\
=& \int_0^\infty \int_0^\infty \mathrm{e}^{-\beta t} \left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin\omega y + \cos\omega y \right] p(t,y)dtdy \\
=& \int_0^\infty \left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin\omega y + \cos\omega y \right] \hat{p}(\beta,y)dy \\
=& \frac{\beta}{c}\int_0^\infty \frac{\sin\omega y}{\omega}\hat{p}(\beta,y)dy + \frac{1}{2c}\int_0^\infty \omega\sin\omega y\hat{p}(\beta,y)dy + \int_0^\infty \cos\omega y\hat{p}(\beta,y)dy
\end{aligned}
$$

For convenience, we drop β from the notation and denote $\hat{p}(\beta,y) = \hat{p}(y)$. Consider the first term,

$$
\begin{aligned}
&\int_0^\infty \omega\sin\omega y\hat{p}(y)dy \\
=& -\int_0^\infty \omega\sin\omega y \int_y^\infty \hat{p}'(u)dudy \\
=& -\int_0^\infty \int_0^u \omega\sin\omega y dy\hat{p}'(u)du \\
=& \int_0^\infty \hat{p}'(u)\cos\omega u du - \int_0^\infty \hat{p}'(u)du
\end{aligned}
$$

Since $\hat{p}(y) = -\int_y^\infty \hat{p}'(u)du$, we have

$$
-\int_0^\infty \hat{p}'(u)du = \hat{p}(0) = 0
$$

An intuitive argument is given here. Since the running maximum of the Brownian motion will not hit the barrier when it is increasing, it means Y_{τ_b} is always larger than zero. Therefore, $p(t,0) \equiv 0$ for all t and also

$$
\hat{p}(\beta,0) = \int_0^\infty \mathrm{e}^{-\beta t} p(t,0)dt = 0
$$

This property can also be seen in the iterative discretization scheme in section (4).

Consider the second term in the expectation,

$$
\begin{aligned}
&\int_0^\infty \frac{\sin \omega y}{\omega} \hat{p}(y) dy \\
&= - \int_0^\infty \frac{\sin \omega y}{\omega} d \left(\int_y^\infty \hat{p}(u) du \right) \\
&= \int_0^\infty \left(\int_y^\infty \hat{p}(u) du \right) \cos \omega y dy
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^\infty \left[(\frac{\beta}{c\omega} + \frac{\omega}{2c}) \sin \omega y + \cos \omega y \right] \hat{p}(\beta, y) dy \\
= &\int_0^\infty \left(\frac{\beta}{c} \int_y^\infty \hat{p}(u) du + \frac{1}{2c} \hat{p}'(y) + \hat{p}(y) \right) \cos \omega y dy
\end{aligned}
$$

For the RHS of (3.4),

$$
\mathrm{e}^{-\frac{b}{2c}\omega^2} = \sqrt{\frac{2}{\pi}} \sqrt{\frac{c}{b}} \int_0^\infty \mathrm{e}^{-\frac{y^2}{2(\frac{b}{c})}} \cos \omega y dy
$$

Denote

$$
\hat{P}(y) = \int_y^\infty \int_0^\infty \mathrm{e}^{-\beta t} p(t, u) dt du = \int_y^\infty \hat{p}(u) du
$$

and hence

$$
\hat{P}'(y) = -\hat{p}(y), \quad \hat{P}''(y) = -\hat{p}'(y)
$$

We obtained an ODE from (3.4),

$$
\frac{-1}{2c} \hat{P}''(y) - \hat{P}'(y) + \frac{\beta}{c} \hat{P}(y) = \sqrt{\frac{2}{\pi} \frac{c}{b}} \mathrm{e}^{-\frac{c}{2b} y^2}
$$

Two boundary conditions are needed to solve the ODE. By definition, $\hat{P}(\infty) = 0$. Another boundary condition is $\hat{P}'(0) = -\hat{p}(0) = 0$. $\square$

4.3 Solution of the Differential Equation and the Laplace Transform

In particular, we are interested in solving $\hat{P}(0) = \mathbb{E}(\mathrm{e}^{-\beta\tau_b^*})$.

Theorem 13. *Denote* $\xi_\pm = \pm\sqrt{c^2+2\beta} - c$,

$$\hat{P}(0) = \frac{4c}{\sqrt{2\beta+c^2}+c}\mathrm{e}^{\frac{b}{2c}\xi_+^2}\Phi\left(-\sqrt{\frac{b}{c}}\xi_+\right)$$

Proof. The general solution of the ODE(12) is given by

$$\begin{aligned}\hat{P}(y) =& \mathrm{e}^{\xi_- y}\int_1^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_- z\right)dz \\ &- \mathrm{e}^{\xi_+ y}\int_1^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_+ z\right)dz \\ &+ k_1\mathrm{e}^{\xi_- y} + k_2\mathrm{e}^{\xi_+ y}\end{aligned}$$

for some constants k_1 and k_2. Consider the boundary conditions,

$$\lim_{y\to\infty}\hat{P}(y) = \lim_{y\to\infty}\mathrm{e}^{\xi_+ y}\left(k_2 - \int_1^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_+ z\right)dz\right) = 0$$

then

$$k_2 = \int_1^\infty \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_+ z\right)dz$$

and

$$\begin{aligned}&\hat{P}(y) = \\ &\quad \mathrm{e}^{\xi_- y}\left(\int_1^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_- z\right)dz + k_1\right) \\ &\quad + \mathrm{e}^{\xi_+ y}\int_y^\infty \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b} - \xi_+ z\right)dz\end{aligned}$$

Consider another boundary condition,

$$\hat{P}'(y) =$$

$$\xi_- e^{\xi_- y}\left(\int_1^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_- z\right)dz + k_1\right)$$
$$+ e^{\xi_- y}\left(\frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cy^2}{2b}-\xi_- y\right)\right)$$
$$+ \xi_+ e^{\xi_+ y}\int_y^\infty \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_+ z\right)dz$$
$$- e^{\xi_+ y}\left(\frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cy^2}{2b}-\xi_+ y\right)\right)$$

Substitute $y = 0$,

$$k_1 =$$

$$-\frac{\xi_+}{\xi_-}\int_1^\infty \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_+ z\right)dz$$
$$-\int_1^0 \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_- z\right)dz$$

Therefore,

$$\hat{P}(y) =$$

$$e^{\xi_- y}\left(\int_0^\infty \frac{-\xi_+}{\xi_-}\frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_+ z\right)dz\right.$$
$$\left.+\int_0^y \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_- z\right)dz\right)$$
$$+ e^{\xi_+ y}\int_y^\infty \frac{c}{\sqrt{c^2+2\beta}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_+ z\right)dz$$

Put $y=0$,

$$\hat{P}(0) = \left(1-\frac{\xi_+}{\xi_-}\right)\frac{c}{\sqrt{2\beta+c^2}}\int_0^\infty \sqrt{\frac{2}{\pi}}\sqrt{\frac{c}{b}}\exp\left(-\frac{cz^2}{2b}-\xi_+ z\right)dz$$
$$=\frac{4c}{\sqrt{2\beta+c^2}+c}\sqrt{\frac{c}{b}}\int_0^\infty \phi\left(\sqrt{\frac{c}{b}}(z+\frac{b}{c}\xi_+)\right)\mathrm{e}^{\frac{b}{2c}\xi_+^2}dz$$

Let $t=\sqrt{\frac{c}{b}}(z+\frac{b}{c}\xi_+)$, the Laplace transform of τ_b^* is recovered,

$$\hat{P}(0) = \frac{4c}{\sqrt{2\beta+c^2}+c}\mathrm{e}^{\frac{b}{2c}\xi_+^2}\int_{\sqrt{\frac{b}{c}}\xi_+}^\infty \phi(t)dt$$
$$=\frac{4c}{\sqrt{2\beta+c^2}+c}\mathrm{e}^{\frac{b}{2c}\xi_+^2}\Phi\left(-\sqrt{\frac{b}{c}}\xi_+\right)$$

□

Chapter 5

Crossing Probabilities for the Running Maximum of Brownian Motion by a Poisson Barrier

In this chapter, we present two importance sampling methods through exponential tilting and a compound Poisson martingale to estimate the crossing probabilities for Running Maximum of Brownian motion by Poisson barriers

5.1 Introduction

In the previous chapters, the first hitting time for the running maximum of a Brownian motion by linear barriers was simulated by an acceptance-rejection method. Here we look at the infinite horizon problem by a compound Poisson barrier and estimate the probability that the barrier is eventually crossed.

We are interested in the simulation method of the barrier crossing probabilities. The running maximum of a drifted Brownian motion is not guaranteed to hit the barrier if the drift is of a different sign of the barrier. In a naive simulation scheme, one often needs to stop the simulation prematurely when the number of iterations or the total time elapsed is large enough. These long trials are counted as a failure to hit the barrier. Therefore, truncation methods underestimate the probabilities of barrier crossing. It is more problematic when the crossing probabilities are very small, for example, when the barrier is far away or the force driving away the process is strong, or both. Most of the sample paths do not cross the barrier and are wasted.

Importance sampling methods are used to reduce both the bias and the variance of the estimated barrier crossing probabilities. The fundamental idea behind importance sampling is the same as that of change of measure. Under certain regularity conditions, the expectation under one probability measure can be expressed as an expectation under another probability measure through the Radon-Nikodym theorem. The correct choice of the alternative probability measure will achieve variance reduction.

An early classic example of importance sampling applied to derivatives pricing is Reider [1993]. In pricing deeply out-of-money call European options, increasing the drift in the underlying geometric Brownian motions substantially decreases the variance in simulations. More examples on applying importance sampling in derivatives pricing include Andersen [1995] and Boyle et al. [1997].

In the context of insurance risk, importance sampling is used to estimate small ruin probabilities. In Asmussen et al. [1985], attention is given to the ruin probability in infinite horizon for compound Poisson risk processes. In Dassios and Zhao [2012], they consider a risk process with the arrival of claims modelled by a dynamic contagion process, a generalization of the Cox process and Hawkes process introduced by Dassios and Zhao [2011]. In both applications, importance sampling is used to change the original measure to a new measure such that large downward jumps occur more often and hence ruins happen more often.

In this chapter, we provide two likelihood ratios, via exponential tilting and compound Poisson martingales, to obtain an upper bound of barrier crossing probabilities and achieve variance reduction through importance sampling. Exponential tilting is a distribution shifting technique commonly used in rare-event simulation, and rejection and importance sampling in particular. Exponential tilting is known as the Esscher transform in mathematical finance Asmussen and Glynn [2007] and is used in such contexts as insurance futures pricing Cruz et al. [2014]. The earliest formalization is often attributed to Esscher, see Butler [2007], with its use in importance sampling being attributed to David Siegmund, see Siegmund [1976]. Another way to look at importance sampling is change of measure Musiela and Rutkowski [1997]. Therefore, we also present a Poisson martingale method.

