The London School of Economics and Political Science

# Essays in Financial Economics

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*To my parents, for their unconditional love and support.*

# Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).



I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

### Statement of conjoint work

I confirm that Chapter 2 was jointly co-authored with Professor Ian Martin and I contributed 50% of this work. I confirm that for Chapter 3 was jointly co-authored with Konstantinos Tokis and Georgios Vichos and I contributed 33% of this work.

I declare that my thesis consists of less than 100,000 words.

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# Abstract

This thesis consists of three essays in financial economics. In the first chapter, I present an asymmetric information model of financial markets that features rational, but uninformed, hedge fund managers who trade against informed and noise traders. Managers are uncertain not only about fundamentals, but also about the proportion of informed to noise traders in the market and use prices to update their beliefs about these uncertainties. Extreme news leads to an increase in both types of uncertainty, while it decreases price informativeness. Prices react asymmetrically to positive and negative news, with higher expected returns at times of increased uncertainty about market composition. The model generates a price-volume relationship that is consistent with established stylized facts. I then extend to a three-period model and study the dynamics of expected returns and volatility.

In the second chapter, we study a dynamic model featuring risk-averse investors with heterogeneous beliefs. Individual investors have stable beliefs and risk aversion, but agents who were correct in hindsight become relatively wealthy; their beliefs are overrepresented in market sentiment, so "the market" is bullish following good news and bearish following bad news. Extreme states are far more important than in a homogeneous economy. Investors understand that sentiment drives volatility up, and demand high risk premia in compensation. Moderate investors supply liquidity: they trade against market sentiment in the hope of capturing a variance risk premium created by the presence of extremists.

In the final chapter, we consider a continuum of potential investors allocating funds in two consecutive periods between a manager and a market index. The manager's alpha, defined as her ability to generate idiosyncratic returns, is her private information and is either high or low. In each period, the manager receives a private signal on the potential performance of her alpha, and she also obtains some public news on the market's condition. The investors observe her decision to either follow a market neutral strategy, or an index tracking one. It is shown that the latter always results in a loss of rep-

utation, which is also reflected on the fund's flows. This loss is smaller in bull markets, when investors expect more managers to use high beta strategies. As a result, a manager's performance in bull markets is less informative about her ability than in bear markets, because a high beta strategy does not rely on it. We empirically verify that flows of funds that follow high beta strategies are less responsive to the fund's performance than those that follow market neutral strategies.

# Contents

# List of Tables

# List of Figures

# Chapter 1

# Trading under Uncertainty about other Market Participants

## 1.1 Introduction

Throughout the history of financial markets there have been numerous occasions where investors were taken by surprise by an extreme price movement. In many of these cases, these large price shocks were described as puzzling and could not be rationalized by many professional investors. The Black Monday, the Flash Crash of 2010, or the more recent Bitcoin boom are a few such examples. At the same time, many hedge funds have been paying closer attention to such events by expending more and more resources to track the fluctuation in market sentiment and learn whether trades represent information or noise. Why is it usually more difficult to interpret extreme shocks? What type of information do they convey? And why is uncertainty increasing during these times?

Most theoretical models in finance literature assume that traders know the degree of rationality of other investors in the market. In this paper, instead, I take the perspective of sophisticated investors ("hedge funds") who are uncertain about the proportion of informed - compared to noise - traders in the market. By studying their investment behaviour and the subsequent learning under this assumption, I contribute to financial research in a number of ways. First, I find that uncertainty about the market composition increases when there is an extreme market outcome (for example, a crash). Second, this increased uncertainty leads to higher uncertainty about fundamentals and higher risk premia. As a result, there is an asymmetry in the price reaction to positive and negative shocks. Third, I establish that the variation in market composition constitutes a type of risk, unrelated to fundamentals, for which investors demand higher expected returns. Moreover, I show that during a crash, traders rely less on cashflow news to update their expectations about future payoffs. Finally, the model generates a price-volume relationship that fits well with the relevant stylized facts established in empirical literature and summarized in Karpoff (1987).

The model consists of three types of agents; there are Hedge Fund managers, Informed investors, and Noise traders. Hedge Funds act as rational uninformed investors who are using the information in prices to make their portfolio choices. Informed traders can be thought of as insiders who hold information about payoffs and trade based on it, and Noise traders are the irrational investors in the market, who trade either because of wrong information or because of sentiment shocks. Importantly, managers are uncertain, not only about the fundamentals, but also about the proportion, of informed (to noise) traders that exist in the market. This makes their inference problem much more challenging since they have to use the signal they are getting through price to learn both about the fundamentals and about the composition of the rest of the traders. The main intuition behind this model is that hedge funds learn whether the rest of the traders are (more likely to be) homogeneous or not, by observing the size of the price signal. When the size is high, then the probability that the rest of the traders are of the same type is also high, and thus their uncertainty regarding the number of informed traders increases. One of the greatest challenges of any model with additional uncertainties is to keep it tractable. This is achieved by assuming that investors have a mean variance utility and by considering noise traders who receive signals that are independent, but identically distributed, to those of informed traders, so that their demand functions have the same functional form.

The first main result of this model is that market crashes (and booms) make hedge funds more uncertain about both the market composition and the fundamentals. This is because such extreme outcomes are actually very informative about the belief dispersion of investors in the market; in the limit, they can only occur when all investors behave in the same way. That is, it is much more likely to observe a crash when investors are either all informed or all noise. However, these two cases lead to very different interpretations of the price movement; in the first case, its informativeness is the highest possible, while in the latter it should be completely ignored. Thus, fund managers become less confident about how to interpret the price and their uncertainty about fundamentals increases. Therefore, we find that during a crash (or a boom) the risk premium part of the price increases.

Another important result is that expected returns are increasing in the uncertainty about the proportion of informed traders. For example, a market with fewer sources of information is naturally perceived to have a lower degree of belief dispersion and is associated with a higher variation in the ratio of informed-to-noise traders. As described above, managers who try to interpret the information that is contained in prices are less confident about their interpretation. This constitutes a type of risk, which they anticipate and hence, when the market is dominated by hedge funds, this uncertainty is translated into a higher expected return on the asset. Furthermore, I analyze the effect of this uncertainty on the sensitivity of price to signal. We find there is an asymmetry in the sensitivity of prices to positive and negative shocks, which becomes more pronounced when the uncertainty about the proportion of informed traders becomes the largest. This is because prices consist of an expectation part that moves prices in the direction of the news, and a risk premium part that is affected by the uncertainty about the dividend. Since this last part is increasing in the size of the news, the price reacts more with negative news than it does with positive news. Moreover, the expected risk premium is larger during these times, therefore leading to a higher expected asymmetry. In a similar way, I show that price instability, defined as the expected price's reaction to a sentiment shock, is higher during a crash.

I run numerical simulations of the model to study the behavior of trading volume. The patterns of volume and price very closely fit established stylized facts. In particular, volume increases when the absolute price increases, because, during that time, the disagreement between fund managers and the rest of the traders is the largest. For example, when there is a crash, managers are reluctant to significantly change their expectation about fundamentals. At the same time, the group of informed and noise traders is very likely to be homogeneous during that period, dominated either by informed or by noise traders; in either case, these traders will then hold a very different opinion, compared to Hedge Funds, about the expected payoff. Thus, there will be a large trading volume. More interestingly, simulations also show that there is a positive correlation between volume and price. This naturally arises in the model because of the abovementioned asymmetry between positive and negative news: that is, a large positive price is associated with an even larger signal than the corresponding negative price. Both of the above effects are amplified when the uncertainty about the proportion of informed traders increases.

Finally, I extend the model to a dynamic setting to discuss resulting implications for the expectation and volatility of returns. Under our assumption, trading in consecutive periods is connected via the updating of beliefs about the mass of informed traders. We then find that a crash (or boom) in period one is associated with higher expected returns, and lower volatility, in period two. This is because sophisticated investors conclude that the rest of the market is dominated either by informed or by noise traders, and so they expect to be less confident about the interpretation of any signal they observe.

Since the work of Radner (1968), it has been understood that uncertainty about payoffs acts in a very different way than uncertainty about other investors' behavior; more recently, there have been attempts by various authors to create such rational expectations models to study the effect of uncertainty about some market parameters. It is on this strand of literature that I build up by extending these ideas, recognizing the relevant uncertainties and introducing general distributional assumptions about the mass of informed traders, that will help us analyze, in a more realistic setting, the corresponding implications. For instance, Romer (1992) and Avery and Zemsky (1998) provide two such models, which can generate price crashes and herding, respectively, caused by uncertainty regarding other traders. Easley et al. (2013) study how expected returns are affected in an economy in which ambiguity-averse traders are uncertain about each other's risk aversion. Finally, some other papers that analyze higher dimensions of uncertainty are those of Yuan (2005), Cao and Ye (2016) and Cao et al. (2002). The abovementioned papers use different models to analyse how non-payoff uncertainties affect prices, while a common characteristic in this literature is that models are often non-tractable. Instead, our focus is on a type of uncertainty that increases during market crashes and can help us explain return dynamics during these times.

The most similar model to ours is that of Banerjee and Green (2015) (BG henceforth), where investors are uncertain whether informed or noise traders are present in the market (but not both). The key novelty of our paper, compared to BG, is that we allow both of these traders to co-exist in the market. This generalization creates the following fundamental difference: the equilibrium price conveys information both about fundamentals and about the composition of traders in the market. In particular, an extreme price movement, in our model, makes Hedge Funds believe that they are trading against either *all Informed* or *all Noise* traders, while a more moderate price will shift their beliefs towards thinking that they face a mixed population of traders. However, in BG there can be no such updating. This allows us to get many results about the way uncertainty about market composition affects expected returns, price informativeness and the slope of the price-volume relationship. Overall, while BG study the role of the first moment of the distribution of the proportion of Informed traders, I study the effect of the second moment and I show how learning about it affects equilibrium results.

Another important relevant paper is that of Gao et al. (2013). The authors consider a Grossman-Stiglitz model, in which the proportion of informed to uninformed traders is unknown; they focus on jumps that may appear due to the multiple equilibria that arise, and they find that there can be complementarity in information acquisition. In contrast, we study the uncertainty in the proportion of informed to noise traders. This type of uncertainty generates very different predictions about return dynamics and our modeling assumptions lead to a unique equilibrium, which is also more tractable.

Methodologically, this paper contributes to the growing literature on non-linear equilibria, in a CARA-normal setting. Building on the Grossman and Stiglitz (1980) paper of asymmetric information, many recent papers, such as Breon-Drish (2015), have shown that by relaxing assumptions about the distribution of dividends such equilibria may exist. In our paper, however, as in Banerjee and Green (2015) this non-linearity arises because of the assumption of uncertainty about a market parameter, while the payoffs remain normally distributed.Moreover, our modeling assumptions resemble that of Mendel and Shleifer (2012). In their paper, they use the same three types of agents and information structure to study the effect of noise traders in the market, even when their mass is negligible. They find that the Outsiders (rational uninformed agents) can rationally amplify the impact of a sentiment shock, leading to prices that significantly diverge from fundamental values. Importantly, their focus is on the price stability and specifically on its behavior as the mass of noise or informed traders changes. In this paper, while we also emphasize the importance of noise trading, our focus is on the behavior of sophisticated investors when they are uncertain about these masses.

Finally, regarding the empirical literature it is worth noting two relevant papers. First, Easley et al. (2002) provides an empirical measure of the probability of information-based trading (PIN) in a market. By estimating PIN through their microstructure model, the authors conclude that informed trading positively predicts returns. In my model, I emphasize that uncertainty about this probability also matters; this issue is discussed in more detail in Section 6. Second, Sadka (2006) provides evidence that the variable component of liquidity risk can explain the momentum and PEAD returns. He further interprets this variable component as representing the *"unexpected variation in the ratio of informed to noise traders"*. My paper presents a theoretical model in which there is uncertainty about this ratio and suggests that this uncertainty is important for the return dynamics and, more specifically, can indeed lead to higher expected returns.

The rest of this paper is organized as follows. In the following section I present the model and its main assumptions. In Section 3 I analyze the resulting equilibrium quantities in the static model. In Section 4, I explore the implications of assuming various distributions for the prior belief about the proportion of informed traders and I analyze some numerical simulations. Section 5 describes a dynamic extension of the model, under some simplifying assumptions. Finally, Section 6 contains a discussion of the paper, while, in Section 7 a conclusion is given.

## 1.2 The Fundamentals of the Model

### 1.2.1 Agents

There are three types of agents in the economy, each endowed with an initial wealth $W$. There is a mass 1 of rational uninformed agents ($H$) who are trying to learn from prices, and there is also a mass $m \in [0, 1]$ of informed (I) agents who observe an informative signal about fundamentals at each period, and a mass of $1 - m$ of noise traders (N) who think they are informed and trade in signals that are actually uncorrelated with the fundamentals.[1] The fact that the mass of $H$ is the same as the mass of $I$ and $N$ combined, is just for simplicity and does not drive any results; later, we consider the general case where the mass of hedge funds is $Q$ and the total mass of $I$ and $N$ is $2-Q$ and we talk about the cases $Q \to 0$ and $Q \to 2$. Our *most important assumption* is that $m$ is not a known parameter, but instead it is a random variable in $[0, 1]$.

### 1.2.2 Timing

The benchmark model is a static model with two periods. During Period 1, trading takes place, while in Period 2, dividends are paid and uncertainty is resolved. In Section 5, we extend this model to a dynamic version with multiple periods.

### 1.2.3 Assets

There are two assets in the market, a risk-free asset, with a return normalized to 1, and a risky asset that pays a dividend $d \sim N(0, \sigma^2)$ and is found in supply $Z$, which is a known constant.

[1] In Appendix B, we discuss some slight alterations of the model, in which we have different types of agents, and which lead to some alternative results and interpretations.

### 1.2.4 Utilities

All agents have mean variance utilities[2] and are price-takers. Traders maximize the utility of their terminal wealth. More specifically, each trader solves the following maximization problem:

$$\max_x \mathrm{E}[W + x(d-p)] - \frac{\alpha}{2}\mathrm{Var}[W + x(d-p)]$$

where $\alpha$ represents the degree of risk-aversion.

### 1.2.5 Information Structure

Informed and Noise traders behave similarly. They both receive a signal, which they both think is the only source of (payoff-relevant) information in the market. $I$'s signal is:

$$s_I = d + \varepsilon_I$$

where $\varepsilon_I$ is Normally distributed with mean 0 and volatility $\sigma_\varepsilon$. The informativeness of I's signal is given by the signal-to-noise ratio: $\lambda = \dfrac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}$.
On the other hand, the signal of Noise traders is:

$$s_N = u + \varepsilon_N,$$

where $u \sim N(0, \sigma^2)$ is independent and identically distributed to $d$, and $\varepsilon_N$ is independent and identically distributed to $\varepsilon_I$. Thus, the perceived informativeness of the noise traders is also equal to $\lambda$. Hedge fund managers are not aware of informed-to-noise traders in the market and have, at $t = 0$, a prior distribution $f(m)$ about $m$. Our model nests the Banerjee and Green model in the case where $f(m)$ is such that $f(1) = \pi_0$ and $f(0) = 1 - \pi_0$.

Our specification for Noise traders is different to that in Grossman and Stiglitz (1980) or De Long et al. (1990), where $N$ have a random inelastic de-

[2] Note that this is *not equivalent* to using CARA utility, as $p$ will be non-linear in the signal and hence will not be normal in equilibrium.

mand. In contrast, our approach to modeling noise traders can be also found in Black (1986) or - in a very similar form - in Mendel and Shleifer (2012). The main advantage of this approach is that it delivers a much more tractable model, which better serves the intuition behind this the paper. Uncertainty about $m$ matters, because it alters the perceived homogeneity in the market, even if the demand functions of the two groups of traders are indistinguishable (from $H$'s point of view). In particular, this assumption, makes all the equilibrium quantities just a function of $ms_I + (1-m)s_N$, and, as we will see in the next section, allow us to find an equilibrium that is *mixed-signal revealing*. It is for the same reason that we assume that the total mass of $I$ and $N$ is known. Otherwise, it would be much more difficult to find an equilibrium.

In this model, we can see that proportion uncertainty is closely related to the belief dispersion in the market. Indeed, Informed and Noise traders form heterogeneous beliefs about the asset's payoff once they receive their signals. From the perspective of the fund managers, who do not know $m$, this *belief dispersion* between $I$ and $N$ traders can be measured by the inverse of the correlation of two random signals in the market. When this correlation is high, this means that it is very likely for all traders to hold the same information and thus the belief dispersion is low, and vice versa. More formally, if $i, j$ are i.i.d. Bernoulli random variables that take the values $I$ and $N$ with probabilities $m, (1-m)$ respectively, then belief dispersion is $\frac{1}{corr(s_i, s_j)}$. This is a function of the second moment of $m$, since it depends on the probability that both $i$ and $j$ are of the same type, which is equal to $m^2 + (1-m)^2$. Therefore we get the following corollary:

**Corollary 1.** *Belief dispersion in the market*[3] *is decreasing in the variance of $m$, $var[m]$, as long as $E[m]$ is a constant.*

Hence, we can see how the uncertainty about the mass $m$ of $I$ traders is related to the belief dispersion between $I$ and $N$ traders. Thereafter, we will

[3]We only consider the belief dispersion between two random traders who are either I or N, because those traders obtain their own signals.

use $var[m]^{-1}$ as a measure of this asymmetry in the market.[4]

In the following section, I solve for the equilibrium price in the static model.

[4] We will be able to do that, as we will only focus on the case where $E[m] = \frac{1}{2}$.

## 1.3 The Equilibrium

### 1.3.1 Two-period model

Solving the maximization problem for $\theta = I, N$ we get:

$$x_\theta = \frac{\mathrm{E}_\theta[d] - p}{\alpha \mathrm{Var}_\theta[d]} = \frac{\lambda s_\theta - p}{\alpha\sigma^2(1-\lambda)}$$

That is, informed and noise traders behave in the same way (but receive different signals) and do not try to use the price to learn any further information about $d$. On the other hand, $H$ try to learn about $d$ by observing the price and residual demand (i.e. $Z - x_H$). When updating their belief about the fundamental, the above two quantities give them a "mixed" signal that has some information because of $s_I$, but is also contaminated by noise ($s_N$).

Using the market clearing condition, we have $x_H + mx_I + (1-m)x_N = Z$. Therefore:

$$x_H + \frac{\lambda\overbrace{(ms_I + (1-m)s_N)}^{\tilde{s}} - p}{\alpha\sigma^2(1-\lambda)} = Z$$

I write $\tilde{s} = ms_I + (1-m)s_N$. Note that the price and the residual demand can *reveal to* $H$ *the mixed signal* $\tilde{s}$. This would not be true if the demand of noise traders was simply a random variable $z$ (instead of being a function of price) and, therefore, we would not be in a position to find the equilibrium.

In order to find hedge funds' demand, $x_H$, we need to find the expectation and variance of $d$ from their perspective after they observe the abovementioned mixed signal. Henceforth, I may write $E_H[\cdot]$, $Var_H[\cdot]$ for $E[\cdot|\tilde{s}], Var[\cdot|\tilde{s}]$ respectively, and I use these notations interchangeably. Using the law of iterated expectations, as well as the formula for the conditional expectation of normally distributed variables, we get:

$$\mathrm{E}[d|\tilde{s}] = \mathrm{E}\left[\frac{m}{m^2+(1-m)^2}|\tilde{s}\right]\lambda\tilde{s}, \tag{1.3.1}$$

To simplify notation, I set $L(m) := \frac{m}{m^2+(1-m)^2}$. The term that multiplies $\tilde{s}$ in (1.3.1) is the expected $\frac{cov(d,\tilde{s})}{var(\tilde{s})}$, which we interpret as the *informativeness* of the signal.

We can now observe that the expectation of $d$ as perceived by the Hedge Funds depends on the conditional probability density function of $m$ given the mixed signal, $f_{m|\tilde{s}}(m|\tilde{s})$. In the next section, I use a prior three-point distribution, $f(m)$, to give the intuition of the results, but I also prove the validity of main results under any symmetric distribution.

Similarly, we can find the perceived variance of $d$ from the perspective of $H$. For that, we will need to use the *law of total variance*. We have:

$$Var[d|\tilde{s}] = \underbrace{\mathrm{E}\left[\mathrm{Var}[d|\tilde{s},m]|\tilde{s}\right]}_{\mathrm{E}[\sigma^2(1-\lambda mL(m))|\tilde{s}]} + \underbrace{\mathrm{Var}\left[\mathrm{E}[d|\tilde{s},m]|\tilde{s}\right]}_{\mathrm{Var}[\lambda L(m)|\tilde{s}](\tilde{s})^2}$$

To simplify further, we will set $c(\tilde{s}) := \lambda^2\mathrm{Var}[L(m)|\tilde{s}]$ and we will examine the function $c(\cdot)$ later on. Moreover, in the Appendix I prove that for any symmetric $f(m)$, we have $E[mL(m)|\tilde{s}] = \frac{1}{2}$. Therefore:

$$\mathrm{Var}_H[d] = \sigma^2(1-\frac{\lambda}{2}) + c(\tilde{s})\tilde{s}^2$$

So we observe that both the expectation and variance depend on $m, s_N$ and $s_I$, only through the mixed signal, $\tilde{s}$.

By using the market clearing condition, we can thus get the following proposition:

**Proposition 1.** *In the two-period model, there exists a mixed signal ($\tilde{s}$) re-*

*vealing equilibrium. The price in this equilibrium is given by:*

$$P = \underbrace{\lambda\tilde{s}\kappa(\tilde{s}) + \lambda\tilde{s}\mathrm{E}\left[L(m)|\tilde{s}\right](1-\kappa(\tilde{s}))}_{\textit{Expectation component}} - \underbrace{\alpha\kappa(\tilde{s})\sigma^2(1-\lambda)Z}_{\textit{Risk premium component}},$$

*where*

$$\kappa(\tilde{s}) = \frac{Var[d|\tilde{s}]}{\sigma^2(1-\lambda) + Var[d|\tilde{s}]}$$

*Proof.* A more general proof, for when the mass of $H$ traders is $Q \in (0,2)$ and the mass of $I$ and $N$ together is $2-Q$, can be found in the Appendix (baseline model corresponds to $Q = 1$). □

In the case where $Q \to 2$, in which Hedge Funds dominate the market, the price takes the simple form:

$$p \approx E[d|\tilde{s}] - \frac{1}{2}\alpha Z Var[d|\tilde{s}].$$

What we can see from the above proposition is that the equilibrium price is not linear in the mixed signal, and, consequently, non-linear in $s_I$. The expectation component is a weighted average of the expectations of each group of agents, and the weight is given by $\kappa(\tilde{s})$. This weight is increasing in $Var_H[d]$, and hence, as I will prove in Section 4, it is also increasing in $|\tilde{s}|$ , whereas in standard models, the posterior variance of fundamentals is independent of the signal, because of the properties of the conditional normal distributions.[5] As in BG, this makes the expectation and risk premium components of the price to not behave in the same way with positive and negative realizations of $s_I$ (or of $s_N$). This creates an asymmetry that makes the derivative of price to $\tilde{s}$ greater for negative realizations of the mixed signal than for positive ones, which I will discuss further in the next section.

[5]That is, if A,B are jointly normal random variables, $var[A|B]$ is not a function of B.

## 1.4 Results

To explore the equilibrium implications of this model we need to make an assumption about the distribution of $m$. In BG (2015), this is assumed to be a Bernoulli distribution that takes the value 1 with $\pi_0$ and 0 with $1-\pi_0$, but, in such a model, the belief dispersion is always constant and there is no uncertainty or learning about it. In contrast, we will assume that this dispersion is unknown, and we will study how Hedge Funds learn about it and how it affects their demand functions. The simplest way to provide the intuition is to use a simple, three-point distribution, which allows us to talk of different levels of belief dispersion between $I$ and $N$ traders in the market. Then, in Section 4.2, we will generalize the main results to any symmetric distribution in $[0,1]$.

### 1.4.1 Three-point distribution

Assume that $m \in \{0, \frac{1}{2}, 1\}$ with $\pi_i = P(m=i)$ for $i \in \{0, 1/2, 1\}$. As I explain later, augmenting the support of $m$ to include values in $(0,1)$ leads to learning about $m$, by observing $\tilde{s}$. Importantly, what we will see is that the *posterior belief* $\hat{\pi}_i = P(m=i|\tilde{s})$ *is in general not equal to* $\pi_i$. This seems to be self-evident; however, it is not true under the simple assumption of the BG model, and this is what drives many of the interesting results that are different. Using Bayes' rule we find that the posterior distribution satisfies $f_{m,\tilde{s}}(i,\tilde{s}) = \mathrm{P}(m=i) f_{\tilde{s}|m}(\tilde{s}|m=i)$, where the conditional distribution of the right hand side is normal, with mean 0 and variance $(i^2+(1-i)^2)(\sigma^2+\sigma_\varepsilon^2)$. Note that $\tilde{s}|(m=1) = s_I$ and $\tilde{s}|(m=0) = s_N$, which are identically distributed. Thus, we can define $h_1(x)$ to be the pdf of $\tilde{s}$ under $m=1$ or $m=0$ and $h_2(x)$ to be the pdf under$m = \frac{1}{2}$. We have:

$$\hat{\pi}_0 = \mathrm{P}(m=0|\tilde{s}) = \frac{\pi_0 h_1(\tilde{s})}{(\pi_0+\pi_1)h_1(\tilde{s}) + \pi_{1/2} h_2(\tilde{s})} \tag{1.4.1}$$

and similarly for $\pi_{1/2}$ and $\pi_1$. We see, therefore, that the posterior probabilities depend on $s$ through $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$, which is decreasing in $|\tilde{s}|$. Let us now see what is the implication of this fact, in the symmetric case where $\pi_0 = \pi_1 = \pi$ (and $\pi_{1/2} = 1 - 2\pi$). We have the following corollary:

**Corollary 2.** *Uncertainty about the proportion of Informed traders, $m$, increases as (mixed) news becomes more extreme, i.e., $|\tilde{s}|$ becomes large.*

Extreme news is naturally associated with extreme prices. Hence, this corollary tells us that, during crashes or booms, uncertainty about market composition spikes. This uncertainty has plenty of implications as we see in the results that follow. Technically, the fact that a mixed signal distribution, corresponding to $m = \frac{1}{2}$, is less fat-tailed than a normal distribution, is what causes $\hat{\pi}$ to increase as $|\tilde{s}|$ increases. In other words, a more extreme signal makes $H$ believe that the other group of traders is (more likely) either all Informed or all Noise. Hence, Hedge Funds believe that extreme market outcomes occur when the rest of the traders are more homogeneous, or, in other words, they believe that the *belief dispersion* in the market is low.[6] In fact, although $H$ becomes more uncertain about $m$, he does become more certain about the deviation of $m$ from its mean, $|m - \frac{1}{2}|$. Especially when $|\tilde{s}| \to \infty$, we know that $P(|m - \frac{1}{2}| = \frac{1}{2}) \to 1$; that is, Hedge Funds learn that all the agents (between $I$ and $N$) in the market are of the same type.

The first implication that we will examine is the effect on the informativeness of the signal in extreme times; managers becomes almost certain that the signal is either very informative or not at all. We get the following corollary:

**Corollary 3.** *The Hedge Funds' informativeness of the price signal is decreasing in the size ($|\tilde{s}|$) of the signal.*

*Proof.* As defined in equation (1.3.1), informativeness is $\lambda \mathrm{E}[L(m)|\tilde{s}]$. Now note that given the formulas we provided for $\hat{\pi}_i$, we can easily see that when

[6]We cannot use Corollary 1 to directly claim that belief dispersion as defined in that lemma decreases, since the conditional covariance $cov(s_i, s_j|\tilde{s}) = 0$. However, we abuse the term belief dispersion here to refer to the the probability that two traders (randomly drawn) are not the same (which has now decreased).

$\pi_0 = \pi_1$ then $\hat{\pi}_0 = \hat{\pi}_1 = \hat{\pi}$ (more generally, if the prior distribution is symmetric with respect to $1/2$ then the posterior distribution will also be symmetric). Therefore:Informativeness $= \lambda(1 - \hat{\pi})$. But since $\hat{\pi}$ is increasing in $\tilde{s}^2$, the expected informativeness must be decreasing in $\tilde{s}^2$. $\square$

The main idea that can be conveyed, even with a three-point distribution, is that an extreme signal (either very positive or very negative) shifts the posterior beliefs about the proportion in such a way that it is now much more likely that the other group of traders is either all Noise or all informed ($m = 0$ or 1). In turn, this causes the informativeness of the price to be decreasing on $|\tilde{s}|$, since managers are now more reluctant to interpret any signal as representing information. More interestingly, simulations show that for very small values of the prior $\pi$, the expectation of fundamentals, $E_H[d]$ is non monotonic on $|\tilde{s}|$; that is, a very high $\tilde{s}$ may even lead a hedge fund manager to lower his expectation about $d$, since the effect of the reduced informativeness may outweigh the effect of the increased $\tilde{s}$.

In addition to the effect on informativeness, when the news is extreme, the perceived variance of informativeness increases. This is because hedge funds understand that it is more likely that either all other traders are informed or all noise, and hence they are most uncertain about the weight they should put on the signal they observe (in one case, this is very informative, and, in the other, they should completely ignore it).

This, in turn, affects the uncertainty of managers about the fundamentals of the assets and leads us to the following proposition.

**Proposition 2.** *Hedge Funds become more uncertain about the fundamentals when they observe more extreme news.*

*Proof.* As detailed in the Appendix, we can get that $c(\tilde{s}) = \lambda^2 \hat{\pi}(1 - \hat{\pi})$ which is increasing in $\tilde{s}^2$. $\square$

That is, $var_H[d]$ and hence the risk premium (and $\kappa(\cdot)$) are increasing in

$|\tilde{s}|$. When the magnitude of the mixed signal increases, the posterior variance of $m$ increases; this makes the investors more uncertain about the composition of the market and, hence, about the informativeness of the signal. In turn, this leads to an increase in the uncertainty about fundamentals.[7] Having established the above results, we now have that $\kappa$, and hence the risk premium component of the price, is also increasing in the magnitude of the mixed signal. This means that a very high signal (seemingly positive news) could yield a lower price than an averagely good signal in two ways; first, by implying a lower expectation of fundamentals (from H's perspective), and second, by increasing the uncertainty of fundamentals. Moreover, as we see in the proof of proposition 2 in the Appendix, hedge funds' perceived variance of fundamentals, $Var_H[d]$, is increasing in the posterior belief $\hat{pi}$; therefore, we can conclude that it is also increasing in their perceived variance of the proportion $m$ of informed traders. This then leads us to the following important lemma:

**Lemma 1.** *Hedge funds' expected uncertainty about fundamentals, $E[Var_H[d]]$, is increasing in the prior variance of $m$.*

This lemma is based on Proposition 2 together with the fact that if $var[m_1] > var[m_2]$, then the corresponding signal $\tilde{s}^2_{m_1}$ stochastically dominates $\tilde{s}^2_{m_2}$. As shown in the Appendix, this stochastic dominance leads us to conclude that the expected risk premium part of the price is increasing in $var[m]$. Therefore, we obtain the following proposition:

**Proposition 3.** *When the market is dominated by Hedge Funds, the expected return of the asset, $E[d-p]$, is increasing in the uncertainty about the proportion of informed traders, $var[m]$ and hence in the perceived homogeneity of the market.*

From Corollary 1, we have seen that the variance of $m$ can be interpreted as the inverse of the level of (perceived) heterogeneity in the market. Therefore,

[7] This result is also true in BG model. However, in our setting it is amplified by the uncertainty about $m$, and it then naturally leads to Proposition 3. Moreover, the fact that uncertainty about informativeness, measured by $c(\tilde{s})$, is increasing in $|\tilde{s}|$ is a novel result.

one can deduce that the higher this heterogeneity, the lower the expected return of the risky asset. In other words, managers who trade in a homogeneous market want to be compensated for the additional risk that they are taking, given that they do not know whether the price contains any information at all. In the next subsection, I use the propositions and corollaries found above, to discuss the sensitivity of prices to changes in the signals.

### Price to signal sensitivity

A measure that is extensively studied in the work of Mendel and Shleifer (2012) is that of price instability, defined as the derivative of price to a sentiment shock (corresponding to $\tilde{s}_N$ here). The higher the price instability, the more fragile the price and the more likely it will lead to an abrupt jump (big change) in the price. We will use the results of the previous section, to study a similar measure in the context of this model in which there is a higher degree of uncertainty.

First of all, we have the following corollary, that is a generalization of a corresponding proposition in Banerjee and Green (2015), which indicates an important asymmetry between positive and negative news:

**Corollary 4.** *For any $\tilde{s}_0 > 0$, the derivative of price to the mixed signal is smaller for $\tilde{s}_0$ than it is for $-\tilde{s}_0$. That is:*

$$\left.\frac{dP}{d\tilde{s}}\right|_{\tilde{s}=\tilde{s}_0} < \left.\frac{dP}{d\tilde{s}}\right|_{\tilde{s}=-\tilde{s}_0} \tag{1.4.2}$$

*Moreover, the expected size of this asymmetry is larger when the variance of $m$ is larger.*

When the mixed signal is positive the expectation component of price is (generally) increasing, while the risk premium component makes the price lower than it would be, making the derivative smaller. On the other hand, when $\tilde{s}$ is negative, under a more extreme signal, both the expectation part and the risk premium part move the price to the same direction (making it more negative), thus increasing the sensitivity of the price movement to the change of $\tilde{s}$. Mathematically, we have that:

$$P(\tilde{s}) + P(-\tilde{s}) = -2\alpha Z\kappa(\tilde{s})\sigma^2(1-\lambda) \tag{1.4.3}$$

which is decreasing for $\tilde{s} > 0$, and hence gives us first part of corollary 4. This result is important, because it shows that, in this setting, negative news can lead to a more extreme drop in price than the corresponding positive news. Finally, by taking expectations in equation (1.4.3) and then differentiating with respect to $\tilde{s}$, we also get the second part of the corollary, since by Proposition 3 we get that $E[\kappa(\tilde{s})]$ is increasing in $var(m)$.

Moreover, we can study the expected sensitivity of price to a sentiment shock (or to a shock in a Noise trader's signal in our model) and see how it varies for different values of $\tilde{s}$. Mendel and Shleifer (2012) relate this measure to the Hedge Funds' demand, and, more particularly, to whether these traders end up chasing noise or not. In this model, the sign of the sensitivity of $H$'s demand to mixed news changes for different values of $\tilde{s}$, and it turns out that $H$ traders may be chasing noise when $|\tilde{s}|$ gets large, and act in the opposite way when $\tilde{s}$ is close to 0.

**Corollary 5.** *Price instability is higher when $|\tilde{s}|$ goes to infinity than when $\tilde{s}$ goes to 0.*

The proof of the above result can be found in the Appendix. Because of the uncertainty about market composition, when the size of the signal is large, Hedge Funds prefer to stay away from the market and do not trade aggressively. Thus, during these times, the effect of noise trading is amplified, and the price becomes more sensitive to any sentiment shock.

### 1.4.2 Any Symmetric Distribution

Having described the main intuition using a simple three-point distribution, we will now work with a continuous distribution for $m$ to make our model more rich and realistic. In fact, we will prove the main propositions using any continuous distribution that is symmetric in $[0, 1]$ (w.r.t $1/2$). This makes the results of the model robust to a variety of distributional assumptions and corresponds a more realistic setting where $m$ can take any value in $[0, 1]$. First of all, when the pdf of $m$ is symmetrically distributed around $\frac{1}{2}$, we can easily get the following corollary:

**Corollary 6.** *When the prior distribution of the proportion $m$ of Informed traders is symmetric w.r.t $\frac{1}{2}$, the posterior distribution from the perspective of $H$ (conditioning on $\tilde{s}$) is also symmetric. Hence:*

$$\mathrm{E}_H[m] = \frac{1}{2}$$

This means that the expectation about the proportion of informed traders remains unchanged, independently of the observation of $\tilde{s}$. However, that does not mean, that the distribution of $m$, and hence the informativeness of the signal, does not change. In Appendix, I explain how we can compute the joint density of $(d, u, m, \tilde{s})$, from which we can find the conditional distribution of $d|\tilde{s}$ or of $m|\tilde{s}$. In short, we can use Bayes' rule to find the posterior distribution $f_{m|\tilde{s}}(m|\tilde{s})$. We first find the joint distribution $g(m, \tilde{s})$, as: $g(m, \tilde{s}) = g(\tilde{s}|m)f_m(m)$. But $\tilde{s}|m$ is a linear combination of normals, hence it is a normal itself, with mean 0 and variance $C(m) = (m^2 + (1-m)^2)(\sigma^2 + \sigma_\varepsilon^2)$. Hence, we get:

$$g(m, \tilde{s}) = \frac{1}{\sqrt{2\pi C(m)}} \exp\left(-\frac{\tilde{s}^2}{2C(m)}\right) f_m(m)$$

Finally, this means that $f_{m|\tilde{s}}(m|\tilde{s}) = \dfrac{g(m, \tilde{s})}{g_{\tilde{s}}(\tilde{s})}$, where $g_{\tilde{s}}(\tilde{s})$ is the density function of $\tilde{s}$. Therefore, we see that

$$f(m) = f(1-m) \Longrightarrow g(m,\tilde{s}) = g(1-m,\tilde{s}) \Longrightarrow f_{m|\tilde{s}}(m|\tilde{s}) = f_{m|\tilde{s}}(1-m|\tilde{s}),$$

i.e., the posterior is symmetric, as required. Using the above, we can now prove the following, proposition.

**Corollary 7.** *For any symmetric prior distribution of $m$, the expected informativeness of the signal is decreasing in the size of the signal.*

*Proof.* Proof can be found in the Appendix and is based on the use of Cauchy-Schwarz inequality. □

How can we interpret this result? As also described in the previous section, when fund managers observe a large realization of $\tilde{s}$ they know this is a good sign for the fundamentals (assuming $\tilde{s} > 0$), but also need to estimate how accurate this sign is. The larger the $\tilde{s}$ gets, the more likely it is that $m$ has an extreme value (closer to 0 or 1). However, since (as I prove in the Appendix) the informativeness decreases more as $m \to 0$ than it increases for $m \to 1$, its expected value ends up decreasing in $|\tilde{s}|$ and this is a key result.[8]. As the mixed signal gets larger the signal-to-noise ratio becomes smaller and smaller, which can even lead H's expectation of fundamentals to be decreasing in $|\tilde{s}|$ (in particular when $\pi_0$ is small, simulations show that $E_H[d]$ can be non-monotonic on $|\tilde{s}|$). All in all, Corollary 7 shows us how the uncertainty about other traders affects the expectation part of the price.

We can also prove the following result, which generalizes Proposition 2, and is one of the main results of this paper.

**Proposition 4.** *For any symmetric prior distribution of $m$, Hedge Funds' perceived variance of fundamentals is increasing in $|\tilde{s}|$. Therefore, the informativeness of the signal, measured by $var[d|\tilde{s}]^{-1}$ is decreasing in $|\tilde{s}|$.*

[8]The case $\tilde{s} < 0$ is totally symmetric, since $\mathrm{E}[d|\tilde{s}] = -\mathrm{E}[d|-\tilde{s}]$.

The importance of this proposition lies in establishing the fact that the risk premium part of the price is increasing in $|\tilde{s}|$, for any prior symmetric distribution (thus covering most of the known *uninformative priors* that are assumed in cases of parameters for which we have no information).

Finally, one more corollary we can obtain concerning $\mathrm{E}_H[d]$, is the following:

**Corollary 8.** *For any symmetric prior distribution of $m$, H's perceived expectation of dividends satisfies the following inequality:*

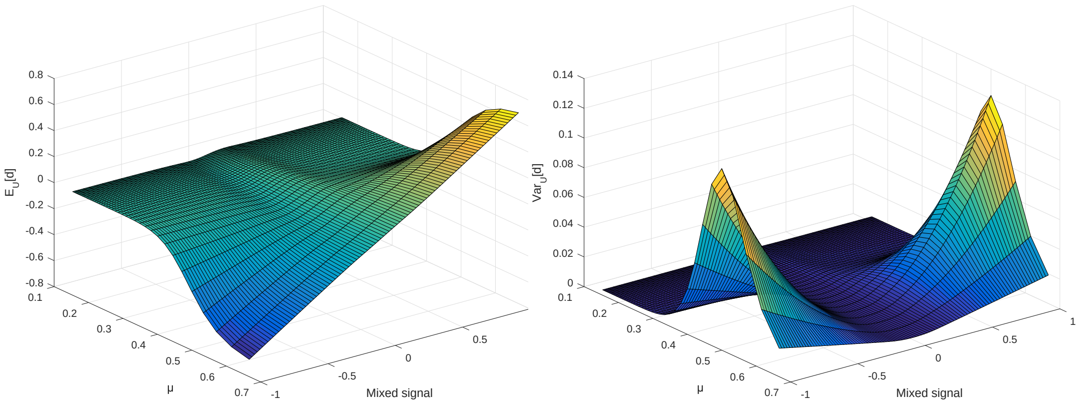
$$\frac{1}{2}\lambda|\tilde{s}| \leq |\mathrm{E}[d|\tilde{s}]| \leq \lambda|\tilde{s}|$$

This corollary helps us to get a grasp of the magnitude of the posterior expectation of the fundamentals from the perspective of Hedge Funds. For example, we can easily see that the left hand side of the inequality implies that the expectation is strictly smaller in size than $\lambda|\tilde{s}|$. This means that, even when $s_I = s_N$, the expectation of the Hedge Funds will be different than the expectation of the other agents.

### 1.4.3 Simulations

One very general distribution in $[0, 1]$ that we can select in order to run simulations and show our results graphically, is the Beta distribution with parameters, $a, b$.[9] Note that the uniform distribution $U[0, 1]$, is just a special case of Beta with parameters $a = 1, b = 1$, while the so called "uninformative" Jeffrey's prior is also a Beta distribution with $a = 1/2, b = 1/2$. The parameters $a, b$ determine the shape of the distribution: the mean is equal to $\mu = \frac{a}{a+b}(= E[m])$, and the distribution is positively skewed iff $a < b$. Using the Beta(a,b) distribution, we can now make various plots, which can help us in the interpretation of the model.

[9]The density function of Beta(a,b) is $f_m(m) = \dfrac{m^{a-1}(1-m)^{b-1}}{B(a,b)}$. Throughout this section, we focus on the case where $a = 1$, unless otherwise stated.




(a) $\mathrm{E}_H[d]$ w.r.t. $\tilde{s}$ and $\mu$ (b) $\mathrm{Var}_H[d]$ w.r.t. $\tilde{s}$ and $\mu$

Figure 1.4.1: Hedge Funds' perceived expectation and variance of dividends, as a function of the mixed signal $\tilde{s}$ and of the prior expectation of $m$ ($\mu = E[m]$).

In particular, Figures 1.4.1a and 1.4.1b show H's perceived expectation and variance of $d$, as a function of the mixed signal, as $\mu$ varies (keeping $a = 1$). The parameters we have used are: $Q = 1$, $\sigma = 0.06, \sigma_\varepsilon = 0.04$, as well as $\alpha = 1$ and $Z = 10$.

When $\mu$ is large, the expectation is increasing in the signal $\tilde{s}$, and in the limit as $\mu \to 1$, then $E_H[d] \to \lambda\tilde{s}$. On the other hand, when $\mu$ small is enough (i.e., small expected number of informed agents) the expectation of $d$ stays almost constant at 0, since hedge fund managers do not think that $\tilde{s}$ can be very useful in updating their prior knowledge about payoffs. It is also interesting to note that for $\mu \leq \frac{1}{2}$ it is not necessarily true that the expectation is increasing in the signal. This is because a higher signal can lead $H$ to update their belief about $m$ (downwards), leading them to believe that the mixed signal they observe is uninformative, thus tilting their expectation about dividends closer to their prior expectation, i.e. 0.

To understand this further, we need to examine the two forces acting against each other in the case where $\tilde{s}$ is increasing. On the one hand, this increase causes the expectation to directly increase, as managers know that the mixed signal can be, at least partially, attributed to good news from informed agents. On the other hand, as $\tilde{s}$ increases it becomes more probable that $m$

is closer to either 0 or 1. A heuristic way to see this is to note that $u$ or $d$ (corresponding to $m = 0$ or $m = 1$ respectively) have a larger variance than, for example, $\frac{1}{2}d + \frac{1}{2}u$ (corresponding to $m = 1/2$). Thus, when the prior is that there will be more noise traders, i.e., small $m$ , the additional information that $\tilde{s}$ is large makes $H$ update their information about $m$ so that they believe that $m$ is closer to 0, or equivalently, that the mixed signal is driven by noise traders and is less informative than $H$ previously thought. So, this effect leads $H$ to trust the signal less, and leads to this non-monotonicity of $\mathrm{E}_H[d]$, with respect to $\tilde{s}$. The above observation leads us to a main difference between this model and the BG model; in the present model, price can be *non-monotonic* in the signal, even in the case of risk-neutral uninformed agents, while in BG this non-monotonicity, could only happen in case of large risk aversion or supply ($\alpha Z$) due to the effect of the risk premium component.

As far as the variance is concerned, we can see in Figure 1.4.1b that it is increasing in the size of the signal as well as symmetric with respect to 0; in combination with the fact that $\mathrm{E}[d|\tilde{s}] = -\mathrm{E}[d| - \tilde{s}]$, this means that the price will behave asymmetrically to positive and negative news (Corollary 4). Finally, we note that $\mathrm{Var}_H[d]$ is almost constant for $\mu$ small enough. This is because, in that case, fund managers expect very few Informed traders to be in the market and, thus, do not use the signal too much to update their beliefs about $d$; hence, their posterior variance of $d$ is close to the prior one, independently of the observed mixed signal.

Moreover, Figure 4.3 shows the equilibrium price[10] with respect to $\tilde{s}$ for two different values of $E[m]$ (corresponding to $Beta(1,1)$ and $Beta(1,3)$ prior distributions). We verify that price is non-linear and that it exhibits the aforementioned asymmetric reaction to positive/negative news. Finally, we see that these effects are largest when $\mu = 0.5$. Instead, a small $\mu$ means that more agents are probably noise, making the signal less important. In that case, we can see more clearly that the price can even decrease with a higher signal (in the neighborhood around $\tilde{s} = 0$), both because of the effect of risk premium and because of the decreased informativeness of the signal.

[10] Prices are negative, because of the parameter values that have been chosen but could be shifted up by a constant $\overline{d}$, simply by assuming that prior mean of $d$ (and $u$) is $\overline{d}$.

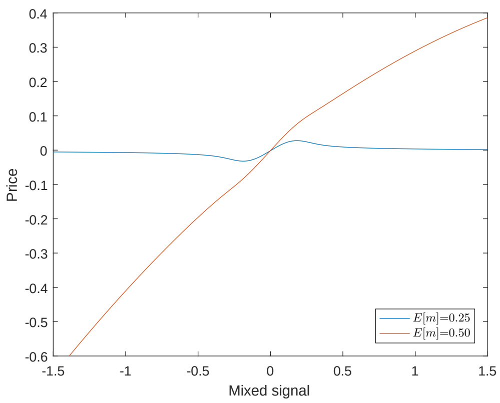



Figure 1.4.2: Equilibrium price as a function of $\tilde{s}$ for $E[m] = 0.50$ and $E[m] = 0.25$

### Volume-price relationship

We will see from the simulations that follow, the volume-price relationship that arises from simulated data of our model fits very well with the empirical facts found in many studies about volume. This is in spite of the fact that our model was not constructed with the intention of matching these specific empirical results. In particular, the survey of Karpoff (1987) establishes the following stylized facts. The first, that can be consistently found in many empirical papers, is that volume and the absolute change in price ($|\Delta p|$) are positively related. The second is that there is a positive correlation between volume and price change *per se*; that is, volume is higher when there is a positive price changes, than when there is a negative one. I verify that this model provides suggestive evidence in favor of both of these predictions,[11] I study how results are affected by the composition uncertainty and I explain the intuition behind them.

I use the baseline model, with $m \sim U[0, 1]$, to simulate a dataset of price and trading volume[12] and I run two main regressions on this data. Table 1.4.1

[11] We will think of price, in the context of our static model, as corresponding to the *price change* in empirical studies.

[12] Trading volume is computed using the equation: Volume $= \frac{1}{2}(|x_H| + m|x_I| + (1 - m)|x_N| + |Z|)$.

Table 1.4.1: Estimation results : Volume on squared price.

Baseline model: Volume $\sim$ (price)$^2$ + price + constant.

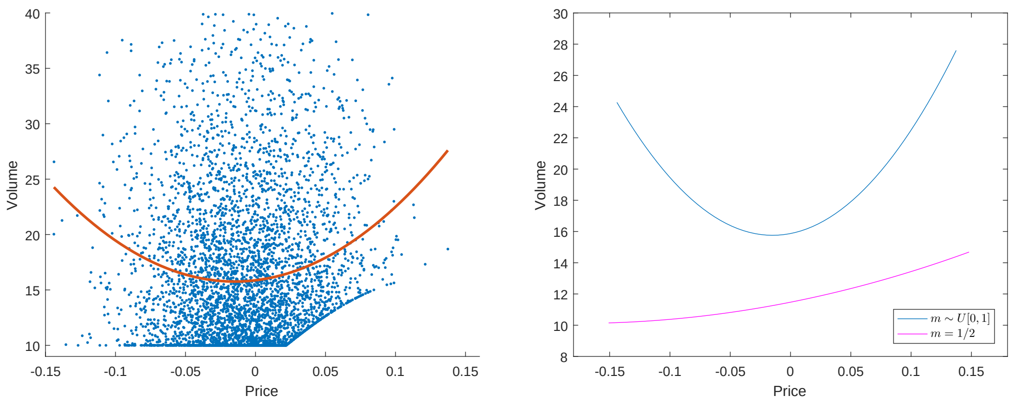
| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| (Price)$^2$ | **510.65**$^{**}$ | (52.108) |
| Price | 15.197$^{**}$ | (2.979) |
| Intercept | 15.87$^{**}$ | (0.12837) |

Significance levels : † : 10% $*$ : 5% $**$ : 1%

shows the results of a regression of trading volume on a quadratic function of price.[13] What we observe is that the coefficient on the quadratic term, (price)$^2$, is positive and significant. In other words, higher volume arises when the absolute price is larger. This means that if we fit a quadratic polynomial of price into the simulated data, the relationship between price and volume is U-shaped, as shown in Figure 1.4.3. This is a result arising in many "Differences of Opinion" (DO) models, such as Harris and Raviv (1993), because disagreement increases in periods where signals are larger (simply because an informed traders' signal is more informative than that of the uninformed).

Although this result in our model has a similar flavour with abovementioned models, we also offer a new insight. More specifically, when price is large in absolute value, the uncertainty of Hedge Funds increases (since $|\tilde{s}|$ is large) and $H$'s expectation about payoffs becomes less sensitive to cashflow news, as proved in Corollary 3. At the same time, it becomes highly likely that the group of $I$ and $N$ is very homogeneous (Corollary 2), and they all hold beliefs in which they have greatly updated their expectation about $d$. This leads to high disagreement between $H$ and the (average of the) rest of the agents, which we can measure using the difference in their expected payoffs;

[13] The regression of volume on absolute price gives qualitatively similar results, but we prefer this specification to emphasize that the coefficient of the linear term is also positive.




(a) Fitting a quadratic polynomial (b) The role of composition uncertainty

Figure 1.4.3: Volume-price relationship

that is

$$|E[d|\tilde{s}] - \lambda\tilde{s}| = \lambda|\tilde{s}|\left(1 - E[L(m)|\tilde{s}]\right), \tag{1.4.4}$$

which is increasing in $|\tilde{s}|$.[14] This is, in turn, translated into high trading volume, which is, in fact, even greater when the prior uncertainty about $m$ is higher. Indeed, when the composition uncertainty is higher the expected disagreement increases since $|\tilde{s}|$ is likely to be higher and $E[L(m)|\tilde{s}]$ is lower; as a result, the expected volume is higher during these times. As we discuss in Section 5, this could have implications for predicting the magnitude of the regression coefficients; in particular, a crash today would make $var[m]$ increase and hence would predict a higher trading volume tomorrow as well as a greater slope of volume on $(\text{price})^2$ because of the $E[d|\tilde{s}]$ term in equation (1.4.4).

To study the effect of $var[m]$, I have also run the same simulations for different distributions of $m$, including the case where $m = \frac{1}{2}$ (no composition uncertainty). The coefficients of the squared price term in these regressions are smaller and in the extreme case, in which Hedge Funds are certain that $m = 1/2$, this coefficient is non-significant. This is presented in Figure 1.4.3(b), where we see that the quadratic curve fitting simulated data for $m = \frac{1}{2}$, almost becomes a line. Indeed, in that case, the difference in beliefs of $H$ with the

[14]Note that $x_H = \frac{E[d|s] - \lambda\tilde{s}}{\alpha(\sigma^2(1-\lambda) + var[d|\tilde{s}])} + Z(1 - \kappa(\tilde{s}))$. As $\tilde{s} \gg 0$ this is likely to be very negative, while when $\tilde{s} \ll 0$ this is likely to be much larger than $Z$; this is because the increase in $|E[d|\tilde{s}] - \lambda\tilde{s}|$ dominates the increase in $var[d|s]$.

(average of the) rest of the traders is 0 since $E[d|\tilde{s}, m = \frac{1}{2}] = \lambda\tilde{s}$, thus disagreement in that case does not depend on $|\tilde{s}|$. That is, we get the prediction that the coefficient on the price-squared term is increasing on the uncertainty about $m$. Finally, further simulations show that the expected volume is also increasing in $var[m]$, as explained in the end of the last paragraph.

Table 1.4.2: Estimation results : Volume on price.

Baseline model: Volume $\sim$ price + constant.

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| Price | **6.5247**$^{*}$ | (2.8715) |
| Intercept | 16.546$^{**}$ | (0.10931) |

Significance levels : $\dagger$ : 10% $\quad *$ : 5% $\quad **$ : 1%

Moreover, an econometrician might want to test the effect of price *per se* on trading volume. For this reason, I run a simple linear regression on the simulated data. What we can clearly see from table 1.4.2 is that there is a positive and significant (on the 2% level) relationship between volume and price. This is directly related to the asymmetry in positive and negative shocks, established in Corollary 4. Indeed, a large positive price is associated with an even larger positive signal. Hence, the intuition described above leads (on average) to an even higher volume for $p > 0$ than for $-p$, since, in this case, the uncertainty about $m$ and the corresponding disagreement is larger. Given that the -expected- asymmetry in positive/negative news is higher in times of greater composition uncertainty (Corollary 4), the model predicts that the slope of this regression is also higher during these times.

Therefore, we can see that composition uncertainty can even generate patterns of volume and price that arise in data and for which theoretical justifications are still not very concrete.

## 1.5 Extension: Dynamic Model

We will now extend our model to a dynamic version to study some of its predictions when there are more than two periods. In this case, learning about $m$ from each period, will yield the future returns partially predictable. We will assume that the market is dominated by Hedge Funds; that is their mass $Q$ is close to 2 (see proof of Proposition 1). As far as the prior distribution of $m$ is concerned, we will keep the simple assumption of a three-point symmetric distribution, as in Section 4.1.

Due to the non-normality of the equilibrium price $p$, it would be very difficult to solve the problem of long-term maximization, where agents maximize consumption over their terminal wealth. Instead, we will assume that agents trade two independent short-dated assets maturing at dates 2 and 3, respectively, with the corresponding dividends denoted by $d_2$ and $d_3$.[15] We also assume that they only observe the realized dividends after they have traded, so they cannot use them for updating their beliefs. The only thing connecting the trading of the two periods is the updating on $m$, because of the observed mixed signals. Moreover, all agents are myopic. First, we have the following characterization of the equilibrium price in each period, as in Proposition 1.

**Corollary 9.** *When the market is dominated by Hedge Funds, the equilibrium price, $p_i$ in each period $i$ is equal to*[16]

$$p \to E[d_{i+1}|\tilde{s}^{(i)}] - \frac{1}{2}\alpha Z var[d_{i+1}|\tilde{s}^{(i)}] \tag{1.5.1}$$

*where $d_i$ is the cash-flow news for the risky asset in the i-Period, $\tilde{s}_i$ is the realization of the mixed signal at Period $i$ and $\tilde{s}^{(i)} = \{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_i\}$, i.e. it is*

[15] Alternatively, we could assume $d_i$, $i = 2, 3$ are independent and identically distributed cash-flow news that arrive in the market at period $i$, such that the final dividend of the asset is $d = d_2 + d_3$. Traders in Period $i$ would choose their investment to maximize their utility as if $d_{i+1}$ was actually their next Period's dividend. This interpretation is preferred when we think of the implications in the stock market.

[16] Note that we have made the simplifying assumption that at the beginning of the 2nd period, agents optimize by assuming they will just get $d_3$ in the future. Hence when forming their expectations and variances at $t = 2$, they do not take $d_2$ into account.

*the history of mixed signal realizations up to time $i$.*

This gives us a very simple characterization of the price, similar to that obtained in our baseline model. Let us now write $E_i[\cdot], var_i[\cdot]$ to denote $E[\cdot|\tilde{s}^{(i)}], var[\cdot|\tilde{s}^{(i)}]$, respectively, as perceived by Hedge Funds. Thus, for instance, $E_1[d_3 - p_2]$ denotes the expectation from the perspective of Hedge Funds of the return they will receive from Period 2 to Period 3, conditioning on the information they have at period 1, $\tilde{s}_1$. Under the above assumptions, we can now establish the following interesting result:

**Proposition 5.** *Assume that market is dominated by Hedge Funds. Then, the expectation of the future return of the asset, as of Period 1, $E_1[d_3 - p_2]$, is increasing in the uncertainty about the proportion of informed traders, and hence also in the size of the mixed signal of the first Period, $|\tilde{s}_1|$.*

*Proof.* First of all note that from equation 1.5.1 we have that:

$$E_1[d_3 - p_2] = E_1[d_3] - \mathrm{E}_1\left[\mathrm{E}[d_3|\tilde{s}^{(2)}]\right] + \frac{\alpha Z}{2}\mathrm{E}_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right] \tag{1.5.2}$$

$$= \frac{\alpha Z}{2}\mathrm{E}_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right] \tag{1.5.3}$$

because of the law of iterated expectations (note that we take the expectation at the end of Period 1 when $H$ has already observed $\tilde{s}_1$). But, using Corollary 2, we know that $E_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right]$ is increasing on the perceived variance of $m$ (that is, $\mathrm{var}[m|\tilde{s}_1]$). Since $\mathrm{var}[m|\tilde{s}_1]$ is increasing on $|\tilde{s}_1|$, the above proposition follows. □

It is worth discussing the above proposition further. It states that the more extreme the price in the first period is (e.g. during a crash) the higher H's expected return about the next period, as agents want to be compensated for the risk they are taking; this additional risk comes from their uncertainty about whether they are trading against signal or noise (increasing variance of $m$). Note that the signal of the first period gives no information at all about the dividends of the next period ($d_3$). Instead, it is the learning about $m$ that happens in the first period, through $\tilde{s}_1$, that carries information about the return of the next period and makes it (partially) predictable.[17] Empirically, this observation implies that during periods that are extreme in terms of the mixed news that arrives in the market (including real and fake news, or market sentiment), we should see that the future expected return of the risky asset becomes higher.

Finally, we would like to examine what happens to the volatility of prices in the second period, in terms of the first period's mixed signal. We can see that this relation will depend on our parameters. In particular, we have the following Corollary:

**Corollary 10.** *For risk considerations ($\alpha Z$) close to* $0$*, we have:*

1. *If the market is dominated by Hedge Funds, then $var_1[p_2]$ is decreasing in $var_1[m]$ and hence in $|\tilde{s}_1|$.*
2. *If the market is dominated by $I$ and $N$ agents, then $var_1[p_2]$ is increasing in $var_1[m]$ and hence in $|\tilde{s}_1|$.*

Note that the first case corresponds to $Q \to 2$, while the second to $Q \to 0$. If there are only $I$ and $N$ agents in the market, a high mixed signal in the first period implies that it is more likely that $m = 0$ or $m = 1$. This, in turn, makes the variance of $\tilde{s}_2$ higher, and thus leads to higher variance of $p_2$. On the other hand, if $H$ traders are setting the price in this market, when they observe a higher signal in the first period, they understand that they should expect a high conditional variance in the next period (see Lemma 1), and adjust their future expectation of dividends, $E[d|\tilde{s}^(2)]$, so that their perceived variance of the price becomes decreasing in $|\tilde{s}_1|$.

It would also be interesting to see what the limiting behavior (and learning) would be in this economy after many Periods. Note that Hedge Funds would then be able to condition on a history of realizations $\tilde{s}^{(n)} = \{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n\}$.

[17] Importantly in a model, such as that of BG(2015), where $H$ cannot update their perceived distribution of $m$ through the mixed signals, this channel cannot exist.

We are interested in finding the posterior distribution $f_{m|\tilde{s}^{(n)}}(m|\tilde{s}^{(n)})$. Note that, as shown in Corollary 7, if the prior distribution $f(m)$ is symmetric then the posterior distribution would remain symmetric, after any number of periods. However, as $n \to \infty$, we can get the following proposition.

**Proposition 6.** *As the number of periods tends to infinity, Hedge Funds learn $|m - \frac{1}{2}|$. Thus, the posterior distribution of $m$ converges to a symmetric two-point distribution.*

*Proof.* Can be found in the Appendix. The main idea is that by the Law of Large numbers we can find the empirical variance of $\tilde{s}$ and then equate it with the actual variance of $\tilde{s}$ (conditional on $m$). $\square$

The above proposition implies that, in the long-run, managers can learn the true distance of $m$ from $\frac{1}{2}$, but since the posterior distribution is always symmetric they can never distinguish between $m$ or $1-m$. If we call $var[m|\tilde{s}^{(n)}]$ the long-run uncertainty about $m$, then this uncertainty is higher when $|m-\frac{1}{2}|$ is larger. This is because, as $n \to \infty$, $var[m|\tilde{s}^{(n)}] \to \frac{1}{2}((m^*)^2 + (1-m^*)^2 - \frac{1}{2})$, where $m^*$ is the realization of the random variable m. As in Lemma 1, we can thus show (see Appendix) that the expected returns are higher when this long run uncertainty, or equivalently, $|m^* - \frac{1}{2}|$ is higher. Finally, using $(corr(s_i, s_j))^{-1}$, which is decreasing in $m^2 + (1-m)^2$, as the measure of belief dispersion in the market (*see Corollary 1*), we conclude that higher belief dispersion leads, in the long run, to lower expected returns.

## 1.6 Discussion

There are two key main points that the model in this paper makes. The first is that a mixed signal that traders observe can change their perceived variance about the proportion of informed traders in the market. In particular, the higher the size of this signal, the higher the uncertainty about this proportion. The second major point is that this variance is a measure of the (perceived) belief dispersion in the market and affects the informativeness of the prices, it creates an asymmetric reaction to positive and negative news, and leads to predictability about future expected returns. We would thus like to interpret the $var[m]$ as a measure of the asymmetric information in the market. By doing so, we could have a way of empirically testing some of the predictions of this model. In particular, there have been empirical studies that use the probability of informed trading (PIN), constructed by Easley et al. (2002), as a measure of the asymmetric information. This paper claims that, controlling for the expectation of PIN, what should also matter is the variance of PIN, or the uncertainty about its value.

In particular, according to our model, one should empirically expect to find that a higher variance of PIN leads to higher expected returns. Moreover, to study the effect of this variance on informativeness one could look into the sensitivity of investment decisions to stock prices (which is a proxy of informativeness used in papers such as Bond et al. (2012)), to see whether times with more uncertain PIN, are associated with lower such informativeness.

A main empirical challenge would be to try find a good proxy for the mixed signal ($\tilde{s}$) used in this paper. I believe that some indices of market sentiment can be used as a proxy for $\tilde{s}$; indeed, a survey-based market sentiment index can contain both information and noise, and hence it could be an appropriate proxy for $\tilde{s}$. Similarly, another proxy we could use for the mixed signal would be the mutual fund flows. Both these proxies have been used as proxies for pure noise (investor sentiment), but have been criticized exactly because they are subject to confounding variables (related to fundamental information).

## 1.7 Conclusion

In this paper I describe a model of asymmetric information in which the proportion of Informed traders to Noise traders is unknown to the Hedge Fund managers who trade in the market. I study how traders learn about this additional uncertainty and I examine the resulting equilibrium quantities. Moreover, I relate this uncertainty to the perceived heterogeneity of beliefs in the market. More specifically, it is shown that Hedge Funds become more uncertain about fundamentals when they observe extreme news (for example, during a market crash), as they becomes less confident in inferring information from the prices. In addition, in this setting, the expected returns are decreasing in the perceived heterogeneity in the market, and there is an asymmetric price reaction in positive and negative news.

I illustrate the intuition of the key findings by assuming at first a three-point prior distribution for the proportion of informed traders, and I extend the results to the case of any symmetric continuous distribution. Furthermore, I find that this model is consistent with the empirical stylized facts concerning the volume-price relationship and I thus offer a possible theoretical explanation for these findings. Overall, the focus of this paper lies on understanding how traders learn about fundamentals, while also learning about their market environment given the signal that equilibrium quantities convey. As shown in the dynamic extension of the model, this setting carries many implications about the information quality of prices and the resulting volatility in the market. Finally, the fact that the equilibrium price does not fully reveal the signal of informed agents, provides a very useful model to work on and makes a dynamic version of the model, in which agents learn from dividends or from stale information, very interesting to investigate further.

## 1.8 Appendix A

**Proof of Proposition 1:**

*Proof.* We will write down a proof for the general case where the mass of Hedge funds is Q, while the total mass of agents remains 2 (for the baseline model Q=1) and the proportion of $I$ to $N$ is still $m : (1-m)$. The first step for this proof is to write down the equilibrium demand function of $H$. We can easily show that under mean variance utility this is:

$$x_H = \frac{E_H[d] - p}{\alpha \mathrm{Var}_H[d]},$$

where as we have described throughout section 2, $H's$ expectation and variance is conditional on the mixed signal $\tilde{s}$ (which in equilibrium is revealed if he observes the price and the residual demand). Therefore, using the market clearing condition, we have:

$$Qx_H + (2-Q)(mx_I + (1-m)x_N) = Q\frac{E_H[d] - p}{\alpha \mathrm{Var}_H[d]} + (2-Q)\frac{\lambda\tilde{s} - p}{\alpha\sigma^2(1-\lambda)} = Z$$

Equivalently

$$p\cdot\left[(2-Q)\mathrm{Var}_H[d] + Q\sigma^2(1-\lambda)\right] = Q\sigma^2(1-\lambda)E_H[d] + (2-Q)\mathrm{Var}_H[d]\lambda\tilde{s} - \alpha Z\mathrm{Var}_H[d]\sigma^2(1-\lambda)$$

.

Finally, defining

$$\kappa(\tilde{s}) = \frac{(2-Q)var[d|\tilde{s}]}{Q\sigma^2(1-\lambda) + (2-Q)var[d|\tilde{s}]},$$

gives us the equilibrium price:

$$P = \lambda\tilde{s}\kappa(\tilde{s}) + \mathrm{E}\left[d|\tilde{s}\right](1-\kappa(\tilde{s})) - \frac{1}{2-Q}\alpha\kappa(\tilde{s})\sigma^2(1-\lambda)Z,$$

□

**Proof of Corollary 1:**

*Proof.* First of all, note that $E[s_i|m] = mE[s_I] + (1-m)E[s_N] = 0$, hence, by the Law of Total covariance, and since $cov(E[s_i|m], E[s_j|m]) = 0$, we have

$$\begin{aligned} cov(s_i, s_j) = E[cov(s_i, s_j|m)] &= E[E[s_i s_j|m]] \\ &= E[m^2 E[s_I^2] + 2m(1-m)E[s_I s_N] + (1-m)^2 E[s_N]^2] \\ &= E[m^2 + (1-m)^2](\sigma^2 + \sigma_\varepsilon^2) \\ &= (2(var[m] + E[m]^2 - E[m]) + 1)(\sigma^2 + \sigma_\varepsilon^2) \end{aligned}$$

Moreover, $var[s_i] = E[E[s_i^2|m]] = E[mE[s_I^2] + (1-m)E[s_N^2]] = \sigma^2 + \sigma_\varepsilon^2$. Therefore, we get:

$$corr(s_i, s_j) = \frac{cov(s_i, s_j)}{\sqrt{var[s_i]var[s_j]}} = 2(var[m] + E[m]^2 - E[m]) + 1$$

Hence, if $E[m]$ is constant (which is the case when the distribution of beliefs about $m$ is symmetric), then a higher $var[m]$ implies a higher covariance between two random signals $s_i, s_j$ and thus also a lower degree of belief dispersion. □

**Proof of Corollary 2:**

*Proof.* We need to show how the posterior beliefs about $m$ depend on $\tilde{s}^2$. We have:

$$\frac{h_2(\tilde{s})}{h_1(\tilde{s})} = \sqrt{2}\exp\left(-\tilde{s}^2 \cdot \frac{1}{2(\sigma^2 + \sigma_\varepsilon^2)}\right)$$

Therefore $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$ is decreasing in $\tilde{s}^2$. But from equation (4.1) we can see that $\hat{\pi}_0$ (and similarly $\hat{\pi}_1$) is decreasing in $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$. Therefore we get that $\hat{\pi}_0$ and $\hat{\pi}_1$ are increasing in $\tilde{s}^2$. In contrast $\hat{\pi}_{1/2} = 1 - \hat{\pi}_0 - \hat{\pi}_1$ and hence it is decreasing in $\tilde{s}^2$. That is to say, a more extreme signal means that is more probable that the other group of traders is either all informed or all noise, while a signal closer to 0 means it is more probable there is a mixture of both. $\square$

**Proof of Proposition 2**:

*Proof.* First of all we have:

$$\begin{aligned} c(\tilde{s}) &= \lambda^2 \text{Var}[L(m)|\tilde{s}] = \\ &= \lambda^2 \left( E[(L(m))^2|\tilde{s}] - (E[L(m)|\tilde{s}])^2 \right) = \\ &= \lambda^2 \left[ ((1-2\hat{\pi}) \cdot \frac{1/2}{1/2} + \hat{\pi} \cdot 1) - (1-\hat{\pi})^2 \right] = \\ &= \lambda^2 \hat{\pi}(1-\hat{\pi}) \end{aligned}$$

which is increasing in $\tilde{s}^2$ since $\frac{d\hat{\pi}}{d\tilde{s}^2} > 0$ (by Corollary 2) and $\hat{\pi} < 1/2$, as $P(m \in 0, 1) = 2\hat{\pi}$.

Moreover, with a few algebraic manipulations we can get:

$$E[mL(m)|\tilde{s}] = \lambda \left[ (1-2\hat{\pi}) \cdot \frac{1/4}{1/2} + \hat{\pi} \cdot 1 \right] = \frac{1}{2}$$

Therefore H's perceived variance of $d$ must be increasing in $\tilde{s}^2$. $\square$

**Proof of Lemma 1**:

*Proof.* First of all, note that:

$$\text{Var}[m] = \pi + (1-2\pi) \cdot \frac{1}{4} - \frac{1}{4} = \frac{\pi}{2}.$$

Thus, it is sufficient to prove that $E[var[d|\tilde{s}]]$ is increasing in $\pi = P(m = 0)$, as $var[m]$ is increasing in $\pi = P(m = 0)$ (for symmetric 3-point distribution of $m$). Moreover, we know that $var[d|\tilde{s}]$ is increasing in $\tilde{s}^2$. Hence it would be sufficient to prove that $\tilde{s}^2(\pi_1)$ first order stochastically dominates $\tilde{s}^2(\pi_2)$ if

$\pi_1 > \pi_2$. Indeed we have, for any $x > 0$:

$$
\begin{aligned}
P(\tilde{s}^2 \leq x) =& P(-\sqrt{x} \leq \tilde{s} \leq \sqrt{x}) = \\
=& 1 - 2P(\tilde{s} \leq -\sqrt{x}) = \\
=& 1 - 2[2\pi P(s_I \leq -\sqrt{x}) + (1-2\pi)P(\frac{s_I + s_N}{2} \leq -\sqrt{x})] = \\
=& 1 - 2\left[P(\frac{s_I + s_N}{2} \leq -\sqrt{x}) + 2\pi\left[P(s_I \leq -\sqrt{x}) - P(\frac{s_I + s_N}{2} \leq -\sqrt{x})\right]\right]
\end{aligned}
$$

But now note that:

$$
\begin{aligned}
P(s_I \leq -\sqrt{x}) - P(\frac{s_I + s_N}{2} \leq -\sqrt{x}) =& P\left(\frac{s_I}{\sigma} \leq \frac{-\sqrt{x}}{\sigma}\right) - P\left(\frac{s_I + s_N}{\sigma\sqrt{2}} \leq \frac{-\sqrt{x})}{\sigma/\sqrt{2}}\right) \\
=& \Phi(\frac{-\sqrt{x}}{\sigma}) - \Phi(\frac{-\sqrt{x}}{\sigma/\sqrt{2}}) > \\
>& 0
\end{aligned}
$$

since $\dfrac{-\sqrt{x}}{\sigma} > \dfrac{-\sqrt{x}}{\sigma/\sqrt{2}}$.

Thus, $P(\tilde{s}^2 \leq x)$ is decreasing in $\pi$ and hence this shows that $\tilde{s}^2(\pi_1)$ first order stochastically dominates $\tilde{s}^2(\pi_2)$ when $\pi_1 > \pi_2$, and the proof is completed. □

**Proof of Proposition 3**:

*Proof.* Combining the fact that $\kappa(\tilde{s})$ is increasing in $\tilde{s}^2$ together with the stochastic dominance established in the proof of Lemma 1, we get that $\kappa(\tilde{s})$, and hence the risk premium is increasing in $var[m]$.

Moreover, when the market is dominated by Hedge Funds, $Q \to 2$ and $p = E[d|\tilde{s}] - \frac{1}{2}\alpha Z var[d|\tilde{s}]$. Therefore, by using the law of iterated expectations we get

$$E[d - p] = \frac{1}{2}\alpha Z E[var[d|\tilde{s}]]$$

which is increasing in $var[m]$ by the abovementioned lemma. □

**Proof of Corollary 5:**

*Proof.* Because of the symmetry of $f(m)$ with respect to 1/2, we can get:

$$E[\frac{dp}{ds_N}] = E[(1-m)\frac{dp}{d\tilde{s}}] = \frac{1}{2}E[\frac{dp}{d\tilde{s}}]$$

where the first equality is because of chain rule, while the second uses the abovementioned symmetry. Taking the derivative of price with respect to the mixed signal and setting $I(s) := E[L(m)|\tilde{s}]$, we get:

$$\frac{dp}{d\tilde{s}} = \lambda(\kappa(\tilde{s})+\tilde{s}\kappa^{'}(\tilde{s})+I(\tilde{s})(1-\kappa(\tilde{s}))+\tilde{s}I^{'}(\tilde{s})(1-\kappa(\tilde{s}))-\tilde{s}I(\tilde{s})\kappa^{'}(\tilde{s}))-\alpha\kappa^{'}(\tilde{s})\sigma^2(1-\lambda)Z$$

When $|\tilde{s}| \to \infty$, we get:

$$\begin{aligned} \kappa(\tilde{s}) &\to 1 \\ \kappa'(\tilde{s}) &\to 0 \\ \tilde{s}\kappa'(s) &\to 0 \\ I(\tilde{s}) &\to \frac{1}{2} \\ I'(\tilde{s}) &\to 0 \\ \tilde{s}I'(\tilde{s}) &\to 0 \end{aligned}$$

where the 3rd and the 6th lines hold because $\kappa'(\tilde{s})$ is of order[18] $\tilde{s}^{-2}$ and $I'(\tilde{s})$ is exponentially decreasing. Therefore we have $\frac{dp}{d\tilde{s}} \to \lambda$, as $|\tilde{s}| \to \infty$.

On the other hand we can see how this derivative behaves close to $\tilde{s} = 0$ and because of continuity, it is sufficient to just calculate the derivative at 0.

[18] In particular, note that $\kappa'(\tilde{s}) = \frac{(var_H[d])'}{(\sigma^2(1-\lambda)+var_H[d])^2}$. As $|\tilde{s}| \to \infty$, the denominator is of order $\tilde{s}^4$ while the numerator is equal to $(\lambda^2\hat{\pi}(1-\hat{\pi})\tilde{s}^2)'$ which is of order smaller than $\tilde{s}^2$ since the derivative of $\hat{\pi}(1-\hat{\pi})$ at infinity, is definitely bounded, since it is an increasing bounded function of $\tilde{s}^2$.

Indeed, we have:

$$\frac{dp}{d\tilde{s}}\Big|_{\tilde{s}=0} = \lambda\left(\kappa(\tilde{s}) + I(\tilde{s})(1-\kappa(\tilde{s}))\right) - \alpha\kappa'(\tilde{s})\sigma^2(1-\lambda)Z < \lambda$$

since $I(\tilde{s}) < 1$, and $1 > \kappa(\tilde{s}) > 0$ (even in the limit as $\tilde{s} \to 0$) and $\kappa'(s) \to 0$ as $\tilde{s} \to 0$, since $\kappa(\tilde{s})$ is increasing in $\tilde{s}^2$ (see Proposition 2), which implies $\kappa'(\tilde{s})$, is positive for $\tilde{s}$ positive, and negative for $\tilde{s}$ negative (and therefore that $\kappa'(0) = 0$ from Darboux theorem). □

**Joint distribution of $d, \tilde{s}$:**
Given a beta prior distribution for $m$, we can use a change of variables transformation to calculate the conditional densities $d|\tilde{s}$ and $m|\tilde{s}$. Using the map $d, u, m, \varepsilon \mapsto d, u, m, \underbrace{d+u+\varepsilon}_{\tilde{s}}$ we get the joint distribution of $d, u, m, \tilde{s}$:

$$g(d,u,m,\tilde{s}) = f_{d,u,m,\varepsilon}(d,u,m,\tilde{s} - md - (1-m)u) \cdot |\det(J)|,$$

where the $J$ is the Jacobian of the inverse map, and it can be easily computed to be an upper triangular matrix with 1's in the main diagonal.

Combining the above with the fact that $d, u, m, \varepsilon$ are independent we get $g(d,u,m,\tilde{s}) = f_d(d)f_u(u)f_m(m)f_\varepsilon(\tilde{s} - md - (1-m)u)$, where $f_u, f_d, f_\varepsilon$ are normal pdfs and $f_m(m)$ is the pdf of a Beta distribution. Integrating out $u$ and $m$, by completing the square where necessary (to get rid of the integral w.r.t. $u$), we get:

$$\begin{aligned} g(d,\tilde{s}) &= \int_0^1 \int_{-\infty}^{\infty} g(d,u,m,\tilde{s})dudm \\ &= \ldots = \\ &= \frac{e^{-\frac{d^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_0^1 \frac{m^{a-1}(1-m)^{b-1}}{B(a,b)} \frac{1}{\sqrt{2\pi V(m)}} e^{-\frac{(\tilde{s}-md)^2}{2V(m)}} dm. \end{aligned}$$

In the same way we can derive the joint distribution of $m, \tilde{s}$ and hence get the conditional density $m|\tilde{s}$ that we need, in order to make simulations for

$m \sim Beta(a, b)$. For the latter, we can write alternatively, using Bayes' rule:

$$g(m, \tilde{s}) = g(\tilde{s}|m) f_m(m)$$

But $\tilde{s}|m$ is a linear combination of normals, hence it is a normal itself, with mean 0 and variance $C(m) = (m^2 + (1-m)^2)(\sigma^2 + \sigma_\varepsilon^2)$. Hence, we get equation (1.4.2).

**Proof of Corollary 7**:

*Proof.* Let $V(m) := m^2 + (1-m)^2$. We will show that the informativeness of the signal, $E_H[\frac{m\lambda}{V(m)}]$, is decreasing in $\tilde{s}^2$. We will use the notation: $g_{\tilde{s}^2}(m, s)$ to refer to partial derivative: $\frac{\partial g(m,\tilde{s})}{\partial \tilde{s}^2}$. Using the formula for $g(m, \tilde{s})$, described in equation (1.4.2), we get: $g_{\tilde{s}^2}(m, \tilde{s}) = -\frac{g(m,\tilde{s})}{2V(m)(\sigma^2+\sigma_\varepsilon^2)}$. We will now use the following auxiliary result:

$$E_H[\frac{m\lambda}{V(m)}] = \frac{1}{2} E_H[\frac{\lambda}{V(m)}]$$

This is because $V(m)$ and $f_{m|\tilde{s}}(m|\tilde{s})$ is unchanged under the change of variables $m \mapsto 1 - m$ (as long as $m$ has symmetric distribution).

Using the above, together with the Leibniz integral rule, which allows us to interchange an integral with a partial derivative, as long as the integrand is a continuous function, we get:

$$\begin{aligned}
\frac{\partial}{\partial \tilde{s}^2} E_H[\frac{m\lambda}{V(m)}] &= \frac{1}{2} \int_0^1 \frac{\lambda}{V(m)} \left[ \frac{g_{\tilde{s}^2}(m, \tilde{s}) g(\tilde{s}) - g(m, \tilde{s}) g_{\tilde{s}^2}(\tilde{s})}{g(\tilde{s})^2} \right] dm \\
&= \frac{\lambda}{2g(\tilde{s})^2} \int_0^1 \frac{g(m, s)}{V(m)} \left[ -\frac{g(\tilde{s})}{2V(m)(\sigma^2 + \sigma_\varepsilon^2)} - g_{\tilde{s}^2}(\tilde{s}) \right] dm
\end{aligned}$$

Now noting that $g(\tilde{s}) = \int_0^1 g(m, s) dm$ and using that to obtain $g_{\tilde{s}^2}(\tilde{s}) = -\int_0^1 \frac{g(m,s)}{2V(m)(\sigma^2+\sigma_\varepsilon^2)} dm$ we get, that the sign of the derivative we want, depends on the sign of:

$$-\int_0^1 g(m,s)dm \int_0^1 \frac{g(m,s)}{V^2(m)}dm + \int_0^1 \frac{g(m,s)}{V(m)}dm \int_0^1 \frac{g(m,s)}{V(m)}dm$$

But using the Cauchy-Schwarz inequality[19] for integrals we know that:

$$\left(\int_0^1 \frac{g(m,s)}{V(m)}dm\right)^2 < \int_0^1 g(m,s)dm \int_0^1 \frac{g(m,s)}{V^2(m)}dm$$

Therefore the sign of the derivative we want is strictly negative. In other words $E_H[\frac{m\lambda}{V(m)}]$ is decreasing in $\tilde{s}^2$, thus *completing the proof.*

□

**Proof of Proposition 4**

*Proof.* We want to prove that the perceived variance of $d$ is decreasing in $\tilde{s}^2$. Firstly, since the posterior of $m$ is symmetric, as we showed before, we will have:

$$E_H[\frac{m^2}{V(m)}] = E_H[\frac{(1-m)^2}{V(m)}]$$

But since $V(m) = m^2 + (1-m)^2$ we get the the above expectations are equal to 1/2 (since their sum is equal to 1). Therefore:

$$E[mL(m)|\tilde{s}] = \frac{1}{2},$$

consistent with what we showed before for the case of the three-point distribution (note: $mL(m) = \frac{m^2}{V(m)}$). As a result, it would be sufficient to prove that

[19] Equality would only hold if $g(m,s)$ and $\frac{g(m,s)}{V^2(m)}$ were proportional, which of course it is not the case.