5.2 Definition

The construction is similar to definition (3.1) but the barrier is no longer deterministic. In this chapter, we are going to define the Compound Poisson process

$$Z_t = \sum_{i=1}^{N_t} U_i$$

where N_t is a Poisson process with rate λ and the jump size $U_i > 0$ has distribution function $H(u)$. The corresponding first hitting time of a compound Poisson barrier is defined as

$$\tau_b = \inf\{t > 0 | M_t - Z_t \leq -b\}$$

with $b > 0$. Therefore, the hitting level is below zero. If the hitting level is above zero, the problem is reduced to the first hitting time of a Brownian motion (not the running maximum of it). Therefore we focus the case when the hitting level is below zero.

The compound Poisson barrier is not continuous and will overshoot the running maximum so the stopping position for $M_t - Z_t$ can exceed $-b$. An illustration graph will be presented in the later section with the simulation algo.

5.3 A Martingale for the Compound Poisson Barrier

Let $\psi(\nu) = \int_0^\infty \mathrm{e}^{\nu u} dH(u)$ be the moment generating function the distribution function of $H(u)$. A martingale for the compound Poisson process $Z_t = \sum_{i=1}^{N_t} U_i$ is

$$g(t, Z) = \mathrm{e}^{-\lambda t(\psi(\nu)-1)+\nu Z}$$

In particular, if U_i are i.i.d. exponential random variables with mean $1/\alpha$, i.e. $U_i \sim \exp(\alpha)$, $\psi(\nu) = \frac{\alpha}{\alpha-\nu}$ for $\nu < \alpha$. The martingale becomes:

$$g(t, Z) = \exp\left(\lambda\left(1 - \frac{\alpha}{\alpha-\nu}\right)t + \nu Z\right)$$

Recall the martingale for (M_t, Y_t) in (3),

$$f(t, M, Y) = \exp\left(\frac{\omega^2}{2}t - \gamma M\right)h(Y)$$

where

$$h(y) = \cos(\omega y) + \frac{\gamma}{\omega}\sin(\omega y)$$

Let $-\beta = \frac{\omega^2}{2} + \lambda\left(1 - \frac{\alpha}{\alpha-\nu}\right)$, we have

$$\nu_\beta = \alpha\left(1 - \frac{\lambda}{\frac{\omega^2}{2} + \lambda + \beta}\right) < \alpha$$

As the compound Poisson process Z_t and the pair (M_t, Y_t) are independent, the process $f \cdot g(t, M, Y, Z)$ is still a martingale.

$$\begin{aligned} f\cdot g(t, M_t, Y_t, Z_t) =& \mathrm{e}^{(\frac{\omega^2}{2}+\lambda(1-\frac{\alpha}{\alpha-\nu_\beta}))t}\mathrm{e}^{-\gamma M_t}\mathrm{e}^{\nu_\beta Z_t}h(Y_t) \\ =& \mathrm{e}^{-\beta t}\mathrm{e}^{-(\gamma-\nu_\beta)M_t}\mathrm{e}^{-\nu_\beta(M_t-Z_t)}h(Y_t) \end{aligned} \tag{5.1}$$

For convenience, we denote $f\cdot g(t, M_t, Y_t, Z_t)$ as $f\cdot g(t)$. With the finite stopping time $\tau_b \wedge T$ for some fixed $T > 0$,

$$\mathbb{E}(f\cdot g(\tau_b \wedge T)) = \mathbb{E}(f\cdot g(\tau_b)\mathbf{1}\{\tau < T\}) + \mathbb{E}(f\cdot g(T)\mathbf{1}\{\tau_b \geq T\})$$

Since $\{M_T - Z_T > -b, \forall T < \tau_b\}$,

$$\mathbb{E}(f\cdot g(T)\mathbf{1}\{\tau_b \geq T\}) < \mathbb{E}(\mathrm{e}^{-\beta T}\mathrm{e}^{-(\gamma-\nu_\beta)M_T}\mathrm{e}^{-\nu_\beta b}h(Y_T)\mathbf{1}\{\tau_b \geq T\})$$

For $\beta > 0$ and $\gamma \geq \nu_\beta$, let $T \to \infty$, as $h(Y_T)$ is bounded,

$$\mathbb{E}(\mathrm{e}^{-\beta T}\mathrm{e}^{-(\gamma-\nu_\beta)M_T}\mathrm{e}^{-\nu_\beta b}h(Y_T)\mathbf{1}\{\tau_b \geq T\}) \to 0$$

Therefore, applying the Optional Stopping Theorem,

$$\mathbb{E}(f\cdot g(\tau)\mathbf{1}\{\tau_b < \infty\}) = \mathbb{E}(f\cdot g(0)) = 1$$

By letting $\beta \to 0$ and $\gamma \to \nu_\beta$,

$$\mathbb{E}(\mathbf{1}\{\tau_b < \infty\}) = 1$$

Therefore, the boundary crossing happens with probability one in the driftless case. Intuitively, it is natural since $\mathbb{E}(M_t) \propto \sqrt{t}$ but $\mathbb{E}(Z_t) \propto t$. With these martingales, we can obtain a differential equation for the Laplace transform of the first hitting time, see Appendix (.1).

5.4 An Iterative Simulation Algorithm for the First Hitting Time

Boundary crossing can only happen when a jump occurs in Z_t. Therefore, a simple simulation method is to simulate the following quantities accordingly,

1. the inter-arrival time τ_i
2. the jump size U_{τ_i}
3. the running maximum M_{τ_i} at the jump time

and check if the boundary is crossed. In Figure 5.1, an illustration graph similar to Figure 3.4 is plotted. The increasing barrier is a compound Poisson process instead of a straight line. The first hitting time in this particular case is $\tau_b = \tau_1 + \tau_2$.

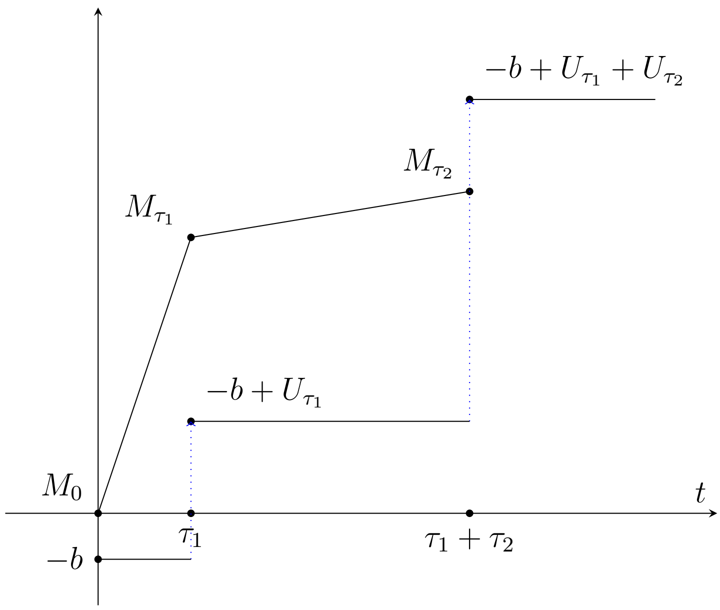

Figure 5.1: An illustration of iterative simulation of compound Poisson barrier crossing

Suppose $X_t^\mu = \mu t + W_t$ is a Brownian motion with drift μ and $M_t^\mu = \max_{0<s<t} W_s^\mu$, the algorithm of simulation $\tau_b^\mu = \inf\{t > 0 | M_t^\mu - Z_t \leq -b\}$ is below.

Algorithm 7 Simulation Method for the Compound Poisson Barrier

1: Generate $\tau_i \sim \exp(1/\lambda)$, with $\tau_0 = 0$
2: Generate $U_i \sim H$
3: Generate Generate $X_i^\mu = X_{i-1}^\mu + \mu\tau_i + N(0,1)$, where $N(0,1)$ is a standard normal r.v.
4: Generate $m_i^\mu = \frac{X_i^\mu + X_{i-1}^\mu + \sqrt{(X_i^\mu - X_{i-1}^\mu)^2 - 2\tau_i ln(V)}}{2}$, where $V \sim U(0,1)$
5: Set $M_i^\mu = \max(m_i^\mu, M_{i-1}^\mu)$
6: Repeat previous steps until $M_i^\mu - \sum_{k=1}^i U_k \leq -b$, return ($\tau_b^\mu$, $M_{\tau_b}^\mu$, $X_{\tau_b}^\mu$, $Z_{\tau_b}^\mu$)=($\sum_{k=1}^i \tau_k$, M_i^μ, X_i^μ, $\sum_{k=1}^i U_k$)

A sample path is plotted in fig 5.2, with $b = 1$, $\mu = 0$, $X_0 = 0$, $M_0 = 0$, $\lambda = 1$ and $U_i \sim \exp(1)$.

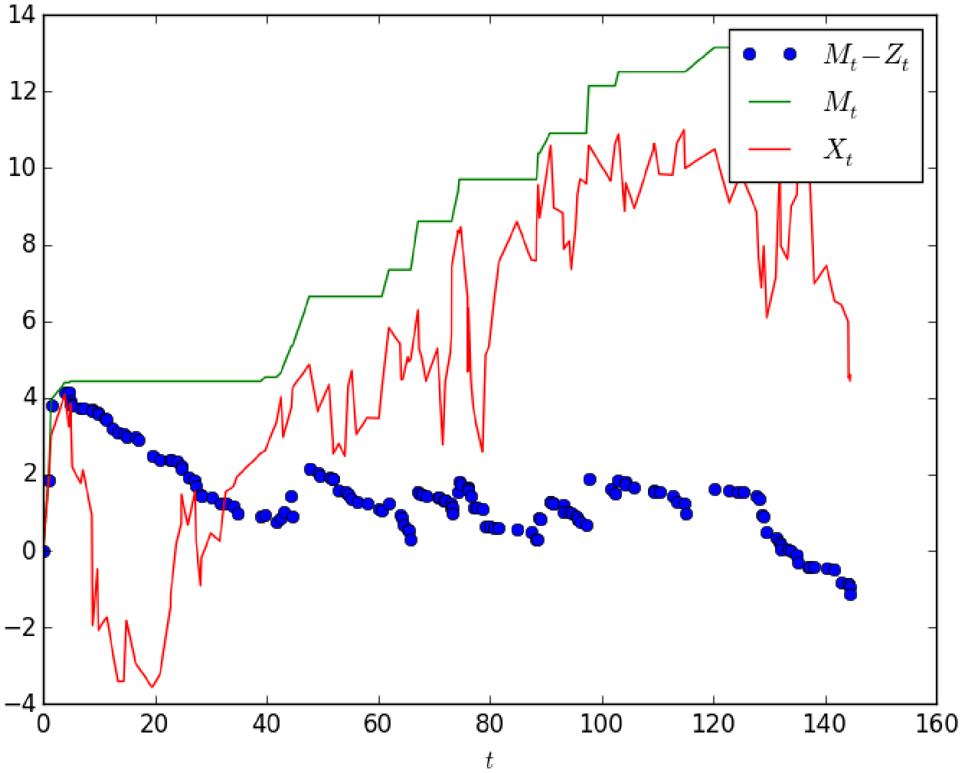

Figure 5.2: A sample path of M_t, X_t and $M_t - Z_t$.

Since the simulation method only generates values when there is a jump, the dots also indicate the locations of jumps.

10000 samples of τ_b and $M_\tau - X_\tau$ are plotted in fig 5.3 and 5.4 respectively, with $b = 1$, $\mu = 0$, $X_0 = 0$, $M_0 = 0$, $\lambda = 1$ and $U \sim \exp(1)$.

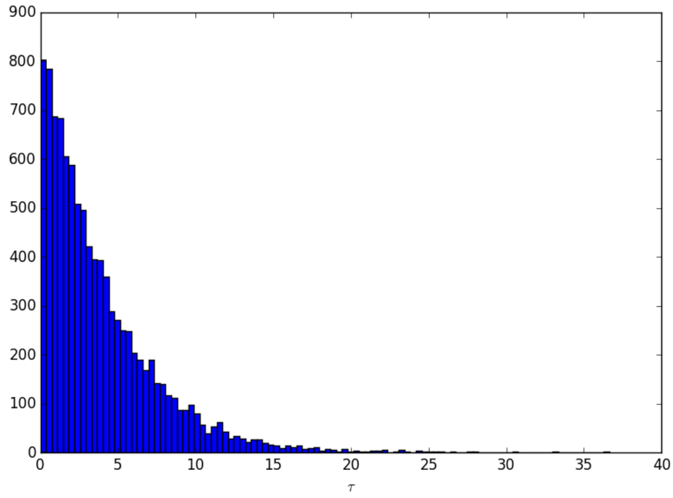

Figure 5.3: Histogram of Stopping Time τ for Poisson Barrier

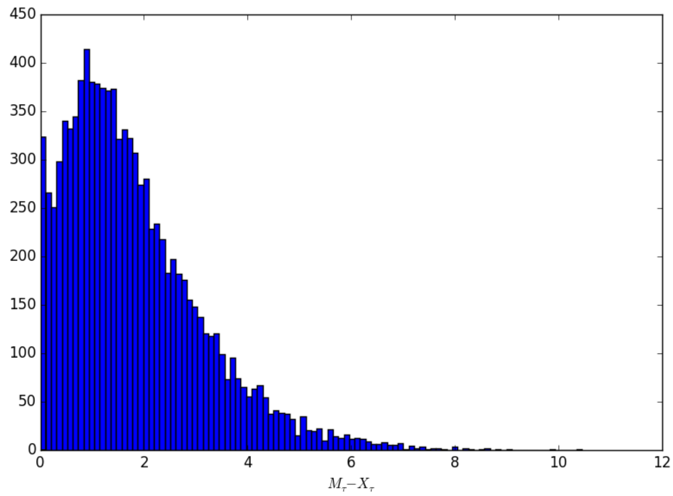

Figure 5.4: Histogram of Stopping Position $M_\tau - X_\tau$ for Poisson Barrier

We observe from the histograms that (1) the distribution of τ might be thin-tailed and (2) there is a peak at zero of the distribution of $M_\tau - X_\tau$, the density function might not be zero at zero.

5.5 Infinite Horizon with Drift

Theorem 14. *Let $X_t = \mu t + W_t$ is a Brownian motion with drift μ, $M_t = \max_{0<s<t} X_s$, $Y_t = M_t - X_t$ and $\tau = \inf\{t > 0 | M_t - Z_t \leq -b\}$*

$$Pr(\tau < \infty) = \frac{\alpha - \nu_\beta}{\alpha} \frac{\mathrm{e}^{-\nu_\beta b}}{\mathbb{E}^\mu \left(\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau - \mu X_\tau} h(Y_\tau) \middle| \tau < \infty \right)}$$

where $h(Y_\tau) = \cos(\omega Y_\tau) + \frac{\gamma}{\omega}\sin(\omega Y_\tau)$.

Proof. Consider the martingale in (5.1) for a standard Brownian Motion X_t, its running maximum M_t and the compound Poisson process Z_t,

$$f{\cdot}g(t) = \mathrm{e}^{-\beta t}\mathrm{e}^{-(\gamma-\nu_\beta)M_t}\mathrm{e}^{-\nu_\beta(M_t - Z_t)}h(Y_t)$$

where $\nu_\beta = \alpha\left(1 - \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right) < \alpha$ and $\nu_\beta > 0$ for $\beta \geq 0$. By Optional Stopping Theorem, for some fixed $T > 0$,

$$\mathbb{E}(f{\cdot}g(\tau \wedge T)) = 1$$

On the other hand,

$$\mathbb{E}(f{\cdot}g(\tau \wedge T)) = \mathbb{E}(f{\cdot}g(\tau)\mathbf{1}\{\tau < T\}) + \mathbb{E}(f{\cdot}g(T)\mathbf{1}\{\tau \geq T\})$$

Applying Girsanov Theorem,

$$\mathbb{E}(f{\cdot}g(T)\mathbf{1}\{\tau \geq T\}) = \mathbb{E}^\mu(\mathrm{e}^{-\mu X_T + \frac{\mu^2}{2}T} f{\cdot}g(T)\mathbf{1}\{\tau \geq T\})$$

Under the measure μ, X_t is a Brownian Motion with drift μ, $M_t = \sup_{0\leq s\leq t} X_s$, $Y_t = M_t - X_t$ is a reflected Brownian Motion with drift $-\mu$ (see Peskir [2006]),

$$\begin{aligned}
&\mathbb{E}^\mu(\mathrm{e}^{-\mu X_T + \frac{\mu^2}{2}T} f{\cdot}g(T)\mathbf{1}\{\tau \geq T\}) \\
=&\mathbb{E}^\mu(\mathrm{e}^{-\mu X_T + \frac{\mu^2}{2}T}\mathrm{e}^{-\beta T}\mathrm{e}^{-(\gamma-\nu_\beta)M_T}\mathrm{e}^{-\nu_\beta(M_T - Z_T)}h(Y_T)\mathbf{1}\{\tau \geq T\}) \\
<&\mathbb{E}^\mu(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})T}\mathrm{e}^{-\mu X_T}\mathrm{e}^{-(\gamma-\nu_\beta)M_T}\mathrm{e}^{\nu_\beta b}h(Y_T)\mathbf{1}\{\tau \geq T\})
\end{aligned}$$

If $-(\gamma-\nu_\beta) \leq \mu$ and $\beta \geq \mu^2/2$, let $T \to \infty$,

$$\begin{aligned}
&\mathbb{E}^\mu(\mathrm{e}^{-\mu X_T+\frac{\mu^2}{2}T} f{\cdot}g(T)\mathbf{1}\{\tau \geq T\}) \\
\leq &\mathrm{e}^{\nu_\beta b}\mathbb{E}^\mu(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})T}\mathrm{e}^{\mu Y_T}h(Y_T)\mathbf{1}\{\tau \geq T\}) \to 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}(f{\cdot}g(\tau)\mathbf{1}\{\tau < T\}) \\
=&\mathbb{E}^\mu(\mathrm{e}^{-\mu X_\tau+\frac{\mu^2}{2}\tau} f{\cdot}g(\tau)\mathbf{1}\{\tau < T\}) \\
=&\mathbb{E}^\mu(\mathrm{e}^{-\mu X_\tau+\frac{\mu^2}{2}\tau}\mathrm{e}^{-\beta\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}\mathrm{e}^{-\nu_\beta(M_\tau-Z_\tau)}h(Y_\tau)\mathbf{1}\{\tau < T\})
\end{aligned}$$

Let U_τ be the overshoot and $U_\tau \sim \exp(\alpha)$,

$$\begin{aligned}
&\mathbb{E}(f{\cdot}g(\tau)\mathbf{1}\{\tau < T\}) \\
=&\mathbb{E}^\mu(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})\tau}\mathrm{e}^{-\mu X_\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}\mathrm{e}^{\nu_\beta(b+U_\tau)}h(Y_\tau)\mathbf{1}\{\tau < T\}) \\
=&\frac{\alpha\mathrm{e}^{\nu_\beta b}}{\alpha-\nu_\beta}\mathbb{E}^\mu(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})\tau}\mathrm{e}^{-\mu X_\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}h(Y_\tau)\mathbf{1}\{\tau < T\})
\end{aligned}$$

Let $T \to \infty$,

$$\begin{aligned}
1 =&\frac{\alpha\mathrm{e}^{\nu_\beta b}}{\alpha-\nu_\beta}\mathbb{E}^\mu(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})\tau}\mathrm{e}^{-\mu X_\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}h(Y_\tau)\mathbf{1}\{\tau < \infty\}) \\
=&\frac{\alpha\mathrm{e}^{\nu_\beta b}}{\alpha-\nu_\beta}\mathbb{E}^\mu\left(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})\tau}\mathrm{e}^{-\mu X_\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}h(Y_\tau)\Big|\tau < \infty\right)Pr(\tau < \infty)
\end{aligned}$$

Rearranging terms,

$$Pr(\tau < \infty) = \frac{\alpha-\nu_\beta}{\alpha}\frac{\mathrm{e}^{-\nu_\beta b}}{\mathbb{E}^\mu\left(\mathrm{e}^{-(\beta-\frac{\mu^2}{2})\tau}\mathrm{e}^{-\mu X_\tau}\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau}h(Y_\tau)\Big|\tau < \infty\right)}$$

Further substituting $\beta = \mu^2/2$,

$$Pr(\tau < \infty) = \frac{\alpha-\nu_\beta}{\alpha}\frac{\mathrm{e}^{-\nu_\beta b}}{\mathbb{E}^\mu\left(\mathrm{e}^{-(\gamma-\nu_\beta)M_\tau-\mu X_\tau}h(Y_\tau)\Big|\tau < \infty\right)}$$

$\square$

Note that if $e^{-(\gamma-\nu_\beta)M_\tau-\mu X_\tau}h(Y_\tau) \geq 1$. We have an inequality for the ruin probability,

$$Pr(\tau < \infty) \leq \frac{\alpha - \nu_\beta}{\alpha}e^{-\nu_\beta b}$$

For example if $\gamma \geq \nu_\beta - \mu \geq 0$ then $e^{-(\gamma-\nu_\beta)M_\tau-\mu X_\tau} \geq e^{\mu Y_\tau} \geq 1$, we can let $\omega \to 0$, and

$$h(Y_\tau) = \cos(\omega Y_\tau) + \frac{\gamma}{\omega}\sin(\omega Y_\tau) \to 1 + \gamma Y_\tau \geq 1$$

But if $\mu >> 0$ and $\alpha \to 0$,

$$\beta >> 0 \quad \text{and} \quad \nu_\beta = \alpha\left(1 - \frac{\lambda}{\frac{\omega^2}{2} + \lambda + \beta}\right) \to \alpha$$

then $\gamma = \alpha - \mu < 0$ and we may not obtain an upper bound between 0 and 1. In this case, the process is very volatile.