$c(\tilde{s})$ is increasing in $\tilde{s}^2$. Indeed, we have

$$\frac{\partial}{\partial \tilde{s}^2}[\mathrm{Var}_H[\frac{m}{V(m)}]] \propto \int_0^1 \frac{m^2 g(m,\tilde{s})}{V^2(m)}\left[-\frac{g(\tilde{s})}{C(m)} - g_{\tilde{s}^2}(\tilde{s})\right] dm-$$
$$-2\int_0^1 \frac{m}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm \int_0^1 \frac{m\cdot g(m,\tilde{s})}{V(m)}\left[-\frac{g(\tilde{s})}{C(m)} - g_{\tilde{s}^2}(\tilde{s})\right]$$

where the proportionality is with respect to a positive integer.

Now if we let $A(m,\tilde{s}) = \frac{g(m,\tilde{s})}{V(m)}$, and simplify $g_{\tilde{s}^2}(\tilde{s})$, as in the proof of corollary 7, we need to prove that the following is positive:

$$-g(\tilde{s})\int_0^1 \frac{m^2}{V^2(m)}A(m,\tilde{s})dm + \int_0^1 A(m,\tilde{s})dm \int_0^1 \frac{m^2}{V^2(m)} g(m,\tilde{s})dm-$$
$$-\frac{2\int_0^1 mA(m,\tilde{s})dm}{g(\tilde{s})}\left[-g(\tilde{s})m\int_0^1 \frac{mA(m,\tilde{s})}{V(m)}dm + \int_0^1 A(m,\tilde{s})dm\int_0^1 \frac{mg(m,\tilde{s})}{V(m)}dm\right]$$

Now we will use our usual transformation of $m \longrightarrow 1-m$ to get the following results:

$$\int_0^1 \frac{m}{V(m)}A(m,\tilde{s})dm = \frac{1}{2}\int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm$$
$$\int_0^1 \frac{m^2}{V^2(m)}A(m,\tilde{s})dm = \frac{1}{2}\int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm$$
$$\int_0^1 \frac{m^2}{V^2(m)}g(m,\tilde{s})dm = \frac{1}{2}\int_0^1 A(m,\tilde{s})dm$$
$$\int_0^1 mA(m,\tilde{s})dm = \frac{1}{2}\int_0^1 A(m,\tilde{s})dm$$

Plugging in the above, and multiplying the result by 2, we see that it is sufficient to prove that the following expression is positive:

$$-g(\tilde{s})\int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm + (\int_0^1 A(m,\tilde{s})dm)^2 + \int_0^1 A(m,\tilde{s})dm\int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm - \frac{(\int_0^1 A(m,\tilde{s})dm)^3}{g(\tilde{s})}$$

But the above expression is equal to

$$\left[\frac{(\int_0^1 A(m,\tilde{s})dm)^2}{g(\tilde{s})} - \int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm\right] \cdot \left[g(\tilde{s}) - \int_0^1 A(m,\tilde{s})dm\right]$$

Finally note that both of the above brackets are less than 0. The first one is less than 0 since by Cauchy-Schwarz inequality:

$$(\int_0^1 A(m,\tilde{s})dm)^2 < (\int_0^1 A(m,\tilde{s})V(m)dm) \int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm$$

while, the second one is negative because $A(m,\tilde{s}) = \frac{g(m,\tilde{s})}{V(m)} \geq g(m,\tilde{s})$, because $V(m) \leq 1$.

Therefore the overall expression is positive, making $c(\tilde{s})$ increasing in $\tilde{s}$ and thus concluding the proof that $\mathrm{Var}_H[d]$ is increasing in $\tilde{s}^2$.

□

**Proof of Corollary 8:**

*Proof.* We firstly observe that: $V(m) \in [\frac{1}{2}, 1]$, as the function $m^2 + (1-m)^2$ defined on the interval $[0,1]$ takes its minimum at $m = 1/2$ and its maximum at $m = 0, 1$. Moreover for $f_m(m)$ symmetric with respect to 0.5, we can easily see that: $f_{m|\tilde{s}}(m|\tilde{s}) = f_{m|\tilde{s}}(1-m|\tilde{s})$ and $V(m) = V(1-m)$. Therefore, using the change of variables $m' = 1-m$ we get, as before:

$$\int_0^1 \frac{m\lambda}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm = \frac{1}{2}\int_0^1 \frac{\lambda}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm$$

To prove the inequalities, we now just need to note that:

$$|\mathrm{E}_H[d]| = |\frac{\tilde{s}\lambda}{2}\int_0^1 \frac{1}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm|$$

Hence, combined with the abovementioned bounds on $V(m)$ we get the required result. □

**Proof of Corollary 10:**

*Proof.* We write $E_i[\cdot], var_i[\cdot]$ to denote $E[\cdot|\tilde{s}^{(i)}], var[\cdot|\tilde{s}^{(i)}]$ respectively. For the first case, $Q \to 2$ and $p_2 \to E[d_3|\tilde{s}^{(2)}]$. Therefore, using the law of total variance, we get:

$$var_1[p] \approx var_1[E[d_3|\tilde{s}^{(2)}]] = var_1[d_3] - E_1[var[d_3|\tilde{s}^{(2)}]]$$

which is decreasing in $|\tilde{s}_1|$, since $var_1[d_3] = var[d_3]$ and $E_1[var[d_3|\tilde{s}^{(2)}]]$ is increasing in $|\tilde{s}_1|$ as shown in Proposition 3, because conditioning on $\tilde{s}_1$ can only affect the distribution of $\tilde{s}_2$ through $m$ (and higher $|\tilde{s}_1|$ implies higher $var_1[m]$).
For the second case, $Q \to 0$ and $p_2 \to \lambda\tilde{s}_2$. Therefore $var_1[p] \approx \lambda^2 var_1[s_2]$, which is increasing in $|\tilde{s}_1|$, since a higher $|s_1|$ leads to a higher $var[m]$ which then leads to higher $var_1[\tilde{s}_2]$ (remember, the intuition, that the variance of $s_I$ or $s_N$ is higher than the variance of $\frac{s_I+s_N}{2}$). □

**Proof of Proposition 6:**

*Proof.* We have that

$$\tilde{s}_j^2 = (ms_{I,j} + (1-m)s_{N,j})^2 = m^2 s_{I,j}^2 + (1-m)^2 s_{N,j} + 2m(1-m)s_{I,j}s_{N,j}$$

Since $s_{I,j}, s_{N,j}$ for $j = 1, \ldots, n$ can be seen as realizations of a $(2 \times 1)$ random variable, with covariance matrix $\begin{pmatrix} \sigma^2+\sigma_\varepsilon^2 & 0 \\ 0 & \sigma^2+\sigma_\varepsilon^2 \end{pmatrix}$ by the Law of Large numbers, as $n \to \infty$ we get:

$$\sum_{j=1}^{n} \frac{\tilde{s}_j^2}{n} \to E[\tilde{s}_j^2] = (m^2 + (1-m)^2)(\sigma^2 + \sigma_\varepsilon^2)$$

That is, Hedge Funds learn the value of $m^2 + (1-m)^2$. By solving this quadratic one can see that there are always 2 solutions of the form $m*, (1-m*)$ and the managers have no way of distinguishing between the two, as the posterior distribution needs to remain symmetric.

Further to the above, we want to show that in the long run, the expected returns are higher when the long run uncertainty (which is proportional to $m^2 + (1-m)^2$) is higher. For that, we will first show that if $m_1^2 + (1-m_1)^2 > m_2^2 + (1-m_2)^2$ then $\tilde{s}^2(m1)$ first order stochastically dominates $\tilde{s}^2(m_2)$.

Indeed $P(\tilde{s}^2 \leq x) = 1 - 2P(\tilde{s} \leq -\sqrt{x})$. Now since $\tilde{s}$ has a symmetric distribution (when $m$ can take only 2 values, $m^*$ or $1-m^*$ with equal probability, which happens in the long run), we get:

$$\begin{aligned} P(\tilde{s} \leq -\sqrt{x}) =& \frac{1}{2} P(m^* s_I + (1-m^*)s_N \leq -\sqrt{x}) + \frac{1}{2} P(m^* s_N + (1-m^*)s_I \leq -\sqrt{x}) \\ =& P(m^* s_I + (1-m^*)s_N \leq -\sqrt{x}) \\ =& \Phi\left(\frac{-\sqrt{x}}{((m^*)^2 + (1-m^*)^2)(\sigma^2 + \sigma_\varepsilon^2)}\right) \end{aligned}$$

which is increasing in $(m^*)^2 + (1-m^*)^2$. This concludes the proof that $\tilde{s}^2(m_1) \succeq \tilde{s}^2(m_2)$. Therefore as in Proposition 3, we get that in the long run, the expected returns are increasing in $(m^*)^2 + (1-m^*)^2$ and hence in $|m^* - \frac{1}{2}|$. $\square$

## 1.9 Appendix B

In this section of the Appendix we would like to briefly discuss some alternative specifications (or interpretations) of the model, that can lead to results similar to these in the main body of the paper.

### 1.9.1 Two groups of Informed Traders

Here, we will discuss the case where instead of the Noise traders, the market is comprised of Hedge Funds, and two distinct groups of Informed traders, say A and B, who obtain signals on $d$ and are *cursed*, in the sense that they do not use the price to update their beliefs further. A traders interpret their signal correctly, but are dogmatic so that they do not take into account B's signal. B on the other hand, are dogmatic but also overconfident about their signal; that is they behave as if $\psi = 1$. Their signals are:

$$\begin{aligned} s_A &= d + \varepsilon_A \\ s_B &= \psi d + \sqrt{1-\psi^2}u + \varepsilon_B \end{aligned}$$

where $\psi \in [0,1]$, $\varepsilon_A, \varepsilon_B$ i.i.d variables $\sim N(0, \sigma_\varepsilon)$ and $u$ is distributed identically to $d$, but $B$ think that their signal is the same as the signal of informed traders. As before, hedge fund managers are uncertain about the ratio of $A$ to $B$ traders. When $\psi = 0$, then $B$ are Noise traders, and we get back to our original model (as $A$ do not even need to be dogmatic; they know they hold all relevant information). An alternative special case is when $\psi = 1$; in that case, $A$ and $B$ are completely symmetric, and can be interpreted as groups of analysts who obtain their own signal through their research and are dogmatic about their signals.

Under this specification, we still have the same price equation, as in Proposition 1. Moreover, the informativeness of the equilibrium quantities is decreasing on the size of the mixed news. This is because the main concept of this paper that extreme outcomes are more likely to occur when traders are more homogeneous still holds. However, Hedge Fund's uncertainty about fundamentals is not monotonically increasing as news get more extreme, contrary to Proposition 2. Indeed, when for instance there is a crash, hedge funds uncertainty about price informativeness is decreasing because they deduce that market consists (with high likelihood) of either all $A$ or all $B$ traders, and in both cases, the informativeness of their signal is the same. In contrast, when there are either all informed or all noise traders, informativeness takes two extreme values (0 and $\lambda$). Therefore all implications that are based on Proposition 2, are no longer true under this specification.

### 1.9.2 Only I and N in the market

Another specification we could think of, would be to have a market in which only $I$ and $N$ trade with each other. Then instead of hedge funds, we can have some managers (M) who (do not trade[20] but) use the price to infer information about the quality of their firms so that they can make better investment decisions, and we can then view the implications from the perspective of $M$. As before, the managers do not know the proportion $m$ of $I$ to $N$ traders.

In this model, market clearing is simply:

$$m\frac{\lambda s_I - p}{\alpha\sigma^2(1-\lambda)} + (1-m)\frac{\lambda s_N - p}{\alpha\sigma^2(1-\lambda)} = Z \tag{1.9.1}$$

and hence price would simply take the form

$$p = \lambda\underbrace{(ms_I + (1-m)s_N)}_{\tilde{s}} - \alpha\sigma^2(1-\lambda)Z$$

Then when the manager wants to get information about the firm using the price, he is faced with the same problem as that of the Hedge Funds of our baseline model. In particular, a larger price is associated with a larger $|\tilde{s}|$ and hence leads to a reduced price informativeness. That is an explanation of why

[20]Managers may not trade, for instance, due to regulatory restrictions.

extreme circumstances with very large (or very small) prices can be bad for real efficiency. In addition, this model has an extra advantage; since, prices are a strictly increasing function of $\tilde{s}$, there is no need for the assumption that agents condition on both the price and the residual demand to get an equilibrium. Starting from this very simple model, we can see how informativeness changes, when noise traders have sentiment shocks as in Mendel and Shleifer (2012), or when noise traders are just usual liquidity traders (with normal inelastic demand). Also, we could analyse whether the informativeness can be computed analytically, in the more general case where both the mass of I, $m_I$, and mass of N, $m_N$, are independent and unknown (in contrast with the original model where $m_I + m_N = 1$). To sum up, under this interpretation we could have a simpler and more tractable way to think of the effect of composition uncertainty on real economic decisions.

# Chapter 2

# Sentiment and speculation in a market with heterogeneous beliefs

*In the short run, the market is a voting machine but in the long run it is a weighing machine.*

—Attributed to Benjamin Graham by Warren Buffett.

In this paper, we study the effect of heterogeneity in beliefs on asset prices. We work with a frictionless dynamically complete market in which uncertainty evolves along a binomial tree. The model is populated by a continuum of risk-averse agents who differ in their beliefs about the probability of good news (i.e., of an "up move" in the binomial tree).

As a result, agents position themselves differently in the market. Optimistic investors make leveraged bets on the market; pessimists go short. If the market rallies, the wealth distribution shifts in favor of the optimists, whose beliefs become overrepresented in prices. If there is bad news, money flows to pessimists and prices more strongly reflect their pessimism going forward. At any point in time, one can define a representative agent who chooses to invest fully in the risky asset, with no borrowing or lending—our analog of Benjamin Graham's "Mr. Market"—but the identity (that is, the level of optimism) of the representative agent changes every period, with his or her beliefs becoming more optimistic following good news and more pessimistic following bad news. Thus market sentiment shifts constantly despite the stability of individual beliefs.

As all agents understand the importance of sentiment and take it into account in pricing, even moderate agents demand higher risk premia than they would in a homogeneous economy: they correctly foresee that either good or bad news will be amplified by a shift in sentiment. The presence of sentiment induces speculation: agents take temporary positions, at prices they believe to be fundamentally incorrect, in anticipation of adjusting their positions in the future. In our model, speculation can act in either direction, driving prices up in some states and down in others. (In fact we show that for a broad class of assets, including the "lognormal" case in which asset payoffs are geometric in the number of up-moves, heterogeneity drives prices down and risk premia up.) This feature is emphasized by Keynes (1936, Chapter 12); in Harrison and Kreps (1978*b*), by contrast, speculation only drives prices above their fundamental value. In our setting it can also happen that an agent—even the representative agent—trades in one direction this period, in *certain* anticipation of reversing his or her position next period.

Extreme states are much more important than they are in a homogeneous-belief economy. Consider a stylized example. The riskless rate is 0%. A risky bond matures in 50 days, and will default (paying \$30 rather than the par value of \$100) only in the "bottom" state of the world, that is, only if there are 50 consecutive pieces of bad news. Investors' beliefs about the probability, $h$, of an up-move are uniformly distributed between 0 and 1. Optimists therefore think default is almost impossible; a pessimistic agent with $h = 0.25$ thinks the default probability is less than $10^{-6}$. Even an agent in the 95th percentile of pessimism, $h = 0.05$, thinks the default probability is less than 8%. Initially, the representative investor is the median agent, $h = 0.5$, who thinks the default probability is less than $10^{-15}$. And yet we show that the bond trades at what might seem a remarkably low price: \$95.63. Moreover, almost half the agents—all agents with beliefs $h$ below 0.478—initially go short at this price, though most will reverse their position within two periods.[1] The low price arises because all agents understand that if there is bad news next period, pessimists' trades will have been profitable: their views will become overrepresented in the market, so the bond's price will decline sharply *in the short run.* Only agents with $h < 0.006$ plan to stay short to the bitter end.

It is interesting to contemplate how an econometrician who experiences multiple repetitions of this economy would think about pricing. Suppose for the sake of argument that the median agent is right, so that the true probability of an up-move is 50%. Econometric tests of short-run return behavior would make pricing look reasonable. Half the time the bond's price increases to \$100 and half the time the price declines to \$91.62, and these facts justify the initial price of \$95.63. But at some point the econometrician might notice a puzzle: measures of long-run value would seem to suggest that a "riskless" bond that "always" pays off nonetheless trades at a substantial discount to par value. With an objective default probability below $10^{-15}$, this conundrum would outlast several econometric careers.

We start by solving the model in discrete time. Terminal payoffs are exogenously specified, and can be arbitrary, subject to being positive at every node so that expected utility is finite. We find the wealth distribution, prices, all agents' investment decisions, and gross leverage at every node. We also characterize the cross-section of subjective perceptions of expected returns, volatilities, and Sharpe ratios. In general we do not take a stance on what the objectively correct beliefs are, nor even on whether there are objectively correct beliefs. But we can relate the equity premium perceived by the representative agent to an objectively measurable quantity, risk-neutral variance, that was proposed as a measure of the equity premium by Martin (2017).

After providing a formula for pricing in the general discrete-time case, we solve the model in a natural continuous-time limit in which the risky asset's

[1] Assuming there are two periods of bad news; if at any stage there is good news, the bond becomes riskless and disagreement vanishes.

terminal payoffs are lognormally distributed. In this limit, the underlying asset price agrees with the corresponding price in the continuous-time model of Atmaz and Basak (2018). As our framework is more tractable, we are able to study various issues that they do not (though, unlike us, they also price the underlying asset in the more general power utility case). We solve for agents' subjective beliefs about expected returns and true ("$\mathbb{P}$") volatility at all horizons; and for option prices at all maturities. Implied ("$\mathbb{Q}$") volatility is higher at short horizons, due to the effect of sentiment; and lower at long horizons, due to the moderating influence of the terminal date at which pricing is dictated entirely by fundamentals. "In the short run, the market is a voting machine but in the long run it is a weighing machine."

High implied volatility in the short run is also reflected in high physical measures of volatility (on which, in this continuous-time limit, all agents agree): there is no short-run variance risk premium. But physical measures of volatility decline more rapidly with horizon, so that there is a long-run variance risk premium.

As different investors have different beliefs but agree on asset prices, they have different stochastic discount factors (SDFs) whose properties help to reveal the interplay of beliefs, expected returns, and volatility. The volatility of any investor's SDF equals the maximum Sharpe ratio that the investor perceives as achievable by trading dynamically in the market (Hansen and Jagannathan, 1991). By comparing this to the Sharpe ratio the investor perceives on the asset if it is statically held—or shorted—to maturity, we can measure the perceived benefit of dynamic trade (i.e., of speculation, as in our setting the only reason to trade dynamically is to exploit differences in beliefs: without belief heterogeneity, agents would hold a static position). We also solve for the entropies of investors' SDFs (Alvarez and Jermann, 2005), which in our setting reveal the dollar value that different agents attach to being able to speculate.

All agents in our economy, particularly those with extreme beliefs, find speculation attractive. Extremists undertake conditional strategies that are increasingly aggressive as the market moves in their direction; in this sense, they are “long volatility.” We show that each investor can be thought of as having an investor-specific target price—the ideal outcome for the investor, given his or her beliefs and hence trading strategy—that can usefully be compared to what the investor expects to happen. The best possible outcome for an extremist is that the market moves by even more than he or she expected.

Conversely, investors with more moderate beliefs are short volatility. Among moderates, there is a particularly interesting *gloomy investor*, whose perception about the maximum attainable Sharpe ratio is most pessimistic among all investors. The gloomy investor is slightly more pessimistic than the median investor, so does not even perceive the market itself as earning a positive risk premium. Among all agents in our economy, the gloomy investor attaches the lowest dollar value to being able to participate in the market, relative to investing at the riskless rate; the (small) maximal Sharpe ratio he perceives can be attained either via a short volatility position or, equivalently, via a contrarian market-timing strategy that exploits what he perceives as irrational exuberance on the up side and irrational pessimism on the down side. The gloomy investor can therefore be thought of as supplying liquidity to the extremists. He hopes to be proved right: in a sense that we make precise, the best outcome for him is the one that he expects.

We make four key modelling choices. The first three are adopted from the model of Geanakoplos (2010) which inspired this paper. First, we assume that agents are dogmatic in their beliefs so that individuals do not experience changes in sentiment as time passes. If we allowed investors to learn over time, we believe that our mechanism would be amplified: that following good news, for example, optimistic agents would become relatively wealthier, as in our model, but all agents would also update their beliefs in an optimistic direction.

Second, we model uncertainty as evolving on a binomial tree so that the market is complete and agents can fully express their disagreement through trading. With an incomplete market, by contrast, agents may have strong differences in beliefs that are not revealed in prices. Market completeness also permits a clean interpretation of some of our results, as it generates a perfect correspondence between the cross-section and the time series. We exploit this fact to interpret our investors' trading behavior both in terms of conditional market-timing strategies and in terms of static positions in derivative securities.

Third, we allow for a continuum of beliefs, unlike papers including Harrison and Kreps (1978*b*), Scheinkman and Xiong (2003), Basak (2005), Banerjee and Kremer (2010), and Bhamra and Uppal (2014). Aside from being realistic, this implies that the identities of the representative investor, and of the investor who chooses to sit out of the market entirely, are smoothly varying equilibrium objects that are determined endogenously in an intuitive and tractable way.

Fourth, and finally, our agents are risk-averse. In this respect we depart from several papers in the heterogeneous beliefs literature—including Harrison and Kreps (1978*b*), Scheinkman and Xiong (2003) and Geanakoplos (2010)—that assume that agents are risk-neutral. Risk-neutrality simplifies matters in some respects, but complicates it in others. For example, short sales must be ruled out for equilibrium to exist. This is natural in some settings, but not if one thinks of the risky asset as representing, say, a broad stock market index. Moreover, the aggressive behavior of risk-neutral investors leads to extreme predictions: every time there is a down-move in the Geanakoplos model, all agents who are invested in the risky asset go bankrupt. From a technical point of view, short-sales constraints and risk-neutrality combine to give agents kinked indirect utility functions. Our agents have smooth indirect utility functions, and ultimately this is responsible for the tractability of our model and for our ability to study the dynamics described above.

## 2.1 Setup

We work in discrete time, with periods running from 0 to time $T$. Uncertainty evolves on a binomial tree, so that whatever the current state of the world, there are two possible successor states next period: "up" and "down." There is a risky asset, whose payoffs at the terminal date $T$ are specified exogenously. We normalize the net interest rate to 0%.

There is a unit mass of agents indexed by $h \in (0, 1)$. All agents have log utility and zero time-preference rate, and are initially endowed with one unit of the risky asset, which we will think of as representing "the market." Agent $h$ believes that the probability of an up-move is $h$; we often refer to $h$ as the agent's *belief*, for short. By working with the open interval $(0, 1)$, as opposed to the closed interval $[0, 1]$, we ensure that the investors' beliefs are all absolutely continuous with respect to each other: that is, they all agree on *what events can possibly happen*. This means in particular that no investor will allow his wealth to go to zero in any state of the world.

The mass of agents with belief $h$ follows a beta distribution governed by two parameters, $\alpha$ and $\beta$, such that the PDF is[2]

$$f(h) = \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha, \beta)}.$$

The parameters $\alpha$ and $\beta$ must be positive, but can otherwise be set arbitrarily. If $\alpha = \beta$ then the distribution of beliefs is symmetric with mean 1/2. If $\alpha = \beta = 1$ then $f(h) = 1$, so that beliefs are uniformly distributed over $(0, 1)$; this is a useful case to keep in mind as one works through the algebra. The case $\alpha \neq \beta$ allows for asymmetric distributions with mean $\alpha/(\alpha + \beta)$ and variance

[2]The beta function $B(\cdot, \cdot)$ is defined by

$$B(x, y) = \int_{h=0}^{1} h^{x-1}(1-h)^{y-1}\, dh\,.$$

If $x$ and $y$ are integers, then

$$B(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!},$$

and more generally the beta function is related to the gamma function as follows:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We will repeatedly use basic facts about the beta function, such as that $B(x, y) = B(y, x)$, and that $B(x+1, y) = B(x, y) \cdot \frac{x}{x+y}$.

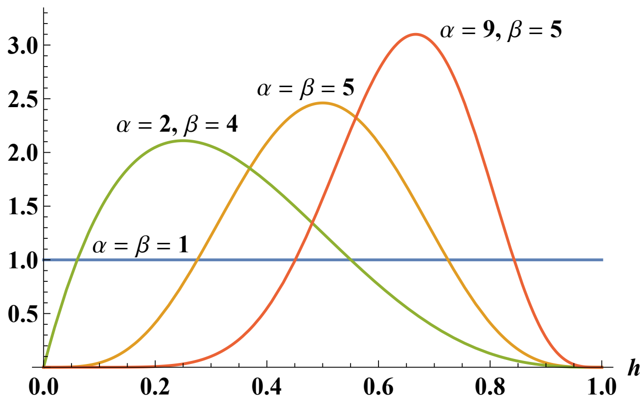



Figure 2.1.1: The distribution of beliefs for various choices of $\alpha$ and $\beta$.

$\alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$. Thus the distribution shifts toward 1 if $\alpha > \beta$ and toward 0 if $\alpha < \beta$, and beliefs are highly concentrated around the mean when $\alpha$ and $\beta$ are large: if, say, $\alpha = 90$ and $\beta = 10$ then beliefs are concentrated around a mean of 0.9, with standard deviation 0.030. Figure 2.1.1 plots the distribution of beliefs, $h$, for a range of choices of $\alpha$ and $\beta$.

## 2.2 Equilibrium

Suppose that the price of the risky asset at the current node is $p$, and that it will be either $p_d$ or $p_u$ next period, where we assume that $p_d \neq p_u$ so that the pricing problem is nontrivial. Suppose also that agent $h$ has wealth $w_h$ at the current node. If he chooses to hold $x_h$ units of the asset, then his wealth next period is $w_h - x_h p + x_h p_u$ in the up-state and $w_h - x_h p + x_h p_d$ in the down-state. So the portfolio problem is to solve

$$\max_{x_h} h \log\left[w_h - x_h p + x_h p_u\right] + (1-h)\log\left[w_h - x_h p + x_h p_d\right].$$

The agent's first-order condition is therefore

$$x_h = w_h\left(\frac{h}{p-p_d} - \frac{1-h}{p_u - p}\right). \tag{2.2.1}$$

The sign of $x_h$ is that of $p - p_u$ for $h = 0$ and that of $p - p_d$ for $h = 1$. These must have opposite signs to avoid an arbitrage opportunity, so there will always be some agents who are short and others who are long. The most optimistic agent[3] levers up as much as possible without risking default, and correspondingly the most pessimistic agent takes on the largest short position possible that does not risk default if the good state occurs. For, the first-order condition (2.2.1) implies that as $h \to 1$, agent $h$ holds $w_h/(p - p_d)$ units of stock. This is the largest possible position that does not risk default: to acquire it, the agent must borrow $w_h p/(p - p_d) - w_h = w_h p_d/(p - p_d)$. If the unthinkable (to this most optimistic agent!) occurs and the down state materialises, the agent's holdings are worth $w_h p_d/(p - p_d)$, which is precisely what the agent owes to his creditors.

It will often be convenient to think in terms of the risk-neutral probability of an up-move, $p^*$, defined by the property that the price can be interpreted as a risk-neutral expected payoff, $p = p^* p_u + (1 - p^*) p_d$. (There is no discounting, as the riskless rate is zero.) Hence

$$p^* = \frac{p - p_d}{p_u - p_d}.$$

In these terms, the first-order condition (2.2.1) becomes

$$x_h = \frac{w_h}{p_u - p_d} \frac{h - p^*}{p^*(1 - p^*)},$$

for example. An agent whose $h$ equals $p^*$ will have zero position in the risky asset: by the defining property of the risk-neutral probability, such an agent perceives that the risky asset has zero expected excess return.

[3] This is an abuse of terminology: there is no 'most optimistic agent' since $h$ lies in the *open* set $(0, 1)$. More formally, this discussion relates to the behavior of agents in the limit as $h \to 1$. An agent for whom $h = 1$ would want to take arbitrarily large levered positions in the risky asset, so there is a behavioral discontinuity at $h = 1$ (and similarly at $h = 0$).

Agent $h$'s wealth next period is therefore

$$w_h + x_h(p_u - p) = w_h(p_u - p_d)\frac{h}{p - p_d} = w_h\frac{h}{p^*} \tag{2.2.2}$$

in the up-state, and

$$w_h - x_h(p - p_d) = w_h(p_u - p_d)\frac{1-h}{p_u - p} = w_h\frac{1-h}{1-p^*} \tag{2.2.3}$$

in the down-state. In either case, all agents' returns on wealth are linear in their beliefs. Moreover, this relationship (which is critical for the tractability of our model) applies at every node. It follows that person $h$'s wealth at the current node must equal

$$\lambda_{\text{path}} h^m (1-h)^n$$

where $\lambda_{\text{path}}$ is a constant that is independent of $h$ but which can depend on the path travelled to get to the current node, which we have assumed has $m$ up and $n$ down steps.

As aggregate wealth is equal to the value of the risky asset—which is in unit supply—we must have

$$\int_0^1 \lambda_{\text{path}} h^m (1-h)^n f(h)\, dh = p.$$

This enables us to solve for the value of $\lambda_{\text{path}}$:

$$\lambda_{\text{path}} = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} p.$$

(This expression can be written in terms of factorials if $\alpha$ and $\beta$ are integers: for example, if $\alpha = \beta = 1$ then $\lambda_{\text{path}} = \frac{(m+n+1)!}{m!n!} p$. See footnote 2.)

Substituting back, agent $h$'s wealth equals

$$w_h = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} h^m (1-h)^n p. \tag{2.2.4}$$

This is maximized by $h \equiv m/(m+n)$: the agent whose beliefs turned out to be most accurate ex post ends up richest.

The wealth distribution—that is, the fraction of aggregate wealth held by type-$h$ agents—is a probability distribution over $h$. Specifically, it is the beta distribution with parameters $\alpha + m$ and $\beta + n$,

$$\frac{w_h f(h)}{p} = \frac{h^{\alpha+m-1}(1-h)^{\beta+n-1}}{B(\alpha+m,\beta+n)}. \tag{2.2.5}$$

We can now revisit Figure 2.1.1 in light of this fact. For the sake of argument, suppose that $\alpha = \beta = 1$ so that wealth is initially distributed uniformly across investors of all types $h \in (0,1)$. If, by time 4, there have been $m = 1$ up- and $n = 3$ down-moves, then equation (2.2.5) implies that the new wealth distribution follows the line denoted $\alpha = 2, \beta = 4$. (Investors with $h$ close to 0 or to 1 have been almost wiped out by their aggressive trades; the best performers are moderate pessimists with $h = 1/4$, whose beliefs happen to have been vindicated ex post.) At time 8, following three more up-moves and one down-move, the new wealth distribution is marked by $\alpha = \beta = 5$. And if by time 12 there have been a further four up-moves then the wealth distribution is marked by $\alpha = 9, \beta = 5$. The changing wealth distribution in this example illustrates a key feature of our model: at any point in time, wealth is concentrated in the hands of investors whose beliefs appear correct in hindsight.

Now we solve for the equilibrium price using the first-order condition

$$x_h = \underbrace{\frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} h^m (1-h)^n p}_{w_h} \left( \frac{h}{p - p_d} - \frac{1-h}{p_u - p} \right).$$

The price $p$ adjusts to clear the market, so that in aggregate the agents hold one unit of the asset:

$$\begin{aligned}\int_0^1 x_h f(h)\, dh &= \frac{p\left[(m+\alpha)(p_u - p) - (n+\beta)(p - p_d)\right]}{(m+n+\alpha+\beta)(p_u-p)(p-p_d)} \\ &= 1.\end{aligned}$$

It follows that

$$p = \frac{(m+\alpha)p_d p_u + (n+\beta)p_u p_d}{(m+\alpha)p_d + (n+\beta)p_u}. \tag{2.2.6}$$

Equivalently, the risk-neutral probability of an up-move must satisfy

$$p^* = \frac{(m+\alpha)p_d}{(m+\alpha)p_d + (n+\beta)p_u}$$

in equilibrium.

These results can usefully be interpreted in terms of *wealth-weighted beliefs*. For example, at time $t$, after $m$ up-moves and $n = t - m$ down-moves, the wealth-weighted cross-sectional average belief, $H_{m,t}$, can be computed with reference to the wealth distribution (2.2.5):

$$H_{m,t} = \int_0^1 h \frac{w_h f(h)}{p}\, dh = \frac{m+\alpha}{t+\alpha+\beta}. \tag{2.2.7}$$

In these terms we can write

$$p^* = \frac{H_{m,t} p_d}{H_{m,t} p_d + (1 - H_{m,t}) p_u}. \tag{2.2.8}$$

It follows that

$$\frac{p_u}{p} = \frac{H_{m,t}}{p^*} \quad \text{and} \quad \frac{p_d}{p} = \frac{1 - H_{m,t}}{1 - p^*}. \tag{2.2.9}$$

Hence $p^*$ is smaller than $H_{m,t}$ if $p_u > p_d$ and larger than $H_{m,t}$ if $p_u < p_d$: in either case, risk-neutral beliefs are more pessimistic than the wealth-weighted average belief.

The share of wealth an agent of type $h$ invests in the risky asset is

$$\frac{x_h p}{w_h} = p\left(\frac{h}{p - p_d} - \frac{1-h}{p_u - p}\right) = \frac{h}{1 - \frac{p_d}{p}} - \frac{1-h}{\frac{p_u}{p} - 1}.$$

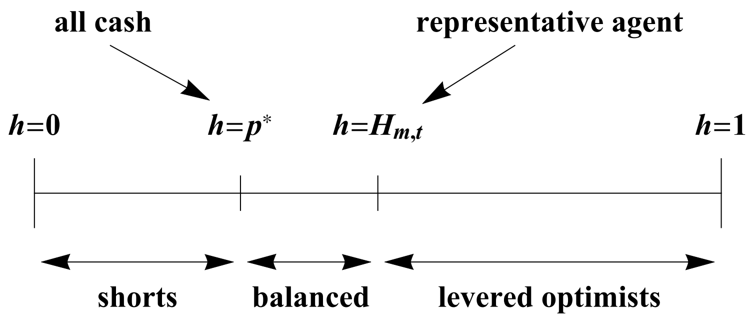



Figure 2.2.1: The range of beliefs in the investor population.

This can be rewritten in a more compact form using (2.2.9):

$$\begin{aligned}\frac{x_h p}{w_h} &\overset{(2.2.9)}{=} \frac{h}{1-\frac{1-H_{m,t}}{1-p^*}} - \frac{1-h}{\frac{H_{m,t}}{p^*}-1} \\ &= \frac{h-p^*}{H_{m,t}-p^*}\,. \end{aligned} \tag{2.2.10}$$

So the agent with $h = H_{m,t}$ can be thought of as *the representative agent*: by equation (2.2.10), this is the agent who chooses to invest her wealth fully in the market, with no borrowing or lending.

The identity of the representative investor therefore moves around over time, as does the identity of the investor with $h = p^*$ who chooses to hold his or her wealth fully in the bond. Figure 2.2.1 illustrates in the case $p_u > p_d$, so that $p^* < H_{m,t}$. Pessimistic investors with $h < p^*$ choose to short the risky asset; moderate investors with $p^* < h < H_{m,t}$ hold a balanced portfolio with long positions in both the bond and the risky asset; and optimistic investors with $h > H_{m,t}$ take on leverage, shorting the bond to go long the risky asset.

In a homogeneous economy in which all agents agree on the up-probability, $h = H$, it is easy to check that

$$p^* = \frac{Hp_d}{Hp_d + (1-H)p_u}. \tag{2.2.11}$$

Comparing equations (2.2.8) and (2.2.11), we see that for *short-run* pricing purposes our heterogeneous economy looks the same as a homogeneous economy featuring a representative agent with belief $H_{m,t}$. But as the identity of the representative agent changes over time, the similarity will disappear when we study the pricing of multi-period claims.

For future reference, the risk-neutral variance of the asset is

$$p^* \left(\frac{p_u}{p}\right)^2 + (1-p^*)\left(\frac{p_d}{p}\right)^2 - 1 = \frac{(H_{m,t}-p^*)^2}{p^*(1-p^*)}. \tag{2.2.12}$$

Below, we will compare this quantity with subjective expected returns, motivated by the results of Martin (2017).

We can also use equation (2.2.10) to calculate the *leverage ratio* of investor $h$, which we define as the ratio of funds borrowed, $x_h p - w_h$, to wealth, $w_h$:

$$\frac{x_h p - w_h}{w_h} = \frac{h - H_{m,t}}{H_{m,t} - p^*}. \tag{2.2.13}$$

If $p_u > p_d$ then $p^* < H_{m,t}$, by (2.2.9); in this case equation (2.2.13) shows that people who are optimistic relative to the representative investor borrow from pessimists. We can define *gross leverage* as the total dollar amount these optimists borrow,[4] scaled by aggregate wealth:

$$\begin{aligned}
\frac{\int_{H_{m,t}}^1 (x_h p - w_h)\, f(h)\, dh}{p} &= \int_{H_{m,t}}^1 \frac{w_h f(h)}{p}\frac{x_h p - w_h}{w_h}\, dh \\
&= \int_{H_{m,t}}^1 \frac{w_h f(h)}{p}\frac{h - H_{m,t}}{H_{m,t}-p^*}\, dh \\
&= \frac{H_{m,t}^{m+\alpha}(1-H_{m,t})^{n+\beta}}{(m+\alpha+n+\beta)B(\alpha+m,\beta+n)(H_{m,t}-p^*)}.
\end{aligned}$$

Conversely, if $p_u < p_d$ then optimists are lenders and pessimists borrowers. In either case, we can define gross leverage as the absolute value of the above expression,

$$\frac{H_{m,t}^{m+\alpha}(1-H_{m,t})^{n+\beta}}{(m+\alpha+n+\beta)B(\alpha+m,\beta+n)\,|H_{m,t}-p^*|}. \tag{2.2.14}$$

[4]The total dollar amount borrowed by *all* investors is zero, as the riskless asset is in zero net supply.

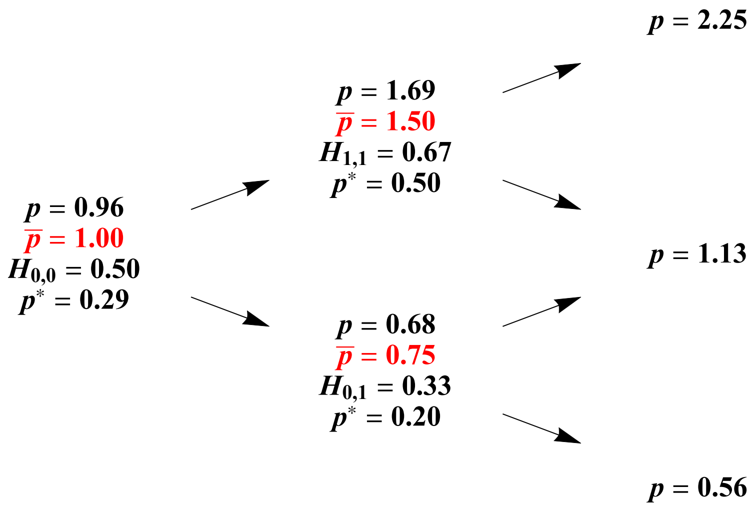



Figure 2.2.2: At each node, $\overline{p}$ denotes the price in a homogeneous economy with $H = 1/2$; $p$ is the price in a heterogeneous economy with $\alpha = \beta = 1$; and $p^*$ and $H_{m,t}$ indicate the risk-neutral probability of an up-move and the identity of the representative agent in the heterogeneous economy. In the homogeneous economy, the risk-neutral probability of an up-move is $1/3$ at every node.

Alternatively, scaling by the wealth of the borrowers and assuming that $p_u > p_d$ for simplicity, we define *borrower fragility*

$$\frac{\int_{H_{m,t}}^{1} (x_h p - w_h)\, f(h)\, dh}{\int_{H_{m,t}}^{1} w_h f(h)\, dh} = \frac{\int_{H_{m,t}}^{1} \frac{w_h f(h)}{p} \frac{x_h p - w_h}{w_h}\, dh}{\int_{H_{m,t}}^{1} \frac{w_h f(h)}{p}\, dh}\,, \tag{2.2.15}$$

which equals gross leverage divided by the fraction of wealth held by borrowers.

Figure 2.2.2 gives a numerical example with uniformly distributed beliefs (i.e., $\alpha = \beta = 1$) and $T = 2$. Terminal payoffs are chosen so that (i) $p_u/p_d = 2$ at the penultimate nodes and (ii) the asset would initially trade at a price of 1 in a homogeneous economy with $H = 1/2$. Initially, sentiment in the heterogeneous belief economy is the same—$H_{0,0} = 1/2$—but the price is lower, at 0.96, because of the anticipated effect of *future* sentiment. If bad news arrives, money flows to pessimists, the identity of the representative agent and risk-neutral beliefs become more pessimistic, and the price declines. Figure 2.2.3 shows the evolution of gross leverage and borrower fragility in the same

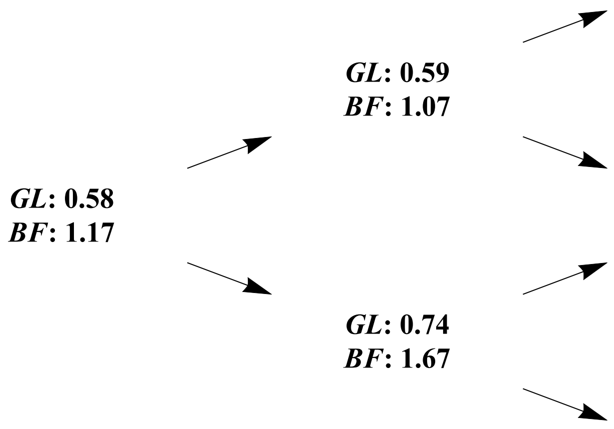



Figure 2.2.3: Gross leverage (GL) and borrower fragility (BF) at each node of the numerical example shown in Figure 2.2.2.

numerical example.