Although we have an expression of the ruin probability via the original drifted measure, the expression is not suitable for simulation as the sample size for the ruin to happen could be small, especially for a large drift.

5.6 Importance Sampling for Infinite Horizon

We also want to simulate the infinite horizon probability $Pr(\tau_b < \infty)$ but we can not let the programme run forever. One way is to stop the programme after a large enough number of iterations N and approximate:

$$Pr(\tau_b^\mu = \infty) \approx Pr(N_{\tau_b} > N)$$

Or stop it when τ_b is larger than a very big number, say T' and approximate:

$$Pr(\tau_b^\mu = \infty) \approx Pr(\tau_b > T')$$

However both methods tend to overestimate $Pr(\tau_b^\mu = \infty)$ and can be very slow if a very large N or T' is used.

We can apply importance sampling to the Brownian motion, the compound Poisson process or both.

5.6.1 Long Horizon Problem

One intuitive idea is to change the drift to zero or even negative so that the barrier is hit faster. However, this can increase the variance of the simulation. This problem is explained in Glasserman [2003].

Define $\tilde{\mathbb{P}}$ by $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = e^{\mu W_t - \mu^2 \frac{t}{2}}$ and denote $A_t = e^{\mu W_t - \mu^2 \frac{t}{2}}$, W_t is a standard Brownian Motion under $\tilde{\mathbb{P}}$. Applying the Girsanov theorem,

$$\begin{aligned} Pr(\tau_b^\mu < T) =& \mathbb{E}^{\mathbb{P}} \left(\mathbf{1} \left\{ \tau_b^\mu < T \right\} \right) \\ =& \mathbb{E}^{\hat{\mathbb{P}}'} \left(\mathbf{1} \left\{ \tau_b^{\mu'} < T \right\} \frac{A_{\tau_b}}{A'_{\tau_b}} \right) \\ =& \mathbb{E}^{\hat{\mathbb{P}}'} \left(\mathbf{1} \left\{ \tau_b^{\mu'} < T \right\} e^{(\mu-\mu')X_{\tau_b}^{\mu'} - \frac{\mu^2 - \mu'^2}{2}\tau_b} \right) \end{aligned}$$

where $X_t^{\mu'}$ is a Brownian motion with a drift μ'. Let $T \to \infty$ and for $\mu' \leq 0$,

$Pr(\tau_b^{\mu'} < \infty) = 1$. For simplicity, we first consider the case when $\mu' = 0$,

$$Pr(\tau_b^{\mu} < \infty) = \mathbb{E}^{\hat{\mathbb{P}}}\left(\mathrm{e}^{\mu W_{\tau_b} - \frac{\mu^2 \tau_b}{2}}\right)$$

Since $Pr(\tau_b < \infty) = 1$, theoretically, the programme will stop eventually. However, even when $\mu' = 0$, if the downward force is not strong enough, i.e. $\lambda\mathbb{E}(U) << b + m_0$, the programme will only stop after many iterations.

We then consider negative drifts. For $\mu >> 0$ and $\mu' \leq 0$, A_t/A'_t will be highly skewed since $\mathbb{E}^{\hat{\mathbb{P}}'}(A_t/A'_t) = 1$ but $\mathbb{E}^{\hat{\mathbb{P}}'}(\ln A_t - \ln A'_t) \to -\infty$ as $t \to \infty$. An example is when μ is large and take $\mu' = -\mu$,

$$Pr(\tau_b^{\mu} < \infty) = \mathbb{E}^{\hat{\mathbb{P}}'}(\mathrm{e}^{2\mu X_{\tau_b}^{\mu'}})$$

where $X_t^{\mu'}$ is a Brownian motion with a large negative drift μ'. The estimator is always very close to zero but the expectation is one. Large values with small but non-negligible probability are generated in the simulation. This can result in a large increase in variance. Therefore, we may need a well-designed change of measure for this case.

5.6.2 Exponential Tilting

Exponential tilting is a distribution shifting technique commonly used in rare-event simulation and importance sampling in particular.

Theorem 15. *Let X_t be a Brownian motion with drift μ, $M_t = \max_{0<s<t} X_s$, $Y_t = M_t - X_t$, $Z_t = \sum_{i=1}^{N_t} U_i$, $N_t \sim Poi(\lambda t)$ and $U_i \sim \exp(\alpha)$ and $\tau^\mu = \inf\{t > 0 | M_t - Z_t \leq -b\}$ under the original measure,*

$$Pr(\tau^\mu < \infty) = \frac{\alpha - \theta_*}{\alpha} \mathrm{e}^{-\theta_* b} \mathbb{E}_\theta^{\mu'} \left(\mathrm{e}^{-\theta_* Y_\tau} \mathbf{1}\{\tau^{\mu'_*} < \infty\} \right)$$

Where under the tilted measure, X_t is a Brownian motion with drift μ'_* , $N_t \sim Poi(\frac{\lambda\alpha}{\alpha-\theta_*}t)$ and $U_i \sim \exp(\alpha - \theta_*)$, with $\mu'_* = \mu - \theta_*$ and
$\theta_* = \frac{1}{2}\left(\alpha + 2\mu - \sqrt{(\alpha - 2\mu)^2 + 4\lambda}\right)$.

Proof. Let X_t be a Brownian Motion with drift μ under the original measure and the tilted density is

$$f_\theta(x) = \mathrm{e}^{(\mu-\mu')x - \frac{\mu^2-\mu'^2}{2}t} f_X(x)$$

which is a normal density function with drift μ' and for the compound Poisson process $Z_t = \sum_{i=1}^{N_t} U_i$ and the tilted density is

$$g_\theta(z) = \mathrm{e}^{-\theta Z_t} M_Z(\theta) g_Z(z)$$

where

$$M_Z(\theta) = \exp\left(\lambda t \left(\frac{\alpha}{\alpha - \theta} - 1\right)\right)$$

Under the tilted measure, by Girsanov Theorem, X_t is a Brownian motion with drift μ'. After the transformation the moment generating function of Z_t,

$$\begin{aligned} M_Z^\theta(r) =& \frac{\mathbb{E}(\mathrm{e}^{(r+\theta)Z})}{M_Z(\theta)} = \frac{M_Z(r+\theta)}{M_Z(\theta)} = \frac{\exp\left(\lambda t \left(\frac{\alpha}{\alpha-r-\theta} - 1\right)\right)}{\exp\left(\lambda t \left(\frac{\alpha}{\alpha-\theta} - 1\right)\right)} \\ =& \exp\left(\frac{\lambda\alpha t}{\alpha - \theta}\left(\frac{\alpha - \theta}{\alpha - \theta - r} - 1\right)\right) \end{aligned}$$

So the transformation inflates the means of both distributions. Now $N_t \sim Poi(\frac{\lambda\alpha}{\alpha-\theta}t)$ and $U_i \sim \exp(\alpha-\theta)$.

The ruin probability for $\mu > 0$ and some μ',

$$\begin{aligned} Pr(\tau^{\mu} < \infty) =& \mathbb{E}_{\theta}^{\mu'} \left(\mathrm{e}^{(\mu-\mu')X_{\tau}} \mathrm{e}^{-\frac{1}{2}(\mu^2-\mu'^2)\tau} \mathrm{e}^{-\theta Z_{\tau}} M_Z(\theta) \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \\ =& \mathbb{E}_{\theta}^{\mu'} \left(\mathrm{e}^{(\mu-\mu')(M_{\tau}-Y_{\tau})} \mathrm{e}^{-\frac{1}{2}(\mu^2-\mu'^2)\tau} \mathrm{e}^{-\theta Z_{\tau}} \mathrm{e}^{\lambda\tau(\varphi(\theta)-1)} \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \end{aligned}$$

where $\varphi(\theta)$ is the MGF of U. Let $\mu - \mu' = \theta > 0$,

$$Pr(\tau^{\mu} < \infty) = \mathbb{E}_{\theta}^{\mu'} \left(\mathrm{e}^{-\theta(b+U_{\tau})} \mathrm{e}^{-\theta Y_{\tau}} \mathrm{e}^{(-\frac{1}{2}(\mu^2-\mu'^2)+\lambda\varphi(\theta)-\lambda)\tau} \mathbf{1}\{\tau^{\mu'} < \infty\} \right)$$

If under the orginial measure, $U \sim \exp(\alpha)$, $\varphi(\theta) = \frac{\alpha}{\alpha-\theta}$. Under the new measure, $U \sim \exp(\alpha-\theta)$,

$$\mathbb{E}_{\theta}^{\mu'} \left(\mathrm{e}^{-\theta U_{\tau}} \right) = \frac{\alpha-\theta}{\alpha-\theta+\theta} = \frac{\alpha-\theta}{\alpha}$$

If further set

$$-\frac{1}{2}(\mu^2-\mu'^2) + \frac{\alpha\lambda}{\alpha-\theta} - \lambda = 0$$

Substituting $\mu'_* = \mu - \theta$ and solve the quadratic equation

$$\theta_{\pm} = \frac{1}{2}\left(\alpha + 2\mu \pm \sqrt{(\alpha-2\mu)^2 + 4\lambda} \right)$$

as we want $\alpha - \theta > 0$ for the exponential distribution, take

$$\theta_{*} = \frac{1}{2}\left(\alpha + 2\mu - \sqrt{(\alpha-2\mu)^2 + 4\lambda} \right)$$

Then

$$Pr(\tau^{\mu} < \infty) = \frac{\alpha-\theta_*}{\alpha} \mathrm{e}^{-\theta_* b} \mathbb{E}_{\theta}^{\mu'} \left(\mathrm{e}^{-\theta_* Y_{\tau}} \mathbf{1}\{\tau^{\mu'_*} < \infty\} \right)$$

□

Note that the ruin probability has an upper bound

$$Pr(\tau^{\mu} < \infty) \leq \frac{\alpha-\theta_*}{\alpha} \mathrm{e}^{-\theta_* b}$$

In particular, if $\mu'_* \leq 0$,

$$Pr(\tau^\mu < \infty) = \frac{\alpha - \theta_*}{\alpha} \mathrm{e}^{-\theta_* b} \mathbb{E}_\theta^{\mu'} \left(\mathrm{e}^{-\theta_* Y_\tau} \right)$$

For $(\lambda, \alpha, b) = (2, 1, 1)$, 10000 samples are generated for every following μ.