### 2.2.1 Subjective beliefs

Investors disagree on the properties of the asset. Consider first moments. Agent $h$'s subjectively perceived expected excess return on the market is

$$\frac{hp_u + (1-h)p_d}{p} - 1 = \frac{(h-p^*)(p_u - p_d)}{p} = \frac{(h-p^*)(H_{m,t} - p^*)}{p^*(1-p^*)}. \quad (2.2.16)$$

Hence the share of wealth invested by agent $h$ in the market (2.2.10) equals the ratio of the subjectively perceived expected excess return on the market (2.2.16) to (objectively defined) risk-neutral variance (2.2.12). In particular, risk-neutral variance reveals the expected excess return perceived by the representative agent, which is given by equation (2.2.16) with $h = H_{m,t}$.

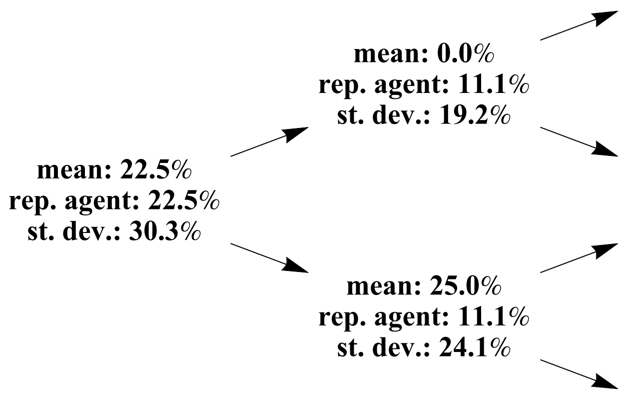



Figure 2.2.4: Mean subjective expected excess returns (2.2.17), the expected excess return perceived by the representative agent (2.2.18), and cross-sectional standard deviation of subjective expected excess returns (2.2.19) in the example shown in Figure 2.2.2. In a homogeneous economy with $H = 1/2$, all agents perceive an expected excess return of 12.5% at every node.

The cross-sectional average expected excess return is

$$\frac{\left(\frac{\alpha}{\alpha+\beta} - p^*\right)(H_{m,t} - p^*)}{p^*(1-p^*)}, \tag{2.2.17}$$

which may be positive or negative. But the wealth-weighted cross-sectional average expected excess return must be positive: by (2.2.7), it equals

$$\int_0^1 \frac{w_h}{p} \frac{(h-p^*)(H_{m,t}-p^*)}{p^*(1-p^*)} f(h)\, dh = \frac{(H_{m,t}-p^*)^2}{p^*\,(1-p^*)}. \tag{2.2.18}$$

Note that this quantity can also be interpreted as the expected excess return perceived by the representative agent $h = H_{m,t}$. The cross-sectional standard deviation of return expectations is

$$\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}}\frac{|H_{m,t}-p^*|}{p^*(1-p^*)}, \tag{2.2.19}$$

using the formula for the standard deviation of the beta distributed random variable $h$ in equation (2.2.16). Figure 2.2.4 shows the evolution of these quantities in the example of Figure 2.2.2.

Next we consider second moments. Person $h$'s subjectively perceived variance of the asset's return is

$$h\left(\frac{p_u}{p}\right)^2+(1-h)\left(\frac{p_d}{p}\right)^2-\left(\frac{hp_u+(1-h)p_d}{p}\right)^2=\frac{h(1-h)\left(H_{m,t}-p^*\right)^2}{p^{*2}(1-p^*)^2},$$

and person $h$'s perceived Sharpe ratio is therefore

$$\frac{h-p^*}{\sqrt{h(1-h)}},$$

which is increasing in $h$ for all $p^*$.

The variance risk premium perceived by investor $h$ (that is, subjective minus risk-neutral variance) is equal to

$$\frac{\left(H_{m,t}-p^*\right)^2}{p^*(1-p^*)}\left[\frac{h(1-h)}{p^*(1-p^*)}-1\right].$$

This is maximized (and weakly positive) for investor $h=1/2$, and negative for agents with beliefs $h$ that are further from $1/2$ than $p^*$ is.

The wealth return for agent $h$ is $h/p^*$ in the up state and $(1-h)/(1-p^*)$ in the down state, as shown in equations (2.2.2) and (2.2.3). So agent $h$'s subjective expected excess return on own wealth is

$$\frac{h^2}{p^*}+\frac{(1-h)^2}{1-p^*}-1=\frac{(h-p^*)^2}{p^*(1-p^*)}.$$

All agents expect to earn a nonnegative excess return on wealth, though they have very different positions. Only agent $h=p^*$ chooses to take no risk, fully invests in the bond, and so correctly anticipates zero excess return.

### 2.2.2 A risky bond

The dynamic that drives our model is particularly stark in the "risky bond" example outlined in the introduction. Suppose that the terminal payoff is 1 in all states apart from the very bottom one, in which the payoff is $\varepsilon$; the price of the asset is therefore 1 as soon as an up-move occurs. Writing $p_t$ for the price at time $t$ following $t$ consecutive down-moves we have, from equation (2.2.6),

$$p_t = \frac{\alpha p_{t+1} + (t+\beta)p_{t+1}}{\alpha p_{t+1} + t + \beta}.$$

Defining $y_t \equiv 1/p_t - 1$, this can be rearranged as

$$y_t = \frac{\beta + t}{\alpha + \beta + t} y_{t+1}. \tag{2.2.20}$$

We can interpret $y_t$ as the inducement to invest in the risky asset at time $t$, following $t$ consecutive down-moves: it is the realized excess return on the asset if there is an up-move from $t$ to $t+1$. Equation (2.2.20) determines the rate at which this inducement must rise in equilibrium.

Solving equation (2.2.20) forward,

$$y_t = \frac{(\beta+t)(\beta+t+1)\cdots(\beta+T-1)}{(\alpha+\beta+t)(\alpha+\beta+t+1)\cdots(\alpha+\beta+T-1)} y_T\,,$$

and the terminal condition dictates that $y_T = (1-\varepsilon)/\varepsilon$. Thus, finally,

$$p_t = \frac{1}{1 + \frac{\Gamma(\beta+T)\Gamma(\alpha+\beta+t)}{\Gamma(\beta+t)\Gamma(\alpha+\beta+T)} \frac{1-\varepsilon}{\varepsilon}}.$$

If $\alpha = \beta = 1$, we can simplify further, to

$$p_t = \frac{1}{1 + \frac{1+t}{1+T}\frac{1-\varepsilon}{\varepsilon}}. \tag{2.2.21}$$

We can calculate the risk-neutral probability of an up-move at time $t$, which we (temporarily) denote by $p_t^*$, by applying (2.2.9) with $p = p_t$, $p_u = 1$,

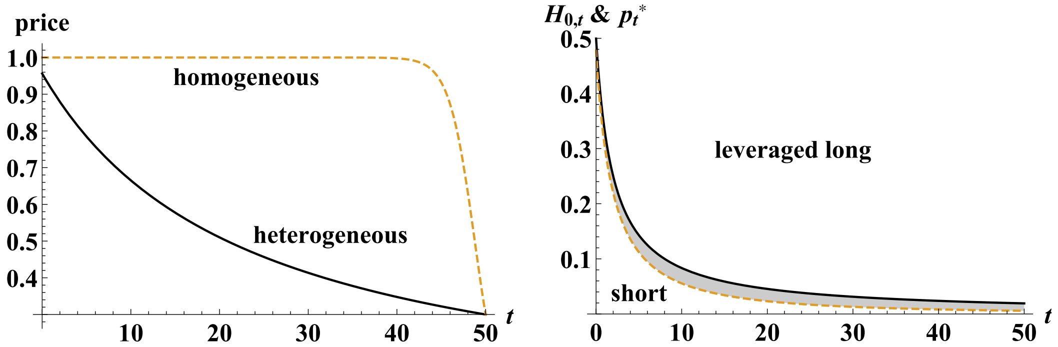



Figure 2.2.5: Left: The risky bond's price over time in the heterogeneous and homogeneous economies following consistently bad news. Right: $H_{0,t}$ reveals the identity of the representative agent at time $t$ following consistently bad news. Investors who are more optimistic, $h > H_{0,t}$, have leveraged long positions in the risky bond. The risk-neutral probability reveals the identity of the investor who is fully invested in the riskless bond at time $t$, with zero position in the risky bond. Investors who are more pessimistic, $h < p_t^*$, are short the risky bond. Investors with $p_t^* < h < H_{0,t}$ (shaded) are long both the risky and the riskless bond.

and $p_d = p_{t+1}$ to find that

$$p_t^* = H_{0,t} p_t = \frac{\alpha p_t}{\alpha + \beta + t}. \tag{2.2.22}$$

Figure 2.2.5 illustrates these calculations in the example described in the introduction, with $T = 50$ periods to go, and a recovery value of $\varepsilon = 0.30$. The panels show how the price and risk-neutral probability evolve if bad news arrives each period. Initially, the bond trades at what might seem a remarkably low price of 0.9563.

By contrast, in a homogeneous economy with $H = 1/2$ the price, $p_t$, and risk-neutral probability, $p_t^*$, following $t$ down-moves would be

$$p_t = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} 0.5^{T-t}} \quad \text{and} \quad p_t^* = \frac{p_t}{2},$$

respectively. Thus with homogeneous beliefs the bond price is approximately 1, and the risk-neutral probability of an up-move is approximately 1/2, until shortly before the bond's maturity.

From the perspective of time 0, the risk-neutral probability of default—call it $\delta^*$—satisfies

$$p_0 = 1 - \delta^* + \delta^* \varepsilon, \quad \text{so} \quad \delta^* = \frac{1 - p_0}{1 - \varepsilon}.$$

In the homogeneous case, therefore,

$$\delta^* = \frac{1}{1 + \varepsilon\,(2^T - 1)} = O(2^{-T})\,;$$

and in the heterogeneous case with $\alpha = 1$,

$$\delta^* = \frac{1}{1 + \varepsilon T} = O(1/T).$$

There is a qualitative difference between the homogeneous economy, in which default is exponentially unlikely, and the heterogeneous economy, in which default is only polynomially unlikely.[5]

To understand pricing in the heterogeneous economy, it is helpful to think through the portfolio choices of individual investors. We use equations (2.2.5), (2.2.7), and (2.2.10), together with the prices and risk-neutral probabilities given in (2.2.21) and (2.2.22) above, to find investors' holdings of the risky asset at each node.

The median investor, $h = 0.5$, thinks the probability that the bond will default—i.e., that the price will follow the path shown in Figure 2.2.5 all the way to the end—is $2^{-50} < 10^{-15}$. Even so, he believes the price is right at time zero (in the sense that he *is* the representative agent) because of the short-run impact of sentiment. Meanwhile, a modestly pessimistic agent with $h = 0.25$ will choose to short the bond at the price of 0.9563—and will remain short at time $t = 1$ before reversing her position at $t = 2$—despite believing that the

[5]This holds more generally for any $\alpha = \beta > 1$: it is easy to show that $\delta^* = O\left(T^{-\alpha}\right)$ by Stirling's formula. It is also true if $\varepsilon > 1$, i.e. in the 'lottery ticket' case. Then, $\delta^*$ is interpreted as the probability of the lottery ticket paying off, which is exponentially small in the homogeneous economy but only polynomially small in the heterogeneous belief economy.

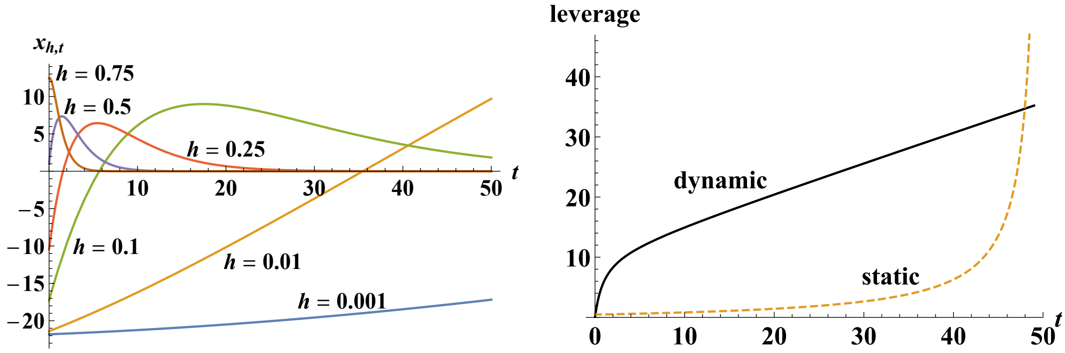



Figure 2.2.6: Left: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time. Right: The evolution of leverage for the median investor under the optimal dynamic and static strategies. Both panels assume bad news arrives each period.

bond's default probability is less than $10^{-6}$. (Recall from equation (2.2.10) that $p_t^*$ is the belief of the agent who is neither long nor short the asset. More optimistic agents, $h > p^*$, are long, and more pessimistic agents, $h < p^*$, are short.) Following a few periods of bad news, almost all investors are long, but the most pessimistic investors have become extraordinarily wealthy.

The left panel of Figure 2.2.6 shows the holdings of the risky asset for a range of investors with different beliefs, along the trajectory in which bad news keeps on coming. The optimistic investor $h = 0.75$ starts out highly leveraged so rapidly loses almost all his money. The median investor, $h = 0.5$, initially invests fully in the risky bond without taking on leverage. If bad news arrives, this investor takes on leverage in order to be able to increase the size of her position despite her losses; after about 10 periods, the investor is almost completely wiped out. Moderately bearish investors start out short. For example, investor $h = 0.25$ starts out short about 10 units of the bond, despite believing that the probability it defaults is less than one in a million, but reverses her position after two down-moves. Investor $h = 0.01$, who thinks that there is more than a 60% chance of default, is initially extremely short but eventually reverses position as still more bearish investors come to dominate the market.

The right panel of Figure 2.2.6 shows how the median investor's leverage changes over time if he follows the optimal dynamic and static strategies. If forced to trade statically, his leverage ratio is initially 0.457. This seemingly modest number is dictated by the requirement that the investor avoid bankruptcy at the bottom node (and in fact the leverage of all investors with $h \geq 0.2$ is visually indistinguishable at the scale of the figure). If the median investor can trade dynamically, by contrast, the optimal strategy is, initially, to invest fully in the risky bond without leverage. Subsequently, however, optimal leverage rises fast. Thus the dynamic investor keeps his powder dry by investing cautiously at first but then aggressively exploiting further selloffs.

All investors perceive themselves as better off if able to trade dynamically, of course. In Appendix 2.6 we analytically characterize the perceived advantage of dynamic versus static trade as a function of each investor's belief $h$.

The volume of trade (in terms of the number of units of the risky asset transacted) in the transition from time $t$ to time $t+1$ is[6]

$$\frac{1}{2}\int_0^1 \left| \frac{(1-h)^t}{\frac{1}{1+t}} \frac{h-p_t^*}{H_{0,t}-p_t^*} - \frac{(1-h)^{t+1}}{\frac{1}{2+t}} \frac{h-p_{t+1}^*}{H_{0,t+1}-p_{t+1}^*} \right| dh = \frac{4(1+t)^{1+t}}{(3+t)^{3+t}} \left(1+t+\frac{1+\varepsilon T}{1-\varepsilon}\right),$$

while gross leverage and borrower fragility, calculated from (2.2.14) and (2.2.15), equal

$$\left(\frac{1+t}{2+t}\right)^{2+t}\left(1+\frac{1+T}{1+t}\frac{\varepsilon}{1-\varepsilon}\right) \quad \text{and} \quad \left(\frac{1+t}{2+t}\right)\left(1+\frac{1+T}{1+t}\frac{\varepsilon}{1-\varepsilon}\right)$$

respectively.

The left panel of Figure 2.2.7 shows the time series of volume, gross leverage, and borrower fragility. In this stylized example there is a burst of trade at first: volume substantially exceeds the total supply of the asset initially, as agents with extreme views undertake highly leveraged trades, but declines rapidly over time as wealth becomes concentrated in the hands of investors with similar beliefs. The right panel shows the corresponding series if $\varepsilon = 0.9$.

[6] We include the factor of 1/2 to avoid double-counting.

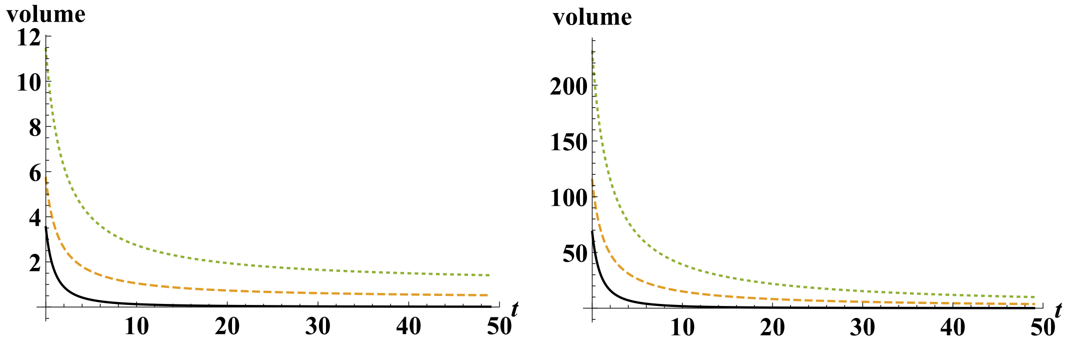



Figure 2.2.7: Volume (solid), gross leverage (dashed), and borrower fragility (dotted) over time, with $\varepsilon = 0.3$ (left) or $\varepsilon = 0.9$ (right). Heterogeneous case only: volume is zero in the homogeneous economy.

In this case disagreement generates more aggressive trading, and more volume, because the relative safety of the asset permits agents to take on more leverage: extremists on both sides of the market are "picking up nickels in front of a steamroller."

### 2.2.3 An example with late resolution of uncertainty

Consider an example with an odd number of periods, $T$, and $\alpha = \beta = 1$; and let $0 < \varepsilon < 1$. If there have been an even number of up-moves at time $T$, the asset pays off $\frac{1}{1+\varepsilon}$; if there have been an odd number of up-moves, the asset pays $\frac{1}{1-\varepsilon}$.

In the homogeneous economy with $H = 1/2$, the asset trades at a price of 1 in every node, and at every period, until the terminal payoff: it is therefore riskless until the final period.

In the heterogeneous economy it follows immediately from Result 1, below, that the asset also trades at 1 initially. But the asset is now volatile: although the payoff of the asset is up in the air until the very last period, the effect of sentiment ripples back so that the asset is volatile throughout its lifetime, and its price therefore embeds a risk premium.[7]

[7] There is also an equilibrium in which the asset's price is 1 until time $T - 1$, as in the

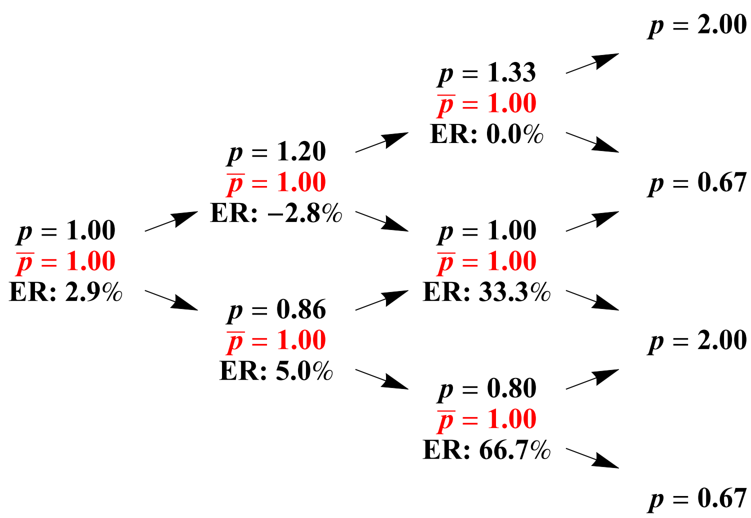



Figure 2.2.8: An example with late resolution of uncertainty. Heterogeneous-economy price ($p$), homogeneous-economy price ($\overline{p}$), and the cross-sectional average perceived excess return in the heterogeneous economy (ER).

Figure 2.2.8 shows an example with $T = 3$ and $\varepsilon = 1/2$. In a homogeneous economy, the asset's price is completely stable until immediately before the terminal date. In the heterogeneous economy, the asset's price is volatile, and it embeds a time-varying risk premium.

### 2.2.4 The general case

Write $z_{m,t} = 1/p_{m,t}$, where $m$ is the number of up moves that have taken place by time $t$. Equation (2.2.6) implies that the following recurrence relation holds at each node:

$$z_{m,t} = H_{m,t} z_{m+1,t+1} + (1 - H_{m,t}) z_{m,t+1}. \tag{2.2.23}$$

That is, the price at each node is the weighted average harmonic mean of the next-period prices, with weights given by the beliefs of the representative agent at the relevant node. By backward induction, $z_{0,0}$ is a linear combination of

homogeneous economy. Then the market is incomplete, and agents have no means of betting against one another. But this equilibrium is not robust to vanishingly small perturbations of the terminal payoffs, which would restore market completeness.

$z_{i,T}$, for $i = 0, 1, \ldots T$:

$$z_{0,0} = \sum_{m=0}^{T} c_m z_{m,T}. \tag{2.2.24}$$

Pricing is not path-dependent in our economy. Indeed, we have

$$\underbrace{\frac{m+\alpha}{t+\alpha+\beta}}_{H_{m,t}} \underbrace{\frac{t-m+\beta}{t+1+\alpha+\beta}}_{1-H_{m+1,t+1}} = \underbrace{\frac{t-m+\beta}{t+\alpha+\beta}}_{1-H_{m,t}} \underbrace{\frac{m+\alpha}{t+1+\alpha+\beta}}_{H_{m,t+1}}.$$

Equivalently, given (2.2.9), the risk-neutral probability of going up and then down (from any starting node) equals the risk-neutral probability of going down and then up. That is,

$$p^*_{m,t}(1 - p^*_{m+1,t+1}) = (1 - p^*_{m,t})p^*_{m,t+1}.$$

These observations allow us to find a general pricing formula that applies for arbitrary terminal payoffs $p_{m,T}$. (The payoffs must be positive so that the expected utility of any agent is well defined.) The proof of the result, and all subsequent results, is in the Appendix.

**Result 1.** *If the risky asset has terminal payoffs $p_{m,T}$ at time $T$ (for $m = 0, \ldots, T$), then its initial price is*

$$p_0 = \frac{1}{\displaystyle\sum_{m=0}^{T} \frac{c_m}{p_{m,T}}}, \tag{2.2.25}$$

*where*

$$c_m = \binom{T}{m} \frac{B(\alpha+m, \beta+T-m)}{B(\alpha, \beta)}. \tag{2.2.26}$$

*The time 0 price of the Arrow–Debreu security that pays off if there have been $m$ up-moves by time $T$ is*

$$q^*_m = c_m \frac{p_0}{p_{m,T}}.$$

The coefficients $c_m$ have a so-called beta-binomial distribution, $BB(T, \alpha, \beta)$. This is a binomial distribution with a random probability of success in each trial given by a $Beta(\alpha, \beta)$ distribution.[8] In the Appendix, we generalize equation (2.2.24) and Result 1 to price the risky asset at any node.

As a corollary of Result 1, we can find the effect of belief heterogeneity on prices for a broad class of assets.

**Result 2.** *If beliefs are symmetric, and the risky asset has terminal payoffs such that* $\frac{1}{p_{m,T}}$ *is convex if viewed as a function of* $m$*, then the asset's time 0 price is decreasing in the degree of belief heterogeneity. In particular, it is sufficient (though not necessary) that* $\log p_{m,T}$ *be concave for the asset's price to be decreasing in the degree of belief heterogeneity.*

Result 2 applies if the terminal payoff is concave in $m$. But it also applies for some convex payoffs. If, for example, the asset's payoffs increase or decrease geometrically in $m$, then the log payoffs are linear in $m$, so that the concavity condition (just) holds. We provide a more extensive analysis of this case in the next section.

## 2.3 A diffusion limit

We consider a natural continuous time limit by allowing the number of periods to tend to infinity and specifying geometrically increasing terminal payoffs. This is the setting of Cox et al. (1979), in which the Black–Scholes formula emerges in the corresponding limit with homogeneous beliefs. We are able to solve for the asset price, risk-neutral probabilities, the volatility term structure, individuals' trading strategies, and other quantities of interest.

Denote by $2N$ the total number of periods (corresponding to time $T$).[9]

[8] In fact, $c_m$ can be interpreted as the *cross-sectional average* (among investors) perceived probability of reaching node $(m, T)$.

[9] The choice of an even number of periods is unimportant, but it simplifies the notation in some of our proofs.

We assume that

$$p_{m,T} = e^{2\sigma\sqrt{\frac{T}{2N}}(m-N)}. \tag{2.3.1}$$

As we will see, $\sigma$ can be interpreted as the volatility of terminal payoffs (on which all agents will turn out to agree). If we set $\lambda = e^{\sigma\sqrt{\frac{T}{2N}}}$, then we see that $p_{m,2N} = \lambda^m(\frac{1}{\lambda})^{2N-m}$, where $\lambda = u = d^{-1}$ and $u, d$ are the up and down percentage movements of the stock price in the Cox–Ross–Rubinstein model. If we now set $\psi = \frac{m-N}{\sqrt{N}}$ then $p_{m,T} = e^{\sigma\sqrt{2T}\psi}$. From the perspective of each agent, $m$ has a binomial distribution; we show, in the Appendix, that in the limit as $N \to \infty$, $\psi$ has an asymptotic normal distribution from the perspective of each investor.

We use Result 1 to price the asset at each node of the tree, then take the limit as $N$ tends to infinity. As the number of up/down steps increases with $N$, the extent of disagreement over any individual step must decline to generate sensible limiting results—that is, we allow the parameters $\alpha, \beta$, which control the belief dispersion in the market, to tend to infinity with $N$. In particular we will write $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$. Small values of $\theta$ correspond to a high belief heterogeneity, while the limit $\theta \to \infty$ corresponds to the homogeneous case; we will refer to $\frac{1}{\theta}$ as capturing the degree of heterogeneity in the market. The level of optimism in the market is captured by $\eta$.

To be more precise, we will introduce a cross-sectional expectation operator $\widetilde{\mathrm{E}}[\cdot]$. So, for example, the cross-sectional mean of $h$ satisfies $\widetilde{\mathrm{E}}[h] = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}}$ and $\widetilde{\mathrm{var}}[h] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{8\theta N+1} + O(\frac{1}{N^2})$. As $\widetilde{\mathrm{E}}\left[\mathrm{E}^{(h)}[\psi]\right] = \frac{\eta}{\theta}$, we can interpret $\eta$ as controlling the cross-sectional mean expected terminal payoff.

In the work of Cox, Ross, and Rubinstein, the central limit theorem is used to approximate a binomial distribution with a normal random variable. A similar, though slightly more convoluted, situation arises in our setting. The argument starts by rewriting equation (2.2.24) as[10]

$$p_0^{-1} = \mathrm{E}_m\left[e^{-\sigma\sqrt{2T}\frac{m-N}{\sqrt{N}}}\right] = M_\psi\left(-\sigma\sqrt{2T}\psi\right).$$

[10]From now on we suppress the explicit dependence of price on state in our notation and write, for example, $p_0$ rather than $p_{0,0}$.

where we write $\mathrm{E}_m$ to indicate that that the expectation is taken over $m$ which, by Result 1, can be viewed as a random variable following the beta-binomial distribution with parameters $2N$, $\alpha$, and $\beta$; and $M_\psi(\cdot)$ denotes the moment generating function (MGF) of $\psi = \frac{m-N}{\sqrt{N}}$. As $\psi$ is asymptotically normal by a result of Paul and Plackett (1978), $M_\psi(\cdot)$ converges to the MGF of a Normal distribution—a known, and simple, function. We provide full details in the Appendix.

**Result 3.** *The price of the asset at time 0 is given by:*

$$p_0 = \exp\left(\frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta+1}{2\theta}\sigma^2 T\right). \tag{2.3.2}$$

If $\eta = 0$, so that the cross-sectional distribution of beliefs is symmetric around $h = 1/2$, then the price at time 0 is decreasing in the degree of heterogeneity, $\theta^{-1}$, consistent with Result 2. But if the cross-sectional average belief is sufficiently optimistic—that is, if $\eta$ is sufficiently positive—then the price may be increasing in the heterogeneity of beliefs.

We now study what this price implies for different agents' expectations about returns. We parametrize an agent by the number of standard deviations, $z$, by which his or her belief deviates from the mean: $h = \widetilde{\mathrm{E}}[h] + z\sqrt{\widetilde{\mathrm{var}}[h]} \approx \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}} + \frac{z}{\sqrt{8\theta N}}$.[11] Thus an agent with $z = 2$ is two standard deviations more optimistic than the mean agent. When we use this parametrization, we write superscripts $z$ rather than $h$ (for example, $\mathrm{E}^{(z)}$ rather than $\mathrm{E}^{(h)}$).

**Result 4.** *The return of the asset from time 0 to time $t$, from the perspective of agent $h = \widetilde{\mathrm{E}}[h] + z\sqrt{\widetilde{\mathrm{var}}[h]}$ has a lognormal distribution with*

$$\mathrm{E}^{(z)} \log R_{0\to t} = \frac{\theta+1}{\theta+\frac{t}{T}}\left(\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta}\sigma^2\right)t$$
$$\mathrm{var}^{(z)} \log R_{0\to t} = \left(\frac{\theta+1}{\theta+\frac{t}{T}}\right)^2 \sigma^2 t\,.$$

[11] Note that $\widetilde{\mathrm{E}}[h] = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}}$ and $\widetilde{\mathrm{var}}[h] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{8\theta N+1} + O(\frac{1}{N^2})$. The lower order terms, $O(1/N^2)$, will not play any role as $N$ approaches infinity.

The expected return on the asset follows immediately:

**Result 5.** *The (annualized) expected return of the asset from 0 to $t$ is*

$$\frac{1}{t}\log \mathrm{E}^{(z)} R_{0\to t} = \frac{\theta+1}{\theta+\frac{t}{T}}\left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta}\frac{2\theta+\frac{t}{T}}{\theta+\frac{t}{T}}\sigma^2\right].$$

*In particular, the instantaneous expected return is*

$$\lim_{t\to 0}\frac{1}{t}\log \mathrm{E}^{(z)} R_{0\to t} = \frac{\theta+1}{\theta}\frac{z\sigma}{\sqrt{\theta T}} + \left(\frac{\theta+1}{\theta}\right)^2\sigma^2$$

*and the expected return to maturity is*

$$\frac{1}{T}\log \mathrm{E}^{(z)} R_{0\to T} = \frac{z\sigma}{\sqrt{\theta T}} + \frac{2\theta+1}{2\theta}\sigma^2. \tag{2.3.3}$$

Results 4 and 5 show that although different agents perceive different expected returns, all agents agree on the (true) volatility of returns.

**Result 6.** *Recall that $\widetilde{\mathrm{E}}$ is the cross-sectional expectation operator. The cross-sectional mean (or median) expected return is*[12]

$$\widetilde{\mathrm{E}}\left[\frac{1}{t}\log \mathrm{E}^{(z)} R_{0\to t}\right] = \frac{(\theta+1)^2\left(\theta+\frac{t}{2T}\right)}{\theta\left(\theta+\frac{t}{T}\right)^2}\sigma^2.$$

*Disagreement is the standard deviation of expected returns $\frac{1}{t}\log \mathrm{E}^{(z)} R_{0\to t}$:*

$$\sqrt{\widetilde{\mathrm{var}}\left[\frac{1}{t}\log \mathrm{E}^{(z)} R_{0\to t}\right]} = \frac{\theta+1}{\theta+\frac{t}{T}}\frac{\sigma}{\sqrt{\theta T}}.$$

[12] One could also measure the cross-sectional average expected return as

$$\frac{1}{t}\log \widetilde{\mathrm{E}}\mathrm{E}^{(z)} R_{0\to t} = \frac{(\theta+1)^2}{\theta\left(\theta+\frac{t}{T}\right)}\sigma^2 = \widetilde{\sigma}^2.$$

It follows from this that $\widetilde{\mathrm{E}}\mathrm{E}^{(z)} R_{0\to t} - 1 = \mathrm{SVIX}_t^2$. However, if $t = 10$ years, as in Cam Harvey's data set, it is somewhat implausible that investors are directly reporting $\mathrm{E}^{(z)} R_{0\to t}$.

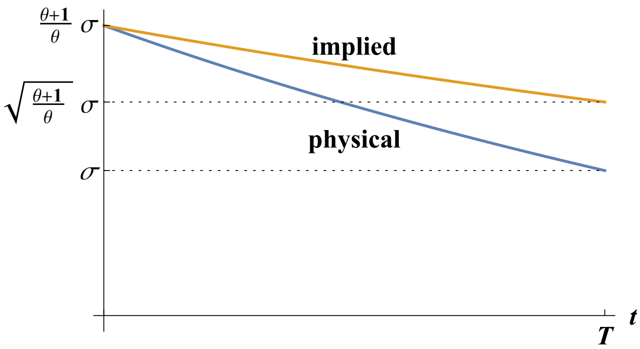



Figure 2.3.1: The term structures of implied volatility and of annualized physical volatility.

Our next result characterizes option prices at all maturities $t \leq T$ and all strikes $K$. As always, options can be quoted in terms of the Black–Scholes formula. What is more unusual is that in our setting, implied volatilities can be expressed in a simple but non-trivial closed form.

**Result 7.** *The time 0 price of an option with maturity $t$ and strike price $K$ is*

$$C(t, K) = p_0 \Phi(d_1) - K\Phi(d_1 - \widetilde{\sigma}\sqrt{t})\,, \tag{2.3.4}$$

*where*

$$d_1 = \frac{\log\left(p_0/K\right) + \frac{1}{2}\widetilde{\sigma}^2 t}{\widetilde{\sigma}\sqrt{t}} \quad \text{and} \quad \widetilde{\sigma} = \frac{\theta+1}{\sqrt{\theta(\theta+\frac{t}{T})}}\,\sigma\,.$$

*In particular, short-dated options (with $t/T \to 0$) have $\widetilde{\sigma} = \frac{\theta+1}{\theta}\sigma$, and long-dated options (with $t = T$) have $\widetilde{\sigma} = \sqrt{\frac{\theta+1}{\theta}}\sigma$.*

Implied volatility is increasing in the degree of heterogeneity $\theta^{-1}$; as the degree of heterogeneity $\theta^{-1}$ goes to 0, we recover the conventional Black–Scholes formula with an implied volatility of $\sigma$. Assuming $\theta^{-1} > 0$, the term structure of implied volatility is downward-sloping. For comparison, recall from Result 4 that all agents agree on physical volatility, which is

$$\frac{1}{\sqrt{t}}\sigma^{(z)}\left(\log R_{0\to t}\right) = \frac{\theta+1}{\theta+\frac{t}{T}}\,\sigma = \sqrt{\frac{\theta}{\theta+\frac{t}{T}}}\,\widetilde{\sigma}.$$

| | Model | Data |
|---|---|---|
| 1mo implied vol | 18.6% | 18.6% |
| 1yr implied vol | 18.2% | 18.1% |
| 2yr implied vol | 17.7% | 17.9% |
| 1yr disagreement | 4.4% | 4.8% |
| 10yr disagreement | 2.9% | 2.9% |
| 1yr mean risk premium | 3.3% | 3.8% |
| 10yr mean risk premium | 1.9% | 3.6% |

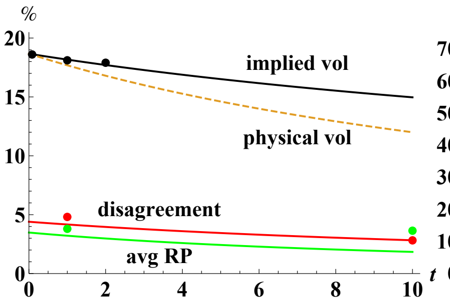
Table 2.3.1: Moments implied by the model's baseline calibration and in the data.

In a homogeneous belief economy, both implied and physical volatilities would be constant, at $\sigma$, at all maturities. The sentiment and speculation induced by heterogeneous beliefs boosts both implied and physical volatility at short horizons, and generates a variance risk premium at long horizons, as shown in Figure 2.3.1.

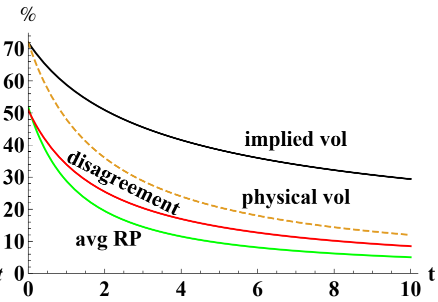
*Two illustrative calibrations.*—In the figures below, we set the horizon over which disagreement plays out to $T = 10$ years, and we set $\sigma$, which equals the volatility of log fundamentals (i.e. payoffs), to 12%. The belief heterogeneity parameter $\theta$ dictates the amount of disagreement, the level of short-run volatility, and the size of the long-run variance risk premium. In our baseline calibration, we set $\theta = 1.8$, which implies that one-month, one-year, and two-year implied volatilities are 18.6%, 18.2%, and 17.7%, respectively, as shown in Table 2.3.1. These numbers are close to their empirically observed counterparts, which are indicated with solid dots in Figure 2.3.2a.

With this value of $\theta$, the model-implied cross-sectional standard deviations of expected returns ("disagreement") are 4.4% and 2.9% at the one- and 10-year horizons. For comparison, in data from the Graham–Harvey Chief Financial Officer surveys, the mean levels of one-year and 10-year disagreement are 4.8% and 2.8%, respectively (indicated with red dots in Figure 2.3.2a).

We also consider a calibration in which $\theta = 0.2$ to explore the behavior




(a) $\theta = 1.8$




(b) $\theta = 0.2$

Figure 2.3.2: Term structures of implied and physical volatility, mean expected returns and disagreement in the baseline (left) and crisis (right) calibrations.

of asset prices under conditions with substantial disagreement, and to discuss some interesting qualitative features of equilibrium that arise once $\theta$ is less than one. The resulting term structures of physical implied volatility, and of average perceived risk premia and disagreement, are shown in Figure 2.3.2b. Heightened belief heterogeneity generates steeply downward-sloping term structures of physical and implied volatility and of risk premia.

### 2.3.1 The perceived value of speculation

An agent's stochastic discount factor (SDF) links his or her perceived true probabilities of events to the associated risk-neutral probabilities. As individuals disagree on true probabilities but agree on risk-neutral probabilities—equivalently, on asset prices, which are directly observable—they have different stochastic discount factors. We now analyze the properties of individuals' SDFs, and hence explore agents' attitudes to speculation.

**Result 8.** *The variance of the SDF of investor $z$ is finite for $\theta > 1$ and is equal to*

$$\operatorname{var}^{(z)} M_{0\to t}^{(z)} = \frac{\theta}{\sqrt{\theta^2 - (\frac{t}{T})^2}} \exp\left\{ \frac{\left[ z\sqrt{\frac{\theta t}{T}} + (\theta+1)\,\sigma\sqrt{t} \right]^2}{\theta\left(\theta - \frac{t}{T}\right)} \right\} - 1\,. \tag{2.3.5}$$

By the Hansen and Jagannathan (1991) bound, this result supplies the maximum Sharpe ratio as perceived by agent $z$, $\mathrm{MSR}^{(z)}_{0\to t}$. Writing $R^e_{0\to t}$ for the excess return on an asset or trading strategy, we have

$$\mathrm{MSR}^{(z)}_{0\to t} = \max_{R^e_{0\to t}} \frac{\mathrm{E}^{(z)} R^e_{0\to t}}{\sigma^{(z)}\left(R^e_{0\to t}\right)} = \frac{\sigma^{(z)}\left(M^{(z)}_{0\to t}\right)}{\mathrm{E}^{(z)} M^{(z)}_{0\to t}} = \sigma^{(z)}\left(M^{(z)}_{0\to t}\right),$$

where we write $\sigma^{(z)}(\cdot) = \sqrt{\mathrm{var}^{(z)}(\cdot)}$ for the standard deviation perceived by investor $z$ (and the final equality follows because we have normalized the interest rate to zero, so $\mathrm{E}^{(z)} M^{(z)}_{0\to t} = \frac{1}{R_{f,0\to t}} = 1$ for all $z$). As the market is complete, there is a strategy that attains the maximal Sharpe ratio (MSR) implied by the Hansen–Jagannathan bound for any agent—and of course different agents will perceive different maximal Sharpe ratios, and different associated trading strategies.

Minimizing (2.3.5) with respect to $z$, we find that the investor who perceives the smallest MSR (at all horizons $t$) has $z = z_0$, where

$$z_0\sqrt{\theta} + (\theta+1)\sigma\sqrt{T} = 0. \tag{2.3.6}$$

**Definition 1.** We refer to investor $z = z_0$, where

$$z_0 = -\frac{\theta+1}{\sqrt{\theta}}\sigma\sqrt{T},$$

as the *gloomy investor*. The gloomy investor perceives that the instantaneous risk premium on the risky asset is exactly zero, by Result 5.