Table 5.1: Importance Sampling by Exponential Tilting (10000 samples)

μ	θ_*	μ'	η	$\hat{\alpha}$	$\bar{p}$	$var(p)$	bound	$\bar{N}$	$\bar{p'}$
2.0000	0.4384	1.5616	3.5616	0.5616	0.2213	0.0070	0.3622	3.0474	1.0000
2.5000	0.5505	1.9495	4.4495	0.4495	0.1488	0.0042	0.2592	2.4990	1.0000
3.0000	0.6277	2.3723	5.3723	0.3723	0.1117	0.0027	0.1987	2.1406	1.0000
3.5000	0.6834	2.8166	6.3166	0.3166	0.0871	0.0019	0.1599	1.9534	1.0000
4.0000	0.7251	3.2749	7.2749	0.2749	0.0725	0.0014	0.1331	1.8044	1.0000
4.5000	0.7574	3.7426	8.2426	0.2426	0.0613	0.0010	0.1138	1.6942	1.0000
5.0000	0.7830	4.2170	9.2170	0.2170	0.0533	0.0008	0.0992	1.6107	1.0000
5.5000	0.8038	4.6962	10.1962	0.1962	0.0475	0.0006	0.0878	1.5368	1.0000
6.0000	0.8211	5.1789	11.1789	0.1789	0.0422	0.0005	0.0787	1.4776	1.0000

Below are the notations in the table.

- μ': drift under the tilted measure
- η: Poisson rate under the tilted measure
- $\hat{\alpha}$: Exponential parameter under the tilted measure
- $\bar{p}$: Estimated ruin probability under the original measure
- $var(p)$: Sample variance of the 10000 samples
- bound: the upper bound of the ruin probability under the original measure
- $\bar{N}$: Average iterations in 10000 samples
- $\bar{p'}$: Estimated ruin probability under the tilted measure

Exponential tilting decreases the drifts of the Brownian motion μ' and increases both the jump intensities η and sizes $1/\hat{\alpha}$ of the compound Poisson process. The simulations stop in a finite time after the change of measure, indicated by $\bar{p'}$.

5.6.3 Change of Measure via compound Poisson Martingale

We may want some flexibility in the importance sampling and hence a Poisson martingale, which is function of the Poisson process N_t, is introduced here.

Let U_i be iid random variables with measure ν and there is an absolutely continuous measure $\hat{\nu}$ w.r.t. ν s.t. h is the Radon-Nikodym derivative, i.e. $h(x) := \frac{d\hat{\nu}}{d\nu}(x)$ and $f(x) := \ln(\frac{\eta}{\lambda}h(x))$. Define

$$L_t := \exp\left(\sum_{i=1}^{N_t} f(U_i) + (\lambda - \eta)t\right)$$

where $\eta > 0$ and L_t is a strictly positive martingale with expectation equal to 1. This martingale allows the change of both jump intensity and size.

In particular, if the distributions of the jump size are unchanged, i.e. $h(x) = 1$ and let the new jump intensity $\eta = (1+\beta)\lambda$,

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := L_t = (1+\beta)^{N_t}\mathrm{e}^{-\lambda\beta t}$$

The process Z_t is a $\mathbb{Q}$-Poisson process with intensity $\eta = (1+\beta)\lambda$.

In our case, we might want to increase the jump size and keep the jump distribution exponential, i.e. $\nu \sim \exp(\alpha)$ and $\hat{\nu} \sim \exp(\hat{\alpha})$, where $1/\alpha$ and $1/\hat{\alpha}$ are the expected jump sizes respectively,

$$L_t = \exp\left(N_t(\ln(\frac{\alpha}{\hat{\alpha}}) + \ln(\frac{\eta}{\lambda})) + (\alpha - \hat{\alpha})Z_t + (\lambda - \eta)t\right)$$

Finally, if we want the jump intensity remains the same, i.e. $\eta = \lambda$,

$$L_t = \exp\left(N_t \ln(\frac{\alpha}{\hat{\alpha}}) + (\alpha - \hat{\alpha})Z_t\right)$$

Theorem 16. *Let $X_t = \mu t + W_t$ is a Brownian motion with drift μ, $M_t = \max_{0<s<t} X_s$, $Y_t = M_t - X_t$ and $\tau = \inf\{t > 0 | M_t - Z_t \leq -b\}$*

$$Pr(\tau^{\mu} < \infty) = \frac{\alpha - (\mu - \mu')}{\alpha} \mathrm{e}^{-(\mu-\mu')b} \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}} \left(\mathrm{e}^{-N_\tau \ln(\frac{\alpha}{\hat{\alpha}} \frac{\eta}{\lambda}) - (\mu-\mu')Y_\tau} \mathbf{1}\{\tau^{\mu'} < \infty\} \right)$$

where μ' satisfies

$$\begin{cases} \mu'^2 < 2\lambda + \mu^2 \\ \mu' > \mu - \alpha \end{cases}$$

Under the measure $\hat{\mathbb{P}}'$ and $\mathbb{Q}$, X_t is a Brownian Motion with drift μ' and $Z_t = \sum_{i=1}^{N_t} U_i$ is compound Poisson process, where N_t is a Poisson process with rate η and the jump size $U_i \sim \exp(\hat{\alpha})$.

Proof. By using the martingales of Brownian motion $A_t = \mathrm{e}^{\mu W_t - \mu^2 \frac{t}{2}}$ and $A'_t = \mathrm{e}^{\mu' W_t - \mu'^2 \frac{t}{2}}$, and the compound Poisson martingale $L_t = \exp\left(N_t \ln(\frac{\alpha}{\hat{\alpha}}) + (\alpha - \hat{\alpha}) Z_t\right)$,

$$\begin{aligned} Pr(\tau^{\mu} < \infty) =& \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'} \left(\frac{A_\tau}{A'_\tau L_\tau} \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \\ =& \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'} \left(\mathrm{e}^{-(\lambda - \eta + \frac{\mu^2 - \mu'^2}{2})\tau + (\mu-\mu')W_\tau - N_\tau \ln(\frac{\alpha}{\hat{\alpha}} \frac{\eta}{\lambda}) - (\alpha - \hat{\alpha}) Z_\tau} \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \end{aligned}$$

Set $\lambda - \eta + \frac{\mu^2 - \mu'^2}{2} = 0$,

$$\begin{aligned} Pr(\tau^{\mu} < \infty) =& \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'} \left(\exp\left((\mu - \mu')W_\tau - N_\tau \ln(\frac{\alpha}{\hat{\alpha}} \frac{\eta}{\lambda}) - (\alpha - \hat{\alpha}) Z_\tau \right) \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \\ =& \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'} \left(\exp\left((\mu - \mu')(M_\tau - Y_\tau) - N_\tau \ln(\frac{\alpha}{\hat{\alpha}} \frac{\eta}{\lambda}) - (\alpha - \hat{\alpha}) Z_\tau \right) \mathbf{1}\{\tau^{\mu'} < \infty\} \right) \end{aligned}$$

Set $\mu - \mu' - (\alpha - \hat{\alpha}) = 0$,

$$
\begin{aligned}
Pr(\tau^{\mu} < \infty) =& \mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'}\left(\exp\left(-(\mu-\mu')(b+U_\tau) - N_\tau \ln(\frac{\hat{\alpha}}{\alpha}\frac{\eta}{\lambda}) - (\mu-\mu')Y_\tau\right)\mathbf{1}\{\tau^{\mu'} < \infty\}\right) \\
=& \frac{\hat{\alpha}}{\hat{\alpha}+\mu-\mu'}\mathrm{e}^{-(\mu-\mu')b}\mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}}\left(\mathrm{e}^{-N_\tau \ln(\frac{\alpha}{\hat{\alpha}}\frac{\eta}{\lambda}) - (\mu-\mu')Y_\tau}\mathbf{1}\{\tau^{\mu'} < \infty\}\right) \\
=& \frac{\alpha-(\mu-\mu')}{\alpha}\mathrm{e}^{-(\mu-\mu')b}\mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}}\left(\mathrm{e}^{-N_\tau \ln(\frac{\alpha}{\hat{\alpha}}\frac{\eta}{\lambda}) - (\mu-\mu')Y_\tau}\mathbf{1}\{\tau^{\mu'} < \infty\}\right)
\end{aligned}
$$

where μ' satisfies

$$
\begin{cases} \eta = \lambda + \frac{\mu^2}{2} - \frac{\mu'^2}{2} > 0 \\ \hat{\alpha} = \alpha + \mu' - \mu > 0 \end{cases} \Leftrightarrow \begin{cases} \mu'^2 < 2\lambda + \mu^2 \\ \mu' > \mu - \alpha \end{cases}
$$

$\square$

Further if $\mu' \le 0$, $Pr(\tau^{\mu'} < \infty) = 1$, then

$$
Pr(\tau^{\mu} < \infty) = \frac{\alpha-(\mu-\mu')}{\alpha}\mathrm{e}^{-(\mu-\mu')b}\mathbb{E}_{\mathbb{Q}}^{\hat{\mathbb{P}}'}\left(\mathrm{e}^{-N_\tau \ln(\frac{\alpha}{\hat{\alpha}}\frac{\eta}{\lambda}) - (\mu-\mu')Y_\tau}\right)
$$

On the other hand, if $\eta/\hat{\alpha} > \lambda/\alpha$ and $\mu' < \mu$, we have an upper bound for the ruin probability

$$
Pr(\tau^{\mu} < \infty) \le \frac{\alpha-(\mu-\mu')}{\alpha}\mathrm{e}^{-(\mu-\mu')b}
$$

Note that we have some flexibility to choose the new drift μ'. Intuitively, if we can achieve a smaller drift for the Brownian motion $\mu' < \mu$, and a larger 'drift' for the compound Poisson process $\eta/\hat{\alpha} > \lambda/\alpha$, the ruin probability under the new measure $Pr(\tau^{\mu'} < \infty)$ will be higher. Therefore, more non-zero samples can be obtained from the simulation to reduce the variance.

For $(\lambda, \alpha, b) = (2, 1, 1)$ and simply take $\mu' = \frac{1}{2}(\mu - \alpha + \sqrt{2\lambda + \mu^2})$ (mid-point of the feasible range), 10000 samples are generated for every following μ.

Table 5.2: Importance Sampling by Poisson Martingale (10000 samples)

μ	μ'	η	$\hat{\alpha}$	$\bar{p}$	$var(p)$	bound	$\bar{N}$	$\bar{p'}$
2.0000	1.9142	2.1679	0.9142	0.2669	0.0561	0.7077	18.2330	1.0000
2.5000	2.3508	2.3619	0.8508	0.1434	0.0300	0.5280	21.7218	1.0000
3.0000	2.8028	2.5722	0.8028	0.0903	0.0164	0.4114	26.0977	0.9997
3.5000	3.2656	2.7930	0.7656	0.0658	0.0104	0.3320	27.8509	0.9989
4.0000	3.7361	3.0209	0.7361	0.0506	0.0069	0.2755	33.6629	0.9977
4.5000	4.2122	3.2536	0.7122	0.0416	0.0049	0.2338	35.3116	0.9971
5.0000	4.6926	3.4898	0.6926	0.0340	0.0037	0.2021	39.1861	0.9940
5.5000	5.1762	3.7286	0.6762	0.0292	0.0027	0.1774	44.4506	0.9946
6.0000	5.6623	3.9693	0.6623	0.0253	0.0022	0.1577	44.7210	0.9925

Below are the notations in the table.

- μ': drift under the new measure
- η: Poisson rate under the new measure
- $\hat{\alpha}$: Exponential parameter under the new measure
- $\bar{p}$: Estimated ruin probability under the original measure
- $var(p)$: Sample variance of the 10000 samples
- bound: the upper bound of the ruin probability under the original measure
- $\bar{N}$: Average iterations in 10000 samples
- $\bar{p'}$: Estimated ruin probability under the new measure

From the numerical results of both methods, we can see

- The jump rates and sizes increases drastically under the new measures.
- The drifts under new measures are not decreased too much compared to the Poisson rates and the exponential parameters.
- The average iterations are smaller than the plain simulations. If the true ruin probability is very small, the average iterations in the plain simulation scheme is usually the maximum iterations allowed.
- The estimated ruin probabilities under the new measures are close to one.

The comparison between two methods are plotted in Figure 5.5. (p1, bound1) are the estimated ruin probability and upper bound for the Poisson Martingale method. (p2, bound2) are the estimated ruin probability and upper bound for the exponential tilting method.

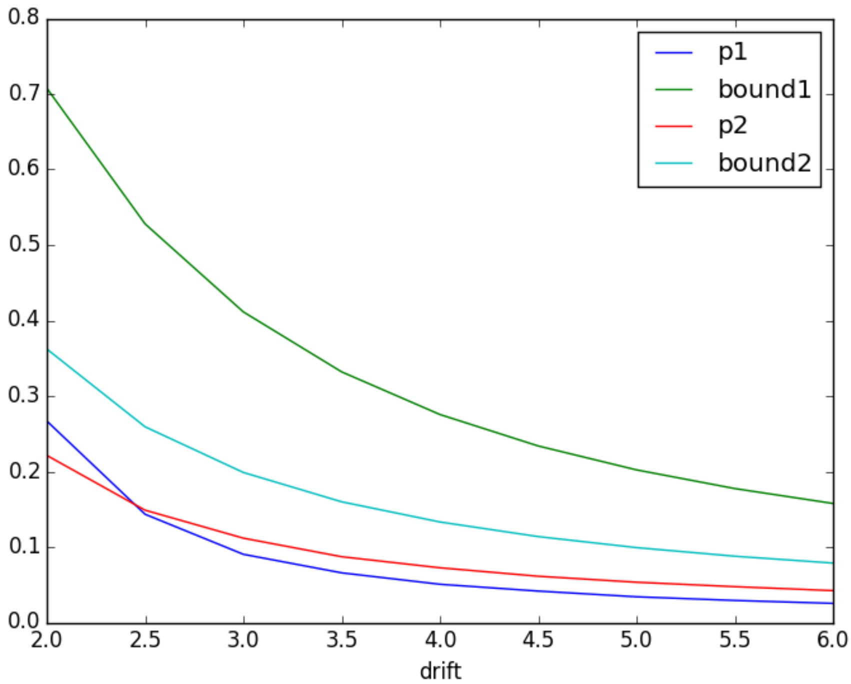

Figure 5.5: Probabilities and Bounds for Importance Sampling

The ruin probabilities estimates are close to each other and smaller than both upper bounds. The variance for the Poisson Martingale method (var(p1)) and the variance for the exponential tilting method (var(p2)) are plotted in Figure 5.6.

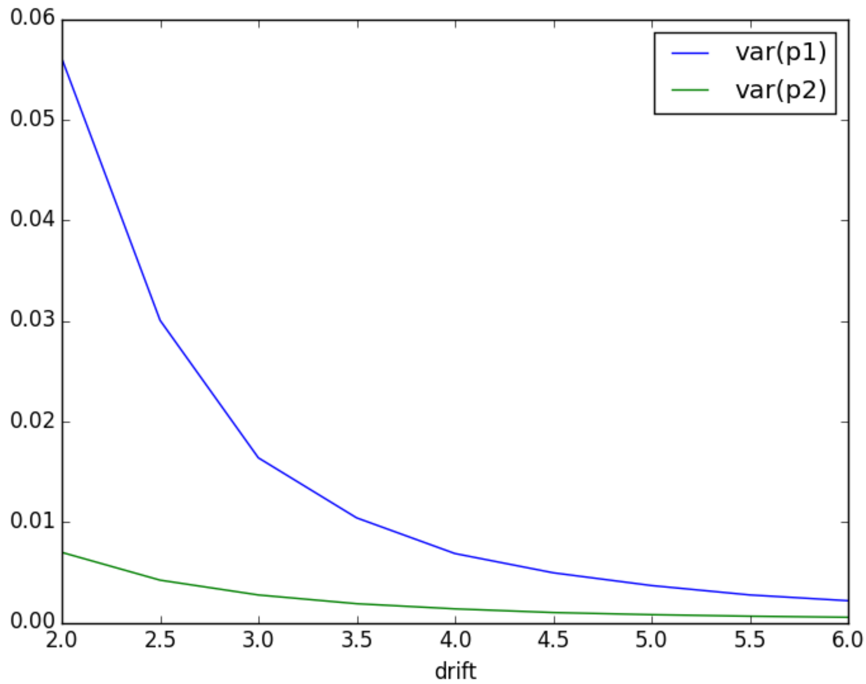

Figure 5.6: Variance for Importance Sampling

In the previous set up, the variance of the exponential tilting are smaller, and also the average iterations. However, the method using Poisson martingale has some flexibility since we can choose μ'. It is possible to outperform exponential tilting. In Figure 5.7, we choose $\mu' = (\mu - \alpha + \sqrt{2\lambda + \mu^2})/2.5$ in the Poisson Martingale method and calculate the empirical variance, var(p1) and the variance for the exponential tilting method, var(p2), with larger original drifts. Note that μ' here is chosen for convenience, one can apply proper optimization.

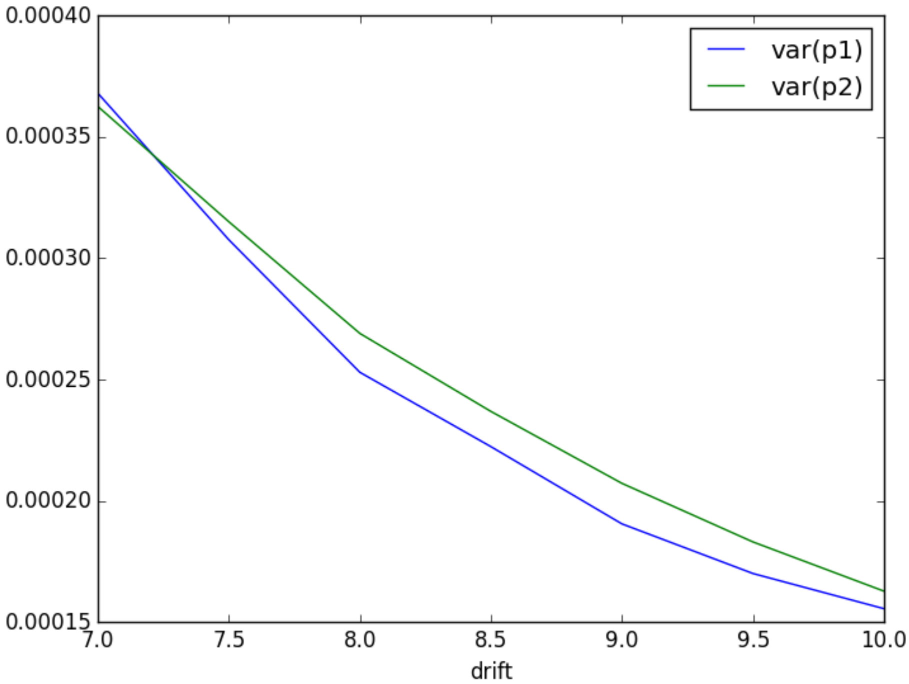

Figure 5.7: Variance for Importance Sampling with Larger Drifts

Chapter 6

Crossing Probabilities for the Running Maximum of Brownian Motion by a Brownian Motion Barrier

6.1 Definition

The set up is similar to definition (3.1) and (5.2). In this chapter, we are going to define another independent Brownian motion as the barrier

$$B_t^{c,\sigma} = ct + \sigma B_t$$

where B_t is a standard Brownian motion independent of W_t, c and σ are constants. The corresponding first hitting time of this barrier is defined as

$$\tau_b = \inf\{t > 0 | M_t - B_t^{c,\sigma} \leq -b\}$$

with $b > 0$. Therefore, the hitting level is below zero. If the hitting level is above zero, the problem is reduced to the first hitting time of a Brownian motion (rather than the running maximum of it). Therefore we focus the case when the hitting level is below zero. There is no overshooting in this case.

6.2 A Simulation Algorithm for the First Hitting Time

The boundary crossing can only happen when the Brownian motion barrier reaches the previous running maximum. Therefore, a natural simulation method will be to simulate the first hitting time for the previous level of the running maximum by the Brownian motion barrier then update running maximum to check if the boundary is crossed. Depending on the sign of the drift c, the first hitting time of the Brownian motion barrier follows different distributions.

- if $c > 0$, the first hitting time for a fixed level $b > 0$ by $B_t^{c,\sigma}$ is distributed according to an inverse-Gaussian, i.e.

$$T_b = \inf\{t > 0 | B_t^{c,\sigma} = b\} \sim IG(\frac{b}{c}, \frac{b^2}{\sigma^2})$$

- if $c = 0$, the first hitting time follows a Lévy distribution, i.e.

$$T_b = \inf\{t > 0 | B_t^{0,\sigma} = b\} \sim IG(\infty, \frac{b^2}{\sigma^2}) \stackrel{\mathcal{L}}{=} L\acute{e}vy(\frac{b^2}{\sigma^2})$$

- if $c < 0$, the probability is defective (see Borodin and Salminen [2002]), i.e.

$$Pr(T_b < \infty) = \mathrm{e}^{\frac{2bc}{\sigma^2}} < 1$$

  Consider $T_b = \inf\{t > 0 | B_t^{c,1} = b\}$,

$$Pr(T_b \in dt) = \frac{b}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(b-ct)^2}{2t}} dt$$

  is not a proper inverse-Gaussian distribution as $c < 0$. By completing the square,

$$Pr(T_b \in dt) = \mathrm{e}^{2cb} \frac{b}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(b+ct)^2}{2t}} dt = Pr(T_b < \infty) \times \frac{b}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(b+ct)^2}{2t}} dt$$

  Therefore,

$$Pr(T_b \in dt | T_b < \infty) \sim IG(-\frac{b}{c}, \frac{b^2}{\sigma^2})$$

Figure 6.1 demonstrates the simulation algorithm for a positively drifted case. The stopping time is $\tau_b = \tau_1 + \tau_2 + \tau_3$.