There are, of course, more *pessimistic* investors ($z < z_0$), but we think of them as being less gloomy in the sense they perceive attractive trading opportunities associated with shorting the risky asset. The MSR perceived by the gloomy investor satisfies

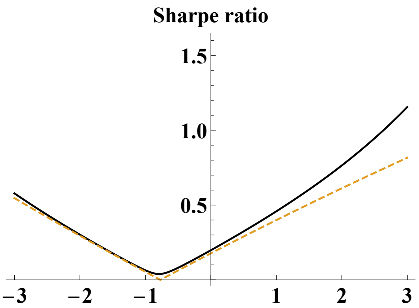
$$\mathrm{MSR}^{(z_0)}_{0\to t} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - \left(\frac{t}{T}\right)^2}} - 1}\,.$$

The dashed lines in the panels of Figure 2.3.3 plot the subjective Sharpe ratio of a *static* position in the risky asset (calculated from Results 4 and 5) against investor type, $z$. The solid lines plot the maximum attainable Sharpe ratio against investor type, $z$. The top panels use the baseline calibration, $\theta = 1.8$, and the bottom panels use the high-disagreement calibration, $\theta = 0.2$. The left panels show perceived Sharpe ratios over the next year; the right panels show annualized Sharpe ratios over the entire 10-year horizon. (We annualize by scaling Sharpe ratios by the square root of horizon; if returns were i.i.d., this would result in a constant Sharpe ratio at all horizons.)

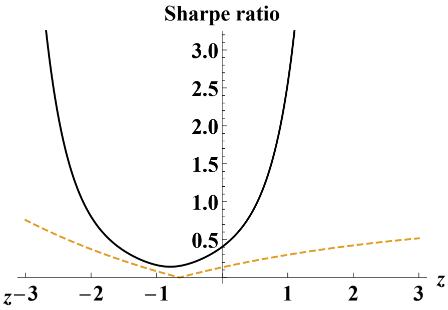
The solid lines lie strictly above the dashed lines, indicating that all investors must trade dynamically to achieve their perceived MSR. In the baseline calibration, the annualized MSR perceived by the gloomy investor $z_0$, is 0.04 at the one-year horizon and 0.14 at the 10-year horizon. All investors perceive attainable Sharpe ratios at least as large as this. Recall that the gloomy investor believes that the risky asset is priced to earn precisely zero risk premium. Loosely speaking, the gloomy investor's maximal-Sharpe-ratio strategy is to go long if the market sells off, and short if the market rallies, thereby exploiting what he views as irrational exuberance on the upside and irrational pessimism on the downside. This is a contrarian, "short vol" strategy. We will expand on this interpretation shortly.

If there is substantial disagreement—as in our calibration with $\theta = 0.2$—agents perceive substantially higher attainable Sharpe ratios. At the one-year horizon depicted in Figure 2.3.3c, even the gloomy investor perceives an MSR of 0.39, while the median investor perceives an MSR of 1.50. Sharpe ratios increase very rapidly for investors with extreme beliefs, and especially so for optimists with extreme beliefs: an investor who is only moderately optimistic, with beliefs one standard deviation above the mean ($z = 1$), perceives an MSR of 8.2. At the 10-year horizon shown in Figure 2.3.3d, all investors perceive that arbitrarily high Sharpe ratios are attainable.

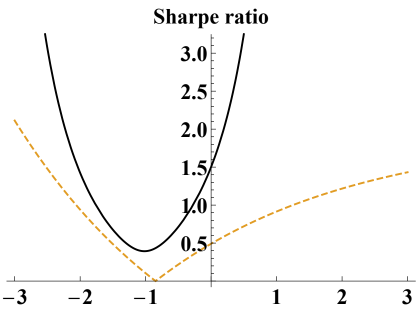
At first sight, this might seem obviously unreasonable. Surely very high Sharpe ratios should not be possible in equilibrium? But our investors are not mean-variance optimizers, so Sharpe ratios do not adequately summarize



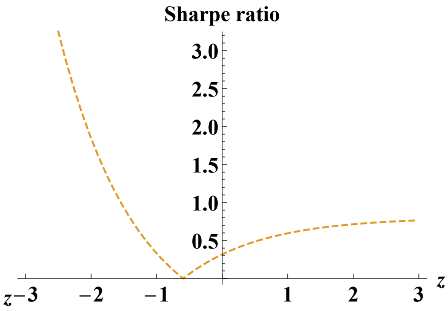

(a) One-year horizon, $\theta = 1.8$.




(b) 10-year horizon, annualized, $\theta = 1.8$.




(c) One-year horizon, $\theta = 0.2$.




(d) 10-year horizon, annualized, $\theta = 0.2$.

Figure 2.3.3: Maximal Sharpe ratios attainable through dynamic (solid) or static (dashed) trading, as perceived by investor $z$. All investors perceive that arbitrarily high Sharpe ratios are attainable dynamically in panel d.

investment opportunities. (And indeed, Sharpe ratios are not considered sufficient measures of the attractiveness of a trading strategy in practice: investors appear to monitor performance measures such as max drawdowns, value at risk, and Sortino ratios, among other things.) In order to measure the attractiveness of dynamic trading strategies in a theoretically motivated way, we calculate the maximum fraction of wealth, $\xi^{(z)}$, that investor $z$ would be prepared to sacrifice in order to avoid being shut out of the market. We assume that when the investor is shut out, he is forced to hold his original position in the risky asset, earning the return $R_{0\to t}$ up to time $t$. Thus $\xi^{(z)}$ satisfies

$$\max_{R^{(z)}_{0\to t}} \mathrm{E}^{(z)} \log \left[ \left(1 - \xi^{(z)}\right) W_0^{(z)} R^{(z)}_{0\to t} \right] = \mathrm{E}^{(z)} \log \left[ W_0^{(z)} R_{0\to t} \right] . \tag{2.3.7}$$

The Alvarez and Jermann (2005) bound states that

$$\max_{R_{0\to t}^{(z)}} \mathrm{E}^{(z)} \log R_{0\to t}^{(z)} = \mathrm{L}^{(z)} \left[ M_{0\to t}^{(z)} \right] , \tag{2.3.8}$$

where the entropy of the SDF, as perceived by investor $z$, is $\mathrm{L}^{(z)} \left[ M_{0\to t}^{(z)} \right] = \log \mathrm{E}^{(z)} M_{0\to t}^{(z)} - \mathrm{E}^{(z)} \log M_{0\to t}^{(z)}$. The bound is attained because the market is complete; we are using the fact that $\log R_{f,0\to t} = 0$ in equation (2.3.8). Combined with equation (2.3.7), this implies that

$$\log \left(1 - \xi^{(z)}\right) = \mathrm{E}^{(z)} \log R_{0\to t} - \mathrm{L}^{(z)} \left[ M_{0\to t}^{(z)} \right] . \tag{2.3.9}$$

**Result 9.** *The subjective entropy of the SDF is*

$$\mathrm{L}^{(z)} \left[ M_{0\to t}^{(z)} \right] = \frac{\left[ z\sqrt{\frac{\theta t}{T}} + (\theta + 1)\,\sigma\sqrt{t} \right]^2}{2\theta\left(\theta + \frac{t}{T}\right)} + \frac{1}{2}\left( \log \frac{\theta + \frac{t}{T}}{\theta} - \frac{\frac{t}{T}}{\theta + \frac{t}{T}} \right),$$

*so that the gloomy investor perceives the minimal SDF entropy.*

*We can also write*

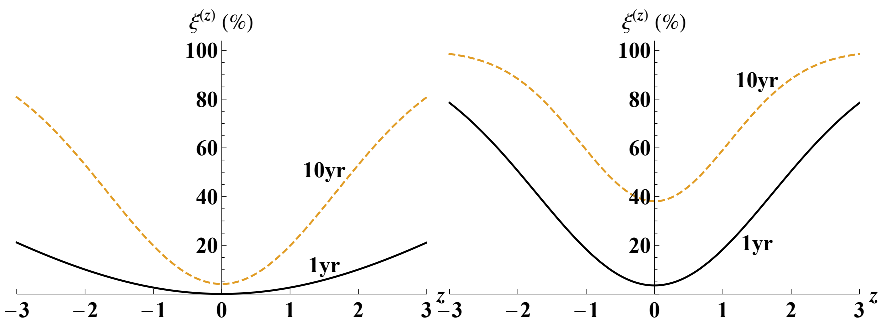
$$\mathrm{L}^{(z)} \left[ M_{0\to t}^{(z)} \right] = \underbrace{\frac{\theta+1}{\theta+\frac{t}{T}} \left( \frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta}\sigma^2 \right) t}_{\mathrm{E}^{(h)} \log R_{0\to t}} + \frac{z^2 \frac{t}{T}}{2\left(\theta + \frac{t}{T}\right)} + \underbrace{\frac{1}{2}\left( \log \frac{\theta + \frac{t}{T}}{\theta} - \frac{\frac{t}{T}}{\theta + \frac{t}{T}} \right)}_{>0}.$$

*It follows that*

$$\xi^{(z)} = 1 - \exp\left\{ -\frac{z^2 \frac{t}{T}}{2\left(\theta + \frac{t}{T}\right)} - \frac{1}{2}\left( \log \frac{\theta + \frac{t}{T}}{\theta} - \frac{\frac{t}{T}}{\theta + \frac{t}{T}} \right) \right\} .$$

*The median investor perceives the minimal $\xi^{(z)}$.*

Figure 2.3.4 plots $\xi^{(z)}$ against $z$ with parameters $\sigma = 0.12$, $T = 10$, and $t = 1$ or $t = 10$. The left panel shows the baseline calibration with $\theta = 1.8$; the right panel shows the high disagreement calibration with $\theta = 0.2$.




(a) $\theta = 1.8$.    (b) $\theta = 0.2$.

Figure 2.3.4: The proportion of wealth investor $z$ would sacrifice to avoid being prevented from trading dynamically for one or 10 years.

### 2.3.2 Investor behavior and the wealth distribution

We now study how the distribution of terminal wealth varies across agents as a function of the terminal payoff of the risky asset. To do so, it is convenient to introduce the notion of an investor-specific *target price* $K^{(z)}$ defined via[13]

$$\log K^{(z)} = \mathrm{E}^{(z)} \log p_T + (z - z_0)\sigma\sqrt{\theta T}\,. \tag{2.3.10}$$

For example, the median and gloomy investors' target prices can be written in terms of the fundamental parameters as

$$\log K^{(0)} = \frac{\eta}{\theta}\sigma\sqrt{2T} + (\theta+1)\sigma^2 T \quad \text{and} \quad \log K^{(z_0)} = \frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta+1}{\theta}\sigma^2 T\,.$$

For comparison, $\log p_0 = \frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{1}{2}\frac{\theta+1}{\theta}\sigma^2 T$, so the median and gloomy investors' target prices are, respectively, above and below the spot price.

As our next result shows, the target price represents the ideal outcome for investor $z$: the value of $p_T$ that maximizes wealth, and hence utility, ex post.

**Result 10.** *The time $T$ wealth of agent $z$ can be expressed as a function of*

[13] If desired, the expected log price, $\mathrm{E}^{(z)} \log p_T = \log p_0 + \mathrm{E}^{(z)} \log R_{0\to T}$, can be written in terms of the fundamental parameters of the model using Results 3 and 4.

*$p_T$:*

$$W^{(z)}(p_T) = p_0\sqrt{\frac{\theta+1}{\theta}}\exp\left\{\frac{1}{2}(z-z_0)^2 - \frac{1}{2(1+\theta)\sigma^2 T}\left[\log\left(p_T/K^{(z)}\right)\right]^2\right\}. \tag{2.3.11}$$

*Thus $W^{(z)}(p_T)$ is maximized when $p_T = K^{(z)}$.*

*This can also be written as a quadratic relationship between an investor's log wealth return, $r^{(z)}_{0\to T} = \log\left(W^{(z)}(p_T)/p_0\right)$, and the log return on the risky asset, $r_{0\to T} = \log R_{0\to T}$:*

$$r^{(z)}_{0\to T} = \frac{1}{2}\log\frac{\theta+1}{\theta} + \frac{1}{2}(z-z_0)^2 - \frac{1}{2(1+\theta)}\left[\frac{r_{0\to T} - \mathrm{E}^{(z)} r_{0\to T}}{\sigma\sqrt{T}} - \sqrt{\theta}\,(z-z_0)\right]^2.$$

*It follows that the expected elasticity of an investor's wealth return with respect to the risky asset return, $\mathrm{E}^{(z)}(\partial r^{(z)}_{0\to T}/\partial r_{0\to T})$, satisfies*

$$\mathrm{E}^{(z)}\frac{\partial r^{(z)}_{0\to T}}{\partial r_{0\to T}} = 1 + \frac{z}{|z_0|}.$$

*In particular, the median investor has an expected elasticity of one and the gloomy investor has an expected elasticity of zero.*

In our model, there is a useful distinction between what investors expect to happen and what they would like to happen. (The distinction also exists, but is uninteresting, in models in which a representative agent statically holds the market, as the target price is infinity in such models.) The gloomy investor would like to be proved right: his log target price equals his expected log price. But targets and expectations differ for all other investors. More optimistic investors have a (log) target price that exceeds their expectations—i.e., they are best off if the risky asset modestly outperforms their expectations—while more pessimistic investors are best off if the risky asset modestly underperforms their expectations. Any investor does very poorly if the asset performs *far* better or worse than anticipated.

Differentiating the expression (2.3.11) twice with respect to $p_T$, we find

$$W^{(z)''}(p_T) = \frac{W^{(z)}(p_T)}{p_T^2}\left\{\left(\frac{\log\left(p_T/K^{(z)}\right)}{(1+\theta)\sigma^2 T} + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{(1+\theta)\sigma^2 T}\right\}. \tag{2.3.12}$$

It immediately follows that any investor's wealth is concave in $p_T$ near their target price, and convex far from their target price. A moderate investor's wealth is also concave near their *expected* log price. But an extreme investor's wealth is convex near their expected log price. These facts follow because if $\log p_T = \mathrm{E}^{(z)} \log p_T$ then we have, after some algebra,

$$\operatorname{sign}\left[W^{(z)''}(p_T)\right] = \operatorname{sign}\left[z^2 - z_0 z - \frac{\theta+1}{\theta}\right],$$

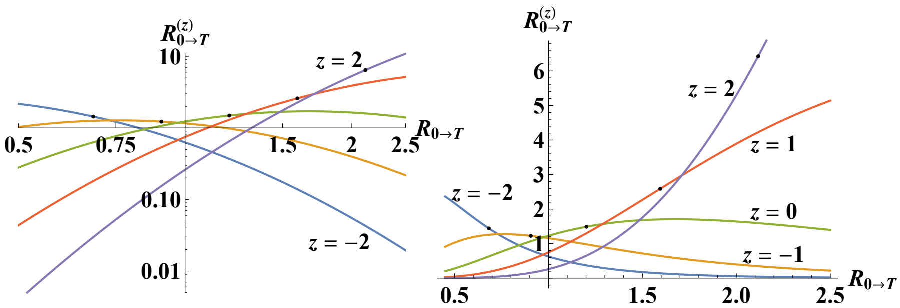
which is positive if $|z|$ is sufficiently large.

Figure 2.3.5 shows how different investors' outcomes depend on the risky asset outcome for $z = -2, -1, \ldots, 2$. The only difference between the two panels is that the left has logarithmic scales and the right linear scales. Dots in each panel indicate the expected gross return on the risky asset perceived by each of the investors. The median ($z = 0$) investor's wealth is a concave function of the risky asset return in the neighbourhood of the investor's expected outcome (indicated in the figure by a dot), while more extreme ($z = \pm 2$) investors have wealth that is convex in the risky asset return in the neighbourhood of their expected outcome.

Equation (2.3.11) can be rewritten as

$$W^{(z)}(p_T) = p_0\sqrt{\frac{\theta+1}{\theta}}\exp\left\{-\frac{1}{2}\left[\frac{\log p_T - \mathrm{E}^{(z)}\log p_T}{\sqrt{\mathrm{var}^{(z)}\log p_T}}\right]^2 + \right.$$
$$\left. + \frac{1}{2(1+\theta)}\left[\sqrt{\theta}\,\frac{\log p_T - \mathrm{E}^{(z)}\log p_T}{\sqrt{\mathrm{var}^{(z)}\log p_T}} + z - z_0\right]^2\right\}.$$

This characterization shows that you get richer if you are an extremist (large $|z|$) whose expectations are realized than you do if you have conventional beliefs




(a) Log scales.

(b) Linear scales.

Figure 2.3.5: Gross return on wealth against the gross return on the risky asset, for a range of investors $z = -2, -1, \ldots, 2$ and $\theta = 1.8$, $T = 10$, and $\sigma = 0.12$. The expected return on the risky asset perceived by each investor is indicated with a dot.

($z$ close to zero) that are realized: it is cheap to purchase claims to states of the world that extremists consider likely because relatively few people are extremists. As a result, there is substantially more wealth inequality in states in which the asset has an extreme positive or negative realized return.

Informally, extreme investors are "long volatility" near the outcome that they expect, while moderate investors are "short volatility" in their corresponding region. To formalize this intuition, we introduce a general result that holds in any frictionless arbitrage-free model in which options are traded. It is in the spirit of the famous result of Breeden and Litzenberger (1978), but the logic operates at the level of payoffs rather than of prices.

**Result 11.** *Let $W(\cdot)$ be such that $W(0) = 0$. Then choosing terminal wealth $W(p_T)$ is equivalent to holding the following portfolio:*

1. *Long $W'(K_0)$ units of the underlying asset (whose price is $p_T$ at time $T$)*
2. *Long bonds with face value $W(K_0) - K_0 W'(K_0)$*
3. *Long $W''(K)\,dK$ put options with strike $K$, for every $K < K_0$*
4. *Long $W''(K)\,dK$ call options with strike $K$, for every $K > K_0$*

*The constant $K_0 > 0$ can be chosen arbitrarily.*

*Proof.* Start from $W(p_T) = \int_0^{p_T} W'(K)\,dK = \int_0^\infty W'(K)\mathbf{1}_{\{p_T>K\}}\,dK$ and integrate by parts. $\square$

We now specialize Result 11 to our setting to identify a static portfolio whose payoffs replicate the investment strategy followed by an arbitrary investor $h$.

**Result 12.** *Investor $z$'s investment strategy is equivalent to the following:*

- *a long position in bonds with face value $W^{(z)}(K^{(z)}) = p_{0,0}\sqrt{\frac{\theta+1}{\theta}}e^{\frac{1}{2}(z-z_0)^2}$;*
- *short positions in options with strikes at and near her target level $K^{(z)}$;*
- *long positions in options with strikes far from $K^{(z)}$.*

*More precisely, the investor holds $W^{(z)''}(K)\,dK$ put options with strike $K$ for all $K < K^{(z)}$, and $W^{(z)''}(K)\,dK$ call options with strike $K$, for all $K \geq K^{(z)}$, where $W^{(z)''}(K)$ is as defined in* (2.3.12). *(Note that $W^{(z)''}(K^{(z)}) < 0$, and that $W^{(z)''}(K) > 0$ if $K$ is sufficiently far from $K^{(z)}$.)*

*The best possible payoff is $W^{(z)}(K^{(z)})$. This occurs if the asset hits its target price, $p_T = K^{(z)}$, in which case all the options expire worthless. Conversely, the investor's wealth approaches zero as $p_T \to 0$ or $p_T \to \infty$.*

*Proof.* It follows from the definition (2.3.10) of $K^{(z)}$, and a direct calculation, that $W^{(z)'}(K^{(z)}) = 0$. The claims in the first paragraph then follow on setting $K_0 = K^{(z)}$ in Result 11. The fact that the best possible payoff is $W^{(z)}(K^{(z)})$ follows from equation (2.3.11). The payoff on the option portfolio must therefore be nonpositive. $\square$

## 2.4 Conclusions

We have presented a frictionless model in which individuals have stable beliefs and risk aversion. All investors are risk-averse; short sales are allowed; all agents avoid bankruptcy; and all agents are on their first-order conditions at all times.

Even so, the model generates a rich set of predictions. Heterogeneity in beliefs gives rise to sentiment, which induces speculation and drives up realized and implied volatility, particularly in the short run. All agents understand these facts, so expected returns are higher than in an otherwise identical homogeneous economy, and securities with payoffs in extreme states of the world are far more highly valued than in otherwise similar economies with homogeneous beliefs. Moderate investors are suppliers of liquidity: they trade in a contrarian manner—they are "short vol"—and capture a variance risk premium created by the presence of extremists.

## 2.5 Appendix: Proofs

**Proof of Result 1:**

*Proof.* Observe from the recurrence relation (2.2.23) that a pricing formula in the form (2.2.24) holds. Each constant $c_m$ is a sum of products of terms of the form $H_{j,s}$ and $1 - H_{j,s}$ over appropriate $j$ and $s$. We noted in the text that $H_{m,t}(1 - H_{m+1,t+1}) = (1 - H_{m,t})H_{m,t+1}$: that is, pricing is path-independent.

Fix $m$ between 0 and $T$. By path independence, all the possible ways of getting from the initial node to node $m$ at time $T$ make an equal contribution to $c_m$. By considering the path that travels down for $T - m$ periods and then up for $m$ periods, and then multiplying by the number of paths, $\binom{T}{m}$, we find that

$$\begin{aligned} c_m &= \binom{T}{m}(1 - H_{0,0}) \cdots (1 - H_{0,T-m-1})H_{0,T-m}H_{1,T-m+1} \cdots H_{m-1,T-1} \\ &= \binom{T}{m}\frac{\beta}{\alpha+\beta} \cdot \frac{\beta+1}{\alpha+\beta+1} \cdots \frac{\beta+T-m-1}{\alpha+\beta+T-m-1} \cdot \frac{\alpha}{\alpha+\beta+T-m} \cdots \frac{\alpha+m-1}{\alpha+\beta+T-1} \\ &= \binom{T}{m}\frac{B(\alpha+m, \beta+T-m)}{B(\alpha,\beta)}. \end{aligned}$$

The risk-neutral probability $q^*_m$ can be determined using the facts that $p^*_{m,t} = H_{m,t}p_{m,t}/p_{m+1,t+1}$ and $1-p^*_{m,t} = (1-H_{m,t})p_{m,t}/p_{m,t+1}$. (We are restating (2.2.9) with subscripts to keep track of the current node.) Thus—using again path-independence in the first line—

$$\begin{aligned} q^*_m &= \binom{T}{m}(1 - p^*_{0,0}) \cdots (1 - p^*_{0,T-m-1}) \cdot p^*_{0,T-m}p^*_{1,T-m+1} \cdots p^*_{m-1,T-1} \\ &= \binom{T}{m}(1 - H_{0,0})\frac{p_{0,0}}{p_{0,1}} \cdots (1 - H_{0,T-m-1})\frac{p_{0,T-m-1}}{p_{0,T-m}} \cdot H_{0,T-m}\frac{p_{0,T-m}}{p_{1,T-m+1}} \cdots H_{m-1,T-1}\frac{p_{m-1,T-1}}{p_{m,T}} \\ &= c_m\frac{p_{0,0}}{p_{m,T}}. \end{aligned}$$

□

We also have the following generalization of Result 1. We omit the proof, which is essentially identical to the above.

**Lemma 2.** *For any node $m, t$:*

$$z_{m,t} = \sum_{j=0}^{T-t} c_{m,t,j} z_{m+j,T}$$

*where $j$ represents the number of further up-moves after time $t$, and*

$$c_{m,t,j} = \binom{T-t}{j} \frac{B(m+\alpha+j, T-m+\beta-j)}{B(m+\alpha, t-m+\beta)}.$$

*Moreover, the risk neutral probability of ending up at $j, T$ starting from node $m, t$ is given by*

$$q^*_{m,t,j} = c_{m,t,j} \frac{p_{m,t}}{p_{m+j,T}}.$$

**Proof of Result 2:**

*Proof.* We will start by proving the following Lemma.

**Lemma 3.** *If $Y_1 \sim BB(\overline{\alpha}, \overline{\alpha}, T)$ and $Y_2 \sim BB(\underline{\alpha}, \underline{\alpha}, T)$, for $\overline{\alpha} > \underline{\alpha}$ then $Y_1$ second order stochastically dominates $Y_2$.*

*Proof.* A sufficient condition for second order stochastic dominance, for variables with the same expectation, is the single crossing dominance. That is, it is sufficient to prove that:

$$F_{\underline{\alpha}}(s) \geq F_{\overline{\alpha}}(s) \Longleftrightarrow s \leq c^* \tag{2.5.1}$$

for some $c^*$, where $F_{\overline{\alpha}}(s), F_{\underline{\alpha}}(s)$ are the cdfs of $Y_1, Y_2$ respectively.[14] Because of symmetry $c^*$ will be just $T/2$. To prove the above it is sufficient to prove that $f_{\underline{\alpha}}(k) - f_{\overline{\alpha}}(k)$ is decreasing in $[0, T/2]$, where $f(\cdot)$ denotes the probability mass function. Then $F_{\underline{\alpha}}(s) - F_{\overline{\alpha}}(s)$ would be decreasing (as a sum of decreasing

[14]See, for instance, Osband & Roy (2018) *"Gaussian-Dirichlet Posterior Dominance in Sequential Learning"*.

functions) and the proof of the lemma would be completed, since this would imply equation 2.5.1. Hence, we need to show that:

$$\binom{T}{k}\left[\frac{B(k+\underline{\alpha},T-k+\underline{\alpha})}{B(\underline{\alpha},\underline{\alpha})}-\frac{B(k+\overline{\alpha},T-k+\overline{\alpha})}{B(\overline{\alpha},\overline{\alpha})}\right]$$

is decreasing in $k$ (in the interval $[0,T/2]$). Equivalently:

$$\Gamma(k+\underline{\alpha})\Gamma(T-k+\underline{\alpha})\left[\frac{1}{\Gamma(T+2\underline{\alpha})B(\underline{\alpha},\underline{\alpha})}-\frac{\Gamma(k+\overline{\alpha})\Gamma(T-k+\overline{\alpha})}{\Gamma(k+\underline{\alpha})\Gamma(T-k+\underline{\alpha})}\frac{1}{\Gamma(T+2\overline{\alpha})B(\overline{\alpha},\overline{\alpha})}\right]$$

is decreasing.

But the above holds because of the following 2 facts:
First, $h(k)=\Gamma(k+\underline{\alpha})\Gamma(T-k+\underline{\alpha})$ is decreasing because

$$[log(h(k))]'=\psi(k+\underline{\alpha})-\psi(T-k+\underline{\alpha})<0$$

where $\psi(\cdot)$ is the digamma function, which is an increasing function since $\Gamma(\cdot)$ is log-convex (and $k<T-k$).

Second, $\frac{\Gamma(k+\overline{\alpha})\Gamma(T-k+\overline{\alpha})}{\Gamma(k+\underline{\alpha})\Gamma(T-k+\underline{\alpha})}$ is increasing. Indeed, assume $k_1>k_2$. Then, we want:

$$\frac{\Gamma(k_1+\overline{\alpha})\Gamma(T-k_1+\overline{\alpha})}{\Gamma(k_1+\underline{\alpha})\Gamma(T-k_1+\underline{\alpha})}>\frac{\Gamma(k_2+\overline{\alpha})\Gamma(T-k_2+\overline{\alpha})}{\Gamma(k_2+\underline{\alpha})\Gamma(T-k_2+\underline{\alpha})}$$

Equivalently:

$$\frac{\Gamma(k_1+\overline{\alpha})\Gamma(T-k_1+\overline{\alpha})}{\Gamma(k_2+\overline{\alpha})\Gamma(T-k_2+\overline{\alpha})}>\frac{\Gamma(k_1+\underline{\alpha})\Gamma(T-k_1+\underline{\alpha})}{\Gamma(k_2+\underline{\alpha})\Gamma(T-k_2+\underline{\alpha})}$$

Now using the property that $\Gamma(z+1)=z\Gamma(z)$ for any $z$ and that $k_1,k_2\in \mathrm{Z}$, we get:

$$\frac{(\overline{\alpha}+k_2)(\overline{\alpha}+k_2+1)\dots(\overline{\alpha}+k_1-1)}{(\overline{\alpha}+T-k_1)(\overline{\alpha}+T-k_1+1)\dots(\overline{\alpha}+T-k_2-1)}>\frac{(\underline{\alpha}+k_2)\dots(\underline{\alpha}+k_1-1)}{(\underline{\alpha}+T-k_1)\dots(\underline{\alpha}+T-k_2-1)}$$

which is true since for example $\frac{(\overline{\alpha}+k_2)}{(\overline{\alpha}+T-k_1+1)}>\frac{(\underline{\alpha}+k_2)}{(\underline{\alpha}+T-k_1+1)}$

Therefore this proves that $Y_1$ single-crossing dominates $Y_2$ and hence it also second order stochastically dominates $Y_2$ and the lemma has been proved. $\square$

Having established the above Lemma, we can now go back to proving Result 2. It is well known that if $Y_1$ second order stochastically dominates $Y_2$ then for any concave function $u(\cdot)$:

$$\mathrm{E}_{Y_1}[u(m)] \geq \mathrm{E}_{Y_2}[u(m)].$$

Pick $u(m) = -\frac{1}{p_{m,T}}$. Then we get: $\mathrm{E}_{Y_1}[\frac{1}{p_{m,T}}] \leq \mathrm{E}_{Y_2}[\frac{1}{p_{m,T}}]$ and therefore:

$$\frac{1}{\mathrm{E}_{Y_1}[\frac{1}{p_{m,T}}]} \geq \frac{1}{\mathrm{E}_{Y_2}[\frac{1}{p_{m,T}}]}$$

That is if $p_1, p_2$ are the corresponding prices (where $p_1$ corresponds to the case with less heterogeneity, as $\overline{\alpha} > \underline{\alpha}$), we have that $p_1 > p_2$.

To show that log-concavity of $p$ implies that $1/p$ is convex, note that log-concavity is equivalent to $(p')^2 \geq pp''$. $\square$

**Proof of Result 3:**

*Proof.* As shown in equation (2.2.24),

$$p_{0,0}^{-1} = \sum_{j=1}^{2N} c_j z_{j,T}$$

From Result 1, $c_j$ equals the probability that a $BB(2N, \alpha, \beta)$ random variable takes the value $j$. Therefore we can equivalently write

$$p_{0,0}^{-1} = \mathrm{E}_j\left[z_{j,T}\right] = \mathrm{E}_j\left[e^{-\sigma\sqrt{2T}\frac{j-N}{\sqrt{N}}}\right] \tag{2.5.2}$$

where the random variable $j$ has a beta-binomial distribution, $BB(2N, \alpha, \beta) \equiv BB(2N, \theta N + \eta\sqrt{N}, \theta N - \eta\sqrt{N})$.

The Paul and Plackett theorem (see Appendix for more details) states that $j$, appropriately shifted and scaled, converges in distribution and in moment generating function to a Normal distribution. More specifically,

$$\Psi_N \equiv \frac{j - N - \frac{\eta}{\theta}\sqrt{N}}{\sqrt{\frac{1+\theta}{2\theta}N}} \longrightarrow N(0,1)$$

where $\mathrm{E}[j] = N + \frac{\eta}{\theta}\sqrt{N}$ and $\mathrm{var}[j] = \frac{1+\theta}{2\theta}N$. As

$$\frac{j-N}{\sqrt{N}} = \Psi_N \sqrt{\frac{1+\theta}{2\theta}} + \frac{\eta}{\theta},$$

we have

$$\begin{aligned} p_{0,0}^{-1} &= \mathrm{E}\left[e^{-\sigma\sqrt{2T}\left(\Psi_N\sqrt{\frac{1+\theta}{2\theta}}+\frac{\eta}{\theta}\right)}\right] \\ &\to \mathrm{E}\left[e^{-\sigma\sqrt{2T}\left(Z\sqrt{\frac{1+\theta}{2\theta}}+\frac{\eta}{\theta}\right)}\right] \\ &= \exp\left(-\frac{\eta}{\theta}\sigma\sqrt{2T} + \frac{\theta+1}{2\theta}\sigma^2 T\right). \end{aligned}$$

From the first to the second line, convergence of expectations follows from the fact that the beta-binomial converges to Normal in moment generating functions. □

**Proof of Result 4**:

*Proof.* We want to find the perceived expectation and variance of returns from 0 to $t$. In order to achieve that, we need to first compute $p_{m,t}$, following the lines of the proof of Result 3, and then find the limiting distribution that it has from the perspective of any investor $h$. We outline the main steps here, and present further details in the Appendix.

Define $\phi = \frac{t}{T}$ and set $m = \phi N + \psi_t\sqrt{\phi N}$, so that $\psi_t$ is a convenient parametrization of $m$. Given that $z_{m+j,2N} = \lambda^{-2(m+j-N)}$, we have, similarly to equation (2.5.2)

$$p_{m,t}^{-1} = \mathrm{E}_j[e^{-\sigma\sqrt{2T}\frac{m+j-N}{\sqrt{N}}}] \tag{2.5.3}$$

where we view $j$ as a random variable with beta-binomial distribution

$$BB\left(2(1-\phi)N,\ (\phi+\theta)N + (\psi_t\sqrt{\phi}+\eta)\sqrt{N},\ (\phi+\theta)N - (\psi_t\sqrt{\phi}+\eta)\sqrt{N}\right).$$

By the Paul and Plackett theorem, the standardized version of $j$ converges in distribution and in moment generating function to a standard Normal random variable. Therefore we can find the (limiting) expectation on the right hand side of (2.5.3), by just considering the expectation under a Normal distribution, with the corresponding mean and variance. As $N$ tends to infinity, we will write $p_{\psi_t} :\equiv p_{m,t}$ (where, $\psi_t = \frac{m-\phi N}{\sqrt{\phi N}}$), to emphasize that we are considering the continuous time limit, in which $\psi_t$ becomes the relevant state variable. We get:

$$p_{\psi_t} = b_t \cdot e^{\frac{\theta+1}{\phi+\theta}\sigma\sqrt{2\phi T}\psi_t} \tag{2.5.4}$$

where $b_t = e^{-\frac{1-\phi}{2}\frac{\theta+1}{\phi+\theta}\sigma^2 T + \frac{1-\phi}{\phi+\theta}\eta\sigma\sqrt{2T}}$.

We then view $p_{\psi_t}$ as a function of $\psi_t$, for which we care about each limiting distribution. We know that $m(=\phi N + \psi_t\sqrt{\phi N})$ has a binomial distribution with mean $2\phi Nh$ and variance $2\phi Nh(1-h)$ from the perspective of agent $h$. Indeed by the Central Limit Theorem (or by De Moivre's theorem), a standardized version of $m$ converges to a standard Normal distribution:

$$\frac{m - 2\phi Nh}{\sqrt{2\phi Nh(1-h)}} \to N(0,1). \tag{2.5.5}$$

Equivalently, we have:

$$\frac{\psi_t - (2h-1)\sqrt{\phi N}}{\sqrt{2h(1-h)}} \to N(0,1), \tag{2.5.6}$$

where $(2h-1)\sqrt{N} = \frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}}$ and $h(1-h) = \frac{1}{4} + O(\frac{1}{N})$. Therefore, the expectation and variance of $\log(p_t)$ are

$$\mathrm{E}^{(z)} \log p_t = \frac{t(\theta+1)\frac{z}{\sqrt{\theta}}\sigma\sqrt{T} - \frac{1}{2}(T-t)(\theta+1)\sigma^2 T}{\theta T + t} + \frac{\eta}{\theta}\sigma\sqrt{2T}$$
$$\mathrm{var}^{(z)} \log p_t = \sigma^2 t \left(\frac{\theta+1}{\theta+\frac{t}{T}}\right)^2. \qquad \square$$

**Proof of Result 5:**

*Proof.* We are interested in finding

$$\mathrm{E}^{(z)}\left[R_{0\to t}\right] = \mathrm{E}^{(z)}\left[\frac{p_{\psi_t}}{p_{0,0}}\right],$$

where as in the proof of Result 4 we use the notation $p_{\psi_t} :\equiv p_{m,t}$, which we have already computed in equation (2.5.4)

$$p_{0,0}^{-1} \cdot b_t \cdot \mathrm{E}^{(z)}\left[e^{\frac{\theta+1}{\phi+\theta}\sigma\sqrt{T}\sqrt{2\phi}\psi_t}\right];$$

and we have established, in equation (2.5.6), that $\psi_t$ converges in distribution and in moment generating function to a Normal (as $m$ does too). Hence asymptotically, the above is the expectation of a log-normal variable. In particular, after some algebra,

$$\mathrm{E}^{(z)}\left[R_{0\to t}\right] = e^{\frac{\phi(\theta+1)}{\theta+\phi}\left[\frac{z}{\sqrt{\theta}}\sigma\sqrt{T} + \frac{\theta+1}{2}\left(\frac{1}{\theta}+\frac{1}{\theta+\phi}\right)\sigma^2 T\right]}. \tag{2.5.7}$$

Setting $\phi = \frac{t}{T}$, the proof is complete. Finally, note that by substituting $\phi = 1$ and $h = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}} + \frac{z}{\sqrt{8\theta N}}$ we obtain equation (2.3.3). $\square$

**Proof of Result 7:**

*Proof.* Note that $2\phi N$ is the number of periods corresponding to $t = \phi T$. Writing $q_{m,t}$ for the risk neutral probability of going from node $(0,0)$ to node

$(m,t)$, we have (as in Lemma 2) $q_{m,t} = \frac{p_{0,0}}{p_{m,t}} c_{m,t}$, where

$$c_{m,t} = \binom{2\phi N}{m} \frac{B(\alpha + m, \beta + 2\phi N - m)}{B(\alpha, \beta)}.$$

As the risk-free rate is 0, it follows that the time zero price of a call option with strike $K$, maturing at time $t$, is

$$\begin{aligned}
C(0,t;K) &= \sum_{m=0}^{2\phi N} q_{m,t}(p_{m,t} - K)^+ \\
&= p_{0,0} \sum_{m=0}^{2\phi N} c_{m,t} \left(1 - \frac{K}{p_{m,t}}\right)^+ \\
&= p_{0,0} \mathrm{E}\left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\phi+\theta}\sigma\sqrt{2\phi T}\psi_t}\right)^+\right]
\end{aligned}$$

where the expectation is taken with respect to the random variable $m$ which follows a $BB(2N\phi, \alpha, \beta)$ distribution and in the last line we have substituted $p_{m,t}$ with its (continuous time limit) value computed at equation (2.5.4) (remember, $\psi_t = \frac{m-\phi N}{\sqrt{\phi N}}$) . By the result of Paul and Plackett, the asymptotic distribution of $m$ satisfies

$$\frac{m - \phi N - \frac{\eta}{\theta}\phi\sqrt{N}}{\sqrt{\frac{\phi+\theta}{2\theta}\phi N}} \to \Psi \sim N(0,1)$$

as $N \to \infty$. Equivalently:

$$\frac{1}{\sqrt{\frac{\phi+\theta}{2\theta}}} \left(\psi_t - \frac{\eta}{\theta}\sqrt{\phi}\right) \to \Psi \sim N(0,1)$$

Thus

$$C(0,t;K) = p_{0,0} \cdot \mathrm{E}\left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2\phi T}(\Psi\sqrt{\frac{\phi+\theta}{2\theta}} + \frac{\eta\sqrt{\phi}}{\theta})}\right)^+\right].$$

(Note that convergence in distribution implies convergence of the expectation by the Helly-Bray theorem, since the function of $\Psi$ inside the expectation is bounded and continuous.) This expectation is now standard, and we have

$$C(0,t;K) = p_{0,0}\left[\Phi\left(-\frac{\log(X)}{\widetilde{\sigma}\sqrt{t}}\right) - e^{\frac{\widetilde{\sigma}^2 t}{2}}\frac{K}{b_t}e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2T}\frac{\eta\phi}{\theta}}\Phi\left(-\frac{\log(X)+\widetilde{\sigma}^2 t}{\widetilde{\sigma}\sqrt{t}}\right)\right]$$

where $X = \frac{K}{b_t}e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2T}\frac{\eta\phi}{\theta}}$ and

$$\widetilde{\sigma}^2 t = \frac{(\theta+1)^2}{\theta(\theta+\phi)}\sigma^2 t = \mathrm{var}\left[\log\left(\frac{K}{b_t}e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2T}(\Psi\sqrt{\frac{\phi+\theta}{2\theta}\phi}+\frac{\eta\phi}{\theta})}\right)\right]$$

Finally, noting that $p_{0,0} = e^{\frac{\widetilde{\sigma}^2 t}{2}}\frac{K}{b_t}e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2T}\frac{\eta\phi}{\theta}}$, we arrive at the Black–Scholes formula

$$C(0,t;K) = p_{0,0}\Phi(d_1) - K\Phi(d_1 - \widetilde{\sigma}\sqrt{t})$$

where

$$d_1 = \frac{\log\left(\frac{p_{0,0}}{K}\right) + \frac{1}{2}\widetilde{\sigma}^2 t}{\widetilde{\sigma}\sqrt{t}}$$

and volatility is determined endogenously via

$$\widetilde{\sigma} = \frac{\theta+1}{\sqrt{\theta(\theta+\frac{t}{T})}}\sigma.$$

□

**Proof of Result 8:**

*Proof.* An agent's SDF links his or her perceived true probabilities to the objectively observed risk-neutral probabilities. Thus

$$M_t^{(h)}(m) = \frac{p_{0,0}}{p_{m,t}}\frac{c_{m,t}}{\pi_t^{(h)}(m)}$$

where $\pi_t^{(h)}(m)$ is the probability that we will end up at node $(m,t)$, as perceived by agent $h$. As $c_{m,t}$ has a beta-binomial distribution and $\pi_t^{(h)}(m)$ has a binomial distribution, they are each asymptotically Normal[15] and we have the following characterization for the SDF $M_T$:

$$M_t^{(h)}(m) \sim \sqrt{\frac{4h(1-h)\theta}{\phi+\theta}}\, p_{0,0} b_t^{-1} e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2\phi T}\psi_t - \frac{\theta(m-\phi N-\frac{\eta}{\theta}\phi\sqrt{N})^2}{(\phi+\theta)\phi N} + \frac{(m-2\phi Nh)^2}{4h(1-h)\phi N}} \tag{2.5.8}$$

where $\psi_t = \frac{m-\phi N}{\sqrt{\phi N}}$ is asymptotically Normal from the perspective of any agent $h$ by the De Moivre–Laplace theorem.[16] Parametrizing further $h$ with $z$ such that $h = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}} + \frac{z}{\sqrt{8\theta N}}$, the right hand side can be rewritten

$$M_t^{(z)}(\psi_t) \sim \sqrt{\frac{\theta}{\phi+\theta}}\, p_{0,0} b_t^{-1} e^{-\frac{\theta+1}{\theta+\phi}\sigma\sqrt{2\phi T}\psi_t - \frac{\theta}{(\phi+\theta)}(\psi_t-\frac{\eta}{\theta}\sqrt{\phi})^2 + (\psi_t-\sqrt{\phi}(\frac{\eta}{\theta}+\frac{z}{\sqrt{8\theta}}))^2}. \tag{2.5.9}$$

Thus $M_t^{(z)}(\psi_t)$ is asymptotically equivalent to a function of the random variable $\psi_t$, and hence of the variable $\Psi^{(z)} = \sqrt{2}(\psi_t - \sqrt{\phi}(\frac{\eta}{\theta} + \frac{z}{\sqrt{8\theta}}))$ which converges in distribution to a standard normal (as $\Psi^{(z)} = \frac{m-2\phi Nh}{\sqrt{2\phi Nh(1-h)}}$). By the continuous mapping theorem, since this function is continuous, it converges in distribution to $f(Z)$ (where $f(\cdot)$ is the corresponding function).