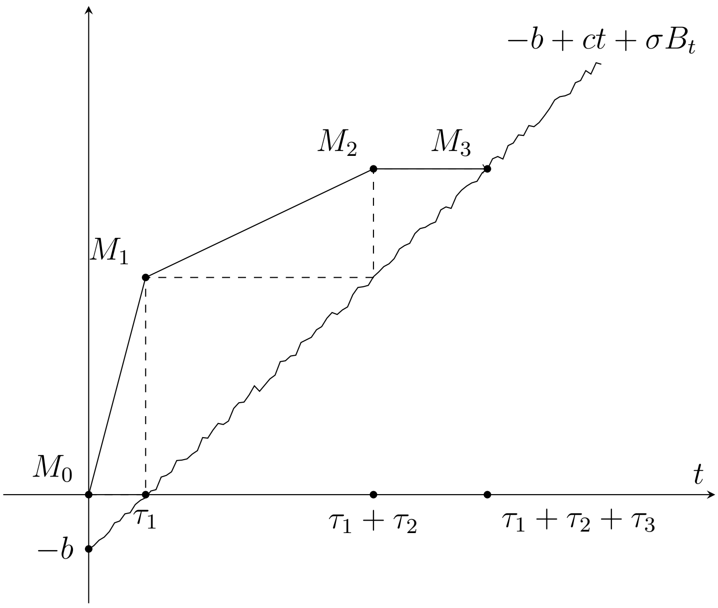

Figure 6.1: An illustration of iterative simulation for Brownian motion barrier crossing

Denote $X_t^\mu = \mu t + W_t$ is a Brownian motion with drift μ and $M_t^\mu = \max_{0<s<t} W_s^\mu$, the algorithm of simulation $\tau_b^\mu = \inf\{t > 0 | M_t^\mu - B_t^{c,\sigma} \leq -b\}$ is below.

Algorithm 8 Simulation Method for the Brownian Motion Barrier

1: Initialize with $b_0 = M_0^\mu + b$. For $i \geq 1$, set $b_i = M_i^\mu - M_{i-1}^\mu$
2: For $c \geq 0$, generate $\tau_i = T_{b_i}$ according to the above distributions
3: For $c < 0$, generate $U \sim U(0,1)$. If $U > \mathrm{e}^{\frac{2b_i c}{\sigma^2}}$, set $\tau_b^\mu = \infty$ and stop. Otherwise, generate $\tau_i = T_{b_i}$ with the distribution $Pr(T_{b_i} \in dt | T_{b_i} < \infty)$ above
4: Generate $X_i^\mu = X_{i-1}^\mu + \mu\tau_i + N(0,1)$, where $N(0,1)$ is a standard normal r.v.
5: Generate $m_i^\mu = \frac{X_i^\mu + X_{i-1}^\mu + \sqrt{(X_i^\mu - X_{i-1}^\mu)^2 - 2\tau_i ln(V)}}{2}$, where $V \sim U(0,1)$
6: Set $M_i^\mu = \max(m_i^\mu, M_{i-1}^\mu)$
7: Repeat previous steps until $M_i^\mu = M_{i-1}^\mu$, return (τ_b^μ, $M_{\tau_b}^\mu$, $X_{\tau_b}^\mu$)=($\sum_{k=1}^i \tau_k$, M_i^μ, X_i^μ)

10000 τ_b^* are generated with (b, c) = (1,1) for the standard Brownian motion case ($\mu = 0$). The histograms are plotted in figure (6.2).

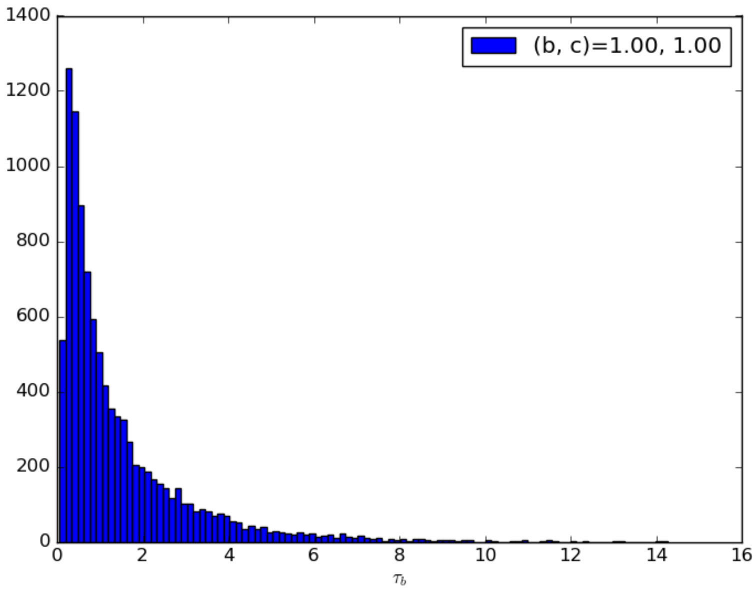

Figure 6.2: Histogram of simulated τ_b with $(b, c) = (1, 1)$

This is a thin-tailed distribution as expected, since $\mu < c$.

Appdx A

.1 An Unsolved Integral Equation for Compound Poisson Barrier

Recall the martingale for $(M_t,\, Y_t,\, Z_t)$ in (5.1),

$$f{\cdot}g(t, M_t, Y_t, Z_t) = \mathrm{e}^{-\beta t}\mathrm{e}^{-(\gamma-\nu_\beta)M_t}\mathrm{e}^{-\nu_\beta(M_t-Z_t)}h(Y_t)$$

We are interested in the stopping time

$$\tau = \inf\{t > 0 | M_t - Z_t \le -b\}$$

Denote the overshoot by U_τ. By the memoryless property of the exponential distribution, $U_\tau \sim \exp(\alpha)$ and is independent of the process $M_\tau - Z_\tau$. By applying the optional stopping theorem,

$$\begin{aligned}1 =&\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\mathrm{e}^{(\nu_\beta-\gamma)M_\tau}\mathrm{e}^{\nu_\beta(b+U_\tau)}h(Y_\tau)\right)\\ =&\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\mathrm{e}^{(\nu_\beta-\gamma)M_\tau}h(Y_\tau)\right)\frac{\alpha}{\alpha-\nu_\beta}\mathrm{e}^{\nu_\beta b}\end{aligned}$$

Rearrange the deterministic terms,

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\mathrm{e}^{(\nu_\beta-\gamma)M_\tau}h(Y_\tau)\right) = \left(1-\frac{\nu_\beta}{\alpha}\right)\mathrm{e}^{-\nu_\beta b}$$

$$= \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\exp\left(-\alpha\left(1-\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)b\right)$$

If $\gamma = \nu_\beta$, we have

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\left(\cos\omega Y_\tau + \frac{\alpha}{\omega}\left(1 - \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)\sin\omega Y_\tau\right)\right)$$
$$= \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\exp\left(-\alpha\left(1-\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)b\right)$$

Therefore,

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\mathrm{e}^{(\nu_\beta-\gamma)M_\tau}h(Y_\tau)\right) = \left(1-\frac{\nu_\beta}{\alpha}\right)\mathrm{e}^{-\nu_\beta b}$$
$$= \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\exp\left(-\alpha\left(1-\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)b\right)$$

If $\gamma = \nu_\beta$, we have

$$\mathbb{E}\left(\mathrm{e}^{-\beta\tau}\left(\cos\omega Y_\tau + \frac{\alpha}{\omega}\left(1 - \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)\sin\omega Y_\tau\right)\right)$$
$$= \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\exp\left(-\alpha\left(1-\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)b\right)$$

LHS=

$$\int_0^\infty\int_0^\infty \mathrm{e}^{-\beta t}p(t,y)\left(\cos\omega y + \frac{\alpha}{\omega}\left(1-\frac{\lambda}{\omega^2/2+\lambda+\beta}\right)\sin\omega y\right)dtdy$$
$$= \int_0^\infty \hat{p}(\beta,y)\left(\cos\omega y + \frac{\alpha}{\omega}\left(1-\frac{\lambda}{\omega^2/2+\lambda+\beta}\right)\sin\omega y\right)dy$$

Consider the last part,

$$\begin{aligned}
&\int_0^\infty \hat{p}(\beta,y)\frac{\alpha}{\omega}\left(\frac{\lambda}{\omega^2/2+\lambda+\beta}\right)\sin\omega y dy \\
=&-\int_0^\infty \frac{\alpha\lambda}{\omega^2/2+\lambda+\beta}\frac{\sin\omega y}{\omega}d\left(\int_y^\infty \hat{p}(u)du\right) \\
=&-\left[\frac{\alpha}{\omega}\left(\frac{\lambda}{\omega^2/2+\lambda+\beta}\right)\sin\omega y\int_y^\infty \hat{p}(u)du\right]_0^\infty \\
&+\int_0^\infty\int_y^\infty \hat{p}(u)du\left(\frac{\lambda\alpha}{\omega^2/2+\lambda+\beta}\right)\cos\omega y dy \\
=&\int_0^\infty\int_y^\infty \hat{p}(u)du\left(\frac{\lambda\alpha}{\omega^2/2+\lambda+\beta}\right)\cos\omega y dy
\end{aligned}$$

Let

$$\hat{P}(y)=\int_y^\infty\int_0^\infty \mathrm{e}^{-\beta t}p(t,u)dtdu=\int_y^\infty \hat{p}(u)du$$

By the Faltung Theorem,

$$\begin{aligned}
&\int_0^\infty\int_y^\infty \hat{p}(u)du\left(\frac{\lambda\alpha}{\omega^2/2+\lambda+\beta}\right)\cos\omega y dy \\
=&\frac{2\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\int_0^\infty \mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}y}\cos\omega y dy\times\int_0^\infty \hat{P}(y)\cos\omega y dy \\
=&\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\int_0^\infty\int_0^\infty \hat{P}(t)\left\{\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}+\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}|t-y|}\right\}dt\cos\omega y dy
\end{aligned}$$

Therefore, LHS=

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \mathrm{e}^{-\beta t}p(t,y)\left(\cos\omega y+\frac{\alpha}{\omega}\left(1-\frac{\lambda}{\omega^2/2+\lambda+\beta}\right)\sin\omega y\right)dtdy \\
=&\int_0^\infty\left(\hat{p}(y)+\alpha\int_y^\infty \hat{p}(u)du\right. \\
&\left.+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\int_0^\infty \hat{P}(t)\left\{\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}+\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}|t-y|}\right\}dt\right)\cos\omega y dy \\
=&\int_0^\infty\left(-\frac{\partial\hat{P}(y)}{\partial y}+\alpha\hat{P}(y)+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\int_0^\infty \hat{P}(t)\left\{\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}+\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}|t-y|}\right\}dt\right)\cos\omega y dy
\end{aligned}$$

Laplace Transform the LHS,

$$\int_0^\infty \hat{P}(t)\left\{\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}+\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}|t-y|}\right\}dt$$
$$=\int_0^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}dt+\int_0^y \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(y-t)}dt+\int_y^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t-y)}dt$$

The first term,

$$\mathcal{L}_y\left\{\int_0^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}dt\right\}(\eta)=\frac{\int_0^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}dt}{\eta+\sqrt{2}\sqrt{\lambda+\beta}}$$

The second term,

$$\begin{aligned}
&\mathcal{L}_y\left\{\int_0^y \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(y-t)}dt\right\}(\eta)\\
=&\int_0^\infty \mathrm{e}^{-\eta y}\int_0^y \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(y-t)}dtdy\\
=&\int_0^\infty \mathrm{e}^{\sqrt{2}\sqrt{\lambda+\beta}t}\hat{P}(t)\int_t^\infty \mathrm{e}^{-(\sqrt{2}\sqrt{\lambda+\beta}+\eta)y}dydt\\
=&\int_0^\infty \mathrm{e}^{\sqrt{2}\sqrt{\lambda+\beta}t}\hat{P}(t)\frac{\mathrm{e}^{-(\sqrt{2}\sqrt{\lambda+\beta}+\eta)t}}{\sqrt{2}\sqrt{\lambda+\beta}+\eta}dt\\
=&\frac{\int_0^\infty \mathrm{e}^{-\eta t}\hat{P}(t)dt}{\sqrt{2}\sqrt{\lambda+\beta}+\eta}
\end{aligned}$$

The third term,

$$\begin{aligned}
&\mathcal{L}_y\left\{\int_y^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t-y)}dt\right\}(\eta)\\
=&\int_0^\infty \mathrm{e}^{-\eta y}\int_y^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t-y)}dtdy\\
=&\int_0^\infty \mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}\hat{P}(t)\int_0^t \mathrm{e}^{-(\eta-\sqrt{2}\sqrt{\lambda+\beta})y}dydt\\
=&\frac{\int_0^\infty \hat{P}(t)(\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}-\mathrm{e}^{-\eta t})dt}{\eta-\sqrt{2}\sqrt{\lambda+\beta}}
\end{aligned}$$

Finally,

$$\mathcal{L}_y\left\{-\frac{\partial\hat{P}(y)}{\partial y}+\alpha\hat{P}(y)+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\int_0^\infty\hat{P}(t)\left\{\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}(t+y)}+\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}|t-y|}\right\}dt\right\}(\eta)$$

$$\begin{aligned}
=&-\eta\int_0^\infty\mathrm{e}^{-\eta y}\hat{P}(y)dy+\hat{P}(0)+\alpha\int_0^\infty\mathrm{e}^{-\eta y}\hat{P}(y)dy\\
&+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\left\{\frac{2\eta}{\eta^2-2(\lambda+\beta)}\int_0^\infty\hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}dt+\frac{2\sqrt{2}\sqrt{\lambda+\beta}}{2(\lambda+\beta)-\eta^2}\int_0^\infty\hat{P}(t)\mathrm{e}^{-\eta t}dt\right\}\\
=&\hat{P}(0)+\left(\alpha-\eta+\frac{2\alpha\lambda}{2(\lambda+\beta)-\eta^2}\right)\int_0^\infty\mathrm{e}^{-\eta y}\hat{P}(y)dy\\
&+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\frac{2\eta}{\eta^2-2(\lambda+\beta)}\int_0^\infty\hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}dt
\end{aligned}$$

Consider the RHS,

$$\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\exp\left(-\alpha\left(1-\frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\right)b\right)=\mathrm{e}^{-\lambda b}\sum_{n=0}^{\infty}\frac{\alpha^n b^n\lambda^{n+1}}{n!(\frac{\omega^2}{2}+\lambda+\beta)^{n+1}}$$

Inverse Cosine Transform,

$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty\mathrm{e}^{-\lambda b}\sum_{n=0}^{\infty}\frac{\alpha^n b^n\lambda^{n+1}}{n!(\frac{\omega^2}{2}+\lambda+\beta)^{n+1}}\cos(\omega y)d\omega\\
=&\frac{2}{\pi}\mathrm{e}^{-\lambda b}\sum_{n=0}^{\infty}\frac{\alpha^n b^n(2\lambda)^{n+1}}{n!}\frac{\sqrt{\pi}}{(2\sqrt{2}\sqrt{\lambda+\beta})^{n+\frac{1}{2}}}\frac{y^{n+\frac{1}{2}}K_{n+\frac{1}{2}}(\sqrt{2}\sqrt{\lambda+\beta}y)}{n!}
\end{aligned}$$

where K is the modified Bessel function of the second kind.

Laplace Transform the RHS,

$$
\begin{aligned}
&\mathcal{L}_y\left\{\frac{2}{\pi}\int_0^\infty \mathrm{e}^{-\lambda b}\sum_{n=0}^\infty \frac{\alpha^n b^n \lambda^{n+1}}{n!(\frac{\omega^2}{2}+\lambda+\beta)^{n+1}}\cos(\omega y)d\omega\right\}(\eta)\\
=&\frac{2}{\pi}\mathrm{e}^{-\lambda b}\sum_{n=0}^\infty \frac{\alpha^n b^n (2\lambda)^{n+1}}{n!}\frac{\sqrt{\pi}}{(2\sqrt{2}\sqrt{\lambda+\beta})^{n+\frac{1}{2}}n!}\mathcal{L}_y\left\{y^{n+\frac{1}{2}}K_{n+\frac{1}{2}}(\sqrt{2}\sqrt{\lambda+\beta}y)\right\}(\eta)\\
=&\frac{2}{\pi}\mathrm{e}^{-\lambda b}\sum_{n=0}^\infty \frac{\alpha^n b^n (2\lambda)^{n+1}}{n!}\frac{\sqrt{\pi}}{(2\sqrt{2}\sqrt{\lambda+\beta})^{n+\frac{1}{2}}n!}\frac{\sqrt{\pi}}{\sqrt{2}(\sqrt{2}\sqrt{\lambda+\beta})^{\frac{1}{2}}}\frac{\Gamma(2n+2)}{(\eta^2-2(\lambda+\beta))^{\frac{1}{2}(n+1)}}\\
&\quad\times P_n^{-(n+1)}\left(\frac{\eta}{\sqrt{2}\sqrt{\lambda+\beta}}\right)\\
=&2\mathrm{e}^{-\lambda b}\sum_{n=0}^\infty \frac{\alpha^n b^n \lambda^{n+1}}{n!n!(\sqrt{2}\sqrt{\lambda+\beta})^{n+1}}\frac{(2n+1)!}{(\eta^2-2(\lambda+\beta))^{\frac{1}{2}(n+1)}}P_n^{-(n+1)}\left(\frac{\eta}{\sqrt{2}\sqrt{\lambda+\beta}}\right)
\end{aligned}
$$

where P is the Legendre function.

The equation becomes

$$
\begin{aligned}
&\hat{P}(0)+\left(\alpha-\eta+\frac{2\alpha\lambda}{2(\lambda+\beta)-\eta^2}\right)\int_0^\infty \mathrm{e}^{-\eta y}\hat{P}(y)dy\\
&\quad+\frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\frac{2\eta}{\eta^2-2(\lambda+\beta)}\int_0^\infty \hat{P}(t)\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t}dt\\
=&2\mathrm{e}^{-\lambda b}\sum_{n=0}^\infty \frac{\alpha^n b^n \lambda^{n+1}}{n!n!(\sqrt{2}\sqrt{\lambda+\beta})^{n+1}}\frac{(2n+1)!}{(\eta^2-2(\lambda+\beta))^{\frac{1}{2}(n+1)}}P_n^{-(n+1)}\left(\frac{\eta}{\sqrt{2}\sqrt{\lambda+\beta}}\right)
\end{aligned}
$$

If $b=0$, inverse cosine transform RHS becomes

$$\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\cos(\omega y)d\omega=\frac{2\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}y}$$

Then the Laplace transform

$$\mathcal{L}_y\left\{\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\frac{\omega^2}{2}+\lambda+\beta}\cos(\omega y)d\omega\right\}(\eta)=\frac{2\lambda}{\sqrt{2}\sqrt{\lambda+\beta}}\left(\frac{1}{\eta+\sqrt{2}\sqrt{\lambda+\beta}}\right)$$

The equation becomes

$$
\begin{aligned}
&\hat{P}(0) + \left(\alpha - \eta + \frac{2\alpha\lambda}{2(\lambda+\beta) - \eta^2}\right) \int_0^\infty \mathrm{e}^{-\eta y} \hat{P}(y) dy \\
&\quad + \frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}} \left(\frac{2\eta}{\eta^2 - 2(\lambda+\beta)}\right) \int_0^\infty \hat{P}(t) \mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t} dt \\
=&\frac{2\lambda}{\sqrt{2}\sqrt{\lambda+\beta}} \left(\frac{1}{\eta + \sqrt{2}\sqrt{\lambda+\beta}}\right)
\end{aligned}
$$

Rearrange LHS, let $\theta = \sqrt{2}\sqrt{\lambda+\beta}$,

$$
\begin{aligned}
&\left(\alpha - \eta + \frac{2\alpha\lambda}{2(\lambda+\beta) - \eta^2}\right) \int_0^\infty \mathrm{e}^{-\eta y} \hat{P}(y) dy \\
&\quad + \frac{\alpha\lambda}{\sqrt{2}\sqrt{\lambda+\beta}} \left(\frac{2\eta}{\eta^2 - 2(\lambda+\beta)}\right) \int_0^\infty \hat{P}(t) \mathrm{e}^{-\sqrt{2}\sqrt{\lambda+\beta}t} dt \\
=&\frac{\theta(\alpha\eta^2 - \alpha\theta^2 + \eta\theta^2 - \eta^3 - 2\alpha\lambda) \int_0^\infty \mathrm{e}^{-\eta y} \hat{P}(y) dy + 2\eta\alpha\lambda \int_0^\infty \mathrm{e}^{-\theta t} \hat{P}(t) dt}{\theta(\eta^2 - \theta^2)}
\end{aligned}
$$

$\eta \to \theta$

$$
\begin{aligned}
\to&\frac{\theta \int_0^\infty (2\alpha\eta - 3\eta^2 + \theta^2 - y(\alpha\eta^2 - \alpha\theta^2 + \eta\theta^2 - \eta^3 - 2\alpha\lambda)) \mathrm{e}^{-\eta y} \hat{P}(y) dy + 2\alpha\lambda \int_0^\infty \mathrm{e}^{-\theta t} \hat{P}(t) dt}{\theta(2\eta - \theta^2)} \\
\to&\frac{2\alpha\lambda \int_0^\infty y \mathrm{e}^{-\theta y} \hat{P}(y) dy + 2(\alpha\theta - \theta^2 + \alpha\lambda) \int_0^\infty \mathrm{e}^{-\theta t} \hat{P}(t) dt}{\theta^2(2 - \theta)}
\end{aligned}
$$

The equation further becomes

$$
\hat{P}(0) + \frac{2\alpha\lambda \int_0^\infty y \mathrm{e}^{-\theta y} \hat{P}(y) dy + 2(\alpha\theta - \theta^2 + \alpha\lambda) \int_0^\infty \mathrm{e}^{-\theta t} \hat{P}(t) dt}{\theta^2(2 - \theta)} = \frac{\lambda}{\theta^2}
$$

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