In order to be able to take expectations of $M_t^2$ (for the rest of the proof, we suppress the dependence on $z$ in our notation) we need one additional condition. In particular we will prove that the above sequence of random variables is uniformly integrable.

For that, rewrite equation (2.5.9) as $(M_t^2)^{(N)} := De^{A(\psi_t^{(N)})^2 + B\psi_t^{(N)} + C}$ to denote a sequence of random variables whose limiting expectation we want to find (we write $\psi_t^{(N)}$, $(M_t^2)^{(N)}$ instead of $\psi_t$, $M_t^2$, to emphasize the dependence on $N$). We want to prove that there exists an $\varepsilon > 0$ such that

$$\sup_N \mathrm{E}[(e^{A(\psi_t^{(N)})^2 + B\psi_t^{(N)} + C})^{1+\varepsilon}] < \infty.$$

[15] Note that the price at 0, is given by Result 3. Moreover the asymptotic distributions of $c_m$ and $\pi^{(h)}(m)$ are given in the proof of Result 4.

[16] The notation $A \sim B$ is used to denote $A$ being asymptotically equivalent to $B$, or in other words: $\lim_{N\to\infty} \frac{A}{B} = 1$.

As $L_p$ convergence for $p > 1$ implies uniform integrability, this will give us the result we want.

By Hoeffding's inequality,[17]

$$\mathbb{P}(|\frac{m-\phi N}{\sqrt{\phi N}}| \geq k) \leq 2e^{-k^2} \tag{2.5.10}$$

for any $k > 0$. As the coefficient, $A$, on $\psi_t^2$ in $M_t^2$ satisfies $A = \frac{2\phi}{\phi+\theta} < 1$, we can set $\varepsilon > 0$ such that $A = 1 - \varepsilon$. Then inequality (2.5.10) implies that

$$\mathbb{P}\left(e^{\frac{1}{1+\varepsilon^2}\frac{(m_t-\phi N)^2}{\phi N}} \geq x\right) \leq 2\frac{1}{x^{1+\varepsilon^2}} \tag{2.5.11}$$

for $x > 0, \gamma > 0$.

Using this inequality together with the fact that $e^{\frac{1}{1+\varepsilon^2}\frac{(m-\phi N)^2}{\phi N}} \geq 1$ we have

$$\begin{aligned}
\mathrm{E}[e^{\frac{1}{1+\varepsilon^2}\psi_t^2}] = \mathrm{E}[e^{\frac{1}{1+\varepsilon^2}\frac{(m_t-\phi N)^2}{\phi N}}] &\leq \int_0^\infty \mathbb{P}\left(e^{\frac{1}{1+\varepsilon^2}\frac{(m_t-\phi N)^2}{\phi N}} \geq x\right) dx \\
&\leq 1 + \int_1^\infty \mathbb{P}\left(e^{\frac{1}{1+\varepsilon^2}\frac{(m_t-\phi N)^2}{\phi N}} \geq x\right) dx \\
&\leq 1 + 2\int_1^\infty \frac{1}{x^{1+\varepsilon^2}} dx \\
&= 1 + \frac{2}{\varepsilon^2} < \infty.
\end{aligned}$$

Finally note that $(1+\varepsilon)A = 1 - \varepsilon^2 < \frac{1}{1+\varepsilon^2}$. Hence there exists a constant, $K$, such that $(1+\varepsilon)(A\psi_t^2 + B\psi_t + C) < \frac{1}{1+\varepsilon^2}\psi_t^2 + K$, and therefore $\mathrm{E}[e^{A\psi_t^{(N)}+B\psi_t^{(N)}+C}] < \mathrm{E}[e^{\frac{1}{1+\varepsilon^2}\psi_t^2+K}] < \infty$. Thus our sequence is uniformly integrable, and hence there is convergence of expectations.[18]

[17]Hoeffding's inequality states that if $Z_1, Z_2, \ldots, Z_n$ are i.i.d. random variables, with $Z_i \in [a,b]$, and $\overline{X} = \frac{1}{n}\sum_{i=1}^n Z_i$, then $\mathrm{E}[|\overline{X} - \mathrm{E}[\overline{X}]| \geq k] \leq 2e^{-\frac{2nk^2}{(b-a)^2}}$. In our case, $m_t$ is the sum of $2\phi N$ i.i.d. Bernoulli variables, so the theorem can be applied.

[18]From equation (2.5.11) one could deduce that our sequence of random variables is dominated by the tail of a Pareto distribution, which has a finite expectation, and then use the dominated convergence theorem to reach the conclusion that there is convergence of expectations.

We can now work towards finding the variance of $M_t$ from the perspective of agent $h$. The results above imply that this problem reduces, in the limit as $N \to \infty$, to finding the expectation of a chi-squared random variable. By computing this expectation we find that

$$\mathrm{E}[M_t^2] = \frac{\theta}{\sqrt{\theta^2 - \phi^2}} \exp\left\{ \frac{\left[z\sqrt{\theta\phi} + (\theta+1)\,\sigma\sqrt{\phi T}\right]^2}{\theta\,(\theta - \phi)} \right\}. \qquad \square$$

**Proof of Result 9:**

*Proof.* We follow the logic of the proof of Result 8. Note, from equation (2.5.9), that $\log M_t$ is a quadratic function of $\psi_t$. Let us assume this quadratic has the form $F\psi_t^2 + G\psi_t + H$ for some constants $F, G, H$. Then this sequence of random variables converges in distribution to the corresponding quadratic of a Normal variable. By the Hoeffding inequality (2.5.10), $\mathbb{P}(2F\psi_t^2 \geq x) = \mathbb{P}(|\psi_t| \geq \sqrt{x/2F}) \leq 2e^{-x/2F}$. Thus $\mathrm{E}[2F\psi_t^2] \leq 2\int_0^\infty e^{-x/2F}dx = 4F < \infty$, and hence $\mathrm{E}\left[F\psi_t^2 + G\psi_t + H\right] < \mathrm{E}\left[2F\psi_t^2 + c\right] < \infty$ for some constant $c$, which implies that the sequence is uniformly integrable. We can thus take the expectation under the corresponding normal distribution. In particular, $\frac{m-2\phi Nh}{\sqrt{2\phi Nh(1-h)}}$ converges to a standard Normal. We can then write $\psi_t$ in terms of this random variable (as in the proof of the previous result) to find

$$\mathrm{E}\log(M_t) = \frac{\left[z\sqrt{\theta\phi} + (\theta+1)\,\sigma\sqrt{t}\right]^2}{2\theta\,(\theta+\phi)} + \frac{1}{2}\left(\log\frac{\theta+\phi}{\theta} - \frac{\phi}{\theta+\phi}\right). \qquad \square$$

**Proof of Result 10:**

*Proof.* Note that $W_T^{(z)} = W_0 \cdot R_{0\to T}^{(z)}$, where $R_{0\to T}^{(z)}$ is the growth optimal return from 0 to $T$ as perceived by investor $z$, and $W_0$ is the initial endowment which equals $p_{0,0}$. As $N \to \infty$,

$$W_T^{(h)} = (M_T^{(h)})^{-1} p_{0,0} \sim p_T \sqrt{\frac{\theta+1}{\theta}}\, e^{\frac{\theta(m-N-\frac{\eta}{\theta}\sqrt{N})^2}{(1+\theta)N} - \frac{(m-2Nh)^2}{4h(1-h)N}}.$$

Substituting $\psi = \frac{m-N}{\sqrt{N}}$ and parametrizing $\sqrt{N}(2h-1) = \frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}}$, we have

$$W_T^{(z)} = \sqrt{\frac{\theta+1}{\theta}} \exp(-\frac{\psi^2}{\theta+1} + \psi(\frac{2\eta}{\theta(\theta+1)} + \frac{2z}{\sqrt{2\theta}} + \sigma\sqrt{2T}) - \frac{z^2}{2\theta} - \frac{2z\eta}{\sqrt{2\theta}\theta} - \frac{\eta^2}{\theta^2(\theta+1)}.$$

Finally, substituting $\log(p_T) = \sigma\sqrt{2T}\psi$, we obtain Result 10. □

## 2.6 Appendix: Static and dynamic trade in the risky bond example

This section contains some further calculations in the risky bond example of Section 2.2.2. Specifically, we ask what happens if agents are not allowed to trade dynamically. Agent $h$ perceives a probability $1-(1-h)^T$ that the bond pays 1, and $(1-h)^T$ that the bond pays $\varepsilon$, so solves

$$\max_{x_h} \left(1-(1-h)^T\right) \log\left(w_h - x_h p + x_h\right) + (1-h)^T \log\left(w_h - x_h p + x_h \varepsilon\right).$$

The first-order condition (after setting $w_h = p$ to account for the fact that all agents are initially endowed with a unit of the risky asset) is

$$x_h = p\left(\frac{1-(1-h)^T}{p-\varepsilon} - \frac{(1-h)^T}{1-p}\right).$$

If $T$ is reasonably large, most agents will have $(1-h)^T \approx 0$, and so will choose $x_h \approx \frac{p}{p-\varepsilon}$; their wealth in the bad state of the world is then approximately zero. Thus, if forced to trade statically most agents will lever up (almost) as much as possible without risking bankruptcy.

For the market to clear, we require $\int_0^1 x_h \, dh = 1$, which implies that $p = \frac{(1+T)\varepsilon}{1+T\varepsilon}$. This is the same as the time-0 price in the case with dynamic trade. It follows that agent $h$'s demand for the asset is

$$x_h = 1 + \left(1-(1+T)(1-h)^T\right) \frac{1+T\varepsilon}{T(1-\varepsilon)}.$$

If an individual investor is forced to trade statically (while everyone else is trading dynamically, so that the price at time $t$ is observed) then the investor's leverage at time $t$, defined as debt-to-wealth ratio, is

$$\text{leverage}_t = \frac{p_0(x_h - 1)}{x_h p_t + p_0 - p_0 x_{h,0}} = \frac{1-(1+T)(1-h)^T}{T - t(1-(1+T)(1-h)^T)} \frac{1+t-t\varepsilon+T\varepsilon}{1-\varepsilon}.$$

For comparison, in the dynamic case investor $h$'s time-$t$ demand will be

$$x_{h,t} = (1-h)^t + \frac{(1-h)^t}{1-\varepsilon}\left[h(2+t)(1+t(1-\varepsilon)+T\varepsilon) - 1 - T\varepsilon\right]$$

and the investor's leverage at time $t$, defined as in equation (2.2.13), is

$$\text{leverage}_t = \frac{x_{h,t}p_t - w_{h,t}}{w_{h,t}} = \frac{(h(2+t)-1)(1+t(1-\varepsilon)+T\varepsilon)}{(1+t)(1-\varepsilon)}.$$

This strategy delivers the dynamic investor higher expected utility. An investor who follows the static strategy has wealth

$$\frac{p_0\left(1-(1-h)^T\right)}{1-(1-p_0^*)\cdots\left(1-p_{T-1}^*\right)}$$

if the bond does not default—which, in the investor's opinion, occurs with probability $1-(1-h)^T$. If the bond does default, the investor ends up with

$$\frac{p_0(1-h)^T}{(1-p_0^*)\cdots\left(1-p_{T-1}^*\right)} = \frac{p_0(1-h)^T(1-\varepsilon)}{1-p_0}.$$

This occurs with probability $(1-h)^T$. The static investor therefore has expected utility

$$EU_{static} = \left[1-(1-h)^T\right]\log\left(\frac{p_0\left(1-(1-h)^T\right)}{1-(1-p_0^*)\cdots\left(1-p_{T-1}^*\right)}\right) + (1-h)^T\log\left(\frac{p_0(1-h)^T(1-\varepsilon)}{1-p_0}\right).$$

Conversely, a dynamic investor ends up with wealth

$$\frac{p_0(1-h)^t h}{(1-p_0^*)\cdots\left(1-p_{t-1}^*\right)p_t^*}$$

if the first up move occurs after $t$ successive down-moves, where $t \in \{0,\ldots,T-1\}$. This outcome has probability $(1-h)^t h$. If the bond defaults, his terminal wealth is

$$\frac{p_0(1-h)^T}{(1-p_0^*)\cdots\left(1-p_{T-1}^*\right)} = \frac{p_0(1-h)^T(1-\varepsilon)}{1-p_0}.$$

Thus his expected utility is

$$EU_{dynamic} = \sum_{t=0}^{T-1}(1-h)^t h \log\left(\frac{p_0(1-h)^t h}{(1-p_0^*)\cdots\left(1-p_{t-1}^*\right)p_t^*}\right) + (1-h)^T \log\left(\frac{p_0(1-h)^T(1-\varepsilon)}{1-p_0}\right).$$

It follows that

$$\begin{aligned} EU_{dynamic} - EU_{static} &= \sum_{t=0}^{T-1}(1-h)^t h \log\left(\frac{h(1-h)^t\left[1-(1-p_0^*)\cdots\left(1-p_{T-1}^*\right)\right]}{\left[1-(1-h)^T\right](1-p_0^*)\cdots\left(1-p_{t-1}^*\right)p_t^*}\right) \\ &= \sum_{t=0}^{T-1}(1-h)^t h \log\left(\frac{(1-h)^t h(1+t)(2+t)T}{\left(1-(1-h)^T\right)(1+T)}\right), \end{aligned}$$

which is independent of $\varepsilon$.

To convert this logic into dollar terms, suppose an investor is indifferent between wealth of $\omega_h w_h$ and being constrained to invest statically, and wealth of $w_h$ and being allowed to invest dynamically. Then $\omega_h$ must satisfy $\mathrm{E}_{static} \log(\omega_h w_h R) = \mathrm{E}_{dynamic} \log(w_h R)$ which implies that $\omega_h - 1 = \exp\{EU_{dynamic} - EU_{static}\} - 1$. Figure 2.6.1 plots this quantity for $\varepsilon = 0.3$ and $T = 50$, as in the main text.

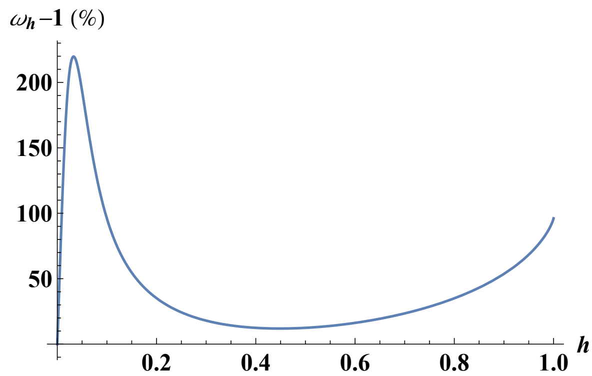



Figure 2.6.1: The attractiveness of dynamic strategies relative to static strategies, for investors of differing levels of optimism $h$.

## 2.7 Appendix: De-Moivre Laplace and Paul and Plackett theorems

First of all, let us write down a version of the De-Moivre Laplace theorem, that refers to the asymptotic approximation of a binomial distribution to the Normal distribution. Note that this theorem is essentially a special case of the Central limit theorem and first appeared in 1716 in De Moivre's "The Doctrine of Chances". We will write here the version shown and proved in the book of Kai Lai Chung, *Elementary probability theory with stochastic processes*, modified in a very slight way such as the proof presented in the book remains unchanged.

**Theorem 1.** *Suppose* $0 < p_n < 1$, $p_n + q_n = 1$, $p_n \to p$ *and*

$$x_k = \frac{k - np_n}{\sqrt{np_nq_n}},\ 0 \le k \le n$$

*Let* $A$ *be an arbitrary, fixed positive number. Then in the range of* $k$ *such that* $|x_k| \le A$ *we get*

$$\binom{n}{k} p_n^k q_n^{n-k} \sim \frac{1}{\sqrt{2\pi np_nq_n}}\, e^{-\frac{x_k^2}{2}}$$

*where the convergence is uniform and the notation* $\sim$ *means that the ratio of* *the right hand side to the left hand side tends to 1 as* $n \to \infty$.[19] *Moreover if* $S_n$ *has the Binomial*$(n, p_n)$ *distribution then, for any 2 constants* $a < b$ *we have:*

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np_n}{\sqrt{np_nq_n}} \le b\right) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{b} e^{-x^2/2}dx$$

The proof uses the Stirling's formula, and then uses a Taylor expansion of $\log(\cdot)$ carefully taking care of the errors. It then uses the definition of a Riemann integral to reach the final statement, which also shows that the standardized binomial distribution converges (in distribution) to a standard normal distribution. One important thing one can notice, relevant to the use of this result in our paper, is that if $k$ takes the form $np_n + c\sqrt{np_nq_n}$ then the approximation holds. Moreover, we have introduced the modification $p_n \to p$ since we will often use this approximation for $h_n = \frac{1}{2} + O(1/\sqrt{N})$. This small change does not alter the proof, since we still have that $k \sim np_n$. [20]

Since we want to be able to take expectations with respect to the standard normal, as $n \to \infty$ we will use one final stronger version of the De-Moivre theorem. This stronger version implies that there is convergence of the moment generating function of a standardized binomial to the standardized normal (there is *convergence of infinite exponential order*), and includes the case where $p_n$ are functions of $n$ as in our case.

Moreover, we introduce here a theorem that appears in Paul and Plackett (1978), that shows that a beta-binomial distribution tends asymptotically to a Normal, but we slightly generalize it to accommodate our needs for our model. In particular, we allow $\alpha, \beta$ to have a $\sqrt{N}$ term as well. The asymptotic normality of the Beta-Binomial distribution has been studied many times in the literature, but many times it has been proved using tools and techniques

[19] The notation $\sim$ for asymptotic equivalence, is first defined by De Bruijn, in his book *Asymptotic Methods in Analysis*. In fact $f(x) \sim g(x)$, as $x \to \infty$ is equivalent to $f(x) = g(x)(1 + o(1))$.

[20] One can also use one of the alternative proofs of this theorem, to see that this extension holds. For example defining *Binomial*$(n, p_n)$ as the sum of *Bernoulli*$(p_n)$ variables one can use the standardized versions of the Bernoulli variables and then use the Central Limit Theorem to show that their sum converges to a standard normal, by showing that the corresponding moment generating functions converge.

that are beyond the scope of this paper; our purpose here is to present an understandable sketch proof of this result. The theorem can be stated as follows:[21]

**Theorem 2.** *If* $Y \sim BB(\overline{\lambda}N, \alpha, \beta)$*, where* $\overline{\lambda} > 0$*,* $\alpha = \theta N + \eta\sqrt{N}, \beta = \theta N - \eta\sqrt{N}$*,*[22] *and we let* $N \to \infty$*, then:*

$$\frac{Y - \frac{1}{2}\overline{\lambda}N - \frac{\eta}{2\theta}\overline{\lambda}\sqrt{N}}{\sqrt{\frac{(\overline{\lambda}+2\theta)}{8\theta}\overline{\lambda}N}} \to N(0,1)$$

*Proof.* We begin by writing the probability density function of the $BB(\overline{\lambda}N, \theta_1 N + \eta\sqrt{N}, \theta_2 N - \eta\sqrt{N})$, as:

$$\binom{\overline{\lambda}N}{m}\frac{B(\theta N + \eta\sqrt{N} + m, \theta N - \eta\sqrt{N} + \overline{\lambda}N - m)}{B(\theta N + \eta\sqrt{N}, \theta N - \eta\sqrt{N})} \tag{2.7.1}$$

which can be rewritten as:

$$\binom{\overline{\lambda}N}{m}\frac{\binom{2\theta N}{\theta N + \eta\sqrt{N}}}{\binom{N(\overline{\lambda}+2\theta)}{\theta N + \eta\sqrt{N}+m}}\frac{\frac{(\theta N + \eta\sqrt{N})(\theta N - \eta\sqrt{N})}{2\theta N}}{\frac{(\theta N + \eta\sqrt{N}+m)(\theta N - \eta\sqrt{N} + \overline{\lambda}N - m)}{N(\overline{\lambda}+2\theta)}} \sim \tilde{Y}_N \frac{2\theta}{2\theta + \lambda} \tag{2.7.2}$$

In fact, writing the distribution in this way, we can now use the De-Moivre Laplace approximation to show that the density of the Beta-Binomial converges to the the density of the normal distribution for all $m = \frac{\overline{\lambda}}{2}N + O(\sqrt{N})$. This is because all binomial coefficients are of the form $\binom{M}{\frac{M}{2}+O(\sqrt{M})}$ for an $M$ that goes to $\infty$. We have denoted by $\tilde{Y}_N$, just the term that includes the binomial coefficients; in fact, one sees that $\tilde{Y}_N$ is just the density of a hypergeometric distribution (which is the conjugate prior of a Beta-Binomial

[21] The limiting distribution of a Polya urn's model has been studied extensively. However, it is often the case that $N \to \infty$ is considered, while the rest of the parameters remain constant. It is known that when only $N \to \infty$ (and $\alpha, \beta$ are fixed) the distribution tends to a Beta distribution. For a general result about the asymptotic normality of a generalized Polya-Eggenberger urn model one may also see Bagchi and Pal (1985).

[22] In fact the theorem holds in the more general case where $\alpha = \theta_1 N + \eta\sqrt{N}, \beta = \theta_2 N - \eta\sqrt{N}$.

distribution) and which we know it converges asymptotically to a normal distribution.In particular, we can complete the proof, as follows. We can use the asymptotic convergence (ratio tends to 1) of all 3 binomial coefficients, to show that $\tilde{Y}_N$ converges to:

$$\sqrt{\frac{2\theta+\overline{\lambda}}{\pi\theta\overline{\lambda}N}}\exp\left(-\frac{2(m-\frac{\overline{\lambda}N}{2})^2}{\overline{\lambda}N}-\frac{(\eta\sqrt{N})^2}{\theta N}+\frac{2(m+\eta\sqrt{N}-\frac{\overline{\lambda}N}{2})^2}{(2\theta+\overline{\lambda})N}\right)$$

From that, we can collect the $m^2, m$, and constant terms in the exponential and re-introduce the terms from equation (2.7.2) which we had left out, to conclude that the BetaBinomial converges asymptotically[23] to

$$2\sqrt{\frac{\theta}{(2\theta+\overline{\lambda})\pi N}}e^{-\frac{4\theta m^2}{(2\theta+\overline{\lambda})\overline{\lambda}N}+\frac{4(\theta N+\eta\sqrt{N})m}{(2\theta+\overline{\lambda})N}-\lambda\theta\frac{(\frac{\eta}{\theta}+\sqrt{N})^2}{(2\theta+\overline{\lambda})}}$$

Finally, this can be rewritten as:

$$\sqrt{\frac{4\theta}{(2\theta+\overline{\lambda})\pi N}}e^{-\frac{4\theta\left(m-\frac{\overline{\lambda}}{2}(N+\frac{\eta}{\theta}\sqrt{N})\right)^2}{(2\theta+\overline{\lambda})\overline{\lambda}N}}$$

and our proof that the Beta-Binomial pdf tends to the Normal has been completed. Finally, Scheffe's theorem gives us that convergence in density functions, implies convergence in distribution and hence we get the desired result. Finally note that in may also want to use that the convergence of moment generating function of Beta-Binomial distribution to the Normal one. In order to do so, we need two extra results. One is that all the moments of the BB distribution converge to that of the Normal, as shown in Bagchi and Pal (1985).[24] Then proof is concluded by noting that if there is convergence in distribution and in all the moments, then there is convergence of the moment generating functions. □

[23] Remember, that asymptotic equivalence means that the ratio tends to 1 as $N \to \infty$.

[24] Also, using the Moment Continuity Theorem, we see that convergence in distribution of subgaussian variables is the same as convergence of moments.

# Chapter 3

# The Effect of Market Conditions and Career Concerns in the Fund Industry

## 3.1 Introduction

In recent years, there has been growing concern in the financial markets about the role of various financial intermediaries such as mutual funds and hedge funds, as the proportion of the institutional ownership of equities has sharply increased and the Global Assets under Management are estimated to exceed $100 trillion by 2020.[1] The managers of these funds are competing with each other, but also with alternative investment vehicles such as market index funds or ETFs, to attract new investors. One of the ways in which they differentiate themselves is through their investment strategy. In particular, managers often signal their confidence by choosing strategies that are highly idiosyncratic,[2] and more importantly their incentive to pick these strategies fluctuates with the general market conditions.

[1] This is according to a research by PWC.

[2] For example, a recent article in Financial Times explains how institutional investors are turning to alternative investments in recent years.

Our first contribution is to build a model in which a manager's investment decision provides an imperfect signal on her ability to generate idiosyncratic returns. To be more precise, the manager will skew her investment choice towards a strategy with low exposure to the market in order to signal her confidence. A highly skilled manager is more likely to invest in her idiosyncratic project, since this will deliver on average superior returns. The investors cannot observe directly the manager's ability, but because of the above they will associate an idiosyncratic strategy with a competent manager; in turn, this will endow such a strategy with a reputational benefit. This asymmetry of information between the manager and her potential investors is the main driving force behind the results of this paper.

Our second contribution is to demonstrate that the signalling value of investing in a low beta strategy depends on the market conditions. Managers have a dual objective; they want to maximise their contemporaneous returns but also their perceived reputation. The better the market (*bull*) is, the more the managers face a trade-off between these two objectives, and the less the investors penalise managers for choosing a high beta strategy. Consequently, there is an interaction between managers' career concerns and market conditions.

To analyse the above interactions we consider a two period model in which there is a continuum of investors and a single fund manager. Each investor chooses between investing his wealth through the manager, or directly in the market index, and this choice is affected by an investor's specific stochastic preference shock. The manager's utility is a function of the fees she collects, which are an exogenous proportion of her fund's assets under management (AUM) at the end of each period. After the investors have allocated their funds, the manager publicly chooses between a high or low beta investment strategy. We model the manager's ability as the ex ante expected return of her idiosyncratic strategy, which is either high or low. In each of the two periods, and before picking an investment strategy, the manager also receives a private signal on the contemporaneous profitability of her idiosyncratic project. Both her ability and this signal are her private information, and she uses them to form her final estimate of the profitability of her contemporaneous idiosyncratic strategy. As a result, a high type manager is more likely to form a high estimate, but this is not always the case.

To model market conditions, we assume that the manager also receives a signal on the market's contemporaneous return. This signal is eventually revealed to the investors, but only after they have made their own investment choice. In some sense, we allow for them to eventually understand the market conditions under which the manager acted. However, at this point it will be too late for them to use this information to trade on their own.[3] In section 3.3.4, we extend our setting by allowing two managers to coexist in the market, in order to study how the competition is affected by market conditions. We focus mainly on the first period, since in the second the manager's investment choice is not affected by her reputational concerns. In fact, the second period is introduced in order to create those concerns.

For our first result, we analyse a refinement of the perfect Bayesian Equilibrium, which we call *monotonic equilibrium* and we prove that this always exists. The only additional restriction that this refinement is imposing is that the manager's reputation is non-decreasing on her performance. In addition, under mild parametric restrictions we demonstrate that the monotonic equilibrium is unique.

In our second result we demonstrate that investing in an idiosyncratic strategy carries a reputational benefit. This is because, the cut-off of the high manager type is smaller than that of the low. In other words the high type is more receptive to the idea of adopting a low beta strategy. Intuitively, the manager's choice is affected by two incentives. On the one hand, she wants to increase her reputation, which skews her preferences towards idiosyncratic investments. On the other hand, she cares about the realised return of her strategy, since her fees depend on it. Hence, for a relatively low private signal even a high type may opt to forfeit the reputational benefit, because investing in the market will generate higher returns, and as a result more fees. Therefore, the investment strategy is informative but it does not fully reveal the manager's

[3]In other words, manager has a superior market-timing ability compared to an investor.

ability, which is a realistic representation of the fund industry.

Our third and most important result is to show that the reputational benefit of investing in the idiosyncratic project is decreasing in the market conditions. In particular, we prove that the expected sensitivity of reputation to performance is higher in bear markets than in bull markets. This is because investors understand the dual objective of managers and the fact that a manager is more likely to invest in the market when the market conditions are good, and thus update their beliefs less aggressively when this is the case; instead, in bad times any change in a fund's performance is much more likely to be attributed to the ability of the manager.

We use the above results to discuss the competition between funds, in terms of their sizes, and its fluctuation depending on market conditions. We predict that the likelihood of changes in the ranking of the funds, measured by assets under management, is hump shaped on the market return, but is also higher during bear markets than during bull markets, due to the higher informativeness of performance; we also find some empirical evidence supporting this prediction. This is in line with the common perception that the industry only rearranges its interaction with its investors during crises.

Finally, as an extension to our model, we study the case where investors cannot observe the managers' investment decision. In this scenario, we assume that the investors cannot observe if the manager had invested on the market or their idiosyncratic portfolio, and we conclude that, under this assumption, the conditions for the existence of a monotonic equilibrium cannot be satisfied.

Academic research in financial intermediaries has so far mainly focused on establishing various empirical results about their structure, returns, flows, managers' skill and many other characteristics; there have been far fewer theoretical papers. One of the seminal papers about mutual funds is from Berk and Green (2004); they construct a benchmark rational model in which the lack of persistence of outperformance, is not due to lack of superior skill by active managers, but is explained by the competition between funds and reallocation of investors' capital between them.

Our paper aims to contribute to various strands of literature that we outline below. First, it relates to many papers that study how managers' concerns about their reputation affect their investment behaviour. Chen (2015) examines the risk taking behaviour of a manager who privately knows his ability and shows that in this model investing in the risky project always makes a manager's reputation higher, thus leading to overinvestment in such risky projects. Dasgupta and Prat (2008) study the reputational concerns of managers, and show how they may lead to herding and can explain some market anomalies; their focus though is mainly on the asset pricing implications of this behaviour. Similarly, Guerrieri and Kondor (2012) build a general equilibrium model of delegated portfolio management to study the asset pricing implications of career concerns; they find that as investors update their beliefs about managers, these concerns lead to a reputational premium, which can change signs depending on the economic conditions. Moreover, Malliaris and Yan (2015) show that career concerns induce a preference over the skewness of their strategy returns, while Hu et al. (2011) present a model of fund industry in which managers alter their risk-taking behaviour based on their past performance and show that this relationship is U-shaped. Huang et al. (2012) on the other hand, build a theoretical framework to show how investors are rationally learning about the managers' skills, and test their predictions about the fund flow-performance relationship empirically; however, they do not take into account any strategic behaviour by the fund managers.

The paper most relevant to our work is that by Franzoni and Schmalz (2017). In their work, they study the relationship between the fund to performance sensitivity and an aggregate risk factor and they find that this is hump shaped. They also build a theoretical model in which investors update their beliefs about the managers' skills while they also learn about the fund's exposure to the market. The second inference in extreme markets is noisier for two reasons. The first is idiosyncratic risk and the second is that investors who are uncertain about risk loadings cannot perfectly adjust fund returns for the contribution of aggregate risk realisations. As a result it becomes harder for investors to judge the managers and update their beliefs, and this is what drives the documented result. The theory we propose differs from that of

Franzoni and Schmalz (2017) because their model describes the fund's loading on aggregate risk ($\beta$) as a preset fund specific exposure, whereas our model gives the ability for managers to strategically choose their investment decision. Also we further investigate how this investment decision will affect the managers' decision if it is observable by the investors or not. Moreover the data source considered for their paper is the CPRSP Mutual Fund Database which is different from the Morningstar CISDM which we use for the empirical part, making it difficult to compare our results. Although the implementation and the structure of their model is completely different to ours and does not imply the same predictions we are making, we conclude that the aggregate risk realisations matter for mutual fund investors and managers.

Another strand of literature in which we contribute to, is the empirical research on the fund flows and characteristics. It is well documented that mutual fund investors chase past returns, Ippolito (1992) and Warther (1995) present empirical evidence supporting our predictions. Sirri and Tufano (1998) show that the flow-performance relationship is convex, and asymmetrically so on the positive side of returns. Furthermore, Chevalier and Ellison (1997), show that managers engage in window dressing their portfolios. More recently, Wahal and Wang (2011) study the competition between funds, by looking at the effect of the entry of new mutual funds on fees, flows and equilibrium prices. Finally, Ma (2013) provides a very comprehensive survey of empirical findings concerning the relationship between mutual fund flows and performance.

The rest of the paper is organised as follows. In section 3.2, we introduce our theoretical framework and our equilibrium. Section 3.3 proves its existence, identifies a condition under which this is unique, and presents our theoretical predictions. In particular, section 3.3.4 discusses the implications of adding a second manager. Subsequently, section 3.4 presents our empirical results. Section 3.5 considers an alternative model where the investment decision is unobservable. Finally, section 3.6 concludes.

## 3.2 The Model

### 3.2.1 Setup

This is a two period model $t \in \{1, 2\}$. There is one fund manager (she) and a continuum of investors (he) of measure one, who collectively form the market. The manager discounts the future with $\delta \in (0, 1]$.

At the beginning of period $t$, each investor decides how to invest a unit of wealth. At the end of period $t$, he consumes all the wealth that this investment generated. The investor is restricted to a binary decision. He can either opt to allocate all his wealth in an index tracking strategy. This has the same returns as the market portfolio, which is given by

$$m_t \sim \mathcal{N}(\mu\,,\, \sigma_m^2) \tag{3.2.1}$$

Or, he can choose to invest all his wealth in the manager's fund.[4] For each unit of wealth invested with the manager let $R_t = \exp(r_t)$ denote its value at the end of this period, where

$$r_t \;=\; (1 - \beta_t) \cdot a_t \;+\; \beta_t \cdot m_t \tag{3.2.2}$$

is the fund's return. This has two components, one of which is the market return $m_t$. The second is given by

$$a_t \sim \mathcal{N}(\alpha,\, \sigma^2\,) \tag{3.2.3}$$

which represents the market neutral component of the manager's investment strategy.[5] Adhering to the fund industry's convention, the manager's ability

[4]Our underlining intuition is that most of the market participants follow a rule of thumb to their investment through intermediaries. For example, they set apart 5% of their wealth and then they decide if they should invest this amount to a fund.

[5]For example think of a long/short equity fund that invests $(1 - \beta_t)$ of its assets on a market neutral portfolio and $\beta_t$ on the S&P 500 index. For the most part we refrain from giving a specific interpretation of the components of the fund's return $r_t$, or which part of its investment strategy they represent. Our framework relies on the simple intuition that

to create idiosyncratic profits is called alpha, and is represented by $\alpha \in \{L, H\}$ where $L < H$. The manager's ability is her private information. The investors share the public prior $\pi = \mathbb{P}(\alpha = H)$.

Finally, $\beta_t$ represents the fund's exposure to the market. This is publicly chosen by the manager after the investors have allocated their wealth. For simplicity we assume that $\beta_t \in \{0, 1\}$. Note that the model's beta $\beta_t$ despite its relevance to the corresponding variable of the CAPM model, is not the same variable. Rather the former represents a deterministic investment decision, whereas the latter its estimate.

In addition, before making her investment decision $\beta_t$, but after the investors have allocated their wealth, the manager receives two signals

$$s_t = a_t + \eta_t \qquad \text{and} \qquad s_t^m = m_t + \eta_t^m. \tag{3.2.4}$$

where $\eta_t \sim \mathcal{N}(0\,,\,\nu^2\,)$ and $\eta_t^m \sim \mathcal{N}(0, \nu_m^2)$. On the one hand, $s_t$ is private and it is associated with the manager's contemporaneous confidence on her alpha.On the other hand, $s_t^m$ is public but it only becomes available after the investors have committed their capital to the manager's fund. This market signal is considered to be the standard piece of information that most institutional participants receive on the market's condition.

To simplify matters, we assume that the manager's fees are exogenously set to a given percentage $f_t \in [0, 1]$ of her asset under management (AUM) at the end of $t$.[6] Even though we do not allow for incentive fees, the plain managerial fees $f_t$ we consider suffice to create direct incentives for the manager to perform in $t$, as her period income per dollar invested is $f_t R_t$.

Two more important assumptions have been made. First, that the man-

some of the return generated by the manager stems from her own ability and some from factor loading. In fact $m_t$ could represent any such factor, and for some funds other choices would be more sensible. For example, a macro fund is more related to the risk-free interest rate than to the equity markets.

[6]Endogenising the choice of fees is left for future research. The complexity of allowing an endogenous choice is that the fees would then serve as a signalling device for the managers' ability, thus making the equilibrium much harder to find.

ager's investment decision is binary. In particular, it allows for either investing all of the fund's assets in the manager's idiosyncratic strategy $a_t$, or all in the market $m_t$. Second, that this decision is observable by the rest of the market participants. The former assumption is imposed mainly to make the model more tractable. We speculate that altering it to allow for $\beta_t \in \{\underline{b}, \overline{b}\}$, where $\underline{b} < \overline{b}$, would not affect our results qualitatively.[7] Regarding the latter assumption, it appears to be reasonable for long investment horizons. This is because the fund's exposure to the market can be ex-ante approximately inferred, either by estimating a multi-factor regression, or by looking at its past portfolio composition, which in many cases is public.

### 3.2.2 Payoffs

Investors are risk-neutral, however each one's decision is influenced by an exogenous preference shock which follows an exponential distribution.

$$z_t^j \sim \exp(\lambda)\,, \quad \text{where} \quad j \in \big[0,1\big] \tag{3.2.5}$$

stands for the shock on investor's $j$ preferences at period $t$. Hence, his payoff from investing in $i \in \{1, m\}$ is

$$v(i, z_t^j) = \begin{cases} \exp(z_t^j - \bar{z}) \cdot (1 - f_t) \cdot R_t & , i = 1 \\ \\ \exp(m_t) & , i = m \end{cases} \tag{3.2.6}$$

where $\bar{z} > 0$ is a constant that we introduce to ensure that under the lowest preference shock $z_t^j = 0$ the investor would opt for the market instead.

There is a plethora of ways to interpret this shock, a valid one being

[7] A possibility that we exclude and is worth mentioning is that of a manager that bets against the market. In particular, in strong bear markets most funds would prefer to short the market portfolio, instead of adopting a strategy that is neutral to it. This would have a significant impact on our analysis. Despite that, it is ignored both to facilitate the exposition and because funds that systematically hold big negative positions are not that common.

that each investor values specific fund characteristics, for example the fund's classification with regards to its investment strategy, its portfolio composition, leverage, etc. An alternative one would be that he is influenced by interpersonal relationships, network effects, word of mouth, or other forms of private information. Our analysis will be silent as to what generates this shock.

Furthermore, note that because $R_t$ comes from a log-normal distribution, we could adopt a CRRA utility function for the investor without altering his decision significantly. However, we opt not to do so in order to maintain our expressions as compact as possible. On the other hand, it will be assumed that the manager has log preferences. In particular, if $A_t$ stands for the AUM the fund in the beginning of $t$, then manager's payoff at $t$ is $\log\big(A_t f_t R_t\big)$. Again we speculate that most of our results would not be significantly different if a generic CRRA was used instead of log, however it turns out that this is the most convenient functional form to work with.

### 3.2.3 Timing

To sum up, the timing in our model is as follows. In each period $t \in \{1, 2\}$, first the preference shock $z_t^j$, $j \in [0, 1]$, is realised and then the investors decide how to allocate their wealth. Second, the manager receives the private and public signals $s_t$ and $s_t^m$, respectively. Third, the investment decision $\beta_t$ is made by the manager, $R_t$ is realised, and both become public. Fourth, the fund's AUM is divided between the manager and her investors, according to the fee $f_t$, and is consumed immediately. Finally, we assume that the investors that are active in the second period observe the public variables of the first period before allocating their wealth. Importantly, they know $(R_1, \beta_1, s_1^m)$ and use them to update their beliefs on the manager's ability $\alpha$. Signal $s_1$ can not be used since it is private information of the manager and it will never be known to the investors.

### 3.2.4 Monotonic equilibrium

We call an equilibrium of our model *perfect Bayesian* (PBE), if all market participants use Bayes' rule to update their beliefs on $\alpha$, whenever possible, and choose their actions in order to maximise their expected discounted payoff at each point they are taking an action. There is a possibility of there being multiple equilibria, which is a common setback for these types of models. For this reason we will further refine the set of equilibria using the following definition, however, the study of these equilibria is beyond the scope of this paper.

**Definition.** Call a PBE a *monotonic equilibrium* if the manager's reputation, for a given choice of investment strategy, is non-decreasing on her performance.

In other words a monotonic equilibrium satisfies $\mathbb{P}(\alpha = H \,|\, r, s^m, \beta)$ is increasing in $r$. Therefore, the only requirement that our refinement imposes is that the manager's reputation is not penalised by the fact that she delivers good returns for her investors. The above definition implies that there exists $\varphi_0$ and $\varphi_1$ such that the public posterior on the manager's ability is given by

$$\begin{aligned} \varphi_0 &= \mathbb{P}(\alpha = H \,|\, r_1, s_1^m, \beta_1), \quad \text{for } \beta_1 = 0 \\ \varphi_1 &= \mathbb{P}(\alpha = H \,|\, r_1, s_1^m, \beta_1), \quad \text{for } \beta_1 = 1 \end{aligned} \tag{3.2.7}$$

We separate the posteriors that follow each choice of $\beta_1$ because those will turn out to have different functional forms.

## 3.3 Analysis

We begin our analysis by first discussing the manager's optimal investment strategy in the second period and how this affects her career concerns in the first period. Second, we characterise the monotonic equilibrium and prove its existence and uniqueness. Third, we present our results on the baseline model with the single manager. Fourth, we discuss the implications of adding a second manager.

### 3.3.1 Investment and AUM in the second period

Here we provide a description of how we solve for the manager's investment decision in the second period and the corresponding AUM that this implies. The interested reader can find a more detailed analysis in Appendix 3.8.

In the second period the manager faces no career concerns. Hence, the objective of her investment decision is to maximise the expected fees she collects at the end of this period. Because those fees are proportional to her fund's AUM at the end of the second period, and we have assumed log preferences, the manager's payoff maximisation problem simplifies to

$$\max_{\beta_2 \in \{0,1\}} \mathbb{E}\big[\log(A_2, f_2, R_2) \,\big|\, \beta_2,\, \alpha,\, s_2,\, s_2^m\big]$$

When opting for her idiosyncratic strategy $\beta_2 = 0$ the above expectation uses the manager's ability $\alpha$ and private signal $s_2$, whereas the index tracking strategy $\beta_2 = 1$ depends only on the market signal $s_2^m$. Since we have assumed that the returns and the corresponding signals are log-normally distributed we can calculate the above expectation for each choice in closed form. This suggests that the manager's optimal second period strategy is to invest in her idiosyncratic project if and only if $s_2 \geq c(\alpha, s_2^m)$ where

$$c(\alpha, s_2^m) \;=\; \frac{\psi_m}{\psi} \cdot s_2^m + \frac{1-\psi_m}{\psi} \cdot \mu - \frac{1-\psi}{\psi} \cdot \alpha \tag{3.3.1}$$

The constants $\psi$ and $\psi_m$ are the weights that the Bayesian updating gives to the signals $s_2$ and $s_2^m$, respectively, and more specifically: $\psi = \frac{\sigma^2}{\sigma^2+\nu^2}$ and $\psi_m = \frac{\sigma_m^2}{\sigma_m^2+\nu_m^2}$. Given the above cut-off strategy we can calculate the expected terminal value of one unit of wealth that is invested by the manager. For a high and low type we will denote those by $u_2^H$ and $u_2^L$, respectively. Therefore, for given posterior reputation $\varphi$, and while ignoring the preference shock $z$, the expected payoff of an investor that opts for the manager is given by

$$[1 - f_2] \cdot [\varphi \cdot u_2^H \; + \; (1 - \varphi)\, u_2^L]$$

This together with the assumed preference shock allows us to calculate the assets of the second period in closed-form.

From (4.6) we have the expected payoff of an investor who chooses to invest in a fund or in the market. He chooses the former if his expected payoff is higher. Since there is a continuum of investors with one unit of wealth, the probability of this event occurring is equal to the assets of fund one. Hence

$$A_2(\varphi) \;=\; \Big(e^{-(\mu+\bar{z}+\sigma_m^2/2)} \cdot [1 - f_2] \cdot [\varphi \cdot u_2^H \; + \; (1-\varphi) \cdot u_2^L]\Big)^{\lambda} \qquad (3.3.2)$$

which is an increasing function of the manager's reputation $\varphi$. One thing we can note is that as long as $\lambda > 1$, the assets under management are a convex function of the reputation $\varphi$. This is a result that has been widely documented in the relevant empirical literature, in slightly different forms.

### 3.3.2 Existence and uniqueness of the monotonic equilibrium

In this section we demonstrate that the monotonic equilibrium exists and is unique under mild conditions. First, we want to understand the manager's incentives in the first period. Her expected discounted payoff at this point is

$$\mathbb{E}_R\Big[\log\left[R_1 f_1\, A_1(\pi)\right] \; + \; \delta \cdot \log\left[R_2 f_2 A_2(\varphi_\beta)\right] \;\Big|\; s^m,\; s,\; \beta\,,\alpha\Big]$$

where the expectation taken with respect to the returns of both periods. $A_1(\pi)$ is the equilibrium allocation of AUM in the first period, which has a functional form similar to that of $A_2(\varphi_\beta)$ and $\beta$ is $\beta_1$.

Hereafter, the focus of the paper shifts to the interactions of the first period. As a result, in order to make our formulas more compact, the time subscript $t$ is dropped, whenever this does not create an ambiguity. Using the properties of the natural logarithm we simplify the manager's payoff maximisation problem in period 1 to

$$\max_{\beta\in\{0,1\}} \mathbb{E}_r \Big[ r \,+\, \delta\cdot\lambda\cdot\log[\varphi_\beta(r,s^m)\cdot(u^H - u^L) \,+\, u^L\,] \,\Big|\, s^m,\, s,\, \beta\,,\alpha \Big] \quad (3.3.3)$$

Therefore, the manager cares both about her returns in the first period $r$, but also on how those affect her posterior reputation $\varphi_\beta(r,s^m)$. This reputation is important because it affects the amount of AUM that the manager will manage to gather in the beginning of the second period.

First, we want to offer a characterisation of the monotonic equilibrium.

**Lemma 4.** *In any monotonic equilibrium the high and low type invest in their idiosyncratic strategy if and only if*

$$s \,\geq\, h(s^m) \quad \textit{and} \quad s \,\geq\, l(s^m)\,, \quad (3.3.4)$$

*respectively, where*

$$l(s^m) - h(s^m) \;=\; \frac{1-\psi}{\psi}\cdot(H-L) \quad (3.3.5)$$

*Proof.* In Appendix 3.7. □

Hence the more confident the manager becomes on her alpha, the more likely she is to use her idiosyncratic strategy, instead of the index tracking one. In addition, the fact that the high type's cutoff is lower captures the fact that a competent manager uses her idiosyncratic investment strategy relatively more often.

Second, we want to calculate the manager's posterior reputation after each investment decision as a fuction of her performance.

**Lemma 5** (Posteriors)**.** *In any monotonic equilibrium the manager's posterior reputation in the beginning of the second period, if she invested on her alpha* $\beta_1 = 0$ *in the first, is*

$$\varphi_0(r, s^m) \;=\; \left(1 + \frac{1-\pi}{\pi} \cdot \rho(r) \cdot \frac{\Phi\left(\frac{r - l(s^m)(1+\psi) + L\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r - h(s^m)(1+\psi) + H\psi}{\nu\sqrt{1+\psi}}\right)}\right)^{-1}, \tag{3.3.6}$$

*where*

$$\rho(r) \;=\; \exp\left(\frac{-2(H-L)\,r + H^2 - L^2}{2\nu^2\psi(1+\psi)}\right).$$

*On the other hand, if she invested in the market* $\beta = 1$ *then this becomes*

$$\varphi_1(s^m) \;=\; \left(1 + \frac{1-\pi}{\pi} \cdot \frac{\Phi\left(\frac{l(s^m) - L)}{\nu}\right)}{\Phi\left(\frac{h(s^m) - H}{\nu}\right)}\right)^{-1} \tag{3.3.7}$$

*We recall that* $r$ *depends on the investment decision* $\beta$*.*

*Proof.* In Appendix 3.7. □

The investors form their posterior belief on the manager's ability by observing her investment decision $\beta$ and the realised return $r$. Note that when using her idiosyncratic investment strategy the manager's performance $r$ is generated by her alpha. Hence, in this case the realisation $r$ carries additional information on the manager's ability. On the other hand, when using the index tracking strategy $r$ is equal to the market's return $m$, which carries no additional information on the manager's ability. This is why $\varphi_0$ is a function of $r$, but $\varphi_1$ is not.

Using the above two lemmas, we prove the main result of this part.

**Proposition 7.** *A monotonic equilibrium always exists. Moreover, a sufficient condition for it to be unique is that*

$$\delta \cdot \lambda \cdot (H - L) \;\leq\; \psi^2 \cdot \nu^2 \tag{3.3.8}$$

*Proof.* In Appendix 3.7. □

We believe that (3.3.8) is satisfied for a wide range of parametric specifications that we would consider natural given the economic setting we study. This translates into two requirements. First, that the difference between the ability of the two types is not too big. Second, that the precision of the signal $s$ is neither so small that it becomes irrelevant, nor so big that the manager's ex-ante ability $\alpha$ becomes irrelevant instead.

### 3.3.3 Results

Here, we present some important properties of the unique monotonic equilibrium. We assume throughout that (3.3.8) holds. To maintain the notation as light as possible keep using $\varphi_0(r, s^m)$ and $\varphi_1(s^m)$ to refer to the equilibrium reputations, which are obtained after substituting the corresponding values for $h(s^m)$ and $l(s^m)$.

**Proposition 8** (Point-wise dominance)**.** *There is a strict reputational benefit for the manager from investing in her alpha, that is*

$$\varphi_0(r, s^m) \; > \; \varphi_1(s^m), \quad \textit{for all} \quad r, s^m \in \mathbb{R}. \tag{3.3.9}$$

*Proof.* In Appendix 3.7. □

We already know that in every monotonic equilibrium $\varphi_0(r, s^m)$ is increasing in $r$, in other words high performance is beneficial for the manager's reputation. The proof demonstrates the result by considering the worst case scenario for the manager $\beta = 0$. In the extreme scenario where return approaches minus infinity her reputation is still greater than choose $\beta = 1$. Hence the equilibrium difference between the cutoffs used by the high and low type is such that the investors' inference on the manager's type relies relatively more on her choice of strategy than on the subsequent performance of her fund.

This may seem counterintuitive at first, but it has a very simple explanation. In the appendix we show that for a monotonic equilibrium to also be rational the difference between the equilibrium cutoffs $l(s^m)$ and $h(s^m)$ cannot be too large. If that was the case, then a low type would have to be so confident in order to invest in her alpha that a very bad performance, under the low beta strategy, would be associated with a high type. An immediate consequence of which would be that the manager's reputation would be non-monotonic on her performance. But those are exactly the type of equilibria that appear to be the less realistic.

The above claim is the most challenging one to verify in the data. This is because for each fund we never observe the counter-factual, that is how the fund's flow would look like if it had chosen a lower, or higher beta strategy. Moreover, the simplifying assumption $\beta \in \{0, 1\}$ makes this result stronger than what an alternative model, where the two betas are closer to each other, would give. Despite that, we can verify empirically that to a certain extent a low beta strategy creates enough signalling value to counter the effect of a low subsequent performance.

As a direct consequence of point-wise dominance, we can now get the following interesting proposition, which characterises the effect of the manager's career concerns on her investment behaviour.

**Proposition 9** (Investment Behaviour)**.** *The equilibrium cutoffs $h(s^m)$ and $l(s^m)$ are decreasing in the discount factor $\delta$. Moreover, there is overinvestment in the manager's idiosyncratic project, that is*

$$h(s^m) \leq c(H, s^m) \qquad \textit{and} \qquad l(s^m) \leq c(L, s^m). \tag{3.3.10}$$

*Proof.* In Appendix 3.7. □

The proof is a simple application of the implicit function theorem on equation (3.7.17), the solution of which is shown in the proof of Proposition 7 to be $h(s^m)$. The corresponding result for $l(s^m)$ is obtained by invoking the fact that in every monotonic equilibrium those two cutoffs are connected through a linear relationship, which was again demonstrated in the above proof.

We use the term *over-investment* to describe the fact that the manager invests in her idiosyncratic strategy more often than in the absence of career concerns. In other words, over-investment exists when the manager lowers her standards with regards to her private signal, i.e. she lowers the confidence level required for her to choose the idiosyncratic investment. Note that the manager's optimal cutoff, in the absence of career concerns, corresponds to that already derived from for the second period in (3.3.1). This is because it is generated by the inefficiency in the investment decision that the manager's career concerns create, which is connected to the underlying parameter $\delta$.

The above proposition demonstrates that there is a bias towards active management in the financial intermediation industry, which is due to its inherent informational asymmetries. To be more precise, we expect managers to get on average less exposure to the market than what would maximise the fund's expected return. Moreover, this action is associated with competence and it is rewarded with an increase in the fund's AUM. Hence, our model provides a theoretical justification for this well documented fact.

Next, we want to see how this bias depends on the unobserved, to the econometricians, market signal $s^m$ and the manager's prior reputation $\pi$.

**Proposition 10.** *The cutoffs $h(s^m)$ and $l(s^m)$ are increasing in the market signal $s^m$. In addition, there exist lower bounds $\bar{s}^m$ and $\bar{\pi}$ such that for every $(s^m, \pi)$ such that $s^m \geq \bar{s}^m$ and $\pi \geq \bar{\pi}$ both cutoffs $h(s^m)$ and $l(s^m)$ are increasing functions of the manager's prior reputation $\pi$.*

*Proof.* The proof of the first statement is similar to that of Proposition 9. The proof of the second follows from Lemma 10, which can be found in Appendix 3.7. □

The first statement is a very intuitive result. The better the manager expects the market portfolio to perform, the more eager she becomes to invest in it, which translates into higher equilibrium cutoffs.

The crucial implication of the proposition's second statement is that the bias created from the signalling value, of investing in the idiosyncratic strategy, is decreasing in the manager's prior reputation. This is because the equilibrium cutoffs are bounded above by the expected return maximising cutoff $c(\alpha, s^m)$, hence the more $\pi$ increases the closer they get to it.

A caveat of this result is that it only holds for a manager that is already relatively recognised in the market, in particular it is shown in the appendix that we need at least $\pi > 1/2$. Intuitively, the closer the prior is to either zero or one, the less it is affected by the actions of the manager. To make this more concrete, think of the extreme case where $\pi \to 1$, in which case it is very difficult for the investors to change their opinion about her ability, as they already know it with almost total certainty. Hence, there is a corresponding result that can be stated for managers of very low reputation. Even though in our model we allow for funds of small size to stay active, in reality most of them would either shut down, or would not even be reported in most datasets, hence we focus just on funds with reputation greater than a 1/2. [8].

Another interesting feature of the presented specification is that it provides a better understanding on how the sensitivity of the fund's asset flows to its performance depend on the market conditions. Let $\varphi^i(r^i, s^m, \beta^i)$ stand for the manager's reputation in either of the two cases and call $\mathrm{d}\varphi^i/\mathrm{d}r^i$ its *sensitivity* with respect to her performance.

**Proposition 11.** *The conditional probability that a manager has invested in the market portfolio* $\mathbb{P}(\beta_t^i = 1 \mid m_t)$ *is increasing in its contemporaneous performance* $m_t$*.*

*In addition, for a sufficiently reputable manager the conditional expected sensitivity of the manager's reputation with respect to her performance, i.e.* $\mathbb{E}_{s^m}[\mathrm{d}\varphi^i/\mathrm{d}r^i \mid m_t]$*, is decreasing in* $m_t$*.*

*Proof.* In Appendix 3.7. □

When markets are expected to perform well, the manager's direct incentives outweigh those of career concerns. Hence we know from Proposition 10

[8]Despite that we hope to test empirically if we can obtain a corresponding result for the flows of small funds.

that she is more likely to give up the reputational benefit of following a low beta strategy. But high beta strategies carry no information with respect to the manager's ability. Hence, even though as noted in Proposition 8 investing in low beta always has a reputational benefit, this benefit is less pronounced in good markets. Therefore investors are expected to rely more on a manager's performance to update their belief about the ability of the manager, when markets are bear than when markets are bull. This result is also supported by the empirical evidence we provide in section 3.4.

### 3.3.4 Discussion on the competition between funds

It follows from the previous discussion that managers will be judged much more strictly on their performance in bear markets than in bull markets. This in turn has some implications for the relative ranking of the various funds with respect to their reputation, or equivalently their AUM.

To study this we extend our model by allowing a second manager to operate in the market. We formally define the investor's preference shock in this case and derive the corresponding AUMs of the two funds in Appendix 3.8. In fact, the whole analysis of this paper and all our results remain unchanged with the addition of a second manager. The reason is that the manager's utility is such that it is only a function $\varphi_\beta(r, s^m) \cdot (u^H - u^L) + u^L$ and is independent of the number of managers that exist in the model.[9]

Our main aim is to study the likelihood of a change in the rank of managers, in terms of investors' beliefs about their ability and relate that to the market conditions. In what follows, we explain why this effect is not monotonic in $m_t$.[10]Indeed, let $\mathbb{P}(i, j \mid s^m) = \mathbb{P}(\beta_{\text{fund 1}} = i, \beta_{\text{fund 2}} = j \mid s^m)$, for $i, j \in \{0, 1\}$.

[9]In particular, equation (39) and thus the determination of the cutoffs $l$ and $h$ will remain the same.

[10]Note, we always condition on $s^m$ as we know that all investors observe this market signal.

In the Appendix it is shown that:

$$\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \;=\; \mathbb{P}(\varphi_0^1 > \varphi_1^2 \,|\, s^m)\,\mathbb{P}(0,1 \,|\, s^m) + \mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m)\,\mathbb{P}(0,0 \,|\, s^m), \tag{3.3.11}$$

What this equation suggests is that the ranks of managers can change through two possible scenarios. In the first scenario, with probability $\mathbb{P}(0,1 \,|\, s^m)$, one of the two managers invests in his idiosyncratic portfolio and the other follows the market. This probability approaches zero for both very large and very small $s^m$, as then both managers invest in the market or both invest in their own project. In turn, this makes the first term of the right hand side of the equation (4.18) hump-shaped in $s^m$. Under this scenario, manager 1 has a reputational benefit from choosing $\beta = 0$ (see Proposition 2) which then makes it possible for his ex-post reputation to be higher than that of manager 2 (despite his initial disadvantage, in terms of the priors $\pi^1, \pi^2$); clearly the smaller the distance between their prior reputations, $\pi^2 - \pi^1$, the larger this likelihood will be.

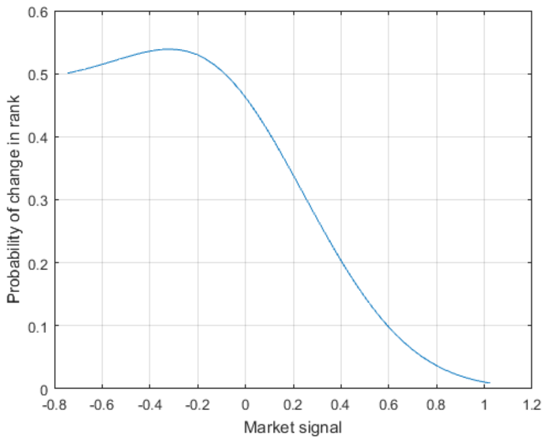
In the second scenario, with probability $\mathbb{P}(0,0 \,|\, s^m)$ both managers invest in their own project and manager one receives a much higher return than the other, thus overcoming the effect of the initial prior reputations; in other words, since $\pi^1 < \pi^2$, in order for the posterior reputations to have the opposite order, what needs to happen is that the realised return of manager 1 is much higher than that of 2. This is clearly not possible if they both invest in the market. However, when they both invest in their idiosyncratic project this can happen either because one is luckier than the other, or simply because manager one has high skill and manager two has low skill. This scenario is less likely to occur as the market conditions get better since $\mathbb{P}(0,0 \,|\, s^m)$ is decreasing in $s^m$, as we can see from Proposition 5. Moreover, we can get the following remark:

**Remark 1.** The likelihood of a change in the ranks of managers is higher in a very bad market, than in a very good market. That is:

$$\lim_{s^m \to -\infty} \mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) > \lim_{s^m \to +\infty} \mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \tag{3.3.12}$$

The proof of this remark is quite simple. As the market becomes really good, the probability of a manager investing in his own project goes to zero, and hence from (4.20) we see that the probability of a rank change will tend to zero. In contrast, for a very negative market signal, this probability is strictly positive, since $\mathbb{P}(0,0|s^m) = 1$ and $\mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m) > 0$.[11]

From the above analysis, it is clear that the overall effect does not have to be monotonic in $s^m$. Hence we use simulations to illustrate the properties of the probability of interest as a function of the market signal, confirming also the observation in the aforementioned remark.[12]




On the y-axis we have the probability of change in rank, and on the x-axis the corresponding market signal. As it can been seen from the graph the total effect is hump-shaped in $s^m$, it is decreasing as the market signal becomes relatively large and also it is smaller when market conditions are good compared to when they are bad.

In the next section, we find empirical evidence supporting our results. This is done by constructing divisions in which each fund is allocated in accor-

[11] Intuitively the return of manager 1 may be much larger than that of manager 2 when they both invest in their own projects (either because one has high skill and the other has low or because one is just luckier than the other) and hence there is a positive probability there will be a change of ranks.

[12] For this simulation we set the parameters as: $\pi^1 = 0.6, \pi^2 = 0.601$, $\alpha^H = 0.16$, $\alpha^L = 0.1, \sigma = \nu = 0.35$, $f^1 = f^2 = 0.01$, $\sigma_m \nu_m = 0.25$, $\lambda_1 = \lambda_2 = 0.8$ and $\delta = 0.5$.

dance with their AUM. Subsequently, we calculate the proportion of funds that changed division from the beginning of each period to its end. Approximately, this measures the probability to which the above proposition refers.

## 3.4 Empirics

### 3.4.1 Data

The data used in this study comes from the Morningstar CISDM database. The time span of our sample is from January 1994 to December 2015. To mitigate survivorship bias we include defunct funds in the sample. We have created a larger group of strategies to accumulate the Morningstar's categories. All fund returns have been converted to USD (U.S dollars) using the exchange rates of each period separately. Observations of performance or assets under management, with more than 30 missing values, have been deleted. All observations are monthly. Our main variable of interest is flows, which gives the proportional in and out flows of the fund with respect to its assets under management. For the market return we consider the S&P 500 and as fund excess returns, the difference of the fund's return with the market. In particular, we use the corresponding Fama-French market factor obtained from the WRDS (or from Kenneth French's website at Darmouth). We also examine the relationship of alpha and beta of a fund as well as their relationship to the flows.

### 3.4.2 Empirical Evidence

The purpose of this section is to empirically test some of the assumptions as well as the results of our model and show that our model can be empirically supported by data. For simplicity we will use CAPM alpha and beta throughout this section, calculated using a 32 month period (which we will define in this section as *one period*).[13] Moreover we will refer to the log of the assets of a fund lagged by one period, simply as the fund's *assets*. First of all, our model assumes that investors get a signal about the market ($s^m$) before everyone else does. This would imply some form of market-timing. We first run

[13] We have also performed a robustness check using the 4-factor alphas and betas.

the following panel regression, with fixed effects:

$$\text{Beta}_t = \lambda_0 + \lambda_1 r_{m,t} + \lambda_2 \text{Assets}_{t-1} + \lambda_3 \text{Age}_t + d_i + \varepsilon_t$$

where $r_{m,t}$ is the period market return (described above) and $d_i$ corresponds to the fixed effects dummy (although the subscript $i$ for the fund has been suppressed in the rest of the variables). The results are shown below:

Table 3.4.1: Estimation results : Beta on Market Return.

The baseline model we run is summarised by beta $\sim r_m +$ assets $+$ controls.

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| $r_m$ | **0.03256**$^{*}$ | (0.01372) |
| assets | 0.01467$^{**}$ | (0.00508) |
| age | 0.00502$^{**}$ | (0.00144) |
| Intercept | 0.02430 | (0.08400) |

Significance levels : † : 10% $*$ : 5% $**$ : 1%

The positive and significant coefficient in front of the market return supports our model assumption (as well as with the prediction of Proposition 9 about over-investment), in the sense that it indicates that when markets are bull, it is more likely that managers choose to get higher exposure to the market. This is consistent with what we would observe if managers had market-timing abilities.

Another result of our model is that in equilibrium $l > h$. Given the definition of the cutoff equilibrium strategies described in (5), this leads to: $P(\beta = 1|L) > P(\beta = 1|H)$. If this is the case, we would expect to see in the data that funds with higher alpha, have on average lower betas, i.e they choose to invest on their idiosyncratic project since they benefit both from potential higher returns thanks to their superior alpha as well as from signalling their skill. Indeed this is the case. We are using the following cross-sectional baseline model, for the last date in our data, December 2015:[14]

$$\text{Alpha}_t = \lambda_0 + \lambda_1 \text{Beta}_t + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + \varepsilon_t,$$

where controls include the age and the strategy of the fund. As shown in Table 2 the coefficient of interest is negative, suggesting that more skilled managers pick a high beta less often.

Table 3.4.2: Cross-sectional Regression of Alphas on Betas and controls, $t =$ 12/2015.The baseline model we run is summarised by alpha $\sim$ beta + assets + controls.

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| beta | **-0.00958**$^{**}$ | (0.00085) |
| assets | 0.00006 | (0.00016) |
| age | 0.00001 | (0.00005) |
| strategy | 0.00003 | (0.00009) |
| Intercept | 0.00130 | (0.00284) |

Significance levels : $\dagger$ : 10% $\quad *$ : 5% $\quad **$ : 1%

Even more importantly, we want to test the second implication of Proposition 11. That is, we want to test whether the data suggest that the sensitivity of flows to performance is higher when beta is 0, or consequently is higher when markets are bear than when they are bull. We will measure the fund flows, as in Sirri and Tufano (1998):

$$\text{Flows}_t = \frac{TNA_t - (1 + R_t)TNA_{t-1}}{TNA_{t-1}}$$

where TNA is the total net assets and R is the return of the fund. We will

[14]We only include funds that report US dollars as their base currency.

use the simple return of the fund, $r_i$, as the measure of performance, as in Clifford et al. (2013). We think that this is the most appropriate measure of performance to test the predictions of our model. The following two tables[15] verify the above finding, and support our predictions[16]. First regression is a cross-sectional one for December 2015.

$$\text{AvFlows}_t = \lambda_1 r_{i,t} \cdot \text{Bigbeta}_t + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + \varepsilon_t$$

where *AvFlows* is the average flows of the previous period, $\text{Bigbeta} = 1_{\{\beta \geq 0.3\}}$, $r_{i,t}$ is the fund's period return and controls include the age, the strategy and the bigbeta dummy of the fund (the intercept $\lambda_0$ is just suppressed in the above equation).

Table 3.4.3: Flows on Performance and Beta, $t = 12/2015$

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| $r_i$·Bigbeta | **-0.12510**$^{**}$ | (0.03433) |
| Bigbeta | 0.01204 | (0.01522) |
| assets | -0.01437$^{**}$ | (0.00389) |
| strategy | 0.00352 | (0.00231) |
| age | -0.00133 | (0.00120) |
| Intercept | 0.25846$^{**}$ | (0.06906) |

Significance levels : † : 10% $*$ : 5% $**$ : 1%

The second table we are presenting is a panel regression with *fixed effects*, where we regress flows on the interaction of annual fund's performance and

[15]Since the funds selected in our model, are only between $\beta = \{0, 1\}$, thus making the implicit assumption that there is no short-selling of the market, we will exclude all observation with negative $\beta$, which are less than 15% of our sample.

[16]This result was only recently documented empirically in a paper by Franzoni and Schmalz (2013).

market return, including the usual controls. That is, our baseline model is:

$$\text{AvFlows}_t = \lambda_1 r_{i,t} \cdot r_{m,t} + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + d_i + \varepsilon_t$$

where controls include the fund's beta and the period return of the market and of the fund itself.

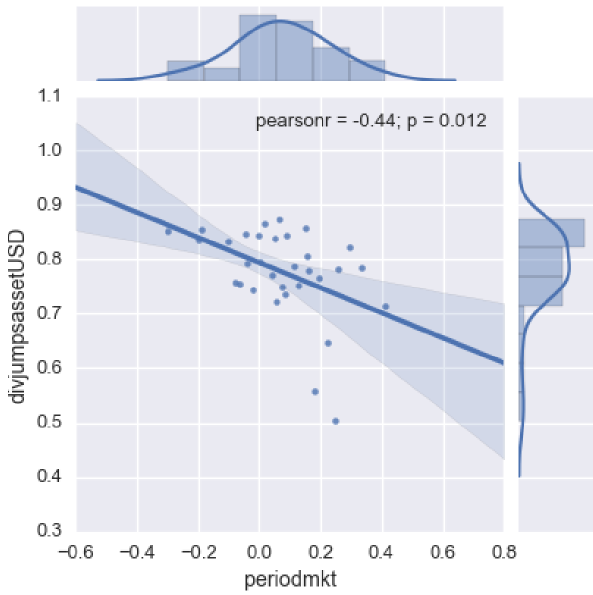
Table 3.4.4: Flows on the interaction of Fund Performance and Market Return

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| $r_i \cdot r_m$ | **-0.15297**$^{**}$ | (0.03697) |
| beta | 0.00423 | (0.00648) |
| $r_i$ | 0.07538$^{**}$ | (0.02036) |
| $r_m$ | 0.02828$^{**}$ | (0.00955) |
| assets | -0.02718$^{**}$ | (0.00224) |
| Intercept | 0.47264$^{**}$ | (0.03964) |

Significance levels : $\dagger$ : 10% $\quad *$ : 5% $\quad **$ : 1%

In both cases we can see that the coefficient of interest is significantly *negative*. The interpretation of these two regressions is the following: the first one shows that funds with higher beta are not judged so much on their performance; that is the higher the beta, the less important the flow performance relationship. On the other hand, the second table supports the statement that the sensitivity of fund flows to performance depends on the state of the market and more specifically it is decreasing on the market return. Under the predictions of our model, these two results are almost equivalent, and we indeed get that the coefficient in both cases is negative and significant, thus supporting one of our main results as well.

Finally, we want to provide some empirical evidence relevant to the discussion on the competition of funds. Namely, we find support for Remark 1, by demonstrating that the probability of changes in the ranking of funds, with respect to their AUM, is higher under adverse market conditions. To achieve this a new variable is constructed. First, the sample is separated in periods of eight months, so that we have thirty periods in total. For each one, seventy divisions (clusters of funds) are created. Funds are allocated in those divisions according to the size of their AUM at the end of each period.[17] Then we define $divjumpassetUSD_t$ as the percentage of funds that changed division from the beginning to the end of period $t$. We are careful to only compare funds that were active during the whole duration of each period. Also, we only consider the US universe of funds to avoid introducing noise created from fluctuations in the exchange rates.




On the y-axis we have our constructed measure of changes between divisions $divjumpassetUSD_t$, and on the x-axis the corresponding total return of the market portfolio during the same period. As it can been seen from the

[17]Our methodology closely follows previous work done by Marathe and Shawky (1999) and Nguyen-Thi-Thanh (2010).

graph there appears to be a negative relationship between the two, which is also statistically significant. Note that this is just an indication of the relationship between the rank of funds and the market conditions, under a simple linear regression, and thus it does not capture any second order effects (or a hump-shaped relationship). Hence, our prediction in Remark 1 is only supported by weak evidence, but we believe that there is much more to explore in this direction in the future.

## 3.5 Extension: Unobservable Investment Decision

In this section, we want to extend our model, and investigate the equilibrium where the investment decision of the fund managers cannot be observed by the investors. In this case, investors use the return of the fund managers' to both update their beliefs about managers skill and to also understand whether or not they invested in their own project. In reality, it is indeed the case that investors do not know exactly the exposure of a fund manager to the systematic risk. Instead they use a history of data of the fund return's comovement with the market return to infer the fund's statistical *beta*. Since the model we are examining here is static, the assumption in this section is that this inference is only made based on the proximity of the market return to the fund's return.

The model considers only one period and it remains the same as before, apart from a few changes outlined below. Firstly, an additional error $\varepsilon$ has been introduced in order to make the manager's choice of investment unobservable by the investors. (Note, that without this tracking error, investors could perfectly observe the decision of managers based on whether or not $r = m$.)

Hence our model becomes:

$$\begin{aligned} r &= (1-\beta)\,a + \beta\,(m+\varepsilon) \\ a &\sim \mathcal{N}(\alpha, \sigma^2\,) \\ m &\sim \mathcal{N}(\mu\,, \sigma_m^2) \\ \varepsilon &\sim \mathcal{N}(0\,, \sigma_\varepsilon^2) \end{aligned} \tag{3.5.1}$$

As before we study only the simple binary case where $\beta \in \{0, 1\}$. The rest of the notation and ideas remain unchanged.

The posterior distribution of $r$, conditional on $(\alpha, \beta, s, s^m)$ is given by

$$\begin{aligned} r \mid \alpha, \beta, s, s^m &\sim \mathcal{N}\Big(\bar{r}(\alpha, \beta, s, s^m),\, \bar{\sigma}^2(\beta)\Big) \\ \bar{r}(\alpha, \beta, s, s^m) &\equiv (1-\beta)[(1-\psi)\alpha + \psi s] \\ &\qquad + \beta[(1-\psi_m)\mu + \psi_m s^m] \\ \bar{\sigma}^2(\beta) &\equiv (1-\beta)^2 \psi \nu^2 + \beta^2(\psi_m \nu_m^2 + \sigma_\varepsilon^2) \end{aligned} \tag{3.5.2}$$

Our goal is to study whether a monotonic cutoff equilibrium (introduced in the previous sections) exists under this alternative assumption. We believe that only such an equilibrium would be interesting and realistic to serve for further study. We move on to find a closed-form expression for the ex-post reputation $\varphi$, which is given by the following lemma.

**Lemma 6.** *The manager's posterior reputation is given by*

$$\varphi(r, m, s^m) = \left(1 + \frac{1-\pi}{\pi} \frac{\rho\big(r, L, l(s^m)\big)}{\rho\big(r, H, h(s^m)\big)}\right)^{-1}, \tag{3.5.3}$$

*where*

$$\rho(r, \alpha, c) = \Phi\left(\frac{r - c(1+\psi) + \alpha\psi}{\nu\sqrt{1+\psi}}\right) \times \frac{\phi\left(\frac{r-\alpha}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}} + \Phi\left(\frac{c-\alpha}{\nu}\right) \frac{\phi\left(\frac{r-m}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}, \tag{3.5.4}$$

*Proof.* In Appendix 3.9. □

Using the above lemma, we can now see whether this model can provide us with an equilibrium where the reputation $\varphi(r, m, s^m)$ is increasing in $r$. In fact, we get the following proposition:

**Proposition 12.** *A monotonic equilibrium under unobservable beta does not exist.*

*Proof.* In Appendix 3.9. □

What this proposition shows is that the reputation $\varphi(r, m, s^m)$ cannot always be increasing in $r$ under the assumption that investors do not observe the investment choices. The reason, as illustrated in the proof in the Appendix, is that in a very bull market, the good performance of a manager may make investors believe that it is more likely that he was following the market, and thus may lead to a loss of reputation. That is to say that the assumption of unobservable investment choice under a static setting can lead us to counter-intuitive equilibrium properties. We believe that in future research it could be interesting to study this realistic case under a dynamic setting where the inference of beta will be indeed based on the co-movement of the market return with the fund's return.

## 3.6 Conclusions

The role of financial intermediaries and their characteristics has been greatly explored in the recent empirical literature. In this article, we have developed a theoretical model that describes how the strategic investment decisions of fund managers is influenced by their career concerns. To sum up our argument, these managers will tend to over-invest in market neutral strategies as a way to signal their ability. Moreover, we have described how managers' reputation depends on the market conditions; in particular, we find that the sensitivity of flows to performance is higher in bear markets than in bull markets and we discuss the competition between funds, measured by the changes in their rankings, as a function of the market conditions. Our model entails predictions about some directly observable fund characteristics such as their size and fees, as well as some indirectly observable quantities such as their reputation or their investment behavior depending on their signals. In our empirical section, we have managed to find support for many of the assumptions as well as predictions of our model. Moreover, we have extended our model to include the case when the manager's investment decision is not observable by the investors.

There are many ways forward with this research. The results of this model do not depend on the specific factor which funds use when they are tracking an index; one, may try to apply the same logic in funds that use factors other than the market return and test the corresponding empirical predictions. Also, using a slightly different interpretation of the investor's decision between allocating funds to a manager or to the market, one could think of an investor choosing between an active and a passive fund and use the closed form solution for the fund's size, to see how the relative (total) size of the passive and active funds, depends on the market conditions.

## 3.7 Appendix: Omitted Proofs

***Proof of Lemma 4.*** Using (3.3.3) it is ease to argue that both idiosyncratic and index tracking strategies have to be played with positive probability. This is because the effect of the reputation $\varphi_\beta(\cdot)$ on the manager's payoff is bounded, whereas that of current return $r$ is not. But this implies that $\varphi_0(\cdot)$ is calculated using Bayesian updating, and as a result it cannot be a function of $r$, since in this case $r$ provides no information on the manager's ability $\alpha$.

Fix $s^m$, then the manager's expected payoff while investing in an index tracking strategy $\beta = 1$ is not a function of $s$. On the other hand, her payoff under the idiosyncratic strategy is a function of $r$. In particular, it follows from the definition of monotonic equilibria that this is increasing in $s$, which proves that the manager's equilibrium strategy is a cut-off one, as presented in (3.3.4).

In addition, the indifference condition that defines $h(s^m)$ is

$$\mathbb{E}_r\Big[a \,+\, \delta\cdot\lambda\cdot\log[\varphi_0(r,s^m)\cdot(u^H-u^L)\,+\,u^L\,]\;\Big|\; s=h(s^m),\,\alpha=H\Big] \\ =\;\mathbb{E}_r\Big[m\,+\,\delta\cdot\lambda\cdot\log[\varphi_0(s^m)\cdot(u^H-u^L)\,+\,u^L\,]\;\Big|\;s^m\Big]$$

while the one that defines $l(s^m)$ is

$$\mathbb{E}_r\Big[a \,+\, \delta\cdot\lambda\cdot\log[\varphi_0(r,s^m)\cdot(u^H-u^L)\,+\,u^L\,]\;\Big|\; s=l(s^m),\,\alpha=L\Big] \\ =\;\mathbb{E}_r\Big[m\,+\,\delta\cdot\lambda\cdot\log[\varphi_0(s^m)\cdot(u^H-u^L)\,+\,u^L\,]\;\Big|\;s^m\Big]$$

But the right hand sides of the above two equations are the same, hence the two expressions on the left hand sides are equal. Therefore, the two conditional normals that are used in the two left hand sides have to be the same,[18] which implies that

$$(1-\psi)\cdot H\,+\,\psi\cdot h(s^m)\;=\;(1-\psi)\cdot L\,+\,\psi\cdot l(s^m)$$

from which (3.3.5) follows. □

***Proof of Lemma 5***. The time subscripts is suppressed, when no ambiguity is created. The same is true for the signal $s^m$ in the cutoffs $h(s^m)$ and $l(s^m)$. To find the posterior $\varphi_0(r)$ calculate

$$\mathbb{P}\left(r,\beta=0\,\middle|\,s^m,H\right)\;=\;\mathbb{P}\left(r\,\middle|\,\beta=0,s^m,H\right)\times\mathbb{P}\left(\beta=0\,\middle|\,s^m,H\right),$$

where

$$\mathbb{P}\left(\beta=0\,\middle|\,s^m,H\right)=\mathbb{P}(s\geq h\,|\,s^m,H)=\Phi\left(-\frac{h-H}{\nu}\right), \tag{3.7.1}$$

[18] Otherwise, one random variable would stochastically dominate the other, since they have the same variance. But then the expectations in the 2 left hand sides could not be the same, since $\varphi_0(r,s^m)$ is increasing in $r$.

and

$$\mathbb{P}\left(r\,|\,\beta=0,s^m,H\right)=\int_h^\infty \phi\left(\frac{r-(1-\psi)H-\psi s}{\sqrt{\psi}\,\nu}\right)\times\frac{1}{\sqrt{\psi}\,\nu}\phi\left(\frac{s-H}{\nu}\right)\frac{1/\nu}{\Phi\left(-\frac{h-H}{\nu}\right)}\,\mathrm{d}s$$

Hence, substituting gives that

$$\mathbb{P}\left(r,\beta=0\,|\,s^m,H\right)\;=\;\int_h^\infty \phi\left(\frac{r-(1-\psi)H-\psi s^i}{\sqrt{\psi}\,\nu}\right)\frac{\phi\left(\frac{s-H}{\nu}\right)}{\sqrt{\psi}\,\nu^2}\,\mathrm{d}s. \tag{3.7.2}$$

Let $\tilde{s}=(s-H)/\nu$, then the above becomes

$$\begin{aligned}
&\int_{\frac{h-H}{\nu}}^\infty \phi\left(\frac{r-H}{\sqrt{\psi}\,\nu}-\sqrt{\psi}\,\tilde{s}\right)\frac{\phi(\tilde{s})}{\sqrt{\psi}\,\nu}\,\mathrm{d}\tilde{s}\\
&=\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\int_{\frac{h-H}{\nu}}^\infty \phi\left(\frac{\tilde{s}-\frac{r-H}{\nu(1+\psi)}}{1/\sqrt{1+\psi}}\right)\sqrt{1+\psi}\,\mathrm{d}\tilde{s} \qquad (3.7.3)\\
&=\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\Phi\left(\frac{r-h(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right).
\end{aligned}$$

Repeat the same process to find $\mathbb{P}\left(r\,|\,\beta=0,s^m,L\right)$ and observe that it follows from Bayes' rule that

$$\varphi_0(r)=\left(1+\frac{1-\pi}{\pi}\frac{\mathbb{P}\left(r,\beta=0\,|\,s^m,L\right)}{\mathbb{P}\left(r,\beta=0\,|\,s^m,H\right)}\right)^{-1}, \tag{3.7.4}$$

from which the provided formula follows. To derive $\varphi_1$ use Bayes' rule to get that

$$\varphi_1=\left(1+\frac{1-\pi}{\pi}\frac{\mathbb{P}\left(\beta=1\,|\,s^m,L\right)}{\mathbb{P}\left(\beta=1\,|\,s^m,H\right)}\right)^{-1}, \tag{3.7.5}$$

where $\mathbb{P}\left(\beta=1\,|\,s^m,\alpha\right)=1-\mathbb{P}\left(\beta=0\,|\,s^m,\alpha\right)$, which has been derived above. □

To prove our existence theorem we need to following three lemmas.

**Lemma 7.** *If $M(\cdot)$ is the normal hazard function, then for $a \geq b$ we have,*

$$M(a) - M(b) \leq a - b \tag{3.7.6}$$

***Proof*.** Since the hazard function is a continuous function, we can use the Mean Value Theorem, which says that for any $a > b$ there exists a $\xi \in (a, b)$ such that $M(a) - M(b) = M'(\xi)(a - b)$. Therefore, it is sufficient to prove that $M'(\xi) < 1$ for any $\xi$. To prove that, note that $M(\cdot)$ is convex, and hence $M'(\cdot)$ is increasing, so it would be sufficient to prove that $\lim_{x\to\infty} M'(x) = 1$. Now we use the following inequality for the normal hazard function. We know that for $x > 0$,

$$x < M(x) < x + \frac{1}{x} \tag{3.7.7}$$

But this easily implies that $M(x)$ has $x$ as its asymptote as $x \to \infty$ (that is $\lim_{x\to\infty} M(x) - x = 0$). Finally this implies that $\lim_{x\to\infty} M'(x) = 1$ and this completes the proof (note the limit exists because $M'(\cdot)$ is increasing and bounded, as $M'(x) = M(x)(M(x) - x) < 1 + \frac{1}{x^2} < 2$). $\square$

**Lemma 8.** *The time subscripted is suppressed. A sufficient condition for $\varphi_0(r, s^m)$ to be increasing in the manager's performance $r$ is that*

$$(H - L) \cdot \frac{1 - \psi}{\psi} \;\geq\; l(s_1^m) - h(s_1^m). \tag{3.7.8}$$

***Proof*.** Suppress inputs $(r, s^m)$, and super/sub-scripts. Differentiating gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{\varphi(1-\varphi)}{\nu\sqrt{1+\psi}}\left[-\frac{H-L}{\nu\psi\sqrt{1+\psi}} + M\left(-\frac{r - l(1+\psi) + L\psi}{\nu\sqrt{1+\psi}}\right) - M\left(-\frac{r - h(1+\psi) + H\psi}{\nu\sqrt{1+\psi}}\right)\right] \tag{3.7.9}$$

Let

$$\begin{aligned} \delta^L &= l(1+\psi) - L\psi \\ \delta^H &= h(1+\psi) - H\psi, \end{aligned} \tag{3.7.10}$$

then the above is positive if and only if

$$\frac{H-L}{\nu\psi\sqrt{1+\psi}} \geq M\left(\frac{\delta^L - r}{\nu\sqrt{1+\psi}}\right) - M\left(\frac{\delta^H - r}{\nu\sqrt{1+\psi}}\right) \tag{3.7.11}$$

But using Lemma 7 we see that the right hand side is bounded above by

$$\frac{\delta^L - \delta^H}{\nu\sqrt{1+\psi}} = \frac{(l-h)(1+\psi)+(H-L)\psi}{\nu\sqrt{1+\psi}}. \tag{3.7.12}$$

Hence, a sufficient condition for the inequality to hold is that

$$\frac{H-L}{\psi} \geq (l-h)(1+\psi)+(H-L)\psi \;\Leftrightarrow\; (H-L)\frac{1-\psi}{\psi} \geq l-h. \tag{3.7.13}$$

□

**Lemma 9.** *For $c > 0$, let*

$$\mu(x) = \left(1 + c\frac{\Phi(a_0 + b\,x)}{\Phi(a_1 + b\,x)}\right)^{-1}. \tag{3.7.14}$$

*Suppose $b > 0$, then $\mu'(x) > 0 \Leftrightarrow a_1 < a_0$, whereas $b < 0$ implies that $\mu'(x) > 0 \Leftrightarrow a_1 > a_0$.*

***Proof***. Differentiating gives

$$\mu'(x) = -b\mu(x)[1-\mu(x)] \times [M(-a_0 - b\,x) - M(-a_1 - b\,x)]\,.$$

Then the statement follows simple from the fact that $M(\cdot)$ is increasing . □

***Proof of Proposition 7***. Suppress time subscript $t$. Also suppress the signal $s^m$ in the cutoffs $h(s^m)$ and $l(s^m)$, and in the reputations $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$.

We start by proving existence. As we have argued in Lemma 4, in any monotonic equilibrium the optimal strategy of a high and low type manager is to pick $\beta = 0$ whenever her signal $s$ is above the cutoffs $h$ and $l$, respectively. In addition, another necessary implication is that $h$ and $l$ satisfy (3.3.5).

But then Lemma 8 together with (3.3.5) give that $\varphi_0(r)$ is indeed increasing in $r$. Hence, the manager's best response to the functional forms of $\varphi_0(\cdot)$ and $\varphi_1$ as given Lemma 5 is to indeed use the cutoff strategies that Lemma 4 describes.

All that remains to prove existence is to shown that those cutoffs always exists. To do this note that the manager's payoff maximisation problem when picking the first period's beta is as given in (3.3.3). Let her expected payoff when picking $\beta = 0$ be denoted by

$$v_0(s,\alpha) \;=\; (1-\psi)\cdot\alpha + \psi\cdot s \;+\; \delta\cdot\lambda\cdot\mathbb{E}_r\left[\log\left(\varphi_0(r)(u^H - u^L) + u^L\right) \;\middle|\; s,\alpha\right],$$

whereas for $\beta = 1$ this becomes

$$v_1 \;=\; (1-\psi_m)\cdot\mu \;+\; \psi_m\cdot s^m \;+\; \delta\cdot\lambda\cdot\log\left(\varphi_1(u^H - u^L) + u^L\right).$$

But then $v_1$ is bounded, while $v(s,\alpha)$ goes from minus to plus infinity. Hence the manager uses both the low and high beta strategy depending on $s$. Next, we provide the equation that defines those cutoffs. Rewrite $l$ as a function of $h$ according to

$$l(h) - L = h - H + \frac{H-L}{\psi},$$

and substitute this equality in $\varphi_0(r)$ and $\varphi_1$ to obtain the following two functions, in which only $h$ appears out of the two equilibrium cutoffs. Substituting in $\varphi_0(r)$ gives

$$\tilde{\varphi}_0(r,h) \;=\; \left(1 + \frac{1-\pi}{\pi}\cdot\rho(r)\cdot\frac{\Phi\left(\frac{r-h\cdot(1+\psi)+H\psi-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r-h\cdot(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right)}\right)^{-1}, \qquad (3.7.15)$$

where $h$ is introduced as an input of the function. Similarly, substituting in $\varphi_1$ gives

$$\tilde{\varphi}_1(h) \;=\; \left(1 + \frac{1-\pi}{\pi}\cdot\frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)}\right)^{-1} \qquad (3.7.16)$$

Then the cutoff $h$ is given by the high types indifference condition $v_0(h,H) =$

$v_1$, which using the above notation becomes

$$\begin{aligned}
&\delta \cdot \lambda \cdot \int \log \left[\tilde{\varphi}_0(r,h)(u^H - u^L) + u^L\right] \cdot \phi\left(\frac{r-(1-\psi)H-\psi h}{\sqrt{\psi\nu}}\right)\frac{1}{\sqrt{\psi\nu}}\,\mathrm{d}r \\
= \; & \delta \cdot \lambda \cdot \log\left[\tilde{\varphi}_1(h)(u^H - u^L) + u^L\right] + (1-\psi_m)\cdot\mu + \psi_m \cdot s^m - (1-\psi)\cdot H - \psi \cdot h
\end{aligned} \tag{3.7.17}$$

where $\phi(\cdot)$ is the density of the standard normal distribution. To prove existence we demonstrate that (3.7.17) equation has at least one solution. Let $LHS(h)$ denote the left hand side of (3.7.17), $RHS(h)$ its right hand side, and $\Delta(h) = LHS(h) - RHS(h)$ their difference. Observe that all the parts of the above equation apart from the last line are bounded. As a result,

$$\begin{aligned}
\lim_{h\to-\infty} \Delta(h) &= -\infty \\
\lim_{h\to+\infty} \Delta(h) &= +\infty.
\end{aligned} \tag{3.7.18}$$

Then it follows from the continuity of this function that there exists at least one point where $\Delta(h) = 0$. Hence we have proven existence.

Next we show that (3.3.8) is indeed a sufficient condition for uniqueness. In particular, we will argue that (3.3.8) implies that $\Delta(h)$ is increasing in $h$. First, note that $LHS(h)$ is increasing in $h$, because $\tilde{\varphi}_0(r,h)$ is increasing in both $r$ and $h$. We have already argued why this is true for $r$. For $h$ the claim is a direct implication of Lemma 9.

Hence it suffices to identify a condition for $RHS(h)$ to be decreasing. Lemma 9 implies that $\tilde{\varphi}_1(h)$ is increasing in $h$. This is the opposite monotonicity, however we can use the fact that the following expression has a relatively simple upper bound

$$\frac{\mathrm{d}}{\mathrm{d}h}\log\left[\tilde{\varphi}_1(h)(u^H-u^L)+u^L\right] = \frac{\tilde{\varphi}_1(h)[1-\tilde{\varphi}_1(h)]/\nu}{\tilde{\varphi}_1(h)+\frac{u^L}{u^H-u^L}}\times\left[M\left(-\frac{h-H}{\nu}\right)-M\left(-\frac{l(h)-L}{\nu}\right)\right]$$

$$\leq \frac{1}{\nu}\left[M\left(-\frac{h-H}{\nu}\right)-M\left(-\frac{l(h)-L}{\nu}\right)\right] = \frac{1}{v}\int_{\frac{L-(1-\psi)H}{\psi}}^{H} M'\left(\frac{x-h}{\nu}\right)\mathrm{d}x \leq \frac{H-L}{\psi\nu^2} \tag{3.7.19}$$

Hence, a sufficient condition for the right hand side to be decreasing, which will imply uniqueness, is that

$$\delta\lambda^i\frac{H-L}{\psi\nu^2} \;\leq\; \psi\,,$$

which equivalently gives (3.3.8). □

***Proof of Proposition 8***. We know that $\varphi_0(r,s^m)$ is increasing in $r$. Hence, it suffices to prove the conjectured result for $r\to-\infty$. The dependence on $s^m$ is suppressed. Let $k=-h(1+\psi)+H\psi$. To find the limit $\lim\limits_{r\to-\infty}\varphi_0(r)$ we first need to calculate.

$$\lim_{r\to-\infty}\frac{\Phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)\exp\left(\frac{2(H-L)r-(H^2-L^2)}{2\nu^2\psi(1+\psi)}\right)}. \tag{3.7.20}$$

Because both the numerator and the denominator go to zero as $r$ goes to minus infinity this limit becomes

$$\frac{e^{\frac{H^2-L^2}{2\nu^2\psi(1+\psi)}}\lim\limits_{r\to-\infty}\frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\nu\sqrt{1+\psi}}}{e^{\frac{(H-L)r}{\nu^2\psi(1+\psi)}}\times\left[\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)\frac{H-L}{\nu^2\psi(1+\psi)}+\frac{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\nu\sqrt{1+\psi}}\right]}.$$

In addition, algebra implies the following simplification

$$\frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)} = \exp\left(\frac{2(r+k)-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right). \tag{3.7.21}$$

This in turn gives

$$e^{-\frac{(H-L)r}{\nu^2\psi(1+\psi)}}\frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}=\exp\left(\frac{2k-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right).$$

Hence the limit becomes

$$\exp\left(\frac{2k+H+L-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right)\times\lim_{r\to-\infty}\left(\frac{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}\frac{H-L}{\nu\psi\sqrt{1+\psi}}+1\right)^{-1},$$

where

$$\lim_{r\to-\infty}\frac{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}=\lim_{x\to\infty}\frac{1-\Phi(x)}{\phi(x)}=0 \tag{3.7.22}$$

Hence, substituting $k$ we obtain that

$$\lim_{r\to-\infty}\varphi_0(r)=\left(1+\frac{1-\pi}{\pi}\exp\left[\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2}\right]\right)^{-1}.$$

Next, we want to show that the above is greater than $\varphi_1(r)$ for every $h$. This holds if and only if

$$\exp\left[\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2}\right]<\frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)} \tag{3.7.23}$$

which can equivalently be rewritten as

$$\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2}<\log\frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)}. \tag{3.7.24}$$

Differentiating the left hand side minus the right hand side we get

$$-\frac{H-L}{\psi\nu^2}+\frac{1}{\nu}M\left(\frac{H-h}{\nu}\right)-\frac{1}{\nu}M\left(\frac{H-h}{\nu}-\frac{H-L}{\nu\psi}\right)\leq-\frac{H-L}{\psi\nu^2}+\frac{H-L}{\psi\nu^2}=0 \tag{3.7.25}$$

Hence it suffices to check that

$$\lim_{h\to-\infty}\frac{\Phi\left(\frac{h-H}{\nu}\right)}{\exp\left(\frac{(H-L)h}{\psi\nu^2}\right)\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}\leq\exp\left[\left(\frac{H-L}{2\psi}-H\right)\frac{H-L}{\psi\nu^2}\right].$$

Similar argumentation with above gives that the limit on the left hand side becomes

$$\begin{aligned}\lim_{h\to-\infty}&\frac{\phi\left(\frac{h-H}{\nu}\right)}{\exp\left(\frac{(H-L)h}{\psi\nu^2}\right)\phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}\\&=\lim_{h\to-\infty}\exp\left(\frac{2(h-H)+\frac{H-L}{\psi}}{2\nu^2}\frac{H-L}{\psi}-\frac{(H-L)h}{\psi\nu^2}\right)\\&\qquad=\exp\left[\left(\frac{H-L}{2\psi}-H\right)\frac{H-L}{\nu^2\psi}\right].\end{aligned} \tag{3.7.26}$$

Hence, the above inequality holds. □

***Proof of Proposition 9***. The Input $s^m$ is suppressed. First, note that $h$ is the solution of (3.7.17), that is the solution of $\Delta(h)=0$, where $\Delta(h)$ is defined under the equation as the difference of its left hand side from its right hand side. Second, the optimal cutoff under no career concerns for the high type $c(H)$ is the one that corresponds to the solution of this equation for $\delta=0$, as this corresponds to the case when the next period is irrelevant. Let $h(\delta)$ denote the solution of (3.7.17) as a function of $\delta$. Then it follows from the implicit function theorem that

$$\frac{\mathrm{d}h(\delta)}{\mathrm{d}\delta}=-\frac{\partial\Delta(h)/\partial\delta}{\partial\Delta(h)/\partial h}\bigg|_{h=h(\delta)}. \tag{3.7.27}$$

But it follows from the limits calculated in (3.7.18) that the unique monotonic equilibrium needs to have $\partial\Delta(h)/\partial h > 0$. Moreover, calculating the derivative on the numerator for some generic $h$ gives

$$\frac{\partial\Delta(h)}{\partial\,\delta} = \lambda\,\mathbb{E}_r\left[\log\left[\tilde{\varphi}_0(r,h)(u^H - u^L) + u^L\right] - \log\left[\tilde{\varphi}_1(h)(u^H - u^L) + u^L\right]\,\middle|\, s = h, H\right],$$

but it follows from Proposition 8 that this is positive, because the difference inside the expectation is positive for every $h$. As a result, for every $\delta \geq 0$ we get that $\mathrm{d}h(\delta)/\mathrm{d}\delta < 0$, which through (3.3.5) implies the same for the cutoff used by the low type.

Finally, note that $\lambda$ and $\delta$ enter (3.7.17) in exactly the same way, hence the same result can be stated for $\lambda$. □

**Lemma 10.** *In the unique monotonic equilibrium, for every prior reputation $\pi > 1/2$ there exists a lower bound $\bar{s}^m(\pi)$, defined as the solution of $\varphi_1(s^m) = 1/2$, such that for every $s^m > \bar{s}^m$ we have $\varphi_1(s^m) > 1/2$, and $\bar{s}^m(\pi^i)$ is increasing in $\pi$.*

*In addition, for every $s^m \geq \bar{s}^m(\pi)$ the cutoffs $h(s^m)$ and $l(s^m)$ are increasing in $\pi$, and the same is true for the posterior reputations $\varphi_0^(r, s^m)$ and $\varphi_1(s^m)$ .*

***Proof.*** In the proof of Proposition 7 is has been shown that in the unique monotonic equilibrium there exists $\tilde{\varphi}_1$ such that $\varphi_1(s^m) = \tilde{\varphi}_1[h(s^m)]$, and its functional form is given in (3.7.16). Moreover, it is an immediate implication of Lemma 9 that this is increasing in $h$, and it is ease to verify that

$$\lim_{h\to+\infty} \tilde{\varphi}_1(h) = \pi. \tag{3.7.28}$$

In addition, it follows from (3.7.17), which defines $h(s^m)$, that

$$(1+\psi)H + \psi h(s^m) + \delta\lambda \log\left(\frac{u^H}{u^L}\right) \geq (1-\psi_m)\mu + \psi_m s^m.$$

This provides a lower bound for $h(s^m)$, which is in an increasing function of

$s^m$, and shows that

$$\lim_{s^m \to +\infty} h(s^m) \;=\; +\infty, \tag{3.7.29}$$

from which the existence of the cutoffs follows. It monotonicity follows from using the implicit function theorem on the equation that defines it

$$\tilde{\varphi}_1\big[\pi, h\big(\bar{s}(\pi)\big)\big] \;=\; 1/2, \tag{3.7.30}$$

where note that $\tilde{\varphi}_1$ is increasing in both $\pi$ and $h$, and it has been argued in Proposition 10 that $h(\cdot)$ is also an increasing function.

For the second statement, it follows from (3.3.5) that is suffices to prove it for $h(s^m)$. Using the implicit function theorem on (3.7.17) we get that

$$\frac{\mathrm{d}h}{\mathrm{d}\pi} \;=\; -\frac{\partial\Delta/\partial\pi}{\partial\Delta/\partial h}, \tag{3.7.31}$$

where direct differentiation gives $\partial\Delta/\partial h = \psi > 0$ and that

$$\frac{\partial\Delta}{\partial\pi} \;=\; \frac{\delta\,\lambda}{\pi(1-\pi)}\mathbb{E}_r\left[\frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\tilde{\varphi}_0 + \frac{u^L}{u^H - u^L}} - \frac{\tilde{\varphi}_1(1-\tilde{\varphi}_1)}{\tilde{\varphi}_1 + \frac{u^L}{u^H - u^L}} \,\middle|\, s = h, H\right], \tag{3.7.32}$$

where the inputs $r$ and $s^m$ have been suppressed. Some basic calculus shows that for every $\tilde{\varphi} \in [1/2, 1]$ the ratio

$$\frac{\tilde{\varphi}(1-\tilde{\varphi})}{\tilde{\varphi} + \frac{u^L}{u^H - u^L}} \tag{3.7.33}$$

is decreasing in $\tilde{\varphi}$. Moreover, we have from Proposition 8 that $\tilde{\varphi}_0(r,h) > \tilde{\varphi}_1(h)$ for every $r \in \mathbb{R}$. But we already showed that $\tilde{\varphi}_1(h) > 1/2$ for every $s^m \geq \bar{s}^m(\pi)$. Hence, we get that $\partial\Delta/\partial\pi < 0$, which implies the second statement.

Finally, the third statement follows trivially from noting that the direct derivative of both the posteriors with respect to $\pi$ is positive, and the fact that both are increasing in $h(s^m)$, implied by Lemma 9, for which it has already being argued that it is increasing in $\pi$. $\square$

***Proof of Proposition 11.*** First, consider the investment decision of a high type manager, for which the probability of choosing the low beta strategy, conditional on the market signal $s^m$, is

$$\mathbb{P}\big(\beta = 0 \,\big|\, s^m\big) \;=\; \mathbb{P}\big(s \geq h(s^m) \,\big|\, s^m\big) \;=\; \mathbb{P}\big(h^{-1}(s) \geq s^m \,\big|\, s^m\big), \tag{3.7.34}$$

since it was shown in Proposition 10 that $h(\cdot)$ is increasing. Moreover, for given $s^m$ the distribution of $m$ is normal and is given by

$$m \,|\, s^m \;\sim\; \mathcal{N}\Big((1-\psi_m)\mu + \psi_m s^m \,,\, \psi_m \nu_m^2\Big). \tag{3.7.35}$$

Let $\tilde{m} = [m - (1-\psi_m)\mu]/\psi^m$. Then

$$\tilde{m} \,|\, s^m \;\sim\; \mathcal{N}\Big(s^m \,,\, \nu_m^2/\psi_m\Big), \tag{3.7.36}$$

while the ex-ante distribution of $s^m$ is

$$s^m \;\sim\; \mathcal{N}\big(\mu \,,\, \sigma_m^2 + \nu_m^2\big), \tag{3.7.37}$$

As a result using again the properties of Bayesian updating with normal distributions we get that

$$s^m \,|\, \tilde{m} \;\sim\; \mathcal{N}\left(\tilde{\psi}\mu + (1-\tilde{\psi})\tilde{m} \,,\, \frac{\tilde{\psi}\nu_m^2}{\psi_m}\right), \tag{3.7.38}$$

where $\tilde{\psi}_m = (\sigma_m^2 + \nu_m^2)/(\sigma_m^2 + \nu_m^2 + v_m^2/\psi_m)$. Hence for for every $\hat{m}$, $m$ such that $\hat{m} > m$, the distribution of corresponding normal that generates $s^m$ conditional on $\hat{m}$ first order stochastically dominates the one of $m$. This immediately implies that

$$\mathbb{P}\big(\beta = 0 \,\big|\, \hat{m}\big) \;<\; \mathbb{P}\big(\beta = 0 \,\big|\, m\big). \tag{3.7.39}$$

Hence under better observed market conditions the manager is less likely to have chosen to invest in her idiosyncratic strategy. The second statement of the proposition follows from noting that

$$\frac{\mathrm{d}\varphi_0(r, s^m)}{\mathrm{d}r} \;\geq\; 0 \;=\; \frac{\mathrm{d}\varphi_1(s^m)}{\mathrm{d}r}, \tag{3.7.40}$$

To calculate the left derivative it is more convenient to use the equivalent $\tilde{\varphi}_0$ function from the proof of proposition 7. The derivative of this can be calculated in a manner similar to that used in the proof of Lemma 8 to be

$$\frac{\mathrm{d}\tilde{\varphi}_0(r,h)}{\mathrm{d}r} = \frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\nu\sqrt{1+\psi}}\left[\frac{H-L}{\nu\psi\sqrt{1+\psi}} - \int_{\underline{x}}^{\overline{x}} M'\left(x + h\sqrt{1+\psi}/\nu\right)\mathrm{d}x\right],$$

where $M(\cdot)$ is the hazard rate of the standard normal distribution,

$$\underline{x} = -\frac{r+H\psi}{\nu\sqrt{1+\psi}} \quad \text{and} \quad \overline{x} = \underline{x} + \frac{(H-L)/\psi}{\nu\sqrt{1+\psi}}. \tag{3.7.41}$$

Next we want to show that this derivative is decreasing in $s^m$. This appears in $\tilde{\varphi}_0$ only indirectly through the cutoff $h(s^m)$, which it has already being shown to be an increasing function. Hence calculate

$$\frac{\mathrm{d}^2\tilde{\varphi}_0(r,h)}{\mathrm{d}r\mathrm{d}h} = \frac{1-2\tilde{\varphi}_0}{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}\left(\frac{\mathrm{d}\tilde{\varphi}_0(r,h)}{\mathrm{d}r}\right)^2 - \frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\nu^2}\int_{\underline{x}}^{\overline{x}} M''\left(x + h\sqrt{1+\psi}/\nu\right)\mathrm{d}x\,,$$

the second line of which is always negative, as $M(\cdot)$ is a convex function. The first line is negative as long as $\tilde{\varphi}_0(r,h) > 1/2$. But we have already argued in Proposition 8 that $\tilde{\varphi}_0(r,h) > \tilde{\varphi}_1(h)$, and in Lemma 10 that there exists lower bound $\bar{s}^m(\pi)$ such that for all $s^m \geq \bar{s}^m(\pi)$ it has to be that $\tilde{\varphi}_1(h) > 1/2$. Moreover, the same Lemma gives that $\bar{s}^m(\pi)$ is an increasing function and it is ease to verify that for bounded $m$

$$\lim_{\pi\to 1} \mathbb{P}(\phi_1(s^m) < 1/2 \,|\, m) = 0. \tag{3.7.42}$$

Hence, indeed $\dfrac{\mathrm{d}\varphi_0(r,s^m)}{\mathrm{d}r}$ is decreasing in $s^m$, from which the second statement of the proposition also follows.

□

***Proof of equation 3.3.11.*** We have:

$$\begin{aligned}\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \;=\; & \mathbb{P}(\varphi^1_1 > \varphi^2_1 \,|\, s^m)\,\mathbb{P}(1,1 \,|\, s^m) \\ & + \mathbb{P}(\varphi^1_1 > \varphi^2_0 \,|\, s^m)\,\mathbb{P}(1,0 \,|\, s^m) + \mathbb{P}(\varphi^1_0 > \varphi^2_1 \,|\, s^m)\,\mathbb{P}(0,1 \,|\, s^m) \\ & + \mathbb{P}(\varphi^1_0 > \varphi^2_0 \,|\, s^m)\,\mathbb{P}(0,0 \,|\, s^m), \end{aligned} \quad (3.7.43)$$

It follows immediately from Lemma 10 that $\varphi^2_1 > \varphi^1_1$. Moreover, Proposition 8 gives that $\varphi^2_0 > \varphi^2_1$, hence we also have that $\varphi^2_0 > \varphi^1_1$. As a result the above becomes

$$\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \;=\; \mathbb{P}(\varphi^1_0 > \varphi^2_1 \,|\, s^m)\,\mathbb{P}(0,1 \,|\, s^m) + \mathbb{P}(\varphi^1_0 > \varphi^2_0 \,|\, s^m)\,\mathbb{P}(0,0 \,|\, s^m), \quad (3.7.44)$$

we can only be certain about the monotonicity of the probability of both managers invest in their idiosyncratic portfolio which is deceasing given large $s^m$. The rest of the terms can not be monotonic as we have observed through simulations.

□

## 3.8 Appendix: Investment and AUM in the Second Period

Here, first we derive the optimal investment decision of a manager in the second period. Second, we use this to calculate her AUM as a function of her posterior reputation, which we later use in order to derive her continuation payoff from period 2. To avoid repetition we consider the extended model in which there are two fund managers. In this the investor's preferences are given by

$$v(i, z^{ij}_t) = \begin{cases} \exp(z^{i1}_t - \bar{z}) \cdot (1 - f^i_t) \cdot R^i_t & , i = 1,2 \\ \exp(m_t) & , i = m \end{cases}$$

Hence, in this case there are two independent preference shocks , one for each fund. The results of the baseline more can be obtained by setting the fees of the second manager equal to one, which will ensure that no investor will invest in her fund.

We solve the second period backwards by first considering the manager's investment decision when the funds have already been allocated. The manager's expected payoff is

$$\mathbb{E}\left[\log\left(A_2^i f_2^i R_2^i\right) \mid s_2^i, s_2^m, \beta_2^i, \alpha\right] \;=\; \log\left(A_2^i f_2^i\right) \,+\, \mathbb{E}\left[r_2^i \mid s_2^i, s_2^m, \beta_2^i, \alpha\right]$$

As a result the manager's objective when choosing her investment strategy $\beta_2^i$ in the second period is to simply maximise the expected return $r_2^i$. Thus, she invests in her alpha only if

$$\mathbb{E}\left[r_2^i \mid s_2^i, s_2^m, \beta_2^i = 0, \alpha\right] \;\geq\; \mathbb{E}\left[r_2^i \mid s_2^i, s_2^m, \beta_2^i = 1, \alpha\right] \tag{3.8.1}$$

It is known that the posterior distributions of $a_2^i$ and $m_2$, after conditioning on $s_2^i$ and $s_2^m$, are also normal distributions with known expected values. Let $\psi = \sigma^2/(\sigma^2+\nu^2)$ and $\psi_m = \sigma_m^2/(\sigma_m^2+\nu_m^2)$. Then (3.8.1) becomes

$$(1-\psi)\cdot\alpha \,+\, \psi\cdot s_2^i \;\geq\; (1-\psi_m)\cdot\mu \,+\, \psi_m\cdot s_2^m\,,$$

which allows us to derive the manager's optimal investment strategy in the second period. This is a cutoff rule such that she invests in her alpha only if $s_2^i \geq c(\alpha, s_2^m)$, where

$$c(\alpha, s_2^m) \;=\; \frac{\psi_m}{\psi}\cdot s_2^m \,+\, \frac{1-\psi_m}{\psi}\cdot\mu \,-\, \frac{1-\psi}{\psi}\cdot\alpha \tag{3.8.2}$$

Thus, for the same market conditions a high type manager invests relatively more frequently on her alpha in the second period, as $c(H, s_2^m) < c(L, s_2^m)$ implies

$$\mathbb{P}[s_2^i \geq c(H, s_2^m)] > \mathbb{P}[s_2^i \geq c(L, s_2^m)] \;\Rightarrow\; \mathbb{P}(\beta_2^i = 0 \mid m_2, \alpha = H) > \mathbb{P}(\beta_2^i = 1 \mid m_2, \alpha = L),$$

where the second line required to infer $s_2^m$ from the realised $m_2$. We will frequently need to condition expectations with respect to $m_t$ instead of $s_t^m$, because we do not have in our data some measure of the latter in our data.

An important point that needs to be made is that the cutoffs $c(\alpha, s_2^m)$ are not the optimal ones for the investors. This is because those are risk-neutral, while the managers are risk-averse. Following the same argumentation as above we can show that the optimal cutoff for the investors is

$$c^*(\alpha, s_2^m) \;=\; c(\alpha, s_2^m) + \frac{\psi_m \sigma_m^2 - \psi \sigma^2}{2\,\psi}. \tag{3.8.3}$$

Thus the investor's optimal cutoff is adjusted by a "risk-loving" factor. For example, suppose that $\psi_m \sigma_m^2 > \psi \sigma^2$, that is investing in the market is relatively more risky conditional on the information that the manager has at her disposal when making the decision. Then an investor would require a higher level of confidence on her alpha $s_2^i$ in order to also agree that relying on it is preferable to 'gambling' with $r_2^m$.

Let $u_2^\alpha$ denote the equilibrium payoff of an investor in the second period, conditional on investing with a manager of type $\alpha$, but net of his preference shock $z_t^{ij}$ and fees $f_2^i$. Then this is given by

$$u_2^\alpha = \mathbb{P}[s_2^i \geq c_\alpha(s_2^m)]\, \mathbb{E}[R_2^i \,|\, s_2^i \geq c_\alpha(s_2^m)] + \mathbb{P}[s_2^i \leq c_\alpha(s_2^m)]\, \mathbb{E}[R_2^i \,|\, s_2^i \leq c_\alpha(s_2^m)], \tag{3.8.4}$$

which has a closed form representation that can be derived using the formulas of the moment generating function of the truncated normal distribution. We avoid providing this here as it does not facilitate the understanding of the model in any meaningful way. However, it is important to point out that when the market's posterior variance $\psi_m \sigma_m^2$ is much bigger than that of the alpha-based strategy $\psi \sigma^2$ then the misalignment between the manager's and the investors' preferences could be so substantial that a low type manager would be preferable simply because she is more reluctant to use her alpha. We exclude that by assuming that $u_2^H > u_2^L$, because if the parameters of the model were such that investing in an index tracking strategy was so attractive, then there would be little need for professional investors.

Let $\varphi^i$ denote the public posterior belief on manager $i$'s ability $\alpha^i$ at the beginning of period two. Then the investor's expected payoff, net of fees and the preferences shock, from opting for fund $i$ is

$$u_2^i = \varphi^i(u_2^H - u_2^L) + u_2^L,$$

and the corresponding actual payoff is $e^{z_t^{ij}}(1 - f_t^i)u_2^i$. In addition, each investor has an outside option, which is to ignore the financial intermediaries and instead invest directly on $m_2$, which gives expected payoff

$$u^m = \mathbb{E}[\exp(m_t)] = e^{\mu + \sigma_m^2/2}.$$

To avoid repetition note that in a manner similar to the one above we can define

$$u_1^i = \pi^i(u_1^H - u_1^L) + u_1^L,$$

as the expected net payoff of an investor active in the first period. However, in this case the functional form of $u_1^\alpha$ will be completely different, as the cutoffs used by the managers in the first period will be influenced by their career concerns. We will derive those under a market equilibrium in the next subsection.

To ensure that when the lowest preference shocks are realised the investor would rather invest directly in the market we will assume that

$$(1 - f_2^i) \cdot u_2^H < u^m \cdot e^{\bar{z}} \tag{3.8.5}$$

We are now ready to derive the AUM of fund $i$ in the beginning of period $t$, as only a function of net expected payoffs and announced fees.

**Lemma 11.** *In any market equilibrium the AUM of fund $i$, competing against fund $k$, in period $t$ is*

$$\left(\frac{(1 - f_t^i)u_t^i}{u^m}\right)^{\lambda^i} \left(1 - \frac{\lambda^i}{\lambda^i + \lambda^k}\left(\frac{(1 - f_t^k)u_t^k}{u^m}\right)^{\lambda^k}\right). \tag{3.8.6}$$

***Proof***. To simplify the algebra drop the investor superscript and time subscripts. Also let $\xi^i = \log(1-f^i)u^i$, $i = 1, 2$ and $\xi^m = \log u^m + \bar{z}$. For an investor to prefer fund 1 to both directly investing in the market and to fund 2, it has to be that

$$\exp(z^1 - \bar{z}) \cdot (1-f^1) \cdot u^1 \geq u^m \Leftrightarrow z^1 \geq \xi^m - \xi^1$$

and

$$\exp(z^1)(1-f^1)u^1 \geq \exp(z^2)(1-f^2)u^2 \Leftrightarrow z^1 + \xi^1 - \xi^2 \geq z^2,$$

respectively. Hence the proportion of the market that fund 1 captures is

$$\begin{aligned}
&\mathbb{P}\left(z^1 \geq \xi^m - \xi^1 \cap z^1 + \xi^1 - \xi^2 \geq z_2\right) \\
&\qquad = \int_{\xi^m-\xi^1}^{\infty} \mathbb{P}\left(z^1 + \xi^1 - \xi^2 \geq z_2 \,\middle|\, z^1\right) \mathrm{d}\mathbb{P}(z^1) \\
&\qquad = \int_{\xi^m-\xi^1}^{\infty} \left(1 - e^{-\lambda^2(z^1+\xi^1-\xi^2)}\right) \lambda^1 e^{-\lambda^1 z^1} \mathrm{d}z^1 \\
&= e^{-\lambda_1(\xi^m-\xi^1)} - e^{-\lambda^2(\xi^1-\xi^2)} \frac{\lambda^1}{\lambda^1+\lambda^2} e^{-(\lambda^1+\lambda^2)(\xi^m-\xi^1)} \\
&= \left(\frac{(1-f^1)u^1}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^1} \cdot \left(1 - \frac{\lambda^1}{\lambda^1+\lambda^2}\left(\frac{(1-f^2)u^2}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^2}\right)
\end{aligned}$$

The proof for fund 2 is equivalent. $\square$

The proof calculates (3.8.6) as the probability of the intersection of two events. The first is that investor $j$ prefers fund $i$ to fund $k$. The second is that fund $i$ is preferred to direct investment in the market.

To obtain the assets for the case where there is only one manager set $f^2 = 1$ to get:

$$\left(\frac{(1-f_t^i) \cdot u_t^i}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^i} \tag{3.8.7}$$

## 3.9 Appendix: Unobservable Investment Decision

***Proof of Lemma 6***. First of all, we will simplify notation by omitting the dependence on $s^m$ both on cutoffs and on expectations. We will follow the proof of Lemma 2 to find the posterior reputation of the manager. In particular, for $\beta = 1$, we have that $r = m + \varepsilon$, hence

$$\Pr\big(r, \beta = 1 \,|\, H, m\big) \;=\; \phi\left(\frac{r-m}{\sigma_\varepsilon}\right)\frac{1}{\sigma_\varepsilon}\Phi\left(\frac{h-H}{\nu}\right) \tag{3.9.1}$$

Moreover, we have

$$\Pr\left(r, \beta = 0 \,|\, H, m\right) = \frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\Phi\left(\frac{r-h(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right). \tag{3.9.2}$$

Hence, we obtain an expression for

$$\Pr(r \,|\, H) \;=\; \Pr\left(\tilde{r} = \frac{r-\beta_0\, m}{1-\beta_0}, \beta_0 \,\middle|\, H\right) + \Pr\big(r, \beta_1 \,|\, H\big) \tag{3.9.3}$$

The expressions for the low type are identical, therefore it is now trivial to use Bayesian updating to derive the posterior reputation of the manager, and complete the proof of this Lemma. □

***Proof of Proposition 12***. We want to investigate if $\phi(r, m, s^m)$ can be always increasing in $r$. From Lemma 3 it is sufficient to see if $\rho$ can always be decreasing in $r$, where, $\rho = \frac{\rho_L}{\rho_H}$. From the previous Lemma we get:

$$\rho = \frac{\Phi\left(\frac{r-l(1+\psi)+L\psi}{\nu\sqrt{1+\psi}}\right)\frac{\phi\left(\frac{r-L}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}} + \Phi\left(\frac{l-L}{\nu}\right)\frac{\phi\left(\frac{r-\mu}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}}{\Phi\left(\frac{r-h(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right)\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}} + \Phi\left(\frac{h-H}{\nu}\right)\frac{\phi\left(\frac{r-\mu}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}} \tag{3.9.4}$$

Firstly, a necessary condition for $\rho$ to be decreasing is: $\nu\sqrt{\psi(1+\psi)} = \sigma_\varepsilon$. After substituting into equation 3.9.4, we get:

$$\rho = \frac{\varepsilon^{A_1 r - C_1}\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + d_1}{\varepsilon^{A_2 r - C_2}\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) + d_2} \tag{3.9.5}$$

where $A_1 = \frac{L-m}{\sigma_\varepsilon^2}, C_1 = \frac{L^2-m^2}{2\sigma_\varepsilon^2}, b_1 = l(1+\psi) - L\psi, d_1 = \Phi\left(\frac{l-L}{\nu}\right)$ and similarly for $A_2, C_2, b_2, d_2$.

Note that $A_1 < A_2$. Then we can take the derivative with respect to $r$, and get the following proportionality:

$$\begin{aligned}
\rho' \propto\ & e^{A_1 r - C_1} e^{A_2 r - C_2}\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)(A_1 - A_2) \\
&+ e^{A_1 r - C_1} e^{A_2 r - C_2}\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)\frac{1}{\nu\sqrt{1+\psi}}\left(M\left(\frac{b_1-r}{\nu\sqrt{1+\psi}}\right) - M\left(\frac{b_2-r}{\nu\sqrt{1+\psi}}\right)\right) \\
&+ d_2\left[\varepsilon^{A_1 r - C_1} A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + \varepsilon^{A_1 r - C_1}\frac{1}{\nu\sqrt{1+\psi}}\phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)\right] \\
&- d_1[\varepsilon^{A_2 r - C_2} A_2 \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) + \varepsilon^{A_2 r - C_2}\frac{1}{\nu\sqrt{1+\psi}}\phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)]
\end{aligned} \tag{3.9.6}$$

Now let $P^*$ denote the first 2 terms of (3.9.6). Then we would want to check whether the derivative of $\rho$ is negative for every $r, m$. We have:

$$\begin{aligned}
\frac{\rho'}{\varepsilon^{A_1 r - C_1}\varepsilon^{A_2 r - C_2}} \propto\ & \frac{P^*}{\varepsilon^{A_1 r - C_1}\varepsilon^{A_2 r - C_2}} + d_2\left[A_1\frac{\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}} + \frac{1}{\nu\sqrt{1+\psi}}\frac{\phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}}\right] \\
&- d_1\left[A_2\frac{\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 - C_1}}\frac{1}{\nu\sqrt{1+\psi}}\frac{\phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 - C_1}}\right]
\end{aligned}$$

We take any $m$ such that $A_1, A_2 < 0$. Intuitively, we consider the case of a good realized market. Then $\frac{P^*}{\varepsilon^{A_1 r - C_1}\varepsilon^{A_2 r - C_2}}$ is finite (as $r \to \infty$) because $\Phi(.) \in [0,1]$ and $M(a) - M(b) \leq a - b$ for $a > b$ (Lemma 4).

We will now show that as $r \to \infty$, the derivative cannot be negative. Indeed, we have that as $\lim_{r\to\infty} \frac{\phi(.)}{e^{A_2 r - C}} = 0$. In addition it is easily shown that, as $r \to +\infty$:

$$\frac{d_2 A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}} - \frac{d_1 A_2 \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 r - C_1}} \sim \frac{d_2 A_1 e^{C_2} e^{r(A_1 - A_2)} - d_1 A_2 e^{C_1}}{e^{r A_1}} \tag{3.9.7}$$

where $\sim$ denotes the asymptotic equivalence of the 2 terms.

We know that $A_1 - A_2 < 0$ so $\lim_{r\to\infty} e^{r(A_1 - A_2)} = 0$, hence in the limit the above expression is asymptotically equivalent to

$$\frac{0 - d_1 A_2 e^{C_1}}{e^r A_1} \tag{3.9.8}$$

Finally, we know that $A_1 < 0$ so $e^r A_1 \to 0$ and therefore the whole expression tends to $+\infty$, since is also $A_2 < 0$.

So we can finally conclude that $\rho'$ cannot always be negative, or in other words, a monotonic equilibrium cannot exist.

$\square$

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